Investigations into the Conditions Necessary for Stochastic Eternal Inflation

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Abstract

Theories of cosmological inflation, an early exponential expansion of the universe, have solved the horizon, flatness, and monopole problems in addition to successfully predicting properties of the fluctuations in the cosmic microwave background. Many of these theories have the property, known as eternal inflation, where inflation never ends everywhere at the same time and where there are always regions of exponentially expanding inflating space. The details of inflation are not known at this time and it would be interesting to estimate how generic eternal inflation is in the space of possible inflaton potentials. Of the several ways that inflation can be eternal, we focus here on the one, known as stochastic eternal inflation, where inflation is prevented from ending everywhere by quantum fluctuations in the inflaton field exceeding its classical motion. We argue that the conditions currently used to classify a trajectory as stochastically eternal are inadequate for general trajectories where the inflaton field may classically have a large velocity or be moving up its potential and are therefore ill-suited to studying how generic stochastic eternal inflation is. We propose an improved condition that takes these possibilities into account as well as more accurately calculating the quantum fluctuations using a perturbative Langevin method developed elsewhere. We investigate this condition in specific inflaton potentials and find examples where this condition deviates significantly from the one usually used in addition to finding examples where the mechanisms for eternal inflation are seemingly met even though space is not inflating.

Contents

Chapter 1

Introduction

1.1 Historical Introduction to Inflation

There were three seemingly unrelated problems with the standard big bang cosmological model in the 1970s [1]. The first, called the horizon problem, is that the cosmic microwave background (CMB) is observed to be extremely homogeneous over distances that would never have been within causal contact with each other in either a radiation or matter dominated universe. The second, called the flatness problem, is that to explain the observed flatness of the universe requires an extremely fine tuned curvature in the early universe. The standard big bang model does not explain how regions that were never in causal contact could be at the same temperature or why the curvature would have the value necessary to explain observations. The third problem, called the monopole problem, is that as the gauge group of grand unified theories is spontaneously broken by phase transitions as the universe cools, magnetic monopoles are produced in quantities that would easily have been detected by experiments [2].

Guth realized that a possible solution to the monopole problem that he was developing with Tye [3] could also solve the horizon and flatness problems at the same time [4]. In this theory, known as inflation, a scalar field is trapped in a false vacuum as in Fig. 1.1. While the field is trapped in the false vacuum, it quickly dominates the energy density and causes space to expand at an exponential rate. This expansion would end when the inflaton tunnels into the true vacuum corresponding to the current universe. In Ref. [3], the scalar field was the Higgs field but current inflationary research refers to the field as the inflaton and even allows for multiple inflaton fields. Inflation solves the horizon problem because the horizon expands more rapidly during inflation than the physical distance between two points. Thus, if the universe experienced an initial period of inflation, the homogeneous regions in the CMB would have been in causal contact. The flatness problem is solved since inflation rapidly drives the universe to the extremely small values of curvature necessary for the subsequent radiation dominated universe to evolve the curvature to its present day value. The monopole problem is solved if monopoles are produced before or during inflation and

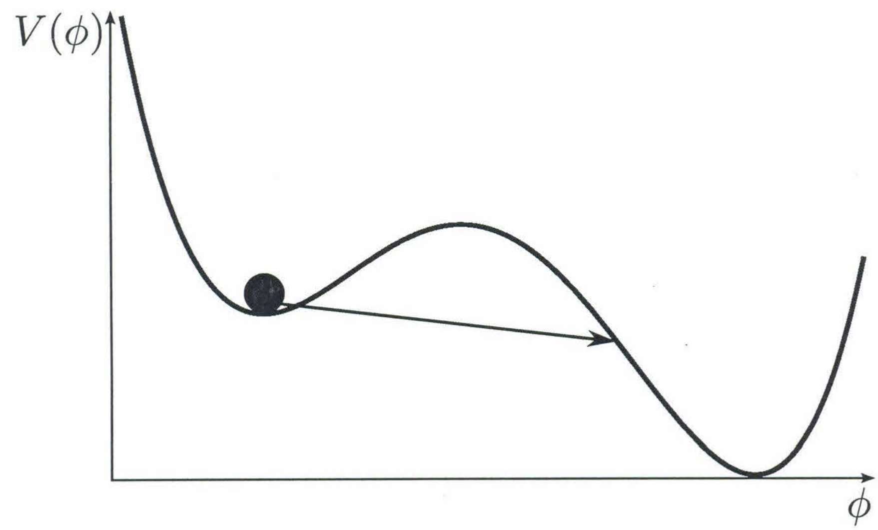

Figure 1.1: False vacuum driven inflation. If the rate for the field to tunnel from the false vacuum to the true vacuum is sufficiently small, inflation will be eternal.

then become very dilute compared to photons as the universe exponentially expands since photons are only produced during reheating as inflation ends. All three problems are solved simultaneously if space expands by more than approximately 60 e-folds during inflation [1].

As was noted in Ref. [4] and was further studied in Refs. [5, 6], the original proposal of Ref. [4] suffered from a problem known as the graceful exit problem. When the field tunnels to the true vacuum, it does so by nucleating a bubble of true vacuum within the background of false vacuum. The bubble of true vacuum proceeds to expand into the false vacuum, which is still exponentially expanding, and in doing so, almost all of its energy is at the bubble walls leaving the interior of the bubble essentially empty. All of the structure of the universe would then be concentrated in the walls of the bubble resulting in a highly inhomogeneous and anisotropic universe. The proposed solution to this problem presented in Ref. [4] is that thermalization occurs as multiple bubbles collide and merge to form a homogeneous and isotropic universe. However, this solution is not viable because bubbles of true vacuum nucleate into regions of inflating false vacuum and the space between bubbles expands too rapidly for the bubbles to collide and thermalize. If the decay rate for tunneling to the true vacuum were sufficiently increased, thermalization could occur; however, an increase in the tunneling decay rate by this amount would not allow for enough inflation to solve the three problems described above [5, 6].

Solutions to the graceful exit problem were quickly found by Linde [7] and Albrecht and Steinhardt [8]. In these theories, called new inflation, the field starts in the false vacuum of the Coleman-Weinberg potential [9] which has a small potential barrier. When bubbles tunnel, they start near the top of the potential and slowly roll down towards the true minimum. During this slow roll down

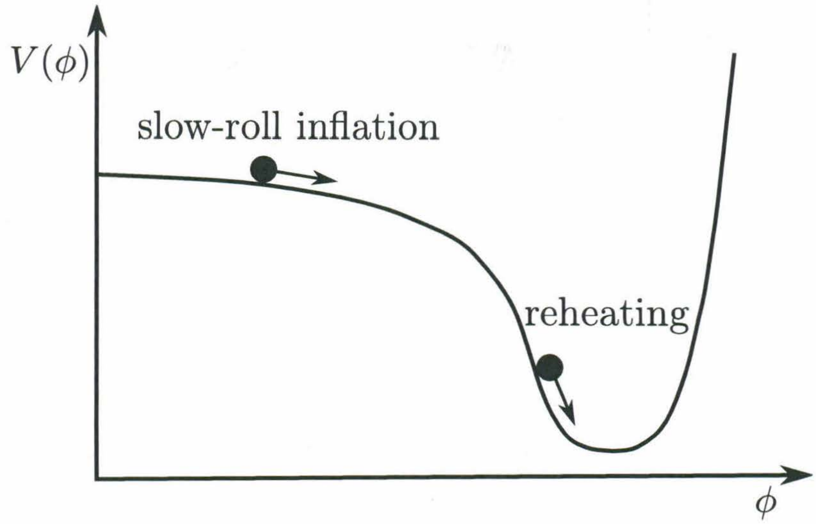

Figure 1.2: Example of an inflaton potential characteristic of new inflation. A false vacuum is sometimes added before the initial period of slow-roll inflation such that the field tunnels into the slow-roll region of the potential as was done in the original new inflation proposals.

the potential, the energy of the field is changing very slowly and space still expands at a nearly exponential rate the still solves all of the problems of Guth's original proposal. Inflation ends as the inflaton reaches the bottom of the potential and decays into standard model particles. This scenario solves the graceful exit problem since the expansion responsible for solving the cosmological problems occurs during the slowly rolling phase and thermalization occurs during reheating resulting in a homogeneous and isotropic universe. Although the original proposals include a false vacuum, the key to the success of new inflation is the period of slow-roll inflation rather than false vacuum driven inflation preceding the inflaton's decay to the current vacuum. Therefore, many new inflation models do not include the initial false vacuum. An example of a new inflationary potential is shown in Fig. 1.2.

1.2 Eternal Inflation

There are several ways for inflation to not end everywhere at once and for there always to be regions of inflating space. This phenomenon is know as eternal inflation and leads to a cosmological multiverse [10]

Eternal inflation and its role in continually creating new universes was first discussed by Vilenkin in the context of false vacuum driven inflation [11]. False vacuum driven eternal inflation occurs when the inflaton gets stuck in a false vacuum as with the original theory of inflation illustrated in Fig. 1.1. As was discussed with the graceful exit problem, if the bubble nucleation rate is sufficiently small, the inflating space between bubbles will expand faster than bubbles can nucleate and expand.

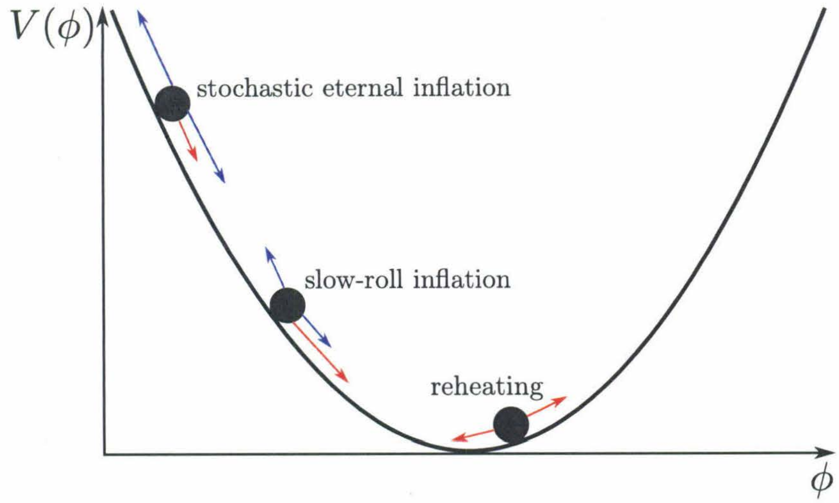

Figure 1.3: Stochastic eternal inflation in a quadratic potential. Quantum fluctuations are schematically shown in blue and classical motion is schematically shown in red.

In this case, there will always be regions of space where the field is in the false vacuum and can subsequently nucleate another bubble and inflation is eternal.

Linde discovered a second mechanism for inflation to be eternal due to quantum fluctuations of the inflaton field exceeding the classical motion of the field [12–14]. This type of eternal inflation is sometimes called chaotic eternal inflation; however, to avoid confusion with other types of chaotic inflation [15], we will refer to this as stochastic eternal inflation. Stochastic eternal inflation is illustrated in Fig. 1.3. Quantum fluctuations in the field are roughly proportional to $\sqrt{V}$. Thus, if the field is high enough on the potential, it will have large quantum fluctuations which could either kick the field down or up the potential. There will be regions of space where the inflaton moves down the potential, experiences the usual slow-roll inflation where quantum effects are small, and reheats as it approaches the minimum. However, there will also be regions of space where the inflaton moves up the potential and continues to expand. The volume of this inflating space continues to expand indefinitely and regions of space where inflation ends and the inflaton reheats will be eternally produced.

The third way inflation can be eternal, illustrated in Fig. 1.4, is due to domain walls in the inflaton field and is called topological eternal inflation [16, 17]. If the field is near the maximum of a potential separating two degenerate vacua, it can classically roll away to either vacua and a domain wall will form separating the two domains. If the domain wall is sufficiently thick, it will expand and new regions of space will continually be created where the field can roll into either domain. Each time a portion of the domain wall rolls into one of the vacua, it reheats as it reaches the minimum and forms a pocket universe similar to the situation with stochastic eternal inflation.

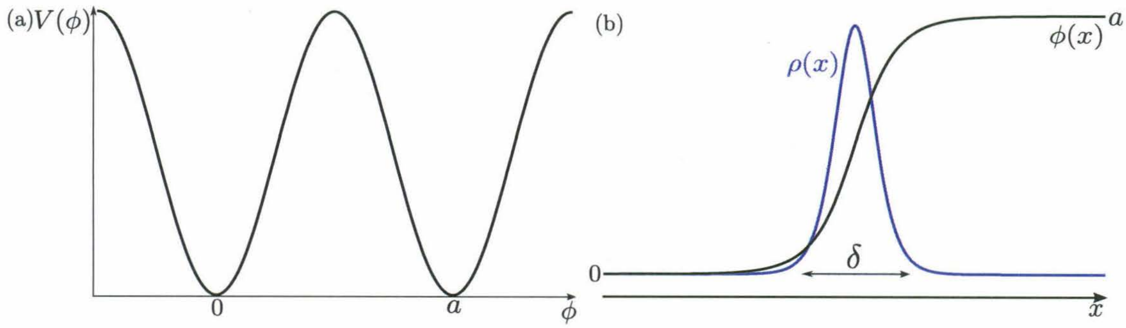

Figure 1.4: Topological eternal inflation. (a): an example of a potential that has vacua at $\phi = 0$ and $\phi = a$. (b): a kink solution interpolates between the vacua at $\phi = 0$ and $\phi = a$ in one spatial dimension. The field is solution is shown in black and the energy density is shown in blue. In more dimensions this kink solution becomes a domain wall and can be the source of eternal inflation provided the thickness δ is sufficiently large.

The details of inflation are not known at this time and the inflaton potential or even whether inflation is driven by a single field or multiple fields are not known. It would be useful to estimate how generic eternal inflation is in the space of possible potentials and initial field conditions to see how likely it is that inflation implies the existence of a multiverse [10]. Due to the general features of eternal inflation described above many people argue that eternal inflation is generic but there have been no quantitative investigations into this question to our knowledge.

There have been many investigations into inflation in random potentials [18–26]. In particular, Refs. [18–20] investigated random Fourier series potentials to study generic inflationary predictions for cosmological observables. It would be interesting to extend these studies to the question of eternal inflation. One could generate many random potentials and numerically find the initial conditions for each potential that would support eternal inflation. By studying a large number of potentials, one could quantitatively estimate how generic each of the three types of eternal inflation are.

1.3 Conditions for Stochastic Eternal Inflation

The program of investigating how generic stochastic eternal inflation is in random potentials described above cannot presently be carried out for two reasons. First, a satisfactory measure on the space of trajectories is not known at this time. The canonical measure on the space of trajectories [27] has a singularity at zero curvature [28] and it is not currently known how to deal with this singularity. How this singularity is dealt with is at least partially responsible for vastly differing estimates for the likeliness of getting 60 e-folds of inflation in, for example, the analysis of a quadratic potential [29, 30]. Ref. [29] finds the probability to be extremely close to unity while Ref. [30] finds the probability to be extremely close to zero. Therefore, even if the regions of phase space supporting stochastic eternal inflation could accurately be found for a large number of random potentials, this knowledge could not be converted into a meaningful probability. Second, we argue that the conditions currently used in the literature to check whether a trajectory is stochastically eternal are not applicable to generic cases where the field could have a large velocity or could be moving up the potential. Both of these cases would be prevalent in the study described above.

In this work, we focus on the second problem and investigate the conditions necessary for eternal inflation. Much of the literature on eternal inflation deals only with slow-roll inflation and, as far as we are aware, all of the conditions for stochastic eternal inflation are only applied when the field is slowly rolling. Inflation when the field has a large velocity have been studied but not in the context of eternal inflation. Refs. [31, 32] discusses fast-roll inflation as a mechanism for generating the initial conditions necessary for inflationary theories relying on slow-roll inflation but do not discuss eternal inflation during a period of fast-roll inflation. Ref. [33] further considers the dynamics of fast-roll inflation, without considering eternal inflation, and Ref. [34] showed that fast-roll inflation models are consistent with observations.

In Sec. 2.3 we generalize the conditions necessary for stochastic eternal inflation so as to take into account the possibility of fields with large velocities and fields classically moving up the potential. We also discuss the generalization of these conditions to the case of multiple inflaton fields. Furthermore, standard methods for dealing with the quantum fluctuations in stochastic eternal inflation usually cite results [11, 35–38] that are derived only for a free field. The justification for this is not discussed and it is not clear that these results are applicable to generic potentials. In addition to the improvements made to the conditions for eternal inflation, we use the perturbative Langevin method of Ref. [39] to more accurately compute these fluctuations. This method, along with a short discussion of the other methods commonly used to deal with these quantum fluctuations, is described in Ch. 3.

In Ch. 4, we apply the conditions derived in Sec. 2.3 to the quadratic and sine-Gordon potentials in addition to a random Fourier series potential typical of the kind studied in Refs. [18–20] that would be useful for studying how generic eternal inflation is. We find that the conditions are not generally accurate for the Fourier series potential but find examples where they are well suited for studying the other potentials. We compare the improved conditions with the conditions typically used in the literature and find examples where the two conditions greatly differ. Additionally, we find examples of trajectories satisfying the conditions for eternal inflation even while the field is not inflating, a possibility not previously discussed to our knowledge. Unfortunately, the approximations needed for the perturbative Langevin method to be valid are not always met so these results serve only as evidence that current methods are inadequate for studying generic inflationary trajectories. Most of the original work for this thesis is contained in Sec. 2.3 and Ch. 4.

In the following, we use natural units where $\hbar = c = 1$ and work with the reduced Planck mass

$\overline{m}_{\rm pl} \equiv m_{\rm pl}/\sqrt{8\pi} = 1/\sqrt{8\pi G} = 2.4 \times 10^{18}$ GeV. We use a metric with signature $(-,+,+,+)$. For spacetime indicies, Greek indicies run over space and time components while lowercase Latin indicies run only over space components. Uppercase Latin indicies label fields within a vector of fields.

Chapter 2

Inflationary Dynamics

2.1 Equations of Motion

Before studying the conditions necessary for stochastic eternal inflation, we review the dynamics of inflation. We will study the evolution of scalar fields in the background of the Friedmann-Robertson-Walker (FRW) metric

$$\mathrm{d}s^2 = -N^2(t)\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right] \tag{2.1}$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor. k is the curvature with $k = 0, +1$, and -1 corresponding to flat, closed, and open geometries respectively. With coordinates $x^\mu = (t, \mathbf{x})$, t and $\mathbf{x}$ are the comoving coordinates while $N(t)t$ and $a(t)\mathbf{x}$ are the physical coordinates. The Hubble parameter is defined as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{2.2}$$

The action for d real scalar fields $\boldsymbol{\phi} = (\phi_1, \dots, \phi_d)$ minimally coupled to the metric is[1]

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left[\frac{\overline{m}_{\mathrm{pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi_I - V(\boldsymbol{\phi})\right] \tag{2.3}$$

where g is the determinant of $g_{\mu\nu}$, R is the Ricci scalar, $V(\boldsymbol{\phi})$ is the potential, and all repeated indicies are summed. More general kinetic terms of the form $-1/2g^{\mu\nu}G^{IJ}(\boldsymbol{\phi})\partial_\mu\phi_I\partial_\nu\phi_J$ have been studied, for example in Ref. [40], and actions that are more general functions of $g^{\mu\nu}\partial_\mu\phi_I\partial_\nu\phi_J$ have been studied in Dirac-Born-Infeld (DBI) inflation motivated by string theory [41, 42]. We will only consider the canonical kinetic term in Eq. (2.3) where the field space metric is δ_{IJ}. As is shown in

[1] If $[A]$ denotes the mass dimension of the quantity A, then $[N(t)] = 0$ and $[a(t)] = -1$. This leads to slightly strange dimensions in the action since, in this notation, $[x^0 = t] = -1$ but $[x^i] = 0$. $[\sqrt{-g}] = -3$ since $[g_{00}] = 0$ and $[g_{ii}] = -2$. Since $[\mathrm{d}^4x] = -1$, the volume form has mass dimension -4 as it should. Similarly, $[\partial_0] = 1$, $[\partial_i] = 0$, $[g^{00}] = 0$, and $[g^{ii}] = 2$ so that the kinetic term has mass dimension 4 and $[S] = 0$, as it should, if $[\phi_I] = 1$.

Appendix B, for the FRW metric,

$$R = 6\left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2} - \frac{\dot{a}\dot{N}}{N^3 a}\right).$$

Thus,

$$S = \int \mathrm{d}^4 x N a^3 \left[3\overline{m}_{\rm pl}^2 \left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2} - \frac{\dot{a}\dot{N}}{N^3 a}\right) + \frac{1}{2N^2}\dot{\phi}^I \dot{\phi}_I - \frac{1}{2a^2}\left(\nabla\phi^I\right)\cdot\left(\nabla\phi_I\right) - V(\phi)\right]$$

where

$$\left(\nabla\phi^I\right)\cdot\left(\nabla\phi_I\right) \equiv a^2 g^{ij}\partial_i\phi^I\partial_j\phi_I.$$

Integrating by parts and assuming boundary conditions such that surface terms vanish

$$\int \frac{\ddot{a}a^2}{N}\,\mathrm{d}t = -\int\left(\frac{2a\dot{a}^2}{N} - \frac{a^2\dot{a}\dot{N}}{N^2}\right)\mathrm{d}t.$$

Therefore, the Lagrangian density is

$$\mathscr{L} = -\frac{3\overline{m}_{\rm pl}^2 a\dot{a}^2}{N} + 3\overline{m}_{\rm pl}^2 Nak + \frac{a^3}{2N}\dot{\phi}^I\dot{\phi}_I - \frac{Na}{2}\left(\nabla\phi^I\right)\cdot\left(\nabla\phi_I\right) - Na^3 V(\phi). \tag{2.4}$$

There are no time derivatives of N which is therefore acting as a constraint. The Euler-Lagrange equations for N give

$$\frac{\partial\mathscr{L}}{\partial N} = \frac{3\overline{m}_{\rm pl}^2 a\dot{a}^2}{N^2} + 3\overline{m}_{\rm pl}^2 ak - \frac{a^3}{2N^2}\dot{\phi}^I\dot{\phi}_I - \frac{a}{2}\left(\nabla\phi^I\right)\cdot\left(\nabla\phi_I\right) - a^3V(\phi) = 0. \tag{2.5}$$

Since N is a constraint, we will set $N = 1$ from now on. Then Eq. (2.5) becomes the Friedmann equation

$$H^2 = \frac{1}{3\overline{m}_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^I\dot{\phi}_I + \frac{1}{2a^2}\left(\nabla\phi^I\right)\cdot\left(\nabla\phi_I\right) + V(\phi)\right] - \frac{k}{a^2}. \tag{2.6}$$

Since

$$\frac{\partial\mathscr{L}}{\partial\phi_I} = -a^3\frac{\partial V}{\partial\phi_I} \qquad \text{and} \qquad \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_I)} = a^3\ddot{\phi}_I + 3a^2\dot{a}\dot{\phi}_I - a\nabla^2\phi_I,$$

the Euler-Lagrange equations for ϕ_I give the scalar field equation

$$\ddot{\phi}_I + 3H\dot{\phi}_I - \frac{1}{a^2}\nabla^2\phi_I + \frac{\partial V}{\partial\phi_I} = 0 \tag{2.7}$$

where

$$\nabla^2\phi_I \equiv a^2\partial_i(g^{ij}\partial_j\phi_I).$$

The expansion of the scale factor gives rise to an effective friction term proportional to H.

During inflation the fields are smoothed out in space very rapidly and the approximation that spacial derivatives of the fields can be neglected is valid. Furthermore, during inflation, a grows exponentially so that the term k/a^2 in Eq. (2.6) rapidly becomes negligible even if $k \neq 0$ and this term can be neglected. In this case, the Friedmann equation Eq. (2.6) becomes

$$H^2 = \frac{1}{3\overline{m}_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^I\dot{\phi}_I + V(\phi)\right] \tag{2.8}$$

and the scalar field equation Eq. (2.7) becomes

$$\ddot{\phi}_I + 3H\dot{\phi}_I + \frac{\partial V}{\partial \phi_I} = 0. \tag{2.9}$$

These are the forms of the equations that we will use. In general, we will be interested in trajectories that are initially not inflating or which are inflating only very briefly. The choice of $k = 0$ is further motivated by the fact that the canonical measure on the space of trajectories [27] diverges for zero curvature [28]. This divergence can be thought of as a delta function at zero curvature [43] even though an explicit form for this measure is not known at this time.

2.2 Slow-Roll Parameters

In this section, we describe the slow-roll parameters useful for characterizing inflationary trajectories. These parameters can be written either in terms of the potential V or in terms of the Hubble parameter H [44]. However, only the parameters written in terms of H are accurate if the field is not slowly rolling, which is the generic situation we wish to consider, so we will only deal with the Hubble slow-roll parameters.

The first slow-roll parameter defined as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \tag{2.10}$$

characterizes whether a trajectory is inflating or not. Since

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2,$$

ϵ is also given by

$$1 - \frac{\ddot{a}}{aH^2}. \tag{2.11}$$

From this form, it is clear that the condition that space is undergoing accelerated expansion $\ddot{a} > 0$ is equivalent to $\epsilon < 1$. In the following analysis, we will consider an inflaton trajectory as inflating at a given time if $\epsilon < 1$ at that time.

To compute ϵ for a given trajectory, it is necessary to know $\dot{H}$. From Eqs. (2.8) and (2.9),

$$2H\dot{H} = \frac{1}{3\overline{m}_{\rm pl}^2}\left(\ddot{\phi}^I\dot{\phi}_I + \frac{\partial V}{\partial \phi_I}\dot{\phi}_I\right) = -\frac{H}{\overline{m}_{\rm pl}^2}\dot{\phi}^I\dot{\phi}_I,$$

so

$$\dot{H} = -\frac{1}{2\overline{m}_{\rm pl}^2}\dot{\phi}^I\dot{\phi}_I. \tag{2.12}$$

Since $\dot{H} \leq 0$, it is clear from Eq. (2.10) that $\epsilon \geq 0$. From Eqs. (2.10) and (2.12),

$$\epsilon = 3\frac{\dot{\phi}^I\dot{\phi}_I/2}{\dot{\phi}^I\dot{\phi}_I/2 + V(\phi)}. \tag{2.13}$$

Therefore $\epsilon < 1$ if and only if $\dot{\phi}^I\dot{\phi}_I/2 < V$. A trajectory is inflating as long as the kinetic energy of the fields is less than the potential energy.

The second slow-roll parameter defined as

$$\eta \equiv \frac{\ddot{H}}{H\dot{H}} \tag{2.14}$$

characterizes the importance of the second derivative terms in Eq. (2.9). Unlike ϵ, η can be positive or negative. If $|\eta| \ll 1$, then $\ddot{\phi}_I$ can be neglected in calculating the classical trajectories. From Eq. (2.12),

$$\ddot{H} = -\frac{1}{\overline{m}_{\rm pl}^2}\ddot{\phi}^I\dot{\phi}_I. \tag{2.15}$$

In the case of a single field, η is proportional to the ratio of the acceleration to the friction term in Eq. (2.9):

$$\eta = \frac{\ddot{\phi}}{2H\dot{\phi}}. \tag{2.16}$$

2.3 Conditions For Stochastic Eternal Inflation

Stochastic eternal inflation occurs when the quantum fluctuations of the inflaton field exceed the classical fluctuations thus allowing the field to get "stuck" up high on the potential in certain regions of space which continue to inflate. For simplicity, we will first consider the case of a single inflaton field ϕ and will generalize the discussion to multiple fields in Sec. 2.3.3.

The change in the field in a Hubble time over the interval $[t_0, t_0 + H_0^{-1}]$ can be decomposed into the classical and quantum contributions as

$$\Delta\phi(t_0) = \Delta\phi_{\rm cl}(t_0) + \Delta\phi_{\rm qu}(t_0) \tag{2.17}$$

where $\Delta\phi_{\rm cl}$ is the classical change in the field and $\Delta\phi_{\rm qu}$ is the change in the field from the quantum noise. $\phi_{\rm cl}(t)$ is the solution to Eq. (2.9) so that

$$\Delta\phi_{\rm cl}(t_0) = \phi_{\rm cl}(t_0 + H_0^{-1}) - \phi_{\rm cl}(t_0). \tag{2.18}$$

The simplest condition for stochastic eternal inflation, and the one frequently used in the literature, is that inflation will be stochastically eternal provided that $|\Delta\phi_{\rm qu}| > |\Delta\phi_{\rm cl}|$. In this case, the quantum fluctuations exceed the classical roll of the field so it is just as likely that the inflaton will roll up the potential as it is that it will roll down. It is often quoted, almost universally in discussions on stochastic eternal inflation, that in one Hubble time , $|\Delta\phi_{\rm qu}| = H/2\pi$ so that the condition for stochastic eternal inflation to occur is

$$\frac{H_0}{2\pi} > |\Delta\phi_{\rm cl}(t_0)|\,. \tag{2.19}$$

There are several problems with this condition. First, it does not take into account the probability of the field moving up the potential or how the expansion of the inflating space affects this probability. As we will see in Sec. 2.3.1, these effects make it easier to have stochastic eternal inflation than the condition Eq. (2.19). Second, it does not take into account the direction of the classical roll; for example, if the field moves up the potential classically without any quantum fluctuations. Finally, using $|\Delta\phi_{\rm qu}| = H/2\pi$ during a Hubble time is not a good approximation for generic potentials. This last point is discussed in detail in Ch. 3.

2.3.1 Accounting for the Expansion of Inflating Space

The conditions for stochastic eternal inflation including the probability of the field moving up the potential are discussed in Ref. [45]. Here we review this argument with slight modifications.

Correlations in the inflaton field extend up to approximately a Hubble length so the field is approximately constant in a Hubble volume. Let ϕ_0 be the average value of the field over a given Hubble volume H_0^{-3} at time t_0. If the field is inflating at t_0, then at $t_1 = t_0 + H_0^{-1}$, the volume of the initial space will have increased by a factor of $\mathrm{e}^3 \approx 20$. Since correlations in the field extend up to a Hubble length, the initial volume will break up into $\mathcal{N}_{\rm dS} = \mathrm{e}^3$ independent Hubble volumes. This process is schematically illustrated in Fig. 2.1. The classical value of the field $\phi_{\rm cl}(t_1)$ can be computed from Eq. (2.9) with the initial conditions ϕ_0 and $\dot{\phi}_0$. Then in the nth independent Hubble volume the average value of the field will be the classical value plus quantum noise $\Delta\phi_{\rm qu}$:

$$\phi_n(t_1) = \phi_{\rm cl}(t_1) + \Delta\phi_{\rm qu}.$$

Thus, if the probability that $V(\phi_n(t_1)) \geq V(\phi_0)$ in one of these e^3 independent regions is greater than e^{-3}, there is at least one region where the field has not moved down the potential and inflation

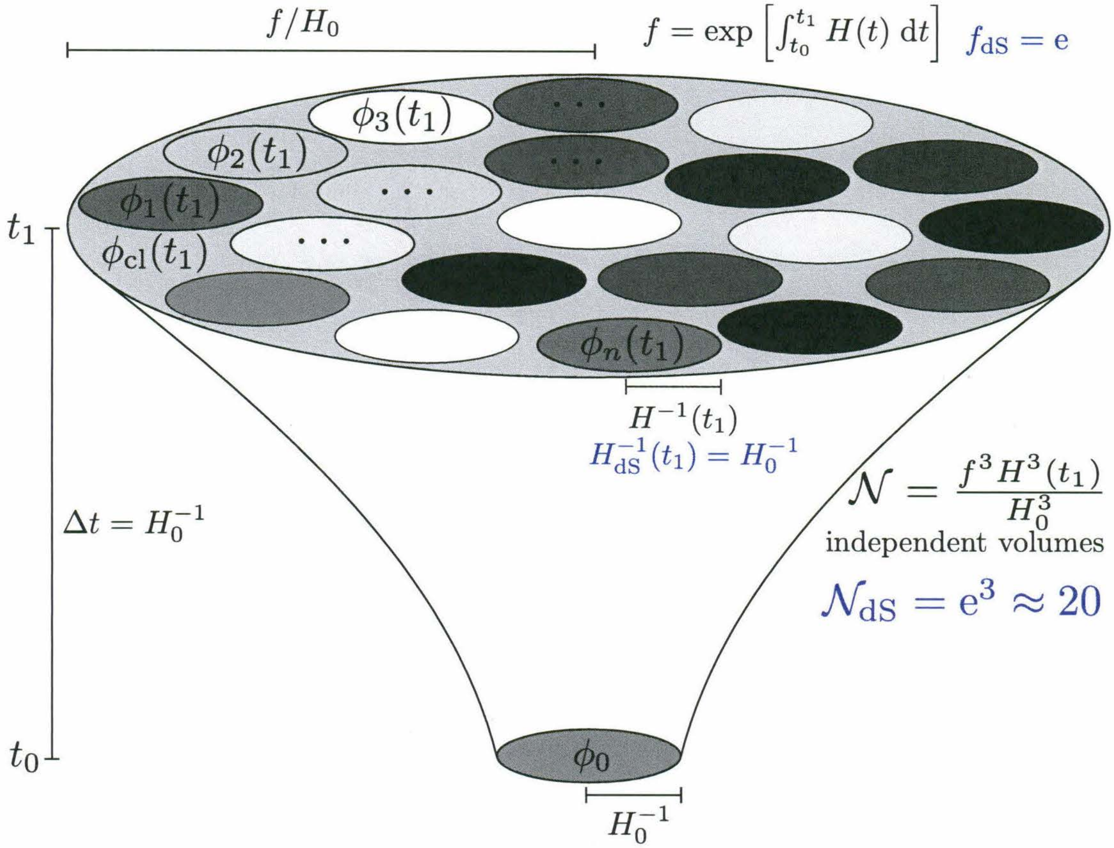

Figure 2.1: Schematic illustration of the evolution of a homogeneous initial Hubble volume. Quantities specific for pure de Sitter space are marked in blue. The value of the field within a Hubble volume is schematically illustrated by the shade of the volume. ϕ_0 is classically evolved to $\phi_{\mathrm{cl}}(t_1)$. The initial volume expands by a factor f^3 and breaks up into $\mathcal{N}$ independent Hubble volumes with an average value of $\phi_n(t_1) = \phi_{\mathrm{cl}}(t_1) + \Delta\phi_{\mathrm{qu}}$ throughout the nth volume. The quantum noise $\Delta\phi_{\mathrm{cl}}$ leads to each independent volume deviating from $\phi_{\mathrm{cl}}(t_1)$. For example, ϕ_1 is larger than, ϕ_2 is approximately the same as, and ϕ_3 is smaller than ϕ_{cl}.

is stochastically eternal.

For convenience, we define the quantity

$$\widetilde{\Delta\phi}(t_0) \equiv \Delta\phi(t_0)\,\mathrm{sgn}\left[\left.\frac{\mathrm{d}V}{\mathrm{d}\phi}\right|_{\phi=\phi(t_0)}\right] \tag{2.20}$$

which has the magnitude of $\Delta\phi(t_0)$ and which is always positive for trajectories moving up the potential $V(\phi) > V(\phi_0)$ and negative for trajectories moving down the potential. The field does not roll down the potential if

$$\widetilde{\Delta\phi}(t_0) \geq 0 \quad \Rightarrow \quad \widetilde{\Delta\phi}_{\mathrm{qu}}(t_0) \geq -\widetilde{\Delta\phi}_{\mathrm{cl}}(t_0). \tag{2.21}$$

Thus, for inflation to be eternal,

$$P_c(\widetilde{\Delta\phi}_{\rm qu} \geq -\widetilde{\Delta\phi}_{\rm cl}) = 1 - \Phi_c(-\widetilde{\Delta\phi}_{\rm cl}) \geq {\rm e}^{-3}. \tag{2.22}$$

The probability distribution for the quantum change in the field during a Hubble time is usually cited as

$$P_c(\Delta\phi_{\rm qu}(t)) = \sqrt{\frac{2\pi}{H^2(t)}}\exp\left\{-\frac{2\pi^2\left[\Delta\phi_{\rm qu}(t)\right]^2}{H^2(t)}\right\}, \tag{2.23}$$

although this is not in general a good approximation. This issue is discussed further in Ch. 3. Using Eq. (2.23), the cumulative distribution function for $\Delta\phi_{\rm qu}$ is

$$\Phi_c(\Delta_{\rm qu}) = \frac{1}{2}\left[1 + {\rm erf}\,\frac{\sqrt{2}\pi\Delta\phi_{\rm qu}}{H(t_0)}\right],$$

so Eq. (2.22) is equivalent to

$$1 - 2{\rm e}^{-3} \geq {\rm erf}\left[-\frac{\sqrt{2}\pi\widetilde{\Delta\phi}_{\rm cl}(t_0)}{H(t_0)}\right].$$

The solution to this inequality gives the condition for stochastic eternal inflation

$$c_{\rm dS} \equiv 1.165 \geq -\frac{\sqrt{2}\pi\widetilde{\Delta\phi}_{\rm cl}(t_0)}{H(t_0)}. \tag{2.24}$$

To compare with Eq. (2.19), this is equivalent to $H(t_0)/2\pi \geq -0.607\widetilde{\Delta\phi}_{\rm cl}(t_0)$. Ref. [45] uses the condition $\Delta\phi_{\rm qu} > |\Delta\phi_{\rm cl}|$ instead of the condition $\widetilde{\Delta\phi}_{\rm qu} \geq -\widetilde{\Delta\phi}_{\rm cl}$ which leads to replacing $-\widetilde{\Delta\phi}_{\rm cl}$ with $|\Delta\phi_{\rm cl}|$ in Eq. (2.24). Both conditions make it slightly easier to have eternal inflation than Eq. (2.19) which requires the standard deviation of the quantum fluctuations to exceed the classical roll. This is too strict since the initial Hubble volume breaks up into e^3 independent Hubble volumes and the field only needs to move up the potential in one of these volumes for inflation to be eternal.

We argue that using $\widetilde{\Delta\phi}_{\rm qu}$ more accurately captures the physics of stochastic eternal inflation, however. The right hand side of Eq. (2.24) can have any sign while it would be non-negative using $|\Delta\phi_{\rm cl}|$. In particular, if the field classically moves up the potential, $\widetilde{\Delta\phi}_{\rm cl} > 0$ and the inequality is satisfied for any value of H. This is a reasonable scenario for generic initial conditions where it is possible for the field to initially be moving up the potential leading to a large $\widetilde{\Delta\phi}_{\rm cl}$ that still results in $\widetilde{\Delta\phi} \geq 0$ even if the magnitude of the classical roll is larger than the magnitude of the quantum fluctuations.

In fact, if the field is initially moving up the potential, both choices of $-\widetilde{\Delta\phi}_{\rm cl}$ and $|\Delta\phi_{\rm cl}|$ predict eternal inflation at some point, while $-\widetilde{\Delta\phi}_{\rm cl}$ will predict eternal inflation the entire time the field is moving up the potential. If the field is moving up the potential there will be some value t^* such that from t^* to $t^* + H^{-1}(t^*)$ the field classically rolls up the potential turns around and returns to its starting point such that $\Delta\phi_{\rm cl}(t^*) = 0$. In this case, there will be a neighborhood B around t^* such that $H(t)/2\pi \geq 0.607\,|\Delta\phi_{\rm cl}(t)|$ for all $t \in B$.

2.3.2 Accounting for Non-Slow-Roll Inflation

The above derivations of Eq. (2.24) assume that the space-time is truly de Sitter space and that the Hubble parameter is constant for the entire trajectory. This is an excellent approximation for slow-roll inflation when $\epsilon \ll 1$ but is unsatisfactory for our general purposes where $\dot{H} = -\dot{\phi}^2/2\overline{m}_{\rm pl}^2$ can be large. Furthermore, numerical investigations show that it is possible, even somewhat likely, for the condition Eq. (2.24) to be met but for the inflaton field to not be inflating. This raises the possibility of having the self-reproducing structure of stochastic eternal inflation without inflation. To properly investigate this possibility and eternal inflation in general when $\dot{\phi}$ can be large, Eq. (2.24) has to be modified to account for the effects of non-constant H.

In general, during the Hubble time length interval from t_0 to $t_1 = t_0 + H_0^{-1}$, the physical volume of the initial space will expand by

$$f^3 = \exp\left[3\int_{t_0}^{t_0+H_0^{-1}} H(t)\;{\rm d}t\right]. \tag{2.25}$$

However, assuming that correlations in the field still last over distances of order H^{-1}, the initial and final Hubble volumes are not the same and the initial Hubble volume will break up into

$$\mathcal{N} = \frac{f^3 H^3}{H_0^3} \tag{2.26}$$

independent Hubble volumes. The condition Eq. (2.22) is thus replaced by

$$P_c(\widetilde{\Delta\phi}_{\rm qu} \geq -\widetilde{\Delta\phi}_{\rm cl}) = 1 - \Phi_c(-\Delta\tilde{\phi}_{\rm cl}) \geq \frac{1}{\mathcal{N}}. \tag{2.27}$$

Now it is possible that $\dot{\phi}$ is so large that H decreases by more than the expansion f can compensate for and $\mathcal{N} < 1$. In this case the right hand side of Eq. (2.27) is greater than 1 and stochastic eternal inflation is not possible regardless of the size of the quantum fluctuations in comparison to the classical roll. This suggests that the slow-roll approximation will be valid for more trajectories supporting stochastic eternal inflation than would be considered with Eq. (2.24). Ref. [39] calculates $P_c(\Delta\phi_{\rm qu}, t)$ for general potentials using the slow-roll approximation by perturbatively solving the Langevin equation as is described in Sec. 3.4. The resulting distribution Eq. (3.30) is normal with mean $\left\langle\widetilde{\delta\phi_2}\right\rangle$ and variance $\langle\delta\phi_1^2\rangle$ given by Eqs. (3.32) and (3.31) respectively. Therefore, the

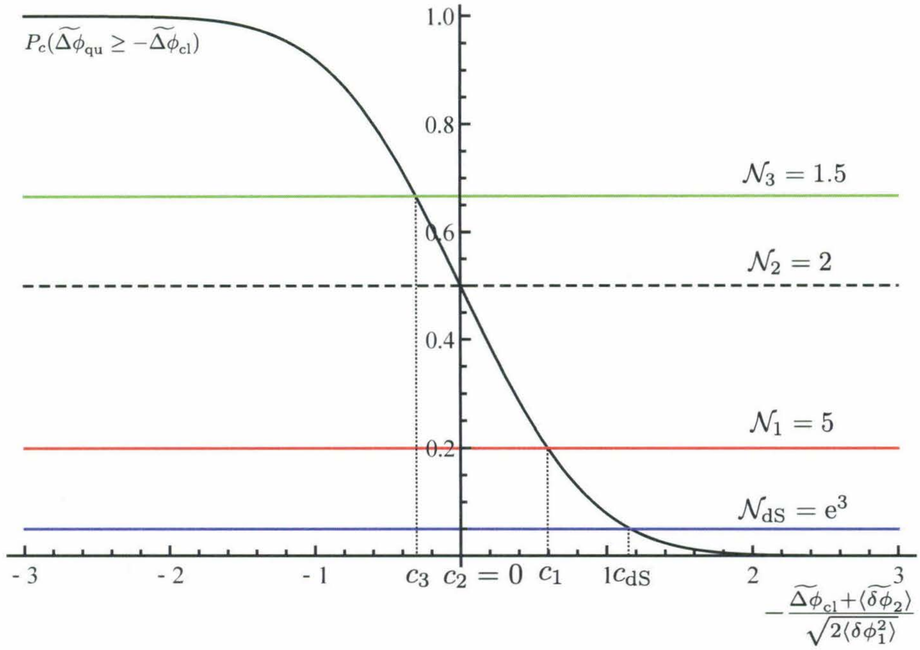

Figure 2.2: Illustrating the condition Eq. (2.27). Inflation is eternal if the probability of $\widetilde{\Delta\phi} > 0$ (the solid black line) is greater than $\mathcal{N}^{-1}$ (the horizontal lines). The dashed horizontal line corresponds to the critical value $\mathcal{N} = 2$. If $\mathcal{N} < 2$, inflation can only be eternal if $\widetilde{\Delta\phi}_{\rm cl} > 0$. The maximum value of $\mathcal{N}$ corresponds to the pure de Sitter space value of ${\rm e}^3$ and is shown in blue.

cumulative distribution function is

$$\Phi_c(\widetilde{\Delta\phi}_{\rm qu}) = \frac{1}{2}\left[1 + {\rm erf}\left(\frac{\widetilde{\Delta\phi}_{\rm qu} - \left\langle\widetilde{\delta\phi}_2\right\rangle}{\sqrt{2\left\langle\delta\phi_1^2\right\rangle}}\right)\right]$$

and, if c is defined as the solution to

$$1 - \frac{2}{\mathcal{N}} = {\rm erf}\, c,$$

then the condition Eq. (2.24) is replaced by

$$c \geq -\frac{\widetilde{\Delta\phi}_{\rm cl} + \left\langle\widetilde{\delta\phi}_2\right\rangle}{\sqrt{2\left\langle\delta\phi_1^2\right\rangle}}. \tag{2.28}$$

As with Eq. (2.24), Eq. (2.28) correctly deals with fields classically moving up the potential but Eq. (2.28) has two new features. First, Eq. (2.28) allows for quantum fluctuations about $\widetilde{\Delta\phi}_{\rm cl}$ with non-zero mean $\left\langle\widetilde{\delta\phi}_2\right\rangle$. However, this effect should be small since, as is discussed in Sec. 3.4, $\sqrt{\left\langle\delta\phi_1^2\right\rangle}$ is first order in quantum noise while $\left\langle\widetilde{\delta\phi}_2\right\rangle$ is second order in quantum noise. If the perturbative expansion is valid so that Eq. (2.28) is valid, then $\left\langle\widetilde{\delta\phi}_2\right\rangle / \sqrt{\left\langle\delta\phi_1^2\right\rangle} \ll 1$.

Second, and more interesting, Eq. (2.28) accounts for Hubble volumes differing in size between

t_0 and $t_0 + H_0^{-1}$. As was discussed above, this condition predicts no eternal inflation if $\mathcal{N} < 1$. Eq. (2.28) also has different behaviors for $\mathcal{N} > 2$ and $\mathcal{N} < 2$ as is illustrated in Fig. 2.2. For simplicity, let $\left\langle \widetilde{\delta\phi_2} \right\rangle = 0$ in this discussion since its effects are negligible. The discussion remains the same for non-zero $\left\langle \widetilde{\delta\phi_2} \right\rangle$ by replacing $\widetilde{\Delta\phi}_{\rm cl}$ with $\widetilde{\Delta\phi}_{\rm cl} + \left\langle \widetilde{\delta\phi_2} \right\rangle$. If $\mathcal{N} > 2$, $c > 0$ so inflation will be eternal if $\widetilde{\Delta\phi}_{\rm cl} > 0$ or if $\widetilde{\Delta\phi}_{\rm cl} < 0$ as long as the quantum fluctuations $\langle \delta\phi_1^2 \rangle$ are sufficiently large. For $\mathcal{N} > 2$, larger quantum fluctuations make it easier to have eternal inflation when $\widetilde{\Delta\phi}_{\rm cl} < 0$. The largest possible values of $\mathcal{N}$ and c are given by pure de Sitter space where $\mathcal{N}_{\rm dS} = {\rm e}^3$ and $c_{\rm dS} = 1.165$. As $\mathcal{N}$ decreases so does c and the ratio $-\widetilde{\Delta\phi}_{\rm cl}/\sqrt{2\langle \delta\phi_1^2 \rangle}$ must increase for inflation to be eternal if $\widetilde{\Delta\phi}_{\rm cl} < 0$. For $\mathcal{N} < 0$, $c < 0$ so inflation will only be eternal if $\widetilde{\Delta\phi}_{\rm cl} > 0$ and the quantum fluctuations $\langle \delta\phi_1^2 \rangle$ are sufficiently small. For $\mathcal{N} < 2$, inflation cannot be eternal if $\widetilde{\Delta\phi}_{\rm cl} < 0$ and larger quantum fluctuations make it harder to have eternal inflation if $\widetilde{\Delta\phi}_{\rm cl} > 0$.

Note that the condition Eq. (2.27) is general and could be used if a more accurate method is developed for computing the distribution of quantum fluctuations than the perturbative Langevin method of Ref. [39]. In this case, the behavior described above and illustrated in Fig. 2.2 would be qualitatively correct (unless the distribution was so extreme that $c_{\rm dS} < 0$).

2.3.3 Multiple Fields

Now consider the case of multiple scalar fields ϕ_I. Let $\mathbf{\Delta\phi} = (\Delta\phi_1, \ldots, \Delta\phi_d)$ be the vector of the $\Delta\phi_{I\rm cl}$s for the d fields. The unit vector pointing in the direction of the increasing gradient of the potential at $\boldsymbol{\phi}_0$ is

$$\hat{\mathbf{n}}(\boldsymbol{\phi}_0) = \left. \frac{\nabla V}{|\nabla V|} \right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_0} \tag{2.29}$$

where the derivatives in ∇ are with respect to the ϕ_I here. If for multiple fields we define

$$\widetilde{\Delta\phi}(t_0) \equiv \mathbf{\Delta\phi}(t_0) \cdot \hat{\mathbf{n}}(\boldsymbol{\phi}(t_0)), \tag{2.30}$$

then the condition for eternal inflation that the field move up the potential Eq. (2.21) is unchanged. As with the single field case, $\mathbf{\Delta\phi}_{\rm cl}$ can be calculated from Eq. (2.9).

The perturbative Langevin method does not allow for an analytic solution for multiple fields; however, the solutions would still be normally distributed for each field ϕ_I. If X_1 and X_2 are two normally distributed random variables with means μ_1 and μ_2 and variances σ_1 and σ_2, then $a_1 X_1 + b_2 X_2$ is a normally distributed random variable with mean $a_1\mu_1 + a_2\mu_2$ and variance $a_2^2\sigma_1^2 + a_2^2\sigma_2^2$ [46]. Therefore, the distribution for $\widetilde{\Delta\phi}_{\rm qu}$ is still given by Eq. (3.30) but where

$$\left\langle \widetilde{\delta\phi}_1^2 \right\rangle = \left(\hat{n}^I\right)^2 \left\langle \widetilde{\delta\phi}_{1I}^2 \right\rangle \qquad \text{and} \qquad \left\langle \widetilde{\delta\phi_2} \right\rangle = \hat{n}^I \left\langle \widetilde{\delta\phi}_{2I} \right\rangle. \tag{2.31}$$

Therefore, with the definition Eq. (2.30) and the parameters Eq. (2.31), the condition Eq. (2.28) is the same for the case of multiple inflaton fields. Note that, as with the case of a single field, this condition takes into account the possibilities of large classical motion still resulting in eternal inflation. For example, Eq. (2.24) accurately describes the case where ϕ is moving with a large velocity in a direction orthogonal to $\hat{\mathbf{n}}$ resulting in almost no change in the value of $V(\phi)$ as an eternal trajectory even if the classical roll exceeds the quantum fluctuations.

2.3.4 Possible Objections to these Conditions

All of the quantum effects leading to Eq. (2.28) are contained in $P_c(\Delta\phi_{\text{qu}})$. This leads to at least two possible sources of error. First, the field equations Eq. (2.9) for the classical roll of the fields use the Hubble parameter computed classically. Thus quantum fluctuations leading to deviations in ϕ and $\dot{\phi}$ are not accounted for in computing the classical trajectory. For Eq. (2.24) and the distributions often computed using the Fokker-Planck equation described in Sec. 3.2 this is a potential problem. However, for the distribution calculated using the perturbative Langevin equation described in Sec. 3.4, the backreaction of the field is accounted for in the perturbative expansion. Therefore, for Eq. (2.28), even though quantum fluctuations in H are not accounted for in the computation of ϕ_{cl}, these fluctuations are included in the calculation of $P_c(\Delta\phi_{\text{qu}})$.

The second possible source of error coming from all of the quantum effects being included in $P_c(\Delta\phi_{\text{qu}})$ occurs if a trajectory does not satisfy Eq. (2.28) for $t_0 = t_{\text{initial}}$ where t_{initial} is the time at which the initial conditions are specified for the trajectory of interest. In this case, it is still possible, and somewhat likely, that the trajectory will satisfy Eq. (2.28) for $t_0 > t_{\text{initial}}$. When Eq. (2.28) is not satisfied for $t_0 = t_{\text{initial}}$ where $\phi_{\text{cl}}(t)$ has been computed for $t \in [t_{\text{initial}}, t_{\text{initial}} + H_{\text{initial}}^{-1}]$, $\phi_{\text{cl}}(t)$ will continue to be computed with initial conditions given by the classical final values of the previously computed portion of the trajectory. Quantum effects leading to deviations from these classical initial conditions are not considered and one could argue that errors in the computation of ϕ_{cl} continue to compound if Eq. (2.28) is not satisfied early in a trajectory.

One could also object to the computation of $\mathcal{N}$ in Eq. (2.26). Quantum fluctuations in H were not considered here and the inclusion of backreation in $P_c(\Delta\phi_{\text{qu}})$ does not enter into the calculation of $\mathcal{N}$. It is also possible to imagine that the division of the initial volume into independent final volumes is more complicated than simply taking a ratio of volumes. However, this is an improvement over current methods and already leads to significant effects as is shown in Ch. 4.

Another possible objection to the condition Eq. (2.28) comes from an assumption that was glossed over in claiming that Eq. (2.27) implies that a trajectory is capable of supporting stochastic eternal inflation. This assumption is that if Eq. (2.27) holds at t_0, it will also hold at $t_0 + H_0^{-1}$ for one of the Hubble volumes for which $V(\phi) \geq V(\phi_0)$, and will continue to hold for one of the fractal volumes created at each iteration. If Eq. (2.27) is satisfied, at least one of the $\mathcal{N}$ Hubble volumes will satisfy $V(\phi) \geq V(\phi_0)$ but Eq. (2.27) does not guarantee that Eq. (2.27) will be satisfied for any of these Hubble volumes for which $V(\phi) \geq V(\phi_0)$. Furthermore, the problem of ignoring quantum effects in the initial conditions for the calculation of $\phi_{\rm cl}$ described above further complicates the analysis of this assumption. This assumption seems reasonable for the typical situations studied in the literature where the field is rolling down the potential and the slow-roll approximation is valid Since both $\epsilon \ll 1$ and $|\eta| \ll 1$, the trajectory satisfying Eq. (2.28) is not changing rapidly and the conditions leading to the satisfaction of Eq. (2.28) seem likely to be returned to in subsequent volumes.

The analysis of this assumption is more delicate in more generic situations where the field could be rolling up the potential, ϵ and $|\eta|$ can be large, or the field may not even be inflating. For example, if Eq. (2.28) is satisfied because ϕ was rolling up the potential and $\widetilde{\Delta\phi}_{\rm cl} \approx 0$ in the neighborhood of the turning point, the quantum fluctuations could be very small. Since the satisfaction of Eq. (2.28) could really be largely due to the classical motion and since that motion is changing rapidly with possibly negligible quantum corrections, it seems possible that the conditions leading to the satisfaction of Eq. (2.28) are never met again. While the field will not roll down the potential in one of the $\mathcal{N}$ Hubble volumes, it seems possible that in the next Hubble volume the field will roll down the potential in every Hubble volume. Again, quantum effects not considered here could play an important role in some of these cases. However, there does not seem to be a straightforward condition to check for whether this assumption is valid and further analysis is needed.

Finally, one could object to using H^{-1} as the length over which correlations in the inflaton field extend and H^{-3} as the volume throughout which ϕ is homogeneous in the calculation of $\mathcal{N}$ Eq. (2.26). Suppose that at t_0 the field is homogeneous throughout some volume, not necessarily the Hubble volume, and that that volume subsequently expands. Then at time t, the correlations in the field will extend over the distance to the particle horizon from t_0 to t:

$$d(t) = a(t) \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')} = \exp\left[\int_{t_0}^{t} H(t')\,\mathrm{d}t'\right] \int_{t_0}^{t} \exp\left[-\int_{t_0}^{t'} H(t'')\,\mathrm{d}t''\right] \mathrm{d}t'. \tag{2.32}$$

To be completely correct, Eq. (2.26) should be replaced by

$$\mathcal{N} = \frac{f^3 d^3(t_0)}{d^3(t)}. \tag{2.33}$$

For pure de Sitter space,

$$d(t) = \mathrm{e}^{Ht} \int_{t_0}^{t} \mathrm{e}^{-Ht'}\,\mathrm{d}t' = \frac{\mathrm{e}^{H(t-t_0)} - 1}{H}.$$

In particular, if we consider expansion after a Hubble time, $d(t_0 + H^{-1}) = (\mathrm{e} - 1)/H$ and the correlations do extend over approximately a Hubble length. Since it is only a ratio of distances to the particle horizon that enters $\mathcal{N}$, it is only important that d be a constant multiple of H^{-1} for the use of H^{-1} instead of d to be correct.

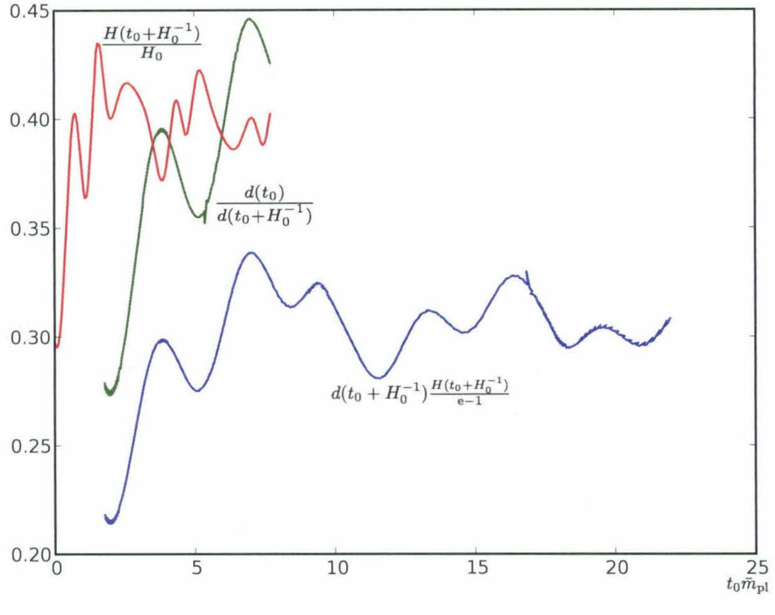

Figure 2.3: Comparison of correlation length scales for an extreme trajectory in a quadratic potential $V(\phi) = \overline{m}_{\rm pl}^2\phi^2/2$. The blue curve is the ratio of the distance to the particle horizon computed by Eq. (2.32) and the distance to the particle horizon in pure de Sitter space. The red curve is $\mathcal{N}/f$ computed using Hubble lengths. The green curve is $\mathcal{N}/f$ computed using the distances to the particle horizon computed by Eq. (2.32).

In the more general case where the expansion is not de Sitter, Eq. (2.32) should be used. While it is straightforward to numerically calculate the integrals in Eq. (2.32) for a given trajectory, it is very inconvenient to use Eq. (2.33) in general since $d(t)$ is now not determined solely by the data at t but for data over the entire interval $[t_0, t]$. Therefore, to more accurately check the condition Eq. (2.28) for a trajectory with initial conditions given at $t_{\rm initial}$, one would have to integrate out to $t_0 = t_{\rm initial} + H_{\rm initial}^{-1}$ and then integrate out to $t_0 + H_0^{-1}$ to before Eq. (2.28) could be checked for the first time. Fig. 2.3 compares $\mathcal{N}$ computed with H and d for an extreme trajectory with large $\dot{\phi}$ that is often not inflating in the steep potential $V(\phi) = \overline{m}_{\rm pl}^2\phi^2/2$. For this trajectory, d is smaller than H^{-1} but $\mathcal{N}/f$ computed with d and H are comparable once the velocity is not so large. $\mathcal{N}/f$ differs by a factor of ~ 1.5 between the two methods at the largest discrepancy for large $\dot{\phi}$.

Chapter 3

Probability Distribution for Quantum Fluctuations

3.1 Fluctuations of a Free Scalar Field in de Sitter Space

The value of $\langle \phi^2 \rangle$ for a free scalar field will be important to normalizing the quantum fluctuations in more general situations described below. In this section, we compute $\langle \phi^2 \rangle$ for a massless free scalar field following Refs. [47, 48].

The field can be expanded as

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3p \left[a(p)\psi_p(t)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + a^\dagger(p)\psi_p^*(t)\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\right] \tag{3.1}$$

where $\mathbf{p}$ and $\mathbf{x}$ are the comoving momentum and position vectors respectively and $a(p)$ and $a^\dagger(p)$ are annihilation and creation operators. In Minkowski space, $\psi_p(t) = \mathrm{e}^{-ip_o t}/\sqrt{2p_0}$. From Eq. (2.7), since $a = \mathrm{e}^{Ht}$ in de Sitter space,

$$\ddot{\psi}_p + 3H\dot{\psi}_p - p^2\mathrm{e}^{-2Ht}\psi_p = 0. \tag{3.2}$$

The solution to Eq. (3.2) is [47, 48]

$$\frac{\sqrt{\pi}}{2} H \left|\eta\right|^{3/2} \left[c_1(p)H_\nu^{(1)}(p\eta) + c_2(p)H_\nu^{(2)}(p\eta)\right] \tag{3.3}$$

with $\nu = 3/2$ and where $\eta = -\mathrm{e}^{-Ht}/H$ is the conformal time and the $H_\nu^{(n)}$ are Hankel functions. The de Sitter space solution that matches the Minkowski solution as $p \to \infty$ is given by $c_1 = 0$ and $c_2 = -1$ [47]:

$$\psi_p(t) = \frac{\mathrm{i}H}{p\sqrt{2p}} \left(1 - \frac{\mathrm{i}p}{H}\mathrm{e}^{-Ht}\right) \exp\left(\frac{\mathrm{i}p}{H}\mathrm{e}^{-Ht}\right). \tag{3.4}$$

With this solution,

$$\langle\phi^2\rangle = \frac{1}{(2\pi)^3}\int |\psi_p(t)|^2 \, \mathrm{d}^3p = \frac{1}{(2\pi)^3}\int \left(\frac{\mathrm{e}^{-2Ht}}{2p} + \frac{H^2}{2p^3}\right) \mathrm{d}^3p = \frac{1}{(2\pi)^3}\int \left(\frac{1}{2k} + \frac{H^2}{2k^3}\right) \mathrm{d}^3k \quad (3.5)$$

where $k = p\mathrm{e}^{-Ht}$ is the physical magnitude of the momentum. The first term is due to the usual vacuum fluctuations of Minkowski space and can be eliminated by renormalization. The second term has the contributions due to inflation. The momenta responsible for the fluctuations are $k_0\mathrm{e}^{-Ht} \lesssim k \lesssim k_0$ [49]. Thus, assuming k_0 is of order H,

$$\langle\phi^2\rangle \approx \frac{H^2}{2(2\pi)^3}\int \frac{\mathrm{d}^3k}{k^3} = \frac{H^2}{4\pi^2}\int_{H\mathrm{e}^{-Ht}}^{H} \frac{\mathrm{d}k}{k} = \frac{H^3t}{4\pi^2}. \quad (3.6)$$

In one Hubble time, this reproduces the result that the variance in the quantum fluctuations is $H^2/4\pi^2$.

The analogous equation to Eq. (3.2) for a free scalar field with mass m, which has the additional term $m^2\psi_p$ on the right hand side, has the same solution Eq. (3.4) with $\nu^2 = 9/4 - m^2/H^2$ [48]. In the limit $m^2 \ll H^2$, a similar, but more complicated, computation gives [47, 48]

$$\langle\phi^2\rangle = \frac{3H^4}{8\pi^2m^2}\left[1 - \exp\left(-\frac{2m^2t}{3H}\right)\right] \quad (3.7)$$

which reproduces Eq. (3.6) for $t \ll H/m^2$.

3.2 The Fokker-Planck Equation

The probability distribution function $P_c(\phi, t)$ to find the field at ϕ at time t within a comoving volume is often calculated using a Fokker-Planck Equation. For the massless free field, the situation is simpler and P_c satisfies the diffusion equation

$$\frac{\partial P_c}{\partial t} = D\frac{\partial^2 P_c}{\partial \phi^2} \quad (3.8)$$

where D is the diffusion coefficient [11, 47]. Since from Eq. (3.6),

$$\langle\phi^2\rangle = \int \phi^2 P_c(\phi, t) \, \mathrm{d}\phi = \frac{H^3t}{4\pi^2},$$

taking time derivatives and using Eq. (3.8),

$$\frac{H^3}{4\pi^2} = \int \phi^2\frac{\partial P_c}{\partial t} \, \mathrm{d}\phi = \int D\phi^2\frac{\partial^2 P_c}{\partial \phi^2} \, \mathrm{d}\phi = -2D\int \phi\frac{\partial P_c}{\partial \phi} \, \mathrm{d}\phi = 2D\int P_c \, \mathrm{d}\phi = 2D.$$

Thus $D = H^3/8\pi^2$. If $\phi = \phi_0$ at $t = 0$, then $P_c(\phi, 0) = \delta(\phi - \phi_0)$ and the solution to Eq. (3.8) is

$$P_c(\phi, t) = \sqrt{\frac{2\pi}{H^3 t}} \exp\left[-\frac{2\pi^2(\phi - \phi_0)^2}{H^3 t}\right]. \tag{3.9}$$

Therefore after a Hubble time $t = H^{-1}$, the distribution of $\Delta\phi = \phi - \phi_0$ is given by Eq. (2.23). Eq. (3.9), which has only been derived for the massless free field, is implicitly used every time it is claimed that the quantum fluctuations in the inflaton field are $H/2\pi$ during one Hubble time.

There is nothing in the above derivation to suggest that Eq. (3.9) is accurate if the inflaton is not massless and free. The generalization of Eq. (3.8) to non-flat potentials is the Fokker-Planck equation [14, 50, 51]

$$\frac{\partial P_c}{\partial t} = \frac{\partial}{\partial \phi}\left[\frac{P_c}{3H}\frac{\mathrm{d}V}{\mathrm{d}\phi} + \frac{1}{8\pi^2}\frac{\partial(H^3 P_c)}{\partial \phi}\right]. \tag{3.10}$$

Eq. (3.10) reduces to Eq. (3.8) for flat potentials. Eq. (3.8) is a good approximation to Eq. (3.10) when $\mathrm{d}V/\mathrm{d}\phi$ is small; however, it is hard to know how small this derivative must be for Eq. (3.8) to be valid without knowing the magnitude of $\partial P_c/\partial t$ and $\partial P_c/\partial \phi$. Refs. [14, 50, 51] solve Eq. (3.10) in a few specific cases that are not widely applicable and it is not clear how to solve Eq. (3.10) for a general potential.

Eq. (3.10) is only applicable in the slow-roll approximation. The most general Fokker-Planck equation is [48]

$$\frac{\partial P_c}{\partial t} = -\frac{\partial(\dot\phi P_c)}{\partial \phi} + 3H\frac{\partial(\dot\phi P_c)}{\partial \dot\phi} + \frac{9H^5}{8\pi^2}\frac{\partial^2 P_c}{\partial \dot\phi^2} + \frac{\partial V}{\partial \phi}\frac{\partial P_c}{\partial \dot\phi}. \tag{3.11}$$

Furthermore, none of the differential equations for P_c presented here Eqs. (3.8), (3.10), and (3.11) take into account the backreaction of the field on the geometry. Therefore, none of these approaches are accurate when the quantum fluctuations are too large. The relevant regime for studying stochastic eternal inflation is when there are large quantum fluctuations is so extra care should be used when using results derived from these equations for this purpose.

3.3 The Langevin Equation

As an alternative to calculating P_c from a Fokker-Planck equation, the effects of quantum fluctuations can be calculated from a Langevin equation [39, 52, 53]. The Langevin equation is obtained by adding a stochastic quantum noise term to the right hand side of the classical equations of motion Eq. (2.9). When the slow-roll approximation is valid, this Langevin equation takes the form

$$\dot\phi + \frac{1}{3H}\frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{H^{3/2}}{2\pi}\xi(t) \tag{3.12}$$

where $\xi(t)$ is a white noise field satisfying

$$\langle \xi(t) \rangle = 0 \tag{3.13a}$$

$$\langle \xi(t)\xi(t') \rangle = \delta(t-t'). \tag{3.13b}$$

The normalization of the right hand side of Eq. (3.12) is chosen so that Eq. (3.6) is reproduced for a free massless field.

Since, from Eq. (2.8),

$$H' \equiv \frac{\partial H}{\partial \phi} = \frac{1}{6\overline{m}_{\rm pl}^2 H}\frac{{\rm d}V}{{\rm d}\phi},$$

Eq. (3.12) can also be written as

$$\dot{\phi} + 2\overline{m}_{\rm pl}^2 H' = \frac{H^{3/2}}{2\pi}\xi(t). \tag{3.14}$$

3.4 Perturbative Langevin Method

In this section, we describe the perturbative method of Ref. [39] used to solve the Langevin equation Eq. (3.12). This method is the first to incorporate the effects of backreation for a general potential. We expand the solution $\phi(t)$ in powers of the noise $\xi(t)$

$$\phi(t) = \phi_{\rm cl}(t) + \delta\phi_1(t) + \delta\phi_2(t) + \ldots \tag{3.15}$$

where $\phi_{\rm cl}(t)$ is the classical solution defined as the solution to Eq. (3.12) with $\xi = 0$ and the terms $\delta\phi_n(t)$ are corrections to this classical solution proportional to ξ^n. Substituting Eq. (3.15) into Eq. (3.14) and keeping terms up to second order in ξ,

$$\frac{\rm d}{{\rm d}t}(\phi_{\rm cl} + \delta\phi_1 + \delta\phi_2) + 2\overline{m}_{\rm pl}^2 H'(\phi_{\rm cl}) + 2\overline{m}_{\rm pl}^2 H''(\phi_{\rm cl})(\delta\phi_1 + \delta\phi_2) + \overline{m}_{\rm pl}^2 H'''(\phi_{\rm cl})\delta\phi_1^2$$
$$= \frac{H^{3/2}(\phi_{\rm cl})}{2\pi}\xi + \frac{3}{4\pi}\sqrt{H(\phi_{\rm cl})}H'(\phi_{\rm cl})\delta\phi_1\xi. \tag{3.16}$$

Note that the backreaction of the field has been accounted for here since H has been expanded in terms of the quantum fluctuations. The terms independent of ξ are by definition the equation for $\phi_{\rm cl}$. The terms linear in ξ give

$$\frac{{\rm d}\delta\phi_1}{{\rm d}t} + 2\overline{m}_{\rm pl}^2 H''(\phi_{\rm cl})\delta\phi_1 = \frac{H^{3/2}(\phi_{\rm cl})}{2\pi}\xi \tag{3.17}$$

and the terms quadratic in ξ give

$$\frac{\mathrm{d}\delta\phi_2}{\mathrm{d}t} + 2\overline{m}_{\rm pl}^2 H''(\phi_{\rm cl})\delta\phi_2 = -\overline{m}_{\rm pl}^2 H'''(\phi_{\rm cl})\delta\phi_1^2 + \frac{3}{4\pi}\sqrt{H(\phi_{\rm cl})}H'(\phi_{\rm cl})\delta\phi_1\xi. \tag{3.18}$$

The differential equations for $\delta\phi_1$ and $\delta\phi_2$ Eqs. (3.17) and (3.18) can be solved with integrating factors. Both equations have the same integrating factor

$$\mu = \exp\left\{2\overline{m}_{\rm pl}^2 \int H''[\phi_{\rm cl}(t)]\,\mathrm{d}t\right\}.$$

Using the fact that

$$\frac{\mathrm{d}\phi_{\rm cl}}{\mathrm{d}t} = -2\overline{m}_{\rm pl}^2 H'(\phi_{\rm cl})$$

to change integration variables,

$$\mu = \exp\left[-\int \frac{H''(\phi)}{H'(\phi)}\,\mathrm{d}\phi\right] = \exp\left(-\ln H'\right) = \frac{1}{H'}.$$

Thus, with the initial conditions $\delta\phi_1(t_0) = \delta\phi_2(t_0) = 0$, the solution to Eq. (3.17) is

$$\delta\phi_1(t) = \frac{H'[\phi_{\rm cl}(t)]}{2\pi}\int_{t_0}^{t}\frac{H^{3/2}[\phi_{\rm cl}(t')]}{H'[\phi_{\rm cl}(t')]}\xi(t')\,\mathrm{d}t' \tag{3.19}$$

and the solution to Eq. (3.18) is

$$\delta\phi_2(t) = -\overline{m}_{\rm pl}^2 H'[\phi_{\rm cl}(t)]\int_{t_0}^{t}\frac{H'''[\phi_{\rm cl}(t')]}{H'[\phi_{\rm cl}(t')]}\delta\phi_1^2(t')\,\mathrm{d}t' + \frac{3H'[\phi_{\rm cl}(t)]}{4\pi}\int_{t_0}^{t}\sqrt{H[\phi_{\rm cl}(t')]}\delta\phi_1(t')\xi(t')\,\mathrm{d}t'. \tag{3.20}$$

From Eq. (3.13), $\langle\delta\phi_1(t)\rangle = 0$ and

$$\langle\delta\phi_1^2(t)\rangle = \frac{H'^2[\phi_{\rm cl}(t)]}{4\pi^2}\int_{t_0}^{t}\frac{H^3[\phi_{\rm cl}(t')]}{H'^2[\phi_{\rm cl}(t')]}\,\mathrm{d}t' = -\frac{H'^2[\phi_{\rm cl}(t)]}{8\pi^2\overline{m}_{\rm pl}^2}\int_{\phi_0}^{\phi_{\rm cl}(t)}\left[\frac{H(\phi)}{H'(\phi)}\right]^3\mathrm{d}\phi \tag{3.21}$$

From Eq. (3.19),

$$\langle\delta\phi_1(t)\xi(t)\rangle = \frac{H^{3/2}[\phi_{\rm cl}(t)]}{4\pi}.$$

Therefore

$$\begin{aligned}\langle\delta\phi_2(t)\rangle &= -\overline{m}_{\rm pl}^2 H'[\phi_{\rm cl}(t)]\int_{t_0}^{t}\frac{H'''[\phi_{\rm cl}(t')]}{H'[\phi_{\rm cl}(t')]}\langle\delta\phi_1^2(t')\rangle\,\mathrm{d}t' + \frac{3H'[\phi_{\rm cl}(t)]}{16\pi^2}\int_{t_0}^{t}H^2[\phi_{\rm cl}(t')]\,\mathrm{d}t' \\ &= \frac{H'[\phi_{\rm cl}(t)]}{2}\int_{t_0}^{t}\frac{H'''(\phi)}{H'^2(\phi)}\langle\delta\phi_1^2\rangle\,\mathrm{d}\phi - \frac{3H'[\phi_{\rm cl}(t)]}{32\pi^2\overline{m}_{\rm pl}^2}\int_{t_0}^{t}\frac{H^2(\phi)}{H'(\phi)}\,\mathrm{d}\phi.\end{aligned}$$

Substituting in Eq. (3.21),

$$\langle\delta\phi_2(t)\rangle = -\frac{H'[\phi_{\rm cl}(t)]}{16\pi^2\overline{m}_{\rm pl}^2}\int_{\phi_0}^{\phi_{\rm cl}(t)} H'''(\phi)\int_{\phi_0}^{\phi}\left[\frac{H(\chi)}{H'(\chi)}\right]^3 {\rm d}\chi\,{\rm d}\phi - \frac{3H'[\phi_{\rm cl}(t)]}{32\pi^2\overline{m}_{\rm pl}^2}\int_{t_0}^{t}\frac{H^2(\phi)}{H'(\phi)}\,{\rm d}\phi.$$

Integrating the first term by parts,

$$\langle\delta\phi_2(t)\rangle = \frac{H'[\phi_{\rm cl}(t)]}{16\pi^2\overline{m}_{\rm pl}^2}\Bigg\{ - H''[\phi_{\rm cl}(t)]\int_{\phi_0}^{\phi_{\rm cl}(t)}\left[\frac{H(\phi)}{H'(\phi)}\right]^3 {\rm d}\phi + \int_{\phi_0}^{\phi_{\rm cl}(t)} H''(\phi)\left[\frac{H(\phi)}{H'(\phi)}\right]^3 {\rm d}\phi - \frac{3}{2}\int_{\phi_0}^{\phi_{\rm cl}(t)}\frac{H^2(\phi)}{H'(\phi)}\,{\rm d}\phi\Bigg\}. \tag{3.22}$$

The second term in Eq. (3.22) can be integrated by parts

$$\int_{\phi_0}^{\phi_{\rm cl}(t)} H''(\phi)\left[\frac{H(\phi)}{H'(\phi)}\right]^3 {\rm d}\phi = -\left.\frac{H^3}{2H'^2}\right|_{\phi_0}^{\phi_{\rm cl}(t)} + \frac{3}{2}\int_{\phi_0}^{\phi_{\rm cl}(t)}\frac{H^2(\phi)}{H'(\phi)}\,{\rm d}\phi$$

and cancels the last term in Eq. (3.22). The first term in Eq. (3.22) is given by Eq. (3.21) so

$$\langle\delta\phi_2(t)\rangle = \frac{H''[\phi_{\rm cl}(t)]}{2H'[\phi_{\rm cl}(t)]} + \frac{H'[\phi_{\rm cl}(t)]}{32\pi^2\overline{m}_{\rm pl}^2}\left\{\frac{H^3(\phi_0)}{H'^2(\phi_0)} - \frac{H^3[\phi_{\rm cl}(t)]}{H'^2[\phi_{\rm cl}(t)]}\right\}. \tag{3.23}$$

We now show that this formalism reproduces the result Eq. (3.6) for a massless free field. In this case, $H' = 0$ so plugging Eq. (3.15) into Eq. (3.14) simply gives

$$\frac{{\rm d}\delta\phi_1}{{\rm d}t} = \frac{H^{3/2}(\phi_{\rm cl})}{2\pi}\xi \tag{3.24}$$

$$\frac{{\rm d}\delta\phi_2}{{\rm d}t} = 0. \tag{3.25}$$

$\delta\phi_2$ is therefore constant and, since $\delta\phi_2(t_0) = 0$, is equal to zero. All of the quantum noise comes from $\delta\phi_1$ in this case. Eq. (3.24) can easily be integrated to give

$$\delta\phi_1(t) = \frac{1}{2\pi}\int_{t_0}^{t} H^{3/2}[\phi_{\rm cl}(t')]\xi(t')\,{\rm d}t' \tag{3.26}$$

and therefore

$$\left\langle\delta\phi_1^2(t)\right\rangle = \frac{1}{4\pi^2}\int_{t_0}^{t} H^3[\phi_{\rm cl}(t')]\,{\rm d}t' = \frac{H^3}{4\pi^2}(t-t_0). \tag{3.27}$$

If $\phi[\xi] = \phi_{\rm cl} + \delta\phi_1 + \delta\phi_2 + \ldots$ is the solution to the Langevin equation Eq. (3.12), then the probability density function is [39, 54, 55]

$$P_c(\phi,t) = \langle\delta(\phi-\phi[\xi])\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle {\rm e}^{{\rm i}k(\phi-\phi[\xi])}\right\rangle {\rm d}k = \frac{1}{2\pi}\int_{-\infty}^{\infty}{\rm e}^{{\rm i}k(\phi-\phi_{\rm cl})}\left\langle {\rm e}^{-{\rm i}k(\delta\phi_1+\delta\phi_2+\ldots)}\right\rangle {\rm d}k. \tag{3.28}$$

To second order in noise

$$\left\langle e^{-ik(\delta\phi_1+\delta\phi_2+\dots)}\right\rangle = \left\langle 1 - ik(\delta\phi_1+\delta\phi_2) - \frac{1}{2}k^2\delta\phi_1^2\right\rangle = \exp\left(-\frac{1}{2}k^2\left\langle\delta\phi_1^2\right\rangle - ik\left\langle\delta\phi_2\right\rangle\right).$$

Therefore,

$$\begin{aligned} P_c(\phi,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}k^2\left\langle\delta\phi_1^2\right\rangle + ik\left(\phi-\phi_{\rm cl}-\left\langle\delta\phi_2\right\rangle\right)\right]\,dk \\ &= \frac{1}{\sqrt{2\pi\left\langle\delta\phi_1^2(t)\right\rangle}}\exp\left\{-\frac{[\phi-\phi_{\rm cl}(t)-\left\langle\delta\phi_2(t)\right\rangle]^2}{2\left\langle\delta\phi_1^2(t)\right\rangle}\right\}. \end{aligned} \tag{3.29}$$

Eq. (3.29) describes the probability $P_c(\phi,t)$ of finding the field at a specific value at a given time but it can easily be used to calculate the probability distribution of the quantum fluctuations $P_c(\widetilde{\Delta\phi}_{\rm qu},t)$ over a given time. Since $\phi = \phi_{\rm cl} + \Delta\phi_{\rm qu}$,

$$P_c(\widetilde{\Delta\phi}_{\rm qu},t) = \frac{1}{\sqrt{2\pi\left\langle\delta\phi_1^2\right\rangle}}\exp\left[-\frac{\left(\widetilde{\Delta\phi}_{\rm qu}-\left\langle\widetilde{\delta\phi_2}\right\rangle\right)^2}{2\left\langle\delta\phi_1^2\right\rangle}\right]. \tag{3.30}$$

The use of $\widetilde{\Delta\phi}_{\rm qu}$ is acceptable as long as $\left\langle\widetilde{\delta\phi_2}\right\rangle$ is used and $\left\langle\widetilde{\delta\phi_1}^2\right\rangle = \left\langle\delta\phi_1^2\right\rangle$ since this multiplies these quantities by either ±1.

In practice, we use

$$\begin{aligned} H(\phi) &= \frac{1}{\sqrt{3}\overline{m}_{\rm pl}}\sqrt{V(\phi)} \\ H''(\phi) &= \frac{1}{2\sqrt{3}\overline{m}_{\rm pl}}\frac{V'(\phi)}{\sqrt{V(\phi)}} \\ H'''(\phi) &= \frac{1}{2\sqrt{3}\overline{m}_{\rm pl}}\left[\frac{V''(\phi)}{\sqrt{V(\phi)}} - \frac{V'^2(\phi)}{2V^{3/2}(\phi)}\right] \end{aligned}$$

to rewrite Eq. (3.21) and compute

$$\left\langle\delta\phi_1^2(t)\right\rangle = -\frac{V'^2[\phi_{\rm cl}(t)]}{12\pi^2\overline{m}_{\rm pl}^4 V[\phi_{\rm cl}(t)]}\int_{\phi_0}^{\phi_{\rm cl}(t)}\left[\frac{V(\phi)}{V'(\phi)}\right]^3 d\phi \tag{3.31}$$

and to rewrite Eq. (3.23) and compute

$$\begin{aligned} \left\langle\delta\phi_2(t)\right\rangle = &\left\{\frac{V''[\phi_{\rm cl}(t)]}{2V'[\phi_{\rm cl}(t)]} - \frac{V'[\phi_{\rm cl}(t)]}{4V[\phi_{\rm cl}(t)]}\right\}\left\langle\delta\phi_1^2(t)\right\rangle \\ &+ \frac{V'[\phi_{\rm cl}(t)]}{48\pi^2\overline{m}_{\rm pl}^4\sqrt{V[\phi_{\rm cl}(t)]}}\left\{\frac{V^{5/2}(\phi_0)}{V'^2(\phi_0)} - \frac{V^{5/2}[\phi_{\rm cl}(t)]}{V'^2[\phi_{\rm cl}(t)]}\right\}. \end{aligned} \tag{3.32}$$

Chapter 4

Stochastic Eternal Inflation in Particular Single Field Potentials

4.1 Overview

In this chapter, we investigate what trajectories are capable of supporting stochastic eternal inflation in specific single field inflaton potentials. We discuss the quadratic potential (Sec. 4.2), the sine-Gordon potential (Sec. 4.3), and a random Fourier series potential (Sec. 4.4). For the quadratic and sine-Gordon potentials, we use the parameters that are consistent with astrophysical observations as well as more generic parameters.

For each potential, we carry out the following procedure to find the trajectories that are capable of supporting stochastic eternal inflation:

1. Choose an initial value of the Hubble parameter H_0 and several initial values of the field ϕ_0 distributed throughout the potential.
2. For each ϕ_0, classically evolve the field using Eq. (2.9) from t_0 to $t_0 + H_0^{-1}$.
3. Check the condition Eq. (2.28). If it is satisfied, that trajectory will support eternal inflation. If it is not satisfied repeat 2 using the $\phi(t_0 + H_0^{-1})$ and $H(t_0 + H_0^{-1})$ as the initial conditions.
4. Continue to repeat 2 and 3 until Eq. (2.28) has been satisfied or until some criteria to stop looking for eternal trajectories is met.
5. Integrate the measure on the space of trajectories over the initial conditions found to support eternal inflation to find the fraction of trajectories supporting stochastic eternal inflation.

In practice we use a flat measure on the space of trajectories since the form of the correct measure is not known. Our main goal is to highlight the differences between the conditions Eqs. (2.28) and (2.24). Thus the correct form of the measure is not as crucial as if we were studying the probability of getting eternal inflation in a given potential. Unless otherwise stated, we stop checking if a trajectory is eternal once it has oscillated about a minimum in the potential more than twice.

It is necessary to choose a value of H_0 so that the same trajectory is not inadvertently counted twice. From Eq. (2.8),

$$\dot{\phi}_0 = 3\overline{m}_{\rm pl}^2 H_0^2 - V(\phi_0). \tag{4.1}$$

Since there are two trajectories with the same value of H_0, for a general potential, the measure on the space of trajectories would need to be divided by two to give the correct measure. The above procedure would also need to be carried out separately for $\dot{\phi}_0 > 0$ and $\dot{\phi}_0 < 0$ and the results combined to give the correct answer. However, for potentials with reflection symmetry, only one of the branches of the solution needs to be calculated if the factor of two in the measure is ignored to give the correct answer. For most of the following, we choose an H_0 corresponding to starting at rest from the maximum of the potential. Thus from Eq. (4.1),

$$H_0 = \sqrt{\frac{V_{\rm max}}{3\overline{m}_{\rm pl}^2}}. \tag{4.2}$$

To better understand the following results, it will be useful to estimate where on a general potential eternal inflation will take place given that the slow-roll approximation is valid. Ref. [45] makes a similar estimate for the quartic potential $V = \lambda\phi^4$. In one Hubble time, $\Delta\phi_{\rm cl} \approx \dot{\phi}_{\rm cl}/H$. If $|\eta| \ll 1$ then from Eq. (2.9)

$$\dot{\phi}_{\rm cl} = -\frac{V'}{3H}.$$

Thus, using Eq. (2.24), inflation will be eternal provided that

$$0.26 \gtrsim \frac{|V'|}{3H^3}.$$

If $\epsilon \ll 1$, $H^2 = V/3\overline{m}_{\rm pl}^2$. Therefore, if the slow-roll approximation is valid and the effects accounted for in Eq. (2.28) are negligible, inflation will be eternal for values of ϕ that satisfy

$$\frac{0.15}{\overline{m}_{\rm pl}^3} \gtrsim \frac{|V'|}{V^{3/2}}. \tag{4.3}$$

4.1.1 Explanation of Figures

In the following sections, we present many figures illustrating the characteristics of various trajectories in different potentials. All of the figures have the same format which we explain here.

Figure (a) is a plot of the classical field $\phi_{\rm cl}$ and Figure (b) is a plot of the classical field's velocity $\dot{\phi}_{\rm cl}$. In both of these figures, the trajectory is plotted in a solid black line if the field is inflating ($\epsilon < 1$) and is plotted in a dashed black line if the field is not inflating ($\epsilon \geq 1$).

Figure (c) is a plot of the eternal conditions Eqs. (2.28) and (2.24). These conditions require a Hubble time worth of data to compute and are plotted as functions of t_0, the initial time for the data spanning t_0 to $t_0 + H_0^{-1}$. The black line is $c_{\rm dS}$ and the red line is c. The green line is the right hand side of Eq. (2.24) and the blue line is the right hand side of Eq. (2.28). Therefore, Eq. (2.24) is satisfied if the green line is below the red line and Eq. (2.28) is satisfied if the blue line is below the red line. If $\mathcal{N} \leq 1$, neither c nor the right hand side of Eq. (2.28) is plotted. If the field is not inflating ($\epsilon \geq 1$) for any time in $[t_0, t_0 + H_0^{-1}]$, the green and black lines are plotted as dashed lines rather than solid lines.

Various quantities quantifying the accuracy of the perturbative Langevin method are plotted in Figure (d). The slow-roll parameters ϵ and $|\eta|$ are plotted as black solid and dashed lines respectively. $|m|^2/H^2$ where $m^2 = \mathrm{d}^2V/\mathrm{d}\phi^2$ is plotted as a blue dashed line. The significance of the remaining quantities plotted in Figure (d) in discussed in Appendix A. The solid blue line is $|\delta\phi_2/\delta\phi_1|$ which should be small if the terms linear and quadratic in quantum noise can be treated separately. The red line marked L_2 and the green line marked R_1 are the ratios

$$\left|\frac{2L_2(\phi_{\rm cl} + \Delta\phi)}{H'''(\phi_{\rm cl})\Delta\phi^2}\right| \quad \text{and} \quad \left|\frac{R_1(\phi_{\rm cl} + \Delta\phi)}{(H^{3/2})'(\phi_{\rm cl})\Delta\phi}\right|$$

respectively defined by Eqs. (A.4) and (A.5). In practice we use $\Delta\phi = \langle\phi_2\rangle \pm \sqrt{\langle\phi_1^2\rangle}$ and plot the above ratios as solids lines for $+$ and dashed lines for $-$. In almost all cases, the $+$ and $-$ plots are indistinguishable. Note that L_2 vanishes for the quadratic potential and is not plotted for this case.

Note that in many of the plots of $\phi_{\rm cl}$, $|\Delta\phi_{\rm cl}/\phi_{\rm cl}| \ll 1$. In this case, the overall scale of the $\phi_{\rm cl}$ axis is labeled at the top of the plot and the difference from that value is labeled along the axis. So, for example, the $\phi_{\rm cl}$ axis is labeled from -2.663158×10^5 to -2.66315×10^5 in Fig. 4.1(a).

4.2 Quadratic Potential

In this section, we consider a quadratic potential of the form

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{4.4}$$

$V(\phi) \leq \overline{m}_{\rm pl}^4$ if $\phi \in [-\sqrt{2}\overline{m}_{\rm pl}^2/m, \sqrt{2}\overline{m}_{\rm pl}^2/m]$ which is the range of ϕ that we will consider. Thus $V_{\max} = \overline{m}_{\rm pl}^4$ and Eq. (4.2) gives $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$. For this potential,

$$V' = m^2\phi \quad \text{and} \quad V^{3/2} = 2^{-3/2}m^3\phi^3.$$

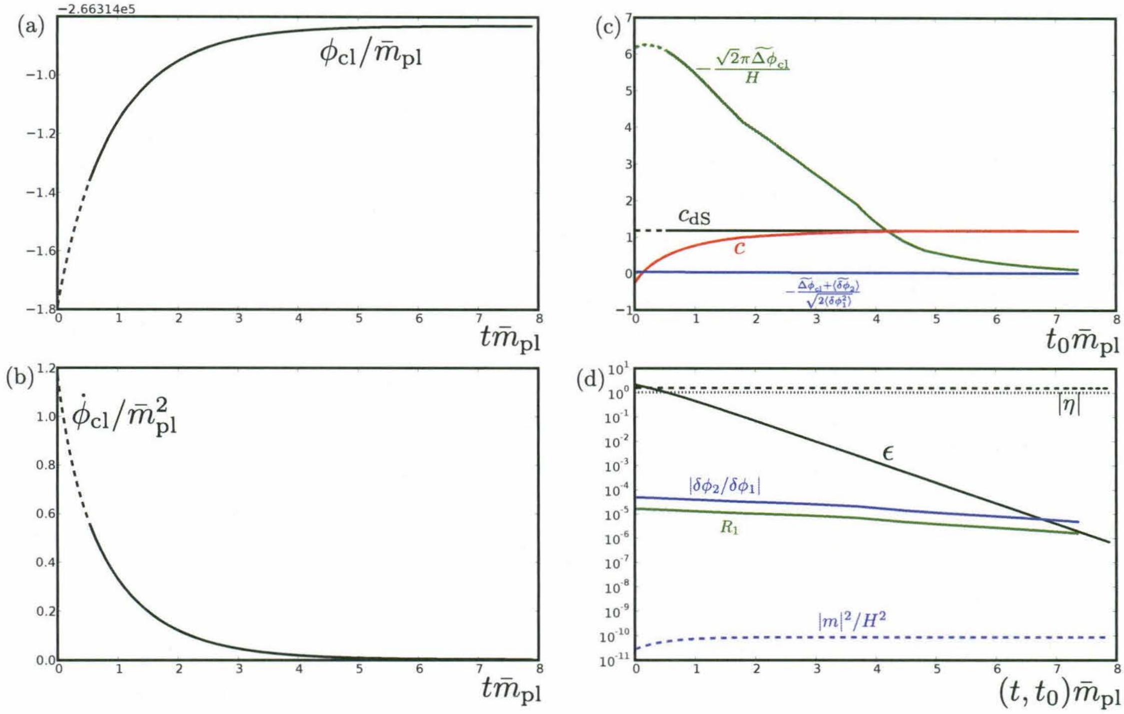

Figure 4.1: Quadratic potential with $m = 3\times 10^{-6}\overline{m}_{\rm pl}$, $\phi_0 = -2.7\times 10^{-5}\overline{m}_{\rm pl}$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

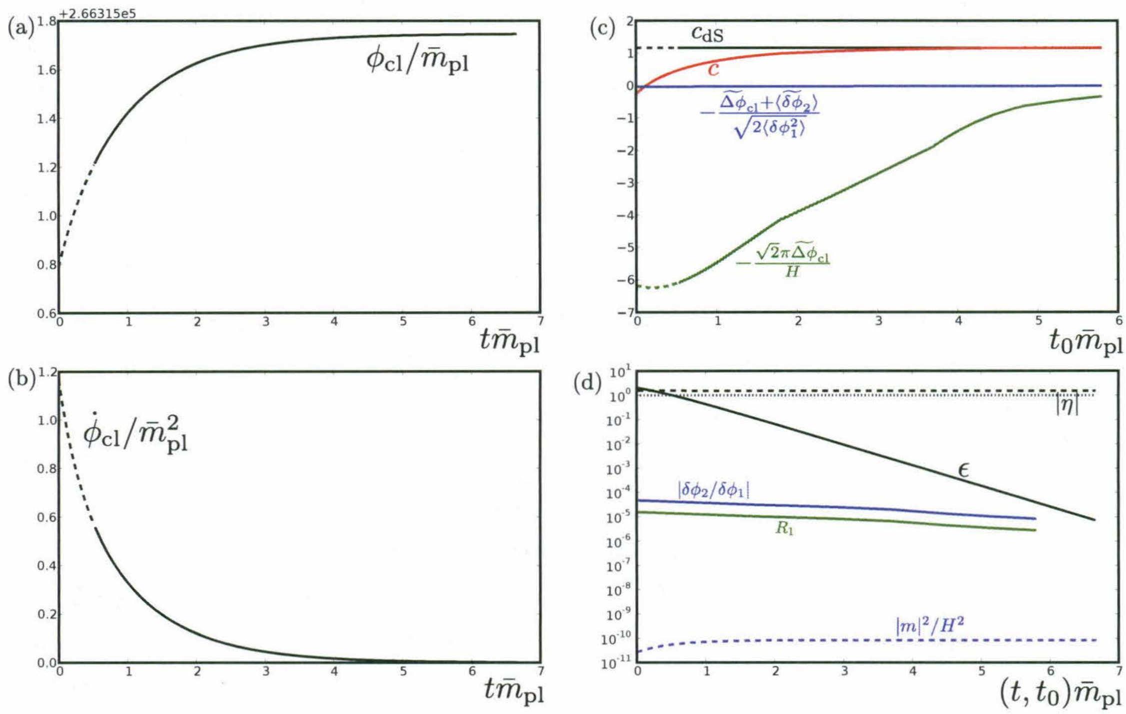

Figure 4.2: Quadratic potential with $m = 3\times 10^{-6}\overline{m}_{\rm pl}$, $\phi_0 = 2.7\times 10^{-5}\overline{m}_{\rm pl}$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

So from Eq. (4.3), we expect inflation to be eternal for

$$|\phi| \gtrsim 4.3\frac{\overline{m}_{\rm pl}^{3/2}}{\sqrt{m}}. \tag{4.5}$$

To be consistent with observations $m = 3 \times 10^{-6}\overline{m}_{\rm pl}$ [43]. In this case, both Eqs. (2.24) and (2.28) predict that the whole range of initial conditions supports eternal inflation carrying out the procedure outlined in Sec. 4.1. Figs. 4.1 and 4.2 show examples of trajectories initially rolling down the potential and initially rolling up the potential respectively. Both of these trajectories show large discrepancies between the right hand sides of Eqs. (2.24) and (2.28) initially which then asymptote to the same value. (Note that the blue line in Figure (c) of both figures is small but nonzero). These large discrepancies are due to the quantum fluctuations as calculated in Eq. (2.28) being larger than those calculated in Eq. (2.24) since $\langle\delta\phi_1^2\rangle$ is larger than $H^2/4\pi^2$. As expected, c differs from $c_{\rm dS}$ initially when $|\dot{H}| \sim |\dot{\phi}|$ is large but approaches $c_{\rm dS}$ as $|\dot{H}|$ becomes small. Except for a brief period in the beginning of the trajectories, Fig. 4.2 shows eternal inflation for both Eqs. (2.24) and (2.28) for the entire trajectory while Fig. 4.1 shows a large period of non-eternal inflation for Eq. (2.24). Both of these trajectories are examples where the conditions for stochastic eternal inflation are met but the field is not inflating the entire time that they are met. This is true of Eq. (2.28) for both trajectories and of Eq. (2.24) for the trajectory of Fig. 4.2. The errors for both of these trajectories are small; however, $|\eta| > 1$ so the slow-roll approximation is not necessarily valid.

Figs. (4.1) and (4.2) are typical of the qualitative behavior for trajectories with initial velocities rolling down and up the potential respectively. For $m = 3 \times 10^{-6}\overline{m}_{\rm pl}$ from Eq. (4.5), we expect inflation to be eternal for $|\phi| \gtrsim 2500\overline{m}_{\rm pl}$. The range of ϕ_0 for which inflation is not eternal is therefore much smaller than the range of values ϕ_0 for which inflation is eternal and such non-eternal trajectories would be difficult to find in a uniform sampling of initial conditions. Even with $|\phi_0| \lesssim 2500\overline{m}_{\rm pl}^4$, from Eq. (4.1) $\dot{\phi}_0$ will be so large that if the field was initially rolling down the potential it will rapidly roll up the other side. As was discussed in Sec. 2.3.2, both Eqs. (2.24) and (2.28) will eventually predict eternal inflation for fields moving up the potential. To illustrate that not all parts of a trajectory are eternal however, we plot the trajectory starting from rest at $\phi_0 = -800\overline{m}_{\rm pl}$ in Fig. 4.3. Eqs. (2.24) and (2.28) both predict no eternal inflation with smaller discrepancies than were seen in Figs. Fig. 4.1 and Fig. 4.2. Since this trajectory also has $|\eta| \ll 1$ in addition to the errors in the perturbative expansion being small, Eq. (2.28) is valid. Note that the trajectory shown in Fig. 4.3 is part of another trajectory with initial conditions given by Eq. (4.1) and therefore should not be counted separately in an investigation of the likelihood for eternal inflation.

We now consider the quadratic potential with $m = \overline{m}_{\rm pl}$ which illustrates the differences between Eqs. (2.24) and (2.28) that could be present in a generic potential with parameters taking values up to the Planck scale. For $\dot{\phi}_0 > 0$, Eq. (2.24) predicts eternal inflation for $\phi_0 \in [-\sqrt{2}\overline{m}_{\rm pl}, \sqrt{2}\overline{m}_{\rm pl}]$

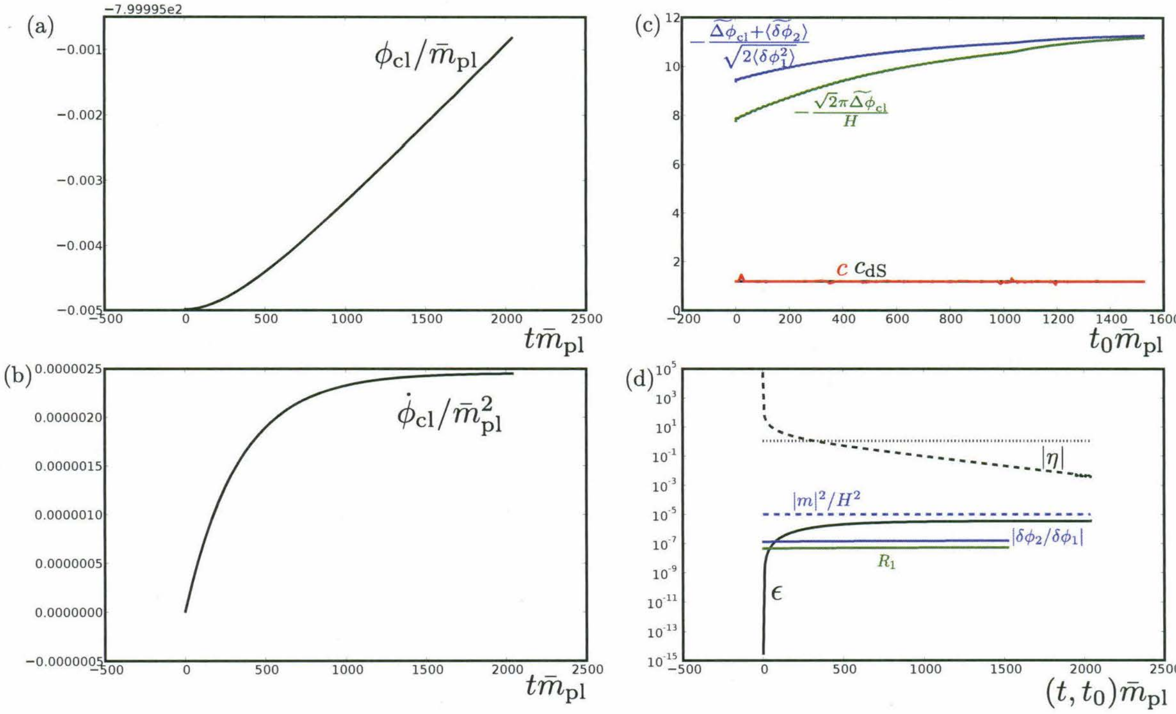

Figure 4.3: Quadratic potential with $m = 3 \times 10^{-6}\overline{m}_{\rm pl}$, $\phi_0 = -800\overline{m}_{\rm pl}$, $\dot{\phi}_0 = 0$.

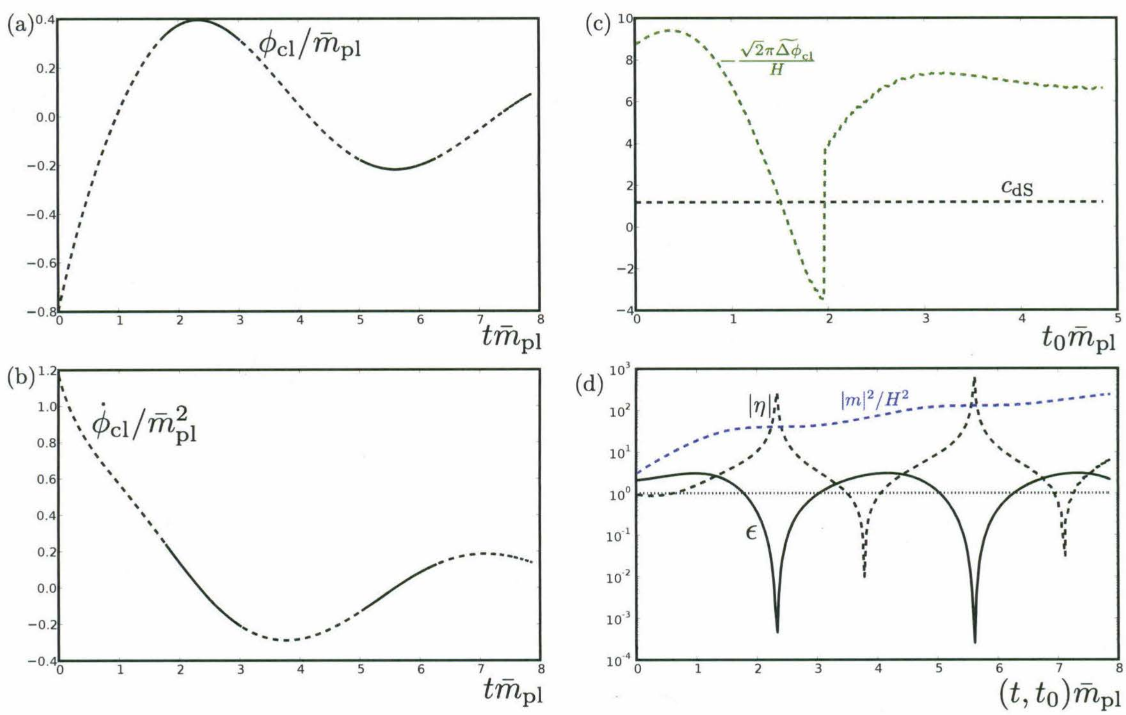

Figure 4.4: Quadratic potential with $m = \overline{m}_{\rm pl}$, $\phi_0 = -0.8\overline{m}_{\rm pl}$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

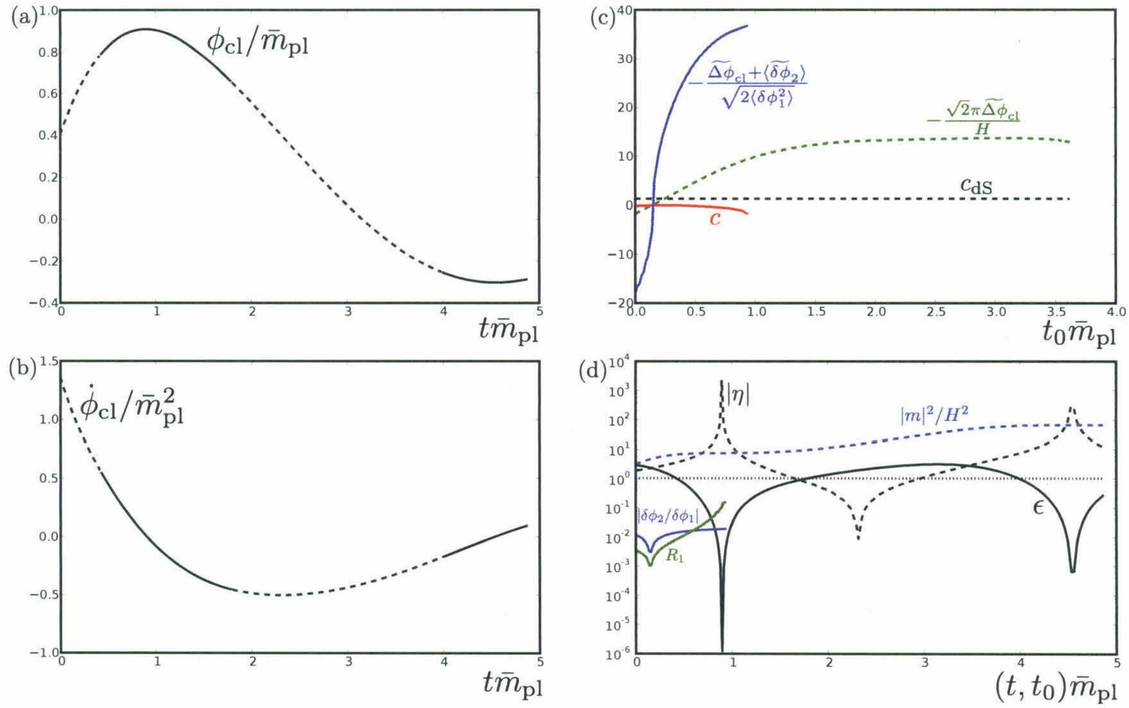

Figure 4.5: Quadratic potential with $m = \overline{m}_{pl}$, $\phi_0 = 0.4\overline{m}_{pl}$, $H_0 = \overline{m}_{pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

corresponding to every trajectory satisfying $V \leq \overline{m}_{pl}^4$ being eternal at some point, while Eq. (2.28) predicts eternal inflation for $\phi_0 \in [0, 0.64\overline{m}_{pl}]$ corresponding to roughly 22% of the allowed trajectories being eternal. To understand these results, note that Eq. (4.5) predicts eternal inflation for $|\phi| \gtrsim 4.3\overline{m}_{pl} > \sqrt{2}\overline{m}_{pl}$. Therefore, we do not expect eternal inflation anytime the slow-roll approximation is valid and the field is rolling down the potential.

Fig. 4.4 shows a typical trajectory initially moving down the potential. $\mathcal{N}$ is never greater than 1 before the field has oscillated about the minimum more than twice and we stop looking for eternal trajectories. Therefore Eq. (2.28) never predicts eternal inflation for the field initially moving down the potential. Eq. (2.24) always predicts eternal inflation after the field passes the minimum of the potential once and begins to move up the potential.

Fig. 4.5 shows a typical trajectory initially moving up the potential for $\phi_0 \in [0, 0.63\overline{m}_{pl}]$. In this case, both Eqs. (2.24) and (2.28) predict eternal inflation initially before the field turns around and begins moving down the potential where inflation is no longer eternal. As with the case of $m = 3 \times 10^{-6}\overline{m}_{pl}$, there are large discrepancies between the two conditions; however, they do not eventually converge to the same solution.

Fig. 4.6 shows a typical trajectory initially moving up the potential for $\phi_0 \in [0.63\overline{m}_{pl}, \sqrt{2}\overline{m}_{pl}]$. In this case, the right hand side of Eq. (2.28) has shifted up and the condition Eq. (2.28) is no longer satisfied initially. However, Eq. (2.24) is still initially satisfied before the field turns around. Again,

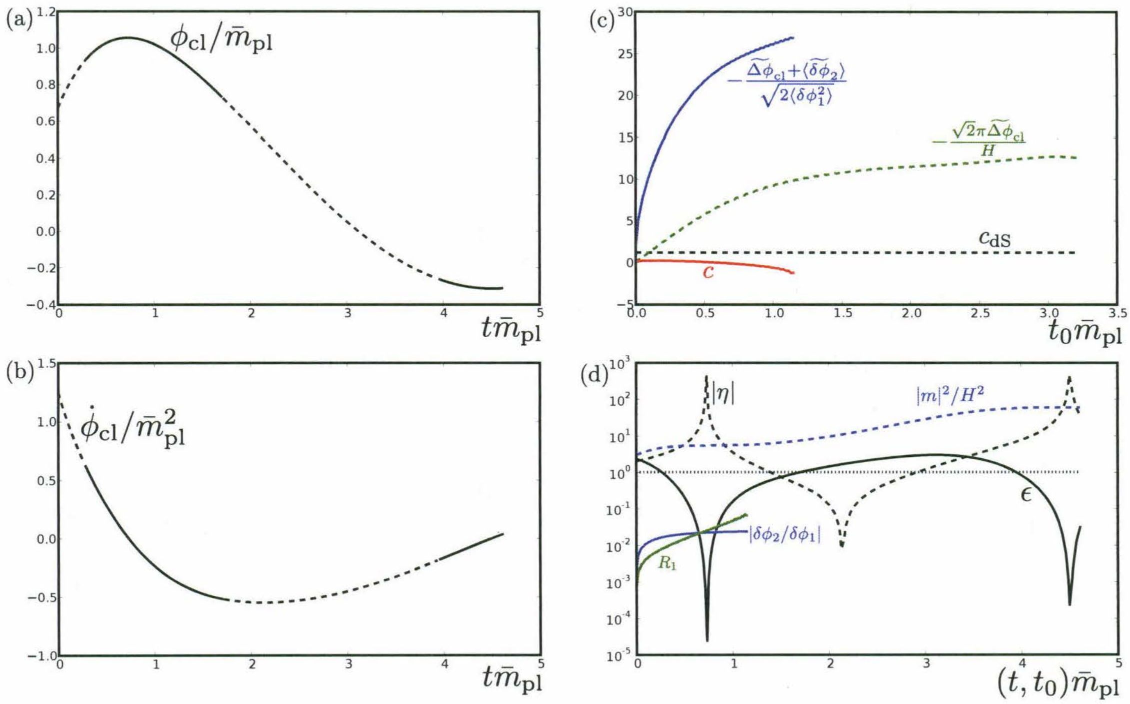

Figure 4.6: Quadratic potential with $m = \overline{m}_{\mathrm{pl}}$, $\phi_0 = 0.67\overline{m}_{\mathrm{pl}}$, $H_0 = \overline{m}_{\mathrm{pl}}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

there are large discrepancies between the two conditions and the solutions do not converge. Note that while the errors in the perturbative expansion, when $\mathcal{N} > 1$, are small, the slow-roll approximation is not valid for $m = \overline{m}_{\mathrm{pl}}$ and $m^2 > H^2$ so these results should be met with skepticism.

4.3 Sine-Gordon Potential

In this section, we consider the sine-Gordon potential, also referred to as natural inflation, of the form

$$V(\phi) = \frac{1}{2}A\left(1 - \cos\frac{\phi}{f}\right). \tag{4.6}$$

We consider $\phi \in [0, 2\pi f]$ so Eq. (4.2) gives $H_0 = \sqrt{A/3\overline{m}_{\mathrm{pl}}^2}$. For this potential,

$$V' = \frac{A}{2f}\sin\frac{\phi}{f} \quad \text{and} \quad V^{3/2} = \left(\frac{A}{2}\right)^{3/2}\left(1 - \cos\frac{\phi}{f}\right)^{3/2}.$$

So from Eq. (4.3), we expect inflation to be eternal for ϕ satisfying

$$0.11\frac{f\sqrt{A}}{\overline{m}_{\mathrm{pl}}^3} \gtrsim \frac{|\sin(\phi/f)|}{[1 - \cos(\phi/f)]^{3/2}}. \tag{4.7}$$

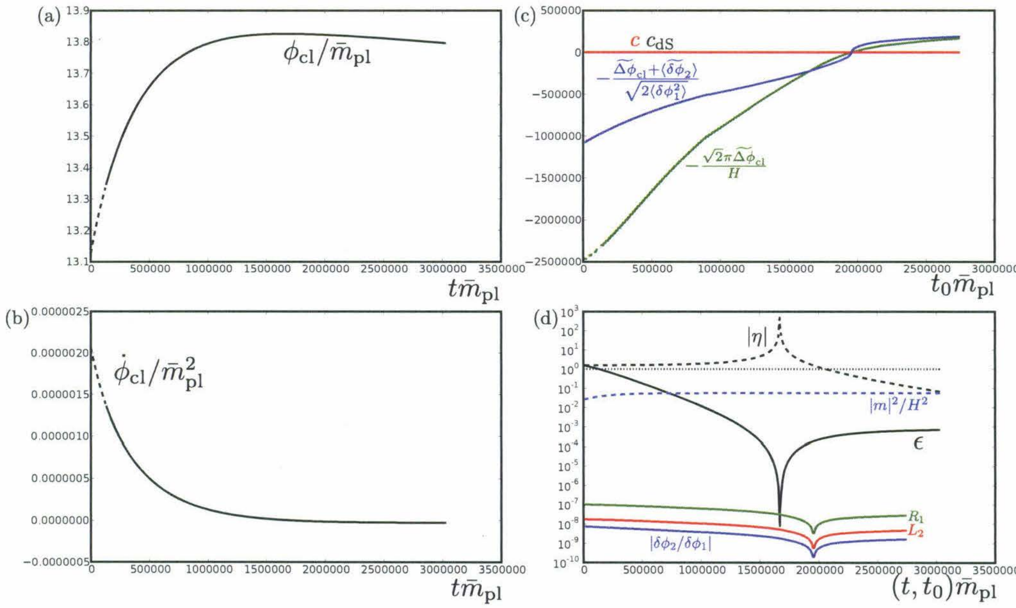

Figure 4.7: Sine-Gordon potential with $A = 2 \times 10^{-12}\overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 5\pi f/6$, $H_0 = \sqrt{A/3\overline{m}_{\rm pl}^2}$, $\dot{\phi}_0 > 0$.

To be consistent with observations, $A = 2 \times 10^{-12}\overline{m}_{\rm pl}^4$ and $f = \sqrt{8\pi}\overline{m}_{\rm pl}$ [43]. In this case for $\dot{\phi}_0 > 0$, both Eqs. (2.24) and (2.28) predict that inflation will be eternal for $\phi_0 \in [0, \pi f]$. In this case, for computational reasons, we only looked for eternal trajectories for the initial part of the trajectories. If the trajectories for $\phi_0 \in [\pi f, 2\pi f]$ were continued to the point where they start to roll back up the potential, both conditions are likely to predict eternal inflation. The only values of ϕ that satisfy Eq. (4.7) are essentially right at $\phi = \pi f$. Therefore, we do not expect eternal inflation if the slow-roll approximation is valid and the field is rolling down the potential and that is what we find. Fig. 4.7 shows a typical trajectory moving up the potential initially. Both conditions predict eternal inflation initially as the field is moving up the potential and predict no eternal inflation around the same point when the field turns around and moves down the potential. The two conditions approach the same solution around the point where the field turns around. Fig. 4.8 shows a typical trajectory starting near the top of the potential and moving down initially. Both conditions asymptote to the same solution eventually but never predict eternal inflation until after the field would have rolled back up the potential. Fig. 4.9 shows a typical trajectory starting near the bottom of the potential and moving down initially. Neither condition predicts eternal inflation but the two solutions do not asymptote to the same solution. Unlike in the quadratic case all of the trajectories in this sine-Gordon potential contain portions of the trajectory where the slow-roll approximation is valid and the errors in the perturbative expansion are small.

We now consider the sine-Gordon potential with $A = \overline{m}_{\rm pl}^4$ and $f = \sqrt{8\pi}$. We find that both

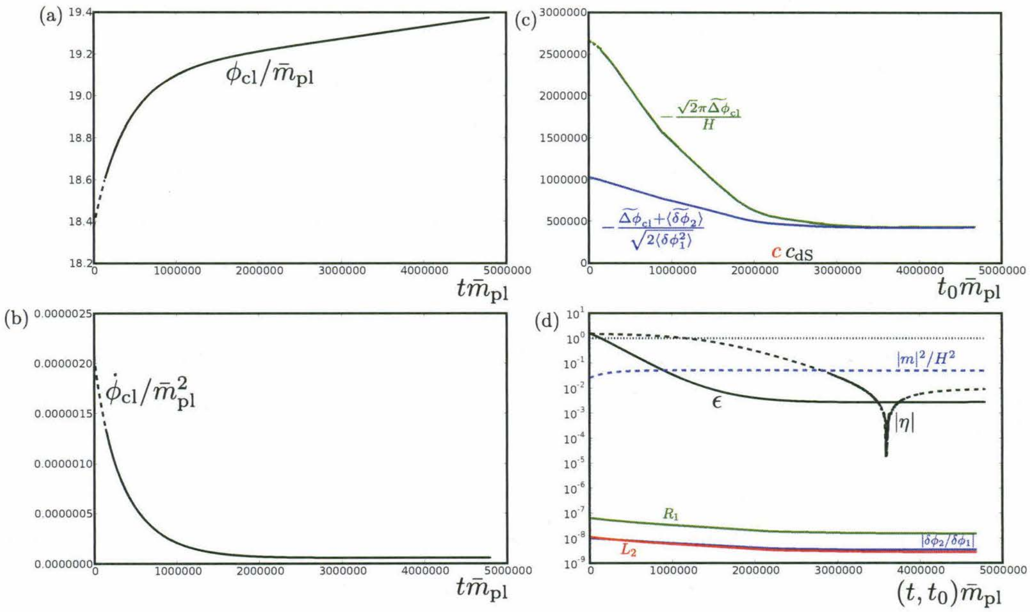

Figure 4.8: Sine-Gordon potential with $A = 2 \times 10^{-12}\overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 7\pi f/6$, $H_0 = \sqrt{A/3\overline{m}_{\rm pl}^2}$, $\dot{\phi}_0 > 0$.

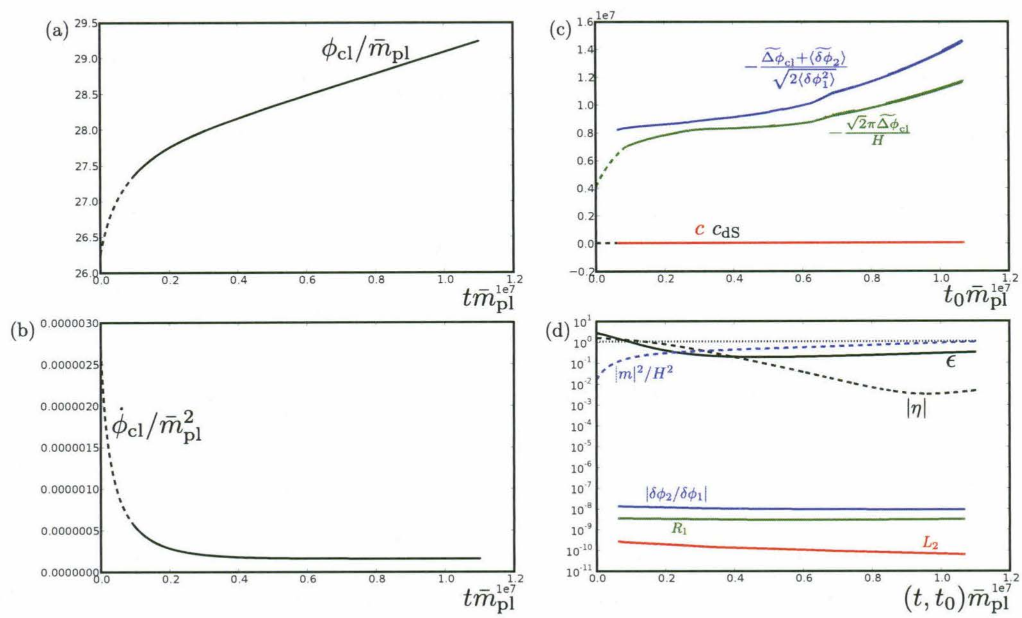

Figure 4.9: Sine-Gordon potential with $A = 2 \times 10^{-12}\overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 5\pi f/3$, $H_0 = \sqrt{A/3\overline{m}_{\rm pl}^2}$, $\dot{\phi}_0 > 0$.

Eqs. (2.24) and (2.28) predict eternal inflation for $\phi_0 \in [0, 20.47\overline{m}_{\rm pl}]$ if $\dot{\phi}_0 > 0$. From Eq. (4.7), we expect inflation to be eternal for $9.9\overline{m}_{\rm pl} \lesssim \phi \lesssim 21.5\overline{m}_{\rm pl}$ if the slow-roll approximation is valid and the field is moving down the potential. There are four types of qualitatively similar trajectories that we illustrate here.

Fig. 4.10 shows a typical trajectory starting low and initially moving up the potential. Both conditions predict eternal inflation while the field is moving up the potential and predict no eternal inflation around when the field turns around and moves down the potential. Fig. 4.11 shows a typical trajectory starting high and initially moving up the potential. Both conditions agree that inflation is eternal while the field is moving up the potential and, since $\phi \gtrsim 9.9\overline{m}_{\rm pl}$, after the field turns around and moves down the potential, both conditions continue to agree that inflation is eternal. Fig. 4.12 shows a typical trajectory starting high and initially moving down the potential. Since $\phi \lesssim 21.5\overline{m}_{\rm pl}$, both conditions predict eternal inflation, although Eq. (2.24) initially predicts no eternal inflation while Eq. (2.28) predicts eternal inflation from the beginning of the trajectory. Finally, Fig. 4.13 shows a typical trajectory starting low and initially moving down the potential. Both conditions predict no eternal inflation since $\phi \gtrsim 21.5\overline{m}_{\rm pl}$.

Both conditions are generally in good agreement in Figs. 4.10, 4.11, and 4.13 while the two conditions initially differ and then asymptote to the same solution in Fig. 4.12. In all of the trajectories, at least after an initial period, both the errors in the perturbative expansion and the slow-roll parameters are small indicating that Eq. (2.28) is a good approximation.

4.4 Random Fourier Series Potential

A generic potential will not be as nice as the quadratic or sine-Gordon potentials. In studying how generic eternal inflation is one might therefore consider potentials of the form

$$V(\phi) = A\overline{m}_{\rm pl}^4 \left[a_0 + B\sum_{n=1}^{N}\left(a_n \cos\frac{n\phi}{\overline{m}_{\rm pl}} + b_n \sin\frac{n\phi}{\overline{m}_{\rm pl}}\right)\right] \tag{4.8}$$

where a_n and b_n are randomly chosen from a uniform distribution with zero mean and standard deviation

$$\sigma_n = \mathrm{e}^{-n^2/2N}.$$

Such potentials, and their multi-field generalizations, have been studied in Refs. [18–20] to calculate the distributions of cosmological observables predicted by generic inflationary potentials. The question of eternal inflation in such potentials has not been studied to our knowledge.

An example of a random potential of the form Eq. (4.8) with $N = 5$ is shown in Fig. 4.14. If we were truly interested in computing the likelihood of eternal inflation in generic potentials, we should set $B = 1$, $a_0 = 0$, and choose A to correspond to different energy scales (as is effectively done in

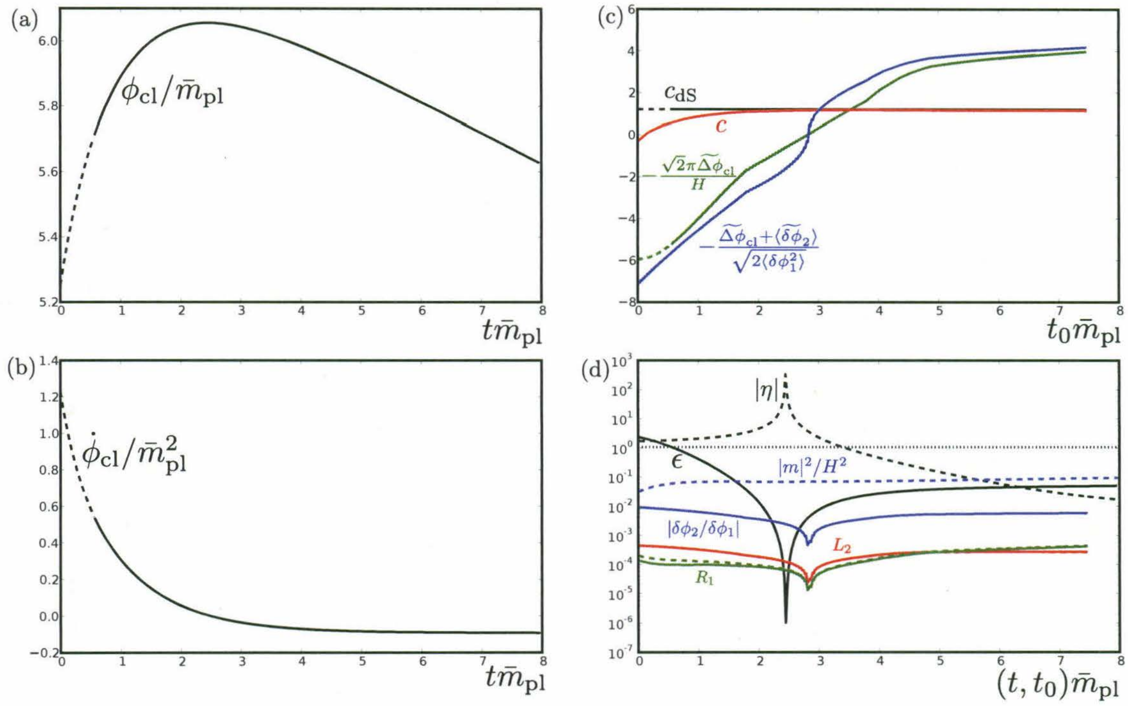

Figure 4.10: Sine-Gordon potential with $A = \overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = \pi f/3$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

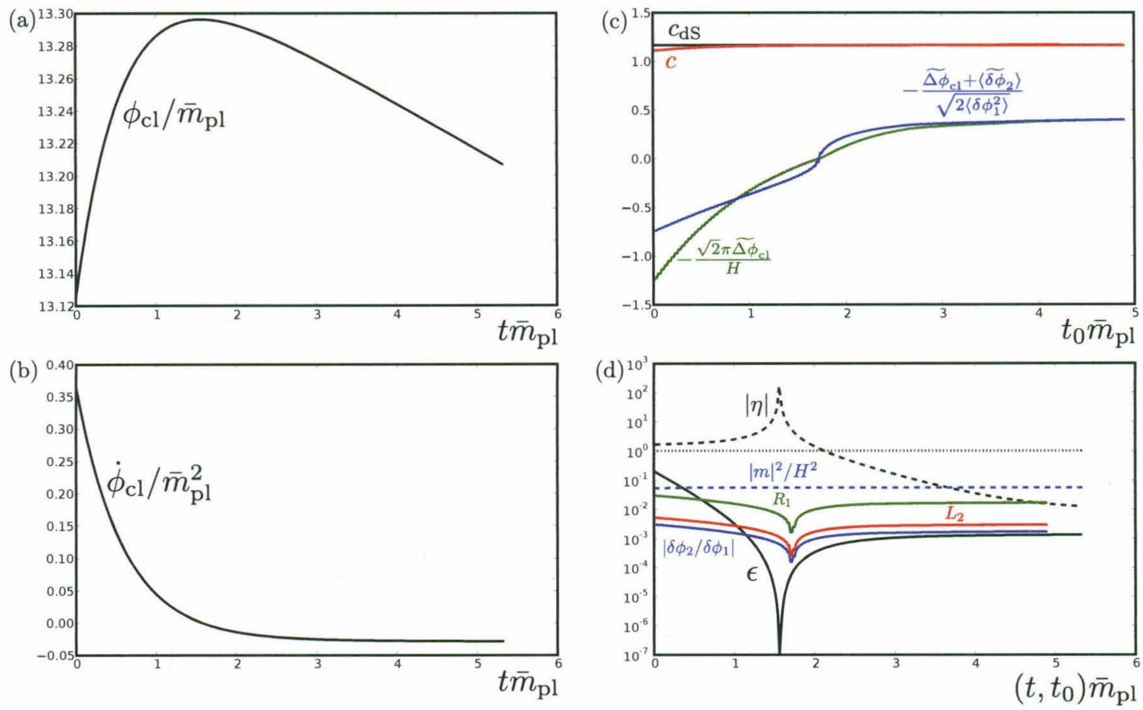

Figure 4.11: Sine-Gordon potential with $A = \overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 5\pi f/6$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

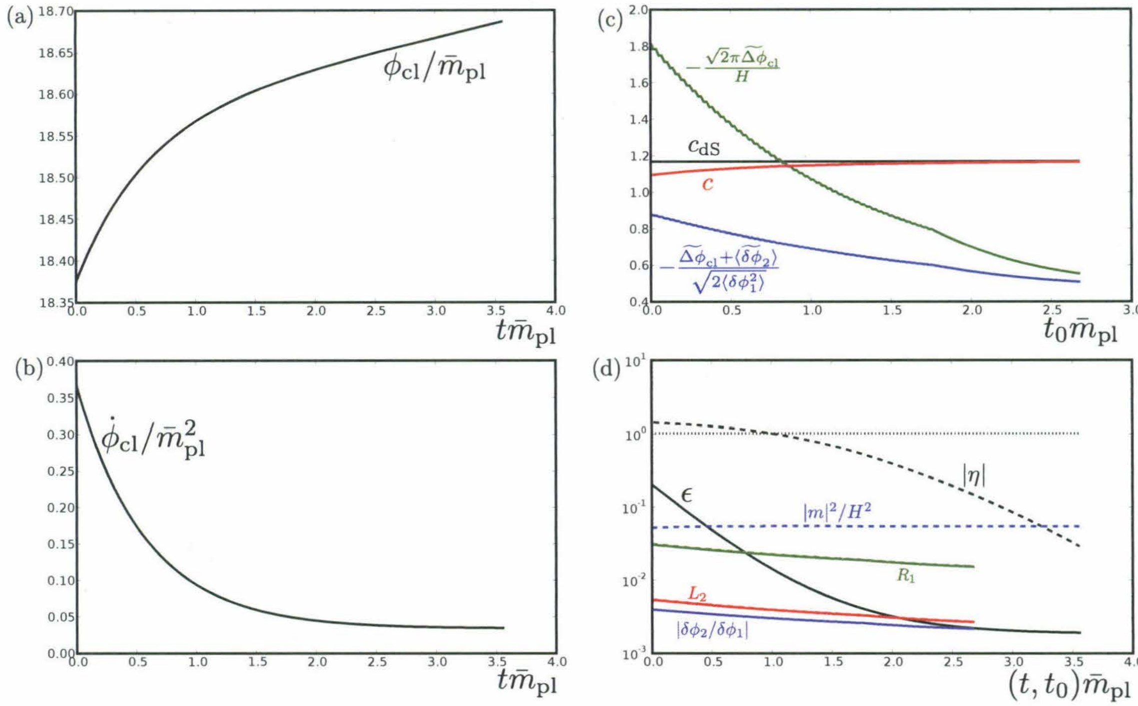

Figure 4.12: Sine-Gordon potential with $A = \overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 7\pi f/6$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

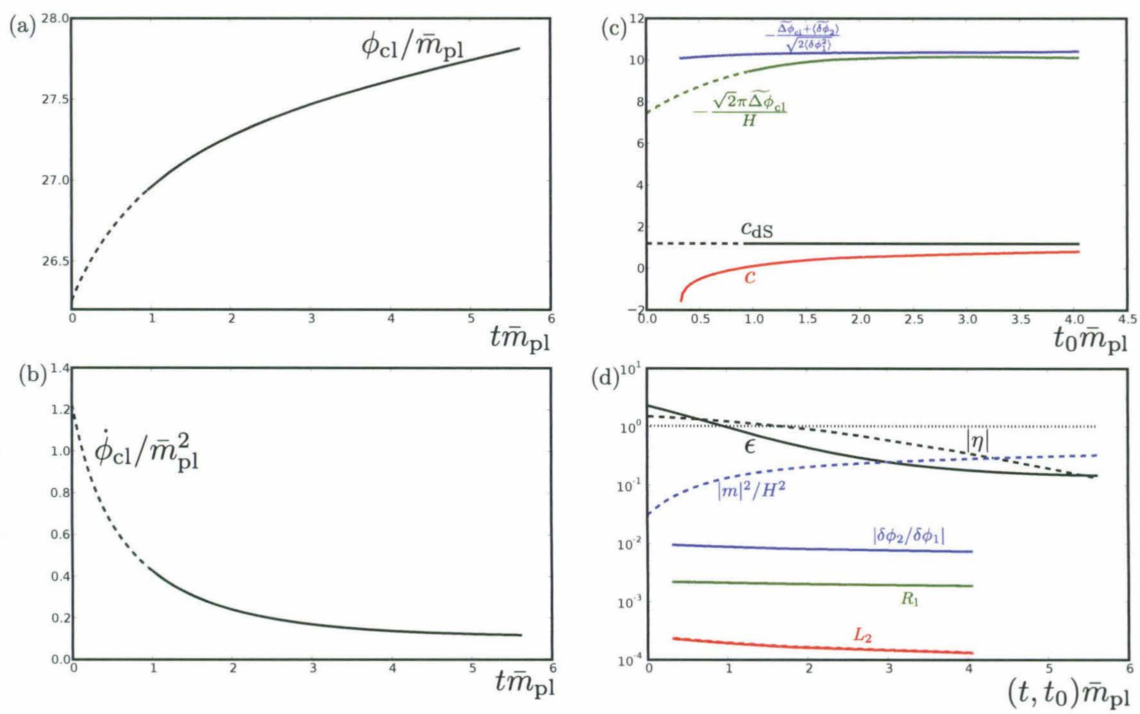

Figure 4.13: Sine-Gordon potential with $A = \overline{m}_{\rm pl}^4$, $f = \sqrt{8\pi}\overline{m}_{\rm pl}$, $\phi_0 = 5\pi f/3$, $H_0 = \overline{m}_{\rm pl}/\sqrt{3}$, $\dot{\phi}_0 > 0$.

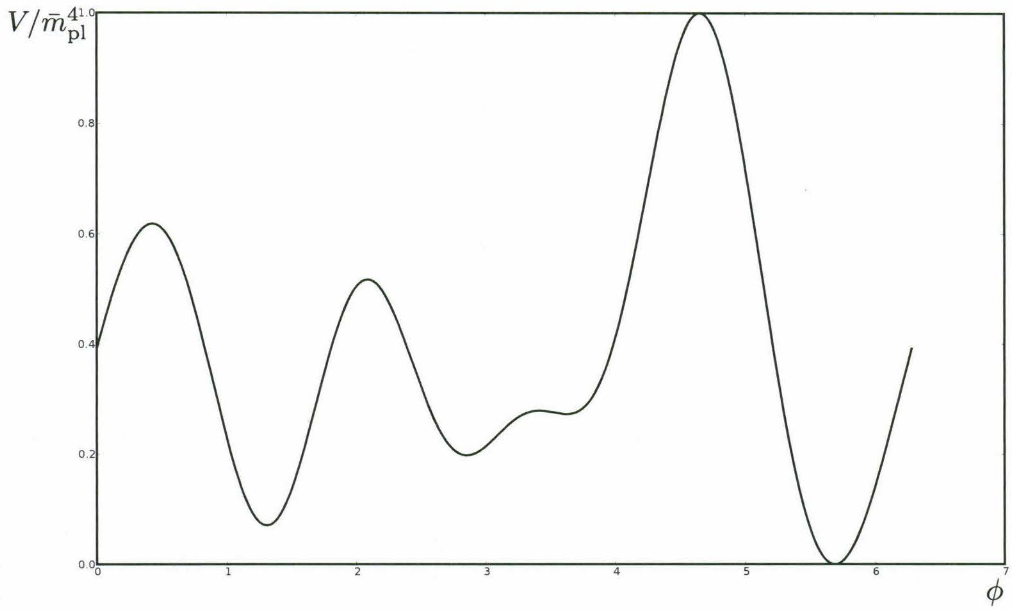

Figure 4.14: Example of a random Fourier series potential of the form Eq. (4.8) with $N = 5$.

Refs. [18–20]). However, for illustrative purposes to avoid complications with potentials for which $V(\phi) < 0$, we choose a_0 and B such that the global minimum is at $V = 0$ and the global maximum is at $V = A\overline{m}_{\rm pl}^4$. A typical trajectory for the potential shown in Fig. 4.14 is shown in Fig. 4.15. Even though the classical trajectory is well behaved, the perturbative expansion completely breaks down for most of the trajectory. This may be partially due to $\langle \delta\phi_1^2 \rangle$ and $\langle \delta\phi_2 \rangle$ Eqs. (3.31) and (3.32) having factors of V' in denominators and diverging near extremal points. However, the perturbative expansion breaks down for this trajectory at points where $V' \neq 0$ and for other similar trajectories where ϕ is never near an extremal point. This example shows that Eq. (2.28) using the perturbative Langevin method of Ref. [39] described in Sec. 3.4 is still inadequate for investigating eternal inflation in generic single field potentials.

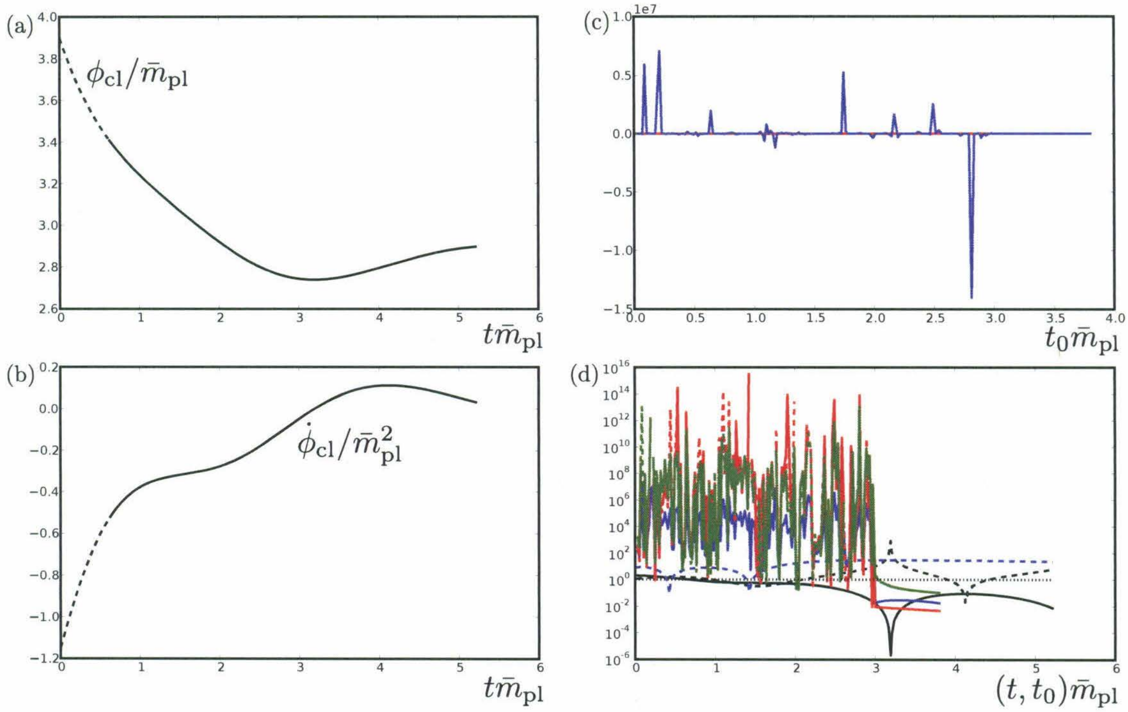

Figure 4.15: The random Fourier series potential shown in Fig. 4.14 with $\phi_0 = 3.9$, $H_0 = \overline{m}_{pl}/\sqrt{3}$, $\dot{\phi}_0 < 0$.

Chapter 5

Summary and Conclusion

While it is widely qouted that eternal inflation is generic, there have been no quantitative studies into this question to our knowledge. The criteria used to support the claim that a potential, or a trajectory in that potential, is capable of supporting stochastic eternal inflation do not account for the generic initial conditions that would be prevalent in a study of how generic stochastic eternal inflation is.

In Sec. 2.3.2, we discussed conditions necessary for stochastic eternal inflation that improve on the conditions normally used in three ways. First, we take into account the direction that the field moves classically which can make it easier for a trajectory to be eternally inflating. Second, we take into account the fact that when $|\dot{\phi}|$, and hence $|\dot{H}|$, is large, the number of independent volumes $\mathcal{N}$ that an initial homogeneous volume expands into can be less than the pure de Sitter value $\mathcal{N}_{\rm dS} = e^3 \approx 20$. Third, we use the perturbative Langevin method of Ref. [39] described in Sec. 3.4 to more accurately compute the distribution of quantum fluctuations.

We compared the our improved conditions and conditions similar to the most sophisticated conditions used in the literature in Ch. 4 for quadratic, sine-Gordon, and random Fourier series potentials. We found examples where these two conditions greatly differ. In particular, taking into account the true value of $\mathcal{N}$ can be crucial since, if it is less than 1, inflation cannot be stochastically eternal even if the quantum fluctuations exceed the classical roll. We also found examples where the conditions for stochastic eternal inflation are met but the field is not inflating which has not previously been discussed to our knowledge.

The results of Ch. 4 should be viewed as evidence that even the most sophisticated conditions currently used in the literature to justify stochastic eternal inflation are inadequate for generic situations and are particularly ill-suited for studying how generic stochastic eternal inflation is. The analysis of Sec. 2.3.2 improves upon these conditions but is still inadequate for this task and more work is required before such an analysis could be made. Further efforts to address the objections made in Sec. 2.3.4 should be made in future work. Work is especially needed for distinguishing when the condition for stochastic eternal inflation is satisfied for every iteration of expansion and when it is satisfied only a finite number of times.

More crucially, a method for accurately calculating the distributions of quantum fluctuations in generic situations needs to be developed. We suggest generalizing the perturbative Langevin method of Ref. [39] to be applicable even when the slow-roll approximation is not valid. This generalization would not have a simple solution that requires only the calculation of single integrals however and would require numerically solving differential equations with stochastic noise terms. This approach has the advantage of easily incorporating the analysis of multiple fields which require such an approach even when the slow-roll approximations are made. Such an approach was used to study hybrid inflation in Ref. [56] where it was found that the differential equations needing to be solved were usually stiff and required care in both the solution and verification that the errors were small. We therefore do not expect this approach to be easily suited to the task of analyzing generic potentials but think it would be useful for analyzing specific cases with generic conditions that do not necessarily satisfy the slow-roll approximation.

We conclude by noting that the analysis presented here, and any improvements made to it in the future, could be useful for studying the conditions necessary for false vacuum driven and topological eternal inflation as well. Ref. [57] studies general tunneling from a false vacuum that could be a combination of Coleman-de Luccia [58–60] and Hawking-Moss [61] tunneling. In a full investigation of eternal inflation, this analysis should be modified to include more accurate computations of the quantum fluctuations responsible for tunneling especially if (possibly fast-rolling) trajectories are being considered. Finally, while quantum fluctuations do not play an important roll in topological eternal inflation, they would likely be important in the formation of a domain wall from a generic trajectory which would need to be considered in a full analysis of topological eternal inflation.

Appendix A

Validity of the Perturbative Langevin Method

The validity of the perturbative method of Ref. [39] for solving the Langevin equation described in Sec. 3.4 is discussed in Ref. [62]. In this appendix, we describe the error analysis of Ref. [62] and slight modifications to it used in the analysis of Ch. 4.

A general function $f(\phi)$ can be expanded about the classical solution $\phi_{\rm cl}$ in powers of $\Delta\phi$ as

$$f(\phi_{\rm cl}+\Delta\phi)=\sum_{k=0}^{n}\frac{f^{(k)}(\phi_{\rm cl})}{k!}\Delta\phi^k+E_n(\phi_{\rm cl}+\Delta\phi) \tag{A.1}$$

where

$$E_n(\phi_{\rm cl}+\Delta\phi)=\frac{1}{n!}\int_{\phi_{\rm cl}}^{\phi_{\rm cl}+\Delta\phi}(\phi_{\rm cl}+\Delta\phi-t)^n f^{(n+1)}(t)\,{\rm d}t \tag{A.2}$$

is the remainder of the expansion truncated to the nth power of $\Delta\phi$ [63]. Therefore, expanding the Langevin equation Eq. (3.14) and using the classical equation of motion

$$\begin{aligned}\frac{{\rm d}\Delta\phi}{{\rm d}t}+2\overline{m}_{\rm pl}^2H''(\phi_{\rm cl})\Delta\phi+\overline{m}_{\rm pl}^2H'''(\phi_{\rm cl})\Delta\phi^2+2\overline{m}_{\rm pl}^2L_2(\phi_{\rm cl}+\Delta\phi)=&\\ \frac{H^{3/2}(\phi_{\rm cl})}{2\pi}\xi+\frac{(H^{3/2})'(\phi_{\rm cl})}{2\pi}\Delta\phi\xi+\frac{R_1(\phi_{\rm cl}+\Delta\phi)}{2\pi}\xi&\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}L_2(\phi_{\rm cl}+\Delta\phi)&\equiv\frac{1}{2}\int_{\phi_{\rm cl}}^{\phi_{\rm cl}+\Delta\phi}(\phi_{\rm cl}+\Delta\phi-t)^2H^{(4)}(t)\,{\rm d}t &\text{(A.4a)}\\ R_1(\phi_{\rm cl}+\Delta\phi)&\equiv\int_{\phi_{\rm cl}}^{\phi_{\rm cl}+\Delta\phi}(\phi_{\rm cl}+\Delta\phi-t)(H^{3/2})''(t)\,{\rm d}t. &\text{(A.4b)}\end{aligned}$$

Note that Eq. (3.16) is Eq. (A.3) truncated to second order in ξ.

Eq. (A.3) determines $\Delta\phi$ exactly. The criterion of Ref. [62] for the perturbative expansion to be valid for $\Delta\phi$ approximated by $\delta\phi_1+\delta\phi_2$ is that L_2 and R_1 be small compared to the other terms in the expansion. Thus the expansion is valid if

$$|L_2(\phi_{\rm cl}+\Delta\phi)| \ll \left|\frac{H'''(\phi_{\rm cl})\Delta\phi^2}{2}\right| \tag{A.5a}$$

$$|R_1(\phi_{\rm cl}+\Delta\phi)| \ll \left|(H^{3/2})'(\phi_{\rm cl})\Delta\phi\right| \tag{A.5b}$$

where $\Delta\phi = \delta\phi_1 + \delta\phi_2$. In practice we use instead $\Delta\phi = \langle\delta\phi_2\rangle \pm \sqrt{\langle\delta\phi_1^2\rangle}$.

Ref. [62] uses a different form of the remainder than Eq. (A.2):

$$E_n(\phi_{\rm cl}+\Delta\phi) = \frac{f^{(n+1)}(\phi_{\rm cl}+\theta\Delta\phi)}{(n+1)!}\Delta\phi^{n+1} \tag{A.6}$$

for some $\theta \in [0,1]$. This leads to the alternative definitions

$$L_2(\phi_{\rm cl}+\Delta\phi) \equiv \frac{H^{(4)}(\phi_{\rm cl}+\theta_L\Delta\phi)}{6}\Delta\phi^3 \tag{A.7a}$$

$$R_1(\phi_{\rm cl}+\Delta\phi) \equiv \frac{(H^{3/2})''(\phi_{\rm cl}+\theta_R\Delta\phi)}{2}\Delta\phi^2 \tag{A.7b}$$

and the alternative conditions

$$\max_{x\in[\phi_{\rm cl},\phi_{\rm cl}+\Delta\phi]} |L_2(x)| \ll \left|\frac{H'''(\phi_{\rm cl})\Delta\phi^2}{2}\right| \tag{A.8a}$$

$$\max_{x\in[\phi_{\rm cl},\phi_{\rm cl}+\Delta\phi]} |R_1(x)| \ll \left|(H^{3/2})'(\phi_{\rm cl})\Delta\phi\right|. \tag{A.8b}$$

Eqs. (A.8) can then be used to solve for the values of $\Delta\phi_{\rm max} > 0$ and $\Delta\phi_{\rm min} < 0$ that provide the strictest bounds for equality in Eqs. (A.8). The approximation will be valid if

$$\Delta\phi_{\rm min} < \langle\phi_2\rangle \pm \sqrt{\langle\phi_1^2\rangle} < \Delta\phi_{\rm max}. \tag{A.9}$$

Eqs. (A.8) can be used to find analytic expressions for $\Delta\phi_{\rm min}$ and $\Delta\phi_{\rm max}$ for the quadratic potential but must be solved numerically for other potentials. We find it easier in practice to compute the integrals in Eqs. (A.4) and directly check Eqs. (A.5) than to solve Eqs. (A.8) for $\Delta\phi_{\rm min}$ and $\Delta\phi_{\rm max}$ and to check Eq. (A.9). To compute the integrals in Eqs. (A.4) we use

$$H^{(4)} = \frac{1}{\sqrt{3}\overline{m}_{\rm pl}}\left(\frac{1}{2}\frac{V^{(4)}}{\sqrt{V}} - \frac{V'''V'}{V^{3/2}} - \frac{3}{4}\frac{V''^2}{V^{3/2}} + \frac{9}{4}\frac{V'^2V''}{V^{5/2}} - \frac{15}{16}\frac{V'^4}{V^{7/2}}\right)$$

$$(H^{3/2})'' = \frac{1}{3^{3/2}\overline{m}_{\rm pl}^{3/4}}\left(\frac{3}{4}\frac{V''}{V^{1/4}} - \frac{3}{16}\frac{V'^2}{V^{5/2}}\right)$$

and to compute the right hand sides of Eqs. (A.5) we use

$$\begin{aligned} H''' &= \frac{1}{\sqrt{3}\overline{m}_{\rm pl}}\left(\frac{1}{2}\frac{V'''}{\sqrt{V}} - \frac{3}{4}\frac{V'V''}{V^{3/2}} + \frac{3}{8}\frac{V'^3}{V^{5/2}}\right) \\ (H^{3/2})' &= \frac{3^{1/4}}{4\overline{m}_{\rm pl}^{3/2}}\frac{V'}{V^{1/4}}. \end{aligned} \tag{A.10}$$

In addition to checking Eqs. (A.5) it is easy to check $\delta\phi_2 \ll \delta\phi_1$. This must be true for $\Delta\phi = \delta\phi_1 + \delta\phi_2$ to be a good approximation for splitting Eq. (3.16) into separate equations for $\delta\phi_1$ and $\delta\phi_2$. In practice, we check $\langle\delta\phi_2\rangle \ll \sqrt{\delta\phi_1^2}$.

In the analysis of Ref. [62], it is also checked that

$$\frac{1}{\sqrt{2\pi\,\langle\delta\phi_1^2\rangle}}\int_{\Delta\phi_{\rm min}}^{\Delta\phi_{\rm max}} \exp\left[-\frac{(\phi - \langle\delta\phi_2\rangle)^2}{2\,\langle\delta\phi_1^2\rangle}\right]\,{\rm d}\phi \approx 1 \tag{A.11}$$

as is required for the probability of $\Delta\phi \in [\Delta\phi_{\rm min}, \Delta\phi_{\rm max}]$ to be close to 1. Since we do not compute $\Delta\phi_{\rm min}$ and $\Delta\phi_{\rm max}$, we do not check this condition in general. However, in the case of the quadratic potential, we computed the integral and found it to be 1 for all cases where the expansion was otherwise considered valid. Furthermore, in all of the cases checked in Ref. [62], this integral only begins to deviate from 1 as $\Delta\phi$ approaches $\Delta\phi_{\rm min}$ or $\Delta\phi_{\rm max}$. We therefore find it unlikely that this condition would not be satisfied when the other conditions for the validity of the expansion are valid.

Appendix B

Calculation of the Curvature of the FRW Metric

B.1 Calculation of the Christoffel Symbols

We calculate the Christoffel symbols using the method described in Ref. [64] where the variation of the integral

$$\begin{aligned} I &= \frac{1}{2}\int g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,\mathrm{d}\tau \\ &= \frac{1}{2}\int\left\{-N^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + a^2\left[\frac{1}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\right\}\mathrm{d}\tau \end{aligned} \tag{B.1}$$

is set to zero and the Christoffel symbols are read off of the appropriate coefficients of the resulting geodesic equation

$$\frac{\mathrm{d}^2x^\rho}{\mathrm{d}\tau^2} + \Gamma^\rho_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{B.2}$$

We start with the variation $t \to t + \delta t$ resulting in

$$\delta I = \frac{1}{2}\int\left\{-2N\dot{N}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\delta t - 2N^2\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta t)}{\mathrm{d}\tau} + 2a\dot{a}\left[\frac{1}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\delta t\right\}\mathrm{d}\tau$$

Integrating by parts and assuming appropriate boundary conditions so that the surface term vanishes,

$$\int N^2\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta t)}{\mathrm{d}\tau}\,\mathrm{d}\tau = -\int\left(2N\frac{\mathrm{d}N}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + N^2\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2}\right)\delta t\,\mathrm{d}\tau.$$

Thus

$$\delta I = \int\left\{N\dot{N}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + N^2\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} + a\dot{a}\left[\frac{1}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\right\}\delta t\,\mathrm{d}\tau = 0$$

and therefore

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{\dot{N}}{N}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{a\dot{a}}{N^2}\left[\frac{1}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right] = 0.$$

We thus have the following Christoffel symbols

$$\Gamma^t_{tt} = \frac{\dot{N}}{N}, \qquad \Gamma^t_{rr} = \frac{a\dot{a}}{N^2(1-kr^2)}, \qquad \Gamma^t_{\theta\theta} = \frac{a\dot{a}r^2}{N^2}, \qquad \Gamma^t_{\phi\phi} = \frac{a\dot{a}r^2\sin^2\theta}{N^2}. \tag{B.3}$$

Next, we consider the variation $r \to r + \delta r$ resulting in

$$\delta I = \frac{1}{2}\int a^2\left[\frac{2}{1-kr^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta r)}{\mathrm{d}\tau} + \frac{2kr}{(1-kr^2)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2\delta r + 2r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2\delta r + 2r\sin^2\theta\left(\frac{\phi}{\tau}\right)^2\delta r\right]\mathrm{d}\tau.$$

Integrating by parts,

$$\int \frac{a^2}{1-kr^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta r)}{\mathrm{d}\tau}\,\mathrm{d}\tau = -\int\left[\frac{a^2}{1-kr^2}\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{2a^2kr}{(1-kr^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{2a}{1-kr^2}\frac{\mathrm{d}a}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right]\delta r\,\mathrm{d}\tau.$$

Thus

$$\delta I = \int\left[-\frac{a^2}{1-kr^2}\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{a^2kr}{(1-kr^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{2a\dot{a}}{1-kr^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} + a^2r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + a^2r\sin^2\,\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\delta r\,\mathrm{d}\tau = 0$$

and therefore

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{kr}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{2\dot{a}}{a}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} - r(1-kr^2)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - r\sin^2\theta\,(1-kr^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 0.$$

We thus have the following Christoffel symbols

$$\Gamma^r_{rr} = \frac{kr}{1-kr^2}, \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2), \qquad \Gamma^r_{\phi\phi} = -r\sin^2\theta\,(1-kr^2), \qquad \Gamma^r_{rt} = \Gamma^r_{tr} = \frac{\dot{a}}{a}. \tag{B.4}$$

Next, we consider the variation $\theta \to \theta + \delta\theta$ resulting in

$$\delta I = \frac{1}{2}\int a^2\left[2r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta\theta)}{\mathrm{d}\tau} + 2r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\delta\theta\right]\mathrm{d}\tau.$$

Integrating by parts,

$$\int a^2r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta\theta)}{\mathrm{d}\tau}\,\mathrm{d}\tau = -\int\left[a^2r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + 2a^2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + 2ar^2\frac{\mathrm{d}a}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right]\delta\theta\,\mathrm{d}\tau.$$

Thus

$$\delta I = \int \left[-a^2 r^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - 2a^2 r \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - 2a\dot{a}r^2 \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + a^2 r^2 \sin\theta\cos\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \right] \delta\theta \, \mathrm{d}\tau = 0$$

and therefore

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{2\dot{a}}{a}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 = 0.$$

We thus have the following Christoffel symbols

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\theta}_{\theta t} = \Gamma^{\theta}_{t\theta} = \frac{\dot{a}}{a}, \qquad \Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{r}. \tag{B.5}$$

Finally, we consider the variation $\phi \to \phi + \delta\phi$ resulting in

$$\begin{aligned}\delta I &= \frac{1}{2}\int 2a^2 r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}(\delta\phi)}{\mathrm{d}\tau}\,\mathrm{d}\tau \\ &= -\int \left[a^2 r^2 \sin^2\theta \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + 2a^2 r^2 \sin\theta\cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2a^2 r \sin^2\theta \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2a\dot{a}r^2 \sin^2\theta \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] \delta\phi\,\mathrm{d}\tau = 0.\end{aligned}$$

Thus

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{2\dot{a}}{a}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 0$$

and we have the following Christoffel symbols

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta, \qquad \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \qquad \Gamma^{\phi}_{t\phi} = \Gamma^{\phi}_{\phi t} = \frac{\dot{a}}{a}. \tag{B.6}$$

All Christoffel symbols not listed in Eqs. (B.3-B.6) vanish.

B.2 Calculation of the Riemann Curvature Tensor

In this section, we calculate the components of the Riemann tensor necessary to calculate the Ricci tensor. The components of the Riemann tensor are

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{B.7}$$

$\Gamma^t_{tr} = 0$ so

$$
\begin{aligned}
R_{trtr} &= g_{tt}\left(\partial_t \Gamma^t_{rr} + \Gamma^t_{tt}\Gamma^t_{rr} - \Gamma^t_{rr}\Gamma^r_{tr}\right) \\
&= -N^2\left[\frac{a\ddot{a}+\dot{a}^2}{N^2(1-kr^2)} - \frac{2a\dot{a}\dot{N}}{N^3(1-kr^2)} + \frac{a\dot{a}\dot{N}}{N^3(1-kr^2)} - \frac{\dot{a}^2}{N^2(1-kr^2)}\right] \\
&= \frac{a}{1-kr^2}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right). \qquad \text{(B.8)}
\end{aligned}
$$

$\Gamma^t_{\phi\phi} = 0$ so

$$
\begin{aligned}
R_{t\theta t\theta} &= g_{tt}\left(\partial_t \Gamma^t_{\theta\theta} + \Gamma^t_{tt}\Gamma^t_{\theta\theta} - \Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta}\right) \\
&= -N^2\left(\frac{a\ddot{a}r^2+\dot{a}^2r^2}{N^2} - \frac{2a\dot{a}\dot{N}r^2}{N^3} + \frac{a\dot{a}\dot{N}r^2}{N^3} - \frac{\dot{a}^2r^2}{N^2}\right) \\
&= ar^2\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right). \qquad \text{(B.9)}
\end{aligned}
$$

$\Gamma^t_{t\phi} = 0$ so

$$
\begin{aligned}
R_{t\phi t\phi} &= g_{tt}\left(\partial_t \Gamma^t_{\phi\phi} + \Gamma^t_{tt}\Gamma^t_{\phi\phi} - \Gamma^t_{\phi\phi}\Gamma^\phi_{t\phi}\right) \\
&= -N\left[\frac{(a\ddot{a}+\dot{a}^2)r^2\sin^2\theta}{N^2} - \frac{2a\dot{a}\dot{N}r^2\sin^2\theta}{N^3} + \frac{a\dot{a}\dot{N}r^2\sin^2\theta}{N^3} - \frac{\dot{a}^2r^2\sin^2\theta}{N^2}\right] \\
&= ar^2\sin^2\theta\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right). \qquad \text{(B.10)}
\end{aligned}
$$

$\Gamma^r_{r\theta} = 0$ so

$$
\begin{aligned}
R_{r\theta r\theta} &= g_{rr}\left(\partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} + \Gamma^r_{rt}\Gamma^t_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta}\right) \\
&= \frac{a^2}{1-kr^2}\left(-1+3kr^2-kr^2+\frac{\dot{a}^2r^2}{N^2}+1-kr^2\right) \\
&= \frac{a^2r^2}{1-kr^2}\left(k+\frac{\dot{a}^2}{N^2}\right). \qquad \text{(B.11)}
\end{aligned}
$$

$\Gamma^r_{r\phi} = 0$ so

$$
\begin{aligned}
R_{r\phi r\phi} &= g_{rr}\left(\partial_r \Gamma^r_{\phi\phi} + \Gamma^r_{rt}\Gamma^t_{\phi\phi} + \Gamma^r_{rr}\Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi}\right) \\
&= \frac{a^2r^2}{1-kr^2}\left[(-1+3kr^2)\sin^2\theta + \frac{\dot{a}^2r^2\sin^2\theta}{N^2} - kr^2\sin^2\theta + (1-kr^2)\sin^2\theta\right] \\
&= \frac{a^2r^2\sin^2\theta}{1-kr^2}\left(k+\frac{\dot{a}^2}{N^2}\right). \qquad \text{(B.12)}
\end{aligned}
$$

$\Gamma^{\theta}_{\theta\phi} = 0$ so

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta}\left(\partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\theta t}\Gamma^{t}_{\phi\phi} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\phi\phi} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}\right) \\ &= a^2 r^2 \left[\sin^2\theta - \cos^2\theta + \frac{\dot{a}^2 r^2 \sin^2\theta}{N^2} - (1-kr^2)\sin^2\theta + \cos^2\theta\right] \\ &= a^2 r^4 \sin^2\theta \left(k + \frac{\dot{a}^2}{N^2}\right). \end{aligned} \tag{B.13}$$

B.3 Calculation of the Ricci Curvature Tensor and Scalar

We use the results of Sec. B.2 to calculate the components of the Ricci curvature tensor necessary to calculate the Ricci curvature scalar. The components of the Ricci curvature tensor are

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\sigma\mu\rho\nu}. \tag{B.14}$$

Thus, from Eqs. (B.8), (B.9), and (B.10),

$$R_{tt} = \frac{3}{a}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right). \tag{B.15}$$

From Eqs. (B.8), (B.11), and (B.13),

$$R_{rr} = -\frac{a}{N^2(1-kr^2)}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right) + \frac{2}{1-kr^2}\left(k + \frac{\dot{a}^2}{N^2}\right). \tag{B.16}$$

From Eqs. (B.9), (B.8), and (B.13),

$$R_{\theta\theta} = -\frac{ar^2}{N^2}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right) + 2r^2\left(k + \frac{\dot{a}^2}{N^2}\right). \tag{B.17}$$

Finally, from Eqs. (B.10), (B.12), and (B.13)

$$R_{\phi\phi} = -\frac{ar^2\sin^2\theta}{N^2}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right) + 2r^2\sin^2\theta\left(k + \frac{\dot{a}^2}{N^2}\right). \tag{B.18}$$

Finally, we can calculate the Ricci curvature scalar. From Eqs. (B.15)-(B.18),

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{6}{a^2}\left(k + \frac{\dot{a}^2}{N^2}\right) - \frac{6}{N^2 a}\left(\frac{\dot{a}\dot{N}}{N} - \ddot{a}\right) = 6\left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2} - \frac{\dot{a}\dot{N}}{N^3 a}\right). \tag{B.19}$$

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