Spin geometry and quantum diffusion

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Sitanshu Gakkhar

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Sitanshu Gakkhar
ORCID: 0000-0002-2895-9921

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ABSTRACT

This thesis studies diffusion processes on spinor endomorphism algebras. The spinor and connection laplacian generated heat semigroups are shown to quantum dynamical semigroups, and after spectral truncation the existence of Evans-Hudson flows is established. The vacuum state expectation of the process is related to spectral action principle in noncommutative geometry. Examples where the flow is proven to exist for untruncated laplacians are given. Convergence of finite dimensional approximations, through discretization and truncation, to spectral triples encoding Riemannian geometry and their statespaces as quantum metric spaces is also considered.

PUBLISHED CONTENT AND CONTRIBUTIONS

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Sitanshu Gakkhar participated in the conception, research, and authoring of the above article.

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Chapter 1

QUANTUM DIFFUSION AND SPIN GEOMETRY

1.1 Introduction

Quantum diffusion is an operator-valued stochastic process specialized to live on the Fock space with the noncommutative probability model fixed to quantum probability. It's associated with a dynamical semigroup on a C^* (or a von Neumann algebra). The initial theory was developed for norm-continuous operator semigroups describing quantum dynamics and the form for generators of such semigroups, which are bounded, is characterized by the work of Gorini, Kossakowski, Sudarshan, and Lindblad. At the same time, the diffusion generated by the unbounded laplacian – Brownian motion, and its associated heat semigroup $e^{-t\triangle}$ on a Riemannian manifold, has deep interactions with the geometry.

This thesis is focused on the intersection of the two: the diffusion processes generated by unbounded elliptic noncommutative operators on endomorphism bundles, specifically, the diffusion generated by the Dirac and connection laplacian on the spinor bundle endomorphism algebra. The work is motivated by the spectral action principle that underlies the spectral standard model where the trace of the spinor heat kernel, defined as the spectral action, plays a central role. The geometry that arises in this context is defined as the almost-commutative geometry, and more generally, it provides an implementation of the program of quantum Riemannian geometry toward noncommutative gauge theory. The Dirac heat kernel and diffusion are studied from this almost-commutative geometry perspective. The related question in discrete differential/finite-noncommutative geometry of how the noncommutative geometric perspective can be discretized is also considered.

Specialized to commutative geometry, the connection between spinor laplacian and Einstein-Hilbert action through the spectral action is well known; the new contribution is the probabilistic formulation. This approach is suggested by observing that the trace of the spinor heat kernel for Robertson-Walker cosmologies can be evaluated by using Brownian bridge integrals. The follow-up question is to see if this can be carried out for almost commutative geometry in general. Because of the noncommutative character, the process now must be a noncommutative diffusion, with quantum probability providing a natural model set on the boson Fock space.

The answer is in the affirmative — the spectral action can be recovered from quantum stochastic flows on the underlying, possibly noncommutative, geometry. This provides an interpretation of the spectral action as being realized by random fluctuations acting on the spinor bundle; more precisely, the spectral action arises as the solution to a quantum stochastic differential equation which can be constructed algorithmically, and carries at least a superficial similarity to the stochastic quantization program.

The epilogue to this work remarks briefly on the possibilities of operator algebraic and geometric methods in studying stochastic dynamics on Hilbert spaces, particularly in the quantum information context, and considers possible future headings.

Outline: The relevant background is introduced as needed, with chapters being self-contained but incremental. The following chapter starts by introducing Dirac bundles and examining the various heat semigroups associated with the Dirac laplacian. The spectral action is shown to be embedded inside the action of the noncommutative laplacian on the spinor endomorphism algebra and can be approximated by the quantum stochastic flow generated by the spectrally truncated laplacian. After this, aspects of quantum stochastic analysis and differential equations are introduced in chapter 3. In chapter 4 infinite dimensional examples where conditions for existence of diffusion generated by the untruncated flow can be established. This includes noncommutative laplacian, the Laplace-Beltrami operator generated diffusion on the compact manifolds where laplacian eigenfunctions follow a growth condition, and the Dirac laplacian generated diffusion in almost commutative geometry over reductive homogeneous spaces. Some tangential ideas are also explored. Finally, before leading into the epilogue, the question of finite approximations and compressions of canonical spectral triples is addressed in chapter 5.

Chapter 2

DIFFUSION ON SPINOR ENDOMORPHISM ALGEBRAS

2.1 Introduction

In this chapter the heat semigroups associated with endomorphism algebras of spinor bundles are studied. These provide the natural generalization of the heat semigroup of a smooth manifold since by Ćaćić [16]'s characterization, spectral triples describing almost commutative geometry are realized as spinor endomorphism sub-bundles. The semigroups are shown to be quantum dynamical semigroups. The existence of a quantum stochastic flows/dilations of Evans-Hudson type is established for spectral truncations of the generators. This question of the existence of such flows — which can be viewed as diffusion on the spectral triple — and realizing an appropriate expectation as a spectral action is in the same vein of results as [21, 32] where von Neumann entropy and the average energy of the Gibbs state are expressed as spectral actions; both are based on fermionic second quantization. The flows considered here, however, live on bosonic Fock space and what they recovered from the dilation on the Fock is the geometry itself. While only the symmetric Fock space stochastic calculus is used here, an antisymmetric Fock space theory is known (a unified construction was given by [45]) and similar constructions on full Fock space have been studied in free probability (see, for instance, [47], and [14] for free stochastic quantization). Since the symmetric Fock space is isomorphic to Wiener spaces by the Wiener-Ito-Segal isomorphism, this approach and the general question of a deeper connection between noncommutative probability and noncommutative geometry, as well as connections to the stochastic quantization program, are suggested by the Brownian bridge integral expansions for the spectral action[20, 34]. In Euclidean fermionic quantum field theory context, stochastic quantization considered by [3] explores related ideas on Grassmann algebras; see also [22]. As an application, the spectral action for an arbitrary compact Riemannian spin manifold is realized from the quantum stochastic flow for the noncommutative laplacian.

Organization

Section 2.2 introduces the quantum dynamical semigroups, dilations, and relevant background; section 2.3 introduces Ćaćić's results and gives the characterization in terms of spinor bundles. The existence of quantum stochastic dilations and flows for associated qsde is considered in 2.4. The results are obtained by first showing that the laplacians define C^*-Dirichlet forms and, therefore, generates a quantum dynamical semigroup, which can be embedded in a conservative quantum dynamical semigroup for which dilations exist after spectral truncation. This is enough to approximate the spectral action up to an arbitrary cutoff as considered in 2.6. Section 2.7 considers heat semigroups on the product almost-commutative spectral triples and general almost-commutative spectral triples, and remarks on the C^*-bundles considered by [28, 24].

Some notational conventions: for Hilbert space, H, $\mathcal{B}(\mathsf{H})$ will denote bounded operators on H, $\mathcal{K}(\mathsf{H})$ compact operators and $\mathcal{H}^2(\mathsf{H})$ the Hilbert-Schmidt operators. i will denote $\sqrt{-1}$. For $m, n \in \mathbb{N}$, $[n]$, $[m:n]$ will denote $\{1, 2 \ldots n\}$ and $\{m, m+1 \ldots n\}$ respectively. $\textsc{Lie}[G]$ will denote the Lie algebra of Lie group G. (M, g) will denote a smooth manifold M with a Riemannian metric g. After fixing a local orthonormal frame, e_j, the connection ∇_j will be used interchangeably for ∇_{e_j}. On bundles carrying Clifford multiplication, $\cdot$ will denote Clifford multiplication; $\cdot$ will be suppressed when Clifford multiplication is clear from context. Following [49], by a Riemannian connection we mean a metric connection not necessarily torsion-free; the canonical Riemannian connection is taken as the torsion-free Riemannian connection. All manifolds will be compact spin manifolds since that is the object in the reconstruction theorems from noncommutative geometry. The boundary is assumed to be empty; this is needed as the Dirac operator may not be symmetric otherwise (see, for instance, [49, eq II.5.7]). For some specialized settings, the assumption of even dimensionality is made. $\mathrm{Cl}(E, q)$ will denote the Clifford algebra (bundle) over the vector space (vector bundle) E with quadratic form q. For Riemannian manifold (M, g), $\mathrm{Cl}(M) = \sqcup_{m \in M} \mathrm{Cl}((T^*_m M), -g)$. $\not{D}$ will denote a geometric Dirac operator, that is, Dirac operator associated to a Clifford connection, with $\not{D}^2$ the associated geometric Dirac laplacian, and D will denote a Dirac-type operator (and its lapacian D^2) which may not be associated to a Clifford connection.

2.2 Quantum stochastic flows

Recall that a $*$-algebra is an algebra endowed with an involution $*$, while a C^*-algebra is a norm-closed $*$-algebra where the norm satisfies the C^*-identity, $\|a^* a\| = \|a\|^2$. Note that if $\mathcal{A}$ is a unital C^*-algebra $\mathcal{A}$, then $\textsc{Mat}_n(\mathcal{A}) \cong \mathcal{A} \otimes \textsc{Mat}_n(\mathbb{C})$. Working with quantum dynamical systems requires a stronger notion of positivity: complete positivity.

Definition 2.2.1. For unital C^*-algebras, $\mathcal{A}_1, \mathcal{A}_2$, with positive cones denoted $(\mathcal{A}_1)_+, (\mathcal{A}_2)_+$,

1. A linear map $T : \mathcal{A}_1 \to \mathcal{A}_2$ is positive if $T((\mathcal{A}_1)_+) \subset (\mathcal{A}_2)_+$.

2. T is completely positive if for all $n \in \mathbb{N}$, $T_n := T \otimes 1_n : \textsc{Mat}_n(\mathcal{A}_1) \cong \mathcal{A}_1 \otimes \textsc{Mat}_n(\mathbb{C}) \to \mathcal{A}_2 \otimes \textsc{Mat}_n(\mathbb{C}) \cong \textsc{Mat}_n(\mathcal{A}_2)$, $T_n([a_{ij}]) = [T(a_{ij})]$, is positive.

Definition 2.2.2 (Quantum Dynamical Semigroups)**.** A semigroup $(T_t)_{t\geq 0}$ on a C^*-algebra $\mathcal{A}$ is strongly continuous if $\lim_{t\to t_0} T_t x = T_{t_0} x$ for all x, t_0. The semigroup is conservative if for all t, $T_t(1) = 1$; equivalently $\mathcal{L}(1) = 0$ for the generator $\mathcal{L}$ of T_t. A quantum dynamical semigroup on a C^*-algebra is a strongly continuous semigroup T_t such that each $T_t : \mathcal{A} \to \mathcal{A}$ is contractive and completely positive. On a von Neumann algebra $\mathcal{M}$, a quantum dynamical semigroup is a semigroup T_t of completely positive, contractive maps such that T_t is normal for each t.

For a C^*or von Neumann algebra, $\mathcal{A}$, $\mathcal{A}''$ will denote the bicommutant. $\mathcal{A}''$ is a von Neumann algebra.

Definition 2.2.3. A conditional expectation is a linear map, $\mathbb{E} : \mathcal{N} \to \mathcal{M}$, between $*$-algebras $\mathcal{M}, \mathcal{N}$, satisfying $\mathcal{M} \subset \mathcal{N}, \mathbb{E}[1] = 1$ and for any $M_i \in \mathcal{M}, N \in \mathcal{N}$ $\mathbb{E}[M_1 N M_2] = M_1 \mathbb{E}[N] M_2$.

Definition 2.2.4 (Stochastic dilation)**.** For quantum dynamical semigroup $(T_t), t \geq 0$ on a C^* $\mathcal{M}$, a (quantum) stochastic dilation is a family of $*$-homomorphisms, $j_t : \mathcal{M} \to \mathcal{N}$, where $\mathcal{N}$ is a $*$-algebra with conditional expectation $\mathbb{E}_0 : \mathcal{N} \to \mathcal{M}$ satisfying $T_t = \mathbb{E}_0[j_t]$.

We will consider stochastic dilations on the Fock space.

Definition 2.2.5. For a Hilbert space H, the free Fock space, $\Gamma^f(\mathsf{H})$ is the sum of the symmetric and antisymmetric Fock spaces, $\Gamma^s(\mathsf{H}), \Gamma^a(\mathsf{H}), \Gamma^f(\mathsf{H}) = \Gamma^a(\mathsf{H}) \oplus \Gamma^s(\mathsf{H})$, where $\Gamma^s(\mathsf{H}), \Gamma^a(\mathsf{H})$ are defined by $\oplus_{\mathbb{Z}\geq 0}\mathsf{H}^{\circ n}$, $\circ$ being the symmetric or tensor product for symmetric Fock space, antisymmetric tensor product for antisymmetric Fock space, and free tensor product for the free Fock space.

The object of interest throughout this work will be the symmetric tensor product (denoted by $\otimes$ again) unless otherwise specified; we will also denote symmetric

Fock space $\Gamma^s(\mathsf{H})$ by $\Gamma(\mathsf{H})$ when clear from the context. The symmetrization operator defines the map from free to symmetric Fock space, $\Gamma^f(\mathsf{H}) \longrightarrow \Gamma^s(\mathsf{H})$ by $\mathrm{Symm}(\otimes_{i\in[n]} g_i) = 1/(n-1)! \sum_{\sigma\in S_n} \otimes_{i\in[n]} g_{\sigma(i)}$. For a subspace $\mathsf{V} \subset \mathsf{H}$, $\mathrm{E}(\mathsf{V}) \subset \Gamma^s(\mathsf{H})$ denotes the $\mathbb{C}$-linear span of exponential vectors $\mathrm{E}(v) = \oplus_{k\in\mathbb{N}} v^k/\sqrt{k!}, v \in \mathsf{V}$. There's an inner product on the $\Gamma(\mathsf{H})$ induced by the inner product on H, $\langle \mathrm{E}(u), \mathrm{E}(v)\rangle = \exp\langle u, v\rangle$.

Example 2.2.6 (Feynman-Kac formula, Brownian motion and stochastic dilation)**.** Viewing Brownian motion on (M, g) as a diffusion generated by the Laplace-Beltrami operator, it's noted that the Feynman-Kac formula for a Riemannian manifold[56, Thm 3.2], (M, g), for the operator $H := \frac{1}{2}\triangle_{C(M)} + V, u \in C^4(M), V \in C(M)$, with Laplace-Beltrami operator, $\triangle_{C(M)}$, acting on the $C^2(M)$ gives

$$(e^{-tH}u)(x) = \int_{W(M)} \frac{e^{\int_0^t V(\omega(s)) - 1/6\cdot\kappa_M(\omega(s))ds} u(x)\, \mathrm{d}\, W_M^x(d\omega)}{N(u, \kappa_M, \mathrm{d}\, W_M^x(d\omega))}$$

where $\mathrm{d}\, W_M^x(d\omega)$ denotes the Wiener measure on $C(M)$, $u \in C^4(M)$ and κ_M is the scalar curvature of M and $N(u, \kappa_M, \mathrm{d}\, W_M^x(d\omega))$ a normalization depending on κ_M, u and $\mathrm{d}\, W_M^x(d\omega)$. This can be thought of as a stochastic dilation of heat semigroup on $C^\infty(M)$ to the Wiener space, $W(M)$, on M, the integral with respect to the Wiener measure playing the role of the conditional expectation.

Quantum stochastic dilation of Evans-Hudson type

On a smooth manifold, M, a homogeneous flow is a smooth map $\phi : \mathbb{R}_{\geq 0} \times M \longrightarrow M$, $\phi_t(m) := \phi(t, m)$, satisfying $\phi(t+s, m) = \phi(s, \phi(t, m))$, $\phi(0, m) = m$. The flow induces a 1-parameter semigroup, $(j_t)_{t\geq 0} : C^\infty(M) \longrightarrow C^\infty(M)$, $j_t(f) := f \circ \phi_t^{-1}$ with the infinitesimal generator $\mathcal{L}$ following the differential equation[44],

$$\frac{d}{dt} j_t(f) = j_t(\mathcal{L}(f)), \text{ with } j_0(f) = f, \; \mathcal{L}(f) = \left.\frac{d}{dt}\right|_{t=0} j_t(f), \; f \in C^\infty(M) \quad (2.1)$$

The classical stochastic flow can be viewed as a stochastic process ψ_t taking values in diffeomorphism group of M which satisfies the flow property almost surely (see, for instance, [48, Ch 3]). Now solutions to stochastic differential equations (sde) on manifolds generate stochastic flows, the stochastic version of flow equation is obtained by introducing Wiener process terms into eq 2.1 yields

$$\frac{d}{dt} j_t(f) = j_t(\mathcal{L}(f)) + \sum_{j\in[n]} j_t(b_j(f)) dB_j$$

for linear maps b_k, and components B_j of n-dimensional Brownian motion B on M with sample space Ω. Algebraically, j_t, are now $*$-algebra homomorphisms, $j_t : \mathcal{B}(M \times \Omega) \supset C^\infty(M) \to \mathcal{B}(M \times \Omega)$ for the space of bounded measurable functions, $\mathcal{B}(M \times \Omega)$, on $M \times \Omega$. Note that $C^\infty(M)$ is embedded in $\mathcal{B}(M \times \Omega)$.

Formulated as integral equations, the quantum analog of this sde can be defined on the Fock space. For a finite dimensional Hilbert space V, set $\mathsf{H} = L^2(\mathbb{R}_{\geq 0}, \mathsf{V}) := L^2(\mathbb{R}_{\geq 0}) \otimes \mathsf{V}$. H decomposes as $\mathsf{H} = \mathsf{H}_t \oplus \mathsf{H}^t$, where $\mathsf{H}_t = L^2([0,t)) \otimes \mathsf{V}, \mathsf{H}^t = L^2([t,\infty)) \otimes \mathsf{V}$. On the Fock space, $\Gamma(\mathsf{H}) = \Gamma(\mathsf{H}_t) \otimes_{\text{alg}} \Gamma(\mathsf{H}^t)$, given an "initial" Hilbert space H_0, set

$$\tilde{\mathsf{H}}_t = \mathsf{H}_0 \otimes \Gamma(\mathsf{H}_t), \tilde{\mathsf{H}}^t = \mathsf{H}_0 \otimes \Gamma(\mathsf{H}^t), \tilde{\mathsf{H}} = \mathsf{H}_0 \otimes \Gamma(\mathsf{H})$$

then for a class of $\mathbb{R}_{\geq 0}$-indexed operator families on $\tilde{\mathsf{H}}$, Λ^i_j, $i, j \in [0 : \dim \mathsf{V}_0]$, called the fundamental processes (or quantum noises, which corresponds to the annihilation, creation, and conservation processes on the Fock space), the quantum stochastic integral $\int_0^t \sum_{i,j} E^j_i d\Lambda^i_j$ can be defined for processes $(E^j_i)_{t \in \mathbb{R}_{\geq 0}}$ that are regular (i.e. the map $t \to (E^j_i)_t(u_0 \otimes \mathrm{E}u)$ is continuous with a growth condition on $\|(E^j_i)_t(u_0 \otimes \mathrm{E}u)\|$) and each $(E^j_i)_t$ is adapted where a process $X_t : \tilde{\mathsf{H}} \to \tilde{\mathsf{H}}$ is adapted if there exists $Y_t : \mathsf{H}_0 \otimes \mathrm{E}(\mathsf{H}_t) \to \mathsf{H}_0 \otimes \Gamma(\mathsf{H}_t)$, so that $X_t = Y_t \otimes 1_{\Gamma(\mathsf{H}^t)}$, that is, X_t does not look into the future — the same as the classical notion of adaptedness. For brevity, the details of quantum stochastic integrals are deferred to in the following chapter 3.

The stochastic calculus can be developed on operator algebras similarly to Hilbert spaces[62, Ch 5] and the stochastic flow can be defined by extending the classical picture: for a dense $*$-algebra $\mathcal{A}_0 \subset \mathcal{A}$ with $\mathcal{A} \subset \mathcal{B}(\mathsf{H}_0)$ unital, the quantum stochastic flow $(j_t)_{t \geq 0}$ is a family of injective $*$-homomorphism, $j_t : \mathcal{A}_0 \to \mathcal{B}(\mathsf{H})$, such that for all $a \in \mathcal{A}$, each $j_t(a)$ is an adapted process and there exists $\{\lambda^i_j : i, j \in [0 : \dim \mathsf{V}]\}$, called the structure maps, with

$$j_t(a) = a \otimes \mathbf{1} + \int_0^t \sum_{i,j} j_t(\lambda^i_j(a)) d\Lambda^j_i$$

Equivalently, in differential form, $dj_t(a) = \sum_{i,j} j_t(\lambda^i_j(a)) d\Lambda^j_i$ with $j_0 = \mathbf{1}$. Flows of this form arising as solutions to a quantum stochastic differential equation, with j_t satisfying an additional dilation property, are called Evans-Hudson flows[58, § 27, 28] and generalize Markov processes to operator algebras[1]. In particular,

[1]From [57] note, any Markov chain on countable state space can be realized as Evans-Hudson flow.

Brownian motion on $\mathbb{R}$ can be realized as Evans-Hudson dilation on the Fock space $\Gamma(L^2(\mathbb{R}_{\geq 0}))$ by specializing to $\mathcal{A} = L^\infty(\mathbb{R})$ viewed as operators on $\mathsf{H} = L^2(\mathbb{R})$, V fixed as trivial and using the Fock space-Wiener space dictionary provided by the Wiener-Ito-Segal isomorphism (more detail in provided in the next chapter).

Definition 2.2.7. Evans-Hudson dilation[62] For a conservative quantum dynamical semigroup $(T_t)_{t\geq 0}$ with generator $\mathcal{L}$ on C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathsf{H})$, a family of $*$-homomorphisms, $(j_t)_{t\geq 0} : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_{\geq 0}) \otimes \mathsf{V}))$ satisfying the following.

- There exist maps $J_t : \mathcal{A} \otimes_{\text{alg}} \mathrm{E}(L^2(\mathbb{R}_{\geq 0}) \otimes \mathsf{V}) \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_{\geq 0}) \otimes \mathsf{V}))$, $J_t(a \otimes e(f))u := j_t(a)(ue(f))$ such that for an ultra-weakly dense subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, $\mathrm{Dom}(\mathcal{L}) \subset \mathcal{A}_0$, on $\mathcal{A}_0 \otimes L^2(\mathbb{R}_{\geq 0}) \otimes \mathsf{V}$ the Evans-Hudson flow qsde
$$dJ_t = J_t(a_\delta(dt) + a_\delta^\dagger(dt) + \Lambda_\sigma(dt) + \mathbf{1}_\mathcal{L}(dt)),\ J_0 = \mathbf{1} \tag{2.2}$$
holds, where $a_\delta, a_\delta^\dagger, \Lambda_\sigma, \mathbf{1}_\mathcal{L}$ are the structure maps (see Chapter 3); J_t as a quantum stochastic process is regular and adapted.

- j_t is a dilation of T_t in the following sense: for all $u, v \in \mathsf{H}, a \in \mathcal{A}$,
$$\langle v\mathrm{E}(0), j_t(a)u\mathrm{E}(0)\rangle = \langle v, T_t(a)u\rangle \tag{2.3}$$
where $\mathrm{E}(0)$ denotes the Fock space vacuum.

The solution J_t to Evans-Hudson flow qsde, equivalently j_t, is defined as the Evans-Hudson flow. The maps $a_\delta, a_\delta^\dagger, \Lambda_\sigma, \mathbf{1}_\mathcal{L}$ and the noise space V are obtained from the structure theory of the generator $\mathcal{L}$. The computation of structure maps for Laplace-Beltrami operator generator flow is considered in chapter 4 and follows the same scheme generally. If the generator is bounded, for example, a spectral truncation of an unbounded generator, then after obtaining the structure maps, the existence (and uniqueness) of the Evans-Hudson flow follows from standard theory (see, for instance, [62, 11]). The unbounded case requires more machinery.

2.3 Almost-commutative spectral triples as spinor bundles

A spectral triple is three basic pieces of data, $(\mathcal{A}, \mathsf{H}, D)$, where D is symmetric operator on the Hilbert space H, and a *-algebra of bounded operators on H, $\mathcal{A} \subset \mathcal{B}(\mathsf{H})$. The operator D is allowed to be self-adjoint and unbounded but with $[D, a]$ bounded for all $a \in \mathcal{A}$. A compact Riemannian spin manifold (M, g) can be characterized by the canonical spectral triple, $\mathfrak{A}_M := (C^\infty(M), L^2(\mathcal{S}), D_M; J_M, \gamma_M)$ where $\mathcal{S}$ is the spinor bundle, $C^\infty(M)$ is the $*$-algebra of smooth functions interpret as operators acting on $L^2(\mathcal{S})$ by multiplication, and D_M is the Dirac operator associated with the Levi-Civita connection on the spinor bundle; the data of a spectral triple has been supplemented with a $\mathbb{Z}_2$ grading operator γ_M on H and an anti-unitary operator $J : \mathsf{H} \to \mathsf{H}$, called the real structure, which makes H an $\mathcal{A} - \mathcal{A}$ bimodule from a left $\mathcal{A}$-module. Such spectral triples can be characterized abstractly; Connes reconstruction theorem recovers the Riemannian spin structure from the abstract spectral triples[41, Thm 11.2].

A finite noncommutative space is the finite spectral triple, $\mathfrak{A}_F := (\mathcal{A}_F, H_F, D_F)$, with $\dim H_F$ finite. This is supplemented with a real structure and a grading, (J_F, γ_F). A product almost-commutative spectral triple is the globally trivial bundle,

$$M \times F := (C^\infty(M) \otimes A_F, L^2(M, S \otimes H_F), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)$$

Ćaćić [16] expands the definition of product almost-commutative spectral triples to include non-trivial algebra bundles over the base space. This is formalized without appeal to the explicit product structure as an abstract almost-commutative spectral triple:

Definition 2.3.1. ([16, Def 2.16]) A spectral triple $(\mathcal{A}, \mathsf{H}, D)$, $\mathcal{B} \subset \mathcal{A}$ a central, unital $*$-subalgebra is an abstract almost-commutative spectral triple over the base $\mathcal{B}$ if $(\mathcal{B}, \mathsf{H}, D)$ is a commutative spectral triple of Dirac type[16], and for all $a \in \mathcal{A}$, $[D, a]^2 \in \mathcal{A}$, additionally

1. For all $a \in \mathcal{A}, b \in \mathcal{B}$, $[[D, b], a] = 0$.
2. $\mathcal{A}$ is an even finitely generated projective $\mathcal{B}$-module and a $*$-subalgebra of the algebra $\mathrm{End}_{\mathcal{B}+i\mathcal{B}}(\mathsf{H}_\infty)$ where $\mathsf{H}_\infty = \cap_{k\in\mathbb{N}} \mathrm{Dom}\, D^k$.

From [18, lemma 4.2.4], the requirement that the spectral triple is of Dirac type can be dropped, since it follows from regularity conditions on the spectral triple (see [17] for the reconstruction theorem in more generality).

The concrete realization of the abstract almost-commutative spectral triple is constructed by appeal to Connes's reconstruction theorem[41, Ch 11], and the following global analytic equivalent formulation is obtained, and this is the formulation that we work with.

Definition 2.3.2. [16, Def 2.3] An almost-commutative spectral triple is a spectral triple of the form

$$(C^\infty(M,A), L^2(M,H), D_0)$$

for a compact oriented Riemannian manifold M, H a self-adjoint Clifford module bundle, A a real unital $*$-algebra sub-bundle of $\mathrm{End}^+_{\mathrm{Cl}(M)}(H)$, and D_0 is a symmetric Dirac-type operator on H, where $\mathrm{End}^+_{\mathrm{Cl}(M)}(H)$ are the even endomorphisms of H that supercommute with the Clifford action $c : T^*M \to \mathrm{End}(H)$ defined by D.

Remark 2.3.3. Recall that for a $\mathbb{Z}_2$ graded $\mathbb{K}$-algebra, $A = A^0 \oplus A^1$, with $A^i \cdot A^j \subset A^{i+j}$, the supercommutator $[\cdot,\cdot]_s$ is the map $[a^i, b^j]_s = a^i b^j - (-1)^{ij} b^j a^i$ for $a^i \in A^i, b^j \in B^j$. As the Clifford action, $c : T^*M \to \mathrm{End}(H)$ and $\mathrm{End}^+_{\mathrm{Cl}(M)}(H)$ consisting of even endomorphisms, $\phi \in \mathrm{End}^+_{\mathrm{Cl}(M)}(H)$ supercommute, $\phi \circ c - c \circ \phi = 0$.

Structure of Dirac bundles

The Clifford algebra $\mathrm{Cl}(V,Q)$ is the algebra generated over the vector space V by the relation $v^2 = Q(v)1$ where Q is a quadratic form on V. It satisfies the following universal property: any linear map $f : V \to \mathcal{A}$, V a vector space, $\mathcal{A}$ a unital associative $\mathbb{K}$-algebra, with $f(v) \cdot f(v) = Q(v)1$ extends uniquely to a $\mathbb{K}$-algebra homomorphism $\tilde{f} : \mathrm{Cl}(V,Q) \to \mathcal{A}$. $\mathrm{Cl}(V,Q)$ comes with a $\mathbb{Z}_2$ grading, $\chi(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k$, that yields the decomposition, $\mathrm{Cl}(V,Q) = \mathrm{Cl}(V,Q)^0 \oplus \mathrm{Cl}(V,Q)^1$. Specializing to $\mathbb{R}^n$, fix $Q_n = \sum_n x_i^2$, define $\mathrm{Cl}_n^+ = \mathrm{Cl}(\mathbb{R}^n, Q_n), \mathrm{Cl}_n^- = \mathrm{Cl}(\mathbb{R}^n, -Q_n)$ and $\mathrm{Cl}^n = \mathrm{Cl}(\mathbb{C}^n, Q_n)$ which is $\mathrm{Cl}_n^+ \otimes_\mathbb{R} \mathbb{C}, \mathrm{Cl}_n^- \otimes_\mathbb{R} \mathbb{C}$. The grading comes from the chirality operator γ_{n+1} on Cl_n is given by $(-\mathrm{i})^m e_1 \cdots e_n$ where e_i's generate Cl_n and $n = 2m$ if even and $n = 2m+1$ for odd. This can be carried over to a vector bundle as follows.

Definition 2.3.4. A Clifford structure on a vector bundle $E \to M$, is a bundle morphism $c : T^*M \to \mathrm{End}(E)$, $\{c(u), c(v)\} = -2g(u,v)1$. $c(v) \in \mathrm{End}(E)$ denotes the "Clifford multiplication by v", and the pair (E, c) is the Clifford bundle. The Clifford bundle $E \to M$ is $\mathbb{Z}_2$ graded if there's a decomposition $E = E^+ \oplus E^-$ such that $c(\alpha)$ for each $\alpha \in T^*M$ is an odd endomorphism: $c(\alpha)(\Gamma(E^\pm)) = \Gamma(E^\mp)$. A vector bundle with a Clifford structure is a Clifford module bundle.

Remark 2.3.5. Note that $c : \Omega^1(M) \to \Gamma(\mathrm{End}(E))$, and by the universal property of Clifford algebras c lifts to the action of the full Clifford algebra, $c : \mathrm{Cl}(M) \to \Gamma(\mathrm{End}(E))$ because $c : T^*M \to \Gamma(\mathrm{End}(E))$ satisfies $\{c(u), c(v)\} = -2g(u,v)$. So (c, E) is a representation of $\mathrm{Cl}(M)$.

On any Riemannian manifold (M, g), there exists a canonical Clifford bundle, $\mathrm{Cl}(T^*M, -g) := \mathrm{Cl}(M)$. A Clifford module bundle is any bundle that carries an action of the Clifford bundle. A Dirac bundle S over a (M, g) is a Clifford module bundle with a connection ∇^S that is compatible with the Clifford multiplication.

- For all $\sigma_i \in S_x, e \in T_xM, \|e\| = 1$, e acting on σ_i by Clifford multiplication, $\langle e \cdot \sigma_1, e \cdot \sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle$ (as $e^2 = -1$, this yields the skew-hermiticity, $\langle e \cdot \sigma_1, \sigma_2 \rangle = -\langle \sigma_1, e \cdot \sigma_2 \rangle$).
- $\nabla^S(\phi \cdot \sigma) = (\nabla^{\mathrm{Cl}(M)}\phi) \cdot \sigma + \phi \cdot \nabla^S \sigma$.

For clarity it is useful to separate out the algebraic Clifford structure from the geometric piece.

Definition 2.3.6. A Dirac bundle, (E, c, h, ∇, M, g), is a Clifford module bundle (E, c) over (M, g) with a hermitian metric h on E and Clifford connection, ∇, compatible with h such that for all $\alpha \in \Omega^1(M)$ the following holds:

- $c(\alpha) \in \mathrm{End}(E)$ is skew-Hermitian
- For $X \in \Gamma(TM), u \in \Gamma^\infty(E), \nabla^M$ the Levi-Civita connection on M, $\nabla_X(c(\alpha)(u)) = c(\nabla^M_X \alpha)u + c(\alpha)(\nabla_X u)$

The Dirac structure is the tuple (∇, h) associated to (E, c).

Definition 2.3.7 (Geometric Dirac operator)**.** A geometric Dirac operator is a Dirac operator, $\not{D}$, that is associated to a (E, c, h, ∇) Dirac structure over (M, g) by

$$\not{D} := c \circ \nabla : \Gamma E \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{c} \Gamma E$$

In local coordinates, after fixing a basis (e^i) of T^*M and the corresponding dual basis (e_i), $\nabla s \in \Gamma(S \otimes T^*M)$ can be expanded in this basis as $\sum_i e^i \otimes \nabla_{e_i} s$. Composed with the clifford action this gives that the geometric Dirac operator acts by $\Gamma(S) \ni \sigma \to \sum_i e^i \cdot \nabla_{e_i}\sigma \in \Gamma(S)$. More generally, Dirac operator can be defined as a first order partial differential operator on the sections of any left $\mathrm{Cl}(M)$ module bundle that squares to a laplacian.

Definition 2.3.8. (Generalized laplacians and Dirac-type operators)

- A generalized laplacian $\triangle$ is a second order differential operator on a vector bundle E with symbol $\sigma_2(L)(x,\xi) = |\xi|^2$.
- A first-order differential operator D on a Clifford module bundle E with Clifford action c over (M,g) satisfying $[D,f] = c(df)$ for all $f \in C^\infty(M)$ is a Dirac-type operator.

Every Dirac operator D on the vector bundle E over M, induces a Clifford action of T^*M on E by $c(df) := [D,f]$ for $f \in C^\infty(M)$, and conversely, associated to any Clifford action c, the operator satisfying $[D,f] = c(df)$ is a Dirac operator (see, for instance, [12, Prop 3.38]).

Definition 2.3.9. (Spinor module) For any oriented vector space V, $\dim V = 2k$, the spinor module is the unique $\mathbb{Z}_2$-graded Clifford module $S = S^+ \oplus S^-$ with $\mathrm{Cl}(V) \otimes \mathbb{C} = \mathrm{End}(S)$.

This generalizes to bundles associated to spin structures.

Definition 2.3.10. For any vector bundle, $E \to M$, with spin structure $\xi : \mathrm{Spin}(E) \to \mathrm{SO}(E)$, the real and complex spinor bundles $S(E), S_{\mathbb{C}}(E)$ are defined by

$$S(E) = P_{\mathrm{Spin}}(E) \times_\mu M,\ S_{\mathbb{C}}(E) = P_{\mathrm{Spin}}(E) \times_\mu M_{\mathbb{C}}$$

where $P_{\mathrm{Spin}}(E)$ is a $\mathrm{Spin}(n)$-principal bundle, $n = \dim E$, and $\mu : \mathrm{Spin}(E) \to \mathrm{SO}(E)$ is the representation given by multiplication by $\mathrm{Spin}(n)$, $M, M_{\mathbb{C}}$ real and complex Clifford modules.

For even-dimensional V, every $\mathbb{Z}_2$ graded complex $\mathrm{Cl}(V)$-module E is isomorphic to $W \otimes S$ where S is the spinor module. Given E, W can be recovered by $W = \mathrm{Hom}_{\mathrm{Cl}(V)}(S,E)$ with trivial $\mathrm{Cl}(V)$ action, that is, the Clifford action on E is the Clifford action on the S component, $e \cdot (w \otimes s) := w \otimes (e \cdot s)$, and $\mathrm{End}(W) \cong \mathrm{End}_{\mathrm{Cl}(V)}(E)$ (see, [12, prop 3.27]). Since the Clifford action is local, this also holds for bundles.

Now for any even-dimensional oriented spin manifold M, denote by $\mathcal{S}$ the unique irreducible complex spinor bundle, then every Clifford module bundle H over M is a twisted bundle $\mathcal{W} \otimes \mathcal{S}$. Working in a local trivialization, it follows from the local version that with $\mathcal{W} \cong \mathrm{End}_{\mathrm{Cl}(M)}(\mathcal{S}, H)$, $\mathrm{End}(\mathcal{W}) \cong \mathrm{End}_{\mathrm{Cl}(M)}(H)$ (see, for instance, [12, Prop 3.35]).

Additionally, on even-dimensional spin manifolds, associated to Clifford structures, Dirac structures exist. This can be seen by working locally: since any Clifford module bundle, H, is a twisting of the bundle $\mathcal{S}$, $H = \mathcal{W} \otimes \mathcal{S}$, the Clifford connection can be defined locally as the tensor product connection of the Levi-Civita connection lifted to $\mathcal{S}$ and any connection on $\mathcal{W}$ compatible with the Clifford action. Note that if $D := \not{D}$ is the geometric Dirac operator for the Dirac structure, and Dirac operators D_0, D give the same Clifford action, then $D - D_0 = A$ for some odd endomorphism, $A \in \Gamma(\mathrm{End}^-(H))$. For a twisted spinor bundle $H = \mathcal{W} \otimes \mathcal{S}$, Dirac operators compatible with given Clifford action are in one–one correspondence with (super)connection on the twisting space (see, for instance, [12, Ch 3]).

The global version follows by a partition of unity argument. Dirac structures exist in odd-diemensional case is well (see, for instance, [54, Prop 11.1.65, 12, Cor 3.41]). The point of noting these details is that being able to recover the twisting space $\mathcal{W}$ in even-dimensional case yields a characterization of almost commutative spectral triples as endomorphism algebra bundles.

With all this at hand, the Clifford module bundle for an almost commutative spectral triple can be given a Dirac structure. Suppose $(C^\infty(M, A), L^2(M, H), D_0)$, $\dim H = 2k$, is an almost-commutative spectral triple with generalized Dirac operator D_0, H a Clifford module bundle over compact spin manifold M. [16, Thm 2.17] gives a metric Q on H which corresponds to the Clifford action associated to D_0 on H. By above, there exists a Dirac structure on H arising from the Clifford action for D_0. H being a Clifford module bundle is a twisted spinor bundle $\mathcal{W} \otimes \mathcal{S}$. Given the metric induced from Q, the connection on $\mathcal{W}$ can be taken to any Riemannian connection on $\mathcal{W}$. Note that if a Dirac structure on H was already known, then the choice to use the Riemannian connections on the twisting space and the spinor bundle is not necessary and the given Dirac structure can be used.

2.4 The noncommutative heat semigroup

For the spinor bundle $S \to M$, we want to consider the complete positivity of semigroup generated by the heat operator e^{-tD^2} on an appropriate algebra $\mathcal{A} \subset \mathcal{B}(L^2(M, \mathrm{End}(S)))$. The algebra $\mathcal{A}$ will contain the Hilbert-Schmidt operators on $L^2(M, S)$, with Hilbert-Schmidt inner product $(f, g)_{HS}$,

$$(f, g)_{HS} := \mathrm{Tr}_{HS}(fg^*) = \sum_i \langle e_i, fg^* e_i \rangle_{L^2(M,E)} = \sum \int (e_i, f(x)g(x)^* e_i)_{E,x} \, \mathrm{d}_{vol}(M) \tag{2.4}$$

where (e_i) is an orthonormal system for $L^2(M,S)$. Such systems are provided by self-adjoint elliptic operators on any vector bundle E. If $P : \Gamma(E) \to \Gamma(E)$ is self-adjoint elliptic operator, then eigenspaces of P, $E_\lambda := \ker(P - \lambda\mathbf{1})$, are finite dimensional, consist of smooth sections, and give a complete orthonormal system for $L^2(E)$, $L^2(E) = \oplus_\lambda E_\lambda$. Additionally, for an elliptic operator $P : \Gamma(E) \to \Gamma(E)$ of order m on vector bundle E over compact X, on any open set $U \subset X, u \in L^2_s(E)$ where $L^2_s(E), s \in \mathbb{R}$ is the Sobolev space, $Pu\big|_U \in C^\infty$ implies $u\big|_U \in C^\infty$. Now the connection laplacian $\nabla^*\nabla$ is an elliptic operator. By [6, Thm 3.7], the closure of the connection laplacian of E, $\overline{\triangle}^E$ is self-adjoint. Since $\overline{\triangle}^E$ restricts to $\triangle^E$ over $\Gamma^\infty(E)$, and the eigenspaces consist of smooth sections, we have a basis for $L^2(E)$ in terms of smooth eigensections of $\triangle^E$ (see [49, Thm III.5.2, III.5.8, Def III.2.3]).

Tr_{HS}, being lower semicontinuous and faithful is permissible in the sense of Albeverio and Høegh-Krohn [2]; this means we can use noncommutative Dirichlet form theory to consider the question of generating completely positive and quantum dynamical semigroups; we introduce this next.

Noncommutative Dirichlet forms

Recall from [2], for a C^*-algebra $\mathcal{A}$ with a lower semicontinuous faithful trace τ, $L^2(\mathcal{A},\tau)$ is the completion on the pre-Hilbert space $\{x : \tau(x^*x) < \infty\}$ with inner product $\langle x, y\rangle_\tau := \tau(y^*x)$. Set $L^2_h(\mathcal{A},\tau) := \{x \in L^2(\mathcal{A},\tau) : x = x^*\}$.

Definition 2.4.1 (Symmetric Markov semigroups)**.** A strongly continuous contraction semigroup, (Φ_t), on $L^2(A,\tau)$ is symmetric if for all x, y, $\langle \Phi_t(x), y\rangle = \langle x, \Phi_t(y)\rangle$. Further, if $0 \le \Phi_t(x) \le 1$ whenever $0 \le x \le 1$ then the semigroup is a Markov semigroup. The semigroup is completely Markov if for all $n \in \mathbb{N}$, $\Phi_t \otimes \mathbf{1}_n$ is Markov semigroup on $L^2(\mathcal{A} \otimes \mathrm{MAT}_n, \tau \otimes \mathrm{Tr}_n)$, Tr_n being the unique normalized trace on MAT_n.

Definition 2.4.2. Suppose $\mathcal{E}(x,x)$ is a closed, quadratic form on $L^2_h(\mathcal{A},\tau)$, with dense domain $\mathrm{Dom}(\mathcal{E})$ with $f(\mathrm{Dom}(\mathcal{E})) = \mathrm{Dom}(\mathcal{E})$ for $f \in \mathrm{Lip}(\mathbb{R}, 0)$, the Banach space of Lipshitz continuous functions that fix zero, $\|f\|_{\mathrm{lip}} := \inf\{m : |f(x) - f(y)| \le m|x-y| \text{ for all } x,y \in \mathbb{R}\}$. Then $\mathcal{E}$ is a Dirichlet form if $\mathcal{E}(f(x), f(x)) \le \|f\|^2_{\mathrm{lip}}\,\mathcal{E}(x,x)$. The form $\mathcal{E}$ is completely Dirichlet if $\mathcal{E} \otimes \mathbf{1}_n$ is Dirichlet for each $n \in \mathbb{N}$.

For a symmetric Markov semigroup $(\Phi_t)_{t\ge0}$ on $L^2(A,\tau)$, with a positive self-adjoint generator $\mathcal{L}$ on $L^2(A,\tau)$, $\Phi_t = e^{-t\mathcal{L}}$, the associated quadratic form[2, thm 2.7] is given by

$$\mathcal{E}_{\mathcal{L}}(x) := \mathcal{E}_{\mathcal{L}}(x,x) = \langle \mathcal{L}^{1/2}x, \mathcal{L}^{1/2}x\rangle = \|\mathcal{L}^{1/2}x\|^2_{L^2(\mathcal{A},\tau)}$$

Observation 2.4.3. It's very useful to note that if $\mathcal{L} = H^2$, with H closed, then $\mathcal{E}_{\mathcal{L}}(x,x) = \langle Hx, Hx\rangle$ is closed. Additionally if H is closed then so is $H \otimes \mathbf{1}_{\mathsf{K}}$ for any Hilbert space K.

Theorem 2.4.4. *([2, Thm 2.7, 3.2]) Dirichlet forms are in one–one correspondence with symmetric Markov semigroups: the positive quadratic form $\mathcal{E}_{\mathcal{L}}$ associated to the positive generator $\mathcal{L}$ for a symmetric Markov semigroup Φ_t is a Dirichlet form. And conversely, if $\mathcal{E}_{\mathcal{L}}(x,x)$ is a Dirichlet form on $L^2_h(A,\tau) \subset L^2(A,\tau)$ then $\mathcal{L}$ generates a Markov semigroup on $L^2(A,\tau)$, $e^{-t\mathcal{L}}$. This extends to complete Markovity: $\mathcal{E}_{\mathcal{L}}$ is completely Dirichlet if and only if $\mathcal{L}$ generates a completely Markov semigroup.*

Given C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$, with a faithful, lower-semicontinuous trace, τ, with $L^2_h(\mathcal{A},\tau) \subset L^2(\mathcal{A},\tau)$ denoting the hermitian elements, a Dirichlet form $E(x,x)$ is a positive closed quadratic form on $L^2_h(\mathcal{A},\tau)$ such that any lipschitz $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ satisfying $f(\mathrm{Dom}(E)) \subset \mathrm{Dom}(E)$, $E(f(x), f(x)) \leq \|f\|^2_{Lip}\, E(x,x)$.

A semigroup T_t on $L^2(\mathcal{A},\tau)$ is τ-symmetric if $\tau(T_t(x)^*y) = \langle x, T_t(y)\rangle_{L^2(\mathcal{A},\tau)}$. Let $\mathcal{L}$ be generator of such T_t with $H = \mathcal{L}^{1/2}$, then the positive quadratic form $E(x,x) := \|Hx\|_2^2 = \|Hx\|^2_{L^2(\mathcal{A},\tau)}$ is a Dirichlet form, while if $E(x,x) := \|Hx\|_2^2$ is a Dirichlet form then e^{-tH^2} is a symmetric Markov semigroup on $L^2(\mathcal{A},\tau)$ [2, Thm 2.7, 2.8]. A completely Dirichlet form is Dirichlet form E such that $\sum_{ij\in[n]} E(x_{ij}, x_{ij})$ is a Dirichlet form on $L^2(\mathcal{A} \otimes \textsc{Mat}_n, \tau \otimes \mathrm{Tr})$ for every $n \in \mathbb{N}$. By [2, Thm 3.2, 62, Prop 3.2.29], τ-symmetric semigroup is completely Markov iff the associated Dirichlet form E is completely Dirichlet.

For a C^*-algebra $\mathcal{A}$, a C^*-Dirichlet form is a completely Dirichlet form $\mathcal{E}$ such that $\mathrm{Dom}(\mathcal{E}) \cap \mathcal{A}$ is norm-dense in $\mathcal{A}$ and form-core for $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$. Note that the C^*-algebra $\mathcal{A}$ is not required to be unital. In the setting considered, $\mathcal{A}$ can be taken to $\mathcal{K}(H)$, compact operators on the given Hilbert space H; this makes checking density hypothesis on $\mathrm{Dom}(\mathcal{E}) \cap \mathcal{A}$ straightforward.

Now any contractive completely-positive map Ψ on $\mathcal{A}$ which is τ-symmetric for every $a, b \geq 0$ in $\mathcal{A}$ (that is, $\tau(\Psi(a)b) = \tau(a\Psi(b))$) extends to an L^2-contraction from $L^2(\mathcal{A},\tau) \cap \mathcal{A}$ to $L^2(\mathcal{A},\tau) \cap \mathcal{A}$ (see [62, § 3.2.3]). This supplies the following correspondence between quantum dynamical semigroups and C^*-Dirichlet forms

([62, propostion 3.2.29]): if $T_t = e^{t\mathcal{L}}$ is a quantum dynamical semigroup symmetric with respect to τ, then T_t viewed as a semigroup of positive contractions on $L^2(\mathcal{A}, \tau)$, with the generator given by the negative operator, $\mathcal{L}_2$, then $\mathcal{E}(x) = \left\|(-\mathcal{L}_2)^{1/2}(x)\right\|_2^2$ with $\mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}((-\mathcal{L}_2)^{1/2})$ is a C^*-Dirichlet form, and conversely, every C^*-Dirichlet form arises from a symmetric quantum dynamical semigroup.

Therefore, any τ-symmetric quantum dynamical semigroup on $\mathcal{A} = \mathcal{K}(H)$ is associated to a C^*-Dirichlet form and its associated contractive semigroup on $L^2(\mathcal{A})$. It's useful to think of $\mathcal{K}(H) \subset \mathcal{B}(H)$, where $\mathcal{B}(H)$ is a unital von Neumann algebra, as continuously embedded inside algebra $\mathcal{B}(L^2(\mathcal{A}, \tau))$.

Example 2.4.5. [2, Corollary 4.4] gives a class of completely Dirichlet forms: for any self adjoint operator $H^2 = M \geq 0$, $m_i \in \mathcal{B}(H), \mathrm{Tr}(m_i^* m_i) < \infty$, $E(x,x) := \mathrm{Tr}(x^2 M) + \sum_i \mathrm{Tr}([x, m_i]^*[x, m_i])$ is a completely Dirichlet form iff $E(x,x)$ is closeable on $L^2(\mathcal{A}, \tau)$. On specializing to $\tau = \mathrm{Tr}$ on $\mathcal{B}(H)$, $L^2(\mathcal{A}, \mathrm{Tr}) = \mathcal{H}^2(H) \subset \mathcal{B}(H)$, the space of Hilbert-Schmidt operators on H, and the Dirichlet form becomes

$$E(x,x) = \|Hx\|_{\mathcal{H}^2}^2 + \sum_i \|[x, m_i]\|_{\mathcal{H}^2}^2$$

Dirichlet forms are in correspondence with bimodule derivations. To make this precise, the following abstract characterization of Dirichlet forms is needed. Note that $L^2(\mathcal{A}, \tau)$ is simply the Gelfand-Naimark-Segal (GNS) Hilbert space for τ; however, the reference to GNS is not necessary since $\mathcal{A}$ is already given as $\mathcal{B}(H)$. This is pointed out since one needs to associate to $\mathcal{A}$ a von Neuman algebra which can be viewed as $L^\infty(\mathcal{A}, \tau)$.

For a C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$, with $\mathcal{M} := \mathcal{A}'' \subset \mathcal{B}(H)$ the von Neumann algebra with unit $\mathbf{1}$, the standard form for a von Neumann algebra $\mathcal{M}$ is the triple $(\mathcal{M}, L^2(\mathcal{A}, \tau), L^2(\mathcal{A}, \tau)_+)$, $L^2(\mathcal{A}, \tau)_+$ being the positive cone induced by the involution J on $\mathcal{M}$ corresponding to $a \to a^*$. The real subspace, $L_h^2(\mathcal{A}, \tau)$, is the subspace of τ-invariant elements. Let $a^\#$ denote the projection of $a \in L_h^2(\mathcal{A}, \tau)$ onto the L^2 closure of the convex set $C = \{a \in L_+^2(\mathcal{A}, \tau) : a \leq \mathbf{1}\}$.

Definition 2.4.6. A Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ on $L^2(\mathcal{A}, \tau)$ is a closed, densely defined, non-negative quadratic form satisfying

1. $a \in \mathrm{Dom}(\mathcal{E})$ implies $J(a) \in \mathrm{Dom}(\mathcal{E})$ with $\mathcal{E}(J(a)) = \mathcal{E}(a)$
2. For $a \in \mathrm{Dom}(\mathcal{E}) \cap L_h(\mathcal{A}, \tau) := \mathrm{Dom}_h(\mathcal{E}), a^\# \in \mathrm{Dom}(\mathcal{E}), \mathcal{E}(a^\#) \leq \mathcal{E}(a)$

With this, there's the following correspondence due to [23].

Theorem 2.4.7. *([23, Theorem 8.3]) Let H be a Hilbert space carrying a $\mathcal{A}-\mathcal{A}$-bimodule structure, J an antilinear (conjugate-linear) involution exchanging right and left $\mathcal{A}$-actions (that is, $J(ahb) = b^*J(h)a^*$) , $\mathcal{B}$ an involutive subalgebra of $\mathcal{A} \cap L^2(\mathcal{A}, \tau)$ dense in both. Then if $\partial : \mathcal{B} \to H$ is a closable derivation, satisfying $J\partial a = \partial a^*$, then the closure of the quadratic form $\mathcal{B} \ni a \to \|\partial a\|_H^2$ is a Dirichlet form.*

The positive generator $\mathcal{L}$ of the semigroup $e^{-t\mathcal{L}}$ associated with the Dirichlet form is given by $\mathcal{L} = \partial^*\bar{\partial}$ where $\bar{\partial}$ is the closure of ∂. The correspondence is one to one: from every Dirichlet form a bimodule and a derivation can be constructed which is unique up to a bimodule map isometry [23, Theorem 8.2].

The noncommutative laplacian

For any Hilbert space H, let $\mathcal{H}^2(H) := L^2(\mathcal{B}(H), \mathrm{Tr})$ be the space of Hilbert-Schmidt operators. Note that $\mathcal{H}^2(H) := L^2(\mathcal{B}(H), \mathrm{Tr}) = L^2(\mathcal{K}(H), \mathrm{Tr})$ where $\mathcal{K}(H)$ are the compact operators since Hilbert-Schmidt operators are compact with norm-closure $\mathcal{K}(H)$ (this is assuming $\mathcal{A} = \mathcal{B}(H)$, otherwise one restricts to $\mathcal{A}$). Recall some background theory:

- $\mathcal{K}(H)$ is the largest norm-closed ideal of $\mathcal{B}(H)$, in particular, it's a C^*-algebra, strongly-dense in $\mathcal{B}(H)$, although not unital unless $\dim H$ is finite.
- The space of Hilbert-Schmidt operators in $\mathcal{B}(H)$, $\mathcal{H}^2(H) := L^2(\mathcal{B}(H), \mathrm{Tr})$ is isometrically isomorphic to the Hilbert space $H^* \otimes H$ with $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle = \langle a_1, a_2\rangle\langle b_1, b_2\rangle$
- For any C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$, $L^2(\mathcal{A}, \mathrm{Tr}) \subset \mathcal{A}$ is a two-sided ideal.

Let $A \in \mathcal{B}(H)$ be a self-adjoint operator, hence closed operator, with spectral decomposition $A = \sum_i \lambda_i.e_i, \lambda_i \leq \lambda_{i+1}$. Denote by $e_{ij} := e^i \otimes e_j$, where $e^i = e_i^*$ and can be taken as $e_i^* = \langle e_i, \cdot\rangle$; e_{ij}'s form a basis for $\mathcal{H}^2(H) = H^* \otimes H$.

Proposition 2.4.8. *The operator $A_{\mathfrak{D}} := A \otimes \mathbf{1} - \mathbf{1} \otimes A$ acting on $H^* \otimes H := \mathcal{H}^2(H)$ where A acts on H^* by $\psi A \psi^{-1}$, $\psi : H^* \to H$ the isomorphism identifying e_i^* and e_i. Then*

1. *$A_{\mathfrak{D}}$ is a derivation and $A_{\mathfrak{D}}(\mathbf{1}) = 0$*

2. *$A_{\mathfrak{D}}$ is the operator $a \to [A, a]$*

Proof. Note that λ_i are real. For the first, by linearity it suffices to check that it's a derivation on the basis $e_{ij} := e_i \otimes e^j$'s. One can assume for $e_{ij'} \circ e_{jk}$, $j = j'$ as otherwise $\mathcal{A}_{\mathfrak{D}}(e_{ij'} \circ e_{jk}) = 0$,

$$A_{\mathfrak{D}}(e_{ij} \circ e_{jk}) = \lambda_i e_{ik} - \lambda_k e_{ik}$$
$$A_{\mathfrak{D}}(e_{ij}) \circ e_{jk} + e_{ij} \circ A_{\mathfrak{D}}(e_{jk}) = \lambda_i e_{ik} - \lambda_j e_{ik} + \lambda_j e_{ik} - \lambda_k e_{ik} = A_{\mathfrak{D}}(e_{ij'} \circ e_{jk})$$

It's straightforward that $A_{\mathfrak{D}}(\mathbf{1}) = A_{\mathfrak{D}}(\sum_i e_i \otimes e_i^*) = 0$.

For second, again by linearity, it's enough to show $[A, e_{ij}](e_{j'k}) = A_{\mathfrak{D}} e_{ij}(e_{j'k})$. If $j \neq j'$ then

$$[A, e_{ij}](e_{j'k}) = Ae_{ij}(e_{j'k}) - e_{ij} A e_{j'k} = 0 = A_{\mathfrak{D}} e_{ij}(e_{j'k})$$

and with $j = j'$,

$$[A, e_{ij}](e_{jk}) = Ae_{ij}(e_{jk}) - e_{ij} A e_{jk} = \lambda_i e_{ik} - \lambda_j e_{ik}$$
$$A_{\mathfrak{D}} e_{ij}(e_{jk}) = (\lambda_i e_{ij} - \lambda_j e_{ij}) e_{jk} = [A, e_{ij}](e_{jk})$$

□

Theorem 2.4.9. *For each n, the operator $-A^2_{\mathfrak{D},n} := -A^2_{\mathfrak{D}}|_{H_n}$, $H_n = Span\{e_i \otimes e_j : i, j \in [0 : n]\}$ defines a completely Dirichlet form, $E_n(x, x) := \left\| \mathrm{i} A_{\mathfrak{D},n} x \right\|_{\mathcal{H}^2}$. The positive generator for the associated semigroup is $A^2_{\mathfrak{D}}$.*

Proof. Consider $\mathrm{i}A_{\mathfrak{D}}$. From proposition 2.4.8, $\mathrm{i}A_{\mathfrak{D}}$ is a derivation on the Hilbert space $\mathcal{H}^2(H)$. Further, the quadratic form $E_n(x, x) = \|\mathrm{i}A_{\mathfrak{D}}\|$ is closed because $A_{\mathfrak{D}}$ is closed as A is self-adjoint, ψ an isometry, making $A \otimes \mathbf{1}, \mathbf{1} \otimes A$ closed. Take $\mathcal{B} = \mathcal{H}^2(H)$ for theorem 2.4.7. $\mathcal{H}^2(H)$ is dense in both $L^2(\mathcal{K}(H), \mathrm{Tr}), \mathcal{K}(H)$, while $*$ on $\mathcal{H}^2(H)$ given by $(\alpha e_{ij})^* = \bar{\alpha} e_{ji}$ for $\alpha \in \mathbb{C}$ exchanges the right and left action of $\mathcal{H}^2(H)$ on itself. The compatibility of $\mathrm{i}A_{\mathfrak{D}}$ with $*$ holds since on the basis elements

$$(\mathrm{i}A_{\mathfrak{D}} \alpha e_{ij})^* = -\mathrm{i}\bar{\alpha}(A_{\mathfrak{D}} e_{ij}) = -\mathrm{i}\bar{\alpha}((\lambda_i - \lambda_j) e_{ij})^* = \mathrm{i}\bar{\alpha}(\lambda_j - \lambda_i) e_{ji}$$
$$\mathrm{i}A_{\mathfrak{D}}(\alpha e_{ij})^* = -\mathrm{i}\bar{\alpha} A_{\mathfrak{D}} e_{ji} = \mathrm{i}\bar{\alpha}(\lambda_j - \lambda_i) e_{ji}$$

therefore, theorem 2.4.7 applies and $E_n(x, x)$ is a Dirichlet form. On replacing H by $H \otimes \mathrm{MAT}_m$, Tr by $\mathrm{Tr} \otimes \mathrm{Tr}_m$ for the normalized trace on Tr_m on MAT_m, the closability of map $A_{n,\mathfrak{D}}$ is unaffected by tensoring the identity, therefore, E_n is completely Dirichlet. The operator $-A^2_{\mathfrak{D}}$ is negative; $A^2_{\mathfrak{D}}$ is the positive generator for the semigroup. □

Recalling that for the spectral triple $(\mathcal{A}, \mathsf{H}, D)$, the Connes' differential 1-forms are defined by

$$\Omega^1_D(\mathcal{A}) := \{\sum a_k[D, b_k] : a_k, b_k \in \mathcal{A}\} \tag{2.5}$$

formally, $[D,[D,\cdot]]$ is defined as the noncommutative laplacian, the intuition being that with respect to the Hilbert-Schmidt inner product, $\langle a, [\mathrm{i}D, b]\rangle_{HS} = -\langle [\mathrm{i}D, a], b\rangle_{HS}$ following that the adjoint of covariant derivative $(\nabla_i)^*$ on a compact manifold without boundary is $-\nabla_i$, and is the generator $\partial^*\bar{\partial}$ for the semigroup associated to the Dirichlet form.

Remark 2.4.10. The noncommutative laplacian $[D,[D,\cdot]]$ is agnostic of the geometric content of the Dirac operator D, and only relies on the associated derivation; the heat semigroups generated by endomorphim lapacians considered later better capture the geometric aspect.

Proposition 2.4.8 yields that the heat semigroup canonically associated to the spectral triple generated by the noncommutative laplacian is a quantum dynamical semigroup. Additionally, the domain for the noncommutative laplacian contains an operator system. Recall that a (not necessarily closed) subspace $S \subset \mathcal{B}(\mathsf{H})$ for any Hilbert space H is an operator system, if it is self-adjoint and unital (see [60, definition 1.36]). The point is that completely positive maps on operator systems extend to the containing unital C^*-algebra by the Arveson's extension theorem (see [60, theorem 1.39]): if S, $S \subset \mathcal{M} \subset \mathcal{B}(\mathsf{K})$ for any Hilbert space K is an operator system in the unital C^*-subalgebra $\mathcal{M}$, then any completely positive map $u : S \to \mathcal{B}(\mathsf{K}')$ for any Hilbert space K' extends to a completely positive map $u : \mathcal{M} \to \mathcal{B}(\mathsf{K}')$ with $\|u\|_{cb} = \|u(\mathbf{1})\|$ where $\|\cdot\|_{cb}$ is the completely-bounded norm, $\|u\|_{cb} = \sup_n \left\|u \otimes \mathbf{1}_{\mathrm{MAT}_n}\right\|$.

Corollary 2.4.11. *For the spectral triple $(\mathcal{A}, H, D)$, the noncommutative laplacian, $-D^2_{\mathfrak{D}}$, $a \to -[D,[D,a]]$ generates a conservative quantum dynamical semigroup on $\mathcal{B}(H)$.*

Proof. This is immediate from theorem 2.4.9 using $A = D$ and then $[D,[D,\cdot]] = (D_{\mathfrak{D}})^2 := \mathcal{L}$, except that $D_{\mathfrak{D}}$ is no longer a bounded operator, and unlike the truncations $D_{\mathcal{D},n}$, the $\mathrm{Dom}(\mathcal{L})$ is not all of $\mathcal{H}^2$. However, since for every $n \in \mathbb{N}$, $\sum_{i,j\in[n]} \alpha_{ij} e_{ij} \in \mathrm{Dom}(\mathcal{L})$, $\mathrm{Dom}(\mathcal{L})$ is norm-dense in $\mathcal{K}(\mathsf{H})$, and strongly dense in $\mathcal{B}(\mathsf{H})$. Now $D_{\mathcal{D}}(\mathbf{1}) = 0$, the semigroup $e^{-t\mathcal{L}}$ is defined on the operator system S generated by $\{\mathbf{1}, e_{ij} : i, j \in [n], n \in \mathbb{Z}_{\geq 0}\}$ and since the $\mathcal{B}(\mathsf{H})$ is the smallest C^*-algebra containing S, $e^{-tD^2_{\mathcal{D}}}$ extends to $\mathcal{B}(\mathsf{H})$ by Arveson's extension theorem. □

This will be used toward realizing spectral action from the Evans-Hudson flow, but since the noncommutative laplacian is not geometric in the sense that while it lives on $\mathsf{H}^* \otimes \mathsf{H} = \mathrm{End}(\mathsf{H})$, it's not associated to a connection on the $\mathrm{End}(\mathsf{H})$. It's useful to consider semigroups generated by elliptic operators associated to connections on $\mathrm{End}(\mathsf{H})$ — for the canonical spinor bundle $\mathcal{S}$ over M, $\mathrm{End}(\mathcal{S}) = \mathrm{Cl}(M)$, so this is an important example; the natural operators also generate quantum dynamical semigroups.

Example 2.4.12. (Matrix geometries and fuzzy spectral triples) A fuzzy spectral triple, (A, H, D, J, γ), is the Clifford algebra, $\mathrm{Cl}_{p,q}$ associated to $\mathbb{R}^{p+q}$ with pseudo-euclidean metric of signature (p, q), along with matrix algebra $\mathrm{MAT}_N(\mathbb{C})$, with $A = \mathrm{MAT}_N(\mathbb{C})$, $H = V_{p,q} \otimes \mathrm{MAT}_N(\mathbb{C})$ with $V_{p,q}$ a $\mathrm{Cl}_{p,q}$-module, $\langle a, b\rangle_{\mathrm{MAT}_N(\mathbb{C})} = \mathrm{Tr}(a^* b)$, with appropriately defined grading γ and real structure J. Observe that since $A = \mathrm{MAT}_N(\mathbb{C})$ and the action of A is defined to be $V_{p,q} \otimes \mathrm{MAT}_N(\mathbb{C}) \ni (\phi \otimes B) \to \phi \otimes AB$, A acts precisely like the algebra of the almost-commutative spectral triple. The Dirac operator D for the fuzzy spectral triple is axiomized to be self-adjoint satisfying $D\gamma = (-1)^{q-p}\gamma D$, $DJ = \epsilon' JD$, $[[D, a], JbJ^{-1}] = 0$ for all $a, b \in A$, ϵ' depending on $q - p$. Since the $\mathrm{Cl}_{p,q} \otimes \mathrm{MAT}_N(\mathbb{C})$ is finite dimensional, the Dirac operator can be parametrized by matrices[8], and the space of all Dirac operators parameterizes the geometries supported over the fuzzy spectral triple. The path integral over the space of Dirac operators is the object of interest. When $q = 0$, that is, signature is euclidean, since the operators are self-adjoint and finite-dimensional the associated flows exist, one can equivalently consider the possible spectral actions. When $q \neq 0$, the underlying spaces are not Hilbert spaces, and the notion of a quantum dynamical semigroups needs to be reformulated.

2.5 Quantum dynamical semigroups on spinor endomorphisms

The quadratic forms associated with the geometric laplacians and their perturbations can be considered using the class of Dirichlet forms introduced earlier in example 2.4.5.

Note that the connection $\nabla^*\nabla$ and Dirac laplacian D^2 are symmetric with respect to the innerproduct structure on spinor bundles and the symmetry of the semigroup follows, making noncommutative Dirichlet theory applicable. As a warm-up, the following result is in the spirit of example 2.4.5, and is obtained directly from the Dirichlet form definition (as compared to [2] which used normal contractions on C^*-algebras). On any vector bundle E, denote by $\mathcal{H}^2(L^2(E))$ the Hilbert-Schmidt operators acting on $L^2(E)$ with Tr_{HS} inner product.

Proposition 2.5.1. *If the quadratic form, $\mathcal{E}_\triangle$, associated with the connection laplacian $\triangle_E$ on a vector bundle $E \to X$ with metric compatible connection, on $\mathcal{H}^2(L^2(E))$, $\triangle_E$ is closed, then the form is Dirichlet and completely Dirichlet. The result also holds if $\triangle_E$ is replaced by the geometric Dirac laplacian, $\not{D}^2$, for any Dirac bundle E, and positive operator $T = Z^*Z$ in general.*

Proof. By definition 2.4.2, with $\mathcal{E}_\triangle(x,x) = \mathrm{Tr}_{HS}(\triangle x^2) = \mathrm{Tr}_{HS}(x \triangle x)$ where $x = x^* \in \mathcal{H}^2_h(L^2(E)), \mathrm{Tr}(x^2) < \infty$. It needs to be checked that for $f \in \mathrm{Lip}(\mathbb{R}, 0)$, $f(\mathrm{Dom}(\mathcal{E})) = \mathrm{Dom}(\mathcal{E})$ and $\mathcal{E}_\triangle(f(x), f(x)) \le \|f\|^2_{\mathrm{lip}}\, \mathcal{E}_\triangle(x,x)$.

The condition $\mathcal{E}_\triangle(f(x), f(x)) \le \|f\|^2_{\mathrm{lip}}\, \mathcal{E}_\triangle(x,x)$ follows by noting that x and x^2 are compact and self-adjoint and, therefore, $x^2 = \sum_i \alpha_i^2 P_i$ where $x = \sum_i \alpha_i P_i, \alpha_i \in \mathbb{R}$ is the representation from the spectral theorem for compact self-adjoint operators. Note that since $f(\alpha_i) \in \mathbb{R}, \nabla f(x) = \nabla \sum_i f(\alpha_i) P_i = \sum_i f(\alpha_i) \nabla P_i$,

$$\|\nabla f(x) e_k\| \le \| \sum_i \|f\|_{\mathrm{lip}} \alpha_i \nabla P_i e_k\| = \|f\|_{\mathrm{lip}}\, \|\nabla x e_k\| \tag{2.6}$$

where it was used that for $r \in \mathbb{R}$, $f(r)/r \le \|f\|_{\mathrm{lip}}$ meaning $f(r) \le \|f\|_{\mathrm{lip}} r$. This means $\mathcal{E}_\triangle(f(x), f(x)) = \mathrm{Tr}_{HS}(f(x) \triangle f(x)) = \sum_i \langle e_i f(x), \nabla^*\nabla f(x) e_i \rangle = \sum_i \|\nabla f(x) e_i\|^2$, and as needed

$$\mathcal{E}_\triangle(f(x), f(x)) \le \|f\|^2_{\mathrm{lip}}\, \mathcal{E}_\triangle(x,x)$$

Since $y \in \mathcal{H}^2(L^2(E))$ can be written as $(y+y^*)/2+(y-y^*)/2$, so to show invariance of the domain, it suffices to show $f(y) \in \mathrm{Dom}(\mathcal{E}_\triangle)$ for self-adjoint $y \in \mathrm{Dom}(\triangle)$. As $y \in \mathrm{Dom}(\triangle)$ means $\mathrm{Tr}_{HS}(y \triangle y) = \|\nabla y\|_{HS} < \infty$, $f(y) \in \mathrm{Dom}(\mathcal{E}_\triangle)$ follows by the estimate in equation 2.6. Therefore, if $\mathcal{E}_\triangle$ is closed, it's Dirichlet.

Set $\triangle_n = \triangle \otimes \mathbf{1}_n$ for $\mathbf{1}_n$ the identity map on $\mathrm{M{\scriptsize AT}}_n$. Since $\triangle_n = (\triangle^* \otimes \mathbf{1}_n)(\triangle \otimes \mathbf{1}_n)$, and any self-adjoint $y \in \mathrm{M{\scriptsize AT}}_n$ is diagonalizable, the same analysis applies. As $\mathbf{1}_n$ is closed, $\mathcal{E}_{\triangle_n}$ is closed if and only if $\mathcal{E}_\triangle$ is closed. This establishes the claim. The same argument applies to the Dirac laplacian $\not{D}^2$ for the spinor bundle E acting on $L^2(M, E)$ and for any T of the specified form. □

The Bochner identity for case when E is Dirac bundle will be useful. In particular, it will allow relating the closedness of the Dirichlet forms for the connection and Dirac laplacians. To set it up, let $R^E_{v_1,v_2}$ denote the curvature transformation of the vector bundle E with connection ∇, $R^E_{v_1,v_2} : \Gamma(E) \to \Gamma(E), e \to (\nabla_{v_1}\nabla_{v_2} - \nabla_{v_2}\nabla_{v_1} - \nabla_{[v_1,v_2]})e \in \Gamma(E)$.

Definition 2.5.2. [The general Bochner identity] For the connection laplacian $\triangle = \nabla^*\nabla$ and the Dirac operator D for any any Dirac bundle S over M, $n = \dim M$, with $R^S_{u,v}$ the curvature transformation of S, (e_i) the orthonormal tangent frame, the general Bochner identity states

$$D^2 = \triangle + \mathfrak{R} \tag{2.7}$$

where $\mathfrak{R}(\phi) := \frac{1}{2}\sum_{j,k\in[n]} e_j \cdot e_k \cdot R^S_{e_i e_j}(\phi)$ is the curvature operator of the bundle.

Now it remains to show that the form $\mathcal{E}_{D^2}$ is closed on Hilbert-Schmidt operators on $L^2(E)$. Note that $\not{D}$ is identified with the Dirac operator extended to the L^2 sections, i.e., acting distributionally, which is self-adjoint, and therefore closed. $\mathcal{E}_{D^2}$ is also identified with the extended version. Then by the Bochner identity and the fact that the curvature operator on a compact manifold is self-adjoint and bounded, and so closed, it follows that the connection laplacian is also closed.

Theorem 2.5.3. *Suppose (S, h) is a Dirac bundle over the compact Riemannian manifold (M, g) with D denoting the self-adjoint extension of the Dirac operator to $L^2(S)$. Let $\mathcal{H}$ be the Hilbert space of Hilbert-Schmidt operators, $\mathcal{H}^2(L^2(S))$ with inner product $\langle x, y\rangle_{HS} = \mathrm{Tr}(x^*y)$. Then the quadratic form $\mathcal{E}_{D^2}(x, y) = q(x, y) := \mathrm{Tr}(x^*yD^2)$ on $\mathcal{H}\times\mathcal{H}$ is closed, and therefore, $\mathcal{E}_{D^2}$ Dirichlet, furthermore, is also completely Dirichlet.*

The proof is immediate by the following observation.

Observation 2.5.4. Since Hilbert-Schmidt operators are isometrically isomorphic to $S^* \otimes S$, if $T \in \mathrm{Lin}(L^2(S), L^2(S))$ viewed as acting on $\mathcal{H}^2(L^2(S))$ by composition, then on a basis element $e_{ij} := e_i^* \otimes e_j$ for $\mathcal{H}^2(L^2(S))$, $T(e_{ij})$ is the same as $e_i^* \otimes T(e_j) = [T \otimes \mathbf{1}](e_i^* \otimes e_j)$, and therefore, if $T = Z^*Z$ with Z closed, the quadratic form $\mathcal{E}_T$ is also closed, and therefore, Dirichlet, and by same argument applied to tensoring with $\mathbf{1}_{\mathrm{MAT}_n}$, completely Dirichlet.

A second proof is included as it illustrates how Sobolev norms naturally appear when the operator is a pseudo-differential operator. This gives intuition for the case where Laplace-Beltrami generated diffusion on the canonical spectral triple is analyzed.

Proof. It's enough to show the claim for x, y self-adjoint, so we work with $q(x,x) = \mathrm{Tr}(x^2D^2))$. Let (e_i) be a basis of $L^2(S)$ consisting of smooth eigensections of the laplacian $\triangle^S = \nabla^*\nabla$ associated with the connection for D. Note that q is semi-bounded, since $\mathrm{Tr}(x^2D^2) = \sum_i \langle Dxe_i, Dxe_i\rangle = \|Dx\|_{HS}^2 \geq 0$ where we used that x is self-adjoint and the trace is cyclic, so $\mathrm{Tr}(x^2D^2) = \mathrm{Tr}(xD^2x)$.

Now suppose (x_n) is a Cauchy sequence in norm $\|a\|_+ := \sqrt{\|a\|_{HS} + q(a,a)}$. So (x_n) is Cauchy sequence in the Hilbert space $(\mathcal{H}, \|\cdot\|_{HS})$, implying $(x_n) \xrightarrow{HS} x \in \mathcal{H}$. And (x_n) being Cauchy in $\|\cdot\|_+$ also gives that $q(x_n - x_m, x_n - x_m) \to 0$. Because $q(a,a) = \|Da\|_{HS}^2$, so (Dx_n) is also Cauchy in $(\mathcal{H}, \|\cdot\|_{HS})$ and therefore convergent with $\lim_{n\to\infty} Dx_n = g \in \mathcal{H}$.

Suppose $g = Dx$, then it follows that q is closed because

$$\lim_{n\to\infty} q(x_n - x, x_n - x) = \lim_{n\to\infty} \|D(x_n - x)\|_{HS} = \lim_{n\to\infty} \|Dx_n - g\|_{HS} = 0$$

Now if xe_i, x_ne_i are weakly (L^2) differentiable (that is, $e_i \in \mathrm{Dom}(x) \cap \mathrm{Dom}(Dx)$, $\mathrm{Dom}(x_n) \cap \mathrm{Dom}(Dx_n)$), since $x_n \xrightarrow{HS} x$, meaning $\|x_n - x\|_{HS} \to 0$, so for all i, $x_ne_i \to xe_i$, then using that D is self-adjoint, and hence closed on $L^2(S)$, yields

$$ge_i = \lim_{n\to\infty} (Dx_n)e_i = \lim_{n\to\infty} D(x_ne_i) = D(xe_i) = (Dx)e_i$$

Since Dx and g agree on the basis (e_i), $Dx = g$.

Finally, the weak differentiability of x_ne_i holds since e_i is smooth and $\|x_ne_i\|^2 \leq \mathrm{Tr}(x_n^2) = \|x_n\|_{HS}^2$ which is finite and similarly $\|Dx_ne_i\|^2 \leq \mathrm{Tr}(x_n^2D^2) = \|Dx_n\|_{HS}^2 < \infty$, so $x_ne_i, Dx_ne_i \in L^2(S)$. The same applies to xe_i, Dxe_i, $\|xe_i\| \leq \|x\|_{HS}$, $\lim_{n\to\infty} \|Dxe_i\| \leq \|g\|_{HS}$, so $xe_i \in \mathrm{Dom}(D)$.

□

Note that the norm $\|a\|_+ := \sqrt{\|a\|_{HS} + q(a,a)} = \sqrt{\|a\|_{HS} + \|Da\|_{HS}}$ is the natural generalization of Sobolev norm to the endomorphism algebra.

Corollary 2.5.5. *The form associated to laplacian, $\mathcal{E}_\triangle$, on the vector bundle S is completely Dirichlet form on $\mathcal{H}^2(L^2(S))$.*

Instead of using Bochner identity, one can also get at the result for the connection laplacian, $\triangle_E$, by adjusting the same reasoning to from D to the closure $\overline{\nabla}$ of ∇ using the results from [6] after accounting for domain and co-domain of ∇ not being the same Hilbert space as for D.

Similarly, the quadratic form for a positive curvature operator, $\mathfrak{R}^S$ can be shown to be completely Dirichlet.

Proposition 2.5.6. *If $\mathfrak{R} \geq 0$ then the associated form, $\mathcal{E}_{\mathfrak{R}}$, is completely Dirichlet on $L^2(A, \tau)$.*

Proof. First note that the $\mathfrak{R}$ at each fiber is a bounded symmetric operator. To see the symmetry, note it can immediately be checked that for any Riemannian connection, the curvature transformation is skew symmetric in the sense $\langle R_{V,W}s, s'\rangle = -\langle s, R_{V,W}s'\rangle$. Consider each term in $\mathfrak{R}$ at $p \in M$, $\langle s, e_l \cdot e_k \cdot R_{e_l e_k} s'\rangle$ for $s, s' \in \Gamma(H)$, and in Riemann normal frame (e_i) centered at p, since $l \neq k$ must hold (otherwise $R_{e_l e_k} = 0$),

$$\langle s, e_l \cdot e_k \cdot R_{e_l e_k} s'\rangle = \langle -R_{e_l e_k} e_k \cdot e_l \cdot s, s'\rangle = \langle R_{e_l e_k} e_l \cdot e_k \cdot s, s'\rangle = \langle e_l \cdot e_k \cdot R_{e_l e_k} s, s'\rangle$$

where to commute e_l, e_k past $\nabla_{e_l}, \nabla_{e_k}$ inside R_{lk}, the product rule was used with the fact that the coordinates are Riemann normal centered at p, so covariant derivatives vanishes at p. As $\mathfrak{R}^H$ varies smoothly, and the manifold is assumed to be compact, it's bounded globally. Everywhere defined symmetric operators are self-adjoint and are closed, therefore, $\mathfrak{R}$ is self-adjoint and closed. If $\mathfrak{R}$ is non-negative, $\mathfrak{R}^{1/2}$ exists and again being bounded is closed; therefore, it follows as before that $\mathcal{E}_{\mathfrak{R}}$ is completely Dirichlet. □

Remark 2.5.7. Note that in the $L^2(A, \|\cdot\|_{HS})$ setting the complete Markovity of the Dirac heat semigroup does not depend on the curvature unlike for C^*-bundles where for Clifford bundles it does[24].

So far $\mathcal{H}^2$ has been considered, but $\mathcal{H}^2$ is not unital which is necessary for existence of quantum stochastic dilations; as before Arveson's extension theorem can be used towards this.

Theorem 2.5.8. *The completely Markov semigroup $e^{-t\mathcal{L}}$ extends from $L^2(\mathcal{A}, \mathrm{Tr}) = \mathcal{H}^2(S) \subset \mathcal{B}(L^2(S))$ to $\mathcal{B}(L^2(S))$ if and only if the $e^{-t\mathcal{L}}$ is completely Markov for each t on the operator system generated by $L^2(\mathcal{A}, \mathrm{Tr})$ and $\mathbf{1}$.*

Proof. If $e^{-t\mathcal{L}}$ is not a completely Markov family of maps on $O(L^2(\mathcal{A},\tau),1)$ then obviously $e^{-t\mathcal{L}}$ does not extend to $\mathcal{B}(L^2(S)) \supset O(L^2(\mathcal{A},\tau),1)$. If it's a completely Markov family, then because $O(L^2(\mathcal{A},\tau),1)$ is an operator system, so as completely positive maps, $e^{-t\mathcal{L}}$, extends to $\mathcal{B}(L^2(S))$ by Arveson's extension theorem . Complete Markovity follows since even though Hilbert-Schmidt operators are not norm dense, they are strongly dense in $\mathcal{B}(L^2(H))$. □

Corollary 2.5.9. *Suppose $\mathcal{L}(1) = 0$ then $e^{-t\mathcal{L}}$ is completely Markov on $O(L^2(\mathcal{A},\tau),1)$ and, therefore, on $\mathcal{B}(L^2(S))$.*

Proof. If $a \in O(L^2(\mathcal{A},\tau),1)$, then $a = \beta 1 + \alpha$ with $\alpha \in L^2(\mathcal{A},\tau)$, $\beta \in \mathbb{C}$, and $\beta 1, \alpha$ commute. $e^{-t\mathcal{L}(\beta 1+\alpha)} = e^{-t\beta\mathcal{L}(1)}e^{-t\alpha} = e^{-t\alpha}$ which is completely Markov. The conclusion follows from the theorem 2.5.8. □

The endomorphism connection

By definition, the Evans-Hudson dilation requires that the semigroup be conservative. However, one quickly notes that acting by composition on $\mathrm{End}(L^2(M,S))$ the laplacian $\triangle$ (or the Dirac laplacian D^2) cannot generate a conservative semigroup. To see this, fix a basis (e_i) of eigensections of $\triangle$ for $L^2(M,S)$ and let λ_i be eigenvalue for $\triangle$ on e_i, then $(e_i \otimes e_j^*)_{i,j}$ is a basis for $\mathrm{End}(H)$. If $\triangle$ acts by composition on $\mathrm{End}(H)$ then it maps $e_i \otimes e_j^*$ to $\lambda_i e_i \otimes e_j^*$ implying $\triangle(\mathbf{1}) = \triangle^H(\sum_i e_i \otimes e_i^*) = 0$ cannot hold. However, there's a natural connection which defines a conservative semigroup.

Observation 2.5.10. On the endomorphism bundle, the canonical connection $\nabla^{\mathrm{End}(H)} = \nabla \otimes 1 + 1 \otimes \hat{\nabla}$, where $\hat{\nabla}$ is the dual connection induced on H^*, is easily seen define a conservative semigroup and is uniquely determined from ∇ since if over (U,ϕ_U) the connection acts locally by $\nabla(\sum \sigma^j \mu_j) = \sum_j (d\sigma^j)\mu_j + \sum_j \sigma^j A\mu_j$ for a matrix of T^*M-valued 1-forms A, then the dual connection acts with matrix $\hat{A} := -A^t$, and $\nabla^{\mathrm{End}(H)}(\sum_{ij} \sigma^i_j \mu_i \otimes \mu^j)$ is given by

$$\nabla^{\mathrm{End}(H)} \sum_{ij} \sigma^i_j \mu_i \otimes \mu^j = \sum_{ij} (d\sigma^i_j)\mu_i \otimes \mu^j + \sum_{jk} [\sigma A - A\sigma]_{jk} \mu_k \otimes \mu^j \qquad (2.8)$$

Additionally, if S^* carries parallel section ϕ_0^*, then an explict computation shows that $\triangle^{\mathrm{End}(H)}(\psi \otimes \phi_0) = [\triangle \otimes 1 + 2\sum_i \nabla_i \otimes \nabla_i + 1 \otimes \triangle](\psi \otimes \phi_0^*) = (\triangle\psi) \otimes \phi_0^*$ so the action of the laplacian embeds inside the action of the endomorphism laplacian. This property of endomorphism laplacian and the fact that it acts by commutator is reminiscent of the operator $A_{\mathcal{D}}$, and will be used for realizing spectral action.

Note that if E is a hermitian (or euclidean) vector bundle with connection ∇^E and H a Dirac bundle with connection ∇^H over M, then the $\phi \cdot (h \otimes e) \longrightarrow (\phi \cdot h) \otimes e$ for $\phi \in \mathrm{Cl}(X)$ defines a Clifford action on $H \otimes E$. The skew hermiticity of the action is obvious and as needed the tensor product connection $\nabla^{H\otimes E}$ satisfies

$$\nabla^{H\otimes E}(\phi \cdot (\sigma \otimes e)) = (\nabla^{\mathrm{Cl}(X)}\phi) \cdot (\sigma \otimes e) + \phi \cdot \nabla^{H\otimes E}\sigma \otimes e$$

Now the Dirac and Clifford structures are local as the Clifford multiplication acts on fibres and the connections can be computed in a chart. The local structures can then be glued to get the global structure. As a special case of tensor product bundles, consider $\mathrm{End}(S)$ for a Dirac bundle S. Suppose local sections $\mu_i : i \in [\dim H])$ form an orthonormal basis of S in chart (U, ϕ_U), and the corresponding dual basis (μ^i) for S^*. Over the U, $\mathrm{End}\big|_U(S)$ is just the bundle $S\big|_U \otimes S^*\big|_U$ with the fibers given by $\textsc{Span}\{e_i \otimes e^j : i, j \in [\dim H]\}$. This yields that if S is a Dirac bundle, then $\mathrm{End}(S), S \otimes E$ are Dirac bundles as well. Relevantly, there's the following observation.

Proposition 2.5.11. *Semigroups generated by laplacians $D^2, \triangle$ on End(S), with respect to the endomorphism connection, are conservative.*

Proof. As $\mathbf{1} = \sum_i \mu_i \otimes \mu^i$ in the local basis μ_i, equation 2.8 is just the commutator with identity, therefore, $\nabla(\mathbf{1})$ vanishes identically over U, making $\triangle(\mathbf{1}) = 0$. Similarly $D^2(\mathbf{1}) = 0$. □

A calculation using this (see chapter 4) leads to the following example which is relevant for canonical spectral triples.

Example 2.5.12. The algebra $C(M)$ acts on $L^2(\mathcal{S})$ by multiplication and $f \in C(M)$ can be identified with $f \cdot \mathbf{1}_{\mathrm{End}(\mathcal{S})}$. Then $\triangle^{\mathrm{End}(\mathcal{S})}$ for the Levi-Civita connection on $\mathcal{S}$ acts on $C(M)$ by sending f to $\triangle^M(f) \cdot 1_{\mathrm{End}(\mathcal{S})}$ where $\triangle^M$ the Laplace-Beltrami operator on functions.

Proposition 2.5.13. *The form $\mathcal{E}_{D^2}$ for the Dirac laplacian D^2 of the bundle End(S) is closed.*

Proof. This follows since the quadratic form $x \longrightarrow \langle Dx, Dx\rangle$ is closed because D is closd. □

Using the same arguments as before, along with proposition 2.5.11 gives the following.

Corollary 2.5.14. *The quadratic forms for both D^2 and $\triangle^{End(S)}$ are completely Dirichlet, they both generate conservative quantum dynamical semigroups.*

Example 2.5.15. The Clifford bundle can be viewed as the endomorphism bundle of the spinor bundle, and the connection as an endomorphism connection. It is a derivation on sections of the bundle, and, therefore, is zero on the identity element of the Clifford bundle.

2.6 Spectral action

The motivating application is introduced next. For the canonical spectral triple over Riemannian spin manifold, (M, g) $(C^\infty(M), L^2(S), D_M)$, $L^2(S)$ being the Hilbert space of square integrable sections of a spinor bundle $S \longrightarrow M$, and D_M the Dirac operator associated to the lift of Levi-Civita connection to the spinor bundle[66, pg 67], the bosonic spectral action is the linear functional, $S_b^M \equiv S_f := \mathrm{Tr}\, f(D/\Lambda)$ for a choice of an even test function $f : \mathbb{R} \to \mathbb{R}, f \geq 0$, which is often taken to be e^{-x^2} and Λ a cutoff parameter[30, § 5.1, 66, § 7.1]. The parameter t of the Dirac heat semigroup e^{-tD^2} corresponds to $t = \Lambda^{-2}$. From the asymptotic expansion $\lim_{\Lambda\to\infty} S_b^M$ for a Riemannian spin 4-manifold M, the spectral action, and therefore, the Einstein-Hilbert action, S_{EH}, can be recovered[30, § 5.3, 66, § 8.3].

From earlier, any self-adjoint operator A on Hilbert space H, the operator $A^2_{\mathcal{D}} := (A \otimes 1 - 1 \otimes A)^2$ acting on $H^* \otimes H = \mathcal{H}^2(L^2(M, S))$, the space of Hilbert-Schmidt operators in $\mathcal{B}(H)$, generates a conservative quantum dynamical semigroup, and the same for any spectral truncation D_n of $D = \sum_i \lambda_i \phi^i$, $D_n := \sum_{i\in[-n:n]} \lambda_i \phi_i$. A simple calculation yields the following, the idea being to use ϕ_0 to kill everything but the Dirac laplacian in the expansion for $D^2_{\mathcal{D}}$.

Lemma 2.6.1. *With $\phi_{i0} := \phi_i \otimes \phi_0 \in \mathcal{H}^2(H)$, $\langle \phi_i e^{-tD^2_{n,\mathcal{D}}}(\phi_{i0})\phi_0\rangle = e^{-t\lambda_i^2}$*

Proof. This follows since

$$e^{-tD^2_{n,\mathcal{D}}}(\phi_{i0}) = \sum_{k\in\mathbb{Z}_{\geq 0}} (-t)^k/k!(D^2_{n,\mathcal{D}})^k \phi_{i0} = \sum_{k\in\mathbb{Z}_{\geq 0}} (-t)^k/k!\lambda_i^{2k}\phi_{i0} = e^{-t\lambda_i^2}\phi_{i0}$$

Therefore, $\langle \phi_i e^{-tD^2_{n,\mathcal{D}}}(\phi_{i0})\phi_0\rangle = \langle \phi_i, e^{-t\lambda_i^2}\phi_i\rangle$ □

This means

$$\mathcal{S}_n := n\langle \sum_{i\in[n]} \phi_i/\sqrt{n}, e^{-tD^2_{n,\mathcal{D}}}(\sum_{i\in[n]} \phi_{i0}/\sqrt{n})\phi_0\rangle = \langle \phi_i, e^{-t\lambda_i^2}\phi_i\rangle = \sum_{i\in[n]} e^{-t\lambda_i^2}$$

$\lim_{n\to\infty}\mathcal{S}_n = S^M_{b,e^{-x^2}}$

Remark 2.6.2. The same calculations can be carried out on the Hilbert space $L^2(\mathcal{K}(H), \mathrm{Tr})$ using the embedding of $\mathcal{K}(H)$ into $\mathcal{B}(L^2(\mathcal{K}(H), \mathrm{Tr}))$.

That is, spectral action arises as a correlation between the state corresponding to the harmonic spinor and a uniformly random state. If D has no harmonic spinors then on replacing D_n with $D'_n := D_n^2 - \lambda^2$ for $\lambda = \lambda_m \in \mathrm{spec}(D), m \leq n$, the following generalization can be obtained.

Corollary 2.6.3. *With $D'_{n,\mathcal{D}} = (D_n^2 - \lambda^2)_{\mathcal{D}}$ for any $\lambda = \lambda_m \in spec(D), m \leq n$, then*

$$\lim_{n\to\infty} \langle \sum_{i\in n} \phi_i, e^{-tD'^2_{n,\mathcal{D}}}(\sum_{i\in[n]} \phi_{i0})\phi_0\rangle = S^M_{b,e^{-(x^2-\lambda^2)^2}}$$

The lemma 2.6.1 can be applied to the semigroup $e^{-t\mathcal{L}}, \mathcal{L} = \triangle^M$ on $C^\infty(M) \subset C(M)$ acting by multiplication on the Hilbert space of L^2 functions, $L^2(M)$, instead of a vector bundle. If ϕ_i's are eigenfunctions of $\triangle^M$ with $\triangle^M\phi_i = \lambda_i^2\phi_i$, $\lambda_i^2 > 0$, then since $C(M)$ has the constant function $\mathbf{1}$ as the unit, $\triangle^M(\mathbf{1}) = 0$ and $\langle\phi_i, \mathbf{1}\rangle = 0$, we immediately have:

Corollary 2.6.4. *The eigenvalues for $\triangle^M$ can be computed from the expectations of heat semigroup:*

$$\langle \phi_j, e^{-t\mathcal{L}}(\phi_i)\mathbf{1}\rangle = \langle \phi_j, e^{-t\lambda_i^2}\phi_i\mathbf{1}\rangle = e^{-t\lambda_i^2}\delta_{ij}$$

Therefore, the heat kernel trace $\sum_i e^{-t\lambda_i^2}$ can be approximated as in 2.6.3. This motivates interest in understanding the Dirac heat semigroup and its Evans-Hudson dilation which yields the expectations of type $\langle u, e^{-t\triangle(x)}v\rangle$ for any u, v in as expectations of a quantum diffusion process. Because the generator $D_{n,\mathcal{D}}$ is bounded, the semigroup generated is norm continuous and, therefore, existing constructions from [11, 62] can be adapted to realize spectral action from a quantum stochastic flow. The flow is associated to a quantum stochastic differential equation (qsde) of Evans-Hudson type. For now the computation of the coeffients of the qsde for $D_{n,\mathcal{D}}$ is deferred since it's covered by existing theory. This leads to the following result.

Theorem 2.6.5. *Let D_n be the Dirac operator, D_n, $f(x) = e^{-x^2}$, $f_\lambda(x) = e^{-(x^2-\lambda^2)}$, $\lambda \in Spec(D)$, the Evans-Hudson flows j_t, j'_t exists for generators $D^2_{n,\mathcal{D}}, D'^2 = (D^2_{n,\mathcal{D}} - \lambda^2)^2$, and satisfies*

1. *If D has a harmonic spinor ϕ_0*

$$\lim_{n\to\infty} \langle \sum_{i\in n} \phi_i E(0), j_t(\sum_{i\in[n]} \phi_{i0})\phi_0 E(0)\rangle = S^M_{b,e^{-x^2}}$$

2. *For eigenspinor associated to λ, ϕ_λ,*

$$\lim_{n\to\infty} \langle \sum_{i\in n} \phi_i E(0), j'_t(\sum_{i\in[n]} \phi_{i\lambda})\phi_\lambda E(0)\rangle = S^M_{b,e^{-(x^2-\lambda^2)^2}}$$

where ϕ_{ij} denotes $\phi_i \otimes \phi^*_j$, $E(0)$ the Fock vacuum for noise space.

In chapter 4, the focus is on computing the trace of the heat kernel for laplacian on functions exactly without truncating: the existence of the flow for the unbouded generator is not apriori clear and even though the spectral action can be approximated it's of interest to consider the existence of the flow for the untruncated laplacian. A growth condition on Sobolev norms of laplacian eigenfunctions on compact manifolds turns out to be sufficient, examples include flat and homogeneous manifolds. This is done by adapting construction from Sinha and Goswami [62].

Note that the $e^{-t\triangle^M}$ defines a positive semigroup on $C^\infty(M)$ and therefore, by commutativity, a completely positive semigroup which is both contractive and conservative. In fact, the laplacian can be formally put in the form of generators from norm continuous quantum dynamical semigroups even though the semigroup generated is only strongly continuous.

2.7 Product and almost commutative spectral triples

The special case of product almost commutative spectral triples is addressed now. While it can be handled with the same Dirichlet form machinery, it's worth noting how the two pieces in the product do not interact. Following this, a brief note is made about C^*-bundles which provides an alternative perspective on bundles of C^*-algebras bundles over Riemannian manifolds.

Complete positivity on products

For real even spectral triples, $(\mathcal{A}_i, \mathsf{H}_i, D_i; J_i, \gamma_i)$, $i \in \{1, 2\}$, that is, the spectral triples come with a real structure J_i and a grading γ_i such that for all $a \in \mathcal{A}_i, \gamma_i a =$

$a\gamma_i, \gamma D_i = -D_i\gamma$, the product is defined by $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2, \mathsf{H} = \mathsf{H}_1 \otimes \mathsf{H}_2, D := D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma = \gamma_1 \otimes \gamma_2, J = J_1 \otimes J_2$. If the second triple is not even then the resulting structure does not have a grading and the adjective even is dropped. Since the first triple is even and D_1, γ_1 anti-commute, $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$. Note that $C^\infty(M, A_F) = C^\infty(M) \otimes A_F$ and the tensor products are $\mathbb{Z}_2$-graded. The algebras $\mathcal{A}_1, \mathcal{A}_2$ are only pre-C^*-algebras, but can be completed in the respective C^*-norm; for the canonical spectral triple, $C^\infty(M)$, will have $C(M)$ as the closure. For a spectral triple, it's not required that $\mathcal{A}_i$ be closed, though a requirement $[D_i, a]$ is bounded needed for $a \in \mathcal{A}_i$. Questions about quantum dynamical semigroups, however, need the algebras to be norm-closed.

The product almost commutative spectral triple is the product of the canonical spectral triple of a Riemannian spin manifold, $\mathfrak{A}_M := (C^\infty(M), L^2(S), D_M; J_M, \gamma_M)$, and a finite noncommutative space, $\mathfrak{A}_F := (A_F, H_F, D_F; J_F, \gamma_F)$,

$$M \times F := (C^\infty(M) \otimes A_F, L^2(M, S \otimes H_F)), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)$$

Recall that on the algebraic tensor product of C^*-algebras $\mathcal{A}_1 \otimes \mathcal{A}_2$, a cross-norm is a norm satisfying $\|a_1 \otimes a_2\|_{\mathcal{A}_1 \otimes \mathcal{A}_2} = \|a_1\|_{\mathcal{A}_1} \|a_2\|_{\mathcal{A}_2}$. A C^*-algebra is nuclear if any tensor product carries a unique cross-norm, and, therefore, the algebraic tensor product has a unique norm completion. The following technical lemma will be used implicitly. The point of this lemma is that such identifications described by it behave well.

Lemma 2.7.1. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are unital C^*-algebras. Suppose at least one of $\mathcal{A}_1, \mathcal{A}_2$ is nuclear, so there's a unique cross-norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$, then the map, $\phi : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_2, a \to 1 \otimes a$, is a completely positive, homeomorphism onto its image.*

Proof. The kernel of ϕ is trivial, and ϕ is positive as a positive in $\mathcal{A}_1$ means $1 \otimes a$ is positive in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Additionally, ϕ is unital. From the $\mathbb{R}$-linearity of the tensor product, it follows the map ϕ preserves norms. It also follows that $\phi \otimes \mathbf{1}_n$ also preserves norms, so ϕ is a unital, completely contractive map, and hence is completely positive. Being contractive also implies continuity. The inverse map on the image, ϕ^{-1}, $1 \otimes a \to a$, is again unital and completely contractive: the same holds for ϕ^{-1} as well. □

We note the following about the complete positivity of the Dirac heat semigroup for product almost commutative spectral triples:

Proposition 2.7.2. *For the product almost-commutative triple, $\mathfrak{A}_F \times \mathfrak{A}_M$,*

- *The norm completion of $C(M) \otimes A_F$ is unique.*
- *The complete positivity of the semigroup does not depend on the order of the product, $\mathfrak{A}_F \times \mathfrak{A}_M$ versus $\mathfrak{A}_M \times \mathfrak{A}_F$.*

Proof. If both spectral triples are $\mathbb{Z}_2$-graded with $\mathbb{Z}_2$-graded tensor product, $C(M)\otimes A_F$, then since commutative C^*-algebra are characterized as nuclear[26] in $\mathbb{Z}_2$ graded tensor product category, all cross-norms on the tensor product agree, then there's no ambiguity on the norm with respect to which to take the norm closure. The independence from the order of the product is simply the symmetry of the norm and therefore of the definition of positivity. □

Theorem 2.7.3. *Let $D^2 = 1 \otimes D_M^2 + D_F^2 \otimes 1$ acting on $A_M \otimes A_F$.*

- *If $e^{-tD_M^2}$ and $e^{-tD_F^2}$ are both completely positive then e^{-tD^2} is as well. The converse holds when e^{-tD^2} is conservative.*
- *If $e^{-tD_M^2}$ and $e^{-tD_F^2}$ are contractive (conservative), then the composition e^{-tD^2} is contractive (conservative). The converse does not hold.*

Proof. Because $1\otimes D_M^2$ and $D_F^2\otimes 1$ commute, therefore, $e^{-tD^2} = e^{-t(1\otimes D_M^2)}e^{-t(D_F^2\otimes 1)} = e^{-t(D_F^2\otimes 1)}e^{-t(1\otimes D_M^2)}$. Now suppose $e^{-tD_M^2}$ and $e^{-tD_F^2}$ are completely positive. The tensor product of completely positive maps extends to a completely positive map with respect to the $\|\cdot\|_{min}$ (see, for instance, [59, Thm 12.3]; the standard result is for ungraded tensor product, but it applies since commutative C^*-algebras are nuclear regardless of the grading and there's only one cross norm across both settings). Since $\mathcal{A}_M$ is nuclear, $1 \otimes e^{-tD_M^2}, e^{-tD_F^2} \otimes 1$ are completely positive on $\mathcal{A}_F \otimes \mathcal{A}_M = \mathcal{A}_F \otimes_{min} \mathcal{A}_M$. Furthermore, $1 \otimes D_M^2, D_F^2 \otimes 1$ commute, and e^{-tD^2} is the composition of completely positive maps and also completely positive. When e^{-tD^2} is conservative, $e^{-tD^2}(1 \otimes \mathcal{A}_M) = 1 \otimes e^{-tD_M}(\mathcal{A}_M)$. Since $1 \otimes \mathcal{A}_M$ generates the C^*-algebra, $\mathbb{K} \otimes_{\mathbb{K}} \mathcal{A}_M \cong \mathcal{A}_M$, with $\mathbb{K} = \mathbb{C}, \mathbb{R}$ depending on the underlying Hilbert space, so $e^{-tD_M^2}$ is completely positive, with symmetric argument for $e^{-tD_F^2}$.

The forward direction in the second part is clear; for the failure of the converse, if on $\mathbf{1}$, $e^{-tD_M^2}$ and $e^{-tD_F^2}$ are multiplication by $a \neq 0, 1$ and $1/a$, their composition can be both conservative and contractive, individually, they do not hold. □

The point of the above characterization is that the generator $-D^2$ can be decomposed into the bounded part $-D_F^2$ and the unbounded part $-D_M^2$, the structure of the bounded generator being completely determined work of Gorini-Kossakowski-Sudarshan-Lindblad[4].

Example 2.7.4. On the product spectral triple, $M \times F = (C^\infty(M) \otimes A_F, L^2(M, S \otimes H_F), D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F)$, it's assumed that D_F does not know about M. This can be generalized slightly following [16] to the picture where $M \times H_F \to M$ is a trivial bundle with a trivial connection ∇^F and $H = S \otimes (M \times H_F)$ is a twisted spinor bundle, i.e., the Clifford action takes place on S. Now the geometric Dirac operator on H is given by

$$\not{D}_{M \times H_F} \equiv \not{D}_H := \not{D}_S \otimes 1 + c \otimes \nabla^F \tag{2.9}$$

where c is the Clifford action. Ćaćić [16] defines the operator $D = \not{D}_H + c \otimes \Omega_F$ where Ω_F is potential associated with D_F, and as the motivating example checks that D is Dirac type operator on the spectral triple $(C(M, A), L^2(M, H), D)$ where $A := \mathsf{L} \otimes (X \times A_F)$ for L a real, unital, trivial sub-bundle of $\mathrm{End}(S)$ given by $\mathsf{L}_x := \mathbb{R}1_{S_x}$. This also illustrates how spinor bundles arise naturally and provides an example of a Dirac operator that is not the geometric operator,

$$D := \not{D}_S \otimes 1 + c \otimes (\nabla^F + \Omega_F), D^2 := \not{D}_S^2 \otimes 1 + c^2 \otimes (\nabla^F + \Omega_F)^2 + (\not{D} \circ c + c \circ \not{D}) \otimes (\nabla^F + \Omega_F) \tag{2.10}$$

In the bundle H, the symmetric operator D_F on the fibers $(H_F)_{m \in M}$ can now vary with $m \in M$. Note the mixed term $(\not{D} \circ c + c \circ \not{D}) \otimes (\nabla^F + \Omega_F)$ that now appears even when D_F is constant. Complete positivity on such bundles is addressed in the following sections with more geometric methods.

Heat semigroups on twisted spinor bundles

On taking tensor products, the product laplacian picks up cross terms, and the heat semigroups on the tensor components no longer commute as the individual generators have non-trivial interaction with the cross-terms. In general, the semigroup for the twisted laplacian $\mathcal{L} = \triangle^{\mathcal{S} \otimes \mathcal{W}}$, $e^{-t\mathcal{L}}$ necessitates using Baker-Campbell-Hausdorff type formula to understand it in terms of the components. The conditions on bundle connections when this may be simplified (for example, like product and almost commutative spectral triples where the two tensor pieces will be established to not interact) requires that the terms in expansion of the laplacian commute.

Suppose $H = \mathcal{S} \otimes E$, that is, H is the spinor bundle $\mathcal{S}$ twisted by E, with Dirac laplacian, D_H^2, while the associated connection laplacian is $\triangle^H$. Using that the connection laplacian $\triangle^{\mathcal{S}\otimes E} = -\operatorname{Tr}((V,V') \to \nabla^2_{V,V'})$ where $\nabla^2_{V,V'} = \nabla_V\nabla_{V'} - \nabla_{\nabla_V V'}$ which in the Riemann normal frame becomes $\triangle^{\mathcal{S}\otimes E} = -\sum_i \nabla_{e_i}\nabla_{e_i}$. Explicitly the tensor connection laplacian is given by:

$$\triangle^{\mathcal{S}\otimes E}\sigma = -\sum_i \nabla^{\mathcal{S}\otimes E}_{e_i}\nabla^{\mathcal{S}\otimes E}_{e_i} = -\sum_i \left(\nabla^{\mathcal{S}}_i \otimes 1 + 1 \otimes \nabla^E_i\right)\left(\nabla^{\mathcal{S}}_i \otimes 1 + 1 \otimes \nabla^E_i\right)\sigma$$
$$= -\sum_i \left(\nabla^{\mathcal{S}}_i\nabla^{\mathcal{S}}_i \otimes 1 + 2\nabla^{\mathcal{S}}_i \otimes \nabla^E_i + 1 \otimes \nabla^E_i\nabla^E_i\right)\sigma = \left(\triangle^{\mathcal{S}} \otimes 1 - 2\sum_i \nabla^{\mathcal{S}}_i \otimes \nabla^E_i + 1 \otimes \triangle^E\right)\sigma \tag{2.11}$$

In general, it can be verified that the terms commute when the curvatures of the bundles E and $\mathcal{S}$ vanish identically.

Lemma 2.7.5. *For any vector bundle, in Riemann normal coordinates centered at $p \in M$, the connection laplacian $[-\triangle^V, \sum_j \nabla_j] = \sum_{ij} R(i,j)\nabla_i + \sum_{ij} \nabla_i R(i,j)$ at p.*

Proof. In Riemann normal frame, $(e_i : i \in [\dim V])$, using $R(i,j) = \nabla_i\nabla_j - \nabla_j\nabla_i$, with shorthand $R(i,j) := R(e_i, e_j)$

$$-\triangle^V \sum_j \nabla_j = \sum_i \nabla_i\nabla_i \sum_j \nabla_j$$
$$= \sum_{ij} \nabla_i\nabla_i\nabla_j = \sum_{ij}(\nabla_i\nabla_j\nabla_i + \nabla_i R(i,j)) = \sum_{ij} \nabla_j\nabla_i\nabla_i + \sum_{ij} R(i,j)\nabla_i + \sum_{ij} \nabla_i R(i,j)$$

That is, $[-\triangle^V, \sum_j \nabla_j] = \sum_{ij} R(i,j)\nabla_i + \sum_{ij} \nabla_i R(i,j)$. Using $R(i,j) = -R(j,i)$ and $R(i,i) = 0$,

$$\sum_{ij} R(i,j)\nabla_i = \sum_{i<j}\left(R(i,j)\nabla_i + R(j,i)\nabla_j\right) + \sum_{i=j} R(i,j)\nabla_i$$
$$= \sum_{i<j}\left(R(i,j)\nabla_i + R(j,i)\nabla_j\right) = \sum_{i<j} R(i,j)\left(\nabla_i - \nabla_j\right)$$

By the second Bianchi identity on bundle E [65], $(\nabla_u R)(v,w) + (\nabla_v R)(w,u) + (\nabla_w R)(u,v) = 0$, when $v = u = w$, $(\nabla_u R)(u,u) = 0$, therefore

$$\sum_{ij} \nabla_i R(i,j) = \sum_{i<j}(\nabla_i R(i,j) + \nabla_j R(j,i)) = \sum_{i<j}(\nabla_i R(i,j) - \nabla_j R(i,j)) = \sum_{i<j}(\nabla_i - \nabla_j)R(i,j)$$

This yields $[-\triangle^V, \sum_j \nabla_j] = \sum_{ij} R(i,j)\nabla_i + \sum_{ij} \nabla_i R(i,j)$

□

Now consider almost-commutative spectral triples. Over an even dimensional Riemannian manifold, (M, g), the algebra for an almost commutative spectral triple, $A \subset \mathrm{End}^{+}_{\mathrm{Cl}(X)}(H)$, where H is Clifford module bundle over M (and therefore a twisting of the complex spinor bundle $\mathcal{S}$, $H = \mathcal{W} \otimes \mathcal{S}$) does not interact with the $\mathcal{S}$-connection.

Proposition 2.7.6. *If $\alpha \in A \subset End^{+}_{\mathrm{Cl}(X)}(H)$ then $\alpha = w_\alpha \otimes 1$ for $w_\alpha \in End(\mathcal{W})$ up to multiplication by $f \in C(X)$. That is, as a module over $C(M)$, $C(M, A)$ is generated by endomorphims of form $w_\alpha \otimes 1$.*

Proof. The proof is basically the observation that locally $\mathrm{End}(\mathcal{W}) \cong \mathrm{End}_{\mathrm{Cl}(X)}(H)$ (see, for instance, [12, Prop 3.27]) (i.e. $A \cong W_A \subset \mathrm{End}(\mathcal{W})$). Now $\mathrm{End}(H)$ is the topological closure of $\mathrm{End}(\mathcal{W}) \otimes \mathrm{End}(\mathcal{S})$, where because $\mathcal{S}$ is the complex spinor bundle, $\mathrm{End}(\mathcal{S}) \cong \mathrm{Cl}(M) \otimes \mathbb{C}$.

Suppose $\alpha = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} \in A \subset \overline{\mathrm{End}(\mathcal{W}) \otimes \mathrm{Cl}(X) \otimes \mathbb{C}}$ where $\alpha_{s,i} \in \mathrm{Cl}(X) \otimes \mathbb{C}$. Consider the Clifford action, $c : \mathrm{Cl}(X) \to \mathrm{End}(H)$, $v \to c(v) := \sum_i w_i \otimes s_i \in \mathrm{End}(H)$ with $w_i \in \mathrm{End}(\mathcal{W})$, $s_i \in \mathrm{Cl}(X) \otimes \mathbb{C}$. By construction of the twisted spinor bundle, the Clifford action on $\mathcal{W}$ piece is trivial so $w_i = 1$ for all i, therefore, $c(v) = 1 \otimes v_s$ with $v_s \in \mathrm{Cl}(X) \otimes \mathbb{C}$.

Since α commutes with the Clifford action

$$\sum_i \alpha_{w,i} \otimes v_s \alpha_{s,i} = (1 \otimes v_s) \circ \sum_i \alpha_{w,i} \otimes \alpha_{s,i} = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} \circ (1 \otimes v_s) = \sum_i \alpha_{w,i} \otimes \alpha_{s,i} v_s$$

In even dimensions, the canonical complex bundle $\mathcal{S}$ in the twisted spinor decomposition, $\mathcal{W} \otimes \mathcal{S}$, is irreducible Clifford module and $\mathrm{Cl}(M)$ is a central simple algebra; therefore, v_s runs over all elements in $\mathrm{Cl}(M) \otimes \mathbb{C}$. As v_s is arbitrary, $\alpha_{s,i}$ lie in the center of $\mathrm{Cl}(M)$. This can be seen locally: choose a basis (e_i) for T^*X, then the basis for $\mathrm{Cl}(T^*M)$ is $(e_I)_{I \subset [\dim T^*X]}$. Expressing α in e_I's gives $\sum_i \alpha_{w,i} \otimes \alpha_{s,i} = \sum_I k_I \alpha_I \otimes e_I$ for k_I. Note that $e_i \cdot e_I = \pm e_I \cdot e_i$ for any i. Suppose $|I| > 0$. If $|I|$ is odd, then there exists $j \notin I$, and $e_I \cdot e_j = -e_j \cdot e_I$ as it commutes past each e_i for $i \in I$. If $|I| = 2k$ with $e_I = e_{i_1} \cdot \ldots e_{i_{2k}}$, then $e_I \cdot e_{i_{2k}} = -e_{i_1} \ldots e_{i_{2k-1}}$, while $e_{i_{2k}} e_I = e_{i_1} \ldots e_{i_{2k-1}}$ because there are $2k - 1$ sign changes on moving across and then a final sign change from $e_{i_{2k}}^2 = -1$. Therefore, $|I| = 0$ for e_I to commute with each e_i but then $e_I \in \mathcal{Z}(\mathrm{Cl}(T^*X))$. The conclusion holds on the algebraic tensor product and also the topological completion. □

This is consistent with the case for commutative spectral triples where the algebra $C^\infty(M)$ acts by multiplication on the spinor bundle $L^2(S)$ and commutes with the Clifford action.

Theorem 2.7.7. *The heat semigroup generated by $\triangle^{\mathcal{W}\otimes\mathcal{S}}$ on $C(M,A)$ is isomorphic to heat semigroup generated by $\triangle^{\mathcal{W}}$ on $\mathcal{W}$, that is, the heat semigroup of an even-dimensional spectral triple is isomorphic to the heat semigroup generated by the laplacian for the twisting space.*

Proof. By proposition 2.7.6, the laplacian of the spinor bundle $\mathcal{S}$ does not interact with $C(M,A)$. It follows from the explicit computation of the Dirac laplacian for the twisted bundle (equation 2.11) $\mathcal{W}\otimes\mathcal{S}$ that the heat semigroup on $C(M,A)$ acts trivially on the $\mathcal{S}$, and therefore is determined solely by action of the $\mathcal{W}$ connection. □

Remark 2.7.8. This is not true for the Dirac laplacian. On expanding, $D^2 = \sum_i e_i\nabla_i \sum_j e_j\nabla_j$ for Clifford action and the connection on the twisted $\mathcal{W}\otimes\mathcal{S}$, D^2 picks up nontrivial action on $\mathcal{S}$ component; this is most easily seen by noting that the curvature operator from the Bochner idenity (definition 2.5.2) does not fix $\mathbf{1}_{\mathcal{S}}\otimes w \in \mathrm{End}(\mathcal{W}\otimes\mathcal{S})$ unless $w = \mathbf{1}_{\mathcal{W}}$ as well.

C^*-bundle point of view

The almost-commutative geometric perspectives works with the infinite-dimensional Hilbert space $L^2(M,S)$. An alternative perspective is to consider the bundle of finite-dimensional C^*-algebras on the fibers. While such bundles parametrize the space of noncommutative gauge fields, after putting a L^2-structure on the fiber and averaging over the fibers, such bundles are necessarily trivial (see, for instance, [7]) and do not see the global structure. Results on complete-positivity of Clifford C^*-bundles were obtained by [28, 24]. The results are recapped, and the difference between the C^*-bundle and noncommutative geometry setting is made precise.

Following [28], a C^*-bundle, $\mathcal{A}$, is a finite dimensional vector bundle over a Riemannian manifold (M,g), compact, without boundary, where the fibers are a finite dimensional C^*-algebra A. The structure group of the bundle is the compact Lie group of $*$-automorphisms of $\mathcal{A}$. A normalized, invariant trace is selected, τ, for A that is used for every fiber. Since fibers are finite-dimensional, the usual normalized matrix trace, $\tau = \mathrm{Tr}$, is most relevant. The trace on the bundle is obtained by integrating over the fibers. The invariance of the trace under automorphisms of the algebra is used as the transition functions need to be trace-preserving.

The space of smooth sections $\Gamma^\infty(M, \mathcal{A})$ can be completed to a C^*algebra of continuous sections, $\Gamma^0(M, \mathcal{A})$ using the C^*-structure on the fibers with the norm $\|f\|_\infty = \sup_{x \in M}\{\|f(x)\|\}$ norm, and the involution defined by pointwise involution on the fibers. The L^p-norm on $\Gamma^\infty(M, \mathcal{A})$ is defined as usual,

$$\|f\|_p^p = \int_M \mathrm{Tr}_x((f(x)f^*(x))^{p/2})\, \mathrm{d}_{vol}\, M \tag{2.12}$$

[28, Thm 17] shows that for any vector bundle V with a metric connection, ∇^V, over M, the Bochner laplacian , $\triangle_B = -(\nabla^V)^*\nabla^V$, generates a completely positive semigroup on $L^2(M, \mathrm{Tr}_x^{\mathrm{Cl}(V)})$. [24] show that on $L^2(M, \mathrm{Tr}_x^{\mathrm{Cl}(T^*M)})$, the Dirac laplacian $\not{D}^2$ generates a completely Markov semigroup if and only if the curvature operator is positive. The idea exploits the form of the curvature operator on $\mathrm{Cl}(T^*M)$ to get it to generate a completely Dirichlet form; then using the general Bochner identity, the complete positivity of the semigroup generated by the Bochner laplacian and the correspondence between completely Markov semigroups and completely Dirichlet forms gives the result[24, Thm 5.1].

Remark 2.7.9. $L^2(M, (\mathrm{End}(S), \|\cdot\|_{HS}))$ and $L^2(M, \mathrm{Tr}_x^{\mathrm{Cl}(V)})$ are different since the norm structures are different. The difference in complete-positivity, i.e., the dependence on the sign of the curvature operator on C^*-bundles appears because the Dirac operator is not symmetric with respect to the trace at the fibers. This difference between the two structures is best captured by noting that $\mathbf{1} \in L^2(M, \mathrm{Tr}_x^{\mathrm{Cl}(E)})$ since at each fiber the normalized trace of the unit will be finite, while $\mathbf{1} \notin L^2(M, (\mathrm{End}(S), \|\cdot\|_{HS}))$ because Hilbert-Schmidt operators are compact and the identity is not compact for the infinite dimensional $L^2(M, (\mathrm{End}(S), \|\cdot\|_{HS}))$.

Chapter 3

MAP-VALUED QSDES

The background to quantum stochastic differential equations is introduced in this chapter. The reference on map-valued qsde's is Goswami and Sinha's monograph[62], while background on quantum probability and Wiener space analysis follows [58, 52, 55]. These constructions can be done in noncommutative probability in general (for instance, see [14] for free probability), the key idea is to replace the σ-algebra of events by the possibly noncommutative algebra of $\mathbb{R}$-indexed random variables, that is, adapted stochastic processes. Note that notion of independence in noncommutative probability is not unique and a model for noncommutative probability needs to be fixed (see [36]). The applications considered in the next chapter deviate from this theory in one regard — the generator for the heat semigroup on the canonical spectral triples does not induce a Frechet structure, and the usual regularity requirement of complete-smoothness (definition 4.5.3) cannot be used. The point of this chapter is to define a more general class of integrable processes than considered by [62]. The integrability of such processes, specifically the guaranteed existence of the quantum stochastic integrals that arise through the Picard scheme is noted in remark 3.2.5

3.1 The Wiener-Segal-Ito correspondence

The operator stochastic integrals are defined in analogy with the classical stochastic integral, the motivating principle being identification between the (boson) Fock space and Wiener space, so that usual Wiener integral is recovered under the identification. In this section, some aspects of this correspondence are summarized.

A remark on notation: H, K will denote Hilbert space. The identity operator on various space and distinguished identity vector will all be denoted by $\mathbf{1}$ when clear from context. The n-fold symmetric tensor product is denoted by $H^{\circ n}$, but clear from context, $\circ$ will be denoted by $\otimes$ and symmetric Fock space $\Gamma_s(H)$ by $\Gamma(H)$. By convention, $H^{\circ 0} = \mathbb{C}$. As is customary, tensor symbol in Hilbert space tensor product will be suppressed when clear, e.g. $H \otimes K = HK$.

Let $\Omega := C(\mathbb{R}^{\geq 0}, \mathbb{R})$ be the space of maps $X : \mathbb{R}^{\geq 0} \to \mathbb{R}$ with $X_t := X(t)$. The space Ω is given the filtration $\mathcal{F}_t$ generated by X_t, and the Wiener measure, $\mathbb{P}_W$ which is the unique measure satisfying $X_0 = 0$ a.s. and the process (X_t) has independent centered Gaussian increments, $\mathbb{E}\left[(X_t - X_s)^2\right] = t - s$.

The symmetric (boson) Fock space over H is the space $\Gamma_s(H) := \oplus_{n\in\mathbb{Z}_{\geq 0}} H^{\circ n}$. The set of exponential vectors $\mathrm{E}(H) := \{\mathrm{E}(v) : v \in H\}$

$$H \ni v \to \mathrm{E}(v) := \mathbf{1} \oplus_{n\in\mathbb{N}} \tfrac{1}{\sqrt{n!}} v^{\circ n} \in \Gamma_s(H)$$

is total in the boson Fock space and compatible with the H-innerproduct, $\langle \mathrm{E}(f), \mathrm{E}(g)\rangle_{\Gamma(H)} = e^{\langle f,h\rangle_H}$. The creation and annihilation operators, a_h^+, a_h^-, $h \in H$, are defined by

$$a_h^- \mathrm{E}(f) := \langle h, f\rangle \mathrm{E}(f) \text{ and } a_f^+ \mathrm{E}(f) := \tfrac{d}{d\epsilon} \mathrm{E}(f + \epsilon h)\Big|_{\epsilon=0} \tag{3.1}$$

The Fock vaccumm vector $\mathbf{1}$ is $\mathbf{1}_{\mathbb{C}} \oplus_{n\in\mathbb{N}} 0$, with $a_f^- \mathbf{1} = 0$.

The second quantization for $A \in \mathcal{B}(H, K)$ is the operator $\Gamma(A) \in \mathcal{B}(\Gamma(H), \Gamma(K))$ defined by $(\circ_{i\in[n]} f_i) = \circ_{i\in[n]} A f_i$, therefore, $\Gamma(A)(\mathrm{E}(f)) := \mathrm{E}(Af)$. The differential second quantization is self-adjoint generator $a^\circ(\mathcal{L}) := d\,\Gamma(\mathcal{L})$ for the unitary group $\Gamma(e^{it\mathcal{L}})$ generated by $\mathcal{L}$ with

$$a^\circ(\mathcal{L})\mathrm{E}(f) := a_{\mathcal{L}f}^+ \mathrm{E}(f)$$

The Wiener-Segal-Ito isomorphism between the Wiener space $\mathcal{W}(\Omega) := (\Omega, \mathbb{P}_W, \mathcal{F})$ and $\Gamma(H)$ for $H = L^2(\mathbb{R}^{\geq 0})$ is the map

$$\mathrm{E}(f) \to e^{M(f)},\ M(f) := \int_0^\infty f(t)dW_t - \tfrac{1}{2}\int_0^\infty f(t)^2 dt$$

where is W_t is one-dimensional Brownian motion. $M(f)$ is the exponential martingale that is unqiue solution to sde $dZ_t = Z_t f(t) dW_t$. The isomorphism is realized through chaos expansions as follows. For a rectangle $H = \times_{i\in[n]}(a_i, b_i]$ in $\Sigma_n = \{(s_i : i \in [n]), s_i < s_{i+1}\} \subset \mathbb{R}^{\geq 0^n}$ (with convention that Σ_0 is $\{\emptyset\}$, J_0 maps to constant random variables), define

$$J_n(\mathbf{1}_H) := \int_{\Sigma_n} \mathbf{1}_H(s_1, \ldots s_n) dX_{s_1} \ldots dX_{s_n} = \prod_i (X_{b_i} - X_{a_i})$$

as the stochastic integral with respect to Brownian motions, X_{s_k}'s. Then J_n extends to $J_n : L^2(\Sigma_n) \to L^2(\mathbb{P}_W)$. The n^{th}-chaos is the image $C_n := J_n(L^2(\Sigma_n))$ with $\oplus_n C_n = L^2(\mathbb{P}_W)$. The order structure of $\mathbb{R}$ is not important for defining J_n: for any symmetric function h,

$$I_n(h) := \int_{\mathbb{R}^n} h(s_1, \ldots s_n) dX_{s_1} \ldots dX_{s_n} = n! J_n(h)$$

The solution $e^{M(f)}$ to $Y_t = 1 + \int_0^t Y_t f(t) dW_t$ (which is sde $dZ_t = Z_t f(t) dW_t$ in integral form) can be constructed as $e^{M(f)} = \sum_n \frac{1}{n!} I_n(f \mathbf{1}_{(0,t])}^{\circ n})$. Since the martingale representation for any Fock vector gives the identification for $\Gamma(L^2(\mathbb{R}^{\geq 0}))$ with $\mathcal{W}(\Omega)$, multiplication by Brownian motion, W_t, in $\mathcal{W}(\Omega)$ can be viewed as an operator on the Fock space, (see, for instance, [13]) and is given by

$$W_t = a_t^+ + a_t^- \text{ where } a_t^{\pm} := a_{\mathbf{1}_{[0,t]}}^{\pm}$$

Because $a_t^- \mathbf{1} = 0$, $a_t^+ \mathbf{1}$ can be identified with Brownian motion W_t.

The Ito-Wiener-Segal isomorphism [55] is defined more generally than $L^2(\mathbb{R}^{\geq 0})$. For a separable Hilbert space H, the H-indexed family $\mathcal{W} = \{W(h), h \in \mathsf{H}\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with each $W(x) \in \mathcal{W}$ a centered Gaussian satisfying $\mathbb{E}(W(h), W(g)) = \langle h, g \rangle_{\mathsf{H}}$, is called an isonormal Gaussian process. When $\mathcal{G}$ is the σ-field generated by $w \in \mathcal{W}$ for an appropriate isonormal Gaussian process $\mathcal{W}$ (see, for instance, [55, § 1.1]), $L^2(\Omega, \mathcal{G}, \mathbb{P})$ is isomorphic to the symmetric Fock space $\Gamma(\mathsf{H})$. Additionally, when H is the space $L^2(T, \mathcal{B}, \mu)$ where μ is σ-finite without atoms over a measure space $(T, \mathcal{B})$, $W(h)$ can be regarded as stochastic integrals, with polynomials in $W(h)$ dense in $L^2(\Omega, \mathcal{G}, \mu)$. The canonical example[58, Ex 19.9] is the one just considered, $\mathsf{H} := L^2(\mathbb{R}_{\geq 0})$ where $\Gamma(L^2(\mathbb{R}_{\geq 0})) \cong L^2(C(\mathbb{R}_{\geq 0}), \mathbb{P}_{\text{Wiener}})$. Through the Ito-Wiener-Segal isomorphism between Wiener space of paths of Brownian motions on a compact manifold M, $W(M)$, and the associated Fock space, the heat semigroup (as considered in example 2.2.6) has a stochastic dilation on the Fock space. This dilation corresponds to a flow for a Evans-Husdon type quantum stochastic differential equation (qsde) introduced next. A process satisfying a qsde of this type is considered as a quantum diffusion process.

3.2 Map-valued Evans-Hudson quantum sde's

In this section the relevant theory for quantum stochastic processes is collected. The exposition is based on the coordinates free formalism developed in [62]. [58] contains a classical treatment, while [52] makes the relationship with Ito calculus and commutative probability clear. Suppose $\mathcal{A}_0$ is a dense $*$-subalgebra inside the C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathsf{H})$. Let k_0 be the noise space, with $\hat{k}_0 := \mathbb{C} \oplus k_0$. Set

- $k := L^2(\mathbb{R}_+, k_0), k_t := L^2([0,t), k_0), k^t := L^2([t, \infty), k_0)$
- $\Gamma = \Gamma_s(k)$. $\Gamma_t := \Gamma(L^2([0,t), k_0), \Gamma^t := \Gamma(L^2([t, \infty), k_0)$

- For $f \in k$, $f_t := f\mathbf{1}_{[0,t)}$, $f^t : f\mathbf{1}_{[t,\infty)}$ are projections onto k^t, k_t

Notice that the algebras $\mathcal{B}(\mathsf{H} \otimes \Gamma_t)$ form an increasing sequence of algebras and define the analog of a filtration in classical probability.

For any map $A : \mathsf{H} \otimes \Gamma_t \to \mathsf{H} \otimes \Gamma_t \otimes k^t$, the creation process is defined by

$$a^\dagger(A)(u \otimes (g^t)^{\otimes n}) = \frac{1}{\sqrt{n+1}} \mathbf{1}_{\mathsf{H}} \otimes \mathrm{Symm}(Au(g^t)^{\otimes n})$$

Intuitively, $a^\dagger$ "creates" a new particle in after time t using the coupling A. $a^\dagger$ can be interpret as a map in $\mathrm{Lin}(\mathsf{H} \otimes \Gamma, \mathsf{H} \otimes \Gamma)$, and this is the usual correspondence (as in equation 3.1).

Given $R \in \mathrm{Lin}(D_0, \mathsf{H} \otimes k_0)$, $D_0 \subset_{\text{dense}} \mathsf{H}$, if $R(u) := a \otimes b$, then for $\Delta \subset (t, \infty)$, using the mapping $k_0 \ni b \to b\mathbf{1}_\Delta \in k_t$, $R_t^\Delta \in \mathrm{Lin}(D_0 \otimes \Gamma_t, \mathsf{H} \otimes \Gamma_t \otimes k^t)$ is defined by

$$D_0 \otimes \Gamma_t \ni u \otimes \psi \to R_t^\Delta(u \otimes \psi) := a \otimes \psi \otimes (b\mathbf{1}_\Delta) \in \mathsf{H} \otimes \Gamma_t \otimes k^t$$

Expressing $R(u)$ as $a \otimes b$ is not possible in general; the intuitive picture is just clearer with this assumption. Formally, R_t^Δ is defined using the canonical unitary isomorphism $\mathrm{Swap}_{23} : A_1 \otimes A_2 \otimes A_3 \to A_1 \otimes A_3 \otimes A_2, \mathrm{Swap}_{23}(a_1 \otimes a_2 \otimes a_3) = a_1 \otimes a_3 \otimes a_2$, $R_t^\Delta(u\psi) = \mathrm{Swap}_{23}((\mathbf{1}_{\mathsf{H}} \otimes \mathbf{1}_\Delta)Ru) \otimes \psi$.

The associated creation process creates k_0 component of R on interval Δ:

$$a_R^\dagger(\Delta) := a^\dagger(R_t^\Delta) \tag{3.2}$$

The corresponding annihilation process is defined by using k_0 component of R to annihilate: for $u_t \in \mathsf{H} \otimes \Gamma_t$,

$$(D_0 \otimes \Gamma_t) \otimes \Gamma^t \ni u_t \mathrm{E}(f^t) \to a_R(\Delta)(u_t \mathrm{E}(f^t)) = \left(\left(\int_\Delta \langle R, f(s) \rangle ds \right) u_t \right) \mathrm{E}(f^t)$$

where $\langle R, f(s) \rangle$ is the adjoint of map $\langle f(s), R \rangle \in \mathrm{Lin}(D_0, \mathsf{H})$ which satisfies $\langle \langle f(s), R \rangle u, v \rangle = \langle Ru, v \otimes f(s) \rangle$. So $\langle R, f(s) \rangle$ is viewed as an operator on H that uses k_0 component of R to annihilate a k_0 particle in Γ_t component of u_t.

The conservation process captures what happens on the tail $[t, \infty)$ driven by a map $T \in \mathrm{Lin}(D_0 \otimes V_0, \mathsf{H} \otimes k_0)$, D_0, V_0 dense. T induces a map $\hat{T}^\Delta : \mathsf{H} \otimes k^t \to \mathsf{H} \otimes k^t$ using identification $\mathsf{H} \otimes k^t \equiv L^2([t, \infty), \mathsf{H} \otimes k_0)$,

$$L^2([t,\infty), \mathsf{H} \otimes k_0) \ni \eta \to \mathbf{1}_\Delta(\cdot) T \circ \eta, \text{ that is, for all } s \geq t, \hat{T}^\Delta(\eta)(s) = \mathbf{1}_\Delta(s) T(\eta(s))$$

Therefore, with $u \in D_0, g_t \in \Gamma_t, f^t \in k^t$, $u_t = ug_t$, the conservation process, $\Lambda_T(\Delta) : D_0 \otimes \Gamma \to \mathsf{H} \otimes \Gamma$, is the creation process driven by $T^{\Delta}_{\cdot}$,

$$\mathsf{H} \otimes \Gamma_t \ni ug_t \to T^{\Delta}_{f^t}(ug_t) := \mathrm{Swap}_{23}(\hat{T}^{\Delta}(uf^t)g_t) \in \mathsf{H} \otimes \Gamma_t \otimes k^t$$
$$\Lambda_T(\Delta)(u_t \mathrm{E}(f^t)) := a^{\dagger}(T^{\Delta}_{f^t})(u_t \mathrm{E}(f^t)) \tag{3.3}$$

Remark 3.2.1. Writing $\mathsf{H} \otimes \Gamma = \mathsf{H} \otimes \Gamma_t \otimes \Gamma^t$, for $u\mathrm{E}(f_t)\mathrm{E}(f^t)$, the component $\mathrm{E}(f^t)$ is the one that parametrizes $\hat{T}^{\Delta}$: f^t is participating the conservation (or exchange) process driven by T. At each $s \in \mathbb{R}_{\geq 0}$, $\hat{T}^{\Delta}(\cdot)(t) \in \mathrm{Lin}(D_0 \otimes V_0, \mathsf{H} \otimes k_0)$, and $\hat{T}^{\Delta}_{f^t}(ug_t)(s)$ is the map given by $T(ag_t(s) \otimes f^t(s))$.

The Hudson-Parthasarathy quantum stochastic calculus on Hilbert spaces is set in the the Schröedinger formalism for quantum dynamics considers stochastic integration with respect to the fundamental processes $a_R, a^{\dagger}_R, t\mathbf{1}, \Lambda_T$. Suppose $(H_t)_{t\geq 0}$ is a family of linear operators on $\mathsf{H} \otimes \Gamma$ with $\{v f_t^{\otimes n} \psi^t\} \subset \mathrm{Dom}(H_t)$ for $v \in D_1 \subset_{\mathrm{dense}} \mathsf{H}$, $f_t \in k_t$, f_t simple, right continuous and valued in $V \subset_{\mathrm{dense}} k_0, \psi^t \in \Gamma^t$, which is adapted in the sense that $H_t = \hat{H}_t \otimes \mathbf{1}_{\Gamma^t}$ for some map $\hat{H}_t : \{\mathsf{H} \otimes \mathrm{E}(k_t)\} \supset \mathrm{Dom}(\hat{H}_t) \to \mathsf{H} \otimes \Gamma$. And additionally, $\sup_{0 \leq s \leq t} \|H_s(u\mathrm{E}(f))\| \leq \|r_t u\|$ for all t, where r_t depending on t, f only is a closable map in $\mathrm{Lin}(D_1, H')$ for some Hilbert space H' depending only on f. Such an adapted process is a regular process. Regularity is saying that $H_t(u\mathrm{E}(f))$ is continuous and

$$\sup_{s \leq t} \|H_t(u\mathrm{E}(f))\| \leq c_{t,f} \|u\| \tag{3.4}$$

for constant $c_{t,f}$ depending on t, f, that is, the operator $\mathsf{H}_t(\cdot \mathrm{E}(f))$ is continuous and point-wise bounded on H.

If H_t is simple, that is, $H_t = \sum_{i=0}^{m} H_{t_i} \mathbf{1}_{[t_i, t_{i+1})}(t)$, $0 = t_0 < \ldots t_m < t_{m+1} = \infty$, then for M as one of the fundamental processes[1] $a_R, a^{\dagger}_R, t\mathbf{1}$,

$$\int_0^t H_s M(ds) = \sum_0^m H_{t_i} M([t_i, t_{i+1}) \cap [0, t]) \tag{3.5}$$

If $X_t = \int_0^t \Lambda_T(ds) + a_R(ds) + a^{\dagger}_S(ds) + H ds$ where $R, S \in \mathrm{Lin}(D_0, \mathsf{H} \otimes k_0), T \in \mathrm{Lin}(D_0 \otimes \mathcal{V}_0, \mathsf{H} \otimes k_0)$, $D_0, \mathcal{V}_0$ dense in H, k_0. Then for any $\mathcal{V}_0$-valued simple functions f, g on $\mathbb{R}^{\geq 0}$, $v, u \in D_0$,

$$\langle X_t v\mathrm{E}(g), u\mathrm{E}(f)\rangle = \int_0^t \langle \Psi(S, R, T, f, g) v\mathrm{E}(g), u\mathrm{E}(f)\rangle ds \tag{3.6}$$

[1] The integral with respect to the conservation process is not treated here, but the treatment is analogous.

where Ψ can be explicit computed using equation 3.5. For general adapted regular processes the integral is defined as a limit of simple processes and still satisfies the above property the maps, if the maps

$$s \to S(s)(v\mathrm{E}(g_s)), \langle R(s), \xi\rangle(v\mathrm{E}(g_s)), T_\xi(v\mathrm{E}(g_s))$$

are continuous and point-wise bounded (as in equation 3.4) in $\xi \in k_0$, $v \in D_0, g$ a $\mathcal{V}_0$-valued simple function, and the T_ξ is the operator defined by $T_\xi(u) = T(u \otimes \xi)$, where R, S, T are all identified with operators they induce on the Fock space (as in equations 3.2,3.3). This statement is the first fundamental lemma ([62, corollary 5.2.7]).

The quantum Ito lemma extends equation 3.6 for the first fundamental lemma to inner-product of adapted regular processes $X_t, X_t', \langle X_t v\mathrm{E}(g), X_t' u\mathrm{E}(f)\rangle$. The explicit form can again be computed from the definition as for Ψ.

The Heisenberg formalism is captured by map-valued processes: for an adapted, regular process, $Y(t) : \mathcal{A} \otimes \Gamma \supset \mathrm{Dom}(Y(t)) \to \mathcal{A} \otimes \Gamma$, $\widetilde{Y(t)} : \mathcal{A} \otimes k_0 \otimes \Gamma_t \supset \mathrm{Dom}(\widetilde{Y(t)}) \to \mathcal{A} \otimes \Gamma_t \otimes k_0$ define $\widetilde{Y(t)} = (Y(t) \otimes \mathbf{1}_{k_0})\mathrm{Swap}_{23}$,

$$\widetilde{Y(s)} : \mathcal{A} \otimes k_0 \otimes \mathrm{E}(k_s) \xrightarrow{\mathrm{Swap}_{23}} \mathcal{A} \otimes \mathrm{E}(k_s) \otimes k_0 \xrightarrow{Y(s)\otimes \mathbf{1}} \mathcal{A} \otimes \Gamma(k_s) \otimes k_0$$

A map-valued process can be viewed in the Hudson-Parthasarathy picture by using the Hilbert space of Hilbert-Schmidt operators. The key difference is that in $X_t = \int_0^t \Lambda_T(ds) + a_R(ds) + a_S^\dagger(ds) + Hds$, the operators S, R, T can now depend on $\mathcal{A}$; this is what makes it possible to describe Markov processes. The dependence is encoded in the structure matrix.

Definition 3.2.2. Given linear maps $\delta : \mathcal{A}_0 \to \mathcal{A} \otimes k_0, \sigma : \mathcal{A}_0 \to \mathcal{A} \otimes \mathcal{B}(k_0), \mathcal{L} : \mathcal{A}_0 \to \mathcal{A}$, the structure matrix is the map

$$\mathcal{A}_0 \ni f \to \Theta(f) = \begin{pmatrix} \mathcal{L}(f) & \delta^\dagger(f) \\ \delta(f) & \sigma(f) \end{pmatrix} \in \mathcal{B}(\mathsf{H} \otimes (\mathbb{C} \oplus k_0))$$

Notice that even if $\mathcal{A}$ is not unital, $\mathcal{L}(\mathbf{1}) = 0$ for $\mathbf{1} \in \mathcal{A}''$ implies that in the von Neumann algebra $\mathcal{A}''$, Θ (and the flow j_t introduced earlier) are defined on the operator system containing $\mathbf{1}, \mathcal{A}$.

Remark 3.2.3. The dependence allows for "iterating" on the structure matrix, Θ. The iteration is not by matrix multiplication: viewing Θ as a map $\mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_2$, to iterate on it $\mathcal{A}_2$ component is ignored, and the $\mathcal{A}_1$ component is fed back to the generator. This is a quantum random walk, e.g., see [11, definition 2.7]). Formally,

$$\Theta^0 = \Theta, \Theta^{n+1} = \Theta \otimes \mathbf{1} \circ \Theta^n \tag{3.7}$$

The structure matrix defines the following fundamental processes:

$$a_\delta(\Delta)(\sum_i x_i \otimes \mathrm{E}(f_i))u := \sum_i a_{\delta(x_i^*)}(\Delta)(u\mathrm{E}(f_i)) \tag{3.8}$$

$$a_\delta^\dagger(\Delta)(\sum_i x_i \otimes \mathrm{E}(f_i))u := \sum_i a_{\delta^\dagger(x_i)}(\Delta)(u\mathrm{E}(f_i)) \tag{3.9}$$

$$I_\mathcal{L}(\Delta)(\sum_i x_i \otimes \mathrm{E}(f_i))u := \sum_i |\Delta|(\mathcal{L}(x_i)u) \otimes \mathrm{E}(f_i) \tag{3.10}$$

$$\Lambda_\sigma(\Delta)(\sum_i x_i \otimes \mathrm{E}(f_i))u := \sum_i \Lambda_{\sigma(x_i)}(\Delta)(u\mathrm{E}(f_i)) \tag{3.11}$$

and the map-valued integrals are defined by them are as below. With $u \in \mathsf{H}, f \in L^4_{\mathrm{loc}}, x \in \mathcal{A}$,

$$\left(\int_0^t Y(s) \circ (a_\delta + I_\mathcal{L})(ds)\right)(x \otimes \mathrm{E}(f))u$$

$$= \int_0^t Y(s)((\mathcal{L}(x) + \langle \delta(x^*), f(s)\rangle \otimes \mathrm{E}(f))u \, ds \tag{3.12}$$

$$\left(\int_0^t Y(s) \circ (a_\delta^\dagger)(ds)\right)(x \otimes \mathrm{E}(f))u = \left(\int_0^t a^\dagger_{\widetilde{Y},x}(ds)\right)u\mathrm{E}(f) \tag{3.13}$$

$$\left(\int_0^t Y(s) \circ (\Lambda_\sigma)(ds)\right)(x \otimes \mathrm{E}(f))u = \left(\int_0^t \Lambda_{\widetilde{Y},x}(ds)\right)u\mathrm{E}(f) \tag{3.14}$$

$$\text{where } a^\dagger_{\widetilde{Y},x}(s)(u\mathrm{E}(f)) = \widetilde{Y(s)}(\delta(x) \otimes \mathrm{E}(f_s))u \tag{3.15}$$

$$\Lambda_{\widetilde{Y},x}(u\mathrm{E}(f) \otimes \xi) = \widetilde{Y(s)}(\sigma(x)_\xi \otimes \mathrm{E}(f_s))u \tag{3.16}$$

$$= \widetilde{Y(s)}(\sigma(x)_{f(s)} \otimes \mathrm{E}(f_s))u \tag{3.17}$$

f_s being the projection of $f \in k$ on k_s since $\widetilde{Y}$ lives on $\mathcal{A} \otimes \Gamma_s$, and in going from second to third equation in equation 3.17, $\xi := f(s) \in k_0$ is set following $\Lambda_T(\Delta)$ in equation 3.3 and remark 3.2.1, and for $\xi \in k_0$, $\sigma(x)_\xi \in \mathrm{Lin}(\mathsf{H}, \mathsf{H} \otimes k_0)$ $\sigma(x) : \mathsf{H} \otimes k_0 \to \mathsf{H} \otimes k_0$ is the map defined by $\sigma(x)_g(u) = \sigma(x)(ug)$.

The stochastic integral $Z(t) = \int_0^t Y(s) \circ (a_\delta + I_\mathcal{L} + a^\dagger(\delta) + \Lambda_\sigma)(ds)$ is the process

$$\int_0^t Y(s) \circ (a_\delta + I_\mathcal{L})(ds) + \int_0^t Y(s) \circ (a^\dagger(\delta) + \Lambda_\sigma)(ds)$$

Let $J_s^{(0)} : \mathcal{A} \otimes \Gamma \to \mathcal{A} \otimes \Gamma$, $J_s^{(0)} = \mathbf{1}$. Define $J^{(n)}(t) := \int_0^t J^{(n-1)} \circ (a_\delta + a\dagger_\delta + I_{\mathcal{L}} + \Lambda_\sigma)(ds)$. The object of interest are the Picard iterates, $S_N(t)$:

$$S_N(t) = \sum_{n \leq N} J_t^{(n)}(x \otimes \mathrm{E}(f)) \tag{3.18}$$

that converge.

When the generator $\mathcal{L}$ for the semigroup $e^{-t\mathcal{L}}$ is unbounded, being able to iterate requires the stochastic integral preserves the domain. Let $\mathcal{A}_\infty$ be a norm-dense algebra, and assume $\mathcal{L}(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$, and that there's a norm-dense subspaces, $(k_0)_\infty \subset k_0$, $(\mathcal{A} \otimes k_0)_\infty \subset \mathcal{A} \otimes k_0$ (where $\mathcal{A} \otimes k_0$ is normed by $\|a \otimes k'\|^2 = \|a\|_{\mathcal{A}}^2 \|k'\|_{k_0}^2$, i.e. as a Hilbert C^*-module). The details are not included as the norm-density is all that is relevant.

Now with $\mathcal{L}(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$, and $\mathcal{V}_0 = (k_0)_\infty$, define

- $\mathcal{V}_t = \{\mathcal{V}_0\text{-valued simple functions in } k_t\}$
- $\mathcal{V} = \{\mathcal{V}_0\text{-valued simple functions}\}$

One defines a map-valued integrable process with respect to $a_\delta, a_\delta^\dagger, I_{\mathcal{L}}, \sigma$ as follows:

Definition 3.2.4. An integrable map-valued process is an adapted process $(Y(s))_{s \geq 0} : \mathcal{A}_\infty \otimes \mathrm{E}(\mathcal{V}) \to \mathcal{A} \otimes \Gamma(k)$ such that:

1. For each $t \geq 0, f \in \mathcal{V}, Y(t)(a \otimes \mathrm{E}(f)) \in (\mathcal{A} \otimes \Gamma(k))_\infty$
2. For every fixed $a \in \mathcal{A}_\infty, f \in \mathcal{V}, \xi \in \mathcal{V}_0$, set $\mathcal{A}_\infty \ni a \to \Omega_{t,f}(a) := Y(t)(a \otimes \mathrm{E}(f)) \in (\mathcal{A} \otimes \Gamma(k))_\infty$ and for any separable Hilbert space H' and the ampilation $\widetilde{\Omega}_{t,f} := \Omega_{t,f} \otimes \mathbf{1}_{\mathsf{H}'}$, with $\widetilde{Y(t)} := \widetilde{\Omega}_{t,f}$ define

$$S_a(s) : \mathsf{H} \otimes \mathrm{E}(\mathcal{V}_s) \ni u\mathrm{E}(f_s) \to \widetilde{Y(s)}(\delta(a) \otimes \mathrm{E}(f_s))u \in \mathsf{H} \otimes \Gamma_s \otimes k_0$$
$$T_a(s) : \mathsf{H} \otimes \mathrm{E}(\mathcal{V}_s) \otimes \mathcal{V}_0 \ni u\mathrm{E}(f_s) \otimes \xi \to \widetilde{Y(s)}(\sigma(a)_\xi \otimes \mathrm{E}(f_s))u \in \mathsf{H} \otimes \Gamma_s \otimes k_0$$

then the maps $s \to S_a(s)(u\mathrm{E}(f))$, $s \to T_a(s)(u\mathrm{E}(f))$, $s \to Y(s)((\mathcal{L}(a)) + \langle \delta(a^*), \xi \rangle)) \otimes \mathrm{E}(f)$ are continuous.

Remark 3.2.5. For the existence of the Evans-Hudson dilation and the convergence of Picard iterates Sinha and Goswami [62] require the boundedness of $\widetilde{\Omega}_{t,f}(x)$ for fixed f, x; we absorb this into the existence theorem for Picard iterates as a bound on $\|\Theta^n\|$, where it's verified given the structure maps, and mirrors the approach taken in [11]. For $Y = J^{(0)} = \mathbf{1}$, the needed continuity of $\widetilde{\Omega}$ can be directly verified as well. They additionally require that the following map is completely smooth (see definition 4.5.3):

$$\mathcal{A}_\infty \ni a \to \Omega_{t,f}(a) : Y(t)(a \otimes \mathrm{E}(f)) \in (\mathcal{A} \otimes \Gamma(k))_\infty \tag{3.19}$$

This is because completely-smooth processes form a class where the Picard iterates can be established to converge using the theory they develop. Complete-smoothness is not applicable in the examples considered next, so the definition of integrable processes needs to be adjusted. The thing to note is that the continuity requirement in the second part of definition 3.2.4 is precisely what is needed to get the existence of the integral (the point-wise boundedness follows from point-wise boundedness of δ, σ). The continuity of the maps in s is equivalent to the (point-wise) boundedness of process $Y, \widetilde{Y}$ on each $a \in \mathcal{A}_\infty$. For $J_t^{(n+1)}$ the boundedness will follow from that of $J_t^{(n)}$ (through lemma 4.4.3), therefore, starting with $J^{(0)} = \mathbf{1}$, each successive Picard iterate is a map-valued integrable process for which the map-valued integral exists, while bounds on Θ^n will yield explicit bounds on $J^{(n)}$.

Chapter 4

UNBOUNDED GENERATORS

4.1 Introduction

In this chapter, we analyze examples to show the existence of the quantum stochastic flows associated to laplacians on spectral triples without requiring truncation. The first example considered is the flow generated by the noncommutative laplacian. The second example deals with geometric laplacians on Clifford and spinor bundles over reductive homogeneous spaces. The flows generated now are covariant with respect to the associated group action.

Organization and overview

In section 4.2, the structure matrix background and an illustrative example are considered. Section 4.3 considers the noncommutative laplacian, while section 4.4 provides details showing the existence of the untruncated flow. The existence of flows for the derived structure matrices is established in section 4.4 by providing estimates that can be plugged into the standard theory. In section 4.5, the existence of Evans-Hudson flow on homogeneous vector bundles over reductive homogeneous spaces is established. This proceeds by showing that the laplacians are completely smooth and utilizing the construction from [62] for such generators. The connections used are not necessarily torsion-free and this needs to be taken into account. Section 4.6 considers growth of Sobolev norms required for convergence in commutative examples.

Some remarks on notation. As before, by Riemannian (M, g), we mean a Riemannian manifold M with metric g. The connection on the tangent bundle of M, TM, is the Levi-Civita connection unless specified otherwise. When clear from context, the same symbol is used for the connection ∇ on a Hermitian or Riemannian bundle E and the dual connection on dual bundle E^*. After fixing a local orthonormal frame about any $p \in M$, $(X_i)_{i \in \dim M}$, ∇_{X_i} will be used interchangeably with ∇_i. By abuse of notation, $g^{ab}\nabla_a\nabla_b\phi$ will denote $g^{ab}(\nabla^2\phi)_{ab}$ where ∇^2 is the iterated covariant derivative, and the same for $\nabla_{i_1}\dots\nabla_{i_k} := (\nabla^k)_{i_1\dots i_k}$ and also the raise indexed version, $\nabla^{i_1}\dots\nabla^{i_k} := (\nabla^k)^{i_1\dots i_k}$. This will be made explicit if not clear from context. For local coordinates (x_i) about any $p \in M$, ∂_i will denote the coordinate vector fields $\frac{\partial}{\partial x_i}$. $[n]$ is the set $\{i \in \mathbb{N}, i \leq n\}$ where $\mathbb{N}$, with convention that

$0 \notin \mathbb{N}$. $[n : m]$ denotes the set $\{n, n+1 \ldots m\}$. The finite linear span is denoted by $\text{FinteLinSpan}(\mathsf{V}) := \{\sum_{i \in [k]} \alpha_i a_i : \alpha_i \in \mathbb{K}, a_i \in \mathsf{V}\}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and is dropped if is clear from context. Throughout $\Gamma(H)$ denotes the symmetric (boson) Fock space over any space H, while $\mathrm{E}(H)$ denotes the exponential vectors given by $\mathrm{E}(v) = \oplus_{n=0}^{\infty} (n!)^{-1/2} v^{\otimes n}$ for $v \in H$. For a self-adjoint operator A on H with discrete spectrum, denote by $\mathcal{S}_A$, an orthonormal basis of eigenfunctions. For $e_i, e_j \in \mathcal{S}_A$, $e_{ij} := e_i \otimes e_j^* \in \mathrm{End}(H)$.

The endomorphism laplacian

To start we note the following sign conventions of the laplacians. Primarily the signs are fixed so the Laplace-Beltrami operator has non-negative spectrum, and signs on all other laplacians cascade from there. On Riemannian (M, g), M compact, without boundary, Tr_g denotes the trace of a covariant tensor taken after identifying with a contravariant tensor via the metric g, $\mathrm{Tr}_g(h) := g^{ij} h_{ij}$. Note that trace on any contravariant tensor, e.g., vector fields, is simply the sum. For $X \in \Gamma(TM)$, $\mathrm{div}(X) = \mathrm{Tr}(\nabla X)$, with ∇ being the connection. The Laplace-Beltrami operator is taken as the operator with non-negative spectrum, that is, $-\mathrm{div}(\nabla) = -\mathrm{Tr}(\nabla_{\cdot,\cdot})$, where $\nabla_{\cdot,\cdot}$ is the second invariant derivative $\nabla^2_{V,W} := \nabla_V \nabla_W - \nabla_{\nabla_V W}$.

So far the scalar and (co)tangent bundle laplacians have been considered. More generally let ∇ be any connection on the vector bundle $E \longrightarrow M$. The connection Laplacian is $\nabla^* \nabla$ where ∇^* is adjoint of the connection $\nabla : \Gamma(E) \longrightarrow \Gamma(E) \otimes T^*M$ with respect to $L^2(\Gamma E)$. Equivalently, $\nabla^* \nabla = -\mathrm{Tr}(\nabla_{\cdot,\cdot})$. Further, $\triangle = -g^{ij} \nabla_i \nabla_j := -g^{ij} (\nabla^2)_{ij}$. The connection Laplacian at $p \in M$, $\triangle = \nabla^* \nabla$ in local coordinates (e_i) is given by $\triangle = -(\sum_i \nabla_i \nabla_i - \nabla_{\nabla_i e_i})$. To evaluate $\triangle \phi$ at any $p \in M$ and $\phi \in \Gamma(E)$, we will use Riemann normal coordinates centered at p so $\nabla_i e_j$ vanish, yielding $\triangle \phi(p) = -\sum_i \nabla_i \nabla_i \phi(p)$.

The endomorphism connection ∇^{End} on the bundle $\mathrm{End}(E) = E \otimes E^*$ associated to a connection ∇ on the Riemannian (or Hermitian) vector bundle E over the Riemannian manifold M is such that for $X \in TM$, $\nabla^{\mathrm{End}}_X = \nabla_X \otimes 1 + 1 \otimes \widetilde{\nabla}_X$, where $\widetilde{\nabla}$ is the dual connection on E^*. The endomorphism Laplacian is defined as usual: at $p \in M$ in Riemann normal coordinates centered at p (denoting $\widetilde{\nabla}^* \widetilde{\nabla}, \widetilde{\nabla}$ by $\triangle, \nabla$ again),

$$\triangle^{\mathrm{End}} = -\sum_i \nabla_i^{\mathrm{End}} \nabla_i^{\mathrm{End}} = \triangle \otimes 1 - 2 \sum_i \nabla_i \otimes \nabla_i + 1 \otimes \triangle \tag{4.1}$$

Note that as $E \otimes E^*$ is balanced over $C(M)$, the action of $C(M)$ on $\mathrm{End}(E)$ can be written as $f \cdot 1_{\mathrm{End}} = \sum_i (f \cdot h_i) \otimes h_i^*$; this convention is used for all computation with Laplacian expressed in this tensor form. It's also very useful to note that in any local coordinates , ∇^{End} acts by commutator: if over chart U, the connection has potential A, $\nabla = d + A$, then for a local orthonormal frame (μ_i) and dual frame (μ^j), $\nabla^{\text{End}} \sum_{ij} \sigma^i_j \mu_i \otimes \mu^j = \sum_{ij} (d\sigma^i_j)\mu_i \otimes \mu^j + \sum_{jk} [\sigma A - A\sigma]_{jk} \mu_k \otimes \mu^j$. In particular, since $\mathbf{1}_{\text{End}}$ is given by the identity matrix locally, it follows (see [38]) that $\nabla^{\text{End}}(\mathbf{1}_{\text{End}}) = 0$. This implies that again in normal Riemann coordinates centered at p yields that for any $f \in C^\infty(M)$, $\triangle^{\text{End}}(f\mathbf{1}_{\text{End}})(p) = -\sum_i \nabla^{\text{End}}_i \nabla^{\text{End}}_i (f \cdot \mathbf{1}_{\text{End}})(p) = \sum_i \nabla^{\text{End}}_i (\partial_i f \mathbf{1}_{\text{End}}) = -\sum_i \partial_i \partial_i f \cdot \mathbf{1}_{\text{End}}(p)$

Proposition 4.1.1. *For $f \in C^\infty(M)$, $\triangle^{End(E)}(f \cdot \mathbf{1}_{End}) = \triangle^M(f) \cdot \mathbf{1}_{End}$.*

Proof. Let Γ^k_{ij} be the Christoffel symbols for Levi-Civita connection, then in local coordinates about $x \in M$, $\triangle^M(f) = -\sum_{ij} g^{ij}(x)(\partial_i\partial_j - \sum_k \Gamma^k_{ij}\partial_k)f$ (see, for instance, [12, pg 66]) and for the endomorphism Laplacian,

$$\begin{aligned}\triangle^{\text{End}}(f \cdot 1) &= -\sum_{ij} g^{ij}(x)(\nabla^{\text{End}}_i \nabla^{\text{End}}_j - \sum_k \Gamma^k_{ij} \nabla^{\text{End}}_k)(f \cdot \mathbf{1}_{\text{End}}) \\ &= -\left(\sum_{ij} g^{ij}(x)(\partial_i\partial_j - \sum_k \Gamma^k_{ij}\partial_k)f\right) \cdot \mathbf{1}_{\text{End}} = \triangle^M(f)\mathbf{1}\end{aligned}$$

where we used $\nabla^{\text{End}}(\mathbf{1}_{\text{End}}) = 0$, $\nabla^{\text{End}}_X(f) \cdot \mathbf{1}_{\text{End}} = X(f) \cdot \mathbf{1}_{\text{End}}$. □

4.2 The structure matrix

From Chapter 2, the heat semigroup $e^{-t\mathcal{L}}$, $\mathcal{L} = \triangle^{\text{End}}$ is a quantum dynamical semigroup on $\text{End}(E) \equiv E \otimes E^*$ with $e^{-\frac{1}{2}t\triangle^{\text{End}}}(\mathbf{1}) = \mathbf{1}$ for all t. We will work with the semigroup living $C(M) \subset \text{End}(E)$. To derive the qsde associated to the heat semigroup, we start by computing the structure matrix for the associated Evans-Hudson flow following the standard prescription (see [62]). The first step is to compute the kernel for the generator $\mathcal{L} = \triangle^{\text{End}}$ on the $\mathcal{A}_\infty = C^\infty(M)$ acting on $\text{End}(E)$ defined by $K_{\mathcal{L}} : X \times X \to \mathcal{B}(E \otimes E^*)$ for $X := \mathcal{A}_\infty \times \mathcal{A}_\infty$, where $K_{\mathcal{L}}$ for any given any $\mathcal{L} : X \to X$ is defined by

$$\begin{aligned}&X \times X \ni ((f_1, f_2), (g_1, g_2)) \to \\ &\quad \mathcal{L}(f_1^* f_2^* g_2 g_1) + f_1^* \mathcal{L}(f_2^* g_2) g_1 - \mathcal{L}(f_1^* f_2^* g_2) g_1 - f_1^* \mathcal{L}(f_2^* g_2 g_1) \in C(M) \quad (4.2)\end{aligned}$$

and from this, using the Kolmogorov decomposition for this kernel, we will obtain the structure matrix.

The Kolmogorov decomposition

We recall some details on reproducing kernel Hilbert modules.

Definition 4.2.1. For any positive kernel $K : X \times X \to \mathcal{B}(H)$, the reproducing kernel Hilbert space R_K is the space of H-valued functions on X such that $R_K = \overline{\text{LinSpan}\{K(\cdot,x)u : x \in X, u \in H\}}$ and $\langle f(x), u\rangle = \langle f, K(\cdot,x)u)\rangle$ for all $f \in R_K, u \in H$. The Kolmogorov decomposition is the Hilbert space R_K with the map

$$V(x) = K_x : H \to R_K, [K_x(u)](y) = K(y,x)u$$

Notice that $\langle K(\cdot,a)u, K(\cdot,b)v\rangle = \langle K(b,a)u, v\rangle = \langle K(u, K(a,b)v\rangle$, $K(a,b) = K(b,a)^*$; the adjoint of $K(\cdot,x)$ is evaluation at x.

The kernel can be computed using proposition 4.1.1. Equivalently it follows by noting that $f \cdot 1_{\text{End}} = \sum_i (f \cdot h_i) \otimes h_i^*$ for $f \in C^\infty(M)$ and (h_i) a basis of eigensections of $\triangle^E$, so in expansion of endomorphism laplacian (equation 4.1), there's no contribution to the kernel from $1 \otimes \triangle$, while the contribution for the term $\nabla \otimes \nabla$ term can only come from ∇ acting on f, but this cancels out. Computing it out yields $K_\triangle((a_1,a_2),(b_1,b_2)) := -2\left(\sum_k da_1(e_k)db_1(e_k)\right) a_2 b_2$.

Normalizing suppresses the extraneous factors of -2. The Kolmogorov decomposition for the kernel associated with $\mathcal{L} = -\frac{1}{2}\triangle^{\text{End}}$ contains the needed data for the structure matrix:

$$K_{\mathcal{L}}((a_1,a_2),(b_1,b_2)) := K^p_{\mathcal{L}}((a_1,a_2),(b_1,b_2)) = \left(\sum_k da_1(e_k)db_1(e_k)\right) a_2 b_2 \tag{4.3}$$

where we are working in Riemann normal coordinates about p, and the superscript $p \in M$ in $K^p_{\mathcal{L}}$ indicates that the expression holds in local coordinates at $p \in M$. The Kolmogorov decomposition can be taken to be the reproducing kernel Hilbert module

$$R_{\mathcal{L}} = \overline{\text{LinSpan}\{K_{\mathcal{L}}(\cdot,b)u : b \in X, u \in E\}}$$

where $X = C^\infty(M) \times C^\infty(M)$, with map $V : X \to \mathcal{B}(\text{E}, R_{\mathcal{L}})$ given by

$$V(x) : E \to R_{\mathcal{L}}, E \ni u \to K_{\mathcal{L}}(\cdot,x)u \in R_{\mathcal{L}}$$

By definition, $K_{\mathcal{L}}(\cdot,x)u$ is total in $R_{\mathcal{L}}$ making the decomposition minimal. The following explicit identification with differential forms allows for interpreting the structure matrix obtained in a meaningful way.

Observation 4.2.2. Note for $x = (a_1, a_2), y = (b_1, b_2)$, in local coordinates,

$$[V(x)u](y) = a_2 b_2 \sum_k da_1(e_k) db_1(e_k) u$$

which can be interpreted as the form $a_2 \sum_k da_1(e_k) de_k$ evaluated on the vector field $b_2 \sum_k db_1(e_k) \frac{\partial}{\partial e_k}$ acting by multiplication on u. Since $K_{\mathcal{L}}(x, y) \in C(M)$, on viewing $x = (a_1, a_2)$ as the 1-form, $a_2 da_1$, $K_{\mathcal{L}}(\cdot, x)$ is viewed as dual at each point $m \in M$, and so is identified with vector fields $\Gamma(TM)$, and therefore, $K_{\mathcal{L}}(\cdot, x)u$ can be viewed as a section of $TM \otimes E$ while $V(x)$ is thought of as 1-form acting by contracting with the TM component. Therefore, $V(x), x = (f_1, f_2)$ is the operator $u \to f_2 df_1 \otimes u$, where $f_2 df_1 \otimes u$ defines an E valued function on X by $f_2 df_1 \otimes u((g_1, g_2)) = f_2 g_2 \langle df_1, dg_1 \rangle u$.

Since $\partial_k(f_1^* f_2^* g_2 g_1) + f_1^* \partial_k(f_2^* g_2) g_1 - \partial_k(f_1^* f_2^* g_2) g_1 - f_1^* \partial_k(f_2^* g_2 g_1) = 0$, this holds, not just in Riemann normal coordinates but in any local orthonormal frame (e_i) over an open set U; additionally, over U $\langle \nabla f, \nabla g \rangle = \sum_i \langle \nabla f, e_i \rangle \langle e_i, \nabla g \rangle = \sum_i df(e_i) dg(e_i) = \sum_i \nabla_i f \nabla_i g$

Hilbert C^*-modules

Recall that a Hilbert C^*-module over C^*-algebra $\mathcal{A}$ (a Hilbert $\mathcal{A}$-module) is right $\mathcal{A}$-module N with a $\mathcal{A}$-linear sesquilinear map $\langle \cdot, \cdot \rangle : N \times N \to \mathcal{A}$, $\langle x, y \rangle a = \langle x, ya \rangle$ for $a \in \mathcal{A}$.

Definition 4.2.3. The standard Hilbert C^*-module $\mathsf{H}_{\mathcal{A}}$ for Hilbert space H is the completion of the algebraic tensor product $\mathcal{A} \otimes \mathsf{H}$, which is Hilbert C^*-module having the right $\mathcal{A}$-action given by $(a \otimes h)a' = aa' \otimes h$, with respect to the norm induced by the $\mathcal{A}$-valued inner product $\langle a \otimes h, b \otimes h' \rangle = a^* b \langle h, h' \rangle$, $\|a \otimes h\|^2 = \|a^* a\| \langle h, h \rangle$. Taking $\mathsf{H} = \ell^2$, $\mathsf{H}_{\mathcal{A}}$ is identified with $(a_i) \in \mathcal{A} \otimes \ell^2, \sum a_i^* a_i < \infty$.

For a Hilbert space k_0, it's convenient to use $\mathcal{A} \otimes_{C^*} k_0 := \mathcal{A} \otimes k_0$ for the standard Hilbert C^*-module. A Hilbert C^*-module generalizes the idea of a vector bundle $E \to M$, where the $\mathcal{A} = C(M)$, the $\mathcal{A}$-valued inner product given by fiberwise contraction. When $\mathcal{A} = C(X)$ for a compact Hausdorff space X, then $\mathsf{H}_{\mathcal{A}}$ is the space of H valued continuous functions. Kasparov stabilization states that for any countably generated Hilbert C^*-module N, there's a unitary map $t' : N \oplus \mathsf{H}_{\mathcal{A}} \to \mathsf{H}_{\mathcal{A}}$.

The structure matrix is the defined by the maps $(\mathcal{L}, \delta, \sigma)$ where $\mathcal{L}$ is the densely defined generator with $\mathrm{Dom}(\mathcal{A}) \subset \mathcal{A}_0$, $\mathrm{Dom}(\mathcal{A}), \mathcal{A}_0$ norm-dense in $\mathcal{A}$, $\sigma(x) =$

$\pi(x) - x \otimes 1$ for a $*$-homomorphism $\pi : \mathcal{A}_0 \to \mathrm{Lin}(\mathcal{A} \otimes_{C^*} k_0)$, δ a π-derivation $\mathcal{A}_0 \to \mathcal{A} \otimes_{C^*} k_0$. These maps in addition to satisfying some structural properties satisfy the compatibility condition

$$\mathcal{L}(xy) - \mathcal{L}(x)y - x\mathcal{L}(y) = \delta^\dagger(x)\delta(y) \tag{4.4}$$

for $x, y \in \mathcal{A}_0$, $\delta^\dagger(f) := \delta(f^*)^*$.

The structure matrix Θ is defined by

$$\Theta(x) = \begin{pmatrix} \mathcal{L}(x) & \delta^\dagger(x) \\ \delta(x) & \sigma(x) \end{pmatrix} \tag{4.5}$$

Note that $\Theta(1) = 0$, and for each fixed x, $\Theta(x) \in \mathcal{B}(H \otimes (\mathbb{C} \oplus (k_0)_\infty))$ where $(k_0)_\infty$ is norm-dense in k_0. The maps δ, σ are extracted from the minimal Kolmogorov decomposition: the decomposition $(R_\mathcal{L}, V)$ induces the maps below on $\mathcal{A}_\infty := C^\infty(M) \subset C(M) = \mathcal{A}$,

$$\begin{aligned} \rho : \mathcal{A}_\infty \to \mathcal{B}(R_\mathcal{L}), \rho(x)(V(\cdot, b)u) = V(\cdot, xb)u \\ \alpha : \mathcal{A}_\infty \to \mathcal{B}(E, R_\mathcal{L}), \alpha(x) = V(x, 1) \end{aligned} \tag{4.6}$$

With observation 4.2.2 and equation 4.3 in mind, $\rho(f)$, $f \in C^\infty(M)$, is multiplication by f on $R_\mathcal{L}$ while $\alpha(f)$ acts by contraction with 1-form $\sum_k df(e_k)de_k$. The representation ρ is the identity map: $C^\infty(M)$ is interpreted as acting by multiplication on $R_\mathcal{L}$, and α is a derivation (by Christensen-Evans theory, α is a ρ-derivation, but ρ is identity).

The construction of the structure maps proceeds as in [62, Thm 6.6.1]. To start define the Hilbert $\mathcal{A}$-module $N = \overline{\{\alpha(x)y : x, y \in \mathcal{A}_\infty\}}$ where the closure is with respect to operator norm for $\mathcal{B}(E, R_\mathcal{L})$. $\mathcal{A}$ has right action on N by multiplication (where the norm density of $\mathcal{A}_\infty \subset \mathcal{A}$ is utilized) and the $\mathcal{A}$-valued inner product is $N \times N \ni (a, b) \to \langle a, b\rangle = a^*b$. By Kasparov's stabilization theorem, there's an isometric embedding into the standard Hilbert module, $t : N \to \mathsf{H}_\mathcal{A}$ where H can be taken to be any infinite-dimensional Hilbert space. The embedding into $\mathcal{A} \otimes \mathsf{H}$ is needed to identify what the structure matrix acts on.

Define $\delta(x) = t(\alpha(x))$. For ρ note ρ induces a left action $\hat{\rho}$ on N, $\hat{\rho}(x)(\alpha(y)) = (\alpha(xy) - \alpha(x)y)$. But as α is a ρ-derivation, $\hat{\rho}(x) = x\alpha(y)$, so $\hat{\rho}(x)$ is multiplication by x and is again identity representation of $\mathcal{A}$ acting by multiplication. Set $\pi(x) = t\hat{\rho}(x)t^*$, again $\pi = \mathbf{1}$ (so the explicit form of t does not come into play).

Given the identification with $\mathcal{A} \otimes R_{\mathcal{L}}$, the compatibility condition (equation 4.4), along with $f^* = f, g^* = g$ as f, g are $\mathbb{R}$-valued yields

$$-\frac{1}{2}\left(\triangle(fg) - \triangle(f)g - f \triangle (g)\right) = -\langle df, dg\rangle = \delta^\dagger(f)\delta(g)$$

Therefore, with equation 4.5 in reference, we have the structure matrix summarized in the following proposition. The embedding t for Laplace-Beltrami operator is described in example 4.4.10 where the remaining details are provided, however, it's a local variant, and given a bundle an explicit global embedding can be found, so general form for that is not included.

Proposition 4.2.4. *For any vector bundle $E \to M$, the structure matrix for the $-\frac{1}{2}\triangle^{End(E)}$ generated flow on $End(E)$ is the map*

$$\mathcal{A}_0 \ni f \to \Theta(f) = \begin{pmatrix} -\frac{1}{2}\triangle^M (f)\cdot \mathbf{1} & \delta^\dagger(f) \\ \delta(f) & 0 \end{pmatrix} \in \mathcal{B}(E \otimes (\mathbb{C} \oplus k_0)) \tag{4.7}$$

where $\mathcal{A}_0 := C^\infty(M)$

- *The multiplicity space $k_0 \subset L^2(\Omega^1(M))$*
- $\delta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes k_0, \delta(f) = t(\alpha(f)), \delta^\dagger(x)\delta(y) = -\langle dx, dy\rangle$.
- $\sigma = 0$ *since* $\sigma(x) = \pi(x) - x \otimes \mathbf{1}_{k_0}$ *and* $\pi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}(k_0)$ *is identity*

4.3 The noncommutative laplacian flow

For noncommutative laplacian $D^2_{\mathcal{D}}$ acting on $\mathrm{End}(S)$ with D the Dirac operator and S the spinor bundle, it's possible to explicitly derive a form for the embedding which is nice enough to get existence of the flow for the untruncated generator.

Consider the noncommutative laplacian $\mathcal{L} := -D^2_{\mathcal{D}} := -[D, [D, \cdot]/2$ acting on $\mathcal{A} = \mathcal{B}(S)$. The kernel (equation 4.2), $K_{\mathcal{L}}$, for $\mathcal{L}$ is given by

$$K_{\mathcal{L}}((f_1, f_2), (g_1, g_2)) := -(D_{\mathcal{D}} f_1^*) f_2^* g_2 (D_{\mathcal{D}} g_1) = -(f_2([D, f_1])^*)^* g_2([D, g_1])$$

with $f_i, g_i \in \mathcal{S}_D = \{\phi_i\}$, ϕ_i's eigenspinors for D forming an orthonormal basis. This is the noncommutative analog of the original calculation (equation 4.3; the difference in sign is due to the usual laplacian having negative spectrum). Denote by $K(\cdot, x) := K_{\mathcal{L}}((\cdot,\cdot), (x, 1)), K(x, y) := K_{\mathcal{L}}((y, 1), (x, 1))$.

Notice that $([D, g_1]g_2)^* = -g_2^*[D, g_1^*]$ is conjugate of the noncommutative differential form, $\Omega^1_D(\mathcal{A})$ (recall chapter 2, equation 2.5). Therefore, as detailed in the following paragraphs, the reproducing kernel Hilbert module is obtained like in observation 4.2.2 and α, ρ, δ are defined in the same manner. The Hilbert C^*-module N can be constructed as in section 4.2, and an explicit embedding into $\mathcal{A} \otimes R_{\mathcal{L}}$ can be obtained (see lemma 4.3.1).

Since $R_{\mathcal{L}}$ is functions on the (right) $\mathcal{A}$-linear span of $[D, a]$, the map $V(y)^* \in \mathcal{B}(R_{\mathcal{L}}, S)$ for $y := (f_1, f_2)$ sends $K_{\mathcal{L}}((\cdot,\cdot),(g_1,g_2))u \to K_{\mathcal{L}}((f_1,f_2),(g_1,g_2))u$, and so is given by evaluation on $-[D, f_1^*]f_2^*$. Viewing $f_2[D, f_1]$ as a noncommutative differential form, $V(y)^*$, acting by evaluation on $-[D, f_1^*]f_2^*$, can be interpreted as analogous to the contraction with a vector field as in observation 4.2.2. This description works because D is symmetric unlike $\nabla^*\nabla$ and captures how the noise space $k_0 \subset R_{\mathcal{L}}$ relates to noncommutative differential forms.

Explicitly, for the kernel $K_{\mathcal{L}} : (\mathcal{S}_D \times \mathcal{S}_D) \times (\mathcal{S}_D \times \mathcal{S}_D) \to \mathcal{B}(S)$, the maps α, ρ are defined by

$$\alpha : x \to V((x,1)), \rho : \mathcal{A} \to \mathcal{B}(R_{\mathcal{L}}), \rho(x)K_{\mathcal{L}}((\cdot,\cdot),(a,b))u \to K_{\mathcal{L}}((\cdot,\cdot),(a,xb))u \tag{4.8}$$

As before the Hilbert $\mathcal{A}$-module $N = \overline{\{\alpha(x)y : x, y \in \mathcal{A}\}}$. The algebra $\mathcal{F}$ is taken as the finite linear span

$$\mathcal{F} = \text{FinteLinSpan}\{e_i \otimes e_j^* : e_i, e_j \in \mathcal{S}_D\}$$

Lemma 4.3.1. *Define the map t by*

$$t : N \ni \alpha(\sum_{ij} a_{ij} e_{ij}) \to \sum_{ij} a_{ij} e_{ij} \otimes K(\cdot, e_{ij}) e_j \in \mathcal{A} \otimes R_{\mathcal{L}} \tag{4.9}$$

and then extend by $\mathcal{A}$-linearity to N. Then t is an $\mathcal{A}$-isometry.

Proof. Note that since $\alpha(x) : u \to K_{\mathcal{L}}((\cdot,\cdot),(x,1))u$. Now $[D, e_{ij}]e_{kl} = 0$ unless $j = k$ so as an operator $[D, e_{ij}] = (\lambda_i - \lambda_j)e_{ij}$. For $x = \sum_{ij} a_{ij}e_{ij}$, $y = \sum_{mn} b_{mn}e_{mn}$,

$$\alpha(x) = \sum_k K(\cdot, x)e_k \otimes e_k^* = \sum_k \sum_{ij} K(\cdot, a_{ij}e_{ij})e_k \otimes e_k^* = \sum_{ij} K(\cdot, a_{ij}e_{ij})e_j \otimes e_j^*$$

$$\begin{aligned} \text{and } \langle \alpha(y), \alpha(x) \rangle &= \sum_{ijmn} \langle K(\cdot, b_{mn}e_{mn})e_n \otimes e_n^*, K(\cdot, a_{ij}e_{ij})e_j \otimes e_j^* \rangle \\ &= \sum_{ijmn} \langle K(\cdot, b_{mn}e_{mn})e_n), K(\cdot, a_{ij}e_{ij})e_j) \rangle e_n \otimes e_j^* \\ &= \sum_{ijmn} \langle e_n, K_{\mathcal{L}}((b_{mn}e_{mn}, 1), (a_{ij}e_{ij}, 1))e_j) \rangle e_n \otimes e_j^* \\ &= \sum_{ijmn} a_{ij}\bar{b}_{mn} \langle e_n, [D, e_{mn}^*][D, e_{ij}])e_j \rangle e_n \otimes e_j^* \end{aligned}$$

Now $[D, e_{nm}][D, e_{ij}]$ forces $m = i$ so $e_n \otimes e_j^* = e_n \otimes e_m^*(e_m) \otimes e_j^*$. Therefore,

$$\langle \alpha(y), \alpha(x) \rangle = \sum_{ijmn} \mathbf{1}_{m=i} (b_{nm} e_{mn})^* a_{ij} e_{ij} \langle K(\cdot, e_{mn}) e_n, K(\cdot, e_{ij}) e_j \rangle$$
$$= \langle \sum_{mn} b_{mn} e_{mn} \otimes K(\cdot, e_{mn}) e_n, \sum_{ij} a_{ij} e_{ij} \otimes K(\cdot, e_{ij}) e_j \rangle = \langle t(\alpha(x)), t(\alpha(y)) \rangle$$

Finally, by $\mathcal{A}$-linearity of t and the inner product

$$\langle t(\alpha(y)), t(\alpha(x) e_{jj'}) \rangle = \langle t(\alpha(x)), t(\alpha(y)) \rangle e_{jj'}$$
$$= \langle \alpha(x), \alpha(y) \rangle e_{jj'}$$

□

To define the structure map $\sigma = \pi - \mathbf{1}$, consider the definition of ρ equation 4.8. Since $\pi(x) = t\hat{\rho}(x)t^*$, where $\hat{\rho}(x)\alpha(y) = \alpha(xy) - \alpha(x)y = \rho(x)\alpha(y)$, since α is a ρ-derivation. Therefore,

$$\pi(x)[e_{ij} \otimes K_{\mathcal{L}}((\cdot, \cdot), (a, b)] = e_{ij} \otimes K_{\mathcal{L}}((\cdot, \cdot), (a, xb)) \tag{4.10}$$

Now FinteLinSpan$\{K_{\mathcal{L}}((\cdot, \cdot), (e_{ij}, e_{mn}) e_k)\}$ is dense $R_{\mathcal{L}}$, and will be chosen as $(k_0)_\infty$. Notice that D being a Dirac operator was not used; so everything holds generally. The following theorem summarizes the structure matrix,

Theorem 4.3.2. *For Hilbert space H, $\mathcal{A} = \mathcal{B}(H)$, the structure matrix for the flow generated by $\mathcal{L} = -(A_{\mathcal{D}})^2$ for any self-adjoint operator A on H, e_i's an orthonormal eigenbasis for A is given by*

1. $k_0 = R_{\mathcal{L}}$, $(k_0)_\infty =$ FinteLinSpan$\{K_{\mathcal{L}}((\cdot, \cdot), (e_{ij}, e_{mn}) e_k)\}$
2. $\alpha(e_{ij}) = K(\cdot, e_{ij}) e_j \otimes e_j^*$, $\pi(x)[e_{ij} \otimes K_{\mathcal{L}}((\cdot, \cdot), (a, b)] = e_{ij} \otimes K_{\mathcal{L}}((\cdot, \cdot), (a, xb)$, $t : \alpha(e_{ij}) \to e_{ij} \otimes K(\cdot, e_{ij}) e_j$

4.4 Quantum Picard iterates

To start, recall the following estimates for map-valued processes with $a_\delta, a_\delta^\dagger, I_{\mathcal{L}}, \sigma$ being the fundamental processes (equations 3.9,3.10) (for background on map-valued qsdes, refer to section 3.2):

Estimate 4.4.1. *[62, Thm 5.4.7, 8.1.37] Define*

- $\Phi^1_{f,s}(x) := (\mathcal{L}(x) + \langle \delta(x^*), f(s) \rangle) \otimes E(f)$

- $\Phi^2_{f,s}(x) := (\sigma_{f(s)}(x) + \delta(x)) \otimes E(f)$
- $\Phi^3_{f,s}(x) := \langle f(s), \sigma_{f(s)}(x) + \delta(x)\rangle \otimes E(f) = \langle \sigma_{f(s)}(x) + \delta(x), f(s)\rangle^* \otimes E(f)$

For a map-valued integrable process Y_s,

$$\left\| \int_0^t Y_s \circ (a_\delta + I_{\mathcal{L}})(ds)(x \otimes E(f))u \right\|^2 \leq e^t \int_0^t \left\| Y_s(\Phi^1_{f,s}(x))u \right\|^2 ds \tag{4.11}$$

$$\left\| \int_0^t Y_s \circ (a^\dagger_\delta)(ds)(x \otimes E(f))u \right\|^2 \leq e^t \int_0^t \left(\left\| \widetilde{Y}_s(\Phi^2_{f,s}(x))u \right\|^2 + \left\| Y_s(\Phi^3_{f,s}(x))u \right\|^2 \right) ds \tag{4.12}$$

The following proposition which is useful to bound the $\widetilde{Y}$ process.

Proposition 4.4.2. *Let $T \in \mathrm{Lin}(K_{\mathcal{W}}, K'_{\mathcal{W}})$ for Hilbert space K, K' and C^*-algebras $\mathcal{W}, \mathcal{W}'$, $K_{\mathcal{W}}, K'_{\mathcal{W}}$ standard Hilbert C^*-modules. Then for any Hilbert space H, $\|T \otimes \mathbf{1}_H(x \otimes k \otimes h)\| \leq \|T(x \otimes k \otimes h)\|$ for any simple tensor $x \otimes k \otimes h$.*

Proof. By linearity of the tensor product, one may assume $\|h\| = 1$, then

$$\begin{aligned} &\|\langle T \otimes \mathbf{1}(x \otimes k \otimes h), T \otimes \mathbf{1}(x \otimes k \otimes h)\rangle\| \\ &\quad = \|\langle T(x \otimes k), T(x \otimes k)\rangle\langle h, h\rangle\| = \|(Tx \otimes k)^*(Tx \otimes k)\| = \|Tx \otimes k\|^2 \end{aligned}$$

□

There's the following characterization for an integrable map-valued process generated by the structure maps in Θ through the Picard iteration scheme, the convergence of which will yields the solution to the qsde needed.

Lemma 4.4.3. *[62, lemma 8.1.37] Let $\mathcal{V} = \{d\mathcal{F}\text{-valued simple functions}\}$, $E(\mathcal{V})$ the exponential vectors, and $J^{(0)} : \mathcal{F} \otimes E(\mathcal{V}) \to \mathcal{A} \otimes \Gamma(k_0)$ be the identity map, then with $J^{(0)} = \mathbf{1}$,*

$$J^{(n+1)}(t) = \int_0^t J^{(n)}(s) \circ (a^\dagger_\delta + a_\delta + I_{\mathcal{L}} + \Lambda_\sigma)(ds), \quad J^{(n+1)} : \mathcal{F} \otimes E(\mathcal{V}) \to \mathcal{A} \otimes \Gamma(k_0) \tag{4.13}$$

each J^n is a map-valued integrable process, Additionally, the following estimates hold ,

$$\begin{aligned}\left\|J_t^{(n+1)}(x \otimes E(f))u\right\|^2 &\leq 2\left(\left\|\int_0^t J_s^{(n)} \circ (a_\delta + I_{\mathcal{L}})(ds)(x \otimes E(f))u\right\|^2\right.\\ &\qquad \left. + \left\|\int_0^t J_s^{(n)} \circ (a_\delta^\dagger + \Lambda_\sigma)(ds)(x \otimes E(f))u\right\|^2\right)\\ &\leq 2e^t \int_0^t \left(\left\|J_s^{(n)}(\Phi_{f,s}^1(x))u\right\|^2 + \left\|\widetilde{J_s^{(n)}}(\Phi_{f,s}^2(x))u\right\|^2 + \left\|J_s^{(n)}(\Phi_{f,s}^3(x))u\right\|^2\right) ds\end{aligned} \tag{4.14}$$

Proof. The continuity requirements for existence of the integral for $J^{(0)}$ are satisfied since for each fixed $\mathrm{E}(f)$ and x the structure maps are bounded (recall remark 3.2.5). The inequalities follow from standard theory (for instance, [62, theorem 5.4.7]). Iterating, one gets that each $J^{(n)}$ integral exists and can be bound by $\left\|J^{(n-1)}\right\|$, and the inequalities hold again. □

By definition a map-valued process is linear, however, the processes $J^{(n)}$ are not completely smooth as the flow generator has much weaker regularity. The Picard iterates defined by

$$S_N(t) = \sum_{n \leq N} J_t^{(n)}(x \otimes \mathrm{E}(f)) \tag{4.15}$$

can be shown to converge on the exponential vectors following a similar scheme as [62, Thm 8.1.38] after plugging in the following estimates which need to be obtained differently as Θ has much less regularity. To motivate the estimates we sketch the convergence arguments.

Convergence for Picard iterates: examples

To establish the convergence of Picard iterates, the growth rate for $\|J_t^{(n+1)}(x \otimes \mathrm{E}(f))u\|^2$ as function of n for fixed x, f needs to be controlled. The idea is to expand $\|J_t^{(n+1)}(x \otimes \mathrm{E}(f))u\|$ recursively. Define $\Psi_{f,s}^i(x)$ such that $\Phi_{f,s}^i(x) := \Psi_{f,s}^i(x) \otimes \mathrm{E}(f)$, factoring out $\mathrm{E}(f)$. Now suppose x belongs to a subspace $\mathcal{A}'$ such that $\Psi_{f,s}^i(\mathcal{A}') \subset \mathcal{A}', i = 1, 3, P_1\Psi_{f,s}^2(\mathcal{A}) = P_1\mathrm{Swap}_{23}\Phi_{f,s}^2(\mathcal{A}') \subset \mathcal{A}'$ for all $x \in \mathcal{A}', f, s$. Additionally, it's required that if $x \in \mathcal{A}', x \geq 0, x$ invertible, $\sqrt{x} \in \mathcal{A}'$, however, this can be avoided if the structure matrix is sufficiently regular (see section 4.4 for an example).

Remark 4.4.4. Denoting any amplilation for $\Psi^i_{f,s} \otimes \mathbf{1}^m_{k_0}$ for $m \in \mathbb{Z}_{\geq 0}$ again by $\Psi^i_{f,s}$, so the composition $\Psi^i_{f,s}\Psi^2_{f,s}(x)$ makes sense: for example if $\Psi^2_{f,s}(x)$ sends x to $x' \otimes k' \in \mathcal{A}' \otimes k_0$ and $\Psi^i_{f,s}$ acts on $\mathcal{A}'$ component, $\Psi^i_{f,s}\Psi^2_{f,s}(x) := \Psi^i_{f,s} \otimes \mathbf{1}_{k_0}(\Psi^2_{f,s}(x))$. By proposition 4.4.2, the ampilation does not affect the norm: $\|\Psi^i_{f,s} \otimes \mathbf{1}(x \otimes e)\| = \|\Psi^i_{f,s}(x)|$, $\|e\| = 1$. By linearity, when input to $J^{(n)}$ is a sum of simple tensors, the bound is applied to each summand. Note that the n nested Ψ^i's that appear in corollary 4.4.5 are the components of iterates of Θ^n as defined in equation 3.7. The ampilation in Ψ^i's come from the ampilations in Θ^n. This yields,

$$\begin{aligned}\left\|\widetilde{J}^{(n)}_s(\Phi^2_{f,s}(x))u\right\| &= \left\|(J^{(n)}_s \otimes 1)(x' \otimes \mathrm{E}(f) \otimes k')u\right\| \\ &= \left\|J^{(n)}_s \otimes 1(x' \|k'\| \otimes \mathrm{E}(f) \otimes k'/\|k'\|)u\right\| = \left\|J^{(n)}_s(\|k'\| x' \otimes \mathrm{E}(f))u\right\|\end{aligned}$$

Therefore, if for fixed f, x, number of summands N doesn't grow too fast, and a uniform bound holds on each, then the additional growth due to each ampilation can be bound by a function of N.

The terms in *r.h.s.* for equation 4.14, can be recursively expanded using estimate 4.4.1 till $J_0 = \mathbf{1}$. Since f is simple, $|\textsc{Range}(f)| = r < \infty$, so for each $s \in [0, t]$, $f(s) \in \{\xi_{i_k} := d\phi_{i_k}, \zeta_i = \phi_{i_k}, k \in [r]\} \equiv \textsc{Range}(f)$. Therefore, all terms depending on f in above can be uniformly bound by a constant $B := B_f$, this yields

$$\begin{aligned}&\left\|J^{(n+1)}_t(x \otimes \mathrm{E}f)\right\|^2 \\ &\leq K_{t,f} \sum_{i_k \in [3]} \int_0^t \int_0^{s_0} \cdots \int_0^{s_{n-1}} \left\|\Psi^{i_0}_{f,s_0}(\Psi^{i_1}_{f,s_1}(\ldots \Psi^{i_{n-1}}_{f,s_{n-1}}(x))\right\| ds_0 ds_1 \ldots ds_{n-1}\end{aligned} \tag{4.16}$$

for $K_{t,f} := (2e^t B)^n \|\mathrm{E}(f)\|^2$. To get the Picard iterates to converge (eq 4.15), one needs to show that $\Psi^{i_0}_{f,s_0}(\ldots \Psi^{i_{n-1}}_{f,s_{n-1}}(x))$ cannot growth too fast as function of n.

Corollary 4.4.5. *Suppose for fixed f, x for any choice $i_k \in [3], k \in \mathbb{N}$, there are constants C, L_x satisfying*

$$\left\|\Psi^{i_0}_{f,s_0}(\Psi^{i_1}_{f,s_1}(\ldots \Psi^{i_{n-1}}_{f,s_{n-1}}(x)))\right\| \leq C^n L_x$$

then $S_N(t)$ defined by eq 4.15 converges.

Proof. In equation 4.16, by above bound,

$$\left\|J^{(n+1)}_t(x \otimes \mathrm{E}f)\right\|^2 \leq K_{t,f} C^n L_x \sum_{i_k \in [3]} \int_0^t \int_0^{s_0} \cdots \int_0^{s_{n-1}} ds_0 ds_1 \ldots ds_{n-1} \leq \frac{K'_{t,f}(C')^n}{n!}$$

so $\sum_n J_t^{(n)}(x \otimes \mathrm{E}f)$ is bounded, giving the convergence. □

Example 4.4.6 (Convergence for the noncommutative laplacian)**.** To get the convergence of Picard iterates for the noncommutative laplacian $\mathcal{L}$, the above bound is needed. Fix $x \in \mathcal{F} = \text{FinteLinSpan}\{e_{ij} : e_i \in \mathcal{S}_D\}$, so $x = \sum_{i,j\in[N]} a_{ij} e_i \otimes e_j^*$, $N < \infty$, and f a simple function taking values in $(k_0)_\infty = \text{FinteLinSpan}\{K_{\mathcal{L}}((\cdot,\cdot),(e_{ij},e_{mn})e_k)\}$ where as usual $e_{ij} := e_i \otimes e_j^*$. Define the smallest N as the grading of x. Let $\mathcal{T}_x$ be all e_i's that appear in x (either as e_i or as e_i^*) . Note that if $x \in \mathcal{F}, x \geq 0$ then as x is bounded and symmetric, it's self-adjoint, therefore, $\sqrt{x}$ is defined by functional calculus. Set

$$\mathcal{V}_{f,x} := \{K_{\mathcal{L}}((\cdot,\cdot),(a,b)y) \in \text{Range}(f)\}$$

Note $\mathcal{T}_x, \mathcal{V}_{f,x} < \infty$ and $\mathcal{T}_{x^*x} \subset \mathcal{T}_{x^*} = \mathcal{T}_x$. Define $\text{Span}(\mathcal{T}_x) = \text{LinSpan}\{e_{ij} : e_i, e_j \in \mathcal{T}_x\}$.

Proposition 4.4.7. *For fixed f, x for any choice $i_k \in [3]$ there exists a constant C,*

$$\left\|\Psi^{i_0}_{f,s_0}(\Psi^{i_1}_{f,s_1}(\ldots\Psi^{i_{n-1}}_{f,s_{n-1}}(x)))\right\| \leq C^n \|x\|$$

Proof. The proof will repeatedly use theorem 4.3.2. Since $x \in \mathcal{F}$, there exists M_x such that $\left\|\mathcal{L}^k x\right\| \leq M_x^k \|x\|$; note $M_x \leq \sup\{2\lambda_i : De_i = \lambda_i e_i \in \mathcal{T}_x\} = M$. This bound also holds for any $y \in \text{Span}(\mathcal{T}_x)$. At the same time, $V_f = \sup_{v\in\mathcal{V}_f} \|v\| < \infty$.

Notice that in Φ^3, Φ^2, $\sigma = \pi - \mathbf{1}$ where π is multiplication by x, and since $x = \sum_{i,j\in[N]} a_{ij} e_{ij}$, $\|\pi(x)\| \leq \|x\|$, hence, $\sigma(x)_{f(s)} \leq 2V_f \|x\|$ where theorem 4.3.2 was used to get form for $\pi(x)$. Furthermore,

$$\left\|\delta^\dagger(x)\delta(x)\right\| = \|\delta(x)\|^2 = \left\|\delta^\dagger(x)\right\|^2 = \|\mathcal{L}(x^*x) - \mathcal{L}(x^*)x - x^*\mathcal{L}(x)\|^2 \leq 3M^2 \|x\|^2$$

By theorem 4.3.2, the $\mathcal{A}$ component of $\mathcal{A} \otimes k_0$ of $\Phi^2_{f,s} \in \text{Span}(\mathcal{T}_x)$. Similarly, $\langle\delta(x^*), f(s)\rangle, \langle\sigma_f(x), f(s)\rangle, \langle f(s), \delta(x)\rangle\mathcal{L}(x) \in \text{Span}(\mathcal{T}_x)$, and since $\|\langle a' \otimes k', a'' \otimes k''\rangle\| \leq |\langle k', k''\rangle| \|a'^* a''\|$,

$$\|\langle\delta(x^*), f(s)\rangle\|, \left\|\langle\sigma_f(x), f(s)\rangle\right\|, \|\langle f(s), \delta(x)\rangle\| \leq \max(3M, 2V_f)\|x\| V_f$$

This means that $\Phi^1_{f,s}(\text{Span}(\mathcal{T}_x)) \subset \text{Span}(\mathcal{T}_x)$ for $= 1, 3$ while $\Phi^2_{f,s}(\text{Span}(\mathcal{T}_x) \otimes k_0^m) \subset \text{Span}(\mathcal{T}_x) \otimes k_0^{m+1}$ (recall note 4.4.4). But all the bounds only depend on M and $\mathcal{V}_f$ and don't change on iterating since Ψ^i's preserve the $\mathcal{A}$ component to be inside $\text{Span}(\mathcal{T}_x)$.

Now there are n-possible ampilations in nested Ψ^i's. The k_0 component is only generated by δ which produces simple tensor for every e_{ij}. Since $x = \sum_{i,j\in[N]} a_{ij} e_{ij}$,

$\mathcal{T}_{e_{ij}} \subset \mathcal{T}_x$ and e_{ij} are preserved by structure maps. Expanding by linearity, and using that the above bounds hold for each of the N summands which are simple tensors on which by proposition 4.4.2 the ampilation does not increase the norm, the triangle inequality on N summands introduces a multiplicative N factor for each ampilation.

Therefore, $C = (\max(3M, 2V_f)V_f + 4M + 2V_f)N$ is sufficient to bound the growth.

□

Remark 4.4.8. Since $\mathcal{F}$ is norm-dense in compact operators, the flow extends to compact operators. Each j_t being a $*$-homomorphism can be extended to enveloping von Neumann algebra as a normal $*$-homomorphism.

Example 4.4.9. This construction can be used for the existence of flow for Laplace-Beltrami operator $\triangle^M$, on any compact manifold, since $\triangle^M$ is positive, therefore, by spectral calculus, $| \triangle |^{1/2} = \lambda_i e_i \otimes e_i^*$ where $E = \{e_i : \triangle^M e_i = \lambda_i^2\}$ then on $\mathrm{End}(E)$, the construction of the flow $(| \triangle |_{\mathcal{D}}^{1/2})^2$ converges.

Example 4.4.10 (Flow for the Laplace-Beltrami operator)**.** Now consider the case where $\triangle$ is acting on $C(M)$ for compact Riemannian manifold (M, h) viewed as operators acting on $L^2(M)$. From earlier $\alpha(g) : f \to \langle \cdot, dg \rangle f$. $\sigma = 0$ as for endomorphism laplacian. The difficulty in this example is that $C(M)$ is not $\mathrm{End}(L^2(M))$, so the global embedding used for noncommutative laplacian does not work. However, a local embedding t can be described about any $p \in M$ and since the flow is generated by local operator $\triangle$, and C^*-norm on $C(M)$ is point-wise, this is enough.

Let $(x_i)'s$ be Riemann normal coordinates about $p \in U \subset M$. Then in the local trivialization over U, $\langle \cdot, df \rangle$ is described the components $df(\partial_i)$, and with $\mathcal{A}|_U = C(U)$, the Hilbert-$\mathcal{A}|_U$-module is $\mathcal{A} \otimes \mathbb{R}^{\dim M}$. The isometry at p (with respect to the $\mathcal{A}$-inner product) embedding is given by

$$N|_p \ni \alpha(f)g \to \langle \nabla f, \partial_i \rangle g \otimes r_i \in \mathcal{A} \otimes \mathbb{R}^{\dim M} \tag{4.17}$$

r_i being the standard basis for $\mathbb{R}^{\dim M}$. This is because p is the center of Riemann normal coordinates where the metric and Christoffel symbols are trivial, the $\mathcal{A}|_U$-valued inner product, $\sum_i dg(\partial_i) df(\partial_i) = \langle \nabla g, \nabla f \rangle_p$.

The embedding t identifies the coordinate vector fields with the noise vector $r \in k_0, r_i \to \partial_i$ in normal coordinates at p, so depends on the chart, and the k_0 contraction in $\mathcal{A} \otimes k_0$ with $\sum_i a_i r_i$ is $\sum_i \langle \nabla f, \partial_i \rangle \otimes \langle r_i, a_i r_i \rangle = \sum_i \langle \nabla f, a_i \partial_i \rangle$. The $\alpha(f), \nabla_i(f) \otimes r_i$

are the $\mathcal{A}$-linear basis locally for $N, \mathcal{A} \otimes \mathbb{R}^{\dim M}$. Define $D_i f := \langle \nabla f, \partial_i \rangle$. Note that iterating on D_i's introduces iterated covariant derivatives,

$$D_i D_j f = \langle \nabla \langle \nabla f, \partial_j \rangle, \partial_i \rangle = \langle \langle \nabla^2 f, \partial_j \rangle + \langle \nabla f, \nabla \partial_j \rangle, \partial_i \rangle$$

and higher order expansions follow the same.

Remark 4.4.11. The isometry can be extended to U either by using an isometric euclidean embedding or defining $t : \alpha(f) g(q) \to H_q^{1/2} [\nabla_i f]_q g(q)$ where H is the matrix representation of the metric h; regardless there's need to control one additional piece of data which is an assumption involving derivatives of the metric.

Now taking $\mathcal{F} = \text{FinteLinSpan}\left(\prod_{i \in [k]} \phi_i : k \in \mathbb{N}\right)$, convergence of the Picard iterates follows mostly like proposition 4.4.7 when $\|O_k O_{k-1} \dots O_1(x)\|_p \leq M_x^k L_x$ where $O_i \in \{\triangle, D_i\}$ for all $x \in \mathcal{F}, p \in M, D_i$'s defined with respect to Riemann normal coordinates centered at p.

Proposition 4.4.12. *For fixed $x \in \mathcal{F}$, f a simple function valued in $\mathbb{R}^{\dim M} = k_0$, the following holds for any choice $i_k \in [3], k \in \mathbb{N}$,*

$$\left\| \Psi^{i_0}_{f,s_0} (\Psi^{i_1}_{f,s_1} (\dots \Psi^{i_{n-1}}_{f,s_{n-1}} (x))) \right\| \leq C^n L_x$$

Proof. Fix $p \in M$ and normal coordinates centered at p. Since $x \in \mathcal{F}$, there exists K such that $\|O_k O_{k-1} \dots O_1(x)\| \leq K^k L_x$. For $\mathcal{R} = \text{Range}(f)$, $V_f = \sup_{v \in \mathcal{R}} \|v\| < \infty$ as $\mathcal{R}$ is finite. Consider the expansion of the nested application $\Psi(x) := \Psi^{i_0}_{f,s_0} (\Psi^{i_1}_{f,s_1} (\dots \Psi^{i_{n-1}}_{f,s_{n-1}} (x)))$. Note $\sigma = 0, x = x^*$, $\Psi^3_{f,s} = \langle f(s), \delta(x) \rangle, \Psi^2_{f,s} = \delta(x), \Psi^1_{f,s} = \triangle(x) + \langle \delta(x), f(s) \rangle$. Additionally, $\delta(x) = D_i(x) \otimes r_i$, so $\langle \delta(x), f(s) \rangle = \sum_i f(s)_i D_i(x)$ with each $f(s)_i$ in $\mathbb{R}$. This gives the form for the inner products, $\langle \delta(x), f(s) \rangle, \langle \delta(x^*), f(s) \rangle$ in Φ^1, Φ^3 which are contraction of $\mathbb{R}^{\dim M}$ component in $\mathcal{A} \otimes \mathbb{R}^{\dim M}$.

Let $\mathcal{G}_0 = \{x\}$, $\mathcal{G}_{k+1} = \text{FinteLinSpan}(\{\triangle(z), D_i(z) : z \in \mathcal{G}_k\})$. Then by above characterization $\Psi^i_{f,s}(z) \in \mathcal{G}_{k+1}$ if $z \in \mathcal{G}_k, i = 1, 3$, while $\Psi^2_{f,s}(z) \in \mathcal{G}_{k+1} \otimes \{r_i\}$. Therefore, at every point $p \in M$ with $m = \dim M$, for any $z \in \mathcal{G}_k$

$$\left\| \Psi^2_{f,s}(z) \right\| \leq mKL_x, \ \left\| \Psi^3_{f,s}(z) \right\| \leq mKL_x \|f(s)\|$$
$$\left\| \Psi^1_{f,s}(z) \right\| \leq mK \|z\| \|f(s)\| + KL_x$$

To track the ampilations, again notice that the δ produces a sum of $\dim M$ simple tensors, $f \to \sum_i df(\partial_i) \otimes r_i$. If x has N summands then on each application it generates at most $T = N \cdot \dim M$ simple tensors, and on each the bounds hold individually, so breaking up in T simple tensors each time, and applying the bounds individually so the ampilation no longer increases the norm, adds on another multiplicative factor of T. Hence, $\Psi^i_{f,s}$'s are bounded operators on each $\mathcal{G}_k$, uniformly bounded in s, k by $C = (2mK)^2 V_f T$ which gives the growth bound. □

For any eigenfunction ϕ, bounds of type $\|O_k O_{k-1} \dots O_1(\phi)\|_p \leq M_\phi^k L_\phi$, require bounds on $\|\nabla^k \phi\|$ (see section 4.6), and $\|\nabla^k \partial_i\|$, which depend on the metric. Alternatively, for any multi-index β, asymptotic bounds in of type $|\partial^\beta \phi_\lambda|_p \leq O(\lambda^{2|\beta|})$ (where λ is the associated eigenvalue) with respect to normal coordinates are known[15], using product rule, the bound $\|O_k O_{k-1} \dots O_1(x)\|_p \leq M_x^k L_x$ is equivalent to a bound on the Christoffel symbols, therefore, as a corollary when Christoffel symbols vanish the Picard iterates converge.

Observation 4.4.13. There's another issue that needs to be dealt with: the algebra $\mathcal{F}$ is not closed under square-roots because they might not be in the finite linear span, but is dense in an appropriate norm. This will require that for $\phi \in \mathcal{S}_\triangle$

$$\langle [\triangle, \nabla^k]\phi, \nabla^k \phi \rangle_{L^2} = 0 \tag{4.18}$$

where $\triangle = \nabla^* \nabla$ now. This also controls the growth of $\|\nabla^k \phi\|$; this is considered in section 4.6.

The extended square-root trick

It's again part of standard theory (for instance, [62, theorem 5.4.9ii]), that for $u, v \in H$, $h, f \in \mathcal{V} \subset L^2(\mathbb{R}_+, k_0) \equiv k$,

$$\langle J_t(a \otimes \mathrm{E}(f))u, J_t(b \otimes \mathrm{E}(h))v \rangle = \langle u\mathrm{E}(f), J_t(a^* b \otimes \mathrm{E}(h))v \rangle \tag{4.19}$$

$$J_t(1 \otimes \mathrm{E}(f))u = u\mathrm{E}(f) \tag{4.20}$$

Define

$$j_t^n(a)(v\mathrm{E}f) := J_t^{(n)}(a \otimes \mathrm{E}f)v \tag{4.21}$$

so j_t^n is unital with the factorization property (equation 4.19), and $j_t^n(a)$ is a linear operator on a dense subspace $\mathcal{K} := H \otimes \mathrm{E}(\mathcal{V}) \subset H \otimes \Gamma(k)$. For any v, f, j_t^n is bounded pointwise on $\mathcal{F}$.

Proposition 4.4.14. *For all $a \in \mathcal{F}$ there exists K such that*

$$\|\Theta(a)\| \leq K \|a\|_{W^{2,\infty}}$$

where $\|\cdot\|_{W^{2,\infty}}$ is the Sobolev norm $\|a\|_{W^{2,\infty}} = \|a\| + \|\nabla a\| + \|\nabla^2(a)\|$

Proof. This follows because for any a, $\Theta(a)$ satisfies $\|\Theta(a)\| \leq C(\|\triangle(a)\| + \|\delta^\dagger(a)\| + \|\delta(a)\|)$ for appropriate C where $\|\triangle(a)\| \leq \dim M \|\nabla^2(a)\|$, $\|\delta^\dagger(a)\| = \|\delta(a)\| \leq \dim M \|\nabla a\|$

□

Now $\mathcal{F} \subset W^{k,p}(M)$ since $\mathcal{F} \subset C^\infty(M)$, M compact for all p, k. Because $\|\Theta(a)\| \leq K \|a\|_{W^{2,\infty}}$, if $J_t^{(n-1)}$ is bounded for each t, then $J_t^{(n)}$ is continuous on $\mathcal{F}$ wit respect to $W^{2,2}$-norm topology with f, v held fixed. We will use this to show that if $J_t^{(n-1)}$ is bounded, then $j_t^{(n)}$ is positive on $\mathcal{K}$. Then using j_t^n is positive on $\mathcal{K}$ it will be checked that for every $a \in \mathcal{F}$, $j_t^n(a) \in \mathcal{B}(\mathcal{K})$ and that it extends from $\mathcal{B}(\mathcal{K})$ to $\mathcal{B}(H \otimes \Gamma(k))$. The base case is $j_t^{(0)} = \mathbf{1} \in \mathcal{B}(\mathcal{K})$ which is obviously positive. Then from $j_t^n : \mathcal{F} \to \mathcal{B}(H \otimes \Gamma(k))$, it extends to $j_t^n : C(M) \to \mathcal{B}(H \otimes \Gamma(k))$.

Lemma 4.4.15. *Suppose $J_t^{(n-1)} \in \mathcal{B}(\mathcal{K})$, then j_t^n is a positive map on $a \in \mathcal{F}, a > 0$.*

Proof. Suppose $a \in \mathcal{F}$ is positive. We want to show $j_t^n(a)$ is positive as well. If $\sqrt{a} \in \mathcal{F}$, then

$$\langle u, j_t^n(a)u\rangle = \langle j_t^n(\sqrt{a})u, j_t^n(\sqrt{a})u\rangle \geq 0 \tag{4.22}$$

for every $u \in \mathcal{K}$, hence $j_t(a)$ is positive.

So assume $\sqrt{a} \notin \mathcal{F}$ where a is positive and invertible, so $a(m) > 0, m \in M$. Since LinSpan$(\mathcal{F})$ is dense in $C(M)$, for any $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that $\|\sqrt{a} - f\| \leq O(\epsilon)$ meaning $\|a - f^2\| \leq O(\epsilon)$. Additionally, f can be chosen so $\|a - f^2\|_{W^{2,\infty}} \leq O(\epsilon)$, so f^2 approximates a in Sobolev $W^{2,\infty}$-norm as well.

To see why this is possible note that since $a > 0, \sqrt{a} \in C^\infty(M)$, therefore, $\sqrt{a} \in L^2(M)$, additionally for each k, $\nabla^k(\sqrt{a}) \in L^2(M)$, with $\sqrt{a} = \sum_i \alpha_i \phi_i$, $\alpha_i = \langle \phi_i, \sqrt{a}\rangle$, then using assumption in equation 4.18 (which can be relaxed if T in proposition 4.6.3 in section 4.6 is independent of the eigenfunction),

$$\|\nabla^k \sum_{i<n} \alpha_i \phi_i\|_{L^2}^2 = \sum_{i<n} \alpha_i^2 \lambda_i^{2k} \leq \sum_i \alpha_i^2 \lambda_i^{2k} = \langle \nabla^k \sqrt{\phi}, \nabla^k \sqrt{a}\rangle_{L^2} < \infty$$

So the sequence $(\sum_{i=1}^n \alpha_i \phi_i)_{n \in \mathbb{N}}$ converging to $\sqrt{a}$ in $L^2(M)$ is a bounded in each $W^{k,2}$ with a bound depending on k. For sufficiently large k, the embedding $W^{k,2}(M) \subset W^{2,2}(M)$ is compact by the Rellich-Kondrachov theorem, that is, $(\sum_{i=1}^n \alpha_i \phi_i)_n$ has a Cauchy, and so a convergent subsequence; wlog let this subsequence be denoted by the same $\sum_{i=1}^n \alpha_i \phi_i := a_n$. For (a_i) to be convergent in

$W^{2,2}(M)$, it must also be convergent in $\|\cdot\|_{L^2(M)}$, so the only possible limit is $\sqrt{a}$. Now suppose the tail $\sum_{i=n}^{\infty} \alpha_i \phi_i$ does not vanish in $W^{2,\infty}$. This means for some $x \in M$ for some $k \in \{0, 1, 2\}$, $\langle \sum_i \alpha_i \nabla^k \phi_i, \sum_i \alpha_i \nabla^k \phi_i \rangle(x) > 0$. But then by the following argument shows that $\langle \sum_i \alpha_i \nabla^k \phi_i, \sum_i \alpha_i \nabla^k \phi_i \rangle_{L^2(U)} > 0$ contradicting the convergence in $W^{2,2}(M)$. So $\sqrt{a}$ can be approximated arbitrarily well in $W^{2,\infty}(M)$.

Claim 4.4.16. *Suppose $a = \sum_i \alpha_i \phi_i \in C^\infty(M)$, then $\langle \nabla^k \sum_i \alpha_i \phi_i, \nabla^k \sum_i \alpha_i \phi_i \rangle(x) > 0$ for some $x \in M$ implies $\langle \nabla^k \sum_i \alpha_i \phi_i, \nabla^k \sum_i \alpha_i \phi_i \rangle_{L^2(M)} > 0$. In particular, this holds for $k = 0$.*

Proof. By smoothness of a, this holds for all $x \in U$ for some open set U. Integrating against compactly supported ψ on U, $\mathbf{1} \geq \psi \geq 0, \psi > 0$ on an open $V \subset U$, $\langle \nabla^k a, \nabla^k a \rangle_{L^2(M)} \geq \int_U \langle \psi \nabla^k \sum_i \alpha_i \phi_i, \nabla^k \sum_i \alpha_i \phi_i \rangle dV_g > 0$ The $k = 0, \nabla^k = \mathbf{1}$ specialization is identical. □

Now define

$$\mathcal{W}_a = \{a\} \cup \{f^2 : f \in \mathcal{F} \text{ with } \left\|a - f^2\right\|_{W^{2,\infty}} \leq 1/n, n \in \mathbb{N}\}$$

then as $\|\Theta(a')\| \leq K \|a'\|_{W^{2,\infty}}$ and $J_t^{(n-1)}$ is bounded on $\mathcal{K}$ by hypothesis, the bound in lemma 4.4.3, implies norm:

$$\|\cdot\| : \mathcal{W}_a \to \mathbb{R}, a' \to \|j_t(a')\|$$

is continuous map with respect to $\|\cdot\|_{W^{2,\infty}}$-topology on $\mathcal{W}_a$

If $j_t^n(a)$ is not positive, then there exists $u \in \mathcal{K}$ such that $\langle u, j_t^n(a)u \rangle < 0$. Since norm is continuous, the map $a' \to \langle u, j_t^n(a')u \rangle$ is also continuous on $\mathcal{W}_a$: by Cauchy-Schwartz inequality, $\langle u, j_t^n(a')u \rangle \leq \|u\| \left\|j_t^n(a')u\right\| \leq \|u\|^2 K'K \|a'\|_{W^{2,\infty}}$ where K' depends on u which we fixed and $\left\|J_t^{n-1}\right\|$. This continuity means $\langle u, j_t^n(\cdot)u \rangle < 0$ on some neighborhood containing a in $\mathcal{W}_a$. However, for any neighborhood U of a in $\mathcal{W}_a$, $w \in U, w \neq a$ implies $w = f^2, f \in \mathcal{F}$, so $\langle u, j_t^n(f^2)u \rangle \geq 0$ by equation 4.22. Therefore, $j_t^n(a)$ must be positive. □

Lemma 4.4.17. *If j_t^n is a positive map on positive $a, a > 0$, then $\left\|j_t^n(a)\right\|^2 \leq \|a\|^2$*

Proof. Let $x \in \mathcal{F}$ so $(1 + \epsilon) \|x\| - x \in \mathcal{F}$ and positive for any $\epsilon > 0$. Define $\Phi_\epsilon(x) := \sqrt{(1 + \epsilon) \|x\| \mathbf{1} - x} \in C(M)$. Approximate $\Phi_\epsilon(x)$ from below by $z \in \mathcal{F}$.

Then $j_t^n(\Phi_\epsilon(x)^2 - z^2) > 0$ because $\Phi_\epsilon(x)^2 - z^2 > 0$ and j_t^n is positive. This yields $\langle \theta, (j_t^n(\Phi_\epsilon(x)^2) - j_t^n(z^2))\theta\rangle \geq 0$ and we have

$$\langle \theta, j_t^n(\Phi_\epsilon(x)^2)\theta\rangle \geq \langle \theta, j_t^n(z^2)\theta\rangle \geq 0$$

Now the usual square-root trick takes over: since j_t^n is unital,

$$0 \leq \left\| j_t^n(z)\theta \right\|^2 = \langle \theta, j_t^n(z^2)\theta\rangle \leq \langle \theta, j_t^{(n)}((1+\epsilon)\left\|x\right\|\mathbf{1} - x)\theta\rangle \tag{4.23}$$

$$\langle \theta, j_t^n(x)\theta\rangle \leq \langle \theta, j_t^n((1+\epsilon)\left\|x\right\|\mathbf{1})\theta\rangle \leq (1+\epsilon)\left\|x\right\|\langle \theta, j_t^n(\mathbf{1})\theta\rangle = (1+\epsilon)\left\|x\right\|\left\|\theta\right\|^2 \tag{4.24}$$

Since ϵ was arbitrary, $\langle \theta, j_t^n(x)\theta\rangle \leq \left\|x\right\|\left\|\theta\right\|^2$. Finally,

$$\left\| j_t^n(x)\theta \right\|^2 = \langle j_t^n(x)\theta, j_t^n(x)\theta\rangle = \langle \theta, j_t^n(x^*x)\theta\rangle \leq \left\|x^*x\right\|\left\|\theta\right\|^2 = \left\|x\right\|^2\left\|\theta\right\|^2 \tag{4.25}$$

. So $\left\| j_t^n(x) \right\|^2 \leq \left\|x\right\|^2$, and the bound on $\left\| j_t^n \right\|$ is uniform. □

Now from density of $\mathcal{F}, \mathcal{V}$ and $\mathcal{K}$, each j_t^n extends from a map $j_t^n : \mathcal{F} \to \mathcal{B}(\mathcal{K})$ to $j_t^n : \mathcal{A} \to \mathcal{B}(H \otimes \Gamma(k))$. Since $S_N(t) = \sum_{n\in[N]} J^n$ converges, so does $S = \lim_{N\to\infty} S_N$, and therefore $\lim_{n\to\infty} \sum j_t^n$ is the needed flow. Precisely, we have the following result:

Theorem 4.4.18. *Following notation from section 4.2, define* $j_t(a)(v_1 Ef_1) := J_t(a \otimes Ef_1)v_1$, *then*

1. $j_t : \mathcal{F} \to \mathcal{B}(H \otimes E(\mathcal{V}))$ *is a unital* $*$*-homomorphism*
2. j_t *extends to* $j_t : \mathcal{F} \to \mathcal{B}(H \otimes \Gamma(k_0))$
3. j_t *extends to* $j_t : \mathcal{A} \to \mathcal{B}(H \otimes \Gamma(k_0))$

Remark 4.4.19. A remark on construction of Sinha and Goswami [62] using Frechet structures and of Belton and Wills [11]: Proposition 4.4.14, along with the growth bounds on $\ell(\nabla^k \phi)$ suggests that convergence of the stochastic integrals can be approached via a generalization of complete smoothness regularity. In absence of the group action, the Frechet space structure on k_0 has to be obtained differently, ∇^k is the natural candidate for defining the Sobolev norms on $d\mathcal{F} \subset k_0$.

Notice that the growth condition in corollary 4.4.5 is similar to one obtained by [11]. However, the algebra is not closed under square-roots and it becomes necessary to use the regularity of the generator with respect to Sobolev norms to push the modified square-root trick through.

4.5 Evans-Hudson dilation on reductive homogeneous spaces

Now a second example where the existence of a quantum stochastic flow associated to an untruncated laplacian is considered: the spectral triple $(\mathcal{A}, L^2(M,S), D)$ where M is a compact reductive homogeneous space, and S a homogeneous Clifford module bundle with Dirac operator D, $\mathcal{A} \subset \mathcal{B}(S)$. The flow is noncommutative and covariant with respect to the group action. Additionally, it realizes the spectral action since the domain of the flow includes finite rank operators.

The construction from [62] uses growth bounds in terms of a family of semi-norms rather than iterates of the structure matrix, and when the semigroup and its generator are covariant with respect to the action of a Lie group, using the equivariant Kasparaov's stabilization theorem, the embedding t can be made equivariant with respect to the semi-norms, which allows for controlling the semi-norms even through the t embedding. Complete smoothness is the regularity condition on the semi-norms that guarantees the convergence of the quantum Picard iterates. It will now be established that both these requirements hold for the heat semigroups over reductive homogeneous spaces.

Complete smoothness

To start, consider C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathsf{H})$ on the Hilbert space H, G is a second countable, compact Lie group with finite dimensional Lie algebra, acting by a strongly continuous representation $G \ni g \to \alpha_g \in \mathrm{Aut}(\mathcal{A})$ on $\mathcal{A}$.

Definition 4.5.1. Suppose $\{\chi_i : i \in [n]\}$ is the basis for the Lie algebra $\textsc{Lie}[G]$, and $\mathrm{d}\,g$ the left Haar measure on G. The smooth algebra is defined by $\mathcal{A}_\infty = \{a : g \to \alpha_g(a)$ is smooth for all $g \in G$ in norm topology$\}$.

Note that $\mathcal{A}_\infty = \cap_{k\in[n]} \mathrm{Dom}(\partial_k)$ where ∂_i is closed $*$-derivation on $\mathcal{A}$ given by the generator of the automorphism group $(\alpha_{t\chi_i})_{t\in\mathbb{R}}$. $\mathcal{A}_\infty$ can be equipped with Sobolev-type norms,

$$\|a\|_n = \sum_{i_1,i_2\ldots i_k : k\leq n} \left\|\partial_{i_1} \cdots \partial_{i_k}(a)\right\| \tag{4.26}$$

with $\|a\|_0 = \|a\|$. $\mathcal{A}_\infty$ is a Frechet algebra. Note that the algebra $\mathcal{A}_\infty$ is also used in [33]; however, the norms $\|\cdot\|_n$ are symmetrized explicitly.

Definition 4.5.2. (Covariant quantum dynamical semigroups) Let G be a locally compact group acting on C^*-algebra by $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ with α_g denoting $\alpha(g)$.

A quantum dynamical semigroup (T_t) is covariant with respect to G if for all $t \geq 0, g \in G, T_t \circ \alpha_g = \alpha_g \circ T_t$, equivalently $\mathcal{L} \circ \alpha_g = \alpha_g \circ \mathcal{L}$ where $\mathcal{L}$ generates (T_t).

If $\mathcal{L}$ is unbounded, but with $\mathcal{L}(\mathcal{A}_\infty) \subset \mathcal{A}_\infty \subset \mathrm{Dom}(\mathcal{L})$ for a dense-subalgebra $\mathcal{A}_\infty$, then one defines covariance for $\mathcal{L}$ by $\mathcal{L}(\alpha_g(a)) = \alpha_g(\mathcal{L}(a))$ for all $a \in \mathrm{Dom}(\mathcal{L})$.

Definition 4.5.3. A map between Frechet algebras $\mathcal{M}_\infty, \mathcal{N}_\infty$ with respect to actions μ_g, η_g of compact Lie group G on C^*-algebras $\mathcal{M}, \mathcal{N}$ is p-smooth if there exists a constant C and $p \in \mathbb{Z}^{\geq 0}$ satisfying that for $\xi \in \mathcal{M}_\infty$,

$$\|\mathcal{L}\xi\|_n \leq C \|\xi\|_{n+p}$$

and it's p-completely smooth if there exists a constant C and $p \in \mathbb{Z}^{\geq 0}$ satisfying for all $n, N \geq 0$ and $\xi \in \mathcal{M}_\infty \otimes \mathrm{MAT}_n$,

$$\left\|\mathcal{L} \otimes 1_{\mathrm{MAT}_N}(\xi)\right\|_n \leq C \|\xi\|_{n+p}$$

$\mathcal{L}$ is called completely smooth if it's p-completely smooth for some p.

Note that bounded operators are completely smooth since from equation (4.26), $\|\cdot\|_l \geq \|\cdot\|_q$ for all $l \geq q$.

Lemma 4.5.4. *Suppose W_i is w_i-completely smooth for $i \in [N]$, then any polynomial in W_i's is completely smooth to some order.*

Proof. First, since W_i is w_i-completely smooth for $i \in [N]$, let $\left\|W_i \otimes 1_{\mathrm{MAT}_N}(\xi)\right\|_n \leq C_i \|\xi\|_{n+w_i}$. By eq 4.26, so we can assume W_i are $w = \max(w_i)$-completely smooth, meaning $C = \max_{[N]} C_i$, $\left\|W_i \otimes 1_{\mathrm{MAT}_N}(\xi)\right\|_n \leq C \|\xi\|_{n+w}$ for all i. This gives

$$\left\|\sum_{i\in[N]} W_i \otimes 1_{\mathrm{MAT}_N}(\xi)\right\|_n \leq \sum_{i\in[N]} \left\|W_i \otimes 1_{\mathrm{MAT}_N}(\xi)\right\|_n \leq NC \|\xi\|_{n+w}$$

For $W_iW_j := W_i \circ W_j$, $\left\|W_iW_j \otimes 1\xi\right\|_n = \left\|W_i \otimes 1(W_j \otimes 1)\xi\right\|_n \leq C_i \left\|W_j \otimes 1\xi\right\|_{n+w_i} \leq C_iC_j \|\xi\|_{n+w_i+w_j}$ and the conclusion follows. □

We note the following version of [62, Thm 8.1.28].

Proposition 4.5.5. *Suppose $\mathrm{LIE}[G]$ has basis $X_i : i \in [m]$, i.e., X_i's generate one-parameter subgroups, then the $\Phi[X_i : i \in [m]]$ be a polynomial degree p in X_i's with coefficients in $\mathcal{B}(H)$, which by the Lie algebra action on $\mathcal{A}_\infty$ defines a map $\Phi : \mathcal{A}_\infty \to \mathcal{A}_\infty$, then Φ is p-completely smooth.*

Proof. Set α as the norm of the largest coefficient of Φ, wlog assume $\alpha \geq 1$. For any monomial Φ_i in Φ, with $\xi = \sum_{[q]} x_l \otimes m_l, \Phi = \Phi[X_i : i \in [m]]$,

$$\begin{aligned}\|\Phi_i \otimes 1\xi\|_n &= \sum_{i_1 \ldots i_k, k \leq n} \left\| \left(\prod_{j \in [k]} X_{i_j} \otimes 1 \right) \sum_{[q]} \Phi_i(x_l) \otimes m_l \right\| \\ &\leq \alpha \sum_{i_1 \ldots i_k, k \leq n+p} \left\| \left(\prod_{j \in [k]} X_{i_j} \otimes 1 \right) \sum_{[q]} x_l \otimes m_l \right\| = \alpha \|\Phi_i \otimes 1\xi\|_{n+p}\end{aligned}$$

This yields $\|\Phi \otimes 1\xi\|_n \leq N\alpha \|\xi\|_{n+p}$ where Φ has N monomials. □

Example 4.5.6. Let $\text{Lie}[G]$ be a semisimple Lie algebra, with universal enveloping algebra $U(\text{Lie}[G])$. The center $\mathcal{Z}(U(\text{Lie}[G]))$ has a distinguished element, the Casimir operator, $\Omega = \sum_i X_i^2$ where X_i is an orthonormal basis for $\text{Lie}[G]$ with respect to the Killing form B. For homogeneous spaces, the Casimir operator induces a laplacian acting on sections of homogeneous vector bundles which is covariant with respect to the group action and completely-smooth (see observation 4.5.9).

Sinha and Goswami [62] construct the Evans-Hudson dilation for semigroups with unbounded generators with structure maps derived from the data of the semigroup. This proceeds like described earlier and the estimates in section 4.4 are motivated by these calculations; complete smoothness is defined precisely to make estimates for *r.h.s.* of equation 4.16 work as needed. The covariance of the flow is required for putting a Frechet structure on the noise space.

Theorem 4.5.7. *(Existence of Evans-Hudson dilation[62, Thm 8.1.38]) If (T_t) is a conservative quantum dynamical semigroup on a unital C^*-algebra $\mathcal{A}$, covariant with respect to action of a second countable compact Lie group G, with possibly unbounded generator $\mathcal{L}$ that is p-completely smooth for some p and $\mathcal{L}(\mathcal{A}_\infty) \subset \mathcal{A}_\infty \subset \mathrm{Dom}(\mathcal{L})$, then the Evans-Hudson dilation exists.*

By theorem 4.5.7, the existence of Evans-Hudson dilation requires that the semigroup be conservative. As remarked before, this does not hold for the semigroups e^{-tL}, $\mathcal{L} = \triangle, D^2$ on a spinor bundle, and one needs to pass to the endomorphism connection, alternatively the commutator. To start, the example of the Clifford bundle is considered where the connection laplacian is conservative. The commutation of the generator with the Lie group action and complete smoothness are tied to the Lie algebra structure. For reductive homogeneous spaces the hypothesis needed can be checked to hold.

Torsion and the canonical connection laplacians

Suppose the homogeneous space $M = K/H$ for compact, connected, Lie group K, closed Lie subgroup $H \subset K$ is reductive with $\textsc{Lie}[K] = \textsc{Lie}[H] \oplus \mathfrak{M}$ as a vector space for an $\textsc{Ad}(H)$ invariant subspace $\mathfrak{M}$. $\mathfrak{M}$ is identified with T_oM where $o = eH$ in the coset manifold K/H. The homogeneous space K/H is principal H-bundle, $\pi : K \to K/H$ and carries a K action. Note that if the K acts effectively on reductive homogeneous space K/H then H is isomorphic to a subgroup of $\mathrm{GL}(\dim M, \mathbb{R})$, and the fiber bundle $\pi : K \to K/H$ is isomorphic to a sub-bundle of the principal frame bundle $F(M, \mathrm{GL}(\dim M, \mathbb{R}))$. The K action is assumed to be effective. The action of $k \in K$ on the T_xM is given by $X \to kX \in T_{kx}M$ by the differential of its left action $L_k : M \to M$, dL_k (which are denoted by k, k_*). The K-action is an isomorphism for all $k \in K, p \in K/H$, that is, the tangent bundle is homogeneous, while H induces automorphism at each fiber, meaning the fibers carry a representation of H.

Additionally, let K be semisimple, so the Killing form B_K defines a positive definite Riemannian metric h on K and an inner product on $\textsc{Lie}[K]$ by $-B_K$ such that the reductive decomposition for K/H satisfies $\mathfrak{M} = \textsc{Lie}[H]^\perp$ with respect to $-B_K$. By left invariance of the Killing form, the inner product on $\textsc{Lie}[G]$ extends to a Riemannian metric on $M = K/H$. Since the Lie group K is compact and connected, the Lie algebra exponential agrees with the Riemannian exponential and is surjective. This means that Casimir laplacian commutes with action of both Lie group and the Lie algebra.

There exists K-invariant connections on the homogeneous space K/H: $k : K/H \to K/H$ means $\nabla_{k_*X}(k_*Y) = k_*(\nabla_X Y)$ for all $X, Y \in TM$. There's a unique K-invariant connection in K such that if $f_t = \exp(tX)$ be the 1-parameter subgroup of K corresponding to $X \in \mathfrak{M}$ with a natural lift of o to u_o in the principal bundle, then the orbit of $\tilde{f}(u_o)$ is horizontal. The connection 1-form for the canonical connection is $\mathrm{Proj}_H \circ \Theta_{MC}$ where Proj_H is the projection onto the $\textsc{Lie}[H]$, and Θ_{MC} is the Maurer-Cartan form, $v \in T_gG \to L_{g^{-1}}v \in T_eG$, and the corresponding horizontal distribution is obtained at o by translating $\mathfrak{M}$ by the left K-action.

The canonical connection is a metric connection, but is not necessarily torsion-free, instead the torsion and curvature are parallel. The canonical connection, therefore, does not agree with the Levi-Civita connection in general; when M is a symmetric homogeneous space, the two connections do agree. To establish complete-smoothness and covariance, to start it needs to be checked that the homogeneous connection acts by the Lie algebra action on the tangent bundle and lifts to a K-invariant connection on the Clifford bundle $\mathrm{Cl}(TM)$.

Lemma 4.5.8. *For all $X \in \mathfrak{M} \subset$ Lie$[K]$, and ϕ given locally about o with orthonormal frame, (e_i), $\sum f_i e_i \; \nabla_X \phi = \sum_i X(f_i) e_i$.*

Proof. If X_i's for a basis for $\mathfrak{M}$, orthonormal to Lie$[H]$, then $\nabla_{X_l}\phi = \sum_i df_i(X_l)e_i + \sum_i f_i \nabla_{X_l} e_i = \sum_i df_i(X_l)e_i + \sum_i f_i \sum_j \omega_i^j(X_l)$ where ω the connection 1-form. Since ω is projection onto Lie$[H]$, $\nabla_X \phi = \sum_i X(f_i)e_i$ at $o \in M$ as needed. □

Now at any $p \in K/H$, by K-invariance, with $p = k \cdot o$, $\nabla_X|_p$ can be written as $k\nabla_{L^*_{k^{-1}}X}|_e k^{-1}$ since $\mathfrak{M}$ and the orthogonal complement are invariant under translation by K. Note that by K-invarince, if (e_i) is Riemann normal frame, $\nabla_{e_i} e_j = 0$ at x, then $k\nabla_{e_i} e_j = \nabla_{ke_i} k e_j = 0$ making (ke_i) Riemann normal frame at kx. The laplacian in at x, $\triangle_x = -\sum_i \nabla_{e_i}\nabla_{e_i} = -\sum_i e_i^2$, and at kx, $\sum_i \nabla_{ke_i}\nabla_{ke_i}$, meaning $\triangle_{kx} = k\triangle_x = -k\sum_i e_i^2$ by K-invariance.

Observation 4.5.9. Note that for homogeneous bundles, $E \longrightarrow K/H$ with the canonical connection, the fibers carry a representation of H, the isotropy representation ρ, while bundle E is the associated bundle to principal bundle K/H for ρ, $K \times_\rho E$. There's an induced representation of K on $\Gamma(E)$. By standard theory (see [51, 19]), $\triangle^E = -C_2(K, \Gamma(E)) + C_2(H, E)$ where $C_2(K, \Gamma(E))$ and $C_2(H, E)$ are Casimir operators for K and H, the representation for K being the induced representation on $\Gamma(E)$ while the H-representation being the isotropy representation acting pointwise. The proofs of these statements are similar to the above lemma. Therefore, the laplacian on K/H is expressed as a Lie algebra action.

Proposition 4.5.10. *The homogeneous connection lifts to a K-invariant connection $\hat{\nabla}$ on $\mathrm{Cl}(TM)$, with $\hat{\nabla}\mathbf{1} = 0$.*

Proof. Since connections are local, working in a local trivialization over $U \subset M$ with an orthonormal frame (e_i) is sufficient. The Clifford bundle over U is the quotient of the tensor bundle $\mathcal{TTM} := \sum \mathbb{C} \oplus_{n\in\mathbb{N}} TM^{\otimes n}$ by the ideal $\mathcal{I}$ generated by $\{v \otimes v + h(v) : v \in TM\}$. The tensor connection ∇ on $TM^{\otimes n}$, $\nabla_X(v \otimes u) = \nabla_X(v) \otimes u + v \otimes \nabla_X(u)$ composed with the quotient $\pi : \mathcal{TTM} \longrightarrow \mathcal{TTM}/\mathcal{I}$ will define a connection $\hat{\nabla} = \pi\nabla\pi^{-1}$ on $\mathrm{Cl}(TU)$ if it's well-defined with respect to the quotient. The K-invariance and being a Lie algebra action are inherited from the canonical connection. To verify it's well-defined one needs that if $\pi u = \pi u'$ then $\hat{\nabla} u = \hat{\nabla} u'$. By linearity, it may be assumed that $u = a_1 \otimes \ldots a_n, u' = a_1 \otimes \ldots a_m$. Without loss of generality, after reordering tensor components, it may also be assumed that $u' = u \otimes y$ for $y = p_{i_1} \otimes p_{i_1} \otimes p_{i_1} \otimes p_{i_1} \ldots \otimes p_{i_l} \otimes p_{i_l}$, that is, the last $2l$ are paired. Since the quadratic for form $\mathrm{Cl}(TM)$ is h, the relation $v \otimes v \sim h(v, v)$ reduces y to $\|y\|^2$. By $C(M)$ linearity of the tensor product, $u' = \|y\|\, u$. Therefore, $\pi u = \pi u'$ means $\|y\| = 1$ identically on U.

Now $\hat{\nabla} u \otimes y = (\hat{\nabla} u) \otimes y + u \hat{\nabla} y$, so to show that $\pi \nabla$ is well-defined, it's enough to show that $\hat{\nabla} v \otimes v = 0$ for $v \in TM$, $\|v\| = 1$. Let $v = \sum f_i e_i$ in the local orthonormal frame, giving that $\sum f_i^2 = 1$ and so $\sum f_i X(f_i) = 0$ for any coordinate vector field X. Finally,

$$\nabla_X(\sum f_i e_i \otimes \sum f_j e_j) = \sum_{i,j} f_j X(f_i) e_i \otimes e_j + f_i X(f_j) e_i \otimes e_j + \sum_{ij} f_i f_j \nabla_X(e_i \otimes e_j)$$

$$\begin{aligned} \text{with } \sum_{i,j} f_j X(f_i) e_i \otimes e_j + f_i X(f_j) e_i \otimes e_j &= \sum_{i,j} f_j X(f_i) e_i \otimes e_j + \sum_{i,j} f_i X(f_j) e_i \otimes e_j \\ &= \sum_{i,j} \left[f_j X(f_i) e_i \otimes e_j + f_j X(f_i) e_j \otimes e_i \right) = \sum_{i,j} f_j X(f_i) \left[e_i \otimes e_j + e_j \otimes e_i \right] \\ &= \sum_{i,j,i \neq j} f_j X(f_i) \left[e_i \otimes e_j + e_j \otimes e_i \right] + 2 \sum_i f_i X(f_i) e_i \otimes e_i \end{aligned}$$

Notice that $\pi(\sum_i 2 f_i X(f_i) e_i \otimes e_i) = 0$ because $\sum_i f_i X(f_i) = 0$ while $\pi(\sum_{i,j,i\neq j} f_j X(f_i)[e_i \otimes e_j + e_j \otimes e_i])$ vanishes because e_i, e_j anti-commute for $i \neq j$. Then $\pi \nabla_X(e_i \otimes e_j) = 0$ as well using the same anti-commutation and that the derivative of the Clifford relation is zero:

$$\nabla(e_i \otimes e_i) = -\nabla h(e_i, e_i) = 0 \tag{4.27}$$

This yields $\hat{\nabla}_X v \otimes v = 0$, and also $\hat{\nabla}_X \mathbf{1} = 0$. □

Proposition 4.5.11. *The connection $\hat{\nabla}$ is Riemannian.*

Proof. The Clifford inner product is given by $\langle a, b \rangle_{\mathrm{Cl}} = (a^* b)_0$ where $(\cdot)_0$ denotes the degree 0 part and $*$ is defined through $(a_{i_1} \otimes \ldots \otimes a_{i_k})^* = (-1)^k (a_{i_k} \otimes \ldots \otimes a_{i_1})$. The Clifford inner product is defined so that the anti-symmetrization map for any vector space E,

$$\mathrm{ASymm} : \Lambda(E) \ni a_1 \wedge a_2 \cdots \wedge a_p \to \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma a_{\sigma(1)} \cdot a_{\sigma(2)} \cdot \cdots \cdot a_{\sigma(p)} \in \mathrm{Cl}(E) \tag{4.28}$$

which is an isomorphism of vector spaces is also an isometry. To show that the connection is Riemannian note that it's sufficient to show it for basis elements,

$a := e_{i_1} \otimes \ldots \otimes e_{i_k}, b := e_{j_1} \otimes \ldots \otimes e_{j_m}$, e_i's being a basis for TM. First note $\hat{\nabla}_X(a^*a)_0 = 0$, while the terms in $((\hat{\nabla}_X a^*)a + a^*\hat{\nabla}_X a)_0$, can be collected to have form $y \otimes \hat{\nabla}(e_i \otimes e_i) \otimes y'$ for some y, y', giving $((\hat{\nabla}_X a^*)a + a^*\hat{\nabla}_X a)_0 = 0$ after applying the Clifford relation.

So consider the case $a \neq b$ with $k + m$ is odd. One can also assume that a, b share no e_i as on reordering it will drop out. This means $(a^*b)_0 = 0$ because a^*b cannot land in degree zero. Thus, $\hat{\nabla}_X \langle a, b \rangle_{\text{Cl}} = 0$. Since $k + m$ is odd, and the Clifford relation reduces degree by 2 each time it's utilized, $(a^*\hat{\nabla}_X b)_0 = 0 = (\hat{\nabla}_X(a^*)b)_0$ as well.

The only case that remains is when $k + m$ is even. Without loss of generality assume that $k = m$ since one can always regroup a^*b. Consider the case $a = e_i, b = e_j, i \neq j$, and that $X = e_r$ for any r. Then

$$\begin{aligned} 2\nabla_r(e_i \otimes e_j) &= \nabla_r(e_i \otimes e_j - e_j \otimes e_i) \\ &= \Gamma^s_{ri} e_s \otimes e_j + e_i \otimes \Gamma^s_{rj} e_s - \Gamma^s_{rj} e_s \otimes e_i - e_j \otimes \Gamma^s_{ri} e_s \end{aligned}$$

so, $\pi(2\nabla_r(e_i \otimes e_j))_0 = \Gamma^j_{ri} e_j \otimes e_j + e_i \otimes \Gamma^i_{rj} e_i - \Gamma^i_{rj} e_i \otimes e_i - e_j \otimes \Gamma^j_{ri} e_j = 0$ where $(e_s \otimes e_i)_0 = 0$ unless $s = i$ (and same for e_j's) was used. Now consider the case where $k \geq 2$; since all e_i's in a^*b are distinct, a, b can be anti-symmetrized which is exactly the isometric identification in equation 4.28, and so the claim follows from metric compatibility of tensor connection on exterior bundle. Explicitly

$$\begin{aligned} a &= \tfrac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma e_{\sigma(i_1)} \cdot e_{\sigma(i_2)} \cdot \cdots \cdot e_{\sigma(i_k)} = e_{i_1} \wedge \cdots \wedge e_{i_k} \\ b &= \tfrac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma e_{\sigma(j_1)} \cdot e_{\sigma(j_2)} \cdot \cdots \cdot e_{\sigma(j_k)} = e_{j_1} \wedge \cdots \wedge e_{j_k} \\ (\nabla ab)_0 &= \langle a, \nabla_r b \rangle_\Lambda + \langle \nabla_r a, b \rangle_\Lambda \end{aligned}$$

Recall that $\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle_\Lambda = \text{Det}(\langle e_s, e_t \rangle_{TM})$. Now expanding with the product rule $\langle \nabla_r a, b \rangle_\Lambda = \sum_{q \in [k]} \langle \nabla^q_r a, b \rangle_\Lambda$ where ∇^q_k denotes ∇_k applied to q tensor component. Each

$$\langle \nabla^q_r a, b \rangle_\Lambda = \langle e_{i_1} \wedge \ldots \nabla_r e_{i_q} \cdots \wedge e_{i_k}, e_{j_1} \wedge \ldots e_{j_q} \cdots \wedge e_{j_k} \rangle$$

is still zero, since it's a determinant of a matrix with every row zero except possibly the q^{th}-row since e_i's and e_j's are all distinct; this is where $k \geq 2$ comes into play. The term $\langle a, \nabla_k b \rangle_\Lambda$ is handled similarly. $\square$

The canonical connection is not torsion-free, i.e., symmetric. This means that for Riemann normal coordinates x_i centered at p and coordinate fields e_i's, $\nabla_i e_i(p) = 0$ but $\nabla_i e_j(p) \neq 0$ for $i \neq j$. For Christoffel symbols Γ^k_{ij} defined by $\nabla_i e_j = \Gamma^k_{ij} e_k$, $\Gamma^k_{ii} = 0$ and $\Gamma^k_{ij} + \Gamma^k_{ji} = 0$ at p (see [46, Prop III.8.4]). Now $\mathrm{div}(v)_p := \sum_i \langle \nabla_i v, e_i \rangle$, so $\mathrm{div}(e_k)_p := \sum_i \langle \nabla_i e_k, e_i \rangle_p = 0$ if ∇ was torison-free. In presence of torsion using $\Gamma^k_{ij} + \Gamma^k_{ji} = 0$ implies

$$\mathrm{div}(e_k)_p := \sum_i \langle \nabla_i e_k, e_i \rangle_p = -\sum_i \langle \nabla_k e_i, e_i \rangle_p \tag{4.29}$$

The non-zero torsion is consequential, the Bochner identity needs to be corrected and the Dirac operator picks up torsion and is no longer self-adjoint; however, it can be corrected to an operator which reduces to the usual Dirac operator as torsion vanishes. Precisely when the Dirac operator for canonical connection on homogeneous bundles is formally self-adjoint is characterized by [1, Proposition 3.1]; the following addresses the modification to the Bochner identity.

Lemma 4.5.12 (Torsion deformed Bochner identity)**.** *Let ∇ be a connection on TM with torsion for manifold M, then the Dirac laplacian $D^2 = \sum_{jk} e_j \nabla_j e_k \nabla_k$ associated to the Clifford bundle with connection obtained from ∇ in normal coordinates (e_i) centered at p, $\mathfrak{R}$ the curvature operator from the usual Bochner identity*

1. *With $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], X, Y \in \Gamma(TM)$, the torsion tensor, $\mathfrak{T} = \frac{1}{2}\sum_{jk} e_j T(e_j, e_k)\nabla_k$*

$$D^2 = \nabla^*\nabla + \mathfrak{R} + \mathfrak{T}$$

2. *If $T \neq 0$, D^2 is not necessarily self-adjoint, the operator*

$$\mathcal{D}^2 := \nabla^*\nabla + \tfrac{1}{2}[\mathfrak{R} + \mathfrak{R}^*] + \tfrac{1}{2}[\mathfrak{T} + \mathfrak{T}^*]$$

is self-adjoint

Proof. At p $D^2\phi = \sum_{jk} e_j \nabla_j e_k \nabla_k \phi$ becomes

$$D^2\phi = \sum_{jk} e_j e_k \nabla_j \nabla_k \phi + e_j \nabla_j(e_k)\nabla_k \phi = \nabla^*\nabla\phi + \mathfrak{R}\phi + \sum_{jk} e_j[\nabla_j(e_k) - \nabla_k(e_j)]\nabla_k \phi$$

where $[e_j, e_k] = 0, \Gamma^s_{kj} = -\Gamma^s_{jk}$ was used to rewrite $2\nabla_j(e_k) = 2\sum_s \Gamma^s_{jk} e_s = \sum_s \Gamma^s_{jk} e_s - \sum_s \Gamma^s_{kj} e_s = \nabla_j(e_k) - \nabla_k(e_j) = T(e_j, e_k)$. The Dirac laplacian fails to be self adjoint because $\mathfrak{R}$ can fail to symmetric if $e_j \nabla_k \phi \neq \nabla_k e_j \phi$ as a consequence of $\nabla_k e_j \neq 0$ at p which is happens exactly when $T \neq 0$. Therefore, when T vanishes $\mathfrak{R} = \mathfrak{R}^*$ giving back the usual Dirac laplacian. The self-adjointness of $\mathcal{D}^2$ is obvious.

When the TM is parallelizable, that is, there's a global orthonormal frame (e_i), then explicitly, using that $\langle u, \nabla_X v \rangle = \langle (-\nabla_X - \mathrm{div}(X))u, v \rangle$ with respect to L^2 inner product, since $\langle u, \sum_i e_i \nabla_i v \rangle_{L^2} = \langle \sum_i e_i(\nabla_i + \mathrm{div}(e_i))v, u \rangle$, the Dirac operator is not self adjoint, but satisfies,

$$D^* = D + \sum_k e_i \mathrm{div}(e_i), \ (D^2)^* = (D + \sum e_i \mathrm{div}(e_i))^2 \tag{4.30}$$

□

For the parallelizable manifold example, the L^2 adjoint for ∇_k was used; the adjoint with respect to the inner product at fiber can be computed using metric compatibility by a straightforward calculation. Using $\langle \sigma', \nabla_k \sigma \rangle_p = \nabla_k \langle \sigma', \sigma \rangle_p - \langle \nabla_k \sigma', \sigma \rangle_p = e_k \langle \sigma', \sigma \rangle_p - \langle \nabla_k \sigma', \sigma \rangle_p$ along with

$$\begin{aligned}
\mathrm{div}(\langle \sigma', \sigma \rangle e_k) &= \sum_j \langle \nabla_j \langle \sigma', \sigma \rangle e_k, e_j \rangle = \sum_j \langle e_j(\langle \sigma', \sigma \rangle) e_k + \langle \sigma', \sigma \rangle \nabla_j e_k, e_j \rangle \\
&= \sum_j e_j(\langle \sigma', \sigma \rangle)(p) \langle e_k, e_j \rangle_p + \langle \sigma', \sigma \rangle \sum_j \langle \nabla_j e_k, e_j \rangle \\
&= e_k(\langle \sigma', \sigma \rangle)_p + \langle \sigma', \sigma \rangle_p \mathrm{div}(e_k)_p
\end{aligned}$$

Therefore, $e_k(\langle \sigma', \sigma \rangle)_p = \mathrm{div}(\langle \sigma', \sigma \rangle e_k)_p - \langle \sigma', \sigma \rangle_p \mathrm{div}(e_k)_p$, implying

$$\langle \sigma', \nabla_k \sigma \rangle_p = \mathrm{div}(\langle \sigma', \sigma \rangle e_k)_p - \langle \sigma', \sigma \rangle_p \mathrm{div}(e_k)_p - \langle \nabla_k \sigma', \sigma \rangle_p \tag{4.31}$$

From this, choosing σ, σ' from an orthonormal frame gives the fiber-wise adjoint in local basis. This calculation gives that $\mathcal{T}, \mathcal{T}^*$ are first order differential operators, meaning the leading symbol of $\mathcal{D}^2$ agrees with the laplacian, and therefore we have the following.

Proposition 4.5.13. *The operator $\mathcal{D}^2$ is a generalized lapacian, formally self-adjoint and elliptic.*

Now note the K-invariance implies that the canonical connection laplacian and curvature operator commute with group action.

Proposition 4.5.14. *For any $\phi \in \Gamma(\mathrm{Cl}(TM))$,*

- $k\mathfrak{R}\phi = \mathfrak{R}k\phi, k\mathfrak{T}\phi = \mathfrak{T}k\phi$.
- *For the canonical connection laplacian,* $k \triangle \phi = \triangle k\phi$

Proof. Let (e_i) be an orthonormal frame for at $x \in M$. Then since the metric is bi-invariant, ke_i is an orthonormal frame at kx, meaning the curvature operators at x, kx are

$$\mathfrak{R}_x := \sum_{ij} e_i \cdot e_j \cdot [\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}],\ \mathfrak{R}_{kx} := \sum_{ij} ke_i \cdot ke_j \cdot [\nabla_{ke_i}\nabla_{ke_j} - \nabla_{ke_j}\nabla_{ke_i}]$$

The K-action Cl_x between Cl_{kx}. Denoting Clifford multiplication at x, kx, by $\cdot_{\mathrm{Cl}_x}, \cdot_{\mathrm{Cl}_{kx}}$, since $\langle ku, kv\rangle_{kx} = \langle u, v\rangle_x$, $k : T^*_x M \to \mathrm{Cl}(T^*_{kx}M)$ satisfies the universal property for Clifford algebras $ku \cdot_{\mathrm{Cl}_{kx}} ku = \langle u, u\rangle 1_{\mathrm{Cl}(T^*_{kx}M)}$, and therefore, k induces a Clifford algebra isomorphism. This means

$$\begin{aligned} k\mathfrak{R}_x\phi &:= k\sum_{ij} e_i \cdot e_j \cdot [\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}]\phi = \sum_{ij} ke_i \cdot ke_j \cdot k[\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}]\phi \\ &= \sum_{ij} ke_i \cdot ke_j \cdot [\nabla_{ke_i}\nabla_{ke_j} - \nabla_{ke_j}\nabla_{ke_i}]k\phi \text{ (by } K\text{-invariance)} \\ &= \mathfrak{R}_{kx}k\phi \end{aligned}$$

which is as needed. The same holds for $\mathfrak{T}$. This works for the laplacian as well. Now if (e_i) is Riemann normal frame at x, then (ke_i) Riemann normal frame at kx, so the laplacian, $\triangle$, at x in Riemann normal frame at x is $-\sum_i \nabla_{e_i}\nabla_{e_i}$, and at kx, $\sum_i \nabla_{ke_i}\nabla_{ke_i}$, meaning $k \triangle_x \phi = \triangle_{kx}k\phi$ as well. □

Quantum stochastic dilation on homogeneous spinor bundles

First note some immediate results that follow from the last section.

Corollary 4.5.15. *The heat semigroup generated by canonical connection laplacian* $\triangle$ *on* $\mathrm{Cl}(TM)$ *are a conservative quantum dynamical semigroup, and the generator* $\triangle$ *is completely smooth.*

Proof. The first claim follows directly from results of chapter 2, the second is because the $\hat{\nabla}$ acts through the Lie algebra; it just needs to be noted that one does not need to change the connection based on the degree of v in $\mathrm{Cl}(TU)$ since one can always tensor with the identity, and apply the tensor product connection for n-fold tensor. □

For connections with torsion, $\mathcal{D}^4$ generates a quantum dynamical semigroup by the same idea. When the space is a symmetric space, the connection is torsion-free and Dirac laplacian D^2 generates a quantum dynamical semigroup; additionally, by Bochner identity, $\mathcal{D}^4, D^2$ are completely smooth.

Example 4.5.16. ([37]) Suppose K/H is a Riemannian symmetric space carrying a homogeneous spin structure with Dirac operator $\not{D}_{K/H}$ associated with the Levi-Civita connection (which for a symmetric space agrees with the canonical connection), then $\not{D}^2_{K/H} = \Omega_K + \kappa/8$ where κ is the scalar curvature and Ω_K the Casimir operator for K.

Example 4.5.17. Kostant's cubic Dirac operator, $D^{1/3}$, is the Dirac operator associated to a linear combination of the canonical and Levi-Civita connection of the reductive space K/H. The laplacian, $(D^{1/3})^2$, can be expressed as the quadratic Casimir operator with an additive scalar (see, for instance, [1, Thm 3.3]). By the same argument it follows that the generated semigroup is a quantum dynamical semigroup.

Corollary 4.5.18. *For the Riemannian symmetric space K/H, the spectral action for the untruncated Dirac operator on the Clifford bundle,* $\mathrm{Cl}(K/H)$*, can be realized from the Evans-Hudson flow.*

Proof. This follows since $D := \not{D}_{K/H,\mathcal{D}}$, is a Lie algebra action, and hence completely-smooth and covariant since it acts though the Casimir operator which is a quadratic element in center of the enveloping algebra for $\text{Lie}[K]$. □

Example 4.5.19. For $K = SU(2)$, i.e. S^3, $H = U(1)$, i.e. S^2, $K/H = S^2$ is the Hopf fibration. S^2 is a symmetric space, so Evans-Hudson flows exists on $\mathrm{Cl}(TS^2)$ and over $\mathrm{End}(S)$ for any homogeneous spinor bundle $S \to S^2$ for a spin-structure.

Observation 4.5.20. More generally, the discussion applies to any finite dimensional homogeneous vector bundles over $M = K/H$, that is, a vector bundle $E \to K/H$ is such that K acts on E, with $kE_x = E_{kx}$, and the action $k : E_x \to E_{kx}$ is an isomorphism for all $k \in K, k \in K/H$. The most relevant setting is that of homogeneous spinor bundle associated to a spin structure. Additionally, one needs the tangent bundle to be homogeneous, with bi-invariant metrics and K-invariant connection on TM, S (the homogeneity and K-invariance for TM are required for homogeneity/invariance of the spin-structure). Because the heat semigroups on the spinor bundle S may not be conservative, one needs to pass to $\mathrm{End}(S)$ and work with the endomorphism Dirac operator or the noncommutative laplacian; this means the complete smoothness and covariance on S^* needs to be wrangled. Note that homogeneous spaces do not carry parallel spinors unless they are Ricci-flat (and so flat) because is required for the existence of parallel spinors[35]; for spin$^{\mathbb{C}}$ bundles there are more parallel spinors[53].

4.6 Uniform Sobolev norms

Now the growth of Sobolev norms are considered. Let (M, g) be a compact Riemannian manifold with Levi-Civita connection ∇. On (p, q)-tensors s, s' there's a natural inner product by contraction with $g^{ij}, g_{ij}, \langle s, s' \rangle = g^{i_1 j_1} \dots g_{m_1 n_1} s'^{m_1 \dots}_{i_1 \dots} s^{n_1 \dots}_{j_1 \dots}$. The Levi-Civita connection has a lift to the tensor bundle and an associated connection laplacian, both also denoted $\nabla, \triangle$. Denote by $\nabla^k u$ the k^{th}-covariant derivative and define the point-wise length with the innerproduct[42, § 2.2.1]:

$$\ell(\nabla^k u)^2 = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^k u)_{i_1 \dots i_k} (\nabla^k u)_{j_1 \dots j_k} = \langle \nabla^k u, \nabla^k u \rangle \tag{4.32}$$

When $f = \Pi_{i \in [N]} \phi_i$ is a finite product of eigenfunctions ϕ_i's, for $m \in M$, since $| \triangle^k f|_m \leq (\dim M)^k \ell(\nabla^{2k} f)$, by product rule, this can be controlled by $\langle \nabla^k \phi_i, \nabla^k \phi_i \rangle$. So one would like to know when a bound like $\langle \nabla^k \phi, \nabla^k \phi \rangle_m \leq C_\phi M_\phi^k$ is possible. To start assume the following, this will be relaxed in proposition 4.6.3. Spaces without curvature provide examples satisfying this; as do some homogeneous spaces.

Assume 4.6.1. *To control the growth of laplacian iterates, first assume* $[\nabla^k, \triangle] u = 0$ *for any laplacian eigenfunction.*

One expects that $\left\| \nabla^k \phi_j \right\|_{L^2(M)}$ should be bounded by λ_j^{2k} when $\triangle = \nabla^* \nabla$ and ∇ almost commute. Since ϕ_j's are smooth this is enough to establish a uniform bound, but this will require leveraging the $\|\cdot\|_{L^2(M)}$ bound locally and the boundary for the local chart will need to be taken into account. Recall the integration by parts formula for tensor fields when M does have a boundary,

$$\int_M \langle \nabla F, G \rangle dV_g = \int_{\partial M} \langle F \otimes N^\flat, G \rangle dV_{\hat{g}} - \int_M \langle F, \operatorname{div}(G) \rangle dV_g \tag{4.33}$$

where $\hat{g}$ is the induced metric on $\partial M, dV_g, dV_{\hat{g}}$ the associated volume forms, $\cdot^\flat$ the musical isomorphism, N the outward unit normal at ∂M, and F, G tensor fields, $\operatorname{div}(G) = \operatorname{Tr}_g(\nabla G)$, the trace being over the last two indices. Note if $G = \nabla H$ then, $-\operatorname{div}(G) = \triangle(H)$.

Proposition 4.6.2. *Assuming* $[\triangle, \nabla] = 0$*, for eigenfunction u with eigenvalue* λ^2*,* $\left\|\nabla^k u\right\|_\infty \leq 2\lambda \left\|\nabla^{k-1} u\right\|_\infty$

Proof. Set $4\lambda^2 \left\|\nabla^{k-1} u\right\|_\infty^2 = K > 0$. Suppose for some $x \in M$, $\langle \nabla^k u, \nabla^k u\rangle_x - K > 0$. Then since u is smooth, there exists an open neighborhood U of x such that on U, $\langle \nabla^k u, \nabla^k u\rangle - K > 0$. Let ψ be such that $\mathrm{supp}(\psi) \subset U$ is compact, $\psi \geq 0$ on U and $\psi > 0$ on open $V \subset U$, then

$$\int_M \psi\langle \nabla^k u, \nabla^k u\rangle - \psi K dV_g = \int_U \langle \psi\nabla^k u, \nabla^k u\rangle - \psi K dV_g > 0 \tag{4.34}$$

Now $\langle \psi\nabla^k u, \nabla^k u\rangle = \langle \nabla(\psi\nabla^{k-1}u), \nabla^k u\rangle - \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle$, and for the first term

$$\int_U \langle \nabla(\psi\nabla^{k-1}u), \nabla^k u\rangle dV_g = \int_{\partial U} \langle \cdot, \cdot\rangle dV_{\hat{g}} + \int_U \langle \psi\nabla^{k-1}u, -\mathrm{div}(\nabla^k u)\rangle dV_g$$

where $\int_{\partial U} \langle \cdot, \cdot\rangle dV_{\hat{g}} = 0$ since $\psi = 0$ on ∂U and outside U, while $-\mathrm{div}(\nabla^k u) = \triangle\nabla^{k-1}u = \nabla^{k-1} \triangle u$ using by assumption 4.6.1. Therefore, we have

$$\begin{aligned}\int_M \psi\langle \nabla^k u, \nabla^k u\rangle &= \int_U \psi\langle \nabla^{k-1}u, \nabla^{k-1} \triangle u\rangle - \int_U \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g \\ &= \lambda^2 \int_U \psi\langle \nabla^{k-1}u, \nabla^{k-1}u\rangle - \int_{\mathrm{supp}(\nabla\psi)} \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g\end{aligned} \tag{4.35}$$

This yields

$$\begin{aligned}0 &< \int_U \psi\langle \nabla^k u, \nabla^k u\rangle - \psi K dV_g \\ &= \lambda^2 \int_U \psi\langle \nabla^{k-1}u, \nabla^{k-1}u\rangle - \int_{\mathrm{supp}(\nabla\psi)} \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g - \int_U \psi K dV_g \\ &= \int_U \psi(\lambda^2 \left\|\nabla^{k-1}u\right\|^2 - K) dV_g - \int_{\mathrm{supp}(\nabla\psi)} \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g\end{aligned} \tag{4.36}$$

Define the linear functional $\omega(\psi) := \int_{\mathrm{supp}(\nabla\psi)} \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g$. Note

$$\begin{aligned}\langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle &= g^{i_1 j_1}\nabla_{i_1}\psi \left(g^{i_2 j_2} \ldots g^{i_k j_k} (\nabla^{k-1}u)_{i_2 \ldots i_k} (\nabla^k u)_{j_1 j_2 \ldots j_k}\right) \\ &= g^{i_1 j_1}\nabla_{i_1}\psi \left(g^{i_2 j_2} \ldots g^{i_k j_k} (\nabla_{i_2} \ldots \nabla_{i_k} u)(\nabla_{j_1}\nabla_{j_2} \ldots \nabla_{j_k} u)\right) \\ &= g^{i_1 j_1}\nabla_{i_1}\psi G_{j_1} = \langle \nabla\psi, G_{j_1}\rangle\end{aligned} \tag{4.37}$$

where $G_{j_1} = g^{i_2 j_2} \ldots g^{i_k j_k} (\nabla_{i_2} \ldots \nabla_{i_k} u)(\nabla_{j_1}\nabla_{j_2} \ldots \nabla_{j_k} u)$.

By showing that there exists a ψ that makes $\omega(\psi) \geq 0$, since $\psi(\lambda^2 \left\|\nabla^{k-1}u\right\|^2 - K) < 0$, it will follow that equation 4.36 cannot hold. Assume that U is small enough to be covered by a Riemann normal coordinates, and consider polar coordinates on U centered at x. Define τ_s^c on U for $c, s \in \mathbb{R}_{>0}$ such that $\tau_s(x) = c$ and then decays linearly in radially outwards direction with slope $-s$ to 0 at $\partial B_R(x)$ with c, s such that $\overline{\text{supp}(\tau_s^c)} \subset U$, R depending on c, s. Then τ_s^c is continuous, piecewise continuously differentiable, with compact support in U, so weakly-differentiable, and $\nabla\tau_s^c = -s\mathbf{1}_{\mathcal{B}_R(x)}$ (there's enough slack to work with mollified versions of τ's, but weak-differentiability suffices for simplicity). If for some τ_s^c, $\omega(\tau_s^c) \geq 0$ then that $\psi = \tau_s^c$ is the required ψ.

If not, then $\omega(\tau_s^c) < 0$ for all c small enough to have support in U. By rescaling wlog assume $c = s = 1$, and set $\tau_1 := \tau_1^1$ (otherwise the constants are messy). For such τ_1, define τ_1' such that $\tau_1'(x) = 0$, and τ_1' increases linearly to 1 at $\partial B_1(x)$, and outside of $\overline{B_1(x)}$, $\tau_1' = 0$. Then $\omega(\tau_1') = -\omega(\tau_1) = \delta > 0$ since $\nabla\tau_1' = -\nabla\tau_1$ on $\text{supp}(\nabla\tau_1') = \text{supp}(\nabla\tau_1)$. It remains to make τ_1' continuous without changing $\omega(\tau_1')$ too much. For this set $\tau_{1,r}'' = \tau_1'$ on $B_1(x)$, $\tau_{1,r}'' = 0$ on $B_{1+r}(x)^c$, and on $B_{1+r}(x)^c \setminus B_1(x)$, $\tau_{1,r}''$ decays linearly to 0 on $\partial B_{1+r}(x)$. Finally, since for all $r > 0$ small enough, $\tau_{1,r}''$ is piecewise continuous, continuously differentiable and compactly supported in U, it remains to check $\left\|\omega(\tau_1') - \omega(\tau_{1,r}'')\right\| \leq \epsilon(r)$ with $\epsilon(r)$ vanishing with r, and there exists $r_\epsilon > 0$ such that for all $r < r_\epsilon$, $\left\|\omega(\tau_s') - \omega(\tau_{1,r}'')\right\| \leq \epsilon$. Note that

$$2\left\|\omega(\tau_1') - \omega(\tau_{1,r}'')\right\| = \| \int_{B_{1+r}(x)(x)\setminus B_1(x)} \langle \tau_{1,r}'', -\text{div}(G_{j_1}\rangle dV_g\| \tag{4.38}$$

using equation 4.37 and that τ'' is compactly supported in U so the boundary term vanishes. Now because as M is compact, $-\text{div}(G_{j_1})$, $\tau_{1,r}''$ are continuous (since $u \in C^\infty(M)$) and so bounded), the $\epsilon(r)$ as needed exists. Notice that $\omega(\tau_1') \geq 0$, meaning $\omega(\tau_{1,r}'') \geq -\epsilon(r)$. Choosing $\psi := \tau_{1,r}''$, since on U $\lambda^2\left\|\nabla^{k-1}u\right\|^2 - K < 0$,

$$\int_U \psi(\lambda^2\left\|\nabla^{k-1}u\right\|^2 - K)dV_g - \omega(\psi) \leq \int_{B_1(x)} \tau_1'(\lambda^2\left\|\nabla^{k-1}u\right\|^2 - K)dV_g + \epsilon(r) := R(\epsilon)$$

Finally, as $K = 4\lambda^2\left\|\nabla^{k-1}\right\|_\infty^2$, choose r such that $\frac{1}{4}|\int_{B_1(x)} \tau_1'(\lambda^2\left\|\nabla^{k-1}u\right\|^2 - K)dV_g| > \epsilon(r) > 0$. This makes $R(\epsilon) < 0$, yielding that equation 4.36 –

$$0 < \int_U \psi(\lambda^2\left\|\nabla^{k-1}u\right\|^2 - K)dV_g - \int_{\text{supp}(\nabla\psi)} \langle\nabla\psi \cdot \nabla^{k-1}u, \nabla^k u\rangle dV_g$$

cannot hold. □

The only place where assumption 4.6.1 was used was equation 4.35 to commute laplacian and covariant derivative, if instead we have that the commutator is somewhat well-behaved then a variant of proposition 4.6.2 holds. The hypothesis of proposition 4.6.3 is geometric, since the commutator will expand in terms of curvature and its covariant derivatives.

Proposition 4.6.3. *If every $k \in \mathbb{N}$ and eigenfunction u for $\triangle$ with eigenvalue λ^2 and $x \in M$, there exists $T = T(x,u), T \geq 0$ such that*

$$\langle \nabla^{k-1}u, \triangle \nabla^{k-1}u \rangle \leq \langle \nabla^{k-1}u, \nabla^{k-1} \triangle u \rangle + T\langle \nabla^{k-1}u, \nabla^{k-1}u \rangle = (\lambda^2 + T)\langle \nabla^{k-1}u, \nabla^{k-1}u \rangle \tag{4.39}$$

that is, for T independent of k, $\langle \nabla^{k-1}u, [\triangle, \nabla^{k-1}]u \rangle \leq T\langle \nabla^{k-1}u, \nabla^{k-1}u \rangle$, then

$$\left\|\nabla^k u\right\|_\infty^2 \leq 2\sqrt{\lambda^2 + T}\left\|\nabla^{k-1}u\right\|_\infty^2$$

Proof. Assume not, then on some open $U \subset M$, for all $x \in U$, for some fixed $c > 1$, $K = 4(\lambda^2 + T)\left\|\nabla^{k-1}u\right\|_\infty^2$, $\langle \nabla^k u, \nabla^k u \rangle_x - cK > 0$ and as in proposition 4.6.2 for some $\psi \geq 0$ compactly supported in $U, \psi > 0$ on an open set, giving

$$\int_U \langle \psi \nabla^k u(x), \nabla^k u(x) \rangle dV_g - \int_U \psi cK dV_g > 0 \tag{4.40}$$

$$\text{with } \int_U \langle \psi \nabla^k u(x), \nabla^k u(x) \rangle dV_g = \int_U \langle \nabla(\psi \nabla^{k-1}u(x)), \nabla^k u(x) \rangle dV_g - \int_U \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u \rangle dV_g$$

$$\int_U \langle \nabla(\psi \nabla^{k-1}u(x)), \nabla^k u(x) \rangle dV_g = \int_U \langle \psi \nabla^{k-1}u(x), \triangle \nabla^{k-1}u(x) \rangle dV_g \leq (\lambda^2 + T)\int_U \langle \psi \nabla^{k-1}u(x), \nabla^{k-1}u(x) \rangle dV_g$$

Therefore, equation 4.40 yields

$$\begin{aligned}&\int_U (\lambda^2 + T)\langle \psi \nabla^{k-1}u(x), \nabla^{k-1}u(x) \rangle dV_g - \int_U \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u \rangle dV_g - \int_U \psi cK dV_g \\ &= \int_U (\lambda^2 + T)\langle \psi \nabla^{k-1}u(x), \nabla^{k-1}u(x) \rangle - \psi cK dV_g - \int_U \langle \nabla\psi \cdot \nabla^{k-1}u, \nabla^k u \rangle dV_g > 0\end{aligned}$$

But choosing ψ as in 4.6.2, since $(\lambda^2 + T)\langle \nabla^{k-1}u(x), \nabla^{k-1}u(x) \rangle - cK < 0$, the last inequality cannot hold. □

If $T := T_k$ depends on k, then the exact behavior of T_k is necessary to know for controlling the bound.

Chapter 5

DISCRETIZATIONS AND TRUNCATIONS

5.1 Introduction

The realization of spectral action from a quantum stochastic flow illustrated the usefulness of truncation for establishing the existence of the flow on a general compact Riemannian manifold. The question of how well such truncations approximate the data of a spectral triple is considered now. A discrete version of the problem is also the same question but from a different perspective: given a compact manifold M and a discrete set $X \subset M$ sampled with respect to a probability measure, how well do the Hilbert space $L^2(X)$ and the operator algebra $C(X)$ approximate $L^2(M), C(M)$, the data of the canonical spectral triple. Similarly to the spectral truncations, $L^2(X), C(X)$ are also finite dimensional. A special setting where X is not random but the 0-skeleton Σ^0 for an embedded simplicial complex Σ for M which is regular in the sense that all k-cells have the same k-volume is independently interesting and more can be said there.

The usual exterior derivative d on exterior bundle $\Lambda(M)$ along with its adjoint d^* defines a Dirac operator $d + d^*$ on $\Lambda(M)$ under the (vectorspace) isomorphism with Clifford bundle $\mathrm{Cl}(M)$. The coboundary operator δ for simplicial complex Σ, at least on $L^2(\Sigma^0)$, acts approximately like the exterior derivative. We show that this has an easy generalization to the higher-dimensional skeletons and differential forms. The metric space Hodge theory introduced by [9] can be modified to apply to Σ and from which a Hodge decomposition theorem for δ is inherited. When the complex is regular, the maps between co-chains on the complex and differential forms are isometric with respect to the natural L^2-structure on co-chains coming from k-skeletons. This L^2-structure is not the usual one considered and differs from the L^2-structure used in discrete exterior calculus (see [43]), which is based on a discretized Hodge dual.

In the setting of spectral truncations, for the algebras, compressed by spectral trunctaion, one does not expect to do better than approximate compact operators, although truncated algebras will usually contain the identity making them operator systems. However, equally relevant is the state-space on the algebra. Adapting ideas from [40], a new class of geometries is given on which the state-space for the compressed algebra converges to one for the uncompressed. Further, the convergence is shown to be with respect to the Lipschitz norm associated to the Dirac operator for the canonical spectral triple.

To introduce some notation, let $\Sigma_E \subset \mathbb{R}^d$ be a finite simplicial complex for the smooth compact manifold $M \subset \mathbb{R}^d$, embedded into $\mathbb{R}^d$ via a homeomorphism $\phi : \Sigma_E \to M$. Denote by $\Sigma = \phi(\Sigma_E)$, that is, the simplicial complex embedded in M. Set $X = \Sigma^0 \subset M, N = |\Sigma^0|$ for the 0-skeleton Σ^0. Note that the top dimensional skeleton is simply M, $\Sigma^{\dim M} = \Sigma = M$. $C^k(\Sigma)$ will denote the space of k-cochains on Σ. Since Σ is identified with M, $\Omega^k(\Sigma) = \Omega^k(M)$ will denote the space of smooth k-forms, and $C^\infty(M) = C^\infty(\Sigma), C(M) = C(\Sigma)$ the space of smooth and continuous functions.

Recall that a map $\phi : \Sigma_E \to M$, with $\Sigma_E \subset \mathbb{R}^d$ a polyhedron, is a piecewise differentiable (PD) homeomorphism when there exists a triangulation Σ'_E for Σ_E such that for every simplex σ in Σ'_E

1. ϕ is a homeomorphism
2. ϕ restricted to σ is smooth
3. $D\phi$ is injective at every $x \in \sigma$

A related notion is that of a piecewise-linear (PL) map: $\phi : K \to \mathbb{R}^k$ is PL for every simplex in some triangulation Σ'_E $\phi|\sigma$ is linear. Every smooth manifold is associated to a PL-manifold by by Whitehead's theorem: for every smooth manifold M there exists a PD-homeomorphism $\phi : K \to M$, K a polyhedron which is piecewise linear (PL) manifold unique up to a PL-homoemoprhism, where a PL-manifold is a polyhedron K such that for all $x \in K$ there exists a neighborhood U^x and a PL-homeomorphism $\phi_x : U^x \to \mathbb{R}^k$. The simplicial complex on K with respect to which $\Phi : K \to M$ is PD, along with Φ is defined as the PL-structure for the manifold.

Assume 5.1.1. *We will assume that the embedding $\phi : \Sigma_E \to M$, with respect to the simplicial structure of Σ_E, is PD.*

Differentials with heat kernel weights

On a finite metric space, (X_n, d), $n = |X_n|$, with a probability measure μ, the point cloud Laplacian can be realized as Hodge Laplacian of a (co)chain complex restricted to functions. This follows by observing that for a finitely supported measure ν on M, the point cloud Laplacian on M is an empirical estimate (via concentration bounds) for the functional approximation to the Laplace–Beltrami operator $\triangle_t f(x) = \int_{X_n}(f(x) - f(y))K_t(x, y)d\nu(y)$. The convergence of the empirical estimate to the Laplace-Beltrami operator then follows using the result from [10].

Consider the picture that n point metric space X_n is n samples from M, d is the distance in ambient euclidean space, d_M the geodesic distance on M, and as n increases we have inclusions $i_n : X_n \to X_{n+1}$, and $X_{n+1} \setminus X_n$ is the one additional sample from M.

Fix $X_n = X$. Barthodi et al[9] consider (co)chain complexes on $L^2(X^l)$ using the coboundary map, $\delta_{l-1} : L^2(X^l) \to L^2(X^{l+1})$,

$$[\delta f](z_0, z_1 \dots z_l) = \sum_{i=0}^{l}(-1)^i \prod_{i \neq j} \sqrt{K(z_i, z_j)} f(z_0, \dots \hat{z}_i \dots z_l) \tag{5.1}$$

where $X^l = \prod_{i\in[l]} X$, $L^\infty(X^2) \ni K : X^2 \to \mathbb{R}$ is symmetric, nonnegative, and measurable; $K := K_t(\cdot, \cdot)$ is taken the t_n scaled heat kernel. The boundary map $\partial_l : L^2(X^{l+1}) \to L^2(X^l)$ is defined by $[\partial g](z_0 \dots z_{l-1}) = \sum_{i=0}^{l}(-1)^i \int_X \prod_{j=0}^{l-1} \sqrt{K(s, z_j)}$ $g(z_0 \dots z_{j-1}, s, z_{j+1} \dots z_{l-1})\ d\mu(s)$ and satisfies $\delta^*_{l-1} = \partial_l$, and the laplacian, $\triangle_l = (\delta^*_l \delta_l + \delta_{l-1}\delta^*_{l-1})$ can be defined. The constructions and results also hold for $L^2_a(X^l) = \{f \in L^2(X^l) : f(x_0, \dots x_l) = (-1)^{\mathrm{sgn}(\sigma)} f(\sigma(x_0), \dots \sigma(x_l)), \sigma \in S_{l+1}\}$. In [9], they also establish that for a Riemannian manifold, (X, g, μ), on restricting this construction to a suitable neighborhood of the diagonal, de Rham cohomology of X can be recovered and a Hodge decomposition exists for each $L^2(X^l)$. Observing that

$$\triangle_0^t(f(x)) = \int_X (f(x) - f(y))K_t(x, y)d\mu(y)$$

i.e. $\triangle_0|_{L^2(X)}$ is exactly the functional approximation to the Laplace-Beltrami operator which in the large sample-small t limit approaches the Laplace-Beltrami operator. Since on restricting to functions, Hodge-de Rham Laplacian agrees with the Laplace-Beltrami operator up to a sign suggests that in this limit $\delta^{(n)}$ associated to the sequence of n-point metric spaces (X_n) must approach the usual exterior derivative d acting on $\Omega^0(X)$. We give a quick intuitive argument using covariant Taylor series (see [5]) with respect to the canonical Riemannian connection ∇.

Proposition 5.1.2. *Suppose $U \subset \mathbb{R}^N$ is such that $M \cap U$ is a normal neighborhood of $x \in M$, and for any $y \in M \cap U$, $y \neq x$, $x(t)$ is the unique unit speed geodesic* *joining x, y, $v := \dot{x}(0)$. Then for $s = d_M(x,y)$ and $K_t(x,y) = \exp(-\|x-y\|_N^2/4t)$, $s = t + O(t^2)$ implies $|\delta f(x,y)/t - df_x(v)| = O(t)$.*

Proof. Since $x(t)$ is unit speed geodesic with $x(0) = x$, so $x(s) = y$. Expanding in a covariant Taylor series about $x(0)$, $f(x(t)) = \sum_{n=0}^{\infty} t^n/n! d^n/d\tau^n f(x(\tau))|_{\tau=0}$, with $d/d\tau = \dot{x}^i(\tau)\nabla_i$, gives $f(y) - f(x) = s \cdot df(v) + O(s^2)$ since first order term is $\dot{x}^i(\tau)\nabla_i f|_{\tau=0} = s \cdot g(v, \nabla f(x)) = s \cdot df_x(v)$. We have $\delta f(x,y) = \sqrt{K_t(x,y)}(f(y) - f(x)) = \sqrt{K_t(x,y)} s \dot{d} f_x(v) + \sqrt{K_t(x,y)} O(s^2)$. For fixed x, using that there exists $\eta \geq 0$, such that $d_M(x,y)^2 - \|x-y\|_N^2 = \eta(y)$ with $|\eta(y)| \leq C d_M(x,y)^4$ for a constant C on the normal neighborhood U, so $\|x-y\|_N^2 = d_M(x,y) - \eta(y)$. Using $e^{\alpha} = 1 + O(\alpha e^{\alpha})$ for $\alpha > 0$, $1/(1+\alpha) \leq 1 + O(\alpha)$ yields the following estimate from which the result follows for $s = t + O(t^2)$:

$$\left|\sqrt{K_t(x,y)}\frac{s}{t} df(v) - df(v)\right| = \left|\left(e^{\eta(y)} e^{-d_M(x,y)^2/8t}\frac{s}{t} - 1\right) df(v)\right|$$
$$\leq \left|\left(\frac{s}{t}(1 + O(s^2/t))(1 + O(s^4/t)) - 1\right) df(v)\right|$$

□

In the large sample limit as the sampled points get closer s/t approaches identity while $s^k/t, k > 1$ terms vanish, and the exterior derivative acting on functions is recovered. To recover the action on differential forms, it's simplest to work with an appropriate discretization of forms which is provided by finite element exterior calculus[29]. This forces adapting L^2-Hodge theory to work with cochains and not alternating functions to approximate the exterior derivative.

5.2 Approximating smooth differentials

L^2 structure on co-chains

For $L^2_a(X^{k+1})$ be the space of alternating L^2 functions in $k+1$ variables. Notice that on viewing each $k+1$ tuple as a k-simplex simplex $(x_0, \dots x_k)$, every k-cochain induces a function in $L^2_a(X^{k+1})$. However, $f \in L^2_a(X^{k+1})$ can be supported on tuples that are not simplicies in the complex. To encode the simplicial structure, the idea is to modify δ slightly. Define $K : X^{k+1} \to \{0,1\}$ by $\lambda_{x_i}(x_0, \dots \hat{x}_i \dots x_k) := K(x_0, x_1 \dots x_k) = 1$ iff $\sigma := (x_0, x_1 \dots x_k)$ is a k-simplex in Σ. As defined $\lambda_{x_i}(x_0, \dots)$ is symmetric in all arguments and positive. This yields a coboundary map generalizing equation 5.1, $\delta_{k-1} : C^{k-1}(\Sigma) \to C^k(\Sigma)$,

$$[\delta f](z_0, z_1 \dots z_l) = \sum_{i=0}^{l} (-1)^i \lambda_{z_i}(z_0, \dots \hat{z}_i \dots z_k) f(z_0, \dots \hat{z}_i \dots z_l) \qquad (5.2)$$

where $C^k(\Sigma)$ is the space of k-co-chains, i.e. functions on k-chains, in particular, on simplicies represented by $k+1$-tuples, X^{k+1}, with $C^k(\Sigma)$ a subspace inside $L^2_a(X^{k+1})$.

Observation 5.2.1. Note that δ_{k-1} is the simplicial coboundary operator: if $\sigma := (z_0, z_1, \ldots, z_k) \notin \Sigma^k$, then $\delta f(\sigma) = 0$, and if $\sigma \in \Sigma^k$, then it's the usual simplicial coboundary, and therefore, $\delta^2 = \delta_k \delta_{k-1} = 0$.

Taking uniform measure μ_X on X, and on X^{k+1} taking the measure $\nu_k = U_{k+1}\mu_X^{\otimes(k+1)}$ which is scalar multiple of the product measure. With respect to $\mu^{\otimes(k+1)}$, each k-simplex has measure $(k+1)!/N^{k+1}$, the scalar normalization U_{k+1} allows for normalizing the measure so that $\nu_k(\sigma) = 1/|\Sigma^k|$ for any $\sigma \in \Sigma^k$, that is, ν_k is the uniform probability measure on the k-simplicies, given by $\nu_k(\sigma) = 1/N_k$, $N_k = |\Sigma^k|$. $C^k(\Sigma)$ becomes a Hilbert space $L^2(C^k)$ by innerproduct,

$$\langle f, g\rangle_{C^k} = 1/N_k \sum_{\sigma\in\Sigma^k} f(\sigma)g(\sigma) = U_{k+1} \int_{X^{k+1}} f(x_0, \ldots x_k)g(x_0, \ldots x_k)d\mu(x_0)\ldots\mu(x_k)$$

where $(k+1)!$ is needed since each $k+1$ tuple gives the same k-simplex, and the fg is invariant under changing orientation.

Observation 5.2.2. Let $\partial_k : C^k \to C^{k-1}$ be δ^*_{k-1}, then $\delta^2 = 0$ implies $\langle \partial^2 g, \partial^2 g\rangle = \langle g, \delta^2\partial^2 g\rangle = 0$, and so $\partial^2 = 0$. That is, $\delta_{k-1} : C^{k-1}(\Sigma) \to C^k(\Sigma)$, $\partial_k : C^k(\Sigma) \to C^{k-1}(\Sigma)$, form a co-chain complex.

A standard computation (e.g. [9]) allows for computing ∂_k explicitly,

Proposition 5.2.3. *With $y \in X^k$,*

$$\partial_k g = \tfrac{U_{k+1}}{U_k} \sum_{i=0}^{k+1} \int_X \lambda_t(y)g(t, y)d\mu(t) = \tfrac{U_{k+1}}{U_k} \sum_{i=0}^{k+1} \int_X \lambda_t(y)g(t, y)d\mu(t)$$

Proof. We have $\langle \delta_{k-1} f, g\rangle$

$$= U_{k+1} \sum_{i=0}^{k} (-1)^i \int_{X^{k+1}} f(x_0 \ldots \hat{x}_i, \ldots x_k)\lambda_{x_i}(x_0 \ldots \hat{x}_i, \ldots x_k)g(x_0, \ldots, x_k) \prod_j \mu(x_j)$$

Setting $G_i(x_0 \ldots x_k) := \left(\int_X \lambda_{x_i}(x_0 \ldots \hat{x}_i, \ldots x_k)g(x_0, \ldots, x_k)d\mu(x_i)\right)$

$$\begin{aligned}\langle \delta_{k-1} f, g\rangle &= U_{k+1} \sum_{i=0}^{k} (-1)^i \int_{X^k} f(x_0 \ldots \hat{x}_i, \ldots x_k)G_i(x_0 \ldots x_k) \prod_{j\neq i} \mu(x_j) \\ &= U_k \int_{X^k} f(x_0 \ldots \hat{x}_i, \ldots x_k) \left(\tfrac{U_{k+1}}{U_k} \sum_{i=0}^{k} (-1)^i G_i(x_0 \ldots x_k)\right) \prod_{j\neq i} \mu(x_j)\end{aligned}$$

With $t = x_i, y = (y_0, \ldots, y_{k-1}) := (x_0, \ldots \hat{x}_i \ldots, x_k)$, using antisymmetry of g to write $g(x_0, \ldots, t, \ldots, x_k) = (-1)^i g(t, y)$ coupled with symmetry of λ,

$$\langle \delta_{k-1} f, g \rangle_{C^k} = U_k \int_{X^k} f(y) \left(\frac{U_{k+1}}{U_k} \sum_{i=0}^{k} \int_X \lambda_t(y) g(t, y) d\mu(t) \right) d\mu(y) = \langle f, \partial_k g \rangle_{C^{k-1}}$$

□

Taking $\lambda_x(y) = 1$ iff $xy \in \Sigma^1$ in equation 5.2 recovers the δ_0 from 5.1 with $\kappa(x, y) = \lambda_x(y)$ and the associated laplacian on functions; the family of kernels, $\kappa_t = e^{-\|x-y\|^2/4t}$ relates to the point cloud laplacian from [10], however this heat kernel weighing is not treated at this moment.

The abstract Hodge lemma from [9] easily yields a Hodge decomposition theorem:

Lemma 5.2.4. *(Abstract Hodge lemma[9, lemma 1]) Suppose the family of Hilbert spaces V_k's, with bounded linear operators δ_k, δ_k^*, define (co)chain complexes, $\ldots \delta_{k-1}^* : V_k \to V_{k-1} \ldots, \ldots \delta_k : V_k \to V_{k+1} \ldots$, with $\delta^2 = 0, \delta^{*2} = 0$, then for $\triangle_l := \delta_l^* \delta_l + \delta_{l-1} \delta_{l-1}^*$, the following are equivalent*

- *δ_k has closed range for all l*
- *δ_k^* has closed range for all l*
- *$\triangle_l$ has closed range for all l*

and if any of the above hold then

$$V_l = Image(\delta_{l-1}) \oplus Image(\delta_l^*) \oplus Kernel(\triangle_l) \tag{5.3}$$

Corollary 5.2.5. *The Hodge decomposition in equation 5.3 applies to the (co)chain complex from observation 5.2.2, $\delta_{k-1} : C^{k-1}(\Sigma) \to C^k(\Sigma)$, $\partial_k : C^k(\Sigma) \to C^{k-1}(\Sigma)$*

Proof. The proof is simply noting that $C^k(\Sigma) \subset L_a^2(X^{k+1})$ are finite dimensional Hilbert spaces so the images of maps δ_k, δ_k^* are closed. □

Remark 5.2.6. L^2 co-chains have been considered in [31], however the innerproduct does not come from the measure on k-skeleton; instead for co-chains c, c', $\langle c, c' \rangle = \langle \mathcal{W} c, \mathcal{W} c' \rangle_{L^2(M)}$ where $\mathcal{W}$ is the Whitney map into differential forms which does not give the L^2-structure for the k-skeleton.

Remark 5.2.7. By combining equations 5.2,5.1, the coboundary operator can be further generalized:

$$\delta_{k-1} : C^{k-1}(\Sigma) \to C^k(\Sigma)$$
$$[\delta f](z_0, z_1 \dots z_l) = \sum_{i=0}^{l} (-1)^i \prod_{i \neq j} \sqrt{\kappa(z_i, z_j)} \lambda_{z_i}(z_0, \dots \hat{z}_i \dots z_k) f(z_0, \dots \hat{z}_i \dots z_l) \tag{5.4}$$

and it can be checked that it defines a (co)chain complex, and therefore the constructions from [9] on $L^2_a(X^{k+1})$ pass to k-cochains.

Local volume forms, de Rham and Whitney maps

Since Σ is identified with M, it inherits a normalized volume form from M which gives a normalized measure on Σ. Now every $\sigma \in \Sigma^k$ inherits a volume form from being embedded in M, so it too inherits a volume measure, $\mathrm{d}_{vol}\,\sigma$ and $\mathrm{Vol}(\sigma)$ is defined. A normalized measure is induced on the k-skeleton, Σ^k, scaling $\mathrm{d}_{vol}\,\sigma$ by $\sum_{\tau \in \Sigma^k} \mathrm{Vol}(\tau)$, so for every k and all $\sigma \in \Sigma^k$, $\int_\sigma \mathrm{d}_{vol}\,\sigma \leq 1$ and $\sum_{\sigma \in \Sigma^k} \int_\sigma \mathrm{d}_{vol}\,\sigma = 1$.

For each k-simplex $\sigma \in \Sigma$ denote by $\mathrm{d}_{vol}\,\sigma$ this normalized volume form. Now $\mathrm{d}_{vol}\,\sigma$ can be viewed as k-form on Σ, not necessarily smooth, by setting $\mathrm{d}_{vol}\,\sigma = \mathbf{1}_{\sigma^\circ}\,\mathrm{d}_{vol}\,\sigma$. Set $\Omega_{lv}(\Sigma) = \oplus_k \mathrm{LinSpan}(\{\mathrm{d}_{vol}\,\sigma : \sigma \in \Sigma^k\})$, the space of local volume forms. Note that $\Omega_{lv}(\Sigma) \subset L^2(\Omega(M))$, but any $\omega \in \Omega^k_{lv}(\Sigma)$ is not necessarily continuous on $\partial\Sigma^k$; therefore, Stokes theorem does not apply since $\omega \in \mathrm{Dom}(d)$ may not hold.

Any k-form $\omega \in \Omega^k(M)$ yields a $f_\omega \in C^k(\Sigma)$ by integration over the $\sigma := (x_0, x_1 \dots x_p) \in \Sigma^k$. This is the de Rham map, $\mathcal{R} : L^2(\Omega(M)) \to C(\Sigma)$,

$$L^2(\Omega^k(M)) \ni \omega \to \mathcal{R}(\omega) := f_\omega \in C^k(X),\ f_\omega(\sum a_i \sigma_i) = \sum_i a_i \int_{\sigma_i} \omega \in C^k(X)$$

Note that $\mathcal{R}(\omega)$ is alternating because simplicies are oriented. Because δ is the simplicial coboundary map, $\mathcal{R}$ is a chain map, $\delta\mathcal{R}\omega = \mathcal{R}d\omega$ for $\Omega(M)$ (see [31, 50]).

Under the following assumption the de Rham map defines an isometry between cochains and local volume forms which allows for passing from uniform distribution over X^{k+1} for L^2-structure on cochains to the uniform distribution over Σ^k for differential forms.

Assume 5.2.8. *For each k assume the volume of each k-simplex is same, wlog assume it's one. Since X carries the uniform measure, let c_k be the constant such that Vol*$(\sigma) := \int_\sigma \mathrm{d}_{vol}(\sigma) = c_k \|\boldsymbol{1}_\sigma\|_{L^2(X^{k+1})}$, $\boldsymbol{1}_\sigma \in C^k(\Sigma)$.

Lemma 5.2.9. *The de Rham map $\mathcal{R}$ is surjective. For any f, $\mathcal{R}^{-1}(f)$ carries a unique representative $\omega_f \in \Omega_{lv}(\Sigma)$.*

Proof. Let $f \in C^k(X)$, then $\omega \in \Omega^p(M)$, not necessarily unique, such that $\mathcal{R}(\omega) = f$ can be constructed by averaging. Let $\mathrm{Support}(f) = \{\sigma : \sigma \in \Sigma^p, |f(\sigma)| > 0\}$. For $\sigma \in \mathrm{Support}(f)$, if ω_σ be any k-form such that ω_σ vanishes outside σ and $\int_\sigma \omega_\sigma = 1$. Then $f = \mathcal{R}(\sum_{\sigma \in \mathrm{Support}(f)} f(\sigma)\omega_\sigma)$.

The unique representative $\omega_f, \mathcal{R}(\omega_f) = f$, is given by

$$\omega_f = \sum_{\sigma \in \mathrm{Support}(f)} f(\sigma)\mathbf{1}_{\sigma^\circ}\, \mathrm{d}_{vol}\, \sigma/\mathrm{Vol}(\sigma) = \sum_{\sigma \in \mathrm{Support}(f)} f(\sigma)\, \mathrm{d}_{vol}\, \sigma \tag{5.5}$$

since $\mathrm{Vol}(\sigma) = 1$ was assumed. The uniqueness is obvious. □

The following corollary is immediate.

Corollary 5.2.10. *$\mathcal{R}$ restricted to $\Omega_{lv}(\Sigma)$, $\hat{\mathcal{R}} = \mathcal{R}|_{\Omega_{lv}(\Sigma)}$ has inverse $\hat{\mathcal{R}}^{-1} : \oplus_k C^k(\Sigma) \to \oplus_k \Omega^k_{lv}(\Sigma)$.*

In fact, it's isometric up to a constant, depending on the grading.

Lemma 5.2.11. *There exists a constant A_k such that $\mathcal{R}''_k : \Omega^k_{lv}(\Sigma) \to C^k(\Sigma)$ given by $\mathcal{R}''_k(\omega) := A_k\hat{\mathcal{R}}(\omega), \omega \in \Omega^k_{lv}(\Sigma)$ is an isometry with respect to $\|\cdot\|_{L^2(\Omega^k)}, \|\cdot\|_{C^k(\Sigma)}$, and therefore, the map*

$$\mathcal{R}'' := \oplus_k \mathcal{R}''_k : \oplus\Omega^k_{lv}(\Sigma) \to \oplus_k C^k(\Sigma), \mathcal{R}''^{-1} : \oplus_k C^k(\Sigma) \to \oplus\Omega^k_{lv}(\Sigma) \tag{5.6}$$

are isometric embeddings

Proof. Note that $f = \mathcal{R}(\sum_{\sigma \in \mathrm{Support}(f)} \omega_f)$ where ω_f is the differential form that takes the constant value $f(\sigma)$ on the interior of σ. Now $\|f\|_{L^2(C^k)} = 1/N_k \sum_{\sigma \in \Sigma^k} f(\sigma)^2$, while the k-form ω_f, $\left\|\omega_f\right\|_{L^2(\Omega^k)} = \sum_\sigma f(\sigma)^2 \int_\sigma \mathrm{d}_{vol}\, \sigma$. Recalling that by assumption 5.2.8 each simplex has unit volume, $A_k = \sqrt{N_k}$ such that $\left\|\omega_f\right\|_{L^2(\Omega^k)} = A_k \|f\|_{L^2(C^k)}$, that is, $A_k\hat{\mathcal{R}}$ is the isometry as needed. □

The space $\Omega^k_{lv}(\Sigma)$ comes with a projection. With Σ fixed, define $\Phi \equiv \Phi_\Sigma : \Omega^k(\Sigma) \to \Omega^k_{lv}(\Sigma)$ $\Phi(\omega) = \sum_{\sigma\in\Sigma^k} \frac{1}{\mathrm{Vol}(\sigma)} \left(\int_\sigma \omega\right) \mathrm{d}_{vol}\,\sigma$. So Φ replaces ω over simplex σ by $\mathrm{d}_{vol}\,\sigma$ scaled by the averaged ω. The following is obvious.

Proposition 5.2.12. *Φ is a projection on to $\Omega^k_{lc}(\Sigma)$, $\Phi^2 = \Phi$ with $\mathcal{R}\Phi(\omega) = \mathcal{R}(\omega)$ for all ω*

The local volume forms are not in domain of d, and therefore, Stokes theorem does not apply and $\mathcal{R}$ is not a chain map on Ω_{lv}.

Observation 5.2.13. Notice that the content of proposition 5.1.2 is encapsulated in the chain-map property of $\mathcal{R}$: for the function f, and associated 0-cochain $\mathcal{R}f$, $[\mathcal{R}df](\sigma) = [\delta\mathcal{R}f](\sigma)$ for any 1-simplex $\sigma = (x_0x_1)$, and proposition 5.1.2 establishes that $df, \delta\mathcal{R}f$ are close. However, since df is a form and $\delta\mathcal{R}f$ a chain, one needs a way to identify $\delta\mathcal{R}f$ with a form, that is, a way to evaluate it at a point $x \in \sigma$. In proposition 5.1.2, this is provided by the covariant Taylor series about x_0; for higher dimensional forms, higher covariant Taylor series are cumbersome, and the Whitney forms offer a cleaner alternative (as opposed to the heuristic argument sketched in [39]).

The Whitney map, $\mathcal{W} : C^k(\Sigma) \to \Omega^k(\Sigma)$, is induced by the barycentric functions, $\lambda_i, i \in [k+1]$, on a k-simplex $\sigma = (v_0, v_1, \ldots, v_k)$. If $k = 0$, then $\mathcal{W}\sigma = \lambda_0$, and otherwise,

$$\mathcal{W}\sigma = k! \sum_{i=0}^{k} \lambda_i d\lambda_0 \wedge \ldots d\hat{\lambda}_i \cdots \wedge d\lambda_k$$

On euclidean polyhedra, the barycentric functions λ_i are the unique affine functions on simplex $(x_0x_1\ldots x_k)$ such that $\lambda_i(x_j) = \delta_{ij}$. So λ_i vanishes on the face opposite to x_i. The barycentric coordinates can be pulled from Σ_E to Σ embedded in M through Φ. The Whitney map $\mathcal{W}$ provides a right inverse to the de Rham map satisfying (see [50, § 2, § 5], [31, corollary 3.27]):

1. $\mathcal{W}$ is a chain map
2. $\mathcal{R}\mathcal{W} = \mathbf{1}_{C^k}$

and additionally –

Proposition 5.2.14. *$\mathcal{W}$ satisfies*

1. $\langle \mathcal{W}\sigma, \mathcal{W}\sigma\rangle = 1$

2. $\|\mathcal{W}\sum_{\sigma\in\Sigma^k} f_\sigma\sigma\|^2_{L^2} = \sum_\sigma f^2_\sigma$

3. $\mathcal{W} := \oplus_k \frac{1}{\sqrt{N_k}}\mathcal{W}'|_{C^k}$ *is an isometry*

Proof. The condition $\mathcal{R}\mathcal{W} = \mathbf{1}$ implies $\int_\tau \mathcal{W}\sigma = \mathbf{1}_{\tau=\sigma}$. This yields an explicit form for $\mathcal{W}\sigma$ since $\int_\tau \mathcal{W}\sigma = \mathbf{1}_{\tau=\sigma}$ is equivalent to using $\mathcal{W}\sigma = \frac{1}{\mathrm{Vol}(\sigma)}\,\mathrm{d}_{vol}\,\sigma$ where $\mathrm{Vol}(\sigma) = 1$ for all σ is assumed. The first now follows simply by

$$\langle \mathcal{W}\sigma, \mathcal{W}\sigma\rangle = \int_\sigma \frac{1}{\mathrm{Vol}(\sigma)^2}\,\mathrm{d}_{vol}\,\sigma \wedge \star\,\mathrm{d}_{vol}\,\sigma = \int_\sigma \frac{1}{\mathrm{Vol}(\sigma)^2}\,\mathrm{d}_{vol}\,\sigma = 1 \tag{5.7}$$

where $\star$ denotes the Hodge dual on σ (not M). And, therefore,

$$\|\mathcal{W}\sum_{\sigma\in\Sigma^k} f_\sigma\sigma\|^2_{L^2(\Omega^k)} = \sum_\sigma f^2_\sigma\langle \mathcal{W}\sigma, \mathcal{W}\sigma\rangle = \sum_\sigma f^2_\sigma = N_k\|\sum_\sigma f_\sigma\sigma\|^2_{L^2(C^k)}$$

This shows that $\frac{1}{\sqrt{N_k}}\mathcal{W}$ is the isometry on C^k. □

Remark 5.2.15. On normalizing the $\mathrm{vol}(\sigma) = 1/N_k$ instead, the Whitney and de Rham maps become isometries without the rescaling.

Uniform approximation for spectral trunctaions

The chain-map property in addition to observation 5.2.13 also allows for implementing d as $\mathcal{W}\delta\mathcal{R}$ and realizing proposition 5.1.2 on p-forms in general. This requires the approximation theorem due to Dodziuk [31] (alternatively a version due to Lohi and Kettunen [50]).

The standard subdivision of a n-simplex $(p_0p_1\dots p_n)$ proceeds by introducing 0-cells $p_{ij} = (p_i + p_j)/2$ for forming 2^n sub-simplicies (see [67, appendeix II, § 4]). Let $\Sigma_{(1)}$ denote the simplicial complex generated by applying the standard subdivision $\mathcal{S}$ to Σ (see [67, appendex II, § 3]), and by $\Sigma_{(n)} = \mathcal{S}\Sigma_{(n-1)}$. The measure $\mu_{\Sigma^0_{(n)}}$ is taken to be uniform on Σ^0_n, and measures on higher skeletons are defined as before. $\mathcal{R}_n, \mathcal{W}_n$ are the de Rham and Whitney maps

$$\oplus_k\mathcal{R}_n : \Omega^k(\Sigma_{(n)}) \to C^k(\Sigma_{(n)}), \;\; \oplus_k\mathcal{W}_n : C^k(\Sigma_{(n)}) \to \Omega^k(\Sigma_{(n)})$$

with $\mathcal{R}'_n, \mathcal{W}'_n$ associated normalized de Rham and Whitney maps.

Dodizuk's approximation theorem ([31, theorem 3.7]) states $\|\omega(x) - \mathcal{W}_n\mathcal{R}_n\omega(x)\|_x < K_\omega\mathrm{dia}(\Sigma), x \in \Sigma_{(n)} \setminus \Sigma^{\dim\Sigma-1}_n$, where $\mathrm{dia}(\Sigma) = \sup_{\sigma\in\Sigma}\mathrm{dia}(\sigma)$ and K_ω the product of a universal constant depending only on M, Σ and maximum of absolutes values of derivatives of components of ω ([31, Corollary 3.27]). From this the $L^2(\Sigma)$ version follows using innerproduct induced by the Riemannian volume form,

$$\langle \omega, \omega' \rangle = \int_\Sigma \omega \wedge \star\omega' = \int_\Sigma \langle \omega, \omega' \rangle \, \mathrm{d}_{vol}\, \Sigma \tag{5.8}$$

On k-skeleton, Σ^k, for $\sigma \in \Sigma^k$, set $\Omega^\bullet(\sigma)$ to be smooth differential forms on σ. Define $\Omega^\bullet(\Sigma^k) = \{\omega \mathbf{1}_{\sigma^o} : \omega \in \Omega^\bullet(\sigma), \sigma \in \Sigma^k\}$. Note that $\omega \in \Omega^\bullet(\Sigma^k)$ is smooth on $\Sigma^k \setminus \Sigma^{k-1}$. The point of introducing $\Omega^\bullet(\Sigma^k)$ is that if τ is a face of σ, then one does not want to consider $d\lambda_i$ on τ for barycentric functions associated to vertices of σ not in τ; when τ is a shared face for σ, σ' such $d\lambda_i$'s from σ and σ' may not agree on τ.

Observation 5.2.16. The $L^2(\Sigma)$ structure considered is with respect to $\mathrm{d}_{vol}\,\Sigma$, that is, $L^2(\Sigma^{\dim M})$, but the approximation theorem can be applied to each k-cell (with Σ^k, for $k < \dim M$ is viewed as a union of its k-cells each in itself a submanifold), and therefore holds for $\omega \in \Omega(\Sigma^k)$. Let m_k be the standard subdivisions required for the k-skeleton, then since the standard subdivsion of a simplex yields standard subdivision of all faces, on $\mathcal{S}_{\max_k\{m_k\}}\Sigma$ forms belonging to all skeletons can be approximated away from a set of zero measure with respect to their volume forms.

With this, the point-wise approximation implies approximation with respect to $L^2(\Sigma^k)$ for $k \leq \dim M$. Therefore,

$$\|\omega - \mathcal{W}\mathcal{R}\omega\|_{L^2(\Sigma^k)} < K'_{k,\omega}\mathrm{dia}(\Sigma) \tag{5.9}$$

where $K'_{k,\omega} = K_\omega \mathrm{Vol}(\Sigma^k)$, $\omega \in \Omega(\Sigma^k)$, and the norm coming from inner product

$$\langle \omega, \omega' \rangle_{L^2(\Omega^k)} = \sum_{\sigma \in \Sigma^k} \int_\sigma \langle \omega, \omega' \rangle \, \mathrm{d}_{vol}\, \sigma \tag{5.10}$$

Remark 5.2.17. Using standard subdivisions can be avoided by using the variant of the result from [50] which holds for euclidean polyhedra where each cell has a lowerbound on ratio of volume to diameter. But the result can be pulled from Σ_E to Σ via ϕ with the constant now dependent on choice of ϕ. In both [50, 31], the constant K_ω depends on partial derivatives of components of ω.

The approximation property (equation 5.9) yields that d can be implemented through $\mathcal{R}_b, \mathcal{W}_n$ using that $\mathcal{R}_n$ is a chain map.

$$\|d\omega - \mathcal{W}_n\mathcal{R}_n d\omega\|_{L^2(\Omega^k)} = \|d\omega - \mathcal{W}_n\delta\mathcal{R}_n\omega\|_{L^2(\Omega^k)} < K_{d\omega}\mathrm{dia}(\Sigma_{(n)}) \tag{5.11}$$

Since on normalizing the $\mathrm{vol}(\sigma) = 1/N_k$ for $\sigma \in \Sigma^k$, the de Rham map (on local volume forms) and Whitney map are isometries, we have the following.

Theorem 5.2.18. *The exterior derivative d on $\Omega^k(\Sigma)$ is implemented by the de Rham and Whitney maps, $\mathcal{R}_n, \mathcal{W}_n$, in the sense that for every $\omega \in \Omega^k(\Sigma)$,*

$$\|(d - \mathcal{W}_n \delta \mathcal{R}_n)\omega\|_{L^2} < K_{d\omega} dia(\Sigma_{(n)})$$

with $\mathcal{R}_n, \mathcal{W}_n$ bounded. If the uniform k-volume assumption (assumption 5.2.8) holds then $\mathcal{R}_n|_{\Omega_{lv}(\Sigma_n)}, \mathcal{W}_n$ isometries.

By the Bochner idenity, $(d + d^*)^2$ is elliptic, and therefore, there exists an eigenbasis $(\omega_i)_{\mathbb{N}}$ of smooth eigenforms for $L^2(\Omega(\Sigma))$ which are also eigenforms for $d + d^*$. Let $E_m := \mathrm{FinteLinSpan}\{\omega_i : i \in [m]\}$, then restricted to finite dimensional subspaces the following uniform variant holds. This realizes both d, d^*, and therefore $d + d^*$, through $\delta, \mathcal{R}, \mathcal{W}$.

Corollary 5.2.19. *For all $\omega \in E_m$, there exists K_m independent of ω such that*

$$\|d\omega - \mathcal{W}_n \delta \mathcal{R}_n \omega\| < K_n dia(\Sigma_{(n)}), \quad \left\|(d^* - \mathcal{R}_n''^* \delta^* \mathcal{W}_n''^*)\omega\right\|_{L^2} < K_n dia(\Sigma_{(n)}) \tag{5.12}$$

Proof. On E_m, $d, \mathcal{W}_n \delta \mathcal{R}_n$ are bounded operators, and since $\|(d - \mathcal{W}_n \delta \mathcal{R}_n)^*\| = \|d - \mathcal{W}_n \delta \mathcal{R}_n\|$, d^* can be approximated as well. Now the claim follows because $K_{d\omega}$ in theorem 5.2.18 can be uniformly bound for $\omega \in E_m$ and there are only finitely many of them. □

Remark 5.2.20. The above result considers $d + d^*$ as an operator on the Hilbert subspace E_m. One is also interested in the action of 0-cochains that play the role of $C(M)$ on k-cochains that are discretized differential forms. This is given by the Whitney product $C^0(\Sigma) \times C^k(\Sigma) \to C^k(\Sigma)$ by $(f, g) \to \mathcal{R}(\mathcal{W}f \wedge \mathcal{W}g)$. Whitney product is nonassociative and defined between k, k'-cochains.

Some comments on uniform k-volume assumption are in order. Notice that since Dodziuk's approximation theorem can be applied to each σ individually, so the measure on the manifold can be rescaled to make each cell have the same volume. Trying to normalize the maps $\mathcal{R}, \mathcal{W}$ leads to failing to maintain the chain map property, but the other properties still hold.

Define the volume normalized de Rham map, $\mathcal{R}'$,

$$\Omega^k(M) \ni \omega \to \mathcal{R}'(\omega) := f_\omega \in C^k(X),\ f_\omega(\sum a_i\sigma_i) = \sum_i a_i \frac{1}{\sqrt{\mathrm{Vol}(\sigma_i)}} \int_{\sigma_i} \omega \in C^p(X)$$

and similarly the volume normalized variant for the Whitney map, $\mathcal{W}'$, which acts on the basis simplices such that $\mathcal{W}'\mathcal{R}'(\sigma) = \mathcal{W}\mathcal{R}(\sigma)$, preserving the approximation property,

$$\mathcal{W}'(\sigma) := \sqrt{\mathrm{Vol}(\sigma)}\mathcal{W}(\sigma)$$

Proposition 5.2.21. *For the volume normalized de Rham map $\mathcal{R}'$*

1. *k-cochain f has canonical representative $\omega_f \in \Omega^k_{lv}(\Sigma)$ given by $\omega_f = \sum_{\sigma\in\Sigma^k} f(\sigma)\frac{1}{\sqrt{Vol(\sigma)}}\mathbf{1}_{\sigma^\circ}\, \mathrm{d}_{vol}\,\sigma$*
2. *$\mathcal{R}'$ induces an isometry on Ω_{lv}, $\mathcal{R}'' = \oplus_k\sqrt{N_k}\mathcal{R}'|_{\Omega^k}$, (as in equation 5.6, which by polarization is unitary).*

Proof. For the first, note that $\mathcal{R}(\omega_f) = f$ since $\mathcal{R}'(\omega_f)(\sigma) = f(\sigma)\frac{1}{\mathrm{Vol}(\sigma)}\int_\sigma \mathrm{d}_{vol}\,\sigma = f(\sigma)$. The isometry property follows by using that

$$\int_\sigma \langle\omega_f, \omega_f\rangle\, \mathrm{d}_{vol}\,\sigma = f(\sigma)^2\frac{1}{\mathrm{Vol}(\sigma)}\int_\sigma \mathrm{d}_{vol}\,\sigma$$

Explicitly, $\left\|\mathcal{R}'(\omega_f)\right\|^2_{L^2(C^k)} = \|f\|^2_{L^2(C^k)} = \frac{1}{N_k}\sum_{\sigma\in\Sigma^k} f^2_\sigma$, and since $\left\|\omega_f\right\|^2_{L^2(\Omega^k)} = \sum_{\sigma\in\Sigma^k} f^2_\sigma$, therefore, $\mathcal{R}''|_{\Omega^k} = \sqrt{N_k}\mathcal{R}'|_{\Omega^k}$ is the isometry. □

Proposition 5.2.22. *$\mathcal{W}'$ satisfies $\langle\mathcal{W}'\sigma, \mathcal{W}'\sigma\rangle = 1$, $\|\mathcal{W}'\sum_{\sigma\in\Sigma^k} f_\sigma\sigma\|^2_{L^2} = \sum_\sigma f^2_\sigma$, and $\mathcal{W}'' := \oplus_k\frac{1}{\sqrt{N_k}}\mathcal{W}'|_{C^k}$ is an isometry*

Proof. Since $\mathcal{W}\sigma = \frac{1}{\mathrm{Vol}(\sigma)}\,\mathrm{d}_{vol}\,\sigma$, so $\mathcal{W}'(\sigma) = \frac{1}{\sqrt{\mathrm{Vol}(\sigma)}}\,\mathrm{d}_{vol}\,\sigma$

$$\langle\mathcal{W}'\sigma, \mathcal{W}'\sigma\rangle = \int_\sigma \frac{1}{\mathrm{Vol}(\sigma)}\,\mathrm{d}_{vol}\,\sigma \wedge \star\,\mathrm{d}_{vol}\,\sigma = \int_\sigma \frac{1}{\mathrm{Vol}(\sigma)}\,\mathrm{d}_{vol}\,\sigma = 1 \tag{5.13}$$

The rest follows as before. □

5.3 Reconstructing $C(M)$

Given $f \in \Omega^0(\Sigma) = C^\infty(M)$, the de Rham map gives a cochain, $\mathcal{R}f \in C^0(\Sigma)$. The Whitney map embeds $C^0(M)$ as a subspace inside $C(M)$. By observation 5.2.16, $\mathcal{W}(C^0(\Sigma))$ is dense in $(C(M), \|\cdot\|_\infty)$ as $\mathrm{Dia}(\Sigma)$ goes to zero. Therefore, $\mathcal{W}C^0(\Sigma)$ as the algebra of multiplication operators acting on $L^2(\Omega(M))$ approximates the action of $C(M)$ acting on $L^2(\Omega(M))$. It remains to answer how $C^0(\Sigma), C(\Sigma)$ approximate $C(M)$ as $\mathrm{Dia}(\Sigma)$ gets smaller, and if the limit can be characterized as an abstract C^*-algebra. The answer to the last question is given through the PL-structure $\phi : \Sigma_E \to M = \Sigma$ by using the noncommutative simplicial complex construction introduced in [27]. At the same time the map ϕ induces an isomorphism of $C(M)$ and $C(\Sigma_E)$. Finally, $\cup_n \mathcal{W}C^0(\Sigma_{(n)})$ where $\Sigma_{(n)}$ is an embedded simplicial complex for M (obtained, for example, as a subdivision of Σ_E) with $\mathrm{Dia}(\Sigma_n) \leq 1/n$, is dense in $C(M)$. In the following subsection, the noncommutative simplicial complex construction from [27] is recapped, and the basic theory of quantum metric spaces is introduced. The state-space over for the algebras $C(\Sigma_E)$ and $C(M)$ are then compared.

State spaces for PL-structures

Notice $\ldots \Sigma^0_{(n)} \hookrightarrow \Sigma^0_{(n_1)} \cdots \hookrightarrow M$ is an increasing sequence of sets, and therefore, $C(M) \hookrightarrow \ldots C(\Sigma^0_{(n)}) \hookrightarrow C(\Sigma^0_{(n-1)}) \ldots$ is an inverse system. Inverse limits of C^*-algebras are delicate since the limit may only be a pro-C^*-algebra and not a C^*-algebra. An approach to question of limits of such finite algebras (which is relevant to limits of finite spectral triples in noncommutative geometry) by [64] side-steps this by taking a dual triangulation and rewriting it as a direct limit. However, note that $C(M)$ can be recovered by pulling back $C(\Sigma_E) : C(M) = \{f \circ \phi^{-1} : f \in C(\Sigma_E)\}$. Since ϕ is PD, the action of exterior derivative can also be pulled back almost everywhere.

Now for the polyhedron Σ_E, let K be the underlying abstract simplicial complex on the vertex set $V_K := \Sigma^0_E$. Σ_E is isomorphic to the geometric realization $|K|$ for K. Define C_K the universal C^*-algebra generated by positive generators $h_i, i \in V_K$, $h_{i_1} h_{i_2} \ldots h_{i_k} = 0$ whenever $\{i_j : j \in [k]\} \not\subset K$ and for all $m \in V_\Sigma$, $\sum_{k \in V_\Sigma} h_m h_k = h_m$. Let C^{ab}_K be the abelianization of C_K, so with the additional constraint $h_k h_m = h_m h_k$. From [27], $C^{ab}_K \cong C_0(|K|) = C(|K|)$ as M, K are compact. The idea is straightforward, by the commutative Gelfand-Naimark theorem, $C^{ab}_K = C_0(X)$ where $X = \mathrm{Spec}(C^{ab}_K)$. $\mathrm{Spec}(C^{ab}_K)$ is exactly the space of map $\{f : V_K \to [0,1] : \sum_{V_k} f(i) = 1\}$ which, by definition, is the geometric realization $|K|$.

The homeomorphism ϕ also allows for mapping states $\sigma : C(\Sigma) \to \mathbb{R}$, $\sigma : C(\Sigma_E) \to \mathbb{R}$. The statespaces, $\mathcal{S}(C(\Sigma)), \mathcal{S}(C(\Sigma_E))$ can be metricized so that their Gromov-Hausdorff distance vanishes. To formalize this some background is needed, for which we follow [40, 63] –

Definition 5.3.1. [40] Let A be a real vector space.

- An ordered vector space is A along with a partial order $\leq$ satisfying $x \leq y, r \in \mathbb{R}_{\geq 0}$ implies $x + z \leq y + z$ and $rx \leq ry$.
- An order-unit space is an ordered vector space $(A, \leq)$ with a distinguished element e, the order-unit, such that $a \in A$ with $a \leq re$ for all $r \in \mathbb{R}_{\geq 0}$ implies $a \leq 0$ and for all $a \in A$, there exists $r_a \in \mathbb{R}$ with $a \leq r_a e$. Morphisms of order-unit spaces are linear maps preserving both $e, \leq$. The order-unit space is normed by $\|a\| = \inf\{t > 0 : -t \leq a \leq t\}$, and morphisms are contractive: $\|\phi(a)\| \leq \|a\|$ for all $a \in A, \phi$ order-unit morphisms.
- The state space $(\mathcal{S}(A), d_L)$ for A is $\mathcal{S}(A) := \{f : A \to \mathbb{R}\ f$ order-unit morphism $\}$. $\mathcal{S}(A)$ is compact in weak* topology. Any semi-norm L on A satisfying $L(a) = 0 \iff a \in \mathbb{R}$ where $\mathbb{R}$ is canonically embedded in A with topology induced by

$$d_L(\sigma, \tau) = \sup\{|\sigma(a) - \tau(a)| : a \in A, L(a) \leq 1\} \tag{5.14}$$

  on $\mathcal{S}(A)$ the weak* topology is called a Lip-norm. The pair (A, L) is a quantum metric space, $(\mathcal{S}(A), d_L)$ is the state-space.

Remark 5.3.2. From [61], note that the topology being weak* is implied by d_L being bounded, $L(a) = 0$ iff $a \in \mathbb{R}$ and the unit Lip-ball, $\mathcal{B}_1 := \{a \in A : L(a) \leq 1, \|a\| \leq 1\}$ being totally bounded in $\|\cdot\|_A$.

Example 5.3.3. For any compact Riemannian manifold (M, g) with geodesic distance d define $\|f\|_{\text{Lip}} = \inf_k\{k > 0, |f(x) - f(y)| \leq k \cdot d(x, y)\}$ and $A = \{f : \|f\|_{\text{Lip}} < \infty\} \subset C(M)$, then $(A, \|\cdot\|_{\text{Lip}})$ is a quantum metric space and associated state space is the state space $\{\mathcal{S}(A), d_{\|\cdot\|_{\text{Lip}}}\}$. Note that as an order-unit space the norm $\|f\|$ for $f \in C(M)$ is the usual $\sup_{x \in M} |f(x)|$.

The Gromov-Hausdorff distance, d_{GH}, between state spaces for two quantum metric spaces $(A_i, L_i), i \in [2]$ is characterized by the following:

Theorem 5.3.4. *[63, Theorem 5] If $\phi_{ij} : (A_i, L_i) \to (A_j, L_j), i \neq j$ are two morphisms such that $\left\|\phi_{ij} \circ \phi_{ji}(a) - a\right\|_i \leq \epsilon L_i(a)$ for all a and $\epsilon > 0$ is the smallest such ϵ, then the Gromov-Hausdorff d_{GH} satisfies*

$$d_{GH}((\mathcal{S}(A_1), d_{L_1}), (\mathcal{S}(A_2), d_{L_2})) < \epsilon$$

The state space over $C(X)$ for any compact metric space X, (so $\mathbf{1} \in C(X)$) is the space of Borel probability measures with Kantorovich-Rubinstein metric ([40]). So if X, Y are homoemorphic then pulling back the probability measures should yield that their state-spaces are also close. In particular, this applies to state-spaces over $C(M), C(\Sigma_E)$. This can be formalized using that PL-structure ϕ.

Proposition 5.3.5. *For quantum metric spaces, $(C(\Sigma)_{Lip}, \|\cdot\|_{Lip}), (C(\Sigma_E)_{Lip}, \|\cdot\|_{Lip})$,*

$$d_{GH}((\mathcal{S}(C(\Sigma)_{Lip}), d_{\|\cdot\|_{Lip}}), (\mathcal{S}(C(\Sigma_E)_{Lip}), d_{\|\cdot\|_{Lip}})) = 0$$

Proof. Setting (A_i, L_i) as $(C(\Sigma)_{\text{Lip}}, \|\cdot\|_{\text{Lip}}), (C(\Sigma_E)_{\text{Lip}}, \|\cdot\|_{\text{Lip}})$, where $C(\cdot)_{\text{Lip}}$ is the subspace of Lipschitz functions. Taking morphisms ϕ_{ij}, ϕ_{ji} as given by pullbacks by the embedding $\phi : \Sigma_E \to \Sigma$ and its inverse. Because ϕ is a PL-diffeomorphism, therefore, ϕ (along with ϕ^{-1}) identifies the Lipshitz functions, $C(\Sigma)_{\text{Lip}}, C(\Sigma_E)_{\text{Lip}}$, while preserving the order and unit. Finally, since ϕ_{ij}, ϕ_{ji} are inverses, therefore, $\left\|\phi_{ij} \circ \phi_{ji}(a) - a\right\|_i = 0$ for all $a \in (C(\Sigma)_{\text{Lip}}, (C(\Sigma_E)_{\text{Lip}}$ which yields the claim theorem 5.3.4. □

Observation 5.3.6. As noted by [40], the Lipschitz norms they consider are not associated to any Dirac operator generally, that is, the L in equation 5.14 is not realizable from a Dirac operator D, $L(a) = \|[D, a]\|$. On $C(M)$, for the Dirac operator $D := d + d^*$, for $f \in C^1(M)$ acting on $g \in C^1(M) \cap L^2(M)$, $[D, f]g = D(fg) - (fD)g = (df)g$ since $d^* f, d^* g = 0$; therefore, it's enough to check that the usual Lipschitz norm on $C(M)$ agrees with the operator norm for $\|[D, f]\|$ and, thus, associated to the Hodge-Dirac operator.

Recall that on $\mathbb{R}$ the Lipschtiz norm for differentiable function F is same as the sup norm for the differential. In $\mathbb{R}^n$ this holds for convex domains where convexity is required to be able to travel along geodesics and reduce it to one-dimensional setting. Using normal coordinates along with geodesic completeness is enough to check that this also holds on compact Riemannian manifolds.

Lemma 5.3.7. *Suppose $f \in C^1(H)$ for a compact Riemannian manifold (H, g). Then $\|f\|_{Lip} = \|df\|_\infty$*

Proof. Since injectivity radius r of the compact Riemannian manifold (H, g) is positive, so for any $x \in H$, let $\exp_p B_r(0) \subset H$ be the normal ball around x. Then the geodesic distance d_g from p to $q = \exp_p(tv)$ where $v \in T_p(H)$, with $\|v\| \leq 1$, is the tangent vector defining geodesic from p to q, $d_g(p, q) = \|tv\|$. Therefore, $f(q) - f(p)/d_g(p, q) = (f(\exp_p(tv)) - f(\exp_p(0)))/\|tv\|$. Note $F = f \circ \exp : [0, t] \to \mathbb{R}$; this gives

$$\frac{\|f(q) - f(p)\|}{\|tv\|} \leq \int_0^t \frac{\left\|\langle \nabla F, v\rangle_{sv}\right\|}{\|tv\|} ds \leq \sup_{s\in[0,t]} \|df_{\exp_p(sv)}\|$$

where it was used that since the differential of the exponential map satisfies $d_0 \exp_p(sv) = sv$ and $\|v\| \leq 1$, so

$$\sup_{s\in[0,t]} |\langle dF_{sv}, v\rangle| \leq \sup_{s\in[0,t]} |\langle df_{\exp_p(sv)} d_0 \exp_p(sv), v\rangle| \leq \sup_{s\in[0,t]} \|df_{\exp_p(sv)}\|\|sv\|$$

Now suppose q is not in the normal neighborhood of p. Then let γ be the geodesic with length $d_g(p, q)$. Pick p_i's on γ, with $p_0 = p, p_n = q$ such that p_{i+1} is in the normal neighnorhood of p_i, this is possible as injectivity radius $r > 0$ on H. Now apply the same argument to each pair of points finally note that $\sum_{i=1}^n d_g(p_{i-1}, p_i) = d_g(p, q)$, yielding $\|f(q) - f(p)\| \leq \sup_{x\in H} \|df_x\| \, d_g(p, q)$. Therefore, $\|f\|_{\text{Lip}} \leq \|df\|_\infty$. The other direction $\|df\|_\infty \leq \|f\|_{\text{Lip}}$ follows from definition since df is limit of a difference quotient.

□

Combinatorial finite and Hodge-de Rham spectral triples

Recall how commutative geometry is encoded in the noncommutative language. The Hodge-de Rham spectral triple, $\mathfrak{A}_M$, for Riemannian manifold (M, g) is the data $\mathfrak{A}_M := (C^\infty(M), \Omega^\bullet(M), d + d*)$ where $d + d*$ is the Hodge-de Rham Dirac operator, d the exterior derivative on differential forms $\Omega^\bullet(X)$, and d^* the adjoint. By Connes' spectral characterization of manifolds[25], (M, g) can be recovered from $\mathfrak{A}_M$.

Now the metric geometry of a finite set X with metric d is encoded by a finite spectral triple $\mathfrak{A}_F$ which is the data $\mathfrak{A}_F := (\mathcal{A}_F, H_F, D_F)$, where $\mathcal{A}_F$ is an unital $*$-algebra represented faithfully on a Hilbert space $H_F, \dim H_F$ finite, and D a symmetric operator on H_F subject to some additional requirements, the explicit form for D encoding the data $d(x, y)$ for $x, y \in X$. The metric d can be used to construct a simplicial complex, for example a Rips complex, or alternatively the simplicial complex could be part of the input geometric data.

To work with spectrally truncated Dirac operator, the definition of a spectral triple needs to be relaxed.

Definition 5.3.8. [63] An operator system spectral triple is a triple (E, H, D) where E is a dense subspace of an operator system $E \subset \mathcal{B}(\mathsf{H})$, on the Hilbert space H and D is a self-adjoint operator on H with compact resolvent satisfying $[D, T]$ is a bounded operator for all $T \in E$.

The operator system carries a natural order-unit structute, and so unit norm positive linear functionals, that is, states, can be considered. Now noting corollary 5.2.19, the combinatorial Dirac operator $\delta+\delta^*$ on the embedded simplicial complex approximates the Hodge-Dirac operator $d+d^*$ uniformly on the truncated space $L^2(\Omega(\Sigma))$, E_m. To encode this as a spectral triple, the algebra $C(\Sigma)$ needs to be compressed to act on E_m, however, the trunctation leads to $mC(\Sigma)m$ being only an operator system since the multiplication in $mC(\Sigma)m$ will not agree with $C(\Sigma)$. This motivates the relaxed notion of operator system spectral triples. Let $\Lambda_m : L^2(\Omega(\Sigma)) \to E_m$ be the projection, then compressed $C(\Sigma)_m = \{\Lambda_m f \Lambda_m : f \in C(\Sigma)\}$. Define the truncated Hodge-de Rham (operator system) spectral triple $\mathfrak{A}_m = (C(\Sigma)_m, E_m, m(d + d^*)m)$. Note that restricted to E_m, $m(d + d^*)m = d + d^*$.

Define the combinatorial finite spectral triples $\mathfrak{A}_{F,n} = (C^0(\Sigma_{(n)}), L^2(C(\Sigma_{(n)})), \delta + \delta^*)$ where, as earlier, $\Sigma_{(n)}$ denotes the n-fold standard subdivison of Σ. The combinatorial finite triples give a finite-dimensional encoding of the finite metric space $\Sigma^0_{(n)}$, different from finite spectral triples. The convergence to the underlying smooth structure can still be made explicit: from before $\delta + \delta^*$ uniformly approximate $d + d^*|_{E_m}$, while on identifying $f \in C^0(\Sigma_{(n)})$ with $f' \in L^\infty(M)$ (for example, by assigning to any k-cell the average of the vertices), the strong density of $C^0(\Sigma_{(n)})$ in $C(M)$ is clear.

For the Hodge-de Rham spectral triple, $\mathfrak{A}_M := (C^\infty(M), \Omega^\bullet(M), d + d^\dagger)$ we have the following.

Theorem 5.3.9. *Let $\mathcal{K}(C(M))$ be the compact operators, then the statespace $\mathcal{S}(C(M)_m)$, is dense in the state space $\mathcal{S}(\mathcal{K}(C(M)))$ in the operator norm $\|\cdot\|_{\mathcal{K}(C(M))\to\mathbb{R}}$.*

Proof. This follows after noting that the continuous dual of $\mathcal{K}(C(M))$ is isometrically isomorphic to the space $\{\mathrm{Tr}(\cdot A) : A \in C(M), \text{ trace-class}\}$, which after normalization become order-unit morphisms. If $f = [f_{ij}]$ in basis ω_i for $L^2(\Omega(\Sigma))$ then $\Lambda_m f \Lambda_m = [(f_m)_{ij}]$ where $(f_m)_{ij} = 0$ unless $i, j \in [m]$, so $C(M)_{(m)}$ are all finite rank operators, and as trace-class operators are compact, the norm-density holds. □

This is in the spirit of [40, proposition 15]; however, the norm on the statespaces is not Lipschitz. Next we adapt the construction from [40] to obtain the same result for Gromov-Hausdorff convergence of statespaces of truncations, first for bitorsors and then for any compact manifold carrying sufficiently nice groups actions.

Spectral truncation on G-spaces

Now let H be compact manifold on which a group G acts continuously and transitively from left and right. Let the left and right actions be $R_g(h') = h'g, L_g(h') = gh'$, and suppose they commute, $L_g R_{g'}(h') = g(xg') = (gx)g' = R_{g'} L_g(h')$. Fix $h \in H$ and for every $y \in H$, let g_y be such that $g_y h = y$ and $\bar{g}_y$ be such that $h\bar{g}_y = y$. Suppose that the geodesic distance d is G-invariant, $d(gh, gh') = d(h, h') = d(hg', h'g')$.

Observation 5.3.10. The identification $H \ni y \to g_y \in G$ is induced by the group action $\phi : G \times H \to H$ as the inverse of the map $\phi_h : G \times \{h\} \to H, g \to gh$. By transitivity of the group action, for any y, g_y exists, but is not necessarily unique unless H is a group. On homogeneous spaces, the assignment $y \to g_y$ can be made continuously (smoothly if the action is smooth) over any local trivialization. The same dicussion applies to $\bar{g}_y$

For simplicity, assume that for the maps $\Psi : H \to G, \Psi(y) = g_y$, $L : G \to \mathrm{Homeo}(H), g \to L_g$, the composition $L \circ \Psi : y \to L_{g_y}$, is 1-Lipschitz continuous with respect to metric $d_{\mathrm{Homeo}(H)}$ on $\mathrm{Homeo}(H)$ where $d_{\mathrm{Homeo}(H)}(\phi, \psi) = \sup_{k \in H} d(\phi(k), \psi(k))$, so $d_{\mathrm{Homeo}}(L_{g_y}, L_{g_z}) \leq d(y, z)$. The same for $\bar{g}_y$.

Let $A := C(H)$, and $A_m := mC(H)m$ where $m : \Omega(H) \to E_m$ is the projection onto E_m, the subspace spanned by first m eigenforms for $d+d^*$. The associated statespaces are $\mathcal{S}(A), \mathcal{S}(A_m)$. Notice that A_m is finite dimensional as it's a subalgebra of endomorphisms of a finite dimensional Hilbert space. Without loss of generality it can be assumed that $\mathbf{1} \in E_m$ for all m, since $\mathbf{1} \in \mathrm{Ker}(d + d^*)$ always holds, so this is just a reordering of the eigenbasis.

Now we proceed as in [40]. The map m is an order-unit morphism and induces $m^* : \mathcal{S}(A_m) \longrightarrow \mathcal{S}(A)$ by $m^*\sigma = \sigma \circ m$. Any Borel measure μ on H defines a linear functional on A, A_m by $I_\mu(f) = \int_H f d\mu$ which preserves the order structure and unit since $f \geq 0$ means $I_\mu(f) \geq 0$, $I_\mu(\mathbf{1}) = 1$. Suppose I_μ is a state on A. The pullback $m^* I_\mu$ is the linear functional given by

$$m^* I_\mu(f) = I_\mu(m(f)) = \int_H [m(f)](y) d\mu(y) = \int_H [m(f)](g_y h) d\mu(y) \tag{5.15}$$

The left action of G on A, A_m is given by $L_g(f)(h') = f(L_g(h')) = f(gh')$, there is also the right action $R_g(f)(h') = f(R_g(h')) = f(h'g)$. Notice L_g acts on $C(H)$ by $f \longrightarrow f \circ L_g$, and that m commutes with L_g. These actions allow for defining the usual Lipschitz norm on $A = C(H)$ in terms of G and any $h' \in H$

$$\begin{aligned}
\mathrm{Lip}(f) := \|f\|_{\mathrm{Lip}} &= \sup_{z,y \in H} \frac{f(y) - f(z)}{d(y,z)} \\
&= \sup_{g,g' \in G, gh' \neq g'h'} \frac{L_g(f)(h') - L_{g'}(f)(h')}{d(gh', g'h')} \\
&= \sup_{g,g' \in G, h'g \neq h'g'} \frac{R_g(f)(h') - R_{g'}(f)(h')}{d(h'g, h'g')}
\end{aligned}$$

where it was used that the actions R_g, L_g are transitive so every $(h', h'') \in H \times H$ can be reached through the action of some $g_{h'}, g_{h''}$. Note $h' \in H$ is arbitrary, so $h' = g_{h'} h$ can be used. Now the following proposition follows directly by using characterization from remark 5.3.2.

Proposition 5.3.11. *The norm $\|\cdot\|_{Lip}$ restricted to A_m is a lip norm, and $m : A \longrightarrow A_m$ is a morphism of quantum metric spaces,* $(A, L), (A_m, L)$.

Next a morphism ν such that lemma 5.3.4 can be used with ν, m is needed. This can be done by adjusting the construction from [40]. Let μ be a probability measure such that μ defines a state on $\mathcal{S}(A_m)$. Define $\nu_{m,\mu}$ by

$$\nu_{m,\mu} : A_m \longrightarrow A, \quad \nu_{m,\mu}(f)(y) := I_\mu(L_{g_y}(f)) \tag{5.16}$$

Proposition 5.3.12. *The map $\nu := \nu_{m,\mu}$ a morphism of quantum metric spaces.*

Proof. The only thing that needs to be checked is that it's contractive. Note that

$I_\mu(L_{g_y}(f)) - I_\mu(L_{g_z}(f)) = I_\mu(L_{g_y}(f) - L_{g_z}(f))$, and therefore,

$$\mathrm{Lip}(\nu(f)) = \sup_{g_z,g_y\in G, g_z h\neq g_y h} \frac{1}{d(y,z)} \int_H (f(g_y w) - f(g_z w))d\mu(w)$$

$$= \sup_{g_z,g_y\in G, g_z h\neq g_y h} \int_H \left(\frac{f(g_y h\bar{g}_w) - f(g_z h\bar{g}_w)}{d(g_y h\bar{g}_w, g_z h\bar{g}_w)} \right) d\mu(w) \leq \mathrm{Lip}(f) \quad (5.17)$$

where the G-invariance was used to get $d(g_y h\bar{g}_w, g_z h\bar{g}_w) = d(hg_y, hg_z)$. □

The following yields the analog of [40, proposition 14] in similar manner.

Proposition 5.3.13. *For all $f \in A$, $f_m \in A_m \subset A$,*

$$\sup_z |\nu \circ m(f)(z) - f(z)| \leq Lip(f) m^* I_\mu(d(g_y h, h))$$

$$\sup_z |m \circ \nu(f_m)(z) - f(z)| \leq Lip(f_m) m^* I_\mu(d(g_y h, h))$$

Proof. Consider $\|\nu \circ m(f) - f\| = \sup_{y\in H} |\nu \circ m(f)(y) - f(y)|$. First

$$(\nu \circ m(f))(z) = I_\mu(L_{g_z}(m(f))) = I_\mu m(L_{g_z}(f)) = \int_H [m(L_{g_z}(f))](y)\mu(y) = m^* I_\mu(L_{g_z}(f))$$

This means

$$|\nu \circ m(f)(z) - f(z)| = |m^* I_\mu(L_{g_z}(f)) - f(z) m^* I_\mu(\mathbf{1})| \quad (5.18)$$

$$= |m^* I_\mu(L_{g_z}(f)) - m^* I_\mu(f(z)\mathbf{1})| \quad (5.19)$$

$$= \int_H m\left([L_{g_z}(f)](y) - f(z)\right) d\mu(y)$$

$$= \int_H m\left(f(g_z g_y h) - f(g_z h)\right) d\mu(y)$$

$$\leq \mathrm{Lip}(f) \int_H m(d(g_z g_y h, g_z h)) d\mu(y)$$

and therefore, $$\sup_z |\nu \circ m(f)(z) - f(z)| \leq \mathrm{Lip}(f) m^* I_\mu(d(g_y h, h)) \quad (5.20)$$

where $f(z)$ moves inside $m^* I_\mu$ since it's a constant with $\mathbf{1} \in E_m$ for all m, and in the last line follows using the G-invariance of metric. Now for $f_m \in A_m$,

$$m \circ \nu(f_m)(z) - f_m(z) = m\left(\int_H [L_{g_z}(f_m)](g_y h) d\mu(y) - f_m(z)\mathbf{1}\right)$$

$$= m\left(\int_H (m[f_m](g_z g_y h) - m[f_m](g_z h)) d\mu(y)\right)$$

$$\leq m\left(\int_H m(\mathrm{Lip}(f_m) d(g_z g_y h, g_z h)) d\mu(y)\right)$$

$$\leq m(\mathrm{Lip}(f_m) m^* I_\mu(d(g_z g_y h, g_z h))) = m(\mathrm{Lip}(f_m) m^* I_\mu(d(g_y h, h)\mathbf{1}))$$

$$\leq \mathrm{Lip}(f_m) m^* I_\mu(d(g_y h, h))$$

since $m(f_m) = f_m$ and in the second last line G-invariance of the metric was used, while the last line used $m(\mathbf{1}) = \mathbf{1}$. □

Notice that the group action on left and right do not need to be the actions of the same group, and if the group is abelian then both left and right actions can be taken to be the same. So the analysis covers any bitorsor with compact base and invariant metric which is a triple (L, H, R), H any compact set equipped with commuting free and transitive groups L and R from left and right respectively.

The rest of the argument proceeds identically to [40]. To recap briefly, the space of probability measures μ such that $m^* I_\mu$ is a state on A_m forms a weak* dense subspace of $\mathcal{S}(A)$. On choosing the state $\delta_h, \delta_h(f) = f(h)$, for the continuous function $f(x) = d(x, h)$, by weak* convergence, there's a μ that is approximates δ_h well, that is, $\int_H d(y, h)\mu \leq \epsilon$. Now note that if μ induces a state on $\mathcal{S}(A_m)$, then it also induces a state on A_n for $n \geq m$, and therefore, ν, m provide the pair of morphisms satisfying requirements for theorem 5.3.4. This allows to conclude the following.

Theorem 5.3.14. *Over any bitorsor (L, H, R), H compact, the metric on H invariant under L and R actions, both 1-Lipschitz continuous, and endowed with continuous maps $\Psi_L : H \to L, \psi_R : H \to R$, the state spaces $\mathcal{S}(A_n)$ converge to $\mathcal{S}(A)$ in d_{GH}: for any $\epsilon > 0$, $d_{GH}(\mathcal{S}(A), \mathcal{S}(A_n)) \leq \epsilon$ for n large enough.*

Example 5.3.15. A group acting on itself from left and right is trivially a bitorsor; theorem 5.3.14 covers the case of any quotient of the flat n-torus by a closed subgroup.

However, needing commuting left and right transitive actions is restrictive. The only place where the existence of a commuting right action was used was to show that ν_m^μ was an order-unit morphism (equation 5.17). Reconsidering

$$\frac{1}{d(y,z)} \int_H (f(g_y w) - f(g_z w))\mu(w) = \int_H \frac{f(g_y w) - f(g_z w)}{d(g_y w, g_z w)} \frac{d(g_y w, g_z w)}{d(y,z)} \mu(w)$$

$$\leq \|f\|_{\text{Lip}} \int_H \frac{d(g_y w, g_z w)}{d(y,z)} \mu(w) \tag{5.21}$$

So ν being an order-unit morphism is implied by $\sup_{w \in H} d(g_y w, g_z w) \leq d(y, z)$. Denoting the map $z \to g_z$ by $\Psi : H \to \text{Homeo}(H)$, then with respect to the metric $d_{\text{Homeo}(H)}$ on $\text{Homeo}(H)$,

$$\sup_{w \in H} \frac{d(g_y w, g_z w)}{d(y,z)} = \frac{d_{\text{Homeo}(H)}(\Psi(y), \Psi(z))}{d(y,z)} \leq 1 \tag{5.22}$$

That is, Ψ is 1-Lipshitz continuous is equivalent to $\sup_{w \in H} d(g_y w, g_z w) \leq d(y, z)$. Therefore, the existence of the right action can be replaced by Lipshitz continuity of the left action.

Remark 5.3.16. The Lipschitz constant 1 is not necessary; any Lipschitz constant C works becuse if $\|\cdot\|_{\mathrm{Lip}}$ is a Lipschitz norm then so is $C\|\cdot\|_{\mathrm{Lip}}$.

Remark 5.3.17. Notice that if H is additionally a group acting on itself from left and right, with $h = e$, the identity, then 1-Lipschitz continuity of the right action also yields $d(za, zb) \leq d(a, b)$ by symmetric argument. Since in proposition 5.3.13, only the bound $d(za, zb) \leq d(a, b)$ is used, the invariance of the metric with respect to left and right actions can be dropped in this setting.

Combining this with lemma 5.3.7 along with choice of $\nu_{m,\mu}(f)(y) = I_\mu(f \circ \Psi(y))$ for $\Psi : H \to G \xrightarrow{L} \mathrm{Homeo}(H)$ where L denotes the action of G, $L : G \to \mathrm{Homeo}(H)$ (analogous to equation 5.16 where $\nu_{m,\mu}(f)(y) = I_\mu(f \circ L_{g_y})$), and using that propositions 5.3.12, 5.3.13 still hold because of the Lipschitz continuity assumption on the action, yields the following result.

Theorem 5.3.18. *Let $\mathfrak{A}_m = (C(H)_m, E_m, m(d + d^*)m)$ be the truncated Hodge-de Rham spectral triple.*

1. *If H is a compact Lie group, and the left and right actions by multiplication of H on itself are 1-Lipschtiz continuous*
2. *Or if H is a compact Riemannian manifold which carries the action L of a group G, $L : G \to Homeo(H)$, a G invariant metric, and a map $\Psi : H \to G$ such that $L \circ \Psi$ is 1-Lipschitz continuous, the G action transitive*

then

$$\lim_{m \to \infty} d_{GH}((\mathcal{S}(C(H)_m), \|\cdot\|_{d_{Lip}}), (\mathcal{S}(C(H)), \|\cdot\|_{d_{Lip}})) = 0$$

and for $f \in C^1(H)$, $\|f\|_{d_{Lip}} := \|[d + d^*, f]\|$.

Chapter 6

EPILOGUE

There are a few possible directions for the ideas developed so far.

In quantum optimal transport literature, gradient flows and noncommutative transport equation have been used to study information theoretic quantities like decay of relative entropy. A natural question is what is the information theoretic content of the infinite-dimensional heat semigroups and how does it reflect the geometry. The methods and vocabulary have been developed in the quantum information community, but the settings considered are finite dimensional and there isn't a dual differential geometric picture (although associating generalized curvature/tangent spaces to semigroups and Markov processes has been explored). Studying the heat semigroups with these tools and connecting with the underlying geometry is one possibility. Additionally, there have been attempts at defining a noncommutative Malliavin calculus; it's interesting to consider if the variational approach will yield something new.

The Dirichlet form machinery breaks in semi-Riemannian geometry because operator algebras over vector bundles with semi-Riemannian metrics are not operator algebras over Hilbert spaces. This was encountered in the $\mathrm{Cl}_{p,q}$ example. Moving between signature $(p, 1)$ and euclidean signature, one expects a Wick rotation to go from a dissipative semigroup to a unitary semigroup. Formalizing this correspondence will aid in understanding how to make sense of quantum dynamics in non-euclidean signature. A more open-ended question is what it means for a stochastic quantization of spectral action; fuzzy spectral triple and related noncommutative geometric approaches to gravity provide a testing ground. Increasingly, noncommutative probability has found utility in stochastic quantization, and a probabilistic perspective on spectral action is one possible way to add geometry to the mix.

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