Pseudo-Normed Linear Sets
over
Valued Rings

Thesis by

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SUMMARY

The main purpose of this thesis has been to investigate to what extent some of the theorems in the study of linear topological spaces are actually dependent on the real number multipliers. This has been done by replacing the real number system by a valued-ring, and, since such a ring may have a discrete topology, the results that we obtain depend only on the algebraic properties of our generalized number system.

In Chapter 2 there is developed a characterization of a topological space in a form which is convenient for the purposes of this paper. Chapters 1 and 3 are taken up with the introduction of the definition of the type of space with which we shall deal. In these chapters the author was guided by Hyers' definition of a pseudo-normed linear space[1] and Michal's definition of a topological abelian group.[2] In Chapter 4 a generalized study of linear functions is considered. Here we obtain results which reduce to known results in the case of Banach spaces. Chapter 5 deals with the concept of a differential and a relationship between this differential and an M_1-differential[2] is established. In Chapter 6 we strengthen our postulates somewhat and carry on the study of linear functions and differentials.

In some respects we have accomplished the purpose of the thesis, though the author feels that much more can be done and hopes that this thesis may prove to be a basis for further work.

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1 See [6] . 2 See [10] .

CONTENTS

BASIC DEFINITIONS

It is, of course, impossible to state here all the definitions to which we shall wish to refer; but, for the sake of being definite and as a matter of convenience to the reader, we shall list those definitions and theorems which are fundamental to this thesis and are not contained in the body of the paper. [1] and [2] are references in algebra and topology which are well suited to the needs of this paper.

The concept of a set shall be assumed to be known, and we shall consider a set S to be defined when given any object p it can be determined whether p is or is not contained in S; i.e. either $p \in S$ or $p \bar{\in} S$. We are also assuming that for each set there is defined a fundamental equivalence relation which we shall call <u>equality</u> and shall write as "=". Two elements p and q of a set S will be said to be <u>distinct</u> if $p \neq q$. All the sets that we shall consider will be assumed to contain at least two distinct elements. Our equality is also to be such that, if $p = q$, then any "statement" that we shall make about p can also be made about q. This means that though we do not require our equality to be identity, we do require that elements which are equal have certain properties in common. These certain properties are any of those which happen to arise in our postulates, since these properties are the only ones which interest us. For instance, if in I(g) below we have $x = y$, then we can state that $(-x) + y = \theta$. We also are able to prove certain theorems about our equality, such as in I(h) and I(i). The concept of a function and an operation which we shall adopt will be the same as that in [1] pp. 2-4, though it is to be remembered that uniqueness is now in terms of our equality.

I <u>Groups</u>. A set G of elements $x, y, z, \cdots$ is called a <u>group</u> with respect to an operation "+" (written additively) if the following conditions are satisfied:

a. $x + y \in G$ for every $x, y \in G$;

b. $x + (y + z) = (x + y) + z$ for all $x, y, z \in G$;

c. there exists an element $\theta \in G$ such that $x + \theta = x$ for all $x \in G$; θ is called the <u>zero element</u> of G;

d. given $x \in G$ there exists an element $-x \in G$ such that $x + (-x) = \theta$.

If, in addition,

e. $x + y = y + x$ for all $x, y \in G$,

the group is said to be <u>abelian</u>.

Then it follows from the above postulates that :

f. $\theta + x = x$ for each $x \in G$;

g. $(-x) + x = \theta$ for each $x \in G$;

h. the zero element θ is unique, i.e. if θ' is any other element of G satisfying (c), then $\theta = \theta'$;

i. the inverse $-x$ of each $x \in G$ is unique;

j. $-(-x) = x$.

II *Rings*. A set A of elements $\alpha, \beta, \gamma, \ldots$ is said to be a *ring* if A is an additive abelian group such that a second operation (written as multiplication) is defined such that:

a. $\alpha\beta \in A$ for all $\alpha, \beta \in A$;

b. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ for all $\alpha, \beta, \gamma \in A$;

c. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ for all $\alpha, \beta, \gamma \in A$.

A ring A is said to have a *unity* element if there exists an element 1 in A such that $1\alpha = \alpha 1 = \alpha$ for each $\alpha \in A$. 1 is then unique.

It can be shown that for any two elements $\alpha, \beta \in A$ that:

d. $0\alpha = \alpha 0 = 0$, where 0 is the zero element of A;

e. $(-\alpha)\beta = \alpha(-\beta) = -(\alpha\beta)$;

f. $(-\alpha)(-\beta) = \alpha\beta$.

A *division ring* (or *quasi-field*) is an additive abelian group whose non-zero elements form a multiplicative group such that II(c) is satisfied.

III *Integral domains*. An *integral domain* A is a commutative ring (i.e. $\alpha\beta = \beta\alpha$ for each $\alpha, \beta \in A$) with a unity element and no divisors of zero(i.e. $\alpha\beta = 0$ implies that $\alpha = 0$ or $\beta = 0$.

IV *Fields*. A *field* A is an additive abelian group whose non-zero elements form a multiplicative abelian group such that $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all $\alpha, \beta, \gamma \in A$.

V *Linear sets over rings*. An additive abelian group G of elements x,y,z,... is called a (left) *linear set over a ring*

A of elements $\alpha, \beta, \gamma, \ldots$ if there exists an operation αx on AG to G such that:

a. $\alpha(\beta x) = (\alpha\beta)x$;

b. $(\alpha + \beta)x = \alpha x + \beta x$;

c. $\alpha(x + y) = \alpha x + \alpha y$

for all $\alpha, \beta \in A$ and $x, y \in G$.

If A has a unity element, we shall also require that $1x = x$ for each $x \in G$.

VI <u>Strongly partially ordered sets</u>. (<u>Directed systems</u>.) A set D of elements d,e,... is said to be a <u>strongly partially ordered set</u> if there exists an order "$>$" defined between some of the elements of D such that:

a. if $e > d$, then it is not true that $d > e$;

b. (transitivity) if $e > d$ and $d > c$, then $e > c$;

c. given e and d from D, there exists a $c \in D$ such that $c \geq e$ and $c \geq d$.

A set D and an order "$>$" satisfying (a) and (b) is said to be <u>partially ordered</u>.

<u>N.B.</u> It is not required that the order "$>$" exist between each pair of elements of D. $e \not> d$ does not necessarily imply that $d \geqq e$.

<u>If E and D are two strongly partially ordered sets, then it may easily be proved that the product space</u>

$$ED = \left[(e,d);\ e \in E,\ d \in D \right]$$

<u>is also a strongly partially ordered set with</u> $(e_1,d_1) \geqslant (e_2,d_2)$

<u>if and only if</u> $e_1 \geqq e_2$ <u>and</u> $d_1 \geqq d_2$ <u>for</u> $d_1, d_2 \in D$ <u>and</u> $e_1, e_2 \in E$. $(e_1, d_1) = (e_2, d_2)$ <u>if and only if</u> $e_1 = e_2$ <u>and</u> $d_1 = d_2$.

Hence, if D is a strongly partially ordered set and R^+ is the set of all positive real numbers with ordering as usual, then

$$E = R^+D = \left[\ rd;\ r > 0,\ d \in D\right]$$

is a strongly partially ordered set with $(e_i = r_i d_i \in E)$ $e_1 \geqq e_2$ if $r_1 \geqq r_2$ and $d_1 \geqq d_2$. $e_1 = e_2$ if $r_1 = r_2$ and $d_1 = d_2$. Define $1d = d$ and $r_1(r_2 d) = (r_1 r_2)d$.

VII <u>Topological spaces</u>. A set T of elements x,y,z,... is said to be a <u>topological space</u> if:

a. To each set M of elements of T there corresponds a set $\overline{M}$ which is called the <u>closure</u> of M.

b. For every $x \in T$, $\overline{x} = x$.

c. If M and N are any two subsets of T, then $\overline{M \cup N} = \overline{M} \cup \overline{N}$.

d. $\overline{\overline{M}} = \overline{M}$ for every subset $M \subset T$.

The elements of T are called <u>points</u>.

VIII <u>Limit points</u>. A point x of a topological space T is called a <u>limit point</u> of a subset $M \subset T$ if $x \in \overline{M - x}$.

IX <u>Open and closed sets</u>. A set M of a topological space T is said to be <u>closed</u> if $\overline{M} = M$. A set $M \subset T$ is said to be <u>open</u> if the complement of M, $CM = T - M$, is closed.

X <u>Complete system of "neighborhoods"</u>. A system Σ of open sets of a topological space T is called a **complete system of "neighborhoods"** of T if every open set of T can be obtained as a sum of open sets belonging to Σ. Every open set of Σ containing a point $x \in T$ is said to be a **"neighborhood"** of x.

<u>N.B.</u> This notion of a neighborhood is stronger than we shall wish to use. See Chapter 2.

XI <u>Limit of a sequence of points</u>. Let $\{x_n\}$ be a sequence of elements of a topological space T. The sequence $\{x_n\}$ is said to have a **limit** if there exists an element $x \in T$ such that for each open set $U_x \subset T$, $x \in U_x$, there exists a positive integer $m = m(U_x)$ such that $x_n \in U_x$ for all $n > m$. We shall denote this by $\lim x_n = x$.

XII <u>Continuous functions (mappings) on a topological space T to a topological space T'</u>. A function(mapping) $f(x)$ on a topological space T to a topological space T' is said to be **continuous** on T to T' if for every set $M \subset T$

$$f(\overline{M}) \subset \overline{f(M)} .$$

XIII <u>Topological groups</u>. A group G is said to be a **topological group** if G is a topological space such that the group operations $x + y$ and $-x$ are continuous in G.

CHAPTER 1

LINEAR SETS OVER VALUED RINGS. VECTOR SPACES.

1.1 Linear sets over rings with a unity element. V-spaces.

DEFINITION 1.11. A linear set[1] T over a ring[2] A with a unity element will be called a *vector space with respect to* A, or briefly, a *V-space w. r. t.* A.

We see immediately that a V-space is a generalization of Banach's linear (vector) space[3], where the real number multipliers have been replaced by multipliers taken from an abstract ring. We shall now proceed to investigate some of the elementary properties of V-spaces.

THEOREM 1.11. *If* T *is a* V-*space w. r. t.* A, *then for all* x T *and* α A

(1.11.) $0x = \Theta$,

(1.12.) $(-\alpha)x = \alpha(-x) = -(\alpha x)$,

(1.13.) $\alpha\Theta = \Theta$.

If A *is a division ring*[4], *then*

(1.14.) $\alpha x = \Theta$ *implies* $\alpha = 0$ *or* $x = \Theta$.

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1 See V. Roman numerals refer to the sections of basic definitions.

2 See II.

3 See [3]. Numbers in brackets refer to the bibliography at the end of the paper.

4 See II.

Proof: By V(b) $\alpha x = (0 + \alpha)x = 0x + \alpha x$, and since T is a group $0x = \Theta$, and (1.11) has been proved. By (1.11) and V(b) $\Theta = 0x = (\alpha - \alpha)x = \alpha x + (-\alpha)x$, and hence $(-\alpha)x = -(\alpha x)$. It then follows from V(d) that $(-1)x = -(1x) = -x$, and from V(a) and II(e) it follows that $\alpha(-x) = \alpha(-1x) = (-\alpha)x$. Therefore (1.12) has been proved. (1.13) follows from V(c) and (1.12) in the following manner: $\alpha\Theta = \alpha(x - x) = \alpha x + \alpha(-x) = \alpha x - (\alpha x) = \Theta$. In order to prove (1.14) let us assume that A is a division ring and that $\alpha x = \Theta$, $\alpha \neq 0$. Then there exists an $\alpha^{-1} \in A$ such that $\alpha^{-1}(\alpha x) = (\alpha^{-1}\alpha)x = x = \Theta$. Hence, if we assume that $\alpha x = \Theta$, $\alpha \neq 0$, and $x \neq \Theta$, we obtain a contradiction, and (1.4) is proved.

DEFINITION 1.12. For $\alpha \in A$ and $S, S_1, S_2, \ldots$ subsets of T, αS, $-S$, and $S_1 + S_2$ are defined respectively by

$$\alpha S = [\alpha x;\ x \in S],$$

$$-S = -1S,$$

and

$$S_1 + S_2 = [x + y;\ x \in S_1, y \in S_2].$$

THEOREM 1.12. <u>For all</u> $x, y \in T$; <u>all</u> $\alpha, \beta \in A$; <u>and all</u> $S, S_1, S_2 \subset T$:

(1.15.) $S + \Theta = S$;

(1.16.) $S_1 + S_2 = S_2 + S_1$;

(1.17.) $S + (S_1 + S_2) = (S + S_1) + S_2$;

(1.18.) $1S = S$, $(-\alpha)S = (\alpha)(-S) = -(\alpha S)$;

(1.19.) $\alpha(S_1 + S_2) = \alpha S_1 + \alpha S_2$;

(1.110.) $\alpha(\beta S) = (\alpha\beta)S$;

(1.111.) $(\alpha + \beta)S = \alpha S + \beta S$;

(1.112.) <u>If</u> $S_1 \subset S_2$, <u>then</u> $\alpha S_1 \subset \alpha S_2$;

(1.113.) <u>If</u> $x - y \in S$, <u>then</u> $x \in y + S$, <u>and</u> , <u>conversely</u>, <u>if</u> $x \in y + S$, <u>then</u> $x - y \in S$.

<u>Proof</u>: (1.15)-(1.17) and (1.13) follow directly from definition 1.12 and the fact that T is an additive abelian group.

(1.18) - (1.112) follow from the fact that T is a V-space and definition 1.12.

DEFINITION 1.13. Define for each $S \subset T$, α , $\beta \in A$, $\beta \neq 0$, $\alpha/\beta\ S = [\, x \,;\, \beta x \in \alpha S \,]$.

<u>N.B.</u> It is clear that if A is a division ring, then $\alpha/\beta\ S = \alpha\beta^{-1}S$.

THEOREM 1.13. For all α , β , γ, $\alpha_1 \in A$; $\beta \neq 0$, $\gamma \neq 0$; and $S \subset T$:

(1.114.) $\alpha/1\ S = \alpha S$;

(1.115.) $\alpha_1/\beta\ (\alpha S) = \alpha_1\alpha/\beta\ S$;

(1.116.) $\alpha/\beta\ S \subset (\gamma\alpha)/(\gamma\beta)\ S$;

(1.117.) $\beta(\alpha/\beta\ S) \subset \alpha S$

(1.118.) <u>if</u> $S \subset \alpha/\beta\ S$, <u>then</u> $\beta S \subset \alpha S$, <u>and</u> <u>conversely</u> $\beta S \subset \alpha S$ <u>implies</u> <u>that</u> $S \subset \alpha/\beta\ S$.

<u>If</u> $\alpha x = \Theta$ <u>implies</u> <u>that</u> $\alpha = 0$ or $x = \Theta$, <u>then</u>

(1.119.) $(\gamma\alpha)/(\gamma\beta)\ S = \alpha/\beta\ S$.

<u>Proof</u>: (1.114) is clearly true.

(1.115): Let $x \in \alpha_1/\beta\ (\alpha S)$; i.e. $x = \alpha_1 y$, where $y = \alpha z$, $z \in S$. Then $\beta x = \alpha_1(\alpha z) = (\alpha_1\alpha)z$, and hence $\alpha_1/\beta\ (\alpha S) \subset (\alpha_1\alpha)/\beta\ S$. Conversely, if $x \in (\alpha_1\alpha)/\beta\ S$, then $\beta x = (\alpha_1\alpha)z$, where $z \in S$, and $\beta x = \alpha_1(\alpha z)$. Then $(\alpha_1\alpha)/\beta\ S \subset \alpha_1/\beta\ (\alpha S)$, and hence $(\alpha_1\alpha)/\beta\ S = \alpha_1/\beta\ (\alpha S)$.

(1.116): If $x \in \alpha/\beta\ S$, then $\beta x \in \alpha S$, and by (1.112) of theorem 1.12 $\gamma\beta x \in \gamma\alpha S$. Hence $x \in (\gamma\alpha)/(\gamma\beta)\ S$, and $\alpha/\beta\ S \subset (\gamma\alpha)/(\gamma\beta)\ S$.

(1.117): If $x \in \alpha/\beta\ S$, then $\beta x \in \alpha S$, and hence $\beta(\alpha/\beta\ S) \subset \alpha S$.

(1.118): This follows almost by definition.

(1.119): If $x \in (\gamma\alpha)/(\gamma\beta)\ S$, then $\gamma\beta x = \gamma\alpha z$, $z \in S$. Hence $\gamma(\beta x - \alpha z) = \Theta$. Since $\gamma \neq 0$, we have by hypothesis that $\beta x = \alpha z$, or $(\gamma\alpha)/(\gamma\beta)\ S \subset \alpha/\beta\ S$. By (1.116) and the above $\alpha/\beta\ S = (\gamma\alpha)/(\gamma\beta)\ S$. This completes the proof of theorem 1.13.

1.2. <u>Valued rings</u>. The next step in our generalization is to introduce the "number system" which is to replace the real number multipliers in the study of linear topological spaces. Such a "number system" will be called a valued ring and is defined as follows.

DEFINITION 1.21. If A is a ring with a unity element such that there is defined on A a real-valued function $/\alpha/$ such that:

(1.21.) $/\alpha/ \geqq 0$ for all $\alpha \in A$;

(1.22.) $/\alpha\beta/ \leqq /\alpha/ \, /\beta/$ for all $\alpha, \beta \in A$;

(1.23.) $/\alpha + \beta/ \leqq /\alpha/ + /\beta/$ for all $\alpha, \beta \in A$;

(1.24.) $/-1/ = 1$;

(1.25.) $/\alpha/ > 1$ for some $\alpha \in A$, $\alpha \neq 0$

then A is said to be a **valued ring.**[1] $/\alpha/$ is called the **valuation** of α .

N.B. _Throughout the remainder of the paper_ A _shall be used to denote a valued ring with elements_ $\alpha, \beta, \gamma \ldots$ _and with_ $/\alpha/$ _the valuation function. Later in the paper we shall put additional restrictions on_ A .

THEOREM 1.21. _If_ A _is a valued ring with_ $/\alpha/$ _the valuation function, then_

(1.26.) $/1/ = 1$,

(1.27.) $/-\alpha/ = /\alpha/$ _for all_ $\alpha \in A$,

(1.28.) $/0/ \neq 0$ _implies that_ $/\alpha/ \geqq 1$ _for all_ $\alpha \in A$.

Proof: $/\alpha/ = /1\alpha/ \leqq /1/ \, /\alpha/$ for all $\alpha \in A$, and hence $1 \leqq /1/$. Also $/1/ = /(-1)(-1)/ \leqq /-1/ \, /-1/ = 1$. Therefore $/1/ = 1$. $/-\alpha/ \leqq /-1/ \, /\alpha/ = /\alpha/$. Similarly, $/\alpha/ \leqq /-1/ \, /-\alpha/ = /-\alpha/$, and hence $/\alpha/ = /-\alpha/$. $/0/ = /0\alpha/ \leqq /0/ \, /\alpha/$ for all $\alpha \in A$, and if $/0/ \neq 0$, then $1 \leqq /\alpha/$ for all $\alpha \in A$.

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[1] It is always possible, without any loss of generality, to define $/0/ = 0$, and throughout this paper we shall assume that $/0/$ has been so defined.

Examples of valued rings.

(1) Valued fields. If A is a field which is a valued ring such that $/\alpha\beta/ = /\alpha/\,/\beta/$ then A is said to be a **valued field**. The concept of a valued field was introduced by Kürschák[1], and the conditions under which the derived field of such a field be isomorphic with either the field of complex numbers or the field of real numbers are well known. Examples of such fields, and hence examples of valued rings are the field of real numbers, the field of complex numbers, and p-adic number fields.

(2) An example of a division ring that is a valued ring is the field of quaternions[2], though here again we have $/\alpha\beta/ = /\alpha/\,/\beta/$.

(3) One of the more interesting instances of valued rings are the "linear normed rings" considered by Michal and Martin in [12] . There is given in that paper an example of a linear normed ring of infinite dimension for which $/\alpha\beta/ \neq /\alpha/\,/\beta/$ for some elements.

(4) An example of an integral domain that is also a valued ring is, of course, the integral domain of all integers, the valuation function being the absolute value.

(5) Let us now construct an example of a valued ring which is such that $/\alpha\beta/ \neq /\alpha/\,/\beta/$ for all $\alpha, \beta \in A$.

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1 See [7] , and also [1] Chapters XI and XII

2 See [14], p. 172.

Let Q with elements p , q, s, ... be a ring with a unity element, and let

$$\alpha = (p_o, p_1, \ldots ,p_n, \ldots)$$

be a sequence of elements of Q . Let A be the set of all sequences of elements of Q which contain only a finite number of non-zero elements of Q . I.e., if $\alpha \in A$, $\beta \in A$ then

$$\alpha = (p_o, p_1, \ldots ,p_n, 0, 0, \ldots)$$

and

$$\beta = (q_o, q_1, \ldots ,q_n, 0, 0, \ldots) .$$

Define $\alpha + \beta = (p_o + q_o, p_1 + q_1, \ldots)$

and

$$\alpha\beta = (s_o, s_1, \ldots , s_{n+m}, 0, 0, \ldots)$$

where $s_i = \sum_{j+k=i} (p_j q_k)$. $\alpha = \beta$ if and only if $p_k = q_k$ for $k = 0, 1, 2, \ldots$.

Then it may be easily verified that A is a ring with a unity element where

$$0 = (0, 0, \ldots)$$

and

$$1 = (1, 0, 0, \ldots).$$

$$-\alpha = (-p_o, -p_1, \ldots) .$$

If $\alpha \in A$, $\alpha \neq 0$, then a last non-zero element p_n of the sequence exists. Define $/\alpha/ = n + 1$, and $/0/ = 0$. Clearly $/\alpha/ = 0$ for all $\alpha \in A$, $/-1/ = 1$, $/\alpha + \beta/ = \max(/\alpha/, /\beta/)$, and there exist $\alpha \in A$ such that $/\alpha/ > 1$.

If $/\alpha/ = n + 1 > 0$, $/\beta/ = m + 1 > 0$, then $/\alpha\beta/ = n + m + 1$. Hence $/\alpha\beta/ \leq /\alpha/ \, /\beta/$, and $/\alpha\beta/ = /\alpha/ \, /\beta/$ implies either $/\alpha/ = /\beta/ = 1$, or $/\alpha/ = 0$ or $/\beta/ = 0$.

Hence we see that the set A of all formal polynomials[1] with coefficients in a ring Q with a unity element is a valued ring, with the valuation as defined above.

(6) It is also possible to value the field R of real numbers in such a manner that $/r/$ is zero or an integer, and also such that there exist $r_1, r_2 \in R$ with $/r_1 r_2/ < /r_1/ \, /r_2/$. Moreover, there will exist no real number $\rho > 0$ such that $/r/ = \rho|r|$ for all $r \in R$. $|r|$ is the absolute value of r .

Given $r \in R$ there exists an integer m such that $m < r \leq m + 1$. Define $/r/ = m + 1$. Hence $/r/ \geq 0$, $/-1/ = 1$, $/r_1 + r_2/ \leq /r_1/ + /r_2/$, $/r_1 r_2/ \leq /r_1/ \, /r_2/$, and there exist $r \in R$ with the property that $/r/ > 1$. $/(1/2)(3/2)/ = 1$, whereas $/(1/2)/ \, /(3/2)/ = 2$, and hence R is valued as stated above.

(7) It should be noted that we have not assumed that the ring A shall have no divisors of zero, and a simple example of a valued ring with divisors of zero is the ring of integers mod 4 . The ring of integers mod 2 is, however, not a valued ring due to postulates (1.24) and (1.25).

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[1] See [1] pp. 17, 18.

These examples of valued rings should indicate the extreme generality of our number system A , and examples (4) and (7) show, in particular, that A is not necessarily a topological space with a non-discrete topology generated by the valuation.

1.3 V_1-<u>spaces</u>.

DEFINITION 1.31. A linear set T over a valued ring A will be said to be a <u>vector space of type one with respect to</u> A or briefly a V_1-<u>space w.r.t.</u> A .

A V_1-space w.r.t. A is then a special case of a V-space, and the set operations defined for V-spaces in definition 1.12 will be applied also to V_1-spaces and will, of course, have the same properties as before.

<u>N.B.</u> It is evident that every additive abelian group is a V_1-space w.r.t. the integral domain of all integers, where the valuation is the absolute value.

CHAPTER 2

NEIGHBORHOOD TOPOLOGIES. N-SPACES.

From now on the spaces that will be considered in this thesis will be topological spaces.[1] We shall not begin by postulating that these spaces are topological, and the aim of this chapter is to develop a new characterization of topological spaces by means of which we can more easily verify that our spaces are topological. This new characterization will also enable us later on to express certain of the topological concepts in a form which will prove to be convenient in what is to follow. The equivalent definition of a topological space that we shall give will be stated in terms of the properties of a special system of subsets of a set, these subsets will be called neighborhoods, though we shall not require that these neighborhoods be open sets, as is done by Fréchet,[2] Hausdorff,[2] and Pontrjagin.[3] However, our neighborhood systems will be equivalent to a complete neighborhood system[4] of the topological space. Then in chapter 3 we shall obtain from this characterization a second one based upon a generalization of Hyers' pseudo-norm.[5] In this manner we show that a generalization of the idea of a norm can be made a basis

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1 See VII.
2 See [15], pp. 33-39.
3 See [14] p. 30, theorem 3.
4 See X.
5 See [6], [8], or [9].

for the study of topological spaces.

2.1. *N-spaces.*

DEFINITION 2.11. Let T be a set of elements x, y, z, ..., and let Σ be a system of subsets of T, the subsets shall be called *neighborhoods*, such that:

(2.11.) There is associated with each element $x \in T$ a non-null subset Σ_x of subsets U_x, called *neighborhoods of* x, of Σ such that for each $U_x \in \Sigma_x$ $x \in U_x$.

(2.12.) If $y \neq x$, there exists a $U_x \in \Sigma_x$ such that $y \bar{\in} U_x$.

(2.13.) If $U_x \in \Sigma_x$ and $V_x \in \Sigma_x$, there exists a $W_x \in \Sigma_x$ such that $W_x \subset U_x \cap V_x$.

(2.14.) Given $x \in T$ and $U_x \in \Sigma_x$, there exists a $V_x \in \Sigma_x$ such that if $y \in V_x$ there exists a $W_y \in \Sigma_y$ such that $W_y \subset U_x$. (Note that this implies that $V_x \subset U_x$.)

The set T will then be called a *space with a neighborhood system* or simply an *N-space w.r.t.* Σ.

Postulates (2.11) - (2.13) are those of Fréchet, whereas, postulate (2.14) is weaker than Fréchet's fourth postulate. Fréchet requires that, if $y \in U_x$, then there exists a $U_y \in \Sigma_y$ such that $U_y \subset U_x$. Pontrajagin's conditions are even stronger, as he considers a system Σ such that if $U \in \Sigma$ and $x \in U$, then U is a neighborhood of x. Hence (2.14) is automatically satisfied.

An example of such a neighborhood system Σ in which the neighborhoods are not necessarily open sets would evidently be furnished by taking as neighborhoods in a Banach space[1] the closed "spheres" generated by the norm.

THEOREM 2.11. <u>If</u> T <u>is a group</u> (written additively) <u>such that there exists a system of subsets</u> $\mathcal{U} = \{U\}$, $U \subset T$, <u>satisfying the following postulates:</u>[2]

(2.15.) <u>the intersection of all</u> $U \in \mathcal{U}$ <u>is</u> Θ;

(2.16.) <u>given</u> $U \in \mathcal{U}$ <u>and</u> $V \in \mathcal{U}$ <u>there exists</u> <u>a</u> $W \in \mathcal{U}$ <u>such that</u> $W \subset U \cap V$;

(2.17.) <u>given</u> $U \in \mathcal{U}$ <u>there exists a</u> $V \in \mathcal{U}$ <u>such that if</u> $y \in V$ <u>there exists a</u> $W \in \mathcal{U}$ <u>such that</u> $y + W \subset U$;

<u>then</u> T <u>is an</u> N-<u>space with</u> $\Sigma_x = [x + U ; U \in \mathcal{U}]$, <u>and</u> $\Sigma = [\Sigma_x ; x \in T]$.

Proof: In order to prove this we shall show that Σ as defined above satisfies (2.11) - (2.14).

(2.11): Since for every $U \in \mathcal{U}$, $\Theta \in U$, we have that $x \in x + U$ for each $U \in \mathcal{U}$.

(2.12): If $x \neq y$, then by (2.15) there exists a $U \in \mathcal{U}$ such that $-x + y \bar{\in} U$. Then $y \bar{\in} x + U$ and (2.12) is satisfied by Σ.

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[1] See [3], p. 53.

[2] We shall call such a group an <u>N-group</u> <u>w.r.t</u> $\mathcal{U}$.

(2.13): This follows immediately from (2.16).

(2.14): Given $U \in \mathcal{U}$ and $x \in T$, we have by (2.17) that there exists a $V \in \mathcal{U}$ such that if $-x + y \in V$, i.e., $y \in x + V$, there exists a $W \in \mathcal{U}$ such that $-x + y + W \subset U$ or $y + W \subset x + U$. Hence (2.14) is satisfied by Σ, and T is an N-space.

2.2. Equivalence of N-spaces and topological spaces.

DEFINITION 2.21. If S is any subset of an N-space T w.r.t Σ, the <u>closure</u> of S, denoted by $\overline{S}$, is defined as follows:

$x \in \overline{S}$ if and only if for each $U_x \in \Sigma_x$, $S \cap U_x \neq 0^\circ$.[1]

LEMMA 2.21. <u>Every N-space T w.r.t. Σ is a topological space with closure defined as in definition</u> 2.21.

Proof: Our proof consists in showing that this operation of closure satisfies VII(a) - VII(d).

(VII(a)): Follows by the definition of $\overline{S}$.

(VII(b)): Follows from (2.12) of definition 2.11.

(VII(c)): Let S_1 and S_2 be any two subsets of T, and assume that $x \in \overline{S}_1 \cup \overline{S}_2$. Then, for each $U_x \in \Sigma_x$, U_x intersects $S_1 \cup S_2$, since each U_x intersects S_1 or S_2. Hence $x \in \overline{S_1 \cup S_2}$ and $\overline{S}_1 \cup \overline{S}_2 \subset \overline{S_1 \cup S_2}$.

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[1] Throughout this paper "0°" will denote the null set.

To prove the converse assume that $x \in \overline{S_1 \cup S_2}$. Then, given $U_x \in \Sigma_x$, $U_x \cap (S_1 \cup S_2) \neq 0°$; i.e., either $U_x \cap S_1 \neq 0°$ or $U_x \cap S_2 \neq 0°$, or both $U_x \cap S_1 \neq 0°$ and $U_x \cap S_2 \neq 0°$. Now assume that there exists a subset $\{U_x\}$ of subsets contained in Σ_x such that for every $U_x \in \{U_x\}$ $U_x \cap S_1 = 0°$ and $U_x \cap S_2 \neq 0°$. Similarly assume the existence of a subset $\{V_x\} \subset \Sigma_x$ such that for every $V_x \in \{V_x\}$ $V_x \cap S_1 \neq 0°$ and $V_x \cap S_2 = 0°$. Given $U_x \in \{U_x\}$ and $V_x \in \{V_x\}$, we have by (2.13) of definition 2.11 that there exists a $W_x \in \Sigma_x$ such that $W_x \subset U_x \cap V_x$. Now W_x either intersects S_1 or S_2 , or intersects both S_1 and S_2 . This contradicts our assumption of the existence of $\{U_x\}$ and $\{V_x\}$, since W_x is contained in both U_x and V_x . Hence every $U_x \in \Sigma_x$ either intersects S_1 or intersects S_2 , or intersects both S_1 and S_2 . Then $x \in \overline{S}_1 \cup \overline{S}_2$, and therefore $\overline{S_1 \cup S_2} \subset \overline{S}_1 \cup \overline{S}_2$. Since we also have shown that $\overline{S}_1 \cup \overline{S}_2 \subset \overline{S_1 \cup S_2}$, $\overline{S_1 \cup S_2} = \overline{S}_1 \cup \overline{S}_2$, and VII(c) is satisfied by our closure.

(VII(d)): From our definition of closure and (2.11) of definition 2.11, we have that $S \subset \overline{S}$ for each $S \subset T$. Hence $S \subset \overline{S} \subset \overline{\overline{S}}$. If $x \in \overline{\overline{S}}$, then every $U_x \in \Sigma_x$ intersects $\overline{S}$. Now by definition 2.11, postulate (2.14), there exists, corresponding to each $U_x \in \Sigma_x$, a $V_x \in \Sigma_x$ such that, if $y \in V_x$, there exists a $W_y \in \Sigma_y$ such that $W_y \subset U_x$. Since we are assuming that $x \in \overline{\overline{S}}$, V_x also intersects

$\bar{S}$, and, for $y \in V_x \cap \bar{S}$, W_y intersects S and $W_y \subset U_x$. Then U_x intersects S and $x \in \bar{S}$. Hence $\bar{\bar{S}} \subset \bar{S}$ and $\bar{\bar{S}} = \bar{S}$. This completes the proof of the theorem.

N.B. Throughout the remainder of the paper when we speak of topological concepts in an N-space T , we shall mean that T is a topological space, closure being that of definition 2.21.

THEOREM 2.21. A necessary and sufficient condition that a subset S of an N-space T w.r.t. Σ be open is that for each $x \in S$ there exists a $U_x \in \Sigma_x$ such that $U_x \subset S$.

Proof: Sufficiency: If $x \in S$, we have by our sufficiency hypothesis that there exist a $U_x \in \Sigma_x$ such that $U_x \subset S$, and hence by definition 2.21 $x \bar{\in} \overline{CS}$, which is equivalent to saying that $S \cap \overline{CS} = 0°$. Since $CS \subset \overline{CS}$, it then follows that $\overline{CS} = CS$. Therefore by definition CS is closed and S is open.[1]

Necessity: Let S be an open set, i.e., CS is closed. Then, if $x \in S$, $x \bar{\in} \overline{CS} = CS$. Suppose that there does not exist a $U_x \in \Sigma_x$ such that $U_x \subset S$. Then $U_x \cap CS \neq 0°$ for each $U_x \in \Sigma_x$, and by definition 2.21 $x \in \overline{CS}$. This is a contradiction, and the necessity is proved.

DEFINITION 2.22 If T is an N-space w.r.t. Σ and also w.r.t. Σ'[2], then the two neighborhood systems Σ and Σ'

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[1] See IX.

[2] More generally, this definition shall apply if Σ and Σ' satisfy (2.11).

will be said to be <u>equivalent</u>, if for each $x \in T \cdot \Sigma_x$ and Σ_x' have the property that given $U_x \in \Sigma_x$ there correspond $U_x' \in \Sigma_x'$ such that $U_x' \subset U_x$, and conversely.

<u>N.B.</u> It may easily be verified that, if T is an N-space w.r.t Σ and also w.r.t. Σ', then the equivalence of Σ and Σ' implies that Σ and Σ' give rise to the same operation of closure.

THEOREM 2.22. <u>If T is an N-space w.r.t. Σ, then Σ is equivalent to a complete system[1] Σ' of neighborhoods of T.</u>

<u>Proof:</u> It follows from (2.14) that given $U_x \in \Sigma_x$ there exist $V_x \in \Sigma_x$ such that, if $y \in V_x$, there exist $W_y \in \Sigma_y$ such that $W_y \subset U_x$. Then, if $y \in V_x$, we see that $y \in \overline{CU_x}$ Hence $y \in C(\overline{CU_x})$, and $V_x \subset C(\overline{CU_x})$, which is an open set. Since $CU_x \subset \overline{CU_x}$, then $C(\overline{CU_x}) \subset U_x$. Therefore, given $U_x \in \Sigma_x$, there corresponds an open set containing x and contained in U_x. The converse follows immediately by theorem 2.21. The set of all open sets is a complete neighborhood system of T, an open set S being a neighborhood of a point x if $x \in S$, and the theorem is proved.

COROLLARY 2.21. <u>If T is an N-space w.r.t. Σ and at the same time an N-space w.r.t. Σ', then a necessary and sufficient that Σ and Σ' give rise to the same</u>

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[1] See X. It is to be remembered that in this case the sets $U \in \Sigma'$ are open and that if $x \in U$ then U is said to be a neighborhood of x; i.e. $\Sigma_x' = [U ; U \in \Sigma', x \in U]$.

operation of closure is that Σ and Σ' be equivalent.

Proof: The sufficiency has been mentioned before, and the necessity follows from theorem 2.22 upon noting that the equivalence of neighborhood systems is an "equivalence relation"[1] and that complete neighborhood systems are equivalent.

As is shown in [14], p. 29, section D, every complete system Σ of neighborhoods of a topological space satisfies (2.12) and (2.13). An open set U is now to be considered a neighborhood of a point x if $x \in U$, and therefore (2.11) and (2.14) are also satisfied by Σ. T is then an N-space w.r.t. Σ, and we may summarize our results in the following theorem.

THEOREM 2.23. Every N-space T w.r.t. Σ is a topological space with closure defined as in definition 2.12, and Σ is equivalent to the set of all open sets in T. Conversely, every topological space T is an N-space w.r.t. any system Σ equivalent to a complete neighborhood system of T.

N.B. It might be pointed out here that if T is an N-space w.r.t. Σ and if each $U \in \Sigma$ is an open set, then Σ is a complete system of neighborhoods of T. This follows immediately from theorem 3, p. 30, in [14].

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1 See [1], p.5. Here we make use of the transitivity of the equivalence relation.

THEOREM 2.24. A necessary and sufficient condition that a function $f(x)$ on an N-space T w.r.t. Σ to any N-space T' w.r.t. Σ' be continuous[1] is that for each $x \in T$ and $U'_{f(x)} \in \Sigma'_{f(x)}$ there correspond $U_x \in \Sigma_x$ such that $f(U_x) \subset U'_{f(x)}$.

Proof: This theorem follows immediately from theorem 2.23 above and theorem 4 of , p. 34.

DEFINITION 2.23. A function $f(x)$ on an open set S_x , $x \in S_x$, of an N-space T w.r.t. Σ to any N-space T' w.r.t. Σ' will be said to be **continuous at a point** $y \in S_x$ if given $U'_{f(y)} \in \Sigma'_{f(y)}$ there exist $U_y \in \Sigma_y$, $U_y \subset S_x$, such that $f(U_y) \subset U'_{f(y)}$.

N.B. It then follows from theorem 2.24 that, if $f(x)$ is a function on an N-space T to an N-space T' , then $f(x)$ is continuous on T to T' if and only if $f(x)$ is continuous at each point $x \in T$.

We saw in theorem 2.11 that every N-group is an N-space, and now we wish to consider the case in which a group T is an N-space w.r.t. Σ and at the same time an N-group w.r.t. Σ_Θ . The question then arises as to whether the two neighborhood systems, Σ and $\Sigma' = \left[x + U_\Theta ;\ x \in T , U_\Theta \in \Sigma_\Theta\right]$, are equivalent or, what

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[1] See XII.

is the same thing, whether the operations of closure that are defined in terms of the two neighborhood systems are the same. By theorem 2.11 and definition 2.23 we may state the following corollary.

COROLLARY 2.21. *If a group* T *is an N-space w.r.t.* Σ *and if* T *is also an N-group w.r.t.* Σ_Θ *, then a necessary and sufficient condition that* Σ *and* $\Sigma' \equiv \left[x + U_\Theta \, ; \; x \in T \, , U_\Theta \in \Sigma_\Theta \right]$, *where* $\Sigma'_x = \left[x + U_\Theta \, ; \; U_\Theta \in \Sigma_\Theta \right]$, *be equivalent is that:*

given $x \in T$, $x + y$ *is continuous in* y *at* $y = \Theta$ *and* $-x + y$ *is continuous in* y at $y = x$.

If T is a topological group, the conditions of corollary 2.21 are satisfied, and hence in this case we are justified in always taking our neighborhood system to be generated by the neighborhoods of the origin in the manner above.

THEOREM 2.25. *A necessary and sufficient condition that* $\lim x_n = x$ [1], *where* $\{x_n\}$ *is a sequence of elements of an N-space* T *w.r.t.* Σ , *is that given* $U_x \in \Sigma_x$ *there exists an integer* $m = m(U_x)$ *such that* $n > m$ *implies that* $x_n \in U_x$.

Proof: The theorem is an immediate consequence of theorem 2.23 .

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1 See XI.

2.3. N_1-spaces.

We shall now introduce our generalization of a linear topological space. As in the development of the study of linear topological spaces,[1] we shall first define the space in terms of a neighborhood system and later consider an equivalent characterization in terms of the existence of a special real-valued function, which we shall call a pseudo-norm.

DEFINITION 2.31. Let T be a V_1-space[2] with respect to A such that there exists a system $\mathcal{U}$ of subsets $U \subset T$ satisfying the following postulates:

(2.31.) the intersection of all $U \in \mathcal{U}$ is Θ ;

(2.32.) given $U \in \mathcal{U}$ and $V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that $W \subset U \cap V$;

(2.33.) given $\alpha \in A$ and $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $\rho V \subset U$ for all $/\rho/ \leqq /\alpha/$;

(2.34.) given $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V + V \subset U$;

(2.35.) given $x \in T$ and $U \in \mathcal{U}$ there exists an $\alpha \in A$ such that $x \in \alpha U$.

We shall call such a space T an <u>N-space of type</u> 1 <u>with respect to</u> A, U or simply an <u>N_1-space w.r.t.</u> A, $\mathcal{U}$.

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1 See [6] .

2 See definition 1.31.

THEOREM 2.31. *Every N_1-space T w.r.t A, $\mathfrak{U}$ is an N-group w.r.t. $\mathfrak{U}$ and is, in particular, a topological abelian group.*[1]

Proof: It follows from (2.31), (2.32), and (2.34) that T is an N-group w.r.t. $\mathcal{U}$. By theorem 2.11 and theorem 2.23 T is a topological space. T is by definition an abelian group. By theorem 2.11 and theorem 2.24 we have that (2.33) and (2.34) imply the continuity of the operations $x + y$ and $-x$. In using (2.33) in the above we take $\alpha = \rho = -1$. Hence T is a topological abelian group, and the theorem is proved.

A further study of this type of space will be carried out in the next chapters when we consider the equivalent characterization of N_1-spaces.

CHAPTER 3

PSEUDO-NORMED SPACES. P-SPACES.

Hyers[1] has shown that the notion of a pseudo-norm, i. e., a non-negative real-valued function defined for each element of a space and each element of a strongly partially ordered set,[2] is sufficient to characterize linear topological spaces. In this chapter we wish to go further and show that a generalization of the form of Hyers' pseudo-norm is fundamental to the study of topological spaces. The pseudo-norm in some cases may seem to be trivial, since it may be only another notation for set inclusion, though the value of our characterization lies in the fact that it takes as the basic concept in topology the extension of a notion which is a very familiar one, namely, that of a norm. This new postulational basis for topological spaces is also in a form which is very convenient for the purposes of this paper, and some of the topological properties imply interesting conditions on the pseudo-norm, as has already been demonstrated in the case of linear topological spaces.[3]

3.1. <u>P-spaces</u>. <u>Equivalence of N-spaces and P-spaces</u>.

DEFINITION 3.11. Let T and D be sets with elements

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1 See [6] .

2 See VI.

3 See for instance [6] , [8] , or [9] .

$x, y, z, \ldots$ and $d, e, c, \ldots$ respectively. Moreover, D will be assumed to be partially ordered.[1] The set T will be said to be a **pseudo-normed space with respect to** D, or briefly a **P-space w.r.t.** D, if there exists a non-negative real-valued function $\|x, d\|$ on TD such that :

(3.11.) there correspond to each $x \in T$ a non-null subset D_x of elements of D such that D_x is a strongly partially ordered set[1] with the same ordering as D, and $\|x, d_x\| \leqq 1$ for each $d_x \in D_x$;

(3.12.) $\|y, d_x\| \leqq 1$ for all $d_x \in D_x$ implies that $y = x$;

(3.13.) given $x \in T$ and $d_x \in D_x$ there exists an $e_x \in D_x$ such that for each y with the property that $\|y, e_x\| \leqq 1$ there corresponds a $d_y \in D_y$ such that $d_y \geqq d_x$;

(3.14.) $d \leqq e$ implies that $\|x, e\| \leqq \|x, d\|$ for all $x \in T$.

$\|x, d\|$ will be called the **pseudo-norm of** x **with respect to** d .

DEFINITION 3.12. If T is a P-space w.r.t. D, we shall define

$$U(d) = \left[x;\ x \in T,\ \|x, d\| \leqq 1 \right],$$

$$\Sigma = \left[U(d)\ ;\ d \in D \right],$$

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[1] See VI.

and

$$\Sigma_x = \left[U(d_x) \ ; \ d_x \in D_x \right] .$$

Σ will be called the <u>neighborhood system generated by the pseudo-norm</u>. If T is also a P-space w.r.t. D' , then the pseudo-norms $//x, d//$ and $//x,d'//'$ will be said to be <u>equivalent</u> if they generate equivalent[1] neighborhood systems.

<u>N.B.</u> A sufficient condition for the equivalence of two pseudo-norms is that given $d_x \in D_x$ there correspond $d'_x \in D'_x$ such that $//y,d_x// \leqq //y,d'_x//'$ for all $y \in T$, and conversely. Later we shall show that in a special case this condition is both necessary and sufficient.

THEOREM 3.11. <u>Every P-space</u> T <u>w.r.t.</u> D <u>is an N-space w.r.t.</u> Σ , <u>where</u> Σ <u>is the neighborhood system generated by the pseudo-norm. Conversely, every N-space w.r.t.</u> Σ <u>is a P-space w.r.t.</u> Σ .

<u>Proof</u>: Let T be a P-space w.r.t. D , and let Σ and Σ_x be as in definition 3.12 . Then (2.11) and (2.12) of definition 2.11 follow respectively from (3.11) and (3.12) of definition 3.11. (3.11), (3.14) and VI(c) imply (2.13). (2.14) is a direct consequence of (3.13) and (3.14). Hence the first part of the theorem has been proved. Now let T be an N-space w.r.t. Σ , and we shall show that

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[1] See definition 2.22.

(3.11) - (3.14) follow from (2.11) - (2.14) . First, it is evident that we can define an order for Σ satisfying VI(a) and VI(b) by taking $U \geqq V$ if $U \subset V$, and $U > V$ if U is properly contained in V , where $U, V \in \Sigma$. Then for each $x \in T$ and $U \in \Sigma$ define

$$//x, U// \leqq 1, \text{ if } x \in U ,$$

and

$$//x, U// \leqq 2 , \text{ if } x \bar{\in} U .$$

Σ_x is a strongly partially ordered space by virtue of (2.13) , and therefore (3.11) is satisfied. (3.12) follows from the definition of $//x, U//$ and (2.12). (2.14) evidently implies (3.13). (3.14) is clearly satisfied, and the theorem has been proved.

3.2. <u>Topological properties of P-spaces.</u>

It then follows from the above theorem and theorem 2.23 that every P-space T w.r.t D is a topological space, and conversely.

<u>COROLLARY</u> 3.21. <u>Every P-space T w.r.t. D is a topological space with closure defined as follows:</u>

<u>if S is any subset of T , then $x \in \bar{S}$, the closure of S , if and only if for each $d_x \in D_x$ there exist $y \in S$ such that $//y, d_x// \leqq 1$, i.e. , $y \in U(d_x)$.</u>

Also Σ , the neighborhood system generated by the pseudo-norm, is equivalent to the set of all open sets of T .

Conversely, every topological space T can be pseudo-normed w.r.t. a complete neighborhood system of T or w.r.t. any system of subsets equivalent to a complete neighborhood system.

Proof: This corollary is an immediate result of theorem 3.11 and theorem 2.23.

COROLLARY 3.22. A necessary and sufficient condition that a subset S of a P-space T w.r.t. D be open is that to each $x \in S$ there corresponds a $d_x \in D_x$ such that $/\!/y, d_x/\!/ \leqq 1$ implies $y \in S$, i.e. $U(d_x) \subset S$.

Proof: That this follows is clear from theorem 3.11 and theorem 2.21.

COROLLARY 3.23. A necessary and sufficient condition that a function $f(x)$ on an open set S_y , $y \in S_y$, of a P-space T w.r.t. D to a P-space T' w.r.t. D' be continuous at $x = y$ is that given $d'_{f(y)} \in D'_{f(y)}$ there exists a $d_y \in D_y$ such that $/\!/f(x), d'_{f(y)}/\!/' \leqq 1$ for all $/\!/x, d_y/\!/ \leqq 1$.

Proof: This result is a consequence of theorem 3.11 and definition 2.23.

COROLLARY 3.24. A necessary and sufficient condition that an element x of a P-space T w.r.t. D be the limit of a sequence $\{x_n\}$ of elements of T is that given $d_x \in D_x$ there exists an integer $m = m(d_x)$ such that for all $n > m$ $//x_n, d_x// \leq 1$.

Proof: The proof of the corollary depends upon theorem 3.11 and theorem 2.25.

3.3. P-groups.[1]

The pseudo-normed spaces that we will be dealing with later on will be groups, and we shall now introduce the concept of P-group which shall correspond to an N-group in the same manner that a P-space corresponds to an N-space.

DEFINITION 3.31. Let T be an additive group with elements $x, y, z, \ldots$, and let D be a strongly partially ordered set of elements $d, e, c, \ldots$. We shall say that T is a pseudo-normed group with respect to D, or simply a P-group w.r.t. D, if there exists a non-negative, real-valued function defined on TD satisfying the following postulates:

(3.31.) $//\Theta, d// \leq 1$ for all $d \in D$;

(3.32.) $//x, d// \leq 1$ for all $d \in D$ implies $x = \Theta$, where Θ is the zero element of T ;

(3.33.) given $d \in D$ there exists an $e \in D$

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[1] For the relation between P-groups and topological abelian groups see section 4.2.

such that if $//y, d// \leqq 1$, then there exists a $c = c(y) \in D$ with the property that $//y + x, e// \leqq 1$ for all $//x, c// \leqq 1$;

(3.34.) $e \geqq d$ implies that $//x, d// \leqq //x, e//$ for all $x \in T$.

THEOREM 3.31. Every P-group w.r.t. D is an N-group w.r.t. $\mathcal{U}$, where $U(d) = [x ; x \in T , //x,d// \leqq 1]$ and $\mathcal{U} = [U(d) ; d \in D]$. Conversely, every N-group w.r.t. $\mathcal{U}$ can be pseudo-normed w.r.t. $\mathcal{U}$.

Proof: The verification of this theorem is quite similar to that of theorem 3.11, and the proof will be omitted.

LEMMA 3.31. Let T be a P-group w.r.t. D , and given $x \in T$ and $d \in D$ define $U_x(d) = [y ; y \in T , //-x + y, d// \leqq 1]$. (We shall always write $U_\Theta(d)$ simply as $U(d)$.) Then $x + U(d) = U_x(d)$.

Proof: Let $y \in x + U(d)$. Then $-x + y \in U(d)$ and $//-x + y, d // \leqq 1$. Hence $y \in U_x(d)$ and $x + U(d) \subset U_x(d)$. Conversely, if $y \in U_x(d)$, then $//-x + y, d// \leqq 1$ and $-x + y \in U(d)$. Therefore $y \in x + U(d)$. Hence $U_x(d) \subset x + U(d)$ and the lemma is proved.

From now on in this thesis we shall be dealing with P-groups, and hence it will be useful to summarize in the following corollary the characterization of certain topological notions in terms of the pseudo-norm of a P-group.

COROLLARY 3.31. (a) Every P-group T w.r.t. D is a topological space with closure defined as follows:

if S is any subset of T, then $x \in \bar{S}$, the closure of S, if and only if for each $d \in D$ there exists a $y \in S$ such that $//-x + y, d// \leq 1$.

Also $\Sigma = \left[U_x(d) \, ; \, x \in T \, , \, d \in D \right]$[1] is equivalent to the set of all open sets of T.

(b) A necessary and sufficient condition that a subset $S \subset T$ be open is that given $x \in S$ there exists a $d \in D$ such that $//-x + y, d// \leq 1$ implies $y \in S$ or what is equivalent $U_x(d) \subset S$.

(c) A necessary and sufficient condition that a function $f(x)$ on an open set $S_y \subset T$, $y \in S_y$, to a P-group T' w.r.t. D' be continuous at $x = y$ is that given $d' \in D'$ there corresponds a $d \in D$ with the property that $//-y + x, d// \leq 1$ implies that $// -f(y) + f(x), d'// \leq 1$.

(d) A necessary and sufficient condition that an element $x \in T$ be the limit of a sequence $\{x_n\}$ of elements of T is that given $d \in D$ there exists an integer $m = m(d)$ such that $//-x + x_n, d// \leq 1$ for all $n > m$.

Proof: This corollary follows from theorem 3.31, lemma 3.31, theorem 2.11, theorem 2.21, theorem 2.23, definition 2.23, and theorem 2.25.

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[1] This neighborhood system will be said to be the neighborhood system generated by the pseudo-norm. This is analogous to definition 3.12.

As an example of a P-group, consider any valued ring A for which the valuation satisfies the additional property that $/\alpha/ = 0$ implies that $\alpha = 0$. Then A can be pseudo-normed w.r.t. the set of all positive real numbers, R^+. This can evidently be done by defining

$$//\alpha, r// = r\ /\alpha/$$

for each $\alpha \in A$ and $r \in R^+$. As we have mentioned before, it is interesting to note that we can find examples of valued rings in which the topology generated by the valuation is trivial in that it is discrete; whereas, the valuation itself, from a more general point of view, is by no means trivial. For instance see examples (4) and (5) of § 1.2.

3.4. P_1-spaces.

In this section of the paper we will define what we shall call a P_1-space and shall show the equivalence of P_1-spaces and N_1-spaces.[1] P_1-spaces will be related to N_1-spaces in much the same manner that pseudo-normed linear spaces[2] are related to linear topological spaces.

DEFINITION 3.41. A V_1-space[3] T w.r.t. A will be said to be a pseudo-normed space of type 1 with respect to A, D, or briefly a P_1-space w.r.t. A, D if T is a P-group[4]

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[1] See § 2.3.
[2] See [6] .
[3] See definition 1.31.
[4] See definition 3.31.

w.r.t D such that the pseudo-norm, $//x, d//$, satisfies the following additional postulates:

(3.41.) $//\alpha x, d// \leq /\alpha/ \, //x, d//$ for all $d \in D$, $\alpha \in A$ and $x \in T$;

(3.42.) given $d \in D$ there exists an $e \in D$ such that $//x, e// \leq 1$ and $//y, e// \leq 1$ implies that $//x + y, d// \leq 1$;[1]

(3.43.) given $r > 0$, $x \in T$, and $d \in D$ there corresponds an $\alpha \in A$ and a $y \in T$ such that $x = \alpha y$, $//y, d// \leq 1$, and $/\alpha/ - //x, d// \leq r$;[2]

(3.44.) given $\alpha \in A$ and $d \in D$ there exists an $e \in D$ such that $//x, e// \leq 1$ implies that $//\rho x, d// \leq 1$ for all $/\rho/ \leq /\alpha/$.

N.B. The postulates which are necessary to determine the properties of the pseudo-norm are then (3.32), (3.34), and (3.41)-(3.44), since (3.41) implies (3.31) and (3.42) implies (3.33).

THEOREM 3.41. *If* T *is a* P_1-*space w.r.t.* A, D, *then*:

(3.45) $//-x, d// = //x, d//$ *for all* $d \in D$ *and* $x \in T$;

(3.46) $//\Theta, d// = 0$ *for all* $d \in D$;

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[1] This postulate is a stronger form of (3.33).

[2] This postulate then makes it unnecessary to assume that $1x = x$ for each $x \in T$. Given $x \in T$, $x = \alpha y$ for some $\alpha \in A$ and $y \in T$. Hence by V(a) $1x = 1(\alpha y) = \alpha y = x$.

(3.47) *given* $\alpha \in A$ *and* $d \in D$ *there exist* $e \in D$ *such that* $//x, e// \leqq 1$ *and* $//y, e// \leqq 1$ *implies that* $//\rho_1 x + \rho_2 y, d// \leqq 1$ *for all* $/\rho_1/ \leqq /\alpha/$ *and* $/\rho_2/ \leqq /\alpha/$.

Proof: (3.45): By (3.41), (1.12) and (1.24)

$$//-x, d// \leqq /-1/ \, //x, d// = //x, d// \; .$$

Similarly,

$$//x, d// \leqq /-1/ \, //-x, d// = //-x, d// \; .$$

Hence $//-x, d// = //x, d//$.

(3.46): This follows from (3.41) and the fact that we are assuming that $/0/ = 0$.[1]

(3.47): By (3.42) we have that given $d \in D$ there exists an $e_1 \in D$ such that $//x, e_1// \leqq 1$ and $//y, e_1// \leqq 1$ implies that $//x + y, d// \leqq 1$. Now by (3.43) we have that given $e_1 \in D$ and $\alpha \in A$ there exists an $e \in D$ such that $//x, e// \leqq 1$ implies that $//\rho x, e_1// \leqq 1$ for all $/\rho/ \leqq /\alpha/$. Hence $//x, e// \leqq 1$ and $//y, e// \leqq 1$ implies that $//\rho_1 x, e_1// \leqq 1$ and $//\rho_2 y, e_1// \leqq 1$ for all $/\rho_1/ \leqq /\alpha/$ and $/\rho_2/ \leqq /\alpha/$, which in turn implies that $//\rho_1 x + \rho_2 y, d// \leqq 1$. This completes the proof of theorem 3.41.

3.5. The equivalence of P_1-spaces and N_1-spaces.

THEOREM 3.51.[2] *Every* P_1*-space* T *w.r.t.* A,D *is an*

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[1] See footnote (1), p. 11 .

[2] Compare with theorem 1 of [6] .

<u>N_1-space w.r.t. A, $\mathcal{U}$, where $\mathcal{U}$ is defined as in theorem 3.31, (i.e., $\mathcal{U}$ is defined to be the neighborhood system generated by the pseudo-norm.). Conversely every N_1-space w.r.t. A, $\mathcal{U}$ can be pseudo-normed with respect to a strongly partially ordered set D in such a manner that the neighborhood system generated by the pseudo-norm is equivalent to $\mathcal{U}$.</u>

Proof: The proof of the first part of the theorem is almost immediate. Let T be a P_1-space w.r.t. A, D and define $\mathcal{U} = [U(d);\ d \in D]$, where $U(d) = [x;\ /\!/x, d/\!/ \leqq 1]$.

(2.31): (2.31) follows from (3.32) and (3.46)

(2.32): Use (3.34) and the fact that D is a strongly partially ordered set.

(2.33): This is immediate from (3.44).

(2.34): Follows from (3.42).

(2.35): A direct result of (3.43).

Hence T is an N_1-space w.r.t. A, $\mathcal{U}$ and the first statement of the theorem has been proved.

Conversely, let T be an N_1-space w.r.t. A, $\mathcal{U}$. Given $U \in \mathcal{U}$ define $U^* = [\alpha x;\ x \in U,\ /\alpha/ \leqq 1]$ and $\mathcal{U}^* = [U^*;\ U \in \mathcal{U}]$. Given $U^* \in \mathcal{U}^*$ we see that $U \subset U^*$. Also given $U \in \mathcal{U}$ we have by postulate (2.33) of definition 2.31 that there exists a $V \in \mathcal{U}$ such that $V^* \subset U$, and hence $\mathcal{U}^*$ is equivalent to $\mathcal{U}$. From this equivalence it follows immediately that $\mathcal{U}^*$ satisfies (2.31) - (2.35) of definition 2.31, and also satisfies:

(3.51) $\alpha U^* \subset U^*$ for $|\alpha| \leqq 1$ and each $U^* \in \mathcal{U}^*$;

(3.52) given $\alpha \in A$ and $U^* \in \mathcal{U}^*$ there exists a $V^* \in \mathcal{U}^*$ such that $\rho_1 V^* + \rho_2 V^* \subset U^*$ for all $|\rho_1| \leqq |\alpha|$ and $|\rho_2| \leqq |\alpha|$.

For let us suppose that $x \in \alpha U^*$ where $|\alpha| \leqq 1$; i.e. $x = \alpha y$ where $y = \rho z$, $|\rho| \leqq 1$ and $z \in U$. Then $x = \alpha\rho z$ and by (1.22) $|\alpha\rho| = |\alpha|\,|\rho| \leqq 1$, and hence $x \in U^*$. Therefore (3.51) is satisfied by $\mathcal{U}^*$. (2.34) of definition 2.31 states that given $U^* \in \mathcal{U}^*$ there exists a $V_1^* \in \mathcal{U}^*$ such that $V_1^* + V_1^* \subset U^*$. Now by (2.33) of definition 2.31 we have that given $\alpha \in A$ there exists a $V^* \in \mathcal{U}^*$ such that $\rho V^* \subset V_1^*$ for $|\rho| \leqq |\alpha|$. Therefore $\rho_1 V^* + \rho_2 V^* \subset V_1^* + V_1^* \subset U^*$ for $|\rho_1| \leqq |\alpha|$ and $|\rho_2| \leqq |\alpha|$. Hence (3.52) is satisfied by $\mathcal{U}^*$. (It is clear that (3.52) holds also for $\mathcal{U}^*$, since we made use of only those properties of $\mathcal{U}^*$ that are also properties of $\mathcal{U}$.)

Define $\|x,U^*\| \doteq$ g.l.b. $\left[\,|\alpha| \;;\; x \in \alpha U\,\right]$. This lower bound exists by (2.35) for each $x \in T$ and $U^* \in \mathcal{U}^*$, and $\|x,U^*\|$ is a non-negative real-valued function defined on $T\mathcal{U}^*$. $\mathcal{U}^*$ is clearly a strongly partially ordered set with $U_2^* \geqq U_1^*$ if $U_2 \subset U_1^*$. This follows by (2.32) of definition 2.31.

In order to show that $/\!/x,U^*/\!/$, as defined above, is a pseudo-norm, we must show that $/\!/x,U^*/\!/$ satisfies (3.32), (3.34) and (3.41)-(3.44).

(3.32): By definition $/\!/x,U^*/\!/ \geq 0$ for all $x \in T$ and $U^* \in \mathcal{U}^*$. Let us assume that $/\!/x,U^*/\!/ \leq 1$ for all $U^* \in \mathcal{U}^*$. Then, given $U^* \in \mathcal{U}^*$ and $\alpha \in A$, we have by (2.33) of definition 2.31 that there exists a $V^* \in \mathcal{U}^*$ such that $\rho V^* \subset U^*$ for all $/\rho/ \leq /\alpha/$. Pick $/\alpha/ > 1$, which is possible by postulate (1.25) of definition 1.21. Then, since $/\!/x,V^*/\!/ \leq 1$, we have that there exist $\rho \in A$ such that $x \in \rho V^*$, $/\rho/ < /\alpha/$. Hence $x \in \rho V^* \subset U^*$, i.e., $x \in U^*$ for every $U^* \in \mathcal{U}^*$. It then follows from (2.31) of definition 2.31 that $x = \Theta$, and (3.32) of definition 3.31 is satisfied by $/\!/x,U^*/\!/$.

(3.41): If $x \in \rho U^*$, then $\alpha x \in \alpha\rho U^*$, and since $/\alpha\rho/ \leq /\alpha/\ /\rho/$, we obtain from the definition of the function $/\!/x,U^*/\!/$ that $/\!/\alpha x,U^*/\!/ \leq /\alpha/\ /\!/x,U^*/\!/$.

(3.42): We have from (3.52), by choosing $\alpha \in A$ and $/\alpha/ > 1$, that given $U^* \in \mathcal{U}^*$ there exists a $V^* \in \mathcal{U}^*$ such that $/\!/x,V^*/\!/ \leq 1$, $/\!/y,V^*/\!/ \leq 1$ implies that $x \in \rho_1 V^*$ and $y \in \rho_2 V^*$, where $/\rho_1/ < /\alpha/$ and $/\rho_2/ < /\alpha/$, and $/\!/x + y,U^*/\!/ \leq 1$. Hence (3.42) is satisfied by $/\!/x,U^*/\!/$.

(3.43): It follows immediately by the definition of

$//x,U^*//$ and (2.35) of definition 2.31 that $//x,U^*//$ satisfies this postulate.

(3.44): Given $\alpha \in A$, $U^* \in \mathcal{U}^*$, there exists a $V^* \in \mathcal{U}^*$ such that $\rho V^* \subset U^*$, for all $/\rho/ \leqq /\alpha/$. This is by postulate (2.33) of definition 2.31. Let α_o be such that $/\alpha_o/ > 1$. Then there exists a $V_o^* \in \mathcal{U}^*$ such that $\rho_o V_o^* \subset V^*$ for all $/\rho_o/ \leqq /\alpha_o/$. If $//x,V_o^*// \leqq 1$, there then exists a $\rho_o \in A$ such that $x \in \rho_o V_o^*$, $/\rho_o/ < /\alpha_o/$. Hence $x \in \rho_o V_o^* \subset V^*$ and $\rho x \in \rho V^* \subset U^*$, $/\rho/ \leqq /\alpha/$. I.e., given $\alpha \in A$ and $U^* \in \mathcal{U}^*$ there exists a $V_o^* \in \mathcal{U}^*$ such that $//x,V_o^*// \leqq 1$ implies that $//\rho x,U^*// \leqq 1$ for all $/\rho/ \leqq /\alpha/$. Hence (3.44) of definition 3.41 is satisfied by $//x,U^*//$.

(3.34): If $U_1^* \subset U_2^*$ and $x \in \alpha U_1^*$, then $x \in \alpha U_2^*$, and hence $U_1 \geqq U_2$ implies $//x,U_1// \geqq //x,U_2//$ for all $x \in T$.

Therefore we have shown that an N_1-space T can be pseudo-normed w.r.t. $\mathcal{U}^*$ in such a manner that T is a P_1-space w.r.t. $A, \mathcal{U}^*$.

It remains to show that the set of all

$$U(U^*) = \left[x;\ //x,U^*// \leqq 1 \right]$$

is equivalent to $\mathcal{U}$. Given $U(U^*)$ it follows immediately from the definition of $//x,U^*//$ and the definition of $U(U^*)$ that $U \subset U(U^*)$. Conversely, given $U \in \mathcal{U}$

we have by (2.33) of definition 2.31 that given $\alpha \in A$ there exists a $V^* \in \mathcal{U}^*$ such that $\rho V^* \subset U^*$ for $/\rho/ \leq /\alpha/$. Pick α such that $/\alpha/ > 1$. If $/\!/x, V^*/\!/ \leq 1$, then there exists a $\rho \in A$ such that $x \in \rho V^*$ and $/\rho/ \leq /\alpha/$. Therefore $x \in U^*$. Hence $U(V^*) \subset U^*$ and the neighborhood system generated by $/\!/x, U^*/\!/$ is equivalent to $\mathcal{U}$, since we have already shown that $\mathcal{U}^*$ is equivalent to $\mathcal{U}$. This completes the proof of the theorem.

An important example of a P_1-space is the "topological abelian group" considered by Michal in [10] . We shall discuss this example in detail later in the paper.

THEOREM 3.52. <u>If</u> T <u>is a P_1-space w.r.t.</u> A, D <u>and also w.r.t.</u> A, D' , <u>then a necessary and sufficient condition that the two pseudo-norms be equivalent[1] is that given</u> $d \in D$ <u>there exists a</u> $d' \in D'$ <u>such that</u> $/\!/x, d/\!/ \leq /\!/x, d'/\!/'$ <u>and conversely</u>.

<u>**Proof**</u>: The sufficiency is evident. The necessity follows from (3.41) of definition 3.41. Suppose that the pseudo-norms do generate equivalent topologies, i.e., given $d \in D$ there exists a $d' \in D'$ such that $/\!/x, d/\!/ \leq 1$ for all $/\!/x, d'/\!/' \leq 1$. Then, given $r > 0$, $x \in T$, and

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[1] See definition 3.12, p. 29.

$d' \in D'$, there exists an $\alpha \in A$ and $y \in T$ such that $x = \alpha y$, $/\alpha/ - //x,d'//' \leq r$ and $//y,d'//' \leq 1$. Hence $//x,d// = //\alpha y,d// \leq /\alpha///y,d// \leq /\alpha/ = //x,d'//' + r_o$, where $r_o \leq r$. Therefore $//x,d// \leq //x,d'//'$ for all $x \in T$. The converse is proved similarly.

We are now able to show that we can start with a weaker set of postulates than those of definition 3.41 and yet retain the possibility of defining a pseudo-norm which will satisfy definition 3.41.

COROLLARY 3.51. *If T is a V_1-space w.r.t. A and D is a strongly partially ordered set, and if a real-valued function $//x,d//$ is defined on TD such that $//x,d//$ satisfies postulates* (3.31), (3.32), (3.34), (3.42), (3.44) *and*

(3.51) *given $x \in T$ and $d \in D$ there exists an $\alpha \in A$ and $y \in T$ such that $x = \alpha y$, $//y,d// \leq 1$,*

then T may be pseudo-normed w.r.t. D in such a manner that T is a P_1-space w.r.t. A,D and the new pseudo-norm $//x,d//'$ is equivalent to $//x,d//$.

Proof: If we define $U(d) = \left[x; \ //x,d// \leq 1 \right]$ and $\mathcal{U} = \left[U(d) \ ; \ d \in D \right]$, the above properties of $//x,d//$ are sufficient to prove that T is an N_1-space A, $\mathcal{U}$. This corollary then follows directly from theorem 3.51.

CHAPTER 4

Φ -LINEAR FUNCTIONS ON P_1-SPACES TO P_1-SPACES

In theorem 3.52 we showed the equivalence of P_1-spaces and N_1-spaces with the pseudo-norm generating the neighborhood system. Hence we can study such spaces either as characterized in terms of a neighborhood system of the origin or in terms of a pseudo-norm. The pseudo-norm is, of course, a generalization of the norm, and for this reason we shall find it more suggestive and also more convenient to work out our theory in terms of the pseudo-norm.

It was shown in theorem 2.31 that every N_1-space is a topological abelian group, and hence by theorem 3.52 every P_1-space is a topological abelian group. Corollary 3.31 will be of importance to us here as it gives the statement of topological concepts in the language of the pseudo-norm.

The purpose of this chapter is to introduce the concept of a linear function which possesses a sort of generalized homogeneity. We know that in the case of Banach spaces that every linear function (i.e., a function which is both additive and continuous) is homogeneous of degree one.[1] Banach's proof of this can also be extended to the case of linear functions in linear topological spaces.[2] Here we shall postulate a generalization of this homogeneity for

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[1] See [3] , p.36, theorem 2. The proof is for a space (F).
[2] See [9] .

our linear functions and shall choose this homogeneity to be such that we shall be able to obtain properties corresponding to the known properties of linear functions in linear topological spaces.

4.1. Φ -Linear functions.

Throughout this section of the paper, unless otherwise stated, T will represent a P_1-space w.r.t. A, D, T' a P_1-space w.r.t. A', D', and $\Phi(\alpha)$ will denote a function on A to A' such that $/\Phi(\alpha)/' \leq /\alpha/$ for all $\alpha \in A$.

DEFINITION 4.11. An additive function[1] $f(x)$ on T to T' will be said to be <u>homogeneous with respect to a function</u> $\Phi(\alpha)$ on A to A' if $f(\alpha x) = \Phi(\alpha) f(x)$ for all $\alpha \in A$ and $x \in T$. We shall say briefly that $f(x)$ is <u>Φ-additive.</u>

DEFINITION 4.12. A function $f(x)$ on T to T' will be said to be Φ -<u>linear</u> if $f(x)$ is Φ-additive and continuous.

DEFINITION 4.13. A Φ-additive function $f(x)$ on T to T' will be said to be <u>bounded</u> if given $d' \in D'$ there exists a $d = d(d') \in D$ such that $//f(x), d'//' \leq //x, d//$ for all $x \in T$.

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[1] A function $f(x)$ is said to be additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in T$.

N.B. Let $f(x)$ be a $\bar{\Phi}$-linear function on T to T'. Then evidently

(4.11) $\bar{\Phi}(m)f(x) = mf(x)$ for all integers m and all $x \in T$,

(4.12) $\bar{\Phi}(\alpha + \beta)f(x) = (\bar{\Phi}(\alpha) + \bar{\Phi}(\beta))f(x)$ for all $\alpha, \beta \in A$ and $x \in T$,

(4.13) $\bar{\Phi}(\alpha\beta)f(x) = \bar{\Phi}(\alpha)\bar{\Phi}(\beta)f(x)$ for all $\alpha, \beta \in A$ and $x \in T$,

and

(4.14) $\bar{\Phi}(m\alpha)f(x) = \bar{\Phi}(\alpha m)f(x) = m\bar{\Phi}(\alpha)f(x)$.

If we assume that T' is such that $\alpha' y' = 0$ implies that $\alpha' = 0$ or $y' = \theta'$, then if $f(x) \neq \mathcal{N}$, $\mathcal{N}$ represents the null function,

(4.15) $\bar{\Phi}(m) = m$,

(4.16) $\bar{\Phi}(\alpha + \beta) = \bar{\Phi}(\alpha) + \bar{\Phi}(\beta)$,

(4.17) $\bar{\Phi}(\alpha\beta) = \bar{\Phi}(\alpha)\bar{\Phi}(\beta)$,

(4.18) $\bar{\Phi}(\alpha m) = \bar{\Phi}(m\alpha) = m\bar{\Phi}(\alpha)$,

for all $\alpha, \beta \in A$ and m any integer.

THEOREM 4.11. <u>A necessary and sufficient condition that a $\bar{\Phi}$-additive function</u> $f(x)$ <u>be $\bar{\Phi}$-linear is that</u> $f(x)$ <u>be bounded</u>.

Proof: The sufficiency follows immediately from Corollary

3.31(c), since $f(x)$ is additive.

In order to prove the necessity let us assume that $f(x)$ is a Φ -linear. Then $f(\alpha x) = \Phi(\alpha)f(x)$, $/\Phi(\alpha)/' \leq /\alpha/$ and given $d' \in D'$ there exists a $d = d(d')$ such that $//f(x),d'//' \leq 1$ for $//x,d// \leq 1$. The latter statement follows from corollary 3.31(c). Hence $//y,d// \leq 1$ implies that $//f(\alpha y),d'//' \leq /\Phi(\alpha)/' \leq /\alpha/$, applying (3.41) of definition 3.41. By (3.43) of definition 3.41 we have that given $d \in D$, $x \in T$, and $r > 0$ there exist $\alpha \in A$ and $y \in T$ such that $x = \alpha y$, $//y,d// \leq 1$ and $/\alpha/ - //x,d// < r$. Then $//f(x),d'//' = //f(\alpha y),d'//' \leq /\alpha/ = //x,d// + r_0$, where $r_0 > r$. Since this holds for any $x \in T$ and $r > 0$, we have that $//f(x),d'//' \leq //x,d//$ for all $x \in T$. This completes the proof of the theorem.

This theorem is then a generalization of Banach's theorem that any linear functional on a Banach space has a modulus.[1]

DEFINITION 4.14. A set $S \subset T$ will be said to be <u>bounded</u> if for each $d \in D$ the set $\left[//x,d// ;\ x \in S \right]$ of real numbers is bounded.

<u>N.B.</u> If S_1 and S_2 are bounded subsets of T, then it is clear that $S_1 \cup S_2$ and $S_1 \cap S_2$ are also bounded subsets of T. It follows from (3.41) of definition 3.41

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[1] See [2]. Also [16] and [9].

that for any $\alpha \in A$ and S a bounded subset of T that αS is a bounded set. Clearly any set with only a finite number of elements is bounded.

COROLLARY 4.11. _A Φ-linear function $F(x)$ on T to T' maps bounded sets into bounded sets._

Proof: This follows immediately from theorem 4.11.

THEOREM 4.12. _A necessary and sufficient condition that a set $S \subset T$ be bounded is that given an open set U containing the zero element there corresponds a real number $r = r(U) > 0$ such that given $x \in S$ there exists an $\alpha \in A$ such that $x \in \alpha U$, $/\alpha/ \leq r$._

Proof: _Sufficiency:_ Assume that the condition is satisfied. Then given $d \in D$ we have by corollary 3.31(a) that there exists an open set containing the zero element and contained in $U(d)$. Let U denote this open set. Hence given $x \in S$ there exists an $\alpha \in A$ such that $/\alpha/ \leq r(U)$ and $x \in \alpha U \subset \alpha U(d)$. I.e., $x = \alpha y$ where $//y, d// \leq 1$. Hence $//x, d// \leq //\alpha y, d// \leq /\alpha/$ by (3.41) of definition 3.41. Therefore for any $x \in S$, $//x, d// \leq r(U)$ and the sufficiency is proved.

Necessity: Let S be a bounded set and U be an open set containing Θ. Then by corollary 3.31(a) there exists

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1 Compare with the discussion on boundedness in [5].

a $U(d) \subset U$. By (3.43) of definition 3.41 we have that given $x \in S$ and $r > 0$ there exists an $\alpha \in A$ and a $y \in T$ such that $x = \alpha y$, $\|y, d\| \leq 1$ and $|\alpha| - \|x, d\| \leq r$. But now, since $x \in S$ and S is bounded, $\|x, d\| \leq r_o(d)$. Hence $|\alpha| \leq r_o(d) + r$. Therefore given $x \in S$, $x \in \alpha U(d) \subset \alpha U$, where $|\alpha| \leq r_o(d) + r$ for any $r > 0$, and this proves the necessity of the condition.

Let F_Φ represent the set of all Φ -linear functions on T to T' . Then if $f_1 , f_2 \in F_\Phi$, we define $f_3 = f_1 + f_2$ to be the function $f_3(x) = f_1(x) + f_2(x)$. With this definition of addition we can prove the following theorem.

THEOREM 4.13. <u>The set F_Φ of all Φ -linear functions on T to T' is a topological abelian group.</u>

<u>Proof:</u> It is evident that the sum of two Φ -additive functions is Φ -additive, since T' is a linear set over A' . Also it follows immediately from corollary 3.31(c) and (3.42) of definition 3.41 that the sum of two continuous functions is a continuous function, and hence $f_1 + f_2 \in F_\Phi$ for every $f_1 , f_2 \in F_\Phi$. The commutative and the associative laws of addition follow from the commutativity and associativity of addition in T' .

The null function $\vartheta(x) = \Theta$ for all $x \in T$ is contained in F_Φ and $f + \vartheta = f$ for all $f \in F_\Phi$. If $f \in F_\Phi$, then $-f$, the function $-f(x)$, is in F_Φ, and $f + -f = \vartheta$. Therefore F_Φ is an additive abelian group.

We wish now to show that F_Φ is a topological group and we shall do this as follows. Let B_T be the set of all bounded sets contained in T. B_T is clearly a strongly partially ordered set with $S_1 \geqq S_2$ if $S_1 \supset S_2$. (S_1, $S_2 \in B_T$.) Define W to be the set of all ordered pairs $w = (d', S)$, $d' \in D'$ and $S \in B_T$. Since B_T and D' are strongly partially ordered sets, W is evidently a strongly partially ordered set[1] with $w_1 \geqq w_2$ if $d_1' \geqslant d_2'$ and $S_1 \geqq S_2$. ($w_i = (d_i', S_i)$ $i = 1, 2, \ldots$)

Define

$$M(f,w) = \text{l.u.b.}\left[\|f(x), d'\|' ;\ x \in S \right],$$

where $w = (d', S) \in W$ and $f \in F_\Phi$. By corollary 4.11 $M(f,w)$ is defined for each $f \in F_\Phi$ and $w \in W$. $M(f,w)$ then has the following properties:

(4.19) $M(f,w) \geqq 0$ for every $w \in W$ and $f \in F_\Phi$; $M(f,w) \leqq 1$ for all $w \in W$ implies that $f = \vartheta$;

(4.110) given $w \in W$ there exist $v \in W$ such that $M(f_1,v) \leqq 1$ and $M(f_2,v) \leqq 1$ implies that $M(f_1 + f_2, w) \leqq 1$;

(4.111) $M(a'f,w) \leqq |a'|'\, M(f,w)$ for each $w \in W$

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[1] See VI.

and each $\alpha' f \in F_{\Phi}$;

(4.112) given $\alpha' \in A'$ and $w \in W$ there exists a $v \in W$ such that $M(f,v) \leq 1$ implies that $M(\rho' f,w) \leq 1$ for all $/\rho'/' \leq /\alpha'/'$ and $\rho' f \in F_{\Phi}$;

(4.113) $w_1 \geq w_2$ implies that $M(f,w_1) \geq M(f,w_2)$ for each $f \in F_{\Phi}$.

These properties of $M(f,w)$ follow from the definition of $M(f,w)$ and the properties of $/\!/f(x),d'/\!/'$.

$M(f,w)$ as defined above clearly satisfies (3.31) - (3.34) of definition 3.31 and hence F_{Φ} is a P-group w.r.t. W. By corollary 3.31(a) F_{Φ} is a topological space. It then follows from corollary 3.31(c) and (4.110) that the operation of addition is continuous. Since $1f = f$ and $-1f = -f \in F_{\Phi}$, it follows from (4.111) and (1.24) that $M(-f,w) = M(f,w)$ for all $f \in F_{\Phi}$ and $w \in W$. Hence by corollary 3.31(c) the inverse operation is continuous and F_{Φ} is a topological abelian group.

Later on in the paper we shall return to the study of such a space F_{Φ} and with stronger conditions on A' we shall be able to show that F_{Φ} is a P_2-space w.r.t. A', W. In preparation for this we shall now show that if A' is a commutative ring, then F_{Φ} is a V_1-space w.r.t. A'.

THEOREM 4.14. *If* F_{Φ} *is the set of all* Φ *-linear functions on a* P_1*-space* T *w.r.t.* A , D *to a* P_1*-space* T' *w.r.t.* A', D' *where* A' *is a commutative*[1] *ring, then the topological abelian group* F_{Φ} *is a* V_1*-space with respect to* A' .

Proof: We have shown in theorem 4.13 that F_{Φ} is a topological abelian group. For $\alpha' \in A'$ and $f \in F_{\Phi}$ we define $\alpha' f$ to be the function $\alpha' f(x)$. Since A' is commutative, it follows that $\alpha_1' f_1(x) + \alpha_2' f_2(x)$ is Φ -additive for each α_1' , $\alpha_2' \in A'$ and f_1, $f_2 \in F_{\Phi}$. It follows from (3.47) of theorem 3.41 and corollary 3.31(c) that $\alpha_1' f_1(x) + \alpha_2' f_2(x)$ is continuous in x. In using (3.47) pick α' to be α_1' or α_2' according as $/\alpha_1'/ \geq /\alpha_2'/$ or $/\alpha_1'/ < /\alpha_2'/$. Hence $\alpha_1' f_1 + \alpha_2' f_2 \in F_{\Phi}$ for any α_1', $\alpha_2' \in A'$ and f_1 , $f_2 \in F_{\Phi}$.

One may then go on and easily verify that F_{Φ} is a V_1- space.

4.2. Sequences of Φ -linear functions.

The first theorem of this section depends only upon T being a P-abelian group which satisfies, in addition, the stronger form of postulate (3.33), namely, (3.42) of definition 3.41, and the following postulate:

(4.21) to each $d \in D$ there corresponds an

e

[1] I.e., $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in A$.

$e \in D$ such that $/\!/x, e/\!/ \leq 1$ implies that $/\!/-x, d/\!/ \leq 1$. It can be shown by use of corollary 3.31 that this is an equivalent form for the definition of a topological abelian group. This equivalence is made clear by §16 and §17 of [14], pp. 52 -57.

THEOREM 4.21. Let T be a topological abelian group. If $\{x_n\}$ and $\{y_n\}$ are sequences of elements of T, it follows that:

(a) if $\lim x_n$ exists, then the limit is unique;

(b) $\lim x_n = x$ and $\lim y_n = y$ implies that $\lim (x_n + y_n) = x + y$.

If T is a P_1-space w.r.t. A, D,

(c) $\lim x_n = x$ implies that for any $\alpha \in A$ $\lim \alpha x_n = \alpha x$.

Proof: As we mentioned above, every topological abelian group can be characterized in terms of the pseudo-norm. We shall give the proof of this theorem using the pseudo-norm.

(a): Let us assume that $\lim x_n = x$ and $\lim x_n = z$. Then by (3.42) and (4.21) we have that given $d \in D$ there exists an $e \in D$ such that $/\!/x - y, d/\!/ \leq 1$ for all $/\!/x, e/\!/ \leq 1$ and $/\!/y, e/\!/ \leq 1$. By corollary 3.31(d) there exists a positive integer $m = m(e)$ such that for $n > m(e)$

$//x - x_n, e// \leq 1$ and $//x_n - z, e// \leq 1$. Hence $//x - z, d// \leq 1$ for all $d \in D$. Therefore, by (3.32) of definition 3.31 $x - z = \Theta$, or $x = z$, and the uniqueness has been proved.

(b): Assume that $\lim x_n = x$ and $\lim y_n = y$. Then, as in the proof of (a), we have that given $d \in D$ there exists an $e \in D$ such that $//x, e// \leq 1$ and $//y, e// \leq 1$ implies that $//x - y, d// \leq 1$. Then by our hypothesis and corollary 3.31(d) there exists a positive integer $m = m(d)$ such that $//x_n - x, e// \leq 1$ and $//y_n - y, e// \leq 1$ for all $n > m$. Hence $//x_n + y_n - x - y, d// \leq 1$ for all $n > m$. By corollary 3.31(d) this completes the proof of (b).

(c): By hypothesis and corollary 3.31(d) we have that given $e \in D$ there exists an $m(e)$ such that $n > m$ implies that $//x - x_n, e// \leq 1$. By (3.44) of definition 3.41 we have that given $\alpha \in A$ and $d \in D$ there exists an $e \in D$ such that $//x_n - x, e// \leq 1$ implies $//\alpha x_n - \alpha x, d// \leq 1$. Hence for $n > m(e) = m(e(\alpha, d))$ we have that $//\alpha x - \alpha x_n, d// \leq 1$, and by corollary 3.31(d) $\lim \alpha x_n = \alpha x$. Thus (c) has been proved.

COROLLARY 4.21. _If_ $\lim f_n = f$, $f_n \in F_\Phi$ _and_ $f \in F_\Phi$, _then_ $\lim f_n(x) = f(x)$ _for all_ $x \in T$.

Proof: As we mentioned at the beginning of this section, every topological abelian group is a special type of P-group. Hence by theorem 4.13 and corollary 3.31(d) we have that given $w \in W$ there exists a positive integer $m = m(w)$ such that $M(f_n - f, w) \leqq 1$ for all $n > m$.[1] Since given $x \in T$ there exists an $S \in B_T$ such that $x \in S$, we see by the definition of $M(f,w)$ that given $d' \in D'$ there exists an integer $m = m(w)$, $w = (d', S)$, such that $n > m$ implies that $/\!/f_n(x) - f(x), d'/\!/' \leqq 1$. Hence by corollary 3.31(d) $\lim f_n(x) = f(x)$ for each $x \in T$.

THEOREM 4.22. If $\lim f_n(x) = f(x)$ for each $x \in T$, $f_n(x) \in F_\Phi$, and if given $d' \in D'$ there exists a $d \in D$ and a positive integer $s = s(d', d)$ such that $/\!/f_s(x) - f(x), d'/\!/' \leqq 1$ for all $/\!/x, d/\!/ \leqq 1$, then $f(x)$ is Φ-linear, i.e., $f \in F_\Phi$.

Proof: Since $\lim f_n(x) = f(x)$, it follows from theorem 4.13 and theorem 4.21 that $f(x)$ is Φ-additive. By 3.41 of definition 3.41 we have that given $d' \in D'$ there exists an $e' \in D'$ such that $/\!/f(x), d'/\!/' \leqq 1$ for $/\!/f_n(x) - f(x), e'/\!/' \leqq 1$ and $/\!/f_n(x), e'/\!/' \leqq 1$. Pick $n = s = s(e', e)$ as in our hypothesis. Then, since $f_s(x)$ is in F_Φ, we have by theorem 4.41 that given $e' \in D'$

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[1] W and $M(f,w)$ are defined in theorem 4.13.

there exists an $e_1 \in D$ such that $/\!/ f_s(x), e' /\!/ \leq 1$ for all $/\!/ x, e_1 /\!/ \leq 1$. Now by hypothesis $/\!/ f_s(x) - f(x), e' /\!/' \leq 1$ for $/\!/ x, e /\!/ \leq 1$. Pick $d \geq e, e_1$. Then $/\!/ f(x), d' /\!/' \leq 1$ for all $/\!/ x, d /\!/ \leq 1$. By corollary 3.31(c) $f(x)$ is continuous. We have already shown that $f(x)$ is Φ-additive and hence $f(x)$ is Φ-linear.

CHAPTER 5

Φ-DIFFERENTIALS OF FUNCTIONS WITH ARGUMENTS AND VALUES IN P_1-SPACES.

Throughout this chapter T, T' and T'' will represent P_1-spaces w.r.t. A, D ; A', D' and A'', D'' respectively, unless otherwise stated. $\Phi(\alpha)$ will denote a function on A to A' such that $/\Phi(\alpha)/' \doteq /\alpha/$ for all $\alpha \in A$.

5.1. Φ -approximation functions.

DEFINITION 5.11. Let $F(x)$ be a function defined on an open set $S_\Theta \subset T$, $\Theta \in S_\Theta$. Then $F(x)$ will be said to be a Φ -approximation function on $S_\Theta \subset T$ to T' if there exists an $e \in D$ such that given $d' \in D'$ there corresponds a $d = d(d') \in D$ for which:

(5.11) $//x, e// \doteq 1$ and $//x, d// \doteq 1$ implies that $//F(x), d'//' \doteq 1$;

and

(5.12) $//\alpha x, e// \doteq 1$ and $//x, d// \doteq 1$ implies that $//\Phi(\alpha)F(x), d'//' \doteq 1$.

N.B. We have by corollary 3.31(a) that there exists a $U(d_1) \subset S_\Theta$, and it is implied in definition 5.11 that $e \doteq d_1$, as we can only speak of (5.11) and (5.12) being satisfied for such $x \in T$ for which $F(x)$ is defined.

LEMMA 5.11. *If $F(x)$ is both a Φ-additive function and a Φ-approximation function on T to T', then $F(x) = \Theta$ for all $x \in T$.*

Proof: By hypothesis we have that there exists an $e \in D$ such that given $d' \in D'$ there exists a $d = d(d') \in D$ such that:

$$(5.13) \quad /\!/F(\alpha y), d'/\!/' = /\!/\Phi(\alpha)F(y), d'/\!/ \leq 1$$

for all $/\!/\alpha y, e/\!/ \leq 1$ and $/\!/y, d/\!/ \leq 1$.

In addition, we have by (3.43) of definition 3.41 that given $x \in T$ and $d \in D$ there exist $y \in T$ and $\alpha \in A$ such that $x = \alpha y$ and $/\!/y, d/\!/ \leq 1$. Hence, if $/\!/x, e/\!/ \leq 1$, we have from (5.13) that

$$(5.14) \quad /\!/F(x), d'/\!/' \leq 1 \text{ for all } d' \in D'.$$

Hence by (3.32) of definition 3.31 $F(x) = \Theta$ for all $/\!/x, e/\!/ \leq 1$. Then by use of (3.43) of definition 3.41 we have that given $x \in T$ there exist $\alpha \in A$ and $y \in T$ such that $/\!/y, e/\!/ \leq 1$ and $x = \alpha y$. Therefore, since $F(x)$ is Φ-additive, $F(x) = \Phi(\alpha)F(y) = \Theta$ for each $x \in T$, and the lemma is proved.

DEFINITION 5.12. Given a valued ring A define

$$A^\circ = \left[\alpha ; \alpha \in A, \ \alpha\rho = \rho\alpha \text{ for each } \rho \in A\right].$$

We see that given any integer m that $m = m1 \in A^\circ$.[1]

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[1] $m\alpha = \alpha + \dots (m \text{ summands}) \dots + \alpha$. A° can easily be shown to be the largest subring of A which is commutative.

We are not in general interested in the open set on which a Φ-approximation function $F(x)$ is defined and so we shall not specify the open set but say only that $F(x)$ is a Φ-approximation function defined on T to T', though it is to be remembered that $F(x)$ may only be defined on an open set $S_\Theta \subset T$, $\Theta \in S_\Theta$.

LEMMA 5.12. <u>If</u> $F_1(x)$ <u>and</u> $F_2(x)$ <u>are</u> Φ-<u>approximation functions on</u> T <u>to</u> T', <u>then</u> $F(x) = \alpha_1' F_1(x) + \alpha_2' F_2(x)$ <u>is a</u> Φ-<u>approximation on</u> T <u>to</u> T' <u>for all</u> α_1', $\alpha_2' \in (A')^\circ$.

<u>Proof</u>: Pick α' to be either α_1' or α_2' depending on whether $/\alpha_1'/' \geqq /\alpha_2'/'$ or $/\alpha_2'/' > /\alpha_1'/'$. Then we have by (3.47) of theorem 3.41 that given $d_1' \in D'$ there exist $d' \in D'$ such that $//F(x), d_1'//' \leqq 1$ for $//F_1(x), d'//' \leqq 1$ and $//F_2(x), d'//' \leqq 1$. Since $F_1(x)$ and $F_2(x)$ satisfy (5.11), we see that there exist $e_i \in D$ such that given $d' \in D'$ there correspond $d_i \in D$ such that

$$//F_i(x), d'//' \leqq 1 \quad \text{for all} \quad //x, e_i// \leqq 1 \quad \text{and}$$

$//x, d_i// \leqq 1$, $i = 1,2$. Hence picking $d \geqq d_1, d_2$ and $e \geqq e_1, e_2$ we have that given $d_1' \in D'$

$$//F(x), d_1'//' \leqq 1 \quad \text{for all} \quad //x, e// \leqq 1 \quad \text{and} \quad //x, d// \leqq 1 .$$

Since α_1', α_2' are in $(A')^\circ$, $\Phi(\alpha)F(x) = \alpha_1' \Phi(\alpha)F_1(x) + \alpha_2' \Phi(\alpha)F_2(x)$. Then in a manner similar to the above it can be shown that

$\|\Phi(\alpha)F(x), d_1'\|' \leqq 1$ for all $\|\alpha x, e\| \leqq 1$ and $\|x, d\| \leqq 1$. (It is to be noted that e and d may have to be rechosen in order to hold for both this equation and the one above. This is possible since D is a strongly partially ordered set.) Therefore $F(x)$ satisfies (5.11) and (5.22). If $F_i(x)$ is defined on $S_i \subset T$, $\theta \in S_i$ and $i = 1, 2$, then $F(x)$ is defined on an open set $S_\theta \subset S_1 \cap S_2$, $\theta \in S_\theta$; since by corollary 3.31(a) such an open set does exist. Hence $F(x)$ is a Φ-approximation function on T to T'.

LEMMA 5.13. *If $F(x)$ is a Φ-approximation function on T to T' and $g(x')$ is a Φ'-linear function on T' to T'', then $G(x) = g(F(x))$ is a Ψ-approximation function on T to T'', where $\Psi(\alpha) = \Phi'(\Phi(\alpha))$.*

Proof: Since $g(x')$ is a Φ'-linear function, we have by theorem 4.11 that given $d'' \in D''$ there exists a $d' \in D'$ such that

$$(5.15) \qquad \|G(x), d''\|'' \leqq \|F(x), d'\|' .$$

Because of the fact that $F(x)$ is a Φ-approximation function, (5.11) is evidently satisfied by $G(x)$.

By the Φ'-linearity of $g(x')$

$$(5.16) \qquad \Psi(\alpha)G(x) = g(\Phi(\alpha)F(x)).$$

Now $G(x)$ satisfies (5.15) and hence

$$(5.17) \qquad \|\Psi(\alpha)G(x), d''\|'' \leqq \|\Phi(\alpha)F(x), d'\|' .$$

$G(x)$ then satisfies (5.12) by virtue of the fact that $F(x)$ does. Clearly $/\Psi(\alpha)/'' = /\Phi(\alpha)/' = /\alpha/$. Hence $G(x)$ is a Ψ-approximation function on T to T'', where $G(x)$ is defined on the same open set in T on which $F(x)$ is defined.

5.2. Φ-**differentiable functions.**

DEFINITION 5.21. Let $f(x)$ be a function on an open set $S_y \subset T$ to T', $y \in S_y$. If there exists a Φ-linear function $f(y;\delta x)$ of δx on T to T' such that

$$(5.21) \quad F(\delta x) = f(y + \delta x) - f(y) - f(y;\delta x)$$

is a Φ-approximation function of δx on an open set $S_\Theta \subset T$, $\Theta \in S_\Theta$, to T', then $f(x)$ will be said to be Φ-**differentiable at** $x = y$ and $f(y;\delta x)$ will be called a Φ-**differential of** $f(x)$ **at** $x = y$ **with increment** δx.

N.B. There exists by corollary 3.31(a), corollary 3.31(b), and lemma 3.31 an open set $S_\Theta \subset T$, $\Theta \in S_\Theta$, such that $y + S_\Theta \subset S_y$. $F(\delta x)$ is then a function on $S_\Theta \subset T$ to T'.

THEOREM 5.21. *If a function $f(x)$ on an open set $S_y \subset T$, $y \in S_y$, to T' is Φ-differentiable at $x = y$ with $f(y;\delta x)$ a Φ-differential of $f(x)$, then $f(y;\delta x)$ is unique.*

Proof: Let us assume that $f_1(y;\delta x)$ and $f_2(y;\delta x)$ are both Φ-differentials of $f(x)$ at $x = y$. Define

$$G(\delta x) = f_1(y;\delta x) - f_2(y;\delta x) .$$

Since

$$G(\delta x) = (f(y + \delta x) - f(y) - f_2(y;\delta x)) - (f(y + \delta x) - f(y) - f_1(y;\delta x)) ,$$

we then have by hypothesis that $G(\delta x)$ is the difference of two Φ-approximation functions. Hence by lemma 5.12 $G(\delta x)$ is itself a Φ-approximation function. $G(\delta x)$ is also the difference of two Φ-linear functions on T to T' and by theorem 4.13 $G(\delta x)$ is a Φ-linear function on T to T'. It then follows by lemma 5.11 that $G(\delta x) = \theta$ for all $\delta x \in T$, and thus the uniqueness of the Φ-differential has been proved.

THEOREM 5.22. _If_ $f(x)$ _is a_ Φ_-differentiable function at_ $x = y$, _then_ $f(x)$ _is continuous at_ $x = y$.

Proof: By (3.42) of definition 3.41 we have that given $d' \in D'$ there exists an $e' \in D'$ such that

$$/\!/ f(y + \delta x) - f(y), d' /\!/' \doteq 1$$

for all $/\!/ f(y + \delta x) - f(y) - f(y;\delta x), e' /\!/' \doteq 1$ and $/\!/ f(y;\delta x), e' /\!/' \doteq 1$. Since $f(y + \delta x) - f(y) - f(y;\delta x)$ is a Φ-approximation function and $f(y;\delta x)$ is a Φ-linear function, it follows that there does exist a $d \in D$ such that $/\!/ x, d /\!/ \doteq 1$ implies that

$$/\!/ f(y + \delta x) - f(y), d'/\!/' \leqq 1 .$$

Therefore by corollary 3.31(c) $f(x)$ is continuous at $x = y$, and the theorem is proved.

LEMMA 5.21. <u>If</u> $f(x)$ <u>on an open set</u> $S_y \subset T$ <u>to</u> T', $y \in S_y$, <u>is</u> Φ-<u>differentiable at</u> $x = y$, <u>and if</u> $H(x')$ <u>is a</u> Φ'-<u>approximation function on an open set</u> $S'_\Theta \subset T'$ <u>to</u> T'', $\Theta \in S'_\Theta$, <u>then</u> $G(\delta x) = H(f(y + \delta x) - f(y))$ <u>is a</u> Ψ-<u>approximation function on an open set</u> $S_\Theta \subset T$ <u>to</u> T'', $\Theta \in S_\Theta$, <u>where</u> $\Psi(\alpha) = \Phi'(\Phi(\alpha))$.

<u>Proof:</u> Since $H(x')$ is a Φ'-approximation function on T' to T'', we have by (5.11) of definition 5.11 that there exists an $e' \in D'$ such that given $d'' \in D''$ there corresponds a $d' \in D'$ such that

$$/\!/ G(\delta x), d''/\!/'' \leqq 1$$

for all

$$/\!/ f(y + \delta x) - f(y), e'/\!/' \leqq 1$$

and

$$/\!/ f(y + \delta x) - f(y), d'/\!/' \leqq 1 .$$

By theorem 5.21 and corollary 3.31(c) there exists an $e_1 \in D$ and $d_1 \in D$ such that

$$/\!/ f(y + \delta x) - f(y), d'/\!/' \leqq 1 \quad \text{for} \quad /\!/ \delta x, d_1/\!/ \leqq 1$$

and

$$/\!/ f(y + \delta x) - f(y), e'/\!/' \leqq 1 \quad \text{for} \quad /\!/ \delta x, e_1/\!/ \leqq 1 .$$

Hence $G(\delta x)$ satisfies condition (5.11) of definition 5.11.

By (5.12) we have that there exist $e' \in D'$ such that given $d'' \in D''$ there correspond $d' \in D'$ such that

$$(5.22) \quad /\!/ \Psi(\alpha)G(\delta x), d''/\!/'' \leqq 1$$

for $/\!/ \Phi(\alpha)(f(y + \delta x) - f(y)), e'/\!/' \leqq 1$ and $/\!/ f(y + \delta x) - f(y), d'/\!/' \leqq 1$.

By theorem 5.12 and corollary 3.31(c) there exist $d_1 \in D$ such that

$$(5.23) \quad /\!/ f(y + \delta x) - f(y), d'/\!/' \leqq 1 \quad \text{for} \quad /\!/ \delta x, d_1/\!/ \leqq 1 .$$

Define $F(\delta x) = f(y + \delta x) - f(y) - f(y;\delta x)$. By hypothesis $F(\delta x)$ is a Φ-approximation function. It then follows from (3.42) of definition 3.41 that given $e' \in D'$ there exist $e_1' \in D'$ such that

$$(5.24) \quad /\!/ \Phi(\alpha)(f(y + \delta x) - f(y)), e'/\!/' \leqq 1$$

for $/\!/ \Phi(\alpha)F(\delta x), e_1'/\!/' \leqq 1$ and $/\!/ \Phi(\alpha)f(y;\delta x), e_1'/\!/' \leqq 1$.

Due to the fact that $F(\delta x)$ is a Φ-approximation function, we have by postulate (5.12) of definition 5.11 that there exists an $e_2 \in D$ such that given $e_1' \in D'$ there corresponds a $d_2 \in D$ such that

$$(5.25) \quad /\!/ \Phi(\alpha)F(\delta x), e_1'/\!/' \leqq 1$$

for $/\!/ \alpha(\delta x), e_2/\!/ \leqq 1$ and $/\!/ \delta x, d_2/\!/ \leqq 1$.

Since, by hypothesis, $f(y;\delta x)$ is Φ-linear, we have that there exists an $e_3 \in D$ such that

$$(5.26) \quad /\!/ \Phi(\alpha) f(y;\delta x), e_1' /\!/ ' = /\!/ f(y;\alpha\delta x), e_1' /\!/ ' \leqq 1$$

for $/\!/ \alpha\delta x, e_3 /\!/ \leqq 1$.

Hence by (5.22) - (5.26) we have for $e_4 \geqq e_2$, $e_3 \in D$ that given $d'' \in D''$ there corresponds $d_4 \geqq d_1$, $d_2 \in D$ such that

$$(5.27) \quad /\!/ \Psi(\alpha) G(\delta x), d'' /\!/ '' \leqq 1$$

for $/\!/ \alpha\delta x, e_4 /\!/ \leqq 1$ and $/\!/ \delta x, d_4 /\!/ \leqq 1$.

Therefore for $e \geqq e_1$, e_4 and $d = d_4$ we have that $G(\delta x)$ satisfies definition 5.11 , and $G(\delta x)$ is a Ψ-approximation function on $S_\Theta \subset T$ to T'' . S_Θ is an open set such that $y + S_\Theta \subset S_y$. As we have shown before, such a set does exist. Clearly

$$/\Psi(\alpha)/'' \leqq /\Phi(\alpha)/' \leqq /\alpha/ .$$

This completes the proof of the theorem.

THEOREM 5.23. <u>If</u> $f(x)$ <u>and</u> $g(x)$ <u>are function on open sets</u> $S_1 \subset T$ <u>and</u> $S_2 \subset T$ <u>respectively</u>, $y \in S_1$, S_2 , <u>to</u> T' , <u>and if</u> $f(x)$ <u>and</u> $g(x)$ <u>are</u> Φ-<u>differentiable at</u> $x = y$, <u>then</u> $h(x) = \alpha_1' f(x) + \alpha_2' g(x)$ <u>is</u> Φ-<u>differentiable at</u> $x = y$ <u>for any</u> α_1' , $\alpha_2' \in A'^\circ$ <u>with</u> $h(y;\delta x) = \alpha_1' f(y;\delta x) + \alpha_2' g(y;\delta x)$.

<u>Proof</u>: By lemma 5.12 $h(y + \delta x) - h(y) - \alpha_1' f(y;\delta x) - \alpha_2' g(y;\delta x)$ is a Φ-approximation function. As in theorem 4.14, $h(y;\delta x) = \alpha_1' f(y;\delta x) + \alpha_2' g(y;\delta x)$ is a Φ-linear function and the theorem is proved.

THEOREM 5.24. *If* $f(x)$ *on an open set* $S_y \subset T$, $y \in S_y$, *to* T' *is* Φ*-differentiable at* $x = y$ *and if* $g(x')$ *on an open set* $S'_{f(y)} \subset T'$ *to* T'', $f(y) \in S'_{f(y)}$, *is* Φ'*-differentiable at* $x' = f(y)$, *then* $h(x) = g(f(x))$ *on* $S_y \subset T$ *to* T'' *is* Ψ*-differentiable at* $x = y$, *where* $\Psi(\alpha) = \Phi'(\Phi(\alpha))$ *and* $h(y;\delta x) = g(f(y);f(y;\delta x))$.

Proof: Define

$$G(\delta x) = h(y + \delta x) - h(y) - g(f(y);f(y;\delta x))$$

Then

$$(5.28) \quad \begin{aligned} G(\delta x) &= g(f(y + \delta x)) - g(f(y)) \\ &\quad - g(f(y);f(y + \delta x) - f(y)) \\ &\quad + g(f(y);f(y + \delta x) - f(y) - f(y;\delta x)) \end{aligned}$$

because, by hypothesis, $g(y';\delta x')$ is additive. Here $y' = f(y)$. Since $g(x')$ is Φ'-differentiable at $x' = y'$, we have that

$$H(\delta x') = g(y' + \delta x') - g(y') - g(y';\delta x')$$

is a Φ'-approximation function on T' to T''. Then by lemma 5.21 $H(f(y + \delta x) - f(y))$ is a Ψ-approximation function in δx on T to T''. It follows from lemma 5.13 that $g(f(y);f(y + \delta x) - f(y) - f(y;\delta x))$ is a Ψ-approximation function in δx on T to T''. Since

$$\begin{aligned} G(\delta x) &= H(f(y + \delta x) - f(y)) \\ &\quad + g(f(y);f(y + \delta x) - f(y) - f(y;\delta x)), \end{aligned}$$

it then follows that $G(\delta x)$ is the sum of two Ψ-approximation functions. Hence by lemma 5.12 $G(\delta x)$ is a

Ψ-approximation function on T to T''. It follows almost immediately from theorem 4.11 that

$$h(y;\delta x) = g(f(y);f(y;\delta x))$$

is a Ψ-linear function on T to T'', and the theorem is proved.

5.3. Φ-differentials in Michal's "topological abelian groups."[1]

A. D. Michal has considered the notion of a differential for functions on an open set of a "topological abelian group" to a "topological abelian group." We wish in this section of the paper to show that the "topological abelian group" which Michal considered is a P_1-space and that every function which is M_1-differentiable[1] is differentiable in the sense of this chapter, the two differentials being equal.

By a "topological abelian group" (t.a.g.) is meant a topological abelian group[2] T which satisfies the following additional postulate:

5.31 given $x \in T$ and any open set $U \subset T$, $\Theta \in U$, there exists a positive integer n such that $x \in nU$. We shall always refer to such a topological abelian group as a t.a.g.

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1 See [10].

2 See XIII.

THEOREM 5.31. *Every t.a.g. T is an N_1-space with respect to A, $\mathcal{U}$, where A is the integral domain of all integers with the absolute value as the valuation and $\mathcal{U} = \{U\}$ is the set of all open sets U of T containing Θ. Conversely every N_1-space with respect to A, $\mathcal{U}$, where A is the absolute-valued integral domain of all integers, is a t.a.g.*

Proof: It is evident that a t.a.g. T is a V_1-space with respect to the integral domain of all positive integers with $/m/ = |m|$. Also it follows from the definition of a t.a.g., theorem 2.23 and theorem 2.24 that:

(5.32) the intersection of all $U \in \mathcal{U}$ is Θ;

(5.33) given $U \in \mathcal{U}$ and $V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that $W \subset U \cap V$;

(5.34) given $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $-V \subset U$;

(5.35) given $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V + V \subset U$;

(5.36) given $x \in T$ and $U \in \mathcal{U}$ there exists a positive integer m such that $x \in mU$;

where $\mathcal{U}$ is the set of all open sets of T containing Θ. Hence (2.31), (2.32), (2.34) and (2.35) of definition 2.31 are satisfied by $\mathcal{U}$.

Now we have by (5.34) that given $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $-V \subset U$ or what is equivalent $V \subset -U$.

By (5.33) there exists a $W \in \mathcal{U}$ such that $W \subset V \cap U \subset -U \cap U$. Hence $W \subset U$, $-W \subset U$. Then by the above and (5.35) we have that given $U \in \mathcal{U}$ there exist $V \in \mathcal{U}$ such that $V + V \subset U$, $V - V \subset U$ and $-V - V \subset U$. Hence given $U \in \mathcal{U}$ and m any integer there exists a $V_n \in \mathcal{U}$ such that $\pm V_n \pm \ldots (2^n \text{ summands}) \ldots \pm V_n \subset U$ and $2^n \geq |m|$. Hence for $|m_1| \leq |m| \leq 2^n$, $m_1 V_n \subset U$. Therefore (2.33) of definition 2.31 is satisfied by $\mathcal{U}$, and the first part of the theorem has been proved.

The converse is an immediate result of theorem 2.31 and theorem 2.24.

It then follows from theorem 3.51 that:

COROLLARY 5.31. Every P_1-space T with respect to A, $\mathcal{U}$, where A is the absolute-valued integral domain of all integers, is a t.a.g. Conversely every t.a.g. T can be pseudo-normed with respect to a strongly partially set D in such a manner that the pseudo-norm generates a neighborhood system equivalent to a complete neighborhood system of the zero element, and T is a P_1-space with respect to A, D.

N.B. It follows from (3.43) of definition 3.41 that the pseudo-norm which may be defined for a t.a.g. T (as in theorem 3.51) is integral-valued and that $//x, d// = 0$

for any $d \in D$ implies that $x = \Theta$. By the above corollary all the theory that we have developed for P_1-spaces is applicable to Michal's "topological abelian groups."

Example.

We shall now show by an example that every P_1-space w.r.t. A, D is not a t.a.g. We know that if T is a P_1-space w.r.t. A, D , then T is a topological abelian group, and though it is true that given $x \in T$ and $U(d) \in \mathcal{U}$ there exist $\alpha \in A$ such that $x \in \alpha U(d)$, this does not imply that there exists an integer $m \in A$ with the above property.

Let R_p be a p-adic number field.[1] Then given $r \in R_p$ the valuation of r , $/r/$, is defined as follows

$$/r/ = \rho^{\omega} \text{ , where } 0 < \rho < 1 \text{ and } \omega \text{ is}$$

the order of r .

This valuation can be shown to be a non-trivial, non-archimedean valuation of R , the field of rational numbers. I.e., for each $r, r_i \in R_p$, $i = 0,1,2,\ldots$

$$(5.37) \quad /0/ = 0 \;;\; /r/ > 0 \;,\; r \neq 0 \;;$$

$$(5.38) \quad /r_1 r_2/ = /r_1/\,/r_2/ \;;$$

$$(5.39) \quad /r_1 + r_2/ \leqq \max(\,/r_1/ \text{ and } /r_2/\,) \;;$$

$$(5.310) \quad /r_o/ \neq 0, 1 \text{ for some } r_o \in R \;;$$

$$(5.311) \quad /n/ \leqq 1 \text{ for all integers } n \;.$$

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1 See [1] , pp. 289-292, for the definition of R_p and all the properties of p-adic number fields referred to here.

R_p is clearly a valued ring, and we shall show that R_p is a P_1-space w.r.t. R_p, R^+, where R^+ is the set of positive real numbers. Evidently R_p is a V_1-space with respect to R_p, and R^+ is a strongly partially ordered set with ordering as usual. Define

$$//r, \delta// = \delta\ /r/ \text{ for each } r \in R_p \text{ and } \delta > 0 .$$

It follows immediately from (5.37)-(5.310) that $//r, \delta//$ satisfies (3.31), (3.32), (3.34), (3.42) and (3.44). From (5.38) and (5.310) it follows that given $\varepsilon > 0$ there exist $r \in R$ such that $/r/ > \varepsilon$. Given $x \in R_p$ and $r \in R_p$, $r \neq 0$, $x = r(r^{-1}x)$. Then from the above we can pick for any given $\delta \in R^+$ and $x \in R_p$ an $r \in R_p$ such that $//r^{-1}x, \delta// \leq 1$, i.e., $/r/ \geq //x, \delta//$. Hence $//r, d//$ satisfies (3.51) of corollary 3.51. Therefore by corollary 3.51 R_p is a P_1-space w.r.t. R_p, R^+. It follows also, since the valuation of R_p is non-archimedean(i.e., $/r/$ satisfies (5.311)), that there exist $x \in R_p$ and $\delta \in R^+$ such that there exists no integer $n \in R_p$ with the property that $x = ny$ and $//y, \delta// \leq 1$. For if we pick x and δ to have the property that $//x, \delta// > 1$, then $1 < //x, \delta// = //ny, \delta// = /n/\ //y, \delta//$. Now $0 < //y, \delta// \leq 1$ requires that $/n/ > 1$. But by (5.311) no such integer exists.

Therefore we see that we have proved the following theorem.

THEOREM 5.32. *Let R_p be any p-adic number field and let R^+ be the set of positive real numbers. Then R_p is a P_1-space w.r.t. R_p, R^+, and R_p is not a t.a.g. in the sense of Michal.*

THEOREM 5.33. *Every additive and continuous function on a t.a.g. T to a t.a.g. T' is a Φ-linear function on T to T' where $\Phi(n) = n$ for any integer n.*

Proof: The proof of this is clear.

In what follows T, T' and T'' will be t.a.g.'s, and we shall deal with them as P_1-spaces pseudo-normed with respect to D, D', and D''. The definition of an M_1-differential[1] which follows is that of Michal translated in terms of the pseudo-norm.

DEFINITION 5.31. Let $f(x)$ be a function on an open set $S_y \subset T$ to T', $y \in S_y$. If

(5.312) there exists a function $f(y;\delta x)$ which is additive and continuous in δx on T to T';

(5.313) there exists a function $\varepsilon(y,x_1,x_2)$ with arguments in T and values in T' such that

$$\text{(a)} \quad \varepsilon(y,\Theta,x) = \Theta \text{ for all } x \in T;$$

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1 See [10].

(b) there exists a $d \in D$ such that

$\mathcal{E}(y, x_1, mx_2) = m\,\mathcal{E}(y, x_1, x_2)$ for all positive integers m, $/\!/x_1, d/\!/ \leq 1$ and all $x_2 \in T$;

(c) there exists an $e \in D$ such that given $d' \in D'$ there exist $d \in D$ such that $/\!/\mathcal{E}(y, x_1, x_2), d'/\!/' \leq 1$ for $/\!/x_2, e/\!/ \leq 1$ and $/\!/x_1, d/\!/ \leq 1$;

(5.314) there exists an open set $S_\Theta \subset T$, $\Theta \in S_\Theta$, such that

$$f(y + \delta x) - f(y) - f(y;\delta x) = \mathcal{E}(y, \delta x, \delta x)$$

for all $\delta x \in S_\Theta$;

then $f(x)$ is said to be <u>M_1-differentiable at</u> $x = y$. $f(y;\delta x)$ is said to be an <u>M_1-differential of</u> $f(x)$ <u>at</u> $x = y$ <u>with increment</u> δx.

THEOREM 5.34. <u>If</u> $f(x)$ <u>is a function on an open set</u> $S_y \subset T$ <u>to</u> T' <u>and if</u> $f(x)$ <u>is M_1-differentiable at</u> $x = y$, <u>then</u> $f(x)$ <u>is</u> Φ <u>-differentiable in the sense of definition</u> 5.21, <u>where</u> $\Phi(m) = m$ <u>for each integer</u> m. <u>In this case we have also that the two differentials are equal.</u>

Proof: In order to prove this theorem we need only show that $F(\delta x) = \mathcal{E}(y, \delta x, \delta x)$ is a Φ-approximation function with $\Phi(m) = m$. This follows since we know by theorem 5.33 that an M_1-differential $f(y;\delta x)$ is Φ-linear in δx.

We have by (5.313(c)) that there exists an $e \in D$ such that given $d' \in D'$ there exist $d \in D$ such that $//F(\delta x), d'//' \leqq 1$ for $//\delta x, e// \leqq 1$ and $//\delta x, d// \leqq 1$. therefore $F(\delta x)$ satisfies (5.11) of definition 5.11. By (5.313(b))

$$//mF(\delta x), d'//' = //\mathcal{E}(y, \delta x, |m| \ \delta x), d'//'$$

for all $//\delta x, d// \leqq 1$ and m any integer. Hence by (5.313(c))

$$//mF(\delta x), d'//' \leqq 1 \quad \text{for} \quad //|m|\delta x, e// = //mx, e// \leqq 1$$

and $//\delta x, d// \leqq 1$, and (5.21) of definition 5.11 is satisfied by $F(\delta x)$. Therefore $\mathcal{E}(y, \delta x, \delta x)$ is a Φ-approximation function. Since we have already shown the uniqueness of the Φ-differential, the M_1-differential, if it exists, must equal the Φ-differential. This completes the proof of this theorem.

CHAPTER 6

PSEUDO-NORMED LINEAR SETS OVER VALUED DIVISION RINGS. P_2-SPACES.

6.1. P_2-spaces.

DEFINITION 6.11. A V_1-space T w.r.t. A will be said to be a **V_2-space w.r.t.** A if A is a valued division ring.[1]

DEFINITION 6.12. Let E be a strongly partially ordered set with positive real numbers associated[2] and T a V_2-space w.r.t. A. Then T will be said to be **pseudo-normed with respect to** E if there exists a real-valued function $//x, e//$ on TE which satisfies the following postulates:

(6.11) $//x, e// \geqq 0$ for all $x \in T$ and $e \in E$; $//x, e// = 0$ for all $e \in E$ implies that $x = \Theta$;

(6.12) $//\alpha x, e// = /\alpha/ \, //x, e//$ for all $\alpha \in A$, $x \in T$ and $e \in E$; $//x, re// = r \, //x, e//$ for all $x \in T$ and $re \in E$ (i.e., $r > 0$ and $e \in D$ or $e \in E$);

(6.13) given $e \in E$ there exist $d \in E$ such that $//x + y, e// \leqq //x, d// + //y, d//$;

(6.14) $e \geqq d$ implies that $//x, e// \geqq //x, d//$ for all $x \in T$.

We shall say briefly that T is a **P_2-space w.r.t.** A, D.

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1 See II.

2 See VI. Also [8] and [9].

LEMMA 6.11. A necessary and sufficient condition that $/\alpha\beta/ = /\alpha/\ /\beta/$ for all α, β in a valued division ring A is that $/\alpha/\ /\alpha^{-1}/ = 1$ for all non-zero elements $\alpha \in A$.

Proof: The necessity is immediate since $/\alpha\beta/ = /\alpha/\ /\beta/$ implies that $/\alpha/\ /\alpha^{-1}/ = /\alpha\alpha^{-1}/ = /1/ = 1$.

Sufficiency: Assume that $/\alpha/\ /\alpha^{-1}/ = 1$ for all $\alpha \neq 0$. By hypothesis $/\alpha\beta/ \leq /\alpha/\ /\beta/$ for all $\alpha, \beta \in A$. Hence

$$/\alpha/ \leq /\alpha\beta/\ /\beta^{-1}/ = /\alpha\beta/\ /\beta/^{-1}$$

and

$$/\beta/ \leq /\alpha\beta/\ /\alpha^{-1}/ = /\alpha\beta/\ /\alpha/^{-1} .$$

Therefore

$$/\alpha/\ /\beta/ \leq /\alpha\beta/^2(/\alpha/\ /\beta/)^{-1} \leq /\alpha\beta/$$

and

$$/\alpha\beta/ = /\alpha/\ /\beta/ .$$

THEOREM 6.11. Every P_1-space T w.r.t. A, D, where A is a valued division ring such that $/\alpha\beta/ = /\alpha/\ /\beta/$ for all $\alpha, \beta \in A$, is a P_2-space w.r.t. to A, $E = R^+D$. Conversely, every P_2-space w.r.t. A, E is a P_1-space w.r.t. A, E with $/\alpha\beta/ = /\alpha/\ /\beta/$ for all $\alpha, \beta \in A$, and the new pseudo-norm $//x, e//^*$ is equivalent to $//x, e//$.

Proof: For the first part of the theorem we have by hypothesis that $/\alpha\beta/ = /\alpha/\ /\beta/$ and hence by lemma 6.11 $/\alpha/\ /\alpha^{-1}/ = 1$ for all $\alpha \neq 0$. Then by (3.41) of definition 3.41

$$/\!/\alpha x, d/\!/ \leqq /\alpha/ \; /\!/x, d/\!/ \quad ,$$

and

$$/\!/x, d/\!/ \leqq /\alpha^{-1}/ \; /\!/\alpha x, d/\!/$$

or

$$/\alpha/ \; /\!/x, d/\!/ \leqq /\!/\alpha x, d/\!/ \; .$$

Therefore $/\!/\alpha x, d/\!/ = /\alpha/ \; /\!/x, d/\!/$ for all $\alpha \in A$, $x \in T$ and $d \in D$, since the equality evidently holds for $\alpha = 0$.

Define

$$E = R^{+}D = \left[rd; \; d \in D \; , \; r > 0 \right] \quad ,$$

and

$$/\!/x, rd/\!/ = r \; /\!/x, d/\!/ \; .$$

Then E is a strongly partially ordered set with $e_1 \trianglelefteq e_2$, $e_i = r_i d_i \; (i = 1,2)$, if $/\!/x, e_1/\!/ \trianglelefteq /\!/x, e_2/\!/$, and $e_1 = e_2$ if $/\!/x, e_1/\!/ = /\!/x, e_2/\!/$ for all $x \in T$. Hence $1d = d$, and $r_1(r_2 d) = (r_1 r_2)d$ for all $d \in D$ and $r_1, r_2 > 0$.

It should then be clear by the definition of $/\!/x, e/\!/$ for $e \in E$ that $/\!/x, e/\!/$ satisfies (6.11), 6.12) and (6.14) of definition 6.12. Therefore in order to verify the first part of the theorem we need only show that $/\!/x, e/\!/$ satisfies (6.13).

By (3.42) of definition 3.41 we have that given $d \in D$ there exists a $d_1 \in D$ such that $/\!/x, d_1/\!/ \leqq 1$ and $/\!/y, d_1/\!/ \leqq 1$ implies that $/\!/x + y, d/\!/ \leqq 1$. Assume that $/\!/x, d_1/\!/ \geqq /\!/y, d_1/\!/$

and $\|x, d_1\| \neq 0$. Then by (3.43) of definition 3.41 we have that given $r > 0$ there exist $\alpha \in A$ such that $0 \leq |\alpha| - \|x, d_1\| \leq r$. Thus

$$\|\alpha^{-1}x, d_1\| = \frac{\|x, d_1\|}{\|x, d_1\| + r_o} \leq 1 , \quad r_o \leq r ,$$

and

$$\|\alpha^{-1}y, d_1\| \leq 1 .$$

Hence $\|\alpha^{-1}(x + y), d\| \leq 1$ or $\|x + y, d \leq |\alpha| \leq \|x, d_1\| + r$ for any $x, y \in T$ and any $r > 0$. Therefore

$$\|x + y, d\| \leq \mathrm{Max}(\|x, d_1\| , \|y, d_1\|)$$
$$\leq \|x, d_1\| + \|y, d_1\| .$$

Given $e = rd$ E we have that $\|x + y, e\| \leq \|x, rd_1\| + \|y, rd_1\|$ for $rd_1 \in E$, and the proof of the first part of the theorem has been completed.

Conversely let T be a P_2-space w.r.t. A, E . It is evident that $\|x, e\|$ satisfies (3.34), (3.41), and (3.42) of definition 3.31 and definition 3.41. In order to show that T is a P_1-space we have by corollary 3.51 that we need only show that $\|x, e\|$ satisfies (3.32), (3.44) and (3.51) .

(3.32): If $\|x, e\| \leq 1$ for all $e \in E$, then $\|x, re\| \leq 1$ or $\|x, e\| \leq r^{-1}$ for all $r > 0$. Hence $\|x, e\| = 0$ for all $e \in E$ and by (6.11) of definition 6.11 $x = \Theta$.

(3.51): Given $x \in T$ and $e \in E$ there exist $\alpha \in A$, $\alpha \neq 0$, such that $|\alpha| \geq \|x, e\|$. Pick $y = \alpha^{-1}x$. Then

$//y, e// \leq /\alpha/^{-1} //x, e// \leq 1$, $x = \alpha y$, and (3.51) is satisfied.

(3.44): Given $\alpha \in A$, and $e \in E$ we can pick $r > /\alpha/$. Hence if $//x, re// \leq 1$, then $//x, e// \leq r^{-1}$ and $//\beta x, e// = r^{-1} /\beta/ = 1$ for all $/\beta/ \leq /\alpha/$. Thus (3.44) of definition 3.41 is satisfied by $//x, e//$.

It then follows by corollary 3.51 that T may be pseudo-normed w.r.t. D in such a manner that T is a P_1-space w.r.t. A, E and the new pseudo-norm $//x, e//^*$ is equivalent to $//x, e//$.

By (6.12) of definition 6.11

$$//x, e// = /\alpha/ \; /\alpha^{-1}/ \; //x, e// \quad \text{for all } x \in T$$

and $e \in E$. If $x \neq \theta$, there exist $//x, e// > 0$. This follows from (6.11). Hence $/\alpha/ \; /\alpha^{-1}/ = 1$ for all $\alpha \in A$, $\alpha \neq 0$. By lemma 6.1 $/\alpha\beta/ = /\alpha/ \; /\beta/$ for all $\alpha, \beta \in A$, and this completes the proof of the theorem.

N.B. Hence we see by theorem 6.11 that every P_2-space is equivalent to an N_2-space[1] in the same manner that a P_1-space is equivalent to an N_1-space. (See theorem 3.51.) We might also note that since A is a division ring that (2.33) of definition 2.31 implies that given $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $\alpha V \subset \beta U$ for all $/\alpha/ \leq /\beta/$. We could consider here a generalization of a normed linear space. The type of definition that we might take is illustrated by theorem 1 and 2 of [8].

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[1] An N_2-space w.r.t. A, is defined to be an N_1-space w.r.t. $A, \mathcal{U}$ where A is a valued division ring such that $/\alpha/ \; /\beta/ = /\alpha\beta/$ for all $\alpha, \beta \in A$.

6.2. Linear functions on P_2-spaces.

If $f(x)$ is a Φ-linear function on a P_2-space T w.r.t. A, E to a P_2-space w.r.t. A', E', then $/\Phi(\alpha)/' \leq /\alpha/$; and if $f(x)$ is not the null Φ-linear function then it follows that $\Phi(\alpha)$ satisfies (4.15)-(4.18). Hence $\Phi(\alpha^{-1}) = (\Phi(\alpha))^{-1}$, and one may easily verify that $\Phi(A)$ is a division subring of A'. It is clear that $\Phi(\alpha)$ is an isomorphic mapping[1] of A on $\Phi(A) \subset A'$.

Hence we see that the existence of a Φ-linear function on T to T' requires that A' contain a division subring isomorphic to A. We shall now introduce a more general concept of a linear function.

N.B. Throughout the remainder of the chapter T, T' and T'' will represent P_2-spaces w.r.t. A, E ; A', E' and A'', E'' respectively. We shall in view of theorem 6.11 consider our pseudo-norms to be the pseudo-norms with respect to which the spaces are also P_1-spaces.

DEFINITION 6.21. An additive function $f(x)$ on T to T' will be said to be **linear**[2] if given $e' \in E'$ there corresponds an $e \in E$ such that $//x, e// \leq 1$ implies $//f(\alpha x), e'//' \leq /\alpha/$ for all $\alpha \in A$.

It follows immediately from theorem 4.11 that every

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[1] See [14], p. 9, definition 5.

[2] This definition could also have been given for P_1-spaces, though it does not seem to follow in P_1-spaces that the sum of two such linear functions is a linear function.

Φ-linear function on T to T' is a linear function on T to T'.

Definition 6.22. An additive function f(x) on T to T' will be said to be bounded if given $e' \in E'$ there correspond $e \in E$ such that

$$//f(x), e'// \leq //x, e// \quad \text{for all} \quad x \in T .$$

THEOREM 6.21. A necessary and sufficient condition that an additive function f(x) on T to T' be linear is that f(x) be bounded.

Proof: The proof is quite similar to that of theorem 4.11, since we are considering our P_2-spaces to be pseudo-normed in such a manner that they are also P_1-spaces. The proof will be left to the reader.

It is then clear by virtue of theorem 6.21 that a linear function maps bounded sets into bounded sets.

THEOREM 6.22. A necessary and sufficient condition that a set $S \subset T$ be bounded is that given an open set U containing the zero element there corresponds an $\alpha \in A$ such that $S \subset \alpha U$.

Proof: The sufficiency follows from theorem 4.12, since T is also a P_1-space.

Necessity: Since $\left[//x, e// \; ; \; x \in S \right]$ is bounded for each $e \in E$, we have that $S \subset \alpha U(e)$ for some $\alpha \in A$. This can be shown in the following manner. There exist $/\alpha/ > 1$ and since $/\alpha\beta/ = /\alpha/\ /\beta/$, it is then true that given $r > 0$ we can find an $\alpha \in A$ such that $/\alpha/ > r$. Given $e \in E$ there exists an $\alpha \in A$ such that $//x, e// \leq /\alpha/$ for all $x \in S$. Hence by (6.12) $//\alpha^{-1}x, e// \leq 1$, i.e., $\alpha^{-1}S \subset U(e)$ or $S \subset \alpha U(e)$. Now given an open set U, $\Theta \in U$, there exists by corollary 3.31(a) a $U(e) \subset U$, and the necessity of the conditon follows from the above.

Let F be the set of all linear functions on a P_2-space T w.r.t. A, E to a P_2-space w.r.t. A', E'. Then $M(f,w)$ can be defined as in theorem 4.13 for all $f \in F$ and $w \in W$, and the following theorem can then be proved.

THEOREM 6.23. <u>The set F of all linear functions on T to T' can be pseudo-normed with respect to the strongly partially ordered set W in such a manner that F is a P_2-space with respect to A', W. $M(f,w)$ is the pseudo-norm.</u>

<u>Proof:</u> It follows from theorem 6.12 and (6.13) of definition 6.12 that for any α_1', $\alpha_2' \in A'$ and f_1, $f_2 \in F$ that $\alpha_1' f_1 + \alpha_2' f_2 \in F$. It can then be easily verified that F is a V_2-space with respect to A'. Define $rw = (re', S)$ and hence

$$M(f,rw) = rM(f,w) \ . \quad (r > 0) \ .$$

It can then easily be verified, using the properties of $//f(x),e'//'$ and the definition of $M(f,w)$ (see theorem 4.13), that F is a P_2-space w.r.t. A', R^+W. In this case $R^+W = W$.

6.3. <u>Φ-differentials of functions with arguments and values in P_2-spaces.</u>

<u>THEOREM</u> 6.31. <u>A necessary and sufficient condition that a function</u> $F(x)$ <u>on</u> $S_y \subset T$ <u>to</u> T' <u>be a</u> Φ-<u>approximation function is that there exists an</u> $e \in E$ <u>such that given a</u> $d' \in E'$ <u>there corresponds a</u> $d \in E$ <u>such that</u>

$$//F(x),d'//' \leq //x,e//$$

<u>for all</u> $//x,d// \leq 1$.

<u>Proof:</u> The sufficiency is clear since

$$//\Phi(\alpha)F(x),d'//' \leq /\alpha/ \ //x,e// = //\alpha x,e//$$

for all $//x,d// \leq 1$. Hence $//\alpha x,e// \leq 1$ and $//x,d// \leq 1$ implies that

$$//\Phi(\alpha)F(x),d'//' \leq 1 \ .$$

Let us assume that $F(x)$ is a Φ-approximation function. Then given $x \in T$, $//x,d// \leq 1$, $r > 0$ and $e \in E$ there exists $\alpha \in A$, $\alpha \neq 0$, and $y \in T$ such that $x = \alpha y$, $//y,e// \leq 1$ and $/\alpha/ - //x,e// \leq r$. Hence

$$//\Phi(\alpha^{-1})F(x),d'//' = 1 \quad ,$$

since $\|\alpha^{-1}x, e\| = \|y, e\| \leqq 1$ and $\|x, d\| \leqq 1$.

Therefore, due to the fact that $\Phi(\alpha^{-1}) = (\Phi(\alpha))^{-1}$,

$$\|F(x), d'\|' \leqq /\Phi(\alpha)/ \leqq /\alpha/ \leqq \|x, e\| + r$$

for $\|x, d\| \leqq 1$. This holds for any $r > 0$, and hence the theorem is proved.

With the aid of theorem 6.31 we can state the following corollary.

COROLLARY 6.31. A necessary and sufficient condition that a function $f(x)$ on an open set $S_y \subset T$, $y \in S_y$, to T' be Φ-differentiable at $x = y$ is that there exists a Φ-linear function $f(y;\delta x)$ of δx on T to T' with the property that there exist an $e \in E$ and to each $d' \in E'$ there corresponds a $d \in E$ for which

$$\|F(\delta x), d'\|' \leqq \|x, e\|$$

for all $\|x, d\| \leqq 1$, where

$$F(\delta x) = f(y + \delta x) - f(y) - f(y;\delta x) .$$

Then $f(y;\delta x)$ is the Φ-differential of $f(x)$ at $x = y$.

THEOREM 6.32. If $\{x_n\}$ is a sequence of elements of T , then a necessary and sufficient condition that $\lim x_n = x$ is that $\lim_{n \to \infty} \|x_n - x, e\| = 0$ for each $e \in E$.

Proof: The sufficiency is clear from corollary 3.31(d). In order to prove the necessity we have by corollary 3.31(d) that given $e \in E$ there exists an integer $m = m(e)$ such that $n > m(e)$ implies that $//x - x_n, e// \leqq 1$. Hence given $r > 0$ we have that for $n > m(r^{-1}e)$ $//x - x_n, e// \leqq r$. Therefore $\lim_{n\to\infty} //x - x_n, e// = 0$ for all $e \in E$.

COROLLARY 6.32. A necessary and sufficient conditionthat $\lim x_n \doteq x$ is that there exist an $r > 0$ such that given $e \in E$ there corresponds an $m = m(e)$ such that $n > m(e)$ implies that $//x - x_n, e// \leqq r$.

Proof: Follows from theorem 6.32, since $r^{-1}e \in E$ for all $r > 0$.

DEFINITION 6.31. If $f(\alpha,y)$ is a function on T' defined for some element $y \in T$ and all $0 < /\alpha/ \leqq r(y)$, then we shall say that

$$\lim_{\alpha\to 0} f(\alpha,y) = g(y)$$ [1]

if given $e' \in E'$ there exists a positive real number $r_o(e')$ such that $0 < /\alpha/ < r_o(e')$ implies that

$$//f(\alpha,y) - g(y), e'//' \leqq 1 .$$

N.B. We see by corollary 6.32 that if $\lim_{\alpha\to 0} f(\alpha,y) = g(y)$, then for any sequence $\{\alpha_n\} \in A$ such that $\alpha_n \neq 0$ (n=1,2,...) and $\lim_{n\to\infty} /\alpha_n/ = 0$ we have that

$$\lim f(\alpha_n,y) = g(y) .$$

Hence $\lim_{\alpha\to 0} f(\alpha,y)$ has the same properties as the limit [2]

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[1] It can be shown that this is equivalent to

$$\lim_{/\alpha/\to 0} //f(\alpha,y) - g(y), e'//' = 0 \quad \text{for all} \quad e' \in E' .$$

[2] See theorem 4.21.

of a sequence of elements of T' .

THEOREM 6.33. *If F(x) on an open set* $S_\Theta \subset T$, $\Theta \in S_\Theta$, *to* T' *satisfies the condition of theorem 6.31, then*

$$\lim_{\alpha \to 0} ((\Phi(\alpha))^{-1} F(\alpha x)) = \Theta \quad \textit{for each} \quad x \in T .$$

Proof: By hypothesis there exists an $e \in E$ such that given $d' \in E'$ there correspond $d(d') \in E$ such that

$$/\!/F(\alpha x), d'/\!/' \leq /\!/\alpha x, e/\!/$$

for all $/\!/x, d(d')/\!/ \leq 1$. For a given $y \in T$ pick $r_o(d')$ such that $r_o(d') > 0$ and

$$(r_o(d'))^{-1} \geq /\!/y, d(rd')/\!/$$

where $r > /\!/x, e/\!/$. Hence for $0 < /\alpha/ < r_o(d')$

$$/\!/\alpha y, d(rd')/\!/ \leq 1$$

and

$$/\!/F(\alpha y), rd'/\!/' \leq /\!/\alpha y, e/\!/ .$$

Since $/(\Phi(\alpha))^{-1}/ = /\Phi(\alpha^{-1})/ = /\alpha/^{-1}$,

$$/\!/(\Phi(\alpha))^{-1} F(\alpha y), d'/\!/' < 1$$

for all $o < /\alpha/ < r_o(d')$. y is an arbitrary element of T and this completes the proof of the theorem.

THEOREM 6.34. *If a function f(x) on an open set* $S_y \subset T$ *to* T' *is* Φ*-differentiable at* $x = y$, *then*

$$\lim_{\alpha \to 0} (\Phi(\alpha))^{-1}(f(y + \alpha\delta x) - f(y)))$$

exists and is equal to $f(y; \delta x)$.

Proof: By hypothesis

$$F(\delta x) = f(y + \delta x) - f(y) - f(y;\delta x)$$

is a Φ-approximation function, and hence by theorem 6.31 and theorem 6.33

$$\lim_{\alpha \to 0} ((\Phi(\alpha))^{-1} F(\alpha\delta x)) = \Theta$$

for all $\delta x \in T$.

Now

$$F(\alpha\delta x) = f(y + \alpha\delta x) - f(y) - \Phi(\alpha) f(y;\delta x)$$

and for $\alpha \neq 0$

$$(\Phi(\alpha))^{-1} F(\alpha\delta x) = (\Phi(\alpha))^{-1} (f(y + \alpha\delta x) - f(y)) - f(y;\delta x) \quad .$$

This last equation combined with the above completes the proof of the theorem.

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