# MEAN VALUE DERIVATIVES

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This thesis is respectfully dedicated to the memory of Professor Harry Bateman who proposed the problem and guided the author in the early stages of the investigation. The author wishes to express his deep gratitude to Professor J.W.Green of the University of California at Los Angeles under whose guidance this thesis was written.

ABSTRACT

Let $L(f;x,y;r)$, $A(f;x,y;r)$ be the mean values of a function $f(x,y)$ of two real variables on the perimeter and on the interior, respectively, of a circle of center $(x,y)$ and radius $r$. The limits

$$\lim_{r\to 0} \frac{L(f;x,y;r)-f(x,y)}{r^2} = f'(x,y) \qquad (1)$$

$$\lim_{r\to 0} \frac{A(f;x,y;r)-f(x,y)}{r^2} = f'_D(x,y) \qquad (2)$$

are called <u>Mean Value Derivatives</u> of $f(x,y)$. This paper is concerned with the investigation of functions with mean value derivatives. These derivatives are essentially generalizations of LaPlace's operator, and , as such, were investigated by Blaschke and Privaloff. In addition Zaremba has investigated another form of generalized LaPlacian, and Plancherel has investigated a generalization of Beltrami's parameter. Many of the results obtained for these last two operators hold true for mean value derivatives.

Chapter I contains some results relating to the mean value derivative as given by eqn. (1) while Chapter II is a similar treatment of eqn. (2). Most of the results given in these two chapters are known for at least one of the four operators, i.e. those of Blaschke, Privaloff, Zaremba, and Plancherel. Chapter III discusses briefly <u>uniform</u> mean value derivatives. Chapter IV is devoted to the use of potential theory in the subject and Chapter V to higher derivatives. Chapter VI is concerned with further problems on the subject and Chapter VII contains a summary of the results of the authors mentioned above. The principle new results obtained are as follows:

(1) If $f'_D(x,y)$ exists then so does $f'(x,y)$. This is a generalization of a result due to Kozakiewicz, who assumed continuity of $f$. This assumption is not necessary.

(2) If (i) $f(x,y)$ is continuous, (ii) $f'(x,y)$ exists and is bounded, (iii) $f'(x,y)=0$ almost everywhere on a domain $D$, then $f(x,y)$ is harmonic on $D$.

(3) If $f(x,y)$ is a logarithmic potential function for which the density of the mass distribution exists at a point $P$ then $f'(P)$ exists.

(4) Expansions in powers of $r^2$ are obtained for the means $L(f;x,y;r)$, $A(f;x,y;r)$ in which the coefficients involve the higher mean value derivatives of in a manner analogous to Taylor's Theorem.

# Chapter I

## FUNCTIONS WITH MEAN VALUE DERIVATIVES

1.1 Introduction: Let $f(x,y)$ be a function of two real variables. Consider the mean value

$$L(f;x,y;r) = \frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\,d\theta$$

of $f(x,y)$ round a circle of radius $r$. The limit

$$\lim_{r\to 0} \frac{L(f;x,y;r) - f(x,y)}{r^2} = f'(x,y)$$

may exist. We propose to investigate the class of functions for which this limit does exist.

Note: Integration will be in the Lebesgue sense, and in all ensuing discussion we shall assume that $f(x,y)$ is summable two dimensionally. In this chapter we shall further assume that $f(x,y)$ is summable along all circles used.

1.2 Notation: Let $P$ be the point $(x,y)$.

$C(P;r) \equiv C(x,y;r)$ is the perimeter of a circle of radius $r$ and center $P$.

$D(P;r) \equiv D(x,y;r)$ is the interior of a circle of radius $r$ and center $P$. $\overline{D}(P;r)$ is the closure of $D(P;r)$.

$$L(f;P;r) = L(f;x,y;r) = \frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\,d\theta = \frac{1}{2\pi r}\int_{C(P;r)} f(P)\,ds_P$$

We shall use the notations in terms of $P$ and of $(x,y)$ interchangeably.

1.3 Preliminary definitions: We define the Mean Value Derivative of $f(P)$ at $P_0$ to be

$$\lim_{r\to 0} \frac{L(f;P_0;r) - f(P_0)}{r^2} = f'(P_0)$$

if the limit exists. If the limit does exist $f(P)$ will be said to be <u>MV differentiable</u> at $P_0$ .

1.4 <u>Preliminary theorems</u>:

1.41 <u>Theorem</u>: If $f(P)$ and $g(P)$ are MV differentiable at $P_0$ then $f(P)+g(P)$ is MV differentiable at $P_0$ and has the mean valu derivative $f'(P_0)+g'(P_0)$ .

<u>Proof:</u> This clearly follows from the definition.

1.42 <u>Theorem</u>: If $f(P)$ has continuous second partial derivatives $f_{xx}, f_{yy}$ at $P(x,y)$, then $f(P)$ is MV differentiable at $P$. Further

$$f'(P) = \frac{1}{4}(f_{xx} + f_{yy})$$

(see Webster (7.13), Plancherel (7.31) )

<u>Proof</u>: We have

$$\lim_{r\to 0} \frac{L(f;P;r)-f(P)}{r^2} = \lim_{r\to 0} \frac{\int_0^{2\pi}\{f(x+r\cos\theta, y+r\sin\theta)-f(x,y)\}\,d\theta}{2\pi r^2}$$

Applying the rule of L'Hospital we get

$$\lim_{r\to 0} \frac{\int_0^{2\pi}\{\left(\frac{\partial f}{\partial x}\right)_1\cos\theta + \left(\frac{\partial f}{\partial y}\right)_1\sin\theta\}\,d\theta}{4\pi r}$$

where $\left(\frac{\partial f}{\partial x}\right)_1, \left(\frac{\partial f}{\partial y}\right)_1$ are the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ respectively, evaluated at $(x+r\cos\theta, y+r\sin\theta)$.

Applying the rule a second time we have

$$\lim_{r\to 0} \frac{\int_0^{2\pi}\{\left(\frac{\partial^2 f}{\partial x^2}\right)_1\cos^2\theta + \left(\frac{\partial^2 f}{\partial y^2}\right)_1\sin^2\theta + 2\left(\frac{\partial^2 f}{\partial x\partial y}\right)_1\sin\theta\cos\theta\}\,d\theta}{4\pi}$$

$$= \lim_{r\to 0} \frac{\left(\frac{\partial^2 f}{\partial x^2}\right)\pi + \left(\frac{\partial^2 f}{\partial y^2}\right)\pi}{4\pi} = \frac{1}{4}(f_{xx} + f_{yy}) = f'(P)$$

1.43 <u>Theorem</u>: If (i) $f(P)$ is continuous on a domain $\mathcal{D}$, (ii) $f(P)$ is MV differentiable everywhere on $\mathcal{D}$ , (iii) $f'(P)=0$ everywhere on $\mathcal{D}$ , then $f(P)$ is harmonic on $\mathcal{D}$ . (see Zaremba 7.22, Plancherel 7.33, Blaschke 7.41)

<u>Proof</u>: Consider a circle $C(P_0;r)$ lying in $\mathcal{D}$ . Let

$$\varphi(P) = \varepsilon\,[f(P)-h(P)] + \kappa\,[|P-P_0|^2 - r^2]$$

where $\varepsilon$ has the values $+1$ and $-1$, $\kappa$ is an arbitrary positive number, $h(P)$ is the harmonic function such that $h(P)=f(P)$ on $C(P_0;r)$. Clearly $\varphi(P)=0$ on $C(P_0;r)$. Now

$$\varphi'(P)=\varepsilon\,[f'(P)-h'(P)]+\{\kappa\,[(x-x_0)^2+(y-y_0)^2-r^2]\}'$$

where $\{\ \}'$ denotes the process of taking the mean value derivative. Applying Theorem 1.42 we see that $h'(P)=0$ and

$$\{\kappa\,[|P-P_0|^2-r^2]\}'=\kappa$$

Further, by hypothesis, $f'(P)=0$. Thus

$$\varphi'(P)=\kappa \tag{1}$$

Now $\varphi(P)\le 0$ for all $P$ in $D(P_0;r)$. For if $\max\varphi(P)=g>0$ then there exists a point $P_1$ in $D(P_0;r)$ such that $\varphi(P_1)=g$. Hence for $\rho$ suffiaiently small

$$\int_{C(P_1;\rho)}[\varphi(P)-\varphi(P_1)]\,ds_p\le 0$$

and thus $\varphi'(P_1)\le 0$. But this contradicts equation (1), hence $\varphi(P)\le 0$ for all $P$ in $D(P_0;r)$. For $\varepsilon=1$ we have

$$f(P)-h(P)\le-\kappa\{|P-P_0|^2-r^2\}$$

For $\varepsilon=-1$ we have

$$-f(P)+h(P)\le-\kappa\{|P-P_0|^2-r^2\}$$

Hence $\qquad |f(P)-h(P)|\le\kappa\,\big||P-P_0|^2-r^2\big|$

But we can take $\kappa$ as small as we please. Therefore $f(P)=h(P)$ within the circle $D(P_0;r)$. We can cover $\mathcal{D}$ by overlapping circles and hence $f(P)$ is harmonic on $\mathcal{D}$.

<u>Note</u>: Continuity is essential in this theorem as is shown by the following example:

$$\begin{aligned} f(x,y) &= 1 & x>0\\ &=-1 & x<0\\ &=0 & x=0\end{aligned}$$

Clearly $f'(x,y)=0$ everywhere, but $f(x,y)$ is not continuous on the line $x=1$. This theorem and its proof are a two dimensional analogue of a theorem of Schwarz concerning his generalized second derivative for functions of one real variable. (H.A. Schwarz: <u>Gesammelte Mathematische Abhandlungen</u>, vol. II, pp 340-3).

1.44 <u>Theorem</u>: If $f(P)$ is continuous and MV differentiable on $\bar{D}(P_0;r)$ then
$$\frac{L(f;P_0;r)-f(P_0)}{r^2}$$
is bounded by the upper and lower bounds of $f'(P)$ on $\bar{D}(P_0;r)$. (See Plancherel 7.32)

<u>Proof</u>: Consider the function
$$\lambda(P) = f(P) - h(P) + L(f;P_0;r) - f(P_0) - \frac{L(f;P_0;r)-f(P_0)}{r^2}|P-P_0|^2$$
where $h(P)$ is the harmonic function such that $h(P)=f(P)$ on $C(P_0;r)$. Clearly $\lambda(P)=0$ on $C(P_0;r)$. Further
$$\lambda(P_0) = f(P_0) - h(P_0) + L(f;P_0;r) - f(P_0) = L(f;P_0;r) - h(P_0)$$
But
$$h(P_0) = \frac{1}{2\pi r}\int_{C(P_0;r)} h(P)\,ds_P = \frac{1}{2\pi r}\int_{C(P_0;r)} f(P)\,ds_P = L(f;P_0;r)$$
Therefore $\lambda(P_0)=0$. Thus the function $\lambda(P)$ has both a maximum and a minimum value on $D(P_0;r)$. Now if $P_1$ is a maximum point of $\lambda(P)$ then $\lambda'(P_1)\leq 0$, for
$$\lambda'(P_1) = \lim_{\rho\to 0}\frac{1}{2\pi\rho^3}\int_{C(P_1;\rho)}\{\lambda(P)-\lambda(P_1)\}\,ds_P \leq 0$$
But we have
$$\lambda'(P) = f'(P) - h'(P) - \frac{L(f;P_0;r)-f(P_0)}{r^2}\{|P-P_0|^2\}'$$
By Theorem 1.42 $\quad h'(P)=0$, $\{|P-P_0|^2\}'=1$.

Therefore
$$\lambda'(P) = f'(P) - \frac{L(f;P_0;r)-f(P_0)}{r^2}$$
But $\lambda'(P_1)\leq 0$, therefore
$$f'(P_1) \leq \frac{L(f;P_0;r)-f(P_0)}{r^2}$$
Similarly if $P_2$ is a minimum point of $\lambda(P)$ on $D(P_0;r)$ we have
$$f'(P_2) \geq \frac{L(f;P_0;r)-f(P_0)}{r^2}$$

Note: This theorem and its proof are a two dimensional analogue of a theorem of Lebesgue relating to Schwarz's second derivative. (Lecons sur les series trigonometriques, by Henri Lebesgue, pp 5-7.)

1.45 Theorem: If (i) $f(P)$ is continuous on a domain $\mathcal{D}$, (ii) $f'(P)$ exists and is zero everywhere on $\mathcal{D}$ save at points of a denumerable set $\mathcal{S}$, (iii) for points on $\mathcal{S}$

$$\lim_{r\to 0} \frac{L(f;P;r)-f(P)}{r} = 0$$

then $f(P)$ is harmonic on $\mathcal{D}$.

Proof: Let $P_0$ be a point in $\mathcal{D}$ not in $\mathcal{S}$. Consider a circle $C(P_0;r)$ lying in $\mathcal{D}$. Let $h(P)$ be the harmonic function such that $f(P)=h(P)$ on $C(P_0;r)$. Suppose $f(P)-h(P)$ has a positive value $p$ at some point $P_1$ in $D(P_0;r)$. Consider the function

$$\phi(P;\kappa) = f(P) - h(P) + \kappa\,|P-P_0|^2 \qquad \text{where } \kappa > 0$$

On $C(P_0;r)$ $\phi(P;\kappa)=\kappa r^2$. Also

$$\phi(P_1;\kappa) = p + \kappa\,|P_1-P_0|^2$$

We pick $\kappa$ so that $\phi(P_1;\kappa) > \kappa r^2$

i.e. $\kappa r^2 < p + \kappa|P_1-P_0|^2$, $\kappa[r^2-|P_1-P_0|^2] < p$

or $$\kappa < \frac{p}{r^2-|P_1-P_0|^2} = K$$

$\phi(P;\kappa)$ thus attains its maximum value on $D(P_0;r)$. Let $P_2$ be a maximum point of $\phi(P;\kappa)$. Then if $P_2$ is not in $\mathcal{S}$ we have

$$\phi'(P_2;\kappa) = \lim_{r\to 0} \frac{L(\phi;P_2;r)-\phi(P_2)}{r^2} \le 0$$

But for points of $D(P_0;r)$ not in $\mathcal{S}$

$$\phi'(P;\kappa) = f'(P) - h'(P) + \kappa = \kappa > 0$$

Hence $P_2$ must be in $\mathcal{S}$.

Now $$\lim_{r\to 0}\frac{L(\phi;P_2;r)-\phi(P_2)}{r} = \lim_{r\to 0}\left\{\frac{L(f;P_2;r)-f(P_2)}{r} + \frac{\kappa[L(|P-P_0|^2;P_2;r)-|P-P_0|^2]}{r}\right\}$$

But $$L(|P-P_0|^2;P_2;r) = \frac{1}{2\pi}\int_0^{2\pi}\left\{(x-x_0+r\cos\theta)^2+(y-y_0+r\sin\theta)^2\right\}d\theta$$

$$= (x-x_0)^2+(y-y_0)^2+r^2$$

Hence $\lim\limits_{r\to 0} \dfrac{L(\phi;P_2;r)-\phi(P_2)}{r} = \lim\limits_{r\to 0}\left\{\dfrac{L(f;P_2;r)-f(P_2)}{r} + kr\right\} = 0$

by hypothesis (iii). But

$$\lim_{r\to 0} \frac{L(\phi;P_2;r)-\phi(P_2)}{r} = \lim_{r\to 0}\left\{\frac{1}{2\pi r}\int_0^{2\pi}[\phi(x_2+r\cos\theta, y_2+r\sin\theta)-\phi(x_2,y_2)]\,d\theta\right\}$$

$$= \lim_{r\to 0}\left\{\frac{1}{2\pi r}\int_0^{\pi}[\phi(x_2+r\cos\theta, y_2+r\sin\theta)-\phi(x_2,y_2)]\,d\theta + \frac{1}{2\pi r}\int_\pi^{2\pi}[\phi(x_2+r\cos\theta, y_2+r\sin\theta) - \phi(x_2,y_2)]\,d\theta\right\}$$

But the two integrals are negative for all small $r$ since $P_2$ is a maximum point of $\phi(P;k)$. Hence

$$\lim_{r\to 0}\frac{1}{2\pi r}\int_0^{\pi}\{\phi(x_2+r\cos\theta, y_2+r\sin\theta)-\phi(x_2,y_2)\}\,d\theta = 0$$

But $\dfrac{1}{2\pi r}\displaystyle\int_0^{\pi}\{\phi(x_2+r\cos\theta,y_2+r\sin\theta)-\phi(x_2,y_2)\}\,d\theta$

$$= \frac{1}{2\pi r}\int_0^{\pi}\{f(x_2+r\cos\theta,y_2+r\sin\theta)-f(x_2,y_2)-h(x_2+r\cos\theta,y_2+r\sin\theta)+h(x_2,y_2)$$
$$+k[(x_2-x_0+r\cos\theta)^2+(y_2-y_0+r\sin\theta)^2]-k[(x_2-x_0)^2+(y_2-y_0)^2]\}\,d\theta$$

$$= \frac{1}{2\pi r}\int_0^{\pi}\{f(x_2+r\cos\theta,y_2+r\sin\theta)-f(x_2,y_2)\}\,d\theta + \frac{k}{2\pi r}\int_0^{\pi}\{2(x_2-x_0)r\cos\theta +$$
$$2(y_2-y_0)r\sin\theta + r^2\}\,d\theta + \frac{1}{2\pi r}\int_0^{\pi}\{h(x_2,y_2)-h(x_2+r\cos\theta,y_2+r\sin\theta)\}\,d\theta$$

$$= \frac{1}{2\pi r}\int_0^{\pi}\{f(x_2+r\cos\theta,y_2+r\sin\theta)-f(x_2,y_2)\}\,d\theta + \frac{2k}{\pi}(y_2-y_0)+kr$$
$$-\frac{1}{2\pi r}\int_0^{\pi}\left[\left(\frac{\partial h}{\partial x}\right)_2 r\cos\theta + \left(\frac{\partial h}{\partial y}\right)_2 r\sin\theta + o(r)\right]d\theta$$

$o(r)$ being uniform in $\theta$, the subscript denoting the values at $(x_2,y_2)$

And so,

$$\lim_{r\to 0}\frac{1}{2\pi r}\int_0^{\pi}\{f(x_2+r\cos\theta,y_2+r\sin\theta)-f(x_2,y_2)\}\,d\theta = \frac{2k}{\pi}(y_2-y_0)-\frac{1}{\pi}\left(\frac{\partial h}{\partial y}\right)_2$$

Thus for each point $P_2$ there is a unique value $k$. Since every $P_2 \in \mathcal{S}$ and $\mathcal{S}$ is denumerable, there exists only a countable number of values of $k$ for which $\varphi(P;k)$ assumes a maximum value on $D(P_0;r)$ ; but this contradicts the fact that $k$ can be chosen arbitrarily $< K$ , and $\varphi(P;k)$ does assume a maximum value on $D(P_0;r)$

Hence the assumption that $f(P)-h(P)$ has a positive value is untrue. Similarly for negative values. Therefore $f(P)=h(P)$ on $D(P_0;r)$. We can cover $\mathcal{D}$ by overlapping circles and hence $f(P)$ is harmonic on $\mathcal{D}$.

Note: This theorem is a two dimensional analogue of a theorem of Hobson on Schwatz's derivative. (E.W.Hobson, The Theory of Functions of a Real Variable, vol.I, p 278.)

1.46 Theorem: If (i) $f(P)$ vanishes on $C(P_0;r)$, (ii) $f(P_0)=0$, (iii) $f(P)$ is continuous on $\bar{D}(P_0;r)$, (iv) $f'(P)$ exists and is continuous on $\bar{D}(P_0;r)$, then there exists a point $Q$ in $D(P_0;r)$ such that $f'(Q)=0$. (i.e. an analogue of Rolle's theorem)

Proof: By Theorem 1.44

$$\frac{L(f;P_0;r)-f(P_0)}{r^2}$$

is bounded above and below by the upper and lower bounds, respectively, of $f'(P)$ on $D(P_0;r)$. But $L(f;P_0;r)-f(P_0)=0$. Hence, since $f'(P)$ is continuous there is a point $Q$ in $D(P_0;r)$ such that $f'(Q)=0$.

1.47 Theorem: If (i) $f(P)$ is continuous on $\bar{D}(P_0;r)$, (ii) $f'(P)$ exists and is continuous on $\bar{D}(P_0;r)$, then there exists a point $Q$ in $D(P_0;r)$ such that

$$L(f;P_0;r)=f(P_0)+r^2 f'(Q)$$

(see Blaschke 7.42)

Proof: We apply Theorem 1.46 to the function

$$\lambda(P)=f(P)-h(P)+\frac{L(f;P_0;r)-f(P_0)}{r^2}\left[r^2-\overline{PP_0}^2\right]$$

where $h(P)=f(P)$ on $C(P_0;r)$, $h(P)$ being harmonic on $D(P_0;r)$.

# Chapter II

## MEAN VALUES OVER CIRCULAR AREAS

2.1 Introduction: In investigating the properties of functions which are MV differentiable it has turned out to be advantageous to consider a mean value derivative of a slightly different type than that defined in Chapter I. This second type involves the mean value of the function in question over the area of a circle rather than around the perimeter. Accordingly we make further definitions.

2.2 Mean value derivative (on D): Let

$$A(f;P;r) = \frac{1}{\pi r^2} \iint_{D(P;r)} f(Q)\,dQ$$

be the mean value of a function $f(P)$ over a circle of center $P$ and radius $r$. The mean value derivative (on D) of $f(P)$ at $P_0$ is then defined to be the limit

$$\lim_{r\to 0} \frac{A(f;P_0;r) - f(P_0)}{r^2} = f'_D(P_0)$$

if it exists. We say that the function is MV(D) differentiable at $P_0$.

2.3 A theorem of Tonelli: In the ensuing discussion we employ a result due to Tonelli:

If $A(f;P;r) = A(f;x,y;r)$ exists, then

$$A(f;x,y;r) = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\rho\,d\theta\,d\rho = \frac{2}{r^2}\int_0^r L(f;x,y;\rho)\rho\,d\rho$$

Note: See S.Saks, Theorie de l'integrale, p 75

2.4 Relation between the MV derivative and the MV(D) derivative

2.41 Theorem: If $f(P)$ is MV differentiable, then it is MV(D) differentiable. Further $f_D'(P) = \frac{1}{2} f'(P)$

(See Kozakiewicz 7.62, 7.63)

Proof: $L(f;P;r)$ exists for small $r$, and further

$$L(f;P;r) - f(P) = r^2\left[f'(P) + \nu(P;r)\right]$$

where $\nu \to 0$ as $r \to 0$. Thus we have

$$A(f;P;r) = \frac{1}{\pi r^2} \iint_{D(P;r)} f(Q)\, dQ = \frac{2}{r^2} \int_0^r L(f;P;\rho)\rho\, d\rho$$

And hence $\dfrac{A(f;P;r) - f(P)}{r^2} = \dfrac{2\int_0^r L(f;P;\rho)\rho\, d\rho - r^2 f(P)}{r^4}$

$$= \frac{1}{r^4}\left\{2\int_0^r [f(P) + \rho^2 f'(P) + \rho^2 v(P;\rho)]\rho\, d\rho - r^2 f(P)\right\}$$

$$= \frac{1}{r^4}\left\{f(P)\cdot\frac{2r^2}{2} + f'(P)\frac{2r^4}{4} + 2\int_0^r \rho^3 v\, d\rho - r^2 f(P)\right\} = \frac{1}{2} f'(P) + \frac{2}{r^4}\int_0^r \rho^3 v\, d\rho$$

The last term is easily shown to approach zero, since $v \to 0$ as $\rho \to 0$. Thus

$$\lim_{r\to 0} \frac{A(f;P;r) - f(P)}{r^2} = \frac{1}{2} f'(P)$$

In view of this theorem when discussing functions for which the mean value derivative as defined in 1.1 exists, we shall be able to utilize expressions involving both $A(f;P;r)$ and $L(f;P;r)$.

2.5 <u>Theorems on MV(D) differentiable functions</u>:

2.51 <u>Theorem</u>: If $f(P)$ has continuous second partial derivatives $f_{xx}$, $f_{yy}$ at $P$, then $f(P)$ is MV(D) differentiable at $P$. Further

$$f'_D = \frac{1}{8}(f_{xx} + f_{yy})$$

(See Webster 7.11, Privaloff 7.51)

<u>Proof</u>: The proof is similar to that of Theorem 1.42.

2.52 <u>Theorem</u>: If (i) $f(P)$ is continuous and MV(D) differentiable everywhere on a domain $\mathcal{D}$, (ii) $f'_D(P) = 0$ everywhere on $\mathcal{D}$, then $f(P)$ is harmonic on $\mathcal{D}$. (See Privaloff 7.52)

<u>Proof</u>: The proof is similar to that of Theorem 1.43.

2.53 <u>Theorem</u>: If $f(P)$ is continuous and MV(D) differentiable on $\bar{D}(P_0;r)$ then

$$\frac{A(f;P_0;r) - f(P_0)}{r^2}$$

is bounded by the upper and lower bounds of $f'_D(P)$ on $\bar{D}(P_0;r)$.

Proof: $L(f;P_0;\rho)$ exists for almost all $\rho \le r$. Consider the function

$$\lambda(P) = f(P) - h(P) + L(f;P_0;\rho) - f(P_0) - \frac{L(f;P_0;\rho) - f(P_0)}{\rho^2}\,\overline{PP_0}^2$$

where $h(P)$ is harmonic and $h(P) = f(P)$ on $C(P_0;\rho)$ and $L(f;P_0;\rho)$ exists. By an arguement similar to that employed in Theorem 1.44 we can show that

$$2m \le \frac{L(f;P_0;\rho) - f(P_0)}{\rho^2} \le 2M$$

where $M$, $m$ are the upper and lower bounds, respectively, of $f'_D(P)$ on $D(P_0;r)$. Thus

$$\frac{2}{r^2}\int_0^r 2m\rho^3 d\rho \le \frac{2}{r^2}\int_0^r L(f;P_0;\rho)\,\rho\, d\rho - \frac{2}{r^2}\int_0^r f(P_0;\rho)\,d\rho \le \frac{2}{r^2}\int_0^r 2M\rho^3 d\rho$$

And hence $\quad m\,r^2 \le A(f;P_0;r) - f(P_0) \le M r^2$

2.54 Theorem: If (i) $f(P)$ is continuous in a domain $\mathcal{D}$, (ii) $f'_D(P)$ exists and is zero everywhere on $\mathcal{D}$ save at points of a denumerable set $\mathcal{d}$, (iii) for points in $\mathcal{d}$

$$\lim_{r\to 0} \frac{A(f;P;r) - f(P)}{r} = 0$$

then $f(P)$ is harmonic on $\mathcal{D}$.

Proof: The proof is similar to that of Theorem 1.45.

2.55 Theorem: If (i) $f(P)$ is continuous on $\overline{D}(P_0;r)$, (ii) $f'_D(P)$ exists and is contiuous on $\overline{D}(P_0;r)$, then there exists a point $\varphi \in D(P_0;r)$ such that $\quad A(f;P_0;r) = f(P_0) + \frac{1}{2}\, r^2 f'(\varphi)$

(See Blaschke 7.42)

Proof: The proof is similar to that of Theorem 1.47.

2.56 Theorem: If (i) $f(P)$ is continuous on a domain $\mathcal{D}$, (ii) $f'_D(P)$ exists everywhere and is bounded on $\mathcal{D}$, (iii) $f'_D(P) = 0$ almost everywhere on $\mathcal{D}$, then $f(P)$ is harmonic on $\mathcal{D}$.

Proof: Consider

$$g(x,y) = A(f;x,y;r) = \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$\frac{A(g;x,y;R) - g(x,y)}{R^2} = \frac{1}{\pi R^4}\int_0^R \int_0^{2\pi} \{g(x+s\cos\phi, y+s\sin\phi) - g(x,y)\}\, s\, d\phi\, ds$$

$$= \frac{1}{\pi R^4}\int_0^R\int_0^{2\pi}\left[\frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\{f(x+s\cos\phi+\rho\cos\theta, y+s\sin\phi+\rho\sin\theta) - f(x+\rho\cos\theta, y+\rho\sin\theta)\}\rho\, d\theta\, d\rho\right]s\, d\phi\, ds$$

$$= \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\rho\, d\theta\, d\rho \cdot \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\{f(x+s\cos\phi+\rho\cos\theta, y+s\sin\phi+\rho\sin\theta) - f(x+\rho\cos\theta, y+\rho\sin\theta)\}\frac{s\, d\phi\, ds}{R^2}$$

$$= \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\rho\, d\theta\, d\rho \cdot \left\{\frac{A(f; x+\rho\cos\theta, y+\rho\sin\theta; R) - f(x+\rho\cos\theta, y+\rho\sin\theta)}{R^2}\right\}$$

Now $\left\{\frac{A-f}{R^2}\right\} \to 0$ as $R \to 0$ almost everywhere. Further, by hypothesis (iii) and Theorem 2.53 $\left\{\frac{A-f}{R^2}\right\}$ is bounded on $\mathcal{D}$. Hence by the Lebesgue theorem of bounded convergence (see Titchmarsh: Theory of Functions, p337) we have

$$\lim_{R\to 0}\frac{A(g; x, y; R) - g(x,y)}{R^2} = 0$$

And so by Theorem 2.52 $g(x,y)$ is a harmonic function on $\mathcal{D}$. But since $f'(P)$ is bounded on $\mathcal{D}$, by Theorem 2.53

$$\left|\frac{A(f; P; r) - f(P)}{r^2}\right|$$

is uniformly bounded on $\mathcal{D}$. And so $A(f; P; r) \to f(P)$ uniformly. Applying Harnack's First Convergence Theorem we see that $f(P)$ is harmonic.

Note: Harnack's First Convergence Theorem: If a sequence of functions harmonic in a region $R$ converges uniformly on the boundary of $R$, then it converges uniformly within $R$ and its limit is harmonic in $R$. (See Kellogg, Foundations of Potential Theory, p 248.)

# Chapter III

## UNIFORMLY MV DIFFERENTIABLE FUNCTIONS

3.1 <u>Definition</u>: If $f(P)$ is MV differentiable on a bounded closed domain $\mathcal{D}$, and if given any $\epsilon>0$ there exists $r_1>0$ such that

$$L(f;P;r) - f(P) = r^2\{f'(P) + v\}$$

where $|v|<\epsilon$, for all $r<r_1$, $r_1$ independent of the choice of $P$ in $\mathcal{D}$, then we say that $f(P)$ is uniformly <u>MV differentiable</u> on $\mathcal{D}$. A similar definition serves for a function which is <u>uniformly MV(D) differentiable</u> on $\mathcal{D}$.

3.2 <u>Theorem</u>: If $f(P)$ is uniformly MV differentiable on $\mathcal{D}$, then it is uniformly MV(D) differentiable on $\mathcal{D}$.

<u>Proof</u>:
$$A(f;P;r) = \frac{2}{r^2}\int_0^r L(f;P;\rho)\rho\,d\rho$$
$$= \frac{2}{r^2}\int_0^r f(P)\rho\,d\rho + \frac{2}{r^2}\int_0^r f'(P)\rho^3 d\rho + \frac{2}{r^2}\int_0^r v\rho^3 d\rho$$
$$= f(P) + \frac{r^2}{2}f'(P) + \frac{2}{r^2}\int_0^r v\rho^3 d\rho$$

If $r<r_1$ then

$$\left|\frac{2}{r^2}\int_0^r v\rho^3 d\rho\right| \leq \frac{2}{r^2}\epsilon\int_0^r \rho^3 d\rho = \frac{\epsilon r^2}{2}$$

3.3 <u>Theorem</u>: If $f(P)$ is uniformly MV differentiable on $\mathcal{D}$, and if $f'(P)$ is bounded on $\mathcal{D}$, then $f(P)$, $f'(P)$ are continuous on $\mathcal{D}$.

<u>Proof</u>: Let $M$ be the upper bound of $|f'(P)|$ on $\mathcal{D}$. Now

$$A(f;P;r) = f(P) + \frac{r^2}{2}f'(P) + r^2 v$$

where $v\to 0$ uniformly as $r\to 0$. Thus, given any $\epsilon>0$ there exists $r_1>0$ such that, for $r<r_1$

$$|A(f;P;r) - f(P)| \leq \frac{r^2}{2}M + r^2\epsilon$$

Therefore $A(f;P;r)\to f(P)$ uniformly as $r\to 0$. But $A(f;P;r)$ is continuous for all $r>0$. Therefore $f(P)$ is continuous. Further, for all $r<r_1$

$$\left|\frac{A(f;P;r)-f(P)}{r^2} - \frac{1}{2}f'(P)\right| \leq \epsilon$$

Thus $f'(P)$ is also the limit of a uniformly convergent sequence of continuous functions and hence is also continuous.

3.4 <u>Theorem</u>: If $f(P)$ is uniformly MV differentiable on $\mathcal{D}$, and if $f'(P)$ is bounded on $\mathcal{D}$, then $f(P)$ has continuous first partial derivatives on $\mathcal{D}$.

<u>Proof</u>: We shall employ the following notation:

3.41 $$A^{(1)}(f;P;r) = A(f;P;r)$$
$$A^{(n)}(f;P;r) = \frac{1}{\pi r^2} \iint\limits_{D(P;r)} A^{(n-1)}(f;Q;r)\, dQ$$

Consider the second mean

$$A^{(2)}(f;P;r) = \frac{1}{\pi^2 r^4} \int_0^r \int_0^{2\pi} \int_0^r \int_0^{2\pi} f(x+\rho\cos\theta + s\cos\phi, y+\rho\sin\theta + s\sin\phi)\rho s \, d\phi\, ds\, d\theta\, d\rho$$

Since $f(P)$ is uniformly MV differentiable and $f'(P)$ is bounded, it follows from Theorem 3.3 that $A^{(2)}(f;P;r)$ has continuous second partial derivatives. In fact

$$\frac{\partial A^{(2)}}{\partial x} = \frac{1}{\pi r^3} \int_0^{2\pi} \int_0^r \int_0^{2\pi} f(x+r\cos\theta + s\cos\phi, y+r\sin\theta + s\sin\phi) \cos\theta \; s\, d\phi\, ds\, d\theta$$

$$\frac{\partial^2 A^{(2)}}{\partial x^2} = \frac{1}{\pi^2 r^2} \int_0^{2\pi} \int_0^{2\pi} f(x+r\cos\theta + r\cos\phi, y+r\sin\theta + r\sin\phi) \cos\theta \cos\phi \, d\phi\, d\theta$$

Hence
$$\nabla^2 A^{(2)} = \frac{1}{\pi^2 r^2} \int_0^{2\pi} \int_0^{2\pi} f(x+r\cos\theta + r\cos\phi, y+r\sin\theta + r\sin\phi) \cos(\theta-\phi)\, d\phi\, d\theta$$

By a transformation of variable (see Appendix I for details) we have

$$\nabla^2 A^{(2)} = \frac{2}{\pi^2 r^4} \int_0^{2r} \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta) \frac{\rho^2 - 2r^2}{\sqrt{4r^2-\rho^2}}\, d\theta\, d\rho = \frac{4}{\pi r^4} \int_0^{2r} L(f;P;\rho) \frac{\rho^2-2r^2}{\sqrt{4r^2-\rho^2}}\, d\rho$$

But $\quad L(f;P;\rho) = f(P) + \rho^2 f'(P) + o(\rho^2)$

where $o(\rho^2) \to 0$ uniformly with respect to $P$. Further

$$\int_0^{2r} \frac{\rho^2 - 2r^2}{\sqrt{4r^2-\rho^2}}\, d\rho = \int_0^{\frac{\pi}{2}} (4r^2\sin^2\phi - 2r^2)\, d\phi = 0$$

$$\int_0^{2r} \frac{\rho^2-2r^2}{\sqrt{4r^2-\rho^2}}\, \rho^2 d\rho = \int_0^{\frac{\pi}{2}} (4r^2\sin^4\phi - 2r^2\sin^2\phi)\, 4r^2 d\phi = \pi r^4$$

Hence
$$\nabla^2 A^{(2)} = 4f'(P) + \frac{4}{\pi r^4} \int_0^{2r} \frac{\rho^2 - 2r^2}{\sqrt{4r^2-\rho^2}}\, o(\rho^2)\, d\rho$$

(We note that $\displaystyle\int_0^{2r} \left| \frac{\rho^2-2r^2}{\sqrt{4r^2-\rho^2}} \right| d\rho$ is finite.)

Therefore $\lim_{r \to 0} \nabla^2 A^{(2)} = 4 f'(P)$ the approach being uniform with respect to $P$.

For every $r > 0$, $A^{(2)}(f; P; r)$ satisfies the relation

$$A^{(2)}(f; P; r) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} A^{(2)}(f; x+\rho\cos\theta, y+\rho\sin\theta; r)\rho \, d\theta \, d\rho$$

$$- \frac{1}{2\pi R^2} \int_0^R \int_0^{2\pi} \left(R^2 \log\frac{R}{\rho} - \frac{R^2-\rho^2}{2}\right) \nabla^2 A^{(2)}(f; x+\rho\cos\theta, y+\rho\sin\theta; r)\rho \, d\theta \, d\rho$$

(Courant and Hilbert: Methoden der Mathematischen Physik, Vol. II p 250)

Now $A^{(2)} \to f(P)$ and $\nabla^2 A^{(2)} \to 4 f'(P)$ uniformly as $r \to 0$. Thus

$$f(x,y) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\rho \, d\theta \, d\rho$$

$$- \frac{2}{\pi R^2} \int_0^R \int_0^{2\pi} \left(R^2 \log\frac{R}{\rho} - \frac{R^2-\rho^2}{2}\right) f'(x+\rho\cos\theta, y+\rho\sin\theta)\rho \, d\theta \, d\rho$$

But this last expression can be differentiated with respect to $x$ or $y$ since $f'(x,y)$ is continuous. In fact by the four step method we obtain:

$$\frac{\partial f}{\partial x} = \frac{1}{\pi R} \int_0^{2\pi} f(x+R\cos\theta, y+R\sin\theta)\cos\theta \, d\theta$$

$$- \frac{2}{\pi R^2} \int_0^R \int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)(R^2-\rho^2)\cos\theta \, d\theta \, d\rho$$

with a similar expression for $\frac{\partial f}{\partial y}$. (See Appendix II for details of the differentation.)

# Chapter IV

## USE OF POTENTIAL THEORY

4.1 Introduction: Functions with mean value derivatives are closely related to logarithmic potential functions. In fact there is a large group of functions which are common to both classes. We establish the relations by the following theorems.

4.2 Theorem: If $\sigma(P)$ is continuous at a point $P_0$, then the logarithmic potential function

$$u(P) = \iint_W \log\frac{1}{PQ}\,\sigma(Q)\,dQ$$

is MV differentiable at $P_0$. ($W$ is the entire plane of $P$.) Further

$$u'(P_0) = -\frac{\pi}{2}\sigma(P_0)$$

Proof:

$$L(u;P_0;r) = \frac{1}{2\pi r}\int_{C(P_0;r)} u(Q)\,ds_Q = \frac{1}{2\pi r}\int_{C(P_0;r)} ds_Q\cdot\iint_W \log\frac{1}{PQ}\,\sigma(P)\,dP$$

$$= \iint_W \sigma(P)\,dP\cdot\frac{1}{2\pi r}\iint_{C(P_0;r)}\log\frac{1}{PQ}\,ds_Q$$

Now

$$\frac{1}{2\pi r}\int_{C(P_0;r)}\log\frac{1}{PQ}\,ds_Q = \log\frac{1}{PP_0} \qquad PP_0 > r$$

$$= \log\frac{1}{r} \qquad PP_0 \le r$$

So

$$L(u;P_0;r) = \iint_{D(P_0;r)}\sigma(P)\,dP\cdot\log\frac{1}{r} + \iint_{W-D}\sigma(P)\,dP\cdot\log\frac{1}{PP_0}$$

where $W-D$ is the complement of $D(P_0;r)$. Hence

$$L(u;P_0;r) = \iint_{D(P_0;r)}\sigma(P)\,dP\cdot\log\frac{1}{r} + \iint_W \sigma(P)\,dP\cdot\log\frac{1}{PP_0} - \iint_{D(P_0;r)}\sigma(P)\,dP\cdot\log\frac{1}{PP_0}$$

$$= u(P_0) + \iint_{D(P_0;r)}\sigma(P)\left\{\log\frac{1}{r} - \log\frac{1}{PP_0}\right\}dP$$

Thus

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\iint_{D(P_0;r)}\sigma(P)\left\{\log\frac{1}{r} - \log\frac{1}{PP_0}\right\}dP$$

But $\sigma(P)$ is continuous at $P_0$, hence $\sigma(P) = \sigma(P_0) + o(r^0)$ on $D(P_0;r)$ where $o(r^0)$ denotes a quantity which approaches zero as $r$ approaches zero. So

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\iint_{D(P_0;r)} \sigma(P_0)\left\{\log\frac{1}{r} - \log\frac{1}{PP_0}\right\} dP$$

$$+ \frac{1}{r^2}\iint_{D(P_0;r)} o(r^0)\left\{\log\frac{1}{r} - \log\frac{1}{PP_0}\right\} dP$$

But $\frac{1}{r^2}\iint_{D(P_0;r)} \left\{\log\frac{1}{r} - \log\frac{1}{PP_0}\right\} dP = \frac{1}{r^2}\int_0^r\int_0^{2\pi}\left\{\log\frac{1}{r} - \log\frac{1}{\rho}\right\}\rho\, d\theta\, d\rho$

$$= \frac{2\pi}{r^2}\int_0^r \left(\log\frac{1}{r} - \log\frac{1}{\rho}\right)\rho\, d\rho = \frac{2\pi}{r^2}\left[\log\frac{1}{r}\cdot\frac{r^2}{2} + \int_0^r \rho\log\rho\, d\rho\right]$$

$$= \frac{2\pi}{r^2}\left[\frac{r^2}{2}\log\frac{1}{r} + \frac{r^2}{2}\log r - \frac{r^2}{4}\right] = -\frac{\pi}{2}$$

Hence $\lim\limits_{r\to 0} \dfrac{L(u;P_0;r)-u(P_0)}{r^2} = -\dfrac{\pi}{2}\sigma(P_0)$

__Note__: If $\sigma(P)$ is continuous the second derivatives of $u(P)$ do not necessarily exist, but this theorem shows that the generalized Laplacian, namely $4u'(P)$, does exists.

4.3 __Theorem__: If $f(P)$ is continuous and MV differentiable on a domain $\mathcal{D}$ and if $f'(P)$ is continuous on $\mathcal{D}$, then $f(P)$ is a logarithmic potential function. In fact

$$f(P) = -\frac{2}{\pi}\iint_W \log\frac{1}{PQ} f'(Q)\, dQ + h(P)$$

where $h(P)$ is harmonic on $\mathcal{D}$.

__Proof__: This follows from Theorem 4.2 and Theorem 1.43.

4.4 __Theorem__: The logarithmic potential function

$$u(P) = \iint_W \log\frac{1}{PQ}\sigma(Q)\, dQ$$

is MV(D) differentiable almost everywhere provided that $|\sigma(P)|^2$ is summable.

__Proof__:

$$A(u;R;r) = \frac{1}{\pi r^2}\iint_{D(R;r)} u(Q)\, dQ = \frac{1}{\pi r^2}\iint_{D(R;r)} dQ\cdot\iint_W \log\frac{1}{PQ}\sigma(P)\, dP$$

$$= \iint_W \sigma(P)\, dP\cdot\frac{1}{\pi r^2}\iint_{D(R;r)} \log\frac{1}{PQ}\, dQ = \iint_W \sigma(P)\, dP\cdot\frac{1}{\pi r^2}\int_0^r d\rho\cdot\int_{C(R;\rho)} \log\frac{1}{PQ}\, ds_Q$$

Now $\frac{1}{2\pi\rho}\int_{C(R;\rho)} \log\frac{1}{P\varphi}\,ds_\varphi = \log\frac{1}{PR}$ $\qquad PR > \rho$

$\qquad\qquad = \log\frac{1}{\rho}$ $\qquad PR \le \rho$

So

$$A(u;R;r) = \iint_{W-D}\sigma(P)\,dP\cdot\frac{1}{\pi r^2}\int_0^r 2\pi\rho\log\frac{1}{PR}\,d\rho + \iint_{D(R;r)}\sigma(P)\,dP\cdot\left\{\frac{1}{\pi r^2}\int_{PR}^r 2\pi\rho\log\frac{1}{\rho}\,d\rho + \frac{1}{\pi r^2}\int_0^{PR}2\pi\rho\log\frac{1}{PR}\,d\rho\right\}$$

$$= \iint_W \sigma(P)\,dP\cdot\frac{1}{\pi r^2}\int_0^r 2\pi\rho\log\frac{1}{PR}\,d\rho + \iint_{D(R;r)}\sigma(P)\,dP\cdot\frac{1}{\pi r^2}\int_{PR}^r\left(2\pi\rho\log\frac{1}{\rho} - 2\pi\rho\log\frac{1}{PR}\right)d\rho$$

$$= u(R) + \iint_{D(R;r)}\sigma(P)\,dP\cdot\frac{2}{r^2}\left[\frac{\rho^2}{2}\log\frac{1}{\rho} + \frac{\rho^2}{4} - \frac{\rho^2}{2}\log\frac{1}{PR}\right]_{PR}^r$$

$$= u(R) + \iint_{D(R;r)}\sigma(P)\,dP\cdot\left\{\log\frac{1}{r} - \log\frac{1}{PR} + \frac{1}{2} - \frac{(PR)^2}{2r^2}\right\}$$

Hence, letting $R \equiv (x,y)$

4.41 $$\frac{A(u;R;r) - u(R)}{r^2} = \frac{1}{r^2}\iint_{D(R;r)}\sigma(P)\,dP\cdot\left\{\log\frac{1}{r} - \log\frac{1}{PR} + \frac{1}{2} - \frac{(PR)^2}{2r^2}\right\}$$

$$= \frac{1}{r^2}\int_0^r\int_0^{2\pi}\sigma(x+\rho\cos\theta, y+\rho\sin\theta)\left\{\log\frac{1}{r} - \log\frac{1}{\rho} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}\rho\,d\theta\,d\rho +$$

But $$\left|\frac{1}{r^2}\int_0^r\int_0^{2\pi}\sigma(x+\rho\cos\theta, y+\rho\sin\theta)\left\{\log\frac{1}{r} - \log\frac{1}{\rho} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}\rho\,d\theta\,d\rho + \frac{\pi}{4}\sigma(x,y)\right|$$

4.42 $$= \left|\frac{1}{r^2}\int_0^r\int_0^{2\pi}\{\sigma(x+\rho\cos\theta, y+\rho\sin\theta) - \sigma(x,y)\}\left\{\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}\rho\,d\theta\,d\rho\right|$$

4.43 since $$\int_0^r\left(\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right)\rho\,d\rho = \int_0^r\left(\log\frac{\rho}{r}\right)\rho\,d\rho + \frac{r^2}{4} - \frac{r^4}{8r^2}$$

$$= \left[\frac{\rho^2}{2}\log\frac{\rho}{r} - \frac{\rho^2}{4}\right]_0^r + \frac{r^2}{4} - \frac{r^2}{8} = -\frac{r^2}{4} + \frac{r^2}{4} - \frac{r^2}{8} = -\frac{r^2}{8}$$

By Schwarz's inequality the expression 4.42 is bounded by

$$\left[\frac{1}{r^2}\int_0^r\int_0^{2\pi}|\sigma(x+\rho\cos\theta, y+\rho\sin\theta) - \sigma(x,y)|^2\rho\,d\theta\,d\rho\right]^{1/2}\left[\frac{1}{r^2}\int_0^r\int_0^{2\pi}\left(\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right)^2\rho\,d\theta\,d\rho\right]^{1/2}$$

The first integral is $o(r^0)$ almost everywhere. (See Appendix IV).

The second can be evaluated. Letting $\frac{\rho}{r} = z$, $d\rho = r\,dz$, we have

$$\frac{2\pi}{r^2}\int_0^r\left(\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right)^2\rho\,d\rho = \frac{2\pi}{r^2}\int_0^1\left(\log z + \frac{1}{2} - \frac{z^2}{2}\right)^2 rz\,r\,dz$$

$$= 2\pi\int_0^1\left(\log z + \frac{1-z^2}{2}\right)^2 z\,dz$$

which is finite and independent of r. Hence the expression 4.42

is $o(r^0)$ almost everywhere and the result follows.

4.44 <u>Theorem</u>: If $\sigma(P)$ is such that

$$\frac{1}{2\pi r}\int_{C(P_0;r)} \sigma(P)\, ds_P = \sigma(P_0) + o(r^0)$$

for almost all small r, then $u(P) = \iint_W \log\frac{1}{PQ}\,\sigma(Q)\,dQ$

is MV(D) differentiable at $P_0$.

<u>Proof</u>: From 4.41

$$\frac{A(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\int_0^r\int_0^{2\pi}\sigma(x_0+\rho\cos\theta, y_0+\rho\sin\theta)\left\{\log\frac{\rho}{r}+\frac{1}{2}-\frac{\rho^2}{2r^2}\right\}\rho\,d\theta\,d\rho$$

$$= \frac{1}{r^2}\int_0^r\left[2\pi\sigma(P_0)+o(\rho^0)\right]\left\{\log\frac{\rho}{r}+\frac{1}{2}-\frac{\rho^2}{2r^2}\right\}\rho\,d\rho = \frac{2\pi\sigma(P_0)}{r^2}\left(-\frac{r^2}{8}\right)+o(r^2) = -\frac{\pi}{4}\sigma(P_0)+o(r^2)$$

using 4.43. Hence $u'_D(P_0) = -\frac{\pi}{4}\sigma(P_0)$

4.5 <u>Theorem</u>: If $u(P)$ is a logarithmic potential function

$$u(P) = \int_W \log\frac{1}{PQ}\,d\mu(Q)$$

where $\mu$ is a mass distribution, and if the density exists at $P_0$, i.e.

$$\lim_{r\to 0}\frac{1}{\pi r^2}\int_{D(P_0;r)} d\mu(P) = a(P_0)$$

exists, then $u(P)$ is MV differentiable at $P_0$ and $u'(P_0) = -\frac{\pi}{2}a(P_0)$

<u>Proof</u>: $$L(u;P_0;r) = \frac{1}{2\pi r}\int_{C(P_0;r)} u(P)\,ds_P = \frac{1}{2\pi r}\int_{C(P_0;r)} ds_P\cdot\int_W \log\frac{1}{PQ}\,d\mu(Q)$$

$$= \int_W d\mu(Q)\cdot\frac{1}{2\pi r}\int_{C(P_0;r)}\log\frac{1}{PQ}\,ds_P$$

Now $$\frac{1}{2\pi r}\int_{C(P_0;r)}\log\frac{1}{PQ}\,ds_P = \log\frac{1}{P_0Q} \qquad P_0Q > r$$

$$= \log\frac{1}{r} \qquad P_0Q \le r$$

Hence $$L(u;P_0;r) = \int_{D(P_0;r)} d\mu(Q)\cdot\log\frac{1}{r} + \int_{W-D} d\mu(Q)\cdot\log\frac{1}{QP_0}$$

$$= \int_W d\mu(Q)\cdot\log\frac{1}{P_0Q} + \int_{D(P_0;r)}\left\{\log\frac{1}{r}-\log\frac{1}{P_0Q}\right\}d\mu(Q) = u(P_0) + \int_{D(P_0;r)}\left\{\log\frac{1}{r}-\log\frac{1}{P_0Q}\right\}d\mu(Q)$$

Therefore

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\int_{D(P_0;r)}\left\{\log\frac{1}{r}-\log\frac{1}{P_0Q}\right\}d\mu(Q)$$

Since the integrand on the right depends only on the distance of $Q$ from $P_0$ we can write

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\int_0^r \left\{\log\frac{1}{r}-\log\frac{1}{\rho}\right\} d\bar{\mu}(\rho)$$

where $\bar{\mu}(\rho) = \int_{D(P_0;\rho)} d\mu(Q)$

Integrating by parts we have

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = \frac{1}{r^2}\left[\left(\log\frac{1}{r}-\log\frac{1}{\rho}\right)\bar{\mu}(\rho)\right]_0^r - \frac{1}{r^2}\int_0^r \bar{\mu}(\rho)\,\frac{d\rho}{\rho}$$

But $\bar{\mu}(\rho) = \int_{D(P_0;\rho)} d\mu(Q) = \pi\rho^2 a(P_0) + o(\rho^2)$

Hence

$$\frac{L(u;P_0;r)-u(P_0)}{r^2} = -\frac{1}{r^2}\int_0^r \left[\pi\rho^2 a(P_0)\right]\frac{d\rho}{\rho} - \frac{1}{r^2}\int_0^r o(\rho^2)\frac{d\rho}{\rho} = -\frac{\pi}{2}a(P_0)+o(r^0)$$

Thus $u'(P_0) = -\frac{\pi}{2}\,a(P_0)$ .

# Chapter V

## HIGHER DERIVATIVES

5.1 Introduction: Suppose the MV derivative $f'(P)$ of a function $f(P)$ is itself MV differentiable. The MV derivative of $f'(P)$ could be considered as a second MV derivative of $f(P)$ . The process could be carried out to n th order derivatives. Accordingly definitions are made.

5.2 Definitions: The n th mean value derivative of $f(P)$ at $P_0$ is defined as

$$f^{(n)}(P_0) = \lim_{r\to 0} \frac{L(f^{(n-1)}; P_0; r) - f^{(n-1)}(P_0)}{r^2}$$

if the limit exists.

The n th mean value derivative (on D) of $f(P)$ at $P_0$ is defined as

$$f_D^{(n)}(P_0) = \lim_{r\to 0} \frac{A(f_D^{(n-1)}; P_0; r) - f^{(n-1)}(P_0)}{r^2}$$

if the limit exists.

5.3 Theorem: If $f^{(n)}(P)$ exists at $P_0$ then so does $f_D^{(n)}(P)$ , and further

$$f_D^{(n)}(P_0) = \frac{1}{2^n} f^{(n)}(P_0)$$

Proof: This follows from the definitions and Theorem 2.41.

5.4 Theorem: If $f(x,y)$ is such that all partial derivatives of order $2s+1$ exist at $(x,y)$, then

(i) $f(x,y)$ has MV derivatives at $(x,y)$ up to order $s$.

(ii) $f^{(k)}(x,y) = \frac{1}{4^k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^k f(x,y) = \frac{1}{4^k} \nabla^{2k} f(x,y)$ for $k = 0, 1, 2, \cdots, s$.

(iii) $L(f; x, y; r) = \sum_{k=0}^{s} \frac{f^{(k)}(x,y)}{(k!)^2} r^{2k} + o(r^{2s})$

(iv) $A(f; x, y; r) = \sum_{k=0}^{s} \frac{f^{(k)}(x,y)}{(k!)^2} \cdot \frac{r^{2k}}{2k+1} + o(r^{2s})$

Proof: (i),(ii). Since $f(x,y)$ has partial derivatives of order $2s+1$ ,all partial derivatives of order $\leq 2s$ are continuous. Hence $f(x,y)$ has continuous second partial derivatives, and thus, by Theorem 1.42 $\quad f'(x,y) = \frac{1}{4} \nabla^2 f(x,y)$

Now $f(x,y)$ has continuous fourth partial derivatives, and hence $f'(x,y)$ has continuous second partial derivatives. Thus

$$f''(x,y) = \frac{1}{4}\nabla^2 f'(x,y) = \frac{1}{4^2}\nabla^4 f(x,y)$$

Proceeding by induction we have

$$f^{(k)}(x,y) = \frac{1}{4^k}\nabla^{2k} f(x,y) \qquad k=0,1,2,\cdots,s.$$

(iii) By Taylor's Theorem we have

$$f(x+r\cos\theta, y+r\sin\theta) = \sum_{m,n=0}^{m+n=2s} A_{m,n}(r\cos\theta)^m(r\sin\theta)^n + o(r^{2s})$$

where $\quad A_{m,n} = \frac{1}{m!\,n!}\left(\frac{\partial^{m+n} f}{\partial x^m \partial y^n}\right)_{x,y}$

and $o(r^{2s})$ is uniform in $\theta$.

So $\quad L(f;x,y;r) = \sum_{m,n=0}^{m+n=2s} A_{m,n}\, r^{m+n}\frac{1}{2\pi}\int_0^{2\pi}\cos^m\theta\sin^n\theta\, d\theta + o(r^{2s})$

Now $\quad \int_0^{2\pi}\cos^m\theta\sin^n\theta\,d\theta = \frac{[1+(-1)^{m+n}+(-1)^m+(-1)^n]}{2}\;\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n}{2}+1)}$

Thus

$$L(f;x,y;r) = \sum_{m,n=0}^{m+n=2s} A_{m,n}\frac{r^{m+n}}{4\pi}\left[1+(-1)^m+(-1)^n+(-1)^{m+n}\right]\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n}{2}+1)} = \sum_{k=0}^{2s} a_k r^k$$

where

$$a_k = \sum_{m=0}^{k}\frac{A_{m,k-m}}{4\pi}\left[1+(-1)^m+(-1)^{k-m}+(-1)^k\right]\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{k-m+1}{2})}{\Gamma(\frac{k}{2}+1)}$$

$$= \sum_{m=0}^{k}\frac{A_{m,k-m}}{4\pi}\left[1+(-1)^n\right]\left[1+(-1)^m\right]\frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{k-m+1}{2})}{\Gamma(\frac{k}{2}+1)}$$

If $k$ is odd then $a_k = 0$. Also the terms in the sum vanish for $m$ odd. Thus

$$a_{2k} = \sum_{m=0}^{k}\frac{A_{2m,2k-2m}}{\pi}\;\frac{\Gamma(m+\frac{1}{2})\Gamma(k-m+\frac{1}{2})}{\Gamma(k+1)}$$

and $\quad L(f;x,y;r) = \sum_{k=0}^{s} a_{2k} r^{2k}$

Now $\quad a_0 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\pi} f(x,y) = f(x,y)$

$$a_2 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\pi\,\Gamma(2)\Gamma(3)}\frac{\partial^2 f}{\partial y^2} + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\pi\,\Gamma(2)\Gamma(3)}\frac{\partial^2 f}{\partial x^2} = \frac{1}{4}\nabla^2 f(x,y)$$

Also $\nabla^2 a_{2k-2} = 4k^2 a_{2k}$. Thus, by induction

$$a_{2k} = \frac{\nabla^2 a_{2k-2}}{4k^2} = \frac{1}{4^k}\frac{\nabla^{2k} f(x,y)}{(k!)^2} = \frac{f^{(k)}(x,y)}{(k!)^2}$$

and $L(f;x,y;r) = \sum_{k=0}^{s} \frac{f^{(k)}(x,y)}{(k!)^2} r^{2k} + o(r^{2s})$

(iv) Since $A(f;x,y;r) = \frac{2}{r^2}\int_0^r L(f;x,y;\rho)\rho\,d\rho$

we have

$$A(f;x,y;r) = \frac{2}{r^2}\sum_{k=0}^{s}\frac{f^{(k)}(x,y)}{(k!)^2}\int_0^r \rho^{2k+1}d\rho + o(r^{2s}) = \sum_{k=0}^{s}\frac{f^{(k)}(x,y)}{(k!)^2}\frac{r^{2k}}{2k+1} + o(r^{2s})$$

5.5 Theorem: If (i) $f(P)$ is continuous on a domain $\mathcal{D}$, (ii) $f''(P)$ exists on $\mathcal{D}$, and (iii) $|f'(P)|^2$ is summable on $\mathcal{D}$, then

$$A(f;P;r) = f(P) + \frac{r^2}{2}f'(P) + \frac{r^4}{12}f''(P) + o(r^4)$$

on $\mathcal{D}$.

Proof: Consider $u(P) = -\frac{2}{\pi}\iint_W \log\frac{1}{PQ}\, f'(Q)\,dQ$

From hypothesis (iii) it follows that $u(P)$ is continuous. (See Appendix III). By Theorem 4.44, since

5.51 $$\frac{1}{2\pi r}\int_{C(P;r)} f'(Q)\,ds_Q = f'(P) + r^2 f''(P) + o(r^2)$$

$u(P)$ has the MV(D) derivative $\frac{1}{2}f'(P)$. Hence $f(P)-u(P)$ has the MV(D) derivative zero. Thus, by Theorem 2.52, $f(P)-u(P)$ is a harmonic function, say $h(P)$. So $f(P) = u(P) + h(P)$ and hence

5.52 $$A(f;P;r) = A(u;P;r) + A(h;P;r)$$

By 4.41 we have

5.53 $$A(u;P;r) = u(P) - \frac{2}{\pi}\iint_{D(P;r)} f'(Q)\,dQ\cdot\left\{\log\frac{PQ}{r} + \frac{1}{2} - \frac{(PQ)^2}{2r^2}\right\}$$

$$= u(P) - \frac{2}{\pi}\int_0^r\left\{\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}d\rho\cdot\int_{C(P;\rho)} f'(Q)\,ds_Q$$

Using 5.51 we have

$$A(u;P;r) = u(P) - 4f'(P)\int_0^r\left\{\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}\rho\,d\rho$$

$$-4f''(P)\int_0^r\left\{\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}\rho^3 d\rho - 4\int_0^r\left\{\log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2}\right\}o(\rho^2)\rho\,d\rho$$

The first integral, by 4.43, has the value $-\frac{r^4}{8}$. For the second we have $\int_0^r \left\{ \log \frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2} \right\} \rho^3 d\rho = r^4 \int_0^1 \left\{ \log z + \frac{1-z^2}{2} \right\} z^3 dz$

$$= r^4 \left[ \int_0^1 z^3 \log z \, dz + \frac{1}{8} - \frac{1}{12} \right] = r^4 \left\{ \left[ \frac{z^4}{4} \log z - \frac{z^4}{16} \right]_0^1 + \frac{1}{24} \right\} = -\frac{r^4}{48}$$

Thus $\quad A(u;P;r) = u(P) + \frac{r^2}{2} f'(P) + \frac{r^4}{12} f''(P) + o(r^4)$

Also $\quad A(h;P;r) = h(P)$ , and so

$$A(f;P;r) = u(P) + \frac{r^2}{2} f'(P) + \frac{r^4}{12} f''(P) + o(r^4) + h(P)$$

$$= f(P) + \frac{r^2}{2} f'(P) + \frac{r^4}{12} f''(P) + o(r^4)$$

5.54 Theorem: If $f(P)$ is continuous, $f''(P)$ exists, and $|f'(P)|^2$ is summable on a domain $\mathcal{D}$ , then

$$L(f;P;r) = f(P) + r^2 f'(P) + \frac{r^4}{4} f''(P) + o(r^4)$$

for every $P$ in $\mathcal{D}$ for almost all small $r$.

Proof: By Theorem 2.3

$$A(f;P;r) = \frac{2}{r^2} \int_0^r L(f;P;\rho) \rho \, d\rho$$

Hence

$$\frac{\partial A}{\partial r} = \frac{2}{r} L(f;P;r) - \frac{4}{r^3} \int_0^r L(f;P;\rho) \rho \, d\rho$$

for almost all small $r$. By Theorem 5.5

$$A(f;P;r) = f(P) + \frac{r^2}{2} f'(P) + \frac{r^4}{12} f''(P) + o(r^4)$$

Hence

$$\frac{\partial A}{\partial r} = r f'(P) + \frac{r^3}{3} f''(P) + o(r^3)$$

for almost all small $r$ . And so

$$\frac{2}{r} L(f;P;r) - \frac{4}{r^3} \int_0^r L(f;P;\rho) \rho \, d\rho = r f'(P) + \frac{r^3}{3} f''(P) + o(r^3) \; ;$$

solving for $L(f;P;r)$ we have

$$L(f;P;r) = \frac{2}{r^2} \int_0^r L(f;P;\rho) \rho \, d\rho + \frac{r^2}{2} f'(P) + \frac{r^4}{6} f''(P) + o(r^4)$$

$$= A(f;P;r) + \frac{r^2}{2} f'(P) + \frac{r^4}{6} f''(P) + o(r^4)$$

$$= f(P) + r^2 f'(P) + \frac{r^4}{4} f''(P) + o(r^4)$$

for almost all small $r$ . It is to be noted that the set of values of $r$ for which the relation is true depends upon $P$.

5.6 Theorem: If (i) $f(P)$ is continuous on a domain $\mathcal{D}$, (ii) $f'(P)$ $f''(P), \cdots, f^{(n-2)}(P)$ are continuous on $\mathcal{D}$, (iii) $|f^{(n-1)}(P)|^2$ is summable on $\mathcal{D}$, (iv) $f^{(n)}(P)$ exists on $\mathcal{D}$, then for $P$ in $\mathcal{D}$

(1) $$A(f;P;r) = \sum_{k=0}^{n} \frac{f^{(k)}(P)}{(k!)^2(k+1)} r^{2k} + o(r^{2n})$$

all small $r$,

(2) $$L(f;P;r) = \sum_{k=0}^{n} \frac{f^{(k)}(P)}{(k!)^2} r^{2k} + o(r^{2n})$$

almost all small $r$.

Proof: The proof is by induction. The Theorem is true for $n=2$. (Theorems 5.5 and 5.54). We assume it true for $n=N$. Thus

$$L(f';P;\rho) = \sum_{k=0}^{N} \frac{f^{(k+1)}(P)}{(k!)^2} \rho^{2k} + o(\rho^{2N})$$

for almost all small . But from 4.41

$$A(f;P;r) = f(P) - \frac{2}{\pi} \int_0^r \left\{ \log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2} \right\} d\rho \cdot \int_{C(P;\rho)} f'(Q)\, ds_Q$$

$$= f(P) - 4\int_0^r \left\{ \log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2} \right\} \rho\, d\rho \cdot L(f';P;\rho)$$

Hence

$$A(f;P;r) = f(P) - 4\int_0^r \left\{ \log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2} \right\} \sum_{k=0}^{N} \frac{f^{(k+1)}(P)}{(k!)^2} \rho^{2k+1} d\rho + o(r^{2N+2})$$

But $$\int_0^r \left\{ \log\frac{\rho}{r} + \frac{1}{2} - \frac{\rho^2}{2r^2} \right\} \rho^{2k+1} d\rho = r^{2k+2} \int_0^1 \left\{ z^{2k+1}\log z + \frac{z^{2k+1}}{2} - \frac{z^{2k+3}}{2} \right\} dz$$

$$= r^{2k+2} \left[ \frac{z^{2k+2}\log z}{2k+2} - \frac{z^{2k+2}}{(2k+2)^2} + \frac{z^{2k+2}}{2(2k+2)} - \frac{z^{2k+4}}{2(2k+4)} \right]_0^1$$

$$= \frac{r^{2k+2}}{4} \left[ -\frac{1}{(k+1)^2} + \frac{1}{(k+1)} - \frac{1}{(k+2)} \right] = -\frac{r^{2k+2}}{4(k+1)^2(k+2)}$$

So

$$A(f;P;r) = f(P) + \sum_{k=0}^{N} \frac{f^{(k+1)}(P)}{[(k+1)!]^2(k+2)} r^{2k+2} + o(r^{2N+2})$$

$$= f(P) + \sum_{k=1}^{N+1} \frac{f^{(k)}(P)}{(k!)^2(k+1)} r^{2k} + o(r^{2N+2})$$

for all small $r$. By the method of Theorem 5.54

$$\frac{\partial A}{\partial r} = \frac{2}{r}\left\{ L(f;P;r) - A(f;P;r) \right\} = \sum_{k=1}^{N+1} \frac{f^{(k)}(P)}{(k!)^2(k+1)} (2k) r^{2k-1} + o(r^{2N+1})$$

and

$$L(f;P;r) = A(f;P;r) + \sum_{k=1}^{N+1} \frac{f^{(k)}(P)}{(k!)^2(k+1)}(k)\, r^{2k} + o(r^{2N+2}) = \sum_{k=0}^{N+1} \frac{f^{(k)}(P)}{(k!)^2}\, r^{2k} + o(r^{2N+2})$$

for almost all small $r$.

# Chapter VI

## FURTHER PROBLEMS CONCERNING MV DERIVATIVES

6.1 Introduction: In this chapter some unsolved problems concerning MV derivatives are discussed. Also some ideas on further developement of the theory of MV derivatives are proposed. It may be that some of these topics are of sufficient interest to merit further investigation.

6.2 The nature of the discontinuities of an MV differentiable function: Continuity is neither a necessary nor a sufficient condition for $f'(P)$ to exist. A few examples will show this: Let $f(x,y) = |x|$ . This function is continous, and

$$L(f;x,y;r) = \frac{1}{2\pi}\int_0^{2\pi}|x + r\cos\theta|\,d\theta$$

If $x>0$, $L(f;x,y;r)=x$ for $r$ sufficiently small, and hence $f'=0$ . Similarly if $x<0$ . But

$$L(f;0,y;r) = \frac{1}{2\pi}\int_0^{2\pi}|r\cos\theta|\,d\theta = \frac{2r}{\pi}$$

Therefore $f'(0,y)$ does not exist.

Again consider $f(x,y) = \dfrac{xy}{x^2+y^2}$ for $x^2+y^2 \neq 0$, $f(0,0)=0$

This function is discontinuos at the origin. At any point other than the origin $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ exist and are continous. Hence $f'(x,y)$ exists everywhere save possibly at the origin. On the other hand we have

$$L(f;0,0;r) = \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2\sin\theta\cos\theta}{r^2\sin^2\theta + r^2\cos^2\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0$$

Therefore $f'(0,0)=0$ .Hence $f'(x,y)$ exists everywhere.

The question arises as to just what is the nature of the discontinuities of a function which has an MV derivative. Clearly the existence of $f'(x,y)$ at a point does not imply continuity at a point. In fact the existence of $f'(x,y)$ almost everywhere does not imply any sort of continuity. For consider the function:

$$f(x,y) = 0 \qquad x \text{ or } y \text{ irrational}$$
$$= 1 \qquad x \text{ and } y \text{ rational.}$$

This function is discontinuous everywhere, but $f'(x,y)$ exists for all irrational points, i.e. points at which $x$ or $y$ is irrational. For clearly $L(f;x,y;r)=0$ and hence if $x$ or $y$ is irrational then $f'(x,y)=0$. Another interesting example:

$$f(x,y) = 1 \qquad x>0$$
$$= -1 \qquad x<0$$
$$= 0 \qquad x=0$$

For $x<0$ $f(x,y)$ is harmonic, and hence $f'(x,y)=0$. Similarly for $x>0$. Further for $x=0$, $L(f;0,y;r)=0$. Hence $f'(0,y)=0$. Thus $f'(x,y)=0$ everywhere for this function which has a set of discontinuities of positive linear measure.

Suppose a function does have an MV derivative everywhere in a domain. Can it have discontinuities on a set of positive measure (two dimensional), or can it have discontinuities on an everywhere dense set ? Little headway has been made on these qu questions. Attempts to construct a function with an MV derivative everywhere but possessing discontinuities on an everywhere dense set by the method of Cantor (see Hobson: <u>Theory of Functions of a Real Variable</u>, vol.II, pp 389-421 ) have failed. The existence of $f'(P)$ of course implies that $A(f;P;r) \to f(P)$ as $r \to 0$ Since $A(f;P;r)$ is continuous in $P$ we see that $f(P)$ is of the first class of Baire, and hence its discontinuities form a set of the first category.

Note: If $f(x,y) = \lim_{n\to\infty} f_n(x,y)$ and $f_n(x,y)$ is continuous for all $n$ on a perfect set $\mathcal{P}$, then $f(x,y)$ is pointwise discontinuous on $\mathcal{P}$. (Baire). A function is pointwise discontinuous on if, given any small $\sigma > 0$, the set of points for which its variation $\omega \geq \sigma$ is non-dense on $\mathcal{P}$. Thus the set of points of discontinuity is a set of the first category.

A condition on the ratio $\dfrac{A(f;P;r) - f(P)}{r^2}$ will imply continuity on a domain $\mathcal{D}$. In fact, if this ratio is bounded on then we have

$$\left| \frac{A(f;P;r) - f(P)}{r^2} \right| < B \qquad \text{on } \mathcal{D}$$

Therefore $|A(f;P;r) - f(P)| < r^2 B$ on $\mathcal{D}$.

Hence $f(P)$ is the limit of a uniformly convergent sequence of continuous functions, and thus is continuous everywhere on $\mathcal{D}$. If we replace the ratio in the preceeding by $\dfrac{L(f;P;r) - f(P)}{r^2}$ the result still follows. For we have $L(f;P;r) = f(P) + r^2\eta(P)$ where $|\eta(P)| < B$ on $\mathcal{D}$. Thus

$$A(f;P;r) = \frac{2}{r^2}\int_0^r L(f;P;\rho)\rho\,d\rho = f(P) + \frac{2}{r^2}\int_0^r \eta(P;\rho)\rho^3\,d\rho$$

Therefore $$\left| \frac{A(f;P;r) - f(P)}{r^2} \right| \leq \frac{2}{r^4}\left| \int_0^r \eta(P;\rho)\rho^3\,d\rho \right| < \frac{2}{r^4}\cdot B \cdot \frac{r^4}{4} = \frac{B}{2}$$

6.3 Sequences of functions with MV derivatives: If a sequence of functions $\{f_n(P)\}$, each member of which has an MV derivative, converges uniformly to a function $f(P)$ it does not follow that $f(P)$ is MV differentiable. Consider $f_n(x,y) = \frac{1}{n}\log\cosh nx$

Clearly $f_n$ has an MV derivative, for $\dfrac{\partial f_n}{\partial x} = \dfrac{d}{dx}\left(\frac{1}{n}\log\cosh nx\right) = \tanh nx$ and $\dfrac{\partial^2 f_n}{\partial x^2} = n\,\mathrm{sech}^2 nx = \dfrac{4n}{(e^{nx}+e^{-nx})^2} \to 0$ as $n \to \infty$.

But $\lim_{n\to\infty} f_n = \lim_{n\to\infty} \dfrac{\log\cosh nx}{n} = |x|$ which has no MV derivative at $x = 0$.

On the other hand suppose $f_k(x,y) \to f(x,y)$ uniformly and that
$$L(f_k; x,y;r) = \sum_{n=0}^{\infty} \frac{f_k^{(n)}(x,y)}{(n!)^2} r^{2n}$$

(see Theorem 5.6). Then
$$\lim_{r\to 0}\lim_{k\to\infty} \frac{L(f_k;x,y;r) - f_k(x,y)}{r^2} = \lim_{r\to 0}\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{f_k^{(n)}(x,y)}{(n!)^2} r^{2n-2}$$
$$= \lim_{r\to 0}\left\{ \lim_{k\to\infty} f_k'(x,y) + \lim_{k\to\infty} \sum_{n=2}^{\infty} \frac{f_k^{(n)}(x,y)}{(n!)^2} r^{2n-2}\right\}$$
If $\lim_{k\to\infty} f_k^{(n)}(x,y) = g_n(x,y)$ exists, all $n$ and $\sum_{n=2}^{\infty} \frac{g_n(x,y)\ r^{2n-2}}{(n!)^2}$ is convergent, then
$$\lim_{r\to 0}\lim_{k\to\infty} \frac{L(f_k;x,y;r) - f_k(x,y)}{r^2} = \lim_{r\to 0}\lim_{k\to\infty} f_k'(x,y) = \lim_{k\to\infty} f'_k(x,y)$$
$$\therefore \quad f'(x,y) = \lim_{k\to\infty} f'_k(x,y)$$

provided this last limit exists. A more thorough examination of this subject seems indicated.

6.4 <u>Partial mean value derivatives:</u>

Let
$$L_x(f;x,y;r) = \frac{1}{2\pi}\int_0^{2\pi} f(x + r\cos\theta, y)\,d\theta$$
$$L_y(f;x,y;r) = \frac{1}{2\pi}\int_0^{2\pi} f(x, y + r\sin\theta)\,d\theta$$

We could then define <u>partial MV derivatives</u>
$$f_x'(x,y) = \lim_{r\to 0} \frac{L_x(f;x,y;r) - f(x,y)}{r^2}$$
$$f_y'(x,y) = \lim_{r\to 0} \frac{L_y(f;x,y;r) - f(x,y)}{r^2}$$
If $f(x,y)$ has continuous second partial derivatives, it is clear from Theorem 1.42 that
$$f_x'(x,y) = \frac{1}{4} f_{xx}, \quad f_y'(x,y) = \frac{1}{4} f_{yy}$$
Hence
$$f'(x,y) = f_x'(x,y) + f_y'(x,y) \tag{1}$$
The question arises as to whether this formula would hold true for all functions which have MV derivatives. It is, of course,

not necessarily so that the existence of $f'$ implies the existence of $f'_x$ , $f'_y$. On the other hand, it would seem likely that the existence of $f_x'$ , $f_y'$ would imply the existence of $f'$. Would this also imply equation (1) ? The advantage to be gained in using partial MV derivatives lies in the fact that then the problem is essentally reduced to an investigation of functions of one variable.There has been quite a bit of work done on the one variable problem with generalized derivatives of various sorts.

Note: See, for instance

(1) A.Khintchine:Recherches sur la structure des fonctions measurables. Fund.Math.9(1927)212-279.

(2) J.C.Burkhill and U.S.Haslem-Jones:The derivates and approximate derivates of measurable functions. Proc.London Math.Soc.32 (1931)346-355.

(3) W.L.C.Sargent:The Borel Derivatives of a function. Proc. London Math.Soc. 38(1934-5)180-96.

(4) J.Marcinkiewicz and A.Zygmund:On the differentiability of functions and summability of trigonometrical series. Fund. Math.26(1936)1-43.

6.5 Generalized second MV derivative: A generalized second MV derivative could be obtained by using the second mean

$$L^{(2)}(f;x,y;r)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x+r\cos\theta + r\cos\phi,\ y+r\sin\theta+r\sin\phi)\,d\theta\, d\phi$$

and defining $f_2(x,y)=\lim_{r\to 0}\dfrac{L^{(2)}(f;x,y;r)-2L(f;x,y;r)+f(x,y)}{r^4}$

The relationships between $f_2$ and $f''$ could then be investigated, i.e. does the existence of one imply the existence of the other ? When are the two equal ? One would suspect that $f_2$ is related to $f''$ in a fashion similar to the relations of the Schwarz second derivative to the ordinary first and second derivatives of functions of one variable. (See Marcinkiewicz and Zygmond, ibid.)

6.6 <u>Applications to functions of a complex variable:</u> If $F(x,y)=f(z)g(\bar{z})$ where $f(z)g(\bar{z})$ are analytic functions of $z=x+iy$ and $\bar{z}=x-iy$ respectively, then $F(x,y)$ can be shown tb have mean value derivatives of all orders:

$$L(F;x,y;r)=\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,g(\bar{z}+re^{-i\theta})\,d\theta$$

By Taylor's Theorem

$$f(z+re^{i\theta})=\sum_{n=0}^{\infty}\frac{f^{(n)}(z)\,r^n e^{in\theta}}{n!}$$

$$g(\bar{z}+re^{-i\theta})=\sum_{m=0}^{\infty}\frac{g^{(m)}(\bar{z})\,r^m e^{-im\theta}}{m!}$$

Thus

$$f(z+re^{i\theta})\,g(\bar{z}+re^{-i\theta})=\sum_{m,n=0}^{\infty}\frac{f^{(n)}(z)g^{(m)}(\bar{z})\,r^{n+m}e^{i(n-m)\theta}}{n!\,m!}=\sum_{k=0}^{\infty}s_k r^k$$

where

$$s_k=\sum_{n=0}^{k}\frac{f^{(n)}(z)g^{(k-n)}(\bar{z})\,e^{i(2n-k)\theta}}{n!\,(k-n)!}$$

Hence

$$L(F;x,y;r)=\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\infty}s_k r^k\,d\theta=\sum_{k=0}^{\infty}r^k\left(\frac{1}{2\pi}\int_0^{2\pi}s_k\,d\theta\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi}s_k\,d\theta=\sum_{n=0}^{k}\frac{f^{(n)}(z)\,g^{(k-n)}(\bar{z})}{n!\,(k-n)!}\,\frac{1}{2\pi}\int_0^{2\pi}e^{i(2n-k)}d\theta$$

But

$$\frac{1}{2\pi}\int_0^{2\pi}e^{i(2n-k)\theta}d\theta=0 \qquad \text{for} \quad 2n\neq k$$

$$=1 \qquad \text{for} \quad 2n=k$$

And so $\displaystyle\frac{1}{2\pi}\int_0^{2\pi}s_{2k+1}\,d\theta=0,\quad \frac{1}{2\pi}\int_0^{2\pi}s_{2k}\,d\theta=\frac{f^{(k)}(z)\,g^{(k)}(\bar{z})}{(k!)^2}$

Therefore

$$L(F;x,y;r)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z)g^{(k)}(\bar{z})}{(k!)^2}\,r^{2k}$$

and

$$\frac{L(F;x,y;r)-F(x,y)}{r^2}=\sum_{k=1}^{\infty}\frac{f^{(k)}(z)\,g^{(k)}(\bar{z})}{(k!)^2}\,r^{2k-2}\rightarrow f'(z)\,g'(\bar{z})$$

We thus see that $F^{(n)}(x,y)=f^{(n)}(z)\,g^{(n)}(\bar{z})$

and further

$$L(F;x,y;r)=\sum_{k=0}^{\infty}\frac{F^{(n)}(x,y)}{(k!)^2}\,r^{2k}$$

The results would still hold if $\displaystyle F(x,y)=\sum_{m=1}^{\infty}f_m(z)\,g_m(\bar{z})$

where the series is uniformly convergent.

Consider the "differential equation" $F'(x,y) = \phi(F)$ where $F(x,y) = f(z)g(\bar{z})$. We have

$$\frac{F'(x,y)}{\phi(F)} = 1$$

Suppose that $\frac{1}{\phi(F)} = \sum_{n=0}^{\infty} a_n F^n$

Then $\sum_{n=0}^{\infty} a_n F^n F' = 1$

But

$$\{F^{n+1}\}' = \{f^{n+1}g^{n+1}\}' = \left[(n+1)f^n \frac{df}{dz}\right]\left[(n+1)g^n \frac{dg}{d\bar{z}}\right] = (n+1)^2 F^n F'$$

Hence "integrating" we have

$$\sum_{n=0}^{\infty} \frac{a_n F^{n+1}}{(n+1)^2} = x\bar{z} + H(x,y)$$

where $H(x,y)$ is harmonic. Denoting $\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^2} F^{n+1}$ by $\alpha(F)$ and $x\bar{z} + H$ by $w$ we have $\alpha(F) = w$ and hence, by Lagranges' Theorem (Whittaker and Watson, Modern Analysis, p132)

$$F = F_0 + \sum_{n=1}^{\infty} \frac{(w-w_0)^n}{n!}\left[\frac{d^{n-1}}{dF^{n-1}}\{\psi(F)\}^n\right]_{F=F_0}$$

where $\psi(F) = \frac{F-F_0}{\alpha(F)-w_0}$ and $w_0 = \alpha(F_0)$

Suppose $F(x,y) = f(z)g(\bar{z})$ where

$$f(z) = a_0 + \sum_{n=1}^{\infty}(a_n z^n + b_n z^{-n}), \quad g(\bar{z}) = c_0 + \sum_{n=1}^{\infty}(c_n \bar{z}^n + d_n \bar{z}^{-n})$$

Then

$$L(f;0,0;r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})g(re^{-i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n r^n e^{in\theta} + \frac{b_n}{r^n}e^{-in\theta}\right)\right]$$
$$\times\left[c_0 + \sum_{n=1}^{\infty}\left(c_n r^n e^{-in\theta} + \frac{d_n}{r^n}e^{in\theta}\right)\right]d\theta$$

But $\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\,d\theta = 0, \quad n \neq 0$

Therefore

$$L(F;0,0;r) = \frac{1}{2\pi}\int_0^{2\pi}\left[a_0c_0 + \sum_{n=1}^{\infty}\left(a_nc_n r^{2n} + \frac{b_nd_n}{r^{2n}}\right)\right]d\theta = a_0c_0 + \sum_{n=1}^{\infty}\left(a_nc_n r^{2n} + \frac{b_nd_n}{r^{2n}}\right)$$

Thus, if we define $F(0,0) = a_0c_0$, then

$$\frac{L(F;0,0;r) - F(0,0)}{r^2} = \sum_{n=1}^{\infty} a_nc_n r^{2n-2} + \sum_{n=1}^{\infty}\frac{b_nd_n}{r^{2n+2}}$$

Hence $F'(0,0)$ does not exist unless $b_nd_n = 0$, all $n$. If $b_nd_n = 0$, then $F'(0,0) = a_1c_1$

# Chapter VII

## HISTORY OF MEAN VALUE DERIVATIVES

7.0 Introduction: Some work has been done on generalized Laplace operators which is closely related to the subject of mean value derivatives. In fact several of the results given in previous chapters have been published by various authors using generalizations of Laplace's operator which are more or less similar to the mean value derivatives. In most cases the results were given for functions of three variables. This chapter is devoted to a short review of the various papers related to the subject.

7.1 A.G.Webster, Dynamics, Leipzig(1925)344-347.

By expanding $V(x,y,z)$ in a Taylor's series it is shown that, if $\bar{V}$ is the mean value of $V$ in a sphere of radius $R$ and center $(x_0,y_0,z_0)$

7.11 then $\lim\limits_{R\to 0} \dfrac{\bar{V}-V_0}{R^2} = \dfrac{1}{10}\nabla^2 V_0$

"Hence the excess of the mean value of $V$ throughout the volume of a small sphere over the value at the center is proportional to the value of $\nabla^2 V$ at the center and is of second order of small quantities. This interpretation is due to Stokes. From this point of view Maxwell called $-\nabla^2 V$ the Concentration of $V$.

7.12 Also $\nabla^2 V = 3\,\overline{\dfrac{\partial^2 V}{\partial r^2}}$ ,i.e. $\nabla^2 V$ is three times the mean of the directional second derivative for all directions. This interpretation is due to Boussinesq."

7.13 It is also shown that if $V_1$ is the mean of $V$ on the surface of a sphere, then $\lim\limits_{R\to 0} \dfrac{V_1-V_0}{R^2} = \dfrac{1}{6}\nabla^2 V_0$

7.2 S.Zaremba, Contribution a la theorie d'une equation fonctionelle de la physique. Rend.Circ.Palermo,19(1905)140-150. The author considers the following generalization of Laplace's operator.

Let $\Delta(f,h) = \dfrac{1}{h^2}\{f(x+h,y,z)+f(x-h,y,z)+f(x,y+h,z)+f(x,y-h,z)$
$+f(x,y,z+h)+f(x,y,z-h)-6f(x,y,z)\}$

Then Zaremba's generalized Laplacian is $\mathcal{L}(f) = \lim_{h\to 0} \Delta(f, h)$ .
The author then proves the following:

7.21 If $\Phi(x,y,z) = \Phi(P) = \int_{(D)} \varphi(Q) \frac{dQ}{PQ}$ where $\varphi$ is continuous, then $\mathcal{L}(\Phi) = -4\pi\varphi$

7.22 If $u$ is continuous in a domain $D$ and $\mathcal{L}(u) = 0$ in $D$ then $u$ is harmonic in $D$.

7.23 If $\mathcal{L}(u) = f(P)$ where $f$ is continuous in a domain $D$, then $\Psi(P) = u(P) + \frac{1}{4\pi}\int_{(D)} f(Q) \frac{dQ}{PQ}$ satisfies $\nabla^2\Psi(P) = 0$ . From this it follows that $u(P)$ has continuous first partial derivatives.

7.24 If $\mathcal{L}(u) = f(P)$ is continuous in $D$ and if

$$f(P) = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} + pu$$

where $a, b, c, p$ are functions of $P$ with continuous first derivatives, then $\nabla^2 u$ exists and $\nabla^2 u = \mathcal{L}(u)$ .

The author remarks that his results hold for $n$ dimensions and states the following: For the solution of the Fredholm integral equation $f(P) = \int_{(D)} G(P;Q)\varphi(Q)\,dQ$;
where $G$ is a Green's function or the Hilbert generalization of this function, one can replace the assumption of the existence and continuity of $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ by the existence and continuity of $f(P)$ and $\mathcal{L}(f)$ .

7.3 Michel Plancherel, Les Problems de Cantor et de du Bois-Reymond, Annales Scientifiques Ecole Normale (3) 31 (1914) 223-262. This paper is a more complete discussion of a previous paper of the author's. (Compte Rendus, 155 (1912) 897-900.) Consider a unit sphere. Let $F(\phi, \theta)$ be a function of points on the sphere. Let

$$\Delta_2 F(\phi, \theta; h) = \frac{1}{2\pi \sin h} \int_{C(\phi,\theta;h)} F(\phi', \theta')\,ds' - F(\phi, \theta)$$

where $C$ is a small circle on the sphere with center at $(\phi, \theta)$

and spherical radius $h$. $2\pi \sin h$ is the perimeter of the circle. The author then defines the generalized Beltrami parameter as

$$\Delta_2^* F(\phi,\theta) = \lim_{h\to 0} \frac{\Delta_2 F(\phi,\theta;h)}{\sin^2 \frac{h}{2}}$$

The name comes from the fact that if $F(\phi,\theta)$ has a total differential of the second order then

$$\Delta_2^* F(\phi,\theta) = \Delta_2 F(\phi,\theta) = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial F}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 F}{\partial\theta^2}$$

the right hand side being the second parameter of Beltrami. In Chapter II of the paper the author proves the following:

7.31 At each point $(\phi,\theta)$ where $F(\phi,\theta)$ has a total differential of second order $\Delta_2^* F(\phi,\theta) = \Delta_2 F(\phi,\theta)$

7.32 If $F(\phi,\theta)$ is continuous on a simply connected spherical domain $\mathcal{D}$ and if $\Delta_2^* F$ is bounded in $\mathcal{D}$, i.e. $m \leq \Delta_2^* F(\phi,\theta) \leq M$ then at each point of $\mathcal{D}$ for $h$ sufficiently small

$$\frac{m}{\cos h} \leq \frac{\Delta_2 F(\phi,\theta;h)}{\sin^2 \frac{h}{2}} \leq \frac{M}{\cos h}$$

7.33 If $F(\phi,\theta)$ is continuous on a spherical domain $\mathcal{D}$ and $\Delta_2^* F = 0$ on $\mathcal{D}$ then $\Delta_2 F(\phi,\theta) = 0$ on $\mathcal{D}$, i.e. $F(\phi,\theta)$ is "harmonic" on $\mathcal{D}$.

7.34 If $F(\phi,\theta)$ is continuous on the whole sphere and if $\Delta_2^* F = 0$ everywhere on the sphere then $F$ is a constant (since it is the potential on the surface).

The author employs the generalized Beltrami parameter in discussing a problem of Cantor and a problem of du Bois-Reymond, both concerning ortho-normal functions. The application is in connection with Legendre functions.

7.4 <u>Papers of W.Blaschke.</u>

7.41 <u>Ein Mittelwertsatz und eine Kennzeichnende Eigenschaft des logarithmischen Potentials</u>.Berichte,Gesellschaft der Wissenschaften zu Leipzig,68(1916) 3-7. We quote here the review given in the Fortschritte as the original paper was not available.

"Der Verf. beweist mittels potential theoretischer Hilfsmittel einige Mittelwertsatze, fur die er spater eine elementarere Herleitung gegeben hat. Ferner wird ein zweidimensionales Analogen des folgenden fundamentalen Satzes von Schwarz bewiesen: Jede stetige Funktion $f(x)$ fur welche

$$\lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

uberall im Intervalle $a \leq x \leq b$ versshwindet, ist dort eine ganze lineare Funktion."

7.42 Mittelwertsatze der Potentialtheorie, Deutsche Math. Vereiningung 27(1918)157-160. The author uses the results given in Webster's Dymamics (see 7.11,7.13). The author proves that if $f(P)$ is continuous in a domain $\mathcal{D}$ and has continuous second partial derivatives, then for a sphere $D(P_0;r)$ there are points $P_1$, $P_2$ in $D$ such that

$$\frac{L(f;P_0;r)-f(P_0)}{r^2} = \frac{1}{6} \nabla^2 f(P_1)$$

$$\frac{A(f;P_0;r)-f(P_0)}{r^2} = \frac{1}{10} \nabla^2 f(P_2)$$

where $L, A$ are the mean values of $f$ on and within , respectively a sphere of radius $r$ and center $P_0$ .

7.5 I.Privaloff,Sur les functions harmonique,Moskowskij Matiemat. Sbornik,32(1924-1925)464-469. Let $f(P) = f(x_1, x_2, \cdots, x_n)$ be a one-valued function in an open domain $\mathcal{G}$ and summable in $\mathcal{G}$ . Let $D(P;r)$ be a sphere of center $P$ , radius $r$, and volume $V$ . Let

$$\Delta_r f(P) = \frac{2(n+2)}{r^2 V} \int_{D(P;r)} [f(Q)-f(P)]\, dQ$$

The author then defines his generalized Laplacians as

$$\overline{\lim_{r \to 0}}\, \Delta_r f(P) = \overline{\Delta}^* f(P)$$

$$\underline{\lim_{r \to 0}}\, \Delta_r f(P) = \underline{\Delta}^* f(P)$$

If $\overline{\Delta}^* f(P) = \underline{\Delta}^* f(P)$ the common value is written $\Delta^* f(P)$. The author proves that:

7.51 If $f$ has a total differential of second order then $\Delta^* f = \nabla^2 f$
The proof is by Taylor's Theorem.

7.52 If $f$ is continuous and $\underline{\Delta}^* f \leq 0 \leq \overline{\Delta}^* f$ on a domain $\mathcal{D}$, then $f$ is harmonic on $\mathcal{D}$.

7.53 If $u(P) = \int_w \sigma(Q) \frac{dQ}{PQ}$ where $\sigma(Q)$ is continuous at $P$, then
$\Delta^* u(P) = -4\pi\sigma(P)$

7.6 Waclaw Kozakiewicz, Un theoreme sur les operateurs et son application a la theorie des Laplacians generalises. Towarzystwa Naukowego Warzawskiego 26(1933) part III, 18-24. The author defines an operator axiomatically, of which the operators of Zaremba, $[Z(f)]$, Blaschke, $[B(f)]$, and Privaloff, $[\nabla^*(f)]$, are special cases. He then proves that:

7.61 If $f$ is continuous and $Z(f)$ exists and is continuous at $P_0$, then $Z(f) = B(f) = \nabla^*(f)$ at $P_0$.

7.62 If $f$ is continuous and $B(f)$ exists and is continuous at $P_0$, then $\nabla^*(f)$ exists and $B(f) = \nabla^*(f)$.

7.63 If $f$ is contionuous and $\nabla^*(f)$ exists and is continuous at $P_0$, then $B(f)$ exists and $B(f) = \nabla^*(f)$.

7.7 S.Saks, On the operators of Blaschke and Privaloff for subharmonic functions. Rec.Math.(Mat.Sbornik)N.S.(51)9(1941) 451-456. According to previous results of Blaschke and of Privaloff if $u$ is subharmonic then

$$\overline{\lim_{r \to 0}}\ 6\left[\frac{L(u;P;r) - u(P)}{r^2}\right] \geq 0, \quad \overline{\lim_{r \to 0}}\ 10\left[\frac{A(u;P;r) - u(P)}{r^2}\right] \geq 0$$

The author improves upon these results by showing that for every subharmonic function $u$ the limits, for $r \to 0$, exist and are equal almost everywhere. He also shows that if $\sigma(E)$ is the non-negative mass distribution in terms of which $u$ can be expressed (according to F.Riesz ) as a potential plus a harmonic function, then the limits exist and are equal to the symmetric derivative of $\sigma(E)$ at every point where this derivative exists.

7.8 Papers of I.I.Privaloff.

7.81 On a theorem of S.Saks Rec.Math.(Mat.Sbornik)N.S. 9(51)(1941) 457-460. Using the notation of 7.7 let $u$ be subharmonic. Let $\underline{\rho}$ , $\bar{\rho}$ denote the lower and upper symmetric derivatives, respectively of $\sigma(E)$ . The author proves that $\underline{\rho} \leq \underline{\Delta}^* u \leq \overline{\Delta}^* u \leq \bar{\rho}$ everywhere in domain $\mathfrak{D}$ , and obtains in this manner a new proof of the results of Saks given previously. (Math.Reviews)

7.82 Sur la definition d'une fonction harmonique,C.R.(Doklady) Acad.Sci. URSS (N.S.) 31(1941)102-103. The author proves the following results:

7.821 If $u(\varphi)$ is continuous in a domain $G$ and if $\bar{\Delta}^* u \geq 0$ a.e. in $G$ , and if $\bar{\Delta}^* u > -\infty$ (save possibly for a closed set of zero capacity)then $u$ is subharmonic in $G$ . The proof runs as follows: Suppose $u(\varphi)$ is not subharmonic in the domain $G$ . There exists a domain $D$ , $\bar{D} \subset G$ and $V(\varphi)$ harmonic in $D$ and continuous in $\bar{D}$ , such that

$$u(\varphi) \leq V(\varphi) \tag{1}$$

on the boundary $\Gamma$ of $D$ , while in $D$ there exists a domain $d$ where

$$u(\varphi) > V(\varphi) \tag{2}$$

Construct a non-negative set function $\mu(e)$ as follows. Enclose the set $E$ (on which $\bar{\Delta}^* u$ is not known to be $\geq 0$ .) in a denumerable system of domains $\alpha$ , such that the sum of the volumes is $< \epsilon$ , then domains $\beta$, such that the sum $< \epsilon_2$, etc. where $\epsilon_1 + \epsilon_2 + \cdots$ is convergent. Take $\mu(e)$ equal to the sum of the volumes $\alpha, \beta, \cdots$ and their portions belonging to $e$. The set function $\mu(e)$ has at each point of $E$ the symmetric derivative $+\infty$. Form now the subharmonic function

$$f(\varphi) = -\int_D g(P;\varphi)\, d\mu(e)$$

which satisfies the following:

a) $\Delta^* f(\varphi) = +\infty$ at points of $E$

b) $\underline{\Delta}^* f(\varphi) \geq 0$ in $D$

(hence $f(\varphi)$ is subharmonic.) Consider a point $\varphi_1$ of $d$ not in the set $E_1$ of zero capacity on which it is not know that $\overline{\Delta}^* u > -\infty$ where $f(\varphi_1)$ is finite. Consider the function $F(\varphi) = u(\varphi) + \epsilon f(\varphi)$ and choose $\epsilon$ so that at $\varphi_1$

$$F(\varphi_1) < U(\varphi_1) \qquad (3)$$

This is possible because of (2). On $\Gamma$, by virtue of (1), $F(\varphi) \leq U(\varphi)$ where for points on $\Gamma$ we take for $F(\varphi)$ the limit superior of its values as $\Gamma$ is approached from within $D$. But $F(\varphi)$ is upper semi-continuous in $\overline{D}$. And $\overline{\Delta}^* F(\varphi) \geq 0$ for $\varphi$ not in $E$ since $\overline{\Delta}^* u(\varphi) \geq 0$. For $\varphi$ in $E$ not in $E_1$, $\overline{\Delta}^* u(\varphi)) > -\infty$, $\Delta^* f(\varphi) = +\infty$, therefore $\overline{\Delta}^* F(\varphi) > 0$. Hence $\overline{\Delta}^* F(\varphi) \geq 0$ in $D$ save perhaps on $E_1$. Thus $F(\varphi)$ is subharmonic by a result of Brelot (1934). This contradicts (3)and the result follows.

7.822 Let $u$ be continuous in $G$. Suppose that (i) $\underline{\Delta}^* u \leq 0 \leq \overline{\Delta}^* u$ a.e. in $G$, (ii) $\overline{\Delta}^* u > -\infty$, $\underline{\Delta}^* u < +\infty$ in $G$(save possibly for a closed set of zero capacity.) Then $u$ is harmonic in $G$. This result clearly follows from the previous one.

7.83 Quelques applications de l'operateur generalise de Laplace, C.R.Acad.Sci.URSS.(N.S.)31(1941)104-105. This paper contains restatements of and corollaries to results given in previous papers. For example: Let $u(Q)$ and $v(Q)$ be subharmonic in $G$. Suppose that (1) $\underline{\Delta}^* u < +\infty$, $\underline{\Delta}^* v < +\infty$ everywhere in $G$, and (2) $\underline{\Delta}^* u = \underline{\Delta}^* v$ a.e. in $G$. Then $u-v$ is harmonic in $G$. (Math.Reviews)

7.84 Sur la definition d'une fonction subharmonique Bull.Acad. Sci.URSS.Ser.Math.(Izvestia Akad.Nauk SSSR)5(1941)281-284. Let $E_1$ be a bounded closed set of capacity zero in a p dimensional Euclidean space. Let $\bar{\Delta}^* u$ be the upper generalized Laplacian of $u$. The author proves that, if $u$ is bounded above and upper semi-continuous in a neighborhood of $E_1$, and if in this neighborhood $\bar{\Delta}^* u(Q) > -\infty$ everywhere and $\bar{\Delta}^* u(Q) \geq 0$ almost everywhere, then $u$ is subharmonic in a domain containing $E_1$. (Math.Reviews)

7.85 Quelques applications de l'operateur generalise de Laplace. Rec.Math.(Mat.Sbornik)N.S. 11(53)(1942)149-154. The author deduces two important results on subharmonic functions and their generalized Laplace operators. In p dimensional space ($p > 2$) he defines the operator by

$$\underline{\Delta}\, u(Q_0) = \lim_{h \to 0} \left[ \frac{1}{w} \int u(P)\, dw - u(Q_0) \right] \frac{\Gamma(\frac{p}{2})(p+2)}{\pi^{\frac{p}{2}} (p-2) h^2}$$

where the integration is taken over a sphere of center $Q_0$, radius $h$ and volume $w$. The main result is the following: If (1) $u$ and $v$ are subharmonic in a domain $G$ and $\underline{\Delta}\, u(Q) < \infty$, $\underline{\Delta}\, v(Q) < \infty$ for all $Q \in G$, and (2) $\underline{\Delta}\, u(Q) = \underline{\Delta}\, v(Q)$ for almost all $Q \in G$, then $u(Q) = v(Q) + h(Q)$, where $h(Q)$ is harmonic in $G$. Another theorem shows that, if $E$ is a closed and bounded set of measure zero, and $u$ is subharmonic and bounded from above in a neighborhood of $E$ and upper semi-continuous at the points of $E$, then $u(Q)$ will be subharmonic in a domain containing $E$

if $\overline{\Delta} u(Q) > -\infty$ everywhere in $E$ except at points of a set of zero capacity. These results yield obviously important corollaries for harmonic functions. The proofs, based on some modern results of the theory of functions of real variables are straight forward. (Math.Reviews)

## Appendix I

<u>Details of the transformation employed in 3.4.</u>

Consider the integral

$$\int_0^{2\pi}\int_0^{2\pi} f(x+r\cos\theta+r\cos\phi,\, y+r\sin\theta+r\sin\phi)\cos(\theta-\phi)\,d\phi\,d\theta$$

The region of integration is a circle of radius $2r$ and center $(x,y)$. To every point $(x+\rho\cos\omega, y+\rho\sin\omega)$ in this region there corresponds two sets of values of $(\phi,\theta)$ as is shown in Figure 1. Thus the region of integration is traversed twice in the above integral. Consider first the set $(\phi_1,\theta_1)$ corresponding to a point $(x+\rho\cos\omega, y+\rho\sin\omega)$. We have (see Figure 2)

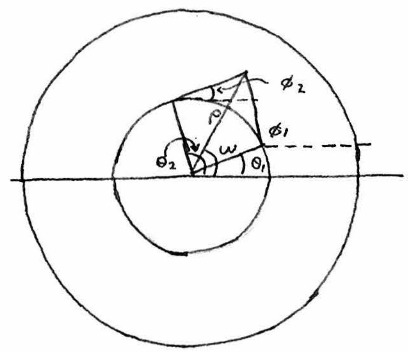


Figure 1

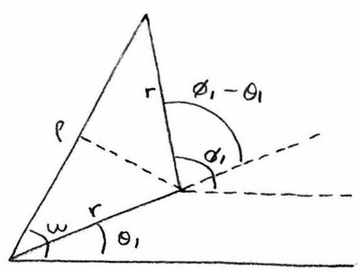


Figure 2

$$\cos(\omega-\theta_1) = \frac{\rho}{2r} = \cos\frac{(\phi_1-\theta_1)}{2}$$

$$\frac{\phi_1-\theta_1}{2} = \cos^{-1}\frac{\rho}{2r}$$

$$\omega = \theta_1 + \frac{\phi_1-\theta_1}{2} = \frac{\phi_1+\theta_1}{2}$$

Hence
$$\phi_1 = \omega + \cos^{-1}\frac{\rho}{2r}$$
$$\theta_1 = \omega - \cos^{-1}\frac{\rho}{2r}$$

And
$$\frac{\partial\phi_1}{\partial\omega} = 1, \quad \frac{\partial\theta_1}{\partial\omega} = 1,$$

$$\frac{\partial\phi_1}{\partial\rho} = -\frac{1}{\sqrt{1-\frac{\rho^2}{2r^2}}}\,\frac{1}{2r} = \frac{-1}{\sqrt{4r^2-\rho^2}}, \quad \frac{\partial\theta_1}{\partial\rho} = -\frac{\partial\phi_1}{\partial\rho} = \frac{1}{\sqrt{4r^2-\rho^2}}$$

Thus

$$\left|J\left(\frac{\partial\phi_1,\partial\theta_1}{\partial\omega,\partial\rho}\right)\right| = \begin{vmatrix} 1 & \dfrac{-1}{\sqrt{4r^2-\rho^2}} \\ 1 & \dfrac{1}{\sqrt{4r^2-\rho^2}} \end{vmatrix}_{abs.} = \frac{2}{\sqrt{4r^2-\rho^2}}$$

Further

$$\cos(\phi_1-\theta_1) = 2\cos^2\left(\frac{\phi_1-\theta_1}{2}\right)-1 = \frac{\rho^2}{2r^2}-1 = \frac{\rho^2-2r^2}{2r^2}$$

And hence for the first set $(\phi_1, \theta_1)$ we have

$$\frac{1}{r^2}\int_0^{2r}\int_0^{2\pi} f(x+p\cos\omega, y+p\sin\omega)\,\frac{p^2-2r^2}{\sqrt{4r^2-p^2}}\,d\omega\,dp$$

For the second set $(\phi_2, \theta_2)$ we have

$$\phi_2 = \omega - \cos^{-1}\frac{p}{2r}\,, \quad \theta_2 = \omega + \cos^{-1}\frac{p}{2r}$$

and we obtain similarly the same integral as before. Adding, we get

$$\frac{2}{r^2}\int_0^{2r}\int_0^{2\pi} f(x+p\cos\omega, y+p\sin\omega)\,\frac{p^2-2r^2}{\sqrt{4r^2-p^2}}\,d\omega\,dp$$

## Appendix II

<u>Details of the differentiation in 3.4</u>

(1) Consider the function

$$g(x,y) = \int_0^r \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho^{2n+1}\,d\theta\,d\rho$$

$$= \iint\limits_{D(x,y;r)} f(\xi,\eta)\,[(\xi-x)^2+(\eta-y)^2]^n\,d\xi\,d\eta$$

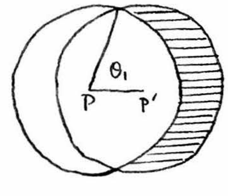
where $f(x,y)$ is continuous. We form the increment $\Delta g = g(x+\Delta x, y) - g(x,y)$

$$= \int_{y-r}^{y+r} d\eta \cdot \left\{ \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} [(\xi-x-\Delta x)^2+(\eta-y)^2)]^n f(\xi,\eta)\,d\xi - \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} [(\xi-x)^2+(\eta-y)^2]^n f(\xi,\eta)\,d\xi \right\}$$

$$= \int_{y-r}^{y+r} d\eta \cdot \left\{ \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \left( [(\xi-x)^2+(\eta-y)^2]^n - 2n\Delta x\cdot(\xi-x) \right) f(\xi,\eta)\,d\xi - \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} [(\xi-x)^2+(\eta-y)^2]^n f(\xi,\eta)\,d\xi \right\}$$

$$= \int_{y-r}^{y+r} d\eta \cdot \left\{ \int_{x+\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} [(\xi-x)^2+(\eta-y)^2]^n f(\xi,\eta)\,d\xi + \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x-\sqrt{r^2-(\eta-y)^2}} [(\xi-x)^2+(\eta-y)^2]^n f(\xi,\eta)\,d\xi - 2n\Delta x \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} (\xi-x)\,f(\xi,\eta)\,d\xi \right\} \qquad (1)$$

neglecting terms $o(\Delta x)$. The first integral is the integration of $[(\xi-x)^2+(\eta-y)^2]^n f(\xi,\eta)$ over the shaded area in the figure. The equation of $C(P';r)$ using $P(x,y)$ as the origin of a polar coordinate system is




$$\rho^2 - 2\rho\Delta x\cos x + (\Delta x)^2 = r^2$$

Thus $\quad \cos\theta_1 = \dfrac{r^2+(\Delta x)^2-r^2}{2r\Delta x} = \dfrac{\Delta x}{2r}$

and $\quad \rho = r + \Delta x\cos\theta + o(\Delta x)$

Setting this integral up in polar coordinates we have

$$\int_{-\theta_1}^{\theta_1} \int_r^{r+\Delta x\cos\theta} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho^{2n+1}\,d\rho\,d\theta$$

Applying the Mean Value Theorem

$$\int_{\theta_1}^{\theta_1} d\theta \cdot f(x+r_1\cos\theta, y+r_1\sin\theta)\int_r^{r+\Delta x\cos\theta} \rho^{2n+1}\, d\rho = \int_{-\theta_1}^{\theta_1} d\theta\cdot f(x+r_1\cos\theta, y+r_1\sin\theta)\left[\frac{\rho^{2n+2}}{2n+2}\right]_r^{r+\Delta x\cos\theta}$$

$$= \int_{-\theta_1}^{\theta_1} f(x+r_1\cos\theta, y+r_1\sin\theta)\, r^{2n+1}\,\Delta x \cos\theta\, d\theta$$

where $r < r_1 < r+\Delta x\cos\theta$. As $\Delta x \to 0$, $\theta_1 \to \frac{\pi}{2}$, $r_1 \to r$, $f(x+r_1\cos\theta, y+r_1\sin\theta) \to f(x+r\cos\theta, y+r\sin\theta)$ and thus, upon division by $\Delta x$ the above integral approaches

$$r^{2n+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

Similarly the second integral in (1) leads to the integral

$$r^{2n+1}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

Adding we get

$$r^{2n+1}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

Now the third integral in (1) is

$$-2n\,\Delta x\int_{y-r}^{y+r} d\eta\cdot\int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} (\xi - x)\, f(\xi,\eta)\, d\xi$$

Dividing by $\Delta x$ and letting $\Delta x \to 0$ we have

$$-2n\int_{y-r}^{y+r} d\eta\cdot\int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} (\xi-x) f(\xi,\eta)\, d\xi = -2n\int_0^r\int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\rho^2\cos\theta\, d\theta\, d\rho$$

Adding this to the previous result we have

$$\frac{\partial g}{\partial x} = r^{2n+1}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta - 2n\int_0^r\int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho^2\cos\theta\, d\theta\, d\rho$$

(2) Consider now

$$g(x,y) = \int_0^r\int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\log\frac{r}{\rho}\cdot\rho\, d\theta\, d\rho$$

$$= \frac{1}{2}\iint_{D(x,y;r)} f(\xi,\eta)\log\frac{r^2}{[(\xi-x)^2+(\eta-y)^2]}\, d\xi\, d\eta$$

where $f(x,y)$ is continuous. We form the increment $\Delta g = g(x+\Delta x, y) - g(x,y)$

$$= \frac{1}{2}\int_{y-r}^{y+r} d\eta\cdot\left\{\int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \log\frac{r^2}{[(\xi-x-\Delta x)^2+(\eta-y)^2]}\cdot f(\xi,\eta)\, d\xi - \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} \log\frac{r^2}{[(\xi-x)^2+(\eta-y)^2]}\cdot f(\xi,\eta)\, d\xi\right\}$$

We have $\log \frac{r^2}{(\xi-x-\Delta x)^2+(\eta-y)^2} = \log r^2 - \log\left[(\xi-x)^2+(\eta-y)^2-2\Delta x(\xi-x)+(\Delta x)^2\right]$

$$= \log r^2 - \log\left[(\xi-x)^2+(\eta-y)^2\right] + \frac{2\Delta x(\xi-x)}{(\xi-x)^2+(\eta-y)^2} + o(\Delta x)$$

and hence

$$\Delta g = \frac{1}{2}\int_{y-r}^{y+r} d\eta \cdot \left\{ \int_{x+\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \log\frac{r^2}{(\xi-x)^2+(\eta-y)^2} f(\xi,\eta)\,d\xi - \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x-\sqrt{r^2-(\eta-y)^2}} \log\frac{r^2}{(\xi-x)^2+(\eta-y)^2} f(\xi,\eta)\,d\xi + \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \frac{2\Delta x(\xi-x)}{[(\xi-x)^2+(\eta-y)^2]} f(\xi,\eta)\,d\xi \right\} + o(\Delta x)$$

But for the first integral we have

$$\left| \int_{x+\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \log\frac{r^2}{(\xi-x)^2+(\eta-y)^2} f(\xi,\eta)\,d\xi \right| \le \max|f| \left| \int_{x+\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \left\{ \log r^2 - \log\left[(\xi-x)^2+(\eta-y)^2\right] \right\} d\xi \right|$$

$$= \max|f| \left| \left[ (\xi-x)\left\{2+\log r^2-\log\left[(\xi-x)^2+(\eta-y)^2\right]\right\} - 2(\eta-y)\tan^{-1}\frac{\xi-x}{\eta-y} \right]_{x+\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} \right|$$

$$= \max|f| \Big| 2\Delta x + \Delta x\log r^2 - \left(\Delta x+\sqrt{r^2-(\eta-y)^2}\right)\log\left[\left(\Delta x+\sqrt{r^2-(\eta-y)^2}\right)^2+(\eta-y)^2\right]$$

$$+ \sqrt{r^2-(\eta-y)^2}\,\log r^2 - 2(\eta-y)\tan^{-1}\frac{\Delta x+\sqrt{r^2-(\eta-y)^2}}{\eta-y} + 2(\eta-y)\tan^{-1}\frac{\sqrt{r^2-(\eta-y)^2}}{\eta-y} \Big|$$

$$= \max|f| \Big| 2\Delta x + \Delta x\log r^2 - \left(\Delta x+\sqrt{r^2-(\eta-y)^2}\right)\left(\log r^2 + \frac{2\Delta x}{r^2}\sqrt{r^2-(\eta-y)^2}\right) + o(\Delta x)$$

$$+ \sqrt{r^2-(\eta-y)^2}\,\log r^2 - 2(\eta-y)\tan^{-1}\frac{\sqrt{r^2-(\eta-y)^2}}{\eta-y} - 2(\eta-y)\frac{\Delta x}{1+\frac{r^2-(\eta-y)^2}{(\eta-y)^2}} + 2(\eta-y)\tan^{-1}\frac{\sqrt{r^2-(\eta-y)^2}}{\eta-y} \Big|$$

$$= \max|f| \left| 2\Delta x - \frac{2\Delta x}{r^2}\left[r^2-(\eta-y)^2\right] - \frac{2\Delta x}{r^2}(\eta-y)^2 + o(\Delta x) \right| = o(\Delta x)$$

Similarly the second integral is $o(\Delta x)$. Thus neglecting terms $o(\Delta x)$

$$\Delta g = \Delta x \int_{y-r}^{y+r} d\eta \cdot \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} f(\xi,\eta)\frac{\xi-x}{(\xi-x)^2+(\eta-y)^2}\,d\xi$$

Hence

$$\frac{\Delta g}{\Delta x} = \int_{y-r}^{y+r} d\eta \cdot \int_{x+\Delta x-\sqrt{r^2-(\eta-y)^2}}^{x+\Delta x+\sqrt{r^2-(\eta-y)^2}} f(\xi,\eta)\frac{\xi-x}{(\xi-x)^2+(\eta-y)^2}\,d\xi$$

and

$$\frac{\partial g}{\partial x} = \int_{y-r}^{y+r} d\eta \cdot \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} f(\xi,\eta)\frac{\xi-x}{(\xi-x)^2+(\eta-y)^2}\,d\xi = \int_0^r \int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\cos\theta\,d\theta\,d\rho$$

(3) Consider now

$$f(x,y) = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$-\frac{2}{\pi R^2}\int_0^R\int_0^{2\pi}\left(R^2\log\frac{R}{\rho} - \frac{R^2-\rho^2}{2}\right) f'(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} f(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$-\frac{2}{\pi}\int_0^R\int_0^{2\pi}\log\frac{R}{\rho}\cdot f'(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$+\frac{1}{\pi}\int_0^R\int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho\, d\theta\, d\rho$$

$$-\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho^3\, d\theta\, d\rho$$

We apply the results of (1) to the first, third , and fourth integrals and apply the results of (2) to the second integral. Thus

$$\frac{\partial f}{\partial x} = \frac{1}{\pi R}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

$$-\frac{2}{\pi}\int_0^R\int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)\cos\theta\, d\theta\, d\rho$$

$$+\frac{R}{\pi}\int_0^{2\pi} f'(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

$$-\frac{R}{\pi}\int_0^{2\pi} f'(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

$$+\frac{2}{\pi R^2}\int_0^r\int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)\,\rho^2\cos\theta\, d\theta\, d\rho$$

$$= \frac{1}{\pi R}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)\cos\theta\, d\theta$$

$$-\frac{2}{\pi R^2}\int_0^R\int_0^{2\pi} f'(x+\rho\cos\theta, y+\rho\sin\theta)\,(R^2-\rho^2)\cos\theta\, d\theta\, d\rho$$

# Appendix III

Consider

$$u(x,y) = u(Q) = \iint_W \log \frac{1}{PQ}\, \sigma(P)\, dP = \int_0^\infty \int_0^{2\pi} \log \frac{1}{\rho} \cdot \sigma(x+\rho\cos\theta, y+\rho\sin\theta)\, \rho\, d\theta\, d\rho$$

Let
$$f_\delta(\rho) = \begin{cases} \log \frac{1}{\rho} & \rho > \delta \\ \log \frac{1}{\delta} + \frac{1}{2} - \frac{\rho^2}{2\delta^2} & \rho \le \delta \end{cases}$$

Let
$$u_\delta(Q) = \int_0^\infty \int_0^{2\pi} f_\delta(\rho)\, \sigma(x+\rho\cos\theta, y+\rho\sin\theta)\, \rho\, d\theta\, d\rho$$

Now
$$|u(Q) - u_\delta(Q)| \doteq \left| \int_0^\infty \int_0^{2\pi} \left\{ \log \frac{1}{\rho} - f_\delta(\rho) \right\} \sigma\, \rho\, d\theta\, d\rho \right|$$
$$= \left| \int_0^\delta \int_0^{2\pi} \left\{ \log \frac{1}{\rho} - \log \frac{1}{\delta} - \frac{1}{2} + \frac{\rho^2}{2\delta^2} \right\} \sigma\, \rho\, d\theta\, d\rho \right|$$
$$\le \sqrt{\int_0^\delta \int_0^{2\pi} \left\{ \log \frac{\delta}{\rho} - \frac{1}{2} + \frac{1}{2}\left(\frac{\rho}{\delta}\right)^2 \right\}^2 \rho\, d\theta\, d\rho} \cdot \sqrt{\int_0^\delta \int_0^{2\pi} |\sigma|^2 \rho\, d\theta\, d\rho}$$

using Schwarz's inequality in the last.

But
$$\int_0^\delta \int_0^{2\pi} \left\{ \log \frac{\delta}{\rho} - \frac{1}{2} + \frac{1}{2}\left(\frac{\rho}{\delta}\right)^2 \right\}^2 \rho\, d\theta\, d\rho = 2\pi \int_0^\delta \left\{ \log \frac{\delta}{\rho} - \frac{1}{2} + \frac{1}{2}\left(\frac{\rho}{\delta}\right)^2 \right\} \rho\, d\rho$$
$$= 2\pi\, \delta^2 \int_0^1 \left\{ \frac{1}{2} r^2 - \frac{1}{2} - \log r \right\}^2 r\, dr = C^2 \delta^2 \qquad C \text{ a constant.}$$

(In the last we made the change of variable $\frac{\rho}{\delta} = r$.)

Therefore
$$|u(Q) - u_\delta(Q)| \le C\delta \sqrt{\int_0^\delta \int_0^{2\pi} |\sigma|^2 \rho\, d\theta\, d\rho}$$

and hence, if $|\sigma|^2$ is summable , then $u_\delta(Q) \to u(Q)$ uniformly as $\delta \to 0$ . But the functions $u_\delta(Q)$ are continuous for all $\delta > 0$. Therefore $u(Q)$ is continuous.

## Appendix IV

Consider $\frac{1}{r^2}\int_0^r\int_0^{2\pi}|\sigma(x+\rho\cos\theta,y+\rho\sin\theta)-\sigma(x,y)|^2\,\rho\,d\theta\,d\rho$ (1)

where $|\sigma|^2$ is summable. Let $Q\equiv(x,y)$ , then this expression can be written

$$\frac{1}{r^2}\iint\limits_{D(Q;r)}|\sigma(P)-\sigma(Q)|^2dP=\frac{1}{r^2}\iint\limits_{D(Q;r)}\{|\sigma(P)|^2-2\sigma(P)\sigma(Q)+|\sigma(Q)|^2\}dP$$

$$=\frac{1}{r^2}\iint\limits_{D(Q;r)}|\sigma(P)|^2dP-\frac{2\sigma(Q)}{r^2}\iint\limits_{D(Q;r)}\sigma(P)dP+\pi|\sigma(Q)|^2$$

But

$$\Phi_1(E)=\iint\limits_E|\sigma(P)|^2dP\,,\quad \Phi_2(E)=\iint\limits_E\sigma(P)dP$$

are completely additive functions of sets. Thus

$$D^*\Phi_1(Q)=\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{D(Q;r)}|\sigma(P)|^2dP=|\sigma(Q)|^2 \qquad \text{almost all } Q.$$

$$D^*\Phi_2(Q)=\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{D(Q;r)}\sigma(P)\,dP=\sigma(Q) \qquad \text{almost all } Q.$$

(See McShane, <u>Integration</u>? p 382, especially Theorem 73.5 )

Hence

$$\lim_{r\to 0}\frac{1}{r^2}\iint\limits_{D(Q;r)}|\sigma(P)-\sigma(Q)|^2dP=\pi|\sigma(Q)|^2-2\pi\sigma(Q)\sigma(Q)+|\sigma(Q)|^2=0$$

for almost all $Q$ . Thus the integral (1) is $o(r^0)$ , for almost all $Q$.