NON-PERTURBATIVE EFFECTS IN DENSE QUARK MATTER

Thesis by

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Abstract:

Spontaneous symmetry violations in dense quark matter with QCD interactions are investigated by means of variational energy minimization procedures and self-consistent field theory. At least three distinct phases are found to exist, a colour symmetric baryon phase around nuclear density, a colour symmetry violating diquark condensate at high densities and low temperatures, and a strictly perturbative phase with all symmetries restored at high temperatures. Phase transitions associated with heavy quark thresholds and the disappearance of meson condensates are also suspected. The gap equation for the superconducting diquark phase is solved explicitly. The critical temperature for its transition to the perturbative phase is found to be far below the big bang temperature. Diquark condensates possess numerous Goldstone modes and other collective excitations with calculable properties.

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1) INTRODUCTION

1.1) Quark Soup--a Dish from the Devil's Kitchen

Nature's menu offers all sort of quark soups, hot and cold, thick and thin. It comes hot at the big bang, and cold in atomic nuclei, neutron stars, and stars collapsing to never-never land inside a black hole. Before ordering, you might care to know the flavour and colour of the soup, and whether it is a solid, superfluid, or gas. Of this I treat.

Perturbation theory is inadequate to calculate the equation of state of quark matter. Even though we may agree to put our trust in QCD, perturbation theory fails in some regimes. The usual bugbears, strong coupling and infrared devils, are not at fault here. By virtue of asymptotic freedom, effective couplings are mercifully weak at high densities.(1) Moreover, a soup can polarize as the vacuum cannot, shielding glue forces, and cutting off infrared divergences.(2) Non-perturbative effects plague us here instead. Superconductivity, crystallization, and like horrors threaten to violate the virginal symmetries of colour, flavour, fermion number, and even momentum conservation.

Phase transitions mark the onsets of symmetry violations, and the study of thermodynamic phases is largely a matter of determining what, if any, symmetry violations are favoured in various regimes. Only the phase with all symmetries intact is correctly described by perturbation theory. The others require more elaborate variational descriptions.

1.2) Approaches and Results

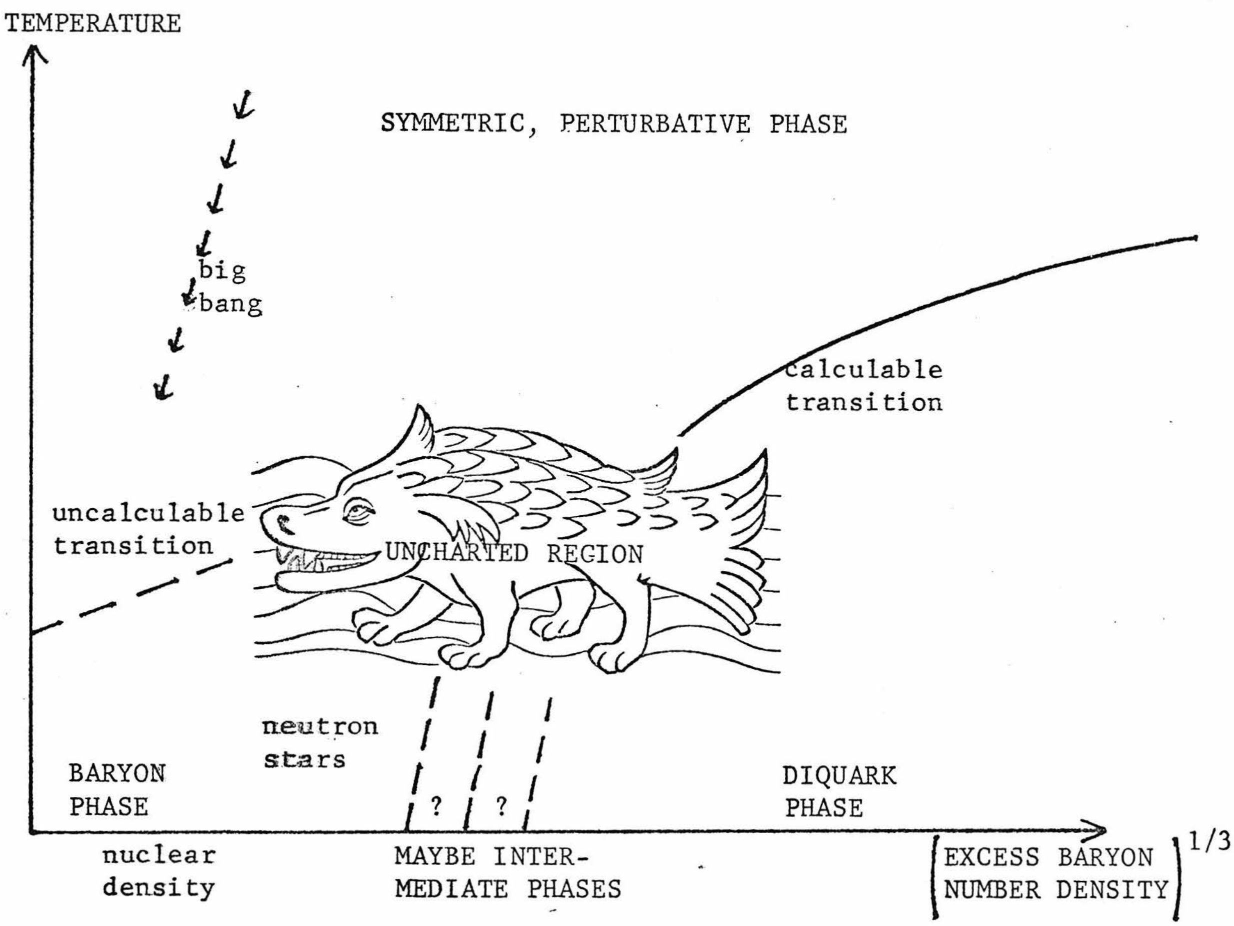
The literature on very dense nuclear matter follows two general lines, either stretching asymptotically free QCD, or squeezing nuclear physics. Each approach has its pitfalls.

In the limit of infinite density, a quark soup can be pictured as several degenerate Fermi seas, one per spin-colour-flavour state. Asymptotic freedom makes the couplings weak, and the perturbative admixture of excited states is consequently small. Many investigators falsely assume that weak coupling is sufficient to justify perturbation theory. Not so; it is necessary but not sufficient. The classic counterexample is BCS superconductivity, where any attraction, no matter how weak, induces non-perturbative pairing. Some authorities have warned that attractive colour forces in certain channels would cause analogous phenomena in quark soups[1], but other investigators have heedlessly ground out high orders of probably meaningless, zero temperature perturbation theory.[3] Perturbation theory generates divergent asymptotic series when the "pre-vacuum" is unstable.* This difficulty manifests itself as a complex energy when certain infinite ladders of perturbation diagrams are summed, but no finite order of the perturbation series shows any danger signals. (In the BCS problem, for example, the unpaired Fermi sea of electrons is unstable against the formation of Cooper pairs, which appear diagramatically as poles in summed ladders.) Perturbation theory still has its place; it is valid at temperatures high enough to restore symmetry, but there special finite temperature Feynman rules must be used to evaluate the diagrams.

Near normal nuclear density, it is reasonable to treat nucleons as distinct, fundamental objects, rather than as three correlated quarks. Such a description is sensible provided that the nucleons have no significant spatial overlap.[4] In nuclei, the mean spacing between nucleons is about twice their root-mean-square charge radius, but the nucleon picture must give way to a quark picture after a density increase of less than an order of magnitude. The greatest difficulty in working up from nuclear density is that phenomenological interaction potentials cannot be reliably extrapolated to higher densities. Their most troublesome feature is the infamous repulsive core, whose existence is inferred by the following logic: if nucleons are fundamental particles, and if 2-body interactions predominate, and if the range of interactions is not significantly affected by shielding at higher densities, then the saturation of nuclear forces requires a repulsive core in the 2-nucleon interaction potential. These assumptions are fictions, but they are all reasonable approximations at low density, and the conclusion has entered the canon of nuclear phenomenology. Slightly above nuclear density, it leads to predictions of crystallization[5], and has led some authors to deny the possibility of superconductivity, which requires predominantly attractive interactions. An intelligent discussion of these effects would have to be conducted in quark language, and would have to explain the 3-body correlation that makes low-density hadronic matter look more like nucleons than like individual quarks. In this language, the repulsive short distance interactions are interpreted as quark kinetic energy.(4) Predictions about superconductivity and crystallization are cast in quite a different light and are dramatically altered.

The moral of this critique is that we need a quark language to discuss physics above nuclear density, and that we must watch out for several non-perturbative effects: diquark pairing à la BCS, 3-body correlations that bind quarks into nucleons, meson condensation, and perhaps even crystallization.

My quest for such non-perturbative effects has yielded the following incomplete map of thermodynamic phases:



The thrust of Chapter 2 is to calculate the properties of the paired diquark phase in the lower right corner of the figure, and to locate the transition to the perturbative phase. These can be done reliably in the limit of infinite density thanks to weak coupling. The critical temperature of the transition turns out to be much lower than the temperature of the big bang, but much higher than stellar temperatures. The phase transition can alternatively be induced by a critical magnetic field, which turns out to be much larger than anything one might expect to find in stars.

Attempts to produce a similarly detailed theory of the baryon phase are sabotaged by strong coupling. Chapter 4 presents the negative result that such a phase does not exist at high densities (weak couplings), and gives suggestive arguments for the possibility of baryon correlations at stronger couplings.

Likewise, investigations of meson condensation in Chapter 5 yield only the negative result that mesons do not condense at high density.

1.3) A Unifying Perspective on Non-Perturbative Effects

All the non-perturbative effects mentioned above involve spontaneous symmetry violations.

1) The paired ground state ("the vacuum") contains an indefinite number of pairs. Various processes tap the vacuum as a reservoir of pairs, and fermion number is only conserved modulo two. Moreover, since there is no way to make a colour singlet diquark pair, colour symmetry is also violated.

2) The three quark correlation is really a 6-body correlation in disguise, since nucleons are known to pair in heavy nuclei, and because the vacuum can only be a condensate of bosons. Fermions are conserved only modulo six, but colour symmetry is preserved.

3) Meson condensation breaks chiral symmetries.

4) Crystallization is the spontaneous breakdown of translation invariance from R^3 to Z^3, i.e. from continuous translations in all directions to translations in integral multiples of three lattice vectors. This is equivalent to momentum conservation only modulo reciprocal-lattice vectors.

In each of these cases, various Green functions forbidden by symmetry are non-zero. They obey Dyson equations determined by the topology of the field theory, which possess non-trivial solutions besides the trivial, perturbative, zero solution. The method of describing forbidden Green functions by their Dyson equations is known as self-consistent field theory.

An equivalent but more cumbersome variational approach consists in parameterizing the ground state by Green functions or occupation amplitudes and of minimizing its energy with respect to them. The Euler equations of this variational scheme are precisely the Dyson self-consistency equations.

The second approach is usually harder, but it is older. BC&S originally described their paired ground state in terms of an occupation amplitude function, and minimized its energy.[6] Self-consistent field theory was later developed by Nambu.[7]

Solving such Dyson equations is an art, unlike conventional perturbation theory, which is a mechanical science. Perturbation theory is a systematic iterative scheme for generating corrections to bare Green functions, but the symmetry breaking problem lacks even the starting point of a bare Green function, and the solution must be pulled out of thin air.* (The closest mathematical analogy to this situation comes from the theory of integral equations, perturbation theory being the analogue of the Neumann series solution for inhomogeneous equations only.)

We will examine suspected symmetry violations by the following general method:

1) Identify non-vanishing, symmetry-forbidden Green functions. Typically, one weird Green function is the granddaddy of all other weird Green functions.

2) Construct its Dyson equation, ordinarily a non-linear integral equation.

3) Seek non-trivial solutions.

4) Calculate the vacuum energy. This will involve a diagrammatic expansion in powers of forbidden Green functions. This energy must be lower than that of any other thermodynamic phase at the same conditions, or else the expansion will have all the diseases typical of perturbation expansions with unstable pre-vacua.

This method is most successful in calculating the properties of paired diquarks, but it is plagued by technical difficulties when applied to dibaryon and meson condensates. Even so, it is useful in constructing proofs that baryons and mesons do not condense at high densities and weak couplings.

The physical interpretation of symmetry breaking is subtle. One kind, famous from ordinary superconductivity theory, is fermion non-conservation. It is a useful mathematical fraud; fermions do not get lost. An illusion of non-conservation is created by the fact that field theory's zeroth-order state has indefinite particle number;

it is a non-unique ground state of $H_o + \mu N$ (see section 1.5 for definitions) constructed by analogy with the grand canonical ensemble of statistical mechanics. It is formally too difficult to constrain particle number exactly, but the approaches are effectively equivalent for very large systems. A different, more tangible kind of broken symmetry prefers certain directions in group space. Colour SU(3) has three distinct if similar base states in its fundamental representation, and colour breaking singles one out for special treatment, causing pairing between two colours but leaving a third Fermi sea degenerate. Realistically, quark soup would develop domains with different preferred colours. However, the ground state that we calculate is a single, infinite domain. Domain boundaries belong to the realm of excitations. We cannot calculate their surface energies, which involve highly non-linear excitations. Only linearized excitations are tractable and are the subject of Chapter 3.

1.4) A Review of Finite Density Perturbation Theory

If it worked, perturbation theory at finite density would be very similar to perturbation theory in the void. The main difference would be in the zeroth-order ground state, but the perturbation series would have its familiar form. The vacuum at zero density is really a Dirac sea with negative-energy states filled. At finite density, all states with energies up to the "chemical potential" (denoted by μ) fill, forming a Fermi sea.

In calculating the amplitudes of processes taking place inside a soup, we must include interactions with particles under the Fermi sea. We need no new Feynman diagrams to take them into account. It suffices to reinterpret hole lines as holes either in the Dirac sea (bona fide positrons) or in the Fermi sea. To effect this reinterpretation algebraically, we must modify the fermion propagator, $S_F(p) = 1/(\not{p} - m - i\epsilon)$, by changing the sign of the $i\epsilon$ to put above-sea particles into the upper half complex p_0 plane, and both kinds of hole poles into the lower half plane. In other words, poles go into the first and third quadrants of the complex p_0-μ plane.

What new physics is there in a soup? For one thing, we lose Lorentz invariance; there is a preferred, "wind still" frame. For another, fermions cannot scatter into many states already occupied.

Particularly interesting is the shielding of boson exchange processes. Exchanged bosons Compton scatter off particles in the Fermi sea. This effect is included in the reinterpreted vacuum-polarization diagram, and is properly called sea-polarization. Due to the lack of Lorentz invariance, the timelike (electric) polarizations of vector gauge bosons are shielded very differently from the spacelike (magnetic) polarizations. The propagator for electric polarizations gets a plasma frequency added to its denominator. Although this shielding arises from one-loop diagrams, it is really a classical effect, and does not vanish in the limit of zero Planck's constant. Magnetism, however, is unshielded and remains long-range;* there is no Meissner effect in perturbation theory. (For a calculation of the shielding length see reference (2).)

Shielding effects make composite hadrons "dissolve" in a dense soup. When the range of glue forces becomes shorter than typical hadronic radii, mesons and baryons fall apart. The demise of meson exchange interaction mechanisms considerably simplifies perturbation theory. Quark-gluon perturbation theory is complete in any case, but the Bethe-Salpeter ladders that sum to hadron poles lose importance. (This triumph is short lived. New and different collective modes arise, as discussed in Chapter 3.)

1.5) A Note on Terminology

It is important to distinguish clearly between three concepts of "vacuum", which refer to the ground states of several variants of the Hamiltonian H. We define H_o to be the kinetic part of H, bilinear in the fields, and N to be the fermion number operator. Then

the "void" is the ground state of H_o ;
the "pre-vacuum" is the ground state of $H_o+\mu N$; and
the "vacuum" is the ground state of $H+\mu N$.

2) DIQUARK CONDENSATION

2.1) The Condensate

It is natural to suspect a diquark condensation in quark soup directly analogous to the dielectron condensation in BCS superconductivity. The substitution of gluons for phonons as the origin of the attractive force alters little since the general character of the phenomenon is insensitive to the details of the interaction. We can therefore steal many results of the BCS theory with impunity.

Cooper devised a sufficient diagnostic test for the invalidity of perturbation theory in finite density problems.[1] If it is energetically advantageous to form bound pairs in the empty states above the Fermi surface, then the degenerate sea is unstable and perturbation theory must diverge. Clever resummations of infinite sets of diagrams indicate an absurd, complex ground state energy, and do not fix the problem.

Cooper's test is satisfied in quark soups. Quarks come in colour triplets, and can combine in two different colour representations of diquarks, in one of which gluon exchange forces are attractive. The rest of the argument is identical to the argument for the instability of the electron sea.

The resolution of this instability is less obvious. In the BCS problem, there are only two kinds of electrons, spin up and down. Since phonon forces are extremely short range, only S-wave Cooper pairs bind, and Fermi statistics requires the spin wave function to be antisymmetric, i.e. spin zero. The quantum numbers of the disease suggest the quantum numbers of the cure. In quark soups, however, there are 3 colours x 2 spins x ever-so-many flavours of quarks, which can make multitudinous varieties of diquarks, and it is not obvious which of them will condense. In fact, even triquarks (baryons), which attract in colour singlet states, might plausibly bind, pair, and condense. We will discuss this possibility in chapter 4.

BC&S resolve the instability by altering the ground state. Their Fermi surface is blurred over an energy range called the gap, and each particle is paired with a partner of exactly opposite spatial momentum. One can express this ground state algebraically as the exponential of a pair creation operator acting on the void, but this construction is not useful for our purposes; we are more interested in Green Functions.

Following this lead, let us simply assume that the ground state is paired. This may not in fact be the lowest energy correlation scheme in all regimes, but we shall ignore this unhappy possibility until we have a chance to compare the energy of this trial state against the energies of states with other sorts of correlations. In the present case, the vacuum is a reservoir of diquarks, which pop in and out of interaction diagrams, so that fermions are only conserved modulo two. The granddaddy of all fermion non-conserving Green functions is just the amplitude for a diquark to pop out of the vacuum, expressible also as an effective Lagrangian term:

$$\langle 0|\ \psi^s_a(x)\ \psi^t_b(y)\ |0\rangle \Rightarrow L_{eff} = \int dx\, dy\ \psi^s_a(x)\ \Delta^{st}_{ab}(x-y)\ \psi^t_b(y) \qquad (2.1\text{-}1)$$

We must now make some decisions about the character of the colour wave function expressed by the indices st and the spin wave function expressed by the spinor indices ab. Since the diquarks that condense in the vacuum are substantially identical to the Cooper pairs that formed above the pre-vacuum's unpaired sea, it is safe to assume that they will be colour anti-triplets, that being the attractive representation. Their other quantum numbers are restriced by Fermi statistics. Colour anti-triplets are antisymmetric, and the space-spin-flavour wave function is consequently totally symmetric. If there is only one flavour, then space and spin are either both antisymmetric (S=0; L=1,3,5...; J=1,3,5...) or both symmetric (S=1; L=0,2,4...; J=1,2,3,4...). In neither case may the diquark be a spinless scalar. The vacuum is not isotropic. Its properties are painful to calculate. If there are several flavours, however, the statistics do admit spinless, S-wave diquarks antisymmetric in flavour, and our favourite symmetry has a fighting chance.

Of course, nature does not share our aesthetic prejudice that spatial anisotropy is ugly, but that flavour antisymmetry is pretty. Nature chooses between these schemes on the basis of energetic advantage as cold-bloodedly as a corporate accountant selects investments on the basis of profitability. Here, beauty pays. Antisymmetric spins are energetically preferable because of chromomagnetic interactions. (Remember: the N is lighter than the Δ.) S-waves are almost universally preferable to orbitally excited states, (except in some exotic cases such as He^3). Flavour symmetry or antisymmetry has no direct energetic implication unless flavour is gauged, in which case the antisymmetric state is preferred.

2.2) The Gap Equation

Now let us set up the self-consistency condition for the symmetry violating amplitude Δ. For lack of masochism, I shall treat only the isotropic case of spinless diquarks.

In setting up the problem, we will have to make a string of approximations. All are "asymptotically justified," i.e., they are accurate in the weak-coupling limit, which coincides with the limit of infinite density thanks to asymptotic freedom.

The propagation of particles suffers an interesting alteration in a paired soup. A propagating hole is ambushed by a diquark that pops out of the vacuum; one member stuffs the hole, and the other continues on as a particle with the hole's original momentum and spin. Then the vacuum resorbs the diquark, and the hole is restored. Such adventures dress the hole.

Mathematically, the bare hole's propagator is $(\not{p} - \not{\mu})^{-1}$, where the energy p_0 is measured relative to the chemical potential μ. (We define $\not{\mu} = \mu\gamma_0$ and assume massless quarks.) Each flip-flop inserts a factor $\Delta\,(\not{p} - \not{\mu})^{-1}\,\Delta^*$.

The mixing of particles and holes by these flip-flops is most conveniently expressed by writing the propagator as a matrix.[(2)]

$$S^{-1} = \left[\begin{array}{cc} \not{p}-\not{\mu} & 0 \\ 0 & \not{p}+\not{\mu} \end{array}\right] + \begin{array}{c} \begin{array}{cc} > & < \end{array} \\ \left[\begin{array}{cc} 0 & \Delta \\ \Delta & 0 \end{array}\right] \end{array} \begin{array}{l} \\ > \text{particles} \\ < \text{holes} \end{array}$$

$$= \quad S_o^{-1} \quad - \quad \Sigma \qquad (2.2\text{-}1)$$

To see the physical quasiparticle spectrum, we must locate the poles of S. These lie where S^{-1} is null, i.e.

$$\det (S^{-1}) = p^2 - \mu^2 + [\not{\mu}, \not{p}] - \Delta^2 = 0 \qquad (2.2\text{-}2)$$

We can rid this equation of Dirac matrices by multiplying it by a duplicate factor with opposite sign commutator. Then

$$(p^2 - \mu^2 - \Delta^2)^2 - 4\mu^2 \underset{\sim}{p}^2 = 0 \quad , \qquad (2.2\text{-}3)$$

where $\underset{\sim}{p}$ denotes the spatial components of p, and $p^2 = E^2 - \underset{\sim}{p}^2$. A minor rearrangement yields the familiar quasiparticle spectrum with gap Δ,

$$E(\underset{\sim}{p}) = (\Delta^2 + (|\underset{\sim}{p}| \pm \mu)^2)^{\frac{1}{2}} \quad , \qquad (2.2\text{-}4)$$

with an extra branch for positrons.

The crucial Dyson equation for the quark propagator is symbolically

$$S^{-1} = S_o^{-1} - \int \gamma S \Gamma D \quad ,^{*} \qquad (2.2\text{-}5)$$

where γ and Γ are bare and dressed vertices , and D is the glue propagator. We make the ladder approximation and calculate to lowest quantum mechanical, one-loop, order. Thus we use γ for Γ, but keep the shielding in D, since it is a classical effect, loops nonwithstanding. (This approximation is weakly but innocuously gauge dependent. The situation is similar to that in the calculation of the renormalization group's β-function, $\beta(g) = dg(M)/d(\log M) = bg^3 + cg^5 + \ldots$, where only the leading coefficient b is gauge invariant. Likewise, our results are correct to leading perturbative order regardless of gauge.)

This equation is itself a matrix equation in the particle-hole basis. Its off-diagonal element is the self-consistency equation for the gap. Let us isolate it. The inhomogeneous term S_o^{-1} contributes nothing.

$$\Delta(p) = -ig^2 \int \frac{d^4k}{h^4} D_{\sigma\tau}(p-k) \frac{\gamma^\sigma \lambda^{\dagger a}}{2} \frac{\Delta(k)}{k^2-\mu^2-\Delta^2-[\not\mu,\not k]} \frac{\gamma^\tau \lambda^a}{2} \qquad (2.2\text{-}6)$$

(triplet colour indices on λ and Δ have been suppressed.) This has the form of a Bethe-Salpeter equation for a diquark bound state with zero energy relative to the chemical potential. We can simplify the equation by using our knowledge of the colour wave function. The equation splits into sextet and anti-triplet moieties. The sextet is repulsive and lacks solutions. Restricted to the anti-triplet portion, the colour index machinery reduces to a Casimir operator as follows:

$$\tfrac{1}{2}\lambda^{a}_{mi}\ \varepsilon_{ijk}\ \tfrac{1}{2}\lambda^{a}_{nj} = -C\ \varepsilon_{mnk} \quad ; \quad C = 2/3 \quad \text{for } SU(3)\ . \qquad (2.2\text{-}7)$$

The spinor and tensor machinery needs work too. For this, we must use the specific form of $D_{\sigma\tau}(q)$, with electric components shielded but magnetic components long range. Let us temporarily indulge in the myth that all components are shielded alike, so that the propagator is Lorentz invariant. This would in fact be right IF colour forces were short range rather than gauge forces. We use $D_{\sigma\tau}(q) = g_{\sigma\tau} / (q^2 - \Omega^2)$.

The gamma contraction deletes the commutator, giving

$$\Delta(p) = Cg^2 \int \frac{d^4k}{h^4} D(p-k) \frac{4\ (k^2-\mu^2-\Delta^2)}{(k^2-\mu^2-\Delta^2)^2-4\mu^2 \underset{\sim}{k}^2} \Delta(k) \qquad (2.2\text{-}8)$$

Now, performing the dk_0 integration and surrounding poles in the upper half complex k_0 plane, we simplify the integration to a conventional three-dimensional convolution. (Approximation: We drop a tiny contribution from positron poles, whose relative importance is as Δ/μ.)

$$\Delta(p) = Cg^2 \int \frac{d^3k}{h^3} D(p-k) \frac{\Delta(k)}{E(k)} \ ; \qquad (2.2\text{-}9)$$

$$E(k) = ((|\underset{\sim}{k}| - \mu)^2 + \Delta^2)^{\frac{1}{2}} \ .$$

Only one thing could make this equation nicer--linearity. This too can be had if we make the approximation of replacing $\Delta(k)$ in $E(k)$ by Δ_o, its value near the Fermi surface, which actually appears as the value of the gap in the quasiparticle energy spectrum.

Even after these drastic simplifications, exact solutions are hard to come by. We will have to settle for limiting forms valid as $g\to 0$, $\Delta\to 0$. First we recognize that Δ/E peaks sharply at the Fermi surface, and has the general character of a delta function there. The unnormalized shape of $\Delta(p)$ is given in this limit by

$$\Delta_{shape}(p) = \int d^3k\; D(p-k)\; \delta(|\underset{\sim}{k}| - \mu) \quad . \tag{2.2-10}$$

With the eigenfunction of the linear, homogeneous integral equation in hand, finding the eigenvalue g^2 is easy.

$$g^{-2} = C \int \frac{d^3q}{h^3} D(q) \; \frac{\Delta_{shape}(q+Q) \;/\; \Delta_{shape}(Q)}{E(q+Q)} \quad , \tag{2.2-11}$$

where Q is a perfectly arbitrary momentum.

Now let us evaluate this to obtain g as a function of Δ_o. (Approximation: Since $\Delta_o \ll \Omega \ll \mu$, we ignore the curvature of the Fermi surface. It is convenient to take $Q = \mu\hat{z}$, exactly on the Fermi surface.) Then

$$g^{-2} = C \int \frac{d^2q_{xy}\; dq_z}{h^3} \; \frac{1}{q^2 + \Omega^2} \; \frac{\Delta_{shape}(q+Q) \;/\; \Delta_{shape}(Q)}{(q_z^{\;2} + \Delta_o^{\;2})^{\frac{1}{2}}} \tag{2.2-12}$$

$$g^{-2} = C \frac{1}{4\pi^2} \; \log \frac{\mu^2}{\Omega^2} \; \log \frac{\Omega^2}{\Delta^2} \tag{2.2-13}$$

(Approximation: We have also ignored the retardation in D(q), which adds a small, ignorable constant to the factor $\log(\mu/\Omega)$.) This inverts to the dramatic dependence

$$\Delta_o^{\;2} \sim \mu^2 \exp\left(-4\pi^2 \;/\; Cg^2 \log(\mu^2/\Omega^2)\right) \quad , \tag{2.2-14}$$

reminiscent of the BCS result. The gap is miniscule, but its philosophical interest is disproportionate to its numerical magnitude.

We must now repent of the false assumption that magnetic as well as electric components of the glue propagator are shielded with plasma frequency Ω, when they are in fact long range. We will now make the opposite extreme assumption that neither electric nor magnetic components are shielded. The truth lies between these extremes.

Since D(0) is now singular, we may no longer use the estimate of Δ_{shape} given by eq.(2.2-10), which diverges logarithmically at the Fermi surface. A more careful convolution of $D \circ (\Delta/E)$ shows that this logarithm is cut off at the Fermi surface give-or-take roughly Δ_o, i.e. very roughly $\Delta_{shape} \approx \log \mu/E$. Whereas previously the only effect of Δ_{shape} in eq.(2.2-12) was to give $o(\mu)$ cutoffs, while remaining constant in the vicinity of the Fermi surface, the new Δ_{shape} varies appreciably there. Recalculating eq.(2.2-12), we get at the intermediate stage of the integration

$$g^{-2} = C \frac{2\pi}{h^3} \int^{o(\mu)} dq_z \log \frac{\mu}{q_z} \frac{1}{(q_z^{\,2} + \Delta_o^{\,2})^{\frac{1}{2}}} \frac{\log (\mu/(q_z^{\,2} + \Delta_o^{\,2})^{\frac{1}{2}})}{\log (\mu/\Delta_o)}$$

$$= C \frac{2\pi}{h^3} \frac{2}{3} \log^2 (\mu/\Delta_o) \qquad (2.2\text{-}15)$$

This leads to the asymptotic behaviour

$$\Delta_o \sim \mu \exp (-6^{\frac{1}{2}}\pi \,/\, C^{\frac{1}{2}} g) \quad , \qquad (2.2\text{-}16)$$

notable for the odd power of g in the exponent.

An honest calculation should take into account the fact that the spatial, magnetic components of the gluon dominate by virtue of their longer range. Such a treatment is tedious and unenlightening, and produces a result boringly similar to eq.(2.2-16). We leave this déjà vu excercise to the appendix.

2.3) The Energy of the Ground State

Three methods exist for calculating the energy density of the vacuum. One is to find the matrix element of T_{00}, an unpleasant procedure because the time direction gets special treatment.

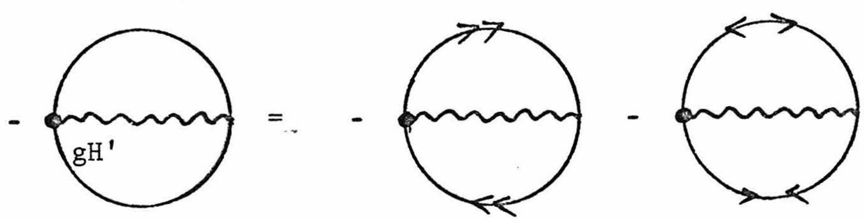
A second method consists in calculating vacuum bubbles.[3] The vacuum energy density is the logarithm of that obnoxious phase which multiplies all S-matrix elements, the vacuum-to-vacuum amplitude, divided by the 4-volume of space-time. Diagrammatically, it is the sum of linked vacuum bubbles, which are easily specified in perturbation theory. With non-perturbative effects, however, diagrammatic specification of bubbles becomes ambiguous.

A third method is suggested by the following little theorem.[4] If $E(g)$ is the ground state energy of H_0+gH', then

$$E(\bar{g}) = E(0) + \int_0^{\bar{g}} dg/g \quad \langle\psi(g)|\ gH'\ |\psi(g)\rangle \quad . \tag{2.3-1}$$

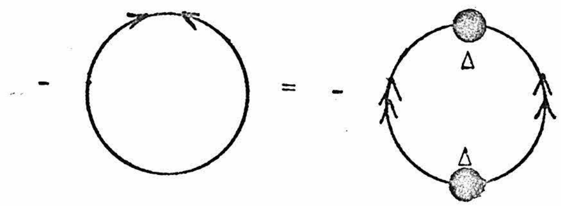
The matrix element of gH' is unambiguous, and is moreover felicitously simple because gH' vertices are precisely what occur naturally in vacuum bubbles.

The simplest matrix element of gH' is the bubble



(2.3-2)

The first term is the obvious perturbative exchange interaction between like quarks, with minor kinematic corrections due to the gap in the energy spectrum. The second term is new. It represents the binding energy of paired particles. It is the heart of this non-perurbative effect, which must be energetically advantageous to be meaningful. By virtue of the gap equation, it reduces to a single loop.



(2.3-3)

$$= i \int \frac{d^4k}{h^4} \, \mathrm{Tr} \, \frac{(\not{k}-\not{\mu}) \; \Delta(k) \qquad (\not{k}+\not{\mu}) \; \Delta(k)}{(k^2-\mu^2-\Delta^2-[\not{\mu},\not{k}]) \; (k^2-\mu^2-\Delta^2-[\not{\mu},\not{k}])}$$

$$= i \int \frac{d^4k}{h^4} \quad \frac{4(k^2-\mu^2) \quad \Delta^2(k)}{(k^2-\mu^2-\Delta^2)^2 - 4\mu^2 \underset{\sim}{k}^2}$$

$$= - \int \frac{d^3k}{h^3} \quad \frac{\Delta^2(k)}{2E(k)} \quad (2 + o(\Delta^2/\mu^2))$$

$$= -\Delta_o^2 \int \frac{d^3k}{h^3} \, \frac{\Delta^2_{\text{shape}}(k) / \; \Delta^2_{\text{shape}}(Q)}{E(k)}$$

$$= -\Delta_o^2 \; \mu^2 \; \log \; (\mu/\Delta_o) \; / \; \pi^2 \qquad \text{in short range approximation}$$

$$= -\Delta_o^2 \; \mu^2 \; \log \; (\mu/\Delta_o) \; / \; 3\pi^2 \qquad \text{in unshielded approximation}$$

This expression must yet undergo a $\int dg/g \ldots$ integration to become the energy density, but this does not alter its form.

The non-perturbative contribution to the energy is unimpressive compared to the perturbative energy, but it is large enough to be important by another standard. It is equal to the energy density of the critical magnetic field that would cause a transition from the superconducting to the normal state. This magnetical field is too large to be attained in reasonable stars.

2.4 Finite Temperatures

Pairing phenomena disappear at a finite critical temperature, above which symmetry is restored and perturbation theory is valid. The value of the gap decreases continuously with increasing temperature, reaching zero at the critical temperature, which consequently marks a second-order transition between the paired and perturbative phases.

A neat formalism exists for finite temperature field theory.[5] Our Dyson equation is altered to read

$$S^{-1} = S_o^{-1} - T\sum_{\omega_n}\int \frac{d^3p}{h^3}\,\gamma S(\omega_{n+\frac{1}{2}})\Gamma D \quad ; \quad p_0=\omega_n=i n h T \qquad (2.4\text{-}1)$$

$$= S_o^{-1} + i\int \frac{d^4p}{h^4}\,\gamma S(\omega_{n+\frac{1}{2}})\Gamma D \coth \beta\omega/2 \quad \text{enclosing coth poles only}$$

$$= S_o^{-1} - i\int \frac{d^4p}{h^4}\,\gamma S(\omega)\Gamma D \tanh \beta\omega/2 \quad \text{enclosing } S(\omega,p) \text{ poles only}$$

The contour at infinity vanishes.

The gap equation becomes simply

$$\Delta(p) = C\int \frac{d^3k}{h^3}\, D(p-k)\,\frac{\Delta(k)}{2E(k)}\tanh \beta E(k)/2 \quad . \qquad (2.4\text{-}2)$$

All that is new is the tanh factor. At zero temperature, the tanh is unity, and the peak of the integrand comes primarily from the factor $1/E = (k^2 + \Delta_o^2)^{-\frac{1}{2}}$, which is reminiscent of $1/k$ with a cutoff at Δ_o. At the critical temperature, on the other hand, $\Delta_o(T_c)=0$, and the corresponding integrand factor is $(1/k)\tanh \beta_c k/2$, again reminiscent of $1/k$ with a cutoff at T_c. The quantities T and Δ_o play similar roles, and a trade-off can be arranged. It is not surprising that T_c and Δ_o are of the same order. For the short range case, we can steal the BCS result as is: $T_c = .57\,\Delta_o(T=0)$. This number comes from the condition

$$\int \frac{dk}{(k^2+\Delta^2)^{\frac{1}{2}}} = \int \frac{dk \ \tanh \beta k/2}{k} \quad , \qquad (2.4\text{-}3)$$

which can be converted into

$$\log \beta\Delta/2 \approx \int_0^{\infty} dx \ \left((x^2+1)^{-\frac{1}{2}} - \tanh(x) / x \right) \quad . \qquad (2.4\text{-}4)$$

This result is valid only for the short range case. No accurate result is available for the unshielded case because the result is disastrously sensitive to Δ_{shape}, which is only crudely known.

The critical temperature is too low for these phenomena to have been important in the early universe, when the temperature was about a thousand times higher than the cube root of the excess baryon number density. Neutron stars are cool, but stability conditions limit their central density to a few times normal nuclear density, hardly enough to justify our assumption of asymptotic freedom (weak coupling). The low temperature, high density limit in which these results apply exists in nature only in collapsing stars, where wise men fear to tread.

2.A) Appendix: Honest Treatment of Shielding

In calculating the gap with the honest glue propagator

$$D_{00} = -1 \,/\, (q^2 - \Omega^2) \;, \quad D_{ii} = 1 \,/\, q^2 \qquad , \tag{2.A-1}$$

we run into one messy complication. Because $D_{\beta\gamma} \neq g_{\beta\gamma}\, D$, we lose the beneficial simplification $\gamma_\sigma\, [\not{u}, \not{p}]\, \gamma^\sigma = 0$, which deleted the commutator from eq.(2.2-8) and justified the use of a scalar gap. We now need a gap with both scalar and commutator parts, which get mixed by the gap equation.

Accordingly, our new quantities are

$$\Delta = \Delta_1 + i\Delta_2\, [\not{u}, \not{k}]\, \mu^{-2} \qquad , \tag{2.A-2}$$

$$\Delta_o^{\,2} = \Delta_1^{\,2} + \Delta_2^{\,2} \qquad , \tag{2.A-3}$$

$$S_{\diamond} = S_o\, \Delta\, S_{\gg} \tag{2.A-4}$$

$$= (\Delta_1 - i\Delta_2\, [\not{u}, \not{k}]\, \mu^{-2})\, \frac{k^2 - \mu^2 - \Delta_o^{\,2} - [\not{u}, \not{k}]}{(k^2 - \mu^2 - \Delta_o^{\,2})^2 - 4\mu^2 \underset{\sim}{k}^2} \qquad ,$$

$$\Delta = \int \gamma_s\, S_{\diamond}\, \gamma^s\, D \tag{2.A-5}$$

$$= \Delta_1 + i\Delta_2\, [\not{u}, \not{k}]\, \mu^{-2} = \int (\,3\, S^{\text{scalar}} + 1\, S^{\text{commutator}})\, D \qquad .$$

Finally, we obtain

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = Cg^2 \int \frac{d^3k}{h^3}\, \frac{D}{2E} \begin{bmatrix} 3/2 & -3i \\ i & -1/2 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \tag{2.A-6}$$

as opposed to the $\gamma_\sigma \ldots \gamma^\sigma$ case with the different matrix

$$\begin{bmatrix} 2 & \\ 0 & 0 \end{bmatrix} \tag{2.A-7}$$

The upper eigenvalue of the new matrix is 5/2 versus the previous value 2 . Thus the gap is really the eigenmixture

$$\Delta = (3i + 5/2 - i[\not{u}, \not{p}]\, \mu^{-2})\, \Delta_{\text{scalar}} \qquad , \tag{2.A-7}$$

and eq.(2.2-16) is to be multiplied by the ratio of new to old eigenvalues.

3) COLLECTIVE EXCITATIONS

3.1) Survey of Models

Quark soups with spontaneously broken symmetries are highly prized for their interesing and abundant collective excitations, especially their Goldstone modes. Quite apart from their astrophysical importance, they are an ideal mathematical laboratory for studying the properties of these excitations.

Goldstone modes are central to our understanding of spontaneously broken gauge theories, where the mass of vector gauge bosons supposedly comes from "eating" Goldstone excitations.[1] They are also put forward as models of featherweight mesons, such as the pion, and elucidate the true meaning of PCAC.

Several models, of varying degrees of realism, reproduce some or all of these interesting phenomena, and permit calculation of their properties. Their successes are summarized in the table below.

The soup models are most successful mathematically--almost all their properties are readily calculable--but they shed little light on the obscure problem of the real $\mu=0$ world, since a soup's difermion condensate is not comparable to the $\mu=0$ world's fermion-antifermion condensate, and the collective excitations of these condensates are in no sense homologous.

The wave functions of ordinary mesons in the $\mu=0$ world and of difermion collective excitations differ considerably. The size of ordinary mesons is largely determined by quark masses and the infrared end of glue forces. The size of difermion collective modes is determined by the smallness of the gap; they are extended objects much larger than the average interparticle distance.

MODEL	PRE-VACUUM	VACUUM	INTERACTION	GOLDSTONE MODES	FAT GAUGE BOSON MASS	FERMION MASS GAP	OTHER COLLECTIVE MODES
HIGGS(1)	the void	φ-condensate	$\varphi^4+\bar{\psi}\varphi\psi$	fundamental	calculable in tree approx.	calculable in tree approx.	none
NAMBU & JONA-LASINIO(2)	the void	$q\bar{q}$-condensate	$\bar{\psi}\psi\bar{\psi}\psi$	pointlike composite pion	not calculable	not calculable	none
CORNWALL & NORTON (3)	the void	$q\bar{q}$-condensate	toy gauge fields	yes, but structure obscure	estimated in 1-loop approx.	scale unknown depends on IR only tail is calculable	QED-like bound states
BCS	Fermi sea	ee-condensate	phonons	one (eaten) fully calculable	London(4) theory	calculable	depends on details of phonons
QUARK SOUP	Fermi sea	qq-condensate	QCD	numerous fully calculable	London theory	calculable	a veritable zoo

Table 3.1-1: Flops to flip over

3.2) Goldstonology

Every spontaneously broken continuous symmetry gives rise to a Goldstone excitation.[5] Gauge bosons of a local symmetry eat their associated Goldstone modes and get fat. Goldstone excitations of global symmetries have arbitrarily small energy, however.

The relevant symmetries of the pre-vacuum are fermion number conservation $U(1)_{\#}$, colour symmetry $SU(M)_c$, flavour symmetry $SU(N)_f$, and spatial isotropy $SO(3)_s$. It is convenient to lump fermion conservation together with the flavour group by replacing SU with U.

In the BCS problem, $U(1)_{\#} \times SO(3)_s$ breaks to $SO(3)_s$. One generator of the underlying group fails to leave the vacuum invariant. The sole corresponding Goldstone mode is eaten by electromagnetism, causing the Meissner effect.

In the quark soup problem, symmetries suffer the following fates:

1) $SU(2)_c \times U(1)_{\#} \times SO(3)_s$ breaks to

$SU(2)_c \times SO(2)_s$. There are three Goldstone modes: a scalar and two polarizations of a vector. Barring local electromagnetism, none are eaten.

2) $SU(2)_c \times U(2N)_f \times SO(3)_s$ breaks to

$SU(2)_c \times (SU(2)_f)^N \times SO(3)_s$. There are one bland scalar and $4N^2$-3N-1 flavour adjoint scalars. Barring local flavour, even these tasty Goldstone modes are not eaten.

Although a given pair involves the antisymmetric combination of just two flavours, quarks of every flavour want a piece of the action. Flavours therefore double up to pair.

3) $SU(2)_c \times U(2N+1)_f \times SO(3)_s$ breaks to

$SU(2)_c \times (SU(2)_f)^N \times SO(2)_s$. The odd flavour forms an anisotropic condensate as in case (1); the rest condense as in case (2).

4) $SU(3)_c \times U(1)_{\#} \times SO(3)_s$ breaks to

$SU(2)_c \times U(1)_{c-\#} \times SO(2)_s$. Colour breaking is laid to the condensation of anti-triplet diquarks, which select a particular direction in group space. In addition to the dull Goldstone modes of case (1), there are five coloured Goldstone modes, which get eaten by their corresponding gluons. Only an SU(2) subgroup of colour forces remains long range.*

5) $SU(3)_c \times U(2)_f \times SO(3)_s$ breaks to

$SU(2)_c \times U(1)_{c-\#} \times SU(2)_f \times SO(3)_s$. This case has Goldstone modes like those of case (2) and fat gluons like those of case (4).

6) $SU(3)_c \times U(2N)_f \times SO(3)_s$ shatters intricately.
Various doubled-up flavours can select differently oriented subgroups of colour in which to pair. We must determine the energetically optimal relative orientation of the $SU(2)_c$ subgroups chosen. Let us suppose that several such subgroups were differently oriented so that $SU(3)_c$ would be fully broken; every gluon would then get fat. As we have shown, the gap and therefore the non-perturbative interaction energy are much smaller for finite range forces than for unshielded forces. The alternative is to pick all $SU(2)_c$ subgroups parallel, thereby preserving an $SU(2)_c \times U(1)_c$ with unshielded gluons. This latter option is energetically superior.

We have slighted the internal symmetry groups by ignoring the possibility of chiral structure. If bare quarks are massless, then these groups would be doubled to $SU(M)_{cL} \times SU(M)_{cR} \times U(N)_{fL} \times U(N)_{fR}$. This would cause the parity-doubling of all internal symmetry Goldstone modes; every scalar would have a pseudoscalar twin. Local vector gauge bosons would eat only the scalars, however.

If quarks are light but not quite massless, i.e., the underlying chiral symmetry is imperfect even in the pre-vacuum and not just spontaneously broken, then the pseudoscalar modes would not be zero energy excitations. This is precisely the situation of the pion in the real world. The pion would be an exactly massless, pointlike particle only in the ideal PCAC limit of perfect underlying chiral symmetry, spontaneously broken. Instead, it is a featherweight meson with discernible structure, which can be ascribed to the imperfection of the chiral symmetry.

3.3) Anatomy of a Goldstone Boson

Group theory predicts the existence of Goldstone modes, but sheds no light on their structure. Several other approaches fill this need.

A diagrammatic demonstration of the existence of Goldstone modes starts from the observation that the gap equation has the form of a Bethe-Salpeter bound state equation, which is the condition for a summed ladder to have a pole. Such an S-matrix pole signifies a collective excitation mode. One solution, the gap function, is then the wave function of a bound state of zero momentum and energy (relative to the chemical potential), which is the endpoint of the Goldstone excitation spectrum.

A related approach utilizes Ward-Takahashi identities. These identities hold for all currents of underlying symmetries, whether spontaneously broken or not.[6]

$$(p-q)^{\sigma}\, \Gamma_{\sigma}^{\ a} = \lambda^{a}\, S^{-1}(p) - S^{-1}(q)\, \lambda^{a} \tag{3.3-1}$$

In particular, if λ^a is the generator of a spontaneously broken symmetry, it will not commute with S. Therefore, as $p-q \to 0$, the right-hand side does not vanish but approaches a constant. We must conclude that Γ_σ contains a pole of the form $(p-q)_\sigma/(p-q)^2$ times a residue representing the amputated Bethe-Salpeter wave function of the Goldstone boson. This residue is $\chi = [\lambda, S^{-1}(p)]$.

The connection between these approaches becomes obvious when we verify that χ satisfies the Bethe-Salpeter equation. In ladder approximation, the equation reads

$$\chi = \int D\gamma S \chi S \gamma \tag{3.3-2}$$

Inserting the alleged solution, we get

$$\int D\gamma S[\lambda, S^{-1}]S\gamma = \int D\gamma[S,\lambda]\gamma = [\lambda, \int D\gamma S\gamma] \tag{3.3-3}$$
$$= [\lambda,\Sigma] = [\lambda, S^{-1}-S_o^{-1}] = [\lambda, S^{-1}] \text{ , q.e.d.}$$

Qualitatively, we may also describe Goldstone modes as bound states of various kinds of quasiparticles. For example, let us classify the five coloured Goldstone modes engendered by the spontaneous breakdown of $SU(3)_c$ to $SU(2)_c$. Let "red" and "white" be the base colours of the preserved SU(2), and "blue" be the colour that does not participate in pairing. R denotes a red particle, and $\bar{R}$ a red hole. Swallowing RW pairs from the vacuum mixes R with $\bar{W}$ and $\bar{R}$ with W. Accordingly, there are five incestuously mixed families, suggestively named after broken generators of SU(3):

4) RB, $\bar{W}$B
5) WB, $\bar{R}$B
6) RB, $\bar{W}$B
7) WB, $\bar{R}$B
8) RW, $\bar{W}$W, $\bar{R}$R, $\bar{W}$R .

These are the only attractive possibilities. Symmetric states repel and cannot bind as Goldstone bosons.

The pieces of the Bethe-Salpeter wave function differ slightly for the blue+quasipink and for the diquasipink bosons, but the ultimate result is the same. The bipedal form of the wave function ($\varphi = S\chi S$) is

$$\begin{array}{cccc} & \underline{S_{\text{left leg}}} & \underline{\chi_{\text{amputee}}} & \underline{S_{\text{right leg}}} \\ \varphi_4 = & S_{>>} & \Delta & S_o \\ \varphi_8 = & S & \begin{bmatrix} 0 & +\Delta \\ -\Delta & 0 \end{bmatrix} & S \end{array} \tag{3.3-4}$$

where S is the matrix form of the propagator, and $S_{>>}$ is its diagonal entry. (See eq.(2.2-1).) Specifically,

$$\varphi_{4-7} = \Delta/((\not p+\not \mu)(\not p-\not \mu) - \Delta^2) \qquad (3.3\text{-}5)$$
$$\varphi_8 = \begin{bmatrix} 0 & +\Delta/((\not p+\not \mu)(\not p-\not \mu)-\Delta^2) \\ -\Delta/((\not p-\not \mu)(\not p+\not \mu)-\Delta^2) & 0 \end{bmatrix}$$

which are the familiar integrands of the gap equation (2.2-6).

3.4) Taxonomy of Collective Excitations

To illustrate the techniques for analyzing the collective excitation spectrum, we shall study one example in detail, the $SU(2)_c \times U(2)_f$ soup, which preserves all but its fermion number symmetry.

The Bethe-Salpeter equation for all its excitations resembles the equation for type (8) Goldstone modes discussed above. Solutions of this equation have a wide variety of Dirac matrix structures, which are multiplied by various orbital tensors built of 3-momenta. The number of loose spatial tensor indices determines the spin. The solutions fall into families:

1) The singlet series:

This series has the simplest possible Dirac structure, namely the unit matrix. As discussed in appendix (2.A), the unit matrix is mixed with a commutator term. The lowest member of this series is our friend the Goldstone mode.

$^1S_0 = (1 \;\&\; \gamma_0 \underset{\sim}{\not p})\ (1)$

$^1P_1 = (\ \text{"}\ )\ (\underset{\sim}{p})$

$^1D_2 = (\ \text{"}\ )\ (\underset{\sim}{p}\,\underset{\sim}{p} - \text{trace})$

etc.

Because of the chiral symmetry, each of these modes is degenerate with a chiral twin containing an extra γ_5 in its wave function.

2) The lower triplet series:

A spin index of the Dirac structure is inner-multiplied with the orbital tensor.

$^3P_0 = (\not{\underset{\sim}{p}}\ \&\ \gamma_0)\ (1)$

$^3D_1 = (\quad "\quad)\ (\underset{\sim}{p})$

etc. and chiral twins

3) The upper triplet series:

The spin index is outer-multiplied with the orbital tensor.

$^3S_1 = (\underset{\sim}{\gamma} \times \underset{\sim}{\gamma}\ \&\ \gamma_5\underset{\sim}{\gamma}\not{p})\ (1)$

$^3P_2 = (\quad "\quad)\ (\underset{\sim}{p})$ - trace

etc. and chiral twins

4) The middle triplet series:

The spin index is cross-multiplied with the orbital tensor.

$^3P_1 = (\underset{\sim}{p} \times \underset{\sim}{\gamma}\ \&\ \underset{\sim}{\gamma}\gamma_5)\ (1)$

etc. and chiral twins

With perfect chiral symmetry, the upper and lower triplet states do not mix, as they would in a deuteron, because the lower series has an odd number of γ-matrices, whereas the upper series has an even number.

3.5) Calculation of Selected Excitations

It is not too difficult to calculate the excitation energies of singlet series modes in the short-range approximation. We solve the Bethe-Salpeter equation by an ansatz for the wave function. It differs from the Goldstone modes' wave functions in several respects: It has a spherical harmonic angular dependence. Its legs carry unequal 4-momenta, $p \pm M/2$, where M is the excitation energy. Finally, its matrix structure is not exactly $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, but it is a good approximation to use just that. Accordingly, we take

$$\chi_L = \begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix} Y_{Lm}(\theta) \tag{3.5-1}$$

$$\varphi_L(p+M/2, p-M/2) = S(p+M/2)\ \chi_L\ S(p-M/2)$$

and insert them into the Bethe-Salpeter equation (3.3-2). The integrand now has poles at $p_0 = \pm M/2 \pm E(\underset{\sim}{p})$. After performing the $\gamma_\sigma \ldots \gamma^\sigma$ contractions and the p_0-integrations, omitting positron poles as in eq.(2.2-9), we get

$$\chi(p) = Cg^2 \int \frac{d^3k}{h^3} D(p-k) \frac{E(k)\quad \chi(k)}{E^2(k) - M^2/4} \tag{3.5-2}$$

This equation is seen to reduce to eq.(2.2-9) as $M \to 0$. The effect of the angular dependence of χ upon the integral can be extracted as follows:

$$\Delta(Q) = Cg^2 \int \frac{dq_z}{h^3} \frac{(q_z^2 + \Delta_o^2)^{\frac{1}{2}}\ \Delta(Q - q_z)}{q_z^2 + \Delta_o^2 - M^2/4} 4\pi\mu^2 A_L \tag{3.5-3}$$

$$A_L = \int d(\cos\theta) \frac{P_L(\cos\theta)}{2\mu^2(1-\cos\theta) + \Omega^2} .$$

This can be seen to reproduce the factors of eq.(2.2-13) when L=0 . The factor A_L is always positive but decreases monotonically as L grows. Meanwhile, $M_L \to 2\Delta_o$, which is, not surprisingly, precisely the two quasiparticle threshold. The integrand becomes very large near $q_z = 0$, compensating for the reduced A_L . Thus we find an infinite number of orbital excitations, leading up to the quasiparticle spectrum.

No radial excitations are known.

The various triplet series modes may be calculated by the same method, but the algebra is made onerous by all the complications of non-commutative Dirac algebra.

It would also be interesting to lift the degeneracy of chiral twins by "manufracturing" the chiral symmetry. Inserting a mass by hand into the bare propagator changes eq.(2.2-1) to

$$S^{-1} = \begin{bmatrix} \not{p}-\not{\mu}-m & \Delta \\ \Delta & \not{p}+\not{\mu}+m \end{bmatrix} \tag{3.5-4}$$

This makes the algebra unbearable because terms with even and odd numbers of Dirac matrices are now mixed, so that most Bethe-Salpeter wave functions combine four different Dirac structures.

The most controversial problem is the calculation of the dispersion law for the energy as a function of non-zero total 3-momentum. Several calculations in the literature disagree, and I do not wish to add to the body of incorrect results.

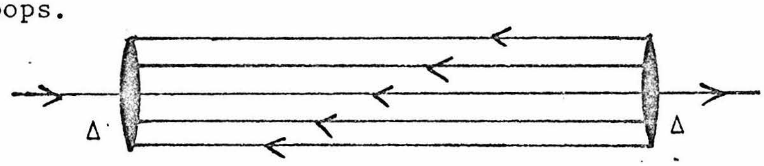
4) DIBARYON CONDENSATION

4.1) Adapting the 2-body Formalism

The correlation scheme of low density nuclear matter is known experimentally. Instead of pairing, quarks clump in threes to form colour singlet baryons. At a higher level, nucleons themselves pair and cluster as alpha particles, but these are much weaker correlations. When, if ever, do these complex correlations give way to simple quark pairing? We find that they are fragile and survive only at low densities.

The formalism of self-consistent field theory is designed to describe the vacuum expectation values of forbidden Bose operators only. We cannot readily discuss a baryon sea; we must discuss a di- or tetra-baryon condensate, even though single baryons appear to be the physically significant unit at low densities, di- and tetra-baryon correlations being so weak as to be almost irrelevant. Consequently, the fundamental, forbidden Green function that we examine is the amplitude for six or twelve quarks to pop out of the vacuum in a colour singlet state. Even if there is only one flavour, Fermi statistics admits spinless clumps, consisting of one particle in each of the 3 colour x 2 spin states. (We will restrict our attention to this simple case.) For convenience, we describe the spatial wave function of six bodies in shell model language; in the simplest conceivable situation, all six quarks occupy the 1S shell.[1] This is not particularly realistic in view of the fact that quarks actually segregate into two almost independent clumps of three rather than one great clump of six, but this inaccuracy is excusable since the thrust of the analysis is not so much to produce a theory of 3-quark correlations as to resolve the clumping versus pairing controversy by showing that any more-than-two-quark correlations are fragile.

As in the diquark condensate, the quark spectrum has a gap. The mechanism for producing this gap is not identical, however. A hole that swallows a dibaryon turns into a pentaquark, which in turn reverts to a hole. The details of pentaquark propagation are complicated, involving five quark propagators and four loops.

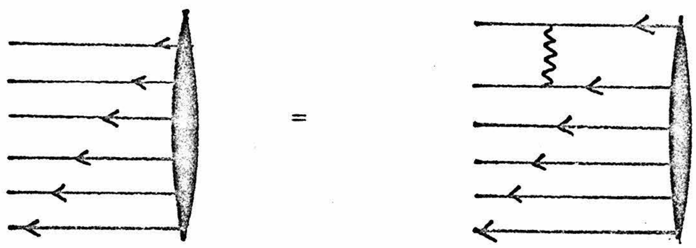


(4.1-1)

The gap in the quasiparticle spectrum will still be called Δ_o , but it is not so simply related to the wave function Δ of the clump.

The derivation of the gap equation carries through as before. In ladder approximation,

$$\Delta \quad = \quad g^2 \quad K\ SSSSSS\ \Delta \tag{4.1-2}$$



This equation is just like the pairing equation plus a few extra legs, but the effect of these legs on the solution is drastic.

The Bethe-Salpeter form of the gap equation is elegant but intractable, and its covariance is a dispensable frill in a non-Lorentz-invariant medium. Moreover, its solution contains a wealth of useless information about off-shell constituents. We would be better served by a Schroedinger version of the same equation. To convert $\Delta = g^2\ K\ S^6\ \Delta$ to Schroedinger form, we consider the residue of SSSSSS Δ at its mass shell pole. All but one of the constituents can be on shell simultaneously. The last, off-shell propagator is a non-trivial factor conventionally absorbed into the Schroedinger wave function. A cavalier simplification of the Dirac algebra renders this last propagator as $(p_{06} - E(\underset{\sim}{p}_6))^{-1}$. The condensed dibaryons have zero total 4-momentum (measured relative to the chemical potential); therefore $p_{06} = -p_{01} - \ldots - p_{05}$. The last propagator thus becomes $-1 / (E_1 + \ldots + E_6)$, where $E_i = ((|\underset{\sim}{p}_i| - \mu)^2 + \Delta_o^{\ 2})^{\frac{1}{2}}$. We identify $\psi_{\text{Schroedinger}} = \Delta / \Sigma E$.

The Schroedinger version of the gap equation is

$$\Delta = \sum_{i<j} c_{ij} \int \frac{d^3q}{h^3}\, D(q)\, \frac{(2P_i+q)_\sigma (2P_j-q)^\sigma}{2P_{0i} \quad 2P_{0j}}\, \psi(\ldots p_i+q \ldots p_j-q \ldots) \tag{4.1-3}$$

(Capitalized momenta include the chemical potential; they are not measured relative to it.) Equivalently,

$$(\Sigma E)\ \psi = Cg^2 \int \frac{d^3q}{h^3}\, D(q)\ (1 - \cos\theta_{12})\ \psi(p_1+q,\ p_2-q,\ldots) \quad (4.1\text{-}4)$$

It is easy to verify that this equation reduces to the 2-body form eq.(2.2-9) . There, partners have exactly opposite 3-momenta; thus $\cos\theta=-1$ and $\Sigma E=2E$. Only the value of the Casimir operator is different. (The agreement of factors of order unity should not be taken too seriously because of our careless simplification of the spinology.)

The gap equation $(\Sigma E)\ \psi = g^2\ K\ \psi$ may also be profitably reformulated as a variation condition,

$$0 = \delta\, \frac{\langle\psi|\,K\,|\psi\rangle}{\langle\psi|\,\Sigma E\,|\psi\rangle} \quad . \quad (4.1\text{-}5)$$

No full solution has ever been constructed for this equation. The best that can be done is to invent variational trial solutions and to determine an approximate relation between the gap and the coupling strength. The danger in using variational estimates is that they invariably overestimate the energy by some amount proportional to our artlessness in inventing a trial wave function. Here, they tend to overestimate the coupling strength needed to produce a given gap. While a variational method can prove binding possible, it can never prove binding impossible because a better trial function might always be found to give a lower energy.

Here, our trial function suggests that 6-body, colour singlet clumping is possible only at strong coupling. Although the estimate of the minimum coupling strength (derived in section 4.2) for such a correlation can certainly be pushed down by better choices of trial functions, backup arguments exist to show that some finite minimum coupling strength does exist. By all accounts, it is rather large, and the phenomenon of clumping is relegated to a zone of mystery, where nothing can be calculated accurately.

4.2) The Shell Approximation

The state of the art in multibody bound state physics is slightly inferior to the Hottentot number system, "1 2 3 ∞". In bound state physics, two's tractable; three's a cloud. Problems with very many degrees of freedom may, however, be successfully treated by the shell model, where each particle is taken to move in an average field of all other particles.

We give specific mathematical content to this assumption by neglecting angular correlations and assuming a variational Hartree form $\psi = f(p_1)\ldots f(p_6)$. This violates one obvious property of the true gap function, namely $\Sigma \underset{\sim}{p}_i = 0$; there is a redundant degree of freedom. The consequent error is mitigated by the large number of particles in the clump.

From the variational reformulation of the gap equation (4.1-5), or by manipulation of the gap equation (4.1-4) itself, we can derive a special equation for f(p). Taking advantage of the factorizable form, we first split the true gap equation into

$$E_1 f_1 \ldots = \frac{1}{6} C \sum_{j \neq 1} \int \frac{d^3 q}{h^3} D(q)\ f(p_1+q)\ldots f(p_j-q)\ldots \qquad (4.2\text{-}1)$$

and then eliminate the dependence on $p_2,\ldots,p_6$ by multiplying by $f_2^*\ldots f_6^*$ and integrating over $d^3p_2\ldots d^3p_6$. Then

$$E_1 f_1 = \frac{1}{6} C \int \frac{d^3 q}{h^3} D(q)\ F(q)\ f(p_1-q) \qquad (4.2\text{-}2)$$

where

$$F(q) = \int d^3p\ f^*(p) f(q-p) \ \Big/ \int d^3p\ f^* f \qquad (4.2\text{-}3)$$

This resembles eq.(2.2-9) with the gluon propagator softened by a form factor F(q), which can be physically interpreted as follows:

In a paired sea, states of opposite momentum are either both full or both empty. When a quark scatters from a full into an empty state, its opposite momentum partner is guaranteed to find an empty destination upon absorbing the recoil. Not so with

6-body correlations, where five partners divide each opposing momentum, and there are no exact full-full or empty-empty correlations. The "form factor" represents the probability that a partner will be available to absorb the recoil.

The modification of the potential by a form factor can also be understood as a consequence of the shell model in coordinate space. The potential in a 2-body bound state is precisely the 2-body interaction, but in a larger bound state, the shape of the potential well is the 2-body interaction folded together with the shape of the cloud of partners.

4.3) No Clumping at Weak Coupling

The gap associated with a non-perturbative effect must have one of three possible behaviours as the coupling strength decreases:

1) The gap can persist out to arbitrarily weak coupling--as does pairing. $\Delta_o \to 0$ as $g \to 0$. Or

2) The gap can vanish continuously at some finite coupling strength. $\Delta_o \to 0$ as $g \to g_{crit} > 0$. Or

3) The gap can vanish discontinuously.

In the context of the shell approximation, we can eliminate the first two possibilities. It is possible to construct a solution to eq.(4.2-2) in the hypothetical limit $\Delta_o \to 0$. We solve the equation by the approximate ansatz $f \approx 1/E$. The full form factor F(q) can be obtained numerically, but an analytic approximation to its tail fills our needs. For momenta in the range $\mu \gg q \gg \Delta_o$, the form factor behaves like $F \propto f \circ f \sim a \log^2 (q/\Delta_o)\, \Delta_o/q$, with a of order unity.

The analogue of eq.(2.2-12) is

$$g^{-2} \approx C \int \frac{d^3 q}{h^3} \frac{F}{\underset{\sim}{q}^2 + \Omega^2} \frac{1}{(q_z^2 + \Delta_o^2)^{\frac{1}{2}}} \qquad (4.3\text{-}1)$$

$$\approx C \int \frac{dq_z}{h^3} \frac{1}{(q_z^2 + \Delta_o^2)^{\frac{1}{2}}} \quad \pi^2 a \frac{\Delta}{\Omega} \log^2 \frac{\Omega}{\Delta}$$

$$g^{-2} \approx C\, h^{-3}\, \pi^{2}\, a \frac{\Delta}{\Omega} \log^{3} \frac{\Omega}{\Delta_o} \qquad (4.3\text{-}1)$$

As $\Delta_o \to 0$, g neither vanishes nor approaches a constant, but blows up pathologically. In fact, it never gets below $g^2_{min} \approx 20$. This pathology is not subject to the facile interpretation that the six-body correlation disappears discontinuously--poof--right at g_{min}, with a first order phase transition. The shell approximation is partially responsible for the pathological behaviour, and a better approximation might conceivably remove the apparent discontinuity.* We could imagine an intermediate phase between the experimentally observed colour-singlet dibaryon condensate at low densities, and the theoretically sound paired state at high densities. This intermediate state might involve 6-body correlations, but break colour symmetry. It is a vain exercise to attempt to calculate the properties of phase transitions near g_{min}, since they are deep inside the strong coupling regime, where diagrammatic methods fail. The intermediate phase can only be an object of speculation.

This variational trial proves two things. It proves that colour singlet clumping can happen. (This is reassuring since it does happen.) It also shows that the most obvious kind of 6-body correlation cannot persist to arbitrarily weak coupling. It does not exhaustively eliminate other hypothetical correlations. For this we must turn to another argument.

4.4) Stability of the Paired State

It can be shown that at sufficiently weak couplings, the paired state has no secondary Cooper-type instability involving formation of 3-body colour singlet bound states. The proof does not rely on difficult comparisons between two and three-body bound states, but rather on a theorem about the persistence of binding to arbitrarily weak coupling in various numbers of spatial dimensions, and on a property of Goldstone excitations of the paired state.

Although the world has three spatial dimensions, and the electrons in practical superconductors are strictly non-relativistic, the Cooper pairing problem is closely analogous to relativistic binding in one dimension. The wave function of a Cooper pair is concentrated at the Fermi surface, which is a two dimensional locus, only one short of the full number of spatial dimensions. The relevant density of states is therefore characteristic of a one-dimensional problem. Moreover, the excitation energy is not $p^2/2m$ but $(q+p_F)^2/2m - E_F = qv_F$, which is typical of a relativistic problem with c replaced by v_F.

The hypothetical 3-body Cooper instability would involve the binding of an unpaired quark to a condensed pair. The colour group SU(3) breaks to SU(2); let us call the base states of the surviving SU(2) "red" and "white," and the non-participating colour "blue." Blue quarks are unpaired and are attracted to pink pairs in total colour singlet states. This may render the paired state unstable if the attraction is strong enough to cause binding. The lowest energy pink pairs lie in the neighborhood of zero momentum, which is just a point, a zero-dimensional locus, and according to the energy spectrum for Goldstone excitations, their energy increases linearly with their momentum. This instability is therefore analogous to relativistic binding in three dimensions.

Relativistic binding persists at arbitrarily weak couplings for $D \leq 1$, as we shall show. Non-relativistic binding persists for $D \leq 2$.

Consider a potential well of depth $-V_o$ and finite range R, which we will attempt to fill with a trial wave function of adjustable width w in coordinate space. The expectation value of the kinetic energy is $\overline{T} \approx 1/2mw^2$ non-relativistically or $\overline{T} \approx 1/w$ relativistically. The expectation value of the potential energy is $\overline{V} \approx -V_o$ for $w \ll R$ but $\overline{V} \sim -V_o(R/w)^D$ for $w \gg R$. Binding is guaranteed if $\overline{T}+\overline{V}<0$ since the trial energy always exceeds the true energy. If $D<2$ non-relativistically or $D<1$ relativistically, $\overline{V}$ dominates $\overline{T}$ as $w \to \infty$, and arbitrarily shallow wells can bind particles by having w sufficiently big. If $D> 2$ or 1, respectively, $\overline{T}$ dominates $\overline{V}$

in both limits $w \to 0$ and $w \to \infty$, and no such compensation is possible; there is a critical minimum depth for the well to bind anything. In the borderline cases, D=2 and D=1, $\overline{T}$ appears to remain competitive with $\overline{V}$, and the variational argument is inconclusive, but it is possible to improve the trial wave function to prove persistent binding. We shall use a wave function in momentum space $\psi(p) = 1/(T(p)+B)$, with B adjustable instead of w. Then $\overline{T} \sim B \log B$ and $\overline{V} \sim -V_o RB \log^2 B$ as $B \to 0$. $\overline{V}$ dominates. (The choice of this trial function is motivated by the fact that it is the exact bound state of a delta function well in one dimension. Any finite-range well resembles a delta function when compared to a very wide spatial wave function.)

From this theorem, we conclude that the paired state is stable against the formation of 3-body and larger clumps up to some mysterious critical value of the coupling. This supports the conclusion drawn in the shell approximation.

5) MESON CONDENSATION

Meson condensation is a familiar phenomenon at zero chemical potential. Some version of this phenomenon is almost certainly the mechanism for quark mass generation and flavour symmetry breaking. Two principal variants have been proposed: the vacuum could be a condensate of fundamental Higgs scalars, coupling to quarks, or it could be a condensate of composite mesons, consisting of quarks. Inserting the quark masses into the Lagrangian "by hand" is generally disdained because it breaks gauge invariance and destroys renormalizability. Spontaneous symmetry breaking schemes eschew this vice.

The fate of a Higgs condensate at high chemical potential has already been investigated.[1] There is little doubt that this phenomenon can coexist with the pairing and clumping effects discussed above. Our present interest is in the fate of a composite meson condensate, however.

A quark mass can be understood as the gap of the $q\bar{q}$ condensate's quasiparticle excitation spectrum. As such, it manifests itself as a density threshold for the appearance of a heavy flavour, the condition for appearance being that the chemical potential exceed the gap.

The threshold for each flavour is indirectly influenced by all the other flavours present, which shield the forces that bind the condensed $q\bar{q}$ pairs. Heavy quark thresholds may therefore be expected to lie lower than any estimate of their current or constituent masses.

Just above threshold, the $q\bar{q}$ condensate is minimally disrupted by the presence of a shallow Fermi sea. Far above threshold, however, there is a deep sea of quarks not paired with antiquark partners. A $q\bar{q}$ condensate becomes energetically unfavourable, because the q is restricted to the empty high-momentum states above the Fermi surface, and its $\bar{q}$ partner has an equally high opposing momentum. The effects of the chemical potential on q and $\bar{q}$ cancel, leaving a huge net energy. The attractive potential energy, small in any case, becomes even smaller the more of momentum space is off limits. We may therefore expect the $q\bar{q}$ condensate to disappear completely somewhere above threshold, with a continuous phase transition marking the restoration of chiral symmetry.

Attempts to locate these transitions fail because the transitions lie in the strong coupling regime. We must content ourselves with a no-go theorem for weak coupling:

In three dimensions, at zero chemical potential, a $D(q)=1/q^2$ or $V(r)=1/r$ potential fails to bind massless particles. This is obvious from the nonsensical zero mass limit of the formulas for hydrogenic bound states as well as from the problem's lack of scale.

The $q\bar{q}$ condensation problem in a soup is complicated by shielding and the exclusion principle, but these do not alter the result.

Shielding weakens the attractive potential. This tends to raise the energy of any trial wave function. The no-binding result therefore applies a fortiori even though a scale is present.

Above threshold, the Fermi sea fills a ball in momentum space, making its states off limits to $q\bar{q}$ pairs. Only trial wave functions that do not use these states are permitted. Any constraint on the wave function necessarily raises the energy. Again, the no-binding result applies a fortiori.

Strong coupling provides the only known loophole to this theorem by undercutting the assumption of a simple Coulombic potential. "Infrared slavery" strengthens the potential at long distances and does provide a scale.

REFERENCES

For chapter (1):

(1) J. C. Collins & M. J. Perry, Phys Rev Lett 34 1353 (1975)

(2) M. Kislinger & P. D. Morley, Phys Rev D13 2765 (1976)

(3) B. A. Freedman & L. McLerran, Phys Rev D16 1130 (1977)

(4) B. C. Barrois, Nuovo Cimento 38A 50 (1977)

(5) V. Canuto, Annual Review of Astronomy and Astrophysics v.13 (1975)

(6) J. R. Schrieffer, Superconductivity (Benjamin, 1964) ch.2

(7) Y. Nambu, Phys Rev 117 648 (1960)

For chapter (2):

(1) Schrieffer, op cit, ch.2

(2) Nambu, op cit.

(3) A. L. Fetter & J. D. Walecka, Quantum Theory of Many Particle Systems, (McGraw Hill, 1971) p.111

(4) ibid., p.70 eq.7.30

(5) ibid., ch.7

For chapter (3):

(1) E. S. Abers & B. W. Lee, Phys Reports 9C 1 (1973)

(2) Y. Nambu & G. Jona-Lasinio, Phys Rev 124 246 (1961)

(3) J. M. Cornwall & R. E. Norton, Phys Rev D8 3338 (1973)

(4) Schrieffer, op cit, ch.8

(5) J. Goldstone et al., Phys Rev 127 965 (1962)

(6) Nambu, op cit.

For chapter (4):

(1) B. C. Barrois, Nuc Phys B129 390 (1977)

For chapter (5):

(1) D. A. Kirzhnits & A. D. Linde, Annals of Phys 101 195 (1976)

NOTES

1.2)*) One might conjecture that such asymptotic series Borel-sum to the non-analytic forms in eqs.(2.2-14 & 16). If so, the form $\exp(-C/g^2)$ comes from a perturbation series $\Sigma\, a_n g^{2n}$ with coefficients growing as $a_n \sim n!\, C^{-n}$.

1.3)*) An iterative scheme might still be useful for refining approximate solutions to the Dyson equations. A crude approximation would be ground through the non-linear integral expression of the self-consistency equation, and son-of-approximation would emerge. Whether successive refinements converge is an open question. The hope for convergence is raised by the example of a related linear problem, the homogeneous Schroedinger equation, $(T-E_o)^{-1}V\,\psi = \psi$. Repeated applications of the operator $(T-E_o)^{-1}V$ do improve approximate ψ's because all its eigenvalues lie between 0 and 1, provided only that T and -V are both positive definite, and that E_o is the ground state energy.

1.4)*) It is an oversimplification to classify forces as long or short range. A trichotomy is widely suspected: short, long, or confining. Shielding bobs electric forces, but the absence of a perturbative Meissner effect leaves open the question of whether magnetism is long or confining. Renormalization group arguments suggest that magnetism is merely long. The infrared growth of the effective coupling comes from the glue loop correction to the glue propagator, which has a logarithmic divergence cut off at the low end by the gluon's momentum or some fixed mass parameter, whichever is greater. According to the Appelquist-Carrazone decoupling theorem, the growth of the coupling from this diagram is arrested when the momentum is smaller than the mass of the loop constitutents. The relevant mass parameter is neither the electric inverse shielding length, Ω, nor the magnetic shielding, zero, but the plasma frequency, which coincidentally equals Ω. This suggests that the best propagator for the magnetic components of the gluon would be an ordinary long-range propagator used with a coupling constant renormalized at a spacelike momentum of order Ω. Since $\Omega \sim g\mu$, we may also renormalize at μ without affecting the results to leading order.

2.2)*) A perturbative derivation of Dyson equations even for forbidden Green functions can be sketched. It relies on the resemblance of the Dyson equation to the Bethe-Salpeter equation for a zero energy and momentum bound state (see section 3.3). Perturbatively, there is no source of diquarks (or other forbidden objects). The amplitude to feel a diquark is, however, the strength of its source times its amplitude to propagate. Since the propagator has a Bethe-Salpeter pole at zero energy and momentum (which is exactly what objects lurking in the vacuum must have), the amplitude to be felt is zero times infinity. The algebra is better defined in a certain limit. Inserting a small diquark source by hand into the Lagrangian breaks the symmetry and moves the Bethe-Salpeter pole slightly away from zero energy. The limit of the product of source and propagator as we turn off our hand-inserted source is the forbidden amplitude obeying the Dyson equation.

This derivation is not very enlightening numerically, but it clears up topological ambiguities in the field theory by prescribing a simple replacement rule: a self-consistent forbidden blob for each Bethe-Salpeter ladder.

3.2)*) The Meissner effect shortens the range of several gluons. Except in short-range approximation, we must discriminate between short and long-range gluons, and we should only use the Casimir operator for the unbroken SU(2) subgroup of colour. This alters eq.(2.2-7) to C=3/4 .

4.3)*) The use of long-range propagators banishes Ω, effectively replacing it by Δ_o in most contexts. Consequently, g^2 approaches a large constant as Δ_o shrinks. This signals a continuous phase transition.

COMMONLY USED SYMBOLS

C	=	Casimir operator (2.2-7)
D	=	gluon propagator
E	=	quasiparticle energy (2.2-4)
f	=	factor of Hartree clump wave function (4.2)
F	=	form factor of Hartree clump (4.2-3)
g	=	density dependent coupling constant, renormalized at a spacelike momentum of order μ
h	=	Planck's constant, 2π
H	=	Hamiltonian operator (1.5)
k	=	fermion momentum
K	=	Bethe-Salpeter interaction kernel, $K... = \int D\gamma ... \gamma$ (4.1-2)
p, P	=	fermion momenta
q	=	gluon momentum
Q	=	momentum on Fermi surface
S	=	particle-hole propagator matrix (2.2-1)
Γ	=	dressed quark-current vertex
γ	=	bare quark-gluon vertex, Dirac matrix
Δ	=	gap function (2.1-1)
Δ_o	=	gap in quasiparticle energy spectrum (2.2-9 ff.)
ε	=	antisymmetric symbol
λ	=	group generator or representation
μ	=	chemical potential, Fermi energy (1.4)
Σ	=	blob in fermion propagator, (2.2-1); summation
φ	=	bipedal Bethe-Salpeter wave function (3.3-4)
χ	=	amputee Bethe-Salpeter wave function (3.3-2)
ψ	=	six-body clump wave function
Ω	=	inverse shielding length, plasma frequency (1.4)