Definable Combinatorics of Graphs and Equivalence Relations

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Let $\mathbf{D} = (X, D)$ be a Borel directed graph on a standard Borel space X and let $\chi_B(\mathbf{D})$ be its Borel chromatic number. If $F_0, \ldots, F_{n-1} : X \to X$ are Borel functions, let $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ be the directed graph that they generate. It is an open problem if $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \in \{1, \ldots, 2n+1, \aleph_0\}$. Palamourdas [25] verified the foregoing for commuting functions with no fixed points. We show here that for commuting functions with the property that there is a path from each $x \in X$ to a fixed point of some F_j, there exists an increasing filtration $X = \bigcup_{m<\omega} X_m$ such that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \restriction X_m\right) \leq 2n$ for each m. We also prove that if $n = 2$ in the previous case, then $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \leq 4$. It follows that the approximate measure chromatic number $\chi_M^{ap}(\mathbf{D}) \leq 2n + 1$ when the functions commute.

If X is a set, E is an equivalence relation on X, and $n \in \omega$, then define $[X]^n_E = \{(x_0, ..., x_{n-1}) \in {}^nX : (\forall i, j)(i \neq j \to \neg(x_i \mathrel{E} x_j))\}$. For $n \in \omega$, a set X has the n-Jónsson property if and only if for every function $f : [X]^n_= \to X$, there exists some $Y \subseteq X$ with X and Y in bijection so that $f[[Y]^n_=] \neq X$. A set X has the Jónsson property if and only for every function $f : (\bigcup_{n\in\omega}[X]^n_=) \to X$, there exists some $Y \subseteq X$ with X and Y in bijection so that $f[\bigcup_{n\in\omega}[Y]^n_=] \neq X$. Let $n \in \omega$, X be a Polish space, and E be an equivalence relation on X. E has the n-Mycielski property if and only if for all comeager $C \subseteq {}^nX$, there is some Borel $A \subseteq X$ so that $E \leq_B E \restriction A$ and $[A]^n_E \subseteq C$. The following equivalence relations will be considered: E_0 is defined on ${}^\omega 2$ by $x \mathrel{E_0} y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_1 is defined on ${}^\omega({}^\omega 2)$ by $x \mathrel{E_1} y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_2 is defined on ${}^\omega 2$ by $x \mathrel{E_2} y$ if and only if $\sum\{\frac{1}{n+1} : x(n) \neq y(n)\} < \infty$. E_3 is defined on ${}^\omega({}^\omega 2)$ by $x \mathrel{E_3} y$ if and only if $(\forall n)(x(n) \mathrel{E_0} y(n))$. Holshouser and Jackson have shown that $\mathbb{R}$ is Jónsson under AD. The present research will show that E_0 does not have the 3-Mycielski property and that E_1, E_2, and E_3 do not have the 2-Mycielski property. Under $\mathsf{ZF} + \mathsf{AD}$, ${}^\omega 2/E_0$ does not have the 3-Jónsson property.

Let $\mathbf{G} = (X, G)$ be a graph and define for $b \geq 1$ its b-fold chromatic number $\chi^{(b)}(\mathbf{G})$ as the minimum size of Y such that there is a function c from X into b-sets of Y with $c(x) \cap c(y) = \emptyset$ if $x \mathrel{G} y$. Then its fractional chromatic number is $\chi^f(\mathbf{G}) = \inf_b \frac{\chi^{(b)}(\mathbf{G})}{b}$ if the quotients are finite. If X is Polish and $\mathbf{G}$ is a Borel graph, we can also define its fractional Borel chromatic number $\chi^f_B(\mathbf{G})$ by restricting to only Borel functions. We similarly define this for Baire measurable and μ-measurable functions for a Borel measure μ. We show that for each countable graph $\mathbf{G}$, one may construct an acyclic Borel graph $\mathbf{G}'$ on a Polish space such that $\chi^f_{BM}(\mathbf{G}') = \chi^f(\mathbf{G})$ and $\chi_{BM}(\mathbf{G}') = \chi(\mathbf{G})$, and similarly for χ^f_μ and χ_μ. We also prove that the implication $\chi^f(\mathbf{G}) = 2 \Rightarrow \chi(\mathbf{G}) = 2$ is false in the Borel setting.

[3] William Chan and Connor Meehan. "Definable Combinatorics of Some Borel Equivalence Relations" In: ArXiv e-prints (Sep. 2017). arXiv: 1709.04567 [math.LO].

Connor Meehan participated in producing the results and writing the paper. Each author contributed equally to this work.

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INTRODUCTION

Descriptive set theory is a branch of set theory that focuses on sets of a particular structure definable via some constructive rules, as opposed to sets whose existence is guaranteed by some consequence of the axiom of choice. A Polish space is a topological space that is separable and completely metrizable. The prototypical example of a definable set is a Borel set: one that belongs to the smallest family of sets containing all open sets and closed under complements and countable unions. To repeat the first sentence more concretely, descriptive set theory concerns the study of definable sets in Polish spaces.

Descriptive set theory is a major research area of set theory and can both be approached from and have results that relate to many other fields of mathematics, such as mathematical logic, topology, and ergodic theory. In this thesis, we focus chiefly on descriptive combinatorics. In particular, we study problems related to graphs and equivalence relations from the viewpoint of descriptive set theory, asking when certain definable objects will satisfy certain definable properties. For example, every acyclic graph can be 2-coloured, but it is possible for the minimum size of the range of a Borel colouring of an acyclic graph to be $2^{\aleph_0}$.

This thesis consists of three separate papers: *Borel Chromatic Numbers of Graphs of Commuting Functions* (with Konstantinos Palamourdas), *Definable Combinatorics of Some Borel Equivalence Relations* (with William Chan), and *Fractional Borel Chromatic Numbers and Definable Combinatorics*. The first two papers have been submitted for publication. For a more detailed introduction to each paper, see the introduction section of each chapter.

In Chapter 2, we study a Borel graph colouring problem; namely, what is the smallest size of a finite set (if one even exists) that can be used for a Borel colouring of a directed graph generated by n Borel functions on a standard Borel space? Palamourdas [25] has previously shown that for such a digraph $\mathbf{D}_{F_0,\ldots,F_{n-1}}$, its Borel chromatic number $\chi_B(\mathbf{D}_{F_0,\ldots,F_{n-1}})$ is in the set $\left\{1, \ldots, \frac{1}{2}(n+1)(n+2), \aleph_0\right\}$ and that if the functions F_j commute and have no fixed points, then this can be improved to $\{1, \ldots, 2n+1, \aleph_0\}$. In Chapter 2, we provide a recursive structure for breaking down subdividing a Borel graph of this structure. We show that in the commuting case, if $\chi_B(\mathbf{D}_{F_0,\ldots,F_{n-1}}) < \aleph_0$ and each point has a forward path to a fixed point, then very large portions of the graph can be coloured with only $2n$ colours: there exists an increasing filtration $X = \bigcup_{m<\omega} X_m$

such that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \restriction X_m\right) \le 2n$ for each m.

It is a standard result of graph theory that any digraph $\mathbf{D}$ with maximum out-degree bounded by n can be $(2n+1)$-coloured. Though the case $\chi_B(\mathbf{D}_{F_0,\ldots,F_{n-1}}) = \aleph_0$ is possible, Kechris, Solecki, and Todorčević [15] have asked whether the Borel analog of this fact is that $\chi_B(\mathbf{D}_{F_0,\ldots,F_{n-1}}) = \{1,\ldots,2n+1,\aleph_0\}$ for arbitrary Borel functions F_j. The results in Chapter 2 give further evidence that this may be true in the case that the functions commute.

In Chapter 3, we study some well-known equivalence relations defined on Polish spaces and whether they satisfy a combinatorial notion known as the n-Mycielski property (so-called due to a theorem of Mycielski showing that $=$ has this property). We show that, though E_0 satisfies the 2-Mycielski property, the relations E_1, E_2, and E_3 do not, and moreover E_0 fails to have the 3-Mycielski property. These equivalence relations occupy special positions in the poset of Borel equivalence relations under Borel reducibility, and theorems regarding the structure of Borel sets A for which $E_i \le_{\mathbf{\Delta}^1_1} E_i \restriction A$ are important in our proofs about the Mycielski property.

One reason that the Mycielski property is interesting is that it aids in the study of another combinatorial notion: the Jónsson property. This is a notion that has been studied by set theorists under various axiom systems; for example, the existence of cardinals with the Jónsson property implies that $0^\sharp$ exists. Holshouser and Jackson [11] used the Mycielski property of $=$ to show that the set ${}^\omega 2$ has the Jónsson property under the axiom system $\mathsf{ZF} + \mathsf{AD}$. We show using the fact that E_0 does not have the 3-Mycielski property that the quotient ${}^\omega 2/E_0$ does not have the Jónsson property in this axiom system.

The exact relation between an equivalence relation possessing the Mycielski property and its quotient having the Jónsson property is still unknown. We ask whether the quotients for the equivalence relations E_1, E_2, and E_3 also fail to have the Jónsson property under the axiom of determinacy.

In Chapter 4, we study a classical combinatorial notion in the definable setting; that of the fractional chromatic number χ^f. We show that the fractional Borel number χ^f_B behaves much like that of its classical counterpart. In particular, the quantities χ, χ^f, χ^f_B can all be made distinct for the same Borel graph (and indeed any values that satisfy some obvious conditions). We construct Borel graphs to prove this idea. These Borel graphs are acyclic but otherwise behave (for the sake of Borel colourings) much like a fixed countable graph. We also show that even though $\chi^f(\mathbf{G}) = 2$ forces $\chi(\mathbf{G}) = 2$, this is not true for Borel colourings: there exists a graph $\mathbf{G}$ for which $\chi^f_B(\mathbf{G}) = 2$ but $\chi_B(\mathbf{G}) = 3$.

No consensus yet exists for how to compute $\chi_B^f(\mathbf{G})$ for Borel graphs $\mathbf{G}$ with $\chi(\mathbf{G}) < \chi_B(\mathbf{G})$. Marks showed [22] that the shift graph $\mathbf{G}_n$ on the free part of $2^{\mathbb{F}_n}$ satisfies $\chi_B(\mathbf{G}_n) = 2n + 1$, and we ask whether it is true that $\chi_B^f(\mathbf{G}_n) = 2$.

BOREL CHROMATIC NUMBERS OF GRAPHS OF COMMUTING FUNCTIONS

(with Konstantinos Palamourdas)

2.1 Introduction

Kechris, Solecki, and Todorčević initiated the field of descriptive graph combinatorics in their seminal paper [15]. The broad objective is to study how combinatorial notions of graphs such as matchings and colourings behave under definability constraints. For a comprehensive review of the subject, see [19].

A **directed graph**, or **digraph**, is a pair $\mathbf{D} = (X, D)$, where X is a set and D is a binary irreflexive relation on X. A **path** in $\mathbf{D}$ is a sequence $\{x_i\}_{i<\alpha}$ in X for some $\alpha \in \{1, 2, \ldots, \omega\}$ such that $x_i \, D \, x_{i+1}$ for all i with $i + 1 < \alpha$. A subset $A \subseteq X$ is **independent** if for any $x, y \in A$, $\neg x \, D \, y$. If $A \subseteq X$, we let $\mathbf{D} \restriction A = (A, D \cap (A \times A))$. For any set Y, a Y-**colouring** on $\mathbf{D}$ is a function $c : X \to Y$ such that if $x \, D \, y$, then $c(x) \neq c(y)$. The **chromatic number** $\chi(\mathbf{D})$ is then the minimum cardinality of Y such that there is a Y-colouring of $\mathbf{D}$.

In case X is a standard Borel space, we can consider analogous definable notions of colouring. We define the **Borel chromatic number** $\chi_B(\mathbf{D})$ to be the minimum cardinality of a standard Borel Y such that there is a Borel Y-colouring of $\mathbf{D}$. If μ is a Borel probability measure on X, we define the **approximate μ-measurable chromatic number** $\chi_\mu^{ap}(\mathbf{D})$ to be the minimum cardinality of a standard Borel Y such that for every $\varepsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a μ-measurable Y-colouring of $\mathbf{D} \restriction A$. We then define the **approximate measure chromatic number** $\chi_M^{ap}(\mathbf{D})$ as the supremum of $\chi_\mu^{ap}(\mathbf{D})$, where μ ranges over all Borel probability measures on X.

If $\mathcal{F} \subseteq X^X$ is a family of functions from X to X, we define $\mathbf{D}_\mathcal{F} = (X, D_\mathcal{F})$ by

$$x \, D_\mathcal{F} \, y \Leftrightarrow x \neq y \wedge \exists f \in \mathcal{F} \, (f(x) = y) \, .$$

If $\mathcal{F} = \{F_i : i < n\}$, we also write $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ for $\mathbf{D}_\mathcal{F}$.

It is a standard graph theory result (for a proof, see [15]) that for the case of finitely many functions, we have $\chi\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq 2n + 1$ for any digraph of this form. Because the $(2n + 1)$-clique (i.e., a digraph of size $2n+1$ in which every x, y have either $x\, D\, y$ or $y\, D\, x$) can be written in this form, this bound is sharp.

In this paper, we will be interested in $\chi_B(\mathbf{D}_{\mathcal{F}})$ in the case when X is a standard Borel space and $\mathcal{F}$ is a finite family of Borel functions on X. Such digraphs arise naturally, for example, when studying the action of a finitely generated group on a standard Borel space. Kechris, Solecki, and Todorčević [15] showed that in the case of a single function F, we have $\chi_B(\mathbf{D}_F) \in \{1, 2, 3, \aleph_0\}$.

Let $X = [\omega]^{\aleph_0}$, the set of all infinite subsets of ω, and let $F : X \to X$ be defined by $F(A) = A \setminus \{\min A\}$. Kechris, Solecki, and Todorčević showed in [15] that $\chi_B(\mathbf{D}_F) = \aleph_0$, even though $\mathbf{D}_F$ is acyclic and therefore $\chi(\mathbf{D}_F) = 2$. This is an example of a major departure of the definable version of a combinatorial graph notion from its classical counterpart.

It follows from $\chi_B(\mathbf{D}_F) \in \{1, 2, 3, \aleph_0\}$ that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \in \{1, \ldots, 3^n, \aleph_0\}$. In [15], Kechris, Solecki, and Todorčević ask whether it is in fact true that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \in \{1, \ldots, 2n+1, \aleph_0\}$. In [25], Palamourdas made improvements and showed that

$$\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \in \left\{1, \ldots, \frac{1}{2}(n+1)(n+2), \aleph_0\right\}$$

He also showed that in the case of only 2 or 3 functions,

$$\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \in \{1, \ldots, 5, \aleph_0\} \text{ and } \chi_B\left(\mathbf{D}_{F_0,F_1,F_2}\right) \in \{1, \ldots, 8, \aleph_0\}.$$

In the case that the functions $\{F_i\}_{i<n}$ commute, Palamourdas [25] claimed that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \in \{1, \ldots, 2n+1, \aleph_0\}$. However, Michael Wheeler noted in private correspondence that his proof requires that the functions do not have any fixed points in X.

In Section 4.5.1 of this paper, we show that Palamourdas's argument lends itself to a natural induction result concerning the general bound for $\chi_B(\mathbf{D}_{F_0,\ldots,F_{n-1}})$. This insight allows us to obtain a slight improvement of the bound:

Corollary 2.2.4. *For $n \geq 3$, we either have $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) = \aleph_0$ or $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq \frac{1}{2}(n+1)(n+2) - 2$.*

From Section 2.3 onward, we revisit the case in which the functions $\{F_i\}_{i<n}$ commute, without assuming anything about the existence of fixed points for the functions. We show that such a digraph can be separated into two smaller digraphs, one for which no fixed points exist, and one for which every point has a path to a fixed point (we say the digraph has a **fixed ceiling**). We then provide some evidence that digraphs of the latter type may actually be easier to colour in a Borel way:

Theorem 2.3.15. *If $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ has a fixed ceiling and $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$, there exists an increasing filtration $X = \bigcup_{i\in\omega} X_i$ such that each $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}|X_i\right) \leq 2n$.*

This particular structuring of X implies a result for measurable colourings of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$:

Corollary 2.3.22. *If $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ has a fixed ceiling and $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$, then we have $\chi_M^{ap}\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq 2n+1$.*

In Section 2.4, we more closely study this setting when there are only 2 functions. In this case, we may drop the requirement to approximate X by X_i:

Theorem 2.4.3. *If $\mathbf{D}_{F_0,F_1}$ has a fixed ceiling and $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) < \aleph_0$, then $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \leq 4$.*

2.2 The General Case

If $\mathbf{D} = (X, D)$ is a digraph, we let $D(x) = \{y \in X : x\, D\, y\}$. The members of $D(x)$ are known as **successors** of x.

We first recall two results from [25].

Theorem 2.2.1. *[25] Fix $n \geq 0$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of Borel functions on X such that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$. Then $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq \frac{1}{2}(n+1)(n+2)$. In particular, there exists a Borel colouring $c : X \to \left\{(i,j) \in (n+1)^2 : i+j \leq n\right\}$ of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ such that if $c(x) = (i_0, j_0)$, then $\{(i_0, j) : j < j_0\} \subseteq c\left[D_{F_0,\ldots,F_{n-1}}(x)\right]$ and for each $i < i_0$, $c\left[D_{F_0,\ldots,F_{n-1}}(x)\right] \cap \{(i,j) : j \leq n-1\} \neq \emptyset$.*

In particular, Theorem 2.2.1 provides a new proof that $\chi_B(\mathbf{D}_F) \in \{1, 2, 3, \aleph_0\}$. The bound provided by this theorem is not optimal in general, however, as the following special improvements show.

Theorem 2.2.2. *[25] Suppose that X is standard Borel and $F_0, F_1, F_2 : X \to X$ are Borel. Then $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \in \{1,2,3,4,5,\aleph_0\}$ and $\chi_B\left(\mathbf{D}_{F_0,F_1,F_2}\right) \in \{1,\ldots,8,\aleph_0\}$.*

The colouring produced by Theorem 2.2.1 divides X into $\frac{1}{2}(n+1)(n+2)$ independent subsets in a very organized way. In particular, if we restrict the digraph to $A = c^{-1}\left[\{(i,j) : i > 0\}\right]$, then each $x \in A$ has at most $n-1$ successors in A, since one successor always has colour $(0, j)$ for some $j < n+1$. By a standard result of descriptive set theory (for a proof, see [17]), we can write a Borel relation $D \subseteq A \times A$ with the property that $\forall x\, (|D(x)| \leq n-1)$ as the disjoint union of $n-1$ Borel functions $\{G_i\}_{i<n-1}$, where G_i has Borel domain $A_i \subseteq A$. By extending G_i to be identity on $A \backslash A_i$, it follows that

$$\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright A = \mathbf{D}_{G_0,\ldots,G_{n-2}}.$$

This observation allows for a natural induction argument that is unnecessary for the proof of Theorem 2.2.1. However, it becomes useful when combined with the base case of Theorem 2.2.2.

Define the function $u : \omega \to \omega$ by letting $u(n)$ be the maximum value of $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right)$, for $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ a digraph of the form described in Theorem 2.2.1. Since a $(2n+1)$-clique can take this form, $u(n) \geq 2n+1$. In particular, $u(1) = 3$, $u(2) = 5$, $u(3) \in \{7, 8\}$ and Theorem 2.2.1 implies that $u(n) \leq \frac{1}{2}(n+1)(n+2)$.

Lemma 2.2.3. *For all $n < \omega$, we have $u(n+1) \leq u(n) + n + 1$.*

Proof. Suppose we have a digraph $\mathbf{D}_{F_0,\ldots,F_n}$ of the form described in Theorem 2.2.1. It suffices to prove that $\chi_B(\mathbf{D}_{F_0,\ldots,F_n}) \leq u(n) + n + 1$.

Let c be the colouring furnished by Theorem 2.2.1. If we define $A = c^{-1}\left[\{(i,j) : i > 0\}\right]$, then as noted above, there are Borel functions $\{G_i\}_{i<n}$ on A such that $\mathbf{D}_{F_0,\ldots,F_n} \upharpoonright A = \mathbf{D}_{G_0,\ldots,G_{n-1}}$. Since $\chi_B(\mathbf{D}_{G_0,\ldots,G_{n-1}}) \leq \chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$, we can let $d : A \to u(n)$ be a Borel colouring of $\mathbf{D}_{F_0,\ldots,F_n} \upharpoonright A$. Then $X = \bigsqcup_{i<u(n)} d^{-1}\left[\{i\}\right] \sqcup \bigsqcup_{j<n+1} c^{-1}\left[\{(0,j)\}\right]$ is a Borel partition of X into $u(n) + n + 1$ independent subsets. □

Corollary 2.2.4. *Fix $n \geq 3$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of Borel functions on X such that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$. Then $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq \frac{1}{2}(n+1)(n+2) - 2$.*

Proof. By Theorem 2.2.2, we have that $u(3) \leq 8$. Hence the result follows by applying Lemma 2.2.3 and induction. □

2.3 Commuting Functions

In this section, we shall examine digraphs generated by commuting functions, which have a number of properties that allow them to be analyzed more easily. For example, Fact 2.3.11 will show that if x satisfies $|D(x)| \leq m$, then so do each of its successors.

Definition 2.3.1. *If $n < \omega$, X is a set, and $\{f_i\}_{i<n}$ is a sequence in X^X, we let $\circ_{i<n} f_i$ be shorthand for the composition function $f_0 \circ \cdots \circ f_{n-1}$. In the case $n = 0$, $\circ_{i<n} f_i$ denotes* id_X. *If $\mathcal{F} = \{f_i\}_{i\in I}$ for some finite index set I instead and $\mathcal{F}$ commutes, we also define $\circ_{i\in I} f_i = \circ_{j<|I|} f_{i_j}$ for any bijection $j \mapsto i_j$ from $|I|$ to I.*

If $\mathcal{F} \subseteq X^X$ is a family of functions, we let $C_{\mathcal{F}} = \left\{\circ_{i<n} f_i \in X^X : n < \omega \wedge \forall i < n \,(f_i \in \mathcal{F})\right\}$. In particular, $\{\mathrm{id}_X\} \cup \mathcal{F} \subseteq C_{\mathcal{F}}$.

Note that if $\mathcal{F}$ is a commuting family of functions on X, then so is $C_{\mathcal{F}}$.

Definition 2.3.2. *If X is a set and $f \in X^X$, we let $\mathrm{Fix}_f = \{x \in X : f(x) = x\}$ be the set of all fixed points of f.*

Given digraphs $\mathbf{D}_0 = (X, D_0)$ and $\mathbf{D}_1 = (Y, D_1)$ on the standard Borel spaces X and Y, a Borel **homomorphism** from $\mathbf{D}_0$ to $\mathbf{D}_1$ is a Borel function $f : X \to Y$ such that for every $a, b \in X$,

$$a \, D_0 \, b \quad \Rightarrow \quad f(a) \, D_1 f(b).$$

In this case, we write $\mathbf{D}_0 \leq_B \mathbf{D}_1$. By composing f with a colouring of $\mathbf{D}_1$, it follows that $\chi_B(\mathbf{D}_0) \leq \chi_B(\mathbf{D}_1)$.

Note that if X is standard Borel and $\mathcal{F} \subseteq X^X$ is a commuting family of Borel functions with no fixed points, then any $H \in C_{\mathcal{F}}$ is a Borel automorphism of $\mathbf{D}_{\mathcal{F}}$.

Let $\mathcal{D}$ be a family of digraphs on standard Borel spaces and $n \geq 1$ an integer. Then it is clear that $\chi_B(\mathbf{D}) \leq n$ for every $\mathbf{D} \in \mathcal{D}$ if and only if $\mathbf{D} \leq_B \mathbf{K}_n$ for every $\mathbf{D} \in \mathcal{D}$, where $\mathbf{K}_n$ is the complete digraph on n points. Below, we construct a much more complicated finite digraph that has chromatic number $2n+1$ which admits Borel homomorphisms for the family of digraphs generated by n commuting Borel functions with no fixed points.

Definition 2.3.3. *For integers $M, n \geq 0$, let $\mathbf{D}_{M,n} = \left(X_{M,n}, D_{M,n}\right)$ be the finite digraph with*

$$X_{M,n} = \left\{x \in (M+1)^{[-M,M]^n} : \forall \bar{s}, \bar{t} \in [-M, M]^n \left(\left|\bar{s} - \bar{t}\right| = 1 \to x(\bar{s}) \neq x(\bar{t})\right)\right\}$$

$$\text{and } x \, D_{M,n} \, y \Leftrightarrow \exists j < n \forall \bar{s} \left(s_j < M \to y(\bar{s}) = x\left(\bar{s} + e_j\right)\right).$$

Note that if $x \, D_{M,n} \, y$, then $x(\bar{0}) \neq y(\bar{0})$.

One can regard the points of the digraph $\mathbf{D}_{M,n}$ as n-dimensional cubic grids of side length $2M+1$, each of whose cells contains one of $M+1$ different colours. A grid x connects to y if it can be shifted onto y, replacing the missing face in an arbitrary way. As we will see in Theorem 2.3.5, we can think of each cell as a point in a digraph of the form $\mathbf{D}_{F_0,\ldots,F_{n-1}}$, with the colour of the cell representing the colour assigned by some fixed finite colouring of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$.

Theorem 2.3.4. *For any $M, n \geq 0$, $\chi\left(\mathbf{D}_{M,n}\right) \leq 2n+1$.*

Proof. For $j < n$ and $k \in \{-1, 1\}$, let $S_j^{(k)} : X_{M,n} \to X_{M,n}$ be defined by

$$S_j^{(k)}(x)(\overline{s}) = \begin{cases} x\left(\overline{s} + ke_j\right) & \text{if } ks_j < M \\ x\left(\overline{s}\right) + 1 \pmod{M+1} & \text{otherwise.} \end{cases}$$

As noted above, $c(x) = x(\overline{0})$ is already an $(M+1)$-colouring for $\mathbf{D}_{M,n}$. We will define new colourings $e_0, \ldots, e_M$ by recursion. First, set $e_0 = c$. Then let

$$e_{i+1}(x) = \begin{cases} \min\,(2n+1) \setminus \left\{ e_i\left(S_j^{(k)}(x)\right) : \ j < n, k \in \{-1, 1\} \right\} & \text{if } x(\overline{0}) = i+1 \\ e_i(x) & \text{otherwise.} \end{cases}$$

We claim that for each $i < M+1$, if $x \restriction [-i, i]^n = y \restriction [-i, i]^n$, then $e_i(x) = e_i(y)$. This is clearly true for $i = 0$. Assume it holds for i and that $x \restriction [-(i+1), i+1]^n = y \restriction [-(i+1), i+1]^n$. If $x(\overline{0}) = y(\overline{0}) \neq i+1$, we are done. Otherwise, note that for all $j < n$ and $k \in \{-1, 1\}$, $S_j^{(k)}(x) \restriction [-i, i]^n = S_j^{(k)}(y) \restriction [-i, i]^n$, and so $e_i\left(S_j^{(k)}(x)\right) = e_i\left(S_j^{(k)}(y)\right)$. This implies that $e_{i+1}(x) = e_{i+1}(y)$.

To complete the proof, we verify that e_M is a $(2n+1)$-colouring of $\mathbf{D}_{M,n}$. Note that for each x, $e_M(x) = e_{x(\overline{0})}(x) < 2n+1$. Suppose that $x \, D_{M,n} \, y$ and $x(\overline{0}) < y(\overline{0})$ (the case $y(\overline{0}) < x(\overline{0})$ is similar). Choose $j < n$ such that $y(\overline{s}) = x\left(\overline{s} + e_j\right)$ for all $\overline{s}$ with $s_j < M$. Then for $\overline{s} \in \left[-\left(y(\overline{0}) - 1\right), y(\overline{0}) - 1\right]^n$, we have that $S_j^{(-1)}(y)(\overline{s}) = y(\overline{s} - e_j) = x(\overline{s})$. It follows from our claim that $e_{y(\overline{0})-1}\left(S_j^{(-1)}(y)\right) = e_{y(\overline{0})-1}(x)$. Therefore, by definition of $e_{y(\overline{0})}$, it follows that

$$e_M(y) = e_{y(\overline{0})}(y) \neq e_{y(\overline{0})-1}\left(S_j^{(-1)}(y)\right) = e_{y(\overline{0})-1}(x) = e_M(x).$$

□

Theorem 2.3.5. *[25] Fix $n \geq 0$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of commuting Borel functions on X with no fixed points. Then if $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$, we have $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq 2n+1$.*

Proof. Choose $M \geq 0$ and a Borel colouring $c : X \to M+1$ of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$. Define $f : X \to X_{M,n}$ by

$$f(x)(\overline{s}) = c\left(\circ_{i<n} F_i^{M+s_i}(x)\right).$$

Then f is clearly Borel. If $\overline{s}, \overline{t} \in [-M, M]^n$ with $\left|\overline{s} - \overline{t}\right| = 1$, then without loss of generality, $\overline{t} = \overline{s} + e_j$ for some $j < n$. So $f(x)(\overline{s}) = c\left(\circ_{i<n} F_i^{M+s_i}(x)\right) \neq c\left(F_j\left(\circ_{i<n} F_i^{M+s_i}(x)\right)\right) = f(x)(\overline{t})$, since $\mathcal{F}$ commutes and F_j has no fixed points, which verifies that $f(x) \in X_{M,n}$. If $x D_{F_0,\ldots,F_{n-1}} y$, choose

$j < n$ such that $y = F_j(x)$. Then for any $\overline{s}$ with $s_j < M$, we have $f(y)(\overline{s}) = c\left(\circ_{i<n} F_i^{M+s_i}(F_j(x))\right) = f(x)(\overline{s} + e_j)$, whence $f(x)\, D_{M,n}\, f(y)$. Hence we have $\mathbf{D}_{F_0,\ldots,F_{n-1}} \leq_B \mathbf{D}_{M,n}$, and so by Theorem 2.3.4, we have $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq 2n+1$. $\square$

Definition 2.3.6. *Let $\mathbf{D} = (X, D)$ be a digraph and let r be any function on X. We say that a Y-colouring c of $\mathbf{D}$ is **restricted by** r if for all $x \in X$, $c(x) \notin r(x)$.*

We will have the need to produce colourings like the one Theorem 2.3.5 furnishes, but are also restricted by Borel functions of a particular kind. Toward this, we will consider a slightly more complicated version of $\mathbf{D}_{M,n}$ below, in which every cell of every grid also has a restriction associated with it.

Definition 2.3.7. *For integers $M, n, m \geq 0$, let $\mathbf{D}_{M,n,m} = \left(X_{M,n,m}, D_{M,n,m}\right)$ be the countable digraph with*

$$X_{M,n,m} = \Big\{x \in ((M+1) \times \{S \subseteq \omega : \ |S| \leq m\})^{[-M,M]^n} :$$
$$\forall \overline{s}, \overline{t} \in [-M, M]^n \left(\left|\overline{s} - \overline{t}\right| = 1 \rightarrow x_0(\overline{s}) \neq x_0(\overline{t})\right)\Big\},$$

where $x = (x_0, x_1)$ is the notation for the component functions. Let $D_{M,n,m}$ be defined as in Definition 2.3.3.

Theorem 2.3.8. *For any $M, n, m \geq 0$, there is a $(2n+m+1)$-colouring c of $\mathbf{D}_{M,n,m}$ restricted by $r(x) = x_1\left(\overline{0}\right)$.*

Proof. As before, $c(x) = x_0(\overline{0})$ is an $(M+1)$-colouring of $\mathbf{D}_{M,n,m}$. This time, define $e_{-1} = c$ and, for $i \leq M$,

$$e_i(x) = \begin{cases} \min\,(2n+m+1) \setminus \left(\left\{e_{i-1}\left(S_j^{(k)}(x)\right) : \ j < n, k \in \{-1, 1\}\right\} \cup x_1(\overline{0})\right) & \text{if } x(\overline{0}) = i \\ e_{i-1}(x) & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 2.3.4, e_M is a $(2n+m+1)$-colouring of $\mathbf{D}_{M,n,m}$. $\square$

The next lemma strengthens Theorem 2.3.5 by showing that we can create a similar style of colouring that is restricted by any function with finite values and that has arbitrarily large invariance under the functions $\{F_i\}_{i<n}$. Below, we regard $\mathcal{P}(\omega)$ as the standard Borel space 2^ω via the identification $A \mapsto \chi_A$.

Lemma 2.3.9. *Fix $n, m \geq 0$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of commuting Borel functions on X with no fixed points. If $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq M+1$ for some integer M, then for any $N \geq M$ and Borel function $t : X \to \{S \subseteq \omega : |S| \leq m\}$, there exists a Borel function $b : X \to 2n+m+1$ such that $B(x) = b\left(\circ_{i<n} F_i^{N-M}(x)\right)$ is a $(2n+m+1)$-colouring of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ restricted by $T(x) = t\left(\circ_{i<n} F_i^N(x)\right)$.*

Proof. Fix a Borel colouring $c : X \to M+1$ of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$. Define $f : X \to X_{M,n,m}$ by

$$f(x)(\overline{s}) = \left(c\left(\circ_{i<n} F_i^{M+s_i}(x)\right), t\left(\circ_{i<n} F_i^{M+s_i}(x)\right)\right).$$

Then, as in Theorem 2.3.5, f is a Borel homomorphism witnessing that $\mathbf{D}_{F_0,\ldots,F_{n-1}} \preceq_B \mathbf{D}_{M,n,m}$. Let d be the colouring of $\mathbf{D}_{M,n,m}$ furnished by Theorem 2.3.8 and let $b = d \circ f$, so that b is Borel. Then $B = b \circ \left(\circ_{i<n} F_i^{N-M}\right)$ is a colouring of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ since $\circ_{i<n} F_i^{N-M} \in C_{\mathcal{F}}$ is an automorphism of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$. For each $x \in X$, we have $B(x) \notin f\left(\circ_{i<n} F_i^{N-M}(x)\right)_1\left(\overline{0}\right) = t\left(\circ_{i<n} F_i^N(x)\right)$. □

We now introduce some definitions useful for analyzing the structure of the digraphs of interest.

Definition 2.3.10. *Let $<_{\mathcal{F}}$ be the transitive closure of $D_{\mathcal{F}}$. If we have a finite set $\{F_i : i < n\}$ of functions and $I \subseteq n$, let $<_I$ be shorthand for $<_{\{F_i : i \in I\}}$. Finally, let $<$ be shorthand for $<_n$. For each relation $<_\xi$, we let $\leq_\xi$ denote $x \leq_\xi y \Leftrightarrow x <_\xi y \vee x = y$.*

Note, in particular, that $<_{\mathcal{F}}$ is not necessarily irreflexive.

Suppose X is a set and $\mathcal{F} \subseteq X^X$. We make the obvious but important observation that for $x, y \in X$, $x \leq_{\mathcal{F}} y \Leftrightarrow \exists H \in C_{\mathcal{F}}\,(H(x) = y)$. In particular, if $P \subseteq X$ is Borel, then so is $\{x \in X : \forall y \geq x\,(y \in P)\}$. This also yields the following two basic properties:

Fact 2.3.11. *Suppose X is a set, $\mathcal{F} \subseteq X^X$ is a family of commuting functions, $H \in \mathcal{F}$, and $x, y \in X$ with $x \leq_{\mathcal{F}} y$. Then*

1. *$H(x) \leq_{\mathcal{F}} H(y)$ and*
2. *if $x \in \mathrm{Fix}_H$, then $y \in \mathrm{Fix}_H$.*

Proof. Let $H' \in C_{\mathcal{F}}$ be such that $y = H'(x)$. Then $H(y) = H(H'(x)) = H'(H(x))$ proves (1). If $x \in \mathrm{Fix}_H$, then $H(y) = H'(H(x)) = H'(x) = y$ proves (2). □

Definition 2.3.12. *Let* $\mathbf{D} = (X, D)$ *be a digraph. We say that a collection* $\{A_i\}_{i\in I}$ *of disjoint subsets of* X *are* ***separated*** *if for all* $i \neq j \in I$ *and* $x \in A_i$, $y \in A_j$, *we have* $\neg x\, D\, y$. *If* $X = \bigsqcup_{i\in I} A_i$, *we say that* $\{A_i\}_{i\in I}$ *is a* ***separation****.*

In particular, if $\bigsqcup_{i\in I} A_i$ is a separation for $\mathbf{D}_{F_0,\dots,F_{n-1}}$, then for each $i \in I$, $\mathbf{D}_{F_0,\dots,F_{n-1}} \restriction A_i = \mathbf{D}_{F_0\restriction A_i,\dots,F_{n-1}\restriction A_i}$.

Definition 2.3.13. *Let* X *be a set and* $\mathcal{F} \subseteq X^X$ *a family of functions. We say that* $\mathbf{D}_\mathcal{F}$ *has a* ***fixed ceiling*** *if for every* $x \in X$, *there exists a* $y \in X$ *with* $x \leq_\mathcal{F} y$ *such that* $y \in \mathrm{Fix}_f$ *for some* $f \in \mathcal{F}$.

Lemma 2.3.14. *Fix* $n \geq 0$. *Suppose that* X *is standard Borel and* $\mathcal{F} = \{F_i\}_{i<n}$ *is a finite family of commuting Borel functions on* X. *Then there exists a Borel separation* $X = A \sqcup B$ *such that for each* $i < n$, $F_i \restriction A$ *has no fixed points and* $\mathbf{D}_{F_0,\dots,F_{n-1}} \restriction B$ *has a fixed ceiling.*

Proof. Define

$$B = \bigcup_{H\in C_\mathcal{F}} \bigcup_{j<n} H^{-1}\left(\mathrm{Fix}_{F_j}\right) \text{ and } A = X\setminus B.$$

Since $C_\mathcal{F}$ is a countable family of Borel functions, $A \sqcup B$ is a Borel partition of X. Note that $x \in B \Leftrightarrow \exists H \in C_\mathcal{F}\, \exists j < n \left(F_j(H(x)) = H(x)\right) \Leftrightarrow \exists y \geq x\, \exists j < n \left(F_j(y) = y\right)$. Therefore, we may conclude $\mathbf{D}_{F_0,\dots,F_{n-1}} \restriction B$ has a fixed ceiling if we show that $\bigcup_{j<n} F_j[B] \subseteq B$. Since $\bigcup_{j<n} \mathrm{Fix}_{F_j} = \bigcup_{j<n} (id_X)^{-1}\left(\mathrm{Fix}_{F_j}\right) \subseteq B$, no $F_i \restriction A$ has a fixed point.

It remains only to show that $A \sqcup B$ is a separation; consider any $x \in A$ and $y \in B$. If $x\, D_{F_0,\dots,F_{n-1}}\, y$, then there is a fixed point z such that $x < y \leq z$. This implies that $x \in B$, a contradiction. If $y\, D_{F_0,\dots,F_{n-1}}\, x$, then choose $j < n$ such that $F_j(y) = x$. Let $z \geq y$ be a fixed point; by Fact 2.3.11, $F_j(z) \geq F_j(y) = x$ is a fixed point too, again implying $x \in B$. □

Lemma 2.3.14 shows us that it is enough to consider digraphs with fixed ceilings, since Theorem 2.3.5 will allow to find a Borel colouring for $\mathbf{D}_{F_0,\dots,F_{n-1}} \restriction A$.

Theorem 2.3.15. *Fix* $n \geq 0$. *Suppose that* X *is standard Borel and* $\mathcal{F} = \{F_i\}_{i<n}$ *is a finite family of commuting Borel functions on* X. *If* $\chi_B\left(\mathbf{D}_{F_0,\dots,F_{n-1}}\right) < \aleph_0$ *and* $\mathbf{D}_{F_0,\dots,F_{n-1}}$ *has a fixed ceiling, then there exists an increasing filtration* $X = \bigcup_{i\in\omega} X_i$ *such that for each* $i \in \omega$, $\chi_B\left(\mathbf{D}_{F_0,\dots,F_{n-1}}|X_i\right) \leq 2n$.

We will prove this theorem after detailing a useful way to decompose the space X of such a digraph. In what follows, we work with a fixed digraph $\mathbf{D}_{F_0,\dots,F_{n-1}}$ with the above properties.

We will define Borel subsets $Q_m \subseteq X$ for each $m < \omega$ by recursion. Let

$$\mathcal{I} = \left\{(I_0, I_1) \in \mathcal{P}(n)^2 : n = I_0 \sqcup I_1 \wedge I_1 \neq \emptyset\right\}.$$

For each $(I_0, I_1) \in \mathcal{I}$, define

$$Q_0^{(I_0,I_1)} = \{x \in X : \forall y \geq x \forall i\, (i \in I_0 \leftrightarrow F_i(y) \neq y)\},$$

and define $Q_0 = \bigcup_{(I_0,I_1)\in\mathcal{I}} Q_0^{(I_0,I_1)}$.

Lemma 2.3.16. *For every $x \in X$, there exists $y \geq x$ with $y \in Q_0$.*

Proof. Fix any $x \in X$. Let $I_1 = \{i < n : \exists y \geq x\, (F_i(y) = y)\}$ and $I_0 = n \backslash I_1$. Since $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ has a fixed ceiling, $I_1 \neq \emptyset$, so that $(I_0, I_1) \in \mathcal{I}$. For each $i \in I_1$ and $j < n$, let $k_j^{(i)} \geq 0$ be such that $\circ_{j<n} F_j^{k_j^{(i)}}(x)$ is a fixed point of F_i. Then if we let $k_j = \max\left\{k_j^{(i)} : i \in I_1\right\}$ for each $j < n$, it follows that $y = \circ_{j<n} F_j^{k_j}(x) \in Q_0^{(I_0,I_1)}$. □

Let $\mathcal{I}' = \left\{(I_0, I_1) \in \mathcal{P}(n)^2 : n \supsetneq I_0 \sqcup I_1\right\}$ and $\mathcal{J} = (\{0\} \times \mathcal{I}) \cup (\omega \backslash \{0\} \times \mathcal{I}')$. For $(I_0, I_1) \in \mathcal{I}'$, let $I_2 = n \backslash (I_0 \sqcup I_1)$. Suppose that for every $i \leq m$ and $I_0, I_1 \subseteq n$ with $(i, I_0, I_1) \in \mathcal{J}$, we have defined $Q_i^{(I_0,I_1)}$. For any $(I_0, I_1) \in \mathcal{I}'$, we define

$$Q_{m+1}^{(I_0,I_1)} = \left\{x \in X \backslash \bigcup_{i\leq m} Q_i : \forall i \in I_0 \forall y \geq_{I_0} x \left(F_i(y) \neq y \wedge F_i(y) \notin \bigcup_{i\leq m} Q_i\right)\right.$$
$$\left. \wedge \forall i \in I_1 \forall y \geq_{I_0} x\, (F_i(y) = y) \wedge \forall i \in I_2 \exists j \leq m \forall y \geq_{I_0} x \left(F_i(y) \in Q_j\right)\right\}.$$

and we define $Q_{m+1} = \bigcup_{(I_0,I_1)\in\mathcal{I}'} Q_{m+1}^{(I_0,I_1)}$. It is immediate from these definitions that if $x \in Q_m$ and $x \leq y$, then $y \in Q_{m'}$ for some $m' \leq m$.

Lemma 2.3.17. $X = \bigcup_{m<\omega} Q_m$.

Proof. For ease of notation, let $Q = \bigcup_{m<\omega} Q_m \subseteq X$. Suppose for sake of contradiction that $x \in X \backslash Q$. Define $x_0 = x$ and $I'_0 = n$. Suppose that $x_m \geq x$ and $I'_m \subseteq n$ have been defined with $x_m \notin Q$ and $F_j(x_m) \in Q$ for all $j \in n \backslash I'_m$. If there exists $y >_{I'_m} x$ with $y \in Q$, define x_{m+1} and I'_{m+1} as follows: for this y, choose x_{m+1} such that $x_m \leq_{I'_m} x_{m+1} <_{I'_m} y$ and $j \in I'_m$ such that $x_{m+1} \notin Q$ but $F_j(x_{m+1}) \in Q$. Let $I'_{m+1} = I'_m \backslash \{j\}$. Note that since $F_j(x_{m+1}) \geq F_j(x_m)$, we have that $F_j(x_{m+1}) \in Q$ for all $j \notin I'_{m+1}$. Continue this process until we reach some $N \leq n$ such that $x_N \notin Q$ but $\forall y >_{I'_N} x$, $y \notin Q$.

By Lemma 2.3.16, there are k_i for $i < n$ such that $y = \circ_{i<n} F_i^{k_i}(x) \in Q_0$. In particular, $N > 0$, and so $I_N \subsetneq n$. For $j \in n \backslash I'_N$, let $m_j = \min \left\{ m < \omega : \exists y' \geq_{I'_N} x \left(F_j(y') \in Q_m \right) \right\}$. For $j \in n \backslash I'_N$ and $i \in I'_N$, let $k_i^{(j)}$ be such that if $y = \circ_{i \in I'_N} F_i^{k_i^{(j)}}(x_N)$, then $F_j(y) \in Q_{m_j}$. Let $I_1 = \left\{ i \in I'_N : \exists y \geq_{I'_N} x \left(F_i(y) = y \right) \right\}$ and $I_0 = I'_N \backslash I_1$. Next, for $j \in I_1$ and $i \in I'_N$, let $\tilde{k}_i^{(j)}$ be such that $\circ_{i \in I'_N} F_i^{\tilde{k}_i^{(j)}}(x_N) \in \mathrm{Fix}_{F_i}$. Then let $k_i = \max \left(\left\{ k_i^{(j)} : j \in n \backslash I'_N \right\} \cup \left\{ \tilde{k}_i^{(j)} : j \in I_1 \right\} \right)$ for each $i \in I'_N$. If we set $y = \circ_{i \in I'_N} F_i^{k_i}(x_N)$, then one can check that $y \in Q^{(I_0, I_1)}_{\max_{j \in n \backslash I'_N}(m_j + 1)}$, even though $y \notin Q$.

□

Lemma 2.3.18. *Suppose that $x \in Q_m^{(I_0, I_1)}$ and $j \notin I_0 \cup I_1$. Then $F_j(x) \in Q_{m'}^{(I'_0, I'_1)}$ for some $m' < m$ with $I_0 \cup I_1 \subseteq I'_0 \cup I'_1$.*

Proof. We need only check that the inclusion $I_0 \cup I_1 \subseteq I'_0 \cup I'_1$ holds. Suppose that $\ell \in I'_2 \backslash I_2$. Then $F_\ell \left(F_j(x) \right) \in Q_{m''}$ for some $m'' < m'$, but $F_j(F_\ell(x)) \in Q_{m'}$, a contradiction. □

Given $m > 0$, $(I_0, I_1) \in \mathcal{I}'$, and a function $a : I_2 \to \left\{ \left(m', I'_0, I'_1 \right) \in \mathcal{J} : m' < m \right\}$, we define

$$Q_m^{(I_0, I_1)}(a) = \left\{ x \in Q_m^{(I_0, I_1)} : \forall j \in I_2 \forall y \geq_{I_0} x \left(a(j) = \left(m', I'_0, I'_1 \right) \to F_j(y) \in Q_{m'}^{I'_0, I'_1} \right) \right\}.$$

Let $a = (a_0, a_1, a_2)$ be the component functions for such an a. We now show that for every $m > 0$, we can separate Q_m into rather homogeneous subsets.

Lemma 2.3.19. *Both $Q_0 = \bigsqcup_{(I_0, I_1) \in \mathcal{I}} Q_0^{(I_0, I_1)}$ and (for any $m > 0$) $Q_m = \bigsqcup_{I_0, I_1, a} Q_m^{(I_0, I_1)}(a)$ are separations.*

Proof. It is clear by the definitions that each is a partition. We will handle the $m > 0$ case. Choose $x \in Q_m^{(I_0, I_1)}(a)$, $x' \in Q_m^{(I'_0, I'_1)}(a')$, and $j < n$ such that $F_j(x) = x'$ and $x \neq x'$. Since $x \neq x'$, we must have $j \in I_0$, which in particular implies that $x \leq_{I_0} x'$. By the definition of $Q_m^{(I_0, I_1)}(a)$, it follows immediately that $a = a'$. Also, $i \in I_1 \Leftrightarrow F_i(x') = x' \Leftrightarrow i \in I'_1$, so that $I_1 = I'_1$, and similarly, $I_0 = I'_0$. The proof of the $m = 0$ case is analogous. □

Fix an integer $M \geq 0$ and a colouring $c : X \to M + 1$ of $\mathbf{D}_{F_0, \ldots, F_{n-1}}$.

Lemma 2.3.20. *For every integer $N \geq 0$, there exists a Borel function $b : Q_0 \to 2n - 1$ such that the function $B(x) = b \left(\circ_{i<n} F_i^N(x) \right)$ is a $(2n - 1)$-colouring of $\mathbf{D}_{F_0, \ldots, F_{n-1}} \restriction Q_0$.*

Proof. Since $Q_0 = \bigsqcup_{(I_0,I_1)\in\mathcal{I}} Q_0^{I_0,I_1}$ is a finite Borel separation, it suffices to find such a $(2n-1)$-colouring for each $Q_0^{I_0,I_1}$. To do this, note that

$$\mathbf{D}_{F_0,\ldots,F_{n-1}} \restriction Q_0^{I_0,I_1} = \mathbf{D}_{\{F_i :\, i\in I_0\}} \restriction Q_0^{I_0,I_1}$$

and that F_i for $i \in I_0$ has no fixed points in $Q_0^{I_0,I_1}$. Therefore, we can apply Lemma 2.3.9 (with $m = 0$) to obtain a Borel $b : Q_0^{I_0,I_1} \to 2\,|I_0| + 1$ such that $B(x) = b\left(\circ_{i\in I_0} F_i^N(x)\right) = b\left(\circ_{i<n} F_i^N(x)\right)$ is a colouring on $Q_0^{I_0,I_1}$. Since $2\,|I_0| + 1 \leq 2n - 1$, we are done. ◻

For $x \in X$, let $I_0(x) = I$ if and only if $x \in Q_m^{(I,I_1)}$ for some $(m, I, I_1) \in \mathcal{J}$.

Lemma 2.3.21. *Let $m \geq 0$ be an integer. Suppose that $N \geq M$ and there is a Borel function $b : \bigcup_{j\leq m} Q_j \to 2n$ such that the function $B(x) = b\left(\circ_{i\in I_0(x)} F_i^N(x)\right)$ is a colouring. Then there is a Borel $b' : \bigcup_{j\leq m+1} Q_j \to 2n$ such that the function $B'(x) = b'\left(\circ_{i\in I_0(x)} F_i^{N-M}(x)\right)$ is a colouring of $\bigcup_{j\leq m+1} Q_j$ and B' extends B.*

Proof. First of all, define b' on $\bigcup_{j\leq m} Q_j$ by $b'(x) = b\left(\circ_{i\in I_0(x)} F_i^M(x)\right)$, so that $B' \restriction \left(\bigcup_{j\leq m} Q_j\right) = B$. Since $Q_{m+1} = \bigsqcup_{I_0,I_1,a} Q_{m+1}^{(I_0,I_1)}(a)$ is a finite Borel separation, it suffices to extend b' to generate a $2n$-colouring on each $Q_{m+1}^{(I_0,I_1)}(a) \cup \bigcup_{j\leq m} Q_j$. To do this, note as in Lemma 2.3.20 that $\mathbf{D}_{F_0,\ldots,F_{n-1}} \restriction Q_{m+1}^{(I_0,I_1)}(a)$ is generated by $\{F_i : i \in I_0\}$, which have no fixed points, so we can apply Lemma 2.3.9. This time, we obtain a $b' : Q_{m+1}^{(I_0,I_1)}(a) \to 2\,|I_0| + |I_2| + 1$ by using the restriction

$$T(x) = \{B'(F_i(x)) : i \in I_2\} = \left\{b\left(\circ_{\ell\in a_0(i)} F_\ell^M(F_i(x))\right) : i \in I_2\right\}$$

$$= \left\{b\left(\circ_{\ell\in a_0(i)\cup a_1(i)} F_\ell^M(F_i(x))\right) : i \in I_2\right\} = \left\{b\left(\circ_{\ell\in a_0(i)\cup a_1(i)\setminus I_0} F_\ell^M \circ F_i\left(\circ_{\ell\in I_0} F_\ell^M(x)\right)\right) : i \in I_2\right\}$$

$$= t\left(\circ_{i\in I_0} F_i^M(x)\right),$$

where $t = \left\{b\left(\circ_{\ell\in a_0(i)\setminus I_0} F_\ell^M \circ F_i(x)\right) : i \in I_2\right\}$ is Borel. Letting B' be as defined in the statement, then B' is a colouring separately on $Q_{m+1}^{(I_0,I_1)}(a)$ and $\bigcup_{j\leq m} Q_j$. If $x\, D_{F_0,\ldots,F_{n-1}}\, y$ and $x \in Q_{m+1}^{(I_0,I_1)}(a)$, $y \in \bigcup_{j\leq m} Q_j$, then $y = F_i(x)$ for some $i \in I_2$, and hence the restriction T ensures $B'(x) \neq B'(y)$. Since $2\,|I_0| + |I_2| + 1 \leq 2n$, we are done. ◻

We now have enough information to complete the proof of Theorem 2.3.15.

Proof of Theorem 2.3.15. Let $X_m = \bigcup_{i\leq m} Q_i$. Beginning with Lemma 2.3.20 with $N = mM$, we can argue by Lemma 2.3.21 and induction that for every $j \leq m$, there is a Borel function $b_j : \bigcup_{i\leq j} Q_i$

such that the Borel function $B_j(x) = b_j\left(\circ_{i\in I_0(x)} F_i^{N-jM}(x)\right)$ is a $2n$-colouring of $\bigcup_{i\leq j} Q_i$. Hence B_m shows that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright X_m\right) \leq 2n$. $\square$

In the case of measurable colourings, this countable filtration of X is enough to provide a bound for the approximate measure chromatic number.

Corollary 2.3.22. *Fix $n \geq 0$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of commuting Borel functions on X. Suppose in addition that $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) < \aleph_0$. Then we have $\chi_M^{ap}\left(\mathbf{D}_{F_0,\ldots,F_{n-1}}\right) \leq 2n+1$.*

Proof. By Lemma 2.3.14 and Theorems 2.3.5 and 2.3.15, we can write $X = A \sqcup \bigcup_{i<\omega} X_i$, where $\{X_i\}_{i<\omega}$ is increasing, A is separated from $\bigcup_{i<\omega} X_i$, and $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright A\right) \leq 2n+1$ and $\chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright X_i\right) \leq 2n$ for all $i < \omega$. So for any probability measure μ and $\varepsilon > 0$, there exists an n such that $\mu\left(X \setminus \left(A \sqcup X_n\right)\right) < \varepsilon$, and $\chi_\mu\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright (A \sqcup X_n)\right) \leq \chi_B\left(\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright (A \sqcup X_n)\right) \leq 2n+1$. $\square$

2.4 Two Commuting Functions

We first recall a result from [25] that concerns digraphs with no infinite paths.

Definition 2.4.1. *Suppose X is a set, and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of functions on X. We say that $A \subseteq X$ is **bounded** if $D_{F_0,\ldots,F_{n-1}}$ is cowellfounded on A.*

Lemma 2.4.2. *[25] Fix $n \geq 0$. Suppose that X is standard Borel and $\mathcal{F} = \{F_i\}_{i<n}$ is a finite family of Borel functions on X. Suppose $A \subseteq X$ is bounded such that for all $x \in X\setminus A$ and $y \in A$, $\neg x\, D_{F_0,\ldots,F_{n-1}}\, y$. If for some integer M, $c : X\setminus A \to M$ is a Borel colouring of $\mathbf{D}_{F_0,\ldots,F_{n-1}} \upharpoonright (X\setminus A)$, then there is a Borel colouring $c' : X \to \max\{n+1, M\}$ of $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ extending c.*

Given a digraph $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ as in Lemma 2.4.2, let

$$B(X) = \left\{x \in X : \exists M\, \forall i \in n^{M+1} \exists k < M\left(F_{i_k}\left(\circ_{j<k} F_{i_j}(x)\right) = \circ_{j<k} F_{i_j}(x)\right)\right\}.$$

So $B(X)$ is the set of $x \in X$ for which there is an M for which there are no paths of length M beginning at x. Therefore, $B(X)$ is bounded and clearly Borel from its definition.

Theorem 2.4.3. *Suppose that X is standard Borel and $F_0, F_1 : X \to X$ are commuting Borel functions. If $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) < \aleph_0$ and $\mathbf{D}_{F_0,F_1}$ has a fixed ceiling, then $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \leq 4$.*

To prove this theorem, we will introduce a decomposition of X different from the Q_i sets considered in Section 2.3. In what follows, we work with a fixed digraph $\mathbf{D}_{F_0,F_1}$ satisfying the conditions of Theorem 2.4.3. Fix an integer $M \geq 0$ and a colouring $c : X \to M + 1$ of $\mathbf{D}_{F_0,F_1}$.

Define $r : X \to \omega$ by

$$r(x) = \min \left\{ k_0 + k_1 : \; F_0^{k_0} \circ F_1^{k_1}(x) \in \mathrm{Fix}_{F_0} \cup \mathrm{Fix}_{F_1} \right\}.$$

Hence $r(x)$ is the minimum length of a path from x to a fixed point. Since $\mathbf{D}_{F_0,F_1}$ has a fixed ceiling, r is finite on all of X. Define $R_i = \{x \in X : \; r(x) = i\}$. In particular, $R_0 = \mathrm{Fix}_{F_0} \cup \mathrm{Fix}_{F_1}$. Clearly, r and hence each R_i is Borel.

Proposition 2.4.4. *Suppose $x \in X$. Then*

1. *for each $i < 2$, $r\left(F_i(x)\right) \in \{r(x), r(x) - 1\}$;*
2. *if $r(x) > 0$, then $r\left(F_i(x)\right) = r(x) - 1$ for at least one $i < 2$; and*
3. *if $r\left(F_0^k(x)\right) = r(x)$ for all $k < \omega$, then $r\left(F_0^{k_0} \circ F_1^{k_1}(x)\right) = \max\{r(x) - k_1, 0\}$.*

Proof. If $H \in C_{\{F_0,F_1\}}$ and $H(x) \in R_0$, then by Fact 2.3.11, $H\left(F_i(x)\right) \in R_0$ for each $i < 2$, showing that that $r\left(F_i(x)\right) \leq r(x)$. Additionally, $F_0^{k_0} \circ F_1^{k_1}(F_0(x)) = F_0^{k_0+1} \circ F_1^{k_1}(x)$ shows that $r(x) \leq r\left(F_0(x)\right) + 1$, proving (1).

Let $k_0 + k_1$ be minimal such that $F_0^{k_0} \circ F_1^{k_1}(x) \in R_0$. If $r(x) > 0$ then some $k_i > 0$; if $i = 1$, then $F_0^{k_0} \circ F_1^{k_1-1}\left(F_1(x)\right) = F_0^{k_0} \circ F_1^{k_1}(x)$ shows that $r\left(F_1(x)\right) = r(x) - 1$, proving (2).

(3) is immediate for $k_1 = 0$. If $k_1 + 1 \leq r(x)$, then since $r\left(F_0^{k_0} \circ F_1^{k_1}(x)\right) = r(x) - k_1$ for any $k_0 < \omega$, it follows from (2) that $r\left(F_0^{k_0} \circ F_1^{k_1+1}(x)\right) = r(x) - (k_1 + 1)$. If $k_1 + 1 > r(x)$, then $r\left(F_0^{k_0} \circ F_1^{k_1+1}(x)\right) \leq r\left(F_0^{k_0} \circ F_1^{k_1}(x)\right) = 0$. $\square$

Lemma 2.4.5. *Suppose F_0 has no fixed points and that for every $x \in X$, $r(F_0(x)) = r(x)$. Then $\chi_B\left(\mathbf{D}_{F_0,F_1}\right) \leq 3$.*

Proof. Let $f : X \to X_{M,1}$ be the Borel function defined by

$$f(x)(s) = c\left(F_0^{M+r(x)+s} \circ F_1^{r(x)}(x)\right).$$

Since F_0 has no fixed points, $f(x) \in X_{M,1}$. If $y = F_0(x)$, then $r(x) = r(y)$, and so for any $s \in [-M, M)$, we have $f(y)(s) = c\left(F_0^{M+r(y)+s} \circ F_1^{r(y)}\left(F_0(x)\right)\right) = f(x)(s+1)$, whence $f(x)\, D_{M,1}\, f(y)$.

If $y = F_1(x)$ and $x \neq y$, then $r(y) = r(x) - 1$, and so $f(y)(s) = c\left(F_0^{M+r(y)+s} \circ F_1^{r(y)}(F_1(x))\right) = c\left(F_0^{M+r(x)-1+s} \circ F_1^{r(x)-1+1}(x)\right) = f(x)(s-1)$ for $s \in (-M, M]$. Therefore, $\mathbf{D}_{F_0,F_1} \leq_B \mathbf{D}_{M,1}$, and so the result follows by Theorem 2.3.4. □

Lemma 2.4.6. *Suppose that F_0 is the same as in Lemma 2.4.5. For every Borel function $t : R_0 \to \{S \subseteq \omega : |S| \leq m\}$, there exists a Borel colouring $b : X \to m+3$ of X restricted by $T(x) = t\left(F_0^M(x)\right)$ for $x \in R_0$.*

Proof. This time, define the Borel function $f : X \to X_{M,1,m}$ by

$$f(x)(s) = \left(c\left(F_0^{M+r(x)+s} \circ F_1^{r(x)}(x)\right), t\left(F_0^{M+r(x)+s} \circ F_1^{r(x)}(x)\right)\right).$$

As in Lemma 2.4.5, f witnesses that $\mathbf{D}_{F_0,F_1} \leq_B \mathbf{D}_{M,1,m}$. If we let $b = d \circ f$, where d is the colouring from Theorem 2.3.8, then b is a Borel colouring and for each $x \in R_0$, we have $b(x) \notin t\left(F_0^{M+r(x)} \circ F_1^{r(x)}(x)\right) = T(x)$. □

Proof of Theorem 2.4.3: For each $i < 2$, define

$$B_i = \{x \in X : \forall y \geq x\,(F_i(y) \neq y \wedge r(F_i(y)) = r(y))\}.$$

We claim that $B(X) \sqcup B_0 \sqcup B_1$ are separated; in fact, that each of the three sets is closed under successors. If $x \in B_i$, then $\left\{F_i^j(x)\right\}_{j<\omega}$ is an infinite path beginning at x, showing that $B(X) \cap B_i = \emptyset$. Since $\mathbf{D}_{F_0,F_1}$ has a fixed ceiling, there exists $y \geq x$ with $y \in \mathrm{Fix}_{F_{1-i}}$, showing that $B_0 \cap B_1 = \emptyset$. If $x\, D_{F_0,F_1}\, y$ and there are no paths of length M beginning at x, then there are no such paths beginning at y, showing that $B(X)$ is closed under successors. It is also immediate from the definition that each B_i is too, proving the claim.

Each B_i is clearly Borel. By applying Lemma 2.4.2 to $B(X)$ and Lemma 2.4.5 to each B_i, we obtain a Borel 3-colouring c of $\mathbf{D}_{F_0,F_1} \upharpoonright (B(X) \sqcup B_0 \sqcup B_1)$. Consider any $x \in R_0$ and assume without loss of generality that $x \in \mathrm{Fix}_{F_1}$. Then either there exists $j < \omega$ with $F_0^j(x) = F_0^{j+1}(x)$, in which case there are no paths of length $j+1$ beginning at x, or $F_0^i(x) \neq F_0^{i+1}(x)$ for all $i < \omega$, in which case $x \in B_0$. This shows that $R_0 \subseteq B(X) \sqcup B_0 \sqcup B_1$.

Let $B' = X \setminus (B(X) \sqcup B_0 \sqcup B_1)$ and for each $i < 2$, define

$$B'_i = \left\{x \in B' : \forall y \geq x\,(r(F_i(y)) = r(y)) \wedge F_i\left(F_{1-i}^{r(x)}(x)\right) = F_{1-i}^{r(x)}(x)\right\}$$

If $x \in B'_0$, then $r(x) > 0$ and so by Proposition 2.4.4, it follows that $r(F_1(x)) = r(x) - 1$, so that $B'_0 \cap B'_1 = \emptyset$. If $x \in B'_i$, $r(y) > 0$, and $x\, D_{F_0,F_1}\, y$, then $\left\{F_i^j(y)\right\}_{j<\omega}$ is an infinite path beginning at y, so that $y \notin B(X)$. Using Proposition 2.4.4, it is clear that $F_i\left(F_{1-i}^{r\left(F_j(x)\right)}\left(F_j(x)\right)\right) = F_{1-i}^{r\left(F_j(x)\right)}\left(F_j(x)\right)$ for each $j < 2$, and so $y \notin B_i$. Hence $y \in B'$ and so $y \in B'_i$. In particular, this shows that B'_0 and B'_1 are separated and that if $x \in B'_i$, then $F_j(x) \in B'_i$ unless $j = 1 - i$ and $r(x) = 1$, in which case it belongs to $B(X) \sqcup B_0 \sqcup B_1$. If we define $F'_1 : B_0 \to B_0$ by

$$F'_1(x) = \begin{cases} F_1(x) & \text{if } r(x) > 1 \\ x & \text{if } r(x) = 1, \end{cases}$$

then it is easy to check that $F_0 \restriction B_0$ commutes with F'_1 and $\mathbf{D}_{F_0,F_1} \restriction B_0 = \mathbf{D}_{F_0 \restriction B_0, F'_1}$. Hence, letting $s(x) = \{c\,(F_1(x))\}$, we can apply Lemma 2.4.6 with the restriction $t(x) = s\left(F_0^M(x)\right) = \left\{c\left(F_1\left(F_0^M(x)\right)\right)\right\} = \left\{c\left(F_0^M\,(F_1(x))\right)\right\} = \{c\,(F_1(x))\}$. The same argument applies to B'_1, and we obtain a 4-colouring c' on $B(X) \sqcup B_0 \sqcup B_1 \sqcup B'_0 \sqcup B'_1$ that extends c.

Finally, $B'' = B' \setminus \left(B'_0 \sqcup B'_1\right)$ is bounded. If not, let $\{x_i\}_{i<\omega}$ be an infinite path in B''. By Proposition 2.4.4 $r\,(x_i)$ is monotonically decreasing in i, and so we may assume that every $x_i \in R_k$ for some fixed $k > 0$. Without loss of generality, assume $r\,(F_1(x_0)) = k - 1$. Then $r\left(F_0^\ell\,(F_1\,(x_0))\right) \leq k - 1$ for every $\ell < \omega$, and so $x_i = F_0^i\,(x_0)$. By Proposition 2.4.4, $r\left(F_0^{k_0} \circ F_1^{k_1}(x_0)\right) = \max\{k - k_1, 0\}$. So since $x_0 \notin B_0$, it must be that $F(y) = y$ for some $y \geq x$. Letting $y = F_0^{k_0} \circ F_1^{k_1}(x)$, it follows that $x_{k_0} = F_0^{k_0}(x)$ is in B'_1, a contradiction. Hence we may apply Lemma 2.4.2 to extend c' to a Borel 4-colouring of all of $\mathbf{D}_{F_0,F_1}$, completing the proof. $\square$

There are straightforward ways to generalize Proposition 2.4.4 and Lemmas 2.4.5 and 2.4.6 for digraphs of the form $\mathbf{D}_{F_0,\ldots,F_{n-1}}$ for $n > 2$. However, it seems difficult to adapt the proof of Theorem 2.4.3, since the additional functions mean there are much fewer bounded subsets of the digraph.

DEFINABLE COMBINATORICS OF SOME BOREL EQUIVALENCE RELATIONS

(with William Chan)

3.1 Introduction

Set theorists have studied the Jónsson property and other combinatorial partition properties of well-ordered sets under the axiom of choice, large cardinal axioms, and the axiom of determinacy. Holshouser and Jackson began the study of the Jónsson property using definability techniques for sets which generally cannot be well-ordered in a definable manner.

Let X be a set and E an equivalence relation on X. For each $n \in \omega$, let $[X]^n_E$ be the collection of tuples $(x_0, ..., x_{n-1}) \in {}^nX$ so that for all $i \neq j$, $\neg(x_i \ E \ x_j)$. Let $[X]^{<\omega}_E = \bigcup_{n\in\omega}[X]^n_E$. For each $n \in \omega$, X has the n**-Jónsson property** if and only if for every function $f : [X]^n_= \to X$, there is some $Y \subseteq X$ with Y in bijection with X and $f[[Y]^n_=] \neq X$. X has the **Jónsson property** if and only if for every function $f : [X]^{<\omega}_= \to X$, there is some $Y \subseteq X$ with Y in bijection with X, and $f[[Y]^{<\omega}_=] \neq X$.

Holshouser and Jackson showed that ${}^\omega 2$ has the Jónsson property under the axiom of determinacy, AD. Let $f : [{}^\omega 2]^{<\omega}_= \to {}^\omega 2$. For each $n \in \omega$, let $f_n : [X]^n_= \to X$ be $f \upharpoonright [X]^n_=$. Their proof has two notable tasks:

(1) Holshouser and Jackson first (assuming all sets have the Baire property) choose comeager sets $C_n \subseteq {}^n({}^\omega 2)$ so that $f_n \upharpoonright C_n$ is continuous. Then a single perfect set $P \subseteq {}^\omega 2$ is found so that for each n, $f_n \upharpoonright [P]^n_=$ is continuous. To obtain this perfect set P, they use a classical theorem of Mycielski which states: If C_n is a sequence of comeager subsets of ${}^n({}^\omega 2)$, then there is some perfect set $P \subseteq {}^\omega 2$ so that $[P]^n_= \subseteq C_n$ for all n.

(2) Since each f_n is continuous on $[P]^n_=$, they use a fusion argument to simultaneously prune P to a smaller perfect set $Q \subseteq P$ so that there exists some real that is missed by each f_n on $[Q]^n_=$.

Holshouser and Jackson ask whether other sets which may not be well-ordered in some choiceless setting like AD could also have the Jónsson property. They observed that under $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$, every set $X \in L_\Theta(\mathbb{R})$ has a surjective function $f : \mathbb{R} \to X$. Define an equivalence relation on $\mathbb{R}$ by $x \ E \ y$ if and only if $f(x) = f(y)$. Then X is in bijection with $\mathbb{R}/E$. The study of the Jónsson property for sets in $L_\Theta(\mathbb{R})$ is equivalent to studying the Jónsson property for quotients of $\mathbb{R}$ by equivalence relations on $\mathbb{R}$. Note that $\mathbb{R}$ is in bijection with $\mathbb{R}/=$.

Through dichotomy results of Harrington, Hjorth, Kechris, Louveau, and others, the equivalence relations $=$, E_0, E_1, E_2, and E_3 occupy special positions in the structure of $\mathbf{\Delta}^1_1$ equivalence relations under $\mathbf{\Delta}^1_1$ reducibilities. $=$ is the identity equivalence relation on ${}^\omega 2$. E_0 is defined on ${}^\omega 2$ by $x \; E_0 \; y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_1 is defined on ${}^\omega({}^\omega 2)$ by $x \; E_1 \; y$ if and only if $(\exists n)(\forall k > n)(x(k) = y(k))$. E_2 is defined on ${}^\omega 2$ by $x \; E_2 \; y$ if and only if $\sum\{\frac{1}{n+1} : n \in x \triangle y\} < \infty$, where $\triangle$ denotes the symmetric difference. E_3 is defined on ${}^\omega({}^\omega 2)$ by $x \; E_3 \; y$ if and only if $(\forall n)(x(n) \; E_0 \; y(n))$.

Holshouser and Jackson asked whether the methods applicable for showing $\mathbb{R}/=$ has the Jónsson property could be used to show the quotients of these other $\mathbf{\Delta}^1_1$ equivalence relations could be Jónsson. An important aspect of their proof for $\mathbb{R}$ was the theorem of Mycielski. They defined the Mycielski property for arbitrary equivalence relations as follows: Let E be an equivalence relation on a Polish space X. For each $n \in \omega$, E has the **n-Mycielski property** if and only if for every comeager $C \subseteq {}^n X$, there exists some $\mathbf{\Delta}^1_1$ $A \subseteq X$ so that $E \leq_{\mathbf{\Delta}^1_1} E \upharpoonright A$ and $[A]^n_E \subseteq C$.

They asked whether any of the $\mathbf{\Delta}^1_1$ equivalence relations mentioned above have the n-Mycielski property for various $n \in \omega$ and whether the Mycielski property could be used to prove the Jónsson property for the quotient of any of these equivalence relations. Holshouser and Jackson began this study by showing that E_0 has the 2-Mycielski property and this can be used to show ${}^\omega 2/E_0$ has the 2-Jónsson property. This paper will show that the Mycielski property fails in most cases:

Theorem 3.8.1. The equivalence relation E_0 does not have the 3-Mycielski property.

Theorem 3.13.4. The equivalence relation E_1 does not have the 2-Mycielski property.

Theorem 3.15.1. The equivalence relation E_2 does not have the 2-Mycielski property.

Theorem 3.18.2. The equivalence relation E_3 does not have the 2-Mycielski property.

These results require understanding the structure of $\mathbf{\Delta}^1_1$ subsets of ${}^\omega 2$ or ${}^\omega({}^\omega 2)$ so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$ (or $E_1 \leq_{\mathbf{\Delta}^1_1} E_1 \upharpoonright A$, etc.) that come from the proofs of the dichotomy results. Kanovei, Sabok, and Zapletal in [13], [14], [29], and [30] have studied the forcing of such $\mathbf{\Delta}^1_1$ sets for each of these equivalence relations.

Given that the Mycielski property fails in general, a reflection on Holshouser and Jackson's proof of the Jónsson property for ${}^{\omega}2$ shows that it is only used to find some perfect set P so that $f_n \upharpoonright [P]^n_=$ is nicely behaved (i.e., continuous). This paper will give a forcing style proof of Holshouser and Jackson results that ${}^{\omega}2$ is Jónsson and ${}^{\omega}2/E_0$ is 2-Jónsson assuming all functions satisfy a certain definability condition expressed in Lemma 3.3.3. This definability condition follows from the Mycielski property for the equivalence relation and the assumption that all sets have the Baire property. All $\mathbf{\Delta}^1_1$ functions have this definability condition and under the axiom of choice and large cardinal assumptions, projective and even more complex sets also satisfy this condition.

Following part (2) of Holshouser and Jackson's template for ${}^{\omega}2$, suppose one could find some $\mathbf{\Delta}^1_1$ set $B \subseteq {}^{\omega}2$ with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B$ and $f \upharpoonright [B]^3_{E_0}$ is continuous for some function f. Could one then somehow prune B to some $C \subseteq B$ so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright C$ and $f[[C]^3_{E_0}] \neq {}^{\omega}2$, or even better, miss an E_0-class? This paper will have some discussion on how these continuity and surjectivity properties for E_0 and E_2 can fail.

This line of reasoning shows both part (1) and part (2) of the proof of Holshouser and Jackson establishing ${}^{\omega}2$ is Jónsson fail for E_0 and several other $\mathbf{\Delta}^1_1$ equivalence relations. Moreover, for E_0, it will in fact be shown that ${}^{\omega}2/E_0$ is not Jónsson under determinacy:

Theorem 3.10.4. ($\mathsf{ZF} + \mathsf{AD}$) ${}^{\omega}2/E_0$ does not have the 3-Jónsson property and hence is not Jónsson.

Here are some historical remarks about the Jónsson property: Under the axiom of choice, mathematicians usually study the Jónsson property on cardinals. Cardinals possessing the Jónsson property are called **Jónsson cardinals**. For $n \in \omega$, let $\mathscr{P}^n(X)$ denote the collection of all n-element subsets of X. Since there is a well-ordering, the Jónsson property is usually defined using $\mathscr{P}^n(X)$ rather than $[X]^n_=$. This paper will use the term "classical Jónsson property" when discussing the Jónsson property using $\mathscr{P}^n(X)$.

Under the axiom of choice, Jónsson cardinals also have model-theoretic characterizations. The existence of Jónsson cardinals imply $V \neq L$. Moreover, it has large cardinal consistency strength: for instance, it implies $0^\sharp$ exists. Erdős and Hajnal ([5] and [4]) showed that if $2^\kappa = \kappa^+$, then κ^+ is not a Jónsson cardinal. Hence under CH, $2^{\aleph_0}$ is not a Jónsson cardinal. Every real valued measurable cardinal is Jónsson (see [4] Corollary 11.1). Solovay showed the consistency of a measurable cardinal implies the consistency of $2^{\aleph_0}$ being real valued measurable. Hence it is consistent relative to a measurable cardinal that $2^{\aleph_0}$ is a Jónsson cardinal. The sets ${}^{\omega}2$, ${}^{\omega}2/E_0$, ${}^{\omega}({}^{\omega}2)/E_1$, ${}^{\omega}2/E_2$, ${}^{\omega}({}^{\omega}2)/E_3$ are all in bijection with each other using the axiom of choice. Hence if CH holds, these quotients do not have the Jónsson property and if $2^{\aleph_0}$ is real valued measurable, then all these quotients do have the Jónsson property.

Studies of the Jónsson property and other combinatorial partition properties of cardinals under AD date back to the 1960s and 1970s. Assuming AD, for each $n \in \omega$, $\aleph_n$ is a Jónsson cardinal ([21]). More recently Woodin had shown that under $\mathsf{ZF} + \mathsf{AD}^+$, every cardinal $\kappa < \Theta$ has the Jónsson property. Also [12] showed that in $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$, every cardinal $\kappa < \Theta$ is Jónsson. [12] asked whether ${}^{\omega}2$, which cannot be well-ordered, has the Jónsson property. In analogy, they asked if every set in $L_{\Theta}(\mathbb{R})$ has the Jónsson property. Holshouser and Jackson's answer to this question for ${}^{\omega}2$ begins the work that is carried out in this paper.

Throughout, results attributed to Holshouser and Jackson can be found in [11] and [10]. We will frequently identify $\mathbb{R}$ with ${}^{\omega}2$ and use $\mathbf{\Delta}^1_1$ as an abbreviation for "Borel".

This paper is organized as follows:

Section 3.2 contains definitions of the main concepts and some basic facts about determinacy.

Section 3.3 will give a proof of the result of Holshouser and Jackson which shows ${}^{\omega}2$ has the Jónsson property if all sets have the Baire property. The proof uses forcing arguments and fusion. This section will have some discussions about how absoluteness available under AD^+ can help prove said result without using the Mycielski property. However, throughout the paper, a flexible fusion argument is necessary for handling the combinatorics. It is unclear what the relation is between properness, fusion, and the Jónsson property for the five equivalence relations considered.

Upon considering the Jónsson property for ${}^{\omega}2$, a natural question is whether there is a function $f : \mathscr{P}^{\omega}({}^{\omega}2) \to {}^{\omega}2$ so that for all $A \subseteq {}^{\omega}2$ with $A \approx {}^{\omega}2$, $f[\mathscr{P}^{\omega}(A)] = {}^{\omega}2$. Such a function is called an **ω-Jónsson function** for ${}^{\omega}2$. Under the axiom of choice, [5] showed that every set has an ω-Jónsson function. Section 3.4 gives an example under $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_{\omega}$ (choice for countable sets of nonempty subsets of ${}^{\omega}2$) of a $\mathbf{\Delta}^1_1$ ω-Jónsson function for ${}^{\omega}2$.

From the effective proof of the E_0-dichotomy, every Σ^1_1 set $A \subseteq {}^{\omega}2$ so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$ contains the body of a perfect tree with certain symmetry restrictions, known as an E_0-tree. Section 3.5 will modify the proof of the E_0-dichotomy using Gandy-Harrington methods to prove a structure theorem for Σ^1_1 sets with the same E_0-saturation on which the restriction of E_0 is not smooth: For example, if A and B are two Σ^1_1 sets with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B$, and $[A]_{E_0} = [B]_{E_0}$, then there are E_0 trees p and q with $[p] \subseteq A$, $[q] \subseteq B$, and p and q are the same except possibly at the stem. This consideration reveals the failure of the weak 3-Mycielski property (see Definition

3.2.24) for E_0.

Section 3.6 will introduce the forcing $\widehat{\mathbb{P}}^2_{E_0}$. We will use this forcing to prove the result of Holshouser and Jackson stating that ${}^\omega 2/E_0$ has the 2-Jónsson property.

Let X and Y be sets. Let $n \in \omega$. Define $X \to (X)^n_Y$ to mean that for any function $f : \mathscr{P}^n(X) \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[\mathscr{P}^n(Z)]| = 1$. Define $X \mapsto (X)^n_Y$ to mean that for any function $f : [X]^n_= \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[[Z]^n_=]| = 1$. Section 3.7 will show that ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$ holds for all $n \in \omega$.

Section 3.8 will show that E_0 does not have the 3-Mycielski property or weak 3-Mycielski property.

Section 3.9 will produce a continuous function $Q : [{}^\omega 2]^3_{E_0} \to {}^\omega 2$ so that for every Σ^1_1 $A \subseteq {}^\omega 2$ with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$, $Q[[A]^3_{E_0}] = {}^\omega 2$. A modification of this function yields a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that on any such set A, $K \restriction [A]^3_{E_0}$ is not continuous.

Section 3.10 will use the function produced in the previous section to show that ${}^\omega 2/E_0$ does not have the 3-Jónsson property or the classical 3-Jónsson property under $\mathsf{ZF} + \mathsf{AD}$. (In particular, ${}^\omega 2/E_0$ is not Jónsson under AD.)

Section 3.11 will use the classical 3-Jónsson map for ${}^\omega 2/E_0$ to show the failure of ${}^\omega 2/E_0 \to ({}^\omega 2/E_0)^3_2$.

The fusion argument related to the proper forcing $\widehat{\mathbb{P}}^2_{E_0}$ established many of the combinatorial properties of E_0 in dimension two. Given the failure of these properties in dimension three, a natural question would be whether the three dimensional analog $\widehat{\mathbb{P}}^3_{E_0}$ is proper and possesses a reasonable fusion. Section 3.12 will show that $\widehat{\mathbb{P}}^3_{E_0}$ is proper by having some type of fusion argument. However, there is far less control of this fusion.

Section 3.13 will show that E_1 does not have the 2-Mycielski property.

Section 3.14 will modify the proof of the E_2-dichotomy result using Gandy-Harrington methods to give structural result about E_2-big Σ^1_1 sets with the same E_2-saturation: For example, if A and B are two Σ^1_1 sets with $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction A$, $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction B$, and $[A]_{E_2} = [B]_{E_2}$, then there are two E_2-trees (perfect trees with certain properties) p and q so that $[p] \subseteq A$, $[q] \subseteq B$, $[[p]]_{E_2} = [[q]]_{E_2}$, and p and q resemble each other in specific ways.

Section 3.15 will use results of the previous section to show E_2 does not have the 2-Mycielski property and the weak 2-Mycielski property.

Section 3.16 will produce a continuous function $Q : [{}^\omega 2]^3_{E_2} \to {}^\omega 2$ so that on any Σ^1_1 set A with

$E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright A$, $Q[[A]^3_{E_2}] = {}^\omega 2$. There is also a $\mathbf{\Delta}_1^1$ function $P' : {}^3({}^\omega 2) \to {}^\omega 2$ so that for any such set A, $P' \upharpoonright A$ is not continuous.

Section 3.17 contains no new results but just gives the rather lengthy characterization of Σ_1^1 sets $A \subseteq {}^\omega({}^\omega 2)$ so that $E_3 \leq_{\mathbf{\Delta}_1^1} E_3 \upharpoonright A$ that comes from the E_3-dichotomy result. This structure result is applied in Section 3.18 to show that E_3 does not have the 2-Mycielski property.

Section 3.19 will study the completeness of non-principal ultrafilters on quotients of Polish spaces by equivalence relations.

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3.2 Basic Information

Definition 3.2.1. *Let $\sigma \in {}^{<\omega}2$. Suppose $|\sigma| = k$. Then $\tilde{\sigma} \in {}^\omega 2$ is defined by $\tilde{\sigma}(n) = \sigma(j)$ where $0 \leq j < k$ and $j \equiv n \mod k$.*

For example, $\tilde{0}$, $\tilde{1}$, $\widetilde{01}$, etc. will appear frequently.

Definition 3.2.2. *Let $\sigma \in {}^{<\omega}2$. Let $N_\sigma = \{x \in {}^\omega 2 : x \supseteq \sigma\}$.*

$\{N_\sigma : \sigma \in {}^{<\omega}2\}$ is a basis for the topology on ${}^\omega 2$.

Let $\sigma, \tau \in {}^{<\omega}2$. Let $N_{\sigma,\tau} = \{(x, y) \in {}^2({}^\omega 2) : x \in N_\sigma \wedge y \in N_\tau\}$.

$\{N_{\sigma,\tau} : \sigma, \tau \in {}^{<\omega}2\}$ is a basis for the topology on ${}^2({}^\omega 2)$.

$N_{\sigma,\tau,\rho}$ is defined similarly for ${}^3({}^\omega 2)$.

Definition 3.2.3. *Let $n \in \omega$ and $\sigma : n \to {}^{<\omega}2$. Let $N_\sigma = \{x \in {}^\omega({}^\omega 2) : (\forall k < n)(\sigma(k) \subseteq x(k))\}$.*

$\{N_\sigma : \sigma \in {}^{<\omega}({}^{<\omega}2)\}$ is a basis for the topology on ${}^\omega({}^\omega 2)$.

Let $\sigma, \tau : n \to {}^{<\omega}2$. Let $N_{\sigma,\tau} = \{(x, y) \in {}^2({}^\omega({}^\omega 2)) : x \in N_\sigma \wedge y \in N_\tau\}$.

$\{N_{\sigma,\tau} : \sigma, \tau \in {}^{<\omega}({}^{<\omega}2) \wedge |\sigma| = |\tau|\}$ is a basis for the topology on ${}^2({}^\omega({}^\omega 2))$.

Definition 3.2.4. *Let A and B be two sets. $A \approx B$ denotes that there is a bijection between A and B.*

Often this paper will consider settings where the full axiom of choice may fail. In such contexts, not all sets have a cardinal, i.e. is in bijection with an ordinal. Similarity of size is more appropriately given by the existence of bijections. Recall the following method of producing bijections between sets which is provable in ZF:

Fact 3.2.5. *(Cantor-Schröder-Bernstein)* (ZF) *Let X and Y be two sets. Suppose there are injections $\Phi : X \to Y$ and $\Psi : Y \to X$. Then there is a bijection $\Lambda : X \to Y$.*

Definition 3.2.6. *Let X and Y be sets. ${}^X Y$ is the set of functions from X to Y.*

$\mathscr{P}(X)$ is the power set of X.

Let $n \in \omega$. Define

$$\mathscr{P}^n(X) = \{F \in \mathscr{P}(X) : F \approx n\}$$

$$\mathscr{P}^{<\omega}(X) = \bigcup_{n\in\omega} \mathscr{P}^n(X)$$

Let E be an equivalence relation on a set X. Let $n \in \omega$. Define

$$[X]^n_E = \{(x_0, ..., x_{n-1}) \in {}^n X : (\forall i, j < n)(i \neq j \Rightarrow \neg(x_i \; E \; x_j))\}$$

$$[X]^{<\omega}_E = \bigcup_{n\in\omega} [X]^n_E$$

Definition 3.2.7. *Let X be a set and $n \in \omega$. A set X has the n-Jónsson property if and only for all functions $f : [X]^n_= \to X$, there is some $Y \subseteq X$ so that $Y \approx X$ and $f[[Y]^n_=] \neq X$. X has the Jónsson property if and only if for all $f : [X]^{<\omega}_= \to X$, there is some $Y \subseteq X$ so that $Y \approx X$ and $f[[Y]^{<\omega}_=] \neq X$.*

A set X has the classical n-Jónsson property (or classical Jónsson property) if and only the above holds with $[X]^=_n$ (or $[X]^{<\omega}_=$) replaced with $\mathscr{P}^n(X)$ (or $\mathscr{P}^{<\omega}(X)$, respectively).

If X is a wellordered set, one can identify a finite set $F \subseteq X$ with the increasing enumeration of its elements. Such a presentation is helpful for defining useful functions on $\mathscr{P}^n(X)$. In the absence of choice, it is easier to define functions when one considers order tuples from $[X]^n_=$. For this reason, the paper mostly concerns the Jónsson property as defined above rather than the classical Jónsson property, although the classical version will be discussed in Section 3.10.

Definition 3.2.8. *Let X be a set. $[X]^\omega_=$ and $\mathscr{P}^\omega(X)$ are defined as above (with ω in place of $n \in \omega$).*

Let $N \in \omega \cup \{\omega\}$. A N-Jónsson function for X is a function $\Phi : [X]^N_= \to X$ so that for any $Y \subseteq X$ with $Y \approx X$, $\Phi[[Y]^N_=] = X$.

A classical N-Jónsson function for X is defined in the same way as the above with $\mathscr{P}^N(X)$ instead of $[X]^N_=$.

With the axiom of choice, [5] showed that every set has an ω-Jónsson map. The existence of ω-Jónsson maps for certain cardinals is where Kunen's original proof of the Kunen inconsistency used the axiom of choice. Note that for $N \in \omega \cup \{\omega\}$, a counterexample to the N-Jónsson property for some set is equivalent to the existence of an n-Jónsson function for that set.

Definition 3.2.9. *Let X and Y be Polish spaces. Let E and F be equivalence relations on X and Y, respectively. A $\mathbf{\Delta}^1_1$ reduction between X and Y is a $\mathbf{\Delta}^1_1$ function $\Phi : X \to Y$ such that for all $a, b \in X$, $a \mathrel{E} b$ if and only if $\Phi(a) \mathrel{F} \Phi(b)$.*

This situation is denoted by $E \leq_{\mathbf{\Delta}^1_1} F$. Define $E \equiv_{\mathbf{\Delta}^1_1} F$ if and only if $E \leq_{\mathbf{\Delta}^1_1} F$ and $F \leq_{\mathbf{\Delta}^1_1} E$.

Definition 3.2.10. *Let E be an equivalence relation on a set X. If $x \in X$, then $[x]_E = \{y \in X : y \mathrel{E} x\}$ is the E-class of x. Let $A \subseteq X$. $[A]_E = \{y \in X : (\exists x \in A)(x \mathrel{E} y)\}$ is the E-saturation of A.*

Definition 3.2.11. *Let X be a Polish space and E be an equivalence relation on X. Let $n \in \omega$. X has the n-Mycielski property if and only if for every $C \subseteq {}^nX$ which is comeager in ${}^nX$, there is a $\mathbf{\Delta}^1_1$ set $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and $[A]^n_E \subseteq C$.*

E has the Mycielski property if and only if for all sequences $(C_n : n \in \omega)$ such that for all $n \in \omega$, $C_n \subseteq {}^nX$ is comeager in ${}^nX$, there is a some set $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and for all $n \in \omega$, $[A]^n_E \subseteq C_n$.

The Mycielski property of equivalence relations comes from the following eponymous result:

Fact 3.2.12. *(Mycielski) Let $(C_n : n \in \omega)$ be a sequence such that for each $n \in \omega$, $C_n \subseteq {}^n({}^\omega 2)$ is a comeager subset of ${}^n({}^\omega 2)$. Then there is a perfect set $P \subseteq {}^\omega 2$ so that for all $n \in \omega$, $[P]^n_= \subseteq C_n$.*

Definition 3.2.13. *Let E be an equivalence relation on a Polish space X. Let $n \in \omega$. E has the n-continuity property if and only if for every function $f : {}^nX \to X$, there is some $\mathbf{\Delta}^1_1$ $A \subseteq X$ so that $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ and $f \restriction [A]^n_E$ is continuous.*

Fact 3.2.14. *Let E be an equivalence relation on a Polish X which has the n-Mycielski property. Then for every function $f : {}^nX \to X$ with the property of Baire (i.e. $f^{-1}[U]$ has the Baire property for every open set U), there is some $\mathbf{\Delta}^1_1$ $A \subseteq X$ with $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ so that $f \restriction [A]^n_E$ is continuous. Hence if every set has the Baire property, then E has the n-continuity property.*

Proof. Let $f : {}^nX \to X$. Since f is Baire measurable, there is some $C \subseteq {}^nX$ so that $f \restriction C$ is continuous. By the n-Mycielski property, there is some $A \subseteq X$ with $E \equiv_{\mathbf{\Delta}^1_1} E \restriction A$ so that $[A]^n_E \subseteq C$. $f \restriction [A]^n_E$ is continuous. □

In place of the axiom of choice, the paper will often use the axiom of determinacy. The following is a quick description of determinacy:

Definition 3.2.15. *Let X be a set. Let $A \subseteq {}^{\omega}X$. The game G_A is defined as follows: Player 1 plays $a_i \in X$, and player 2 plays $b_i \in X$ for each $i \in \omega$. At turn $2i$, player 1 plays a_i, and at turn $2i+1$, player 2 plays b_i. Let $f \in {}^{\omega}X$ be defined by $f(2i) = a_i$ and $f(2i+1) = b_i$. Player 1 wins this play of G_A if and only if $f \in A$. Player 2 wins otherwise.*

A winning strategy for player 1 is a function $\tau : {}^{<\omega}X \to X$ so that for any $(b_i : i \in \omega)$ if $(a_i : i \in \omega)$ is defined recursive by $a_0 = \tau(\emptyset)$ and $a_{n+1} = \tau(a_0 ... a_n b_n)$, then player 1 wins the resulting play of G_A. One may define a winning strategy for player 2 similarly.

The Axiom of Determinacy for X, denoted AD_X, is the statement that for all $A \subseteq {}^{\omega}X$, G_A has a winning strategy for some player.

AD *refers to* AD_2 *or equivalently* AD_ω. $\mathsf{AD}_\mathbb{R}$ *will also be used. Note that* $\mathsf{AD}_\mathbb{R}$ *often will refer to* $\mathsf{AD}_{{}^{\omega}2}$ *or* $\mathsf{AD}_{{}^{\omega}\omega}$.

AD implies classical regularity properties for sets of reals: Every set of reals has the Baire property and is Lebesgue measurable. Every uncountable set of reals has a perfect subset. Every function on the reals is continuous on a comeager set.

Uniformization, however, is more subtle:

Definition 3.2.16. *Let $R \subseteq {}^{\omega}2 \times {}^{\omega}2$. Let $R^x = \{y : (x, y) \in R\}$. Suppose for all $x \in {}^{\omega}2$, $R^x \neq \emptyset$. R is uniformizable if and only if there is some function $f : {}^{\omega}2 \to {}^{\omega}2$ so that for all $x \in {}^{\omega}2$, $(x, f(x)) \in R$. Such a function f is called a uniformization of R.*

Fact 3.2.17. ($\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$) *Every relation is uniformizable.*

Proof. Suppose $R \subseteq {}^{\omega}2 \times {}^{\omega}2$ with the property that for all x, $R^x \neq \emptyset$. Consider the two-step game where player 1 plays $a \in {}^{\omega}2$ and player 2 responds with $b \in {}^{\omega}2$. Player 2 wins if and only if $(a, b) \in R$. Clearly player 1 cannot have a winning strategy. Any winning strategy for player 2 yields a uniformization of R. □

Woodin has shown that if there is a measurable cardinal with infinitely many Woodin cardinals below it, then $L(\mathbb{R}) \models \mathsf{AD}$. Solovay showed in [27] Lemma 2.2 and Corollary 2.4 that the relation $R(x, y)$ if and only y is not ordinal definable from x is not uniformizable in $L(\mathbb{R})$. Hence $\mathsf{AD}_\mathbb{R}$ is stronger than than AD. AD is not capable of proving full uniformization.

Definition 3.2.18. *Let E be an equivalence relation on ${}^\omega 2$ and $n \in \omega$. Let $f : ({}^\omega 2/E)^n \to {}^\omega 2/E$. A lift of f is a function $F : {}^n({}^\omega 2) \to {}^\omega 2$ with the property that for all $(x_0, ..., x_n) \in {}^n({}^\omega 2)$, $[F(x_0, ..., x_{n-1})]_E = f([x_0]_E, ..., [x_{n-1}]_E)$.*

Fact 3.2.19. *Let E be an equivalence relation on ${}^\omega 2$ and $n \in \omega$. Let $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$. Define $R_f(x_0, ..., x_{n-1}, y) \Leftrightarrow y \in f([x_0]_E, ..., [x_{n-1}]_E)$. If F is a uniformization of R_f (with respect to the last variable), then F is a lift of f.*

Under $\mathsf{AD}_{\mathbb{R}}$, every such function has a lift.

Many natural models of AD such as $L(\mathbb{R})$ are not models of $\mathsf{AD}_{\mathbb{R}}$. However, functions on quotients of equivalence relations with all classes countable are still uniformizable: AD^+ is a strengthening of AD which holds in all known models of AD (in particular $L(\mathbb{R})$). See [28] Definition 9.6 for the definition of AD^+. It is open whether AD and AD^+ are equivalent. Also $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ implies AD^+, and it is open whether this holds without DC.

Fact 3.2.20. *(Countable Section Uniformization) (Woodin)* (AD^+) *Let $R \subseteq {}^\omega 2 \times {}^\omega 2$ have the property that for all $x \in {}^\omega 2$, R^x is countable. Then R is uniformizable.*

Proof. See [20] Theorem 3.2 for a proof. □

Fact 3.2.21. (AD^+) *Let E be an equivalence relation on ${}^\omega 2$ with all classes countable. Let $n \in \omega$ and $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$. Then f has a lift.*

For the results of this paper, we replace all results that require lifts by lifts on some comeager set. The benefit is that such lifts follows from comeager uniformization which is provable in just $\mathsf{ZF} + \mathsf{AD}$.

Fact 3.2.22. *(Comeager Unformization)* ($\mathsf{ZF} + \mathsf{AD}$) *Let $R \subseteq {}^\omega 2 \times {}^\omega 2$ be a relation such that $(\forall x)(\exists y)R(x, y)$, then there is a comeager $C \subseteq {}^\omega 2$ and some function $f : C \to {}^\omega 2$ so that $(\forall x \in C)R(x, f(x))$.*

By shrinking to an appropriate comeager set, one can assume that the uniformizing function is also continuous.

Often, in order to use the techniques of forcing over countable elementary structures, we will need to augment the axiom of determinacy by dependent choice (DC). Kechris [16] proved that AD and

$\mathsf{AD} + \mathsf{DC}$ have the same consistency strength by showing if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \mathsf{DC}$. However, Solovay [27] showed that $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ has strictly stronger consistency strength than $\mathsf{AD}_{\mathbb{R}}$.

If one is ultimately interested in functions $F : {}^n({}^\omega 2) \to {}^\omega 2$ which are lifts of some function $f : {}^n({}^\omega 2/E) \to {}^\omega 2/E$ only in order to infer information about f, then the demand in the Mycielski property that one considers tuples coming from a single set $A \subseteq {}^\omega 2$ such that $E \equiv_{\mathbf{\Delta}^1_1} E \upharpoonright A$ seems restrictive. If one ultimately will collapse back to the quotient, two sets A and B with the same E-saturation should work equally well. These parameters motivate the following concepts:

Definition 3.2.23. *Let $n \in \omega$ and E be an equivalence relation on some Polish space X. Let $(A_i : i < n)$ be a sequence of subsets of X. Define*

$$\prod_{i<n}^{E} A_i = \{(x_0, ..., x_{n-1}) : (\forall i)(x_i \in A_i) \wedge (\forall i \neq j)(\neg(x_i \; E \; x_j))\}.$$

This set will sometimes be denoted $A_0 \times_E ... \times_E A_{n-1}$.

Definition 3.2.24. *Let E be an equivalence relation on a Polish space X. Let $n \in \omega$. E has the n-weak-Mycielski property if and only if for any $C \subseteq {}^n X$ which is comeager in ${}^n X$, there are $\mathbf{\Delta}^1_1$ sets $(A_i : i < n)$ with the property that for each $i < n$, $E \equiv_{\mathbf{\Delta}^1_1} E \upharpoonright A_i$ and $\prod_{i<n}^{E} A_i \subseteq C$.*

3.3 ${}^\omega 2$ Has the Jónsson Property

This section will give a forcing style proof of Holshouser and Jackson's result that ${}^\omega 2$ has the Jónsson property under some determinacy assumptions. The Jónsson property for ${}^\omega 2$ will follow from a flexible fusion argument for Sacks forcing and the fact that under determinacy assumptions, every function is definable (on some perfect set) with certain absoluteness properties between countable structures and the real universe. Continuous functions will satisfy this property. Thus, the Baire property and the Mycielski property for $=$ suffice to show that every function has such a definition on some perfect set. Absoluteness phenomena that occur under AD^+ can also induce this definability. Later, we will see that the Mycielski property fails for all the other simple equivalence relations considered; the hope is that such a definability and absoluteness approach could establish Jónsson type properties without the Mycielski property. In the following, the fusion argument is essential for the combinatorics of the forcing argument. It is unclear what the relation is between fusion (or properness), the Mycielski property, and the Jónsson property.

Definition 3.3.1. *A tree p on 2 is a subset of ${}^{<\omega}2$ so that if $s \in p$ and $t \subseteq s$, then $t \in p$. p is a perfect tree if and only if for all $s \in p$, there is a $t \supseteq s$ so that $t\hat{}0, t\hat{}1 \in p$.*

Let $\mathbb{S}$ denote the collection of all perfect trees on 2, $\leq_{\mathbb{S}}=\subseteq$, and $1_{\mathbb{S}} = {}^{<\omega}2$. $(\mathbb{S}, \leq_{\mathbb{S}}, 1_{\mathbb{S}})$ is Sacks forcing, denoted by just $\mathbb{S}$.

Let $p \in \mathbb{S}$. $s \in p$ is a split node if and only if $s\hat{}0, s\hat{}1 \in p$. $s \in p$ is a split of p if and only if $s \upharpoonright (|s| - 1)$ is a split node of p. For $n \in \omega$, s is a n-split of p if and only if s is a $\subseteq$-minimal element of p with exactly n-many proper initial segments which are split nodes of p.

Let $\mathrm{split}^n(p)$ denote the set of n-splits of p. Note that $|\mathrm{split}^n(p)| = 2^n$ and $\mathrm{split}^0(p) = \{\emptyset\}$.

If $p, q \in \mathbb{S}$, define $p \leq^n_{\mathbb{S}} q$ if and only if $p \leq_{\mathbb{S}} q$ and $\mathrm{split}^n(p) = \mathrm{split}^n(q)$.

If $p \in \mathbb{S}$ and $s \in p$, then define $p_s = \{t \in p : t \subseteq s \vee s \subseteq t\}$.

Let $p \in \mathbb{S}$. Let Λ be defined as follows:

(i) $\Lambda(p, \emptyset) = \emptyset$.

(ii) Suppose $\Lambda(p, s)$ has been defined for all $s \in {}^n2$. Fix an $s \in {}^n2$ and $i \in 2$. Let $t \supseteq \Lambda(p, s)$ be the minimal split node of p extending $\Lambda(p, s)$. Let $\Lambda(p, s\hat{}i) = t\hat{}i$.

Let $\Xi(p, s) = p_{\Lambda(p,s)}$.

For $n \in \omega$, let $\mathbb{S}^n$ denote the n-fold product of $\mathbb{S}$. If $p \in \mathbb{S}$, then let $p^n \in \mathbb{S}^n$ be defined so that for all $i < n$, $p^n(n) = p$.

Let $n \in \omega$ and $m < n$. There is an $\mathbb{S}^n$-name $x^{n,m}_{\mathrm{gen}}$ which names the m^{th} Sacks-generic real coming from an $\mathbb{S}^n$-generic filter.

Fact 3.3.2. *A fusion sequence is a sequence $\langle p_n : n \in \omega\rangle$ in $\mathbb{S}$ so that for all $n \in \omega$, $p_{n+1} \leq^n_{\mathbb{S}} p_n$. The fusion of this sequence is $p_\omega = \bigcap_{n\in\omega} p_n$.*

p_ω is a condition in $\mathbb{S}$.

Lemma 3.3.3. *Let $f : [{}^\omega 2]^{<\omega}_{=} \to {}^\omega 2$. Let $f_n = f \upharpoonright [{}^\omega 2]^n_{=}$. Suppose there is a countable model M of some sufficiently large fragment of ZF, $p \in \mathbb{S} \cap M$, and a $\mathbb{S}^n$-name $\tau_n \in M$ so that $p^n \Vdash \tau_n \in {}^\omega 2$ and whenever $G^n \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M with $p^n \in G^n$, $\tau_n[G^n] = f_n(x^{n,0}_{\mathrm{gen}}[G^n], ..., x^{n,n-1}_{\mathrm{gen}}[G^n])$. Then there exists a $q \in \mathbb{S}$ so that $f[[[q]]^{<\omega}_{=}] \neq {}^\omega 2$.*

Proof. For each $n \in \omega$, let $(D^n_m : m \in \omega)$ be a sequence of dense open subsets of $\mathbb{S}^n$ in M so that for all m, $D^n_{m+1} \subseteq D^n_m$ and if D is a dense open subset of $\mathbb{S}^n$ in M, then there is some m so that $D^n_m \subseteq D$.

Let $z \in {}^\omega 2 \setminus ({}^\omega 2)^M$.

A fusion sequence $\langle p_n : n \in \omega \rangle$ with $p_0 = p$ will be constructed with the following properties:

For all $n > 0$, $m \leq n$, and $(\sigma_0, ..., \sigma_{m-1}) \in {}^m({}^n2)$ so that $\sigma_i \neq \sigma_j$ if $i \neq j$:

(i) $(\Xi(p_n, \sigma_0), ..., \Xi(p_n, \sigma_{m-1})) \in D_n^m$.

(ii) There are some $k \in \omega$ and $i \in 2$ so that $z(k) \neq i$ and $(\Xi(p_n, \sigma_0), ..., \Xi(p_n, \sigma_{m-1})) \Vdash_{\mathbb{S}^m}^M \tau_m(\check{k}) = \check{i}$.

Suppose this fusion sequence $\langle p_n : n \in \omega \rangle$ could be constructed. Let q be its fusion. Fix $m > 0$. Suppose $(x_0, ..., x_{m-1}) \in [[q]]^m_=$. Let $G^m_{(x_0,...,x_{m-1})} = \{(p_0, ..., p_{m-1}) \in \mathbb{S}^m \cap M : (\forall i < m)(x_i \in [p_i])\}$. Note that $G^m_{(x_0,...,x_{m-1})}$ is a $\mathbb{S}^m$ generic filter over M: There is some L so that for all $k \geq L$, there are $\sigma_i^k \in {}^k2$ with the property that for all $i < m$, $x_i \in \Xi(p_k, \sigma_i^k)$ and for all $i \neq j$, $\sigma_i^k \neq \sigma_j^k$. Then for all $k \geq L$, $(\Xi(p_k, \sigma_0^k), ..., \Xi(p_k, \sigma_{m-1}^k)) \in G^m_{(x_0,...,x_{m-1})}$. (i) asserts that this element belongs to D_k^m. Hence $G^m_{(x_0,...,x_{m-1})}$ is $\mathbb{S}^m$-generic over M.

By (ii), $\tau_m[G^m_{(x_0,...,x_{m-1})}] \neq z$. Also

$$\tau_m[G^m_{(x_0,...,x_{m-1})}] = f_m(x^{m,0}_{\text{gen}}[G^m_{(x_0,...,x_{m-1})}], ..., x^{m,m-1}_{\text{gen}}[G^m_{(x_0,...,x_{m-1})}]) = f_m(x_0, ..., x_{m-1}).$$

Hence $z \notin f_m[[[q]]^m_=]$. Thus $f[[[q]]^{<\omega}_=] \neq {}^\omega 2$.

The construction of the fusion sequence remains: Let $p_0 = p$.

Suppose p_n has been constructed with the above properties. For some $J \in \omega$, let $(\bar{\sigma}_k : k < J)$ enumerate all tuples of strings $(\sigma_0, ..., \sigma_{m-1})$ where $m \leq n+1$, $\sigma_i \in {}^{n+1}2$, and if $i \neq j$, $\sigma_i \neq \sigma_j$.

Next, one constructs a sequence $r_{-1}, ..., r_{J-1}$ as follows: Let $r_{-1} = p_n$. Suppose we have constructed r_k for $k < J - 1$ and $\bar{\sigma}_{k+1} = (\sigma_0, ..., \sigma_{m-1})$.

(Case I) There is some $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$ and $c \in ({}^\omega 2)^M$ so that

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m = \check{c}.$$

Also as D^m_{n+1} is dense open in $\mathbb{S}^m$, one may choose $(u_0, ..., u_{m-1})$ satisfying the above and $(u_0, ..., u_{m-1}) \in D^m_{n+1}$. Note that since $z \notin M$ and $c \in M$, there must be some $j \in \omega$ and $i \in 2$ so that $c(j) \neq i$ and $z(j) = i$. Now let $r_{k+1} \in \mathbb{S}$ be so that for all $\sigma \in {}^{n+1}2$

$$\Xi(r_{k+1}, \sigma) = \begin{cases} u_i & (\exists i)(0 \leq i \leq m-1)(\sigma = \sigma_i) \\ \Xi(r_k, \sigma) & \text{otherwise} \end{cases}.$$

(Case II) For all $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$,

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m \notin M.$$

Hence there are $(u_0, ..., u_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$, $(v_0, ..., v_{m-1}) \leq_{\mathbb{S}^m} (\Xi(r_k, \sigma_0), ..., \Xi(r_k, \sigma_{m-1}))$, and $j \in \omega$ so that

$$(u_0, ..., u_{m-1}) \Vdash^M_{\mathbb{S}^m} \tau_m(\check{j}) = \check{0}$$

$$(v_0, ..., v_{m-1}) \Vdash^m_{\mathbb{S}^m} \tau_m(\check{j}) = \check{1}$$

Without loss of generality, suppose that $z(j) = 1$. Moreover since D^m_{n+1} is dense open, one may assume that $(u_0, ..., u_{m-1}) \in D^m_{n+1}$. Define r_{k+1} in the same way as in Case I.

Finally, let $p_{n+1} = r_{J-1}$. $p_{n+1} \leq^n_{\mathbb{S}} p_n$ and condition (i) and (ii) are satisfied. This completes the construction. □

Fact 3.3.4. *(*$\mathsf{ZF} + \mathsf{DC}$*) Let $p \in \mathbb{S}$ and $n \in \omega$. If $f_n : [[p]]^n_= \to {}^\omega 2$ is continuous, then there is a countable elementary $M \prec V_\Xi$ (for Ξ some sufficiently large cardinal) and a name τ_n so that M, τ_n, and p satisfy the conditions of Lemma 3.3.3.*

Proof. If f_n is continuous, then f_n has Σ^1_1 and Π^1_1 formulas with parameters from ${}^\omega 2$ defining it. Let $M \prec V_\Xi$ be a countable elementary substructure containing p and all the parameters used to define f_n. (This requires DC.) Using Mostowski's absoluteness, f_n (as defined by this formula) continues to define a function in the forcing extension $M[G]$, where $G \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M. So there is some $\mathbb{S}^n$-name $\tau_n \in M$ so that $p^n \Vdash^M_{\mathbb{S}^n} \tau_n = f_n(x^{n,0}_{\text{gen}}, ..., x^{n,n-1}_{\text{gen}})$. Suppose $G^n \subseteq \mathbb{S}^n$ is $\mathbb{S}^n$-generic over M and contains p^n. Then $M[G^n] \models \tau_n[G^n] = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])$. Let $\pi : M[G^n] \to N$ be the Mostowski collapse of $M[G^n]$. Since reals are not moved by the Mostowski collapse map π, $\pi(f_n)$ is still defined by the same formula. So

$$N \models \pi(\tau_n[G^n]) = \tau_n[G^n] = \pi(f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])) = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n]).$$

Then applying Mostowski absoluteness, $\tau_n[G^n] = f_n(x^{n,0}_{\text{gen}}[G^n], ..., x^{n,n-1}_{\text{gen}}[G^n])$. □

Theorem 3.3.5. *(Holshouser-Jackson) Assume* $\mathsf{ZF} + \mathsf{DC}$ *and all sets of reals have the Baire property. Then ${}^\omega 2$ has the Jónsson property.*

Proof. Let $f : [{}^\omega 2]^{\leq\omega}_= \to {}^\omega 2$. Let $f_n : [\mathbb{R}]^n_= \to \mathbb{R}$ be defined by $f_n = f \upharpoonright [{}^\omega 2]^n_=$. Since all sets of reals have the Baire property, there are comeager subsets $C_n \subseteq {}^n({}^\omega 2)$ so that $f_n \upharpoonright C_n$ is continuous. By the theorem of Mycielski (i.e. $=$ has the Mycielski property), there is a perfect tree p so that $[[p]]^n_= \subseteq C_n$ for all $n \in \omega$. Hence for all $n \in \omega$, $f_n \upharpoonright [[p]]^n_=$ is a continuous function. Then Fact 3.3.4 and Lemma 3.3.3 imply that ${}^\omega 2$ has the Jónsson property. □

Remark 3.3.6. As a consequence of phrasing this argument using forcing, one needed to introduce countable elementary substructures. DC is necessary in general to obtain useful countable elementary substructures. One can use a more direct topological argument to avoid DC.

3.4 ω-Jónsson Function for ${}^\omega 2$

Let $\mathsf{AC}^{\mathbb{R}}_\omega$ be the axiom of countable choice for ${}^\omega 2$: If $\mathcal{E}$ is a countable set of nonempty subsets of ${}^\omega 2$, then $\mathcal{E}$ has a choice function.

Note that $\mathsf{ZF} + \mathsf{AD}$ implies $\mathsf{AC}^{\mathbb{R}}_\omega$.

Using the axiom of choice, every set has an ω-Jónsson function. However, just $\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$ implies there is a Δ^1_1 classical ω-Jónsson function for ${}^\omega 2$. In fact, a slightly stronger statement holds:

Theorem 3.4.1. ($\mathsf{ZF} + \mathsf{AC}^{\mathbb{R}}_\omega$) *There is a Δ^1_1 function $\Phi : \mathscr{P}^\omega({}^\omega 2) \to {}^\omega 2$ so that if $B \subseteq {}^\omega 2$ is uncountable, then $\Phi[\mathscr{P}^\omega(B)] = {}^\omega 2$.*

There is a Δ^1_1 classical ω-Jónsson function for ${}^\omega 2$.

Proof. Let A be a countable subset of ${}^\omega 2$.

Let $a^A_\emptyset$ be the longest element of ${}^{<\omega}2$ which is an initial segment of every element of A.

If a^A_σ is undefined, then $a^A_{\sigma\hat{\ }i}$ is undefined for $i \in 2$. If a^A_σ is defined, let $a^A_{\sigma\hat{\ }i}$ be the longest element of ${}^{<\omega}2$ which is an initial segment of every element of $A \cap N_{a^A_\sigma\hat{\ }i}$, if it exists. Otherwise $a^A_{\sigma\hat{\ }i}$ is undefined. (Note this happens if and only if $A \cap N_{a_\sigma\hat{\ }i}$ is a singleton.)

For $A \in \mathscr{P}^\omega({}^\omega 2)$, let $\Psi(A)$ be the collection of $\sigma \in {}^{<\omega}2$ so that a^A_σ is defined. $\Psi(A)$ is an infinite tree on 2 with possibly dead nodes and is not perfect. Ψ is a Δ^1_1 function.

Using some recursive coding, let X be the collection of reals coding infinite binary trees (which may have dead branches). X is an uncountable Π^0_1 set.

Let $T \in X$. Let $\hat{T} = \{\sigma\hat{\ }\tilde{0}, \sigma\hat{\ }1\hat{\ }\tilde{0} : \sigma \in T\}$. $\hat{T} \in \mathscr{P}^\omega({}^\omega 2)$. One seeks to show that $\Psi(\hat{T}) = T$. To see this, the following claim is helpful: If $\sigma \in T$, then $a^{\hat{T}}_\sigma = \sigma$ and if $\sigma \notin T$, then $a^{\hat{T}}_\sigma$ is undefined.

This claim is proved by induction: $\emptyset \in T$ so $\tilde{0}, 1\hat{\ }\tilde{0} \in \hat{T}$. $a^{\hat{T}}_\emptyset = \emptyset$. Suppose this holds for σ. Suppose $\sigma\hat{\ }i \in T$. Then $\sigma\hat{\ }i\hat{\ }0\hat{\ }\tilde{0}$ and $\sigma\hat{\ }i\hat{\ }1\hat{\ }\tilde{0}$ are both in $\hat{T}$. By induction, $a^{\hat{T}}_\sigma = \sigma$. The longest string which is an initial segment of every element of $\hat{T} \cap N_{a^{\hat{T}}_\sigma\hat{\ }i} = \hat{T} \cap N_{\sigma\hat{\ }i}$ is $\sigma\hat{\ }i$. This shows $a^{\hat{T}}_{\sigma\hat{\ }i} = \sigma\hat{\ }i$. Suppose $\sigma\hat{\ }i \notin T$. Either $a^{\hat{T}}_\sigma$ is undefined or $a^{\hat{T}}_\sigma$ is defined. If $a^{\hat{T}}_\sigma$ is undefined, then $a^{\hat{T}}_{\sigma\hat{\ }i}$ is undefined. Suppose $a^{\hat{T}}_\sigma$ is defined. By induction, $a^{\hat{T}}_\sigma = \sigma$. $\sigma \in T$ implies that $\sigma\hat{\ }0\hat{\ }\tilde{0}$ and $\sigma\hat{\ }1\hat{\ }\tilde{0}$ are both in $\hat{T}$. Since $\sigma\hat{\ }i \notin T$, $N_{\sigma\hat{\ }i} \cap \hat{T} = N_{a^{\hat{T}}_\sigma\hat{\ }i} \cap \hat{T} = \{\sigma\hat{\ }i\hat{\ }\tilde{0}\}$. $a^{\hat{T}}_{\sigma\hat{\ }i}$ is undefined. This completes the proof of the claim.

This shows $\Phi(\hat{T}) = T$. Hence $\Psi[\mathscr{P}^\omega({}^\omega 2)] = X$. Let $\Gamma : X \to {}^\omega 2$ be a Δ^1_1 bijection. Let $\Phi = \Gamma \circ \Psi$.

Let B be an uncountable subset of ${}^\omega 2$. Then there is an uncountable $C \subseteq B$ which has no isolated points. One way to see this is to note that using a countable basis, the Cantor-Bendixson process must stop at a countable ordinal. The fixed point starting from B would be an uncountable set with no isolated points.

Fix such a set C. Let $\mathcal{E} = \{N_\sigma \cap C : \sigma \in {}^{<\omega}2 \wedge N_\sigma \cap C \neq \emptyset\}$. $\mathcal{E}$ is a countable set. Using $\mathsf{AC}^{\mathbb{R}}_\omega$, let Λ be a choice function for $\mathcal{E}$. Let T be any infinite binary tree on 2.

The following objects will be constructed:

(I) $c_s \in C$ for each $s \in {}^{<\omega}2$.

(II) A strictly increasing sequence $(k_i : i \in \{-1\} \cup \omega)$ of integers.

For each $n \in \omega$, let $A_n = \{c_s : s \in {}^n 2\}$. The objects above will satisfy the following properties:

(i) If $s \in T$, then $a_s^{A_{|s|+1}}$ is defined and has length less than $k_{|s|}$. If $s \notin T$, then $A_{|s|+1} \cap N_{c_s \restriction k_{|s|}} = \{c_s\}$.

(ii) If $s \in T$, then $c_{s\hat{}i} \supseteq a_s^{A_{|s|+1}}\hat{}i$ for each $i \in 2$.

(iii) For all m, if $n > m$, then $\{x \restriction k_m : x \in A_m\} = \{x \restriction k_m : x \in A_n\}$.

Let $c_\emptyset$ be any element of C. Let $k_{-1} = 0$. Let $A_0 = \{c_\emptyset\}$.

Suppose for $m \in \omega$, $c_s \in C$ for all $s \in {}^m 2$ and k_{m-1} have been defined. Suppose properties (i) to (iii) hold for $t \in {}^{<\omega}2$ with $|t| < m$. Let $s \in T \cap {}^m 2$. Since $c_s \in C$ and C has no isolated points, there is some $m_s > k_{m-1}$ so that $N_{(c_s \restriction m_s)\hat{}(1-c_s(m_s))} \cap C \neq \emptyset$. Let $c_{s\hat{}c_s(m_s)} = c_s$ and let $c_{s\hat{}(1-c_s(m_s))} = \Lambda(N_{(c_s \restriction m_s)\hat{}(1-c_s(m_s))})$. If $s \notin T$, then let $c_{s\hat{}i} = c_s$ for each $i \in 2$. Let $k_m = \sup\{m_s + 1 : s \in {}^m 2 \cap T\}$.

Since for each $s \in T \cap {}^m 2$, $m_s > k_{m-1}$, (i) to (iii) still hold for t with $|t| < m$. Let $s \in T$. Using the induction hypothesis for (ii) on $s \restriction m-1$, one has that $c_s \supseteq a^{A_m}_{s\restriction m-1}\hat{}s(m-1)$. $c_{s\hat{}0}$ and $c_{s\hat{}1}$ extend $a^{A_m}_{s\restriction m-1}\hat{}s(m-1)$. This shows that $a_s^{A_{m+1}}$ is defined. In fact, $a_s^{A_{m+1}} = c_s \restriction m_s$. If $s \notin T$, (i) is clear from the construction. Properties (i) to (iii) hold for $s \in {}^m 2$.

Let $A = \bigcup_{n\in\omega} A_n$. Note that A is countably infinite and $A \subseteq C \subseteq B$. From the above properties, if $s \in T$, then a_s^A is defined and in fact equal to $a_s^{A_{|s|+1}}$. Suppose $s \notin T$. Let $t \subseteq s$ be maximal with $t \in T$. The above properties imply that $A \cap N_{a_t^A\hat{}s(|t|)} = \{c_{t\hat{}s(|t|)}\} = \{c_s\}$. Hence $a^A_{t\hat{}s(|t|)}$ is not defined and hence a_s^A is not defined. Thus $T = \Psi(A)$. This shows that $\Phi[\mathscr{P}^\omega(B)] = {}^\omega 2$. Φ is an ω-Jónsson function for ${}^\omega 2$. $\square$

Question 3.4.2. *Under* $\mathsf{ZF} + \neg\mathsf{AC}^{\mathbb{R}}_{\omega}$*, can there be a classical* ω*-Jónsson function for* ${}^{\omega}2$*?*

The first statement of Theorem 3.4.1 may not be true without $\mathrm{AC}^{\mathbb{R}}_{\omega}$*: Let* $\mathbb{C}_\omega$ *denote the finite support product of Cohen forcing* $\mathbb{C}$*. Let* $G \subseteq \mathbb{C}_\omega$ *be* $\mathbb{C}_\omega$*-generic over* L*. For each* $n \in \omega$*, let* c_n *be the* n^{th}*-Cohen generic real naturally added by* G*. Let* $A = \{c_n : n \in \omega\}$*. Let* $H = (\mathrm{HOD}(A \cup \{A\}))^{L[G]}$*.* H *is called the Cohen-Halpern-Lévy model. In* H*,* A *has no countably infinite subsets. Hence the first statement of Theorem 3.4.1 cannot hold. However* A *is not in bijection with* ${}^{\omega}2$*. This suggest the following natural question: In* H*, is there a classical* ω*-Jónsson function for* ${}^{\omega}2$*?*

3.5 The Structure of E_0

Definition 3.5.1. E_0 *is the equivalence relation defined on* ${}^{\omega}2$ *by* $x\ E_0\ y$ *if and only if* $(\exists n)(\forall k > n)(x(k) = y(k))$*.*

Definition 3.5.2. *Let* $\{s, v^i_n : i \in 2 \wedge n \in \omega\} \subseteq {}^{<\omega}2$ *have the property that for all* $n \in \omega$ *and* $i \in 2$*,* $\langle i\rangle \subseteq v^i_n$ *and* $|v^0_n| = |v^1_n|$*.*

Let $\varphi(\emptyset) = s$*. If* $\sigma \in {}^{<\omega}2$ *and* $|\sigma| > 0$*, then let* $\varphi(\sigma) = s\hat{\ } v_0^{\sigma(0)}\hat{\ }...\hat{\ }v_{|\sigma|-1}^{\sigma(|\sigma|-1)}$*.*

A perfect tree p *is an* E_0*-tree if and only if there is a sequence* $\{s, v^i_n : i \in 2 \wedge n \in \omega\}$ *with the above properties so that* p *is the* $\subseteq$*-downward closure of* $\{\varphi(\sigma) : \sigma \in {}^{<\omega}2\}$*.*

Let $\mathbb{P}_{E_0}$ *be the collection of all perfect* E_0 *trees. If* $p, q \in \mathbb{P}_{E_0}$*, then* $p \leq_{\mathbb{P}_{E_0}} q$ *if and only if* $p \subseteq q$*. Let* $1_{\mathbb{P}_{E_0}} = {}^{<\omega}2$*.* $(\mathbb{P}_{E_0}, \leq_{\mathbb{P}_{E_0}}, 1_{\mathbb{P}_{E_0}})$ *is forcing with perfect* E_0*-trees.*

If $p \in \mathbb{P}_{E_0}$*, then the notation* s^p *and* $v^{i,p}_n$ *will be used to denote the strings witnessing* p *is a perfect* E_0*-tree.*

Let $\Phi : {}^{\omega}2 \to [p]$ *be defined by* $\Phi(x) = \bigcup_{n\in\omega} \varphi(x \upharpoonright n)$*, where* φ *is associated with the* E_0*-tree* p *as above.* Φ *is the canonical homeomorphism of* ${}^{\omega}2$ *onto* $[p]$*, and* Φ *is a reduction witnessing* $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright [p]$*.*

Fact 3.5.3. *Suppose* B *is a* Σ^1_1 *set so that* $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B$*. Then there is an* E_0*-tree* p *so that* $[p] \subseteq B$*.*

Proof. This is implicit in [7]. See [29] Lemma 2.3.29 and [13] Theorem 10.8.3. □

The weak Mycielski property for E_0 considers E_0-products of $\mathbf{\Delta}^1_1$ sets $A_0, ..., A_{n-1}$ so that $E_0 \equiv_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_i$ and $[A_i]_{E_0} = [A_j]_{E_0}$. Showing the failure of the weak Mycielski property requires finding some structure shared by all of the sets $A_0, ..., A_{n-1}$. For instance, are there perfect E_0-trees p_i so that $[p_i] \subseteq A_i$ and $[[p_i]]_{E_0} = [[p_j]]_{E_0}$? How similar can p_i and p_j be chosen to be?

We will first consider a simpler solution using the σ-additivity of the E_0-ideal, which follows Fact 3.5.3. A stronger result giving more information using effective methods will follow.

Fact 3.5.4. *For each $n \in \omega$, suppose $A_n \subseteq {}^\omega 2$ is Σ^1_1 and there is no E_0-tree p so that $[p] \subseteq A_n$. Then there is no E_0 tree p so that $[p] \subseteq \bigcup_{n\in\omega} A_n$.*

Proof. Suppose there is some E_0-tree p so that $[p] \subseteq \bigcup_{n\in\omega} A_n$. Let $\Phi : {}^\omega 2 \to [p]$ be the canonical injective reduction witnessing $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright [p]$. For each $n \in \omega$, $\Phi^{-1}[A_n]$ is a Σ^1_1 set, and ${}^\omega 2 = \bigcup_{n\in\omega} \Phi^{-1}[A_n]$. There is some $m \in \omega$ so that $\Phi^{-1}[A_m]$ is nonmeager. Therefore, there is some continuous injective function $\Psi : {}^\omega 2 \to \Phi^{-1}[A_m]$ which witnesses $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright \Phi^{-1}[A_m]$. $\Phi \circ \Psi$ witnesses $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_m$. This implies there is some E_0-tree q so that $[q] \subseteq A_m$. Contradiction. $\square$

Definition 3.5.5. *If $x \in {}^\omega 2$ and $n \in \omega$, let $x_{\geq n} \in {}^\omega 2$ be defined by $x_{\geq n}(k) = x(n + k)$.*

If $A \subseteq {}^\omega 2$, then let $(A)_{\geq n} = \{z : (\exists x \in A)(z = x_{\geq n})\}$.

Definition 3.5.6. *Let $s \in {}^{<\omega}2$. Define $\mathsf{switch}_s : {}^\omega 2 \to {}^\omega 2$ by*

$$\mathsf{switch}_s(x)(n) = \begin{cases} s(n) & n < |s| \\ x(n) & \text{otherwise} \end{cases}.$$

Also if $\sigma \in {}^{<\omega}2$, $\mathsf{switch}_s(\sigma) \in {}^{|\sigma|}2$ is defined as above just for $n < |\sigma|$.

Theorem 3.5.7. *Let $n \in \omega$. For $k < n$, let $A_k \subseteq {}^\omega 2$ be Σ^1_1 so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_k$ and for all $k < n - 1$, $[A_k]_{E_0} \subseteq [A_{k+1}]_{E_0}$. Then there exists E_0-trees p_k so that $[p_k] \subseteq A_k$ and for all $a, b < n$, $|s^{p_a}| = |s^{p_b}|$, and $v^{i,p_a}_m = v^{i,p_b}_m$ for all $m \in \omega$ and $i \in 2$.*

Proof. For each $c \in \omega$, let

$$E_c = \left\{(x_0, ..., x_{n-1}) \in \prod_{k<n} A_k : (\forall i, j < n)((x_i)_{\geq c} = (x_j)_{\geq c})\right\}.$$

For each $c \in \omega$, E_c is a Σ^1_1 set. For $k < n$, let $\pi_k : {}^n({}^\omega 2) \to {}^\omega 2$ be the projection map onto the k^{th} coordinate. $\pi_0[E_c]$ is Σ^1_1 for each $c \in \omega$. Since $[A_i]_{E_0} \subseteq [A_{i+1}]_{E_0}$, $\bigcup_{c\in\omega} \pi_0[E_c] = A_0$. By Fact 3.5.4, there is some $m \in \omega$ so that $\pi_0[E_m]$ contains the body of an E_0-tree q_0. By choosing an appropriate subtree, one may assume that $|s^{q_0}| > m$. Let $s_0 = s^{q_0} \upharpoonright m$.

Fix $k < n - 1$. Suppose the E_0-tree q_k and $s_k \in {}^m2$ have been constructed so that $s_k \subseteq s^{q_k}$ and $[q_k] \subseteq \pi_k[E_m]$. Then

$$\left[\bigcup_{s \in {}^m2} \mathsf{switch}_s[[q_k]] \cap \pi_{k+1}[E_m]\right]_{E_0} = [[q_k]]_{E_0}.$$

By Fact 3.5.4, there is some $s_{k+1} \in {}^m2$ so that $\mathsf{switch}_{s_{k+1}}[[q_k]] \cap \pi_{k+1}[E_m]$ contains an E_0-tree. Let q_{k+1} be such an E_0-tree. Note that $\mathsf{switch}_{s_k}[q_{k+1}]$ is an E_0-subtree of q_k.

For $i < n$, let $p_i = \mathsf{switch}_{s_i}[q_{n-1}]$. Note that $[p_i] \subseteq \pi_i[E_m] \subseteq A_i$. □

The rest of this section will prove a result that implies Theorem 3.5.7 using an effective definability condition. We will use the methods from [13] Theorem 10.8.3 to simultaneously produce E_0-trees, which are very similar to each other, through several sets.

Definition 3.5.8. *Let $z \in {}^\omega 2$. Let $\mathbb{P}_z$ be the forcing of nonempty $\Sigma^1_1(z)$ sets ordered by inclusion with largest element $\mathbb{P}_z = {}^\omega 2$. $\mathbb{P}_z$ is z-Gandy-Harrington forcing.*

Fact 3.5.9. *There is a z-recursive (in a suitable sense) collection $\mathcal{D} = \{D_n : n \in \omega\}$ of dense open subsets of $\mathbb{P}_z$ so that if $G \subseteq \mathbb{P}_z$ is generic for $\mathcal{D}$, then $\bigcap G \neq \emptyset$.*

Proof. See [13] Theorem 2.10.4. □

Fact 3.5.10. *Let $z \in {}^\omega 2$. There is a $\Pi^1_1(z)$ set $D \subseteq \omega$, $\Sigma^1_1(z)$ set $P \subseteq \omega \times {}^\omega 2$, and $\Pi^1_1(z)$ set $Q \subseteq \omega \times {}^\omega 2$ with the following properties:*

(i) For all $e \in D$, $P^e = Q^e$, where if $X \subseteq \omega \times {}^\omega 2$, then $X^e = \{x \in {}^\omega 2 : (e, x) \in X\}$.

(ii) If $X \subseteq {}^\omega 2$ is $\Delta^1_1(z)$, then there is some $e \in D$ so that $X = P^e = Q^e$.

Definition 3.5.11. *Let $z \in {}^\omega 2$. Let S_z be the union of all $\Delta^1_1(z)$ sets C so that for all $x, y \in C$, $\neg(x \; E_0 \; y)$.*

Let $H_z = {}^\omega 2 \setminus S_z$.

Fact 3.5.12. *Let $z \in {}^\omega 2$. S_z is $\Pi^1_1(z)$. H_z is $\Sigma^1_1(z)$. If $X \cap H_z \neq \emptyset$ and X is $\Sigma^1_1(z)$, then there exists $x, y \in X$ with $x \neq y$ and $x \; E_0 \; y$. H_z is E_0-saturated.*

Proof. Let D, P, and Q be the sets from Fact 3.5.10. Note that

$$x \in S_z \Leftrightarrow (\exists e)(e \in D \wedge x \in Q^e \wedge (\forall f, g)((f \neq g \wedge f, g \in P^e) \Rightarrow \neg(f \; E_0 \; g))).$$

S_z is $\Pi^1_1(z)$. Hence H_z is $\Sigma^1_1(z)$.

Let $\mathcal{A}$ be the collection of all $\Sigma^1_1(z)$ subsets of ${}^\omega 2$ whose elements are pairwise E_0-inequivalent. Let $U \subseteq \omega \times {}^\omega 2$ be a universal $\Sigma^1_1(z)$ set.

$$\{e : U^e \in \mathcal{A}\} = \{e : (\forall f, g)((f, g \in U^e \wedge f \neq g) \Rightarrow \neg(f\ E_0\ g))\}$$

The above is a $\Pi^1_1(z)$ set. So $\mathcal{A}$ is a collection of $\Sigma^1_1(z)$ sets which is $\Pi^1_1(z)$ in the codes. By $\Sigma^1_1(z)$-reflection (see [13], Theorem 2.7.1), every $\Sigma^1_1(z)$ set X whose elements are E_0-inequivalent has a $\Delta^1_1(z)$ set C whose elements are E_0-inequivalent and $X \subseteq C$.

Suppose X is $\Sigma^1_1(z)$, $X \cap H_z \neq \emptyset$, and the elements of X are pairwise E_0-inequivalent. By the previous paragraph, there is some $\Delta^1_1(z)$ set C which also E_0-inequivalent and $X \subseteq C$. Then $X \subseteq S_z$. Contradiction.

Suppose $x \in H_z$, $y \notin H_z$, and $x\ E_0\ y$. Let $n \in \omega$ be so that $x_{\geq n} = y_{\geq n}$. $y \in S_z$ implies that there is some $\Delta^1_1(z)$ E_0-inequivalent set X so that $y \in X$. $\mathsf{switch}_{x \restriction n}[X]$ is a $\Delta^1_1(z)$ E_0-inequivalent set containing x. This contradicts $x \in H_z$. This shows H_z is E_0-saturated. □

Lemma 3.5.13. *Let $z \in {}^\omega 2$ and $n, \ell \in \omega$. Suppose $(\hat{B}_i : i < n)$ is a collection of nonempty $\Sigma^1_1(z)$ sets. Suppose for all $i, j < n$, $(\hat{B}_i)_{\geq \ell} = (\hat{B}_j)_{\geq \ell}$. Let $D \subseteq \mathbb{P}_z$ be a dense open subset of the forcing $\mathbb{P}_z$. Then there is a collection $(B_i : i < n)$ of nonempty $\Sigma^1_1(z)$ sets so that for all $i, j < n$, $B_i \in D$, $B_i \subseteq \hat{B}_i$, and $(B_i)_{\geq \ell} = (B_j)_{\geq \ell}$.*

Proof. For $k < n$, $\Sigma^1_1(z)$ sets $\{B^k_i : -1 \leq k < n \wedge 0 \leq i < n\}$ will be constructed with the properties that

(i) For all $i < n$, $B^i_i \in D$.

(ii) If $-1 \leq k < n - 1$ and $0 \leq i < n$, then $B^{k+1}_i \subseteq B^k_i$.

(iii) For all $-1 \leq k < n$ and $0 \leq i, j \leq n$, $(B^k_i)_{\geq \ell} = (B^k_j)_{\geq \ell}$.

Note that this implies that if $k \geq i$, then $B^k_i \in D$.

Let $B^{-1}_i = \hat{B}_i$. (iii) is satisfied.

Suppose for $-1 \leq k < n - 1$ and $0 \leq i < n$, B^k_i has been constructed with the desired properties. Since D is dense open, there is some nonempty $\Sigma^1_1(z)$ set, denoted B^{k+1}_{k+1}, so that $B^{k+1}_{k+1} \subseteq B^k_{k+1}$ and $B^{k+1}_{k+1} \in D$. For $0 \leq i < n$, let $B^{k+1}_i = \{x \in B^k_i : (\exists z)(z \in B^{k+1}_{k+1} \wedge x_{\geq \ell} = z_{\geq \ell})\}$. All the conditions are satisfied.

Finally, let $B_i = B^{n-1}_i$.

□

Theorem 3.5.14. *Let $z \in {}^{\omega}2$ and $n \in \omega$. Let $(A_a : a < n)$ be a collection of $\Sigma^1_1(z)$ sets so that $\bigcap_{a<n}[A_a \cap H_z]_{E_0} \neq \emptyset$. Then there are E_0-trees $(p_a : a \in n)$ so that for all $a, b < n$, $k \in \omega$ and $i < 2$,*

(i) $|s^{p_a}| = |s^{p_b}|$ and $v_k^{i,p_a} = v_k^{i,p_b}$

(ii) $[p_a] \subseteq A_a$.

Proof. We will construct the following objects: For each $a < n$, $k \in \omega$, $i \in 2$, and $t \in {}^{<\omega}2$,

(a) $w^a_t, s^a, v^i_k \in {}^{<\omega}2$

(b) $\ell_k \in \omega$ and for all $k \in \omega$, $\ell_k \leq m_k < \ell_{k+1}$

(c) $\Sigma^1_1(z)$ nonempty sets X^a_t

with the following properties

(i) For all $a < n$ and $t \in {}^{<\omega}2$, $|w^a_t| = \ell_{|t|}$. For all $a < n$ and $k \in \omega$, $|s^a| = |s^b|$ and $|v^0_k| = |v^1_k|$. For all $k \in \omega$ and $i \in 2$, $\langle i\rangle \subseteq v^k_i$. For all $a < n$ and $t \in {}^{<\omega}2$, if $|t| = 1$, then $s^a \subseteq w^a_t$ and if $t > 1$, then $s^a{}^\frown v_0^{t(0)}{}^\frown \dots {}^\frown v_{|t|-2}^{t(|t|-2)} \subseteq w^a_t$.

(ii) If $a \in n$, $t \in {}^{<\omega}2$, $u \in {}^{<\omega}2$, and $t \subseteq u$, then $X^a_u \subseteq X^a_t$, $X^a_t \subseteq N_{w^a_t}$, and $X^a_\emptyset \subseteq A_a \cap H_z$.

(iii) Let $\mathcal{D} = (D_n : n \in \omega)$ be the collection of dense open subset of $\mathbb{P}_z$ from Fact 3.5.9. For all $t \in {}^{<\omega}2$, $X^a_t \in D_{|t|}$.

(iv) For all $k < \omega$, $\ell_k < \ell_{k+1}$. For all $k < \omega$, $t, u \in {}^k2$, and $a, b \in n$, $(X^a_t)_{\geq \ell_k} = (X^b_u)_{\geq \ell_k}$.

Suppose we can construct the objects with these properties. For $a < n$, let p^a be the E_0-tree given by $s^{p_a} = s^a$ and $v_k^{i,p_a} = v^i_k$. Let $\Phi^a : {}^{\omega}2 \to [p_a]$ be the canonical map associated with the E_0-tree p_a. For each $x \in {}^{\omega}2$, let G^a_x be the $\mathbb{P}_z$-filter generated by the upward closure of $\{X^a_{x \upharpoonright k} : k \in \omega\}$. $G^a_x \cap D_k \neq \emptyset$ since $X^a_{a \upharpoonright k} \in D_k$. G^a_x is a filter generic for $\mathcal{D}$. By Fact 3.5.9, $\bigcap G^a_x \neq \emptyset$. By (i) and (ii), $\bigcap G^a_x = \{\Phi^a(x)\}$. Thus $\Phi^a(x) \in X^a_\emptyset \subseteq A_a \cap H_z$. Therefore, $[p_a] \subseteq A_a$.

Next, we will describe the construction. Since $\bigcap_{a<n}[A_a \cap H_z]_{E_0} \neq \emptyset$, let $(x_a : a < n)$ be elements of ${}^{\omega}2$ so that for all $a, b < n$, $x_a \in A_a \cap H_z$ and $x_a \mathrel{E_0} x_b$. Choose $\ell_0 \in \omega$ so that for all $a, b < n$, $(x_a)_{\geq \ell_0} = (x_b)_{\geq \ell_0}$. Let $w^a_\emptyset = x_a \upharpoonright \ell_0$. Let $Z = \{r \in {}^{\omega}2 : (\exists y_0, \dots, y_{n-1})(\bigwedge_{a<n} y_a \in A_a \cap H_z \wedge w^a_\emptyset{}^\frown r = y_a)\}$. Z is a nonempty $\Sigma^1_1(z)$ set. Let $B_a = \{x \in A_a \cap H_z : (\exists r)(r \in Z \wedge x \supseteq w^a_\emptyset \wedge x_{\geq \ell_0} = r)\}$. For each $a < n$, B_a is a nonempty $\Sigma^1_1(z)$ set. Note that for all $a, b < n$, $(B_a)_{\geq \ell_0} = (B_b)_{\geq \ell_0}$. Applying Lemma 3.5.13, find sets $(X^a_\emptyset : a \in \omega)$ so that $X^a_\emptyset \subseteq B_a$ and $X^a_\emptyset \in D_0$.

Suppose X^a_t has been constructed for $a < n$ and $t \in {}^k2$. $X^0_{0^k} \subseteq H_z$ is nonempty and $\Sigma^1_1(z)$. By Fact 3.5.12, there are $x, y \in X^0_{0^k}$ so that $x \neq y$ and $x \mathrel{E_0} y$. By (i) and (ii), $x \upharpoonright \ell_k = y \upharpoonright \ell_k$. Therefore, there is some $\ell_{k+1} > \ell_k$ so that $x_{\geq \ell_{k+1}} = y_{\geq \ell_{k+1}}$. Let m_k be smallest m so that $\ell_k \leq m < \ell_{k+1}$ and $x(m) \neq y(m)$. Without loss of generality, suppose that $x(m_k) = 0$ and $y(m_k) = 1$. For $i \in 2$, let $w^0_{0^k\hat{}0} = x \upharpoonright \ell_{k+1}$ and $w^0_{0^k\hat{}1} = y \upharpoonright \ell_{k+1}$. For $a \in n$ and $t \in {}^{<\omega}2$, let $w^a_{t\hat{}i} = \mathrm{switch}_{w^a_t} w^0_{0^k\hat{}i}$.

If $k = 0$, then let $s^a = w^a_{\langle 0\rangle} \upharpoonright m_0$. If $k > 0$, then let $L_k = |s^0| + \sum_{0\leq j\leq k-1} |v^0_j|$. Let v^i_k be the string of length $m_k - L_k$ defined by $v^i_k(j) = w^0_{0^k\hat{}i}(L_k + j)$.

Let $Z = \{z \in {}^\omega 2 : (\exists x, y)(x, y \in X^0_{0^n} \wedge w_{0^k\hat{}0} \subseteq x \wedge w_{0^k\hat{}1} \subseteq y \wedge z = x_{\geq \ell_{k+1}} = y_{\geq \ell_{k+1}})\}$. Z is a nonempty $\Sigma^1_1(z)$. For $t \in {}^{k+1}2$ and $a < n$, let $B^a_t = \{x \in X_{t\upharpoonright k} : (\exists z)(z \in Z \wedge x = w^a_t\hat{}z)\}$. B^a_t is a nonempty $\Sigma^1_1(z)$ set and $(B^a_t)_{\geq \ell_{k+1}} = (B^b_u)_{\geq \ell_{k+1}}$ for all $a, b \in n$ and $t, u \in {}^{k+1}2$. Apply Lemma 3.5.13 to get $X^a_t \subseteq B^a_t$ so that $X^a_t \in D_{k+1}$ and $(X^a_t)_{\geq \ell_{k+1}} = (X^b_u)_{\geq \ell_{k+1}}$. This completes the construction. □

3.6 ${}^\omega 2/E_0$ Has the 2-Jónsson Property

Theorem 3.6.1. *(Holshouser-Jackson) E_0 has the 2-Mycielski property.*

One can prove this by producing an E_0-tree p using an E_0-tree fusion argument to ensure that $[[p]]^2_{E_0}$ meets a fixed countable sequence of dense (topologically) open subsets of ${}^2({}^\omega 2)$. The fusion argument is quite similar to the fusion argument used in the forcing style proof of the 2-Jónsson property for ${}^\omega 2/E_0$ in this section.

By Theorem 3.5.7, if $(A_a : a < n)$ is a sequence of Σ^1_1 sets so that $[A_a]_{E_0} = [A_b]_{E_0}$ for all $a, b < n$, then there is a sequence of E_0-trees $(p_a : a \in n)$ so that for all $a, b < n$ and $i < 2$, $|s^{p_a}| = |s^{p_b}|$ and $v^{p_a}_i = v^{p_b}_i$. This motivates the definition of the following forcings:

Definition 3.6.2. *Let $n > 0$, let $\widehat{\mathbb{P}}^n_{E_0}$ be the collection of n-tuples of E_0-trees $(p_0, ..., p_{n-1})$ so that for all $a, b < n$ and $i < 2$, $|s^{p_a}| = |s^{p_b}|$ and $v^{p_a}_i = v^{p_b}_i$. Let $\leq_{\widehat{\mathbb{P}}^n_{E_0}}$ be coordinatewise $\leq_{\mathbb{P}_{E_0}}$. Let $1_{\widehat{\mathbb{P}}^n_{E_0}} = (1_{\mathbb{P}_{E_0}}, ..., 1_{\mathbb{P}_{E_0}})$. $(\widehat{\mathbb{P}}^n_{E_0}, \leq_{\widehat{\mathbb{P}}^n_{E_0}}, 1_{\widehat{\mathbb{P}}^n_{E_0}})$ is forcing with n E_0-trees with the same E_0-saturation.*

Let x^0_{gen} and x^1_{gen} be the $\widehat{\mathbb{P}}^2_{E_0}$ names for the left and right generic real added by a generic filter for $\widehat{\mathbb{P}}^2_{E_0}$.

Definition 3.6.3. *If $p, q \in \mathbb{P}_{E_0}$, then let $p \leq^n_{\mathbb{P}_{E_0}} q$ be defined in the same way as $p \leq^n_{\mathbb{S}} q$, when p and q are considered as conditions in Sacks forcing $\mathbb{S}$.*

Let $\leq^n_{\widehat{\mathbb{P}}^2_{E_0}}$ be the coordinate-wise ordering using $\leq^n_{\mathbb{P}_{E_0}}$.

A sequence $\langle p_n : n \in \omega\rangle$ of conditions of $\mathbb{P}_{E_0}$ is a fusion sequence if and only if $p_{n+1} \leq^n_{\mathbb{P}_{E_0}} p_n$ for all $n \in \omega$. Similarly, a sequence $\langle (p_n, q_n) : n \in \omega\rangle$ of conditions in $\widehat{\mathbb{P}}^2_{E_0}$ is a fusion sequence if and only if $(p_{n+1}, q_{n+1}) \leq^n_{\widehat{\mathbb{P}}^2_{E_0}} (p_n, q_n)$ for all $n \in \omega$.

Suppose $p \in \mathbb{P}_{E_0}$. Let $n \in \omega$ with $n > 0$. Let $u, v \in {}^n2$ with $u(n-1) \neq v(n-1)$. Suppose $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$ with the property that $p' \leq_{\mathbb{P}_{E_0}} \Xi(p, u)$ and $q' \leq_{\mathbb{P}_{E_0}} \Xi(p, v)$. Let $A = \{s \in {}^n2 : s(n-1) = u(n-1)\}$ and $B = \{s \in {}^n2 : s(n-1) = v(n-1)\}$. Then define $\mathsf{prune}^{(u,v)}_{(p',q')}(p) \in \mathbb{P}_{E_0}$ by

$$\mathsf{prune}^{(u,v)}_{(p',q')}(p) = \bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(p,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(p,t)}}(q')$$

Suppose $(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$. Let $n \in \omega$, $n > 0$, and $u, v \in {}^n2$ so that $u(n-1) \neq v(n-1)$. Let $(p', q') \leq_{\widehat{\mathbb{P}}^n_{E_0}} (p, q)$ so that $p' \leq_{\mathbb{P}_{E_0}} \Xi(p, u)$ and $q' \leq_{\mathbb{P}_{E_0}} \Xi(q, v)$. Define $\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q)$ by

$$\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q) = \left(\mathsf{prune}^{(u,v)}_{(p',\mathsf{switch}_{s^p}(q'))}(p), \mathsf{prune}^{(u,v)}_{(\mathsf{switch}_{s^q}(p'),q')}(q)\right)$$

Perhaps more concretely: Let $A = \{s \in {}^n2 : s(n-1) = u(n-1)\}$ and $B = \{s \in {}^n2 : s(n-1) = v(n-1)\}$.

$$\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q) = \left(\bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(p,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(p,t)}}(q'), \bigcup_{t \in A} \mathsf{switch}_{s^{\Xi(q,t)}}(p') \cup \bigcup_{t \in B} \mathsf{switch}_{s^{\Xi(q,t)}}(q')\right)$$

Fact 3.6.4. *Suppose (p, q), (p', q'), n, u, and v are as in Definition 3.6.3. Then $\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$ and if $\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q) = (x, y)$, then $s^x = s^p$ and $s^y = s^q$.*

If $|u| = n$, then $\mathsf{2prune}^{(u,v)}_{(p',q')}(p, q) \leq^n_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$.

If $\langle p_n : n \in \omega\rangle$ is a fusion sequence of conditions in $\mathbb{P}_{E_0}$, then $p_\omega = \bigcap_{n \in \omega} p_n$ is a condition in $\mathbb{P}_{E_0}$ and is called the fusion of the fusion sequence.

Similarly, if $\langle (p_n, q_n) : n \in \omega\rangle$ is a fusion sequence of conditions in $\widehat{\mathbb{P}}^2_{E_0}$, then $(p_\omega, q_\omega) = (\bigcap_{n \in \omega} p_n, \bigcap_{n \in \omega} q_n)$ is a condition in $\widehat{\mathbb{P}}^2_{E_0}$ and is called the fusion of the fusion sequence.

Fact 3.6.5. *([14] Proposition 7.6) $\widehat{\mathbb{P}}^2_{E_0}$ is a proper forcing. In fact, for any countable model M and $(p, q) \in (\widehat{\mathbb{P}}^2_{E_0})^M$, there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ which is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition and $[p'] \times_{E_0} [q']$ consists of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M.*

Moreover, if τ is a $\widehat{\mathbb{P}}^2_{E_0}$-name in M such that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in {}^\omega2$, then one can even find (p', q') with the above properties and so that either

(i) there is some $z \in {}^\omega2 \cap M$ with $z \; E_0 \; \tilde{0}$ so that $(p', q') \Vdash_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{z}$

or

(ii) $(p', q') \Vdash_{\widehat{\mathbb{P}}^2_{E_0}} \neg(\tau \; E_0 \; \tilde{0})$.

Proof. Let $(D_n : n \in \omega)$ be a decreasing sequence of dense open subsets of $\widehat{\mathbb{P}}^2_{E_0}$ in M with the property that if D is a dense open subset of $\widehat{\mathbb{P}}^2_{E_0}$ in M, then there is some $k \in \omega$ so that $D_k \subseteq D$.

There are two cases: (Note that $\tilde{0}$ can be replaced by any real in M in the following argument and hence as well in the statement of the fact.)

(Case I) There is some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $(p', q') \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau\ E_0\ \tilde{0}$. Then there is some $z \in ({}^{\omega}2)^M$ and (p'', q'') so that $(p'', q'') \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{z}$. Since D_0 is dense open in $\widehat{\mathbb{P}}^2_{E_0}$, one may assume $(p'', q'') \in D_0$. Let $(p_0, q_0) = (p'', q'')$.

(Case II) $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \neg(\tau\ E_0\ \tilde{0})$. Let (p', q') be any condition below (p, q) so that $(p', q') \in D_0$. Let $(p_0, q_0) = (p', q')$.

In either case, we will construct a fusion sequence $((p_n, q_n) : n \in \omega)$ of conditions in $(\widehat{\mathbb{P}}^2_{E_0})^M$ with the following property:

(i) For all $n \in \omega$, $(u, v) \in {}^n2 \times {}^n2$ so that $u(n-1) \neq v(n-1)$, $(\Xi(p_n, u), \Xi(q_n, v)) \in D_n$.

Suppose this can be done to produce a fusion sequence $\langle (p_n, q_n) : n \in \omega \rangle$. Let (p', q') be the fusion of this fusion sequence. First, it will be shown that (i) implies that (p', q') is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition with the property that $[p'] \times_{E_0} [q']$ consists entirely of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M.

Let D be a dense open subset of $\widehat{\mathbb{P}}^2_{E_0}$ with $D \in M$. By the choice of $(D_n : n \in \omega)$, there is some $k \in \omega$ so that $D_k \subseteq D$. Let $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ be $\widehat{\mathbb{P}}^2_{E_0}$-generic over M containing (p', q'). Let $K = |\Lambda(p_k, 0^k)|$. Let E_K be the collection of $(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$ so that $|s^p| > K$ and there is some j with $K < j < |s^p|$ so that $s^p(j) \neq s^q(j)$. E_K is dense open. Since G is generic, $G \cap E_K \neq \emptyset$. As G is generic and $(p', q') \in G$, one may assume that there is some $(p'', q'') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p', q')$ with $(p'', q'') \in G \cap E_k$. So there is some $J > k$ and $u, v \in {}^J2$ with $u(J-1) \neq v(J-1)$ so that $(p'', q'') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(p', u), \Xi(q', v)) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(p_J, u), \Xi(q_J, v))$. Since $(p'', q'') \in G$ and G is a filter, $(\Xi(p_J, u), \Xi(q_J, v)) \in G$. However $(\Xi(p_J, u), \Xi(q_J, v)) \in D_J \subseteq D_k$ by (i). Note that $(\Xi(p_J, u), \Xi(q_J, v)) \in M$. Since G was an arbitrary generic containing (p', q'), it has been shown that $(p', q') \Vdash^V_{\widehat{\mathbb{P}}^2_{E_0}} \dot{G} \cap \check{M} \cap \check{D} \neq \emptyset$. (p', q') is a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition. $\widehat{\mathbb{P}}^2_{E_0}$ is proper.

Now suppose $(a, b) \in [p'] \times_{E_0} [q']$. Let $G_{(a,b)} = \{(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M : (a, b) \in [p] \times [q]\}$. Let D be a dense open set. There is some k so that $D_k \subseteq D$. Since $\neg(a\ E_0\ b)$, there is some $j > k$ and some $u, v \in {}^j2$ with $u(j-1) \neq v(j-1)$ so that $\Lambda(p', u) \subseteq a$ and $\Lambda(q', v) \subseteq b$. Then

$(a, b) \in [\Xi(p_j, u)] \times_{E_0} [\Xi(q_j, v)]$. Therefore $(\Xi(p_j, u), \Xi(q_j, v)) \in G_{(a,b)} \cap D_k \subseteq G_{(a,b)} \cap D$. $G_{(a,b)}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M. (a, b) is a $\widehat{\mathbb{P}}^2_{E_0}$-generic pair of reals over M.

It is clear using the forcing theorems that if Case I holds, then statement (i) of the fact holds and if Case II holds, then statement (ii) of the fact holds.

Now it remains to construct the fusion sequence:

(p_0, q_0) is already given depending on the case. The rest of the construction is the same for both cases.

Suppose (p_n, q_n) have been constructed with the desired properties. For some $J \in \omega$, let (u_0, v_0), ..., (u_{J-1}, v_{J-1}) enumerate all $(u, v) \in {}^n2 \times {}^n2$ with $u(n-1) \neq v(n-1)$.

We will construct sequence of conditions in $\widehat{\mathbb{P}}^2_{E_0}$, (x_{-1}, y_{-1}), ..., (x_{J-1}, y_{J-1}):

Let $(x_{-1}, y_{-1}) = (p_n, q_n)$.

Suppose one has (x_k, y_k) for $k < J - 1$.

Since D_{n+1} is dense open, find some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (\Xi(x_k, u_k), \Xi(y_k, v_k))$ so that $(p', q') \in D_{n+1}$. Let $(x_{k+1}, y_{k+1}) = \mathsf{2prune}^{(u_k,v_k)}_{(p',q')}(x_k, y_k)$. ◻

Lemma 3.6.6. *Let $f : [{}^\omega2]^2_{E_0} \to {}^\omega2$. Suppose there is a countable model M of some sufficiently large fragment of* ZF*, $(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M$, and $\tau \in M^{\widehat{\mathbb{P}}^2_{E_0}}$ so that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in {}^\omega2$ and whenever $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M with $(p, q) \in G$, then $\tau[G] = f(x^0_{\text{gen}}[G], x^1_{\text{gen}}[G])$. Then there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $[f[[p'] \times_{E_0} [q']]]_{E_0} \neq {}^\omega2$.*

Proof. Let $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$ be given by Fact 3.6.5. Then exactly one of the two happens:

(i) For all $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ which are generic over M, $M[G] \models \tau[G]\ E_0\ \tilde{0}$. By absoluteness, $\tau[G]\ E_0\ \tilde{0}$.

(ii) For all $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ which are generic over M, $M[G] \models \neg(\tau[G]\ E_0\ \tilde{0})$. By absoluteness, $\neg(\tau[G]\ E_0\ \tilde{0})$.

Let $(a, b) \in [p'] \times_{E_0} [q']$. By Fact 3.6.5, there is some $\widehat{\mathbb{P}}^2_{E_0}$-generic filter $G_{(a,b)}$ so that $x^0_{\text{gen}}[G_{(a,b)}] = a$ and $x^1_{\text{gen}}[G_{(a,b)}] = b$.

If (i) holds, then $f(a, b) = f(x^0_{\text{gen}}[G_{(a,b)}], x^1_{\text{gen}}[G_{(a,b)}]) = \tau[G_{(a,b)}]$ which is E_0 related to $\tilde{0}$. So $[f[[p'] \times_{E_0} [q']]]_{E_0} = [\tilde{0}]_{E_0} \neq {}^\omega2$.

If (ii) holds, then $f(a, b) = f(x^0_{\text{gen}}[G_{(a,b)}], x^1_{\text{gen}}[G_{(a,b)}]) = \tau[G_{(a,b)}]$, but $\neg(\tau[G_{(a,b)}]\ E_0\ \tilde{0})$. So $\tilde{0} \notin [f[[p'] \times_{E_0} [q']]]_{E_0}$. ◻

Theorem 3.6.7. *(Holshouser-Jackson)* (ZF + DC + AD) ${}^\omega 2/E_0$ *has the* 2*-Jónsson property.*

Proof. Let $F : [{}^\omega 2/E_0]^2_= \to {}^\omega 2/E_0$. Define the relation $R \subseteq {}^\omega 2 \times {}^\omega 2 \times {}^\omega 2$ by

$$R(x, y, z) \Leftrightarrow z \in F([x]_{E_0}, [y]_{E_0})$$

AD can prove comeager uniformization (Fact 3.2.22): There is a comeager set $C \subseteq [{}^\omega 2]^2_{E_0}$ and a continuous function $f : C \to {}^\omega 2$ which uniformizes R on C. Since E_0 has the 2-Mycielski property, let p be an E_0-tree so that $[[p]]^2_{E_0} \subseteq C$. So $f \upharpoonright [[p]]^2_{E_0}$ is a continuous function. By a similar argument as in Fact 3.3.4, one can find a name τ satisfying Lemma 3.6.6 using the condition $(p, p) \in \widehat{\mathbb{P}}^2_{E_0}$. Then Lemma 3.6.6 gives some $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, p)$ so that $[f[[p'] \times_{E_0} [q']]]_{E_0}$ is either $[\tilde{0}]_{E_0}$ or does not contain $\tilde{0}$. Since $(p', q') \in \widehat{\mathbb{P}}^2_{E_0}$, $[p']$ and $[q']$ have the same E_0-saturation. Let $A = [[p']]_{E_0} = [[q']]_{E_0}$. Note that $E_0 \equiv_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$. Let $B = A/E_0$. There is a bijection between B and ${}^\omega 2/E_0$. Moreover, $F[[B]^2_=]$ is either the singleton $\{[\tilde{0}]_{E_0}\}$ or is missing the element $[\tilde{0}]_{E_0}$. In either case, $F[[B]^2_=] \neq {}^\omega 2/E_0$. ◻

3.7 Partition Properties of ${}^\omega 2/E_0$ in Dimension 2

Definition 3.7.1. *Let X and Y be sets. Let $n \in \omega$. Denote $X \to (X)^n_Y$ to mean that for any function $f : \mathscr{P}^n(X) \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[\mathscr{P}^n(Z)]| = 1$.*

Denote $X \mapsto (X)^n_Y$ to mean that for any function $f : [X]^n_= \to Y$, there is some $Z \subseteq X$ with $Z \approx X$ and $|f[[Z]^n_=]| = 1$.

Fact 3.7.2. *(Galvin) Assuming* ZF *and all sets of reals have the Baire property, ${}^\omega 2 \to ({}^\omega 2)^2_n$ for all $n \in \omega$.*

Note that ${}^\omega 2 \mapsto ({}^\omega 2)^2_2$ is not true: If $x, y \in {}^\omega 2$ and $x \neq y$, then define $d(x, y) = \min\{n : x(n) \neq y(n)\}$. Define $f : [{}^\omega 2]^2_= \to 2$ by $f(x, y) = x(d(x, y))$. Note that $f(x, y) \neq f(y, x)$. It is impossible to find a homogeneous set for this coloring of $[{}^\omega 2]^2_=$.

However, under AD, ${}^\omega 2/E_0 \mapsto ({}^\omega 2/E_0)^2_n$ does hold:

Lemma 3.7.3. *Let $n > 1$. Let $F : [{}^\omega 2]^2_{E_0} \to n$ be a function. Suppose there is a countable model M of some sufficiently large fragment of* ZF*, $(p, q) \in \widehat{\mathbb{P}}^2_{E_0} \cap M$, and τ is a $\widehat{\mathbb{P}}^2_{E_0}$-name in M so that $(p, q) \Vdash^M_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in \check{n}$ and whenever $G \subseteq \widehat{\mathbb{P}}^2_{E_0}$ is $\widehat{\mathbb{P}}^2_{E_0}$-generic over M with $(p, q) \in G$, $\tau[G] = F(x^0_{\text{gen}}[G], x^1_{\text{gen}}[G])$. Then there is a $(p', q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p, q)$ so that $|F[[p'] \times_{E_0} [q']]| = 1$.*

Proof. Since $(p,q) \Vdash^{M}_{\widehat{\mathbb{P}}^2_{E_0}} \tau \in \hat{n}$, there is some $(r,s) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p,q)$ and some $k \in n$ so that $(r,s) \Vdash^{M}_{\widehat{\mathbb{P}}^2_{E_0}} \tau = \check{k}$. Using Fact 3.6.5, let $(p',q') \leq_{\widehat{\mathbb{P}}^2_{E_0}} (r,s)$ be a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition so that $[p'] \times_{E_0} [q']$ consists of pairs of reals which are $\widehat{\mathbb{P}}^2_{E_0}$-generic over M. Using the forcing theorem and the assumptions, (p',q') works. □

Theorem 3.7.4. (ZF + DC + AD) ${}^{\omega}2/E_0 \mapsto ({}^{\omega}2/E_0)^2_n$ *for all* $n \in \omega$.

Proof. Let $f : [{}^{\omega}2/E_0]^2_{=} \to n$. Define a relation $R \subseteq {}^{\omega}2 \times {}^{\omega}2 \times n$ by

$$R(x,y,k) \Leftrightarrow k = f([x]_{E_0}, [y]_{E_0}).$$

AD can prove comeager uniformization (Fact 3.2.22) so there is some comeager $C \subseteq [{}^{\omega}2]^2_{E_0}$ and a continuous function $F : C \to n$ which uniformizes R on C. Since E_0 has the 2-Mycielski property, let p be an E_0-tree so that $[[p]]^2_{E_0} \subseteq C$. $F \restriction [[p]]^2_{E_0}$ is a continuous function. As before, one can find a $\widehat{\mathbb{P}}^2_{E_0}$-name τ satisfying Lemma 3.7.3 using the the condition (p,p). Then using Lemma 3.7.3, there is a $(r,s) \leq_{\widehat{\mathbb{P}}^2_{E_0}} (p,p)$ so that $|F[[r] \times_{E_0} [s]]| = 1$. Let k be the unique element in this set.

Let $A = [r]_{E_0} = [s]_{E_0}$. $A/E_0 \approx {}^{\omega}2/E_0$. Suppose $x, y \in A/E_0$ and $x \neq y$. There is some $a, b \in A$ with $a \in [r]$, $b \in [s]$, $a \in x$, and $b \in y$. $F(a,b) = k$. Therefore, $R(a,b,k)$. By definition, $f(x,y) = k$. So $|f[[A]^2_{=}]| = 1$. □

Remark 3.7.5. Before, we mentioned that ${}^{\omega}2 \mapsto ({}^{\omega}2)^2_2$ is not true. Using the notation from the above proof: Observe that the function F produced in the above proof is E_0-invariant in the sense that if $x \; E_0 \; x'$ and $y \; E_0 \; y'$ then $F(x,y) = F(x',y')$. Also since $(r,s) \in \widehat{\mathbb{P}}^2_{E_0}$, if $a \in A$, then there is some $a' \in [r]$ and $a'' \in [s]$ with $a \; E_0 \; a' \; E_0 \; a''$. These two facts are essential in proving ${}^{\omega}2/E_0 \mapsto ({}^{\omega}2/E_0)^2_n$.

Later it will be shown that the partition relation for ${}^{\omega}2/E_0$ will fail in higher dimension. The counterexample is closely connected to the failure of the 3-Jónsson property.

3.8 E_0 Does Not Have the 3-Mycielski Property

An earlier section mentioned that Holshouser and Jackson proved E_0 has the 2-Mycielski property and ${}^{\omega}2/E_0$ has the 2-Jónsson property. The next few sections will show that dimension 2 is the best possible for these combinatorial properties. This section will show the failure of the 3-Mycielski property and the weak 3-Mycielski property for E_0.

Theorem 3.8.1. *Let* $D \subseteq {}^{3}({}^{\omega}2)$ *be defined by*

$$D = \{(x,y,z) \in {}^{3}({}^{\omega}2) : (\exists n)(x(n) \neq y(n) \wedge x(n) \neq z(n) \wedge y(n+1) \neq z(n+1))\}.$$

D is dense open in ${}^3({}^\omega 2)$.

If p is an E_0-tree with associated objects $\{s, v_n^i : i \in 2 \wedge n \in \omega\}$, φ, and Φ as in Definition 3.5.2, then

$$(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \notin D.$$

E_0 does not have the 3-Mycielski property.

Proof. Suppose $(x, y, z) \in D$. Then there is an $n \in \omega$ so that $x(n) \neq y(n)$, $x(n) \neq z(n)$, and $y(n+1) \neq z(n+1)$. Let $\sigma = x \upharpoonright (n+2)$, $\tau = y \upharpoonright (n+2)$, and $\rho = z \upharpoonright (n+2)$. Then $N_{\sigma,\tau,\rho} \subseteq D$. D is open.

Suppose $\sigma, \tau, \rho \in {}^{<\omega}2$ and $|\sigma| = |\tau| = |\rho| = k$. Let $\sigma' = \sigma\hat{}00$, $\tau' = \tau\hat{}10$, and $\rho' = \rho\hat{}11$. Then $N_{\sigma',\tau',\rho'} \subseteq N_{\sigma,\tau,\rho}$ and $N_{\sigma',\tau',\rho'} \subseteq D$. D is dense open.

Let $L_{-1} = |s|$. For $n \in \omega$, let $L_n = |s| + \sum_{k \leq n} |v_k^0|$. Note that if $x(n) = y(n)$, then for all $L_{n-1} \leq k < L_n$, $\Phi(x)(k) = \Phi(y)(k)$. If $x(n) \neq y(n)$, then $\Phi(x)(L_{n-1}) \neq \Phi(y)(L_{n-1})$, and there may be other $L_{n-1} \leq k < L_n$ so that $\Phi(x)(k) \neq \Phi(y)(k)$.

Suppose $\Phi(\widetilde{010})(n) \neq \Phi(\widetilde{110})(n)$ and $\Phi(\widetilde{010})(n) \neq \Phi(\tilde{1})(n)$. Then there exists some k so that $L_{3k-1} \leq n < L_{3k}$. If $n \neq L_{3k} - 1$, then $n + 1 < L_{3k}$. Since $\widetilde{110}(3k) = 1 = \tilde{1}(3k)$, $\Phi(\widetilde{110})(n+1) = \Phi(\tilde{1})(n+1)$. Hence if $n \neq L_{3k} - 1$, then n cannot be used to witness that $(\Phi(\widetilde{101}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \in D$. Suppose $n = L_{3k} - 1$. Since $\widetilde{110}(3k+1) = 1 = \tilde{1}(3k+1)$, one has that $\Phi(\widetilde{110})(n+1) = \Phi(\widetilde{110})(L_{3k}) = \Phi(\tilde{1})(L_{3k}) = \Phi(\tilde{1})(n+1)$. If $n = L_{3k} - 1$, then n does not witness membership in D. Hence $(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \notin D$.

Since $\neg(\widetilde{010} \; E_0 \; \widetilde{110})$, $\neg(\widetilde{010} \; E_0 \; \tilde{1})$, and $\neg(\widetilde{110} \; E_0 \; \tilde{1})$, $(\Phi(\widetilde{010}), \Phi(\widetilde{110}), \Phi(\tilde{1})) \in [[p]]_{E_0}^3$. Hence $[[p]]_{E_0}^3 \not\subseteq D$.

Suppose B is $\mathbf{\Delta}_1^1$ so that $E_0 \upharpoonright B \equiv_{\mathbf{\Delta}_1^1} E_0$. By Fact 3.5.3, there is some E_0-tree p so that $[p] \subseteq B$. By the above, $[B]_{E_0}^3 \not\subseteq D$. E_0 does not have the 3-Mycielski property. □

The 3-Mycielski property asks for a single $\mathbf{\Delta}_1^1$ set A with $E_0 \leq_{\mathbf{\Delta}_1^1} E_0 \upharpoonright A$ so that $[A]_{E_0}^3 = A \times_{E_0} A \times_{E_0} A$ is contained inside a comeager set. If one is interested in combinatorial properties of the quotient ${}^\omega 2/E_0$, such as the Jónsson property, then one is only concerned with three sets A, B, and C with the same E_0-saturation. With this consideration, the 3-Mycielski property seems unnecessarily restrictive. The weak 3-Mycielski property was defined to remove this demand.

One other curiosity of the 3-Mycielski property is that Theorem 3.8.1 allows a (topologically) dense open subset of ${}^3({}^\omega 2)$ to be a counterexample to the 3-Mycielski property. Let $D \subseteq {}^3({}^\omega 2)$ be any dense open set. There are three strings σ, τ, and γ of the same length so that $N_{\sigma,\gamma,\tau} \subseteq D$. Let p, q, r be any three perfect E_0-trees so that $s^p = \sigma$, $s^q = \tau$, $s^r = \gamma$, and for all $n \in \omega$ and $i \in 2$, $v_n^{i,p} = v_n^{i,q} = v_n^{i,r}$. Then $[p] \times_{E_0} [q] \times_{E_0} [r] \subseteq D$. Also $[[p]]_{E_0} = [[q]]_{E_0} = [[r]]_{E_0}$. So no dense open set can be a counterexample to the weak 3-Mycielski property.

Using the more informative structure theorem for E_0 proved above and the argument in Theorem 3.8.1, a comeager subset of ${}^3({}^\omega 2)$ is used to show E_0 does not have the weak 3-Mycielski property.

Theorem 3.8.2. *For each $k \in \omega$, let*

$$D_k = \{(x, y, z) \in {}^3({}^\omega 2) : (\exists n \geq k)(x(n) \neq y(n) \wedge x(n) \neq z(n) \wedge y(n+1) \neq z(n+1))\}.$$

Each D_k is dense open.

Let $C = \bigcap_{n\in\omega} D_n$. C is comeager.

For any $\mathbf{\Delta}^1_1$ sets A_0, A_1, and A_2 such that

(I) $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_0$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_1$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_2$

(II) $[A_0]_{E_0} = [A_1]_{E_0} = [A_2]_{E_0}$,

$A_0 \times_{E_0} A_1 \times_{E_0} A_2 \nsubseteq C$.

C does not have the weak 3-Mycielski property.

Proof. Let A_0, A_1, A_2 be any three $\mathbf{\Delta}^1_1$ sets so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_i$ and have the same E_0-saturation. By Theorem 3.5.7, there are E_0-trees, p, q, and r so that

(i) $|s^p| = |s^q| = |s^r| = k$

(ii) $v_n^{i,p} = v_n^{i,q} = v_n^{i,r}$ for all $n \in \omega$ and $i \in 2$

(iii) $[p] \subseteq A_0$, $[q] \subseteq A_1$, and $[r] \subseteq A_2$.

Note that the only differences among the three E_0-trees occurs in the stems. Hence by the same argument as in Theorem 3.8.1, $[p] \times_{E_0} [q] \times_{E_0} [r] \nsubseteq D_k$. Hence $A_0 \times_{E_0} A_1 \times_{E_0} A_2 \nsubseteq D_k$. So $A_0 \times_{E_0} A_1 \times_{E_0} A_2 \nsubseteq C$. □

3.9 Surjectivity and Continuity Aspects of E_0

As we have seen, based on Holshouser and Jackson's proof of the Jónsson property for ${}^\omega 2$, the Mycielski property primarily shows that an arbitrary function f on ${}^n({}^\omega 2)$ could be continuous on

$[[p]]^n_=$ for some perfect tree p. Using the continuity of $f \restriction [[p]]^n_=$, they show that there is some perfect subtree $q \subseteq p$ so that $f[[[q]]^n_=] \neq {}^\omega 2$.

As the previous section shows that E_0 does not have the 3-Mycielski property, it is natural to ask if by some other means it is possible to find for any $f : {}^3({}^\omega 2) \to {}^\omega 2$, some $\mathbf{\Delta}^1_1$ set A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction A$ and $f \restriction [A]^3_{E_0}$ is continuous. Also if the function $f \restriction [A]^3_{E_0}$ is continuous, is it possible to find some $\mathbf{\Delta}^1_1$ $B \subseteq A$ with $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \restriction B$ so that $f[[B]^3_{E_0}]$ does not meet all E_0 equivalence classes? This section will provide an example to show both of these properties can fail. This example will also be modified in the next section to show the failure of the 3-Jónsson property for E_0.

Fact 3.9.1. *Let $A = \{x \in {}^\omega 3 : (\forall n)(x(n) \neq x(n+1))\}$. There is a continuous function $P : [{}^\omega 2]^3_{E_0} \to A$ so that for any E_0-tree p, $P[[[p]]^3_{E_0}] = A$. Moreover, if p, q, and r are E_0-trees so that $|s^p| = |s^q| = |s^r|$ and for all $i \in 2$ and $n \in \omega$, $v^{i,p}_n = v^{i,q}_n = v^{i,r}_n$, then $P[[[p]] \times_{E_0} [[q]] \times_{E_0} [[r]]]$ meets all E_0-classes of A, where the latter E_0 is defined on ${}^\omega 3$.*

Proof. Let $(x, y, z) \in [{}^\omega 2]^3_{E_0}$. Let L_0 be the largest $N \in \omega$ so that $x \restriction N = y \restriction N = z \restriction N$. Define

$$a_0 = \begin{cases} 0 & x(L_0) = y(L_0) \\ 1 & x(L_0) = z(L_0) \\ 2 & y(L_0) = z(L_0) \end{cases}.$$

Suppose one has defined L_n and a_n. Let L_{n+1} be the smallest $N > L_n$ so that $x(N) \neq y(N)$ if $a_n = 0$, $x(N) \neq z(N)$ if $a_n = 1$, and $y(N) \neq z(N)$ if $a_n = 2$. Define

$$a_{n+1} = \begin{cases} 0 & x(L_{n+1}) = y(L_{n+1}) \\ 1 & x(L_{n+1}) = z(L_{n+1}) \\ 2 & y(L_{n+1}) = z(L_{n+1}) \end{cases}.$$

Define $P(x, y, z) \in A$ by $P(x, y, z)(n) = a_n$. P is continuous.

Now let p be an E_0 tree. Let s and v^i_n, for $n \in \omega$ and $i \in 2$, be associated with the E_0-tree p. Let $\Phi : {}^\omega 2 \to [p]$ be the canonical homeomorphism.

Let $v \in A$. Let

$$(a_i, b_i) = \begin{cases} (0,1) & v(i) = 0 \\ (1,0) & v(i) = 1 \\ (1,1) & v(i) = 2 \end{cases}.$$

Let $a, b \in {}^\omega 2$ be defined by $a(n) = a_n$ and $b(n) = b_n$. Then $(\Phi(\tilde{0}), \Phi(a), \Phi(b)) \in [[p]]^3_{E_0}$ and $P((\Phi(\tilde{0}), \Phi(a), \Phi(b))) = v$. Hence $P[[[p]]^3_{E_0}] = A$. A similar procedure proves the second statement after noting the three E_0-trees are the same after their stems. $\square$

Theorem 3.9.2. *There is a continuous $Q : [{}^\omega 2]^3_{E_0} \to {}^\omega 2$ so that for any E_0-tree p, $Q[[[p]]^3_{E_0}] = {}^\omega 2$.*

Proof. Let $t_0 = 00$, $t_1 = 01$, and $t_2 = 10$.

Let $Q' : {}^\omega 3 \to {}^\omega 2$ be defined by $Q'(x) = t_{x(0)} \hat{\ } t_{x(1)} \hat{\ } \ldots$ Q' is a continuous injection.

$Q'[A]$ is a perfect subset of ${}^\omega 2$. $Q'[A] = [T]$ for some perfect tree T. Let $Q'' : Q'[A] \to {}^\omega 2$ be the continuous bijection naturally induced by T.

Let $Q = Q'' \circ Q' \circ P$. Q has the desired property. $\square$

Corollary 3.9.3. *There is a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that on any Σ^1_1 set A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$, $K[[A]^3_{E_0}] = {}^\omega 2$. Moreover, for any Σ^1_1 sets A_0, A_1, and A_2 so that for all $i < 3$, $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A_i$ and $[A_0]_{E_0} = [A_1]_{E_0} = [A_2]_{E_0}$, $[K[\prod^{E_0}_{i<3} A_i]]_{E_0} = {}^\omega 2$.*

Fact 3.9.4. *There is a $\mathbf{\Delta}^1_1$ function $P' : {}^3({}^\omega 2) \to A$ so that for all E_0-tree p, $P' \upharpoonright [[p]]^3_{E_0}$ is not continuous.*

Proof. Let P be the function from Fact 3.9.1. Define P' by

$$P'(x, y, z) = \begin{cases} \widetilde{01} & (x, y, z) \notin [{}^\omega 2]^3_{E_0} \\ P(x, y, z) & (\forall k)(\exists n > k)(P(x, y, z)(n) = 2) \\ \widetilde{01} & (\exists k)(\forall n > k)(P(x, y, z)(n) < 2) \end{cases}.$$

Suppose P' is continuous on some $[[p]]^3_{E_0}$. Let $s \in {}^{<\omega}2$ be so that for all $n < |s| - 1$, $s(n) \neq s(n+1)$ and there exists some $n < |s|$ so that $s(n) = 2$. By continuity, $(P')^{-1}[N_s] \cap [[p]]^3_{E_0}$ is open in $[[p]]^3_{E_0}$. There is some $u, v, w \in {}^{<\omega}2$ so that $|u| = |v| = |w|$ and $N_{\varphi(u),\varphi(v),\varphi(w)} \cap [[p]]^3_{E_0} \subseteq (P')^{-1}[N_s] \cap [[p]]^3_{E_0}$.

Let $x = u \hat{\ } \tilde{0}$, $y = v \hat{\ } \widetilde{01}$, and $z = w \hat{\ } \widetilde{10}$. Then $(\Phi(x), \Phi(y), \Phi(z)) \in (P')^{-1}[N_s] \cap [[p]]^3_{E_0}$. However, there is a k so that for all $n > k$, $P(\Phi(x), \Phi(y), \Phi(z))(n) < 2$. Therefore, $P'(\Phi(x), \Phi(y), \Phi(z)) = \widetilde{01}$. However, $\widetilde{01} \notin N_s$ since there is some n so that $s(n) = 2$. $\square$

Theorem 3.9.5. *There is a $\mathbf{\Delta}^1_1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that for all Σ^1_1 sets A so that $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright A$, $K \upharpoonright [A]^3_{E_0}$ is not continuous.*

Proof. Let Q' be the function from the proof of Theorem 3.9.2. Let P' be the function from Fact 3.9.4. $K = Q' \circ P'$ works. $\square$

As a consequence, one has another proof of the failure of the 3-Mycielski property for E_0.

Corollary 3.9.6. *E_0 does not have the 3-Mycielski property.*

Proof. Let $C \subseteq {}^3({}^\omega 2)$ be any comeager set so that $K \upharpoonright C$ is a continuous function, where K is from Fact 3.9.5. Then C witnesses the failure of the 3-Mycielski property for E_0. $\square$

3.10 ${}^\omega 2/E_0$ Does Not Have the 3-Jónsson Property

Definition 3.10.1. *For $n \in \omega$, let E^n_{tail} be the equivalence relation defined on ${}^\omega n$ by $x\ E^n_{\text{tail}}\ y$ if and only if $(\exists r)(\exists s)(\forall a)(x(r + a) = y(s + a))$.*

Fact 3.10.2. *The function $P : [{}^\omega 2]^3_= \to A$ from Fact 3.9.1 is E_0 to E^3_{tail} invariant, which means for all $(x, y, z), (x', y', z') \in [{}^\omega 2]^3_{E_0}$ such that $x\ E_0\ x'$, $y\ E_0\ y'$, and $z\ E_0\ z'$, $P(x, y, z)\ E^3_{\text{tail}}\ P(x', y', z')$.*

Proof. Using the notation from Fact 3.9.1, let $(L_k : k \in \omega)$ and $(a_k : k \in \omega)$ be the L and a sequences for (x, y, z) and let $(J_k : k \in \omega)$ and $(b_k : k \in \omega)$ be the L and a sequences for (x', y', z').

Let $M \in \omega$ be so that $x_{\geq M} = x'_{\geq M}$, $y_{\geq M} = y'_{\geq M}$, and $z_{\geq M} = z'_{\geq M}$. Let $r \in \omega$ be largest so that $L_r < M$, and let $s \in \omega$ be largest so that $J_s < M$.

(Case I) Suppose $L_{r+1} = J_{s+1}$. Then $a_{r+1} = b_{s+1}$. Since for all $n \geq 1$, $L_{r+n}, J_{s+n} \geq M$, $L_{r+n} = J_{s+n}$ and $a_{r+n} = b_{s+n}$. Hence $P(x, y, z)\ E^3_{\text{tail}}\ P(x', y', z')$.

(Case II) Suppose $a_r = b_s$. Then one must have for all $n \in \omega$, $L_{r+n} = J_{s+n}$ and $a_{r+n} = b_{s+n}$. Hence $P(x, y, z)\ E^3_{\text{tail}}\ P(x', y', z')$.

(Case III) Suppose $a_r \neq b_s$ and $L_{r+1} \neq J_{s+1}$. Without loss of generality, $L_{r+1} < J_{s+1}$, $a_r = 0$, and $b_s = 2$. This implies that for all $M \leq k < L_{r+1}$, $x(k) = x'(k) = y(k) = y'(k) = z(k) = z'(k)$. However, $x(L_{r+1}) \neq y(L_{r+1})$ and $y(L_{r+1}) = y'(L_{r+1}) = z'(L_{r+1}) = z(L_{r+1})$ because $L_{r+1} < J_{s+1}$. Hence $a_{r+1} = 2$. For any k so that $L_{r+1} \leq k < J_{s+1}$, $y(k) = y'(k) = z(k) = z'(k)$. However, $y'(J_{s+1}) \neq z'(J_{s+1})$ hence $y(J_{s+1}) \neq z(J_{s+1})$ and J_{s+1} is the smallest $N > L_{r+1}$ for which this happens. Hence $L_{r+2} = J_{s+1}$. Also $a_{r+2} = b_{s+1}$. Hence for all $n \in \omega$, $a_{(r+2)+n} = b_{(s+1)+n}$. This implies $P(x, y, z)\ E^3_{\text{tail}}\ P(x', y', z')$. $\square$

Fact 3.10.3. *$E^2_{\text{tail}} \leq_{\Delta^1_1} E^3_{\text{tail}} \upharpoonright A$. Hence $E_0 \equiv_{\Delta^1_1} E^3_{\text{tail}} \upharpoonright A$.*

Proof. Let $\Phi : {}^{\omega}2 \to A$ be defined by $\Phi(x) = x \oplus \tilde{2}$, where

$$(x \oplus y)(n) = \begin{cases} x(k) & n = 2k \\ y(k) & n = 2k+1 \end{cases}$$

Suppose $x \; E^2_{\text{tail}} \; y$. Then there are some $a, b \in \omega$ so that for all n, $x(a+n) = y(b+n)$. For all $n \in \omega$, $\Phi(x)(2a+n) = \Phi(y)(2b+n)$.

Suppose $\neg(x \; E^2_{\text{tail}} \; y)$. Let $a, b \in \omega$. Suppose a is even and b is odd. Then $\Phi(x)(a+0) \in 2$ but $\Phi(y)(b+0) = 2$. The same argument works if a is odd and b is even. Suppose a and b are both even. Let $a = 2a'$ and $b = 2b'$. Since $\neg(x \; E^2_{\text{tail}} \; y)$, there is some k so that $x(a'+k) \neq y(b'+k)$. Then $\Phi(x)(a+2k) = x(a'+k) \neq y(b'+k) = \Phi(y)(b+2k)$. Suppose a and b are both odd. $a = 2a'+1$ and $b = 2b'+1$. Since $\neg(x \; E^2_{\text{tail}} \; y)$, there is some k so that $x((a'+1)+k) \neq y((b'+1)+k)$. Hence $\Phi(x)(a+(1+2k)) \neq \Phi(y)(b+(1+2k))$. This shows $\neg(\Phi(x) \; E^3_{\text{tail}} \; \Phi(y))$.

Since $E^2_{\text{tail}} \equiv_{\mathbf{\Delta}^1_1} E_0$ and $E^3_{\text{tail}} \upharpoonright A \leq_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \equiv_{\mathbf{\Delta}^1_1} E_0$, one has $E_0 \equiv_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$. $\square$

Theorem 3.10.4. $(\mathsf{ZF} + \mathsf{AD})$ *${}^{\omega}2/E_0$ does not have the 3-Jónsson property.*

Proof. Let P be the function from Fact 3.10.2. Let $\bar{P} : [{}^{\omega}2/E_0]^3_{=} \to A/E^3_{\text{tail}}$ be defined by $\bar{P}(a, b, c) = d$ if and only if

$$(\forall x, y, z)((x \in a \wedge y \in b \wedge z \in c) \Rightarrow P(x, y, z) \in d).$$

By Fact 3.10.2, $\bar{P}$ is a well defined function. Since $E_0 \equiv_{\mathbf{\Delta}^1_1} E^3_{\text{tail}} \upharpoonright A$ by Fact 3.10.3, let $U : A/E^3_{\text{tail}} \to {}^{\omega}2/E_0$ be a bijection (given by Fact 3.2.5).

Let $F : [{}^{\omega}2/E_0]^3_{=} \to {}^{\omega}2/E_0$ be defined by $U \circ \bar{P}$.

Let $X \subseteq {}^{\omega}2/E_0$ be such that there is a bijection $B : {}^{\omega}2/E_0 \to X$. By Fact 3.2.22, AD implies B has a lift $B' : D \to \bigcup X$, where $D \subseteq {}^{\omega}2$ is some comeager set. Since B was a bijection, B' is a reduction of $E_0 \upharpoonright D$ to $E_0 \upharpoonright \bigcup X$. Using AD, there is a comeager set $C \subseteq D$ so that $B' \upharpoonright C : C \to \bigcup X$ is a continuous reduction of $E_0 \upharpoonright C$ to $E_0 \upharpoonright \bigcup X$. There is a continuous function witnessing $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright C$. By composition, there is a continuous reduction witnessing $E_0 \leq_{\mathbf{\Delta}^1_1} E_0 \upharpoonright B'[C]$. There is an E_0-tree p so that $[p] \subseteq B'[C] \subseteq \bigcup X$. By Fact 3.9.1, $P[[[p]]^3_{E_0}] = A$. This implies that $\bar{P}[[X]^3_{=}] = A/E^3_{\text{tail}}$. Since U is a bijection, $U[\bar{P}[[X]^3_{=}]] = F[[X]^3_{=}] = {}^{\omega}2/E_0$. F witnesses ${}^{\omega}2/E_0$ does not have the 3-Jónsson property. $\square$

As an earlier section mentions, since this paper is often concerned with sets without well-orderings, we defined the Jónsson property using sets of tuples $[A]^n_=$. The usual definition of the Jónsson property (of cardinals) involve the n-elements subsets of A, $\mathscr{P}^n(A)$. This paper calls this the classical n-Jónsson property. With a slight modification, one can also obtain the failure of the classical 3-Jónsson property for ${}^\omega 2/E_0$.

Definition 3.10.5. *Let S_3 be the permutation group on $3 = \{0, 1, 2\}$. S_3 acts on ${}^\omega 3$ in the natural way: if $p \in S_3$ and $x \in {}^\omega 3$, then $(p \cdot x)(n) = p(x(n))$.*

Let F be an equivalence relation on ${}^\omega 3$ defined by $x\ F\ y$ if and only if $(\exists p \in S_3)(p \cdot x\ E^3_{\text{tail}}\ y)$.

Fact 3.10.6. *Let $A = \{x \in {}^\omega 3 : (\forall n)(x(n) \neq x(n + 1))\}$. $F \upharpoonright A \equiv_{\boldsymbol{\Delta}^1_1} E_0$.*

Proof. Note that $E^3_{\text{tail}} \upharpoonright A$ is hyperfinite by Fact 3.10.3. Note that $E^3_{\text{tail}} \upharpoonright A \subseteq F \upharpoonright A$ and each $F \upharpoonright A$ equivalence class is a union of at most six $E^3_{\text{tail}} \upharpoonright A$ equivalence classes. By a result of Jackson, $F \upharpoonright A$ is hyperfinite. Hence, $F \upharpoonright A \leq_{\boldsymbol{\Delta}^1_1} E_0$.

Next a reduction $\Phi : {}^\omega 2 \to A$ will be produced witnessing $E^2_{\text{tail}} \leq_{\boldsymbol{\Delta}^1_1} F \upharpoonright A$:

$$\Phi(x) = x(0)\hat{\ }2012102\hat{\ }x(1)\hat{\ }2012102\hat{\ }x(2)\ldots$$

If $x\ E^2_{\text{tail}}\ y$, then $\Phi(x)\ F\ \Phi(y)$.

Suppose $\Phi(x)\ F\ \Phi(y)$. This means there is some $g \in S_3$ so that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$. Consider what happens for each $g \in S_3$: g will be presented in cycle notation.

$g = \text{id}$: It is clear that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$ implies that $x\ E^2_{\text{tail}}\ y$.

$g = (0, 1)$: Then a portion of $g \cdot \Phi(x)$ looks like

$$\ldots\hat{\ }g(x(i))\hat{\ }2102012\hat{\ }g(x(i + 1))\hat{\ }2102012\hat{\ }g(x(i + 2))\hat{\ }\ldots$$

$g = (0, 2)$:

$$\ldots\hat{\ }g(x(i))\hat{\ }0210120\hat{\ }g(x(i + 1))\hat{\ }0210120\hat{\ }g(x(i + 2))\hat{\ }\ldots$$

$g = (0, 1, 2)$:

$$\ldots\hat{\ }g(x(i))\hat{\ }0120210\hat{\ }g(x(i + 1))\hat{\ }0120210\hat{\ }g(x(i + 2))\hat{\ }\ldots$$

$g = (0, 2, 1)$:

$$\ldots\hat{\ }g(x(i))\hat{\ }1201021\hat{\ }g(x(i + 1))\hat{\ }1201021\hat{\ }g(x(i + 2))\hat{\ }\ldots$$

In all these cases, $\Phi(y)$ will contain a block of 2012102, but $g \cdot \Phi(x)$ cannot possibly contain such a block. So it is impossible that $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(y)$.

$g = (1, 2)$:

$$...\hat{}g(x(i))\hat{}1021201\hat{}g(x(i+1))\hat{}1021201\hat{}g(x(i+2))\hat{}...$$

The only way that some tail of $g \cdot \Phi(x)$ contains blocks of 2012102 is if $x\ E^2_{\text{tail}}\ \tilde{1}$. This however forces $g \cdot \Phi(x)\ E^3_{\text{tail}}\ \Phi(\tilde{1})$. This implies that both $x\ E^2_{\text{tail}}\ \tilde{1}$ and $y\ E^2_{\text{tail}}\ \tilde{1}$. $x\ E^2_{\text{tail}}\ y$.

This shows that Φ is a reduction of E^2_{tail} into $F \upharpoonright A$.

This completes the proof that $E_0 \equiv_{\mathbf{\Delta}^1_1} F \upharpoonright A$. □

Theorem 3.10.7. ($\mathsf{ZF} + \mathsf{AD}$) *${}^{\omega}2/E_0$ does not have the classical 3-Jónsson property.*

Proof. Let $P : [{}^{\omega}2]^3_{E_0} \to A$ be the function from Fact 3.9.1. Fact 3.10.2 shows that P is E_0 to E^3_{tail} invariant. Note that if $(x_0, x_1, x_2) \in [{}^{\omega}2]^3_{E_0}$ and $g \in S_3$, then there is some other $h \in S_3$ so that $P(x_{g(0)}, x_{g(1)}, x_{g(2)}) = h \cdot P(x_0, x_1, x_2)$.

Define a function $\Psi : \mathscr{P}^3({}^{\omega}2/E_0) \to A/F$ as follows: Let $D \in \mathscr{P}^3({}^{\omega}2/E_0)$. Choose any $(x_0, x_1, x_2) \in [{}^{\omega}2]^3_{E_0}$ so that $D = \{[x_0]_{E_0}, [x_1]_{E_0}, [x_2]_{E_0}\}$. Let $\Psi(A) = [P(x_0, x_1, x_2)]_F$.

By the above observations, Ψ is a well-defined surjection onto A/F. By Fact 3.10.6, there is a bijection $\Gamma : A/F \to {}^{\omega}2/E_0$. Let $\Phi = \Gamma \circ \Psi$. By an argument similar to Theorem 3.10.4, Φ witnesses ${}^{\omega}2/E_0$ does not have the classical 3-Jónsson property. □

3.11 Failure of Partition Properties of ${}^{\omega}2/E_0$ in Dimension Higher Than 2

This section will use the failure of the classical 3-Jónsson property to show that the classical partition property in dimension three fails for $\mathbb{R}/E_0$. Note that for any Y, the failure of ${}^{\omega}2/E_0 \to ({}^{\omega}2/E_0)^3_Y$ implies the failure of ${}^{\omega}2/E_0 \mapsto ({}^{\omega}2/E_0)^3_Y$.

Theorem 3.11.1. ($\mathsf{ZF} + \mathsf{AD}$) *For any set Y with at least two elements, ${}^{\omega}2/E_0 \to ({}^{\omega}2/E_0)^3_Y$ fails.*

In fact, if Y is a set so that there is a partition of ${}^{\omega}2/E_0$ by nonempty sets indexed by elements of Y, then there is map $f : \mathscr{P}^3({}^{\omega}2\ /E_0) \to Y$ with the property that for all $C \subseteq {}^{\omega}2/E_0$ with $C \approx {}^{\omega}2/E_0$, $f[\mathscr{P}^3(C)] \approx Y$.

Proof. Let $a, b \in Y$. Partition ${}^{\omega}2/E_0$ into two nonempty disjoint sets A and B. Let $\Lambda : {}^{\omega}2/E_0 \to Y$ be defined by

$$\Lambda(x) = \begin{cases} a & x \in A \\ b & x \in B \end{cases}$$

Let Φ be the classical 3-Jónsson function from the proof of Theorem 3.10.7.

Define $f : \mathscr{P}^3({}^\omega 2/E_0) \to Y$ by $f = \Lambda \circ \Phi$.

Suppose $C \subseteq {}^\omega 2/E_0$ and $C \approx {}^\omega 2/E_0$. Suppose $a_0 \in A$ and $b_0 \in B$. Since Φ is a classical 3-Jónsson map, there are some $R, S \in \mathscr{P}^3(C)$ so that $\Phi(R) = a_0$ and $\Phi(S) = b_0$. Since $f(R) = a$ and $f(S) = b$. $|f[\mathscr{P}^3(C)]| = 2$.

For the second statement, suppose $(A_y : y \in Y)$ is a partition of ${}^\omega 2/E_0$ into nonempty sets. Define $\Lambda : {}^\omega 2/E_0 \to Y$ by $\Lambda(x) = y$ if and only if $x \in A_y$. Let $f = \Lambda \circ \Phi$. This maps works by an argument like above. ☐

Given a set X and $n \in \omega$, one can define $d_X(n)$ to be the smallest element of ω, if it exists, such that for every k and every function $f : \mathscr{P}^n(X) \to k$, there is some $S \subseteq k$ with $|S| \leq d_X(n)$ and $A \subseteq X$ with $A \approx X$ so that $f[\mathscr{P}^n(A)] \subseteq S$. Say that $d_X(n)$ is infinite if no such integer exists.

[1] showed that assuming the appropriate sets have the Baire property, for every $n, k \in \omega$ and function $f : \mathscr{P}^n({}^\omega 2) \to k$, there is an $S \subseteq k$ with $|S| \leq (n-1)!$ and a set $A \subseteq {}^\omega 2$ with $A \approx {}^\omega 2$ so that $f[\mathscr{P}^n(A)] \subseteq S$. Hence for $n > 0$, $d_{{}^\omega 2}(n) \leq (n-1)!$ assuming AD.

Under AD^+, $d_{{}^\omega 2/E_0}(2)$ is finite and equal to 1, but for $n \geq 3$, $d_{{}^\omega 2/E_0}(n)$ is infinite.

3.12 $\widehat{\mathbb{P}}^3_{E_0}$ Is Proper

Fact 3.6.5 shows that $\widehat{\mathbb{P}}^2_{E_0}$ is proper by having a very flexible fusion argument. Moreover, below any condition $(p, q) \in \widehat{\mathbb{P}}^2_{E_0}$ and countable elementary submodels M, one can find a $(M, \widehat{\mathbb{P}}^2_{E_0})$-master condition (p', q') so that every element of $[p'] \times_{E_0} [q']$ is a $\widehat{\mathbb{P}}^2_{E_0}$-generic real over M. This fusion argument for $\widehat{\mathbb{P}}^2_{E_0}$ is also used to prove numerous combinatorial properties in dimension 2. The analog of most of these properties in dimension 3 fails. No fusion with the type of property that $\widehat{\mathbb{P}}^2_{E_0}$ has can exist for $\widehat{\mathbb{P}}^3_{E_0}$. The natural question to ask would be whether $\widehat{\mathbb{P}}^3_{E_0}$ is proper at all.

This section will show that $\widehat{\mathbb{P}}^3_{E_0}$ is proper via a fusion argument. However, one loses control of when exactly conditions meet dense sets.

Definition 3.12.1. *Suppose $(p, q, r) \in \widehat{\mathbb{P}}^3_{E_0}$. Let (u, v, z) be a triple of strings in ${}^{<\omega}2$ of the same length $n+1$ so that $\{0, 1\} = \{u(n), v(n), z(n)\}$. Suppose $(p', q', r') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p, u), \Xi(q, v), \Xi(r, z))$. Let $\mathsf{3prune}^{(u,v,z)}_{(p',q',r')}(p, q, r)$ be the unique condition $(a, b, c) \leq^{n+1}_{\widehat{\mathbb{P}}^3_{E_0}} (p, q, r)$ so that $\Xi(a, u) = p'$, $\Xi(b, v) = q'$ and $\Xi(c, z) = r'$.*

In the above, we defined the relation $\leq^n_{\widehat{\mathbb{P}}^3_{E_0}}$ as coordinate-wise $\leq^n_{\mathbb{P}_{E_0}}$. $\mathsf{3prune}^{(u,v,z)}_{(p',q',r')}(p,q,r)$ has an explicit definition that is obtained by copying p', q' and r' below the appropriate part of (p,q,r) like in Definition 3.6.3.

Fact 3.12.2. *(With Zapletal) $\widehat{\mathbb{P}}^3_{E_0}$ is a proper forcing.*

Proof. Let $(p,q,r) \in \widehat{\mathbb{P}}^3_{E_0}$. Let Ξ be some large regular cardinal. Let $M \prec V_\Xi$ be a countable elementary substructure containing (p,q,r). Let $(D_n : n \in \omega)$ enumerate all the dense open subsets of $\widehat{\mathbb{P}}^3_{E_0}$ that belong to M. One may assume that $D_{n+1} \subseteq D_n$ for all $n \in \omega$.

If there exists some condition $(p',q',r') \leq^0_{\widehat{\mathbb{P}}^3_{E_0}} (p,q,r)$ with $(p',q',r') \in D_0$, then by elementarity there is such a condition in M. Let (p_0,q_0,r_0) be such a condition in M. Otherwise, let $(p_0,q_0,r_0) = (p,q,r)$.

Suppose one has defined (p_n,q_n,r_n). Let $\{(u_i,v_i,z_i) : i < K\}$ enumerate all the strings in ${}^{<\omega}2$ with length $n+1$ so that $\{u(n),v(n),z(n)\} = \{0,1\}$.

Let $(a_{-1},b_{-1},c_{-1}) = (p_n,q_n,r_n)$. For i with $-1 \leq i < K-1$, suppose one has defined (a_i,b_i,c_i).

Let $(a^{-1}_{i+1},b^{-1}_{i+1},c^{-1}_{i+1}) = (a_i,b_i,c_i)$. Suppose for j with $-1 \leq j < n+1$, one has defined $(a^j_{i+1},b^j_{i+1},c^j_{i+1})$. If there exists some condition below $(\Xi(a^j_{i+1},u_{i+1}),\Xi(b^j_{i+1},v_{i+1}),\Xi(c^j_{i+1},z_{i+1}))$ that belongs to D_{j+1}, then choose, by elementarity, such a condition $(a',b',c') \in M \cap D_{j+1}$. Let $(a^{j+1}_{i+1},b^{j+1}_{i+1},c^{j+1}_{i+1}) = \mathsf{3prune}^{(u_{i+1},v_{i+1},z_{i+1})}_{(a',b',c')}(a^j_{i+1},b^j_{i+1},c^j_{i+1})$. If no such condition exists, then let $(a^{j+1}_{i+1},b^{j+1}_{i+1},c^{j+1}_{i+1}) = (a^j_{i+1},b^j_{i+1},c^j_{i+1})$.

Let $(a_{i+1},b_{i+1},c_{i+1}) = (a^{n+1}_{i+1},b^{n+1}_{i+1},c^{n+1}_{i+1})$. Let $(p_{n+1},q_{n+1},r_{n+1}) = (a_{K-1},b_{K-1},c_{K-1})$. Note that $(p_{n+1},q_{n+1},r_{n+1}) \leq^{n+1}_{\widehat{\mathbb{P}}^3_{E_0}} (p_n,q_n,r_n)$.

$\langle (p_n,q_n,r_n) : n \in \omega\rangle$ forms a fusion sequence in $\widehat{\mathbb{P}}^3_{E_0}$. Let $(p_\omega,q_\omega,r_\omega)$ be the fusion of this fusion sequence. The claim is that this is a $(M,\widehat{\mathbb{P}}^3_{E_0})$-master condition below (p,q,r).

It is necessary to show for each n that $(p_\omega,q_\omega,r_\omega) \Vdash_{\widehat{\mathbb{P}}^3_{E_0}} \check{M} \cap \check{D}_n \cap \dot{G} \neq \emptyset$. Let G be any $\widehat{\mathbb{P}}^3_{E_0}$-generic over M containing $(p_\omega,q_\omega,r_\omega)$. There is some $(p',q',r') \in G \cap D_n$. Since G is a filter, there is some $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (p_\omega,q_\omega,r_\omega)$ so that $(p'',q'',r'') \in G \cap D_n$. By genericity, one may assume there is some $m > n$ and some $(u,v,z) \in {}^m2$ so that $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p_\omega,u),\Xi(q_\omega,v),\Xi(r_\omega,z))$. During the construction while producing (p_m,q_m,r_m), the strings $(u,v,z) = (u_i,v_i,z_i)$ for some i in the chosen enumeration of strings. Note that $(p'',q'',r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(a^{n-1}_i,u_i),\Xi(b^{n-1}_i,v_i),\Xi(c^{n-1}_i,z_i))$ and $(p'',q'',r'') \in D_n$. At this stage, one would have chosen some $(a',b',c') \in M \cap D_n$ below $(\Xi(a^{n-1}_i,u_i),\Xi(b^{n-1}_i,v_i),\Xi(c^{n-1}_i,z_i))$ and set $(a^n_i,b^n_i,c^n_i) = \mathsf{3prune}^{(u_i,v_i,z_i)}_{(a',b',c')}(a^{n-1}_i,b^{n-1}_i,c^{n-1}_i)$. Note that

$(p'', q'', r'') \leq_{\widehat{\mathbb{P}}^3_{E_0}} (\Xi(p_\omega, u), \Xi(q_\omega, v), \Xi(r_\omega, z)) \leq_{\widehat{\mathbb{P}}^3_{E_0}} (a', b', c')$. Since G is a filter and $(p'', q'', r'') \in G$, $(a', b', c') \in G \cap M \cap D_n$. This shows that $\widehat{\mathbb{P}}^3_{E_0}$ is a proper forcing. ◻

In the proof, one extends a portion of the three trees to get into a dense set D only if it was possible and otherwise ignored D. Because of this, one cannot prove that $[p_\omega] \times_{E_0} \times [q_\omega] \times_{E_0} [r_\omega]$ consists entirely of reals which are $\widehat{\mathbb{P}}^3_{E_0}$-generic over M.

3.13 E_1 Does Not Have the 2-Mycielski Property

This section will give an example to show E_1 does not have the 2-Mycielski property. The notation of Definition 3.2.3 will be used in the following.

As in earlier sections, an understanding of the structure theorem of E_1-big Σ^1_1 sets is essential:

Definition 3.13.1. *E_1 is the equivalence relation on ${}^\omega({}^\omega 2)$ defined by $x \; E_1 \; y$ if and only if there exists a k so that for all $n \geq k$, $x(n) = y(n)$.*

Definition 3.13.2. *[14] Let s be an infinite subset of ω. Let $\pi_s : \omega \to s$ be the unique increasing enumeration of s. A homeomorphism $\Phi : {}^\omega({}^\omega 2) \to {}^\omega({}^\omega 2)$ is an s-keeping homeomorphism if and only if the following hold:*

1. For all $n \in \omega$, if $x(n) \neq y(n)$, then $\Phi(x)(\pi_s(m)) \neq \Phi(y)(\pi_s(m))$ for all $m \leq n$.

2. For all $n \in \omega$, if for all $m > n$, $x(m) = y(m)$, then for all $k > \pi_s(n)$, $\Phi(x)(k) = \Phi(y)(k)$.

Fact 3.13.3. *Let $B \subseteq {}^\omega({}^\omega 2)$ be Σ^1_1. $E_1 \restriction B \equiv_{\mathbf{\Delta}^1_1} E_1$ if and only if there is some infinite $s \subseteq \omega$ and s-keeping homeomorphism Φ so that $\Phi[{}^\omega({}^\omega 2)] \subseteq B$.*

Proof. This result is implicit in [18]. See [14], Section 7.2.1. ◻

Theorem 3.13.4. *Let $D = \{(x, y) \in {}^2({}^\omega({}^\omega 2)) : (\exists n)(x(n)(0) \neq y(n)(0))\}$. D is dense open and for all $\mathbf{\Delta}^1_1$ B such that $E_1 \restriction B \equiv_{\mathbf{\Delta}^1_1} E_1$, $[B]^2_{E_1} \not\subseteq D$.*

E_1 does not have the 2-Mycielski property.

Proof. Suppose $(x, y) \in D$. There is some $n \in \omega$ so that $x(n)(0) \neq y(n)(0)$. Let $\sigma, \tau : (n+1) \to {}^1 2$ be defined by $\sigma(k) = x(k) \restriction 1$ and $\tau(k) = y(k) \restriction 1$. Then $(x, y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Let $\sigma, \tau : m \to {}^{<\omega}2$. Define $\sigma', \tau' : (m+1) \to {}^{<\omega}2$ by

$$\sigma'(k) = \begin{cases} \sigma(k) & k < m \\ \langle 0 \rangle & k = m \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k < m \\ \langle 1 \rangle & k = m \end{cases}.$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Let $s \subseteq \omega$ be infinite. Let $\pi_s : \omega \to s$ be the unique increasing enumeration of s. Let $\Phi : {}^{\omega}({}^{\omega}2) \to {}^{\omega}({}^{\omega}2)$ be an s-keeping homeomorphism.

Let $\delta_n : (\pi_s(n) + 1) \to {}^{1}2$ be defined by $\delta_n(k) = \Phi(\bar{0})(k) \upharpoonright 1$. A strictly increasing sequence $\langle m_n : n \in \omega \rangle$ of natural numbers and functions $\sigma_n : m_n \to {}^{m_n}2$ satisfying the following for all $n \in \omega$ will be defined:

1. $\sigma_n(k) \subseteq \sigma_{n+1}(k)$ for each $k < m_n$.

2. $\Phi[N_{\sigma_n}] \subseteq N_{\delta_n}$.

3. There exists a $j < m_n$ such that $\sigma_n(n)(j) = 1$ and for all $k > n$ and $i < m_n$, $\sigma_n(k)(i) = 0$.

Let $m_{-1} = 0$ and $\sigma_{-1} = \delta_{-1} = \emptyset$.

Suppose m_n and σ_n have been defined and satisfy conditions 2 and 3 if $n \geq 0$. Define $y \in N_{\sigma_n}$ by $y(i)(j) = 0$ if $(i, j) \notin m_n \times m_n$. Then $\Phi(y) \in N_{\delta_n}$. Since $y(k) = \bar{0}(k)$ for all $k > n$ and Φ is an s-keeping homeomorphism, $\Phi(y)(k) = \Phi(\bar{0})(k)$ for all $k > \pi_s(n)$. Thus $\Phi(y) \in N_{\delta_{n+1}}$. By continuity of Φ, there is some $M \geq m_n$ so that if $\tau : M \to {}^{M}2$ is defined by $\tau(i) = y(i) \upharpoonright M$, then $\Phi(N_\tau) \subseteq N_{\delta_{n+1}}$. Let $m_{n+1} = M + 1$ and define

$$\sigma_{n+1}(i)(j) = \begin{cases} 1 & i = n + 1 \wedge j = M \\ y(i)(j) & \text{otherwise} \end{cases}.$$

m_{n+1} and σ_{n+1} satisfy conditions 1, 2, and 3.

Let $x \in {}^{\omega}({}^{\omega}2)$ be so that $\{x\} = \bigcap_{n\in\omega} N_{\sigma_n}$. $\neg(\bar{0}\ E_1\ x)$ since for all n, there exists a j so that $x(n)(j) = 1$ by condition 3. However since $\Phi(x) \in N_{\delta_n}$ for all n, $(\Phi(\bar{0}), \Phi(x)) \notin D$. From Definition 3.13.2, Φ is a E_1 reduction so $\neg(\Phi(\bar{0})\ E_1\ \Phi(x))$. Hence $[\Phi({}^{\omega}({}^{\omega}2))]^2_{E_1} \not\subseteq D$.

So it has been shown that for all infinite $s \subseteq \omega$ and s-keeping homeomorphisms Φ, $[\Phi[{}^{\omega}({}^{\omega}2)]]^2_{E_1} \not\subseteq D$. By Fact 3.13.3, every $\mathbf{\Delta}^1_1$ set B so that $E_1 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_1$ contains $\Phi[{}^{\omega}({}^{\omega}2)]$ for some s and some s-keeping homeomorphism Φ. Therefore, $[B]^2_{E_1} \not\subseteq D$ for all such B. E_1 does not have the 2-Mycielski property. □

3.14 The Structure of E_2

This section will give a proof of a result about the structure of E_2-big sets necessary for analyzing the weak-Mycielski property for E_2. The proof is similar to but a bit a more technical than the argument of [13] Theorem 15.4.1. Some of the notation and terminology come from [13].

Definition 3.14.1. *Suppose $x, y \in {}^{\omega}2$. Define*

$$\delta(x, y) = \sum_{k \in x \triangle y} \frac{1}{k+1}.$$

Suppose $m < n \leq \omega$. Suppose $x, y \in {}^{N}2$ where $n \leq N \leq \omega$. Define

$$\delta^n_m(x, y) = \sum \left\{ \frac{1}{k+1} : (m \leq k < n) \wedge (k \in x \triangle y) \right\}.$$

Let $A, B \subseteq {}^{\omega}2$. Let $m < n \leq \omega$. Let $\epsilon > 0$. Define $\delta^n_m(A, B) < \epsilon$ if and only if for all $x \in A$, there is some $y \in B$ so that $\delta^n_m(x, y) < \epsilon$ and for all $y \in B$, there exists some $x \in A$ so that $\delta^n_m(x, y) < \epsilon$.

Definition 3.14.2. *E_2 is the equivalence relation on ${}^{\omega}2$ defined $x \; E_2 \; y$ if and only if $\delta(x, y) < \infty$.*

Lemma 3.14.3. *Let $m < n \leq \omega$. Fix N so that $n \leq N \leq \omega$. If $n < \omega$, then δ^n_m is a pseudo-metric on ${}^{N}2$. If $n = \omega$, then δ^n_m is a pseudo-metric on any E_2 equivalence class.*

Lemma 3.14.4. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma^1_1(z)$ subsets of ${}^{\omega}2$. Let $(x_i : i < p)$ be a sequence in ${}^{\omega}2$ with the property that $x_i \in \hat{X}_i$ and $\delta^{\omega}_m(x_0, x_i) < q$. Then there exists a sequence $(X_i : i < p)$ of $\Sigma^1_1(z)$ sets with $x_i \in X_i$ and $\delta^{\omega}_m(X_0, X_i) < q$.*

Proof. Let

$$X_0 = \left\{ x \in \hat{X}_0 : (\exists z_1, ..., z_{p-1}) \left(\bigwedge_{1 \leq i < p} z_i \in \hat{X}_i \wedge \delta^{\omega}_m(x, z_i) < q \right) \right\}.$$

For $1 \leq i < p$, define

$$X_i = \left\{ x \in \hat{X}_i : (\exists z)(x \in X_0 \wedge \delta^{\omega}_m(x, z) < q) \right\}.$$

□

Lemma 3.14.5. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma^1_1(z)$ sets with $\delta^{\omega}_m(\hat{X}_0, \hat{X}_i) < q$ for all $i < p$. Let $j < p$. Suppose $A \subseteq \hat{X}_j$ is a $\Sigma^1_1(z)$ set. Then there exists a sequence $(X_i : i < p)$ of $\Sigma^1_1(z)$ sets with the property that for all $i < p$, $X_i \subseteq \hat{X}_i$, $X_j = A$, and $\delta^{\omega}_m(X_0, X_i) < q$.*

Proof. Let

$$X_0 = \{x \in \hat{X}_0 : (\exists z)(z \in A \wedge \delta^{\omega}_m(x, z) < q)\}.$$

For all $i < p$ and $i \neq j$, let

$$X_i = \{x \in \hat{X}_i : (\exists z)(z \in X_0 \wedge \delta^{\omega}_m(x, z) < q)\}$$

Let $X_j = A$. □

Lemma 3.14.6. *Let $m, p \in \omega$. Let $q \in \mathbb{Q}^+$. Let $(\hat{X}_i : i < p)$ be a sequence of $\Sigma^1_1(z)$ sets with $\delta^\omega_m(\hat{X}_0, \hat{X}_i) < q$ for all $i < p$. Let D be a dense open subset of $\mathbb{P}_z$. Then there exists a sequence $(X_i : i < p)$ of $\Sigma^1_1(z)$ sets with $X_i \subseteq \hat{X}_i$, $X_i \in D$, and $\delta^\omega_m(X_0, X_i) < q$ for all $i < p$.*

Proof. Let $Y_i^{-1} = \hat{X}_i$.

One seeks to define $\Sigma^1_1(z)$ sets Y_i^j for all $-1 \leq j < p$ with the property that if $-1 \leq j < p-1$, then $Y_i^{j+1} \subseteq Y_i^j$, and for any $-1 \leq j < p$ and $0 \leq i < p$, $Y_j^j \in D$ and $\delta^\omega_m(Y_0^j, Y_i^j) < q$.

Suppose one has defined Y_i^j with the desired properties for $j < p-1$ and all $i < p$. Since D is dense open in $\mathbb{P}_z$, pick some $A \subseteq Y_{j+1}^j$ so that $A \in D$. Use Lemma 3.14.5 with $\{Y_i^j : i < p\}$ and $A \subseteq Y_i^j$ to obtain $\{Y_i^{j+1} : i < p\}$ with the desired properties.

Let $X_i = Y_i^{p-1}$. $\square$

In the previous three lemmas, the first set was distinguished. In the following argument, we index the sets by strings and so in applications of the three lemmas, one will need to indicate what this distinguished set is.

Lemma 3.14.7. *Let $z \in {}^\omega 2$. Let $k, m, p \in \omega$. Let $r, v \in \mathbb{Q}^+$. Let $(B_s^i : s \in {}^k 2 \wedge i < p)$ be a sequence of $\Sigma^1_1(z)$ sets. Let $(b_s^i : s \in {}^k 2 \wedge i < p)$ be a sequence in ${}^\omega 2$ with $b_s^i \in B_s^i$. Suppose for all $i < p$, $\delta^\omega_m(b^0_{0^k}, b^i_{0^k}) < r$. Suppose for each $i < p$ and for all $s \in {}^k 2$, $\delta^\omega_m(b^i_{0^k}, b^i_s) < v$. Let D be a dense open subset of $\mathbb{P}_z$.*

Then there is a sequence $(C_s^i : s \in {}^k 2 \wedge i < p\}$ of $\Sigma^1_1(z)$ sets so that for all $i < p$ and $s \in {}^k 2$, $\delta^\omega_m(C^0_{0^k}, C^i_{0^k}) < r$, $\delta^\omega_m(C^i_{0^k}, C^i_s) < v$, and $C^i_s \in D$.

Proof. For each $i < p$, apply Lemma 3.14.4 to $\{B^i_s : s \in {}^k 2\}$ and $\{b^i_s : s \in {}^k 2\}$ using 0^k as the distinguished index to obtain a sequence of $\Sigma^1_1(z)$ sets $\{E^i_s : s \in {}^k 2\}$ with the property that $E^i_s \subseteq B^i_s$, $b^i_s \in E^i_s$, and $\delta^\omega_m(E^i_{0^k}, E^i_s) < v$.

Now apply Lemma 3.14.4 to $\{E^i_{0^k} : i < p\}$ and $\{b^i_{0^k} : i < p\}$ with 0 as the distinguished index to obtain $\Sigma^1_1(z)$ sets $A_i \subseteq E^i_{0^k}$ so that $\delta^\omega_m(A_0, A_i) < r$.

For each $i < p$, apply Lemma 3.14.5 to $\{E^i_s : s \in {}^k 2\}$ with 0^k as the distinguished index and $A_i \subseteq E^i_{0^k}$ to obtain $\Sigma^1_1(z)$ sets $G^{i,-1}_s$ with the properties that $G^{i,-1}_{0^k} = A_i$ and $\delta^\omega_m(G^{i,-1}_{0^k}, G^{i,-1}_s) < v$.

Note that since $G^{i,-1}_{0^k} = A_i$, the sequence $\{G^{i,-1}_s : i < p \wedge s \in {}^k 2\}$ has the property that for all $i < p$, $\delta^\omega_m(G^{0,-1}_{0^k}, G^{i,-1}_{0^k}) < r$ and for each $i < p$ and $s \in {}^k 2$, $\delta^\omega_m(G^{i,-1}_{0^k}, G^{i,-1}_s) < v$.

One wants to create $G_s^{i,j}$ for $-1 \leq j < p$, $i < p$, and $s \in {}^k2$ so that

(i) For each $i \in \omega$, $s \in {}^k2$, and $-1 \leq l \leq j < p$, $G_s^{i,j} \subseteq G_s^{i,l}$.

(ii) For all $i < p$, $\delta_m^\omega(G_{0^k}^{0,j}, G_{0^k}^{i,j}) < r$.

(iii) For each $i < p$ and $s \in {}^k2$, $\delta_m^\omega(G_{0^k}^{i,j}, G_s^{i,j}) < v$.

(iv) If $j \geq 1$ and $0 \leq l \leq j$, then $G_s^{l,j} \in D$.

This already holds for $j = -1$. Suppose the construction worked up to stage $j < p - 1$ producing objects with the above properties. Apply Lemma 3.14.6 to $\{G_s^{j+1,j} : s \in {}^k2\}$ to get sets $\{G_s^{j+1,j+1} : s \in {}^k2\}$ each in D and $\delta_m^\omega(G_{0^k}^{j+1,j+1}, G_s^{j+1,j+1}) < v$.

Next apply Lemma 3.14.5 to $\{G_{0^k}^{i,j} : i < p\}$ with 0 as the distinguished index and $G_{0^k}^{j+1,j+1} \subseteq G_{0^k}^{j+1,j}$ to obtain sets $G_{0^k}^{i,j+1} \subseteq G_{0^k}^{i,j}$ with $\delta_m^\omega(G_{0^k}^{0,j+1}, G_{0^k}^{i,j+1}) < r$. (Note it is acceptable to use the notation $G_{0^k}^{j+1,j+1}$ since it is the same set as before by the statement of Lemma 3.14.5.)

For each $i \neq j + 1$, apply Lemma 3.14.5 on $\{G_s^{i,j} : s \in {}^k2\}$ with 0^k as the distinguished index and $G_{0^k}^{i,j+1} \subseteq G_{0^k}^{i,j}$ to obtain sets $\{G_s^{i,j+1} : s \in {}^k2\}$ with the property that $\delta_m^\omega(G_{0^k}^{i,j+1}, G_s^{i,j+1}) < v$. This completes the construction at stage $j + 1$.

Let $C_s^i = G_s^{i,p-1}$. □

Definition 3.14.8. *([13] Definition 15.2.2) Let $q > 0$ be a rational number. Let $A \subseteq {}^\omega2$. Let $a \in A$. The q-galaxy of a in A, denoted $\mathrm{Gal}_A^q(a)$, is the set of all $b \in A$ so that there exists $a_0, ..., a_l \in A$ with $a = a_0$, $b = a_l$, and $\delta(a_i, a_{i+1}) < q$ for all $0 \leq i < l - 1$.*

$A \subseteq {}^\omega2$ is q-grainy if and only if for all $a \in A$ and $b \in \mathrm{Gal}_A^q(a)$, $\delta(a, b) < 1$. A is grainy if and only if A is q-grainy for some positive rational number q.

Fact 3.14.9. *Let $z \in {}^\omega2$ and rational $q > 0$. If A is a $\Sigma_1^1(z)$ q-grainy set, then there is some $B \supseteq A$ which is $\Delta_1^1(z)$ q-grainy.*

Proof. (See [13], Claim 15.2.4 for a more constructive proof.)

Let $U \subseteq \omega \times {}^\omega2$ be a universal $\Sigma_1^1(z)$ set. The relation in variables e, a, and b expressing $b \in \mathrm{Gal}_{U^e}^q(a)$ is $\Sigma_1^1(z)$.

Let $\mathcal{A}$ be the collection of all $\Sigma_1^1(z)$ q-grainy subsets of ${}^\omega2$.

$$\{e : U^e \in \mathcal{A}\} = \{e : (\forall a)(\forall b)(b \in \mathrm{Gal}_{U^e}^q(a) \Rightarrow \delta(a, b) < 1)\}$$

This shows that $\mathscr{A}$ is a collection of $\Sigma^1_1(z)$ sets which is $\Pi^1_1(z)$ in the code. By $\Sigma^1_1(z)$ reflection, every $\Sigma^1_1(z)$ q-grainy set is contained inside of a $\Delta^1_1(z)$ q-grainy set. $\square$

Definition 3.14.10. *Let $z \in {}^{\omega}2$. Let S_z be the union of all $\Delta^1_1(z)$ grainy sets. Let $H_z = {}^{\omega}2 \setminus S_z$.*

Fact 3.14.11. *Let $z \in {}^{\omega}2$. S_z is $\Pi^1_1(z)$. Hence H_z is $\Sigma^1_1(z)$.*

Every nonempty $\Sigma^1_1(z)$ subset of H_z is not grainy.

Proof. Similar to Fact 3.5.12. $\square$

Theorem 3.14.12. *Let $z \in {}^{\omega}2$. Let $p \in \omega$. Suppose $(X_i : i < p)$ is a collection of $\Sigma^1_1(z)$ subsets of ${}^{\omega}2$ with the property that $\bigcap_{i<p}[X_i \cap H_z]_{E_2} \neq \emptyset$. Then there is a strictly increasing sequence $(m_k : k \in \omega)$ and functions $g^i : {}^{<\omega}2 \to {}^{<\omega}2$ for each $i < p$ with the following properties:*

1. If $|s| = k$, then for all $i < p$, $g^i(s) \in {}^{m_k}2$.

2. If $s \subseteq t$, then for all $i < p$, $g^i(s) \subseteq g^i(t)$.

3. If $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) < 2^{-(k+1)}$,

4. If $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) - \frac{1}{k}| < 2^{-(k+1)}$.

5. For $i, j < p$ and $s \in {}^{k}2$ with $k > 0$, $\delta^{m_k}_{m_{k-1}}(g^i(s), g^j(s)) < 2^{-(k+1)}$.

6. Define $\Phi^i(x) = \bigcup_{n\in\omega} g^i(x \upharpoonright n)$. $\Phi^i : {}^{\omega}2 \to X_i \cap H_z$ is a reduction witnessing $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright X_i \cap H_z$. Moreover, for $i, j < p$, $[\Phi^i[{}^{\omega}2]]_{E_2} = [\Phi^j[{}^{\omega}2]]_{E_2}$.

Proof. During the construction, one will seek to create

(i) a strictly increasing sequence $(m_k : k \in \omega)$,

(ii) for each $i < p$ and $s \in {}^{<\omega}2$, $\Sigma^1_1(z)$ sets A^i_s,

(iii) and for each $i < p$, $g^i(s) \in {}^{<\omega}2$.

These objects will satisfy the following properties:

(I) If $|s| = k$, then $|g^i(s)| = m_k$. $s \subseteq t$ implies $g^i(s) \subseteq g^i(t)$.

(II) $\emptyset \neq A^i_s \subseteq X^i \cap H_z \cap N_{g^i(s)}$. $s \subseteq t$ implies $A^i_t \subseteq A^i_s$.

(III) If $k > 0$, $|s| = k$, then $\delta^{\omega}_{m_k}(A^i_{0^k}, A^i_s) \leq 2^{-(k+4)}$, where $0^k : k \to 2$ is the constant 0 function.

(IV) If $k > 0$, $|s| = |t| = k$, and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) \leq 2^{-(k+1)}$.

(V) If $k > 0$, $|s| = |t| = k$, $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g^i(s), g^i(t)) - \frac{1}{k}| < 2^{-(k+1)}$.

(VI) If $|s| = k$ and $i < p$, $\delta^{\omega}_{m_k}(A^0_s, A^i_s) < 2^{-(k+6)}$.

(VII) If $|s| = k$ and $k > 0$, then $\delta^{m_k}_{m_{k-1}}(g^i(s), g^j(s)) < 2^{-(k+1)}$.

(VIII) Let $\mathcal{D} = (D_n : n \in \omega)$ be the countable collection of dense open subsets of $\mathbb{P}_z$ from Fact 3.5.9. For all $s \in {}^{<\omega}2$ and $i < p$, $A^i_s \in D_{|s|}$.

Suppose it is possible to construct these objects having the above properties. It only remains to verify 6: $\{\Phi^i(x)\} = \bigcap_{k\in\omega} A_{x\restriction k}$ by (II) and (VIII). Hence Φ^i maps into $X_i \cap H_z$ by (II). Note that $\delta(\Phi^i(x), \Phi^j(y)) = \lim_{k\to\infty} \delta^{m_k}_0(g^i(x \restriction k), g^j(y \restriction k))$. Also using (IV) and (V), for any k

$$|\delta^{m_k}_0(g^i(x \restriction k), g^i(y \restriction k)) - \delta^k_0(x \restriction k, y \restriction k)| < \sum_{j<k} 2^{-(j+1)} < 1$$

Hence $\Phi^i(x)\ E_2\ \Phi^i(y) \Leftrightarrow \delta(\Phi^i(x), \Phi^i(y)) < \infty \Leftrightarrow \delta(x, y) < \infty \Leftrightarrow x\ E_2\ y$. This shows each Φ^i witnesses $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction X_i \cap H_z$. Using (VII), for each $i, j < p$ and $x \in {}^{\omega}2$,

$$\delta^{m_k}_{m_0}(g^i(x \restriction k), g^j(x \restriction j)) < \sum_{j<k} 2^{-(j+1)} < 1$$

Hence $\Phi^i(x)\ E_2\ \Phi^j(x)$. Hence $[\Phi^i[{}^{\omega}2]]_{E_2} = [\Phi^j[{}^{\omega}2]]_{E_2}$.

Next the construction: Since $\bigcap_{i<p}[X_i \cap H_z]_{E_2} \neq \emptyset$, let $(b^i_\emptyset : i < p)$ be such that for all $i, j < p$, $b^i_\emptyset\ E_2\ b^j_\emptyset$ and $b^i_\emptyset \in X_i \cap H_z$. Therefore, choose $m_0 \in \omega$ so that for all $i < p$, $\delta^{\omega}_{m_0}(b^0_\emptyset, b^i_\emptyset) < 2^{-6}$. For each $i < p$, let $g^i(\emptyset) = b^i_\emptyset \restriction m_0$.

Let $B^i_\emptyset = X_i \cap H_z \cap N_{g^i(\emptyset)}$. Apply Lemma 3.14.7 to $\{B^i_\emptyset : i \in p\}$, $\{b^i_\emptyset : i < p\}$, and the dense open (in $\mathbb{P}_z$) set D_0 (where $r = 2^{-6}$ and $v = 1$) to obtain sets $A^i_\emptyset$ with the desired properties.

Suppose the objects from stage k have been constructed with the desired properties.

As $A^0_{0^k} \subseteq H_z$, Fact 3.14.11 implies that $A^0_{0^k}$ is not $2^{-(k+5)}$-grainy. Hence there is a sequence $a_0, ..., a_M$ of points in $A^0_{0^k}$ so that for each $0 \leq j < M-1$, $\delta(a_j, a_{j+1}) < 2^{-(k+5)}$ but $\delta(a_0, a_M) > 1$. Hence there is some I so that $\delta(a_0, a_I) > \frac{1}{k+1}$ and $\delta(a_0, a_I) - \frac{1}{k+1} < 2^{-(k+5)}$.

Let $b^0_{0^k\hat{}0} = a_0$ and $b^0_{0^k\hat{}1} = a_I$. Since $\delta^{\omega}_{m_k}(A^0_s, A^i_s) < 2^{-(k+6)}$ by (VI), find $b^i_{0^{k+1}}, b^i_{0^k\hat{}1} \in A^i_{0^k}$ so that $\delta(b^0_{0^{k+1}}, b^i_{0^{k+1}}) < 2^{-(k+6)}$ and $\delta(b^0_{0^k\hat{}1}, b^i_{0^k\hat{}1}) < 2^{-(k+6)}$.

The claim is that for all $i < p$, $|\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\hat{}1}) - \frac{1}{k+1}| < 2^{-(k+4)}$: To see this,

$$|\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k\hat{}1}) - \frac{1}{k+1}|$$

$$\leq |\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k{}^\frown 1}) - \delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^0_{0^k{}^\frown 1})| + |\delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^0_{0^k{}^\frown 1}) - \frac{1}{k+1}|$$
$$< |\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k{}^\frown 1}) - \delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^i_{0^k{}^\frown 1})| + |\delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^i_{0^k{}^\frown 1}) - \delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^0_{0^k{}^\frown 1})| + 2^{-(k+5)}$$
$$\leq \delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^0_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^i_{0^k{}^\frown 1}, b^0_{0^k{}^\frown 1}) + 2^{-(k+5)}$$
$$\leq 2^{-(k+6)} + 2^{-(k+6)} + 2^{-(k+5)} = 2^{-(k+4)}$$

This proves the claim.

Now fix a $i < p$. By (III), $\delta^{\omega}_{m_k}(A^i_{0^k}, A^i_t) < 2^{-(k+4)}$ for each $t \in {}^k2$. For each $s \in {}^{k+1}2$, let $b^i_s \in A^i_{s\restriction k}$ be such that $\delta^{\omega}_{m_k}(b^i_{0^k{}^\frown s(k)}, b^i_s) < 2^{-(k+4)}$.

Suppose $s \in {}^{k+1}2$ and $s(k) = 1$.

$$|\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_s) - \frac{1}{k+1}|$$
$$\leq |\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_s) - \delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k{}^\frown 1})| + |\delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^i_{0^k{}^\frown 1}) - \frac{1}{k+1}|$$
$$< \delta(b^i_s, b^i_{0^k{}^\frown 1}) + 2^{-(k+4)} \leq 2^{-(k+4)} + 2^{-(k+4)} = 2^{-(k+3)}$$

By (I), (II), and the fact that $b^i_{0^{k+1}}, b^i_{0^k{}^\frown 1} \in A^i_{0^k}$, there exists some $m_{k+1} > m_k$ so that

(i) $|\delta^{m_{k+1}}_{m_k}(b^i_{0^{k+1}}, b^i_s) - \frac{1}{k+1}| < 2^{-(k+3)}$ for all $s \in {}^{k+1}2$ with $s(k) = 1$.

(ii) $\delta^{\omega}_{m_{k+1}}(b^0_{0^{k+1}}, b^i_{0^{k+1}}) < 2^{-(k+7)}$ for all $i < p$.

(iii) $\delta^{\omega}_{m_{k+1}}(b^i_{0^{k+1}}, b^i_s) < 2^{-(k+5)}$ for all $s \in {}^{k+1}2$.

Let $g^i(s) = b^i_s \restriction m_{k+1}$. Suppose $s(k) = t(k)$. Without loss of generality, suppose $s(k) = t(k) = 1$. Then

$$\delta^{m_{k+1}}_{m_k}(g^i(s), g^i(t)) \leq \delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^k{}^\frown 1}) + \delta^{m_{k+1}}_{m_k}(b^i_{0^k{}^\frown 1}, b^i_t) \leq 2^{-(k+4)} + 2^{-(k+4)} = 2^{-(k+3)} < 2^{-(k+2)}$$

This establishes (IV).

Suppose $s(k) \neq t(k)$. Without loss of generality, suppose $s(k) = 1$. Hence $t(k) = 0$.

$$|\delta^{m_{k+1}}_{m_k}(g^i(s), g^i(t)) - \frac{1}{k+1}|$$
$$\leq |\delta^{m_{k+1}}_{m_k}(b^i_s, b^i_t) - \delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^{k+1}})| + |\delta^{m_{k+1}}_{m_k}(b^i_s, b^i_{0^{k+1}}) - \frac{1}{k+1}|$$
$$< \delta^{m_{k+1}}_{m_k}(b^i_t, b^i_{0^{k+1}}) + 2^{-(k+3)} < 2^{-(k+4)} + 2^{-(k+3)} < 2^{-(k+2)}$$

This establishes (V).

Let $s \in {}^{k+1}2$. Without loss of generality suppose $s(k) = 0$. Suppose $i, j < p$. Observe:

$$\delta^{m_{k+1}}_{m_k}(g^i(s), g^j(s)) \leq \delta^{\omega}_{m_k}(b^i_s, b^j_s) \leq \delta^{\omega}_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^j_s)$$

$$\leq \delta^{\omega}_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^j_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^j_{0^{k+1}}, b^j_s)$$

$$\leq \delta^{\omega}_{m_k}(b^i_s, b^i_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^i_{0^{k+1}}, b^0_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^0_{0^{k+1}}, b^j_{0^{k+1}}) + \delta^{\omega}_{m_k}(b^j_{0^{k+1}}, b^j_s)$$

$$\leq 2^{-(k+4)} + 2^{-(k+6)} + 2^{-(k+6)} + 2^{-(k+4)} \leq 2^{-(k+3)} + 2^{-(k+5)} < 2^{-(k+2)}$$

This establishes (VII).

For $s \in {}^{k+1}2$, let $B^i_s = A^i_{s \restriction k} \cap N_{g^i(s)}$. Apply Lemma 3.14.7 on $\{B^i_s : i < p \wedge s \in {}^{k+1}2\}$, $\{b^i_s : i < p \wedge s \in {}^{k+1}2\}$, $r = 2^{-(k+7)}$, $v = 2^{-(k+5)}$, and D_{k+1} to obtain the desired objects $(A^i_s : i < p \wedge s \in {}^{k+1}2)$ which satisfy the remaining conditions.

This completes the proof. □

By relativizing to the appropriate parameter, one can obtain the following result.

Corollary 3.14.13. *Let $p \in \omega$. Suppose $(X_i : i < p)$ is a collection of Σ^1_1 subsets of ${}^{\omega}2$ with the property that for all $i < p$, $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction X_i$ and for all $i, j < p$, $[X_i]_{E_2} = [X_j]_{E_2}$. Then there exists a sequence of strictly increasing integers $(m_k : k \in \omega)$ and maps $g^i : {}^{<\omega}2 \to {}^{<\omega}2$ satisfying conditions 1 - 5 of Theorem 3.14.12.*

Fact 3.14.14. *Let $B \subseteq {}^{\omega}2$ be a Σ^1_1 so that $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction B$. There there exists a strictly increasing sequence $(m_k : k \in \omega)$ with $m_0 = 0$ and function $g : {}^{<\omega}2 \to {}^{<\omega}2$ with the following properties:*

1. If $|s| = k$, then $|g(s)| = m_k$.

2. If $s \subseteq t$, then $g(s) \subseteq g(t)$.

3. If $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g(s), g(t)) < 2^{-(k+1)}$.

4. If $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g(s), g(t)) - \frac{1}{k}| < 2^{-(k+1)}$.

5. Let $\Phi : {}^{\omega}2 \to {}^{\omega}2$ be defined by $\Phi(x) = \bigcup_{n \in \omega} g(x \restriction n)$. Then Φ is a $\mathbf{\Delta}^1_1$ function such that $\Phi[{}^{\omega}2] \subseteq B$ and Φ witnesses $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \restriction B$.

Proof. This is implicit in [8]. Also see [13] Theorem 15.4.1 and [14] Theorem 7.43. The proof is quite similar to Theorem 3.14.12. □

3.15 E_2 Does Not Have the 2-Mycielski Property

Theorem 3.15.1. *Let $D \subseteq {}^2({}^\omega 2)$ be defined by*

$$D = \{(x, y) \in {}^2({}^\omega 2) : (\exists i < j)(\delta_i^j(x, y) > 2 \wedge (\forall n)(i \leq n < j \Rightarrow x(n) \neq y(n)))\}$$

D is dense open.

Let $(m_k : k \in \omega)$, g, and Φ be as in Fact 3.14.14. $(\Phi(\tilde{0}), \Phi(\widetilde{01})) \notin D$*.*

For any $\mathbf{\Delta}_1^1$ set B so that $E_2 \upharpoonright B \equiv_{\mathbf{\Delta}_1^1} E_2$, $[B]^2_{E_2} \not\subseteq D$.

E_2 does not have the 2-Mycielski property.

Proof. Let $(x, y) \in D$. There is some $i < j$ so that for all n with $i \leq n < j$, $x(n) \neq y(n)$ and $\delta_i^j(x, y) > 2$. Let $\sigma = x \upharpoonright (j + 1)$ and let $\tau = y \upharpoonright (j + 1)$. Then $(x, y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Let $\sigma, \tau \in {}^{<\omega}2$ with $|\sigma| = |\tau|$. Let $i = |\sigma|$. Find a $j > i$ so that $\sum_{i \leq n < j} \frac{1}{n+1} > 2$. Let $\sigma', \tau' \in {}^{j+1}2$ be defined by

$$\sigma'(k) = \begin{cases} \sigma(k) & k < i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k < i \\ 1 & \text{otherwise} \end{cases}.$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Note that one must have

$$\sum_{m_{k-1} \leq m < m_k} \frac{1}{m+1} \geq 2^{-k-1}$$

because $\frac{1}{k} - 2^{-k-1} \geq 2^{-k} - 2^{-k-1} = 2^{-k-1}$ and so otherwise, condition 4 could not hold for any $s, t \in {}^k 2$ with $s(k-1) \neq t(k-1)$.

This implies that for any s and t so that $s(k-1) = t(k-1)$, there must be some m with $m_{k-1} \leq m < m_k$ so that $g(s)(m) = g(t)(m)$.

Note that for all $s, t \in {}^{k+1}2$.

$$\delta_{m_{k-1}}^{m_{k+1}}(g(s), g(t)) = \delta_{m_{k-1}}^{m_k}(g(s), g(t)) + \delta_{m_k}^{m_{k+1}}(g(s), g(t)) \leq \frac{1}{k} + 2^{-k-1} + \frac{1}{k+1} + 2^{-k-2} < 2.$$

Hence for any s, t, if there exists $i < j$ so that $\delta_i^j(g(s), g(t)) > 2$, then there is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$.

Now suppose that there is some $i < j$ so that $\delta_i^j(\Phi(\tilde{0}), \Phi(\widetilde{01})) > 2$. There is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$. Without loss of generality, suppose k is even. $\tilde{0}(k) = 0 = \widetilde{01}(k)$.

By the above, there is some l with $m_k \leq l < m_{k+1}$ so that $\Phi(\tilde{0})(l) = \Phi(\widetilde{01})(l)$. This shows $(\Phi(\tilde{0}), \Phi(\widetilde{01})) \notin D$. $\neg(\tilde{0}\ E_2\ \widetilde{01})$ so $\neg(\Phi(\tilde{0})\ E_2\ \Phi(\widetilde{01}))$. Hence $[\Phi[{}^{\omega}2]]^2_{E_2} \not\subseteq D$.

It has been shown that for all $(m_k : k \in \omega)$, g, and associated Φ, $[\Phi[{}^{\omega}2]]^2_{E_2} \not\subseteq D$. If $B \subseteq {}^{\omega}2$ is $\mathbf{\Delta}^1_1$ with the property that $E_2 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_2$, then Fact 3.14.14 implies that there is some $(m_k : k \in \omega)$ and g so that $\Phi[{}^{\omega}2] \subseteq B$. This shows that for all such B, $[B]^2_{E_2} \not\subseteq D$. E_2 does not have the 2-Mycielski property. ◻

Theorem 3.15.2. *For each $n \in \omega$, let*

$$D_n = \{(x, y) \in {}^2({}^{\omega}2) : (\exists i < j)(n \leq i < j \wedge \delta_i^j(x, y) > 3 \wedge (\forall m)(i \leq m < j \Rightarrow x(m) \neq y(m)))\}.$$

Each D_n is a dense open subset of ${}^2({}^{\omega}2)$. Hence $C = \bigcap_{n \in \omega} D_n$ is a comeager subset of ${}^2({}^{\omega}2)$.

Suppose $(m_k : k \in \omega)$, g^0, g^1, Φ^0, and Φ^1 have properties 1 - 6 from Theorem 3.14.12. Then $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin C$.

For any $\mathbf{\Delta}^1_1$ sets B_0 and B_1 with $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright B_0$, $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright B_1$, and $[B_0]_{E_2} = [B_1]_{E_2}$, $B_0 \times_{E_2} B_1 \not\subseteq C$.

E_2 does not have the weak 2-Mycielski property.

Proof. Using Theorem 3.14.12, if $|s| = |t| = k > 0$ and $s(k-1) \neq t(k-1)$, then $|\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(t)) - \frac{1}{k}| < 2^{-k}$. To see this:

$$\begin{aligned} &|\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(t)) - \frac{1}{k}| \\ &\leq |\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(t)) - \delta^{m_k}_{m_{k-1}}(g^1(s), g^1(t))| + |\delta^{m_k}_{m_{k-1}}(g^1(s), g^1(t)) - \frac{1}{k}| \\ &\leq \delta^{m_k}_{m_{k-1}}(g^0(s), g^1(s)) + 2^{-(k+1)} \leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k} \end{aligned}$$

Also if $|s| = |t| = k > 0$ and $s(k-1) = t(k-1)$, then $\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(t)) < 2^{-k}$. To see this:

$$\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(t)) \leq \delta^{m_k}_{m_{k-1}}(g^0(s), g^1(s)) + \delta^{m_k}_{m_{k-1}}(g^1(s), g^1(t)) = 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}$$

D_n is dense open by the same argument as in Theorem 3.15.2.

Note that if $k > 0$, then

$$\sum_{m_{k-1} \leq m < m_k} \frac{1}{m+1} \geq 2^{-k}$$

because $\frac{1}{k} - 2^{-k} \geq 2^{-(k-1)} - 2^{-k} = 2^{-k}$ and so otherwise $|\delta^{m_k}_{m_{k-1}}(g^0(s), g^1(s)) - \frac{1}{k}| < 2^{-k}$ could not hold.

Therefore if $|s| = |t| > 0$, and $s(k-1) = t(k-1)$, then there must be some m with $m_{k-1} \leq m < m_k$ so that $g^0(s)(m) = g^1(t)(m)$.

Note that for all $s, t \in {}^{k+1}2$ with $k > 0$,

$$\delta_{m_{k-1}}^{m_{k+1}}(g(s), g(t)) = \delta_{m_{k-1}}^{m_k}(g(s), g(t)) + \delta_{m_k}^{m_{k+1}}(g(s), g(t)) \leq \frac{1}{k} + 2^{-k} + \frac{1}{k+1} + 2^{-(k+1)} < 3.$$

Hence for any s, t, if there exists $i < j$ so that $\delta_i^j(g^0(s), g^1(t)) > 3$, then there is some $k \geq 1$ so that $i \leq m_{k-1} < m_k < m_{k+1} \leq j$.

Now by essentially the same argument as in Theorem 3.15.1, $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin D_{m_0}$. Hence $(\Phi^0(\tilde{0}), \Phi^1(\widetilde{01})) \notin C$.

Now suppose that B_0 and B_1 are some $\mathbf{\Delta}_1^1$ sets so that $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_0$, $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B_1$, and $[B_0]_{E_2} = [B_1]_{E_2}$. By Corollary 3.14.13, there is a sequence $(m_k : k \in \omega)$, g^0, g^1, Φ^0 and Φ^1 as above so that $\Phi^i[{}^\omega 2] \subseteq B_i$. By the earlier argument, $B_0 \times_{E_2} B_1 \nsubseteq C$. Hence E_2 does not have the weak 2-Mycielski property. ◻

3.16 Surjectivity and Continuity Aspects of E_2

Fact 3.14.14 states that every Σ_1^1 set $B \subseteq {}^\omega 2$ so that $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \upharpoonright B$ has a closed set $C \subseteq B$ so that $E_2 \equiv_{\mathbf{\Delta}_1^1} E_2 \upharpoonright C$. Fact 3.14.14 even asserts that C is the body of a tree on 2 with a specific structure:

Definition 3.16.1. *A tree $p \subseteq {}^{<\omega}2$ is an E_2-tree if and only if there is some sequence $(m_k : k \in \omega)$ and map $g : {}^{<\omega}2 \to {}^{<\omega}2$ satisfying the conditions of Fact 3.14.14 so that p is the downward closure of $g[{}^{<\omega}2]$. Note that if Φ is the map associated with $(m_k : k \in \omega)$ and g, then $[p] = \Phi[{}^\omega 2]$.*

The following notation is used to avoid some very tedious superscripts and subscripts in the following results:

Definition 3.16.2. *If $x, y \in {}^\omega 2$ and $m, n \in \omega$ with $m \leq n$, then let $\varsigma(m, n, x, y) = \delta_m^n(x, y)$.*

Fact 3.16.3. *There is a continuous function $P : [{}^\omega 2]^3_{E_2} \to {}^\omega 3$ so that for any E_2-tree p, $P[[[p]]^3_{E_2}] = {}^\omega 3$.*

Proof. For $(x, y) \in [{}^\omega 2]^2_{E_2}$ and any $n, m \in \omega$, define

$$S_{n,m}(x, y) = \min\{k \in \omega : \delta_n^k(x, y) > 3^{m+2}\}$$

Each $S_{n,m}$ is continuous on $[{}^\omega 2]^3_{E_2}$.

If $(x, y, z) \in [{}^{\omega}2]^3_{E_2}$, then define a strictly increasing sequence of integers $(L_n : n \in \omega)$ by recursion as follows: Let $L_0 = 0$. Given L_n, let

$$L_{n+1} = \min\{S_{L_n,n}(x, y), S_{L_n,n}(x, z), S_{L_n,n}(y, z)\}$$

(It is implicit that L_n depends on the triple (x, y, z).) By induction, it can be shown that each L_n as a function of (x, y, z) is continuous on $[{}^{\omega}2]^3_{E_2}$.

Define

$$P(x, y, z)(n) = \begin{cases} 0 & S_{L_n,n}(x, y) \le S_{L_n,n}(x, z) \text{ and } S_{L_n,n}(x, y) \le S_{L_n,n}(y, z) \\ 1 & S_{L_n,n}(x, z) < S_{L_n,n}(x, y) \text{ and } S_{L_n,n}(x, z) \le S_{L_n,n}(y, z) \\ 2 & S_{L_n,n}(y, z) < S_{L_n,n}(x, y) \text{ and } S_{L_n,n}(y, z) < S_{L_n,n}(x, z) \end{cases}$$

P is continuous on $[{}^{\omega}2]^3_{E_2}$.

Also define the sequence of integers $(N_n : n \in \omega)$ by recursion as follows: Let $N_0 = 0$ and if N_n has been defined, then let

$$N_{n+1} = \min\left\{k \in \omega : \sum_{N_i \le i < k} \left(\frac{1}{i+1} - 2^{-(i+2)}\right) > 3^{n+2}\right\}$$

Note that $N_{n+1} > N_n + 2$ for each $n \in \omega$. By the definition of N_{n+1}, one has that

$$\sum_{N_n \le i < N_{n+1}-1} \left(\frac{1}{i+1} - 2^{-i-2}\right) \le 3^{n+2}.$$

These two facts imply

$$\sum_{N_n \le i < N_{n+1}} \frac{1}{i+1} \le 3^{n+2} + \frac{1}{N_{n+1}} + \sum_{N_n \le i < N_{n+1}-1} 2^{-i-2} < 3^{n+2} + 1 \tag{3.1}$$

Let $k_n = N_{n+1} - N_n$. Fix a $v \in {}^{\omega}3$. Define $\sigma_n, \tau_n \in {}^{k_n}2$ by

$$\sigma_n = \begin{cases} \tilde{1} \restriction k_n & v(n) = 0 \\ \widetilde{01} \restriction k_n & \text{otherwise} \end{cases} \qquad \tau_n = \begin{cases} \tilde{1} \restriction k_n & v(n) = 1 \\ \widetilde{10} \restriction k_n & \text{otherwise} \end{cases}$$

Let $x = \tilde{0}$, $y = \sigma_0\hat{}\sigma_1\hat{}\sigma_2\hat{}...$, and $z = \tau_0\hat{}\tau_1\hat{}\tau_2....$ Note $(x, y, z) \in [{}^{\omega}2]^3_{E_2}$.

Fix an E_2-tree p. Let $(m_k : k \in \omega)$, $g : {}^{<\omega}2 \to {}^{<\omega}2$, and $\Phi : {}^{\omega}2 \to {}^{\omega}2$ be the associated objects of p coming from the definition of an E_2-tree.

Suppose $v(n) = 0$, then

$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) = \sum_{N_n \le i < N_{n+1}} \varsigma(m_i, m_{i+1}, \Phi(x), \Phi(y)) > \sum_{N_n \le i < n_{n+1}} \left(\frac{1}{i+1} - 2^{-i-2} \right) > 3^{n+2} \tag{3.2}$$

using the definition of N_{n+1}. Also

$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) < \sum_{N_n \le i < N_{n+1}} \left(\frac{1}{i+1} + 2^{-n-2} \right) < 3^{n+2} + 1 + \sum_{N_n \le i < N_{n+1}} 2^{-i-2} < 3^{n+2} + \frac{3}{2} \tag{3.3}$$

using equation (3.1) for the second inequality.

Note also

$$\begin{aligned} \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) &= \varsigma(m_{N_n}, m_{N_n+1}, \Phi(x), \Phi(y)) + \sum_{N_n < i < N_{n+1}} \varsigma(m_i, m_{i+1}, \Phi(x), \Phi(z)) \\ &< \frac{1}{N_n + 1} + 2^{-N_n - 2} + \frac{1}{2} \sum_{N_n < i < N_{n+1}} \frac{1}{i+1} + \sum_{N_n < i < N_{n+1}} 2^{-i-2} \\ &= \frac{1}{2} \sum_{N_n \le i < N_{n+1}} \frac{1}{i+1} + \frac{1}{2} \left(\frac{1}{N_n + 1} \right) + \sum_{N_n \le i < N_{n+1}} 2^{-i-2} < \frac{1}{2}(3^{n+2}) + \frac{3}{2} \end{aligned}$$

using equation (3.1). In summary,

$$\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2} \tag{3.4}$$

Similarly, $\varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(y), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}$. The case for $v(n) = 1$ and $v(n) = 2$ are similar.

It remains to show that $P(\Phi(x), \Phi(y), \Phi(z)) = v$. In the following, let $(L_n : n \in \omega)$ be the sequence defined as above using $(\Phi(x), \Phi(y), \Phi(z))$. The following statements will be proved by induction on n:

(I) $m_{N_n} < L_{n+1} \le m_{N_{n+1}}$.

(II) $P(\Phi(x), \Phi(y), \Phi(z))(n) = v(n)$.

(III) The following holds:

$$\max\{\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(y)), \varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(z)), \varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(y), \Phi(z))\} < \frac{1}{2}(3^{n+2}) + \frac{3}{2}.$$

Suppose properties (I), (II), and (III) holds for all $k < n$. Suppose $v(n) = 0$. (The other cases are similar.)

$L_n \leq m_{N_n}$ by definition if $n = 0$ and by the induction hypothesis otherwise. Therefore,

$$\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(y)) \geq \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) > 3^{n+2}$$

using equation (3.2). This shows $L_{n+1} \leq S_{L_n,n}(\Phi(x), \Phi(y)) \leq m_{N_{n+1}}$. Using the induction hypothesis or the definition when $n = 0$,

$$\max\left\{\varsigma(L_n, m_{N_n}, \Phi(x), \Phi(y)), \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(z)), \varsigma(L_n, m_{N_n}, \Phi(y), \Phi(z))\right\} < \frac{1}{2}(3^{n+1}) + \frac{3}{2} < 3^{n+2}.$$

Hence $L_{n+1} \leq m_{N_n}$ is impossible. This proves (I).

Observe that

$$\begin{aligned}\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(z)) &= \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(z)) + \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) \\ &< \frac{1}{2}(3^{n+1}) + \frac{3}{2} + \frac{1}{2}(3^{n+2}) + \frac{3}{2} \leq 3^{n+2}\end{aligned}$$

using the induction hypothesis and equation (3.4). This shows $S_{L_n,n}(\Phi(x), \Phi(z)) > m_{N_{n+1}}$. Similarly, $S_{L_n,n}(\Phi(y), \Phi(z)) > m_{N_{n+1}}$. $S_{L_n,n}(\Phi(x), \Phi(y)) \leq m_{N_{n+1}}$ has already been shown above. Thus

$$P(\Phi(x), \Phi(y), \Phi(z))(n) = 0 = v(n).$$

This shows (II).

Note that

$$\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(x), \Phi(z)) \leq \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}$$

using equation (3.4). Similarly,

$$\varsigma(L_{n+1}, m_{N_{n+1}}, \Phi(y), \Phi(z)) < \frac{1}{2}(3^{n+2}) + \frac{3}{2}.$$

Finally,

$$\begin{aligned}\varsigma(L_n, m_{N_{n+1}}, \Phi(x), \Phi(y)) &= \varsigma(m_{N_n}, m_{N_{n+1}}, \Phi(x), \Phi(y)) - \big(\varsigma(L_n, L_{n+1}, \Phi(x), \Phi(y)) - \varsigma(L_n, m_{N_n}, \Phi(x), \Phi(y))\big) \\ &< 3^{n+2} + \frac{3}{2} - 3^{n+2} + \frac{1}{2}(3^{n+1}) + \frac{3}{2} < \frac{1}{2}(3^{n+2}) + \frac{3}{2}\end{aligned}$$

using equation (3.3), the definition of the sequence $(L_n : n \in \omega)$, and the induction hypothesis. This proves (III). $\square$

Theorem 3.16.4. *There is a continuous function $Q : [{}^{\omega}2]^3_{E_2} \to {}^{\omega}2$ so that for any E_2-tree p, $Q[[[p]]^3_{E_2}] = {}^{\omega}2$.*

There is a Δ^1_1 function $K : {}^3({}^{\omega}2) \to {}^{\omega}2$ so that on any Σ^1_1 set A with $E_2 \leq_{\mathbf{\Delta}^1_1} E_2 \upharpoonright A$, $K[[A]^3_{E_2}] = {}^{\omega}2$ (and in particular the image meets each E_2-equivalence class).

Proof. One can obtain Q by composing the function from Fact 3.16.3 with a homeomorphism from ${}^{\omega}3 \to {}^{\omega}2$.

One can obtain K by mapping elements of ${}^{3}({}^{\omega}2) \setminus [{}^{\omega}2]^3_{E_2}$ to $\tilde{0}$ and mapping elements in $[{}^{\omega}2]^3_{E_2}$ according to Q. Note that by Fact 3.14.14, every such set A contains an E_2-tree. $\square$

Lemma 3.16.5. *Let X and Y be topological spaces. Let $\mathscr{B} \subseteq \mathscr{P}(X)$ be a nonempty family of subsets of X. Let $A \subseteq Y$ be Borel. Suppose $f : X \to Y$ is a Borel function with the property that for all $B \in \mathscr{B}$ and open $U \subseteq X$ with $U \cap B \neq \emptyset$, there exists an $x \in B$ with $f(x) \notin A$ and an $x' \in U \cap B$ with $f(x') \in A$. Then there is a Borel function $g : X \to Y$ such that $g \restriction B$ is not continuous for all $B \in \mathscr{B}$.*

Proof. The assumption above implies that A is Borel but not equal to either $\emptyset$ or Y. The topology of Y is not $\{\emptyset, Y\}$. There exists some $y_1, y_2 \in Y$ and open set $V \subseteq Y$ with $y_1 \in V$ and $y_2 \notin V$. Define

$$g(x) = \begin{cases} y_1 & f(x) \notin A \\ y_2 & f(x) \in A \end{cases}.$$

Suppose there was some $B \in \mathscr{B}$ so that $g \restriction B$ is a continuous function. By the assumptions, there is a $x \in B$ so that $f(x) \notin A$. So $g(x) = y_1$. By continuity, $g^{-1}[V] \cap B$ is a nonempty open set containing $x \in B$. There is some $U \subseteq X$ open so that $g^{-1}[V] \cap B = U \cap B$. By the assumptions, there some $x' \in U \cap B$ so that $f(x') \in A$. Hence $g(x') = y_2 \notin V$. Contradiction. $\square$

Fact 3.16.6. *There is a $\mathbf{\Delta}^1_1$ function $P' : [{}^{\omega}2]^3_{E_2} \to {}^{\omega}3$ so that on any E_2-tree p, $P' \restriction [[p]]^3_{E_2}$ is not continuous.*

Proof. Apply Lemma 3.16.5 to $X = [{}^{\omega}2]^3_{E_2}$, $Y = {}^{\omega}3$, $\mathscr{B} = \{[[p]]^3_{E_2} : p \text{ is an } E_2\text{-tree}\}$, f the function P from Fact 3.16.3, and $A = \{z \in {}^{\omega}3 : (\exists k)(\forall n > k)(z(n) = 0)\}$. It remains to show that these objects satisfy the required properties of Lemma 3.16.5.

Fix an E_2-tree p. Let $(m_k : k \in \omega)$, $g : {}^{<\omega}2 \to {}^{<\omega}2$, and Φ be the objects associated with p from the definition of an E_2-tree. Fact 3.16.3 implies $P[[[p]]^3_{E_2}] = {}^{\omega}3$. Hence there is some $(x, y, z) \in [[p]]^3_{E_2}$ so that $P(x, y, z) \notin A$. Let $U \subseteq [{}^{\omega}2]^3_{E_2}$ be open so that $U \cap [[p]]^3_{E_2} \neq \emptyset$. There are some $s, t, u \in {}^{<\omega}2$ so that $\emptyset \neq N_{s,t,u} \cap [[p]]^3_{E_2} \subseteq U \cap [[p]]^3_{E_2}$. Let $x' = s\hat{\ }\tilde{0}$, $y' = t\hat{\ }\tilde{1}$, and $z' = u\hat{\ }\widetilde{01}$. Using the computation from the proof of Fact 3.16.3, if k is chosen so that $L_k \geq m_{|s|}$, then for all $n > k$, $P(\Phi(x'), \Phi(y'), \Phi(z'))(n) = 0$. Hence $(\Phi(x'), \Phi(y'), \Phi(z')) \in U \cap [[p]]^3_{E_2}$ and $P(\Phi(x'), \Phi(y'), \Phi(z')) \in A$. $\square$

Theorem 3.16.7. *There is a $\mathbf{\Delta}_1^1$ function $K : {}^3({}^\omega 2) \to {}^\omega 2$ so that for any Σ_1^1 set A with $E_2 \leq_{\mathbf{\Delta}_1^1} E_2 \restriction A$, $K \restriction A$ is not continuous.*

Proof. Use the usual arguments to adjust the domain and range of the function from Fact 3.16.6. Then apply Fact 3.14.14. □

Corollary 3.16.8. *E_2 does not have the 3-Mycielski property.*

Proof. Let $C \subseteq {}^3({}^\omega 2)$ be any comeager set so that $K \restriction C$ is continuous. Then C witnesses the failure of the 3-Mycielski property for E_2. □

3.17 The Structure of E_3

This section will give the characterization of E_3-big Σ_1^1 sets coming from its dichotomy result. See the references mentioned below for the details.

Definition 3.17.1. *E_3 is the equivalence relation on ${}^\omega({}^\omega 2)$ defined by $x\ E_3\ y$ if and only if $(\forall n)(x(n)\ E_0\ y(n))$.*

Definition 3.17.2. *Let $\langle \cdot, \cdot \rangle : {}^2\omega \to \omega$ be some recursive pairing function.*

Let $\pi_1, \pi_2 : {}^2\omega \to \omega$ be projections onto the first and second coordinate, respectively.

Let $A \subseteq \omega$. Define $\text{dom}(A) = \{(i, j) : \langle i, j \rangle \in A\}$.

If $A \subseteq \omega$ is finite, let $L(A) = \sup \pi_1[\text{dom}(A)]$.

If $s \in {}^n 2$, let $\text{grid}(s) : \text{dom}(n) \to 2$ be defined by $\text{grid}(s)(i, j) = s(\langle i, j \rangle)$.

Definition 3.17.3. *$\mathbb{Z}_2$ is the group $(2, +^{\mathbb{Z}_2}, 0)$ where $+^{\mathbb{Z}_2}$ is modulo 2 addition and 0 denotes the identity element.*

Let ${}^\omega\mathbb{Z}_2 = ({}^\omega 2, +^{{}^\omega\mathbb{Z}_2}, \tilde{0})$ where $+^{{}^\omega\mathbb{Z}_2}$ is the coordinate-wise addition of $+^{\mathbb{Z}_2}$ and $\tilde{0}$ is the constant 0 function.

Let ${}^\omega({}^\omega\mathbb{Z}_2) = ({}^\omega({}^\omega 2), +^{{}^\omega({}^\omega\mathbb{Z}_2)}, \bar{0})$ where $+^{{}^\omega({}^\omega\mathbb{Z}_2)}$ is the coordinate-wise addition of $+^{{}^\omega\mathbb{Z}_2}$ and $\bar{0} \in {}^\omega({}^\omega 2)$ is defined by $\bar{0}(k)(j) = 0$ for all $k, j \in \omega$.

Let $Z \subseteq {}^\omega 2$ be defined by $Z = \{x \in {}^\omega 2 : (\exists k)(\forall j > k)(x(j) = 0)\}$.

$\bigoplus_{n\in\omega} \mathbb{Z}_2 = (Z, +^{{}^\omega\mathbb{Z}_2}, \tilde{0})$ is the ω-direct product of $\mathbb{Z}_2$.

${}^\omega(\bigoplus_{n\in\omega} \mathbb{Z}_2) = ({}^\omega Z, +^{{}^\omega({}^\omega\mathbb{Z}_2)}, \bar{0})$ is the ω-product of $\bigoplus_{n\in\omega} \mathbb{Z}_2$.

If $g \in {}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$*, define* $\operatorname{supp}(g) = \{n \in \omega : g(n) \neq \tilde{0}\}$.

${}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ *acts on* ${}^{\omega}({}^{\omega}2)$ *by left addition when* ${}^{\omega}({}^{\omega}2)$ *is considered as* ${}^{\omega}({}^{\omega}\mathbb{Z}_2)$*. That is,* $g \cdot x = g +^{{}^{\omega}({}^{\omega}\mathbb{Z}_2)} x$.

Fact 3.17.4. *$x \, E_3 \, y$ if and only if there is a $g \in {}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ so that $x = g \cdot y$.*

Definition 3.17.5. *A grid system is a sequence $(g_{s,t} : s,t \in {}^{<\omega}2 \wedge |s| = |t|\})$ in ${}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)$ with the following properties:*

(I) If $s,t,u \in {}^{n}2$ for some $n \in \omega$, then $g_{s,u} = g_{t,u} +^{{}^{\omega}(\bigoplus_{n\in\omega}\mathbb{Z}_2)} g_{s,t}$.

(II) For all $m \leq n$, $s,t \in {}^{n}2$, $u,v \in {}^{m}2$ and $l \in \pi_1[\mathrm{dom}(n)]$ with $u \subseteq s$, and $v \subseteq t$, if

$$\mathrm{grid}(s) \restriction \mathrm{dom}(n \setminus m) \cap ((l+1)\times\omega) = \mathrm{grid}(t) \restriction \mathrm{dom}(n \setminus m) \cap ((l+1)\times\omega)$$

then for all $i \leq l$, $g_{s,t}(i) = g_{u,v}(i)$.

Fact 3.17.6. *Let $B \subseteq {}^{\omega}({}^{\omega}2)$ be Σ^1_1 so that $E_3 \restriction B \equiv_{\mathbf{\Delta}^1_1} E_3$. Then there is a continuous injective map $\Phi : {}^{\omega}({}^{\omega}2) \to {}^{\omega}({}^{\omega}2)$, a grid system $(g_{s,t} : s,t \in {}^{<\omega}2 \wedge |s| = |t|)$, and sequences $(k_i : i \in \omega)$ and $(p_{m,i} : m,i \in \omega)$ in ω with the following properties:*

(i) $\Phi[{}^{\omega}({}^{\omega}2)] \subseteq B$.

(ii) If $s,t \in {}^{n}2$, then $\operatorname{supp}(g_{s,t}) \subseteq (k_{L(n)} + 1)$.

(iii) For each $m \in \omega$, $(k_i : i \in \omega)$ and $(p_{m,i} : i \in \omega)$ are strictly increasing sequences.

(iv) For all $x,y \in {}^{\omega}({}^{\omega}2)$ and $m,j \in \omega$, if $x(m)(j) = 0$ and $y(m)(j) = 1$, then $\Phi(x)(k_m)(p_{m,j}) = 0$ and $\Phi(y)(k_m)(p_{m,j}) = 1$.

(v) Let $x,y \in {}^{\omega}({}^{\omega}2)$ and $l \in \omega$. Suppose

$$\Big(\forall (i,j) \in ((l+1)\times\omega) \setminus dom(n)\Big)\Big(x(i)(j) = y(i)(j)\Big).$$

Let $s,t \in {}^{n}2$ be such that for all $(i,j) \in \mathrm{dom}(n)$, $\mathrm{grid}(s)(i,j) = x(i)(j)$ and $\mathrm{grid}(t)(i,j) = y(i)(j)$. Then $(g_{s,t} \cdot \Phi(x))(l) = \Phi(y)(l)$.

Proof. This is implicit in [9]. See the presentation in [13] Chapter 14, especially Section 14.5 and 14.6. □

Note that Φ as above is an E_3 reduction.

3.18 E_3 Does Not Have the 2-Mycielski Property

Definition 3.18.1. *For each $s \in {}^{<\omega}2$, let $N_{\text{grid}(s)} = \{x \in {}^{\omega}({}^{\omega}2) : (\forall(i,j) \in \text{dom}(|s|))(x(i)(j) = s(\langle i, j\rangle))\}$.*

Each $N_{\text{grid}(s)}$ is an open neighborhood of ${}^{\omega}({}^{\omega}2)$ and also the collection $\{N_{\text{grid}(s)} : s \in {}^{<\omega}2\}$ forms a basis for the topology of ${}^{\omega}({}^{\omega}2)$.

When $\sigma : m \to {}^{<\omega}2$, then N_σ will refer to the usual basic open neighborhood of ${}^{\omega}({}^{\omega}2)$. Both types of open sets will be used in the proof of the following result.

Theorem 3.18.2. *Let $D = \{(x, y) \in {}^2({}^{\omega}({}^{\omega}2)) : x(0) \neq y(0)\}$. D is dense open.*

For all Σ^1_1 sets $B \subseteq {}^{\omega}({}^{\omega}2)$ with $E_3 \upharpoonright B \equiv_{\mathbf{\Delta}^1_1} E_3$, $[B]^2_{E_3} \not\subseteq D$.

E_3 does not have the 2-Mycielski property.

Proof. Suppose $(x, y) \in D$. There is some n so that $x(0)(n) \neq y(0)(n)$. Define $\sigma, \tau : 1 \to {}^{<\omega}2$ by $\sigma(0) = x(0) \upharpoonright (n + 1)$ and $\tau(0) = y(0) \upharpoonright (n + 1)$. Then $(x, y) \in N_{\sigma,\tau} \subseteq D$. D is open.

Suppose $\sigma, \tau : m \to {}^{<\omega}2$ have the property that for all $k < m$, $|\sigma(k)| = |\tau(k)|$. Define $\sigma', \tau' : m \to {}^{<\omega}2$ by

$$\sigma'(k) = \begin{cases} \sigma(k) & k \neq 0 \\ \sigma(k)\hat{\ }0 & k = 0 \end{cases} \quad \text{and} \quad \tau'(k) = \begin{cases} \tau(k) & k \neq 0 \\ \tau(k)\hat{\ }1 & k = 0 \end{cases}$$

$N_{\sigma',\tau'} \subseteq N_{\sigma,\tau}$ and $N_{\sigma',\tau'} \subseteq D$. D is dense open.

Fix Φ and the other objects specified by Fact 3.17.6. Note that for any $s \in {}^{<\omega}2$, $g_{s,s} = \bar{0}$. In particular, $g_{\emptyset,\emptyset} = \bar{0}$.

Let $\rho_n : 1 \to {}^{<\omega}2$ be defined by $\rho_n(0) = \Phi(\bar{0})(0) \upharpoonright n$. If $s \in {}^{<\omega}2$, then define $x_s \in {}^{\omega}({}^{\omega}2)$ by

$$x_s(i)(j) = \begin{cases} s(\langle i, j\rangle) & (i, j) \in \text{dom}(|s|) \\ 0 & \text{otherwise} \end{cases}.$$

Let $s_0 = \emptyset$.

Suppose one has defined $s_n \in {}^{<\omega}2$ so that $x_{s_n}(0) = \tilde{0}$ and $\Phi(x_{s_n})(0) = \Phi(\bar{0})(0)$. By continuity, find some $u \in {}^{<\omega}2$ with $s_n \subseteq u$ and $x_{s_n} \in N_{\text{grid}(u)}$ so that $N_{\text{grid}(u)} \subseteq \Phi^{-1}[N_{\rho_{n+1}}]$. Now find the least

$k > |u|$ so that $k = \langle 1, q \rangle$ for some $q \in \omega$. Define $s_{n+1} \supseteq u$ of length $k + 1$ by

$$s_{n+1}(j) = \begin{cases} u(j) & j < |u| \\ 1 & j = k \\ 0 & \text{otherwise} \end{cases}.$$

Note that since $x_{s_{n+1}}(0) = \tilde{0} = \bar{0}(0)$, $(g_{\emptyset,\emptyset} \cdot \Phi(x_{s_{n+1}}))(0) = \Phi(\bar{0})(0)$ by condition (v) of Definition 3.17.6. This implies that $\Phi(x_{s_{n+1}})(0) = \Phi(\bar{0})(0)$.

Now define $x \in {}^{\omega}({}^{\omega}2)$ by

$$x(i)(j) = \left(\bigcup_{n \in \omega} \text{grid}(s_n) \right) (i, j).$$

Since $N_{\text{grid}(s_n)} \subseteq \Phi^{-1}[N_{\rho_n}]$ for all $n \in \omega$, $\Phi(x)(0) = \Phi(\bar{0})(0)$. Hence $(\Phi(x), \Phi(\bar{0})) \notin D$. However, there are infinitely many $q \in \omega$ so that $x(1)(q) = 1$. Since Φ is an E_3 reduction, $\neg(\Phi(x)\ E_3\ \Phi(\bar{0}))$.

We have shown that for any map Φ as in Fact 3.17.6, $[\Phi[{}^{\omega}({}^{\omega}2)]]^2_{E_3} \nsubseteq D$. Since any Σ^1_1 set $B \subseteq {}^{\omega}({}^{\omega}2)$ with the property that $E_3 \upharpoonright B \equiv_{\Delta^1_1} E_3$ has some such map Φ so that $\Phi[{}^{\omega}({}^{\omega}2)] \subseteq B$, no such B can have the property that $[B]^2_{E_3} \subseteq D$. E_3 does not have the 2-Mycielski property. □

3.19 Completeness of Ultrafilters on Quotients

Without the axiom of choice, one needs to define the notion of completeness of ultrafilters with care.

Definition 3.19.1. *Let X be a set. Let U be an ultrafilter on X. Let I be a set. U is I-complete if and only if for any set J which inject into I but is not in bijection with I, and any injective function $f : J \to U$, $\bigcap_{j \in J} f(j) \in U$.*

U is I^+-complete if and only if for all J which inject into I and all injective functions $f : J \to U$, $\bigcap_{j \in J} f(j) \in U$.

$\aleph_1$-complete is often called countably complete. A well-known result is that there are no countably complete ultrafilters on ${}^{\omega}2$. There are countably complete ultrafilters on quotients of Polish spaces by equivalence relations.

Fact 3.19.2. *Let $C \subseteq \mathcal{P}({}^{\omega}2/E_0)$ be defined by $A \in C$ if and only if $\bigcup A$ belongs to the comeager filter on ${}^{\omega}2$. C is a countably complete ultrafilter on ${}^{\omega}2/E_0$.*

Proof. C is an ultrafilter follows from the generic ergodicity of E_0. Countable completeness is clear; in fact under AD, every ultrafilter is countably complete. $\square$

A natural question is whether this ultrafilter or any ultrafilter on ${}^{\omega}2/E_0$ could be more than just countably complete. $\mathbb{R}$ injects into ${}^{\omega}2/E_0$. Is C $\mathbb{R}^+$-complete? Note the function f in Definition 3.19.1 is required to be injective. Otherwise this notion becomes clearly trivial using the function $f : \mathbb{R} \to C$ defined by $x \mapsto ({}^{\omega}2/E_0) \setminus \{[x]_{E_0}\}$. The next fact will show using a modification of the above function that there are no nonprincipal $\mathbb{R}^+$-complete ultrafilters on quotients of Polish spaces.

Fact 3.19.3. ($\mathsf{ZF} + \mathsf{AD}$) *Suppose E is an equivalence relation on a Polish space X so that $=\leq E$ (where $\leq$ denotes the existence of a reduction). Then no nonprincipal ultrafilter on X/E is $\mathbb{R}^+$-complete.*

Proof. Let U be a nonprincipal $(\mathbb{R})^+$-complete ultrafilter on X/E. Let $\Psi : {}^{\omega}2 \to X$ be a reduction witnessing $=\leq E$. Let $\Phi : {}^{\omega}2 \to X/E$ be defined by $\Phi(x) = [\Psi(x)]_E$. Φ is an injective function.

Let $\tilde{L} = (X/E) \setminus \Phi[{}^{\omega}2] = \bigcap_{x \in {}^{\omega}2} (X/E) \setminus \{\Phi(x)\}$. $\tilde{L} \in U$ since U is both nonprincipal and $\mathbb{R}^+$-complete.

Let $L = \bigcup \tilde{L}$. L must be uncountable. Hence L is in bijection with ${}^{\omega}2$. Define $f : L \to (X/E)$ by

$$f(x) = (X/E) \setminus \{[x]_E, \Phi(x)\}$$

To show f is injective, it suffices to show that the map on L defined by $x \mapsto \{[x]_E, \Phi(x)\}$ is injective: Suppose $x \neq y$ and $\{[x]_E, [\Psi(x)]_E\} = \{[y]_E, [\Psi(y)]_E\}$. Since Ψ is a reduction, $\neg(\Psi(x)\ E\ \Psi(y))$. Therefore, one must have that $x\ E\ \Psi(y)$. This is impossible since $[x]_E \in (X/E) \setminus \Phi[{}^{\omega}2]$. This shows f is injective.

Since for all $x \in L$, $[x]_E \notin f(x)$, $\tilde{L} \cap \bigcap_{x \in L} f(x) = \emptyset$. Since $\tilde{L} \in U$, $\bigcap_{x \in L} f(x) \notin U$. U is not $\mathbb{R}^+$-complete. Contradiction. $\square$

Fact 3.19.4. ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ *or* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$) *Let X be a Polish space and E be an equivalence relation on ${}^{\omega}2$. If ${}^{\omega}2/E$ is not well-ordered, then there is no $\mathbb{R}^+$-complete nonprincipal ultrafilter on X/E.*

Proof. Under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, results of Woodin and Martin show that every set of reals is κ-Suslin for some $\kappa < \Theta$. So the complement of E is κ-Suslin for some $\kappa < \Theta$. In $\mathsf{ZF} + \mathsf{AD}$, [6] showed that if the complement of E is κ-Suslin, then either the identity reduces into E or ${}^{\omega}2/E$ is in bijection with a cardinal less than or equal to κ (and hence can be well-ordered).

Under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, [2] Theorem 1.4 (along with [2] Corollary 3.2) states that for any set X, either X is wellordered or $\mathbb{R}$ injects into X.

In either case, the result now follows from 3.19.3. $\square$

3.20 Conclusion

This section includes some questions.

Question 3.20.1. *Under* $\mathsf{ZF} + \neg\mathrm{AC}_{\omega}^{\mathbb{R}}$*, can there be* ω*-Jónsson functions for* ${}^{\omega}2$*?*

In particular, is there an ω*-Jónsson function for* ${}^{\omega}2$ *in the Cohen-Halpern-Lévy model* H *(see Question 3.4.2)?*

Question 3.20.2. *It was shown that* E_2 *does not have the* 2*-Mycielski property. An interesting question would be: what is the relation between the n-Mycielski property, n-Jónsson property, and the surjectivity properties in dimension n for* E_2*?*

In particular, does the 2-dimensional version of the results in Section 3.16 hold?

Does ${}^{\omega}2/E_2$ *have the 2-Jónsson property, 3-Jónsson property, or full Jónsson property?*

Question 3.20.3. *For* E_1 *and* E_3*, this paper only considers the Mycielski property. One can ask about some of the other properties of* E_1 *or* E_3 *which had been studied for* E_0 *and* E_2*. For example:*

Does ${}^{\omega}({}^{\omega}2)/E_1$ *or* ${}^{\omega}({}^{\omega}2)/E_3$ *have the Jónsson property?*

Question 3.20.4. *Assuming determinacy, if* R/E_0 *injects into a set* X*, can* X *have the Jónsson property? More specifically, if* E *is a* $\boldsymbol{\Delta}_1^1$ *equivalence relation on* $\mathbb{R}$ *so that* $E_0 \leq_{\boldsymbol{\Delta}_1^1} E$*, can* $\mathbb{R}/E$ *have the Jónsson property?*

FRACTIONAL BOREL CHROMATIC NUMBERS AND DEFINABLE COMBINATORICS

4.1 Introduction

A **graph** is an ordered pair $\mathbf{G} = (X, G)$, where X is a set and $G \subseteq X \times X$ is symmetric and irreflexive. For any set Y, a Y-**colouring** on $\mathbf{G}$ is a function $c : X \to Y$ such that if $x \, G \, y$, then $c(x) \neq c(y)$. The **chromatic number** $\chi(\mathbf{G})$ is then the minimum cardinality of Y such that there is a Y-colouring of $\mathbf{G}$. A slight generalization of this is as follows: if $b \geq 1$ is an integer and Y is a set of colours, a b**-fold colouring** of a graph $\mathbf{G} = (X, G)$ is an assignment $C : X \to [Y]^b$ (the subsets of Y of size b) such that if $x \, G \, y$, then $C(x) \cap C(y) = \emptyset$. If $|Y| = a \in \mathbb{N}$, then C is an $a : b$**-colouring**. The b**-fold chromatic number** $\chi^{(b)}(\mathbf{G})$ is then defined exactly as the regular chromatic number. Since $\chi^{(b)}(\mathbf{G}) = \chi(\mathbf{G})$ whenever $\chi(\mathbf{G})$ is infinite, the notion is only interesting when $\chi(\mathbf{G})$ is finite. In this case, the sequence $\left\{\chi^{(b)}(\mathbf{G})\right\}_{b \geq 1}$ is subadditive [26]. The **fractional chromatic number** of such a graph $\mathbf{G}$ is $\chi^f(\mathbf{G}) = \lim_{b \to \infty} \frac{\chi^{(b)}(\mathbf{G})}{b} = \inf_b \frac{\chi^{(b)}(\mathbf{G})}{b}$.

As with many combinatorial notions, the field of descriptive set theory is interested in how the study changes when one restricts the focus to only definable objects. The subfield known as descriptive graph combinatorics began in the paper of Kechris, Solecki, and Todorčević [15] and is comprehensively reviewed in [19].

In case X is a standard Borel space, we can consider analogous definable notions of colouring. If α is a class of functions between standard Borel (or Polish) spaces (such as Borel functions), we define the α**-chromatic number** $\chi_\alpha(\mathbf{G})$ to be the minimum cardinality of a standard Borel Y such that there is a Y-colouring of $\mathbf{G}$ in the class α. In this paper, we will be interested in the cases $\alpha = B$, the Borel functions, $\alpha = BM$, the Baire measurable functions, and $\alpha = \mu$, the μ-measurable functions for μ a Borel probability measure on X. It will occasionally be helpful to also denote the classical chromatic number χ as χ_c to fit with this convention. In case $\chi_\alpha(\mathbf{G})$ is finite, we can analogously define $\chi_\alpha^{(b)}(\mathbf{G})$ for every $b \geq 1$, in addition to $\chi_\alpha^f(\mathbf{G})$.

Given graphs $\mathbf{G}_0 = (X, G_0)$ and $\mathbf{G}_1 = (Y, G_1)$, a **homomorphism** from $\mathbf{G}_0$ to $\mathbf{G}_1$ is a function $f : X \to Y$ such that for every $a, b \in X$, $a \, G_0 \, b$ implies that $f(a) \, G_1 \, f(b)$. In this case, we write $\mathbf{G}_0 \leq \mathbf{G}_1$, and we write $\mathbf{G}_0 \leq_B \mathbf{G}_1$ if X, Y are standard Borel and f is Borel. It is easy to see that $\mathbf{G}_0 \leq_B \mathbf{G}_1$ implies that $\chi_B(\mathbf{G}_0) \leq \chi_B(\mathbf{G}_1)$.

The classical chromatic number is a celebrated invariant of graph theory. The fractional chromatic number uncovers some answers to related combinatorial problems. Scheinerman and Ullman [26] give the following example: suppose we have a set of committees who require hour-long meetings with graph relation G determining that two committees cannot meet at the same time. If $\mathbf{G} = \mathbf{C}_5$, the cycle on 5 vertices, then since $\chi(\mathbf{C}_5) = 3$, all meetings can take place within 3 hours. However, since there is a $5 : 2$-colouring of $\mathbf{C}_5$, they may take place in 2.5 hours if the hour-long meetings can be split into 2 parts.

The fractional chromatic number χ^f satisfies $\chi^f \leq \chi$ by its definition. It also can take any value in $\{0\} \cup \{1\} \cup [2, \infty)$. Lastly, if $\chi^f(\mathbf{G}) = 2$, it follows that $\chi(\mathbf{G}) = 2$ since both are equivalent to the characterization that $\mathbf{G}$ has no odd cycles.

In descriptive graph combinatorics, it is common to examine which theorems of classical combinatorics remain true in the definable setting and to determine what the analog is for those that do not. In this paper, we begin what appears to be the first examination of the fractional Borel chromatic number.

In section 4.2, we review basic facts about the fractional chromatic number that we will require in the paper. In particular, we show that for any values $2 < \chi^f \leq \chi$, there is a countable graph attaining these values.

In sections 4.3 and 4.4, we prove that an analog of this is true for both the Baire measurable and the μ-measurable chromatic number. In fact, this is due to a more general phenomenon. Given a countable graph, one may recreate it as a graph on a Polish space that is acyclic but such that any Baire measurable colouring must essentially follow the rules of classical colouring on the original graph.

The exact statements are:

Theorem 4.3.1. *Let $\mathbf{G} = (Y, G)$ be a countable graph. There exists a Borel graph $\mathbf{G}' = \left({}^{\omega}2 \times Y, G'\right)$ such that*

1. *if $y_0 \, G \, y_1$, then for all nonmeager $B_0, B_1 \subseteq {}^{\omega}2$, there exist $x_0 \in B_0$ and $x_1 \in B_1$ with $(x_0, y_0) \; G' \; (x_1, y_1)$; and*
2. *if $\neg y_0 \, G \, y_1$, then for all $x_0, x_1 \in {}^{\omega}2$, we have $\neg(x_0, y_0) \, G' \, (x_1, y_1)$.*

Theorem 4.4.1. *Let $\mathbf{G} = (Y, G)$ be a countable graph and let μ be Lebesgue measure on $[0, 1]$. There exists a Borel graph $\mathbf{G}' = ([0, 1] \times Y, G')$ such that*

1. *if $y_0 \, G \, y_1$, then for all non μ-null $B_0, B_1 \subseteq [0,1]$, there exist $x_0 \in B_0$ and $x_1 \in B_1$ with $(x_0, y_0) \; G' \; (x_1, y_1)$; and*

2. *if $\neg y_0 \, G \, y_1$, then for all $x_0, x_1 \in [0,1]$, we have $\neg(x_0, y_0) \, G' \, (x_1, y_1)$.*

These theorems essentially allow one to strip away the original combinatorics of the graph at a classical scale (by making it acyclic and hence bipartite) while leaving the original combinatorics intact at a definable scale.

In section 4.5, we then show that these two theorems can be freely combined:

Theorem 4.5.1. *For any possible way of assigning values from the set $[2, \infty)$ to χ, χ_μ, χ_{BM}, χ_B, and their fractional counterparts that is consistent with the obvious (classical) conditions, there is a Borel graph G on a Polish space X with a probability measure μ realizing these eight values.*

Lastly, in Section 4.6, we show perhaps the most interesting fact: the rule that $\chi^f(\mathbf{G}) = 2$ implies that $\chi(\mathbf{G}) = 2$ does not transfer over to the definable case.

Theorem 4.6.1. *There exists a Borel graph on a Polish space X with $\chi_B^f(\mathbf{G}) = 2$, but $\chi_B(\mathbf{G}) = 3$.*

In light of this fact, we theorize that one should be able to replace 3 in the above theorem with n for any $n \geq 4$.

4.2 Elementary Facts about the Fractional Chromatic Number

First, we will observe that one can make the gap between χ^f and χ arbitrarily large with finite graphs.

Definition 4.2.1. *Given positive integers a, b with $a \geq 2b$, define $K_{a:b} \subseteq [a]^b \times [a]^b$ by*

$$s \, K_{a:b} \, t \qquad \Leftrightarrow \qquad s \cap t = \emptyset$$

and let $\mathbf{K}_{a:b} = \left([a]^b, K_{a:b}\right)$. Graphs of the form $\mathbf{K}_{a:b}$ are known as Kneser graphs.

It is known (see Section 3.2 of [26]) that $\chi^f(\mathbf{K}_{a:b}) = \frac{a}{b}$ and $\chi(\mathbf{K}_{a:b}) = a - 2b + 2$.

Lemma 4.2.2. *Given any integers $n \geq 1$ and $m \geq 3$, there is a finite graph $\mathbf{G}$ with $\chi^f(\mathbf{G}) = 2 + \frac{1}{n}$ and $\chi(\mathbf{G}) = m$.*

Proof. Let $\mathbf{G} = \mathbf{K}_{(m-2)(2n+1):(m-2)n}$. $\square$

If $\mathbf{C}_m$ is the cycle graph on m vertices, then $\chi^f(\mathbf{C}_{2m+1}) = 2 + \frac{1}{m}$ [26]. Hence, for an arbitrary graph $\mathbf{G}$, $\chi^f(\mathbf{G}) = 2$ implies the lack of odd cycles and hence $\chi^f(\mathbf{G}) = 2 \Rightarrow \chi(\mathbf{G}) = 2$.

It will be useful to define a special graph. Let $X = [0, 1)$ regarded as the circle, i.e. equipped with the metric

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Definition 4.2.3. *Given any real number $r \geq 2$, define $G_r \subseteq X^2$ by*

$$x\, G_r\, y \qquad \Leftrightarrow \qquad d(x, y) \geq \frac{1}{r}.$$

Let $\mathbf{G}_r = (X, G_r)$. Clearly, $\mathbf{G}_r$ is closed and hence Borel.

Lemma 4.2.4. *Given any real $r \geq 2$ and positive integers c, d such that $r \leq \frac{c}{d}$, there exists a Borel $c : d$-colouring of $\mathbf{G}_r$. In particular, $\chi^f_B(\mathbf{G}_r) \leq r$ and $\chi_B(\mathbf{G}_r) \leq \lceil r \rceil$.*

Proof. Define the map $f : [0, d) \to c$ by $f(x) = i \Leftrightarrow x \in \left[\frac{i}{r}, \frac{i+1}{r}\right)$. Since $\frac{c}{r} \geq d$, f indeed maps into c. Then define $g : X \to [c]^d$ by

$$g(x) = \{f(x), f(x + 1), \ldots, f(x + d - 1)\}.$$

Clearly, g is a Borel map. It is straightforward to check by a case analysis that g is a $c : d$-colouring of $\mathbf{G}_r$. $\square$

Definition 4.2.5. *Given positive integers a, b with $a \geq 2b$, let*

$$i\, G_{a,b}\, j \qquad \Leftrightarrow \qquad j \in \{i + b, i + b + 1, \ldots, i + a - b\}\ (mod\ a)$$

for $i, j < a$ and let $\mathbf{G}_{a,b} = (a, G_{a,b})$.

It is known [26] that $\chi^f(\mathbf{G}_{a,b}) = \frac{a}{b}$. In fact, we can slightly improve this statement.

Proposition 4.2.6. *For all positive integers a, b with $a \geq 2b$, $\mathbf{G}_{a,b} \leq \mathbf{G}_{\frac{a}{b}}$. In particular, if c, d are positive integers with $\frac{a}{b} \leq \frac{c}{d}$, then there is a $c : d$-colouring of $\mathbf{G}_{a,b}$.*

Proof. Define $\phi : a \to X$ by $\phi(i) = \frac{i}{a}$. It is straightforward to check that ϕ is an embedding of $\mathbf{G}_{a,b}$ into $\mathbf{G}_{\frac{a}{b}}$. $\square$

The reason that Proposition 4.2.6 is useful is that we can find relatively nice d-fold colourings for a countable disjoint union of graphs of the form $\mathbf{G}_{a,b}$. This yields the following result:

Lemma 4.2.7. *Given any real $\alpha > 2$ and any integer $m \geq \alpha$, there exists a countable graph* $\mathbf{G}$ *with* $\chi^f(\mathbf{G}) = \alpha$ *and* $\chi(\mathbf{G}) = m$.

Proof. Choose sequences p_i, q_i of positive integers such that $2 \leq \frac{p_i}{q_i} \leq \alpha$ and $\frac{p_i}{q_i} \to \alpha$. Let $\mathbf{G}_1 = \bigsqcup_{i<\omega} \mathbf{G}_{p_i,q_i}$. Then $\chi^f(\mathbf{G}_1) \geq \chi^f\left(\mathbf{G}_{p_i,q_i}\right) = \frac{p_i}{q_i}$, so $\chi^f(\mathbf{G}_1) \geq \alpha$. Given $\frac{c}{d} \geq \alpha$, since $\frac{c}{d} \geq \frac{p_i}{q_i}$, we can find a $c : d$-colouring of each $\mathbf{G}_{p_i,q_i}$ and hence a $c : d$-colouring of $\mathbf{G}$ by joining these together, showing $\chi^f(\mathbf{G}_1) = \alpha$. It follows that $\chi(\mathbf{G}_1) = \lceil \alpha \rceil$ by using the above argument with $c = \lceil \alpha \rceil$ and $d = 1$. Then choose $m \geq 1$ so that $2 + \frac{1}{n} \leq \alpha$ and let G_2 be a finite graph with $\chi^f(\mathbf{G}_2) = 2 + \frac{1}{n}$ and $\chi(\mathbf{G}_2) = m$, as furnished by Lemma 4.2.2. □

4.3 The Baire Measurable Case

We first handle the Baire measurable case, showing that we can construct an acyclic graph that sets χ^f_{BM} and χ_{BM} to any possible values we wish them to be. Our construction follows that of Miller [23], who showed that the equivalence relation E_0 on ${}^{\omega}2$ can be treed by a graph relation $\mathbf{G}$ such that $\chi_{BM}(\mathbf{G}) = \kappa$ for any $\kappa \in \left\{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\right\}$. In effect, for finite n, Miller divided the space ${}^{\omega}2$ into n subspaces such that every Baire measurable colouring could not assign the same colour to nonmeager subsets of different subspaces. We generalize this below; as Miller's graph was to $\mathbf{K}_n$, so ours is to an arbitrary countable graph $\mathbf{G}$.

Theorem 4.3.1. *Let* $\mathbf{G} = (Y, G)$ *be a countable graph. There exists a Borel graph* $\mathbf{G}' = \left({}^{\omega}2 \times Y, G'\right)$ *such that*

1. *if $y_0 \, G \, y_1$, then for all nonmeager $B_0, B_1 \subseteq {}^{\omega}2$, there exist $x_0 \in B_0$ and $x_1 \in B_1$ with $(x_0, y_0) \; G' \; (x_1, y_1)$; and*
2. *if $\neg y_0 \, G \, y_1$, then for all $x_0, x_1 \in {}^{\omega}2$, we have $\neg(x_0, y_0) \, G' \, (x_1, y_1)$.*

In particular, we have that $\chi(\mathbf{G}') = \chi^f(\mathbf{G}') = 2$, $\chi_{BM}(\mathbf{G}') = \chi_B(\mathbf{G}') = \chi(\mathbf{G})$, *and* $\chi^f_{BM}(\mathbf{G}') = \chi^f_B(\mathbf{G}') = \chi^f(\mathbf{G})$.

Proof. As mentioned above, most of our construction works due to Miller's argument in [23]. Enumerate Y as $\{y_i\}_{i<|Y|}$. Because ${}^{<\omega}2$ is countable, we can pick sequences $\{u_n\}$, $\{v_n\}$ in ${}^{<\omega}2$ such that for each n, u_n, v_n are of length $n+1$, $u_n(n) \neq v_n(n)$, there exist $i \neq j$ such that $0^i{}^\frown 1 \subseteq u_n$ and $0^j{}^\frown 1 \subseteq v_n$, and for any such pair $u, v \in {}^{<\omega}2$, there exists $n \in \mathbb{N}$ with $u \subseteq u_n$ and $v \subseteq v_n$. We then define G' by

$$(x_0, y_i)G'(x_1, y_j) \Leftrightarrow$$
$$y_i G y_j \wedge \exists n \left(u_n \subseteq 0^i{}^\frown 1^\frown x_0 \wedge v_n \subseteq 0^j{}^\frown 1^\frown x_1 \wedge \forall m \geq n+1 \left(\left(0^i{}^\frown 1^\frown x\right)(m) = \left(0^j{}^\frown 1^\frown y\right)(m)\right)\right).$$

By our choice of sequences $\{u_n\}$, $\{v_n\}$, $\mathbf{G}'$ is acyclic. Property (2) is also immediately clear from the definition of the graph.

To prove property (1), suppose that $y_i \; G \; y_j$. Choose $u, v \in {}^{<\omega}2$ such that B_0 is comeager in N_u and B_1 is comeager in N_v. Then by our choice of $\{u_n\}$, $\{v_n\}$, there exists an n with $0^i{}^\frown 1^\frown u \subseteq u_n$ and $0^j{}^\frown 1^\frown v \subseteq v_n$. Define u', v' so that $0^i{}^\frown 1^\frown u' = u_n$ and $0^j{}^\frown 1^\frown v' = v_n$. If we choose $x \in {}^\omega 2$ such that $u'^\frown x \in B_0$ and $v'^\frown x \in B_1$, then we have $(u'^\frown x, y_i) \; G' \; \left(v'^\frown x, y_j\right)$.

Colouring $\mathbf{G}'$ according to the colouring of $\mathbf{G}$ shows that $\chi_B(\mathbf{G}') \leq \chi(\mathbf{G})$ and $\chi_B^f(\mathbf{G}') \leq \chi^f(\mathbf{G})$. If c is a Baire measurable k-fold colouring of $\mathbf{G}'$, then define an induced function c' on Y by setting $c'(y)$ to be some k-set whose preimage under c is nonmeager in ${}^\omega 2 \times \{y\}$. Then property (1) shows that c' must be a k-fold colouring of $\mathbf{G}$. Hence $\chi_{BM}^{(k)}(\mathbf{G}') \geq \chi^{(k)}(\mathbf{G})$ for every $k \in \mathbb{N}$. This shows that $\chi_{BM}(\mathbf{G}') \geq \chi(\mathbf{G})$ and $\chi_{BM}^f(\mathbf{G}') \geq \chi^f(\mathbf{G})$. □

One can view the graph from Theorem 4.3.1 in the following way: each vertex from the original countable graph Y has been enlarged from a single point to become the Polish space ${}^\omega 2$. Then "almost all" of the points in a pair of these "enlarged vertices" are connected in the graph $\mathbf{G}'$ if and only if they were connected in $\mathbf{G}$. The connections are in a definable way such that the resulting graph is acyclic. So from a classical viewpoint, the graph is bipartite and hence has lost its original structure. But from a definable view, the combinatorics of the graph have been preserved.

Corollary 4.3.2. *For any assignment of the quantities $\chi, \chi^f, \chi_{BM}, \chi_{BM}^f \in [2, \infty)$ that obeys the conditions $\chi^f \leq \min\left\{\chi, \chi_{BM}^f\right\}$, $\max\left\{\chi, \chi_{BM}^f\right\} \leq \chi_{BM}$, $\chi, \chi_{BM} \in \mathbb{N}$, and $\chi_\alpha^f = 2 \Rightarrow \chi_\alpha = 2$ for $\alpha \in \{c, BM\}$, there exists a Borel graph $\mathbf{G}$ on a Polish space X whose chromatic numbers satisfy this assignment.*

Proof. Let $\mathbf{G}_0, \mathbf{G}_1$ be countable graphs with chromatic numbers χ_{BM} and χ and fractional chromatic number χ_{BM}^f and χ^f, respectively. Then let $\mathbf{G}'$ be the Borel graph obtained from $\mathbf{G}_0$ as in Theorem 4.3.1 and let $\mathbf{G}$ be the disjoint union of $\mathbf{G}'$ and $\mathbf{G}_1$. □

4.4 The μ-Measurable Case

We wish to repeat Section 4.3 for μ-measurable colourings, and so we require a new construction. This time, we modify a graph constructed by Laczkovich as noted in the appendix of [15]. Laczkovich divides the interval $[0, 1]$ into n subspaces in such a way that any Lebesgue measurable colouring must restrict each colour almost everywhere to a particular subspace. This was an early example of a graph for which $\chi(\mathbf{G}) = 2$ but $\chi_B(\mathbf{G}) = n$ for any n. Again, we modify the graph to do the same to any countable graph as the original one did to $\mathbf{K}_n$.

Theorem 4.4.1. *Let $\mathbf{G} = (Y, G)$ be a countable graph and let μ be Lebesgue measure on $[0, 1]$. There exists a Borel graph $\mathbf{G}' = ([0, 1] \times Y, G')$ such that*

1. *if $y_0 \, G \, y_1$, then for all non μ-null $B_0, B_1 \subseteq [0, 1]$, there exist $x_0 \in B_0$ and $x_1 \in B_1$ with $(x_0, y_0) \; G' \; (x_1, y_1)$; and*
2. *if $\neg y_0 \, G \, y_1$, then for all $x_0, x_1 \in [0, 1]$, we have $\neg(x_0, y_0) \, G' \, (x_1, y_1)$.*

In particular, if we let μ' be a product measure on we have that $\chi\,(\mathbf{G}') = \chi^f\,(\mathbf{G}') = 2$, $\chi_\mu\,(\mathbf{G}') = \chi_B\,(\mathbf{G}') = \chi\,(\mathbf{G})$, and $\chi^f_\mu\,(\mathbf{G}') = \chi^f_B\,(\mathbf{G}') = \chi^f\,(\mathbf{G})$.

Proof. We will instead construct our Borel graph on a Borel subset $X \subseteq [0, 1]$ and argue that it is isomorphic to a graph with the properties above. Let $U = \left\{(a, b, c, d) \in \mathbb{R}^4 : ad - bc \neq 0\right\}$. Then U is open and hence we may find a sequence $\{(a_k, b_k, c_k, d_k)\}$ that is dense in U and such that $\{a_k, b_k, c_k, d_k : k \in \mathbb{N}\}$ is algebraically independent over the rationals. Define f_k on $\mathbb{R}$ by $f_k(x) = \frac{a_k x + b_k}{c_k x + d_k}$. By a theorem of J. von Neumann [24], the linear fractional transformations f_k generate a free group H under composition. Let $X \subseteq [0, 1)$ be the cocountable set of all real numbers that are not fixed points of any non-identity member of H.

Enumerate Y as $\{y_i\}_{i<|Y|}$. Define G' on $X \times X$ by

$$G' = \left(\bigcup_{k\in\mathbb{N}} \operatorname{graph}(f_k)\right) \cap \left(\bigcup_{(i,j)\in\left\{(i,j) : i,j<|Y| \wedge y_i \, G \, y_j\right\}} [1 - 2^{-i}, 1 - 2^{-(i+1)}) \times [1 - 2^{-j}, 1 - 2^{-(j+1)})\right).$$

We can convert $\mathbf{G}'$ to be of the form required in the theorem statement by fixing a measure-preserving Borel isomorphism of $X \cap [1 - 2^{-i}, 1 - 2^{-(i+1)})$ with $[0, 1] \times \{y_i\}$. Hence we prove the associated statements for the graph $\mathbf{G}' = (X, G')$.

Property (2) is again immediately clear from the definition of the graph. If we have two subsets $B_i \subseteq X \cap [1-2^{-i}, 1-2^{-(i+1)})$ and $B_j \subseteq X \cap [1-2^{-j}, 1-2^{-(j+1)})$ with $y_i \, G \, y_j$ that have positive Lebesgue measure, then we can find points $t_i \in B_i$ and $t_j \in B_j$ that are Lebesgue points of density. Choose $\varepsilon > 0$ such that if $k \in \{i, j\}$ and $0 < h < \varepsilon$, then $[t_k, t_k + h) \subseteq [1-2^{-k}, 1-2^{-(k+1)})$ and $\mu(B_k \cap [t_k, t_k + h)) > \frac{9}{10}h$. Then for any C^1 function f on $[0,1]$ such that $|f' - 1| < \frac{1}{10}\varepsilon$ and $\left|f(t_i) - t_j\right| < \frac{1}{10}\varepsilon$, it follows that $f(B_i) \cap B_j \neq \emptyset$. For (a, b, c, d) sufficiently close to $(1, t_j - t_i, 0, 1)$, these conditions hold for the associated linear fractional transformation $\frac{ax+b}{cx+d}$. Hence by density there is some k such that $f_k(B_i) \cap B_j \neq \emptyset$. Choosing $x_0 \in B_i$ with $x_1 = f(x_0) \in B_j$, we prove the analog of property (1).

If we colour each point in $X \cap [1-2^{-i}, 1-2^{-(i+1)})$ according to the colour of y_i, we obtain that $\chi_\mu(\mathbf{G}') \le \chi(\mathbf{G})$ and $\chi_\mu^f(\mathbf{G}') \le \chi^f(\mathbf{G})$. If c is a μ-measurable k-fold colouring of $\mathbf{G}'$, then define an induced function c' on Y by setting $c'(y)$ to be some k-set whose preimage under c is not μ-null in $X \cap [1-2^{-i}, 1-2^{-(i+1)})$. Then property (1) shows that c' must be a k-fold colouring of $\mathbf{G}$. Hence $\chi_\mu^{(k)}(\mathbf{G}') \ge \chi^{(k)}(\mathbf{G})$ for every $k \in \mathbb{N}$. This shows that $\chi_\mu(\mathbf{G}') \ge \chi(\mathbf{G})$ and $\chi_\mu^f(\mathbf{G}') \ge \chi^f(\mathbf{G})$. □

As before, we have the following immediate corollary, which is proved exactly as Corollary 4.3.2 was.

Corollary 4.4.2. *For any assignment of the quantities $\chi, \chi^f, \chi_\mu, \chi_\mu^f \in [2, \infty)$ that obeys the conditions $\chi^f \le \min\left\{\chi, \chi_\mu^f\right\}$, $\max\left\{\chi, \chi_\mu^f\right\} \le \chi_\mu$, $\chi, \chi_\mu \in \mathbb{N}$, and $\chi_\alpha^f = 2 \Rightarrow \chi_\alpha = 2$ for $\alpha \in \{c, \mu\}$, there exists a Borel graph $\mathbf{G}$ on a standard Borel space X and a probability measure μ on X whose chromatic numbers satisfy this assignment.*

4.5 Combining All Quantities

We can combine Theorems 4.3.1 and 4.4.1 to create a Borel graph that handles all quantities considered in this paper at the same time.

Theorem 4.5.1. *For any possible way of assigning values from the set $[2, \infty)$ to χ, χ_μ, χ_{BM}, χ_B, and their fractional counterparts that is consistent with the conditions*

1. *$\chi \le \min\left\{\chi_\mu, \chi_{BM}\right\}$ and $\max\left\{\chi_\mu, \chi_{BM}\right\} \le \chi_B$, and similarly for the fractional counterparts;*

2. *$\chi_\alpha^f \le \chi_\alpha$ for every $\alpha \in \{c, \mu, BM, B\}$;*

3. *χ, χ_μ, χ_{BM}, $\chi_B \in \mathbb{N}$; and*

4. *$\chi_\alpha^f = 2 \Rightarrow \chi_\alpha = 2$ for every $\alpha \in \{c, \mu, BM, B\}$;*

there is a Borel graph G on a Polish space X with a probability measure μ realizing these eight values. If $\chi = 2$, the graph can be chosen to be acyclic.

Proof. Let $\mathbf{G}_c = (X_c, G_c)$ be a countable graph with χ and χ^f as desired (in particular, take $\mathbf{G}_c$ to be acyclic if $\chi = 2$). For each $\alpha \in \{\mu, BM, B\}$, use Corollaries 4.3.2 and 4.4.2 to create a graph $\mathbf{G}_\alpha = (X_\alpha, G_\alpha)$ on a Polish space satisfying the specified values of χ_α and χ_α^f. Then define $\mathbf{G} = \mathbf{G}_c \sqcup \mathbf{G}_{BM} \sqcup \mathbf{G}_\mu \sqcup \mathbf{G}_B$, where X_c, X_μ, and X_B are considered meager, and X_c, X_{BM}, and X_B are considered μ-null. Because $\mathbf{G}_\alpha$ is acyclic for $\alpha \in \{\mu, BM, B\}$, it follows that $\chi(\mathbf{G}) = \chi(\mathbf{G}_c) = \chi$ and $\chi^f(\mathbf{G}) = \chi^f$. Because $\mathbf{G}_\mu$ and $\mathbf{G}_B$ have Baire measurable 2-colourings (when considered subgraphs of $\mathbf{G}$), $\chi_{BM}(\mathbf{G})$ and $\chi_{BM}^f(\mathbf{G})$ are as desired. Similarly, $\chi_\mu(\mathbf{G})$ and $\chi_\mu^f(\mathbf{G})$ are as desired because χ_{BM} and χ_B have μ-measurable 2-colourings. Finally, $\chi_B(\mathbf{G})$ and $\chi_B^f(\mathbf{G})$ are as desired because these are the largest quantities. □

4.6 The $\chi_B^f(\mathbf{G}) = 2$ Case

We noted in Section 4.1 that if $\chi^f(\mathbf{G}) = 2$ for a graph $\mathbf{G}$, this implies that $\chi(\mathbf{G}) = 2$ as well, since both conditions are equivalent to the lack of odd cycles. This classical reasoning has no obvious counterpart when moved to the definable setting. In fact, as the next example shows, there exist Borel graphs $\mathbf{G}$ for which $\chi_B^f(\mathbf{G}) = 2$ but $\chi_B(\mathbf{G}) > 2$.

Theorem 4.6.1. *There exists a Borel graph on a Polish space X with $\chi_{BM}^f(\mathbf{G}) = \chi_B^f(\mathbf{G}) = 2$ (and so $\chi^f(\mathbf{G}) = \chi(\mathbf{G}) = 2$), but $\chi_{BM} = \chi_B(\mathbf{G}) = 3$.*

Proof. Let $\mathbb{Z}$ act on the Polish space $2^{\mathbb{Z}}$ by the shift action: $(n * p)(k) = p(k-n)$ and let $\mathrm{Free}\left(2^{\mathbb{Z}}\right) = \left\{p \in 2^{\mathbb{Z}} : \forall n \neq 0\,(n * p \neq p)\right\}$. Letting $S(p) = 1 * p$, consider the graph $\mathbf{G} = (X, \mathrm{graph}(S))$. Then $\mathbf{G}$ is acyclic (in fact each connected component is isomorphic to $\mathbb{Z}$), and so $\chi(\mathbf{G}) = 2$. Kechris, Solecki, and Todorčević [15] show that $\chi_B(\mathbf{G}) = 3$ and that for each $m \geq 1$, there exists a Borel subset $A_m \subseteq X$ such that for every $x \in X$, there exist $i \geq 0$ and $j > 0$ such that $S^i(x), S^{-j}(x) \in A_m$, and for the minimal such i and j, $i - j \in \{2m, 2m+1\}$. It follows that for every $m \geq 1$ $\mathbf{G}_1 \leq_B \mathbf{H}_m$, where $\mathbf{H}_m$ consists of a copy of $\mathbf{C}_{2m}$ and $\mathbf{C}_{2m+1}$ intersecting at exactly one point. Since $\chi^f(\mathbf{H}_m) = 2 + \frac{1}{m}$, it follows that $\chi_B^f(\mathbf{G}) = 2$. □

This is the first example of a major departure in the definable theory from the classical theory with regard to this branch of combinatorics. Since it is false that $\chi_B^f = 2 \Rightarrow \chi_B = 2$, it is natural to wonder if χ_B can be made arbitrarily large (but finite) while keeping χ_B^f fixed at 2. This seems likely, but no other examples are currently known.

Question 4.6.2. *Does there exist a Borel graph on a standard Borel space X satisfying $\chi_B^f(\mathbf{G}) = 2$ but $\chi_B(\mathbf{G}) = m$ for $m \geq 4$? Is Theorem 4.5.1 true but with condition 4 relaxed to only hold for $\alpha = c$?*

One natural way to generalize the graph in Theorem 4.6.1 is to consider the (acyclic) shift graph $\mathbf{G}_n$ on Free $\left(2^{\mathbb{F}_n}\right)$. Marks [22] has shown that $\chi_B(\mathbf{G}_n) = 2n + 1$.

Question 4.6.3. *Is $\chi_B^f(\mathbf{G}_n) = 2$ for each $n \geq 1$?*

Theorem 4.6.1 shows the answer is affirmative in the case $n = 1$. Unfortunately, the proof of Theorem 4.6.1 seems to be too dependent on the group $\mathbb{Z}$ having a single generator to generalize.

[1] Andreas Blass. A partition theorem for perfect sets. *Proc. Amer. Math. Soc.*, 82(2):271–277, 1981. ISSN 0002-9939. doi: 10.2307/2043323. URL `http://dx.doi.org/10.2307/2043323`.

[2] Andrés Eduardo Caicedo and Richard Ketchersid. A trichotomy theorem in natural models of AD^+. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 227–258. Amer. Math. Soc., Providence, RI, 2011. doi: 10.1090/conm/533/10510. URL `http://dx.doi.org/10.1090/conm/533/10510`.

[3] William Chan and Connor Meehan. Definable combinatorics of some borel equivalence relations. *arXiv preprint arXiv:1709.04567*, 2017.

[4] Keith J. Devlin. Some weak versions of large cardinal axioms. *Ann. Math. Logic*, 5:291–325, 1972–1973. ISSN 0168-0072.

[5] P. Erdős and A. Hajnal. On a problem of B. Jónsson. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 14:19–23, 1966. ISSN 0001-4117.

[6] L. Harrington and R. Sami. Equivalence relations, projective and beyond. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math.*, pages 247–264. North-Holland, Amsterdam-New York, 1979.

[7] L. A. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *J. Amer. Math. Soc.*, 3(4):903–928, 1990. ISSN 0894-0347. doi: 10.2307/1990906. URL `http://dx.doi.org/10.2307/1990906`.

[8] G. Hjorth. Actions by the classical Banach spaces. *J. Symbolic Logic*, 65(1):392–420, 2000. ISSN 0022-4812. doi: 10.2307/2586545. URL `http://dx.doi.org/10.2307/2586545`.

[9] Greg Hjorth and Alexander S. Kechris. Recent developments in the theory of Borel reducibility. *Fund. Math.*, 170(1-2):21–52, 2001. ISSN 0016-2736. doi: 10.4064/fm170-1-2. URL `http://dx.doi.org/10.4064/fm170-1-2`. Dedicated to the memory of Jerzy Łoś.

[10] Jared Holshouser. *Partition Properties for Non-Ordinal Sets*. PhD thesis, University of North Texas, 2017.

[11] Jared Holshouser and Stephen Jackson. Partition properties for hyperfinite quotients. To appear.

[12] S. Jackson, R. Ketchersid, F. Schlutzenberg, and W. H. Woodin. Determinacy and Jónsson cardinals in $L(\mathbb{R})$. *J. Symb. Log.*, 79(4):1184–1198, 2014. ISSN 0022-4812. doi: 10.1017/jsl.2014.49. URL `http://dx.doi.org/10.1017/jsl.2014.49`.

[13] Vladimir Kanovei. *Borel equivalence relations*, volume 44 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. ISBN 978-0-8218-4453-3. doi: 10.1090/ulect/044. URL `http://dx.doi.org/10.1090/ulect/044`. Structure and classification.

[14] Vladimir Kanovei, Marcin Sabok, and Jindřich Zapletal. *Canonical Ramsey theory on Polish spaces*, volume 202 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. ISBN 978-1-107-02685-8. doi: 10.1017/CBO9781139208666. URL `http://dx.doi.org/10.1017/CBO9781139208666`.

[15] A. S. Kechris, S. Solecki, and S. Todorcevic. Borel chromatic numbers. *Adv. Math.*, 141(1): 1–44, 1999. ISSN 0001-8708. doi: 10.1006/aima.1998.1771. URL `http://dx.doi.org/10.1006/aima.1998.1771`.

[16] Alexander S. Kechris. The axiom of determinacy implies dependent choices in $L(\mathbb{R})$. *J. Symbolic Logic*, 49(1):161–173, 1984. ISSN 0022-4812. doi: 10.2307/2274099. URL `http://dx.doi.org/10.2307/2274099`.

[17] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. ISBN 0-387-94374-9. doi: 10.1007/978-1-4612-4190-4. URL `http://dx.doi.org/10.1007/978-1-4612-4190-4`.

[18] Alexander S. Kechris and Alain Louveau. The classification of hypersmooth Borel equivalence relations. *J. Amer. Math. Soc.*, 10(1):215–242, 1997. ISSN 0894-0347. doi: 10.1090/S0894-0347-97-00221-X. URL `http://dx.doi.org/10.1090/S0894-0347-97-00221-X`.

[19] Alexander S. Kechris and Andrew S. Marks. Descriptive graph combinatorics. `http://www.math.caltech.edu/~kechris/papers/combinatorics16.pdf`.

[20] Richard Ketchersid, Paul Larson, and Jindřich Zapletal. Ramsey ultrafilters and countable-to-one uniformization. *Topology Appl.*, 213:190–198, 2016. ISSN 0166-8641. doi: 10.1016/j.topol.2016.08.018. URL `http://dx.doi.org/10.1016/j.topol.2016.08.018`.

[21] Eugene M. Kleinberg. *Infinitary combinatorics and the axiom of determinateness*. Lecture Notes in Mathematics, Vol. 612. Springer-Verlag, Berlin-New York, 1977. ISBN 3-540-08440-1.

[22] Andrew Marks. A determinacy approach to borel combinatorics. *arXiv preprint arXiv:1304.3830*, 2013.

[23] Benjamin D Miller. Measurable chromatic numbers. *The Journal of Symbolic Logic*, 73(4): 1139–1157, 2008.

[24] J von Neumann. Zur allgemeinen theorie des masses. *Fundamenta Mathematicae*, 13(1): 73–116, 1929.

[25] K. Palamourdas. $1, 2, 3, \ldots, 2n + 1, \infty!$. PhD thesis, UCLA, 2012.

[26] Edward R Scheinerman and Daniel H Ullman. *Fractional graph theory: a rational approach to the theory of graphs*. Courier Corporation, 2011.

[27] Robert M. Solovay. The independence of DC from AD. In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 171–183. Springer, Berlin, 1978.

[28] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010. ISBN 978-3-11-019702-0. doi: 10.1515/9783110213171. URL `http://dx.doi.org/10.1515/9783110213171`.

[29] Jindřich Zapletal. Descriptive set theory and definable forcing. *Mem. Amer. Math. Soc.*, 167 (793):viii+141, 2004. ISSN 0065-9266. doi: 10.1090/memo/0793. URL `http://dx.doi.org/10.1090/memo/0793`.

[30] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008. ISBN 978-0-521-87426-7. doi: 10.1017/CBO9780511542732. URL `http://dx.doi.org/10.1017/CBO9780511542732`.