AN INVESTIGATION OF THE EFFECT OF THE MUZZLE ON THE MOTION OF THE PROJECTILE IN A RECOILLESS GUN

Thesis by

Frank Arthur Woodward

In Partial Fulfillment of the Requirements

for the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1952

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. H. S. Tsien, Robert H. Goddard Professor of Jet Propulsion, California Institute of Technology, for suggesting this research; and for his valuable assistance and criticism while the work was being carried out.

SUMMARY

Expressions for the position and velocity of a projectile at any time in a recoilless gun are derived. The one dimensional wave equation is assumed to be valid in describing the motion of the gas. The muzzle velocity of the projectile is calculated, and compared with that obtained for the case in which no waves are reflected back from the muzzle. It is shown that the effect of the reflected waves is to increase the muzzle velocity. In the limiting case, when the ratio of the mass of gas initially in the barrel to the mass of the projectile is infinite, the theoretical muzzle velocity of the projectile is increased to twice the value that would be obtained if the effect of the reflected waves was neglected.

PART I

INTRODUCTION

In conventional solutions of internal ballistic problems[1],[2],[3], it is assumed that the barrel of the gun is infinitely long, so that no wave is reflected back to the projectile from the muzzle. This simplification is usually justified by the statement that any such reflected wave would be at most of the second order of magnitude, and that in view of the lack of knowledge concerning several presumably larger effects, such as, for example, the magnitude of the friction acting on the projectile in the barrel, no appreciable error will result from its omission.

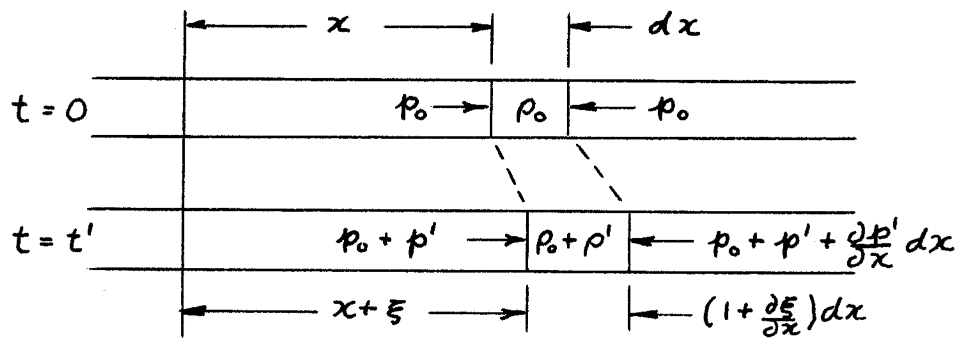
In this thesis, an attempt is made to evaluate the magnitude of this "muzzle effect" on the exit velocity of the projectile. In order to simplify the analysis, the equations are linearized by restricting the motion in the gas to waves of small amplitude. In addition, a type of recoilless gun was chosen such that the boundary condition at the breech, or nozzle end of the chamber, corresponded to that of a semi-infinite tube, so that no wave would be reflected forward to the projectile. In this way the effect of the waves reflecting between the projectile and the muzzle on the resulting motion of the projectile can be clearly seen. In the appendix to this thesis, a more general example has been worked out which includes the effects of having the breech either completely open, or closed off as in a conventional gun. As can be seen from these examples, the problem of separating the muzzle effect from the complex wave pattern is not so readily handled as it is in the simple case of the recoilless gun given below.

PART II

DERIVATION OF THE EQUATIONS

Consider a tube of unit cross-sectional area containing a non-viscous, non-heat conducting gas of density ρ and pressure p. Initially, the gas is everywhere at rest. Let ρ_o and p_o denote the initial constant values of the density and pressure.

The motion of an element of gas contained between parallel planes at distances x and x + dx from some arbitrary origin is to be investigated.



At time t = 0 the mass of the element is $\rho_o dx$. Following a disturbance, the density changes to $\rho_o + \rho'$ where $\rho' = \rho'(x, t)$.

From the principle of conservation of mass

$$\rho_o dx = \rho_o \left(1 + \frac{\rho'}{\rho_o}\right)\left(1 + \frac{\partial \xi}{\partial x}\right) dx$$

Thus if $\rho' \ll \rho_o$

$$\frac{\rho'}{\rho_o} = -\frac{\partial \xi}{\partial x} \qquad (1)$$

The forces acting on this element of mass are

$$p_o + p' - \left(p_o + p' + \frac{\partial}{\partial x}(p_o + p') dx\right)$$

Thus

$$\rho_o dx \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p'}{\partial x} dx$$

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho_o} \frac{\partial p'}{\partial x} \qquad (2)$$

where p' = p'(x, t).

Now if no heat is gained or lost during the disturbance, the process must be isentropic, so

$$p_0 + p' = K(\rho_0 + \rho')^{\gamma} \tag{3}$$

where $K = p_0 \rho_0^{-\gamma}$

$\gamma = \frac{c_p}{c_v}$

$$p_0\left(1 + \frac{p'}{p_0}\right) = \frac{p_0}{\rho_0^{\gamma}}\rho_0^{\gamma}\left(1 + \frac{\rho'}{\rho_0}\right)^{\gamma}$$

$$\therefore \frac{p'}{p_0} = \frac{\gamma \rho'}{\rho_0} \tag{4}$$

and

$$\frac{dp'}{d\rho'} = \frac{\gamma p_0}{\rho_0} = a_0^2 \tag{5}$$

where ao is, by definition, the velocity of sound in the undisturbed gas.

So, from (2)

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\frac{dp'}{d\rho'}\frac{\partial \rho'}{\partial x}$$

$$= -\frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x}$$

But

$$\frac{\partial \rho'}{\partial x} = -\rho_0 \frac{\partial^2 \xi}{\partial x^2} \qquad \text{from (1)}$$

$$\therefore \frac{\partial^2 \xi}{\partial t^2} = a_0^2 \frac{\partial^2 \xi}{\partial x^2} \tag{6}$$

Similarly, from (1) and (4)

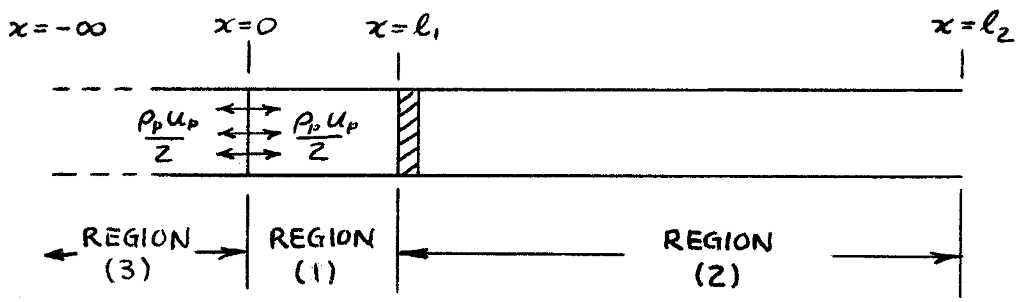
$$\left.\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} &= a_0 \frac{\partial^2 \rho'}{\partial x^2} \\ \frac{\partial^2 p'}{\partial t^2} &= a_0 \frac{\partial^2 p'}{\partial x^2} \end{aligned}\right\} \tag{7}$$

Thus ξ , ρ', p' are seen to satisfy the wave equation under the restriction that the amplitude of the disturbance is small.

PART III

THE BOUNDARY CONDITIONS IN THE GUN

The hypothetical recoilless gun considered in this example is made up of a semi-infinite tube of unit cross-sectional area fitted with a frictionless piston or projectile of mass M at a distance ℓ_1 from the origin. The muzzle is located at $x = \ell_2 > \ell_1$



Gas is added to the tube in a one dimensional fashion through a plane perpendicular to the axis at the origin. This plane corresponds to the propellant surface in the actual gun, and thus the mass flow of gas through this plane is to be equal to that given off by the propellant. This plane is such that it offers no resistance to the flow of the gas in the tube, nor does it cause a discontinuity in the pressure or density at the origin. It acts only as a surface across which a change in mass flow occurs.

Thus if $\rho_p u_p$ is the mass flow given off by the propellant at any time, the boundary condition at the origin becomes

$$\left.\begin{aligned} \rho_0\left(\frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_3}{\partial t}\right) &= \rho_p u_p \\ p_1' &= p_3' \end{aligned}\right\} \qquad (8)$$

where the subscripts refer to the regions indicated on the diagram.

At the projectile, initially located at $x = \ell_1$, the boundary conditions are

$$\left.\begin{aligned} \xi_1 &= \xi_2 \\ M\frac{\partial^2 \xi_{1,2}}{\partial t^2} &= p_1' - p_2' \end{aligned}\right\} \tag{9}$$

At the muzzle, $p_2' = 0$ (10)

In region (3), $\xi_3 \neq \infty$ for $x = -\infty$ (11)

Collecting results, the set of equations to be satisfied are

$$\frac{\partial^2 \xi_i}{\partial t^2} = a_0^2 \frac{\partial^2 \xi_i}{\partial x^2} \qquad i = 1,\ 2,\ 3$$

At $x = 0$

$$\frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_3}{\partial t} = \frac{\rho_p}{\rho_0} u_p = u_0 \qquad t > 0$$

$$p_1' = p_3'$$

or $$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_3}{\partial x}$$ since $p' = -\rho_0 a_0^2 \frac{\partial \xi}{\partial x}$ (12)

from (1), (4), (5)

At $x = \ell_2$, $\frac{\partial \xi_2}{\partial x} = 0$

At $x = -\infty$ $\xi_3 \neq \infty$

At the piston,

$$\xi_1 = \xi_2$$

$$\frac{\partial^2 \xi_{1,2}}{\partial t^2} = -\frac{\rho_0 a_0^2}{M}\left(\frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_2}{\partial x}\right)$$

At $t = 0$

$$\xi = \frac{\partial \xi}{\partial t} = 0$$

$$p = p_0 \quad , \quad \rho = \rho_0$$

PART IV

SOLUTION BY MEANS OF THE LAPLACE TRANSFORM[4]

For non-stationary problems such as this, especially when conditions must be satisfied on moving boundaries, it is convenient to make use of the operatinnal calculus in obtaining the solution. In this way the resulting wave motion can be easily visualized, and the boundary conditions satisfied in a step by step numerical process if necessary, in much the same way as the method of characteristics is used in treating the exact hydrodynamical equations.

Define $$\bar{\xi}(x,s) = \int_0^\infty e^{-st}\,\xi(x,t)\,dt = \mathcal{L}(\xi)$$

Operating on the set (12) in this manner the equations become in terms of the variable x and parameter s, where s is in general complex,

$$\frac{d^2\bar{\xi}_i}{dx^2} - \frac{a_0^2}{s^2}\bar{\xi}_i = 0 \qquad i = 1, 2, 3.$$

At $x = 0$, $$\begin{cases} \bar{\xi}_1 - \bar{\xi}_3 = \dfrac{u_0}{s^2} \\ \dfrac{d\bar{\xi}_1}{dx} = \dfrac{d\bar{\xi}_3}{dx} \end{cases}$$

At $x = \ell_2$, $$\frac{d\bar{\xi}_2}{dx} = 0$$

At $x = -\infty$ $$\bar{\xi}_3 \neq \infty$$

(13)

At $x = \ell_1 + \xi_p(t)$
$$\bar{\xi}_1 = \bar{\xi}_2 = -\frac{\rho_0 a_0^2}{M s^2}\left(\frac{d\bar{\xi}_1}{dx} - \frac{d\bar{\xi}_2}{dx}\right)$$

It is to be noted that the boundary conditions to be satisfied at the piston in the transformed problem are in terms of $\bar{\xi}(x,s)$ while the position of the piston is $x_p = \ell_1 + \xi_p(t)$ which is not a function of s. For shortness of notation, put $x_p = \ell_1$, where it is understood $\ell_1 = \ell_1 + \xi_p(t)$.

The solution to the first three ordinary differential equations in (13) may be written down directly.

$$\begin{aligned} \bar{\xi}_1 &= A_1 e^{\frac{s}{a_0}x} + A_2 e^{-\frac{s}{a_0}x} \\ \bar{\xi}_2 &= B_1 e^{\frac{s}{a_0}x} + B_2 e^{-\frac{s}{a_0}x} \\ \bar{\xi}_3 &= C_1 e^{\frac{s}{a_0}x} + C_2 e^{-\frac{s}{a_0}x} \end{aligned} \tag{14}$$

where A_i, B_i, C_i are in general arbitrary functions of the parameters ℓ_1, ℓ_2, u_0, s. Substituting in values at the boundaries:

At $x = -\infty$

$$\bar{\xi}_3 = \lim_{x \to -\infty} \left[C_1 e^{\frac{s}{a_0}x} + C_2 e^{-\frac{s}{a_0}x} \right] \neq \infty$$

$$\therefore \bar{\xi}_3 = C e^{\frac{s}{a_0}x} \tag{15}$$

At $x = \ell_2$

$$\frac{d\bar{\xi}_2}{dx} = \frac{s}{a_0}\left[B_1 e^{\frac{s\ell_2}{a_0}} - B_2 e^{-\frac{s\ell_2}{a_0}} \right] = 0$$

$$\therefore B_1 = B_2 e^{-\frac{2s\ell_2}{a_0}} \tag{16}$$

$$\therefore \bar{\xi}_2 = B\left[e^{-\frac{sx}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2 - x)} \right]$$

At $x = 0$ $\quad \dfrac{d\bar{\xi}_1}{dx} = \dfrac{d\bar{\xi}_3}{dx}$

$$\therefore \quad A_1 - A_2 = C$$

Also
$$\bar{\xi}_1 - \bar{\xi}_3 = \frac{u_0}{s^2}$$

$$\therefore \quad A_1 + A_2 - C = \frac{u_0}{s^2}$$

$$A_1 = C + \frac{u_0}{2s^2} \quad ; \quad A_2 = \frac{u_0}{2s^2}$$

$$\therefore \bar{\xi}_1 = C e^{\frac{sx}{a_0}} + \frac{u_0}{2s^2}\left[e^{\frac{sx}{a_0}} + e^{-\frac{sx}{a_0}}\right] \qquad (17)$$

At $x = \ell_1$

$$\bar{\xi}_1 = \bar{\xi}_2 = \bar{\xi}_P$$

$$\therefore C e^{\frac{s\ell_1}{a_0}} + \frac{u_0}{2s^2}\left[e^{\frac{s\ell_1}{a_0}} + e^{-\frac{s\ell_1}{a_0}}\right] = B\left[e^{-\frac{s\ell_1}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2-\ell_1)}\right]$$

$$C + \frac{u_0}{2s^2}\left[1 + e^{-\frac{2s\ell_1}{a_0}}\right] = B\left[e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}}\right]$$

$$\therefore \frac{C}{B} = e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}} - \frac{u_0}{2s^2B}\left[1 + e^{-\frac{2s\ell_1}{a_0}}\right] \qquad (18)$$

Also at $x = \ell_1$

$$\bar{\xi}_P = -\frac{\rho_0 a_0^2}{Ms^2}\left[\frac{d\bar{\xi}_1}{dx} - \frac{d\bar{\xi}_2}{dx}\right]$$

$$\therefore B\left[e^{-\frac{s\ell_1}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2-\ell_1)}\right] = -\frac{\rho_0 a_0}{Ms}\left[C e^{\frac{s\ell_1}{a_0}} + \frac{u_0}{2s^2}\left(e^{\frac{s\ell_1}{a_0}} - e^{-\frac{s\ell_1}{a_0}}\right)\right]$$

$$+ \frac{B\rho_0 a_0}{Ms}\left[e^{-\frac{s}{a_0}(2\ell_2-\ell_1)} - e^{-\frac{s\ell_1}{a_0}}\right]$$

$$\frac{Ms}{\rho_0 a_0}\left[e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}}\right] = e^{-\frac{2s\ell_2}{a_0}} - e^{-\frac{2s\ell_1}{a_0}} - \frac{C}{B} - \frac{u_0}{2Bs^2}\left[1 - e^{-\frac{2s\ell_1}{a_0}}\right]$$

$$\therefore \frac{C}{B} = \left(1 - \frac{Ms}{\rho_0 a_0}\right)e^{-\frac{2s\ell_2}{a_0}} - \left(1 + \frac{Ms}{\rho_0 a_0}\right)e^{-\frac{2s\ell_1}{a_0}} - \frac{u_0}{2Bs^2}\left[1 - e^{-\frac{2s\ell_1}{a_0}}\right]$$

Equating with (18)

$$\frac{u_0\, e^{-\frac{2s\ell_1}{a_0}}}{s^2 B} = 2e^{-\frac{2s\ell_1}{a_0}} + \frac{Ms}{\rho_0 a_0}\left[e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}}\right]$$

$$\frac{u_0}{2s^2 B} = 1 + \frac{Ms}{2\rho_0 a_0}\left(1 + e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right)$$

$$\therefore B = \frac{u_0}{2s^2\left[1 + \frac{Ms}{2\rho_0 a_0}\left(1 + e^{-\frac{2s}{a_0}(\ell_2-\ell_1)}\right)\right]} \qquad (19)$$

$$\therefore \bar{\xi}_2 = \frac{u_0}{2s^2}\,\frac{e^{-\frac{sx}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2 - x)}}{1 + \frac{Ms}{2\rho_0 a_0}\left(1 + e^{-\frac{2s}{a_0}(\ell_2-\ell_1)}\right)} \qquad (20)$$

In particular, at $x = \ell_1$

$$\bar{\xi}_P = \frac{u_0}{2s^2}\,\frac{e^{-\frac{s\ell_1}{a_0}}\left(1 + e^{-\frac{2s}{a_0}(\ell_2-\ell_1)}\right)}{1 + \frac{Ms}{2\rho_0 a_0}\left(1 + e^{-\frac{2s}{a_0}(\ell_2-\ell_1)}\right)}$$

$$\bar{\xi}_p = \frac{u_0 a_0 \rho_0}{s^2 M\left(s + \frac{2a_0\rho_0}{M}\right)} \frac{e^{-\frac{s\ell_1}{a_0}}\left(1 + e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right)}{\left[1 + \frac{s}{s + \frac{2a_0\rho_0}{M}} e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right]}$$

$$= \frac{u_0 r}{s^2(s+2r)} \frac{e^{-\frac{s\ell_1}{a_0}}\left(1 + e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right)}{\left[1 + \frac{s}{s+2r} e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right]} \qquad (21)$$

where $r = \frac{a_0\rho_0}{M}$

This is the equation for the displacement of the projectile in terms of s, where it is remembered that $\ell_1 = \ell_1 + \xi_p(t)$. In a similar manner the velocity of the projectile is given by

$$\bar{\dot{\xi}}_p = s\bar{\xi}_p = \frac{u_0 r}{s(s+2r)} \frac{e^{-\frac{s\ell_1}{a_0}}\left(1 + e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right)}{\left[1 + \frac{s}{s+2r} e^{-\frac{2s}{a_0}(\ell_2 - \ell_1)}\right]} \qquad (22)$$

If the expressions for $\bar{\xi}_p$ and $\bar{\dot{\xi}}_p$ are expanded, and each term of the resulting series transformed back to the original variables by use of the inversion theorem, defined as

$$\xi(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\,\bar{\xi}(x,s)\,ds$$

where the integral is to be evaluated in the complex plane by means of the calculus of residues; the transformed expressions become:

$$\xi_p = \frac{u_0}{2}\left\{\left[(t-\tau_1) - \frac{1}{2r}\left(1 - e^{-2r(t-\tau_1)}\right)\right] H(t-t_1)\right.$$

$$\left. + \left[(t-\tau_2)\left(1 + e^{-2r(t-t_2)}\right) - \frac{1}{r}\left(1 - e^{-2r(t-\tau_2)}\right)\right] H(t-t_2)\right.$$

$$+\left[e^{-2r(t-t_3)}\left\{(t-t_3)+r(t-t_3)^2\right\}-\frac{1}{2r}\left(1-e^{-2r(t-t_3)}\right)\right]H(t-t_3)$$

$$+\sum_{n=4}^{\infty}(-1)^n e^{-2r(t-t_n)}\left[\sum_{m=4}^{n}\frac{(n-4)!\,(-2r)^{m-2}(t-t_n)^{m-1}}{(m-1)!\,(m-4)!\,(n-m)!}\right]H(t-t_n)\Bigg\}$$

(23)

where $r = \frac{a_0 \rho_0}{M}$

$$t_n = t_{n-1} + \frac{1}{a_0}\left(2\ell_2 - 2\ell_1 - \xi_n - \xi_{n-1}\right)$$

and $t_1 = \frac{\ell_1}{a_0}$; $\xi_1 = 0$

$H(t-t_n) = 0$ for $t < t_n$

$= 1$ for $t \geq t_n$

$$\dot{\xi}_p = \frac{u_0}{2}\Bigg\{\left[1-e^{-2r(t-t_1)}\right]H(t-t_1)$$

$$+\left[\left(1-e^{-2r(t-t_2)}\right)-2r(t-t_2)e^{-2r(t-t_2)}\right]H(t-t_2)$$

$$+\sum_{n=3}^{\infty}(-1)^n e^{-2r(t-t_n)}\left[\sum_{m=3}^{n}\frac{(n-3)!\,(-2r)^{m-1}(t-t_n)^{m-1}}{(m-1)!\,(m-3)!\,(n-m)!}\right]H(t-t_n)\Bigg\} \quad (24)$$

Thus the displacement and velocity of the projectile can, in principle, be determined at any time. The determination of these quantities from the formulae is not straightforward however. As each successive wave is reflected back from the muzzle to the projectile, an additional term must be added to the expansion, resulting in an infinite series of terms at the time that the projectile reaches the muzzle. In addition, the time at which each reflection occurs depends on the time of the previous reflection, thus the displacement at any time is a function of the displacement up to that time. This makes it necessary to compute the time and displacement of each reflection successively before proceeding to the next. In this particular problem it is necessary to solve a transcendental equation at each step in the motion. As a consequence, in order to find the time at which the projectile reaches the muzzle in any particular case, an infinite number of transcendental equations must be solved. Due to this feature of the equations, further analytical treatment of the equations is not rewarding. Fortunately however, the series of terms for the displacement and velocity converge fairly rapidly so that not more than about ten terms are required in order to calculate a numerical solution of reasonable accuracy.

For ease in making calculations, the formulae (23) and (24) are best put in a non-dimensional form.

Define $X = \frac{\rho_0 x}{M}$, $\Xi = \frac{\rho_0 \xi}{M}$, $T = \frac{a_0 \rho_0}{M} t$

$$U = \frac{u_0}{a_0} \; , \quad R = \frac{\rho_0 \ell_2}{M} \; , \quad k = \frac{\ell_1}{\ell_2}$$

$$\Xi_p = \frac{U}{2}\Big\{\Big[(T-T_1) - \big(1 - e^{-(T-T_1)}\big)\Big] H(T-T_1)$$

$$+\Big[(T-T_2)\big(1+e^{-(T-T_2)}\big) - 2\big(1-e^{-(T-T_2)}\big)\Big] H(T-T_2)$$

$$+\Big[e^{-(T-T_3)}\Big\{(T-T_3)+\tfrac{1}{2}(T-T_3)^2\Big\} - \big(1-e^{-(T-T_3)}\big)\Big] H(T-T_3)$$

$$+\sum_{n=4}^{\infty}(-1)^n e^{-(T-T_n)}\Bigg[\sum_{m=4}^{n}\frac{(-1)^m (n-4)!\,(T-T_n)^{m-1}}{(m-1)!\,(m-4)!\,(n-m)!}\Bigg] H(T-T_n)\Bigg\} \qquad (25)$$

$$\dot{\Xi}_p = \frac{U}{2}\Big\{\Big[1 - e^{-(T-T_1)}\Big] H(T-T_1)$$

$$+\Big[1 - e^{-(T-T_2)} - (T-T_2)\,e^{-(T-T_2)}\Big] H(T-T_2)$$

$$+\sum_{n=3}^{\infty}(-1)^n e^{-(T-T_n)}\Bigg[\sum_{m=3}^{n}\frac{(-1)^{m-1}(n-3)!\,(T-T_n)^{m-1}}{(m-1)!\,(m-3)!\,(n-m)!}\Bigg] H(T-T_n)\Bigg\} \qquad (26)$$

and

$$T_n = T_{n-1} + 2R(1-k) - \Xi_n - \Xi_{n-1}$$

or $$\Xi_n = 2R(1-k) - \Xi_{n-1} - (T_n - T_{n-1}) \qquad (27)$$

where $T_1 = kR \quad , \quad \Xi_1 = 0$

Thus the non-dimensional velocity and displacement are functions only of U, the Mach number of the incoming gas, R, the ratio of the mass of the gas initially in the right-hand section of the tube to the mass of the projectile, and k, the ratio of the distance from the initial position of the projectile to the origin, and the distance to the muzzle.

PART V

CALCULATIONS

The calculations have been carried out for the case U = 0.2

In figure (1), the projectile path Ξ (T) is shown for the case R = 0.2 for various values of k. These calculations were carried out in order to obtain the time that the projectile reached the muzzle, T_{ℓ}. The type of wave motion resulting from a disturbance of this type in a semi-infinite tube is clearly seen from the figure.

In figure (2), the muzzle velocity of the projectile $\dot{\Xi}_{\ell}$ is plotted as a function of R for various values of k.

In figure (3), the fractional difference between the muzzle velocity given by (26) and that given by the first term of (26)(corresponding to the case with $\ell_2 = \infty$) is shown. This corresponds to the fractional increase in muzzle velocity when the reflected waves are considered over that when no wave is reflected.

PART VI
DISCUSSION

As can be seen from the preceding analysis and the figures, the muzzle velocity tends to approach asymptotically to the inlet velocity of the propellant gases, and is limited in attaining this velocity only by the finite length of the barrel. The most important feature that this analysis brings out is the fact that the solution of the same problem using the assumption of an infinite barrel, (i.e., no reflected waves) tends to a limiting muzzle velocity of only half this amount. This effect may bring in serious errors in predicting the performance of high velocity guns, and indicates that an investigation into the problem using the exact hydrodynamical equations would be well worth while.

In order to obtain the highest muzzle velocity in a gun with fixed barrel length and projectile mass, the optimum initial position of the projectile is apparently as near as possible to the source of propellant gases. This gives the longest acceleration time for the projectile.

The calculations given in this thesis consider only the simplest case of muzzle reflections. More general examples of the method are worked out in the appendix, for the case in which the breech end of the gun is either fully open; or closed, as in a conventional recoilling gun. Owing to the large number of terms resulting from a series expansion of the expressions for $\bar{\xi}$ and $\dot{\bar{\xi}}$, a calculation such as carried out above would require an almost prohibitive amount of effort.

Further extensions of the method might include the case where the mass flow into the gun depended on the pressure acting on the propellant surfaces, as it does in a real gun; and the case where the initial temperature of the incoming gas is not the same as the air in the tube. Either of these cases is, in principle, simply handled by the techniques described above.

The application of a linearized theory to the solution of problems in internal ballistics should be made with caution. In actual guns, it is very unlikely that the simplifying assumptions made in this analysis would be true even in the earliest stages of the motion. However, in view of the extremely laborious calculations required to solve similar problems by means of the exact equations[(1), (2), (3)], the linearized theory may have some value in at least showing general trends, with a minimum amount of effort.

REFERENCES

(1) Corner, J.: Theory of the Interior Ballistics of Guns, John Wiley & Sons, 1950.

(2) Love, A.E.H., and Pidduck, F. B.: Lagrange's Ballistic Problem, Phil. Trans. Roy. Soc. London A222 (1921-22), p. 167.

(3) Carriere, P.: The Method of Characteristics Applied to the Problems of Internal Ballistics. Proc. 7th Intern. Congr. Applied Mech., (London) (1948), 3, 139.

(4) Carslaw, H. S., and Jaeger, J. C.: Operational Methods in Applied Mathematics, Oxford 1941.

TABULATION OF RESULTS

U = 0.2

(1) R = 0.2; k = 0.9

n	1	2	3	4	5	6	7	8	9	10
T_n	.180	.220	.260	.299	.337	.374	.410	.444	.476	.507
Ξ_n	.0000	.0001	.0003	.0007	.0011	.0017	.0026	.0033	.0042	.0049

$T_e = .85$ $\quad \Xi_e = .02 \quad$ $\dot{\Xi}_e = .057$

$T_o = .71$ $\quad\quad\quad\quad\quad\quad$ $\dot{\Xi}_o = .050$

(2) R = 0.2; k = 0.8

n	1	2	3	4	5	6	7	8	9
T_n	.160	.239	.317	.394	.466	.534	.598	.658	.713
Ξ_n	.0000	.0007	.0012	.0026	.0045	.0076	.0088	.0111	.0138

$T_e = 1.14$ $\quad \Xi_e = .04 \quad$ $\dot{\Xi}_e = .076$

$T_o = 1.05$ $\quad\quad\quad\quad\quad\quad$ $\dot{\Xi}_o = .065$

(3) R = 0.2; k = 0.6

n	1	2	3	4	5	6	7	8	9
T_n	.120	.279	.433	.579	.715	.837	.946	1.042	1.126
Ξ_n	.000	.001	.004	.009	.015	.022	.029	.035	.041

$T_e = 1.56$ $\quad \Xi_e = .08 \quad$ $\dot{\Xi}_e = .100$

$T_o = 1.60$ $\quad\quad\quad\quad\quad\quad$ $\dot{\Xi}_o = .080$

(4) R = 0.2; k = 0.4

n	1	2	3	4	5	6	7	8	9
T_n	.080	.317	.545	.756	.946	1.112	1.254	1.375	1.476
Ξ_n	.000	.002	.010	.020	.031	.043	.055	.064	.075

$T_e = 1.90$ $\qquad \Xi_e = .12 \qquad \dot{\Xi}_e = .115$

$T_o = 2.07$ $\qquad\qquad\qquad\quad \dot{\Xi}_o = .087$

(5) R = 0.2; k = 0.2

n	1	2	3	4	5	6	7	8	9
T_n	.040	.355	.655	.927	1.166	1.369	1.538	1.677	1.789
Ξ_n	.000	.0045	.0160	.0719	.0496	.0672	.0834	.0981	.1097

$T_e = 2.21$ $\qquad \Xi_e = .16 \qquad \dot{\Xi}_e = .127$

$T_o = 2.52$ $\qquad\qquad\qquad\quad \dot{\Xi}_o = .092$

(6) R = 0.2; k = 0

n	1	2	3	4	5	6	7	8	9
T_n	.000	.393	.763	1.092	1.374	1.607	1.799	1.952	2.073
Ξ_n	.000	.007	.024	.047	.072	.094	.114	.132	.147

$T_e = 2.49$ $\qquad \Xi_e = .20 \qquad \dot{\Xi}_e = .135$

$T_o = 2.95$ $\qquad\qquad\qquad\quad \dot{\Xi}_o = .095$

(7) R = 0.40; k = 0.25

n	1	2	3	4	5	6	7	8
T_n	.100	.685	1.224	1.688	2.070	2.375	2.613	2.796
Ξ_n	.000	.015	.047	.088	.129	.166	.196	.221

$T_e = 3.33$ $\qquad \Xi_e = .300 \qquad \dot{\Xi}_e = .152$

$T_o = 3.98$ $\qquad\qquad\qquad\quad \dot{\Xi}_o = .098$

APPENDIX A

A solution of the equations (12) for two special cases

(a) Conventional gun

(b) Recoilless gun with venting area equal to area of bore.

The breech (or vent) is to be located at $x = -\ell_3$, so the length of the chamber is $\ell_1 + \ell_3$, and the overall length of the barrel is $\ell_2 + \ell_3$.

$$\frac{\partial^2 \xi_i}{\partial x^2} = \frac{1}{a_0^2}\frac{\partial^2 \xi_i}{\partial t^2} \qquad i = 1, 2, 3.$$

At $x = 0$
$$\frac{\partial \xi_1}{\partial t} - \frac{\partial \xi_3}{\partial t} = u_0 \qquad t > 0$$
$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_3}{\partial x}$$

At $x = \ell_2$
$$\frac{\partial \xi_2}{\partial x} = 0 \tag{13a}$$

At $x = -\ell_3$ Case (a) $\xi_3 = 0$

Case (b) $\frac{\partial \xi_3}{\partial x} = 0$

At the piston, $x_p = \ell_1 + \xi_p(t)$
$$\xi_1 = \xi_2$$
$$\frac{\partial^2 \xi_p}{\partial t^2} = -\frac{a_0 \rho_0}{M}\left(\frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_2}{\partial x}\right)$$

At $t = 0$ $\quad \xi = \frac{\partial \xi}{\partial t} = 0$ for all x.

Carrying out the Laplace transformation, the equations become:

$$\frac{d^2\bar{\xi}_i}{dx^2} - \frac{s^2}{a_0^2}\bar{\xi}_i = 0 \qquad i = 1, 2, 3$$

At x = 0,
$$\bar{\xi}_1 - \bar{\xi}_3 = \frac{u_0}{s^2}$$
$$\frac{d\bar{\xi}_1}{dx} = \frac{d\bar{\xi}_3}{dx}$$

At x = ℓ_2,
$$\frac{d\bar{\xi}_2}{dx} = 0$$

At the piston $x = \ell_1(t)$
$$\bar{\xi}_1 = \bar{\xi}_2 = \bar{\xi}_p$$
$$\bar{\xi}_p = -\frac{\rho_0 a_0^2}{M s^2}\left(\frac{d\bar{\xi}_1}{dx} - \frac{d\bar{\xi}_2}{dx}\right)$$

At x = $-\ell_3$
$$\bar{\xi}_3 = 0 \qquad \text{Case (a)}$$
$$\frac{d\bar{\xi}_3}{dx} = 0 \qquad \text{Case (b)}$$

As in the previous example, the solutions of the first equation in the three regions may be written

$$\bar{\xi}_1 = A_1 e^{\frac{sx}{a_0}} + A_2 e^{-\frac{sx}{a_0}}$$

$$\bar{\xi}_2 = B_1 e^{\frac{sx}{a_0}} + B_2 e^{-\frac{sx}{a_0}}$$

$$\bar{\xi}_3 = C_1 e^{\frac{sx}{a_0}} + C_2 e^{-\frac{sx}{a_0}}$$

In region (3), at $x = -\ell_3$

$$\bar{\xi}_3 = 0$$

$$\therefore 0 = C_1 e^{-\frac{s\ell_3}{a_0}} + C_2 e^{\frac{s\ell_3}{a_0}}$$

or $$C_2 = -C_1 e^{-\frac{2s\ell_3}{a_0}}$$ for Case (a)

or $$\frac{d\bar{\xi}_3}{dx} = 0 = \frac{s}{a_0}\left(C_1 e^{-\frac{s\ell_3}{a_0}} - C_2 e^{\frac{s\ell_3}{a_0}}\right)$$

$$\therefore C_2 = +C_1 e^{-\frac{2s\ell_3}{a_0}}$$ for case (b)

In general, then, $C_2 = \pm C_1 e^{-\frac{2s\ell_3}{a_0}}$ the plus and minus signs corresponding to Cases (b) and (a) respectively.

$$\therefore \bar{\xi}_3 = C\left[e^{\frac{sx}{a_0}} \pm e^{-\frac{s}{a_0}(2\ell_3+x)}\right]$$

In region (2), at $x = \ell_2$

$$\frac{d\bar{\xi}_2}{dx} = 0 = B_1 e^{\frac{s\ell_2}{a_0}} - B_2 e^{-\frac{s\ell_2}{a_0}}$$

$$\therefore B_1 = B_2 e^{-\frac{2s\ell_2}{a_0}}$$

$$\bar{\xi}_2 = B\left[e^{-\frac{sx}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2-x)}\right] \qquad (*)$$

In region (1), at $x = 0$

$$\bar{\xi}_1 - \bar{\xi}_3 = \frac{u_0}{s^2}$$

$$\therefore A_1 + A_2 - C\left[1 \pm e^{-\frac{2s\ell_3}{a_0}}\right] = \frac{u_0}{s^2}$$

Also $$\frac{d\bar{\xi}_1}{dx} = \frac{d\bar{\xi}_3}{dx}$$

$$\therefore A_1 - A_2 = C\left[1 \mp e^{-\frac{2s\ell_3}{a_0}}\right]$$

$$2A_2 + C\left(1 \mp e^{-\frac{2sl_3}{a_0}}\right) - C\left(1 \pm e^{-\frac{2sl_3}{a_0}}\right) = \frac{u_0}{s^2}$$

$$2A_2 \mp 2C e^{-\frac{2sl_3}{a_0}} = \frac{u_0}{s^2}$$

$$\therefore A_2 = \frac{u_0}{2s^2} \pm C e^{-\frac{2sl_3}{a_0}}$$

$$A_1 = \frac{u_0}{2s^2} + C$$

$$\therefore \bar{\xi}_1 = \left[\frac{u_0}{2s^2} + C\right] e^{\frac{sx}{a_0}} + \left[\frac{u_0}{2s^2} \pm C e^{-\frac{2sl_3}{a_0}}\right] e^{-\frac{sx}{a_0}}$$

$$= \frac{u_0}{2s^2}\left[e^{\frac{sx}{a_0}} + e^{-\frac{sx}{a_0}}\right] + C\left[e^{\frac{sx}{a_0}} \pm e^{-\frac{s}{a_0}(2l_3+x)}\right]$$

At $x = l_1$, $\bar{\xi}_1 = \bar{\xi}_2$

$$\therefore \frac{u_0}{2s^2}\left[e^{\frac{sl_1}{a_0}} + e^{-\frac{sl_1}{a_0}}\right] + C\left[e^{\frac{sl_1}{a_0}} \pm e^{-\frac{s}{a_0}(2l_3+l_1)}\right]$$

$$= B\left[e^{-\frac{sl_1}{a_0}} + e^{-\frac{s}{a_0}(2l_2-l_1)}\right]$$

$$\frac{u_0}{2s^2}\left[1 + e^{-\frac{2sl_1}{a_0}}\right] + C\left[1 \pm e^{-\frac{2s}{a_0}(l_3+l_1)}\right]$$

$$= B\left[e^{-\frac{2sl_1}{a_0}} + e^{-\frac{2sl_2}{a_0}}\right]$$

$$\therefore \frac{C}{B} = \frac{e^{-\frac{2sl_1}{a_0}} + e^{-\frac{2sl_2}{a_0}} - \frac{u_0}{2Bs^2}\left[1 + e^{-\frac{2sl_1}{a_0}}\right]}{1 \pm e^{-\frac{2s}{a_0}(l_3+l_1)}}$$

Also, at $x = \ell_1$

$$\frac{d\bar{\xi}_1}{dx} - \frac{d\bar{\xi}_2}{dx} = -\frac{Ms^2}{\rho_0 a_0^2}\bar{\xi}_p = -\frac{s^2}{r a_0}\bar{\xi}_2$$

where $r = \frac{a_0 \rho_0}{M}$

$$\therefore \left[\frac{u_0}{2s^2} + C\right] e^{\frac{s\ell_1}{a_0}} - \left[\frac{u_0}{2s^2} \pm C e^{-\frac{2s\ell_3}{a_0}}\right] e^{-\frac{s\ell_1}{a_0}}$$

$$-B\left[e^{-\frac{s}{a_0}(2\ell_2-\ell_1)} - e^{-\frac{s\ell_1}{a_0}}\right] = -\frac{sB}{r}\left[e^{-\frac{s\ell_1}{a_0}} + e^{-\frac{s}{a_0}(2\ell_2-\ell_1)}\right]$$

$$\therefore \frac{u_0}{2Bs^2} + \frac{C}{B} - e^{-\frac{2s\ell_1}{a_0}}\left[\frac{u_0}{2Bs^2} \pm \frac{C}{B} e^{-\frac{2s\ell_3}{a_0}}\right]$$

$$= -\frac{s}{r}\left[e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}}\right] + e^{-\frac{2s\ell_2}{a_0}} - e^{-\frac{2s\ell_1}{a_0}}$$

$$\therefore \frac{C}{B}\left[1 \mp e^{-\frac{2s}{a_0}(\ell_1+\ell_3)}\right] + \frac{u_0}{2Bs^2}\left[1 - e^{-\frac{2s\ell_1}{a_0}}\right]$$

$$= \left(1-\frac{s}{r}\right) e^{-\frac{2s\ell_2}{a_0}} - \left(1+\frac{s}{r}\right) e^{-\frac{2s\ell_1}{a_0}}$$

$$\therefore \frac{C}{B} = \frac{\left(1-\frac{s}{r}\right) e^{-\frac{2s\ell_2}{a_0}} - \left(1+\frac{s}{r}\right) e^{-\frac{2s\ell_1}{a_0}} - \frac{u_0}{2Bs^2}\left[1 - e^{-\frac{2s\ell_1}{a_0}}\right]}{1 \mp e^{-\frac{2s}{a_0}(\ell_1+\ell_3)}}$$

Equating:

$$\frac{\left(1-\frac{s}{r}\right) e^{-\frac{2s\ell_2}{a_0}} - \left(1+\frac{s}{r}\right) e^{-\frac{2s\ell_1}{a_0}} - \frac{u_0}{2Bs^2}\left[1 - e^{-\frac{2s\ell_1}{a_0}}\right]}{1 \mp e^{-\frac{2s}{a_0}(\ell_1+\ell_3)}}$$

$$= \frac{e^{-\frac{2s\ell_1}{a_0}} + e^{-\frac{2s\ell_2}{a_0}} - \frac{u_0}{2Bs^2}\left[1 + e^{-\frac{2s\ell_1}{a_0}}\right]}{1 \pm e^{-\frac{2s}{a_0}(\ell_1+\ell_3)}}$$

$$\therefore \quad e^{-\frac{2s\ell_1}{a_0}}\left[2 + \frac{s}{r} - \frac{u_0}{Bs^2}\right] + \frac{s}{r}\, e^{-\frac{2s\ell_2}{a_0}}$$

$$= \pm\, e^{-\frac{2s}{a_0}(\ell_1+\ell_3)}\left[\left(2-\frac{s}{r}\right)e^{-\frac{2s\ell_2}{a_0}} - \frac{s}{r}\, e^{-\frac{2s\ell_1}{a_0}} - \frac{u_0}{Bs^2}\right]$$

$$\frac{u_0 r}{Bs^2}\left[1 \mp e^{-\frac{2s\ell_3}{a_0}}\right] = (2r+s)\left\{1 + \frac{s}{s+2r}\left[e^{-\frac{2s}{a_0}(\ell_2-\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_3+\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}\right] \mp \frac{2r}{s+2r}\, e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}\right\}$$

$$B = \frac{\dfrac{u_0 r}{s^2(s+2r)}\left[1 \mp e^{-\frac{2s\ell_3}{a_0}}\right]}{1 + \dfrac{s}{s+2r}\left[e^{-\frac{2s}{a_0}(\ell_2-\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_3+\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}\right] \mp \dfrac{2r}{s+2r}\, e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}}$$

Substituting in (*), at $x = \ell_1$

$$\xi_p = \frac{\dfrac{u_0 r}{s^2(s+2r)}\, e^{-\frac{s\ell_1}{a_0}}\left[1+e^{-\frac{2s}{a_0}(\ell_2-\ell_1)}\right]\left[1 \mp e^{-\frac{2s\ell_3}{a_0}}\right]}{1 + \dfrac{s}{s+2r}\left[e^{-\frac{2s}{a_0}(\ell_2-\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_3+\ell_1)} \pm e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}\right] \mp \dfrac{2r}{s+2r}\, e^{-\frac{2s}{a_0}(\ell_2+\ell_3)}}$$

where the upper sign refers to Case (b) and the lower to Case (a). The transformation of this expression may be made in the usual way by expanding the denominator and using the inversion theorem of the Laplace transform. However, due to the complexity of the expression a simple formula for ξ cannot be obtained.

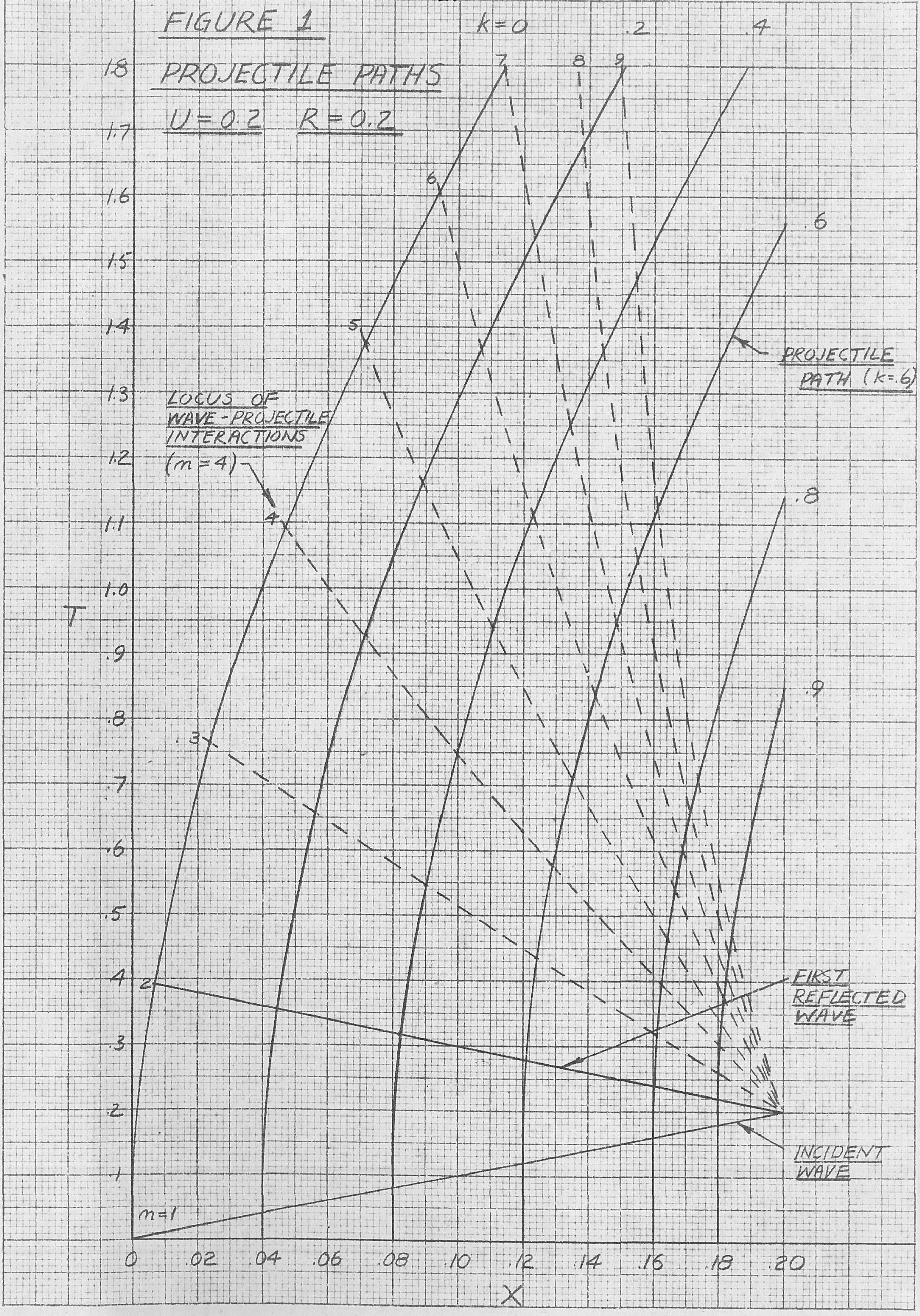


FIGURE 1
PROJECTILE PATHS
U = 0.2   R = 0.2
k = 0
.2
.4
.6
.8
.9
LOCUS OF WAVE-PROJECTILE INTERACTIONS (m = 4)
PROJECTILE PATH (k = .6)
FIRST REFLECTED WAVE
INCIDENT WAVE
m = 1
2
3
4
5
6
7
8
9
T
X
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8

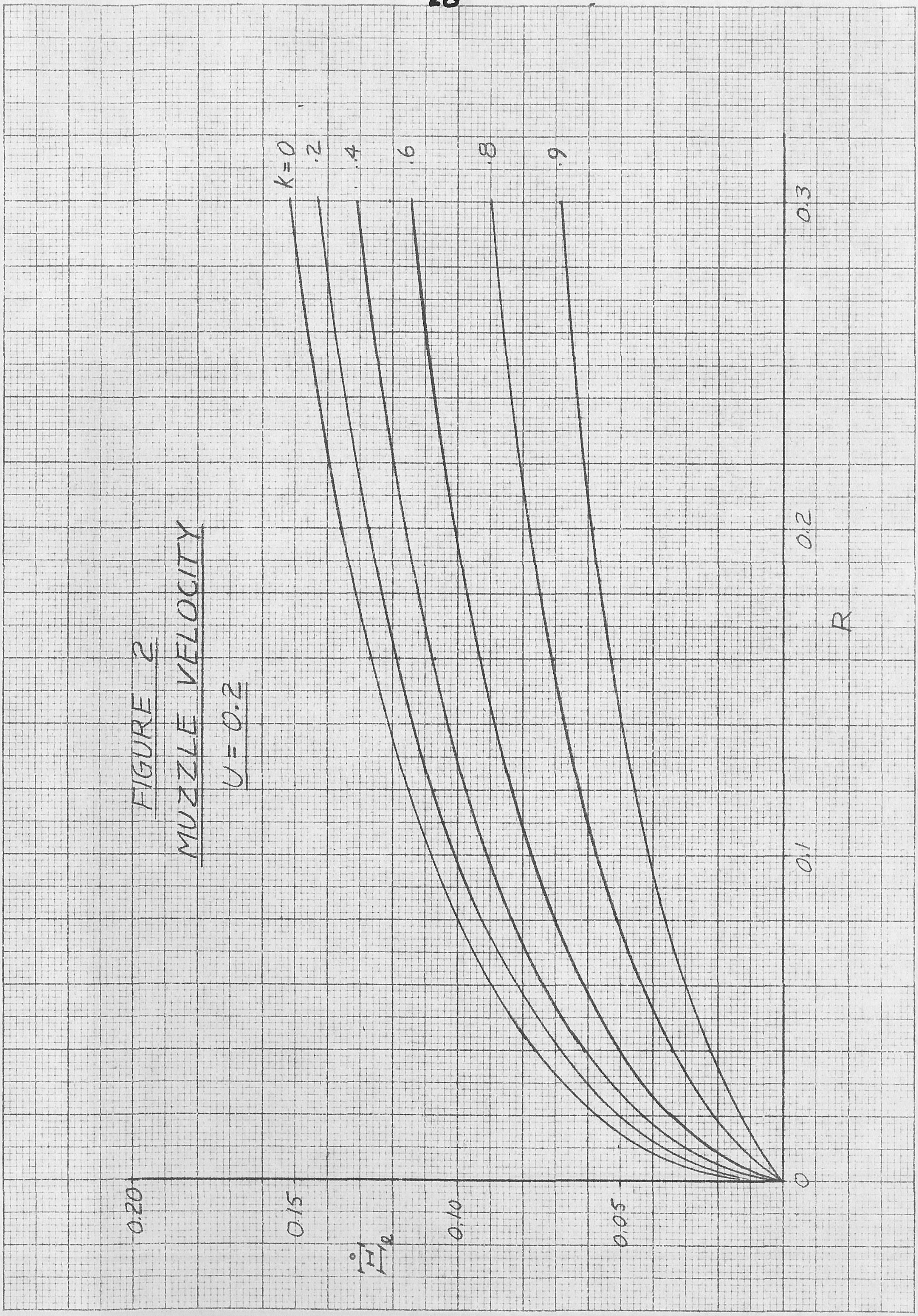


FIGURE 2

MUZZLE VELOCITY

U = 0.2

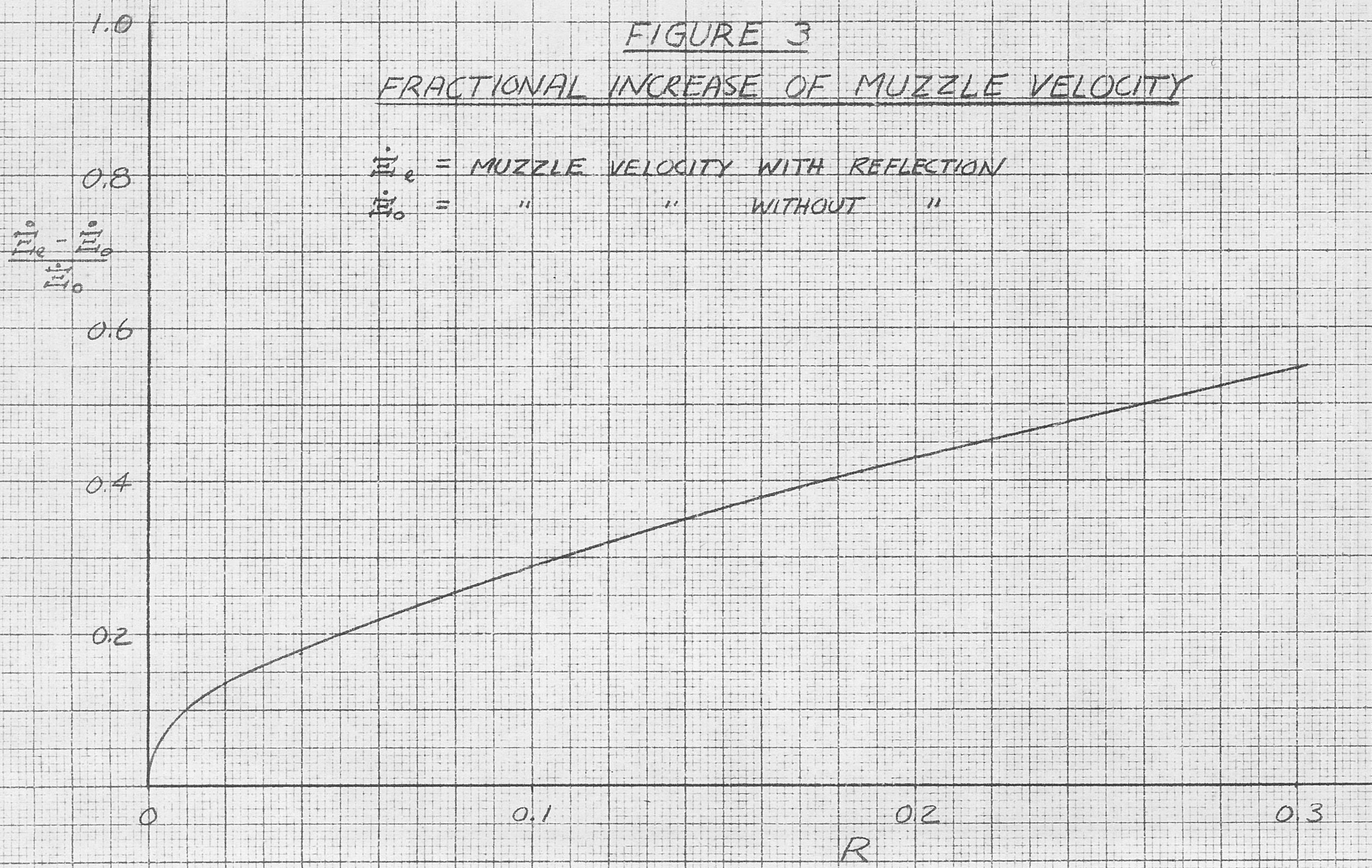


FIGURE 3

FRACTIONAL INCREASE OF MUZZLE VELOCITY