TEMPERATURE PERTURBATIONS AND THEIR EFFECT ON THE TEMPERATURE MAXIMA AND MINIMA IN THE INTERIOR OF THE EARTH

Thesis submitted in partial fulfilment
of the requirements for the degree of
Master of Science

by

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May 29, 1941

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Abstract

This thesis investigates the effect of temperature perturbations in the earth on the change of position of local temperature maxima and minima prevailing in the earth's interior. The final solutions (mathematical) give the position and the velocity of the point of maximum temperature as a function of time. The solutions indicate that for certain limits (discussed) the position of the point of maximum temperature and its velocity increase exponentially with time. A method of correlating the rate of energy transfer in a given direction with the velocity of the point of maximum temperature in that direction is outlined. Suggestions dealing with the application of these solutions to geological problems are given.

Introduction

The problem was investigated in order to answer the following questions:

(1) What effect do temperature perturbations surrounding temperature maxima and minima in the earth have on their position, direction of motion, and velocity?

(2) What effect do temperature perturbations have on the rate of energy transfer from hot to cold regions in the earth?

(3) Applications of results from items one and two for interpretation of geological facts.

The problem was analyzed as a one dimensional one. The principal of the results can be applied to higher dimensional problems with slight corrections.

Analysis

Consider the equation for heat flow in dimension X

$$\frac{\partial^2 V'_{(x,t)}}{\partial x^2} = a^2 \frac{\partial V'_{(x,t)}}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots 1$$

Where:

$V'_{(x,t)}$ = Temperature as $f(x,t)$

x = Linear coordinate

t = Time

$a^2 = \frac{\rho c}{K}$

ρ = Density

c = Specific heat

K = conductivity

A solution for equation (1) can be obtained as follows:

Let

$$V'_{(x,t)} = V_{(x)} e^{-z^2 t} + c_1 x + c_2 \quad \ldots\ldots 2$$

Substituting this solution in equation (1), we get

$$\frac{\partial^2 V_{(x)}}{\partial x^2} e^{-z^2 t} = -z^2 a^2 V_{(x)} e^{-z^2 t}$$

$$= \frac{\partial^2 V_{(x)}}{\partial x^2} = -z^2 a^2 V_{(x)} \quad \ldots\ldots\ldots\ldots 3$$

If we can find a function V(x) to satisfy equation (3) then equation (2) will be a satisfactory solution for the fundamental differential equation (1). This follows, since if the solution given by equation (2) is satisfactory, it establishes the existence of equation (3). Therefore, if we find a solution for equation (3), the solution will satisfy equation (1); and, therefore, solution (2) would be satisfactory.

Let

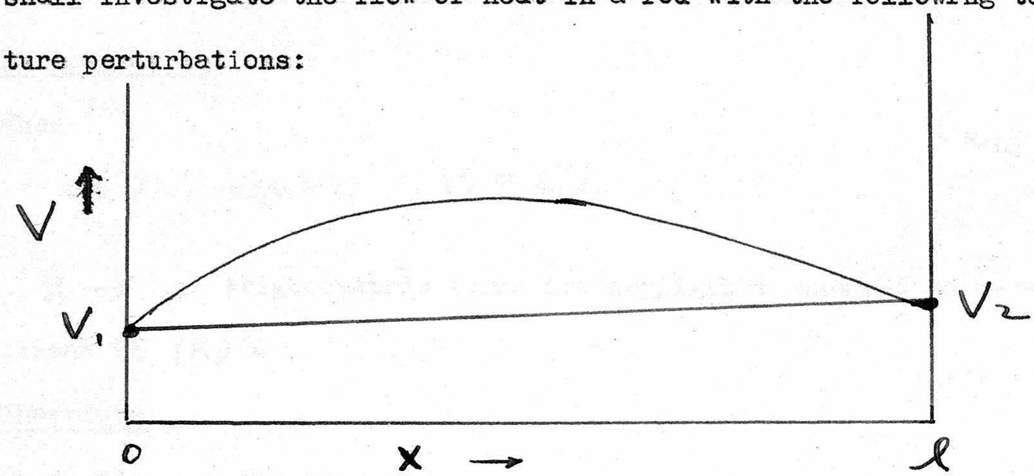
$$V(x) = A_N \sin Z_N a x + B_N \cos Z_N a x$$

$$\frac{\partial V}{\partial x} = Z_N a A_N \cos Z_N a x - Z_N a B_N \sin Z_N a x$$

$$\frac{\partial^2 V}{\partial x^2} = -Z_N^2 a^2 A_N \sin Z_N a x - Z_N^2 a^2 B_N \cos Z_N a x$$

$$= -Z_N^2 a^2 V(x) \qquad \ldots\ldots\ldots 4$$

This shows that equation (4) is a solution to equation (3). We shall investigate the flow of heat in a rod with the following temperature perturbations:



Boundary Conditions

$$t=0\ ,\quad V'(x) = V_1 + \frac{(V_2 - V_1)x}{\ell} + \sum_{N=1}^{\infty} \alpha_n \sin \frac{N\pi x}{\ell}$$

$$x=0\ ,\quad V' = V_1$$

$$x=\ell\ ,\quad V' = V_2 \qquad \ldots\ldots\ldots 5$$

For $0 < x < \ell$ we get a maximum which is represented by the Sine

Series. By inspection it is clear that the boundary conditions are satisfied.

Referring to equation (2) it follows from the analysis thus far that:

$$V'(x) = \sum_{N=1}^{\infty} A_N \sin Z_N a x + B_N \cos Z_N a x \qquad \ldots\ldots\ldots 6$$

$$+ C_1 X + C_2$$

$$V'(x) = V_1 + \frac{(V_2 - V_1)x}{\ell} + \sum_{N=1}^{\infty} \alpha_N \sin \frac{N\pi\ell}{\ell}$$

must equal equation Number (5) which satisfies the boundary conditions, as chosen.

By setting $X=0$ + $X=\infty$ and by equating $C_2 = V_1$ since they are both independent of (X), I get the following identities between the terms on the right and left hand sides of equation Number (6):

By inspection:

When

$$x = 0 \quad , \quad B_N = 0 \quad , V_1 = C_2$$

$X \rightarrow \infty$ trigonometric terms are negligible compared to terms linear in (X).

Therefore:

Which also implies that

$$Z_N a = \frac{N\pi}{\ell}$$

Therefore returning to our original solution for $V'(x,t)$ as a function of position and time, we get:

$$V'(x,t) = V_1 + \frac{(V_2 - V_1)x}{\ell} + \sum_{N=1}^{\infty} \alpha_N e^{-\frac{\pi^2 t}{\ell^2 a^2}} \sin \frac{n\pi x}{\ell} \qquad \ldots\ldots\ldots 7$$

Since the exponential terms decrease with the second power of N, it will be a plausible approximation to neglect terms in the Series for $N > 1$.

Rewriting equation Number (7) for

$$V'_{(x,t)} = V_1 + \frac{(V_2 - V_1)x}{\ell} + \alpha_1 e^{-\frac{\pi^2 t}{a^2 \ell^2}} \sin \frac{\pi x}{\ell} \qquad \ldots\ldots 8$$

By taking the gradient of V' in equation Number (8) and setting it equal to zero, we shall find the point of maximum temperature, i.e.: point of zero heat flow.

It is essentially the behavior of the surface of zero heat flow in relation to types of temperature perturbations already discussed, that this thesis is trying to investigate. By taking the gradient and equating it to zero, we have identified this point of maximum temperature mathematically and we may proceed to study its dynamic characteristics.

Taking the gradient of V' in equation (8) we get:

$$\frac{\partial V'}{\partial x} = \frac{V_2 - V_1}{\ell} + \frac{\pi}{\ell} \alpha_1 e^{-\frac{\pi^2 t}{a^2 \ell^2}} \cos \frac{\pi x}{\ell} \qquad \ldots\ldots 9$$

Where α_1 is maximum amplitude of the superimposed temperature perturbation.

Let

$$\frac{\partial V'_{(x,t)}}{\partial x} = 0$$

Therefore

$$V_1 - V_2 = \pi \alpha e^{-\frac{\pi^2 t}{a^2 \ell^2}} \cos \frac{\pi x}{\ell}$$

$$\therefore \cos \frac{\pi x}{\ell} = \frac{V_1 - V_2}{\pi \alpha} e^{\frac{\pi^2 t}{a^2 \ell^2}} \qquad \ldots\ldots 10$$

$$\therefore x = \mathrm{arc.}\cos \left(\frac{V_2 - V_1}{\pi \alpha} e^{\frac{\pi^2 t}{a^2 \ell^2}} \right) \frac{\ell}{\pi} \qquad \ldots\ldots 11$$

Equation (11) gives the position of the point of maximum temperature at a given time. In order to understand the manner in which "X" changes with time I shall rewrite "X" in terms of the arc cosine series of $\left(\frac{V_1 - V_2 e^{\frac{\pi^2 t}{a^2 l^2}}}{\pi \alpha}\right)$

If $y = \cos^{-1} x$ the expansion of the $\cos^{-1} x$ series is given by the following:

$$\cos^{-1} x = \left[\frac{\pi}{2} - \left(x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots\cdots\right)\right.$$

NOTE, THE ODD POWERS OF "X"

Since $V_2 > V_1$ it is clear that the term $\left(\frac{V_1 - V_2 e^{\frac{\pi^2 t}{a^2 l^2}}}{\pi \alpha}\right)$ in equation (11) which corresponds to the "X" term in the series expansion, is always negative. Since all the powers of "X" in the series are odd, it is clear that the

$$\text{arc. } \cos \left(\frac{V_1 - V_2 e^{\frac{\pi^2 t}{a^2 l^2}}}{\pi \alpha}\right)$$

series will increase exponentially with time. It is also clear from equation (11) that "X" is directly proportional to the difference in temperature $(V_1 - V_2)$ and is inversely proportional to the maximum temperature of the superposed Sine perturbation denoted by the α term. Even/though the periodicity of the arc cos term is of general importance; it is not important in this specific problem since the problem itself is limited to only one cycle. The purpose of the series expansion was simply to determine the trend of variation of the position of the point of maximum temperature with time.

I shall next calculate the rate of motion of the point of maximum temperature. Differentiating equation (10) with respect to time, we obtain the following:

$$-\frac{\pi}{l}\sin\frac{\pi x}{l}\,dx = \frac{(V_1-V_2)\pi^2 e^{\frac{\pi^2 t}{a^2l^2}}}{\pi\alpha a^2 l^2}\,dt$$

Therefore:

$$\frac{dx}{dt} = \frac{X\pi(V_2-V_1)e^{\frac{\pi^2 t}{a^2l^2}}}{\alpha a^2 l \sin\frac{\pi x}{l}}$$

By using equation (10) and substituting

in order to eliminate "X" in the right side of the equation we get:

$$\frac{dx}{dt} = \frac{\pi(V_2-V_1)e^{\frac{\pi^2 t}{a^2l^2}}}{a^2 l\left[\pi^2\alpha^2-(V_1-V_2)^2 e^{\frac{2\pi^2 t}{a^2l^2}}\right]} \quad \ldots\ldots 12$$

By inspecting equation (12) carefully it is possible to judge the trend in the variation of velocity with time. It is clear that the solution given by equation 12 becomes imaginary when:

$$\pi\alpha^2 = (V_1-V_2)^2 e^{\frac{2\pi^2 t}{a^2l^2}}$$

In order to find out if such a condition of a physically meaningless solution could arise in the present problem, I shall introduce some boundry condition and study equation (12).

When $t=0$ the solution is real, since by definition $\pi\alpha^2 > (V_1-V_2)^2$ The solution does become imaginary for "t" sufficiently large. From the continuity of the phenomena of the problem there must exist values for "t" for which equation (12) is real. In this range of time it is clear from equation (12) that the velocity increases exponentially with time. The next thing to be done is to determine the range in "X" corresponding to the range in time (t) for which equation (12) is real.

The points along the rod for which equation (12) becomes imaginary are the two points which are maintained at temperatures V_1 & V_2. This can be proven by substituting respectively o & 1 for "X" in equation (10).

I get:

$$V_1 - V_2 = \pi\alpha e^{-\frac{\pi^2 t}{a^2 l^2}}$$

$$\text{SINCE, } \cos\frac{\pi X}{l} = \pm 1, \text{ FOR } X=0, \& X=l$$

$$\therefore (V_1 - V_2) e^{\frac{\pi^2 t}{a^2 l^2}} = \pi\alpha \qquad \ldots\ldots\ldots 10'$$

Equation (10') proves that equation (12) becomes imaginary at 0 & l since equation (10') is the same equation that is obtained from the denominator of equation (12) when it becomes imaginary. This is a proof that $X=0$ & $X=l$ is a sufficient condition. The proof that it is a necessary condition, can be found by reversing the line of reasoning. This result is of utmost importance, for it permits us to draw the following conclusions.

The velocity of the point of maximum or minimum temperature increases exponentially with time for the interval $X=0$, $X=l$.

Equation (12) also shows that:

(1) The velocity of the point of maximum temperature is directly proportional to the temperature differential

(2) The velocity of "X" is inversely proportional to "l"--the distance between the points maintained at temperatures V_1 & V_2.

Of outstanding importance is the fact that the velocity of the point of maximum temperature increases exponentially with time.

In order to illustrate an application of our final solution, let us calculate a specific case in the earth in c.g.s. units:

$\rho = 3$	density of Granite
$C = .2$	specific heat
$K = .004$	conductivity
$5 \times 10^5 = l$	Centimeters - diameter of a Batholith
$V_2 - V_1 = 200^\circ$	Centigrade
$V_M = \alpha = 2000^\circ$	Centigrade - temperature of Batholith

With these parameters the solution for $\frac{dx}{dt}$ is approximately 1 CM. per day after 5 years. As time increases, the velocity will increase exponentially with it as shown by equation Number (12).

Next, I will show a correlation between the motion of this point and the energy transfer in both directions from which the net energy transfer in either direction may be calculated. The procedure for calculating energy transfer in terms of the point of maximum temperature is as follows:

The energy stored in a rod maintained at a temperature "T_1" is given by

$$E_1 = C \int_0^l \int_{t_0}^{t} T_1 \, dx \, dt \qquad \ldots\ldots 13$$

<u>Where</u>

dx	=	dimension along rod
dt	=	time (Δt)
C	=	specific heat

In this case we are interested in knowing the amounts of energy stored in the rod on either side of the point of maximum temperature along the rod, which is a function of both position and time. We can carry out the integration by setting the upper limit for X in the integral as is determined by equation Number (10) for a given time, "t"[1]. On one side of the point, the integration will be from 0 to X, AND ON OTHER SIDE THE INTEGRATION WILL BE FROM, $\ell - x$, TO ℓ i.e.

$$E_2 = C \int_{\ell-x}^{\ell} \int_{t_0}^{t} T_2 \, dx \, dt \qquad \ldots\ldots 14$$

I have hereby attempted to establish a mathematical correlation between the velocity of the surface and the energy transfer. A more direct procedure would be to solve equations Number (7) and (10) simultanuously. This would automatically give "V" as a function of (X, t) of the point of maximum temperature. We are primarily interested in the difference in energy transfer, i.e.: $E_1 - E_2 = \Delta E$. It is also clear by inspection that this energy transfer is proportional to the velocity of the point of maximum temperature along the rod.[2] This is true since the energy stored in an element of rod "dx" is directly proportional to dx which is directly proportional to the velocity of the point of maximum temperature[2], on one side of the maximum, and inversely proportional on its other side.

If we consider the difference in energy transfered from the point of maximum temperature to both sides of the rod, we can calculate qualitatively how the energy difference will change with time assuming other factors to remain constant with velocity.

1. Substitute this "X" into temperature distribution function.
2. PROOF, WILL FOLLOW

Since on one side of the point of maximum temperature "X" increases with velocity and on the other side it decreases inversely to it, we may write

$$E_1 - E_2 = \Delta E = \left(K_1 V_1 - \frac{K_2}{V_1}\right) = \frac{K_1 V_1^2 - K_2}{V_1} \quad \ldots\ldots 15$$

Where K_1 & K_2 are independent of velocity, This equation shows that the difference in energy transfer (E_1-E_2) as defined increases with velocity.

Geological Application of the Analysis

(1) Periodic Volcanism, i.e., rejuvenated volcanos

(2) Local metamorphosis of country rock.

(3) Clarification of the Pocket Hypothesis and its possibilities.

(4) Local Thermal Ore deposites. Daly points out that the pocket hypothesis is inconsistent with the theory of Cooling of the earth and Volcanism active, dormant, and extinguished.

The results of the analysis can be applied to the problem of the source region for magma. From field and laboratory observations, it appears the subcooling of natural magmas occur primarily at the surface, where rapid radiation gives the vitreous state represented by the Obsidian and the Trachylite. The hypothesis that each region acting as a heat source is a local one rather than a continuous earth shell of Basalt feeding the distributed channels of magma, seems inconsistant with the theory of a cooling earth. The pocket hypothesis is not in agreement with the occurance of BOTH active and inactive volcanos.

The analytical solution that we have obtained indicates that both hopotheses are plausible under adverse circumstances. In case there is an isotropic cooling condition which would exist were the temperature distribution of the rock surrounding the magma the same, the pocket hypothesis would seem reasonable. Otherwise, the pocket hypothesis seems unlikely. We may reason that since temperature perturbations are known to exist, the pocket magma hypothesis is unreasonable as a result of the theoretical conclusion of the thesis. This line of reasoning is consistant with the earth's Cooling hypothesis. The pocket magma hypothesis is inconsistant with the earth's Cooling hypothesis and is also inconsistant with the theory of this thesis, if cooling is anisotropic.

Conclusion

(a) When the boundaries of a temperature maximum or minimum are maintained at more or less constant but different temperatures, the velocity of the point or surface of maximum temperature will increase exponentially with time.

(b) The velocity of the point of maximum temperature in a given direction is proportional to the energy transfer in that direction.

(c) The velocity of points of maxima and minima of temperatures in the earth are directly proportional to the difference in the temperatures maintained at its boundaries. The rate of motion of this surface is inversely proportional to the magnitude of the maximum temperature.

(d) According to Chapman, in the deeper layers of the earth the Electric conductivity is high. It follows from the Franz-Weideman relationship[1] that the thermal conductivity is also high in such layers. Where the conductivity is high, the rate of motion of the point of maximum temperature is large.

(e) We should expect the points or surfaces of maxima to shift most rapidly in the deeper layers of the earth, when other favorable circumstances exist.

(f) Economic geological theories pertaining to thermal deposits can be interpreted more clearly and accurately with the results indicated by the analysis.

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1. The Franz-Weideman relationship has been verified only in metals which does imply that it may be applied to rock.

APPENDIX

Thus far it was assumed that the temperatures V_1 & V_2 remain constant throughout the time of motion of the point of maximum temperature. This assumption would be valid were the conductivity at the points of temperature V_1 & V_2 infinite. Since my assumption, i.e., boundry conditions deviate from the actual boundry condition, I shall give a method of better approximation to true conditions.

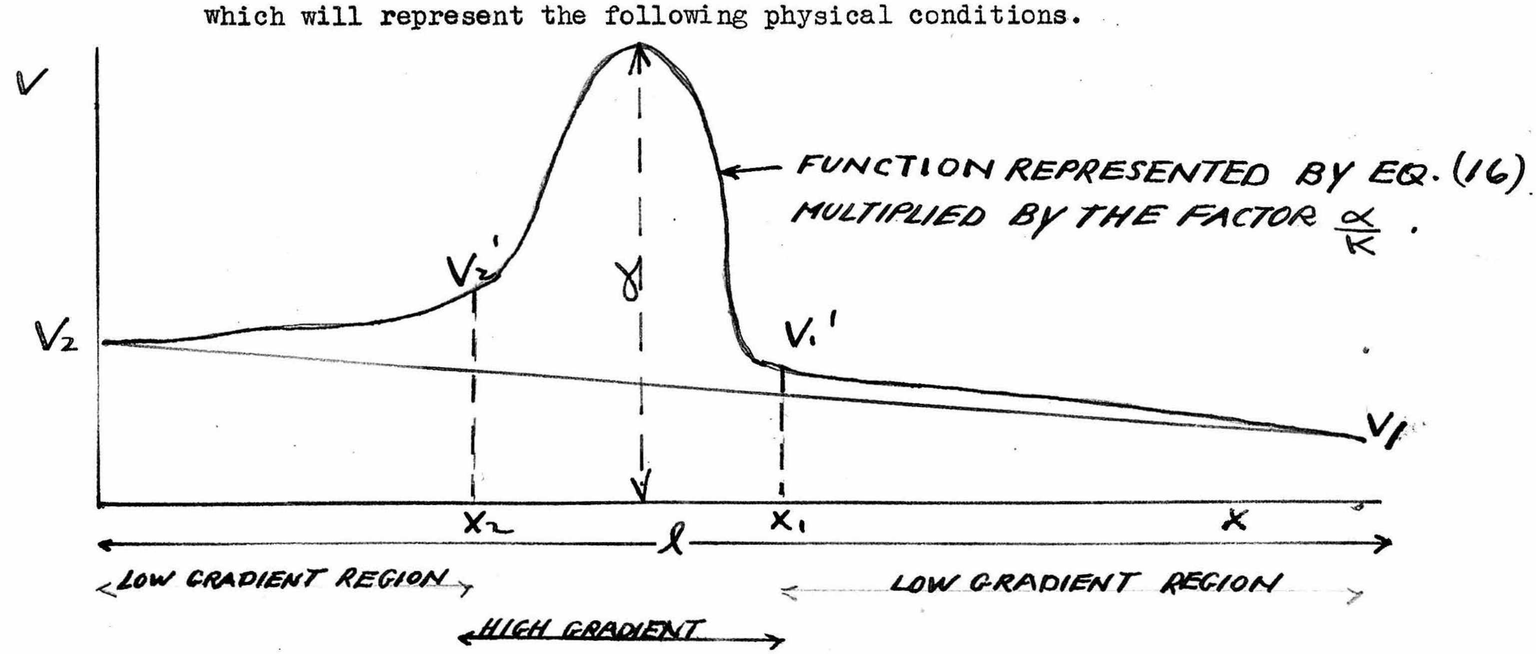
<u>Procedure</u>

The principal mathematical procedures and solutions used in the thesis thus far will remain unaltered. The primary modification will rest in the Sine function that was superimposed on the $\frac{(V_2 - V_1)X}{l}$ term in EQ. (5).

Instead of superimposing a simple Sine function, I can use an odd Fourier series of the form

$$y = \sum_{N=1}^{K} (-1)^N \sin \frac{(2N+1)\pi X}{l} \quad \ldots\ldots\ldots\ldots 16$$

which will represent the following physical conditions.



Here "1" again represents the distance between the points corresponding to V_1 & V_2. In this case assume that at $t=0$ x_1 + x_2 are points along the rod maintained at temperatuer V_2' & V_1' respectively. The boundry conditions are the same at 0 and 1 as they were originally. In order to investigate the changes in temperatures V_1 & V_2 with time, a Fourier series will give a much better approximation than would a Sine function. For this problem an odd Fourier series of the form

$$y = \sum_{N=0}^{K} (-1)^N \sin \frac{(2N+1)\pi x}{l}$$

is appropriate. It gives a maxima at $x=\frac{1}{2}$ and all of its harmonics also have their maxima at $x=\frac{1}{2}$. Since the series is a diverging series the approximation of the first few terms will have to suffice. I could construct a convergance factor but physically it would not improve the analysis. Note that in using the Fourier series we have two regions, both of a low temperature gradient, i.e., from 0 to x_1', from x_2' to 1. By using the Fourier series as given by the equation in equation (16) I eventually obtained the following equation:

$$V_1 - V_2 = \frac{\alpha}{K} \sum_{N=1}^{K} (2N+1)\pi \, e^{-\frac{(2N+1)\pi^2 t}{a^2 l^2}} \cos \frac{(2N+1)\pi x}{l}$$

This equation corresponds to equation Number 10 in the previous analysis where only the Sine perturbation was used. In equation Number (6)

α = Maximum temperature of the superimposed Fourier perturbation

K = The sum of the series to the K^{TH} term for $x=\frac{1}{2}$

The solution given by equation (17) can be obtained by substituting

$T = \frac{\alpha}{K} \sum (-1)^N \sin \frac{(2N+1)\pi x}{l}$ . . . . into equation Number (6), taking its gradient and equating it to zero. Handling this equation will involve great difficulty. In order to solve for values of x_1 & t for which V_2-V_1 constant, I would suggest the following procedure:

(1) Let $V_2-V_1=$ constant

(2) Plot this constant on a graph for $t=0$

(3) Let "t" = constant and by varying x on the same graph, we obtain AN intersection with the V_2-V_1 constant curve. This process can be repeated for various values of "t". In order to check the validity of the original assumption made in this thesis concerning a constant value for V_2-V_1 we can observe by the suggested graphical solution how V_2-V_1 deviate from its initial value at $t=0$. The actual graphical solution of this problem will be left for the future.