LINEARIZED SUPERSONIC FLOW

Thesis by

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Summary

This thesis is a presentation of the methods and concepts of the theory of linearized supersonic flow. The fundamental theory which serves as a basis for this investigation is discussed in the first two chapters. Special emphasis is placed upon the study of planar systems.

A system of conical coordinates is introduced in which the method of separation of variables is applied. The resultant solutions have the Mach cone as a natural boundary and involve a family of hypergeometric functions related to the Legendre functions.

Basic integral relations for planar systems are obtained between the normal velocity component and the component giving the pressure. The behavior of planar systems relative to the planform configuration is discussed and the concept of problems of the first and second kind is introduced. The lift problem is treated with particular reference to the behavior of the leading edge singularity and to the concept of the Kutta condition as applied to a planform in supersonic flow.

The nature of drag in linearized supersonic systems is investigated and the separation of the drag into types is discussed. For planar systems the drag may be divided into basic and induced parts. For general systems the basic division may be made into wave drag and vortex drag. Two fundamental reversed flow theorems are obtained which state that the drag of a system is the same as that of the system with the flow reversed in direction.

The theory of conical flow as applied to planar systems is developed and the results for a basic thickness distribution and various lifting triangles are presented.

The method of the separation of the lateral variable is investigated using Schlömilch series.

The flow about bodies of revolution is discussed and the application of the Riemann method to the problem is given.

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LINEARIZED SUPERSONIC FLOW

Introduction

Many of the important problems that arise in the study of the steady-state flow of a perfect gas at supersonic velocities are concerned with the disturbances caused by a body placed in a uniform stream of gas flowing at a constant supersonic velocity and with the forces on such a body. Under the assumption that the disturbance velocities are small compared to the basic flow velocity and that the viscosity is small, shock wave phenomena do not enter the problem and the velocity potential equation for the then necessarily irrotational flow can be linearized. As a result of this linearization and the resulting properties of superposition the solutions to a great variety of problems are greatly simplified and may be obtained readily. These linearized solutions are good approximate solutions to many practical problems and are the more accurate the smaller are the disturbances from the uniform flow and the less important are real fluid effects. General references to the flow of compressible fluids and the linearized equations are 1 to 8. This thesis deals with the study of linearized supersonic flow, the development of the theory, the formulation of new concepts in this field, and the application of the theory to various examples.

The first application of the concept of linearized supersonic flow was made by Ackeret (9,10) in 1925 in the solution of the two-dimensional problem. In this he obtained expressions for the lift, drag, and moment coefficients of a two-dimensional supersonic airfoil in closed form in terms of the airfoil shape. This two-dimensional solution can be considered as a linearization of the precise characteristic method (11) or as a superposition of the linearized form of the fundamental Meyer flow (12). In 1932 von Karman and Moore (14) presented the application of the theory to bodies of revolution at zero incidence. More recently von Karman (15), Ferrari (16), and Tsien (17) have continued this development. Busemann, at the 1935 Volta Congress (18), introduced the important concept of normal Mach number in his treatment of cylindrical flow, and in the same contribution he described the second and third approximations beyond Ackeret's theory showing that the effect of shock waves does not enter the first and second approximations. In this paper he also introduced the important concept of conical flow. In a recent paper (20) Busemann showed that linearized supersonic conical problems can be reduced to a two-dimensional potential problem and solved several examples. Prandtl (21) introduced the concept of the acceleration potential and showed how it could be applied to Ackeret's problem. Schlichting (22), using the concepts developed by Prandtl, developed a supersonic wing theory and applied it to a few examples.

Very recently a large number of contemporary investigators have worked in the field of linearized supersonic flow. Among these investigators are: Puckett (35) who solved the planar conical thickness problem with lateral symmetry by a method similar to Schlichting's and applied his results to the pressure distribution, lift, and drag of various delta-shaped wings; Stewart (36)who completed the boundary problem for a lifting delta-wing with leading edges inside the Mach cone; Jones (27) who introduced a system of oblique coordinates and applied them to certain wing problems; von Karman (37) and Chang (47) who developed a theory for planer systems using Fourier integrals and applied it to several examples; Lighthill (34) who continued the development for bodies of revolution. The work of Taunt, Snow, Beskin, and Lagerstrom may also be cited. Due to security restrictions references to some of this recent work in linearized supersonic flow may not be included here. In a recent lecture von Karman (23) summarized the results of these investigations.

This thesis is divided into several chapters, each one developing a different topic. The first presents the fundamental theory on which is based the remainder of the work and gives certain of the results of other investigators which are needed. Other chapters discuss various methods of obtaining solutions of the potential equation and the application of these methods. Special emphasis is placed on the study of planar systems and most of the examples worked out are examples of such systems. One chapter deals with the flow about bodies of revolution. The last consists of a general discussion of results obtained and of the outlook for future investigations in this field. Some of the results of this thesis have already been published (40,41,42).

It is the general purpose of this thesis to present a reasonably complete development of the theory of linearized supersonic flow. However, the works of other investigators are presented with only sufficient detail for continuity and the purposes of the analysis, so that the thesis is for the greatest part the writer's own development. This groundwork is found in the first two chapters of the thesis. As a general principle the development and explanation of the concepts essential to the theory are stressed.

I. Fundamental Theory

1. Basic Flow Equations. The fundamental assumptions that are made for the steady-state compressible potential flow of a perfect gas are the following:

a. The flow is steady state; the partial derivatives of all the flow variables with respect to time are identically zero.

b. The flow is frictionless; the viscosity is zero.

c. There is no transfer of heat between fluid elements.

d. There are no body forces acting on the fluid.

e. There are no discontinuous shock wave phenomena.

f. The flow is isentropic; there exists a unique relation between the pressure and the density.

g. The flow is irrotational.

It should be pointed out that the last two assumptions are really consequences of the first five assumptions (24,p.36;25).

From the above assumptions may be derived the Eulerian equation of motion, the definition of the velocity potential, the equation of continuity, and the definition for the speed of sound:

$$\bar{q} \cdot \nabla \bar{q} = -\frac{1}{\rho} \nabla p = (\nabla \times \bar{q}) \times \bar{q} + \frac{1}{2} \nabla (q^2) \tag{1.1}$$

$$\nabla \times \bar{q} = 0 \qquad ; \qquad \bar{q} = \nabla \phi \tag{1.2}$$

$$\rho \nabla \cdot \bar{q} + \bar{q} \cdot \nabla \rho = 0 \tag{1.3}$$

$$a^2 = \frac{dp}{d\rho} \tag{1.4}$$

Consequences of the above are the familiar potential equation

$$\nabla\phi \cdot \nabla\left(\tfrac{1}{2}\nabla\phi^2\right) = a^2 \nabla^2\phi \tag{1.5}$$

and Bernoulli's equation

$$q\,dq + \frac{1}{\rho}\,dp = 0 \tag{1.6}$$

For the case of the perfect gas are obtained

$$p = C\rho^\gamma \quad , \quad a^2 = \frac{\gamma p}{\rho} \tag{1.7}$$

$$a^2 + \frac{\gamma-1}{2}q^2 = a_0^2 \tag{1.8}$$

The acceleration of a particle in a compressible fluid flow is given by

$$\bar{b} = -\frac{1}{\rho}\nabla p \tag{1.9}$$

Hence the quantity

$$\psi = -\int\frac{dp}{\rho} = \frac{1}{2}q^2 \tag{1.10}$$

is a potential whose gradient is the acceleration vector and may be termed the acceleration potential. This concept was introduced by Prandtl (21).

2. The Scale Transformation. Since only the derivatives with respect to the cartesian coordinates appear in the equations of section 1, these equations are invariant under translation. It is also clear from the vector form of these equations that these equations are also invariant under a pure rotation. Another fundamental but slightly less obvious transformation which is of use is the scale transformation.

If the transformation

$$\phi_2 = K\,\phi_1 \tag{2.1a}$$

$$x_2 = K\,x_1 \tag{2.1b}$$

$$y_2 = K\,y_1 \tag{2.1c}$$

$$z_2 = K\,z_1 \tag{2.1d}$$

is made in the potential equation (1.5), this equation is found to be invariant. This means that if $\phi(x,y,z)$ is a solution of the potential equation, so also is $K\phi(\frac{x}{K},\frac{y}{K},\frac{z}{K})$. This is a change of one solution to another similar solution of different scale, the factor K describing the change in scale. This transformation is termed a scale transformation.

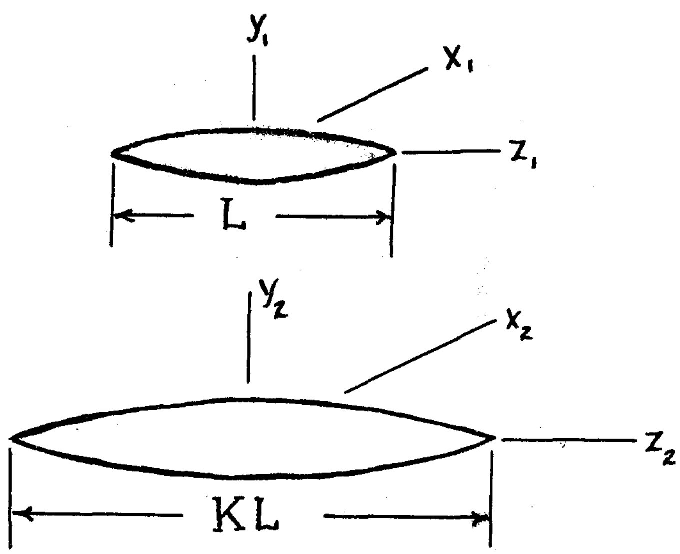


Fig. 2.1. The Scale Transformation

This scale transformation, here derived for the general potential flow, will also hold for rotational flow with shock waves providing the shock waves are considered to be infinitesimally thin.

3. Cylindrical and Conical Flow. If a solution to the potential equation is unchanged by an arbitrary translation in a given direction the flow is two-dimensional and may be reduced to a solution in two variables by an appropriate rotation. If a solution changes by an additive constant proportional to the translation under a translation of arbitrary magnitude, an appropriate rotation will change it to the form

$$\phi = V_t x + f(y,z) \quad (3.1)$$

The concept of such cylindrical flow was introduced by Busemann (18). A cylindrical flow consists of a two-dimensional flow on which is superposed a uniform velocity in the direction normal to the plane of the two-dimensional flow. The general flow about an infinite cylindrical body under appropriate boundary conditions will be a flow of this type.

An infinite conical body with its vertex at the origin will remain invariant in shape under a scale transformation. If in a problem involving the flow about a conical body the pertinent boundary conditions at infinity are unchanged by the transformation the solution will be unchanged by the transformation.

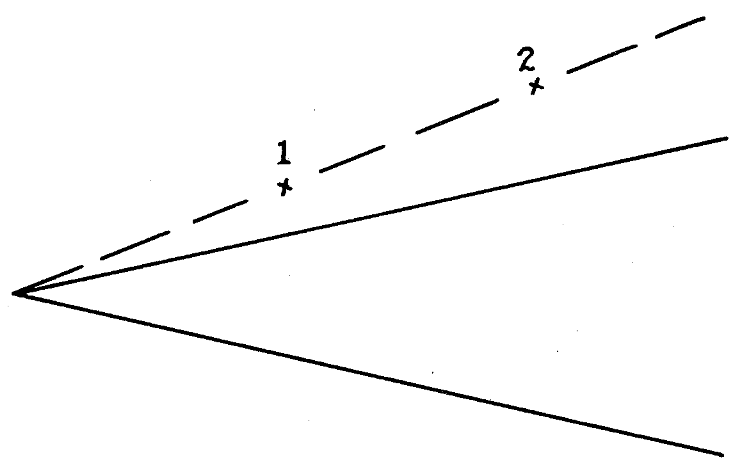


Fig. 3.1. Conical Flow

Thus, in Fig. 3.1, the scale transformation changes point 1 to point 2, and the velocity components and pressure will be the same at these two points. Extending this result to other points with the general scale transformation it is seen that the velocity components and the pressure will be constant along any ray extending from the vertex. These conditions at infinity cannot be met in general for subsonic flow but may be met within certain limitations by finite conical bodies in supersonic flow. Such flow is termed conical flow. The concept of conical flow was introduced by Busemann (18).

4. Linearized Equations. The assumption is made that the velocity components vary but slightly from those for a uniform flow of velocity V, which is taken in the positive z direction. Neglecting terms of higher order is equivalent to using the velocity of uniform flow in the first derivative terms of the potential equation (1.5), including the a^2 term. The equation thus becomes an equation in the second derivatives of ϕ with constant coefficients. This equation is

$$\phi_{xx} + \phi_{yy} - (M^2-1)\phi_{zz} = 0 \tag{4.1}$$

where M denotes the ratio of the free-stream velocity to the speed of sound in the undisturbed stream and is the Mach number of the flow. The fundamental uniform flow is given by the potential $\phi_o = Vz$. This part of the solution will be generally excluded from ϕ and equation (4.1) will always be considered as yielding velocity deviations which are to be added to the fundamental solution to describe the net flow. The pressure difference from the pressure of the uniform flow is given by equation (1.6) to be

$$p = -\rho\left(Vw + \frac{u^2+v^2}{2}\right) \tag{4.2}$$

where w is the disturbance velocity component in the z direction and u and v are mutually perpendicular velocity components in the plane normal to the direction of flow. Henceforth the term p will refer to the variation in pressure from the free stream pressure, while ρ will refer to the density of the fluid in the free stream, unless otherwise particularly specified as in section 5. For most applications the last two terms in the above equation may be dropped giving the more customary equation

$$p = -\rho V w \qquad (4.3)$$

The acceleration potential under this linearized theory is given by the expression

$$\psi = -\frac{p}{\rho} = V w + \frac{u^2+v^2}{2} \qquad (4.4)$$

The assumptions necessary for the validity of the linearized theory should be expressly stated:

a. The lateral velocity components must be small compared with V. This assumption is not necessary for the linearization of the equations themselves, but is necessary in order that the usual geometrical assumptions as to small angles may be satisfied so that the boundary conditions may be linearized. Except for low subsonic speeds this is ensured automatically by assumptions b and c.

b. The lateral velocity components must be small compared with the velocity of sound in the free stream. This is necessary in order that the terms which are dropped in equation (4.1) are small. This assumption prevents flows in the hypersonic region from being linearized. That the axial velocity component satisfies the same condition is ensured either by assumption b or c.

c. The axial velocity component w must be small compared with $|V-a|$. This is necessary in order that the factor $\sqrt{M^2-1}$ may be considered to be a constant. This assumption prevents flows in the transonic region from being linearized.

The remainder of this investigation will deal with solutions to the linearized potential equation (4.1) when the flow is supersonic, when M is greater than 1.

A small disturbance in such a supersonic flow can only affect the flow at points within a cone lying downstream from the source of the disturbance. This cone has its vertex at the source of the disturbance, its axis in the flow direction, and semi-vertex angle $\csc^{-1} M$. This cone defines the zone of action and is termed the downstream Mach cone. In a similar fashion the flow at a given point can only be affected by disturbances lying within a similar cone upstream from the point in question. This defines the zone of influence and is termed the upstream Mach cone. This concept of zones of action and influence marks the principal distinction between subsonic and supersonic flow, and is due to the hyperbolic form of the supersonic flow equations. In subsonic flow a disturbance at a given point can affect the flow at all other points and conversely the flow at a given point may be affected by disturbances at all other points.

It is convenient to have the equations of irrotationality and continuity for the linearized case in explicit form. The equations of irrotationality are

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \tag{4.5a}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{4.5b}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.5c}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - (M^2-1)\frac{\partial w}{\partial z} \quad . \tag{4.6}$$

5. <u>Momentum and Acoustical Energy</u>. The force on a body in a steady-state fluid flow equals minus the force necessary to maintain a system including the body and the fluid within a given volume at equilibrium. This may be expressed in terms of the absolute pressure, density, and velocities used in section 1 by

$$\bar{F} = -\iint \left[\bar{q}(\rho\bar{q}\cdot\bar{n}) + p\bar{n}\right] dS \tag{5.1}$$

in which the force on the system is equated to the momentum transfer through the boundaries of the system. This may be re-expressed in terms of a dyadic quantity

$$\bar{F} = \iint \bar{\Phi}\cdot\bar{n}\, dS - mV\bar{k} \tag{5.2}$$

where the dyadic is given by

$$\bar{\Phi} = -(p-p_o)I - \rho(\bar{q}-V\bar{k})\bar{q} \tag{5.3}$$

the quantity I is the idemfactor, and m is the net mass flow of fluid out of the closed surface, given by

$$m = \iint \rho\,\bar{q}\cdot\bar{n}\, dS \tag{5.4}$$

This net mass flow will be zero for the applications considered here. The dyadic may be expressed

$$\bar{\Phi} = \bar{i}\,\bar{A} + \bar{j}\,\bar{B} + \bar{k}\,\bar{C} \tag{5.5}$$

where the three terms give separately the three components of the force.

The third of these components is the drag and may be obtained by applying the law of conservation of energy from the point of view of an observer fixed with respect to the undisturbed fluid. Considering the same system as before, the total energy per unit time leaving the system will be

$$\mathcal{E} = \iint \left[\left(\tfrac{1}{2} \rho (\bar{q} - V\bar{k})^2 + \tfrac{1}{\gamma - 1} p \right) \bar{q} + p(\bar{q} - V\bar{k}) \right] \cdot \bar{n}\, dS \tag{5.6}$$

where the first term is the transport of kinetic energy out of the system, the second the transport of internal energy, and the third the work done on the surroundings. This may be expressed as

$$\mathcal{E} = V \iint \bar{C} \cdot \bar{n}\, dS + m\left(E_o - \tfrac{1}{2} V^2\right) \tag{5.7}$$

where m is the net mass flow leaving the region, E_0 is the total energy per unit mass of this net flow given by

$$E_o = \frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \tag{5.8}$$

and $\bar{C}$ is the vector

$$\bar{C} = -(p - p_o)\bar{k} - \rho w \bar{q} \tag{5.9}$$

where $\bar{C}$ is the same vector as in equation (5.5). This energy equals the work done against the drag plus the rest energy of the ejected mass plus the kinetic energy of this mass with respect to the stationary observer

$$\mathcal{E} = DV + m\left(E_0 + \tfrac{1}{2}V^2\right). \tag{5.10}$$

Combining equations (5.7) and (5.10) yields

$$D = \iint \bar{C}\cdot\bar{n}\, dS - mV \tag{5.11}$$

in complete agreement with equation (5.2).

In the presentation above the dyadic Φ is of first order in the disturbance velocities. Hence the substitution of a linearized solution will give the force on the body correct to this first order. However, the terms giving the drag cancel out in the linearized theory. The drag on a body in supersonic flow is a force of the second order.

In order to obtain the drag it is necessary to obtain an expansion of $\bar{C}$ to terms of the second order. This will be valid for the linearized theory because the first order terms cancel identically. The density is given to first order terms by

$$\rho = \rho_0 + \frac{p-p_0}{a_0^2} \cdots = \rho_0\left(1 - M^2\frac{w}{V}\right). \tag{5.12}$$

The pressure is given to second order terms by considering Bernoulli's integral

$$\int\frac{dp}{\rho} = -\left(Vw + \frac{u^2+v^2+w^2}{2}\right)$$

$$= \frac{p-p_0}{\rho_0} - \frac{(p-p_0)^2}{2\rho_0^2 a_0^2} \cdots$$

whence the pressure is given by

$$p - p_0 = -\rho\left(Vw + \frac{u^2+v^2-(M^2-1)w^2}{2}\right). \tag{5.13}$$

Substituting these second order expressions into the $\bar{C}$ part of Φ yields the correct linearized expression for Φ. In matrix form this is

$$\Phi = \rho \begin{bmatrix} Vw & 0 & -Vu \\ 0 & Vw & -Vv \\ -uw & -vw & \dfrac{u^2+v^2+(M^2-1)w^2}{2} \end{bmatrix} \tag{5.14}$$

where ρ_0 has been replaced by ρ in consistency with the linearized notation. It may be checked by means of the divergence theorem with the aid of equations (4.5) and (4.6) that the integral of Φ in (5.2) over a contour not enclosing a singularity or body is zero.

Linearized flow satisfies the customary acoustical assumptions for an observer stationary with respect to the undisturbed fluid. Thus it is permissible to use the concepts of acoustical theory in the linearized theory and it is convenient in particular to utilize those concepts describing the energy in the fluid. From the acoustical point of view the reference state of the fluid is its undisturbed state and the pressure and potential energy are defined as zero at this state. The pressure is then the linearized pressure given by equation (4.3). The kinetic energy per unit volume is

$$KE = \frac{\rho}{2}\left(u^2+v^2+w^2\right) \tag{5.15}$$

and the potential energy is (26, vol. II, p. 15)

$$PE = \frac{p^2}{2\rho a^2} = \frac{\rho}{2} M^2 w^2 \tag{5.16}$$

From this point of view the vector $\bar{C}$ in the energy consideration is given by

$$\bar{C} = (KE + PE)\bar{k} - \rho w(u\bar{i} + v\bar{j} + w\bar{k}) \tag{5.17}$$

where the first terms denote the transport of energy and the latter terms represent the work done on the surroundings. In matrix form consistent with equation (5.14) this is

$$\bar{C} = \rho \begin{bmatrix} -uw & -vw & \dfrac{u^2+v^2+(M^2-1)w^2}{2} \end{bmatrix} \tag{5.18}$$

These concepts will be applied to the calculation of drag in a later section.

The remarks of Theodorson (50) with respect to the impulse and momentum in an infinite fluid apply with similar validity in linearized supersonic flow for the lateral force components. The relative contribution of the various parts of the integral giving the side force or lift generally depend on the shape of the closed contour even if this contour is considered to be infinitely large.

6. The Fundamental Similitude in $\sqrt{M^2-1}$. The potential equation (4.1) includes the parameter (M^2-1). It is advantageous for analytical work to remove this factor. The transformation

$$\phi = \frac{1}{\sqrt{M^2-1}} \phi' \tag{6.1a}$$

$$x = x' \tag{6.1b}$$

$$y = y' \tag{6.1c}$$

$$z = \sqrt{M^2-1}\; z' \tag{6.1d}$$

changes equation (4.1) into the equation

$$\phi'_{x'x'} + \phi'_{y'y'} - \phi'_{z'z'} = 0 \tag{6.2}$$

with

$$u' = \sqrt{M^2-1}\; u \tag{6.3a}$$

$$v' = \sqrt{M^2-1}\; v \tag{6.3b}$$

$$w' = (M^2-1)\, w \tag{6.3c}$$

This correlates a linearized supersonic flow at a given Mach number with a similar flow at a Mach number of $\sqrt{2}$. A body in the original flow will be transformed into a single body with the same transverse dimensions but of different length, with an aspect ratio, thickness ratio, and angle of attack that are given by

$$\mathcal{R}' = \sqrt{M^2-1}\,\mathcal{R} \tag{6.4}$$

$$TR' = \sqrt{M^2-1}\,TR \tag{6.5}$$

$$\alpha' = \sqrt{M^2-1}\,\alpha \tag{6.6}$$

The lift coefficients of such bodies will be connected by the relation

$$C_L' = (M^2-1)\,C_L \tag{6.7}$$

and the drag coefficients will be connected by

$$C_D' = (M^2-1)^{3/2}\,C_D. \tag{6.8}$$

In these equations the primed expressions are for the equivalent system at $M = \sqrt{2}$. If these coefficients are based on a cross-sectional area instead of a plan form area equations (6.7) and (6.8) will be changed by the factor $\sqrt{M^2-1}$.

In planar systems, for which the boundary conditions may be satisfied on a plane, the factor between ϕ and ϕ' may be set equal to one, yielding instead of equations (6.5 to 6.8) the following relations

$$TR'' = TR \tag{6.9}$$

$$\alpha'' = \alpha \tag{6.10}$$

$$C_L'' = \sqrt{M^2-1}\ C_L \tag{6.11}$$

$$C_D'' = \sqrt{M^2-1}\ C_D . \tag{6.12}$$

In this similitude the aspect ratio is still given by equation (6.4). Due to its greater simplicity this form of the similitude is more useful than the strict one given above, but its limitation to planar systems must be kept in mind.

It is thus unnecessary to consider variations in Mach number in a general analysis, and it is most convenient to fix the value of the Mach number at $\sqrt{2}$. In an actual problem the method will be to consider the equivalent problem according to the above similitude at $M = \sqrt{2}$ and then transform the results of this problem back to the given Mach number. In future analysis the equation considered will be equation (6.2) with the primes dropped:

$$\phi_{xx} + \phi_{yy} - \phi_{zz} = 0 . \tag{6.13}$$

This equation is cylindrical coordinates is

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} - \phi_{zz} = 0 . \tag{6.14}$$

7. The Oblique Transformation. A very convenient transformation pointed out by R. T. Jones (27) is the oblique transformation. In the form presented

by Jones these transformations do not form a group unless the general scale transformation is included. For this reason the form of the transformation is changed in this presentation so that these transformations alone will form a group. This transformation is

$$X = \frac{1}{\sqrt{1-m^2}} X' - \frac{m}{\sqrt{1-m^2}} Z' \tag{7.1a}$$

$$Z = \frac{1}{\sqrt{1-m^2}} Z' - \frac{m}{\sqrt{1-m^2}} X' \tag{7.1b}$$

$$y = y' \tag{7.1c}$$

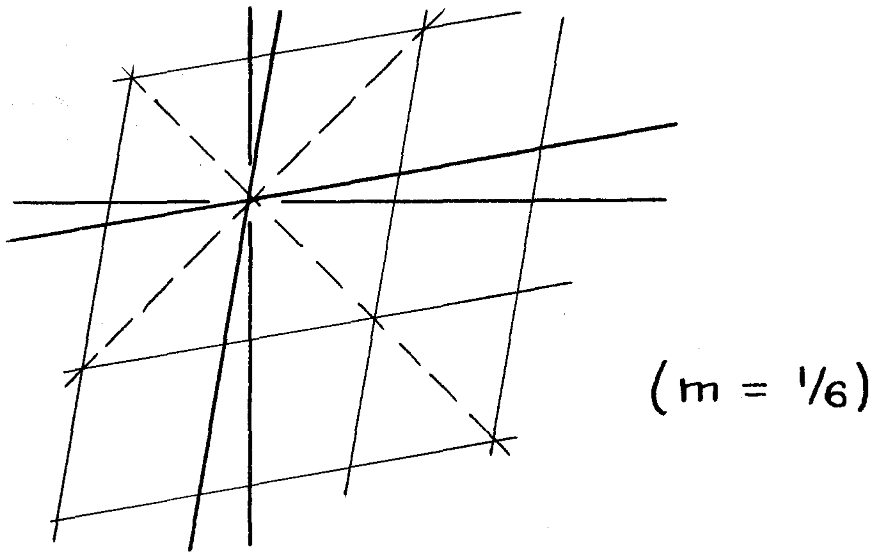


Fig. 7.1 The Oblique Transformation

It may be seen that the potential equation (6.7) is invariant under this transformation, that the transformations form a group, that the hyperbolic distance defined by

$$R = \sqrt{z^2 - x^2 - y^2} \tag{7.2}$$

is left unchanged, and that cylindrical and conical flows retain their character. The entire class of transformations for which the potential equation (6.13) remains invariant consists of translation, the scale transformation, rotation about the z axis, and the oblique transformation.

The derivatives of the velocity potential are related by the following equations:

$$u = \frac{1}{\sqrt{1-m^2}} u' + \frac{m}{\sqrt{1-m^2}} w' \tag{7.3a}$$

$$w = \frac{1}{\sqrt{1-m^2}} w' + \frac{m}{\sqrt{1-m^2}} u' \tag{7.3b}$$

$$v = v' \tag{7.3c}$$

The Jacobian of the variables (x, z) with respect to (x', z') is unity, whence the area element dx dz may be replaced by dx' dz'. Another property of this transformation is given by the relation

$$\left(\frac{x}{z}\right) = \frac{\left(\frac{x'}{z'}\right) - m}{1 - m\left(\frac{x'}{z'}\right)} \tag{7.4}$$

that the ratio x/z transforms with a homographic transformation which leaves the points $\pm$ 1 invariant.

8. Two-Dimensional Flow. The equation for two-dimensional flow may be taken from equation (6.13) by considering the velocity potential to be independent of the variable x

$$\phi_{yy} - \phi_{zz} = 0 \tag{8.1}$$

The general solution of this equation is

$$\phi = f(y+z) + g(y-z) \tag{8.2}$$

so that

$$v = f' + g' \tag{8.3a}$$

$$w = f' - g' \tag{8.3b}$$

For a wing only the g solution is taken on the upper surface and only the f solution is taken on the lower surface. This is done in order that there may be no disturbance ahead of the zone of action of the wing, which will be the envelope of Mach cones from the leading edge of the wing. If the contour of the upper surface of the wing is given by y_1 (z) the boundary condition may be satisfied on the plane y = 0, and will be

$$V y_1'(z) = g'(-z) \tag{8.4a}$$

Similarly if y_2 (z) is the contour of the lower surface the boundary condition in the lower surface will be

$$V y_2'(z) = f'(z) \tag{8.4b}$$

From expression (4.3) the pressures on the two surfaces will be

$$p_1 = \rho V^2 \; y_1'(z) \tag{8.5a}$$

$$p_2 = -\rho V^2 \; y_2'(z) \tag{8.5b}$$

The lift coefficient can be derived from these expressions in terms of the angle of attack of the chord line drawn from the leading edge of the airfoil to the trailing edge.

$$C_L = 4\alpha \tag{8.6}$$

The drag coefficient of the airfoil may be expressed

$$C_D = 2\left(\overline{y_1'^2} + \overline{y_2'^2}\right) \tag{8.7}$$

where $\overline{y_1'^2}$ and $\overline{y_2'^2}$ are the average values of the quantities $y_1'^2$ and $y_2'^2$ along the chord. These are the original results of Ackeret (9).

If the new quantities are introduced

$$y_m = \frac{1}{2}(y_1 + y_2) \tag{8.8a}$$

$$y_t = \frac{1}{2}(y_1 - y_2) \tag{8.8b}$$

the drag coefficient may be expressed

$$C_D = 4\left(\overline{y_m'^2} + \overline{y_t'^2}\right) \tag{8.9}$$

In the above expression the first term is the drag due to camber distribution and the second term is the drag due to thickness distribution. Thus it is seen that for a single airfoil the drag due to camber and lift and the drag due to thickness are separate. In general this is not true for a system of airfoils, such as a biplane combination. An example of this is the Busemann biplane (18) where both elements of the biplane have lift and thickness (although no angle of attack) but the net drag is zero. The lifts on the two elements are equal in magnitude but opposite in sense, so that the system has zero net lift.

The drag of a single airfoil may be separated further conveniently by expanding the thickness slope distribution and the camber slope distribution in series of Legendre polynomials. A new variable is defined

$$\zeta = \frac{2z}{c} - 1 \tag{8.10}$$

such that ζ equals -1 at the leading edge ($z = 0$), and equals +1 at the trailing edge ($z = c$). Then y_m' and y_t' are expressed

$$y_m' = \sum_{j=0}^{\infty} a_j \sqrt{2j+1}\; P_j(\zeta) \tag{8.11}$$

$$y_t' = \sum_{j=1}^{\infty} b_j \sqrt{2j+1}\; P_j(\zeta) \tag{8.12}$$

The drag coefficient is given by

$$C_D = 4a_0^2 + \sum_{j=1}^{\infty} (a_j^2 + b_j^2) \tag{8.13}$$

The angle of attack is equal to the coefficient-a_0 so that the lift coefficient is

$$C_L = -4a_0 \tag{8.14}$$

Similarly the moment coefficient about the half-chord point is

$$C_{M_0} = \frac{2}{\sqrt{3}} a_1 \tag{8.15}$$

A volume coefficient may be defined as being the ratio of the cross-sectional area of the airfoil to the square of the chord. This is

$$C_V = \frac{1}{\sqrt{3}} b_1 . \tag{8.16}$$

It should be noted that the P_0 term is omitted from the thickness distribution. This is equivalent to the condition that the airfoil must be a closed body.

It is clear that for an airfoil of given volume coefficient and moment coefficient the minimum drag will be obtained by using only the P_1 contours, giving an airfoil with circular arcs. For such an airfoil the drag will be

$$C_{D_{min.}} = \frac{1}{4} C_L^2 + 3 C_{M_0}^2 + 12 C_V^2 . \tag{8.17}$$

It may be pointed out that for a supersonic two-dimensional airfoil the only possible practical reason for camber is to obtain a moment coefficient and that this is always most efficiently carried out by a circular arc shape. The procedure of minimizing the drag with given volume and moment coefficients is easily done by the calculus of variations but the analysis with Legendre polynomials gives a more complete picture of the composition of the drag.

II. Methods for Solving the Potential Equation

9. The Principle of Superposition. Since equation (6.13) is considered to yield potentials whose gradient is the perturbation velocity and since this equation is linear, two valid solutions may be added together to obtain another valid solution. This principle is called the principle of superposition and may be applied in a number of different ways. For example, if a given solution is displaced an infinitesimal distance in a given direction, changed in sign, and then superposed upon the original solution, the result is equivalent to a cartesian differentiation of the solution in the given direction and produces a new valid solution. Similarly a solution multiplied by a variable factor and applied at points along a line in the flow field may be superposed by integration to give a valid solution. This may be further generalized to a multiple integral; thus if ϕ_1 is a solution

$$\phi = \iiint f(x_0, y_0, z_0)\, \phi_1(x-x_0, y-y_0, z-z_0)\, dx_0\, dy_0\, dz_0 \tag{9.1}$$

is also a solution, where f is an arbitrary function.

Since the cartesian velocity components are obtained from the potential by cartesian differentiation they will satisfy individually the potential equation (6.13) and will have the same properties of superposition. It should be noted that the pressure and the velocity potential cannot be superposed in all cases although the velocity component w can be superposed. This is particularly true for such solutions as the point source and the infinitesimal horseshoe vortex that have no individual physical significance.

10. General Methods. For a complete presentation of the general methods available for the wave equation (6.13) the reader is referred to references (51, 52, 53, 54). The principal methods which will be used here are that of

Riemann for two-dimensional hyperbolic equations and that of Hadamard, similar to that of Volterra, for the three-dimensional wave equation.

First it will be necessary to introduce the concept of the co-normal. The normal, or normal vector, is a unit vector perpendicular to the line or surface element under consideration. The co-normal for the cases considered here is a unit vector related to the normal having the same component as the normal in the z direction but having the components perpendicular to the z direction changed in sign. Thus if a, b, and c are the direction cosines of the normal, the vectors representing the normal and co-normal are

$$\bar{n} = a\bar{i} + b\bar{j} + c\bar{k} \tag{10.1a}$$

$$\bar{\nu} = -a\bar{i} - b\bar{j} + c\bar{k} \tag{10.1b}$$

The symbol n refers to the normal and the symbol ν refers to the co-normal. For the two-dimensional case the relative orientations of the line element, the normal, and the co-normal are illustrated in the following figure:

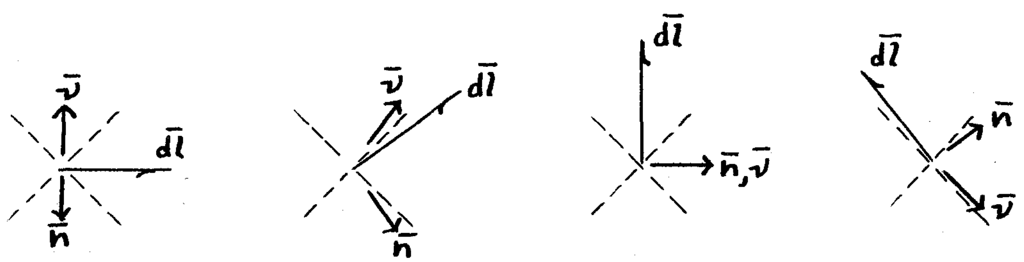


Fig. 10.1 The Co-normal

The form of the method of Riemann that will be used here will be that for the self-adjoint homogeneous equation

$$\frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 U}{\partial z^2} + C(y,z)\,U = 0 \tag{10.2}$$

If U and V are two valid solutions to equation (10.2) the following relation will hold

$$\oint \left(U \frac{dV}{d\nu} - V \frac{dU}{d\nu} \right) dl = 0 \tag{10.3}$$

the integral being taken about a closed contour. The essence of the Riemann integration method is the use of equation (10.3) with V a particularly chosen solution to equation (10.2). This solution must have the property

$$V = 1 \qquad \text{on} \quad z - y = z_0 - y_0 \qquad \text{and} \quad z + y = z_0 + y_0 \tag{10.4}$$

Since the directions of the line element and the co-normal coincide along a characteristic the quantity $\frac{dV}{d\nu}$ will be zero along such a contour and the quantity $V\frac{dU}{d\nu}$ will be simply the derivative of U in the direction of the contour. The contribution of the integral along a portion of a characteristic to (10.3) will be given by the values of U at the end points of this portion. The classical application of Riemann's method is to an initial value problem and the contour chosen consists of portions of two intersecting characteristics and a line joining a point on one of the characteristics to a point on the other. However, initial value problems will not be of particular interest here and various other contours will be used. The function V, called the Riemann function, has the property of being unchanged on exchanging (y_0, z_0) for (y,z). It is convenient for the derivation of the Riemann function to make a linear transformation of the variables to a coordinate system in the characteristics as in the classical statement of the Riemann method.

The result of Hadamard for equation (6.13) as will be used in this analysis is

$$\pi\phi_0 = \left|\overline{\iint\left(\frac{1}{R}\frac{d\phi}{d\nu} - \phi\frac{d}{d\nu}\frac{1}{R}\right)dS}\right. \tag{10.5}$$

where

$$R = \sqrt{(z-z_0)^2 - (x-x_0)^2 - (y-y_0)^2} \tag{10.6}$$

is the hyperbolic distance, and the symbol $\left|\overline{\quad\quad}\right.$ means "the finite part of" in the sense introduced by Hadamard. The reader is referred to Hadamard's text (51) for a complete discussion of this concept and especially to section 128, page 205, for the particular closed surface considered. The surface integral in equation (10.5) is taken over a plane passing through the point (x_0, y_0, z_0) parallel to the flow direction and over a surface closing the cone formed by the plane and half of the upstream Mach cone.

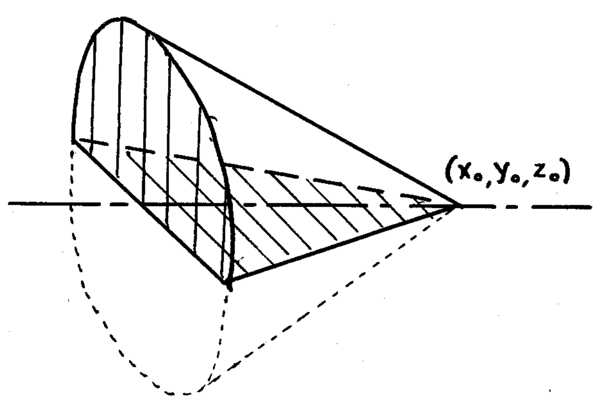


Fig. 10.2. Surface for Hadamard Integral

The customary closed surface used for initial value problems includes the entire Mach cone and has a factor 2π in equation (10.5) instead of π . The plane parallel to the flow direction is a non-duly inclined surface while the other surface is generally a duly inclined surface, these terms being in the sense defined by Hadamard. In both the Riemann theory and the theory of Hadamard a distinction must be made between duly inclined and non-duly inclined boundaries as to the question of sufficiency and necessity of boundary conditions.

The concept introduced by Hadamard as to the finite part of an integral with an infinity of fractional order has been found to be useful. It is advantageous to explain this concept from another point of view. In order to have a finite integral a function with a -3/2 power singularity must be considered to arise from the differentiation of a function with a -1/2 power singularity. The differentiation from the null side of the singularity to the infinity corresponds in the function with the -3/2 power singularity to a pulse similar to the familiar delta function but with an infinite integral. Thus if a -3/2 power singularity is always associated with such a pulse the integral across such a singularity will be finite and will represent the finite part in the sense of Hadamard. In a similar way a -5/2 power singularity is considered to have associated with it a double pulse so that integration across the singularity will give the -3/2 power singularity with its single pulse. The concept may be extended to integration across singularities of even integral order, odd integral orders being taken care of by a Cauchy principal value. Thus if a singularity of order -2 is considered to have associated with it a pulse and if the -1 power singularity associated with it is integrated with a Cauchy principal value the integral across this singularity may be considered to remain finite.

As a further explanation for integrals of this type and as a justification for them they may be considered as integrals of a complex variable. An integral of a real variable may be considered under reasonable restrictions as an integral of a complex variable along the real axis. Real integrals across singularities of half fractional order may be considered from this point of view as the real part of the complex integral where the contour is taken around the singularity to the real axis on the other side. In a similar way the integral across a singularity of integral order may be taken as the real part of the corresponding complex integral with the contour deformed around the corresponding pole. These integrals will be found to be the same as those using the pulse concept or the finite part concept of Hadamard. Such an integral about a simple pole gives the Cauchy principal value for the -1 power singularity.

11. Solutions by Analogy from Subsonic Theory. A great many solutions are known for the corresponding linearized subsonic or incompressible fluid theory, these being solutions to Laplace's equation. It is possible by multiplying the variables x and y by the factor i to change Laplace's equation to the wave equation (6.13). Correspondingly the solutions to Laplace's equation are changed to solutions of the wave equation, the radial distance changing to the hyperbolic distance (7.2). An example of this is the solution for a unit source in incompressible flow

$$\phi = -\frac{1}{4\pi\sqrt{z^2+x^2+y^2}} \qquad (11.1)$$

The corresponding solution to the wave equation is real inside the Mach cones and is imaginary in the region between the Mach cones. The corresponding solution for supersonic flow is obtained by discarding both the imaginary part of the solution and the real part in the upstream Mach cone. Due to the fact that half of the real part of the solution is discarded the numerical factor must be multiplied by two for the solution to represent a unit source.

$$\phi = - \frac{1}{2\pi\sqrt{z^2 - x^2 - y^2}} \tag{11.2}$$

Considering the real solution before the part in the upstream Mach cone is discarded there is a question of sign arising from the square root. The complete real solution with the same sign in both Mach cones and the factor $\frac{1}{4\pi}$ represents a unit source. The same solution with the sign in the upstream Mach cone changed represents a possible solution of zero source strength. The solution (11.2) may be considered as the superposition of these two solutions.

The solution for an infinitesimal trailing vortex of unit lift in incompressible flow is given by

$$\phi = \frac{1}{4\pi\rho V} \frac{\sin\theta}{\sqrt{x^2+y^2}} \left(\frac{z}{\sqrt{z^2+x^2+y^2}} + 1 \right) \tag{11.3}$$

In this solution the first term in the parentheses represents a solution symmetric with respect to the axial variable. The second term in the parentheses is the term which must be added to remove the upstream vortex system. In the supersonic system this second term is not necessary as the upstream vortex system is removed by the same stratagem as for the source, by discarding the solution in the upstream Mach cone. This solution is

$$\phi = \frac{1}{2\pi\rho V} \frac{\sin\theta}{\sqrt{x^2+y^2}} \left(\frac{z}{\sqrt{z^2-x^2-y^2}} \right) \tag{11.4}$$

Other solutions to supersonic flow may be obtained by the same method, as for example that for a ring source. The discarding of the imaginary part of the solution is easily seen to be justified when it is observed that each of the real and imaginary parts of a complex solution to equation (6.13) will satisfy the equation individually.

12. Separation of the Axial Variable. The general method of separation of variables may be applied to the wave equation in many ways. One method applied by von Karman and Chang (15,23,37) consists in separating the axial variable and superposing solutions of the resultant equation. If the substitution is made

$$\phi = e^{isz} \Phi(x,y) \tag{12.1}$$

the variable z is separated and the resulting function satisfies the equation

$$\Phi_{xx} + \Phi_{yy} + s^2 \Phi = 0 \tag{12.2}$$

This equation in polar coordinates

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + s^2 \Phi \tag{12.3}$$

may be further separated by the substitution

$$\Phi = \sin(m\theta + \beta)\, R(r) \tag{12.4}$$

where β is an arbitrary constant. The function R satisfies the equation

$$R'' + \frac{1}{r}R' + \left(s^2 - \frac{m^2}{r^2}\right)R = 0 \tag{12.5}$$

which has the solution in terms of Bessel functions

$$R = Z_m(sr) \tag{12.6}$$

where Z_m represents a general solution of parameter m. In order that this solution represent a wave inclined downstream, with no energy going inward the particular solution that must be chosen is the second Hankel function $H_m^{(2)}$, due to the fact that it is asymptotic to e^{-isr}. The application of this separation will not be discussed here, although an example will be given later.

It is possible to complete the separation of equation (12.2) in terms of elliptic coordinates or in terms of parabolic coordinates. The first of these could be applied to the solution on a strip of constant width either lying in the flow direction or made askew by the oblique transformation. Similarly the second of these could be applied to solutions arising at the edge of a half infinite sheet.

13. Separation of the Lateral Variable. In this method the lateral variable x is separated by the equation

$$\phi = \sin(kx+\beta)\ \bar{\Phi}(y,z) \tag{13.1}$$

which yields from equation (6.13)

$$\bar{\Phi}_{yy} - \bar{\Phi}_{zz} - k^2\bar{\Phi} = 0 . \tag{13.2}$$

This equation is of the form (10.2) for which Riemann's method may be applied, with $U = \bar{\Phi}$ and the Riemann function

$$V = J_0\left(k\sqrt{(z-z_0)^2-(y-y_0)^2}\right) . \tag{13.3}$$

The derivation of this Riemann function will not be described as it is well known and is given in various texts.

This method may be applied for the investigation of planar systems, periodic or not, with the use of Fourier's series or Fourier integrals.

14. Separation of the Azimuthal Variable. If in the cylindrical coordinate form of the potential equation (6.14) a substitution is made

$$\phi = \sin(m\theta + \beta)\ \Phi(r, z) \tag{14.1}$$

the function Φ satisfies the equation

$$\Phi_{rr} + \frac{1}{r}\Phi_r - \Phi_{zz} - \frac{m^2}{r^2}\Phi = 0 \tag{14.2}$$

If the further substitution is made

$$\Phi = \frac{1}{\sqrt{r}}\,U \tag{14.3}$$

the resulting equation is in the form (10.2)

$$U_{rr} - U_{zz} - \frac{m^2 - 1/4}{r^2}U = 0. \tag{14.4}$$

The Riemann function for this equation will be derived in a later section. This method will be used for the investigation of bodies of revolution.

15. Separation in Conical Coordinates. The new coordinate will be introduced into the cylindrical form (6.14) to replace the coordinate r

$$t = r/z. \tag{15.1}$$

This quantity is the ratio of the tangent of the polar angle of the point in question relative to the origin to the tangent of the Mach angle. Equation (6.14) with r eliminated and t introduced becomes

$$(1-t^2)\,\phi_{tt} + \frac{1}{t}(1-2t^2)\,\phi_t + \frac{1}{t^2}\,\phi_{\theta\theta} + 2tz\,\phi_{tz} - z^2\phi_{zz} = 0. \quad (15.2)$$

The substitution

$$\phi = z^n\,\Phi(t,\theta) \quad (15.3)$$

yields the equation

$$(1-t^2)\,\Phi_{tt} + \frac{1}{t}\Big(1+2(n-1)t^2\Big)\Phi_t - n(n-1)\,\Phi + \frac{1}{t^2}\,\Phi_{\theta\theta} = 0. \quad (15.4)$$

This function may be called the velocity potential for generalized conical flow. If n = 1 the function describes conical flow; if in place of ϕ is placed one of its cartesian derivatives, n = 0 describes conical flow.

If in the equation for n = 0 the substitution is made

$$t = \frac{2s}{1+s^2} \quad (15.5)$$

equation (15.4) reduces to the polar coordinate form of the two-dimensional Laplace's equation. This fact serves as the basis for the later treatment of conical flow. If the same substitution is made in the equation for n = -1 the function $\sqrt{1-t^2}\,\Phi$ satisfies the same equation. It was pointed out by R Jones (49) that these results were obtained in essence by W. Donkin in 1857 (52, p 357).

If the further substitution is made

$$\Phi = \sin(m\theta+\beta)\,T(t) \quad (15.6)$$

the resulting equation for T is

$$(1-t^2)\,T'' + \frac{1}{t}\left(1+2(n-1)t^2\right)T' - \frac{1}{t^2}\left(m^2+n(n-1)t^2\right)T = 0. \quad (15.7)$$

The solutions of this equation will be investigated in the next section. It should be noted that the axial velocity component is no longer in this system of coordinates by the partial derivative of the velocity potential with respect to z, as this derivative is now taken with t constant instead of r.

$$w = \phi_z - \frac{t}{z} \phi_t . \tag{15.8}$$

The velocity components in the radial and azimuthal directions are given by

$$q_r = u = \frac{1}{z} \phi_t \tag{15.9a}$$

$$q_\theta = v = \frac{1}{tz} \phi_\theta \tag{15.9b}$$

where u and v in this case are not cartesian velocity components but are nevertheless mutually perpendicular velocity components normal to the flow direction. This variation in notation will be used only when there will be no confusion with the more widely used notation for which u and v are the cartesian velocity components.

III. Fundamental Flows in Conical Coordinates

16. Solutions of the Differential Equation. The differential equation for T, (15.7), has four regular singular points. However, it is possible to change the independent variable to t^2 without introducing radicals. This brings the equation into hypergeometric form, with the singularities in terms of t:

Singularity	Exponents	Exponent difference in t^2
$t = 0$	$m, -m$	m
$t = \pm 1$	$0, n + 1/2$	$n + 1/2$
$t = \infty$	$-n, 1 - n$	$1/2$

The exponent difference in t^2 is of importance in establishing when logarithmic solutions may occur. According to the theory of linear differential equations, logarithmic solutions may occur at $t = 0$ when m is an integer, at $t = \pm 1$ when $n + 1/2$ is an integer, but never at $t = \infty$. For $m = 0$ or $n + 1/2 = 0$ logarithmic solutions must appear.

The singularity at $t = \pm 1$ represents the two Mach cones extending from the origin. Various ranges of t correspond to various regions of flow:

Range of t	Region of Flow
$0 \leq t < 1$	inside downstream cone
$-1 < t \leq 0$	inside upstream cone
$1 < t \leq \infty$ $-\infty \leq t < -1$	outside both cones

Of course t may be considered to change sign within a Mach cone in the same manner as r in polar coordinates.

Since m does not enter into the values of the exponents at $t = \pm 1$ and $t = \infty$ the solutions are of the same exponents regardless of the value of m. Hence these exponents may be considered to be characteristic of the solutions for Φ from equation (15.4). An examination of this equation shows that it is similar to Laplace's equation in two dimensions in the vicinity of $t = 0$, whence there is no singularity in equation (15.4) other than that fundamental in polar coordinates. Similarly it is clear that the singularity at $t = \infty$ also arises only from the structure of the coordinate system. Hence the equation for Φ has a fundamental singularity only at $t = \pm 1$, with exponents 0 and $n + 1/2$.

It is essential to classify solutions obtained from either equation (15.4) or (15.7) according to their behavior at the point $t = \pm 1$. Two types of solutions will be distinguished for the case where $n + 1/2$ is not an integer: the first, designated as type I, has the exponent 0 at $t = \pm 1$ and has the property of being real and single-valued throughout the range of t; the second, designated as type II, has the exponent $n + 1/2$ at $t = \pm 1$ and has the property that it may represent a solution which exists only within the Mach cones. Where $n + 1/2$ is an integer the situation is more complicated and must be investigated separately in each particular case. The only general statements that may be made are that for $n + 1/2 \geq 0$ the solution of that exponent is not logarithmic and is of type II in the sense above, and that for $n + 1/2 < 0$ the solution of exponent zero is not logarithmic and is of type I.

The solutions about the origin are

$$T = t^m F\left(\frac{-n+m}{2}, \frac{-n+m+1}{2}; 1+m; t^2\right) \tag{16.1a}$$

$$= t^m (1-t^2)^{n+\frac{1}{2}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}; 1+m; t^2\right) \tag{16.1b}$$

$$T = t^{-m} F\left(\frac{-n-m}{2}, \frac{-n-m+1}{2} ; 1-m ; t^2\right) \tag{16.2a}$$

$$= t^{-m}(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n-m+1}{2} ; \frac{n-m+2}{2} ; 1-m ; t^2\right) \tag{16.2b}$$

where the second form of each solution is obtained through the identity for the hypergeometric function

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \tag{16.3}$$.

In order to separate the solutions of the two types it is convenient to express the solutions about $t^2 = 1$. These solutions are

$$\text{I)} \quad T = t^{m} F\left(\frac{-n+m}{2}, \frac{-n+m+1}{2} ; -n+\tfrac{1}{2} ; 1-t^2\right) \tag{16.4a}$$

$$= t^{-m} F\left(\frac{-n-m}{2}, \frac{-n-m+1}{2} ; -n+\tfrac{1}{2} ; 1-t^2\right) \tag{16.4b}$$

$$\text{II)} \quad T = t^{m}(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2} ; n+3/2 ; 1-t^2\right) \tag{16.5a}$$

$$= t^{-m}(1-t^2)^{n+\frac{1}{2}} F\left(\frac{n-m+1}{2}, \frac{n-m+2}{2} ; n+3/2 ; 1-t^2\right). \tag{16.5b}$$

The solutions about $t = \infty$ are of less interest but are given here

$$T = t^{n} F\left(\frac{-n+m}{2}, \frac{-n-m}{2} ; \frac{1}{2} ; \frac{1}{t^2}\right) \tag{16.6}$$

$$T = t^{n+1} F\left(\frac{-n+m+1}{2}, \frac{-n-m+1}{2} ; 3/2 ; \frac{1}{t^2}\right). \tag{16.7}$$

By introducing the variable iz as a new variable into equation (6.14), Laplace's equation in three dimensions is obtained (36). The Legendre function solutions for this equation are essentially the same as the solutions in terms of T. Hence the T functions may be expressed in terms of Legendre functions of argument either real and larger than one or imaginary. These solutions are

$$T = (1-t^2)^{n/2}\; P_n^m\left[(1-t^2)^{-1/2}\right] \tag{16.8a}$$

$$T = (1-t^2)^{n/2}\; Q_n^m\left[(1-t^2)^{-1/2}\right]. \tag{16.8b}$$

These have been extensively investigated for general values of the parameters, particularly by Barnes. Reference may be made to Bateman (<u>53</u>).

The solutions of type I in equation (16.4) for which $m = |n|$ are given by $T = t^n$. In cylindrical coordinates the corresponding complete solutions are

$$\phi = r^n \sin\left(|n|\theta + \beta\right) \tag{16.9}$$

and represent two-dimensional cross-flow. Since for these solutions $w = 0$ the flow appears as an incompressible flow and the pressure is given only by the quadratic terms in equation (4.2).

17. <u>Relations between the Solutions</u>. As mentioned in section 9 equation (6.13) is invariant under cartesian differentiation. Solutions of the type of equations (15.3) and (15.6) expressed in cartesian coordinates and differentiated with respect to these coordinates are still solutions of equation (6.13) or (15.2). This fact permits a given solution of parameters n and m to yield solutions of parameters $n - 1$ and m, $m + 1$, or $m - 1$:

$$T(n-1, m) = nT - tT' = -t^{n+1}\frac{d}{dt}\left(t^{-n}T\right) \tag{17.1}$$

$$T(n-1, m+1) = \frac{m}{t}T - T' = -t^m\frac{d}{dt}\left(t^{-m}T\right) \tag{17.2a}$$

$$T(n-1, m-1) = \frac{m}{t}T + T' = t^{-m}\frac{d}{dt}\left(t^mT\right). \tag{17.2b}$$

In a similar manner solutions with the parameter n increased by one may be obtained by reversing equations (17.1) and (17.2) with suitable integrations. In such a process the constant of integration is not arbitrary but must be chosen so that the result is a valid solution. For $n + 1/2$ not an integer the type of the solution is preserved in these processes.

These relations between contiguous solutions are not to be considered as recurrence relations, as no system has been established for specifying solutions with respect to the multiplicative constant. The relations between contiguous hypergeometric functions may be used to establish relations between T functions but these do not seem to be particularly useful.

An integral relation connecting two solutions whose parameters differ in value may be obtained either from the corresponding relation for the Legendre functions or directly from equation (15.7). If T_1 denotes a solution corresponding to n_1 and m_1 and T_2 a solution corresponding to n_2 and m_2, the relation is

$$\begin{aligned}(m_1^2 - m_2^2)\int_a^b \frac{1}{t}(1-t^2)^{-\frac{1}{2}(n_1+n_2+1)} T_1 T_2 \, dt \\ + (n_1-n_2)(n_1+n_2+1)\int_a^b t\,(1-t^2)^{-\frac{1}{2}(n_1+n_2+3)} T_1 T_2 \, dt \\ = \Bigg[t\,(1-t^2)^{-\frac{1}{2}(n_1+n_2-1)}\left(T_2 \frac{dT_1}{dt} - T_1 \frac{dT_2}{dt}\right) \\ + (n_1-n_2)\,t^2(1-t^2)^{-\frac{1}{2}(n_1+n_2+1)} T_1 T_2 \Bigg] \Bigg|_a^b \end{aligned} \tag{17.3}$$

Setting $n_1 = n_2$ or $m_1 = m_2$, we obtain simpler equations as special cases which may be used to obtain orthogonality relations between solutions.

18. <u>Solutions with n and m Integral</u>. Since the general solution to equation

(15.7) involves both values of m it is convenient to consider m to be positive and to express separately the solutions of positive and negative exponent at the origin. For m and n integral three special cases are distinguished according to the relative values of n and m:

case A: $-\infty < n \leq -m-1$,

case B: $-m \leq n \leq m-1$,

case C: $m \leq n < \infty$

The distribution of these cases for small values of m and n is shown in the table:

m \ n	-3	-2	-1	0	1	2
0	A	A	A	C	C	C
1	A	A	B	B	C	C
2	A	B	B	B	B	C
3	B	B	B	B	B	B

From a consideration of equations (16.1) to (16.5) the forms of the two types of solutions in the various cases may be found. For all solutions except solutions I-A (i.e., solutions of type I in case A) and solutions II-C, the form is explicit in terms of a polynomial in t^2 or in $(1 - t^2)$. Solutions I-A and II-C have logarithmic singularities at $t = 0$ and are discussed later. The polynomial forms are expressed as follows where P indicates a polynomial:

Solutions	Form	Order of $P(t^2)$ whichever is integral of	Equation for Calculation
I-A	logarithmic	———	———
II-A	$t^m(1 - t^2)^{n+\frac{1}{2}} P(t^2)$	$\frac{1}{2}(-n - m - 1)$ or $\frac{1}{2}(-n - m - 2)$	(16.1b) or (16.5a)
I-B	$t^{-m} P(t^2)$	$\frac{1}{2}(n + m)$ or $\frac{1}{2}(n + m - 1)$	(16.2a) or (16.4b)
II-B	$t^{-m}(1 - t^2)^{n+\frac{1}{2}} P(t^2)$	$\frac{1}{2}(-n + m - 1)$ or $\frac{1}{2}(-n + m - 2)$	(16.2b) or (16.5b)
I-C	$t^m P(t^2)$	$\frac{1}{2}(n - m)$ or $\frac{1}{2}(n - m - 1)$	(16.1a) or (16.4a)
II-C	logarithmic	———	———

It should be observed that the solutions (16.2) are not well-defined when m is an integer.

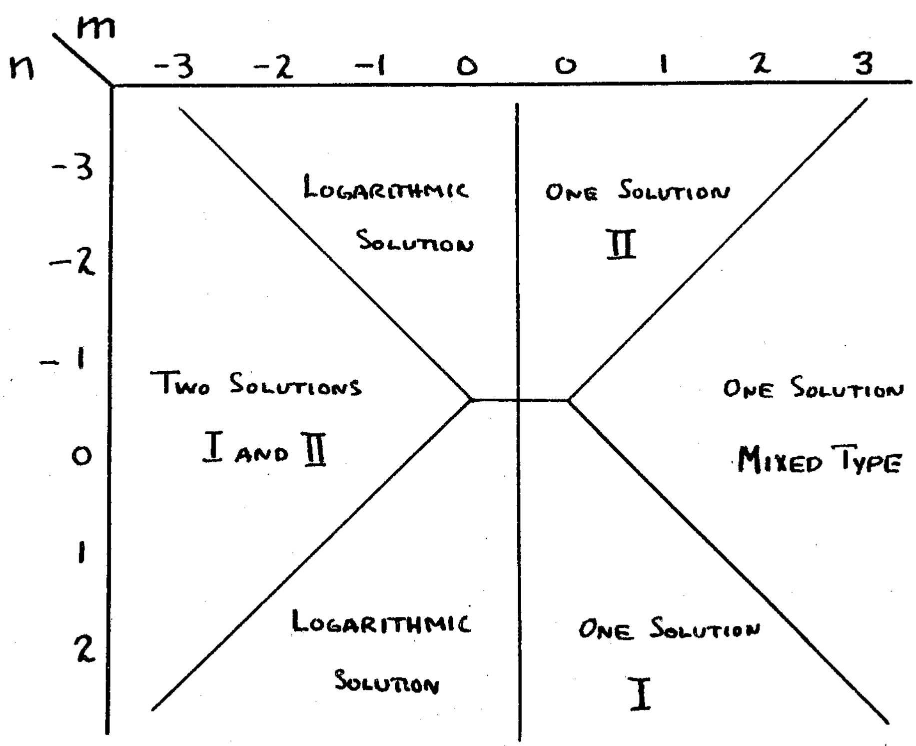
The logarithmic solutions I-A and II-C are most easily expressed in terms of the Legendre functions, as in equation (16.8b). They are

$$\text{I-A} \qquad T = (1-t^2)^{n/2}\, Q^m_{-n-1}\left[(1-t^2)^{-\frac{1}{2}}\right] \tag{18.1a}$$

$$\text{II-C} \qquad T = (1-t^2)^{n/2}\, Q^m_{n}\left[(1-t^2)^{-\frac{1}{2}}\right] \tag{18.1b}$$

Where n is negative -n - 1 is used in place of n. The corresponding solutions II-A and I-C are given by the related Legendre polynomials. Other forms for the solutions may be obtained, including explicit forms for the logarithmic solution (41).

The distribution of solutions with respect to the exponent m may be seen from the following chart where m is allowed to be negative to indicate a negative exponent:



19. Solutions with n and m Half-Integral. The term half-integral will be used to denote quantities differing from an integer by one half. Thus when m and n are half-integrals the quantities m - 1/2 and n + 1/2 are integers. For this case it will be seen that either equation (16.4) or (16.5) is not well-defined, and that since n + 1/2 is integral the question as to the division of the solutions into types I and II is undetermined as discussed in section 16. In contrast with the case where m and n are integral the solutions of different exponents at the origin are now well-defined. The same classification with respect to the relative values of n and m is made as in the previous section. However, the behavior of the solutions is entirely different for the various cases.

From an examination of the hypergeometric forms of the solutions given in section 6 it is clear that there is a direct analogy between the solutions for m and n integral and for m and n half-integral. This is expressed by considering the corresponding solution of different parameters given by the relation

$$n+\tfrac{1}{2} = m' \qquad ; \qquad m = n'+\tfrac{1}{2} \tag{19.1}$$

Cases A and C now correspond to case B' for the related solution with changed parameters and case B corresponds to cases A' and C'. Equations (16.1) and (16.2) take the place of (16.4) and (16.5). The final relation between the solutions obtained is given by

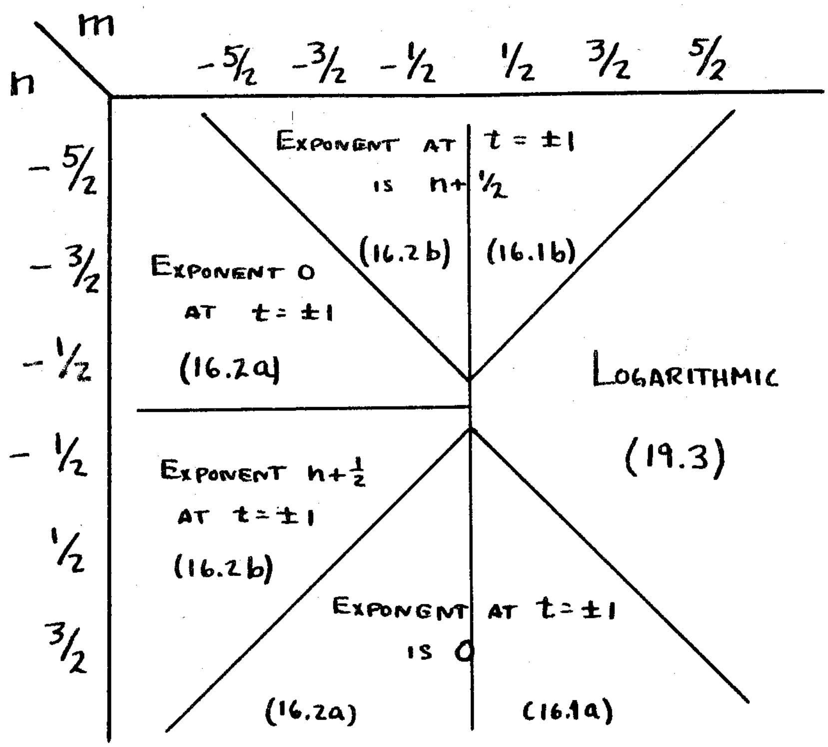
$$T(m,n,t) = t^{-m}(1-t^2)^{\frac{n+\frac{1}{2}}{2}}\, T\left(n+\tfrac{1}{2}, m-\tfrac{1}{2}, \sqrt{1-t^2}\right) \tag{19.2}$$

In this equation the solution of negative exponent at the origin is given by the solution of type I' and the solution of positive exponent at the origin is given by the solution of type II'. These relations between solutions of integral exponents and solutions of half-integral exponents are analogous to the corresponding relations for the Legendre functions as described by Bateman (53).

The solutions in the various cases may be characterized as follows: In case A all solutions may be expressed in polynomial form analogous to the polynomial forms of the preceding section. The solutions of both positive and negative exponents at the origin are of exponent n + 1/2 at t = ± 1. The solution of 0 exponent at t = ± 1 is mixed with respect to the exponents at the origin. In case C as in case A all solutions may be expressed in polynomial form. The solutions of both positive and negative exponents at the origin are of exponent 0 at t = ± 1. The solution of exponent n + 1/2 at t = ± 1 is mixed with respect to the exponents at the origin. In case B the solutions of negative exponent at the origin are of exponent 0 or n + 1/2, whichever is greater, at t = ± 1. The solutions of positive exponent at the origin are logarithmic and are given by

$$T = t^{-\frac{1}{2}}(1-t^2)^{\frac{n+\frac{1}{2}}{2}} Q_{m-\frac{1}{2}}^{|(n+\frac{1}{2})|}\left(\frac{1}{t}\right) \qquad (19.3)$$

The distribution of solutions may be seen from the following chart where n is allowed to be negative to indicate a negative exponent:



20. <u>Methods of Obtaining other Solutions</u>. The scale transformation and rotation about the flow axis applied to the solutions obtained above will not give essentially new solutions. On the other hand the oblique transformation will produce new solutions in most cases. These solutions can be expressed as solutions to equation (15.4) but not as solutions to equation (15.7); in other words, they are Φ solutions but not T solutions. The method of superposition using equation (9.1) may be used to extend these solutions.

For the solutions of type II a general relation may be given between solutions having the same value of m and different values of n. The new solution will be formed from the equation analogous to (9.1)

$$\phi_{n+q} = \int_r^z (z-z_1)^{q-1}\, \phi_1(z_1)\, dz_1 \tag{20.1}$$

Fig. 20.1 is given to illustrate the manner in which equation (20.1) is obtained.

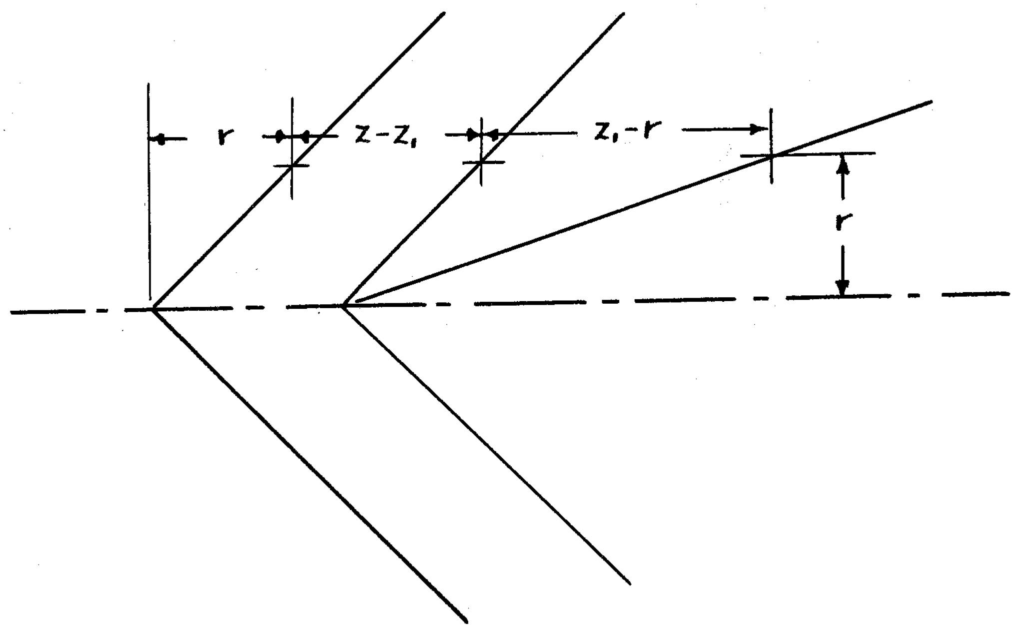


Fig. 20.1. Superposition of Type II Solutions

Equation (20.1) may be re-expressed using the separation (15.3) as

$$\Phi_{n+l} = t^{n+l}\int_1^{1/t}\left(\frac{1}{t}-\frac{1}{t_1}\right)^{l-1}\left(\frac{1}{t_1}\right)^n \Phi_1(t_1)\; d\left(\frac{1}{t_1}\right) \qquad (20.2)$$

where l is the quantity by which the parameter n is increased. This gives the relation between the corresponding T functions

$$T_{n+l} = t^{n+l}\int_1^{1/t}\left(\frac{1}{t}-\frac{1}{t_1}\right)\left(\frac{1}{t_1}\right)^n T_1(t_1)\; d\left(\frac{1}{t_1}\right) \qquad (20.3)$$

In general it is more convenient to leave the integration in terms of the variable 1/t. The function that is expressed by the term in parentheses to the power $l - 1$ is defined in such a way as to be null for negative values of the argument. Thus for $l = -1/2$ the function must have a pulse of the type discussed in section 10. For $l = 1$ the well-known unit step function is to be used. For $l - 1$ negative the function that must be used consists of a pulse system at the origin, that for $l = 0$ being the Dirac delta function, and the others yielding differentiations. Some of these are illustrated in Fig. 20.2.

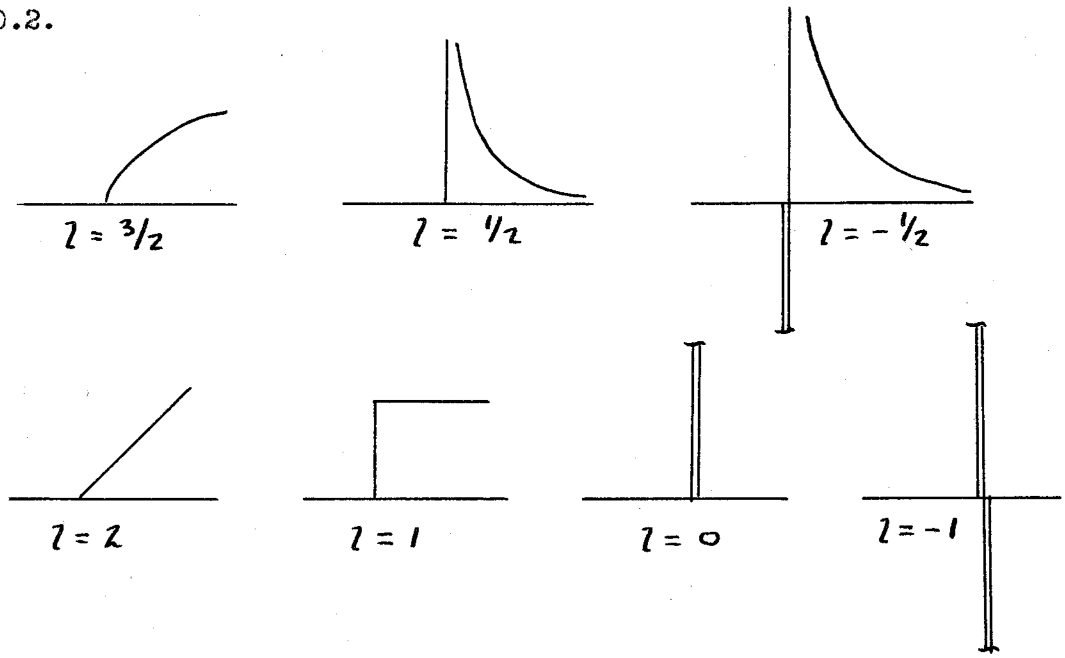


Fig. 20.2. Functions used in the Superposition

21. Interpretation of the Solutions. The solutions have been divided in principle into two types, type I representing a solution throughout the entire flow field, type II representing a solution which may be restricted to lie only in the Mach cones. Interest in the solutions of type I is very limited and only a very few elementary solutions such as those giving cross-flow are used. The principal interest lies in the solutions of type II which,

by the simple expedient of dropping the solution in the upstream Mach cone as in section 11, represent solutions existing only within the zone of action of a single point.

The solution (II, -1, 0), that of type II with n = -1 and m = 0, represents the source described by (11.2). Similarly the solution (II, -2, 0) represents a dipole and the solution (II, -1, 1) is that for the infinitesimal horseshoe vortex described by (11.4). The solution (I, 1, 1) gives a uniform cross-flow that is useful for representing angle of attack.

The solutions for which n = 1 give conical flow. The solution (II, 1, 0) gives the flow about a right circular cone at zero incidence, while the solution (II, 1, 1) with the (I, 1, 1) solution mentioned above represents the effect of angle of attack. These two type II solutions are

$$(\mathrm{II}, 1, 0) \qquad T = \sqrt{1-t^2} - \cosh^{-1}\frac{1}{t} \tag{21.1a}$$

$$(\mathrm{II}, 1, 1) \qquad T = \frac{1}{t}\sqrt{1-t^2} - t\cosh^{-1}\frac{1}{t} \tag{21.1b}$$

An example of a solution with m half-integral which corresponds to a case of interest is obtained by letting m = -1/2, applying an oblique transformation, and extending the solution by equation (20.3) with l = 1. The solutions of this type will be the eigensolutions found later in the study of conical flow.

The solutions may also be considered to yield any of the cartesian derivatives of the velocity potential with the value of n decreased by 1. In particular the solution for w may be obtained through equation (17.1) or the same equation for Φ in place of T.

IV. Planar Systems

22. General Considerations. A planar system is defined as a system for which the boundary conditions may be satisfied on a plane parallel to the flow direction. For the purposes of this development the plane is taken to be the x-z plane. The assumptions which are necessary for a system to be regarded as a planar one are first that the inclination of the surface of the body with respect to the plane is small and second that the distance between the surface and the plane is small. This latter condition is not unambiguously stated as it is necessary to specify a reference dimension for the purpose of comparison. This reference dimension is essentially the wave length of the surface disturbances for the case considered here, with $M = \sqrt{2}$. These considerations are the same as for linearized planar systems in subsonic flow.

With bodies that are relatively thick in comparison with the wave length of the surface disturbances it is still possible to apply the concepts of planar systems, except that the view which must be taken for the flow field close to the system is different from that for the flow field at a distance from the system. For the local field the plane considered must be a plane parallel to the fundamental one but translated laterally so as to approximate the surface position. For the distant field the plane considered may be the fundamental one but with the disturbances translated in the axial direction. This point is illustrated in Fig. 22.1.

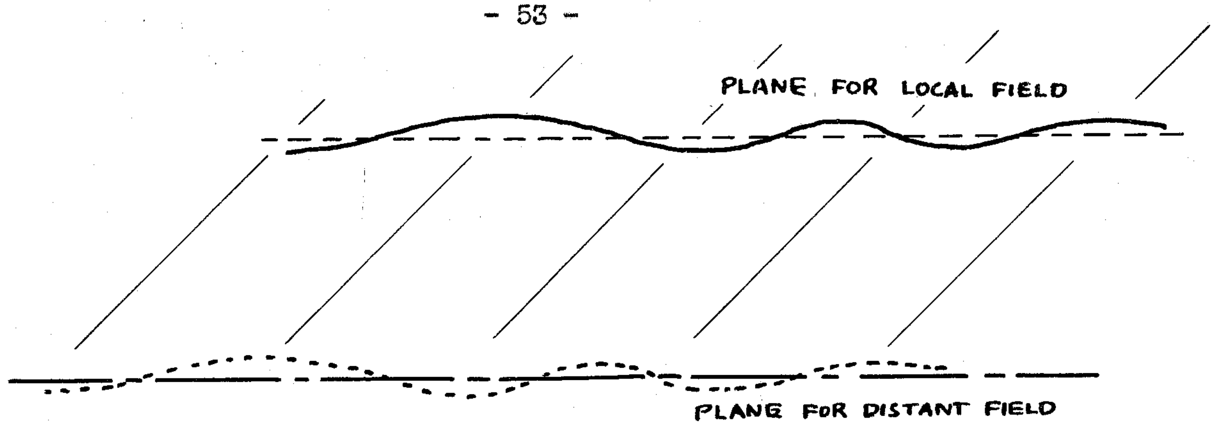


Fig. 22.1. Planes for Local and Distant Flow Fields

It is advantageous to divide the flow field for a planar system into symmetric and antisymmetric parts with respect to the x-z plane. This separation is given by

$$\phi_s = \frac{1}{2}\left[\phi(y) + \phi(-y)\right] \qquad (22.1)$$

$$\phi_a = \frac{1}{2}\left[\phi(y) - \phi(-y)\right] \qquad (22.2)$$

Thus the net flow field may be given by the superposition of the field arising from a symmetric velocity potential and the field arising from an antisymmetric potential. The velocity components u and w have the same symmetry properties as the velocity potential from which they are derived as they arise from differentiations in the x-z plane. The velocity component v has the opposite symmetry properties from the velocity potential from which it is derived as it arises from differentiation with respect to the direction normal to the x-z plane.

A discontinuity in a quantity across the x-z plane can only occur when that quantity is antisymmetric. Thus for the symmetric velocity potential, which will be referred to as describing a symmetric problem, a discontinuity can occur only in the velocity component v. Since there is no discontinuity in the quantity w which describes the pressure a symmetric problem or system is one which may have no lift distribution. From the discontinuity in v there may be a variation in the thickness of the body describing the system. Hence the symmetric problem describes a thickness distribution. Similarly, for an antisymmetric system discontinuities may occur in the velocity components u and w but not in v. Therefore, such a system will correspond to a zero thickness distribution and will represent a lift distribution.

The thickness and lift properties of a planar system are thus separated by these considerations of symmetry. The two systems have separate boundary conditions, the symmetric problem being determined by the thickness distribution of the system, the antisymmetric problem being determined either by the mean camber distribution of the system or by the lift distribution. An example of this division has already been given in section 8 for a two-dimensional planar system.

There must be a body in order that a discontinuity in either v or w may be sustained. However, this is not true for the quantity u and this feature describes an important characteristic of flow systems, that of the existence of vortex sheets. From the vorticity relation (4.5b) the strength of such a discontinuity in u will be constant in the z direction in any region of the x-z plane for which w = 0, as for the case where there is no body which may sustain lift. Hence, in linearized flow the strength of such a vortex system will extend unchanged to infinity downstream from a body. The general considerations on influence regions and the original undisturbed flow preclude any vortex system upstream from a single body.

An important result of the existence of such vortex sheets is that the oblique transformation may change the region over which a body is necessary for a given solution, because of the fact that the quantities u and w become linearly transformed as described by equations (7.3). In a similar fashion the requirement that a body be closed with respect to its thickness distribution may be changed by the oblique transformation. Thus for a body with a given planform the oblique transformation may be applied only where there is no vortex sheet arising from the lift distribution and where the condition of closure of the body will not be changed. The significant consequence of these considerations is that the z direction is a characteristic direction of the flow field. This direction in the x-z plane, together with the two directions which give the wave systems, describe three characteristic directions which are important in the consideration of the planforms of planar systems.

By arbitrarily changing the sign of the velocity potential for y negative a symmetric problem is changed into an antisymmetric problem and vice versa. This reversal of the symmetry of a system is often helpful. Thus there is an equivalence between two systems of opposite symmetry, the planforms describing the two bodies not necessarily being the same. As a result of this equivalence it is necessary to consider solutions only in the upper half space for which y is positive.

23. Basic Integral Relations. The method of Hadamard described in section 10 will be applied to the derivation of basic integral relations connecting the velocity components on the upper side of the plane of a planar system. The contour described will be extended sufficiently far upstream to place the surface closing the half Mach cone in a region of no disturbance, so that the integration may be taken only over the x-z plane. Since for this plane the direction of the conormal is perpendicular to the plane the quantity in equation (10.5) which involves the derivative of the inverse of the hyperbolic distance is zero and equation (10.5) reduces to

$$\phi_o = -\frac{1}{\pi}\iint \frac{1}{R}\,\frac{\partial \phi}{\partial y}\,dx\,dz \qquad (23.1)$$

where the minus sign comes from the fact that the direction of the conormal is opposite to the y direction. The quantity R in this expression is given by

$$R = \sqrt{(z_o-z)^2-(x_o-x)^2}\,. \qquad (23.2)$$

This integral is essentially a finite integral, as the region of integration need only be taken a sufficient distance upstream to cover the distribution of disturbances. Thus the order of integration may be changed arbitrarily, no difficulty arising from the singularity in $1/R$ with reasonable conditions imposed on the integrand. The integration may be carried out in either order, the limits being

$$\Bigg|_{x_o-(z_o-z)}^{x_o+(z_o-z)}\;\Bigg|_{-\infty}^{z_o} \qquad \text{or} \qquad \Bigg|_{-\infty}^{z_o-|x_o-x|}\;\Bigg|_{-\infty}^{+\infty}$$

for integration with respect to x first or z first, respectively.

Equation (23.1) is re-expressed in terms of the z component of the velocity

$$\phi_{z_o} = -\frac{1}{\pi}\iint \frac{\phi_{yz}}{R}\, dx\, dz \quad . \tag{23.3}$$

This expression may be integrated by parts twice, the first time taking the x derivative of the term in ϕ and the second time taking the z integral of this term. The result may be expressed

$$w_o = -v_o + \frac{1}{\pi}\iint \frac{x_o - x}{R(z_o - z)} \frac{\partial v}{\partial x}\, dx\, dz \quad . \tag{23.4}$$

In a similar fashion the relation for the y component of the velocity may be expressed with the aid of equation (6.13)

$$\phi_{y_o} = -\frac{1}{\pi}\iint \frac{\phi_{zz}}{R}\, dx\, dz + \frac{1}{\pi}\iint \frac{\phi_{xx}}{R}\, dx\, dz \quad . \tag{23.5}$$

Integrating this expression twice by parts and combining the two resultant integrals results in

$$v_o = -w_o - \frac{1}{\pi}\iint \frac{R}{(x_o - x)(z_o - z)} \frac{\partial w}{\partial x}\, dx\, dz \tag{23.6}$$

where a Cauchy principle value is to be taken across the singularity in x.

The two relations (23.4) and (23.6) are the fundamental integral relations desired, the first giving w as a function of v and the second giving v as a function of w. The first may be considered an integral equation in v for which the second is the solution and the second may be considered an integral equation in w for which the first is the solution. The expression (23.4) was first obtained by considering a distribution of sources on a plane and (23.6) was first obtained by considering a distribution of infinitesimal horseshoe vortices on a plane. In this process it was necessary to use some of the concepts described in section 10 as to integration across singularities. The method using Hadamard's integral is more direct.

It is evident that the relations obtained reduce to equations (8.3) with f = 0 for two-dimensional flow. Another simple application of these relations is to the problem of cylindrical flow. Two cases must be distinguished relative to the inclination of the cylindrical system to the characteristic wave direction. If the system is inclined at more than 45 degrees with respect to the flow direction the influence region of a single point includes only a finite portion of the cylindrical body. For this case the normal velocity component V_n corresponds to a supersonic flow. Conversely, if the cylindrical system is inclined at less than 45 degrees to the flow direction the influence region of a point will include an infinite portion of the system lying upstream and the normal velocity corresponds to a subsonic flow. These two cases are illustrated in Fig. 23.1. For the second case it is necessary to include the infinite portion of the system by a suitable limiting process, by assuming the system finite and extending it to infinity. The results of this application are not included here but they agree with the results obtained by considering the cylindrical flow from the simpler point of view of section 3 and thus serve as a check on the validity of the method.

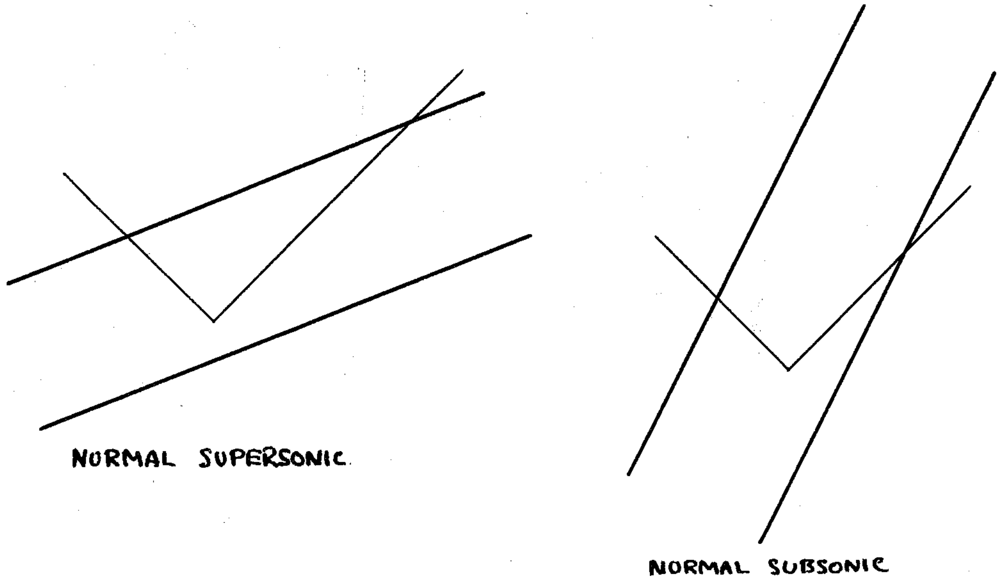


Fig. 23.1. Cylindrical Flow

24. Classification of Planform Edges. The bounding edges of a planform describing the body for a planar system will be classified with respect to their orientation to the characteristic directions on the plane. This classification is illustrated in Fig. 24.1.

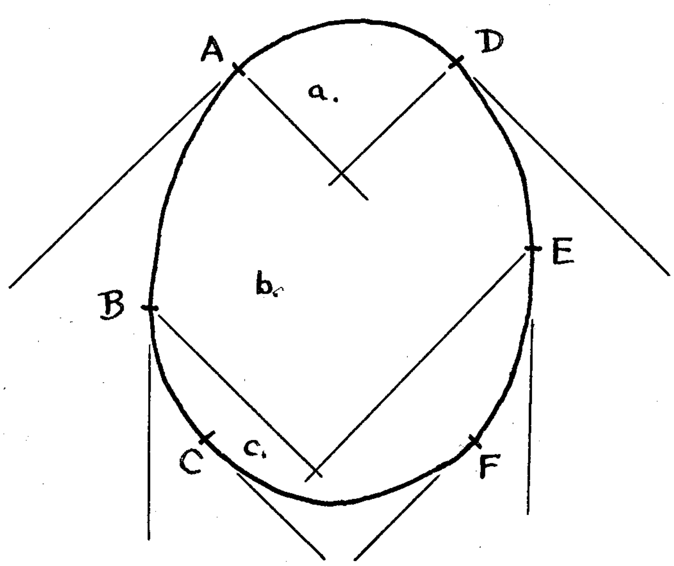


Fig. 24.1. Supersonic Planform

The normal Mach number for an edge is defined as the Mach number corresponding to the inclination of the edge according to the concepts of cylindrical flow. Thus if ψ is the inclination of the edge with respect to the flow direction and the Mach number of the uniform flow is $\sqrt{2}$ the normal Mach number is given by $\sqrt{2}\ \sin\psi$. This normal Mach number will be defined so that it is positive for a leading edge where the flow direction passes from a region off the planform onto the planform and will be defined as negative for a trailing edge. Thus it is possible to divide all the edges of a given planform into four types. These are: the supersonic leading edge, edge AD in Fig. 24.1; the subsonic leading edge, edges AB and DE; the subsonic trailing edge, BC and EF; and the supersonic trailing edge, CF. With the definition of normal Mach number as given above these edges are classified as shown in the table:

Supersonic Leading	$1 < M_n$
Subsonic Leading	$0 < M_n < 1$
Subsonic Trailing	$-1 < M_n < 0$
Supersonic Trailing	$M_n < -1$

The special boundary cases which occur between those classified above may be given special names. The case $M_n = 1$ may be denoted by the term sonic leading edge; the case $M_n = 0$ may be called a side edge; and the case $M_n = -1$ may be called a sonic trailing edge.

The planforms which may be constructed embrace a large variety of these types of edges. In general planforms will belong to one of three types: those having only supersonic edges, those having only subsonic edges, and those having both supersonic and subsonic edges. The only fundamental distinction that will be made is with respect to the first of these types. A planform having only supersonic edges will be termed a "simple" planform. A planform having some subsonic edges will be termed a "non-simple" planform. The significance of this classification will be made clear in the next section. Examples of this classification of planforms are given in Fig. 24.2.

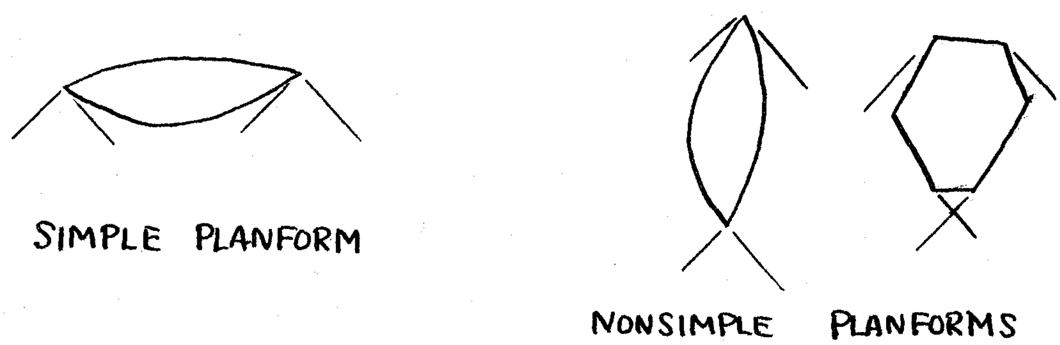


Fig. 24.2. Planform Classification

25. Problems of the First and Second Kind. A fundamental division of the problems arising in the study of planar systems is made with respect to the direct applicability of the integral relations derived in section 23. The x-z plane is divided into two regions, region 1 consisting of the interior of the planform and region 2 consisting of the remainder of the plane.

For the symmetrical or thickness case the problem of the first kind will be the problem of determining the pressure distribution where the shape of the body is known. For the antisymmetric or lift case the problem of the first kind is the problem of determining the camber distribution where the lift distribution is known. In the symmetric case the shape of the body determines the value of v in region 1. The condition of symmetry requires that v = 0 in region 2. Hence the quantity v is known over the entire plane and relation (23.4) gives the pressure distribution directly in both regions. Similarly in the antisymmetric case the quantity w is known over the entire plane and relation (23.6) gives directly the camber distribution in region 1 and the downwash distribution in region 2.

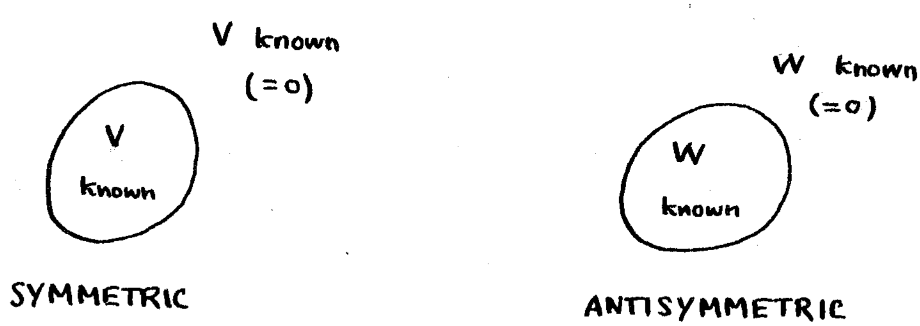


Fig. 25.1. Problems of the First Kind

For the symmetric case the problem of the second kind is the problem of determining the thickness distribution required to produce a given pressure distribution on the body. For the antisymmetric case the problem of the second kind is the problem of determining the lift distribution where the camber distribution of the body is known. In both of these cases neither the quantity v nor the quantity w is known over the entire plane and the integral relations of section 23 cannot be applied diredtly. The problem here is mixed with respect to which quantity is known and which is unknown.

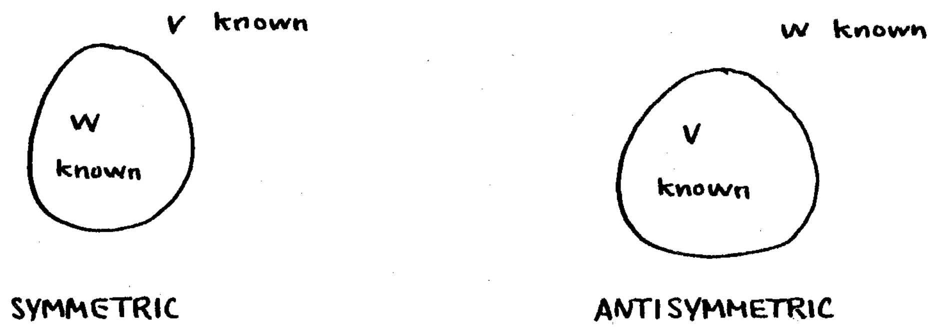


Fig. 25.2. Problems of the Second Kind

For the case of a simple planform, as defined in the previous section, the region which does not lie in the zone of action of the planform is completely undisturbed, so that the quantity w is known in the symmetric case and the quantity v is known in the antisymmetric case for this region. The process of reversing the symmetry described in section 22 may be applied. The zone of influence of the planform has no points in common with the zone of action. Hence if the value of the quantity w in the symmetric case and the quantity v in the antisymmetric case are set equal to zero in this zone of action no change will be made in the solution of the problem on the planform. This modification of the problem with the reversal of symmetry changes the problem of the second kind into a problem of the first kind. Thus any problem of the second kind for a simple planform may be transformed into a problem of the first kind for the purpose of obtaining a solution on the planform itself. This solution is of course incorrect in the zone of action of the planform but the correct solution there may be obtained by first finding the solution on the planform as described above and considering this as a known quantity in a problem of the first kind.

These concepts as to problems of the first and second kind apply with equal validity to linearized subsonic flow. Of course the concept of a simple planform does not exist for this case.

26. Methods for the Problem of the First Kind. There are several methods available for the solution of problems of the first kind. These may be classified briefly as:

a. The method of singularities. In this method a distribution of singularities is assumed over the plane in order to represent the body. Sources and sinks are used for the description of the thickness properties and horseshoe vortices or dipoles are used for the description of the lift properties. The resultant field from these singularities is obtained through the method of superposition. This is the method used by Schlichting, Puckett, Jones, and others. A modification of this method which yields only the drag will be described in a later section.

b. The method of integral relations. This is the method in which the relations (23.4) and (23.6) obtained above are applied directly.

c. The method of von Karman. This is the method briefly described above in section 12 in which the axial variable is separated and the flow is obtained through the use of Fourier integrals.

d. The method of separation of the lateral variable. This is the method introduced in section 13 for which a brief development will be given later.

e. The method of superposition of simple solutions. In this method the desired system is obtained by the superposition of known solutions. These are generally solutions of conical type. Recently Lagerstrom has applied this method to the solution of some problems of the second kind.

V. Wing Theory - The Lift Problem

27. Integral Equations for the Problem of the Second Kind. The relations between v and w given in section 23 are not directly applicable to the problem of the second kind which was discussed in section 25. It is possible to employ the concept of a change of variables in which the quantities that are known are considered to be one variable and the quantities which are unknown are considered to be another variable. Thus, for the lift problem that will be considered in this chapter one function will be defined as v in region 1 and w in region 2. The other function will be defined as w in region 1 and v in region 2. The problem of the second kind is solved if a relation is obtained expressing the second function in terms of the first.

The mathematical question that arises in such a problem is first whether sufficient information is available in principle for a complete solution, and second whether the solution obtained is unique. For the simple planform defined in section 24 these questions are answered by the solution procedure which was described in section 25.

The mathematical problem may be discussed in terms of the matrix representations analogous to the integral expressions. Thus the integral expressions (23.4) and (23.6) may be represented by

$$w = Av \quad ; \qquad v = Bw \qquad (27.1)$$

These expressions may be rewritten in terms of the two regions as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{27.4}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} . \tag{27.3}$$

For the mixed problem which constitutes the problem of the second kind, the formal solutions may be given

$$\begin{pmatrix} w_1 \\ v_2 \end{pmatrix} \quad \begin{pmatrix} B_{11}^{-1} & -B^{-1}B_{12} \\ -A_{22}^{-1}A_{21} & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ w_2 \end{pmatrix} \tag{27.4}$$

$$\begin{pmatrix} v_1 \\ w_2 \end{pmatrix} \quad \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ -B_{22}^{-1}B_{21} & B_{22}^{-1} \end{pmatrix} \begin{pmatrix} w_1 \\ v_2 \end{pmatrix} . \tag{27.5}$$

These expressions are valid providing the diagonal sub-matrices involved are nonsingular, which for the integral equations involved requires that the corresponding limited kernels have resolvents. However, if (27.5) has zero eigensolutions equation (27.4) does not exist formally. For this case some distributions of (v_1 w_2) may not be allowed and to any valid solution of the form (27.4) may be added any of the zero eigensolutions of (27.5).

The answer to the mathematical problem as stated above may be inferred from these considerations of the corresponding matrix equations, and they apply with equal validity to linearized planar systems in either subsonic or supersonic flow. Apart from the possibility of eigensolutions, sufficient information is available in the problem of the second kind for the existence of a solution. Eigensolutions to the problem may exist and hence the solution obtained is not necessarily unique. Since the character of the solution depends upon the way in which the x-z plane is divided into two regions, this character depends distinctively on the planform shape.

For the application of these concepts to the problem of determining the lift on a wing with a known camber distribution, $w_2 = 0$ and the equation which must be considered may be simplified to

$$w_1 = B_{11}^{-1} v_1 . \tag{27.6}$$

Hence in terms of the integral equation the problem amounts to finding the resolvent to the limited kernel represented by B_{11}. This is equivalent to finding the resolvent to A_{22} as the two inverse matrices are connected by a simple relation.

28. General Behavior of the Lift Solution. In general, the flow in the immediate vicinity of a planform edge can be considered to have locally the properties of cylindrical flow except at corners or where a wave disturbance intersects the edge. The justification for this point of view is made clear by considering that a scale transformation will make any edge approach a straight line and will make the conditions along the edge approach those for cylindrical flow provided reasonable continuity conditions are met. Considering conditions locally is the same as considering them at a given point as the scale transformation increases the size of the body indefinitely. Thus the terms used to classify the planform edges in section 24 should describe roughly the local flow conditions.

As a result of the general considerations on zones of influence, the conditions for locally cylindrical flow are met strictly for a supersonic leading edge and the magnitude of the pressure at the edge is given immediately by the local angle of attack. From the fact that no part of the zone of action of a supersonic trailing edge intersects the region upstream the solution on the planform is not affected by the presence of the edge.

In analogy with two-dimensional subsonic flow there will be in general for a subsonic leading edge a $-1/2$ power singularity in the pressure on the wing and a corresponding singularity in the upwash velocity ahead of the wing. As with the subsonic flow, the strength of the singularity is not determined by the local angle of attack. This singularity is termed a bound vortex and has associated with it a force acting in the x-z plane perpendicular to the edge. This force is termed the leading edge thrust.

A subsonic trailing edge mathematically may have the same type of singularities as the subsonic leading edge. In two-dimensional subsonic flow the Kutta condition is applied whereupon no $-1/2$ power singularity may exist at the trailing edge. Hence, by analogy from the subsonic case, it may be expected that no bound vortex exists at a subsonic trailing edge, and that the singularity in that vicinity generally should be of $+1/2$ order.

29. <u>The Leading Edge Singularity</u>. The nature of the singularity at a subsonic leading edge will be investigated. In order to show the behavior of the dependency on Mach number the flow will be considered at a general value of M instead of at $M = \sqrt{2}$. The geometry of the system which is considered is shown in Fig. 29.1. The normal Mach number of the edge considered is given by the equation

$$M_n = M \sin \psi \tag{29.1}$$

A useful relation in terms of this normal Mach number is

$$\sqrt{1-M_n^2} = \cos\psi \sqrt{1-t_o^2} \tag{29.2}$$

where

$$t_o = \sqrt{M^2-1} \tan\psi . \tag{29.3}$$

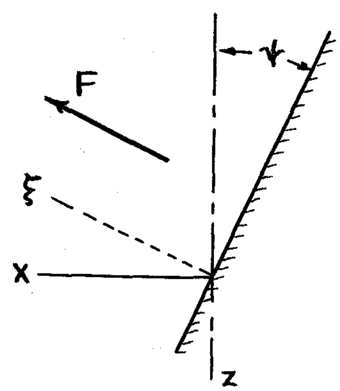


Fig. 29.1. The Leading Edge Singularity

The strength of the singularity is defined by a quantity C in terms of the variation in upwash velocity

$$v = C x^{-1/2} = C \sqrt{\cos\psi} \; \xi^{-1/2} . \tag{29.4}$$

The velocity component in the x-z plane normal to the edge is given by the expression

$$q_\xi = \frac{C}{\sqrt{1-M_n^2}} (-x)^{-1/2} \tag{29.5}$$

The velocity component w is then given with the aid of (29.2) by

$$w = q_\xi \sin\psi = \frac{C}{\sqrt{M^2-1}} \frac{t_o}{\sqrt{1-t_o^2}} (-x)^{-1/2} \tag{29.6}$$

The factor in M in equation (29.6) disappears for $M = \sqrt{2}$.

The leading edge thrust may be expressed from equation (29.4) and the results of subsonic theory. The force acting normal to the edge per unit distance along the edge is equal to

$$F = S_z = \pi\rho \frac{C^2 \cos\psi}{\sqrt{1-M_n^2}} = \pi\rho \frac{C^2}{\sqrt{1-t_o^2}} \tag{29.7}$$

The component of this force in the x direction per unit distance in the z direction is denoted by S_z and has the same value as F. The component of the force in the -z direction per unit distance in that direction is the leading edge thrust for the supersonic planform and is given by

$$T_z = F \tan\psi = \frac{\pi\rho}{\sqrt{M^2-1}} \frac{C^2 t_o}{\sqrt{1-t_o^2}} \tag{29.8}$$

In this expression the factor in M disappears for $M = \sqrt{2}$.

An expression will be obtained for the continuity of this leading edge singularity at a corner. The geometry of such a corner is shown in Fig. 29.2. The continuity of the leading edge singularity is obtained by means of the condition that no discontinuity across a wave system may be permitted for a velocity component parallel to the wave surface. In this case, there may be no singularity in the upwash velocity along the

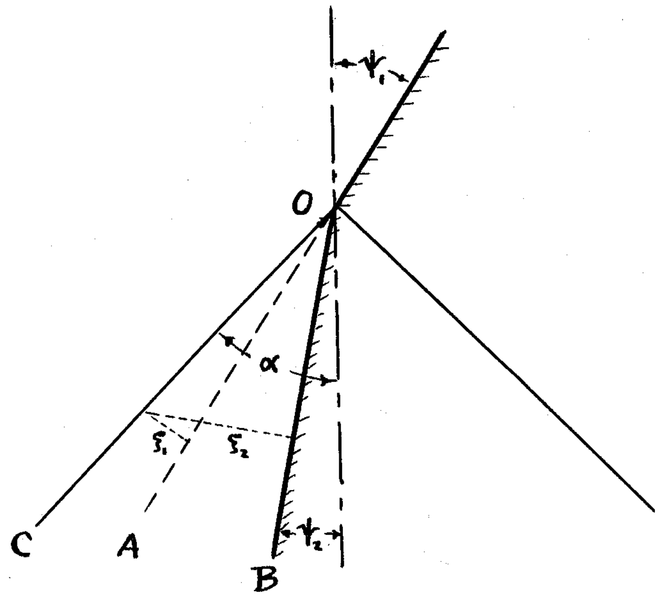


Fig. 29.2. Continuity of the Leading Edge Singularity

line OC. This velocity is expressed in terms of the strength of the singularity before the corner by

$$V = C_1 \sqrt{\cos\psi_1}\ \xi_1^{-1/2} = C_1 \sqrt{\frac{\cos\psi_1}{\sin(\alpha-\psi_1)}}\ l^{-1/2} \tag{29.9}$$

where l denotes the distance along the line OC. Equating the velocity as obtained from the singularity of the forward portion of the edge to that obtained from the downstream portion the relation is obtained

$$\frac{C_1^2 \sin\alpha \cos\psi_1}{\sin(\alpha-\psi_1)} = \frac{C_1^2}{1-t_{o_1}} = \frac{C_2^2}{1-t_{o_2}} . \tag{29.10}$$

This result shows how ths strength of the leading edge singularity changes at a corner. The side forces from equation (29.7) are related by

$$\left(S_z \sqrt{\frac{1+t_o}{1-t_o}}\right)_1 = \left(S_z \sqrt{\frac{1+t_o}{1-t_o}}\right)_2 \tag{29.11}$$

and the leading edge thrusts are related by

$$\left(T_z \frac{1}{t_o}\sqrt{\frac{1+t_o}{1-t_o}}\right)_1 = \left(T_z \frac{1}{t_o}\sqrt{\frac{1+t_o}{1-t_o}}\right)_2 \qquad (29.12)$$

This continuity of the leading edge singularity could also have been derived by the same method applied to the velocity component parallel to the wave direction on the body. The results of this analysis confirm those obtained by the method above. The validity of this relation has not been checked over the Mach cone but should hold true. The flow pattern given will not be invariant under the scale transformation but will reproduce itself with a factor dependent upon K. The flow presented is for an infinite system with uniform singularities in the portions ahead and behind the downstream Mach cone extending from the corner. Although this is a conical body the boundary conditions of section 3 are not satisfied and the flow described is not a conical flow. For a general case where the strength of the singularities are not uniform the results given for the change of strength of the singularities will hold locally at the corner.

30. Eigensolutions and the Kutta Condition. If the solution to a wing problem is not unique there must exist eigensolutions for the planform. This is evident from the fact that where the solution is not unique there must exist at least two distinct solutions for which the difference will be an eigensolution. A planform will be considered to be divided into three regions, a, b, and c. Region a consists of all points on the planform which have only supersonic leading edges within their zones of influence. In general this region is the zone of action of the supersonic

leading edge. Region b is comprised of all points on the planform which have subsonic leading edges in their zones of influence, but no subsonic trailing edges. Region c consists of all points which have subsonic trailing edges in their zones of influence. These regions are shown in Fig. 24.1. The solution for region a may be obtained by the method given in section 25 for a simple planform and therefore no question of uniqueness arises. Any eigensolutions which occur must lie only in the remaining regions.

Due to the hyperbolic nature of the flow, an eigensolution may be considered to start at a point and extend into its zone of action. However, an eigensolution may not start from a point in the interior of a planform. This is true because the conditions at such a point are uniquely determined if the solution in the zone of influence is given and this solution cannot be affected by an eigensolution starting from the point. Therefore, any eigensolution which exists must start at a point on the subsonic edge.

The possibility of an eigensolution starting from a point on a subsonic edge will be considered from a point of view of the local flow. It will be assumed that the form of the eigensolution chosen is such that it will be reproduced under a scale transformation with a factor proportional to a power of K. With this assumption it is possible to superpose the eigensolution along the edge so as to produce a locally conical flow. Thus, the existence of an eigensolution may be taken to be equivalent to the existance of a locally conical eigensolution.

As will be shown later in the chapter on conical flow an eigensolution may exist only where there is a subsonic trailing edge. These eigensolutions have a $-1/2$ power singularity on the edge in analogy with the two-dimensional subsonic circulation flow. Thus an eigensolution may not start from a point on a subsonic leading edge but may start from any point on a subsonic trailing edge. Eigensolutions with singularities of order $-3/2$ or higher are not considered.

With reference to our classification of the regions on the planform it may be stated that in region a the solution is unique and is obtainable by methods for problems of the first kind, in region b the solution is unique, and in region c the solution is not unique.

The Kutta condition for a supersonic planform specifies that no infinity of $-1/2$ order or higher may exist in the pressure distribution in the vicinity of a subsonic trailing edge. This condition is permissible from the versatility of the eigensolutions in removing a $-1/2$ power singularity. It provides a uniqueness to the solution since once the solution with the Kutta condition is obtained no eigensolutions may be superposed without destroying the condition. The Kutta condition influences the solution only in region c.

An interesting point of discussion is with respect to the conditions at a side edge. For a side edge, even though there is a $-1/2$ power singularity in the sense of a bound vortex, the singularity in the velocity component w is of order $+1/2$ since the z direction is now parallel to the edge. No conical eigensolution exists for a side edge. However, if a side edge is changed in angle very slightly to form a trailing edge, the continuity law indicates that the singularity in w which appears is weak. The eigensolution which must be added to remove this singularity is weak with regard to the magnitude of w.

31. Methods for Solution. One method for the solution of the problem of the second kind is the method indicated by the development of section 27, the solution of the integral equation. This involves getting the general solution to the mixed integral equation for the planform by finding the resolvent to the limited kernel represented by the matrix B_{11}. This method corresponds to method b. of section 26, but is of different order of difficulty.

Because of the hyperbolic nature of the problem it is possible to apply a step-by-step method to the lift problem. Since the only difficulties in the problem are associated with the solutions near an edge, it is unnecessary to consider the entire planform at one time. The geometry considered for this discussion is shown in Fig. 31.1.

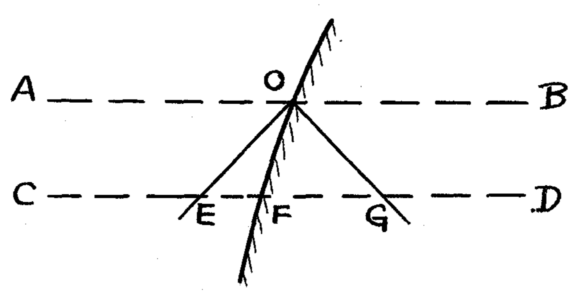


Fig. 31.1. Step-by-Step Method

It is assumed that a complete solution has been obtained as far as the line AB, and that it is desired to extend this complete solution to the line CD. The step that is taken is assumed to be sufficiently short that the line OF may be considered to be straight. The solution in the region AOEC may be obtained directly by equation (23.6). The solution in the region BOGD may be obtained directly from equation (23.4). This leaves only the triangle OEFG to be solved. A solution for this conical region must be obtained which satisfies the proper boundary conditions in that region itself and matches the other solutions on the lines OE and OG in the sense used in obtaining the continuity for the leading edge singularity. For a trailing edge the -1/2 power singularity must be destroyed at the start and subsequent solutions must be used which have a +1/2 power singularity. The details of this solution method have not been worked out, and the computational problem promises to be difficult.

It is possible to obtain the solution to certain problems of the second kind by the method of superposition of simple solutions analogous to method e of section 26. The simplest examples of such solutions are those for triangular wings (36,42,46). A less trivial example is one due to Lagerstrom (48) which is depicted in Fig. 31.2.

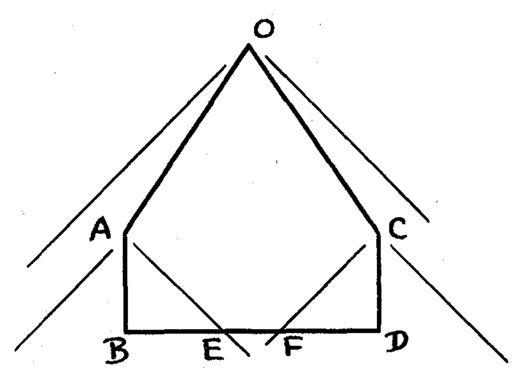


Fig. 31.2. Lagerstrom's Example

In this case, the solution for the region AOCFE is obtained from the conical triangular wing solution with vertex O. The solution for the regions ABE and CDF are obtained by superposing conical flows with vertices distributed along AB and CD which cancel the lift in the region off the planform assumed in the original triangular wing solution.

VI. The Drag of Arbitrary Systems

32. The Concept of Drag. The drag of a body in supersonic flow may be considered to be composed of a perfect fluid drag and a real fluid drag. The real fluid drag consists of drag which arises from shear forces and from pressure forces resulting from boundary layer and separation. This type of drag is not considered in this analysis. The perfect fluid drag actually arises from energy losses through shock waves or turbulence but for the purposes of the linearized theory no energy degradation is involved. This perfect fluid drag for subsonic flow results only from the distribution of lift and generally is termed the induced drag. It may be considered to arise either from the effect of velocities induced on the system by itself or from the energy transported away from the system by the trailing vortex system. However, for supersonic flow perfect fluid drag exists without the necessity for a lift distribution and may not be considered to arise only from the energy transported away by the trailing vortex system.

The application of the methods derived in section 5 may be used with any closed contour about the system in question. It is generally convenient to consider only two contours, one very close to the body and the other at a great distance from the body. It is of conceptual value to separate the drag of a supersonic system into various parts. However, this must be done either on the basis of considering the local field with the contour close to the body or the distant field with the contour at a great distance from the body. The term induced which is familiar from drag considerations in subsonic flow implies a connection with velocities induced in the vicinity of the system and hence should be used only with the local flow concept of drag. The term wave which has been used in connection with supersonic drag implies a relation with the wave system emanating from a body and hence should be used only with the distant flow concept of drag. These two concepts should not be mixed and hence the terminology wave-induced is inadvisable.

The drag of a body as considered from the point of view of the local field is the drag which is calculated from the pressure forces acting directly on the body, modified for the existence of a net mass flow if necessary. For most systems it is possible to express a body or system in supersonic flow by means of a distribution of singularities. In this case the drag from the point of view of the local flow arises from the mutual interaction of these singularities.

Following the concepts of section 5 the drag from the point of view of the distant field may be considered as either the energy or the momentum transported away from the system. In an actual case the disturbances are attenuated and the drag appears as shock wave loss but the linearized analysis is correct on the basis of unattenuated waves. The flow in the distant field may be considered to arise from the totality of singularities representing the system.

The singularities that will be used in the analysis have already been described in section 11. One of these elements is the unit source represented by the potential

$$\phi = \frac{-1}{2\pi z\sqrt{1-t^2}} \tag{32.1}$$

from which a local drag arises from the effect of an induced axial velocity. This is used to represent the thickness distribution of a planar system. The condition of closure of a planar body is that the integral of the source strength in the z direction be zero everywhere across the body. The other singularities is the infinitesimal horseshoe vortex or lift element represented by the potential

$$\phi = \frac{\sin\theta}{2\pi\rho V\ z t \sqrt{1-t^2}} \qquad (32.2)$$

from which a drag arises from the effect of an induced velocity in the direction of the lift. This represents a singularity with a unit lift and is used to represent a planar lift distribution.

The reason for the existence of supersonic form drag is seen from a consideration of the mutual influence of two sources. In subsonic flow the mutual effect of two sources on one another will exactly cancel so that no interaction drag can arise. In supersonic flow if one source is in the zone of action of a second, the second cannot be in the zone of action of the first and the interaction drag does not cancel in this case.

33. The First Reversed Flow Theorem. A theorem relating the drags of various subsonic systems is well-known as Munk's stagger theorem. This theorem states that the drag of a subsonic system remains unchanged if the lift elements representing the lift distribution of the system are staggered arbitrarily in the axial direction. Such a theorem does not hold in supersonic flow but it is possible to obtain theorems relating the behavior of a system with that of the same system with the flow reversed. The theorem

obtained in this section is for a system composed of sources and lift elements.

Under a reversal of the flow about a system of sources and lift elements the strength of a source is defined to be the same in magnitude but opposite in sign. A lift element under a reversal of the flow is so defined that the lift vector produced by the singularity remains unchanged. In either case, this is equivalent to taking the negative of the solution upstream which was canceled in the process described in section 11. The drag is considered to arise from the totality of interactions between the elements of the system. If the drag arising from the interaction between two elements remains unchanged when the flow is reversed the total drag remains unchanged. Therefore, it is only necessary to consider the mutual effect of two singularities.

The two singularities are considered with respect to the plane passing through the two points tangent to the flow direction. If neither of the points lies in the zone of action of the other the same is true with the flow reversed and no interaction drag can occur for either case. If a point is in the zone of action of a second point when the flow is reversed this second point lies in the zone of action of the first. Three singularities must be considered: the source, designated by A; a lift element normal to the plane including the points, designated by B; the lift element parallel to the plane, designated by C. Any lift element may be considered to be composed of a component B and a component C.

No interaction producing drag may occur between elements A and B; this may be stated that the AB interaction is zero. Similarly the BC interaction is zero. That the interactions AA, BB, and CC have the same drag when the flow is reversed follows directly from the symmetry of the types of singularity. The interaction AC may be shown to have the same drag with the flow reversed; this would be expected from the fact that the lift singularity may be obtained from the source singularity by differentiating laterally and integrating axially.

The first reverse flow theorem may be stated: If a collection of sources and lift elements is placed in a reversed flow with the signs of the sources changed and the lift vectors unchanged the drag of the system will remain unchanged. It should be noted that in general the geometry of a body represented by the system will change under such a reversal of flow. It will apply with unchanged geometry to the thickness distribution of a planar system or to a body of revolution with the slender body approximation. The theorem is essentially limited to finite systems. Another proof will be given in section 38.

For planar systems there is no interaction between the thickness and lift distributions so that the consideration of interactions of the type AC is unnecessary. For the special case of planar two-dimensional flow the pressure due to thickness is changed in sign. The pressure due to lift is unchanged but the angles of the camber distribution giving the lift are changed in sign. From the point of view of the method of von Karman discussed in section 12 the pressure distribution on a planar thickness system consists of two parts, one due to the Y_0 terms in the Hankel functions, and one due to the J_0 terms. The terms in the velocity potential in Y_0 satisfy the boundary conditions of themselves. The pressure distribution in Y_0

produces no drag, and the entire drag of the system arises from the J_o pressure distribution. If the flow is reversed the first Hankel function takes the place of the second. The pressure distribution in Y_o is unchanged while the pressure distribution in J_o is changed in sign, giving the same drag with the reversed flow. It is necessary that the system considered be finite for this analysis. The reversed flow theorem for the thickness case has been checked in detail for the delta wing of Puckett (35) by L. Friedman and others of the theoretical aerodynamics group at the Aerophysics Laboratory of North American Aviation, Inc. It should be noted that the thickness and lift drags of a planar system satisfy the reversed flow theorem separately.

For the nonplanar case two examples in two-dimensional flow are presented for illustration in Fig. 33.1 and 33.2.

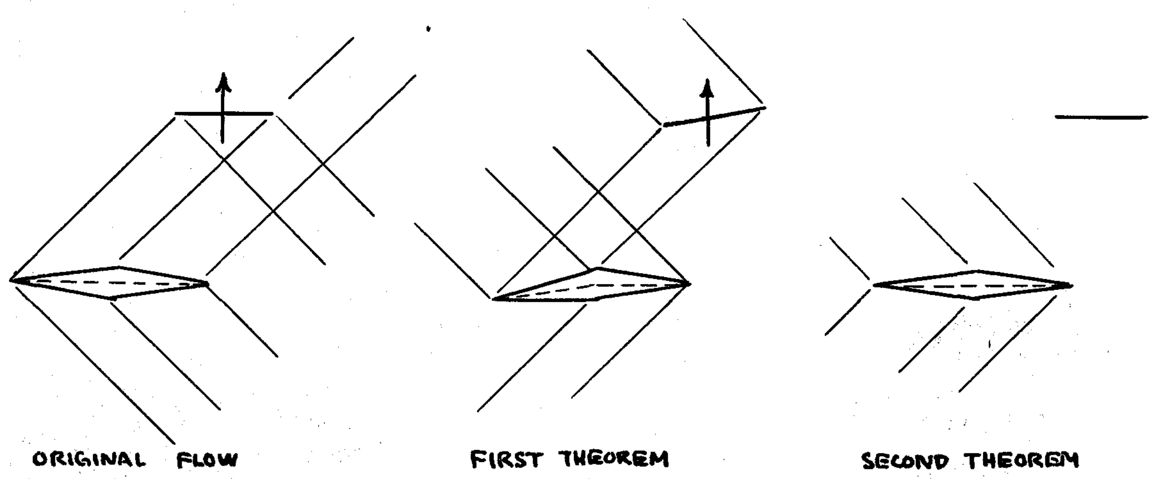


Fig. 33.1. Example of Reversed Flow Theorem

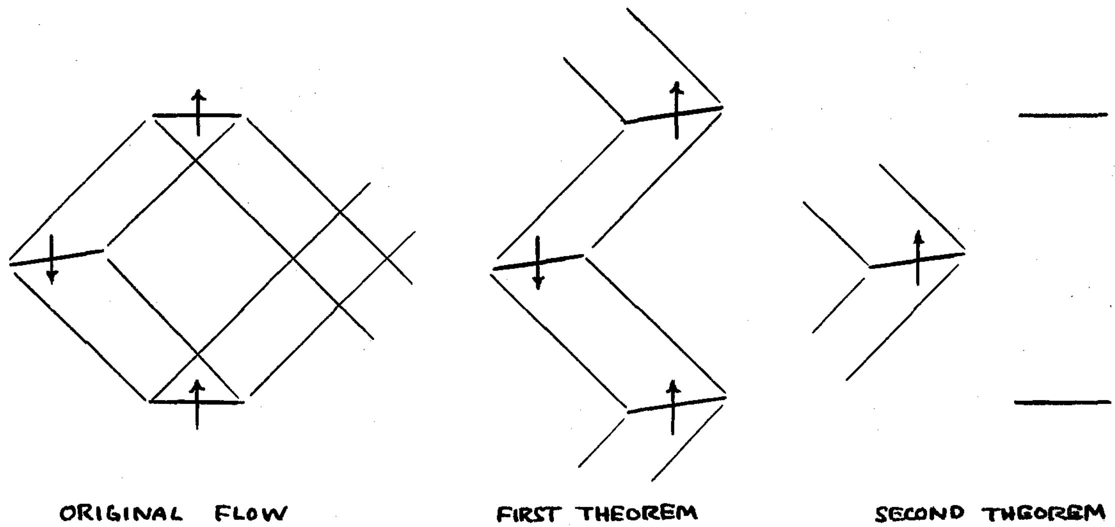


Fig. 33.2. Second Example

The second example is of particular interest in that it demonstrates that negative lift may result from a positive angle of incidence.

34. The Second Reversed Flow Theorem. A reversed flow theorem relating systems with the same geometry rather than the same lift would be desirable. Such a theorem may be obtained but is more limited than the first reversed flow theorem.

For a planar system the first reversed flow theorem provides the same geometry for a thickness distribution. The second reverse flow theorem for the camber distribution holds only for simple planforms. For a simple planform there can be no interaction between the upper and lower surfaces. Hence, the method of reversing the symmetry of the system used in section 25 gives the theorem from the application of the first reversed flow theorem to the system with reversed symmetry. The failure of the closure condition in this case is unimportant. For planar two-dimensional flow the theorems are the same except for a change in the sign of the lift distribution.

The lift curve slope of a simple planform is equal to the drag of a flat plate divided by the square of the angle of attack. Since such a planform will have the same drag with the flow reversed the lift curve slopes will be identical. This has been checked for the kite-shaped planform of the German Zitterrochen wing by R. Lew of North American (58). Lagerstrom has given several examples of nonsimple planforms for which this is also true. This suggests the possibility of a more general reversed flow theorem relating to lift curve slopes.

The second reverse flow theorem holds generally for nonplanar systems in two dimensional flow. This may be obtained easily from a consideration of the interaction between planar systems comprising the system as a whole. Examples of the application of this theorem are depicted in Fig. 33.1 and 33.2.

A general nonplanar system can be considered to consist of isolated interacting planar systems. For the application of the second reverse flow theorem three conditions must be satisfied. The first condition is that each of the individual planar systems must consist of a simple planform. The second condition is that there may be interaction between single surfaces only, that none of the leading edges of the simple planforms lie in any of the zones of action of the other planforms. The third condition is that it is possible to symmetrize the system in the sense used in considering simple planforms without introducing nonexistent interactions. The second condition is not necessary for two-dimensional flow as there is no interaction between an element and the opposite surface of a planar system in this case.

35. Basic and Induced Drag for Planar Systems. The drag of a planar system is divided fundamentally into the drag due to thickness and the drag due to lift. It is possible to make a separation of each of these drags into two types on the basis of the local flow field. This separation is made on a different basis for the lift drag from the thickness drag and is derived from the fundamental relations (23.4) and (23.6). Consequently, on the basis of a solution only in the upper half space two distinct separations are possible. From the results of section 5 the drag may be obtained by integrating the quantity $-\rho vw$ over a plane immediately above the planar system. If a thickness system is being considered the drag is obtained by multiplying (23.4) by v and integrating; similarly for a lift system the drag is obtained by multiplying (23.6) by w and integrating.

In deriving relation (23.4) from a planar distribution of sources it is helpful to consider it as a limiting case of a spatial distribution of sources. The first term in the expression for the axial velocity arises from the limiting process by which the spatial distribution is changed into a planar one. This term depends only upon the local conditions and is denoted as the "basic" term. The second term arises from the velocities induced by the planar distribution of sources within the zone of influence of the point in question. This term is an integral over a portion of the plane and is denoted as the "induced" term. The terms in the expression for the upwash velocity in relation (23.6) are obtained from a distribution of lift elements in a similar fashion.

The drag as calculated from these fundamental relations is naturally divided into two terms, a basic drag and an induced drag. The basic drag is the drag that would be obtained from a two-dimensional calculation. The induced drag may be either positive or negative. This separation of the drag is for problems of the first kind, no immediate separation in terms of the known quantities being available for problems of the second kind. The first reversed flow theorem holds for each of these separated drag portions.

For a nonsimple planform with $-1/2$ power singularities the separation on either basis fails because the integrals do not converge. For this case the only possibility is to integrate the drag from expression (23.4) over the planform only and to correct this result for the leading edge thrust.

36. Wave and Vortex Drag. A consideration of the distant flow field permits a separation of the drag of a system into two types. For the purposes of this section the drag is considered to arise from the transport of acoustical energy away from the system. The system is considered to consist of sources and lift elements.

Since the only singularity in the flow field from a source lies on the downstream Mach cone and from a lift element on the Mach cone and on the axis, a finite system may be considered to be concentrated at a point for a sufficiently distant observer not near either the Mach cone or the axis. An investigation is made of the portion of drag represented by the energy transported away within the cone $t = 1 - \epsilon$ and for the lift element exterior to the cone $t = \epsilon$.

The distant surface considered is a plane perpendicular to the flow direction at a distance z from the system, for which the area element is $z^2\, t\, dt\, d\theta$. Since both the source and the lift element are singularities for which $n = -1$, the velocities are of the form $f(t,\theta)\ z^{-2}$. Squaring these velocities and integrating across the area shows that the contribution of this portion of the field to the drag is of the order z^{-2}. For a source the integration in t is from 0 to $1 - \epsilon$; no matter how small the quantity ϵ is chosen the drag represented by this integral may be made as small as desired by making z sufficiently large. Thus the drag from a finite system of sources may be considered to be concentrated as energy near the Mach cone. For a lift element the integration in t is from ϵ to $1 - \epsilon$. The drag for a finite system of lift elements or for a general finite system may be considered to consist of two parts, one concentrated as energy near the Mach cone and the other as energy near the axis. These drags are termed the "wave" and "vortex" drags, respectively. The condition that the drag of a system of the type considered in chapter III be thus concentrated is that $n < 0$.

Methods for the direct calculation of the wave drag are presented in the next two sections. The energy in the vicinity of the Mach cone is essentially that in a plane wave system; the flow field near the Mach cone approaches a system of plane waves with increasing distance. For such a wave system, $v = 0$ and $u = -w$ in the radial notation. The wave is weakly attenuated as a result of the radial expansion and the velocities in the wave decrease in magnitude as $r^{-\frac{1}{2}}$. From the result of Raleigh (26) half of the energy in the system is acoustical kinetic energy and half is potential energy.

The vortex drag is so called because it is associated with the energy in the trailing vortex system. It is sometimes called the induced drag in analogy with the equivalent subsonic drag, but as was pointed out in section 32 this is really a misnomer as there is no connection with the effect of local induced velocities. Munk's stagger theorem holds for the vortex drag as is evident from the fact that staggering the lift elements arbitrarily has no effect on the composition of the trailing vortex system. This drag may be obtained by standard methods (24) for calculating the energy in an incompressible two-dimensional flow, essentially by integrating the quantity $\rho\phi\frac{\partial\phi}{\partial n}$ around line contours enveloping the individual vortex systems which comprise the system as a whole.

37. Linear Source and Lift Distributions. As a preliminary to the method of calculation of the wave drag of an arbitrary finite system the wave drag of a linear distribution of sources and lift elements is investigated. The source strength per unit length is denoted by f and the lift strength per unit length is denoted by $\frac{\rho}{\sqrt{M^2-1}}Vg$. The lift distribution is considered to be divided into two components, denoted by g_y and g_x. The differential expressions for the potential and the axial velocity in terms of z and t are

$$d\phi = \frac{-f + t^{-1}\sin\theta\, g_y + t^{-1}\cos\theta\, g_x}{2\pi z\sqrt{1-t^2}}\, dz \qquad (37.1)$$

$$dw = \frac{-f' + t^{-1}\sin\theta\, g_y' + t^{-1}\cos\theta\, g_x'}{2\pi z\sqrt{1-t^2}}\, dz \qquad (37.2)$$

Since only conditions very near the Mach cone are to be considered the approximations may be made:

$$V = 0 \quad , \quad -uW = W^2 \tag{37.3a}$$

$$t^{-1} = 1 \quad , \quad \sqrt{1-t^2} = \sqrt{2}\,\sqrt{1-t} \tag{37.3b}$$

The variation with respect to aximuth angle is to be considered last and the abbreviation

$$h = f - g_y \sin\theta - g_x \cos\theta = f - g \tag{37.3c}$$

is made, where g here denotes the component of the lift distribution in the direction considered.

The surface used for the calculation of the wave drag is a cylinder of radius r and the integration is taken over the range near the Mach cone excluded in the considerations of section 36. The radius must be chosen such that ϵr is large compared with the length of the linear distribution. The distribution is assumed to start at z = 0 and the variable z' = z - r is introduced for the purpose of notation. The geometry of the system is depicted in Fig. 37.1.

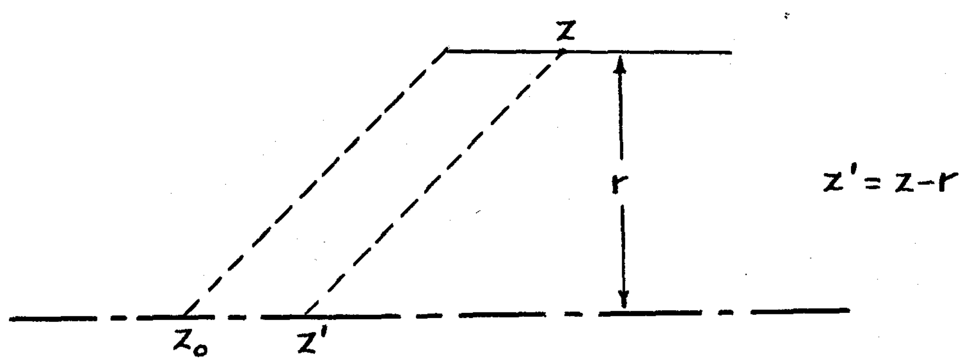


Fig. 37.1. Linear Source and Lift Distribution

Using equations (37.3) the axial velocity component may be expressed

$$w = -\frac{1}{2\pi\sqrt{2r}}\int_0^{z'}\frac{h_0'\,dz_0}{\sqrt{z'-z_0}} \qquad (37.4)$$

The drag per unit aximuth angle is given by

$$\frac{dD}{d\theta} = \int_0^{\epsilon r}\rho w^2 r\,dz' = \frac{\rho}{8\pi^2}\int_0^{\epsilon r}\left[\int_0^{z'}\frac{h_0'\,dz_0}{\sqrt{z'-z_0}}\right]\left[\int_0^{z'}\frac{h_1'\,dz_1}{\sqrt{z'-z_1}}\right]dz' \qquad (37.5)$$

which by rearrangement of the order of integration becomes

$$\frac{dD}{d\theta} = \frac{\rho}{4\pi^2}\int_0^{\epsilon r}\int_0^{z_0}h_0'h_1'\left[\int_{z_0}^{\epsilon r}\frac{dz'}{\sqrt{z'-z_0}\sqrt{z'-z_1}}\right]dz_1\,dz_0 \qquad (37.6)$$

The integral in the brackets may be evaluated

$$\int_{z_0}^{\epsilon r}\frac{dz'}{\sqrt{(z'-z_0)(z'-z_1)}} = \cosh^{-1}\frac{z'-\frac{z_0+z_1}{2}}{\frac{z_0-z_1}{2}}\Bigg|_{z_0}^{\epsilon r} = -\log(z_0-z_1)+\log C\epsilon r \qquad (37.7)$$

The constant term in (37.7) gives no contribution to the drag, and the wave drag per unit azimuth angle is

$$\frac{dD}{d\theta} = -\frac{\rho}{4\pi^2}\int_0^{\infty}\int_0^{z_0}h_0'h_1'\log(z_0-z_1)\,dz_1\,dz_0 \qquad (37.8)$$

This is the same result as that obtained by von Karman (15) for a linear source distribution. His method is not applicable to a lift distribution as it is based on a close contour rather than a distant one. An important feature of this drag is that it is independent of the Mach number.

For a true linear distribution the total wave drag may be obtained by expressing the product

$$h_0' h_1' = f_0' f_1' + g_{y_0}' g_{y_1}' \sin^2\theta + g_{x_0}' g_{x_1}' \cos^2\theta + A \sin\theta + B\cos\theta + C \sin\theta\cos\theta \tag{37.9}$$

and integrating with respect to θ. All the cross terms vanish in the integration and the total wave drag may be expressed

$$D = D_f + D_{g_y} + D_{g_x} . \tag{37.10}$$

The source distribution and the two components of the lift distribution are noninteracting with respect to the wave drag. The vortex drag for a true linear distribution of lift elements is infinite unless the net lift is zero.

38. The Calculation of Wave Drag. In the calculation of the wave drag of an arbitrary system it is convenient to consider z as a time variable for a distant observer at a given value of r and θ. The upstream Mach cones from the positions of the observer represent surfaces of coincident signals in the sense of Huygen's principle, disturbances from all points of which arrive simultaneously. These surfaces are parallel planes in the vicinity of the system under study. From the point of view of the observer it is impossible to distinguish between two sources or lift elements of the system which lie on the same plane and it is possible to represent the system by

a linear distribution of singularities along a reference axis. All the singularities lying on one of the planes of coincident signals are considered to be concentrated at the intersection of the plane with the axis. The wave drag contribution in terms of drag per unit aximuth angle may be obtained by the equations of the preceding section for the direction given by the coordinate θ of the observer. This quantity may be obtained as a function of θ and integrated to obtain the total wave drag of the system.

The wave drag as calculated with the flow reversed according to the definition of section 33 is the same as the original drag. The only differences in the calculation are that the angles are changed by π, that there is a change in the direction of the integration, and that there is a change in the sign of the distribution. Since the distribution strength appears twice in the integrand and there is no effect from the other factors, the result for the drag is the same.

The vortex drag remains unchanged in a reversed flow as the vortex system is changed only by a reflection. Thus the first reversed flow theorem holds for the wave and vortex drags separately and therefore for the total drag.

These considerations pertaining to the wave drag permit some general conclusions as to the behavior of a system as the Mach number approaches unity. As $M \to 1$, the planes of coincident signals all approach planes perpendicular to the flow direction, that is, to the same set of planes. For the purposes of wave drag the system approaches a true linear distribution which from the preceding section has the following properties: the wave drag calculation is simplified to that of a slender body of revolution in that the integration with respect to θ is unimportant; the drags from the source and lift component distributions become independent according to (37.10).

For a planar system the distribution in f is the same for a given direction as for the opposite direction, with θ changed by π. The distribution in g_y is the same in magnitude for a given direction as for the opposite direction but is changed in sign. There is no distribution in g_x. The contributions to the drag of opposite directions may be combined, in which case it is seen that the cross terms cancel in the expression for $h_o h_1$. This shows the noninteraction of the thickness and lift distributions of a planar system in producing wave drag.

For a planar system the surfaces of coincident signals reduce to the lines which are the intersections of these surfaces with the fundamental plane. For the calculation of the wave drag it is necessary to obtain an expression for the orientation of these lines to the flow direction. This is measured by the angle β between the intersections and a perpendicular to the flow direction. A surface of coincident signals is represented by a line on a plane perpendicular to the flow direction. This line is tangent to the circle which represents the Mach cone of which the vertex is the intersection of the surface with the axis, the point of tangency being at the specified value of θ. The radius of this Mach circle is $z/\sqrt{M^2-1}$ and the distance from the axis to the intersection of the surface of coincident signals with the fundamental plane is equal to sec θ times this value. The tangent of the angle β is the ratio of z to this distance and is given by

$$\tan \beta = \sqrt{M^2-1} \quad \cos \theta \tag{38.1}$$

for a general value of the Mach number.

As an instructive example of the method of calculating the drag the analysis for the half-arrowhead wing treated by von Karman by the axial variable method is given. This example is for a semi-infinite system and shows the necessity for care in applying these drag concepts to a system which is not finite. The configuration treated is shown in Fig. 38.1.

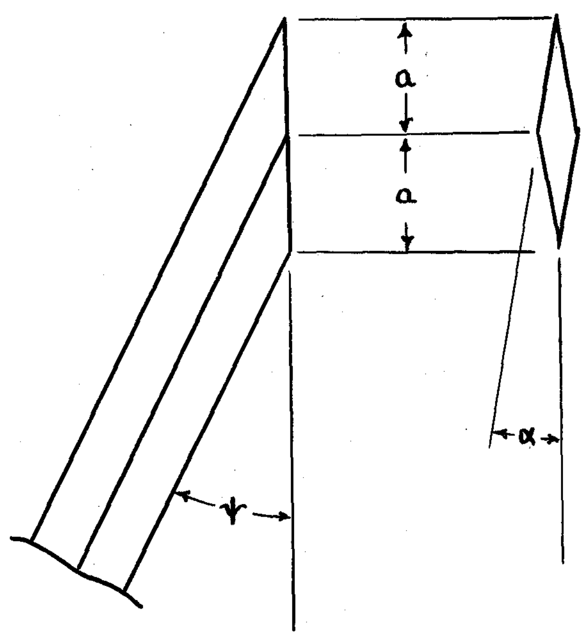


Fig. 38.1. Half-Arrowhead Wing

This body has a diamond cross section with a chord length 2a in the axial direction and a maximum thickness equal to $2a\alpha$. The body has no lift and is represented by a distribution of sources of strength $\pm 2\alpha V$ per unit area. The equivalent linear source strength may be expressed as the product of two factors, one containing the azimuthal dependence and the other the axial variation. Thus the source strength is

$$f = \frac{f_{\pi/2}}{1 - \tan\beta \tan\psi} = \frac{f_{\pi/2}}{1 - t_0 \cos\theta} \qquad (38.2)$$

where

$$f_{\pi/2} = 2\alpha V \tan\psi \quad , \quad 0 < z < a$$

$$= -2\alpha V \tan\psi \quad , \quad a < z < 2a \qquad (38.3)$$

The latter distribution substituted into equation (37.8) gives the drag per unit azimuth angle at $\theta = \pi/2$

$$\left(\frac{dD}{d\theta}\right)_{\frac{\pi}{2}} = \left(\frac{-\rho}{4\pi^2}\right)\left(4\alpha^2 V^2 \tan^2\psi\right)\left(-2a^2 \log 2\right)$$

$$= \frac{2\log 2}{\pi^2}\rho V^2 a^2 \alpha^2 \tan^2\psi \qquad (38.4)$$

The integration of the factor containing the dependence on the azimuth angle gives

$$2\int_0^{\pi}\left(\frac{1}{1-t_o\cos\theta}\right)^2 d\theta \quad = \quad \frac{2\pi}{(1-t_o^2)^{3/2}} \tag{38.5}$$

whence the total wave drag may be expressed

$$D_w \quad = \quad \frac{4\ \log 2}{\pi}\ \rho V^2 a^2 \alpha^2 \tan^2\psi \ \frac{1}{(1-t_o^2)^{3/2}} \tag{38.6}$$

In this form the dependence on M appears only in the term t_o. As the Mach number approaches one the drag approaches a value dependent only on the axial variation of the cross-sectional area. It must be emphasized that for a system which is not finite the wave drag does not in general comprise the entire drag that would be calculated from the pressure distribution which is true in this case.

A finite system is considered with the half-arrowhead at both ends of a long cylindrical body. This body is assumed to be sufficiently long that there is no interference between the disturbances from the two ends of the body. No disturbance is propagated from the main part of the body as the flow there is essentially a subsonic cylindrical flow. Since the system is now finite the wave drag is the total drag and amounts to $2D_w$, half coming from the nose of the body and half from the tail. An investigation of the pressure drag of the body gives the same result except that the pressure drag on the nose is $2D_w$ and there is no net pressure drag on the tail. This apparent discrepancy disappears with the concept of energy transport along the body, as energy in the subsonic cylindrical field which is being transported by the tangential velocity component of the principal flow. An investigation of this cylindrical field shows that the drag represented by this energy transport is equal to D_W. The finite system with the energy flows is illustrated in Fig. 38.2.

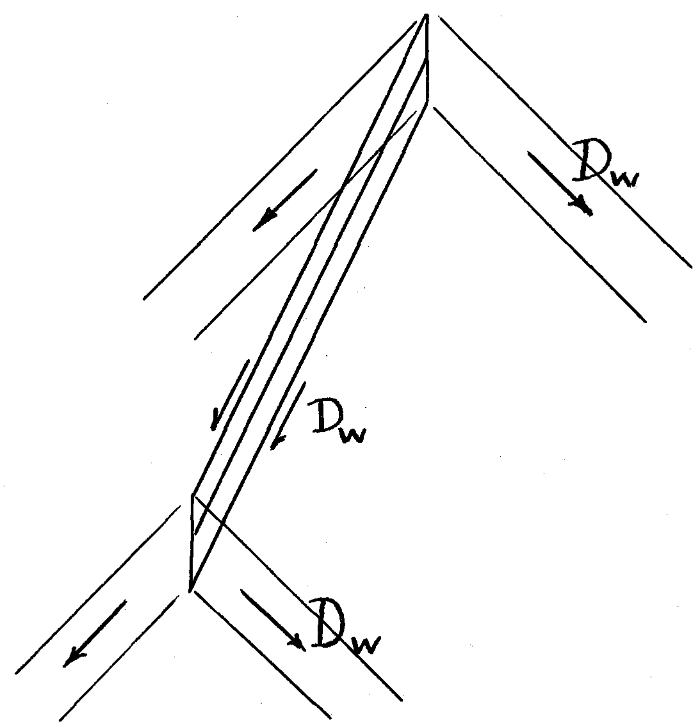


Fig. 38.2. Energy Flow for Finite System

It may be noted that the similarity between equation (37.8) and the expression for the energy of a subsonic two-dimensional source distribution thus aquires a physical significance.

The results of this analysis agree in all comparable details with those of von Karman and Chang. The drag calculated from the J_0 distribution for the semi-infinite system agrees with the wave drag and thus does not give the correct pressure drag. The Y_0 terms which produce no drag for a finite system do give a drag for the semi-infinite system which corresponds to the drag represented by the energy transported along the body. The results for the complete arrowhead wing check in a similar way.

VII. Conical Flow

39. Basic Relations between Velocity Components. The first development of linearized conical supersonic flow was presented by Busemann (20) in 1942. Since that time various other presentations have appeared (36,39, 42,44,45,48,57) in which the only fundamental differences lie in the derivation of the basic equations and the methods of application to various examples. The treatment given here is essentially that of (42) and is made with special reference to planar systems.

The equation for the velocity potential in conical flow is obtained from (15.3) and (15.4) by setting n = 1

$$(1-t^2)\phi_{tt} + \frac{1}{t}\phi_t + \frac{1}{t^2}\phi_{\theta\theta} = 0 \qquad (39.1)$$

From the relation (15.8) may be obtained the relation between the velocity potential and the axial velocity w

$$\phi = -zt\left[\int_1 \frac{w\,dt}{t^2} + \Theta\right] \qquad (39.2)$$

where Θ is an arbitrary function of θ and the integration is taken from t = 1. After an integration by parts the radial velocity component is given by

$$q_r = \frac{1}{z}\phi_t = -\left[\int_1 \frac{w_t\,dt}{t} + w_1 + \Theta\right] \qquad (39.3)$$

where w_1 is the value of w at t = 1, on the Mach cone. Similarly the azimuthal velocity component is given by

$$q_\theta = \frac{1}{2t}\phi_\theta = -\left[\int_1 \frac{w_\theta\, dt}{t^2} + \Theta_\theta\right] . \tag{39.4}$$

If t is set equal to 1 in equation (39.1) a condition satisfied by ϕ on the Mach cone is obtained

$$\phi_t + \phi_{\theta\theta} = 0 \tag{39.5}$$

from which can be derived the condition on

$$\Theta_{\theta\theta} + \Theta + w_1 = 0 . \tag{39.6}$$

This condition gives the relations

$$d\left[(w_1 + \Theta)\cos\theta - \Theta_\theta \sin\theta\right] = \cos\theta \; dw_1 \tag{39.7a}$$

$$d\left[(w_1 + \Theta)\sin\theta + \Theta_\theta \cos\theta\right] = \sin\theta \; dw_1 . \tag{39.7b}$$

From the above relation the cartesian velocity components are given by

$$u = q_r\cos\theta - q_\theta \sin\theta = -\left[\int_1 \frac{w_t\cos\theta - \frac{1}{t}w_\theta \sin\theta}{t}dt + \int \cos\theta\, dw_1\right] \tag{39.8a}$$

$$v = q_r \sin\theta + q_\theta\cos\theta = -\left[\int_1 \frac{w_t \sin\theta + \frac{1}{t}w_\theta\cos\theta}{t}dt + \int \sin\theta\, dw_1\right] \tag{39.8b}$$

with suitable arbitrary constants. These equations give the two cross velocity components in terms of the axial component. The path of integration is around the Mach circle to the desired azimuth angle and thence radially to the desired value of t.

The substitution (15.5), re-expressed as

$$t = \frac{2s}{1+s^2} \quad , \qquad s = \frac{t}{1 \pm \sqrt{1-t^2}} \tag{39.9}$$

transforms equation (15.4) for the axial velocity component into Laplace's equation in polar coordinates

$$W_{ss} + \frac{1}{s} W_s + \frac{1}{s^2} W_{\theta\theta} = 0 \tag{39.10}$$

The relation between t and s is shown in Fig. 39.1. The quantity s is a double-valued function of t and is real only for $t < 1$.

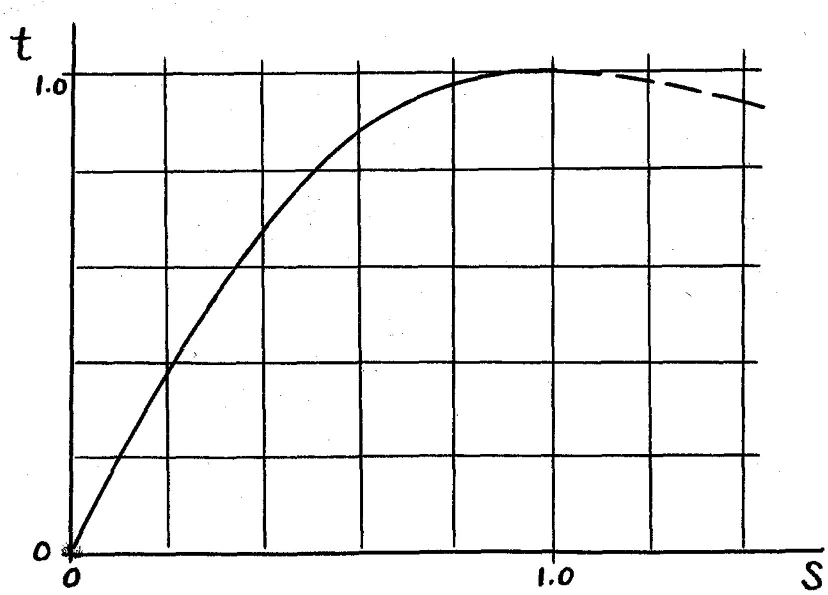


Fig. 39.1. Relation between t and s

Introducing the complex variable

$$\epsilon = s e^{i\theta} \tag{39.11}$$

w must be the real part of an analytic function of ϵ , the same being true for u and v. Letting w' be the function conjugate to w so that

$$w + i w' = f(\epsilon) \tag{39.12}$$

the Cauchy-Riemann conditions may be expressed

$$w_s = \frac{1}{s} w'_\theta \tag{39.13a}$$

$$\frac{1}{s} w_\theta = - w'_s \tag{39.13b}$$

Using the relations between t and s, the Cauchy-Riemann conditions (39.13), and the definitions of ϵ and f, the cartesian velocity components may be expressed by the integrals in the complex ϵ plane

$$u = - \mathcal{R}\ell \int \frac{1+\epsilon^2}{2\epsilon} df \tag{39.14a}$$

$$v = - \mathcal{R}\ell \; i \int \frac{1-\epsilon^2}{2\epsilon} df \tag{39.14b}$$

with suitable arbitrary constants. The path of integration is the same as for equations (39.8). This contour may be deformed arbitrarily within restrictions imposed by the singularities of the integrands. These integral relations are the same as the one integral relation of Busemann.

Only the region inside the unit circle on the t plane, the region for which the differential equation (39.10) is elliptic, is mapped on the s plane, once inside the unit circle of the s plane and once as a reflection outside the unit circle. The region outside the unit circle of the t plane is a region for which equation (39.10) is hyperbolic.

For the region outside the Mach circle the real characteristics of equation (39.10) are given by

$$(t^2-1)\,d\theta^2 - \frac{1}{t^2}\,dt^2 = 0 \tag{39.15}$$

which integrates into

$$\theta + \sec^{-1} t = \theta_1 \tag{39.16a}$$

$$\theta - \sec^{-1} t = \theta_2 \tag{39.16b}$$

These are the equations for straight lines lying tangent to the Mach circle in the t plane. Thus the Mach circle is the envelope of the characteristics. These characteristics are illustrated in Fig. 39.2.

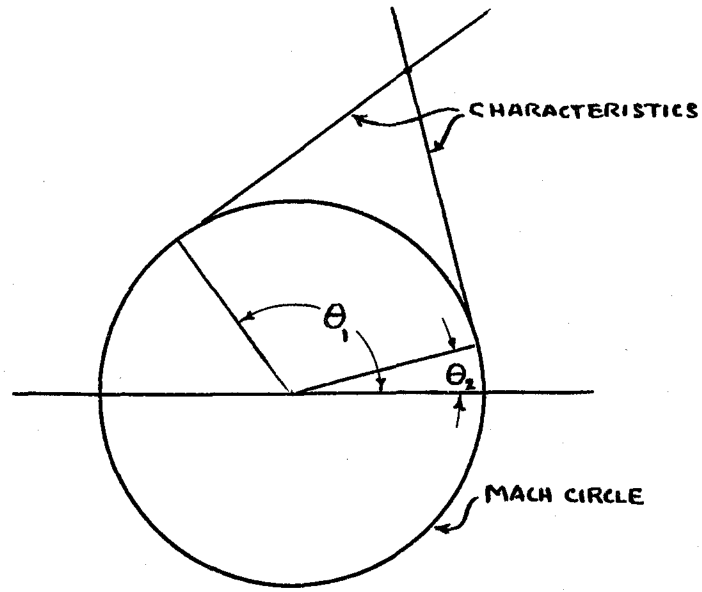


Fig. 39.2. Characteristics for Conical Flow

Introducing θ_1 and θ_2 as new independent variables into equation (39.10) results in the equation

$$\frac{\partial^2 w}{\partial\theta_1 \partial\theta_2} = 0 \tag{39.17}$$

which has a general solution of the form

$$w = f_1(\theta_1) + f_2(\theta_2) . \tag{39.18}$$

Since u and v satisfy equation (39.10) they must also have general solutions of the same form. However, these solutions are not independent. Using equations (39.8) expressions analogous to (39.14) may be derived

$$u = -\int (\cos\theta_1 \, df_1 + \cos\theta_2 \, df_2) \tag{39.19a}$$

$$v = -\int (\sin\theta_1 \, df_1 + \sin\theta_2 \, df_2) \tag{39.19b}$$

with suitable arbitrary constants. The path of integration is the same as for equations (39.8). Since the integrands are perfect differentials the contour may be deformed arbitrarily within reasonable restrictions.

A convenient point of view which helps to clarify equations (39.19) is to consider u and v as being the components of a two-dimensional vector cross velocity. If w changes on crossing a characteristic there occurs a change in this vector velocity equal to the change in w and of direction perpendicular to the characteristic.

The characteristic method given above is very similar to the same method for linearized supersonic two-dimensional flow. There the two velocity components are related and are each given as the sum of an arbitrary function of a variable determining one family of characteristics and another function of a variable determining the other family. Such phenomena as the reflection of a characteristic from a solid boundary are similar in the two cases.

The above analysis has been established for the space which lies downstream from the vertex with both z and t positive. For the space upstream from the vertex the most consistent way of extending the analysis is to keep the definition (15.1) unchanged and to let t be negative. A physical section taken normal to the flow upstream of the vertex will show radius vectors proportional to $-t$ and angles changed by the additive constant π. The sign of $\sec^{-1} t$ in definitions (39.16) is also changed. This apparent change in the definition of the angle must be kept in mind in all cases where part of the body extends upstream from the vertex. Of course no part of the body may lie within the upstream Mach cone.

In the problem of determining the flow about a given conical body the boundary conditions which are given are conditions on the cross velocity, on the velocity components u and v or q_r and q_θ. Two general procedures are open: (1) to determine the solution directly in terms of one of the cross velocity components or of an appropriate function of them, applying the boundary conditions directly and obtaining the axial velocity component where desired from the relations between the velocity components; or (2) to translate certain of the boundary conditions into conditions on the axial velocity component and to determine an appropriate solution in terms of this component with arbitrary factors, evaluating the factors later using the cross velocity boundary conditions. The first procedure might appear to be the better as it is more direct. However, the second method has the advantage that if only the pressure distribution is desired and not the other cross velocity distributions, only one function throughout space need be determined. Also, for many practical examples encountered the functions giving w inside the Mach circle are of much simpler form than those giving the cross velocity components.

The procedure of solution is as follows:

(a) The solution in the region upstream from the vertex is obtained by using the method of characteristics if any part of the body extends into this region. As a result of the condition that no part of the conical body in question extends into the upstream Mach cone there will be a zero solution for all velocity components within and on the Mach circle in the upstream t plane.

(b) Using the results of (a) for boundary conditions at infinity the solution in the region downstream from the vertex but outside of the downstream Mach cone is obtained by the method of characteristics. If a solution from (a) is used the change in sign of t must be kept in mind.

(c) Using the results of (b) for boundary conditions on the Mach cone the solution within the downstream Mach cone is obtained by the solution of a two-dimensional Laplace's equation using the relations expressed above.

This procedure will yield any desired information on the flow around a body of given shape provided the boundary conditions may be established simply in terms of the cross velocity and the necessary quadratures in the relations between velocity components may be carried through.

40. <u>Solution Method for Planar Systems</u>. For planar systems a relation between the velocity components v and w on the plane may be obtained from the vorticity relation (4.5a). This relation on the t plane is

$$\frac{\partial w}{\partial n} = -t\,\frac{\partial v}{\partial t} \qquad (40.1)$$

and shows that a knowledge of v on the plane implies a knowledge of On the axis, at $\epsilon = 0$, the relations (39.14) give the conditions that

$$\frac{\partial w}{\partial n} = 0 \text{ for v finite and u continuous,}$$

$$\frac{\partial w}{\partial t} = 0 \text{ for u finite and v continuous.}$$

Hence if these conditions are to be met $df = 0$ at $\epsilon = 0$. In the case of a vortex sheet lying on one side of the axis $\frac{\partial w}{\partial n} \neq 0$ and there is a discontinuity in u. If the body is such that there is a discontinuity in v, $\frac{\partial w}{\partial t} \neq 0$.

It is convenient to apply a conformal transformation to the plane for the purposes of studying planar systems. This is the inverse Joukowski transformation suggested by (39.9)

$$z = \frac{2\epsilon}{1+\epsilon^2} \quad , \qquad \frac{1-z}{1+z} = \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \tag{40.2}$$

where the new variable z should not be confused with the axial coordinate. Under this transformation the relations (39.14) become

$$u = -\mathcal{R}\ell \int \frac{1}{z}\, df \tag{40.3a}$$

$$v = -\mathcal{R}\ell\; i \int \frac{\sqrt{1-z^2}}{z}\, df \tag{40.3b}$$

With this mapping there is a return to physical coordinates on the real axis which represents the plane and the mapping of the interior of the unit circle of the t plane becomes one-to-one. For this z plane the relation (40.1) becomes

$$\frac{\partial w}{\partial n} = -\frac{t}{\sqrt{1-t^2}} \frac{\partial v}{\partial t} \tag{40.4}$$

It is of interest to consider the mapping of the unit circle of the t plane onto the z plane in more detail. One method of graphically carrying out the mapping for individual points may be devised easily from the relation

$$\frac{1}{z^*} = \frac{1}{t}\cos\theta + i\,\frac{\sqrt{1-t^2}}{t}\sin\theta \qquad (40.5)$$

A more instructive method is obtained from a consideration of the families of curves defined by

$$t\cos\theta = a \qquad (40.6a)$$

$$t\sin\theta = b\sqrt{1-a^2} \qquad (40.6b)$$

The first family of lines consists of vertical lines in the t plane and is specified by a = constant. These lines transform into circles in the z plane with centers at $\frac{1+a^2}{2a}$ and radii equal to $\frac{1-a^2}{2a}$. The mapping of this family is illustrated in Fig. 40.1.

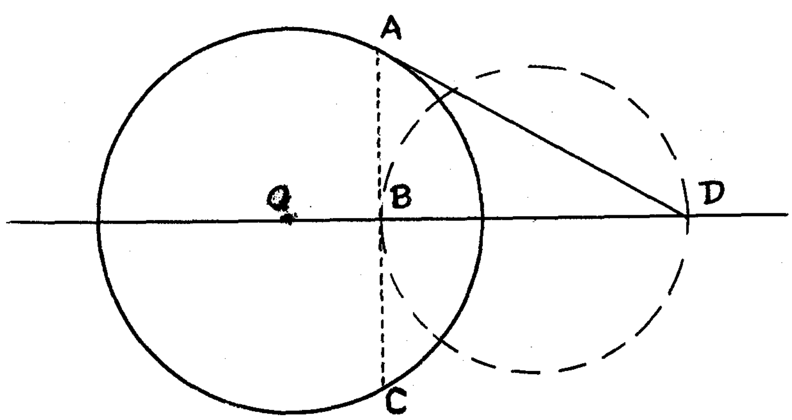


Fig. 40.1. Mapping of First Family

In this figure the line ABC in the t plane is transformed into the circle with diameter BD in the z plane. The representative curve of the second family defined by b = constant is the upper half of an ellipse with major axis between -1 and 1 on the t plane and minor axis b, with which is associated for convenience the lower half of a similar ellipse with minor axis $-\sqrt{1-b^2}$. These two curves transform into a circle in the z plane with center at $i\left(\frac{2b^2-1}{2b\sqrt{1-b^2}}\right)$ and radius $\left(\frac{1}{2b\sqrt{1-b^2}}\right)$. This mapping is illustrated in Fig. 40.2.

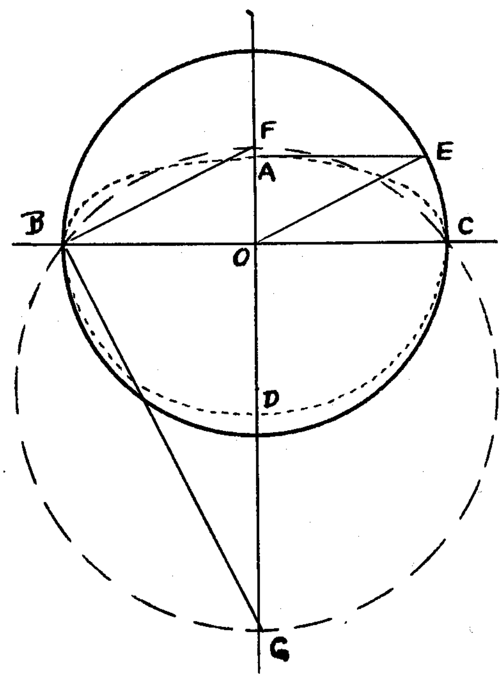


Fig. 40.2. Mapping of Second Family

In this figure the semiellipses BAC and BDC in the t plane are transformed into the circle in the z plane with diameter FG. The line BF is parallel to OE and the distance AE is equal to OD. Since the two families of circles in the z plane are invariant under the transformation (40.2) a similar mapping exists for the ϵ plane.

The Mach circle is transformed into the real axis external to the interval (-1,1), points on the circle going to the intersection of a tangent with the real axis. Thus in Fig. 40.1 point A is transformed into D. If no part of the solution extends upstream from the vertex so that the solution is null at infinity in the t plane the method of characteristics gives an exact correspondence between the solution on the plane outside the Mach circle and the solution on the circle. For such a case the solution on the real axis of z is identical with the solution on the physical plane over its entire range.

It should be noted that equation (40.3b) does not give an immediate connection between the values of v and w on the plane since v depends upon the imaginary part of f and, conversely, w depends upon the function conjugate to v. Since the real and imaginary parts of an analytic function are not independent it is possible to obtain relations connecting v and w on the plane of the form

$$w = \int K \, dv \tag{40.7a}$$

$$v = \int \mathcal{H} \, dw \tag{40.7b}$$

where the range of integration is over the entire real axis of the z plane or around the upper Mach semicircle on the t plane. These relations are somewhat cumbersome and are not presented here but are analogous in conical flow theory to the basic relations (23.4) and (23.6).

Problems of the first and second kind arise in the same manner as described for the general case in section 25, problems of the first kind having solutions given explicitly by equations (40.7). The analogy to a simple planform is one which includes either all or none of the interval within the Mach circle. From the fact that v and w satisfy the potential equation problems of the second kind with nonsimple planforms are fundamentally simpler than in the general case. In fact such problems can be said to be trivial in the sense of the general problem.

The oblique transformation applied to conical systems leaves each curve of the family b = constant unchanged and transforms each curve of the family a = constant into another curve of the same family. As indicated by equation (7.4) this amounts to a homographic transformation in the z plane

$$z = \frac{z' - m}{1 - m z'} \tag{40.8}$$

where m is real. Equations (40.3) may be shown to be invariant with the aid of equations (7.3). The restriction of section 22 with regard to vortex sheets may be translated into the df = 0 condition.

41. Planar Thickness Problems. The solution to the general conical thickness problem may be obtained by superposition from the solution of a basic thickness distribution. This basic distribution represents a triangular wedge-shaped body with the thick edge parallel to the flow direction, illustrated in Fig. 41.1.

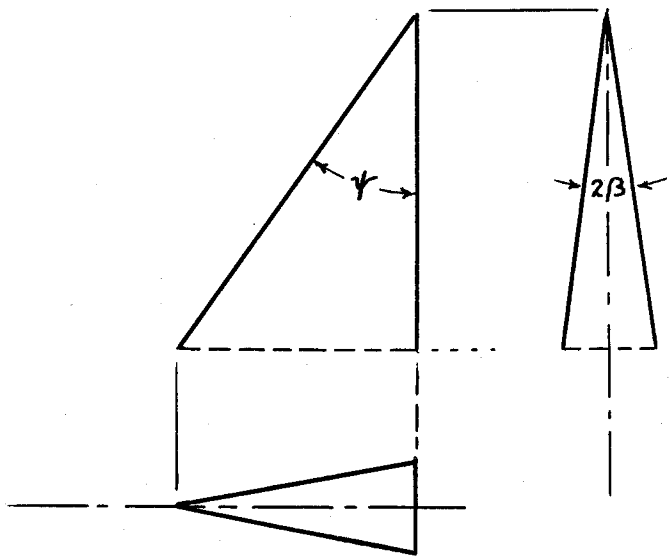


Fig. 41.1. Basic Thickness Distribution

The boundary condition is that on the upper surface $v = V\beta$ for $0 < t < t_o$ and $v = 0$ elsewhere. Two methods of solution are available, to obtain the solution for v first and then for w, or to obtain the solution for w first and then evaluate the multiplicative constant in terms of the boundary condition.

Two cases may be distinguished for convenience as to whether the leading edge is subsonic or supersonic, whether t_o is less or greater than one. For the first case $0 < t_o < 1$ and the body is entirely within the Mach cone. For this case the solution is

$$f = -\frac{2V\beta}{\pi} \frac{t_o}{\sqrt{1-t_o^2}} \tanh^{-1}\sqrt{\frac{1-t_o}{1+t_o} \cdot \frac{1+z}{1-z}} \quad (41.1)$$

Consequently the solution for w on the plane is

$$w = -\frac{2V\beta}{\pi} \frac{t_o}{\sqrt{1-t_o^2}} \tanh^{-1}\sqrt{\frac{1-t_o}{1+t_o} \cdot \frac{1+t}{1-t}}, \quad -1 < t < t_o \quad (41.2a)$$

$$w = -\frac{2V\beta}{\pi} \frac{t_o}{\sqrt{1-t_o^2}} \tanh^{-1}\sqrt{\frac{1+t_o}{1-t_o} \cdot \frac{1-t}{1+t}}, \quad t_o < t < 1 \quad (41.2b)$$

For the second case $1 < t_o$, the solution is

$$f = -\frac{2V\beta}{\pi}\frac{t_o}{\sqrt{t_o^2-1}}\tan^{-1}\sqrt{\frac{t_o-1}{t_o+1}\cdot\frac{1+z}{1-z}} \tag{41.3}$$

and the solution for w on the plane is

$$w = -\frac{2V\beta}{\pi}\frac{t_o}{\sqrt{t_o^2-1}}\tan^{-1}\sqrt{\frac{t_o-1}{t_o+1}\cdot\frac{1+t}{1-t}}\,, \qquad -1<t<1 \tag{41.4a}$$

$$w = -\frac{V\beta\, t_o}{\sqrt{t_o^2-1}}\,, \qquad 1<t<t_o\,. \tag{41.4b}$$

For either of these solutions the quantity df/dz at the origin depends only upon $V\beta$ so that the condition df = 0 is automatically satisfied if the angle of inclination of the final surface is continuous across the axis. The solution outside the Mach cone in the second case (41.4b) is the solution that would be obtained in that region by cylindrical flow methods. These solutions have been obtained previously by Jones (27) and for the case with lateral symmetry by Puckett (35) using different methods.

The basic conical relation (40.7a) giving w in terms of v may be obtained directly using these solutions by superposing them with the same vertex to obtain a general conical symmetric system. As this is a problem of the first kind superposition may be used without particular restriction and the solutions for a number of closed bodies may be obtained by superpositions of conical flows with different vertices.

42. Planar Lift Problems. The problem of the first kind in the lift case is analogous to that in the thickness case and is not discussed here in

detail. The only essential difference lies in the consideration of the condition $df = 0$. In the thickness case continuity of v across the axis assures that the condition is satisfied. In the lift case it is necessary to impose the condition if a singularity in v or u is to be avoided. The general solution to the lift problem of the first kind will yield the form of the relation (40.7b).

Only the problem of the second kind is treated here, which is the problem of determining the lift on a triangular wing where the camber is known. The problems must be divided into four major cases according to the leading and trailing edge configuration. These cases are:

I. Both edges lie inside the Mach cone. The designation I refers to the case where both edges are leading edges, while I-K refers to the case where one of the edges is a trailing edge. The designation K implies that the Kutta condition is to be applied.

II. One edge lies inside the Mach cone and the other edge lies outside the Mach cone. The designation II refers to the case where the edge inside the Mach cone is a leading edge, while II-K refers to the case where this edge is a trailing edge.

III. Both edges lie outside and on the same side of the Mach cone. One of the edges is thus a trailing edge, and no part of the wing lies in the Mach cone. Such planforms are simple.

IV. The two edges lie on opposite sides outside of the Mach cone. Thus both are leading edges and the wing entirely traverses the Mach cone. These are simple planforms.

These cases are depicted in Fig. 42.1. Any of the limiting cases that arise when an edge lies on the Mach cone or when an edge is parallel to the flow direction may be derived from either of the two cases described above which bound the limiting case. The flat plate solutions for the four cases are discussed below.

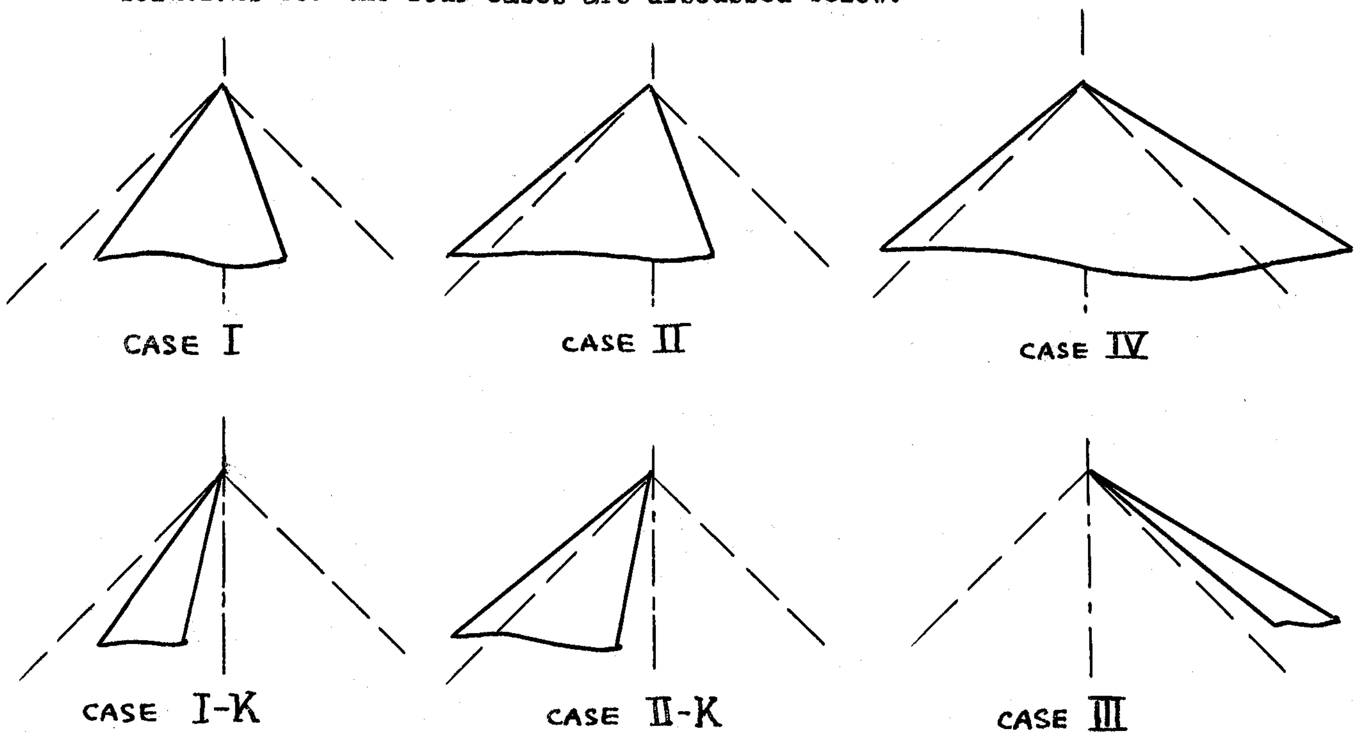


Fig. 42.1. Cases of Planar Lift Problems

Cases III and IV which are cases of simple planforms, are treated first.

Case III: In this case the solution on the wing is obtained completely by the method of characteristics. The lift for a wing with a supersonic trailing edge depends only upon the leading edge position and is given by

$$\frac{dC_L}{d\alpha} = 4\,\frac{t_o}{\sqrt{t_o^2-1}} \tag{42.1}$$

Since the trailing edge is supersonic the Kutta condition is not applicable but there is a vortex sheet behind the wing. The downwash is constant behind the wing outside the Mach circle.

Case IV: Since this is a simple planform the solution may be obtained by applying directly the methods of section 41. The solution for the flat plate at an angle of attack β is obtained by superposing two of the second case thickness solutions. For a wing with a straight supersonic trailing edge the lift curve slope is independent of the positions of the two leading edges and depends only upon the trailing edge angle. This is

$$\frac{dC_L}{d\alpha} = 4\,\frac{|t_1|}{\sqrt{t_1^2-1}} \tag{42.2}$$

where t_1 here describes the inclination of the trailing edge. This result of course follows from the second reverse flow theorem for lift curve slopes. This was observed first by Puckett (35) for a triangular wing with complete lateral symmetry.

Case I: For this case the flat plate solution has a $-1/2$ power singularity on both leading edges and satisfies the condition $df = 0$ at the origin. The complete solution is

$$f = \frac{V\alpha}{2E(k)r}\left[t_1\sqrt{\frac{t_2+z}{t_1-z}} + t_2\sqrt{\frac{t_1-z}{t_2+z}}\right] \tag{42.3}$$

where

$$r = \sqrt{\frac{1+t_1t_2+\sqrt{1-t_1^2}\sqrt{1-t_2^2}}{2}} \tag{42.4a}$$

$$k = \frac{\sqrt[4]{(1-t_1^2)(1-t_2^2)}}{r} \tag{42.4b}$$

and E is the complete elliptic integral of the second kind. The form of this solution was first found independently by W. Hayes and P. Lagerstrom. An interesting feature of the solution is that the form is independent of the Mach number.

The factor which determines the magnitude of the solution is obtained by applying a homographic transformation of the form (40.8) with

$$-m = b' = \frac{t_1 - t_2}{1 - t_1 t_2 + \sqrt{1-t_1^2}\sqrt{1-t_2^2}} \tag{42.5}$$

and applying Jacobi's imaginary transformation to the integral for v. This factor was first obtained for the case with lateral symmetry by Stewart (36). For a wing with a straight trailing edge normal to the flow direction the lift curve slope is

$$\frac{dC_L}{d\alpha} = \frac{2\pi(t_1+t_2)}{2E(k)r} . \tag{42.6}$$

The details of the analysis for this and subsequent cases may be found in (57). The solution for the case with the t's very small was obtained by Jones (30) by different methods.

For the case with lateral symmetry with $t_1 = t_2$ the results of Stewart may be expressed

$$f = \frac{V\alpha}{2E\left[\sqrt{1-t_1^2}\right]}\left[\frac{2t_1^2}{\sqrt{t_1^2 - z^2}}\right] \tag{42.7}$$

$$\frac{dC_L}{d\alpha} = \frac{2\pi t_1}{E\left[\sqrt{1-t_1^2}\right]} . \tag{42.8}$$

The leading edge thrust may be calculated by the methods of section 29. For the wing with the straight trailing edge normal to the flow direction the net drag may be expressed

$$C_D = C_L \alpha \left[1 - \frac{t_1\sqrt{1-t_1^2} + t_2\sqrt{1-t_2^2}}{2\,E(k)\,r\,(t_1+t_2)} \right] \tag{42.9}$$

which for the wing with lateral symmetry becomes

$$C_D = C_L \alpha \left[1 - \frac{\sqrt{1-t_1^2}}{2\,E\left[\sqrt{1-t_1^2}\right]} \right]. \tag{42.10}$$

<u>Case I-K</u>: This is similar to case I but differs in that the Kutta condition is applied and the df = 0 condition is not met. The solution is

$$f = \frac{V\alpha}{\left[E(k) + b'k'\Pi_0(k,-b^2)\right] r} \left[t_1 \sqrt{\frac{z-t_2}{t_1-z}} \right] \tag{42.11}$$

where

$$r = \sqrt{\frac{1-t_1t_2+\sqrt{1-t_1^2}\sqrt{1-t_2^2}}{2}} \tag{42.12a}$$

$$k = \frac{1}{r}\sqrt[4]{(1-t_1^2)(1-t_2^2)} \tag{42.12b}$$

$$k' = \sqrt{1-k^2} = \frac{t_1-t_2}{2r^2} \tag{42.12c}$$

$$b' = \frac{t_1+t_2}{1+t_1t_2+\sqrt{1-t_1^2}\sqrt{1-t_2^2}} \tag{42.12d}$$

$$b = \sqrt{1-b'^2} \tag{42.12e}$$

Relations (42.10a,b,d) are the same as relations (42.4a,b) and (42.5) for case I but with the sign of t_2 changed. The function Π_0 is Legendre's complete elliptic integral of the third kind defined by

$$\Pi_0(k,-b^2) = \int_0^1 \frac{du}{(1-b^2u^2)\sqrt{(1-u^2)(1-k^2u^2)}} \qquad (42.13)$$

This quantity may be evaluated in terms of the incomplete elliptic integrals of the first and second kinds by the formula for the interchange of argument and parameter in Jacobi's integral of the third kind (55, p 523). For a wing with a straight trailing edge normal to the flow direction the lift curve slope is

$$\frac{dC_L}{d\alpha} = \frac{2\pi t_1}{(E + b'k'\Pi_0)r} \qquad (42.14)$$

Taking into account the leading edge thrust the drag coefficient for this wing is

$$C_D = C_L\,\alpha\left[1 - \frac{\sqrt{1-t_1^2}}{(E+b'k'\Pi_0)r}\right] \qquad (42.15)$$

Case II: For this case $t_1 > 0$ and the condition $df = 0$ is applied. The solution is

$$f = \frac{2V\alpha}{\pi}\left[\frac{t_1}{\sqrt{t_1^2-1}}\tan^{-1}\sqrt{\frac{t_1-1}{t_1+t_2}\cdot\frac{t_2+z}{1-z}} + \frac{t_2}{1+t_2}\sqrt{\frac{t_1+t_2}{t_1+1}}\sqrt{\frac{1-z}{t_2+z}}\right] \qquad (42.16)$$

For a wing with a straight trailing edge normal to the flow direction and the leading edge not upstream from the vertex the lift curve slope is

$$\frac{dC_L}{d\alpha} = 4\sqrt{\frac{t_1+t_2}{t_1+1}} \tag{42.17}$$

and the drag coefficient is

$$C_D = C_L\alpha\left[1 - \frac{2t_2\sqrt{1-t_2^2}}{\pi(1+t_2)\sqrt{(t_1+t_2)(t_1+1)}}\right]. \tag{42.18}$$

Case II-K: Here the Kutta condition is applied and df $\neq$ 0 at the origin. The solution is

$$f = \frac{2\bar{V}\alpha}{\pi}\frac{t_1}{\sqrt{t_1^2-1}}\tan^{-1}\sqrt{\frac{t_1-1}{t_1-t_2}\cdot\frac{z-t_2}{1-z}}. \tag{42.19}$$

For a wing with a straight trailing edge normal to the flow direction and the leading edge not lying upstream from the vertex the lift curve slope is

$$\frac{dC_L}{d\alpha} = 4\frac{t_1}{\sqrt{(t_1+1)(t_1-t_2)}}. \tag{42.20}$$

There is no leading edge thrust for this case and the drag equals the lift times the angle of attack.

The solution for the general problem of a conical wing with an arbitrary camber distribution may be obtained for case III by the method of characteristics and for case IV by superposition of the solutions given in section 41. For the other cases a general solution must consist of a superposition of basic solutions with the same planform configuration within the Mach cone. As a general rule it is necessary in superposing solutions for problems of the second kind to use solutions for the same planform. Only the supersonic leading edge configurations may be different.

The eigensolutions which appear in connection with the Kutta condition are, for case I-K

$$f = A\sqrt{\frac{t_1 - z}{z - t_2}} \tag{42.21}$$

and for case II-K

$$f = A\sqrt{\frac{1 - z}{z - t_2}}\,. \tag{42.22}$$

The first of these is a particular solution which depends on the leading edge position. The second of these is the general solution of section 30 which may be applied locally at any point on a subsonic trailing edge. These solutions lie only within the Mach cone and have a zero distribution of v along the real axis of z. If the condition df = 0 had been imposed in cases I-K and II-K instead of the Kutta condition there would have been no vortex sheets for these cases and the chordwise integral of the lift would have been zero.

Of particular interest are certain of the solutions of Lagerstrom (48) in that they are solutions to problems of the second kind which are of a different nature from those considered here. In these problems the lift distribution desired is known over a portion of the span and it is required that the upwash distribution be zero over the remainder. Such a problem is identical with the problem in the thickness case of determining the thickness distribution for a desired pressure distribution. These solutions may be used to remove the pressure distribution over a portion of a wing and thus effectively change the planform.

43. Nonplanar Problems. A general conical body is represented by a contour in the t plane. The perpendicular distance between the origin in the t plane and a tangent to the contour at a point is a measure of the inclination of the body at that point to the flow direction. The linearizing assumptions require that this quantity is small. Thus it may be seen that except for the portion of the body near the axis a general body must lie along planes passing through the axis. Boundary conditions of the type used for planar systems may be applied on the body at a distance from the axis and the general boundary condition of Busemann (20) needs to be applied only near the axis. This general condition, that the relative velocity

$$\Omega_\epsilon = u(1+\epsilon^2) + iv(1-\epsilon^2) - 2V\epsilon \tag{43.1}$$

has the body for a streamline in the ϵ plane, is simplified near the axis in that the relative velocity may be taken to be

$$\Omega = u + iv - 2V\epsilon . \tag{43.2}$$

A conical body may be termed "almost planar" if its only deviation from a true planar system is in its shape near the axis. For such a body the relative velocity in the z plane

$$\Omega_z = u\sqrt{1-z^2} + iv(1-z^2) - Vz\sqrt{1-z^2} \tag{43.3}$$

may be expressed as

$$\Omega = u + iv - Vz \tag{43.4}$$

for application near the axis. The boundary conditions over the remainder of the plane are the same as those treated in the previous three sections.

For general nonplanar conical bodies the planar similitude may not be used and the general similitude of section 6 gives no change in the shape of a body in the t plane assuming that t is defined so as to be one on the Mach cone. It may be necessary to include the $u^2 + v^2$ terms in the pressure near the axis.

The simplest example of nonplanar flow is the flow about a cone, for which solutions were obtained first by Karman and Moore (14) and Ferrari (16) by other methods. The solution with angle of attack is

$$f = V\beta^2 \log \epsilon + V\alpha\beta\left(\frac{1-\epsilon^2}{\epsilon}\right) \tag{43.5}$$

where the angle of attack is provided by a uniform side flow instead of by moving the cone. The $u^2 + v^2$ terms in the pressure are necessary in this solution.

Examples of almost planar problems arise in the problem of the interaction of a triangular wing with a conical body. Some of these are being studied by the Aerophysics Laboratory at North American Aviation.

VIII. The Method of Separation of the Lateral Variable

44. Fundamental Relations. This method consists of the representation of a planar system by the superposition of solutions that are periodic in the lateral variable x. Such a solution is the sinusoidal source whose potential distribution in its zone of action is given directly by the Riemann function of section 13. The Riemann method is applied to a particular contour in the y-z plane, along the line $y = 0$ from $z = -\infty$ to $z = z_0$ and thence along a characteristic. The contour is considered to be closed by a line sufficiently far upstream that there is no disturbance on the line. The integral along the z axis is expressed as an integral from $-\infty$ but it is to be understood that this is really from a finite distance upstream. The contour for the Riemann method is shown in Fig. 44.1

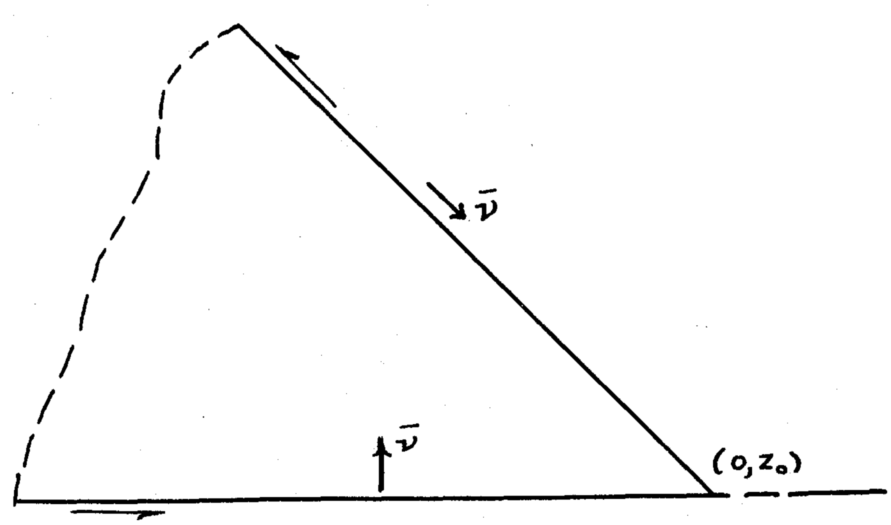


Fig. 44.1. Contour for Riemann Method

Using equation (10.3) and the Riemann function (13.3) the result may be expressed

$$\Phi_0 = -\int_{-\infty}^{z_0} J_0\left(k(z_0-z)\right)\frac{\partial\Phi}{\partial y}\,dz\,. \tag{44.1}$$

By differentiating this expression with respect to z_0 the relation is obtained

$$\Phi_{z_0} = -\Phi_{y_0} + k\int_{-\infty}^{z_0} J_1\left(k(z_0-z)\right)\Phi_y\,dz\,. \tag{44.2}$$

Applying the result (44.1) to the y derivative of Φ results using equation (13.2) in

$$\Phi_{y_0} = -\int_{-\infty}^{z_0} J_0\left(k(z_0-z)\right)\left[\Phi_{zz} + k^2\Phi\right]dz \tag{44.3a}$$

which by two integrations by parts becomes

$$\begin{aligned}\Phi_{y_0} &= -\Phi_{z_0} - k^2\int_{-\infty}^{z_0}\left(J_0'' + J_0\right)\Phi\,dz \\ &= -\Phi_{z_0} - k\int_{-\infty}^{z_0}\frac{J_1\left(k(z_0-z)\right)}{(z_0-z)}\Phi\,dz\end{aligned} \tag{44.3b}$$

A further integration by parts yields the relation

$$\Phi_{y_0} = -\Phi_{z_0} - k\int_{-\infty}^{z_0}\mathcal{J}_1\left(k(z_0-z)\right)\Phi_z\,dz \tag{44.4}$$

where

$$\mathcal{J}_1(x) = \int_0^x \frac{J_1(s)}{s}\, ds \qquad (44.5)$$

Equation (44.4) may be obtained directly from equation (44.2) by solving it as a Volterra integral equation by means of Laplace transformations.

Equations (44.2) and (44.4) are the basic relations for the sinusoidal solutions and are directly analogous to the basic relations for planar systems (23.4) and (23.6). In each of these relations there is a separation into a basic term and an induced term in the sense of section 35. The induced terms give the variation from two-dimensional flow results. If $k = 0$ the flow becomes two-dimensional and the induced terms vanish. The periodicity may be made skew by means of the oblique transformation.

45. Periodic Solutions. If a number of identical finite systems are considered to be placed along a line in the x direction with a uniform spacing the result is a periodic system with a periodic solution. If the spacing is made sufficiently great that no interaction between the systems is possible the solution in the vicinity of any one of the systems is identical with the solution that would hold for the isolated system. Thus it is possible to study the behavior of a finite system through a study of periodic solutions.

The period of the system is chosen to be $2l$ laterally. This quantity is completely arbitrary as it may be changed for a given system by the scale transformation. Solutions with this periodicity are those with $k = m\pi/l$, where m is an integer. The new variable θ is introduced in place of x

$$\theta = \frac{\pi x}{l} \tag{45.1}$$

This should not be confused with the azimuthal variable. The complete periodic solution is expressed in terms of a Fourier series

$$\phi = \sum_{m=0}^{\infty} A(y,z,m) \cos m\theta + \sum_{m=1}^{\infty} B(y,z,m) \sin m\theta \tag{45.2}$$

where the A's and B's are functions of the type Φ. A separation of the solution on the basis of lateral symmetry is an immediate result, the cosine terms giving the symmetric part

$$\phi_s = \frac{1}{2}\left[\phi(x) + \phi(-x)\right] \tag{45.3a}$$

and the sine terms giving the antisymmetric part

$$\phi_a = \frac{1}{2}\left[\phi(x) - \phi(-x)\right] \tag{45.3b}$$

These two parts may be considered separately if desired.

The drag per period for the solution on the upper half space is obtained by integrating $-\rho\, vw$ over the plane

$$D = -\rho \int_{-l}^{l} \int_{-\infty}^{\infty} v_0 w_0 \, dz_0 \, dx_0 = D_b + D_i \tag{45.4}$$

The quantities v and w may be obtained from (45.2) and the derivatives of the A's and B's expressed in terms of relation (44.2) or (44.4). The calculation for the drag gives the basic drag D_b and the induced drag D_i separately. Since the basic drag is easily calculated by integrating the two-dimensional drag expression only the induced drag is considered. This is given by

$$D_i = -\rho \int_{-\infty}^{\infty} \int_{-\infty}^{z_0} \sum_{m=1}^{\infty} m\pi \, J_1\left(\frac{m\pi}{l}(z_0 - z)\right) \left[A_y A_{y_0} + B_y B_{y_0}\right] dz \, dz_0 \quad (45.5)$$

for the thickness case and by

$$D_i = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{z_0} \sum_{m=1}^{\infty} m\pi \, J_1\left(\frac{m\pi}{l}(z_0 - z)\right) \left[A_z A_{z_0} + B_z B_{z_0}\right] dz \, dz_0 \quad (45.6)$$

for the lift case.

If the summation is carried out before the integration the series involved are Schlömilch series. Some of the results of the theory of these series may be helpful in their evaluation. The original theory may be expressed: if

$$f(\theta) = \sum_{m=0} a_m \cos m\theta \quad (45.7)$$

the corresponding series with the Bessel function of zero order is

$$g(\theta) = \sum_{m=0} a_m J_0(m\theta) = \frac{2}{\pi} \int_0^{\pi/2} f(\theta \sin\psi) \, d\psi . \quad (45.8)$$

The solution of this integral equation for f is

$$f(\theta) = g(0) + \theta\int_0^{\pi/2} g'(\theta\sin\psi)\,d\psi . \tag{45.9}$$

For the purposes of the induced drag calculations two modified Schlömilch series are needed. For these the theory may be expressed: if

$$F(\theta) = \sum_{m=1} b_m \sin m\theta \tag{45.10}$$

the Schlömilch series are

$$G(\theta) = \sum_{m=1} b_m J_1(m\theta) = \frac{2}{\pi}\int_0^{\pi/2} F(\theta\sin\psi)\sin\psi\,d\psi \tag{45.11}$$

and

$$\mathcal{J}(\theta) = \sum_{m=1} b_m \mathcal{J}_1(m\theta) = \frac{2}{\pi}\int_0^{\pi/2} F(\theta\sin\psi)\frac{\cos^2\psi}{\sin\psi}\,d\psi . \tag{45.12}$$

The solutions to the integral equations for F are

$$F(\theta) = \frac{d}{d\theta}\left[\theta\int_0^{\pi/2} G(\theta\sin\psi)\,d\psi\right] \tag{45.13}$$

$$F(\theta) = \frac{d}{d\theta}\left[\theta^2\int_0^{\pi/2} \mathcal{J}'(\theta\sin\psi)\sin\psi\,d\psi\right] . \tag{45.14}$$

The procedure that may be followed in evaluating a Schlömilch series is to evaluate the corresponding Fourier series (45.7) or (45.10) first and to obtain the Schlömilch series through the appropriate integral relation.

These results may be applied directly to the expression for the potential on the axis, with x = 0. The terms in B do not enter and the potential may be expressed

$$\phi_o = -\sum_{m=o} \int_{-\infty}^{z_o} J_o\left(\frac{m\pi}{l}(z_o-z)\right) A_y \, dz \, . \tag{45.15}$$

Evaluating the corresponding Fourier series by means of the cosine part of equation (45.2) and using relation (45.8) the potential is given by

$$\phi_o = -\frac{2}{\pi}\int_{-\infty}^{z_o}\int_{0}^{\pi/2} \phi_{sy}\left((z_o-z)\sin\psi\right) d\psi \, dz \, . \tag{45.16}$$

This equation is identical with equation (23.1) and this analysis serves as an alternate derivation. The equation is independent of l and is therefore unchanged if l is changed or allowed to become very large. In a similar way expressions identical with equations (23.4) and (23.6) may be obtained using the modified Schlömilch relations.

As an example the induced drag of a symmetric prismatic body of length l in the x direction and of width a is calculated with the restriction that $a < l$. This restriction is necessary in setting up the system with the periodicity $2l$ in order that there be no interference. Such a body is depicted in Fig. 45.1. If the surface of this body is given by a function y(z) the boundary condition on the body may be expressed

$$\phi_y = V y'(z) \, . \tag{45.17}$$

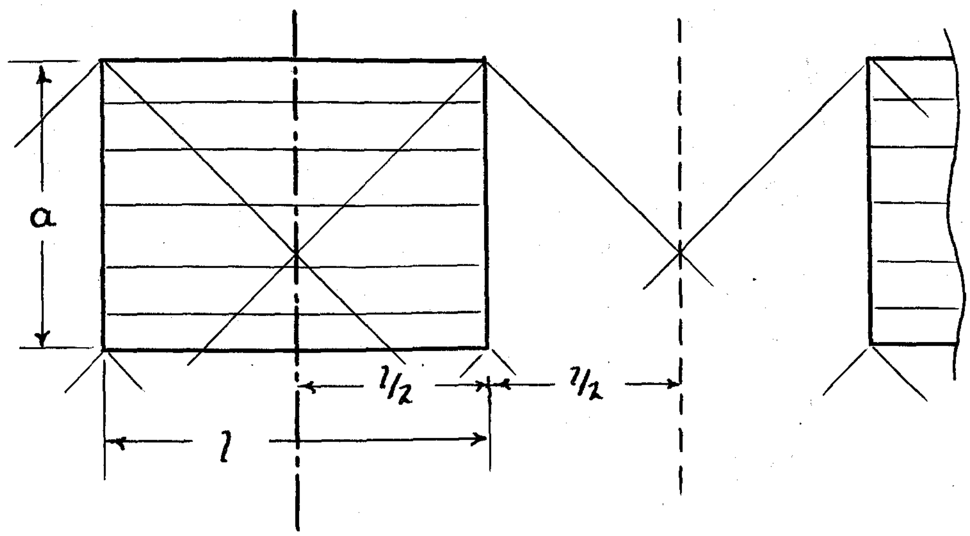


Fig. 45.1. Finite Prismatic Body

Except for the m = 0 term which does not enter the expression for the induced drag the even order terms in the Fourier expansion of Φ_y are zero and the other terms are given by

$$A_y(2n+1) = \frac{2}{\pi} V y' \frac{(-1)^n}{2n+1} \quad (45.18)$$

The Schlömilch series that appears in the induced drag expression is evaluated by equation (45.11) from the corresponding Fourier series (45.10)

$$\sum \frac{J_1\left(\frac{(2n+1)\pi}{l}(z_0-z)\right)}{2n+1} = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{4}\right)\sin\psi\, d\psi \quad (45.19)$$

The induced drag may be evaluated

$$\begin{aligned} D_i &= \frac{-2\rho V^2}{\pi}\int_0^a\int_0^{z_0} y_0' y' \, dz\, dz_0 \\ &= \frac{-\rho V^2}{\pi}\left(y_0^2\right)\Big|_0^a = 0 \end{aligned} \quad (45.20)$$

From the condition for closure of the body the induced drag is zero. This result, that the drag of such a body is given by the two-dimensional expression alone, may be derived from conical flow theory by superposing conical solutions along the edge.

46. Lateral Fourier Integrals. The Fourier series of the previous section may be extended to Fourier integrals in the customary manner by letting approach infinity. The correspondence is made

$$l A \longrightarrow \sqrt{2\pi}\; g(y,z,k) \tag{46.1a}$$

$$l B \longrightarrow \sqrt{2\pi}\; h(y,z,k) \tag{46.1b}$$

$$\Sigma' \pi/l \longrightarrow \int_0 dk \tag{46.1c}$$

and the expression for the velocity potential (45.2) becomes

$$\phi = \frac{2}{\sqrt{2\pi}}\int_0^\infty g \cos kx \, dk + \frac{2}{\sqrt{2\pi}}\int_0^\infty h \sin kx \, dk \tag{46.2}$$

where g and h are even and odd functions of k, respectively. This may be expressed in the complex form

$$\phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (g + ih)\, e^{-ikx} \, dk \tag{46.3}$$

with g and h given by

$$g + ih = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi \, e^{ikx} \, dx \tag{46.4}$$

The induced drag for the entire system is analogous to that of equations (45.5) and (45.6) and is given by

$$D_i = -2\rho \int_{-\infty}^{\infty} \int_{-\infty}^{z_0} \int_{0}^{\infty} k\, J_1(k(z_0 - z)) \left(g_y g_{y_0} + h_y h_{y_0} \right) dk\, dz\, dz_0 \quad (46.5)$$

for the thickness case and by

$$D_i = 2\rho \int_{-\infty}^{\infty} \int_{-\infty}^{z_0} \int_{0}^{\infty} k\, \mathcal{Y}_1(k(z_0 - z)) \left(g_z g_{z_0} + h_z h_{z_0} \right) dk\, dz\, dz_0 \quad (46.6)$$

for the lift case. It should be kept in mind that this is the drag for the solution in the upper half space and should be doubled for the complete drag of a finite system.

The Schlömilch series of the previous section become integrals related to the Fourier-Bessel integrals. Analogous relations to the equations (45.7) to (45.14) hold and a similar procedure may be used for the evaluation. For an integral in J_0 the corresponding Fourier integral with the cosine in place of J_0 may be evaluated and the original integral obtained by the relation (45.8). Similarly integrals in J_1 or $\mathcal{Y}_1$ may be obtained through equations (45.13) and (45.14) by evaluating the corresponding sine integrals. Equation (45.16) and the similar relations identical with equations (23.4) and (23.6) may be obtained on this basis.

IX. Flow about Bodies of Revolution

47. General Considerations. For the flow about a body of revolution the body is assumed to be placed at zero incidence to the flow direction and the fundamental coordinates are taken to be cylindrical coordinates with the same axis as the body. With such a system the boundary conditions imposed are upon the radial velocity component q_r or u, and are independent of the azimuth angle either in magnitude or in the value of r at which the conditions are taken. No boundary conditions are imposed upon the azimuthal velocity component q_θ or v. The flow field in which the body is considered may consist of the fundamental flow alone or may include a disturbance flow of unspecified origin.

It is convenient to consider the entire flow divided according to the separation of the azimuthal variable into superposed flows. The solution for the flow about the body with no exterior disturbance contains only terms for which m = 0, and the boundary conditions on the radial velocity u are satisfied with this solution. This flow is termed the principal flow. For a ducted body the internal and external solutions are to be obtained separately and matched at the exit.

An external disturbance flow is divided into a number of parts, each with a different value of m. For each of these the effect of the body is to create an additional disturbance with the same value of m such that the boundary condition u = 0 is met everywhere on the body.

Since bodies of revolution are not planar systems it is necessary in general to include the quadratic cross velocity terms in the pressure.

The inclusion of these terms is justified in that the linearization of the pressure expression is not necessary and in general is not permitted for the linearization of the equation for the velocity potential. The flow about a cone is an example where the quadratic terms are necessary.

For the problem of a body of revolution at an angle of incidence there are two systems which may be considered: 1. The body may be considered to be placed in the uniform flow tilted at the angle of attack. For this system the body does not fit the cylindrical coordinate system and the boundary conditions are not simple. 2. The body may be considered to lie in the uniform flow at zero incidence with a disturbance flow consisting of a uniform cross flow of magnitude $V \sin\alpha$ and an axial flow $-V(1-\cos\alpha)$. This disturbance flow may be characterized by the equations

$$m = 0 \qquad w_0 = -\frac{1}{2}V\alpha^2 \tag{47.1a}$$

$$m = 1 \qquad \begin{cases} u_0 = V\alpha \sin\theta & (47.1b) \\ v_0 = V\alpha \cos\theta & (47.1c) \end{cases}$$

These two systems give consistent results for the pressure including the quadratic terms. Because of its much greater simplicity the second method is used. An example of the comparison in simplicity is between the result of Busemann (20) for the cone at an angle of incidence and the result (43.5). It should be noted that the axial disturbance flow w_0 strictly should be included in the pressure. Since this pressure term is constant there is no effect on any of the forces on the body. This flow automatically satisfies the $u = 0$ condition for the disturbance flow so that the entire angle of attack problem is one for which $m = 1$.

A sufficiently slender body in curved flight of a radius which is large compared with the length may be considered as a body with a variable angle of attack.

The lift distribution on a body of revolution at an angle of attack may be taken to be the force distribution normal to the axis. The drag on the body is given by the pressure force in an axial direction corrected for the axial component of the net lift. However, the drag is given correctly by the energy flow considerations at a great distance without the correction for the lift component. For a body which closes to a point on the axis no net lift is possible since the vortex system which this requires would be concentrated on the axis and would give an infinite vortex drag. This apparently anomalous result, that a closed body without a duct produces no net lift, simply means that the linearized theory is inadequate to explain the lift on such a body. The theory is apparently quite adequate for the lift distribution on a body which is increasing in size for a sufficiently small angle of attack. For angles of attack which are large compared with the wall angles or for bodies which are decreasing in size, vortices are shed from the body before it closes and extend downstream beside the body. This gives a finite vortex system and provides a net lift. No criterion is available in this theory to predict the manner of the shedding of these vortices as there is no sharp edge such as the trailing edge of a planar wing to serve as a separation point for the vortex system. The actual situation depends upon real fluid effects, upon boundary layer distribution and separation.

For a ducted body the situation is different. Although shedding of vortices may occur in the manner described in the previous paragraph it is possible for the system to have a finite net lift following the linearized theory. For this case a finite vortex system is shed from the tail of the body, this system consisting of variable cylindrical vortex sheets.

The solution to the flow about a given body by any of the available methods usually involves integral equations of the Volterra type. It should be noted that in the previous application of the Hadamard method to general planar systems and the Riemann method for the separation of the lateral variable the integral terms in the unknown quantities vanish. Such integral equations of the Volterra type are always soluble in principle by the use of step-by-step methods. This is unlike the subsonic case for which the integral equations are not of the Volterra type.

The methods that are available for the solution of the flow about a body of revolution are described briefly:

a. The classic method. In this method the system is assumed to consist of a linear distribution of sources on the axis to represent the principal flow and with dipoles or lift elements to represent the angle of attack disturbance (14,15,16,17). The basic equations for this distribution are given in section 37 by equations (37.1) and (37.2). The approximations made in that section in order to obtain conditions at a distance from the body are not valid for the local flow. Recently further contributions have been made by Jones and Margolis (32) and by Lighthill (34). The interior solutions for ducted bodies are obtained by superposing the solutions for zero source strength or zero lift which form the imaginary parts of the solutions of section 11.

For a body of revolution a singularity in the source strength along the axis and the corresponding singularity in the velocity component u differ in order by 1/2 unless this occurs on the axis in which case the difference in order is 1. This is in contrast with a planar system for which the singularity in the source strength distribution is the same as that for the velocity component normal to the plane.

b. Superposition of solutions of the T type. A body of revolution may be represented by a superposition of solutions of the type considered in Chapter III for which $m = 0$ for the principal flow and $m = 1$ for the skew flow. For the interior of a ducted body solutions are chosen which have no contribution in the upstream Mach cone but which exist between the Mach cones. This method is essentially the same as method a. It has advantages for numerical calculation over method a in that polynomial fitting may be used to obtain a smooth solution and that solutions of appropriate type may be superposed at various stations as desired to conform with irregularities in the body contour.

c. Method of Fourier integrals. The method of separation of the axial variable may be applied through the use of Fourier integrals or Laplace transformations. This method was applied by von Karman (15) to the calculation of the flow about a cylinder with ring-shaped corrugations.

d. The method of characteristics. The nonlinearized method of Sauer (7), Tollmien, or Guderley may be applied to the linearized theory. This method is practical only when a numerical calculation of the entire flow field is desired and is feasible only for the principal flow with $m = 0$.

e. Superposition of ring sources. A body of revolution may be represented by a distribution of ring sources placed on the surface of the body. For the skew flow appropriate variable source rings are used. This method is essentially the same as f.

f. The Riemann method. The method of Riemann described in section 10 may be applied to the calculation of the flow about bodies of revolution. This method is described later in this chapter.

48. The Slender Body Approximation. For slender bodies of revolution an approximation may be made in the method of satisfying the boundary conditions which is analogous to the method used in planar systems. In this case the boundary conditions are satisfied with respect to conditions in the vicinity of the axis. The boundary conditions are not satisfied on the axis. The treatment will be made for a general value of the Mach number since information on the Mach number dependency of the solution is obtained.

The relation between the source and lift element distributions and the geometry of the body is obtained directly. The total source strength at a given value of z may be equated to the velocity times the cross-sectional area of the body

$$\int f\,dz = V\pi r^2 \tag{48.1a}$$

or the radial velocity component at a distance r from the axis may be equated to the velocity times the slope of the body

$$\frac{1}{2\pi r} f = u = V \frac{dr}{dz} \tag{48.1b}$$

The total lift element strength at a given value of z is equal to the linear dipole distribution strength. The velocity from this line dipole at a distance r from the axis may be equated to the velocity times the angle of attack

$$\frac{1}{2\pi r^2} \int \frac{g_y}{\sqrt{M^2-1}} dz = u_{\pi/2} = V\alpha . \tag{48.1c}$$

This yields the expressions for the distribution of f and g

$$f = 2\pi V r \frac{dr}{dz} \tag{48.2a}$$

$$\frac{1}{\sqrt{M^2-1}} g_y = 4\pi V r \frac{dr}{dz} \alpha \tag{48.2b}$$

$$g_x = 0 \tag{48.2c}$$

The results of this theory may be summarized:

1. The pressure distribution due to the principal flow is dependent upon the Mach number in both distribution and magnitude.

2. The drag on the body due to the principal flow is independent of the Mach number.

3. The pressure distribution due to angle of attack and the resultant lift distribution is independent of the Mach number in both magnitude and distribution.

4. The net lift is zero for a closed body and is independent of the Mach number for a body which is not closed.

5. The wave drag due to lift may be neglected in comparison with the wave drag due to the principal flow.

The condition of applicability for the slender body approximation may be stated that the radius of the body must be small compared with the wave length of the disturbances. This wave length may be defined roughly as the ratio of an average absolute value of the slope taken over a length of the order of magnitude of the radius to the maximum value of the second derivative of the radius with respect to the axial distance. The condition may be expressed

$$r \ll \frac{\left(\frac{dr}{dz}\right)_{av.}}{\frac{d^2r}{dz^2}} \qquad (48.3)$$

As examples of the failure of the slender body approximation may be cited the cylinder with ring-shaped corrugations of von Karman and the finite cone which ends in a semi-infinite cylinder. In the latter case the drag which would be given by the slender body approximation is infinite and the actual drag which is given directly by the pressure on the cone does have a Mach number dependency.

49. <u>The Linearized Method of Characteristics</u>. The linearized method of characteristics is applied only to the principal flow. For this case the

velocity potential is independent of θ and from equation (6.14) satisfies the equation

$$\phi_{rr} + \frac{1}{r}\phi_r - \phi_{zz} = 0 \quad (49.1)$$

It should be noted that it is possible to express the velocity potentials for $m \neq 0$ in similar forms. If the substitution

$$\Phi = r^{\pm m}\Psi \quad (49.2)$$

is made in equation (14.2) the resultant equation is

$$\Psi_{rr} + \frac{1 \pm 2m}{r}\Psi_r - \Psi_{zz} = 0 \quad (49.3)$$

This is of the same form as (49.1). However, the boundary conditions are not expressed in terms of the first derivative of this new function with respect to r but in the derivative of the velocity potential

$$\Phi_r = r^{\pm m}\left(\Psi_r \pm \frac{m}{r}\Psi\right) \quad (49.4)$$

so that the method must be modified to include the calculation of Ψ.

The transformation of the independent variables is made

$$\eta = z + r \quad (49.5a)$$

$$\xi = z - r \tag{49.5b}$$

$$z = \frac{1}{2}(\eta + \xi) \tag{49.5c}$$

$$r = \frac{1}{2}(\eta - \xi) \tag{49.5d}$$

the new coordinates η and ξ being constant along the two families of characteristics. With this transformation the velocity components are expressed

$$w = \phi_z = \phi_\eta + \phi_\xi \tag{49.6a}$$

$$u = \phi_r = \phi_\eta - \phi_\xi \tag{49.6b}$$

$$\phi_\eta = \frac{1}{2}(w + u) \tag{49.6c}$$

$$\phi_\xi = \frac{1}{2}(w - u) \tag{49.6d}$$

and the equation for the velocity potential (49.1) becomes

$$\phi_{\xi\eta} = \frac{\phi_\eta - \phi_\xi}{2(\eta - \xi)} \tag{49.7}$$

This equation permits a step-by-step computation of the flow field by means of the equations

$$d\phi_\eta = \frac{1}{2} \cdot \frac{\phi_\eta - \phi_\xi}{\eta - \xi}\, d\xi \quad , \quad d\eta = 0 \tag{49.8a}$$

$$d\phi_\xi = \frac{1}{2} \cdot \frac{\phi_\eta - \phi_\xi}{\eta - \xi}\, d\eta \quad , \quad d\xi = 0 \tag{49.8b}$$

the first equation being used to continue the solution along characteristics for which η is constant and the second along the similar lines ξ is constant. Equations (49.8) may be expressed in terms of the original quantities

$$d(w+u) = \frac{1}{2}\,\frac{u}{r}(dz-dr) \quad , \quad dz+dr=0 \tag{49.9a}$$

$$d(w-u) = \frac{1}{2}\,\frac{u}{r}(dz+dr) \quad , \quad dz-dr=0 \tag{49.9b}$$

for the purpose of calculation without the transformation of the coordinate system.

Equations (49.8a) and (49.8b) may be integrated directly to obtain ϕ_η as an integral of ϕ_ξ with a general function of η and to obtain ϕ_ξ as an integral of ϕ_η with a general function of ξ. These may be combined and the boundary conditions applied to give integral equations for these two quantities. However, these integral equations are in two variables and do not appear to be of any particular value in computation.

50. <u>Derivation of the Riemann Functions</u>. The substitution (14.3) in the separated equation for the velocity potential (14.2) is repeated here

$$\bar{\Phi} = \frac{1}{\sqrt{r}}\, U \tag{50.1}$$

with the resulting equation (14.4)

$$U_{rr} - U_{zz} + \frac{1/4 - m^2}{r^2} U = 0 \tag{50.2}$$

The transformation of the previous section is made and yields the equation for U

$$U_{\xi\eta} + \frac{m^2 - 1/4}{(\eta - \xi)^2} U = 0 \tag{50.3}$$

The Riemann function is a solution of this equation which is equal to one on the two characteristics leading from a point, on ξ = a and on η = b. The function equals one when

$$(\xi - a)(\eta - b) = 0 \tag{50.4}$$

The form of the Riemann function was discovered originally by considering the product of equation (50.4) as an independent variable together with r and finding a solution as a power series in this new variable with coefficients that were functions of r. This process is not necessary and the form of the Riemann function may be inferred from general considerations. The Riemann function is assumed to be a function of a single variable, which is to be zero on the two characteristics through the point (a,b). The considerations that are necessary to obtain the form of this variable are:

a. The quantity in equation (50.4) is a factor of the variable.

b. Equations (50.2) and (50.3) are invariant under a scale transformation. Hence the variable is homogeneous of degree zero in the coordinates.

c. The Riemann function is unchanged if the variables a and b are exchanged with ξ and η, respectively.

d. The desired variable may be expected to have a singularity in $(\eta - \xi)$. This behavior leads to a variable of the form

$$\zeta = \frac{(\xi - a)(\eta - b)}{(b-a)(\eta - \xi)} . \tag{50.5}$$

It is found that solutions for U which are functions of this variable alone exist and satisfy the equation

$$\zeta(1-\zeta)U'' + (1-2\zeta)U' - (\tfrac{1}{4} - m^2)U = 0 . \tag{50.6}$$

This gives immediately the Riemann function as a hypergeometric function

$$V = F(\tfrac{1}{2} - m, \tfrac{1}{2} + m ; 1 ; \zeta) . \tag{50.7}$$

For negative values of the argument the identity for the hypergeometric function

$$F(a,b;c;z) = (1-z)^{-a} F(a, c-b; c; \tfrac{-z}{1-z}) \tag{50.8}$$

may be employed to yield the alternative expressions for the Riemann function

$$V = (1-\zeta)^{-\frac{1}{2}+m} F(\tfrac{1}{2} - m, \tfrac{1}{2} - m ; 1 ; \tfrac{-\zeta}{1-\zeta}) \tag{50.9a}$$

$$V = (1-\zeta)^{-\frac{1}{2}-m} F(\tfrac{1}{2} + m, \tfrac{1}{2} + m ; 1 ; \tfrac{-\zeta}{1-\zeta}) . \tag{50.9b}$$

The range of the variable with respect to the geometry of the (z,r) plane is shown in Fig. 50.1.

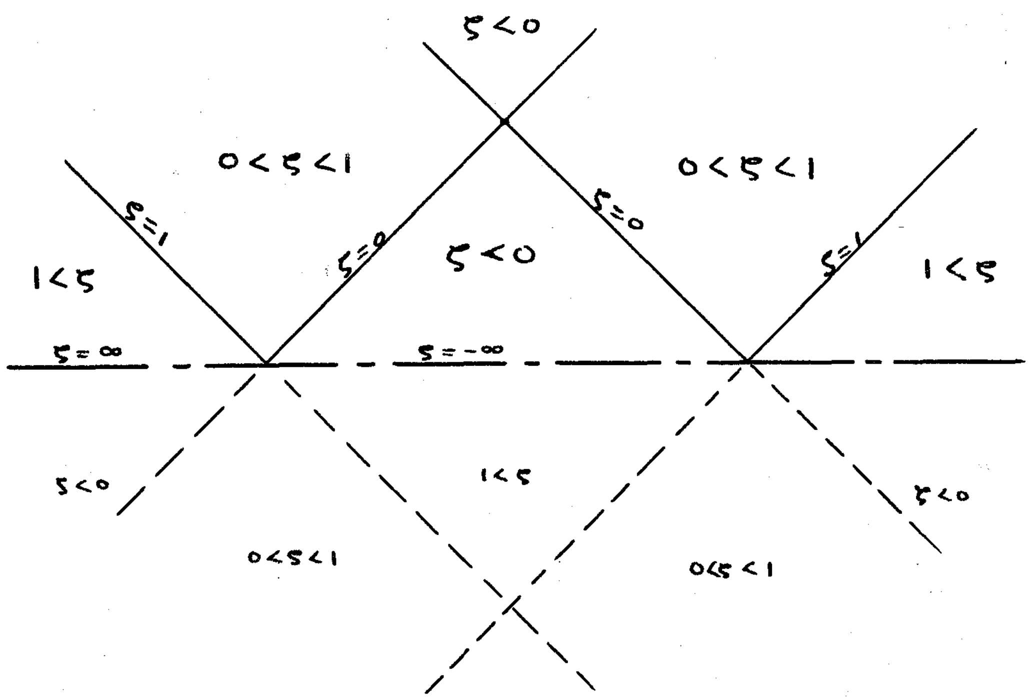


Fig. 50.1. Range of the Variable ζ

The solution is assumed to be continued across the axis with r negative. Expressing the results in terms of the old variables with

$$b = z_0 + r_0 \tag{50.10a}$$

$$a = z_0 - r_0 \tag{50.10b}$$

the fundamental variable is

$$\zeta = \frac{(z-z_0)^2 - (r-r_0)^2}{4rr_0} \tag{50.11}$$

and its derivatives are

$$\zeta_z = \frac{z - z_0}{2 r r_0} \tag{50.12a}$$

$$\begin{aligned} \zeta_r &= -\frac{r - r_0}{2 r r_0} - \frac{\zeta}{r} \\ &= -\frac{(z-z_0)^2 + (r^2 - r_0^2)}{4 r^2 r_0} \end{aligned} \tag{50.12b}$$

Of particular interest is the value of this variable with the sign of r changed. This is given by

$$\zeta(-r) = -\frac{(z-z_0)^2 - (r+r_0)^2}{4 r r_0} = 1 - \zeta . \tag{50.13}$$

The Riemann functions have logarithmic singularities at $\zeta = 1$ and half-integral singularities at $\zeta = \infty$. They are all expressable in terms of the complete elliptic integrals of the first and second kind

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; \zeta) = \frac{2}{\pi} K(\zeta) \tag{50.14a}$$

$$F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; \zeta) = \frac{2}{\pi} E(\zeta) \tag{50.14b}$$

with the aid of the identities for the hypergeometric function

$$(b-a) F = b F_{b+} - a F_{a+} \tag{50.15a}$$

$$(b-a)(1-\zeta) F = (b-c) F_{b-} - (a-c) F_{a-} \tag{50.15b}$$

The derivative of the hypergeometric function may be expressed

$$\zeta F' = a(F_{a+} - F) = b(F_{b+} - F) \tag{50.16}$$

These equations give all the relations necessary to connect the hypergeometric functions with parameter c = 1 and their derivatives.

51. Application of the Riemann Method. The contour taken for the application of the Riemann method to an exterior flow field is similar to the contour of section 44 with the important difference that the point (z_0, r_0) may be taken at any point on the characteristic. This contour is illustrated in Fig. 51.1.

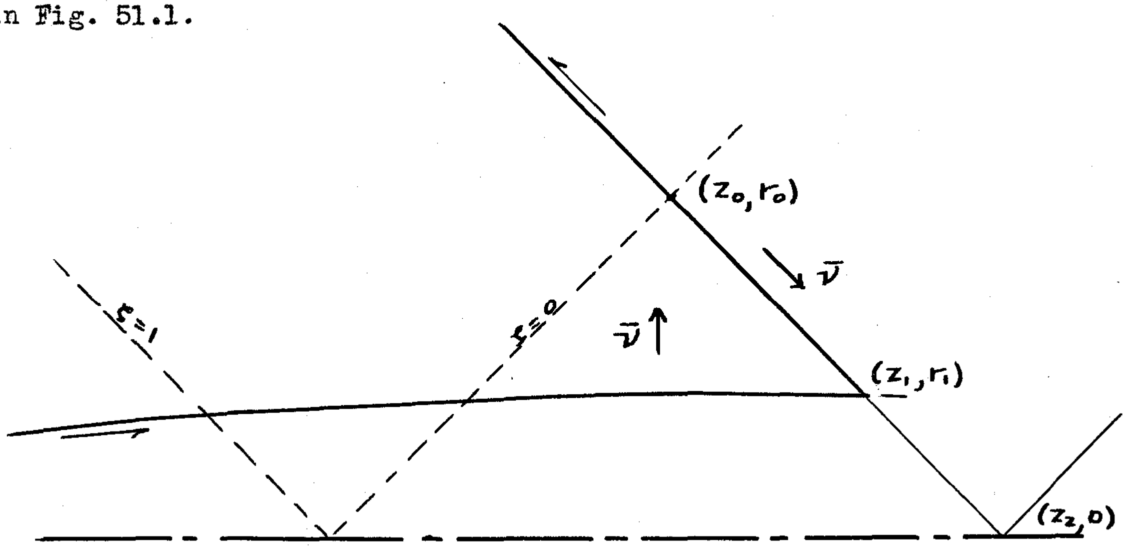


Fig. 51.1. Riemann Contour for Exterior Field

The first part of the contour follows the body and is sufficiently close to a line for which r is constant that the conormal may be taken in the r direction. From the Riemann method the value of U at the point (z_1, r_1) is given by

$$U_1 = \int_{-\infty}^{z_1}\left(U\frac{\partial V}{\partial r} - V\frac{\partial U}{\partial r}\right)dz \quad . \tag{51.1}$$

With the substitution (50.1) and the further substitution

$$V = \sqrt{r}\,\Psi \tag{51.2}$$

equation (51.1) may be expressed

$$\sqrt{r_1}\,\Phi_1 = \int_{-\infty}^{z_1}\left(\Phi\frac{\partial \Psi}{\partial r} - \Psi\frac{\partial \Phi}{\partial r}\right)r\,dz \quad . \tag{51.3}$$

This equation is an integral equation for Φ with the nucleus

$$r\Psi_r = \sqrt{r}\left(V'\zeta_r - \frac{1}{2r}V\right) . \tag{51.4}$$

This nucleus has different forms dependent upon the relative values of r_o and r_1.

The coordinates of the fixed points satisfy the condition

$$z_o + r_o = z_1 + r_1 = z_2 \quad . \tag{51.5}$$

The parameter λ is defined by the equation

$$\begin{aligned}\lambda &= (z_1 - r_1) - (z_o - r_o) \\ &= 2(r_o - r_1) \quad .\end{aligned} \tag{51.6}$$

Along the contour of the body $(r - r_1)$ is small in magnitude compared with

$(z - z_1)$ and the variable ζ may be expressed along this line from equation (50.11) as

$$\zeta = \frac{(z-z_1)(z-z_1+\lambda)}{2r\ (2r_1+\lambda)} \tag{51.7}$$

and its r derivative

$$r\zeta_r = \frac{\lambda}{2(2r_1+\lambda)} - \zeta \tag{51.8}$$

Two particular cases are of interest. These are $\lambda = 0$ for which

$$\zeta = \frac{(z-z_1)^2}{4rr_1} \tag{51.9a}$$

$$r\zeta_r = -\zeta \tag{51.9b}$$

and $\lambda = \infty$ for which

$$\zeta = \frac{z-z_1}{2r} \tag{51.10a}$$

$$r\zeta_r = \frac{1}{2} - \zeta \tag{51.10b}$$

Any value of the parameter λ may be used in the integral equation except the value $-2r_1$. The value $\lambda = \infty$ has no singularity in the finite field of integration and gives an integral equation for m = 0 similar to that of Lighthill (34). Where the path of integration crosses $\zeta = 1$ a Cauchy principal value must be taken across the singularity.

For the interior flow in a ducted body a different contour is necessary. This contour is illustrated in Fig. 51.2.

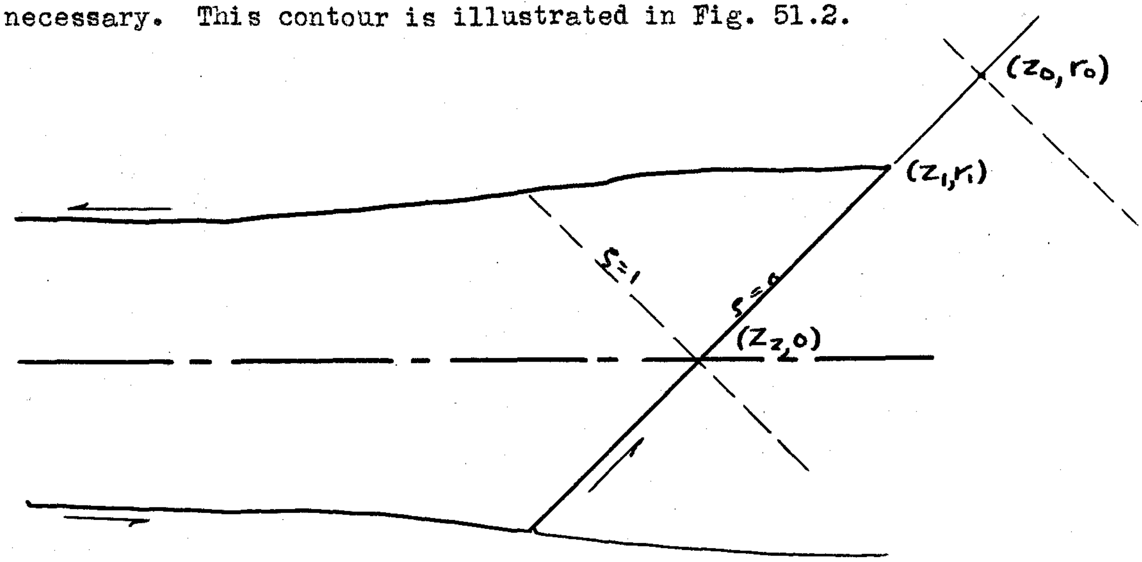


Fig. 51.2. Riemann Contour for Interior Field

In this case the characteristic belongs to the other family and equations (51.5) to (51.10b) may be retained with a change in sign of all the z terms. For this case the integration across the $\zeta = 1$ singularity cannot be avoided and the singularity has a physical significance. The integration passes along the contour of the duct twice, once with r negative and once with r positive. The solution is symmetric with respect to r for m even and is antisymmetric for m odd. Hence these two integrals may be combined.

These solution methods have not yet been developed in detail. It is hoped that analytic solutions to the Volterra integral equations for a cylinder r = constant may eventually be found. These would extend the solution of Lighthill and permit an attack on more general problems.

52. Wing-Body Interaction. A fundamental problem in the general question of wing-body interaction is that of the influence of a cylinder with its axis in the flow direction upon the disturbance field of a source or lift element. With the solution to this problem more general cases for which the planar problem is one of the first kind may be obtained by superposition.

One method would be to expand the flow field from the source or lift element by Fourier series around the axis of the cylinder and satisfy the boundary condition $u = 0$ on the cylinder for each value of m. This would require either the analytic solutions to the integral equations of the previous section or an axial superposition of solutions by Fourier integrals according to the method of the separation of the axial variable.

The influence of a cylinder on the flow field of an axial sinusoidal distribution may be obtained readily. The geometry is depicted in Fig. 52.1.

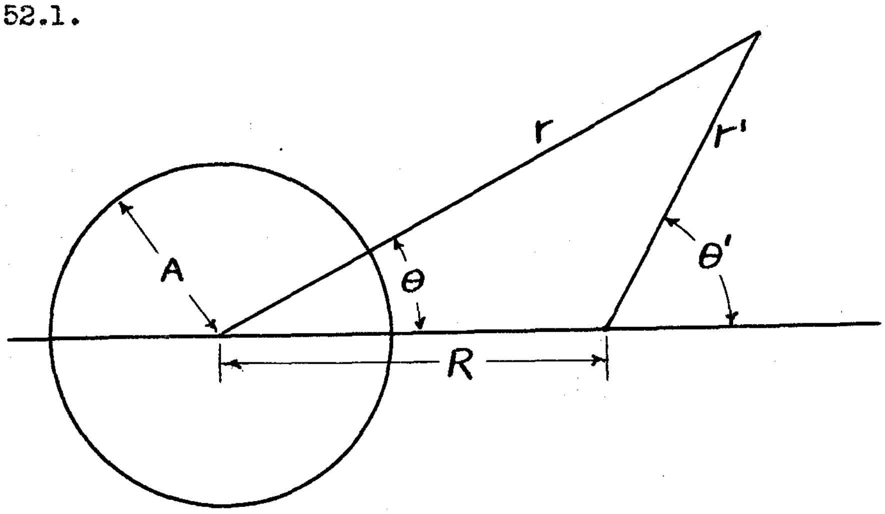


Fig. 52.1. Cylinder and Sinusoidal Source

The element lies at a distance R from the axis of the cylinder and has a wave number s. The cylinder has a radius A. The velocity potential function without the cylinder may be expressed from the results of section 12 for a general value of m

$$\phi = e^{isz}\bar{\Phi} = e^{isz}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, m\theta'\right\} R(r') \tag{52.1}$$

the function R being a Bessel function

$$R(r') = H_m^{(2)}(sr') = J_m(sr') - i\,Y_m(sr'). \tag{52.2}$$

The addition theorem for the Bessel function (56) may be expressed for this case

$$\begin{aligned}\bar{\Phi}_o &= H_m^{(2)}(sr')\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, m\theta'\right\}\\ &= \sum_{n=-\infty}^{\infty} H_{n-m}^{(2)}(sR)\,J_n(sr)\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right\}, \quad r<R\end{aligned} \tag{52.3a}$$

$$= \sum_{n=-\infty}^{\infty} J_{n-m}(sR)\,H_n^{(2)}(sr)\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right\}, \quad r>R. \tag{52.3b}$$

An additional velocity potential caused by the cylinder is needed to satisfy the u = 0 boundary condition on the cylinder

$$\Delta\bar{\Phi} = \sum_{n=-\infty}^{\infty} H_{n-m}^{(2)}(sR)\frac{J_n'(sA)}{H_n^{(2)\prime}(sA)}\, H_n^{(2)}(sr)\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right\} \tag{52.4}$$

For the region r > R the complete solution may be expressed using (52.1)

$$\Phi = \sum_{n=-\infty}^{\infty} J_{n-m}(sR)\left\{1 - \frac{\dfrac{Y_{n-m}(sR)}{J_{n-m}(sR)} + i}{\dfrac{Y_n'(sA)}{J_n'(sA)} + i}\right\} H_n^{(2)}(sr)\left\{\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right\} \tag{52.5}$$

A sinusoidal source is given by the case m = 0 and the cosine terms are utilized. A lift element normal to the plane including the element and the axis is given by the case m = 1 with the sine terms.

The problem of combining these solutions to represent a wing involves integration with respect to s and R. It may be possible to obtain the effect of the cylinder on the wave drag by simpler integrations.

X. General Comments

53. Evaluation of the Theory. The conditions of applicability for the linearized theory which were discussed in section 4 may be restated here. These conditions are:

a. The inclination of the body surface α at every point must be small. This limits the theory to bodies which have surfaces inclined at not more than about 5 degrees to the flow direction.

b. The transonic range must be avoided. This requires that the quantity

$$K = \frac{\alpha}{(M-1)^{3/2}}\left(\frac{\gamma+1}{2}\right) \tag{53.1}$$

be small. This further limits the permissible surface inclination where the Mach number is close to unity.

c. The hypersonic range must be avoided. This requires that the inclination of the surface times the Mach number αM be small. This limits the permissible inclination where the Mach number is large. These considerations indicate that the theory of linearized supersonic flow is of sufficient accuracy for quantitative application only within a quite limited range of Mach numbers and body shapes. The results should be valuable qualitatively over a larger range.

The principal experimental results that may be compared with the linearized theory are not presented here as they are of a classified nature.

However, it may be said that the experimental verification so far available is satisfactory and within the expected accuracy.

The results of the linearized theory serve as a guide in the design of bodies which must be capable of supersonic flight. The principal conclusions are that such bodies must have small fineness ratios and should be arranged in such a way that the interference effects will not be too detrimental. As the transonic range is approached a rough estimate of the behavior of a system may be obtained from the linearized considerations by letting M approach 1. The concept of supersonic and subsonic leading edges discussed in section 24 controls the leading edge shape in design.

Of the new concepts introduced in this thesis the most important include the concept of the application of the Kutta condition to supersonic planforms, the concept of the reversed flow theorems and the concept of wave drag and vortex drag. Amont the new methods introduced in the thesis are the Legendre polynomial method for two-dimensional flow, the application of the Hadamard method to planar systems, the procedure for obtaining solutions in conical coordinates, the method for the calculation of the wave drag, and the application of the Riemann method to laterally periodic systems and to solutions for the flow about bodies of revolution.

54. The Field for Future Study. Although many of the methods discussed in this thesis need further refinement for detailed application to specific problems, the theory presented is reasonably complete except with regard to two major problems. One of these is the problem of the second kind for antisymmetric planar systems which was termed the lift problem and treated

in an introductory fashion in chapter V. The other is the problem of the flow about general nonplanar systems. Some specialized cases of nonplanar systems are discussed in sections 43 and 52. The general case probably requires a three-dimensional method of characteristics somewhat analogous to the method for bodies of revolution.

In order to evaluate properly the linearized theory further investigations toward the nonlinear theory are desirable. Of particular advantage would be a second approximation theory for planar systems and for bodies of revolution analogous to the second approximation of Busemann for two-dimensional problems. In analogy with Busemann's results it is probable that these theories would be independent of the existence of shock waves. Real fluid effects upon the linearized solutions should be investigated, for example the effect of the boundary layer on the pressure distribution of a body and the effect of shed vortices on a body of revolution at an angle of attack.

Many of the methods presented in the thesis may be applied to nonstationary linearized problems. Of these the principal problems that may be attacked are those which are periodic in time. Such solutions may be applied to flutter problems. It is expected that important differences would exist between the stationary and the periodic solutions as in the phenomena of drag concentration.

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