Appendix A

Governing equations for the generation of strain, vorticity, dissipation, and enstrophy

A.1 Generation of strain and dissipation

From the Navier-Stokes equations for incompressible flow ($\nabla \cdot \mathbf{u} = 0$):

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \tag{A.1}$$

decompose the velocity gradient tensor $\partial u_i / \partial x_j$ in its symmetric (strain-rate tensor, S_{ij}) and anti-symmetric (rotation-rate tensor, Ω_{ij}) parts:

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}, \qquad S_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{A.2}$$

Take $[\partial(A.1)_i/\partial x_j + \partial(A.1)_j/\partial x_i]/2$ and use

$$\frac{1}{2}\left(\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right) = S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj} \tag{A.3}$$

to obtain:

$$\frac{\mathrm{D}S_{ij}}{\mathrm{D}\,t} = -S_{ik}S_{kj} - \Omega_{ik}\Omega_{kj} - \rho^{-1}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 S_{ij}}{\partial x_k \partial x_k}, \tag{A.4}$$

where $\mathrm{D}/\mathrm{D}\,t$ denotes the substantial derivative operator ($\mathrm{D}/\mathrm{D}\,t = \partial/\partial t + u_k\partial/\partial x_k$).

Multiply equation A.4 by S_{ji} and use its symmetry to obtain:

$$\frac{\mathrm{D}}{\mathrm{D}\,t}\left(\frac{1}{2}S_{ij}S_{ij}\right) = -S_{ik}S_{kj}S_{ji} - \Omega_{ik}\Omega_{kj}S_{ji} - \rho^{-1}S_{ji}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu\, S_{ji}\frac{\partial^2 S_{ij}}{\partial x_k \partial x_k}. \tag{A.5}$$

To find the equivalent relations in terms of the vorticity, $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$, instead of the rotation-rate tensor, $\boldsymbol{\Omega}$, use $\Omega_{ij} = -\epsilon_{ijk}\,\omega_k/2$, where ϵ_{ijk} is the Levi-Civita symbol and the contraction epsilon identity, $\epsilon_{lik}\,\epsilon_{jmk} = \delta_{lj}\delta_{im} - \delta_{lm}\delta_{ij}$, to express

$$\Omega_{ik}\Omega_{kj} = (\omega_i\omega_j - \delta_{ij}\,\omega_m\omega_m)/4. \tag{A.6}$$

Thus, equations A.4 and A.7 can be rewritten as:

$$\frac{\mathrm{D}S_{ij}}{\mathrm{D}\,t} = -S_{ik}S_{kj} - \frac{1}{4}\left(\omega_i\omega_j - \delta_{ij}\omega_k\omega_k\right) - \rho^{-1}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu\,\frac{\partial S_{ij}}{\partial x_k \partial x_k}, \tag{A.7}$$

$$\frac{\mathrm{D}}{\mathrm{D}\,t}\left(\frac{1}{2}\,S_{ij}S_{ij}\right) = -S_{ik}S_{kj}S_{ji} - \frac{1}{4}\,\omega_i S_{ij}\omega_j - \rho^{-1}S_{ji}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu\,S_{ji}\frac{\partial S_{ij}}{\partial x_k \partial x_k}, \tag{A.8}$$

where the symmetry of S_{ji} and the incompressibility condition ($\nabla\cdot\mathbf{u} = \mathrm{tr}(\nabla\mathbf{u}) = \mathrm{tr}(\mathbf{S}) = \delta_{ij}S_{ji} = 0$) have been considered.

A.2 Generation of vorticity and enstrophy

Apply the curl operator to equation A.1, obtaining:

$$\frac{\partial(\nabla\times\mathbf{u})}{\partial t} + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla\times(\nabla p) + \nu\,\nabla^2(\nabla\times\mathbf{u}). \tag{A.9}$$

From tensorial algebra, the following identities hold:

$$\nabla \times (\nabla \phi) \equiv 0, \qquad \nabla \cdot (\nabla \times \mathbf{u}) \equiv 0, \qquad \nabla u^2/2 \equiv \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \times \nabla \times \mathbf{u}, \tag{A.10}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) \equiv (\boldsymbol{B} \cdot \nabla + \boldsymbol{B} \times \nabla \times)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla + \boldsymbol{A} \times \nabla \times)\boldsymbol{B}, \tag{A.11}$$

where ϕ is a scalar field, $\mathbf{u}$ is a vector field with modulus $u \equiv \|\mathbf{u}\|$, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are second-order tensor fields. Use these tensor identities and the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$ to rewrite equation A.9 as:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{A.12}$$

The first term of the right-hand side of equation A.12 is responsible for the vortex-stretching.

Dot product equation A.12 with $\boldsymbol{\omega}$, decompose the velocity gradient tensor as $\nabla \mathbf{u} = \boldsymbol{S} + \boldsymbol{\Omega}$ in the first term of the right-hand side, and use the antisymmetry of $\boldsymbol{\Omega}$ [1] to obtain:

$$\frac{\mathrm{D}}{\mathrm{D}\,t}\left(\frac{1}{2}\,\omega_i \omega_i\right) = \omega_i S_{ik} \omega_k + \nu\, \omega_i \frac{\partial^2 \omega_i}{\partial x_k \partial x_k}. \tag{A.13}$$

To express equation A.13 in terms of $\boldsymbol{\Omega}$, particularize

$$\Omega_{ij} \frac{\partial^2 \Omega_{kj}}{\partial x_p \partial x_p} = \frac{1}{4}\, \epsilon_{ijl}\, \omega_l\, \epsilon_{kjm} \frac{\partial^2 \omega_m}{\partial x_p \partial x_p} = \frac{1}{4}\left(\omega_m \frac{\partial^2 \omega_m}{\partial x_p \partial x_p}\, \delta_{ik} - \omega_k \frac{\partial^2 \omega_i}{\partial x_p \partial x_p}\right) \tag{A.14}$$

for $k = i$, obtaining:

$$\Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_p \partial x_p} = \frac{1}{2}\, \omega_i \frac{\partial^2 \omega_i}{\partial x_p \partial x_p}. \tag{A.15}$$

Then, from equation A.6 and the antisymmetry of $\boldsymbol{\Omega}$ it results $\omega_i \omega_i = 2\,\Omega_{ij}\Omega_{ij}$, $\omega_i \omega_j = 4\,\Omega_{ik}\Omega_{kj} + 2\,\delta_{ij}\Omega_{mn}\Omega_{mn}$, which can be substituted, along with the relation A.15, into equation A.13 to obtain:

$$\frac{\mathrm{D}}{\mathrm{D}\,t}\left(\frac{1}{2}\,\Omega_{ij}\Omega_{ij}\right) = 2\,\Omega_{ik}\Omega_{kj} S_{ji} + \nu\, \Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_k \partial x_k}, \tag{A.16}$$

[1] If $\boldsymbol{A}$ is an antisymmetric second-order tensor, then $\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{a} = 0$, for any vector field $\boldsymbol{a}$.

where the symmetry of $\boldsymbol{S}$ and the incompressibility relation expressed in terms of $\boldsymbol{S}$, $\delta_{ij}S_{ij} = 0$, have been used to rewrite the first term of the right-hand side.