ALIAS-FREE SPECTRAL ESTIMATION OF STOCHASTIC PROCESSES

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ABSTRACT

A scheme for the practical estimation of power spectrum from randomly-timed samples is proposed and investigated for wide-sense stationary stochastic processes. The sampling process $\{t_n\}$ is assumed to be a stationary point process statistically independent of the sampled process $X(t)$. Stationarity of $\{t_n\}$ admits that joint statistics of t_k, t_{k+n} do not depend on k. Closed form analytical formulae are derived for the spectral window $Q_m(f)$ and for $\mathrm{cov}\{\hat{S}(f_r), \hat{S}(f_q)\}$, $\mathrm{var}\{\hat{S}(f_r)\}$ for the particular case of independent identically distributed sampling intervals. Results confirm the alias-free character of the Poisson sampling scheme even for non-bandlimited spectra. It is shown further that for gaussian processes with very smooth spectra Poisson sampling process can yield more reliable estimates (i.e., with a smaller variance) than the well-known method of periodic sampling.

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INTRODUCTION

Power spectral techniques have found many useful applications not only in communication engineering but also in such diverse fields as Astronomy, Meteorology, Structural Dynamics, Oil Exploration and Economics. From measurements of power spectra, it has been possible in the case of linear systems to obtain useful estimates of the stochastic relations generating such time series (see, for example, reference [19]). In turn, identification of linear systems via spectral analysis provides a means of constructive fault-finding and subsequent modification of the design of radio receivers, aircraft and other linear (or approximately linear) systems which are subject to stochastic excitation. In areas of application involving natural systems (which cannot be modified) scientists and engineers have been able by extrapolation to predict responses of those systems to well-defined random excitations. Symbols from a communication source [10], radar echoes from distant planets [12],[13] , swell recordings from distant storms [11], reflections from seismic explosions [19], wind velocities in atmospheric turbulence, and even day-to-day price fluctuation in the stock market are all examples of random signals with which engineers and scientists have to cope. In general, amplitudes of these signals are random and in some cases significant readings of these amplitudes arrive at random times.

For many of the applications mentioned above analog methods of spectral estimation have proved inadequate due, for the most part, to the degree of frequency resolution demanded by modern noise studies. Since the advent of digital computers, discrete-time estimation of power spectra has been widely carried out using the standard procedure of R. B. Blackman and J. W. Tukey [1]. For a certain class of signals, the standard procedure utilizing amplitude readings taken periodically has produced useful results, while for others it has led to erroneous estimates. Of particular concern is the error due to aliasing or unwanted contribution (by "folding over" replicas of the true spectrum) from frequencies which are even multiples of the Nyquist frequency ($f_N = 1/2\Delta t$). The attractive possibility of using randomly-timed samples as a means of eliminating aliasing was suggested by Blackman and Tukey [1] and investigated by H. S. Shapiro and R. A. Silverman [9], F. J. Beutler and O. A. Z. Leneman [2]-[5], all of whom have shown that Poisson sampling is alias-free even for non-bandlimited spectra. Whereas the works cited above have provided the groundwork for this investigation, much-desired processing algorithms utilizing this particular technique together with qualitative analysis are still lacking.

In this thesis, a scheme for the practical estimation of power spectra from randomly-timed samples is outlined and fully investigated for a real wide-sense stationary stochastic process $X(t)$. The scheme requires the sampling process $\{t_n\}$ to be a stationary point process whose statistics are independent of those of $X(t)$. Stationarity of $\{t_n\}$ admits that joint statistics of t_{k+n}, t_k are independent of k. Chapter I provides the basic formulation of the problem together with relevant assumptions on the sampling process. An algorithm for practical estimation of power spectra from non-uniform sampling is outlined briefly. In Chapter II the first-order statistics of the estimator are fully investigated. An analytical expression for the characteristic spectral window is developed in its most general form. Methods of modifying spectral windows are discussed and windows arising from Hanning, Hamming and Barlett modifications are derived analytically. The notion of aliasing is generalized and testing criteria are discussed. Results on covariability and variability of estimates are presented in Chapter III. Closed form expressions of the Blackman-Tukey type are derived for the Poissson sampling scheme. It is further shown that for very smooth spectra, Poisson sampling scheme can achieve a better variance than the method of periodic sampling. For wildly fluctuating spectra, on the other hand, results show that Poisson sampling is equally unreliable.

Chapter I

EMPIRICAL SPECTRAL ESTIMATION

1.1 Statement of the Problem

In the empirical estimation of power spectral density $S_x(f)$ of a real stationary stochastic process $X(t)$, the following procedure is customary:

(i) $X(t)$ is sampled at prescribed times $t_1, t_2 \cdots t_N$.

(ii) A processing algorithm is set up to utilize the sampled data $X(t_1), X(t_2) \cdots X(t_N)$ along with appropriate information on the sample intervals to obtain an estimate $\hat{S}_x(f_r)$ of the noise spectrum at some frequency f_r.

(iii) The estimator $\hat{S}_x(f_r)$ is evaluated and modified appropriately. Evaluation consists in an analysis of the mean and variance of $\hat{S}_x(f_r)$. The need to modify the estimate usually arises during the analysis of the mean of $\hat{S}_x(f_r)$.

Sampling may be done periodically as in the Blackman and Tukey algorithm or in a non-uniform manner as has been suggested and studied in the literature [1]-[5],[9]. In practice, a statistical description can be imposed on these intervals by either (a) sampling the process $X(t)$ according to some well known distribution like Poisson, periodic, rectangular; or (b) observing, in the case of natural phenomena like ocean waves, the arrival times of records and approximating their statistical behavior as best as possible.

The goodness of estimating spectra by methods described above depends to some extent on how accurate the sampling interval statistics have been approximated. In general, one does not need to know the actual sampling times for practical estimation, but a statistical description is required for a qualitative analysis. The scheme to be developed later in this thesis admits of data whose sampling intervals are additively random. Thus, it will be less sensitive to sampling jitter than the procedure of R. B. Blackman and J. W. Tukey in which the sampling intervals are fixed and equal.

The equations fundamental to spectral estimation procedures for a real stationary (wide-sense) stochastic process $X(t)$ are:

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \cos \omega\tau \, d\tau \tag{1.11}$$

$$R_x(\tau) = E\left\{X(t+\tau)\, X(t)\right\} \tag{1.12}$$

where

$S_x(f)$ is the power spectral density

$R_x(\tau)$ is the autocorrelation function of $X(t)$

For ergodic processes ensemble averaging implied in equation 1.12 may be replaced with time averaging defined by

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t+\tau)\, X(t)\, dt \tag{1.13}$$

Equations 1.11 and 1.13 underlie theoretical spectral estimation in continuous time or in discrete time with infinite data. Since empirical spectral estimation is carried out in discrete time with finite data, a great deal of the efforts in the area is directed to developing improved sampling schemes. In what follows, we will develop some processing algorithms for randomly sampled data, modify one of them and evaluate its first and second order statistics.

1.2 Designing Estimation Algorithms for Practical Use

The first step in designing a spectral estimator for a stationary process $X(t)$ which is assumed ergodic is to obtain a suitable numerical approximation to the defining equation:

$$S(f_r) = 2 \int_0^{\infty} \lim_{T \to \infty} \frac{1}{T-\tau} \int_0^{T-\tau} X(t+\tau)\, X(t) \cos 2\pi\, f_r \tau \; dt\, d\tau \tag{1.21}$$

Numerical integration schemes for doing this abound in the literature, varying in complexity from the rectangular and trapezoidal approximations to quadrature formulas of the Lagrangian and Gaussian types (cf. Todd [22]). In this section some estimation algorithms are developed from first and second order integration schemes. The most suitable of these is selected and modified for subsequent evaluation.

1.2.1 Review of Numerical Integration Schemes

Let $f(t)$ be Riemann integrable in the interval $[a,b]$ and let $f(t_n), n = 0,1,2\cdots m$ be samples of $f(t)$ at the times $t_o, t_1 \cdots t_m$. The area defined by the integral

$$I \triangleq \int_a^b f(t)\, dt$$

can be approximated by either of the following sums of rectangles

$$I_F \triangleq \sum_{n=0}^{m-1} f(t_n)(t_{n+1} - t_n) \tag{1.22a}$$

$$I_B \triangleq \sum_{n=1}^{m} f(t_n)(t_n - t_{n-1}) \tag{1.22b}$$

The subscripts on I indicate that the base of corresponding rectangle is obtained by taking the forward step (F) or the backward step (B). A better approximation to the integral is obtained from an average of the last two sums, namely:

$$I_T \triangleq \frac{1}{2} \sum_{n=0}^{m-1} \{f(t_{n+1}) + f(t_n)\} (t_{n+1} - t_n) \tag{1.23}$$

which is none other than the trapezoidal approximation of an integral. To see this, we write out the sum (letting $\alpha_{n+1} \triangleq t_{n+1} - t_n$, $f_n \triangleq f(t_n)$)

$$I_T = \frac{1}{2} \{(f_o + f_1)\alpha_1 + (f_1 + f_2)\alpha_2 + \cdots + (f_{m-1} + f_m)\alpha_m]$$

and rearrange to have

$$I_T = \frac{1}{2} \{f_o\alpha_1 + \sum_{n=1}^{m-1} f_n(\alpha_n + \alpha_{n+1}) + f_m \alpha_m\}$$

Higher order integration schemes result from fitting the sampled data to polynomials of degree less than the number of sample points. Details leading to well known quadrature formulas can be found in texts on numerical analysis (see, for example, Todd [22], pp. 59-61). As a rule polynomial fitting can be regarded as weighting the rectangles in the sums defined by 1.22a, 1.22b.

1.2.2 Algorithms for Estimating from Randomly Sampled Data

Let $X(t_i)$, $i = 0,1,2,\cdots,N$ be samples from a wide-sense stationary stochastic (real) process $\{X(t)\}$ and t_i, $i = 0,1,2,\cdots,N$ be points of the stationary point process $\{t_k\}$ whose properties we will define later. Numerical approximations to the double integral in equation 1.21 will proceed as follows. For simplicity, we will approximate the inner integral as (letting $X_k \triangleq X(t_k)$)

$$\frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)\, X(t)\, dt \cong \frac{1}{(N-n)\alpha}\sum_{k=1}^{N-n} X_{k+n}\, X_k\, \alpha_k \quad , \quad T \equiv N\alpha$$

and apply equations 1.22a, 1.22b and 1.23 to the outer integral to obtain respectively $(\omega_r = 2\pi f_r)$:

$$\hat{S}_F(f_r) = 2\sum_{n=0}^{m-1}\frac{1}{(N-n)\alpha}\sum_{k=1}^{N-n} X_{k+n}\, X_k \cos\omega_r(t_{k+n} - t_k)\alpha_k\ \alpha_{k+n+1} \quad (1.24a)$$

$$\hat{S}_B(f_r) = 2\sum_{n=1}^{m}\frac{1}{(N-n)\alpha}\sum_{k=1}^{N-n} X_{k+n}\, X_k \cos\omega_r(t_{k+n} - t_k)\alpha_k\ \alpha_{k+n} \quad (1.24b)$$

and

$$\begin{aligned}\hat{S}_T(f_r) = {} & \frac{1}{N\alpha}\sum_{k=1}^{N} X_k^2\, \alpha_k^2 \\ & + \sum_{n=1}^{m-1}\frac{1}{(N-n)\alpha}\sum_{k=1}^{N-n} X_{k+n}X_k \cos\omega_r(t_{k+n} - t_k)(\alpha_{k+n} + \alpha_{k+n+1})\alpha_k \\ & + \frac{1}{(N-m)\alpha}\sum_{k=1}^{N-m} X_{k+m}X_k \cos\omega_r(t_{k+m} - t_k)\alpha_{k+m}\ \alpha_k \qquad (1.25)\end{aligned}$$

Each of the above is a valid estimator as far as the integral in 1.21 is concerned and, in fact, $\hat{S}_T(f_r)$ is Blackman and Tukey's estimator for equi-spaced samples. (To see this, put $t_{k+n} - t_k = n\Delta t$,

$\alpha_{k+n} = \alpha_{k+1} = \Delta t$, $t_k = k\Delta t$ in equation 1.25.) We remark in passing that there are infinitely many such valid estimators corresponding to the various integration schemes already mentioned. However, in order to obtain useful averaging filters or spectral windows, as they are commonly referred to in the literature, it is sometimes necessary to modify these estimators somewhat. For example, an earlier pilot analysis showed that using either $\hat{S}_F(f_r)$ or $\hat{S}_T(f_r)$ for general non-uniform sampling schemes is equivalent to averaging with windows which are not integrable. $\hat{S}_B(f_r)$, on the other hand, averages with a Poisson window whose area is zero and whose bandwidth is therefore infinite. Detail discussion on the averaging capability of the spectral windows is presented in Chapter 2.

In general, modification of estimators can be viewed as conveniently weighting the rectangles in the sum of 1.22a, 1.22b. A systematic way of choosing these weights is that provided by higher order schemes of polynomial fitting. (See, for example, Todd, pp. 59-61). Since higher order schemes will usually present analytical difficulties, any choice of weights which does not hinder analysis considerably will be quite acceptable. Further accuracy in the integration scheme can then be achieved by taking samples closer on the average. A simple such modification which we present here for further investigation assigns an approximate weight of 2 to the first rectangle and a weight of 1 to the other rectangles in the sum of equation 1.22b. Thus we have

$$\hat{S}(f_r) = 2 \sum_{n=1}^{m} \frac{1}{(N-n)} \sum_{k=1}^{N-n} X_{k+n} X_k \cos \omega_r (t_{k+n} - t_k)(\alpha_{k+n} + \alpha\delta_{n1})\alpha_k \tag{1.26}$$

where

$$\delta_{n1} = \begin{cases} 1 & , n = 1 \\ 0 & , n \neq 1 \end{cases}$$

The Blackman and Tukey estimator is easily derived from equation 1.26 above, by excluding the term $\alpha\delta_{n1}$ and replacing the sum $\sum_{n-1}^{m}$ with the sum $\mathop{Z}_{n=0}^{m}$ defined by

$$\mathop{Z}_{n=0}^{m} a_n \triangleq \frac{1}{2} \{a_o + 2 \sum_{n=1}^{m-1} a_n + a_m\}$$

1.3 The Sampling Process $\{t_n\}$

The sampling sequence $\{t_n\}$ employed in the scheme outlined above is assumed stationary with independently distributed intervals. Furthermore, the processes $\{t_n\}$ and $\{X(t_n)\}$ are assumed statistically independent. It is implied by the stationarity of the $\{t_n\}$ that the joint statistics of the respective number of points in any set of intervals are invariant under a translation of these intervals. Additional details on the theory of stationary point processes are provided by Beutler and Leneman, Ref. [3]. Summarized below are some relevant properties and assumptions on the sampling sequence $\{t_n\}$.

(i) $t_{\mu+1} - t_\mu \triangleq \alpha_\mu$, $\mu = 1,2,\cdots$ are independent and identically distributed with mean α , common probability density $p(\tau)$, and characteristic function

$$\phi(i\omega) \triangleq E[\exp(-i\omega\alpha\mu)]$$

(ii) $t_{\mu+\nu} \geq t_\mu$, $\nu \geq 0$ with equality iff $\nu = 0$.

(iii) $E[X(t_{\mu+\nu})\, X(t_\mu)] \equiv E_t\{E_x[X(t_{\mu+\nu})\, X(t_\mu)]\}$ with E_t, E_x interchangeable.

(iv) $\phi^*(i\omega) \triangleq \phi(-i\omega)$ where * implies complex conjugate.

Furthermore, $\phi(i\omega)$ satisfies the following:

(v) $|\phi(i\omega)| < |\phi(0)| = 1$, $\omega \neq 0$

(vi) $\lim_{\omega \to \pm\infty} \phi(i\omega) = 0$

The above conditions (v) and (vi) can be derived easily from definition

$$\phi(i\omega) = \int_0^\infty \exp(i\omega\tau)\, p(\tau)\, d\tau$$

and the Riemann-Lebesgue lemma.

Chapter II

FIRST ORDER STATISTICS

2.1 Introduction

In this chapter we proceed to examine the first basic property of the scheme outlined in Section 1.2--the expected value of the estimator defined by equation 1.26. In particular, we will derive a general analytical expression for the spectral window $Q_m(\omega)$ and investigate how certain of its properties affect the resulting spectral estimates. The notion of aliasing for periodic sampling is reviewed and generalized to the non-periodic case. Testing criteria for alias-free estimation simpler than those of Beutler [5], and Shapiro and Silverman [9], are discussed and presented without proof.

2.2 Analytical Derivation of the Spectral Window

To derive an expression for the spectral window we apply the expectation operator

$$E \equiv E_t \, E_x$$

to $\hat{S}(f_r)$ in equation 1.26. Thus,

$$E\{\hat{S}(f_r)\} = 2 \sum_{n=1}^{m} \frac{1}{(N-n)\alpha} \sum_{k=1}^{N-n} E_t \, \{E_x[X(t_{k+n}) \, X(t_k)] \cos \omega_r(t_{k+n} - t_k) \, \alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\} \tag{2.21}$$

Interchange of expectation and summation is justified, since we are dealing with finite sums here. By the stationarity of the sampling process $\{t_n\}$, the summand in equation 2.21 does not depend on k , so that

$$E\{\hat{S}(f_r)\} = \frac{2}{\alpha} \sum_{n=1}^{m} E_t \{E_x[X(t_{k+n})\, X(t_k)]\cos \omega_r(t_{k+n} - t_k)\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\}$$

Next, we take expectation with respect to X and introduce the Wiener-Khinchin relations to have:

$$E\{\hat{S}(f_r)\} = \frac{2}{\alpha} \sum_{n=1}^{m} \int_{-\infty}^{\infty} S(f)\, E_t\{\cos \omega(t_{k+n} - t_k)\cos \omega_r(t_{k+n} - t_k)$$
$$\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\}\, df$$

Now, (Re ≡ "Real part of")

$$E_t\{\cos \omega(t_{k+n} - t_k)\, \cos \omega_r(t_{k+n} - t_k)\, \alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\}$$
$$= \frac{1}{2} E_t \left\{\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\, \mathrm{Re}\left(\exp[i(\omega+\omega_r)(t_{k+n} - t_k)]\right.\right.$$
$$\left.\left. + \exp[i(\omega-\omega_r)(t_{k+n} - t_k)]\right)\right\} = \frac{1}{2}\, [q_n(\omega+\omega_r) + q_n(\omega-\omega_r)]$$

where

$$q_n(\Omega) \triangleq \mathrm{Re}\, E_t\{\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})\, \exp[i\Omega(t_{k+n} - t_k)]\}$$
$$= \alpha \mathrm{Re}\, E_t\, \{\alpha_{k+n}\exp\, (-i\Omega \sum_{\mu=k+1}^{k+n} \alpha_\mu) + \alpha\, \delta_{n1}\exp(-i\Omega\alpha_{k+1})\}$$
$$= \alpha \mathrm{Re}\, \{(\frac{\partial\, \ell n\, \phi}{\partial i\Omega} + \alpha\delta_{n1})\phi^n(\Omega)\}$$

The last equality follows from the assumption that the sampling intervals are independent and identically distributed. Finally, we write

$$E\{\hat{S}(f_r)\} = \int_{-\infty}^{\infty} S(f)\, H_m(\omega;\omega_r)\, df \qquad (2.22)$$

where

$$H_m(\omega;\omega_r) \triangleq \frac{1}{2}\{Q_m(\omega+\omega_r) + Q_m(\omega-\omega_r)\}$$

$$Q_m(\Omega) \triangleq 2\,\mathrm{Re}\{\sum_{n=1}^{m} \phi^n(\Omega)\frac{\partial\,\ell n\,\phi}{\partial i\Omega} + \alpha\phi\} \tag{2.22a}$$

$$= 2\,\mathrm{Re}\,\{\frac{\phi(1-\phi^m)}{1-\phi}\;\frac{\partial\,\ell n\,\phi}{\partial i\Omega} + \alpha\phi\} \tag{2.23}$$

Equation 2.23 gives an analytical expression for the so-called spectral window for non-uniform sampling in which the intervals are independent and identically distributed. For the uniform sampling scheme of Blackman and Tukey, the spectral window takes the form:

$$Q_m^u(\Omega) \triangleq \mathrm{Re}\{\frac{(1+\phi)(1-\phi^m)}{1-\phi}\;\frac{\partial}{\partial i\Omega}\,\ell n\,\phi\} \tag{2.24}$$

2.3 Desired Properties of the Spectral Window

By equation 2.22 $E[\hat{S}(f_r)]$ can be interpreted as a convolution of the true spectral density $S(f_r)$ and the window function $H_m(\omega,\omega_r)$. It is also customary to regard the mean value of $\hat{S}(f_r)$ as a weighted average of the true spectrum over the bandwidth of the spectral window when the latter's main lobe is at f_r. For the purpose of useful interpretation we can rewrite equation 2.22 in the following more general form:

$$E\{\hat{S}(f_r)\} = \frac{\int_{-\infty}^{\infty} S(f)\,H_m(\omega;\omega_r)\,df}{\int_{-\infty}^{\infty} H_m(\omega;\omega_r)\,df} \tag{2.31}$$

Whence it is expected that

$$\text{(i)} \quad Q_m(\omega) \leqq Q_m(0) \quad \text{with equality iff} \quad \omega = 0$$

$$\text{(ii)} \quad \int_{-\infty}^{\infty} Q_m(\omega)\, d\omega = \sigma \ , \qquad 0 < \sigma < \infty \quad , \text{ any } m$$

For absolute convergence of the convolution integral (see, for example, Apostol [25] p. 490), and since $S(f)$ is assumed bounded on $(-\infty,+\infty)$, it is sufficient that

$$\text{(iii)} \quad \int_{-\infty}^{\infty} |Q_m(\omega)|\, d\omega < \infty$$

Finally, it is desirable to have spectral windows $Q_m(\omega)$, indexed on m , which form a defining sequence for the generalized function $\delta(\omega)$ i.e.,

$$\text{(iv)} \quad \lim_{m \to \infty} Q_m(\omega) = \sigma\, \delta(\omega)$$

This last requirement guarantees that $\hat{S}(f_r)$ is asymptotically unbiased. Throughout the preceding it has been taken for granted that $Q_m(\omega)$ is continuous, real and even in ω .

2.3.1 Bandwidth of Spectral Windows

The bandwidth over which averaging of the true spectrum is done plays a major role in the stability of spectral estimates. For example, if the spectral window is "too wide", satisfactory resolution becomes rather difficult to achieve. In particular for spectra with bandwidths of the order of the window bandwidth, estimates at all frequencies will be very highly correlated. We are not concerned, at the moment, with how wide or how narrow the spectral window should be. Discussion on the trade off between resolution and stability will be taken up later. Rather, we are interested in establishing a working definition of bandwidth.

Of the several definitions of bandwidth that abound in the litterature, the one that lends itself to easy calculation is that of Parzen [8]. It is simply that the bandwidth $\beta_m(Q)$ of a spectral window $Q_m(\omega)$ is the base of the rectangle whose height is the peak of $Q_m(\omega)$ and whose area equals that of $Q_m(\omega)$. Symbolically we write:

$$\beta_m(Q) = \frac{\int_{-\infty}^{\infty} Q_m(\omega)\, d\omega}{\max\limits_{\omega} Q_m(\omega)} \tag{2.32}$$

The above definition assumes property (ii) of Section 2.3. Use will be made of equation 2.32 and property (i) of Section 2.3 to compute the bandwidths of periodic and Poisson windows later in this chapter.

2.4 Windows Resulting from Some Common Sampling Densities

In Section 2.2 we derived general analytical formulae for $Q_m(\omega)$ under the assumption that the sampling intervals are independent and identically distributed. By suitably defining the characteristic function $\phi(\omega)$ in equations 2.23 and 2.24, particular expressions can be obtained for spectral windows resulting from any sampling scheme for which our basic assumption is valid. In what follows, we shall derive particular expressions for the periodic, Poisson and rectangular sampling densities, and exhibit some graphical plots as visual aids.

2.4.1 Periodic Sampling: The Blackman-Tukey Window

When sampling is done periodically without jitter as required for the Blackman-Tukey algorithm, we have

$$t_k = k\Delta t \quad , \quad \Delta t \equiv \text{sampling period, with sampling density}$$

$$p_{\alpha_k}(\tau) = \delta(\tau - \Delta t)$$

and characteristic function

$$\phi(i\Omega) = \exp(i\Omega\Delta t)$$

which, with

$$\frac{\partial}{\partial i\Omega} \ln \phi(i\Omega) = \Delta t$$

puts equation 2.24 in the form:

$$Q_m^u(\Omega) = \Delta t \text{ Re } \frac{(1 + \exp(i\Omega\Delta t))\ (1 - \exp(i\Omega m\Delta t))}{1 - \exp(i\Omega\Delta t)}$$

Multiplying the R.H.S. by $\frac{1 - \exp(-i\Omega\Delta t)}{1 - \exp(-i\Omega\Delta t)}$ before taking the real part leads to

$$Q_m^u(\Omega) = \Delta t \sin m\Omega\Delta t \cot \frac{\Omega\Delta t}{2} \tag{2.41}$$

which is Blackman and Tukey's result $Q_o(\omega)$ (cf. [1], p.35).

2.4.2 Poisson Sampling Process: The Poisson Window

Here the sampling times are the occurrence times of events in a Poisson process (e.g., shot noise, radioactive decay). The interval between successive events α_k has the density

$$p_{\alpha_k}(\tau) = \begin{cases} \alpha^{-1} \exp(-\tau/\alpha) , & \tau \geqq 0 \\ 0 , & \tau < 0 \end{cases}$$

and characteristic function

$$\phi(i\Omega) = (1 - i\Omega\alpha)^{-1} \tag{2.41a}$$

where α^{-1} is the mean sampling rate and α the mean sampling interval. The joint density for n successive intervals is

$$p_{\alpha_{k+n},\alpha_{k+n-1},\ldots,\alpha_{k+1}}(\tau) = \frac{\tau^{n-1} \exp(-\tau/\alpha)}{\alpha^n (n-1)!} , \quad \tau \geqq 0$$

from where we have the probability $F(n,\tau)$ that there are n samples in an interval of length τ , given as

$$F(n,\tau) = \frac{\tau^n \exp(-\tau/\alpha)}{\alpha^n \, n!}$$

These last two expressions can be derived easily from the inverse Fourier transform of the joint characteristic function

$$\phi_n(i\Omega) = (1 - i\Omega\alpha)^{-n}$$

Now, from equation 2.41a, we have

$$\frac{\partial}{\partial_{i\Omega}} \ln \phi = \alpha\phi$$

and consequently, (from equation 2.23)

$$Q_m^P(\Omega) = 2\alpha \text{ Re } \{\phi^2(1 - \phi^m)(1 - \phi)^{-1} + \phi\}$$

$$= 2\alpha \text{ Re } \{\phi(1 - \phi^{m+1})(1 - \phi)^{-1}\}$$

Multiply R.H.S. by $(1 - \phi^*)(1 - \phi^*)^{-1}$ to get

$$Q_m^P(\Omega) = 2\alpha \text{ Re } \{\phi(1 - \phi^{m+1})(1 - \phi^*)(1 - 2 \text{ Re } \phi + |\phi|^2)^{-1}\}$$

where $|\phi| \equiv$ magnitude of the complex function ϕ . To simplify, note that we can write

$$\phi = \cos \psi \exp(i\psi)$$

with

$$\psi = \text{arc tan } (\Omega\alpha)$$

so that

$$Q_m^P(\Omega) = 2\alpha \{\cos^{m+2}\psi \sin(m+1)\psi / \sin \psi\} \qquad (2.42)$$

2.4.3 Rectangular Sampling Process: Rectangular Window

As a final example we consider the rectangular sampling process by which sampling intervals are uniformly distributed in $[0,2\alpha]$ with mean α , i.e.,

$$p_{\alpha_k}(\tau) = \begin{cases} 1/2\alpha & 0 \leq \tau \leq 2\alpha \\ 0 & \text{otherwise} \end{cases}$$

The corresponding characteristic function is

$$\phi(\Omega) = (2i\Omega\alpha)^{-1} [\exp(i2\Omega\alpha) - 1]$$

and after carrying out the indicated differentiation,

$$\frac{\partial}{\partial i\Omega} \ell n\ \phi = \alpha[1 + i(\frac{1}{\Omega\alpha} - \cot \Omega\alpha)]$$

whence equation 2.23 gives

$$Q_m^R(\Omega) = 2\alpha \ \mathrm{Re}\left\{(1-\phi)^{-1}\left[\phi(1-\phi) + \phi(1-\phi^m)\ \left(1 + i(\frac{1}{\Omega\alpha} - \cot \Omega\alpha)\right)\right]\right\}$$

The above expression for $Q_m^R(\Omega)$ does not lend itself easily to further analytical investigation. However, we shall use the form

$$Q_m^R(\Omega) = 2\alpha\left\{\frac{\sin 2\Omega\alpha}{2\Omega\alpha} + \sum_{n=1}^{m} (\frac{\sin \Omega\alpha}{\Omega\alpha})^n\ [\cos n\Omega\alpha + \sin n\Omega\alpha(\cot \Omega\alpha - \frac{1}{\Omega\alpha})]\right\} \tag{2.43}$$

for obtaining graphical plot of the rectangular window.

Shown in Figure 1 are the periodic, rectangular and Poisson windows as given in equations 2.41, 2.43 and 2.42 respectively. They are plotted for $m = 25$ and $\alpha = 1$. Based on the figure, one is inclined to deduce that both Poisson and rectangular sampling processes are "alias free" for non-bandlimited spectra, while the periodic process is not. Analytical confirmation of this observation will proceed in subsequent sections for periodic and Poisson windows. As was pointed out earlier, the rectangular window is not tractable analytically.

2.5 Details on the Periodic and Poisson Windows

In what follows we take a closer look at the periodic and Poisson windows (the rectangular window will not be investigated further due to lack of a suitable closed form expression for $Q_m^R(\omega)$).

In particular, we will verify properties (i) - (iv) of Section 2.3 and compute analytically the bandwidth and sidelobes for each of the windows for comparative analysis.

2.5.1 Integrability and Absolute Integrability

(a) Periodic window. To see that $Q_m^u(\omega)$ has repeated major lobes at multiples of $2\pi/\Delta t$, we write

$$Q_m^u(\omega) = \Delta t \left\{1 + 2 \sum_{n=1}^{m-1} \cos n\omega\Delta t + \cos m\omega\Delta t\right\}$$

and apply $\cos \theta = \cos (\theta + 2n\pi)$, $n = 1,2,3,\cdots$ to have

$$Q_m^u(\omega) = Q_m^u(\omega + \frac{2k\pi}{\Delta t}) \tag{2.51}$$

Equation 2.51 suggests restricting the periodic window to the band $|\omega| \leq \pi/\Delta t$ otherwise $Q_m^u(\omega)$ is neither integrable nor absolutely integrable. On the other hand,

$$Q_m^u(\omega) \text{ rect } \frac{\omega}{2\pi/\Delta t}$$

is both integrable and absolutely integrable,

$$\text{rect } \frac{t}{2T} = \begin{cases} 1 & |t| < T \\ 0 & t < T \end{cases}$$

In particular, we have from tables

$$\int_{-\pi/\Delta t}^{\pi/\Delta t} \frac{\sin m\omega \Delta t}{\tan \frac{\omega\Delta t}{2}} d\omega = 2\pi, \qquad \text{(cf Ref. [21], p. 366)}$$

Absolute integrability follows from the finiteness of the integration limits and ordinary integrability demonstrated above. Further, we note

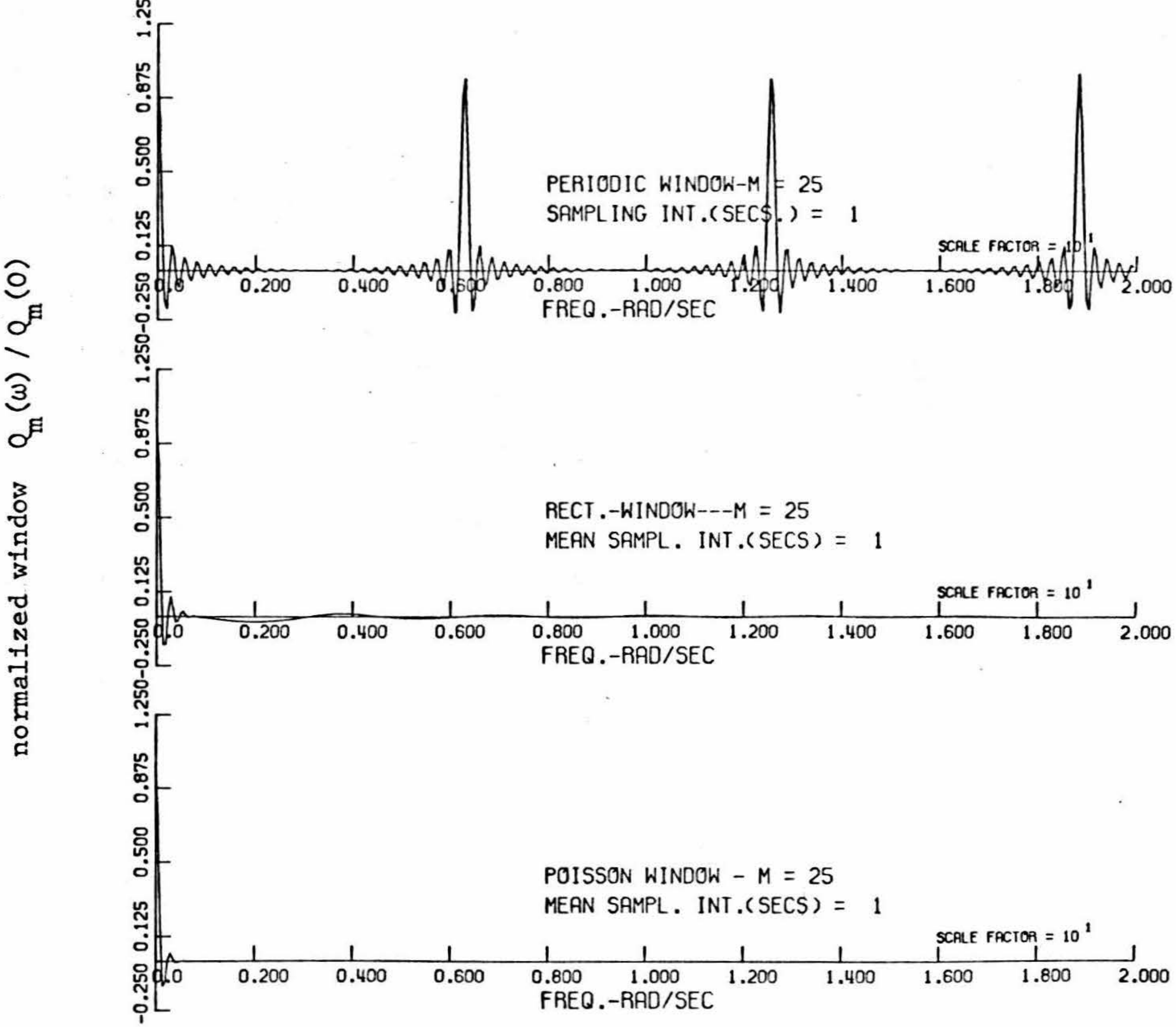

Fig. 1. Spectral Windows Arising from Periodic Rectangular and Poisson Sampling Schemes

that

$$Q_m^u(\omega) \leqq Q_m^u(0) = 2m\Delta t \qquad |\omega| \leq \pi/\Delta t$$

<u>(b) Poisson window.</u> From equation 2.2a, we have

$$Q_m^P(\omega) = 2\alpha \sum_{n=0}^{m} \cos^{n+1}\psi \cos(n+1)\psi, \qquad \psi = \tan^{-1}\omega\alpha$$

whence it follows that

$$Q_m^P(\omega) \leqq Q_m^P(0) = 2(m+1)\alpha$$

Further, $Q_m^P(\omega)$ is integrable, since

$$\int_{-\infty}^{\infty} Q_m^P(\omega)\,d\omega = 2\int_{-\pi/2}^{\pi/2} \frac{\cos^m\psi \sin(m+1)\psi}{\sin\psi}\,d\psi$$

$$= 2\pi \qquad \text{(cf. [21], p.377)}$$

It follows too that $Q_m^P(\omega)$ is absolutely integrable over any finite interval. Now, using the result

$$\lim_{\omega\to\infty} Q_m^P(\omega) = 0(1/\omega^6)$$

derived in the appendix, we can write for large L

$$\int_{-\infty}^{\infty} |Q_m^P(\omega)|\,d\omega = \int_{-L}^{L} |Q_m^P(\omega)|\,d\omega + 0\left(\int_{-\infty}^{-L} \frac{d\omega}{\omega^6} + \int_{L}^{\infty} \frac{d\omega}{\omega^6}\right)$$

$$= \int_{-L}^{L} |Q_m^P(\omega)|\,d\omega + 0(1/L^5)$$

whence it follows that $Q_m^P(\omega)$ is absolutely integrable.

2.5.2 Behavior of $Q_m^u(\omega)$, $Q_m^P(\omega)$ for Large m

The main interest here is to see whether or not the window functions approach the delta function as m increases without bound. For our purpose it will be sufficient if for $F(\omega)$ continuous, bounded and absolutely integrable,

$$\lim_{m \to \infty} \int_{-\infty}^{\infty} Q_m(\omega)\; F(\omega)d\omega = \sigma\; F(0)$$

where σ is the area under $Q_m(\omega)$.

(a) Consider the integral

$$I_u = \int_{-\pi/\Delta t}^{\pi/\Delta t} Q_m^u(\omega)\; F(\omega)\; d\omega$$ and assume that $F(\omega)$ is "good" in

the sense described above. Then

$$I_u = \int_{-\pi/\Delta t}^{\pi/\Delta t} \sin m\omega\,\Delta t \cot \frac{\omega\Delta t}{2}\; F(\omega)\; d\omega$$

$$= \int_{-m\pi}^{m\pi} F(y/m)\; \frac{\sin y}{\tan(y/2m)}\; \frac{dy}{m}$$

i.e.,

$$\lim_{m \to \infty} I_u = 2F(0) \int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy = 2\pi F(0)$$

Consequently,

$$\lim_{m \to \infty} Q_m^u(\omega) \;\mathrm{rect}\; \frac{\omega}{2\pi/\Delta t} = 2\pi\delta(\omega)$$

(b) In a similar manner

$$\int_{-\infty}^{\infty} Q_m^P(\omega)\ F(\omega)\ d\omega$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos^m\psi\ \sin(m+1)\psi}{\sin\ \psi}\ F(\tan\ \psi)\ d\psi$$

$$= 2 \int_{-(m+1)\frac{\pi}{2}}^{(m+1)\frac{\pi}{2}} \frac{\cos^m(\frac{y}{m+1})\ \sin\ y}{\sin\ \frac{y}{m+1}}\ F(\tan\ \frac{y}{m+1})\ \frac{dy}{m+1}$$

$$\lim_{m \to \infty} \text{RHS} = 2 \int_{-\infty}^{\infty} \frac{\sin\ y}{y}\ F(0)\ dy = 2\pi\ F(0)$$

which implies that

$$\lim_{m \to \infty} Q_m^P(\omega) = 2\pi\delta(\omega)$$

Additional evidence is provided by the observation that the major peaks of $Q_m^u(\omega)$, $Q_m^P(\omega)$ given respectively by $2m\alpha$, $2(m+1)\alpha$ tend to infinity with m . Also the bandwidths obtained from equation 2.32 as $1/2m\Delta t$ cps, $1/2(m+1)\alpha$ cps respectively, tend to zero with increasing m .

2.5.3 Computation of Side Lobes

(a) Periodic. $Q_m^u(\omega) = \Delta t\ \sin\ m\omega\ \Delta t\ \cot\ \frac{\omega\Delta t}{2}$

has zeros near $\omega_k = k\pi/m\Delta t$, $k=1,2,3$ so that its side lobes have peaks near

$$\omega_p = (2p+1)\ \frac{\pi}{2m\Delta t}\ ,\quad p = 1,2,\cdots$$

given approximately by

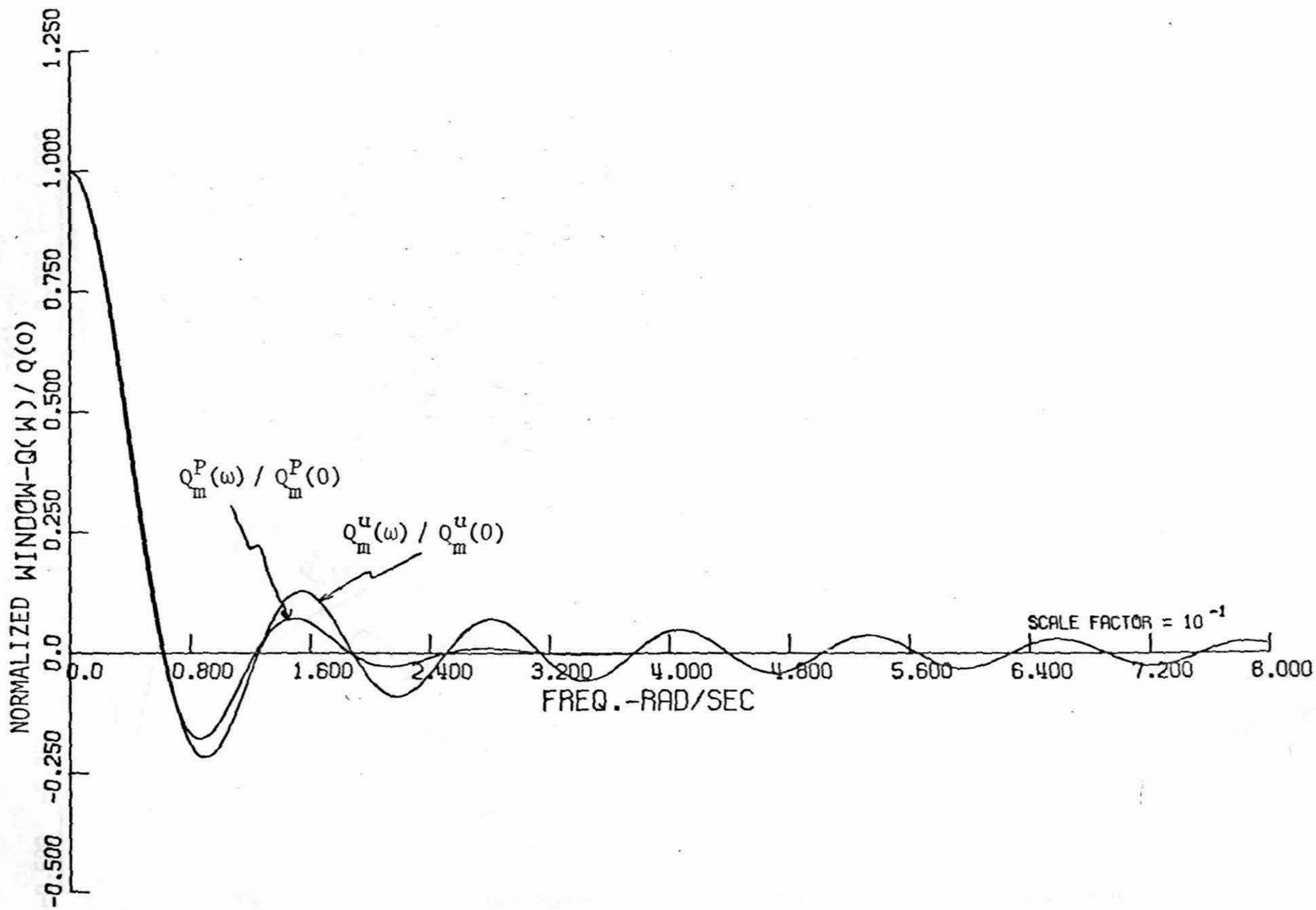

Fig. 2. Periodic and Poisson Spectral Windows for $m = 50$, $\alpha = 1$

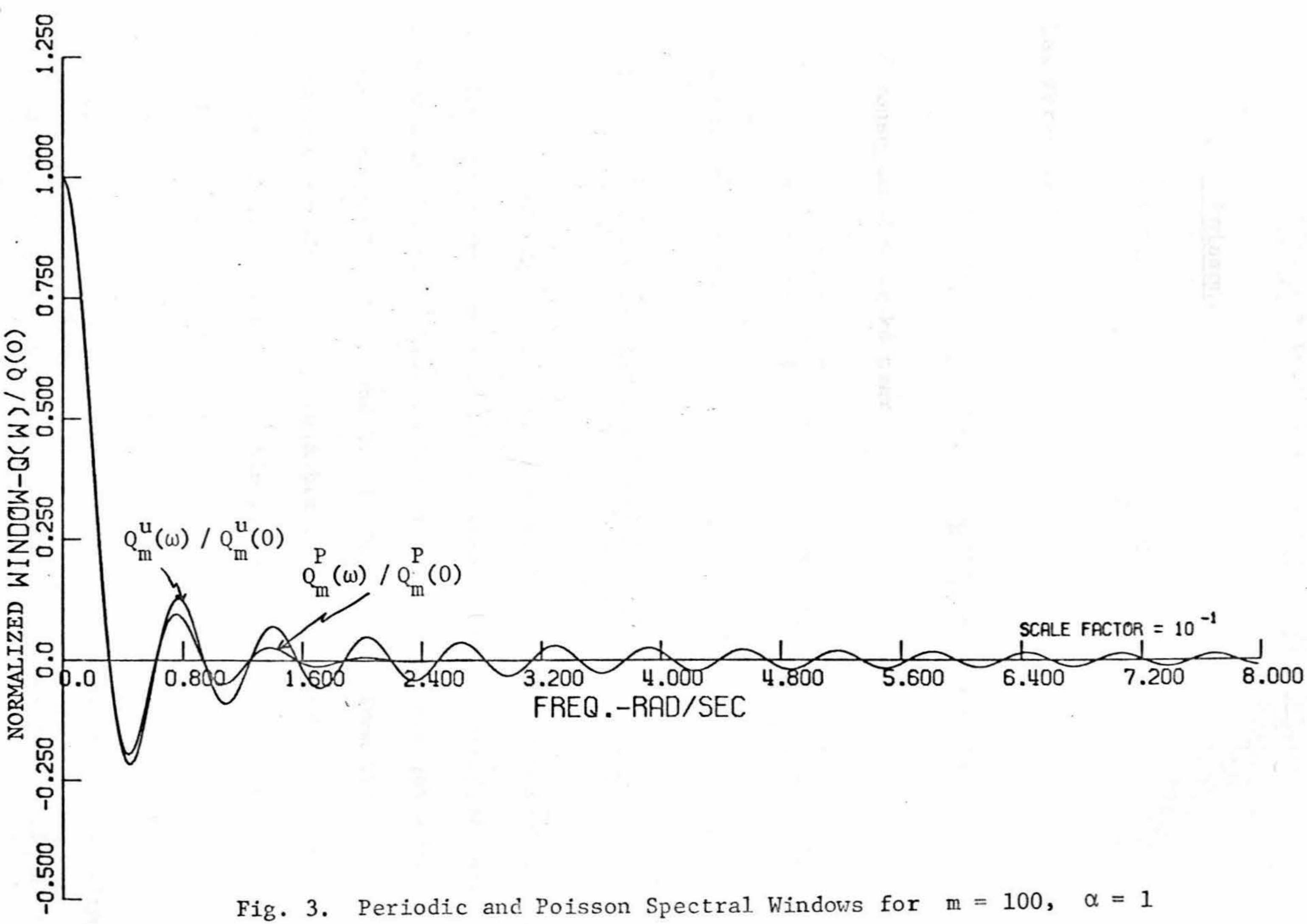

Fig. 3. Periodic and Poisson Spectral Windows for $m = 100$, $\alpha = 1$

$$Q_m^u(\omega_p) \stackrel{\sim}{=} (-1)^p \Delta t \ \text{cotan}[(\frac{2p+1}{2m\Delta t}) \frac{\pi}{2}] \tag{2.52}$$

(b) Poisson.

$$Q_m^p(\omega) = \frac{2\alpha \cos^{m+2}\psi \sin(m+1)\psi}{\sin \psi}$$

has zeros near

$$\psi_k = k\pi/m+1 \qquad \text{or} \qquad \omega_k = \frac{1}{\alpha} \tan(k\pi/m+1),\ k=1,2,\cdots$$

and consequently, peaks near

$$\psi_p = (\frac{2p+1}{2(m+1)}) \frac{\pi}{2} \qquad \text{or} \qquad \omega_p = \frac{1}{\alpha} \tan(\frac{2p+1}{m+1}) \frac{\pi}{2},\ p=1,2,\cdots$$

approximately equal to

$$Q_m^p(\omega_p) \stackrel{\sim}{=} 2\alpha(-1)^p \cos^{m+2}(\frac{2p+1}{m+1}) \frac{\pi}{2} \Big/ \sin(\frac{2p+1}{m+1}) \frac{\pi}{2} \tag{2.53}$$

Using equations 2.52, 2.53 we have computed some coordinates for the first three side lobes (corresponding to $p=1,2,3$) and $m = 50,100,\cdots,300$. Results for $m = 50,100$ agree quite well with those shown in Figures 2 and 3. It is observed from Table 1 that side lobes move closer to the origin but do not decrease with increasing m as is expected. Rather, the side lobes increase with m but do not exceed a fixed fraction (namely, about 1/5 for the first side lobe) of the main lobe. Figures 2 and 3 lead to the conclusion that side lobe contributions are less pronounced with Poisson sampling than with periodic sampling. However, this discrepancy disappears if m is sufficiently large. Finally, we remark that the results of this section confirm the known result that Poisson sampling is alias free even

(a) First Side-lobe p = 1

m	ω_1^n	$Q_m^u(\omega_1)$	ω_1^P	$Q_m^P(\omega_1)$
50	0.0942	-0.21205	0.0927	-0.16945
100	0.0471	-0.21217	0.0467	-0.18980
150	0.0314	-0.21219	0.0312	-0.19702
200	0.0235	-0.21220	0.0234	-0.20072
250	0.0188	-0.21220	0.0188	-0.20297
300	0.0156	-0.21220	0.0156	-0.20449

(b) Second side-lobe p = 2

m	ω_2^u	$Q_m^u(\omega_2^u)$	ω_2^P	$Q_m^P(\omega_2^P)$
50	0.1570	0.12706	0.1552	0.06761
100	0.0784	0.12726	0.0779	0.09329
150	0.0522	0.12729	0.0521	0.10357
200	0.0392	0.12731	0.0391	0.10908
250	0.0314	0.12731	0.0313	0.11252
300	0.0261	0.12732	0.0261	0.11486

(c) Third Side-lobe p = 3

m	ω_3^u	$Q_m^u(\omega_3^u)$	ω_3^P	$Q_m^P(\omega_3^P)$
50	0.2191	-0.09058	0.2190	-0.02567
100	0.1099	-0.09085	0.1093	-0.04929
150	0.0732	-0.09090	0.0729	-0.06062
200	0.0549	-0.09092	0.0548	-0.06714
250	0.0439	-0.09093	0.0438	-0.07136
300	0.0366	-0.09093	0.0365	-0.07432

Table 1. Some Coordinates for the first three side-lobes.

for non-bandlimited spectra (c.f., Beutler [5], Shapiro and Silverman [9]).

2.6 Aliasing of Spectral Estimates

In the practical estimation of spectra from uniformly sampled data, errors have been known to occur in the estimates due to the periodic nature of the sampling scheme. These errors come as unwanted addition of estimates of the true spectrum at certain integral multiples of the Nyquist frequency ($f_n = 1/2\Delta t$, Δt = sampling interval). Furthermore, estimates at these multiples of the Nyquist frequency are indistinguishable from one another. For bandlimited spectra, this problem, often referred to as aliasing, is easily surmounted by taking samples at or above the Nyquist rate. For non-bandlimited spectra, random sampling schemes (namely, the Poisson sampling scheme) have been found to reduce or even eliminate aliasing effects. In this section we derive the basic concept of aliasing for periodic sampling directly from the corresponding window function $Q_m^u(f)$ and generalize to nonuniform sampling schemes.

To this end, let us rewrite equation 2.22 as:

$$E\,\{\hat{S}(f_r)\} = \int_0^\infty S(f)\, Q_m(f - f_r)\, df \tag{2.61}$$

and assume for the moment that $S(f)$ is non-bandlimited. With uniform sampling interval we showed in Section 2.5.1 that $Q_m(f)$ is periodic with period $f_p = 1/\Delta t$ cps. Consequently, we may write

$$Q_m^u(f) = Q_m^u(f)\ \mathrm{rect}\,\frac{f}{1/\Delta t} \circledast \sum_{\nu=-\infty}^{\infty} \delta(f + \nu f_p) \tag{2.62}$$

where ⊛ defines the convolution integral. Now, using 2.62 in 2.61 and recalling (from Section 2.5.2) that

$$\lim_{m \to \infty} \{ Q_m^u(f) \text{ rect } \frac{f}{1/\Delta t} \} = 2\pi\delta(f)$$

we get

$$\lim_{m \to \infty} E\{\hat{S}(f_r)\} = 2\pi \sum_{\nu=-\infty}^{\infty} S(f_r + \nu f_p) \tag{2.63}$$

and

$$\lim_{m \to \infty} E\{\hat{S}(f_r + kf_p)\} = 2\pi \sum_{\nu=-\infty+k}^{\infty+k} S(f_r + \nu f_p) \tag{2.64}$$

Equations 2.63, 2.64 together define aliasing for uniform sampling scheme. Both equations show that on the average, estimates at f_r, $f_r + kf_p$ $k=1,2,\cdots$ are indistinguishable. Each estimate $\hat{S}(f_r)$ is the true spectral density $S(f_r)$ at the frequency of interest f_r, plus magnitudes of $S(f)$ at $f_r + \nu f_p$ $\nu=1,2,\cdots$. The latter interpretation becomes obvious when we write equation 2.63 (leaving out the 2π for convenience) as:

$$\lim_{m \to \infty} E\{\hat{S}(f_r)\} = S(f_r) + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} S(f_r + \nu f_p)$$

To recover $S(f_r)$ from $\hat{S}(f_r)$ with periodic sampling, the above equations suggest that we make f_p large enough (by sampling closer together) so that $S(f)$ is zero outside $(-f_p, f_p)$. For bandlimited spectra this is possible to within the capability of the sampling equipment. However, for non-bandlimited spectra nonuniform sampling patterns must be used, since it is practically impossible to sample uniformly at an infinite rate.

Definition 1: An estimate is said to be aliased in the ordinary sense iff for some integer k and sampling interval Δt

$$\lim_{m \to \infty} E\left|\hat{S}(f_r) - \hat{S}(f_r + k/\Delta t)\right| = 0$$

In terms of definition 1 above, a sufficient condition for estimates to be aliased in the ordinary sense is that the spectral window $Q_m(\omega)$ be a periodic function of ω, with period $\omega_p = 2\pi/\Delta t$.

2.7 Modification of Spectral Windows

One peculiarity of the estimator under investigation is the term $\alpha\delta_{n1}$ which was included to insure that the resulting averaging filter $Q_m(\omega)$ is integrable in $(-\infty,\infty)$. As we have shown earlier, integrability is only one of the essential features of a usable averaging window. The need for modification of this type did not arise for the periodic sampling, since the latter scheme is restricted to bandlimited spectra by which integrability of the spectral windows is virtually guaranteed. All efforts in the past have been directed primarily at de-emphasizing side lobes of windows and forcing $Q_m(\omega)$ to become positive definite--a necessary condition for realizability of filters and other transfer functions. Tailoring of the spectral window in this manner is usually accomplished by weighting the mean-lagged products with an even, positive definite function. Several weighting functions bearing the names of their innovators abound in the literature for the periodic sampling. In this section we extend these tailoring techniques to the nonuniform sampling scheme and examine in detail the effects on the Poisson window of Hann's and Bartlett's weighting functions.

A general modification of the scheme under study can be written as:

$$\hat{S}(f_r) \triangleq 2 \sum_{n=1}^{m} \frac{1}{N-n} \sum_{k=1}^{N-n} D_\nu(t_{k+n} - t_k)\, X(t_{k+n})\, X(t_k) \cos \omega_r (t_{k+n} - t_k) \times \alpha_k(\alpha_{k+n} + \alpha\delta_{n1}) \tag{2.71}$$

where

$$D_\nu(\tau) = \begin{cases} D_\nu(-\tau) & |\tau| < T_m \\ 0 & |\tau| \geqq T_m \end{cases}$$

By appropriately defining $D_\nu(\tau)$, the resulting modified window can be derived analytically as was done in Section 2.2. To illustrate, we will derive the Poisson-Hanning and Poisson-Bartlett windows.

2.7.1 Application of the Bartlett Function to Nonperiodic Sampling Scheme

The weighting function suggested by Bartlett is given as

$$D_B(\tau) = \begin{cases} 1 - |\tau|/T_m & |\tau| < T_m \\ 0 & |\tau| \geqq T_m \end{cases}$$

and the corresponding spectral estimator is

$$\hat{S}_B(f_n) \triangleq 2 \sum_{n=1}^{m} \frac{1}{N-n} \sum_{k=1}^{N-n} \left(1 - \frac{t_{k+n} - t_k}{m\,\alpha}\right) X(t_{k+n})\, X(t_k) \times \cos \omega_r (t_{k+n} - t_k)\, \alpha_k(\alpha_{k+n} + \alpha\delta_{n1}) \tag{2.72}$$

Proceeding as before, we get

$$E\,\{\hat{S}(f_r)\} = 2\sum_{n=1}^{m}\int_{-\infty}^{\infty} S(f)\,E_t\left\{(1-\frac{t_{k+n}-t_k}{m\alpha})\cos\omega_r(t_{k+n}-t_k)\right.$$

$$\left.\times\ \cos\omega(t_{k+1}-t_k)\alpha_k(\alpha_{k+n}+\alpha\delta_{n1})\right\}\,df$$

$$= \int_{-\infty}^{\infty} S(f)\,H_m^B(\omega;\omega_r)\,df$$

with

$$H_m^B(\omega;\omega_r) = \frac{1}{2}\,\{Q_m^B(\omega+\omega_r) + Q_m^B(\omega-\omega_r)\}$$

$$Q_m^B(\Omega) \quad \stackrel{\Delta}{=} \quad Q_m(\Omega) - Q^B(\Omega)$$

where $Q_m(\Omega)$ is the unmodified window derived in Section 2.2 and

$$Q^B(\omega) = \frac{2}{m\alpha}\sum_{n=1}^{m} q_n^B(\Omega)$$

with

$$q_n^B(\Omega) \stackrel{\Delta}{=} E_t\,\{\,\alpha_k(\alpha_{k+n}+\alpha\delta_{n1})(t_{k+n}-t_k)\ \mathrm{Re}[\exp i\Omega(t_{k+n}-t_k)]\,\}$$

$$= E_t\left\{\mathrm{Re}\left[\alpha_k(\alpha_{k+n}+\alpha\delta_{n1})(\sum_{k+1}^{k+n}\alpha_\mu)\exp(i\Omega\sum_{\mu=k+1}^{k+n}\alpha_\mu)\right]\right\}$$

Taking expectations as indicated, and noting that

$$E_t\ \{\alpha_k\ e^{i\Omega\alpha_k}\} = \frac{\partial}{\partial i\Omega}\ \phi(i\Omega)\ ,\quad E_t\{e^{i\Omega\alpha_k}\} = \phi(i\Omega)$$

and

$$E_t\ \{\alpha_k^2\ e^{i\Omega\alpha_k}\} = \frac{\partial^2}{\partial i\Omega^2}\ \phi(i\Omega)$$

it can be shown that

$$q_n^B(\Omega) = \alpha\left\{\frac{\partial^2\phi}{\partial i\Omega^2}\ \phi^{n-1}(i\Omega) + (n-1)\left[\left(\frac{\partial\phi}{\partial i\Omega}\right)^2\ \phi^{n-2}(i\Omega)\right] + \alpha\delta_{n1}\ \frac{\partial\phi}{\partial i\Omega}\right\} \tag{2.73}$$

Finally, we have the Bartlett window arising from nonuniform sampling given by

$$Q_m^B(\Omega) = 2\ \text{Re}\ \sum_{n=1}^{m}\ \{\phi^{n-1}\ \frac{\partial\phi}{\partial i\Omega} + \alpha\delta_{n1}\phi - \frac{1}{m\alpha}\ q_n^B(\Omega)\} \tag{2.74}$$

<u>The Poisson-Bartlett Window</u>

With Poisson sampling intervals,

$$\phi(i\Omega) = (1 - i\Omega\alpha)^{-1}\quad ,\quad \frac{\partial\phi}{\partial i\Omega} = \alpha\phi^2(i\Omega),\quad \frac{\partial^2\phi}{\partial i\Omega^2} = 2\alpha^2\phi^3(i\Omega)$$

so that the Poisson-Bartlett window can be written as (substitute above into 2.73, 2.74):

$$Q_m^{PB}(\Omega) = 2\alpha\ \text{Re}\{\ \sum_{n=0}^{m}\ \phi^{n+1} - \frac{\alpha}{m}\ \sum_{n=0}^{m}\ (n+1)\ \phi^{n+2}(i\Omega) \tag{2.74a}$$

$$= 2\alpha\ \text{Re}\{\ \sum_{n=0}^{m}\ (1 - \frac{n\alpha}{m})\ \phi^{n+1} - (\frac{m+1}{m})\alpha\ \phi^{n+2}\}$$

Now the second summation on the R.H.S. of 2.74a can be carried out by rewriting it as

$$S_m = \sum_{n=1}^{m+1} n\phi^{n+1}$$

and noting that

$$\phi S_m = \sum_{n=2}^{m+2} (n-1)\ \phi^{n+1}$$

Subtracting the latter from the former, and manipulating accordingly, we obtain

$$S_m = (1 - \phi)^{-2}\ \{\phi^3(1 - \phi^{m+1}) + \phi^2(1 - \phi) - (m+2)\phi^{m+3}\ (1 - \phi)\}$$

Next, substitute

$$\phi(i\Omega) = \cos\psi\, e^{i\psi} \quad ; \quad \psi = \tan^{-1}\Omega\alpha$$

and take real parts as before, to obtain

$$Q_m^{PB}(\Omega) = Q_m^{P}(\Omega) - (m \sin^2\psi)^{-1}\ \{2\alpha^2\cos^{m+3}\psi\ [(m+2)\ \cos(m+1)\psi$$

$$- (m+1)\ \cos\ \psi\ \cos(m+2)\psi] - 2\alpha^2\ \cos^2\psi\ \}$$

$$= (m \sin^2\psi)^{-1}\ \{2\alpha\ \cos^{m+2}\psi\ [m\ \sin(m+1)\psi\ \sin\psi - \alpha(m+2)\cos(m+1)\psi\ \cos\ \psi$$

$$+ \alpha(m+1)\ \cos(m+2)\ \psi\cos^2\psi] + 2\alpha^2\cos^2\psi\} \qquad (2.75)$$

Shown graphically in Figures 4 and 5 are the normalized Poisson window, $Q_m^P(\omega)/Q_m^P(0)$ and its Bartlett modification $Q_m^{PB}(\omega)$ / $Q_m^{PB}(0)$ for $m = 50$ and $m = 100$ respectively. The main peaks are respectively, $Q_m^P(0) = 2(m+1)\alpha$ and

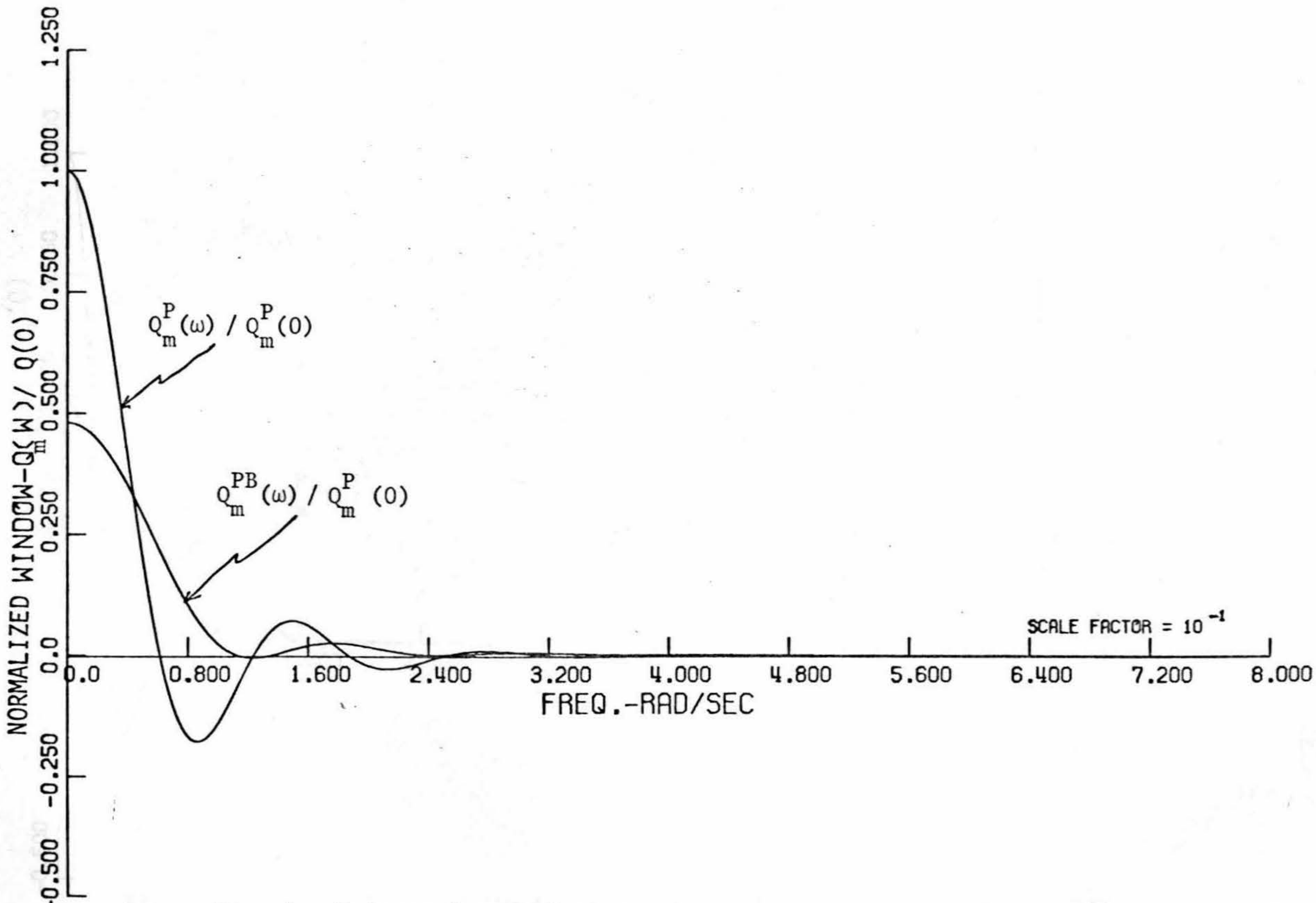

Fig. 4. Poisson Spectral Window and Its Bartlett Modification $m = 50$, $\alpha = 1$

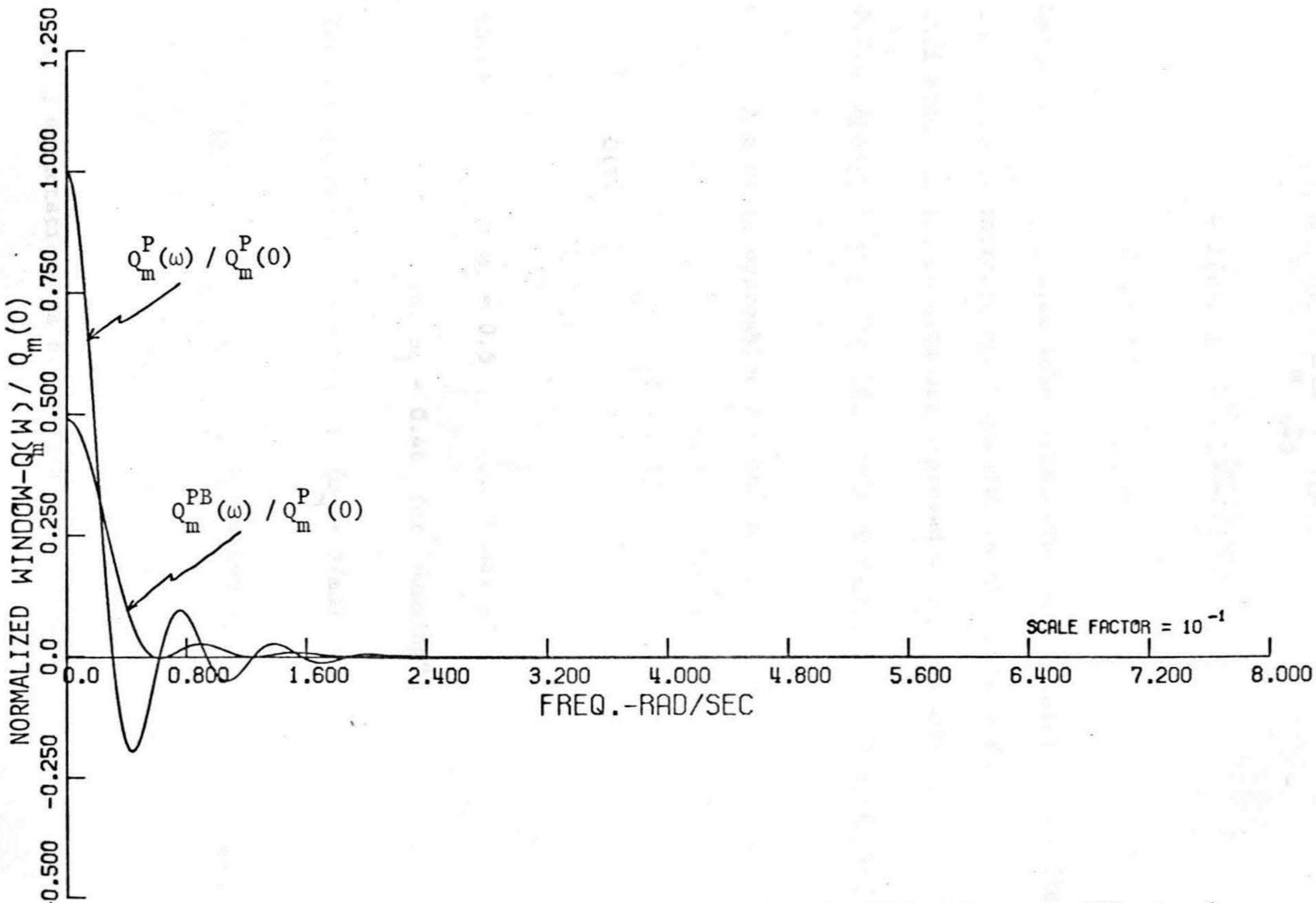

Fig. 5. Poisson Spectral Window and Its Bartlett Modification, $m = 100$, $\alpha = 1$

$$Q_m^{PB}(0) = Q_m^P(0) - \frac{2\alpha^2}{m} \sum_{n=0}^{m} (n+1)$$

$$= 2(m+1)\alpha - \frac{2\alpha^2}{m} [\frac{(m+1)(m+2)}{2}]$$

$$= 2(m+1)\ \alpha[1 - \frac{m+2}{2m}\alpha]$$

While reducing the side lobes considerably, the Bartlett modification tends also to increase the bandwidth as the plots in Figures 4 and 5 indicate. Again, results are improved with increased m .

2.7.2 Application of Hanning, Hamming Functions to Nonperiodic Sampling Schemes

A general expression for the weighting function of the type proposed by Julius Von Hann and R. W. Hamming can be written as:

$$D(\tau) = \begin{cases} a_o + a_1 \cos \pi\ \tau/T_m & |\tau| < T_m \\ 0 & |\tau| \geq T_m \end{cases}$$

where $a_o = a_1 = 0.5$ for "Hanning"

$a_o = 0.54,\ a_1 = 0.46$ for "Hamming"

The corresponding estimator is $(\omega_m = \pi/m\alpha)$

$$\hat{S}_{hh}(f_r) = 2 \cdot \sum_{n=1}^{m} \frac{1}{N-n} \sum_{k=1}^{N-n} (a_o + a_1 \cos \omega_m(t_{k+n} - t_k))\ X(t_{k+n})$$

$$X(t_k) \cos \omega_r(t_{k+n} - t_k)\ \alpha_k(\alpha_{k+n} + \alpha\delta_{n1})$$

Taking expectations as before and noting that

$$E_t\{\cos \omega_m(t_{k+n}-t_k)\cos \omega(t_{k+n}-t_k)\cos \omega_r(t_{k+n}-t_k)\}$$

$$= \frac{1}{4}\ [\cos(\omega+\omega_r+\omega_m)(t_{k+n}-t_k) + \cos(\omega+\omega_r-\omega_m)(t_{k+n}-t_k)$$

$$+ \cos(\omega-\omega_r+\omega_m)(t_{k+n}-t_k) + \cos(\omega-\omega_r-\omega_m)(t_{k+n}-t_k)]$$

we have

$$Q_m^H(\omega) = a_o Q_m(\omega) + \frac{1}{2} a_1 [Q_m(\omega+\omega_m) + Q_m(\omega-\omega_m)] \tag{2.73}$$

where $Q_m(\omega)$ has been defined in equations 2.22a and 2.23.

The Poisson-Hanning Window

Application of the Hanning function to the Poisson scheme leads to the Poisson-Hanning window defined as:

$$Q_m^{HP}(\omega) = \frac{1}{2} Q_m^P(\omega) + \frac{1}{4}[Q_m^P(\omega+\omega_m) + Q_m^P(\omega-\omega_m)]$$

where as before

$$Q_m^P(\omega) = \frac{2\alpha \cos^{m+2}\psi \sin(m+1)\psi}{\sin\psi} \quad , \ \psi = \tan^{-1}\omega\alpha$$

Further,

$$Q_m^{HP}(0) = \frac{1}{2}[Q_m^P(0) + Q_m^P(\pi/m\alpha)]$$

$$\cong m\alpha \qquad\qquad \text{for large } m$$

since for $\psi \cong \pi/m$ and very large m, $\cos^{m+2}\pi/m \cong 1$ and $\sin(m+1)\pi/m = -\sin \pi/m$.

The plots of $Q_m^{HP}(\omega)/Q_m^{HP}(0)$, $Q_m^P(\omega)/Q_m^P(0)$ shown in Figures 6 and 7 for m = 50, 100 respectively, show a similarity between the Bartlett effect and the Hanning effect on the Poisson scheme. The Hanning modification leads to smaller side lobes but larger bandwidth than the Bartlett modification.

2.8 Aliasing: Extension to Nonuniform Sampling and Criteria for Alias-Free Estimation

In references [4] and [9] the authors have laid down some criteria for a sampling process to lead to alias-free estimates. Beutler [5] defines alias-free sampling in terms of the capability to recover the true spectrum of $x(t)$ from the "correlation sequence" $r(n)$ via

$$\hat{S}(f_r) = \sum_{n=1}^{N} C_{nN}(f_r)\ r(n) \tag{2.81}$$

where

$$\begin{aligned} r(n) &= E\left\{X(t_{k+n})\ X(t_k)\right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\ \phi_n^*(i\omega)\ d\omega \end{aligned} \tag{2.82}$$

According to Beutler, alias-free recovery of spectra from correlation sequence requires that $\hat{S}(f_r)$ be uniquely defined by equation 2.81. Equation 2.82 relates spectral recoverability from $r(n)$ to the sampling process by means of the joint characteristic function $\phi_n^*(i\omega)$ of n successive intervals. From 2.82 it is inferred that sampling is alias-free if a one-to-one mapping between $r(n)$ and $S(\omega)$ is implied by equation 2.82. Some criteria for alias-free sampling have been laid down by the authors mentioned above. Whereas Shapiro and Silverman restrict their findings to spectra which are square

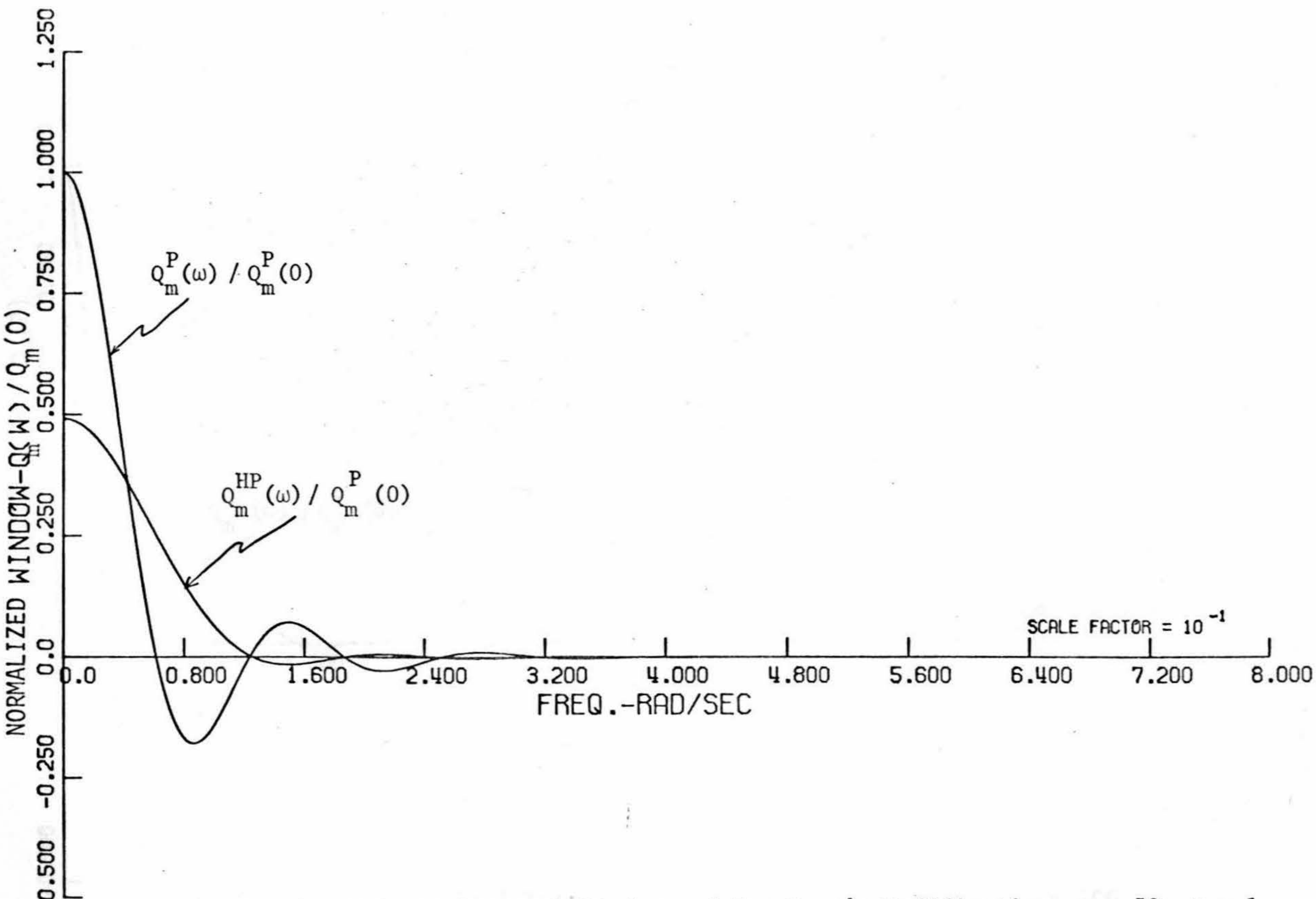

Fig. 6. Poisson Spectral Window and Its Hann's Modification, $m = 50$, $\alpha = 1$

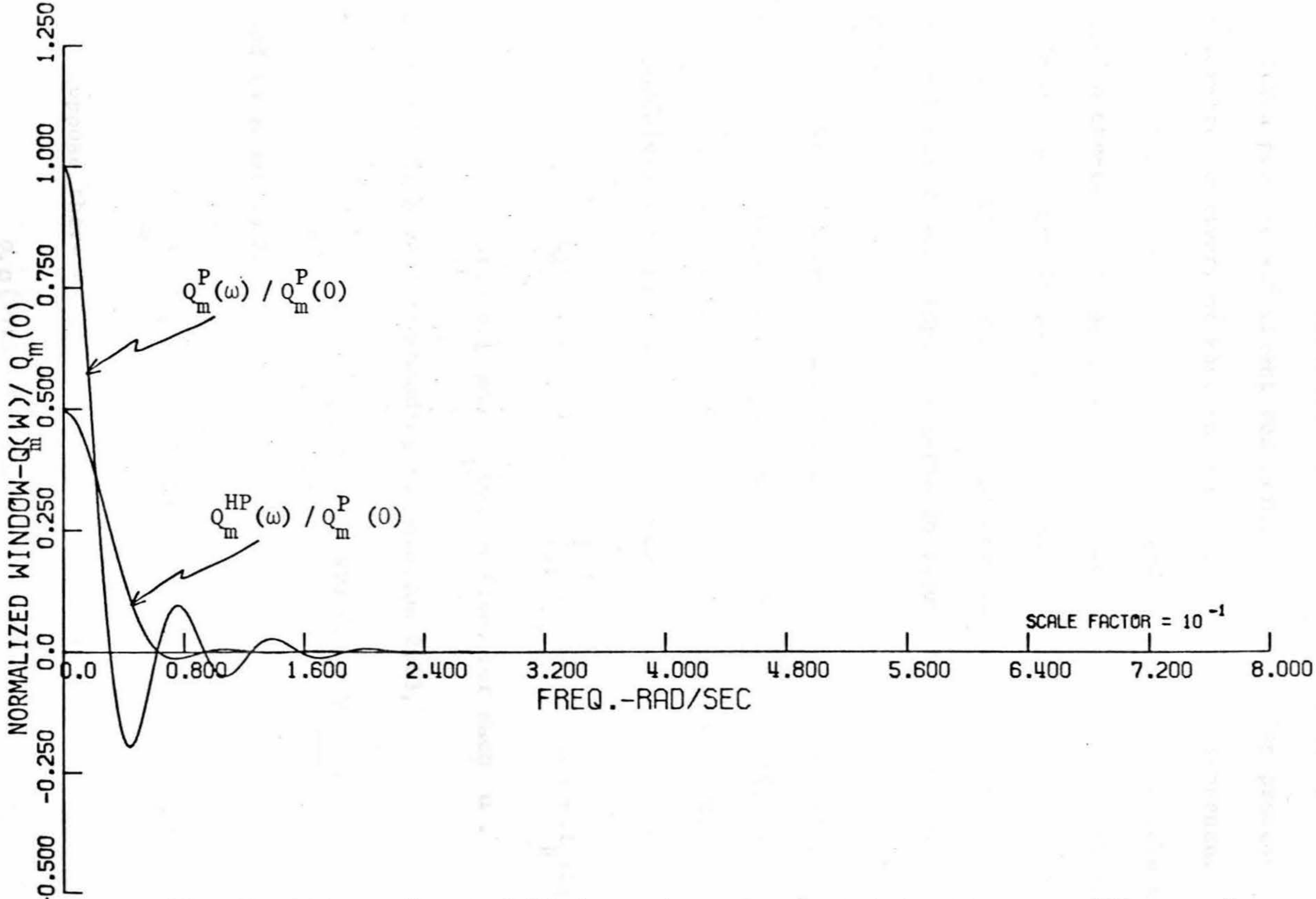

Fig. 7. Poisson Spectral Window and Its Hann's Modification, $m = 100$, $\alpha = 1$

integrable, Beutler considers a more general class of spectra. Their criteria provide sufficient conditions for a sampling process to yield alias-free recovery of spectra from the correlation sequence. In other words, they provide conditions on $\phi_n(i\omega)$ sufficient to make equation 2.82 a one-to-one mapping of $r(n)$ into $S(\omega)$. Further details on this can be found in the cited references. In what follows we shall define alias-free estimation of spectra from the sampled data $X(t_n)$ and establish some simple criteria in terms of the spectral window $Q_m(f)$.

We refer to Section 2.6 and note that, whereas equations 2.63, 2.64 together imply that on the average $\hat{S}(f_r)$ and $\hat{S}(f_r + kf_p)$ are identical, this does not necessarily hold in general. For example, it is conceivable to have a spectral window function $Q_m(f)$ of the form:

$$Q_m(f) = Q_m(f) \text{ rect } \frac{f}{f_p(\alpha)} \circledast \sum_{\nu=-\infty}^{\infty} \rho(\nu)\ \delta(f + \nu f_p(\alpha)) \tag{2.83}$$

where $\rho(\nu) \neq 1$ and $f_p(\alpha)$ is fixed for each α .

Then we will have corresponding to equation 2.63,

$$\lim_{m \to \infty} E\left\{\hat{S}(f_r)\right\} = \sum_{\nu=-\infty}^{\infty} \rho(\nu)\ S(f_r + \nu f_p(\alpha)) \tag{2.84}$$

and to equation 2.64

$$\lim_{m \to \infty} E\left\{\hat{S}(f_r + kf_p(\alpha))\right\} = \sum_{\nu=-\infty+k}^{\infty+k} \rho(\nu+k) S(f_r + \nu f_p(\alpha)) \tag{2.85}$$

Now suppose there exist constants σ_k such that

$$\sigma_k \rho(\nu+k) = \rho(\nu)$$

then it follows that

$$\lim_{m \to \infty} E\left\{\sigma_k \hat{S}[f_r + kf_p(\alpha)]\right\} = \sum_{\nu=-\infty+k}^{\infty+k} \rho(\nu)\ \hat{S}[f_r + kf_p(\alpha)] \tag{2.86}$$

Thus we have

Definition 2: A spectral estimate is said to be aliased in the generalized sense, if for some integer k and average sampling interval α, there exist σ_k, $f_p(\alpha)$ such that

$$\lim_{m \to \infty} E\left|\hat{S}(f_r) - \sigma_k S(f_r + kf_p(\alpha))\right| = 0$$

The above generalizes Definition 1 to include spectra which are identical to within a multiplicative constant. Thus a sufficient condition for aliasing in the generalized sense is that the spectral window $Q_m(f)$ exhibits main lobes at $f_k = kf_p(\alpha)$; $k = \cdot, \cdot\cdot, -2, -1, 0, 1, 2, \cdots$.

Finally we have:

An estimation scheme is said to be alias-free with respect to non-bandlimited spectra iff the characteristic window has one and only one main lobe at $f = 0$.

Chapter III

VARIABILITY AND COVARIABILITY

3.1 Introduction

In the last chapter we investigated the first-order properties of our estimation scheme. One consequence of the subsequent analysis is that the question of whether an estimate is aliased or not can be answered directly from the characteristic spectral window $Q_m(\omega)$. At the first order level there is also the question of bias. Because of the finiteness of data, $\hat{S}(f_r)$ given in equation 1.26 is necessarily biased. However, it can be shown from property (iv) of Section 2.3 that $\hat{S}(f_r)$ is asymptotically unbiased, i.e.,

$$\lim_{m \to \infty} E\{\hat{S}(f_r)\} = S(f_r)$$

Thus far, the quality of our estimation scheme has not been well related to the amount of data N used. It has, however, been related through m , where m is chosen to be always less than N . In the last chapter we found that for certain classes of spectra (band-limited for periodic, non-bandlimited for Poisson sampling) estimates, $\hat{S}(f_r)$ on the average approach the true spectra $S(f_r)$ as m increases without bound. We will see later in this chapter that unless N is increased accordingly, the stability of estimates will be adversely affected. Consequently the pertinent questions to be investigated in this chapter are:

(a) How close to the true value is the estimate available from a finite data size, N ; and what is the covariability of the estimates

$\hat{S}(f_r), \hat{S}(f_q)$?

(b) What is the effect of increased data on the quality of our estimation scheme?

In particular, we will derive analytical formulae for $\mathrm{cov}\{\hat{S}(f_r), \hat{S}(f_q)\}$ and $\mathrm{var}\{\hat{S}(f_r)\}$.

3.2 Analytical Derivation of $\mathrm{cov}[\hat{S}(f_r), \hat{S}(f_q)]$, $\mathrm{var}\{\hat{S}(f_r)\}$

To derive a formula for the covariance of two spectral estimates $\hat{S}(f_r), \hat{S}(f_q)$ obtained from the same record via equation 1.26, we will assume that an effective length Ne, of data has been selected. Details on how to choose Ne are given in [1] (see for example, [1], p.102) but for our purposes here it suffices to point out that $N - m < Ne < N$. Also, we shall make use of

$$\mathrm{cov}\ \{X_1X_2X_3X_4\} = E[X_1X_2]\ E[X_3X_4] + E[X_1X_4]\ E[X_2X_3] \tag{3.21}$$

Equation 3.21 assumes that X_1, X_2, X_3, X_4 are joint gaussian variates with zero means and is derived in Parzen [23], pp. 92-93.

For analytical convenience we shall make use of the following approximately equivalent form of equation 1.26

$$\hat{S}(f_r) = \frac{2}{Ne\alpha} \sum_{n=1}^{m} \sum_{k=1}^{Ne} X(t_k - \frac{\tau_{kn}}{2})\ X(t_k + \frac{\tau_{kn}}{2})\ \cos(\omega_r \tau_{kn}) \alpha_k (\alpha_{k+n} + \alpha\delta_{n1})$$

$$\omega_r = 2\pi f_r\ ,\quad \tau_{kn} \triangleq t_{k+n} - t_k$$

to write

$$\mathrm{cov}\{\hat{S}(f_r),\hat{S}(f_q)\} = \frac{4}{Ne^2\alpha^2}\sum_{j,n=1}^{m}\sum_{i,k=1}^{Ne} E_t\{\mathrm{cov}[X(t_k-\frac{\tau_{kn}}{2})\, X(t_k+\frac{\tau_{kn}}{2}),$$

$$X(t_i-\frac{\tau_{ij}}{2})\, X(t_i+\frac{\tau_{ij}}{2})] \cos\omega_r\tau_{kn}\cos(\omega_q\tau_{ij})\alpha_i\alpha_k(\alpha_{i+j}+\alpha\delta_{j1})$$

$$\times\ (\alpha_{k+n}+\alpha\delta_{n1})\} \tag{3.22}$$

Now, assume that $X(t_k)$'s are gaussian with zero means and apply equation 3.21 to get:

$$\mathrm{cov}[X(t_k-\frac{\tau_{kn}}{2})\, X(t_k+\frac{\tau_{kn}}{2})\ ,\ X(t_i-\frac{\tau_{ij}}{2})\, X(t_i+\frac{\tau_{ij}}{2})]$$

$$= R_x[t_i-t_k-\tfrac{1}{2}(\tau_{ij}-\tau_{kn})]\, R_x[t_i-t_k+\tfrac{1}{2}(\tau_{ij}-\tau_{kn})]$$

$$+ R_x[t_i-t_k-\tfrac{1}{2}(\tau_{ij}+\tau_{kn})]\, R_x[t_i-t_k+\tfrac{1}{2}(\tau_{ij}+\tau_{kn})]$$

where $R_x(\tau) = E_x(X(t+\tau)\, X(t))$ is the autocorrelation function of the process $X(t)$. Next substitute the Wiener-Khinchin relations to get

$$\mathrm{RHS} = \iint_{-\infty}^{\infty} S(f_1)\, S(f_2)\, \{\cos\omega_1[t_i-t_k-\tfrac{1}{2}(\tau_{ij}-\tau_{kn})]$$

$$\cos\omega_2[t_i-t_k+\tfrac{1}{2}(\tau_{ij}-\tau_{kn})] + \cos\omega_1[t_i-t_k-\tfrac{1}{2}(\tau_{ij}+\tau_{kn})]$$

$$\cos\omega_2[t_i-t_k+\tfrac{1}{2}(\tau_{ij}+\tau_{kn})]\}\ \ df_1df_2$$

which, upon expanding the cosine functions, becomes

$$\frac{1}{2}\iint_{-\infty}^{\infty} S(f_1)\, S(f_2)\, \{\cos[(\omega_1+\omega_2)(t_i-t_k)-\tfrac{1}{2}(\omega_1-\omega_2)(\tau_{ij}-\tau_{kn})]$$

$$+\cos[(\omega_1-\omega_2)(t_i-t_k)-\tfrac{1}{2}(\omega_1+\omega_2)(\tau_{ij}-\tau_{kn})]$$

$$+\cos[(\omega_1+\omega_2)(t_i-t_k)-\tfrac{1}{2}(\omega_1-\omega_2)(\tau_{ij}+\tau_{kn})]$$

$$+ \cos[(\omega_1 - \omega_2)(t_i - t_k) - \tfrac{1}{2}(\omega_1 + \omega_2)(\tau_{ij} + \tau_{kn})] \} \, df_1 df_2$$

The last result suggests the following substitution $\omega_1 = \omega' + \omega$, $\omega_2 = \omega' - \omega$ and consequently $df_1 df_2 = 2 df df'$, whereby

$$\mathrm{cov}[X(t_k - \frac{\tau_{kn}}{2}) X(t_k + \frac{\tau_{kn}}{2}) \, , \; X(t_i - \frac{\tau_{ij}}{2}) \, X(t_i + \frac{\tau_{ij}}{2})]$$

$$= \iint_{-\infty}^{\infty} S(f+f') \, S(f-f') \, \{\cos[2\omega'(t_i - t_k) - \omega(\tau_{ij} - \tau_{kn})]$$

$$+ \cos[2\omega(t_i - t_k) - \omega'(\tau_{ij} - \tau_{kn})]$$

$$+ \cos[2\omega'(t_i - t_k) - \omega(\tau_{ij} + \tau_{kn})]$$

$$+ \cos[2\omega(t_i - t_k) - \omega'(\tau_{ij} + \tau_{kn})]\} \, df \, df' \tag{3.23}$$

Using equation 3.23 we can now write an expression for the covariability of two spectral density estimates $\hat{S}(f_r)$, $\hat{S}(f_q)$ in the form:

$$\mathrm{cov}\{\hat{S}(f_n) \, \hat{S}(f_q)\} = 4 \iint_{-\infty}^{\infty} S(f' + f) \, S(f' - f) \, \{\Lambda_1(\omega', \omega, \omega_r, \omega_q)$$

$$+ \Lambda_2(\omega', \omega, \omega_r, \omega_q)\} \, df \, df'$$

where

$$\Lambda_1(\omega', \omega, \omega_r, \omega_q) \triangleq \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t \Big\{ \Big(\cos[2\omega'(t_i - t_k) - \omega(\tau_{ij} - \tau_{kn})]$$

$$+ \cos[2\omega(t_i - t_k) - \omega'(\tau_{ij} - \tau_{kn})] \Big) \, (\cos \omega_r \tau_{kn} \cos \omega_q \tau_{ij})$$

$$\alpha_i \alpha_k (\alpha_{i+j} + \alpha\delta_{j1})(\alpha_{k+n} + \alpha\delta_{n1}) \Big\} \tag{3.24}$$

and

$$\Lambda_2(\omega', \omega; \omega_r, \omega_q) \triangleq \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t \, \{ (\cos[2\omega'(t_i - t_k) - \omega(\tau_{ij} + \tau_{kn})]$$

$$+ \cos[2\omega(t_i - t_k) - \omega'(\tau_{ij} + \tau_{kn})]\big)(\cos\omega_r\tau_{kn}\cos\omega_q\tau_{ij})\alpha_i\alpha_k(\alpha_{i+j} + \alpha\delta_{j1})$$

$$\times \quad (\alpha_{k+n} + \alpha\delta_{n1})\} \qquad (3.24a)$$

Again, we expand the cosine functions using $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$ to rewrite defining equations 3.24, 3.24a compactly as:

$$\Lambda_\mu(\omega', \omega, \omega_r, \omega_q)$$

$$= \frac{1}{4}\{\lambda_\mu(\omega', \omega+\omega_r, \omega+\omega_q) + \lambda_\mu(\omega', \omega-\omega_r, \omega+\omega_q)$$

$$+ \lambda_\mu(\omega', \omega+\omega_r, \omega-\omega_q) + \lambda_\mu(\omega', \omega-\omega_r, \omega-\omega_q)$$

$$+ \lambda_\mu(\omega, \omega'+\omega_r, \omega'+\omega_q) + \lambda_\mu(\omega, \omega'-\omega_r, \omega'+\omega_q)$$

$$+ \lambda_\mu(\omega, \omega'+\omega_r, \omega-\omega_q) + \lambda_\mu(\omega, \omega'-\omega_r, \omega'-\omega_q)\} \quad , \quad \mu = 1,2$$

$$\lambda_1(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$\triangleq \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t\{\alpha_i\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})(\alpha_{i+j} + \alpha\delta_{j1}) \cos[2\omega'(t_i - t_k)$$

$$+ (\omega+\omega_r)\tau_{kn} - (\omega+\omega_q)\tau_{ij}]\}$$

$$\lambda_2(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$\triangleq \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t\{\alpha_i\alpha_k(\alpha_{k+n} + \alpha\delta_{n1})(\alpha_{i+j} + \alpha\delta_{j1}) \cos[2\omega'(t_i - t_k)$$

$$- (\omega+\omega_r)\tau_{kn} - (\omega+\omega_q)\tau_{ij}]\}$$

Finally, we have

$$\lambda_1(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$= \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t\{\alpha_i\alpha_k\alpha_{k+n}\alpha_{i+j} \cos[2\omega'(t_1-t_k) + (\omega+\omega_r)\tau_{kn}$$

$$- (\omega+\omega_q)\tau_{ij}] + \alpha_i\alpha_k\alpha_{k+n}\alpha \cos[2\omega'(t_i-t_k) + (\omega+\omega_r)\tau_{kn} - (\omega+\omega_q)\alpha_{i+1}]$$

$$+ \alpha_i\alpha_k\alpha_{i+j}\alpha \cos[2\omega'(t_i-t_k) + (\omega+\omega_r)\alpha_{k+1} - (\omega+\omega_q) \tau_{ij}]$$

$$+ \alpha_i\alpha_k \alpha^2 \cos[2\omega'(t_i-t_k) + (\omega+\omega_r)\alpha_{k+1} - (\omega+\omega_q) \alpha_{i+1}] \} \qquad (3.25)$$

$$\lambda_2(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$= \frac{1}{Ne^2\alpha^2} \sum_{i,j,k,n} E_t\{\alpha_i\alpha_k\alpha_{k+n}\alpha_{i+j} \cos[2\omega'(t_i-t_k) - (\omega+\omega_r)\tau_{kn}$$

$$- (\omega+\omega_q)\tau_{ij}] + \alpha_i\alpha_k\alpha_{k+n}\alpha \cos[2\omega'(t_i-t_k) - (\omega+\omega_r)\tau_{kn} - (\omega+\omega_q)\alpha_{i+1}]$$

$$+ \alpha_i\alpha_k\alpha_{i+j}\alpha \cos[2\omega'(t_i-t_k) - (\omega+\omega_r)\alpha_{k+1} - (\omega+\omega_q) \tau_{ij}]$$

$$+ \alpha_i\alpha_k\alpha^2 \cos[2\omega'(t_i-t_k) - (\omega+\omega_r)\alpha_{k+1} - (\omega+\omega_q)\alpha_{i+1}]\} \qquad (3.25a)$$

In taking expectation as indicated in equations 3.25, 3.25a, overlapping of the intervals t_i-t_k, $t_{k+n}-t_k$, $t_{i+j}-t_j$ must be taken into account. One way to get around this is to consider a given permutation of the times t_i, t_k, t_{i+j}, t_{k+n} and break up the intervals t_i-t_k, $t_{i+j}-t_i$, $t_{k+n}-t_k$ into non-overlapping intervals so that we can use the assumed independence of disjoint intervals. Subject to the requirement that the sampling instants form an ordered sequence, it is shown in the appendix that there are only six permutations of the times t_i, t_k, t_{k+n}, t_{i+j} , namely:

(i) $t_{i+j} > t_i \geq t_{k+n} > t_k$

(ii) $t_{k+n} > t_k \geq t_{i+j} > t_i$

(iii) $t_{i+j} \geq t_{k+n} > t_i > t_k$

(iv) $t_{k+n} > t_{i+j} > t_k > t_i$

(v) $t_{k+n} > t_{i+j} > t_i \geq t_k$

(vi) $t_{i+j} \geq t_{k+n} > t_k \geq t_i$

Now for each permutation given above, we can proceed to take expectation as stated earlier. This is done in detail in the appendix for all six permutations. The results are summarized in the following expression for the covariability of spectral estimates.

$$\text{cov}\left\{\hat{S}(\omega_r)\ \hat{S}(\omega_q)\right\}$$

$$= \iint_{-\infty}^{\infty} S(f'+f)\ S(f'-f)\ \left\{\Lambda(\omega',\omega;\omega_r,\omega_q) + \Lambda(\omega,\omega',\omega_r,\omega_q)\right\}\ df\ df'$$

where

$$\Lambda(\omega,\omega',\omega_r,\omega_q)$$

$$= \frac{1}{4}\left\{\lambda(\omega',\omega+\omega_r,\omega+\omega_q) + \lambda(\omega',\omega+\omega_r,\ \omega-\omega_q)\right.$$

$$\left. + \lambda(\omega',\ \omega-\omega_r,\ \omega+\omega_q) + \lambda(\omega',\ \omega-\omega_r,\ \omega-\omega_q)\right\}$$

$$\lambda(\omega',\ \omega+\omega_r,\ \omega+\omega_q)$$

$$= \frac{1}{N_e^2\ \alpha^2}\mathrm{Re}\Bigg\{\sum_{R_1(i,j,k,n)} \xi_{i-k-n}(2\omega')\left(\xi_j^*(\omega+\omega_q) + \alpha\delta_{j1}\ \Phi^*(\omega+\omega_q)\right) \cdot$$

$$[\xi_n(2\omega'+\omega+\omega_r) + \xi_n(2\omega'-\omega-\omega_r) + \alpha\delta_{n1}\left(\Phi(2\omega'+\omega+\omega_r) + \Phi(2\omega'-\omega-\omega_r)\right)]$$

$$+ \sum_{R_2(k,j,k,n)} \xi^*_{k-i-j}(2\omega')\left(\xi_j^*(2\omega'+\omega+\omega_q) + \alpha\delta_{j1}\ \Phi(2\omega'+\omega+\omega_q)\right)\ [\xi_n(\omega+\omega_r)$$

$$+ \xi_n^*(\omega+\omega_r) + \alpha\delta_{n1}\left(\Phi(\omega+\omega_r) + \Phi^*(\omega+\omega_r)\right]$$

$$+ \sum_{R_3(i,j,k,n)} \left(\xi^*_{i+j-k-n}(\omega+\omega_q) + \alpha\delta_{j1}\Phi^{i-k-n+1}(\omega+\omega_q)\right)\ [\xi_{i-k}(2\omega'+\omega+\omega_r) \cdot$$

$$\left(\xi_{k+n-i}(\omega_r-\omega_q) + \alpha\delta_{n1}\Phi^{k-i+1}(\omega_r-\omega_q)\right) + \xi_{i-k}(2\omega'-\omega-\omega_r)\left(\xi^*_{k+n-1}(2\omega+\omega_r+\omega_q)\right.$$

$$\left. + \alpha\delta_{n1}\Phi^{*k-i+1}(2\omega+\omega_r+\omega_q)\right)]$$

$$+ \sum_{R_4(i,j,k,n)} [\xi_{i-k}(2\omega'+\omega+\omega_r)\left(\xi_{k+n-i-j}(\omega+\omega_r) + \alpha\delta_{n1}\ \Phi^{k-i-j+1}(\omega+\omega_r)\right)$$
$$\times\ \left(\xi_j(\omega_r-\omega_q) + \alpha\delta_{j1}\ \Phi(\omega_r-\omega_q)\right)$$

$$+ \xi_{i-k}(2\omega'-\omega-\omega_r)\left(\xi^*_{k+n-i-j}(\omega+\omega_r) + \alpha\delta_{n1}\ \Phi^{*k-i-j+1}(\omega+\omega_r)\right)\left(\xi_j^*(2\omega+\omega_r+\omega_q)\right.$$

$$\left. + \alpha\delta_{j1}\ \Phi^*(2\omega+\omega_r+\omega_q)\right)] + \sum_{R_5(i,j,k,n} \xi^*_{k-i}(2\omega'+\omega+\omega_q)\ [\left(\xi_{k+n-i-j}(\omega+\omega_r)\right.$$

$$\left. + \alpha\delta_{n1}\ \Phi^{k-i-j+1}(\omega+\omega_r)\right)\ \xi_{i+j-k}(\omega_r-\omega_q) + \alpha\delta_{j1}\Phi^{i-k+1}(\omega_r-\omega_q))$$

$$+ \left(\xi^*_{k+n-i-j}(\omega+\omega_r) + \alpha\delta_{n1}\ \Phi^{*k-i-j+1}(\omega+\omega_r)\right)\left(\ \xi^*_{i+j-k}(2\omega+\omega_r+\omega_q)\right.$$

$$\left. + \alpha\delta_{j1}\ \Phi^{*i-k+1}(2\omega+\omega_r+\omega_q)\right)\ ]$$

$$+ \sum_{R_6(i,j,k,n)} \xi^*_{k-i}(2\omega'+\omega+\omega_q)\left(\xi^*_{i+j-k-n}(\omega+\omega_q) + \alpha\delta_{j1}\Phi^{i-k-n+1}(\omega+\omega_q)\right)$$

$$[\xi_n(\omega_r-\omega_q) + \xi_n^*(2\omega+\omega_r+\omega_q) + \alpha\delta_{n1}\left(\Phi(\omega_r-\omega_q) + \Phi^*(2\omega+\omega_r+\omega_q)\right)]\Big\} \quad (3.27)$$

where $\quad \xi_\mu(\Omega) \triangleq \Phi^{\mu-1}(\Omega)\,\frac{\partial\Phi}{\partial i\Omega} \quad , \quad \Phi(\Omega) \triangleq \phi(i\Omega)$

$$R_1(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{i+j} > t_i \geq t_{k+n} > t_k$$

$$R_2(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{k+n} > t_k \geq t_{i+j} > t_i$$

$$R_3(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{i+j} \geq t_{k+n} > t_i > t_k$$

$$R_4(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{k+n} > t_{i+j} > t_k > t_k$$

$$R_5(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{k+n} > t_{i+j} > t_i > t_k$$

$$R_6(i,j,k,n) \triangleq (i,j,k,n) \quad : \quad t_{i+j} \geq t_{k+n} > t_k \geq t_i$$

$$\left.\begin{array}{l} i,k = 1,2,\cdots,Ne \\ n,j = 1,2,\cdots,m \end{array}\right\} \triangleq \bigcup_{\mu=1}^{6} R_\mu$$

Note: $\quad R_\mu \cap R_\nu = 0 \quad , \quad \mu \neq \nu$

3.3 Derivation of the Blackman-Tukey Result for Periodic Sampling

The Blackman-Tukey result can be derived from the general result given in equation 3.27 by setting the terms containing $\alpha\delta_{j1}$, $\alpha\delta_{n1}$ equal to zero and replacing the ordinary sums $\sum_{n=1}^{m} a_n \sum_{j=1}^{m} a_j$ with the "trapezoidal sums",

$$\mathop{Z}_{n=0}^{m} a_n = a_o + 2\sum_{n=1}^{m-1} a_n + a_m$$

in accordance with the Blackman Tukey estimator. In addition, we will need the following properties of the periodic sampling process:

$$\Phi(\Omega) = e^{i\omega\Delta t}$$

$$\frac{\partial\Phi(\Omega)}{\partial i\Omega} = \Phi(\Omega)\ \Delta t$$

$$\xi_{\mu}(\Omega) = \Phi^{\mu-1}(\Omega)\ \frac{\partial}{\partial i\Omega}\ \Phi(\Omega) = \Delta t\ \ \Phi^{\mu}(\Omega) \tag{3.31}$$

$$\Phi(\Omega_1+\Omega_2+\Omega_3) = e^{i\Delta t(\Omega_1+\Omega_2+\Omega_3)}$$

$$= \Phi(\Omega_1)\ \Phi(\Omega_2)\ \Phi(\Omega_3) \tag{3.32}$$

$$\Phi^{-\mu}(\Omega) = e^{-i\mu\Omega\Delta t} = \Phi^{\mu}(-\Omega) = \Phi^{*\mu}(\Omega) \tag{3.33}$$

Substitute 3.31, 3.32, 3.33 into the expression so derived from 3.27 to get

$$\lambda(\omega',\omega+\omega_r,\ \omega+\omega_q)$$

$$= \frac{\Delta t^2}{Ne^2}\ \mathrm{Re}\left\{ Z\sum_{R_1} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)\ [\Phi^{n}(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r)]\right.$$

$$+\ Z\sum_{R_2} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)[\Phi^{n}(\omega+\omega_r) + \Phi^{n*}(\omega+\omega_r)]$$

$$+\ Z\sum_{R_3} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)[\Phi^{n}(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r)]$$

$$+\ Z\sum_{R_4} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)[\Phi^{n}(\omega+\omega_r) + \Phi^{*^{n}}(\omega+\omega_r)]$$

$$+\ Z\sum_{R_5} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)[\Phi^{n}(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r)]$$

$$\left.+\ Z\sum_{R_6} \Phi^{i-k}(2\omega')\ \phi^{*j}(\omega+\omega_q)[\Phi^{n}(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r)]\right\}$$

Consequently:

$$\lambda(\omega',\omega;\omega_r,\omega_q)$$

$$= \frac{\Delta t^2}{Ne^2} \sum_{n,j=0}^{m} \sum_{i,k=1}^{Ne} \Phi^{i-k}(2\omega')\ \Phi^{*j}(\omega+\omega_q)\left(\Phi^n(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r)\right)$$

and

$$\Lambda(\omega,\omega';\omega_r,\omega_q)$$

$$= \frac{\Delta t^2}{4} \operatorname{Re}\Bigg\{ \Big[\frac{1}{Ne^2} \sum_{i,k=1}^{Ne} \Phi^{i-k}(2\omega')\Big] \sum_{j=0}^{m} [\Phi^{*j}(\omega+\omega_q)$$

$$+ \Phi^{*j}(\omega-\omega_q)] \sum_{n=0}^{m} [(\Phi^n(\omega+\omega_r) + \Phi^{*n}(\omega+\omega_r) + \Phi^n(\omega-\omega_r) + \Phi^{*n}(\omega-\omega_r)] \Bigg\}$$

Now, with equation 3.33 it can be shown easily that

$$K_u(\omega',Ne) \triangleq \frac{1}{Ne^2} \sum_{i,k=1}^{Ne} \Phi^{i-k}(2\omega') = \left(\frac{\sin Ne\omega'\Delta t}{Ne \sin \omega'\Delta t}\right)^2$$

is real, whence we write:

$$\Lambda(\omega,\omega';\omega_r,\omega_q) = \frac{1}{2} K_u(\omega',Ne)\ H_m(\omega;\omega_q)\ H_m(\omega;\omega_r)$$

where

$$H_m(x,y) \triangleq \frac{1}{2}\left\{Q_m^u(x+y) + Q_m^u(x-y)\right\}$$

$$Q_m^u(x) \triangleq 2\Delta t \operatorname{Re} \sum_{n=0}^{m} \Phi^n(x)$$

$$= \Delta t \sin mx\,\Delta t \cot \frac{x\Delta t}{2} \quad \text{(from 2.41)}$$

Finally, we use the evenness of $S(f)$ to write

$$\mathrm{cov}\{\hat{S}(f_r),\hat{S}(f_q)\} = \int_{-\infty}^{\infty} H(\omega;\omega_q)\ H(\omega;\omega_r)\ \Gamma(\omega)\ df$$

with

$$\Gamma(\omega) \triangleq \int_{-\infty}^{\infty} S(f'+f)\ S(f-f')\ K_u(\omega,Ne)\ df'$$

Also,

$$\mathrm{var}\{\hat{S}(f_r)\} = \int_{-\infty}^{\infty} [H(\omega;\omega_r)]^2\ \Gamma(\omega)\ df$$

Both of these are Blackman-Tukey's results (see, for example, p. 125 of [1]).

Note further, that the periodic variance kernel $K_u(\omega',Ne)$ is periodic with period

$$\omega_p' = \pi/\Delta t \qquad \text{or} \qquad f_p' = 1/2\Delta t$$

and

$$\int_{-\pi/2\Delta t}^{\pi/2\Delta t} K_u(\omega',Ne)\ d\omega' = \frac{1}{\Delta t}\int_{-\pi/2}^{\pi/2} \left(\frac{\sin Ne\ x}{Ne\ \sin x}\right)^2 dx$$

$$= \pi/Ne\ \Delta t$$

3.4 Results for the Poisson Sampling Scheme

For the Poisson distributed sampling intervals

$$\phi(i\Omega) = \frac{1}{1 - i\Omega\alpha}$$

$$\frac{\partial\phi}{\partial i\Omega} \quad \frac{\alpha}{(1 - i\Omega\alpha)^2} = \alpha\phi^2(i\Omega)$$

$$\xi_\mu(\Omega) = \alpha\phi^{\mu+1}(\Omega)$$

so that

$$\lambda(\omega';\omega+\omega_r,\ \omega+\omega_q)$$

$$= \frac{\alpha^2}{Ne^2}\,\mathrm{Re}\Bigg\{ \sum_{R_1(i,j,k,n)} \Phi^{i-k-n+1}(2\omega')\left(\Phi^{*j+1}(\omega+\omega_q) + \delta_{j1}\Phi^*(\omega+\omega_q)\right)$$

$$[\Phi^{n+1}(2\omega'+\omega+\omega_r) + \Phi^{n+1}(2\omega'-\omega-\omega_r) + \delta_{n1}\left(\Phi(2\omega'+\omega+\omega_r) + \Phi(2\omega'-\omega-\omega_r)\right)]$$

$$+ \sum_{R_2(i,j,k,n)} \left(\Phi^{*k-i-j+1}(2\omega')\left(\Phi^{*j+1}(2\omega'+\omega+\omega_q) + \delta_{j1}\ \Phi(2\omega'+\omega+\omega_q)\right)\right)[\Phi^{n+1}(\omega+\omega_r)$$

$$+ \Phi^{*n+1}(\omega+\omega_r) + \delta_{n1}\left(\Phi(\omega+\omega_r) + \Phi^*(\omega+\omega_r)\right)] + \sum_{R_3(i,j,k,n)} \left(\Phi^{*i+j-k-n+1}(\omega+\omega_q)\right.$$

$$\left. + \delta_{j1}\Phi^{i-k-n+1}(\omega+\omega_q)\right)\Big[\Phi^{i-k+1}(2\omega'+\omega+\omega_r)\left(\Phi^{k+n-i+1}(\omega_r-\omega_q)\right.$$

$$\left. + \delta_{n1}\ \Phi^{k-i+1}(\omega_r-\omega_q)\right) + \Phi^{i-k+1}(2\omega'-\omega-\omega_r)\ \left(\Phi^{*k+n-i+1}(2\omega+\omega_r+\omega_q)\right.$$

$$\left. + \delta_{n1}\ \Phi^{*k-i+1}(2\omega+\omega_r+\omega_q)\right)\Big]$$

$$+ \sum_{R_4(i,j,k,n)} \Big[\Phi^{i-k+1}(2\omega'+\omega+\omega_r)\ \left(\Phi^{k+n-i-j+1}(\omega+\omega_r) + \delta_{n1}\Phi^{k-i-j+1}(\omega+\omega_r)\right)$$

$$\left(\Phi^{j+1}(\omega_r-\omega_q) + \delta_{j1}\ \Phi(\omega_r-\omega_q)\right) + \Phi^{i-k+1}(2\omega'-\omega-\omega_r)\left(\Phi^{*k+n-i-j+1}(\omega+\omega_r)\right.$$

$$\left. + \delta_{n1}\ \Phi^{*k-i-j+1}(\omega+\omega_r)\right)\ \left(\Phi^{*j+1}(2\omega+\omega_r+\omega_q) + \delta_{j1}\Phi(\omega_r-\omega_q)\right)\Big]$$

$$+ \sum_{R_5(i,j,k,n)} \Phi^{*k-i+1}(2\omega'+\omega+\omega_q)\ \Big[\ \left(\Phi^{k+n-i-j+1}(\omega+\omega_r) + \delta_{n1}\Phi^{k-i-j+1}(\omega+\omega_r)\right)$$

$$\left(\Phi^{i+j-k+1}(\omega_r-\omega_q) + \delta_{j1}\Phi^{i-k+1}(\omega_r-\omega_q)\right) + \left(\Phi^{*k+n-i-j+1}(\omega+\omega_r)\right.$$

$$\left. + \delta_{n1}\Phi^{k-i-j+1}\ (\omega+\omega_r)\right)\ \left(\Phi^{*i+j-k+1}(2\omega+\omega_r+\omega_q) + \delta_{j1}\Phi^{*i-k+1}(2\omega+\omega_r-\omega_q)\right)\Big]$$

$$+ \sum_{R_6(i,j,k,n)} \phi^{*k-i+1}(2\omega'+\omega+\omega_q)\left(\phi^{*i+j-k-n+1}(\omega+\omega_q)+\delta_{j1}\phi^{*i-k-n+1}(\omega+\omega_q)\right)$$

$$\left.\left[\phi^{n+1}(\omega_r-\omega_q) + \phi^{*n+1}(2\omega+\omega_r+\omega_q) + \alpha\delta_{n1}\left(\Phi(\omega_r-\omega_q)+\Phi^*(2\omega+\omega_r+\omega_q)\right)\right]\right\} \tag{3.41}$$

The ranges of summation $R_\mu(i,j,k,n)$ $\mu = 1,\cdots,6$ are those given following equation 3.27.

3.4.1 An Exact Result for $\mathrm{cov}[\hat{S}(f_r),\hat{S}(f_q)]$, $\mathrm{var}\{\hat{S}(f_r)\}$

Equation 3.41 can be simplified a step further by noting that

$$\delta_{n1}\Phi = \left.\phi^{n+1}\right|_{n=0}$$

By appropriately replacing

$$\sum_{n=1}^{m} \text{ with } \sum_{n=0}^{m}$$

we have for the Poisson sampling scheme

$$\mathrm{cov}\{\hat{S}(f_r),\hat{S}(f_q)\} = \iint_{-\infty}^{\infty} S(f+f')\ S(f-f')\ \{\Lambda(\omega',\omega,\omega_r\omega_q)$$

$$+ \Lambda(\omega,\omega';\omega_r,\omega_q)\}\ df'\ df$$

where

$$\Lambda(\omega,\omega';\omega_r,\omega_q) = \frac{1}{4}\{\lambda(\omega',\omega+\omega_r,\ \omega+\omega_q) + \lambda(\omega',\omega+\omega_r,\omega-\omega_q)$$

$$+ \lambda(\omega';\omega-\omega_r,\omega+\omega_q) + \lambda(\omega',\omega-\omega_r,\omega-\omega_q)\}$$

and

$$\lambda(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$= \frac{\alpha^2}{Ne^2} \mathrm{Re}\left\{ \sum_{R_1(i,j,k,n)} \left[\Phi^{i-k-n+1}(2\omega')\ \Phi^{*j+1}(\omega+\omega_q)\ \left(\Phi^{n+1}(2\omega'+\omega+\omega_r)\right.\right.\right.$$

$$\left.\left. + \Phi^{n+1}(2\omega'-\omega-\omega_r)\right)\right] + \sum_{R_2(i,j,k,n)} \left[\Phi^{*k-i-j+1}(2\omega')\ \Phi^{*j+1}(2\omega'+\omega+\omega_q)\right.$$

$$\left.\left(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\right)\right] + \sum_{R_3(i,j,k,n)} \left[\Phi^{*i+j-k-n+1}(\omega+\omega_q)\right.$$

$$\left(\Phi^{i-k+1}(2\omega'+\omega+\omega_r)\ \Phi^{k+n-i+1}(\omega_r-\omega_q) + \Phi^{i-k+1}(2\omega'-\omega-\omega_r)\right.$$

$$\left.\left.\Phi^{*k+n-i+1}(2\omega+\omega_r+\omega_q)\right)\right] + \sum_{R_4(i,j,k,n)} \left[\Phi^{i-k+1}(2\omega'+\omega+\omega_r)\Phi^{j+1}(\omega_r-\omega_q)\right.$$

$$\left.\Phi^{k+n-i-j+1}(\omega+\omega_r) + \Phi^{i-k+1}(2\omega'-\omega-\omega_r)\Phi^{*j+1}(2\omega+\omega_r+\omega_q)\Phi^{*k+n-i-j+1}(\omega+\omega_r)\right]$$

$$+ \sum_{R_5(i,j,k,n)} \left[\Phi^{*k-i+1}(2\omega'+\omega+\omega_q)\left(\Phi^{i+j-k+1}(\omega_r-\omega_q)\Phi^{k+n-i-j+1}(\omega+\omega_r)\right.\right.$$

$$\left.\left. + \Phi^{*i+j-k+1}(2\omega+\omega_r+\omega_q)\Phi^{*k+n-i-j+1}(\omega+\omega_r)\right)\right]$$

$$+ \sum_{R_6(i,j,k,n)} \left[\Phi^{*k-i+1}(2\omega'+\omega+\omega_q)\ \Phi^{*i+j-k-n+1}(\omega+\omega_q)\ \left(\Phi^{n+1}(\omega_r-\omega_q)\right.\right.$$

$$\left.\left.\left. + \Phi^{*n+1}(2\omega+\omega_r+\omega_q)\right)\right]\right\} \tag{3.42}$$

$$i,k = 1,2,\cdots,Ne \quad ; \quad n,j = 0,1,\cdots,m$$

To obtain an expression for the variance of spectral estimates, we simply note that

$$\mathrm{var}\ \{\hat{S}(f_r)\} = \mathrm{cov}\ \{\hat{S}(f_r), \hat{S}(f_r)\}$$

3.4.2 $\lim_{|f_r - f_q| \to \infty} \operatorname{cov}[\hat{S}(f_r), \hat{S}(f_q)]$

From the Blackman-Tukey result derived in Section 3.3, we can again relate the covariability of spectral density estimates $\hat{S}(f_r)$, $\hat{S}(f_q)$ to the spectral window $Q_m(\omega)$. For example, if $H_m(\omega,\omega_r)$ and $H_m(\omega,\omega_q)$ do not overlap , $\operatorname{cov}\{\hat{S}(f_r),\hat{S}(f_q)\} = 0$. With Poisson sampling intervals, the results obtained so far do not make this so obvious, but intuitively we expect a similar correspondence between covariability of estimates and overlapping of spectral windows.

Now consider $\lambda(\omega',\omega+\omega_r,\omega+\omega_q)$ as given in equation 3.42 and note that

$$\begin{aligned}
&|\lambda(\omega',\omega+\omega_r,\omega+\omega_q)| \\
&\leq \frac{\alpha^2}{Ne^2}\left\{\sum_{R_1(i,j,k,n)}\left|\operatorname{Re}\left[\Phi^{i-k-n+1}(2\omega')\Phi^{*j+1}(\omega+\omega_q)\left(\Phi^{n+1}(2\omega'+\omega+\omega_r)\right.\right.\right.\right. \\
&\left.\left.\left.+\ \Phi^{*n+1}(2\omega'-\omega-\omega_r)\right)\right]\right| + \sum_{R_2(i,j,k,n)}\left|\operatorname{Re}\left[\ \right]\right| + \cdots \\
&\left. + \sum_{R_6(i,j,k,n)}\left|\operatorname{Re}\left[\ \right]\right|\right\} \leq \frac{\alpha^2}{Ne^2}\left\{\sum_{R_1}\left|\left[\ \right]\right| + \cdots + \sum_{R_6}\left|\left[\ \right]\right|\right\}
\end{aligned}$$

The last inequality follows, since

$$|\operatorname{Re} z| \leq |z|$$

Making use of

(i) $\Phi(\omega) \cong 1/i\omega\alpha \qquad , \quad \omega \to \infty$

(ii) $\Phi(\omega+\omega_r) = \Phi(\omega+\omega_q+\omega_r-\omega_q)$

and taking the leading term in each summation it can be shown that

$$\lim_{|\omega_r-\omega_q| \to \infty} \lambda(\omega',\omega+\omega_r,\omega+\omega_q) = \frac{1}{Ne^2|\omega_r-\omega_q|^4\alpha^2}$$

and consequently,

$$\mathrm{cov}\{\hat{S}(f_r),\hat{S}(f_q)\} \sim 0\left(\frac{1}{(f_r-f_q)^4}\right)$$

as $|f_r-f_q| \to \infty$.

3.4.3 An Approximation for $\mathrm{var}\{\hat{S}(f_r)\}$

In Section 3.3 we were able to derive a closed-form expression for $\mathrm{cov}\{\hat{S}(f_r), \hat{S}(f_q)\}$ for the periodic sampling scheme. The procedure outlined in Section 3.3 exploited certain properties of the corresponding characteristic function (cf. equations 3.31, 3.32, 3.33) which do not hold for the general nonuniform sampling scheme. In particular, the characteristic function for the Poisson sampling intervals does not satisfy equations 3.31, 3.32 and 3.33 exactly. However, an approximate closed form expression for $\mathrm{var}\{\hat{S}(f_r)\}$ can be derived from equation 3.41 for values of ω' near $\omega' = 0$ and ω near $\omega = \pm\,\omega_r$. Such an approximation is the most natural one to seek since the effective bandwidths of $Q_m(\omega)$, $K(\omega',Ne)$ are of the order of $1/m$ and $1/Ne$ respectively, where m, Ne are typically very large. Specifically, we will derive an approximation for $\mathrm{var}\{\hat{S}(f_r)\}$ valid in $|\omega'| \leq 1/Ne\alpha$ and $|\omega \pm \omega_r| \leq 1/2m\alpha$.

We begin by assuming the following form for $\lambda(\omega',\omega+\omega_r,\omega+\omega_r)$:

$$\lambda_a(\omega',\omega+\omega_r,\omega+\omega_r)$$

$$= \frac{2\alpha^2}{Ne^2} \sum_{i,k=1}^{Ne} \sum_{n,j=0}^{m} \mathrm{Re}\Big[\Phi^{i+k}(2\omega')\ \Phi^{*k+1}(2\omega')\ \Phi^{*j+1}(\omega+\omega_r)\ \big(\Phi^{n+1}(\omega+\omega_r)$$

$$+ \Phi^{*n+1}(\omega+\omega_r)\big)\Big] + R(\omega',\omega+\omega_r) \qquad \textbf{(3.43)}$$

and proceed to derive an approximation for the remainder function, $R(\omega',\omega+\omega_r)$ valid in $|\omega'| \leq 1/Ne\alpha$ and $|\omega + \omega_r| < 1/2m\alpha$. From equations 3.42, 3.43 we can write the remainder function exactly as:

$$R(\omega',\omega+\omega_r)$$

$$= \frac{2\alpha^2}{Ne} \mathrm{Re}\Big\{ \sum_{R_1} \Phi^{*j+1}(\omega+\omega_r) \Big[\Phi^{i-k-n+1}(2\omega')\ \big(\Phi^{n+1}(2\omega'+\omega+\omega_r)$$

$$+ \Phi^{n+1}(2\omega'-\omega-\omega_r)\big) - \Phi^{i+1}(2\omega')\ \Phi^{*k+1}(2\omega')\ \big(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\big)\Big]$$

$$+ \sum_{R_2} \big(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\big) \Big[\Phi^{*k-i-j+1}(2\omega')\ \Phi^{*j+1}(2\omega'+\omega+\omega_r)$$

$$- \Phi^{i+1}(2\omega')\ \Phi^{*k+1}(2\omega')\ \Phi^{*j+1}(\omega+\omega_r)\Big]$$

$$+ \sum_{R_3}\Big[\big(\Phi^{*i+j-k-n+1}(\omega+\omega_r)\ \Phi^{i-k+1}(2\omega'+\omega+\omega_r) - \Phi^{*j+1}(\omega+\omega_r)\Phi^{n+1}(\omega+\omega_r)$$

$$\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\big) + \big(\Phi^{*i+j-k-n+1}(\omega+\omega_r)\Phi^{i-k+1}(2\omega'-\omega-\omega_r)$$

$$\Phi^{*k+n-i+1}(2\omega+2\omega_r) - \Phi^{*j+1}(\omega+\omega_r)\Phi^{*n+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\big)\Big]$$

$$+ \sum_{R_4} \Big[\big(\Phi^{i-k+1}(2\omega'+\omega+\omega_r)\Phi^{k+n-i-j+1}(\omega+\omega_r)$$

$$- \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\Phi^{*j+1}(\omega+\omega_r)\Phi^{n+1}(\omega+\omega_r)\big) \quad +$$

$$+ \left(\Phi^{i-k+1}(2\omega'-\omega-\omega_r)\ \Phi^{*j+1}(2\omega+2\omega_r)\ \Phi^{*k+n-i-j+1}(\omega+\omega_r)\right.$$

$$\left.- \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\Phi^{*j+1}(\omega+\omega_r)\Phi^{*n+1}(\omega+\omega_r)\right)\Big]$$

$$+ \sum_{R_5} \Big[\Big(\Phi^{*k-i+1}(2\omega'+\omega+\omega_r)\ \Phi^{k+n-i-j+1}(\omega+\omega_r) - \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')$$

$$\Phi^{*j+1}(\omega+\omega_r)\Phi^{n+1}(\omega+\omega_r)\Big) + \Big(\Phi^{*k-i+1}(2\omega'+\omega+\omega_r)\Phi^{*i+j-k+1}(2\omega+2\omega_r)$$

$$\Phi^{*k+n-i-j+1}(\omega+\omega_r) - \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\ \Phi^{*j+1}(\omega+\omega_r)\Phi^{*n+1}(\omega+\omega_r)\Big)\Big]$$

$$+ \sum_{R_6} \Big[\Big(\Phi^{*k-i+1}(2\omega'+\omega+\omega_r)\Phi^{*i+j-k-n+1}(\omega+\omega_r)$$

$$- \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\Phi^{*j+1}(\omega+\omega_r)\Phi^{n+1}(\omega+\omega_r)\Big)$$

$$+ \Big(\Phi^{*k-i+1}(2\omega'+\omega+\omega_r)\Phi^{*i+j-k-n+1}(\omega+\omega_r)\Phi^{*n+1}(2\omega+2\omega_r)$$

$$- \Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\Phi^{*j+1}(\omega+\omega_r)\Phi^{*n+1}(\omega+\omega_r)\Big)\Big]\Bigg\} \tag{3.44}$$

where the ranges $R_\mu(i,j,k,n)$ $\mu = 1,2,\cdots,6$ are as previously specified.

Now consider the sum over $R_1(i,j,k,n)$ which, for convenience, we rewrite as

$$\sum_{R_1} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\left[\frac{\Phi^{i-k+2}(2\omega')}{\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')}\left(\left(\frac{\Phi(2\omega'+\omega+\omega_r)}{\Phi(2\omega')}\right)^{n+1}\right.\right.$$
$$\left.\left.+\left(\frac{\Phi(2\omega'-\omega-\omega_r)}{\Phi(2\omega')}\right)^{n+1}\right) - \Phi^{n+1}(\omega+\omega_r) - \Phi^{*n+1}(\omega+\omega_r)\right]$$

and note that

$$\Phi(2\omega'+\omega+\omega_r) - \Phi(2\omega')\ \Phi(\omega+\omega_r)$$

$$= \frac{1}{1 - i(2\omega'+\omega+\omega_r)\alpha} - \frac{1}{(1 - 2i\omega'\alpha)(1 - i(\omega+\omega_r)\alpha)}$$

$$= \frac{-2\omega'(\omega+\omega_r)\alpha^2}{[1 - 2\omega'(\omega+\omega_r)\alpha^2 - i(2\omega'+\omega+\omega_r)\alpha][1 - i(2\omega'+\omega+\omega_r)\alpha]}$$

By maximizing the numerator and minimizing the denominator of the above expression in $|\omega'| \leq 1/Ne\alpha$, $|\omega'+\omega_r| \leq 1/m\alpha$, obtain that

$$|\Phi(2\omega'+\omega+\omega_r) - \Phi(2\omega')\Phi(\omega+\omega_r)|$$

$$= \frac{2\omega'(\omega+\omega_r)\alpha^2}{[(1 - 2\omega'(\omega+\omega_r)\alpha^2)^2 + (2\omega'+\omega+\omega_r)^2\alpha^2]^{1/2}\ [1 + (2\omega'+\omega+\omega_r)^2\alpha^2]^{1/2}}$$

$$\leq \frac{2}{Nem}\left(1 - \frac{2}{Nem}\right)^{-1} = \frac{2}{Nem}\left(1 + \frac{2}{Nem} + \frac{4}{Ne^2m^2} + \cdot\cdot\cdot\right)$$

whence we have

$$\Phi(2\omega'+\omega+\omega_r) \cong \Phi(2\omega')\Phi(\omega+\omega_r) - \frac{2}{Nem} \qquad (3.45a)$$

Similarly,

$$\frac{\Phi^{i-k+2}(2\omega')}{\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')} \cong 1\ ,\ \text{near}\quad \omega' = 0 \qquad (3.45b)$$

Next, we write, using 3.45a and 3.45b, the sum over $R_1(i,j,k,n)$ approximately as

$$\sum_{R_1} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\ \Big[\Big(\Phi(\omega+\omega_r) - \frac{2}{Nem\ \Phi(2\omega')}\Big)^{n+1}$$
$$- \Phi^{n+1}(\omega+\omega_r) + \Big(\Phi^*(\omega+\omega_r) - \frac{2}{Nem\ \Phi(2\omega')}\Big)^{n+1} - \Phi^{*n+1}(\omega+\omega_r)\Big]$$

$$\cong -\frac{2}{Ne}\sum_{R_1}\Big[\Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k}(2\omega')\ \big(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\big) + 0(1/Ne)\Big] \tag{3.46}$$

The latter is obtained from the former by taking the first term in the binomial expansion of

$$\left(\Phi(\omega+\omega_r) - \frac{2}{Nem\ \Phi(2\omega')}\right)^{n+1}$$

and noting that $n+1/m \leq \frac{m+1}{m} \rightarrow 1$.

Approximations similar to equation 3.46 can be derived for each of the composite summations in equation 3.44 by first establishing approximations similar to equations 3.45a, 3.45b and then substituting into appropriate terms. Summarized below are some useful approximations whose derivations follow closely that of equation 3.45a:

$$\frac{\Phi(2\omega'+\omega+\omega_r)\Phi^*(\omega+\omega_r)}{\Phi^*(2\omega+2\omega_r)} \cong \Phi(2\omega') + 0(1/Nem) \tag{3.45c}$$

$$\Phi(2\omega'+\omega+\omega_r)\ \Phi^*(\omega+\omega_r) \cong \Phi(2\omega') + 0(1/Nem) \tag{3.45d}$$

$$\Phi^{*-1}(\omega+\omega_r) \cong \Phi(\omega+\omega_r) + 0(1/m^2) \tag{3.45e}$$

$$\frac{\Phi^*(2\omega+2\omega_r)}{\Phi^*(\omega+\omega_r)} \cong \overset{*}{\Phi}(\omega+\omega_r) + 0(1/m^2) \tag{3.45f}$$

They are valid in $|\omega'| \leq 1/Ne\alpha$, $|\omega \pm \omega_r| \leq 1/m\alpha$ and can be derived quite easily. To illustrate further, we derive an approximation for the sum over $R_3(i,j,k,n)$ which, for convenience, we write as

$$\sum_{R_3(i,j,k,n)} \Phi^{*j+1}(\omega+\omega_r)\ \Phi^{i+1}(2\omega')\ \Phi^{*k+1}(2\omega')$$

$$\left[\left(\frac{\Phi^{i-k+1}(2\omega'+\omega+\omega_r)}{\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')}\ \frac{\Phi^{*i-k+1}(\omega+\omega_r)}{\Phi^{*n+1}(\omega+\omega_r)} - \Phi^{n+1}(\omega+\omega_r)\right)\right.$$

$$\left. + \left(\frac{\Phi^{*i-k+1}(\omega+\omega_r)\Phi^{i-k+1}(2\omega'-\omega-\omega_r)\Phi^{*n+2}(2\omega+2\omega_r)}{\Phi^{*i-k+1}(2\omega+2\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\Phi^{*n+1}(\omega+\omega_r)} - \Phi^{*n+1}(\omega+\omega_r)\right)\right]$$

Introducing equations 3.45, we have

$$\sum_{R_3(i,j,k,n)} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\left[\left(\frac{\left(\Phi(2\omega') + 0(1/Nem)\right)^{i-k+1}}{\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')}\right.\right.$$

$$\left.\left(\Phi(\omega+\omega_r) + 0(1/m^2)\right)^{n+1} - \Phi^{n+1}(\omega+\omega_r)\right)$$

$$\left. + \frac{\left(\Phi(2\omega') + 0(1/Nem)\right)^{i-k+1}}{\Phi^{i+1}(2\omega')\ \Phi^{*k+1}(2\omega')} \left(\Phi^*(\omega+\omega_r) + 0(1/m^2)\right)^{n+1} - \Phi^{*n+1}(\omega+\omega_r)\right)\right]$$

which, on taking the first terms in the binomial expansions as before, reduces approximately to

$$\frac{1}{m} \sum_{R_3(i,j,k,n)} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i}(2\omega')\Phi^{*k}(2\omega')\Big[\Phi^{n+1}(\omega+\omega_r)$$

$$+ \Phi^{*n+1}(\omega+\omega_r) + 0(1/Ne)\Big] \qquad (3.4)$$

Except for the multiplicative constants $1/m$, $-2/Ne$, equations 3.47 and 3.46 are identical. In a similar manner, it can be shown that approximations to the other sums are identical to equation 3.46 to within multiplicative constants and that these constants are of the order of $1/m$, $1/Ne$. Since m is typically a fraction of Ne (i.e., $m = 0(Ne)$) we can write finally:

$$R(\omega',\omega+\omega_r) \cong 0(1/Ne)\left\{ \frac{\alpha^2}{Ne^2}\,\mathrm{Re} \sum_{R_3(i,j,k,n)} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\right.$$

$$\left. \Phi^{*k+1}(2\omega')\left(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\right) + 0(1/Ne)\right\}$$

Consequently:

$$\lambda_a(\omega',\omega+\omega_r,\omega+\omega_r)$$

$$= \frac{2\alpha^2}{Ne^2}\,\mathrm{Re}\left\{\sum_{i,k=1}^{Ne} \sum_{n,j=0}^{m} \Phi^{*j+1}(\omega+\omega_r)\Phi^{i+1}(2\omega')\Phi^{*k+1}(2\omega')\right.$$

$$\left.\left(\Phi^{n+1}(\omega+\omega_r) + \Phi^{*n+1}(\omega+\omega_r)\right)\left(1 + 0(1/Ne)\right)\right\}$$

which, by the method of Section 3.3, leads to

$$\Lambda_a(\omega';\omega,\omega_r) = \frac{1}{2} K_p(\omega',Ne)H_m^2(\omega;\omega_r)\left(1 + 0(1/Ne)\right)$$

where, as before:

$$K_p(\omega',Ne) = 1/Ne^2 \sum_{i,k=1}^{Ne} \Phi^{i+1}(2\omega')\,\Phi^{*k+1}(2\omega')$$

$$H_m(\omega;\omega_r) = Q_m^P(\omega+\omega_r) + Q_m^P(\omega-\omega_r)$$

$$Q_m^P(\omega) = 2\alpha\,\mathrm{Re}\sum_{n=0}^{m} \Phi^{n+1}(\omega)$$

Thus we have

$$\mathrm{var}\{\hat{S}(f_r)\} \cong \iint_{-\infty}^{\infty} S(f+f')\,S(f-f')K_p(f',Ne)H_m^2(f;f_r)[1+0(1/Ne)]\,df\,df' \tag{3.48}$$

Substitute $\Phi(2\omega') = \cos\varphi\, e^{i\varphi}$, $\varphi = \tan^{-1}2\omega'\alpha$ to obtain the following closed-form expression for the Poisson variance kernel $K_p(\omega',Ne)$:

$$K_p(\omega',Ne) = \frac{\cos^4\varphi\,(1 - 2\cos^{Ne}\varphi\cos Ne\varphi + \mathrm{Cos}^{2Ne}\varphi\,)}{Ne^2\sin^2\varphi}$$

Obtain further,

$$\int_{-\infty}^{\infty} K(\omega',Ne)\,d\omega' = \frac{1}{2Ne^2\alpha}\int_{-\pi/2}^{\pi/2}\sum_{i,k=1}^{Ne}\cos^{i+k}\varphi[\cos(i-k)\varphi + i\sin(i-k)\varphi]\,d\varphi$$

and integrate term by term using (cf. tables, [21])

$$\int_0^{\pi/2}\cos^{p+q-2}x\,\cos(p-q)x\,dx = \frac{\pi}{2^{p+q-1}(p+q-1)B(p,q)}$$

and the fact that $K(\omega',Ne)$ is real

to get

$$\int_{-\infty}^{\infty} K_p(f',N_e)\,df' = \frac{1}{N_e^2\alpha}\sum_{i,k=1}^{Ne}\frac{\pi}{2^{i+k+1}(i+k+1)\,B(i+1,k+1)} \tag{3.49}$$

where $B(p,q)$ is the well known beta function, viz.,

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

Shown graphically in Figures 8a,b,c are the variance kernels $K_u(\omega',Ne)$, $K_p(\omega',Ne)$ for $Ne = 200, 500, 1000$ respectively. Observe that for sufficiently large data (namely, $Ne > 1000$) both kernels coincide.

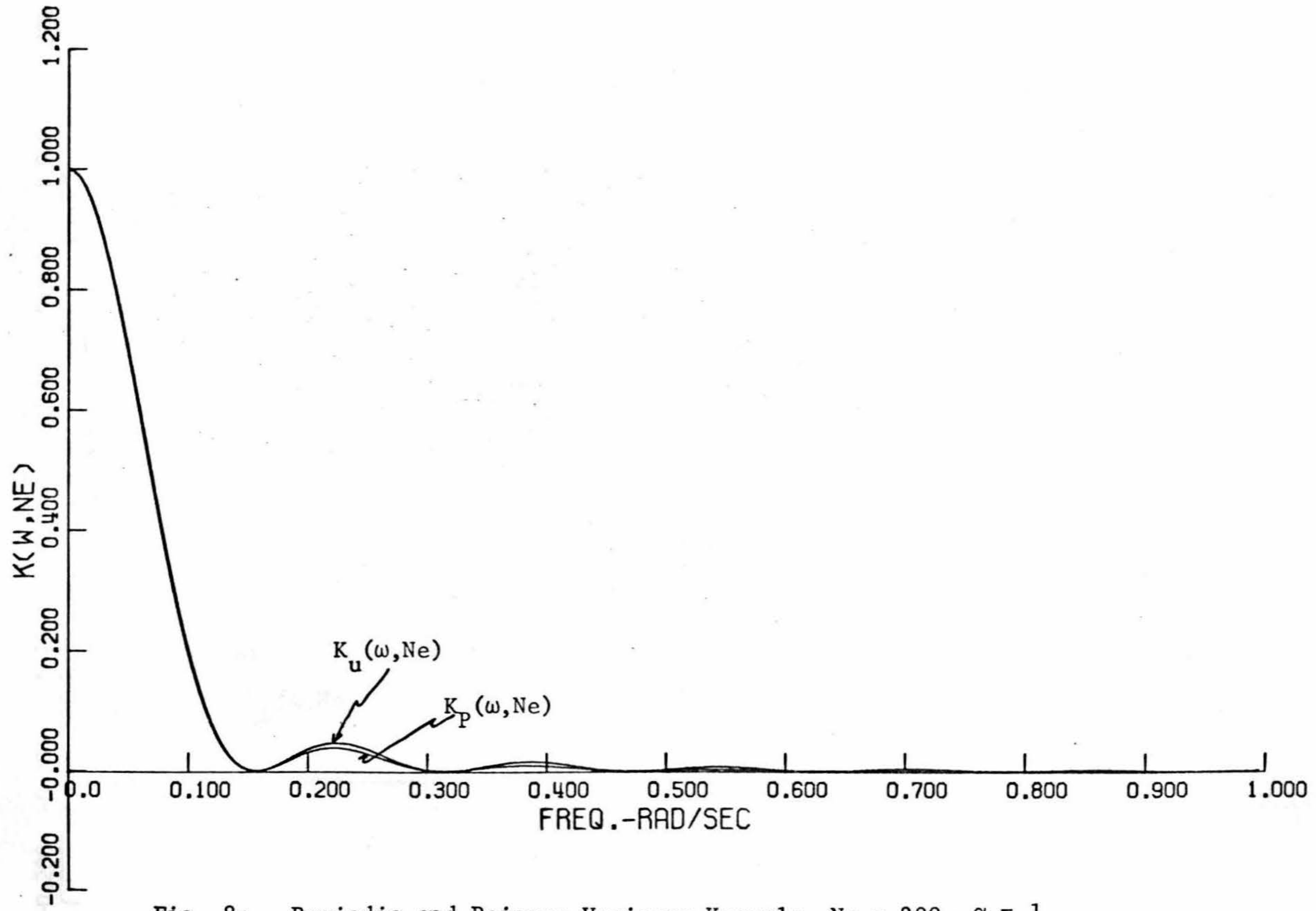

Fig. 8a. Periodic and Poisson Variance Kernels, Ne = 200, $\alpha = 1$

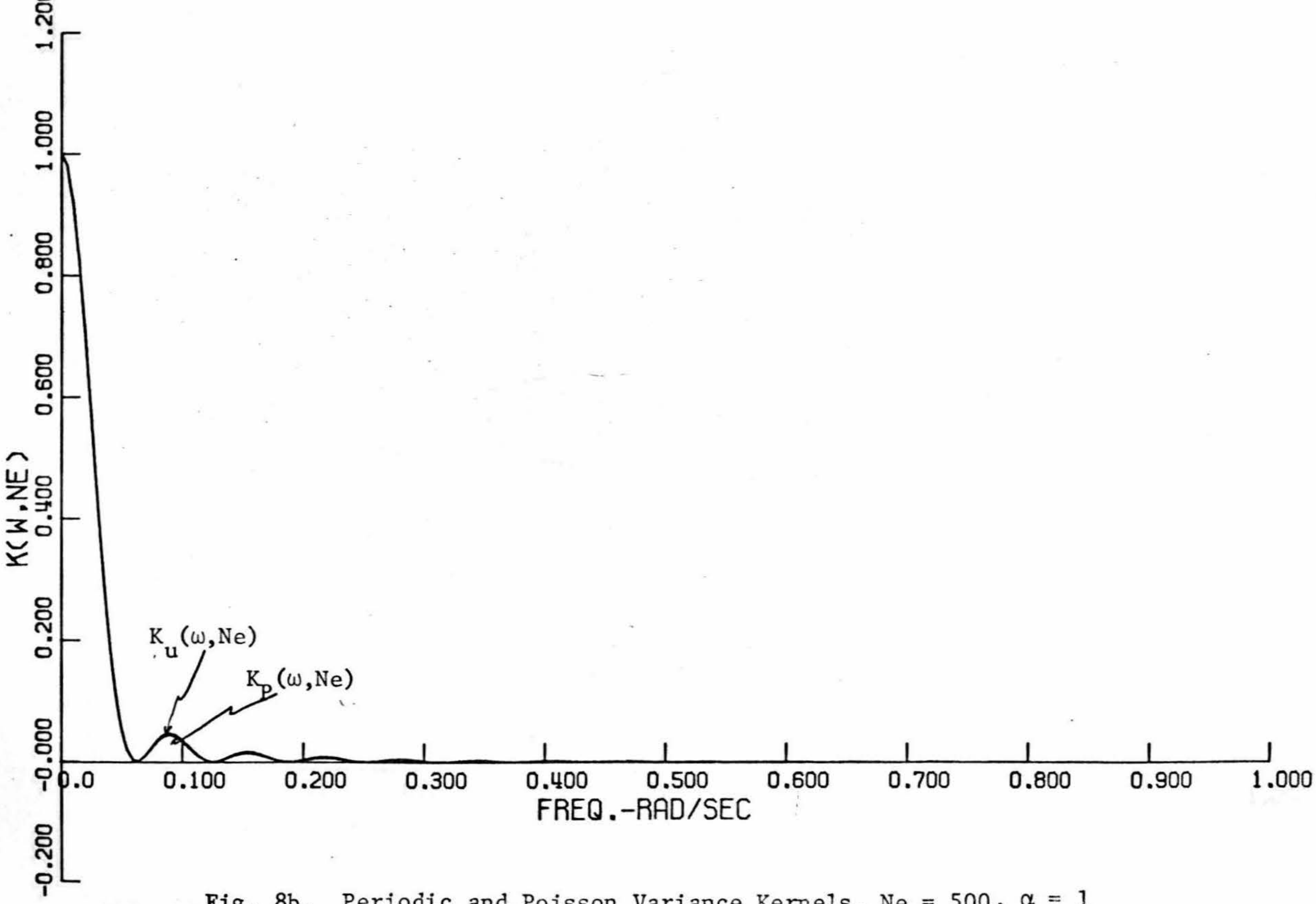

Fig. 8b. Periodic and Poisson Variance Kernels, Ne = 500, α = 1

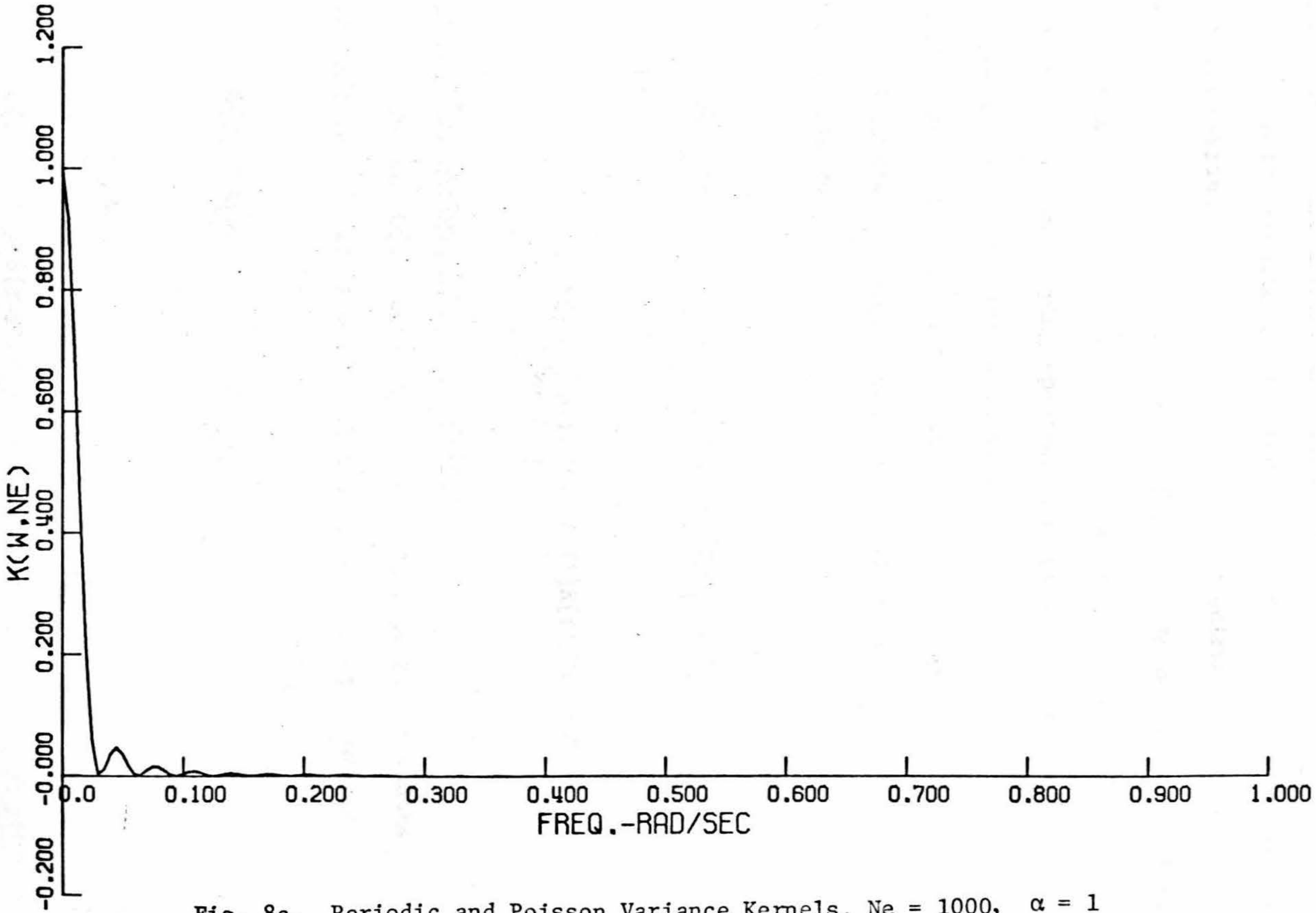

Fig. 8c. Periodic and Poisson Variance Kernels, Ne = 1000, $\alpha = 1$

3.5 Special Cases of Practical Interest

From the results of the preceeding sections, a number of useful results bearing on assumptions about the spectrum $S(f)$, can be extracted. In [1] pp. 104-106, we find results based on four such assumptions. Here we shall examine the two most useful of these, namely: (i) Slow-varying spectra, and (ii) Spectral spikes. Usually in practice, we are faced with estimating spectra which are fairly smooth except for one or two jumps; in which case (i) and (ii) can be applied piecewise. Analytically we shall be deriving approximations to the integrals:

$$\text{Cov}\left\{\hat{S}(f_r)\ \hat{S}(f_q)\right\} = \int_{-\infty}^{\infty} H_m(f,f_r)H_m(f,f_q)\Gamma(f)\ df$$

$$\Gamma(f) = \int_{-\infty}^{\infty} S(f+f')S(f-f')K(f',Ne)\ df'$$

3.5.1 Estimation of Spectral Spikes

Suppose $S(f)$ consists of very sharp peaks (of widths $\ll$ width of $K(f',Ne)$) at $f = \pm f_o$ with area = A, so that we can write:

$$S(f) \cong A\left\{\delta(f+f_o) + \delta(f-f_o)\right\}$$

and

$$S(f+f') \cong A\left\{\delta(f+f'+f_o) + \delta(f+f'-f_o)\right\}$$

$$S(f-f') \cong A\left\{\delta(f-f'+f_o) + \delta(f-f'-f_o)\right\} \quad .$$

$$S(f+f')S(f-f')$$

$$= \quad A^2\{\delta(f+f'+f_o)\delta(f-f'+f_o) + \delta(f+f'-f_o)\delta(f-f'+f_o)$$

$$+ \delta(f+f'+f_o)\ \delta(f+f'-f_o) + \delta(f+f'-f_o)\delta(f-f'-f_o)\}$$

Now, let

$$I(f,f_o) \stackrel{\Delta}{=} \int_{-\infty}^{\infty}\delta(f+f'+f_o)\delta(f-f'+f_o)K(f',Ne)\ df'$$

$$= \delta(2f+2f_o)K(-f-f_o',Ne)$$

$$= \underline{\tfrac{1}{2}\ \delta(f+f_o)K(f+f_o',Ne)}$$

The last equality follows from the evenness of $K(f,Ne)$ and its boundedness and absolute integrability (cf. Lighthill),

so that,

$$\Gamma(f) \cong \frac{A^2}{2}\{\delta(f+f_o)K(f+f_o',Ne) + \delta(f)K(f-f_o,Ne)$$

$$+ \delta(f)K(f+f_o,Ne) + \delta(f-f_o)K(f-f_o,Ne)\}$$

$$\cong \frac{A^2}{2}\{\delta(f+f_o)K(0,Ne) + \delta(f)K(-f_o,Ne)$$

$$+ \delta(f)K(f_o,Ne) + \delta(f-f_o)K(-0,Ne)\}$$

Where the last equality follows from

$$\phi\ (x)\ \delta\ (x) \quad = \quad \phi\ (0)\ \delta\ (x)$$

and the observation that $K(f,Ne)$ is a fairly good function (cf Lighthill p. 42) Next, substitute $K(0,Ne) = 1$ to have

$$\Gamma(f) \quad \cong \quad \frac{A^2}{2}\ \{\delta(f+f_o) + \delta(f-f_o) + 2\ \delta(f)K(f_o,Ne)\} \quad ;$$

Whence

$$\text{Cov}\left\{\hat{S}(f_r),\hat{S}(f_q)\right\} = \int_{-\infty}^{\infty} H_m(f;f_r)H_m(f;f_q)\Gamma(f)\,df$$

$$= \frac{A^2}{2}\left\{H_m(-f_o;f_r)H_m(-f_o;f_q) + H_m(f_o;f_r)H_m(f_o;f_q)\right.$$

$$\left. + \quad 2H_m(0;f_r)H_m(0;f_q)K(f_o;Ne)\right\}$$

By the evenness of $H_m(f_1;f_2)$ we can write:

$$\text{Cov}\left\{\hat{S}(f_r),\hat{S}(f_q)\right\}$$

$$= A^2\left\{H_m(f_o;f_r)H_m(f_o;f_q) + H_m(0;f_r)H_m(0;f_q)K(f_o;Ne)\right\}$$

For $f_o >> {}^1/_{Ne\alpha}$, $K(f_o,Ne) \cong 0$ so that

$$\text{Cov}\left\{\hat{S}(f_r),\hat{S}(f_q)\right\}$$

$$\approx A^2\left\{H_m(f_o;f_r)H_m(f_o;f_q)\right\}$$

and

$$\text{Var}\left\{\hat{S}(f_r)\right\} \cong A^2 H_m^2(f_o;f_r)$$

In particular,

$$\text{Var}\left\{\hat{S}(f_o)\right\} \cong A^2 H^2(f_o;f_o)$$

$$= A^2\left[Q^2(0) + Q^2(2f_o) + 2Q(0)Q(2f_o)\right]$$

$$\cong 4A^2 m^2\alpha^2 \quad , \quad f_o >> \frac{1}{m\alpha}$$

Thus, we see that even with large data size reliable estimates of spectral spikes are difficult to obtain, regardless of the sampling scheme being used.

3.5.2 Estimation of Slow-varying Spectra

Here we are considering $S(f)$ which vary slowly enough so that within the bandwidth of $K(f',Ne)$ the quadratic terms in its Taylor series expansion may be neglected. Then for $f' \le {}^1/_{Ne\alpha}$, $Ne \to \infty$ we write:

$$S(f+f') \cong S(f) + f'S_{f'}(f) + \frac{f'^2}{2!} S_{f'f'}(f) + \ldots\ldots\ldots$$

$$S(f-f') \cong S(f) - f'S_{f'}(f) + \frac{f'^2}{2!} S_{f'f'}(f) + \ldots\ldots\ldots$$

and consequently

$$S(f+f')S(f-f') \cong S^2(f) + f'^2\{S(f)S_{f'f'}(f) - S^2_{f'}(f)\} + \ldots\ldots$$

$$\cong S^2(f) + O({}^1/_{Ne^2\alpha^2})$$

The last approximate equality follows since $S(f)$ is assumed bounded and continuous and $f' \le {}^1/_{Ne\alpha}$, $Ne \to \infty$. Consequently for $S(f)$ bandlimited to $|f| \le B$, we have

$$\Gamma(f) \cong S^2(f)\int_{-(B+f)}^{(B+f)} K(f',Ne)df' + O({}^1/_{Ne^2\alpha^2})$$

and in particular,

$$\mathrm{Var}\{\hat{S}(f_r)\} \cong \sigma(N_e)\int_{-\infty}^{\infty}\left[S(f)H_m(f,f_r)\right]^2 df + O({}^1/_{Ne^2\alpha^2})\int_{-\infty}^{\infty}H_m^2(f;f_r)\, df$$

where

$$\sigma(Ne) \triangleq \int_{-(B+f)}^{(B+f)} K(f',Ne)df \cong \int_{-\infty}^{\infty} K(f',Ne)df\ , \quad \text{for} \quad B >> {}^{1}/Ne\alpha$$

Note that,

$$\int_{-\infty}^{\infty} H_m^2(f,f_r)df \;=\; 4Q_m^2(0) \int_{-\infty}^{\infty}\left[Q_m(f)/Q_m(0)\right]^2 df$$

$$\leq 4\ Q_m(0)\int_{-\infty}^{\infty}|Q_m(f)|\ df \quad ;$$

so that

$$\mathrm{Var}\left\{\hat{S}(f_r)\right\} \cong \sigma(Ne)\int_{-\infty}^{\infty}\left[S(f)H_m(f,f_r)\right]^2 df + O({}^{m}/_{Ne}2)$$

$$= \frac{\sigma(Ne)\ E^2\left[\hat{S}(f_r)\right]}{We} \quad + \quad O(\frac{m}{Ne^2})\ ,$$

where the equivalent width, We , is defined as:

$$We \triangleq \left(\int_{-\infty}^{\infty} S(f)H_m(f;f_r)\ df\right)^2 \Bigg/ \int_{-\infty}^{\infty}\left[S(f)H_m(f;f_r)\right]^2 df \quad .$$

Now, for both the periodic and Poisson sampling schemes, we saw in Chapter II that $E\left\{\hat{S}(f_r)\right\}$ is approximately the same when the spectrum is restricted to the Nyquist band, and m is very large. Also, the equivalent width of estimate can be shown to approximate the corresponding window bandwidths. Since the bandwidths of the windows are for all practical purposes equal, we look to the quantity $\sigma(N_e)$ for a measure of the variability of estimates which we define as

$$\frac{\operatorname{var}\{\hat{S}(f_r)\}}{E^2\{\hat{S}(f_r)\}} \cong \frac{\sigma(Ne)}{We} + O(\frac{m^2}{Ne^2})$$

where for both schemes, $We \cong 1/_{m\alpha}$.

For the periodic scheme we have from section 3.3

$$\sigma_u(Ne) = 1/_{Ne\alpha} ,$$

and for the Poisson (cf eqn. 3.49)

$$\sigma_p(Ne) = \frac{1}{Ne^2\alpha} \sum_{i,k=1}^{Ne} \frac{1}{2^{i+k+1}(i+k+1)\ B(i+1,k+1)}$$

where $B(p,q)$ is the beta function.

Figure 9 exhibits graphical plots of $\sigma_p(Ne)$ and $\sigma_u(Ne)$ in the range $10 \leq Ne \leq 100$; computation of $\sigma_p(Ne)$ for large values of Ne consumes excessive machine time. However, to see that $\sigma_p(Ne)$ is monotone decreasing, note from figures 8 a,b,c, that $K(f',Ne)$ is positive definite and narrows with increasing Ne while its peak remains equal to 1 for all Ne . Consequently we conclude that $\sigma_p(Ne)$ is asymptotically bounded from above by $\sigma_u(Ne)$. Thus we can (cf section 4.2 below) say that it is possible to achieve a smaller variability with Poisson sampling than with the method of periodic sampling.

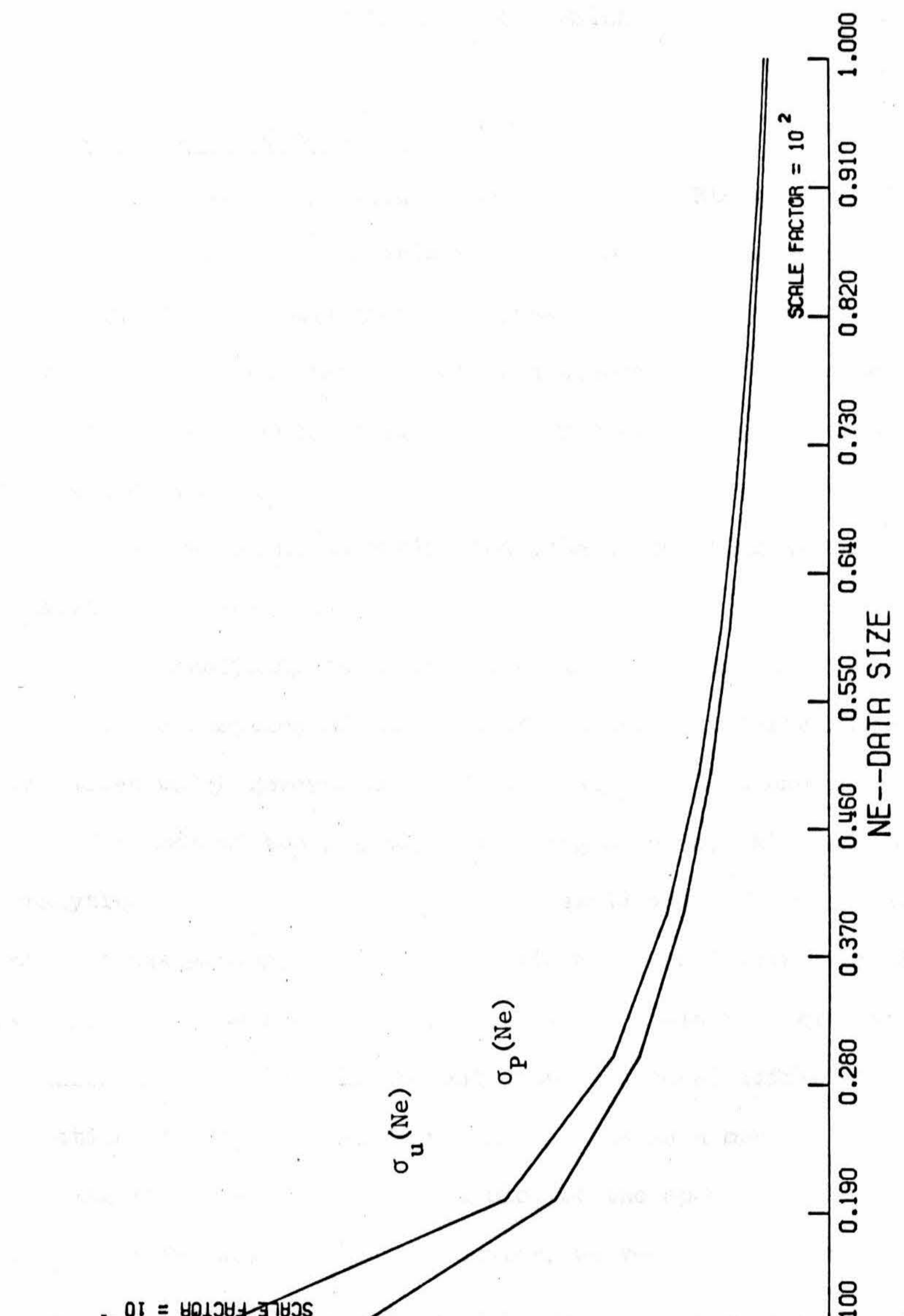

Fig. 9. Scaled Variance versus Data Size

CHAPTER IV

SUMMARY AND CONCLUSIONS

4.1 Summary and Conclusions

For readers unfamiliar with the work of Blackman and Tukey [1], the research reported in this thesis is by no means complete. Details on 'planning for measurement' - 'prewhitening', 'post greening', etc. have been purposely left out of this account to avoid undue repetition of the main reference cited above. Primarily this thesis has been concerned with:

(i) Outlining an estimation scheme admitting of data with random sampling intervals, and,

(ii) Analyzing the mean and variance of such a scheme.

The assumption of independent, identically distributed random variables which governs most of our analysis, is a useful one since it includes most of the practical sampling schemes. Although the analytical results contained in Chapters II and III seem to take advantage of this assumption, similar expressions can be derived for the most general sampling scheme. For this case, closed-form expressions for windows and kernels will present great analytical difficulties. On the question of aliasing, we found in this thesis a more practical way of testing for aliasing - obtain a plot of the spectral window, $Q_m(\omega)$ and check for maxima. In particular, we verified via an estimating algorithm that the Poisson sampling process is alias-free even for non-bandlimited spectra. One shortcoming of our algorithm is its non-general nature, since, in fact, it was tailored for the Poisson sampling process. As was pointed out in Chapter I, it is quite possible using techniques of numerical calculus to design algorithms to suit some particular sampling schemes. Our algorithm which invokes the rectangular approximation (instead of Blackman and Tukey's trapezoidal approximation) to the fourier cosine integral was found unsuitable for a detailed investigation of the rectangular sampling process.

The gaussian assumption to which most of the analysis in Chapter III has been subjected, is not merely for analytical convenience since we are dealing with very large data size and the central limit theorem validates this assumption even when the process is not gaussian. On variability and covariability of estimates we were able to obtain closed form analytical expressions of the Blackman-Tukey type for the case when sampling intervals are Poisson distributed. These results, though approximate, are just as useful as any others derived in this thesis since they make use of the governing assumption of very large data size. Making use of these results we found further, that the Poisson sampling process achieves a smaller variability than the periodic sampling process for spectra which are very smooth. On the other hand, for rapidly varying spectra the Poisson is just as unreliable as the periodic scheme. However, this is not to say that the algorithm of Blackman and Tukey should be discarded as we will see presently.

4.2 Poisson Vs Periodic Sampling

In the last section we inferred that for the same number of samples of the process the Poisson sampling scheme achieves a smaller variance than the periodic scheme. This presumes that data are available for as long as we want. In some cases of practical interest, the duration of data is limited somewhat, so that one is constrained to sample more closely to increase the number of samples or estimate from fewer record samples than needed. Already, we know that increasing the data size achieves smaller variance, so that it is quite possible that in a given time T we can obtain more samples by sampling periodically than with Poisson sampling. Even when signal duration is not limited, the question of how much longer one waits for say, N samples by sampling one way instead of the other, is worth looking into. Answers to these questions can be sought only in a probabilistic sense such as outlined below.

Let $T_N = \alpha_{i+1} + \alpha_{i+2} + \ldots + \alpha_{i+N}$ be the length of time it takes to obtain N data samples in a general non-uniform sampling scheme with mean sampling interval equal to α. If sampling is done uniformly every α secs., $T_N = N\alpha$ secs. For large N, T_N is asymptotically normal with mean

$$\mu(T_n) = E\left\{\sum_{\nu=k}^{k+n} \alpha_\nu\right\}$$

$$= \frac{\partial}{\partial_{i\Omega}} \Phi^N(\Omega) \Big|_{\Omega=0} = N\alpha \quad ,$$

$$E\left\{T_N^2\right\} = \frac{\partial^2}{\partial i\Omega^2} \Phi^N(\Omega) \Big|_{\Omega=0}$$

$$= N(N+1)\,\alpha^2$$

and variance,

$$\sigma^2(T_n) = E(T_n^2) - \mu^2(T_n) = N\,\alpha^2$$

From statistical tables we have

$$\text{Prob}\left\{ \mid T_n - N\alpha \mid < 3\,\sigma\,(T_n) \right\} = 0.99$$

i.e.

$$N\alpha - 3\alpha\sqrt{N} < T_n < N\alpha + 3\alpha\sqrt{N}$$

with probability 0.99.

Alternately, we write

$$1 - \frac{3}{\sqrt{N}} < \frac{T_n}{N\alpha} < 1 + \frac{3}{\sqrt{N}}$$

with probability 0.99.

Now, for the case when the waiting time T, is fixed, the probability of obtaining between $N-K$ and $N+K$ samples by Poisson sampling with mean rate $^1/_\alpha$ is given by

$$\Pr\left\{ N - K \leq N_T \leq N + K \right\} = \sum_{k = N-K}^{N+K} (^T/_\alpha)^k \, e^{-T/\alpha}/k\,!$$

In particular for $^T/_\alpha = 100$ we have from tables (cf [24]) :

$$\Pr\left\{ 90 \leq N_T \leq 110 \right\} = 0.683094$$

$$\Pr\left\{ 80 \leq N_T \leq 120 \right\} = 0.954319$$

$$\Pr\left\{ 70 \leq N_T \leq 130 \right\} = 0.997057$$

4.3 Practical Implications of the Random Sampling Scheme

In certain fields of application, prior statistical information on the sample times is not readily available, in which case, it may be necessary to record these sample times simultaneously. In some cases this will require modification of the existing recording hardware and possibly more computation time. On the other hand, the scheme allows a greater flexibility in data acquisition and errors due to small jitter are less significant in this scheme than they are in the Blackman and Tukey scheme. In what follows we outline the relevance of random sampling in some areas of application.

4.3.1 Communications Technology

In some coded communication systems the time of channel availability are random. By sampling a time sequence in a random manner corresponding to the random availability of the channel rather than sampling periodically, the need for buffer storage can be eliminated.

4.3.2 Seismic Data Processing

In future design of field experiments to investigate random seismic noise by means of correlation or spectral techniques, geophone arrays may now be set up randomly spaced in a manner that will optimize seismic data acquisition. It may also be possible to filter out, using appropriate geophone distributions, the propagating modes or coherent noise as is sometimes called. This will eliminate the need for delay lines as is the usual practice (cf [19]).

4.3.3 Oceanography

Whether it be swells from distant storms or ocean waves, the times of arrival are a random phenomenon and it will no doubt be more expedient and space-saving to take readings whenever they are available rather than periodically as is usually done. In this application arrival times have been known to follow certain well known distributions.

4.3.4 Structural Design

In the design of tall structures and aircraft, destructive power of atmospheric turbulence is of very great concern. Engineers must design to accomodate oscillations due to wind gusts whose magnitude and direction are random phenomena with random occurence times. Estimation of spectra of wind velocities and of ground motion (velocity and acceleration) due to earthquakes normally preceeds actual structural design.

Appendix A

A-1 : Behavior of $Q_m^P(\omega)$ for large ω .

Recall that $\varphi = \arctan \omega\alpha$ and write

$$\lim_{\omega \to \infty} Q_m^p(\omega) = 2\alpha \lim_{\varphi \to \pi/2} \sum_{n=o}^{m} \cos^{n+1}\varphi \cos(n+1)\varphi$$

$$2\alpha \lim_{\varphi \to \pi/2} \Big\{ \cos^2\varphi(1+\cos 2\varphi) + \cos^3\varphi\cos 3\varphi$$

$$+ \cos^4\varphi \cos 4\varphi + \ldots\ldots\ldots + \cos^{m+1}\varphi\cos(m+1)\varphi \Big\}$$

$$\lim_{m \to \infty} \text{RHS} \cong 2\alpha \lim_{\varphi \to \pi/2} \Big\{ 2\cos^4\varphi + \cos^3\varphi\cos 3\varphi + \cos^4\varphi\cos 4\varphi \Big\}$$

$$= 2\alpha \lim_{\epsilon \to o} \Big\{ 2\cos^4(\pi/_2-\epsilon) + \cos^3(\pi/_2-\epsilon)\cos(\tfrac{3\pi}{2}-3\epsilon)$$

$$+ \cos^4(\pi/_2-\epsilon)\cos(2\pi-4\epsilon) \Big\}$$

Expand the cosine functions noting that $\sin{}^\pi/_2 = 1$, $\cos{}^\pi/_2 = 0$ etc., and get

$$\lim_{\omega \to \infty} Q_m^p(\omega) = 2\alpha \lim_{\epsilon \to o} \Big\{ 2\sin^4\epsilon + \sin^4\epsilon\cos 4\epsilon - \sin^3\epsilon\sin 3\epsilon \Big\}$$

$$\cong 2\alpha \Big\{ 2\epsilon^4 + \epsilon^4(1 - 8\epsilon^2 + \ldots.) - 3\epsilon^4 \Big\}$$

$$= -16\,\alpha\,\epsilon^6$$

Now, $\pi/_2 - \epsilon = \tan^{-1}\omega\alpha \Rightarrow \epsilon \cong 1/_{\omega\alpha}$ so that

$$Q_m^p(\omega) = 0(-\epsilon^6)$$

$$= 0(1/\omega^6\alpha^6)$$

A-2 : Behavior of $Q_m^{PB}(\omega)$ for large ω .

$$Q_m^{PB}(\omega) = 2\alpha \left\{ \sum_{n=1}^{m} (1-\frac{n}{m}) \cos^{n+1}\varphi \cos(n+1)\varphi + \cos^2\varphi \right.$$

$$\left. - (1+{}^1/m) \cos^{m+2}\varphi \cos(m+2)\varphi \right\}$$

Note from A-1 that we can write

$$Q_m^{PB}(\omega) = Q_m^P(\Omega) - \frac{2\alpha}{m} \sum_{n=1}^{m} n \cos^{n+1}\varphi \cos(n+1)\varphi$$

$$- 2\alpha(1+{}^1/m) \cos^{m+2}\varphi \cos(m+2)\varphi$$

Proceeding as in A-1 we write

$$\lim_{\omega\to\infty} Q_m^{PB}(\omega) = \lim_{\omega\to\infty} Q_m^P(\omega) - \frac{2\alpha}{m} \lim_{\varphi\to\pi/2} \left\{ \cos^2\varphi\cos2\varphi \right.$$

$$\left. + 2\cos^3\varphi\cos3\varphi + 3\cos^4\varphi\cos4\varphi + \ldots\ldots \right\}$$

Now, $\lim_{\varphi\to\pi/2} \left\{ \cos^2\varphi\cos2\varphi + 2\cos^3\varphi\cos3\varphi + 3\cos^4\varphi\cos4\varphi \right\}$

$$= \lim_{\epsilon\to 0} \left\{ \cos^2({}^\pi/_2-\epsilon)\cos(\pi-2\epsilon) + 2\cos^3(\frac{\pi}{2}-\epsilon)\cos(\frac{3\pi}{2}-3\epsilon) \right.$$

$$\left. + 3\cos^4({}^\pi/_2-\epsilon)\cos(2\pi-4\epsilon) \right\}$$

$$= \lim_{\epsilon\to 0} \left\{ -\sin^2\epsilon \cos2\epsilon - 2\sin^3\epsilon\sin^3\epsilon + 3\sin^4\epsilon\cos4\epsilon \right\}$$

$$= \qquad - \epsilon^2(1-2\epsilon^2+\ldots) - 6\epsilon^4 + 3\epsilon^4(1-8\epsilon^2+\ldots)$$

$$\approx - \epsilon^2$$

Thus,

$$\lim_{\omega \to \infty} Q_m^{PB}(\omega) = \lim_{\epsilon \to 0} \left\{ - 16\alpha\epsilon^6 + \epsilon^2 \right\}$$

whence we say

$$Q_m^{PB}(\omega) = 0 \left(\frac{1}{m\omega^2\alpha^2} \right)$$

A-3 $\max\limits_{\Omega} \; Q_m^{PB}(\Omega)$

$$Q_m^{PB}(\omega) = 2\alpha \mathrm{Re} \left\{ \sum_{n=0}^{m} (1-\frac{n\alpha}{m}) \cos^{n+1}\varphi \cos(n+1)\varphi \right.$$

$$\left. - \alpha\ (1+{}^1/m) \cos^{m+2}\varphi \cos(m+2)\varphi \right\}$$

$$\frac{\partial Q_m^{PB}(\omega)}{\partial \varphi} = 2\ \alpha\ \mathrm{Re} \left\{ \Sigma\ (1 - \frac{n\alpha}{m})(n+1) \left[-\cos^n\varphi \cos(n+1)\ \varphi\ \sin\varphi \right. \right.$$

$$\left. - \alpha(1+{}^1/_m)(m+2) \left[\cos^{m+1}\varphi \cos(m+2)\varphi \sin\varphi - \cos^{m+2}\varphi \sin(m+2)\varphi \right] \right\}$$

$$= 0 \quad \text{when} \quad \varphi = 0\ ,\ \pi\ ,\ 2\pi\ ,\ 3\pi\ ,\ \ldots\ldots.$$

$$\frac{\partial^2 Q_m^{PB}(\omega)}{\partial\ \varphi^2} = -\ 2\alpha \left\{ \sum_{n=0}^{m} (1-\frac{n\alpha}{m})(n+1)(n+2)\cos^{n+1}\varphi \cos(n+1)\varphi \right.$$

$$\left. - (m+1)(m+2)^2\ \alpha\ \cos^{m+2}\varphi \cos(m+2)\ \varphi \right\}$$

It can be deduced from above that

$$\left.\frac{\partial^2 Q}{\partial \varphi^2}\right|_{\varphi = o,\ 2k\pi} < 0$$

Further, we note that

$$\sum_{n=o}^{m} \left(1-\frac{n\alpha}{m}\right) \cos^{n+1}\varphi \cos(n+1)\varphi < \sum_{n=o}^{m} \left(1-\frac{n\alpha}{m}\right)$$

which implies

$$\max_{\Omega} Q_m^{PB}(\Omega) = Q_m^{PB}(0)$$

Appendix B

B-1 Permutations of t_i, t_k, t_{k+n}, t_{i+j} subject to:

(1) $t_i \leq t_{i+j}$ and $t_k \leq t_{k+n}$, $n,j > 0$

First, assume also that

(2) $t_k \leq t_{i+j}$ and $t_i \leq t_{k+n}$

so that only the following four permutations are possible (subject to (1) and (2) above):

(3) $t_i \leq t_k < t_{i+j} < t_{k+n}$ (v)

(4) $t_i \leq t_k < t_{k+n} \leq t_{i+j}$ (vi)

(5) $t_k < t_i \leq t_{i+j} < t_{k+n}$ (iv)

(6) $t_k < t_i < t_{k+n} \leq t_{i+j}$ (iii)

Now relax restrictions in (2) to have from (3) and (6) respectively,

(7) $t_i < t_{i+j} \leq t_k < t_{k+n}$ (ii)

(8) $t_k < t_{k+n} \leq t_i < t_{i+j}$ (i)

An interval tree for the above permutations is illustrated in Figure B1. Note that the permutations are unique only as far as absolute inequalities are concerned.

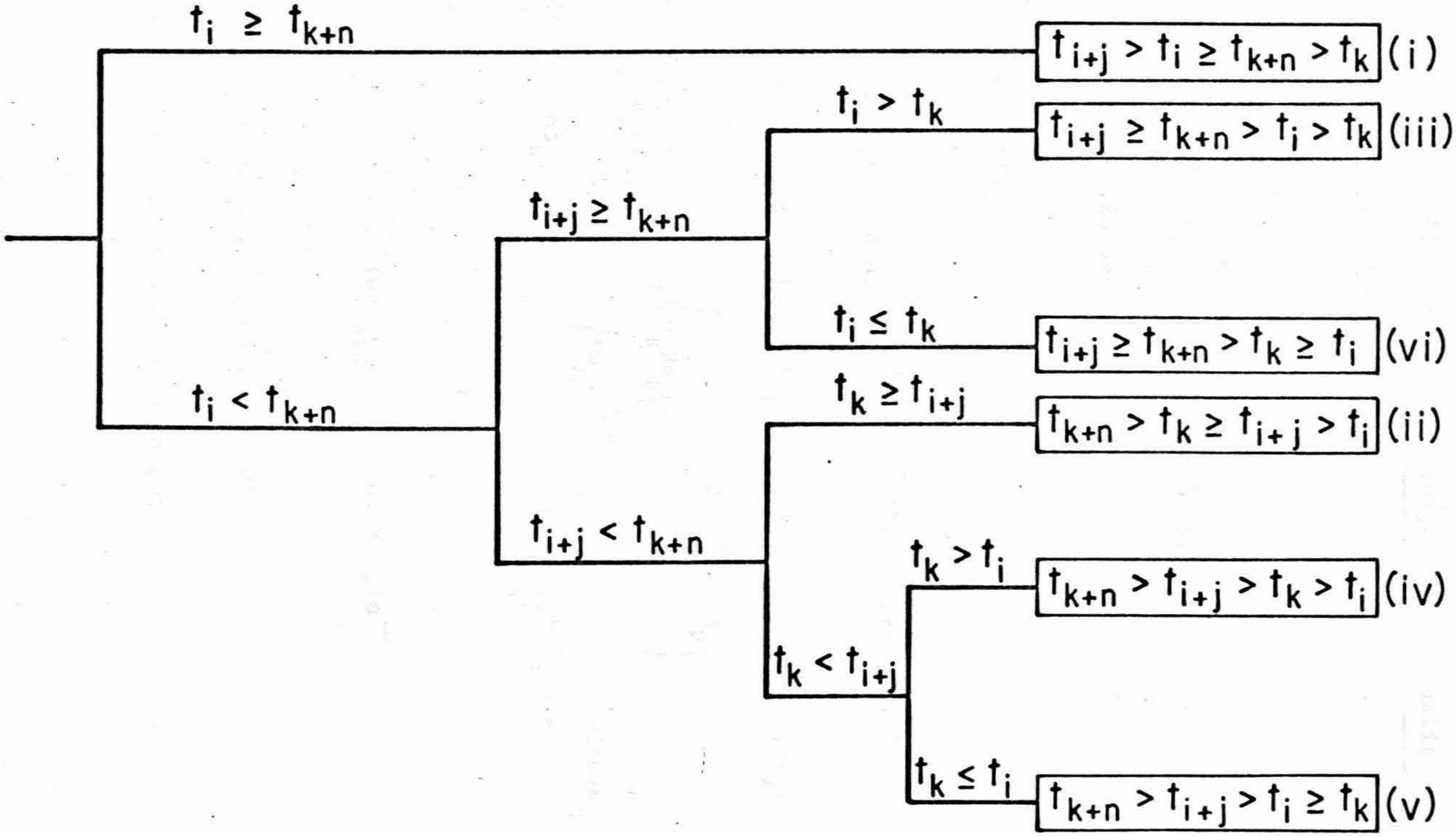

Fig. B1 A representative interval tree for the permutations derived in appendix B-1.

B-2 Derivation of $\lambda(\omega', \omega+\omega_r, \omega+\omega_q)$ subject to inequalities (i) through (vi) of Section B-1

(i) $\quad t_{i+j} > t_i \geq t_{k+n} > t_k$

$$\Longrightarrow t_i - t_k = t_i - t_{k+n} + t_{k+n} - t_k$$

so that equation 3.25 becomes:

$$\lambda_1^{(i)}(\omega', \omega+\omega_r, \omega+\omega_q)$$

$$= \frac{1}{N_e^2 \alpha^2} \sum_{R_1(i,j,k,n)} E_t \Big\{ \alpha_i \alpha_k (\alpha_{k+n} + \alpha\delta_{n1})(\alpha_{i+j} + \alpha\delta_{j1})$$

$$\cos[2\omega'(t_i - t_{k+n}) + (2\omega' + \omega + \omega_r)(t_{k+n} - t_k) - (\omega+\omega_q)(t_{i+j} - t_i)] \Big\}$$

$$= \frac{1}{N_e^2 \alpha^2} \sum_{R_1(i,j,k,n)} E_t \Big\{ \alpha_k \mathrm{Re}\Big[\Big(\alpha_i \, e^{i2\omega' \sum_{k+n+1}^{i} \alpha_\mu} \Big)$$

$$\Big((\alpha_{k+n} + \alpha\delta_{n1}) \, e^{i(2\omega' + \omega + \omega_r) \sum_{k+1}^{k+n} \alpha_\mu} \Big) \Big((\alpha_{i+j} + \alpha\delta_{j1}) \, e^{-i(\omega+\omega_q) \sum_{i+1}^{i+j} \alpha_\mu} \Big) \Big] \Big\}$$

Define:

$$\xi_\mu(\Omega) \triangleq \Phi^{\mu-1}(\Omega) \frac{\partial \Phi}{\partial i\Omega} , \quad \Phi(\Omega) = E_t\{e^{i\Omega\alpha}\}$$

and

$$\xi^*(\Omega) \triangleq -\Phi^{\mu-1}(-\Omega) \frac{\partial}{\partial i\Omega} \Phi(-\Omega)$$

and take expectation as indicated to get

$$\lambda_1^{(i)}(\omega',\omega+\omega_r,\omega+\omega_q)$$

$$= \frac{1}{N_e^2\,\alpha} \sum_{R_1(i,j,k,n)} \mathrm{Re}\quad \xi_{i-k-n}(2\omega')\ \xi_n(2\omega'+\omega+\omega_r) + \alpha\delta_{n1}\Phi(2\omega'+\omega+\omega_r)\Big)$$

$$\times\quad \Big(\xi_j^*(\omega+\omega_q) + \alpha\delta_{j1}\ \Phi^*(\omega+\omega_q)\Big)$$

Similarly, from equation 3.25a, obtain

$$\lambda_2^{(i)}(\omega',\omega+\omega_r,\omega+\omega_q)$$

$$= \frac{1}{N_e^2\,\alpha} \sum_{R_1(i,j,k,n)} \mathrm{Re}\Big\{\ \xi_{i-k-n}(2\omega')\Big(\xi_n(2\omega'-\omega-\omega_r)$$

$$+\ \alpha\delta_{n1}\ \Phi(2\omega'-\omega-\omega_r)\Big)\ \Big(\xi_j^*\ (\omega+\omega_q) + \alpha\delta_{j1}\Phi^*(\omega+\omega_q)\Big)\Big\}$$

To obtain equation 3.27, follow the above procedure and derive $\lambda_1^{(\ell)}$, $\lambda_2^{(\ell)}$ ℓ = (ii),$\cdots$,(vi), then note that equation 3.27 is given by

$$\lambda(\omega',\omega+\omega_r,\omega+\omega_q) = \sum_{\ell=1}^{6} \sum_{R_\ell(i,j,k,n)} \{\lambda_1^{\ell}(\omega',\omega+\omega_r,\omega+\omega_q) + \lambda_2^{\ell}(\omega',\omega+\omega_r,\omega+\omega_q)\}$$

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