A Theory of Projections in Complex Banach Spaces

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<u>Introduction</u>. A Projection is defined as a linear operator P, such that $P^2 = P$. A theory of projections has been developed in complex Banach Spaces, which are reflexive. This was done in 1939 by E. R. Lorch* A complex Banach Space is a complex linear vector space in which a norm, with the usual properties, has been defined. This paper develops the usual theorems concerning projections, which may be found for Hilbert Space in M. H. Stone** The method of proof consists essentially in the use of an inter-space inner product, similar to that of A. D. Michal and D. H. Hyers for real Banach Spaces***

If f is an element of a complex Banach Space B, and F an element of the space (B) of all complex valued linear functionals defined on B, then (B) can be shown to be a space of the same type as B, the norm of F, $\|F\|$ being defined as the bound of the functional F(f), $|F(f)| \leq \|F\| \|f\|$. The interspace inner product [F,f] $f \in B$, $F \in (B)$ has certain demonstrable properties, see II below.

The first paragraph is devoted to an existence proof for complex number valued linear functionals defined on a complex Banach Space. Before the theory of projections is discussed, two paragraphs discuss some of the properties of the inner product and of closed linear manifolds in Banach Spaces. The theory of projections developed in the central portion of the paper is applied in the last two paragraphs; firstly to show that

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* E. R. Lorch. A Calculus of Operators in Reflexive Vector Spaces. Trans. A.M.S. 45:217-234 (1939).

** M. H. Stone. Linear Transformations in Hilbert Space.

*** A. D. Michal and D. H. Hyers. General Differential Geometries with Coordinate Interspace Inner Product. The Tôhoku Math. Jour. 46:309-318 (1940).

$[F, \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} A^i f] = [F, Pf]$, A being a linear operator and P a projection; secondly to developing a theory of reducibility of linear operators defined on B. This latter is essentially an examination of the set of points in B which a linear operator A leaves invariant.

I. Linear Functionals

It is the purpose of this paragraph to demonstrate the existence and to develop certain of the properties of complex valued linear functionals defined on a complex Banach Space.

Let E be a complex vector space, that is a linear space with complex number multipliers. Let F be a complex number valued functional defined on E, which is additive

$$F(f + g) = F(f) + F(g), \quad f, g \varepsilon E,$$

and homogeneous

$$F(\sigma f) = \sigma F(f) , \qquad \sigma \text{ - a complex number.}$$

<u>Theorem 1.1</u>. If there exists a functional p(f) on E to the complex numbers

$$|p(f + g)| \leq |p(f)| + |p(g)| \qquad |p(\sigma f)| = |\sigma| |p(f)|$$

and if there exists an additive and homogeneous functional G(f) defined on a complex linear subset $E_1 \subset E$, such that $|G(f)| \leq |p(f)|$ for all $f \varepsilon E_1$, then there exists an additive and homogeneous functional F(f), such that

$$|F(f)| \leq |p(f)| \text{ for all } f \varepsilon E, \quad F(f) = G(f) \text{ for all } f \varepsilon E_1.$$

<u>Proof</u>. If $E_1 = E$ the theorem is trivial. Let $f_0 \varepsilon E - E_1$. If f_1, $f_2 \varepsilon E_1$, then

$$|G(f_1) - G(f_2)| = |G(f_1 - f_2)| \leq |p(f_1 - f_2)|$$

$$\leq |p(f_1 + f_0)| + |p(-f_2 - f_0)|$$

$$|(|G(f_1)| - |G(f_2)|)| \leq |p(f_1 + f_0)| + |p(-f_2 - f_0)|$$

Then, since

$$|(|G(f_1)| - |G(f_2)|)| \geq |G(f_1)| - |G(f_2)| ,$$

$$|G(f_1)| - |G(f_2)| \leq |p(f_1 + f_0)| + |p(-f_2 - f_0)|$$

and

$$-|p(-f_2 - f_0)| - |G(f_2)| \leq |p(f_1 + f_0)| - |G(f_1)|.$$

Thus there exists some complex constant ρ_0, such that for all $f \varepsilon E_1$

(1) $-|p(-f - f_0)| - |G(f)| \leq |\rho_0| \leq |p(f + f_0)| - |G(f)|$.

Consider all elements of the form

(2) $g = f + \sigma f_0$

where $f \varepsilon E_1$, $f_0 \varepsilon E - E_1$, σ is complex number, and designate by E_0 the class of all such elements. Evidently E_0 is a complex vector space. Let

(3) $\phi(g) = G(f) + \sigma\rho_0$ then

(3') $|\phi(g)| \leq |G(f)| + |\sigma\rho_0|$

Since $f_0 \varepsilon E - E_1$ and $f \varepsilon E_1$, all $g \varepsilon E_0$ admit exactly one representation of the form (2). Thus the functional $\phi(g)$ is defined in E_0 in a unique manner.

In the inequality

$$|\rho_0| \leq |p(f + f_0)| - |G(f)|$$

write $\frac{f}{\sigma}$ in place of f, then

$$|\sigma\rho_0| \leq |p(f + \sigma f_0)| - |G(f)|.$$

Substituting this in (3')

$$|\phi(g)| \leq |p(f + \sigma f_0)| = |p(g)|$$

(4) $|\phi(g)| \leq |p(g)|$ for $g \varepsilon E_0$.

The functional $\phi(g)$ is evidently additive and homogeneous.

When $f_0 = 0$, the inequality (1) shows that $|\rho_0| = 0$ since $p(0) = 0$ is a particular case. Thus

(5) $\phi(g) = G(g)$, $g \varepsilon E_1$.

Thus $\phi(f)$ is the required functional for the space E_0, (the space E_1 to which have been added those elements which consist of a complex number times a fixed element f_0 in $E - E_1$). Repeating the above construction for $E - E_0$ and a fixed element $f_0' \varepsilon E - E_0$ a further extension of G(f) may be obtained. Whence by transfinite induction, the function $F(f)$ exists satisfying the condition of the theorem.

Consider the class of all complex valued linear functionals defined on a complex Banach Space B, $f, g \varepsilon B$. Then if the norm of $F, \|F\|$, is the bound of $F(f)$, $|F(f)| \leq \|F\| \|f\|$. The class of linear functionals so defined form a Banach Space $(\bar{B})$ of the same type as B.

<u>Theorem 1.2</u>. Given a linear functional $G(f)$ defined on a complex vector space $E \subset B$, there exists a linear functional $F(f)$ defined on B and satisfying the conditions

$$F(f) = G(f) \text{ for } f \varepsilon E \text{ and } \|F\| = |G|_E.$$

$|G|_E$ is the bound of the functional $G(f)$ on E.*

<u>Proof</u>. In Theorem 1.1 substitute $p(f) = |G|_E \|f\|$. It evidently satisfies the conditions placed on $p(f)$. Then as a consequence of Theorem 1.1 there exists a functional defined on B such that $F(f) = G(f)$ for all $f \varepsilon E$. Since the two functionals are equal for all $f \varepsilon E$, their bounds are equal, $\|F\| = |G|_E$.

<u>Theorem 1.3</u>. For each $f_0 \varepsilon B$ there exists a linear functional $F(f_0)$ defined on B, such that

$$|F(f_0)| = \|f_0\|, \ \|F\| = 1.$$

<u>Proof</u>. In Theorem 1.2 let E be the set of elements of the form τf_0, and substituting $|G(f)| = |\tau| \|f_0\|$; $|F(f_0 \tau)| = |\tau| \|f_0\|$; $|F(f_0)| = \|f_0\|$

Since by Theorem 1.2 $\|F\| = |G|_E$; $\|\tau F\| = |\tau|$ and $\|F\| = 1$.

The above theorem shows that the set of complex number valued linear

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* Bohnenblust and Sobcyk have given a different proof of this generalization of the Hahn - Banach Theorem. Bull. A.M.S. 44:91-93 (1938).

functionals defined on a complex Banach Space is non-null.

II. The Inter-Space Inner Product

Using Theorem 1.3, it is possible to demonstrate the existence of an inter-space inner product with the following properties. If B is a complex Banach Space and (B) the set of complex number valued linear functionals defined on B to the complex numbers, then for $f \varepsilon B$, $F \varepsilon (B)$:

(1) $[F,f]$ is bilinear on (B), B to the complex numbers,

(2) If $[F,f] = 0$ for all f then $F = 0$,

(3) If $[F,f] = 0$ for all F then $f = 0$.

The first two properties are an immediate result of the correspondence

$$F(f) = [F,f].$$

The third property follows from Theorem 1.3, since we can write for each $f_0 \varepsilon B$

$$|[F,f_0]| = \| f_0 \|$$

Thus $[F,f] = 0$ for all F implies $f = 0$.

<u>Definition 2.1.</u> An <u>operator</u> is an additive and continuous or linear transformation whose range and domain are both in B or both in (B). The letters A, B, P,... are used to represent linear operators; $|A|$, $|B|$, $|P|$,..., their bounds, if they exist.

The space (B) is said to be the adjoint or conjugate space to B.

<u>Definition 2.2.</u> The operator A_1 defined in B and the operator A_2 defined in (B) are said to be <u>adjoint</u>, if

$$[A_2 F,f] = [F,A_1 f]$$

for all f in the domain of definition of A_1, and all F in the domain of definition of A_2. The adjoint of an operator A is written A^*.

III. Linear Manifolds in Complex Banach Spaces

Definition 3.1. A set of elements M such that for $f, g \in M$, $f+g \in M$, $\rho f \in M$ is said to be a linear manifold. Manifolds contained in B are designated by $\mathfrak{M}, \mathfrak{N}$, ... those contained in (B) by $(\mathfrak{M})$, $(\mathfrak{N})$, ...

Definition 3.2. The elements $F \in (B)$, $f \in B$ are said to be orthogonal if $[F,f] = 0$.

Definition 3.3. A linear manifold $\mathfrak{M}$ is said to be closed if the limit point f of any sequence $\{f_n\}$, $f_i \in \mathfrak{M} (i = 1, \ldots n, \ldots)$ is also contained in $\mathfrak{M}$. In that which follows, a closed linear manifold will be designated by the initials c.l.m.

Definition 3.4'. The orthogonal complement of a c.l.m. $\mathfrak{M} \subset B$ is the set of all elements $F \in (B)$ such that $[F,f] = 0$ for all $f \in \mathfrak{M}$. The orthogonal complement of $\mathfrak{M}$ is designated $\mathfrak{M}^{\perp}$, and the orthogonal complement of $\mathfrak{M}^{\perp}$, $\mathfrak{M}^{\perp\perp}$.

It may be shown that the orthogonal complement of a c.l.m. is a c.l.m.

Lemma 3.1. $\mathfrak{N}^{\perp\perp} = \mathfrak{N}$, for $\mathfrak{N} \subset B$.

Proof. Clearly $\mathfrak{N}^{\perp\perp} \supset \mathfrak{N}$. Assume there exists an $f \in \mathfrak{N}^{\perp\perp}$, $f \notin \mathfrak{N}$. Then there exists an element $F \in (B)$, such that F is orthogonal to $\mathfrak{N}$, $F \perp \mathfrak{N}$, $[F,f] \neq 0$. But $[F,f]$ must equal zero for $F \perp \mathfrak{N}$. This contradicts the assumption. Therefore $\mathfrak{N}^{\perp\perp} = \mathfrak{N}$.

Lemma 3.2. $(\mathfrak{N})^{\perp\perp} = (\mathfrak{N})$ for $(\mathfrak{N}) \subset (B)$.

Proof. Clearly $(\mathfrak{N})^{\perp\perp} \supset (\mathfrak{N})$. Assume there exists an $F \in (\mathfrak{N})^{\perp\perp}$, $F \notin (\mathfrak{N})$. Then there exists an element $f \in B$, $f \perp (\mathfrak{N})$ such that $[F,f] \neq 0$. But $[F,f]$ must equal zero for $f \perp (\mathfrak{N})$. This contradicts the assumption. Therefore $(\mathfrak{N})^{\perp\perp} = (\mathfrak{N})$.

Definition 3.4. Two closed linear manifolds $\mathfrak{M}$ and $\mathfrak{N}$ $((\mathfrak{M})$ and $(\mathfrak{N}))$ will be said to be disjoint if there exists a constant $k>0$ $(k'>0)$ such that for every $f \in \mathfrak{M}$ and every $g \in \mathfrak{N}$ (for every $F \in (\mathfrak{M})$ and every $G \in (\mathfrak{N})$), $\|f+g\| \geq k\|f\|$, $(\|F + G\| \geq k'\|F\|)$.

This definition does not differentiate between the manifolds, since $\|f + g\| \geq k\|f\|$, implies $\|f + g\| \geq k''\|g\|$. Since $\|f + g\| \geq \|g\| - \|f\|$ and it is given that $\|f + g\| \geq k\|f\|$, multiplying the first of these inequalities by k and adding them,

$$(1 + k) \|f + g\| \geq k\|g\|.$$

Thus

$$\|f + g\| \geq k''\|g\|, \text{ where } k'' = k/(k + 1).$$

In an entirely analogous manner it may be shown that $\|F + G\| \geq k'\|F\|$, implies $\|F + G\| \geq k'''\|G\|$, where $k''' = k'/(k' + 1)$.

Theorem 3.1. Two closed linear manifolds $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint if and only if they satisfy the following conditions:

(1) The manifolds have only the zero element in common,

(2) The set of all elements of the form f + g, $f \varepsilon \mathfrak{M}$, $g \varepsilon \mathfrak{N}$ is a closed linear manifold.

Proof of (1). Let $f \varepsilon \mathfrak{M}\cdot\mathfrak{N}$, then, since if $\mathfrak{M}$ and $\mathfrak{N}$ are linear manifolds, $\mathfrak{M}\cdot\mathfrak{N}$ is a linear manifold, $-f \varepsilon \mathfrak{M}$. By Definition 3.4

$$0 = \|f - f\| \geq k\|f\|$$

and since $k>0$, $\|f\| = 0$, $f = 0$.

Proof of (2). The set of elements of the form f + g, $f \varepsilon \mathfrak{M}$, $g \varepsilon \mathfrak{N}$ are readily shown to be linear. Let $h_n = f_n + g_n$ ($n = 1, 2, \ldots$), and let $h_n \to h$. Then by the definition of disjointness,

$$\|h_n - h_m\| = \|(f_n + g_n) - (f_m + g_m)\| \geq k\|f_n - f_m\|,$$

and since $\|h_n - h_m\| \to 0$, $\|f_n - f_m\| \to 0$. Similarly, since $\|h_n - h_m\| \geq k'''\|g_n - g_m\|$, $\|g_n - g_m\| \to 0$. Since $\mathfrak{M}$ and $\mathfrak{N}$ are closed $f_n \to f$, $g_n \to g$, thus

$$h = f + g$$

and the set of all elements of the form f + g form a closed linear manifold.

Proof of the Converse. Since $\mathfrak{M}$ and $\mathfrak{N}$ have only the zero element in common, elements of B can be expressed in only one way in the form f + g, $f \varepsilon \mathfrak{M}$, $g \varepsilon \mathfrak{N}$.

There exists a transformation A such that, $A(f + g) = f$, which is distributive. The conditions $h_n = f_n + g_n$ $(n = 1, 2, \ldots)$, $h_n \rightarrow h$, $f_n \rightarrow f$, imply that $g_n \rightarrow g \varepsilon \mathfrak{N}$, where $h = f + g$. Thus A is closed, and a closed distributive operator is by definition linear. Let $k > 0$ be a constant such that

$$\|Ah\| \leq \|h\|/k,$$

then

$$\|A(f + g)\| = \|f\| \leq 1/k\|f + g\|.$$

Thus $\|f + g\| \geq k\|f\|$, which is the definition of disjointness, completing the proof of the theorem.

The following theorem may be proved in an entirely analogous manner to that of Theorem 3.1.

Theorem 3.2. Two closed linear manifolds $(\mathfrak{M})$ and $(\mathfrak{N})$ are disjoint if and only if they satisfy the following conditions:

(1) They have only the zero element in common.

(2) The set of all elements of the form F + G, $F \varepsilon (\mathfrak{M})$, $G \varepsilon (\mathfrak{N})$ is a closed linear manifold.

IV. The Theory of Projections

Definition 4.1. An additive and continuous, or linear, operator P is called a projection if $P^2 = P$.

The operations + and · referred to manifolds will be interpreted as the set sum and set intersection respectively.

Theorem 4.1. If P is any projection in B, and $\mathfrak{M} \subset B$ the set of elements f for which $Pf = f$ and $\mathfrak{N} \subset B$ the set of elements g for which $Pg = 0$, then

$\mathfrak{M}$ and $\mathfrak{N}$ are disjoint closed linear manifolds such that $\mathfrak{M} + \mathfrak{N} = B$. Conversely, if $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint closed linear manifolds for which $\mathfrak{M} + \mathfrak{N} = B$, there exists a unique projection P which satisfies the equations $Pf = f$, $f \varepsilon \mathfrak{M}$; $Pg = 0$, $g \varepsilon \mathfrak{N}$.

Proof. If $f, g \varepsilon \mathfrak{M}$, then it follows from the equations $P\rho f = \rho f$ and $P(f + g) = f + g$, that $\mathfrak{M}$ is a linear manifold. Since P is continuous, $\mathfrak{M}$ is closed. Similarly $\mathfrak{N}$ is a c.l.m.

For $f \varepsilon B$, $f = Pf + (f - Pf)$. Since $P^2 = P$, $Pf \varepsilon \mathfrak{M}$, $f - Pf \varepsilon \mathfrak{N}$. Thus $\mathfrak{M} + \mathfrak{N} = B$.

Since $\|Pf\| \leq |P| \|f\| = |P| \|Pf + (f - Pf)\|$, and $f - Pf = g \varepsilon \mathfrak{N}$, $Pf = f \varepsilon \mathfrak{M}$, it follows that

$$\|f + g\| \geq k\|f\|.$$

This shows that $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint, by Definition 3.4.

Proof of the Converse. Let $\mathfrak{M}$ and $\mathfrak{N}$ be disjoint c.l.m. such that $\mathfrak{M} + \mathfrak{N} = B$. Let $h \varepsilon B$, $h = f + g$, $f \varepsilon \mathfrak{M}$, $g \varepsilon \mathfrak{N}$. Then the operator P for which $Ph = f$ is distributive and closed, hence linear. This may be proved as the converse of Theorem 3.1 was proved. Also, $P^2 h = P(Ph) = Pf = f = Ph$, thus $P^2 = P$. This completes the proof of the theorem.

Theorem 4.2. If P is a projection, then the adjoint P* (cf. Definition 2.2) is also a projection. If $\mathfrak{M}$, $\mathfrak{N}$ and $(\mathfrak{M})$, $(\mathfrak{N})$ are the manifolds associated with P and P* respectively, then $(\mathfrak{M}) = \mathfrak{N}^{\perp}$ and $(\mathfrak{N}) = \mathfrak{M}^{\perp}$.

Proof. From Definition 2.2 $[P^*F, f] = [F, Pf]$ and since P is a projection

$$[P^*F, Pf] = [F, P^2 f] = [F, Pf].$$

Thus

$$[P^{*2} F, f] = [P^*F, f],$$

which shows that $P^{*2} = P^*$, and P^* is thus a projection.

Let $G \varepsilon (\mathfrak{M})$, then $[P^*G,f] = [G,f]$, but $[P^*G,f] = [G,Pf]$. Thus $Pf = f$ and $f \varepsilon \mathfrak{M}$. If $[P^*G,f] = 0$, then for $G \varepsilon (\mathfrak{M}), [G,f] = 0$. Then $f \varepsilon \mathfrak{N}$ since $Pf = 0$ in this case. Thus $(\mathfrak{M}) \perp \mathfrak{N}$ or $(\mathfrak{M}) \subset \mathfrak{N}^{\perp}$.

Let G be any element orthogonal to $\mathfrak{N}$, and let $f \varepsilon B$. Then

$$[G,f] = [G,Pf + (f - Pf)] = [G,Pf] + [G,f - Pf]$$

but $f - Pf \varepsilon \mathfrak{N}$, therefore $[G,f - Pf] = 0$. Thus

$$[G,f] = [P^*G,f]$$

and $G \varepsilon (\mathfrak{M})$, and it follows that $(\mathfrak{M}) = \mathfrak{N}^{\perp}$.

It may be proved in an entirely analogous manner that $(\mathfrak{N}) = \mathfrak{M}^{\perp}$.

<u>Theorem 4.3</u>. If P_1 and P_2 are projections and $\mathfrak{M}_1, \mathfrak{N}_1$; $\mathfrak{M}_2, \mathfrak{N}_2$ their associated manifolds, then

(1) $P_1 P_2$ is a projection except if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have only the zero element in common.

(2) $P_1 + P_2$ is a projection if and only if $P_1 P_2 = 0$.

<u>Proof of (1)</u>. If $\mathfrak{M}_1 \cdot \mathfrak{M}_2 = 0$, then the operator $P_1 P_2$ is the null operator for all $f \varepsilon B$. This will be seen to be a degenerate case. Let $f \varepsilon \mathfrak{M}_1 \cdot \mathfrak{M}_2 \neq 0$, then $P_1 P_2 f = f$, and for all $g \varepsilon B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$, $P_1 P_2 g = 0$. Thus if $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$ are disjoint c.l.m; there exists a projection (cf. Theorem 4.1) P_3 which has $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$ as its associated manifolds. It is thus required to show that $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$ are disjoint closed linear manifolds. They are evidently closed and linear. In order to prove them disjoint, let $f \varepsilon \mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $g \varepsilon B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$, then since $f,g \varepsilon B$

$$\|f + g\| \geq \|f\| - \|g\|.$$

Operating on both sides of this inequality with $P_1 P_2$

$$\|P_1 P_2 (f + g)\| \geq \|P_1 P_2 f\| - \|P_1 P_2 g\|,$$

since $g \varepsilon B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $f \varepsilon \mathfrak{M}_1 \cdot \mathfrak{M}_2$,

$$\|P_1 P_2 (f + g)\| \geq \|f\|$$

Also since P_1 and P_2 are by definition bounded

$$\|P_1 P_2 (f + g)\| \leq k\|f + g\|$$

where $k > 0$ is the bound of $P_1 P_2$. Thus

$$\|f + g\| \geq 1/k\|f\|.$$

Then by Definition 3.4 $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ and $B - \mathfrak{M}_1 \cdot \mathfrak{M}_2$ are disjoint closed linear manifolds.

Since $\mathfrak{M}_1 \cdot \mathfrak{M}_2 = \mathfrak{M}_2 \cdot \mathfrak{M}_1$ and $P_1 P_2 = P_3$ and $P_2 P_1 = P_3$, the projections as here defined are necessarily commutative. If $\mathfrak{M}_1 \cdot \mathfrak{M}_2 = 0$ then $P_1 P_2 = P_2 P_1 = 0$.

Proof of (2). If $P_1 P_2 = P_2 P_1 = 0$, then

$$(P_1 + P_2)^2 = P_1 + P_1 P_2 + P_2 P_1 + P_2 = P_1 + P_2,$$

thus $P_1 + P_2$ is a projection. Conversely, if $P_1 + P_2$ is a projection,

$$P_1 + P_1 P_2 + P_2 P_1 + P_2 = P_1 + P_2$$

thus $P_1 P_2 + P_2 P_1 = 0$. From this

$$P_1 (P_1 P_2 + P_2 P_1) = P_1 P_2 + P_1 P_2 P_1 = 0$$

$$(P_1 P_2 + P_1 P_2 P_1 \quad)P_1 = P_1 P_2 P_1 + P_1 P_2 P_1 = 0$$

Thus $P_1 P_2 = P_2 P_1 = 0$ completing the proof of the theorem.

Theorem 4.4. If P_1 and P_2 are projections, $P_1 - P_2$ is a projection if and only if $P_1 P_2 = P_2$ or $P_2 P_1 = P_2$.

Proof. If P is a projection, then $I - P$ is also a projection, since

$$(I - P)^2 = I - 2P + P = I - P.$$

Thus if $P_1 - P_2$ is a projection, $I - (P_1 - P_2)$ is also a projection.

$$I - (P_1 - P_2) = (I - P_1) + P_2$$

By Theorem 4.3 the sum of two operators is a projection if $(I - P_1)P_2 = 0$ or if $P_2(I - P_1) = 0$, thus $P_1 P_2 = P_2$ or $P_2 P_1 = P_2$.

Conversely, since by Theorem 4.3 $P_1 P_2 = P_2$ implies $P_2 P_1 = P_2$ and

$$(P_1 - P_2)^2 = P_1 - P_1 P_2 - P_2 P_1 + P_2 = P_1 - P_2 .$$

This completes the proof of the theorem.

V. Infinite Systems of Projections

Lemma 5.1. If $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint and if there exists an $f(\neq 0) \varepsilon B$, $f \notin \mathfrak{M} + \mathfrak{N}$, then there exists an $F(\neq 0) \varepsilon \mathfrak{M}^{\perp} \cdot \mathfrak{N}^{\perp}$. Conversely, if $\mathfrak{M}^{\perp}$ and $\mathfrak{N}^{\perp}$ are disjoint, $\mathfrak{M} + \mathfrak{N} = B$.

Proof. Let $f \varepsilon B, \notin \mathfrak{M} + \mathfrak{N}$; then there exists an $F \varepsilon (B)$ such that $[F,f] = 0$, thus $F \perp \mathfrak{M} + \mathfrak{N}$. Thus $F(\neq 0) \varepsilon \mathfrak{M}^{\perp}$ and $F \varepsilon \mathfrak{N}^{\perp}$. Thus $F \varepsilon \mathfrak{M}^{\perp} \cdot \mathfrak{N}^{\perp}$.

If $\mathfrak{M}^{\perp}$ and $\mathfrak{N}^{\perp}$ are disjoint then there exists no element outside the zero element satisfying the condition $[F,f] = 0$. Then the zero element is the only element $\perp \mathfrak{M} + \mathfrak{N}$. Thus $\mathfrak{M} + \mathfrak{N} = B$.

Theorem 5.1. Let $\{P_n\}$ be a sequence of projections for which $P_n < P_{n+1}$ (i.e. the characteristic manifold of P_n, $\mathfrak{M}_n$ is contained in the characteristic manifold of P_{n+1}, $\mathfrak{M}_{n+1}$), $|P_n| \leq k$, $(n = 1, 2, \ldots)$. Let the adjoint of P_n be P_n^*; and let $\mathfrak{M}_n, \mathfrak{N}_n$; $(\mathfrak{M})_n$, $(\mathfrak{N})_n$ denote the manifolds associated with P_n and P_n^* respectively. Let $\mathfrak{M} = \sum_1^{\infty} \mathfrak{M}_k$; $\mathfrak{N} = \prod_1^{\infty} \mathfrak{N}_k$; $(\mathfrak{M}) = \sum_1^{\infty} (\mathfrak{M})_k$; $(\mathfrak{N}) = \prod_1^{\infty} (\mathfrak{N})_k$. Then

(1) $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint; $(\mathfrak{M})$ and $(\mathfrak{N})$ are disjoint.

(2) $(\mathfrak{M}) = \mathfrak{N}^{\perp}$, $(\mathfrak{N}) = \mathfrak{M}^{\perp}$; $\mathfrak{M} = (\mathfrak{N})^{\perp}$, $\mathfrak{N} = (\mathfrak{M})^{\perp}$.

(3) $\mathfrak{M} + \mathfrak{N} = B$; $(\mathfrak{M}) + (\mathfrak{N}) = (B)$.

Proof of (1). Let $f \varepsilon \mathfrak{M}$, $g \varepsilon \mathfrak{N}$. Then there exist elements $f_n \varepsilon \mathfrak{M}_n$ such that $f_n \to f$. Thus

$$\| P_n(f_n + g) \| = \| f_n \| \leq k \| f_n + g \|. \text{ Since}$$

$$\| f_n + g \| \geq 1/k \| f_n \| , \| f + g \| \geq 1/k \| f \|,$$

$\mathfrak{M}$ and $\mathfrak{N}$ are by Definition 3.4 disjoint manifolds.

The case $|P_n| = 0$ is trivial, it has thus been assumed that $k > 0$. From $[P_n^* F,f] = [F,P_n f]$ it follows that $|P_n^*| [F,f] = |P_n| [F,f]$. Thus $|P_n^*| = |P_n| \leq k$. Proceeding similarly $F_n \in (\mathfrak{M})_n$, $F \in (\mathfrak{M})$, $G \in (\mathfrak{N})$,

$$\|P_n^* (F_n + G)\| = \|F_n\| \leq k\|F_n + G\|$$

Since $F_n \to F$, and $k > 0$,

$$\|F + G\| \geq 1/k\|F\|.$$

Thus $(\mathfrak{M})$ and $(\mathfrak{N})$ are disjoint manifolds.

Proof of (2). Since by Theorem 4.2 $(\mathfrak{N})_n = \mathfrak{M}_n^\perp$, $\mathfrak{M}_n \perp (\mathfrak{N}) \subset (\mathfrak{N})_n$. Hence $(\mathfrak{N}) \perp \sum_1^\infty \mathfrak{M}_\alpha = \mathfrak{M}$ or $(\mathfrak{N}) \subset \mathfrak{M}^\perp$. Let $F \perp \mathfrak{M}_n$, thus $F \in (\mathfrak{N})_n$, $F \in \prod_1^\infty (\mathfrak{N})_\alpha$, $F \in (\mathfrak{N})$. Thus $(\mathfrak{N}) = \mathfrak{M}^\perp$, proving the second statement of part (2).

Since by Lemma 3.1 $\mathfrak{M}^{\perp\perp} = \mathfrak{M}$, from $(\mathfrak{N}) = \mathfrak{M}^\perp$; $(\mathfrak{N})^\perp = \mathfrak{M}^{\perp\perp} = \mathfrak{M}$, which proves the third statement of part (2).

Since by Theorem 4.2 $(\mathfrak{M})_n \perp \mathfrak{N}_n$; $(\mathfrak{M})_n \perp \mathfrak{N}$ (n = 1, 2, ...) since $\mathfrak{N} \subset \mathfrak{N}_n$, (n = 1, 2, ...). Then

$$(\mathfrak{M}) = \sum_1^\infty (\mathfrak{M})_\alpha \perp \mathfrak{N},$$

thus $(\mathfrak{M})^\perp \supset \mathfrak{N}$. Let $f \in (\mathfrak{M})^\perp$, then $f \perp (\mathfrak{M})_n$, $f \in (\mathfrak{N})_n$, $f \in (\mathfrak{N})$. Hence $\mathfrak{N} = (\mathfrak{M})^\perp$. By Lemma 3.2 $(\mathfrak{M})^{\perp\perp} = (\mathfrak{M})$. Thus $\mathfrak{N}^\perp = (\mathfrak{M})$ which completes the proof of part (2) of the theorem.

Proof of (3). $\mathfrak{M}^\perp$ and $\mathfrak{N}^\perp$ are disjoint, since by part (2) of this theorem $(\mathfrak{M}) = \mathfrak{N}^\perp$ and $(\mathfrak{N}) = \mathfrak{M}^\perp$. Thus by Lemma 5.1 $\mathfrak{M} + \mathfrak{N} = B$. Similarly, $(\mathfrak{M}) + (\mathfrak{N}) = (B)$. This completes the proof of the theorem.

Theorem 5.2. If $\{P_n\}$ is a sequence of projections for which $|P_n| \leq k$, $P_n < P_{n+1}$, (n = 1, 2, ...), then there exists a projection P having the following properties:

(1) $\mathfrak{M}_P = \sum_1^\infty \mathfrak{M}_\alpha$, $\mathfrak{N}_P = \prod_1^\infty \mathfrak{N}_\alpha$.

(2) $|P| \leq k$.

(3) For any $f \varepsilon B$, $\|(P - P_n) f\| \to 0$.

(4) $P > P_n$ $(n = 1, 2, \ldots)$. If Q is a projection such that $Q > P_n$ $(n = 1,2,\ldots)$ then $Q > P$.

(5) If P_n is permutable with a linear operator A, then P is permutable with A.

<u>Proof of (1)</u>. P is by definition the projection whose associated manifolds are $\mathfrak{M}_P$ and $\mathfrak{N}_P$. By Theorem 3.1 $\mathfrak{M}_P$ and $\mathfrak{N}_P$ are disjoint and $\mathfrak{M}_P + \mathfrak{N}_P = B$. By Theorem 4.1 P is uniquely defined.

<u>Proof of (2)</u>. Since $|P_n| \leq k$ $(n = 1, 2, \ldots)$, it is evident from the definition of P that $|P| \leq k$.

<u>Proof of (3)</u>. If $f \varepsilon \mathfrak{N}_P$ then $Pf = 0$ and $P_n f = 0$, thus $\|(P - P_n)f\| \to 0$. If $f \varepsilon \mathfrak{M}_P$, then $Pf \varepsilon \mathfrak{M}_P$ and there exists a $g_n \varepsilon \mathfrak{M}_n$ $(n = 1, 2, \ldots)$ such that $g_n \to Pf$.

Now

$$Pf - P_n f = Pf - g_n + P_n(g_n + f - Pf) - P_n f$$

and

$$\| Pf - P_n f\| \leq \|Pf - g_n\| + \|P_n(g_n - Pf)\| \to 0$$

since $|P_n| \leq k$.

<u>Proof of (4)</u>. That $P > P_n$ follows immediately from its definition and the associated manifolds. If $Q > P_n$ $(n = 1, 2, \ldots)$ then $\mathfrak{M}_Q \supset \mathfrak{M}_n$ $(n = 1, 2, \ldots)$, hence $\mathfrak{M}_Q \supset \mathfrak{M}_P$ and similarly $\mathfrak{N}_Q \subset \mathfrak{N}_P$. Thus $P < Q$.

<u>Proof of (5)</u>. Since $P_n Af = AP_n f \to APf$ and $P_n Af \to PAf$, $PAf = APf$. This completes the proof of the theorem.

VI. The Mean of the Iteration of Linear Operators

It is the purpose of this paragraph to show that the mean of the iteration of a linear operator approaches a linear operator which is a projection.

<u>Definition 6.1.</u> A sequence $\{f_n\}$, $f_i \varepsilon B$ $(i = 1, 2, \ldots)$, is said to be ~~strongly~~ <u>convergent</u> toward $h \varepsilon B$ if n_0 exists such that

$$\| f_n - h \| < \varepsilon, \text{ for } n \geq n_0(\epsilon),$$

ϵ an arbitrary positive number.

<u>Definition 6.2.</u> The complex number λ is said to be a <u>proper value</u> for the operator A, if $Af = \lambda f$ for some $f \varepsilon B$.

<u>Definition 6.3.</u> The operators A^n are said to be <u>equi-bounded</u> if $\| A^n f \| \leq M \| f \|$, $i = 1, 2, \ldots, n$.

<u>Lemma 6.1.</u> If 1 is not a proper value for A, and if the operators A^n are equi-bounded, then for any $f \varepsilon B$

$$1/n(Af + A^2 f + \ldots + A^n f) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

<u>Proof.</u> For a fixed element f let

$$g_n = 1/n \{ Af + A f + \ldots + A f \}$$

then

$$Ag_n = 1/n \{ A^2 f + A^3 f + \ldots + A^{n+1} f \}.$$

Thus

$$Ag_n - g_n = 1/n\ (A^{n+1} f - A f)$$

and

$$\| Ag_n - g_n \| = 1/n \| A^{n+1} f - A f \|$$

and by the properties of the norm

$$\| Ag_n - g_n \| \leq 1/n \{ \| A^{n+1} f \| + \| A f \| \}$$

Since A^n's are equi-bounded

$$\| Ag_n - g_n \| \leq 1/n \{ M \| f \| + M \| f \| \}$$

Thus in the limit, as $n \to \infty$

$$\| Ag_n - g_n \| = 0$$

$$Ag_n = g_n \quad n \to \infty.$$

But 1 is not a proper value for A, thus $g_n = 0$ as $n \to \infty$, and

$$1/n \{Af + A^2 f + \dots + A^n f\} \to 0 \quad \text{as } n \to \infty$$

Theorem 6.1. If A is a linear operator on B, and A^n is equi-bounded and $\mathfrak{M}$ the set of all elements $f \varepsilon B$, such that $Af = f$, and if P is the projection whose characteristic manifold is $\mathfrak{M}$, then

$$1/n \,(Af + A^2 f + \dots + A^n f) \longrightarrow Pf \quad \text{as } n \to \infty$$

Proof. Let $\mathfrak{M}$ be the set of elements such that $Af = f$, $\mathfrak{N}$ the set of elements disjoint to $\mathfrak{M}$. Then by Theorem 4.1 $\mathfrak{M} + \mathfrak{N} = B$ and P is the projection such that $Pf = f$, $f \varepsilon \mathfrak{M}$; $Pg = 0$, $g \varepsilon \mathfrak{N}$. Since $\mathfrak{M}$ is the set of all $f \varepsilon B$ such that $Af = f$, for the set of elements $\mathfrak{N}$, A does not have 1 as a proper value.

Let $f_1 = Pf$ and $f_2 = f - Pf$, then $f_1 \varepsilon \mathfrak{M}$ and $f_2 \varepsilon \mathfrak{N}$. The operator A transforms $\mathfrak{M}$ into itself since for all $f_1 \varepsilon \mathfrak{M}$, $Af_1 = f_1$

$$A^n f = A^n f_1 + A^n f_2 = f_1 + A^n f_2$$

It has been shown that all elements $f \varepsilon B$ may be represented in the form $f = f_1 + f_2$, $f_1 \varepsilon \mathfrak{M}$, $f_2 \varepsilon \mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint c.l.m. (cf. Theorem 3.1).

By Lemma 6.1 since for $f_2 \varepsilon \mathfrak{N}$ A does not have 1 as a proper value,

$$1/n \,(Af_2 + A^2 f_2 + \dots + A^n f_2) \to 0, \text{ as } n \to \infty.$$

Substituting $A^n f = f_1 + A^n f_2$, $f_1 = Pf$

$$1/n \,(Af + A^2 f + \dots + A^n f) \to Pf \quad \text{as } n \to \infty.$$

This completes the proof of the theorem.

VII. Reducibility

It is the purpose of this paragraph to characterize a linear operator in terms of the manifolds which it leaves invariant.

Definition 7.1. Let A be a bounded linear operator, $\mathfrak{M}$ a closed linear manifold and P the projection whose characteristic manifold is $\mathfrak{M}$. $\mathfrak{M}$ is said to reduce A if A and P are permutable in the following sense: whenever f is in the domain of A, Pf is also in the domain of A and APf = PAf.

Theorem 7.1. If the closed linear manifold $\mathfrak{M}$ reduces the linear operator A, then A leaves $\mathfrak{M}$ invariant in the sense that it carries every element common to its domain and the manifold $\mathfrak{M}$ into an element of $\mathfrak{M}$.

Proof. Let $f \varepsilon \mathfrak{M}$ and in the domain of A. Then since P is the projection whose characteristic manifold is $\mathfrak{M}$, Pf = f and APf = Af. Since $\mathfrak{M}$ reduces A, by Definition 7.1

$$Af = APf = PAf$$

Af is some element of B, thus $PAf \varepsilon \mathfrak{M}$ or PAf = 0, the latter is true only in the trivial case, A is the null-operator. Thsu A leaves $\mathfrak{M}$ invariant in the sense of the theorem.

Theorem 7.2. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are non-intersecting closed linear manifolds which both reduce A then $\mathfrak{M}_1 + \mathfrak{M}_2$ reduces A.

Proof. Let P_1 and P_2 be the projections whose characteristic manifolds are $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Since $\mathfrak{M}_1 \cdot \mathfrak{M}_2 = 0$, $P_1 P_2 = 0$ and $P_1 + P_2$ is a projection (cf. Theorem 4.3). Thus for $f \varepsilon \mathfrak{M}_1 + \mathfrak{M}_2$ and f contained in the domain of A.

$$(P_1 + P_2)f = f \text{ and } Af = A(P_1 + P_2)f = (P_1 + P_2)Af:$$

Thus $\mathfrak{M}_1 + \mathfrak{M}_2$ reduces A if both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ reduce A.

Theorem 7.3. If $\mathfrak{M}_1$ and $\mathfrak{M}_2 \subseteq \mathfrak{M}_1$ are c.l.m. both of which reduce A, then $\mathfrak{M}_1 - \mathfrak{M}_2$ reduces A.

Proof. Let P_1 and P_2 be the projections whose characteristic manifolds are $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Since the condition $P_2P_1 = P_2$ of Theorem 4.4 is satisfied, $P_1 - P_2$ is a projection. Thus

$$Af = A(P_1 - P_2)f = (P_1 - P_2)Af$$

and since $\mathfrak{M}_1 - \mathfrak{M}_2$ is the characteristic manifold of $P_1 - P_2$, $\mathfrak{M}_1 - \mathfrak{M}_2$ reduces A.

Evidently B and null set in B reduce any linear operator which is defined throughout B or the null set.

Definition 7.2. A bounded linear operator A is said to be irreducible if it is reduced by no c.l.m. other than the entire space or the null set.

Theorem 7.4. If A is a bounded linear operator with domain B and A* is its adjoint, and $\mathfrak{M}$ a c.l.m. such that if $f \varepsilon \mathfrak{M}$, $Af \varepsilon \mathfrak{M}$, then $\mathfrak{M}$ reduces A and $(\mathfrak{M})$ reduces A*, where $(\mathfrak{M}) = \mathfrak{M}^{\perp}$ in the notation of Theorem 4.2.

Proof. If A has B as its domain, then A* has (B) as its domain. Let P be the projection whose characteristic manifold is $\mathfrak{M}$, and P* its adjoint. Then by Theorem 4.2 $(\mathfrak{M}) = \mathfrak{M}^{\perp}$ is the characteristic manifold of P*, for $f \varepsilon \mathfrak{M}$, Pf = f and APf = f. Consider the expression

$$[F, APf - PAf] = [F, Af - PAf] = [F, PAf - P^2Af]$$

for $F \varepsilon (B)$, $f \varepsilon \mathfrak{M} \subset B$, thus since $P^2 = P$

$$[F, APf - PAf] = 0$$

and APf = PAf. By Definition 7.1 $\mathfrak{M}$ reduces A. Since

$$[F, APf] = [F, PAf],$$

by the definition of the adjoint

$$[P^*A^*F, f] = [A^*P^*F, f].$$

Thus for $F \varepsilon (\mathfrak{M}) = \mathfrak{M}^{\perp}$, $P^*A^*F = A^*F$. As a result, if $F \varepsilon (\mathfrak{M})$, $A^*F \varepsilon (\mathfrak{M})$; if $F \varepsilon (\mathfrak{M})$ then $P^*A^*F = 0$, and A^*F is zero for all $F \varepsilon (\mathfrak{M})$. Thus $(\mathfrak{M})$ reduces A^*, where $(\mathfrak{M}) = \mathfrak{M}^{\perp}$.

<u>Theorem 7.5.</u> Let $\{P_n\}$ be a bounded sequence of projections for which $P_n < P_{n+1}$; let $\mathfrak{M}_n, \mathfrak{M}_{n+1}$ be their associated manifolds; and let A be a linear operator which is reduced by $\mathfrak{M}_n$ (n = 1, 2, ...); then if $\mathfrak{M} = \sum_1^{\infty} \mathfrak{M}_k$, then $\mathfrak{M}$ reduces A.

<u>Proof.</u> By Theorem 5.2, there exists a projection P such that $\mathfrak{M}_p = \sum_1^{\infty} \mathfrak{M}_k$, $\mathfrak{M}_p$ the characteristic manifold of P. For $f \varepsilon \mathfrak{M}_n$

$$P_n Af = AP_n f \quad (n = 1, 2, \ldots)$$

since $\mathfrak{M}_n$ (n = 1, 2, ...) reduces A. In order to prove the theorem it is necessary to show that

$$PAf = APf .$$

Since there is no element $f \varepsilon \mathfrak{M}$, which is not contained in one of $\mathfrak{M}_n$ (n = 1, 2, ...) $PAf \neq APf$ would mean that for some $f \varepsilon \mathfrak{M}$ and some n = 1, 2, ...

$$P_n Af \neq AP_n f.$$

Thus $PAf = APf$ completing the proof of the theorem.

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