AXIALLY SYMMETRIC THERMAL STRESSES IN A SEMI-INFINITE SOLID

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Summary

In a semi-infinite solid the problem of determination of the thermal stresses caused by a known initial heat source throughout the solid and a known time dependent heat source on the surface is considered. The heat sources are taken as axially-symmetric, and the physical properties of the solid are assumed to be constant.

The equation of heat conduction is first integrated in order to obtain the temperature distribution throughout the solid. This result is then used in the equations of elastic equilibrium expressed in terms of the displacements. The latter equations are integrated and integral expressions for the displacements are obtained. Conditions for convergence are discussed, and the integrals are evaluated for special choices of the initial and boundary heat sources.

Notation

v_i	elastic displacement in the i direction
e_i	unit strain in i direction
e_{ij}	unit shearing strain between i and j directions
s_i	unit normal stress
s_{ij}	unit shearing stress
κ	thermal diffusivity, equals $k/\rho\sigma$ where k is the thermal conductivity, ρ is the density, and σ is the specific heat.
E	Young's Modulus
ν	Poisson's Ratio
G	modulus of rigidity $= E/[2(1+\nu)]$
λ	$= \dfrac{\nu E}{(1+\nu)(1-2\nu)}$
Δ	$= e_1 + e_2 + e_3$
α	coefficient of linear thermal expansion
T	temperature
β	$= \dfrac{\alpha E}{1-2\nu}$
K	$= \dfrac{4\kappa^2 \beta T_0}{\pi(\lambda+2G)}$

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I. Introduction.

The problem of determining the elastic stresses and displacements in a semi-infinite body is historically associated with the name of Boussinesq. A solution of this problem -- also known as the Problem of the Plane -- was first obtained by Lord Kelvin for the fundamental boundary condition of a load concentrated at a point.[1][1] Boussinesq, using potentials, succeeded in solving the problem for more general boundary conditions.[2] Subsequent writers, particularly Cerruti, generalized the solution further and devised new methods of attack.[3] A method of solving the problem, applicable when the surface loadings possess axial symmetry, was studied by H. Lamb[4], and K. Terazawa.[5] This method can be extended further to obtain a solution when there are body forces present which can be expressed as gradients of a potential function, and can be modified, as shown in this paper, to give a solution to the problem of determining thermal stresses caused by axially symmetric heat sources. Since many heat sources are of circular section, the case of axially symmetric distribution is important physically.

In the present paper it will be assumed that the temperature of the plane surface bounding the solid is known as an axially symmetric function of time and radius. It will also be assumed that the initial distribution of temperature throughout the solid is a known function of the depth and the radius, and that the initial condition of strain is that

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1. Numbers in square brackets refer to the notes at the end of the paper. See page 78.

which would be caused by this initial temperature distribution. Aside from the foregoing initial and boundary heat distributions, the solid will be taken as free of heat sources. Finally, the solid will be taken as a homogeneous and isotropic elastic body, the thermal conductivity, the density, and the specific heat being constant throughout.

II. The Thermo-Elastic Boundary Value Problem.

Whenever the stresses in an elastic body are caused by the unequal distribution of heat it is necessary that the fundamental elastic equations be modified. In order to define the notation and have these equations for ready reference, we shall restate them here. Since the present problem is one involving axially symmetric quantities, the elastic equations will be given in the form they assume in cylindrical coordinates.

We shall let $\bar{V}$ be the displacement vector,

$$\bar{V} \equiv \bar{1}_1 v_1 + \bar{1}_2 v_2 + \bar{1}_3 v_3$$

where v_1, v_2, and v_3 are the displacements in the radial, angular, and axial directions respectively, and $\bar{1}_1$, $\bar{1}_2$, and $\bar{1}_3$ are the respective fundamental unit vectors. The unit strains in the respective directions are given by the equations, [6],

$$\text{(A)} \qquad e_1 = \frac{\partial v_1}{\partial r}, \qquad e_2 = \frac{v_1}{r} + \frac{1}{r}\frac{\partial v_2}{\partial \theta}, \qquad e_3 = \frac{\partial v_3}{\partial z}$$

The unit shearing strains between the respective coordinate directions designated by the subscripts are given by,

$$\text{(A')} \quad e_{12} = e_{21} = \frac{1}{r}\frac{\partial v_1}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{v_2}{r}, \quad e_{13} = e_{31} = \frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z}, \quad e_{23} = e_{32} = \frac{1}{r}\frac{\partial v_3}{\partial \theta} + \frac{\partial v_2}{\partial z}$$

The respective normal stress components will be designated by s_1, s_2, and s_3, and the shearing stress components will be designated by $s_{12} = s_{21}$, $s_{13} = s_{31}$, and $s_{23} = s_{32}$, where s_{ij} is the unit shearing stress in the i direction on a surface whose normal is in the j direction. The fundamental relations between the stress components and the strain components are given by Hooke's Law [7], which states:

$$\text{(B)}\quad \begin{aligned} e_1 &= \frac{1}{E}[s_1 - \nu(s_2 + s_3)] & \qquad e_{12} &= \frac{1}{G}\, s_{12} \\ e_2 &= \frac{1}{E}[s_2 - \nu(s_1 + s_3)] & e_{13} &= \frac{1}{G}\, s_{13} \\ e_3 &= \frac{1}{E}[s_3 - \nu(s_1 + s_2)] & e_{23} &= \frac{1}{G}\, s_{23} \end{aligned}$$

where E is Young's Modulus, ν is Poisson's ratio, and G is the modulus of rigidity which is equal to

$$\frac{E}{2(1+\nu)}$$

Equations (B) can be solved for the stress components in terms of the strain components, giving

$$\text{(B')}\quad \begin{aligned} s_1 &= \lambda\Delta + 2Ge_1 \\ s_2 &= \lambda\Delta + 2Ge_2 \\ s_3 &= \lambda\Delta + 2Ge_3 \end{aligned}$$

where $\lambda = \nu E/(1+\nu)(1-2\nu)$ and $\Delta = e_1 + e_2 + e_3 = \overline{\nabla}\cdot\overline{V}$, $\overline{\nabla}$ being the operator, $\frac{I_1}{r} + I_1\frac{\partial}{\partial r} + I_2\frac{1}{r}\frac{\partial}{\partial\theta} + I_3\frac{\partial}{\partial z}$.

If $S = s_1 + s_2 + s_3$, it follows from (B') that

$$\text{(C)}\qquad \Delta = \frac{1-2\nu}{E}\, S$$

In the case that the displacements are axially symmetric, v_2 and the partials with respect to θ vanish.

Equations (A) and (A') become [8],

$$(A'') \quad e_1 = \frac{\partial v_1}{\partial r}, \quad e_2 = \frac{v_1}{r}, \quad e_3 = \frac{\partial v_3}{\partial z}, \quad e_{13} = \frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z}$$

The equations of elastic equilibrium in terms of the unit stresses then take the form [9],

$$(D) \qquad \begin{cases} \dfrac{\partial s_1}{\partial r} + \dfrac{s_1 - s_2}{r} + \dfrac{\partial s_{13}}{\partial z} + W_1 = 0 \\[2ex] \dfrac{\partial s_3}{\partial z} + \dfrac{\partial s_{13}}{\partial r} + \dfrac{s_{13}}{r} + W_3 = 0 \end{cases}$$

where $\overline{W} = \mathbf{1}_1 W_1 + \mathbf{1}_3 W_3$ is the axially symmetric body force. If equations (A") and (B') are substituted in (D), the equilibrium equations in terms of the displacements for the case of axial symmetry are obtained:

$$(E) \qquad \begin{cases} (\lambda + G)\dfrac{\partial \Delta}{\partial r} + G\left(\nabla_{-\theta}^2 v_1 - \dfrac{v_1}{r^2}\right) + W_1 = 0 \\[2ex] (\lambda + G)\dfrac{\partial \Delta}{\partial z} + G\nabla_{\theta}^2 v_3 + W_3 = 0 \end{cases}$$

where $\nabla_{-\theta}^2$ is the operator $\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$.

Boussinesq's problem consists in solving equations (D) or (E) for $z>0$ subject to boundary conditions on the surface $z = 0$ in terms of either the stresses or displacements. For example, we may be given that

$$(F) \qquad \begin{cases} s_{13} = Ge_{13} = R & \text{for } z = 0 \\ s_3 = \lambda\Delta + 2Ge_3 = Z & \text{for } z = 0 \end{cases}$$

where R and Z are functions of r. These conditions together with (D) or (E) suffice to determine a unique set of stresses and displacements throughout the solid $z > 0$.

When the solid is subjected to heating which results in a temperature distribution which is a non-linear point function, the relations (B) for Hooke's Law are no longer valid. The strains will not only be due to external stressing but also to temperature changes. If $T = T(r,z)$ is a point function giving the value of the temperature, measured from some reference level, and α is the coefficient of linear thermal expansion, then the strains due to temperature variation are [10],

$$e_1 = e_2 = e_3 = \alpha T,$$

provided that the magnitude of T is such that the expansion is linear. Since no shearing strains will be caused by temperature changes in an isotropic solid, the quantities e_{ij} will be given by their equations in (B). Adding the above thermal strains to the elastic strains of equations (B), we get the form of Hooke's Law which holds in thermoelastic problems, viz.,

$$(2.1)\qquad \begin{aligned} e_1 &= \frac{1}{E}[s_1 - \nu(s_2 + s_3)] + \alpha T & \qquad Ge_{12} &= s_{12} \\ e_2 &= \frac{1}{E}[s_2 - \nu(s_3 + s_1)] + \alpha T & Ge_{13} &= s_{13} \\ e_3 &= \frac{1}{E}[s_3 - \nu(s_1 + s_2)] + \alpha T & Ge_{23} &= s_{23} \end{aligned}$$

Replacing Hooke's Law (B) by equations (2.1) causes a modification in the other elastic equations. Solving (2.1) for the stresses, we find that equations (B') become,

$$(2.2)\qquad \begin{aligned} s_1 &= \lambda\Delta + 2Ge_1 - \beta T \\ s_2 &= \lambda\Delta + 2Ge_2 - \beta T \\ s_3 &= \lambda\Delta + 2Ge_3 - \beta T \end{aligned}$$

where $\beta = \dfrac{\alpha E}{1 - 2\nu}$

From (2.2), the thermal form of equation (C) is found to be

$$(2.3)\qquad \Delta = \frac{\alpha}{\beta}(S + 3\beta T)$$

In the case the body forces $\overline{W}$ are everywhere zero, substitution of (2.2) and (A") in equations (D) gives for the thermal equilibrium equations in terms of the displacements:

$$(2.4)\qquad \begin{cases} (\lambda + G)\dfrac{\partial\Delta}{\partial r} + G(\nabla_0^2 v_1 - \dfrac{v_1}{r^2}) - \beta\dfrac{\partial T}{\partial r} = 0 \\ (\lambda + G)\dfrac{\partial\Delta}{\partial z} + G\nabla_0^2 v_3 - \beta\dfrac{\partial T}{\partial z} = 0 \ . \end{cases}$$

If the surface tractions are everywhere zero, the boundary condition (F) becomes

$$(2.5)\qquad \begin{cases} e_{31} = 0 & \text{for} \quad z = 0 \\ \lambda\Delta + 2Ge_3 = \beta T & \text{for} \quad z = 0 \end{cases}$$

To solve the thermo-elastic boundary value problem, we must then solve equations (2.4), for $z > 0$, subject to the boundary conditions (2.5) on $z = 0$. Comparing equations (2.4) with (E) and (2.5) with (F), we see that the boundary value problem in thermo-elasticity with no body forces or surface tractions is the elastic boundary value problem with body forces $\overline{W} = -\beta\overline{\nabla}T$ and a normal surface tension of amount βT. The total normal stresses can be found by using the displacements from equations (2.4) and (2.5) in equations (2.2). Physically this means adding a uniform pressure of amount βT to the elastic normal stresses.

If T is a known function throughout the solid $z > 0$ and on the surface $z = 0$, the complete problem is contained in equations (2.4) and (2.5). However, it may be that T is known only on the surface $z = 0$, it then becomes necessary to solve a boundary value problem in heat conduction. In the general case the temperature is not only a function of position, but also of time. T will then be a solution of the equation

$$\nabla_{\theta}^{2} T = \frac{1}{\kappa} \frac{\partial T}{\partial t} \quad , \qquad \kappa \text{ a constant,} \tag{2.6}$$

subject to a boundary condition which is usually of the type $T = F(r,t)$ on $z = 0$, and subject to an initial condition of the type $T = H(r,z)$ at the time $t = 0$.

The thermo-elastic equations (2.4) were derived under the assumption that T was a function of position only. In the case that the temperature is time dependent, it follows that all of the displacements and stresses will also be functions of time. The thermo-elastic equations for this case will take the more general form [11] ,

$$\begin{cases} (\lambda + G) \dfrac{\partial \Delta}{\partial r} + G(\nabla_{\theta}^{2} v_{r} - \dfrac{v_{r}}{r^{2}}) - \beta \dfrac{\partial T}{\partial r} = \rho \dfrac{\partial^{2} v_{r}}{\partial t^{2}} \\[2ex] (\lambda + G) \dfrac{\partial \Delta}{\partial z} + G(\nabla_{\theta}^{2} v_{z}) - \beta \dfrac{\partial T}{\partial z} = \rho \dfrac{\partial^{2} v_{z}}{\partial t^{2}} \end{cases} \tag{2.7}$$

where ρ is the density. In most cases of physical importance the thermal variation is gradual so that the stress configuration is always in equilibrium with the heat distribution. The time then enters the equilibrium equations only as a parameter and the right hand members of equations (2.7) are zero. It follows that whenever the surface conditions are such that a temperature-stress equilibrium is maintained, the complete boundary value problem consists of equations (2.6) subject to the conditions,

$$\begin{cases} T = F(r,t) & \text{on} \quad z = 0 \\ T = H(r,z) & \text{at} \quad t = 0 \end{cases} \tag{2.8}$$

together with equations (2.4) subject to the boundary conditions (2.5). However, Goodier [12] has shown that the latter equations hold to a good degree of approximation even in those cases where temperature fluctuations lead elastic conformation. It must be assumed, however, that the initial stresses are those which are in equilibrium with the initial temperature distribution $H(r,z)$, and that the elastic quantities E, ν, G, and α do not vary with the temperature. (This can be assumed to hold in most practical cases. See Goodier loc. cit.) Under these conditions, the solution of the problem will be unique. For equations (2.6), (2.8) possess a unique solution [13], which in turn assures the uniqueness of the solution to the elastic problem (2.4), (2.5). [14].

A fairly complete list of references to the literature on thermo-elasticity is included in the notes. [15]

III. Heat Conduction in a Semi-Infinite Solid.

The solution of the thermal stress problem was seen to first involve the solution of equation (2.6) together with the conditions (2.8). In order to derive an expression which is a solution to this system of equations, it will be simplest to employ a method analogous to that first used by Minnigerode. [16]

We shall first define a function $u = u(r,\theta,z;\rho,\phi,\zeta;t-\tau)$ as the temperature at the point $P = P(r,\theta,z)$ in the solid $\mathcal{U}$ at the time t due to an instantaneous point source of heat of unit strength[17] located at the point $\pi = \pi(\rho,\phi,\zeta)$ generated at the time τ, where $\tau < t$. The solid $\mathcal{U}$ is assumed to be initially at a temperature zero throughout, and the surface $\mathcal{S}$ of the solid is to be kept at zero temperature. The function u will be called the Green's function for heat conduction. It has the property that $\lim\limits_{t\to\tau} u = 0$ at all points in $\mathcal{U}$ except at the point π. Further, by definition of unit instantaneous heat source, the total quantity of heat in the neighborhood of π as $t \to \tau$ is unity. We thus have that

$$\kappa \nabla^2 u = \frac{\partial u}{\partial t} \quad \text{for } t > \tau \tag{3.1}$$

where ∇^2 is the operator $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$.

Since $\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial \tau}$, u must also satisfy the equation,

$$\kappa \nabla^2 u + \frac{\partial u}{\partial \tau} = 0 \quad \text{for } t > \tau \tag{3.2}$$

Next we shall define the function $w = w(r,\theta,z,t)$ as the temperature in the solid $\mathcal{U}$ at the point $P = P(r,\theta,z)$ at the time t due to an initial temperature $h(r,\theta,z)$ throughout $\mathcal{U}$, and a temperature $f(r,\theta,z,t)$ on $\mathcal{S}$. The function w will satisfy the equations,

$$\begin{cases} \kappa \nabla^2 w = \frac{\partial w}{\partial t} & , \quad t > 0 \\ w = h(r,\theta,z) & , \text{ at } t = 0 \text{ throughout } \mathcal{U} \\ w = f(r,\theta,z,t) & , \text{ on } \mathcal{S} \text{ for } t > 0 \end{cases} \tag{3.3}$$

Since equations (3.3) hold for all $t > 0$, and since τ is restricted to the interval $0 \leq \tau \leq t$, it follows that the function $w = w(r,\theta,z,\tau)$ will satisfy

$$\begin{cases} \kappa \nabla^2 w = \frac{\partial w}{\partial \tau} & , \quad \tau < t \\ w = h(r,\theta,z) & , \text{ at } \tau = 0 \text{ throughout } \mathcal{U} \\ w = f(r,\theta,z,\tau) & , \text{ on } \mathcal{S} \text{ for } \tau > 0 \end{cases} \tag{3.4}$$

Next, let us form the product of the functions u and w, both being considered as functions of r, θ, z, and τ, then by equations (3.2) and (3.4), we have

$$\frac{\partial (u \cdot w)}{\partial \tau} = u\frac{\partial w}{\partial \tau} + w\frac{\partial u}{\partial \tau} = \kappa \left[u \nabla^2 w - w \nabla^2 u\right]. \tag{3.5}$$

Integrating right and left members over the volume $\mathcal{U}$ gives,

(3.6)
$$\int_{\mathcal{U}} \frac{\partial(u\cdot w)}{\partial\tau}\, d\gamma = \kappa\int (u\nabla^2 w - w\nabla^2 u)\, d\gamma$$

where $d\gamma = r\,dr\,d\theta\,dz$ is the differential of volume.

By means of Green's theorem [18] the right hand member can be transformed into a surface integral, giving

(3.7)
$$\int_{\mathcal{U}} \frac{\partial(u\cdot w)}{\partial\tau}\, d\gamma = \kappa\sum_i \int_{\mathcal{S}_i} \left(u\frac{\partial w}{\partial n_i} - w\frac{\partial u}{\partial n_i}\right) d\sigma_i$$

where $\int_{\mathcal{S}_i}$ denotes integration over the surface $\mathcal{S}_i$ whose surface differential is $d\sigma_i$, and $\frac{\partial}{\partial n_i}$ denotes differentiation along the outward drawn normal. In the case that is the semi-infinite solid $z > 0$, there will be only one surface $\mathcal{S}$, namely, the surface $z = 0$, $d\sigma$ will equal $r\,dr\,d\theta$, and $\frac{\partial}{\partial n} = \frac{\partial}{\partial(-z)} = -\frac{\partial}{\partial z}$.

Hence, for the solid $z > 0$, (3.7) reduces to

(3.8)
$$\int_{\mathcal{U}} \frac{\partial(u\cdot w)}{\partial\tau}\, d\gamma = \kappa\int_{\mathcal{S}} \left(w\frac{\partial u}{\partial z} - u\frac{\partial w}{\partial z}\right) d\sigma$$

If ϵ is an arbitrary small positive quantity, integrating (3.8) from $\tau = 0$ to $\tau = t-\epsilon$ gives

$$\int_{\tau=0}^{t-\epsilon}\int_{\mathcal{U}} \frac{\partial(u\cdot w)}{\partial\tau}\, d\gamma\, d\tau = \kappa\int_{\tau=0}^{t-\epsilon}\int_{\mathcal{S}} \left(w\frac{\partial u}{\partial z} - u\frac{\partial w}{\partial z}\right) d\sigma\, d\tau$$

Reversing the order of integration in the left member and proceeding formally,

(3.9)
$$\int_{\mathcal{U}}\int_{\tau=0}^{t-\epsilon} \frac{\partial(u\cdot w)}{\partial\tau}\, d\tau\, d\gamma = \int_{\mathcal{U}}\left\{[u]_{\tau=t-\epsilon}\cdot[w]_{\tau=t-\epsilon} - [u]_{\tau=0}\cdot[w]_{\tau=0}\right\} d\gamma =$$
$$= \kappa\int_{\tau=0}^{t-\epsilon}\int_{\mathcal{S}} \left(w\frac{\partial u}{\partial z} - u\frac{\partial w}{\partial z}\right) d\sigma\, d\tau$$

and since u vanishes on $\mathcal{S}$ this last integral is equal to

$$\kappa\int_{\tau=0}^{t-\epsilon}\int_{\mathcal{S}} w\frac{\partial u}{\partial z}\, d\sigma\, d\tau$$

Next consider the limit in equation (3.9) as $\epsilon \longrightarrow 0$. Since $\lim_{\tau \to t} u = 0$ everywhere except at the point π, the volume integral over $\mathcal{U}$ is equal to a volume integral over an elemental volume η containing the point π. The function w may be considered as a constant throughout η, with a value equal to its value at the point π at time $t = \tau$. Hence from (3.9),

$$[w_\pi]_{\tau=t} \int_\eta \lim_{\tau \to t} [u_\pi]\, d\gamma = \int_{\mathcal{U}} [w]_{\tau=0} [u]_{\tau=0}\, d\gamma + \kappa \int_{\tau=0}^{t} \int_{\mathcal{S}} w \frac{\partial u}{\partial z}\, d\sigma\, d\tau$$

where $\int_\eta$ is the volume integral over the elemental volume containing the point π. But by the properties of instantaneous point sources, [19],

$$\int_\eta \lim_{\tau \to t} [u_\pi]\, d\gamma = 1$$

Therefore $[w_\pi]_t$, the temperature at the point $\pi = \pi(\rho, \phi, \zeta)$ at the time t due to an initial temperature distribution $h(r,\theta,z)$ throughout $\mathcal{U}$ and a boundary temperature $f(r,\theta,t)$ on z=0, (see equation (3.4)), is given by

$$(3.10) \quad w(\rho,\phi,\zeta,t) = \int_{z=0}^{\infty}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} [u]_{\tau=0}\, h(r,\theta,z)\, r\, dr\, d\theta\, dz \; + \\ + \kappa \int_{\tau=0}^{t}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} f(r,\theta,\tau) \left[\frac{\partial u}{\partial z}\right]_{z=0} r\, dr\, d\theta\, d\tau$$

To complete the solution, the Green's function,

$$u = u(r,\theta,z;\rho,\phi,\zeta;t-\tau)$$

for the solid $z > 0$ must be determined. For an infinite solid we have that the Green's function, or the temperature at $P = P(r,\theta,z)$ at time t due to an instantaneous heat source at $\pi = \pi(\rho,\phi,\zeta)$ at time τ, is [20]

$$u = \frac{1}{8[\pi\kappa(t-\tau)]^{3/2}} \exp\left(-\frac{R^2}{4\kappa(t-\tau)}\right)$$

where for cylindrical coordinates

$$R^2 = r^2 + \rho^2 - 2r\rho\cos(\theta - \phi) + (z-\zeta)^2 = Q^2 + (z-\zeta)^2$$

If a negative instantaneous unit source is placed at the point $\pi' = \pi'(\rho,\phi,-\zeta)$, the principle of superposition of heat sources gives for the temperature at P,

(3.11)

$$\frac{1}{8[\pi\kappa(t-\tau)]^{3/2}}\left[\exp\left(-\frac{Q^2+(z-\zeta)^2}{4\kappa(t-\tau)}\right) - \exp\left(-\frac{Q^2+(z+\zeta)^2}{4\kappa(t-\tau)}\right)\right]$$

But this expression vanishes for all points on the surface $z = 0$, it is zero everywhere in $\mathcal{U}$ as $t \to \tau$ except at the point π, and its integral over $\mathcal{U}$ is unity [21]. Therefore, (3.11) is the Green's function u for the solid $z > 0$.

By differentiation of (3.11) with respect to z, we find that

$$(3.12) \qquad \left[\frac{\partial u}{\partial z}\right]_{z=0} = \frac{\zeta}{8\pi^{3/2}[\kappa(t-\tau)]^{5/2}}\exp\left(-\frac{Q^2+\zeta^2}{4\kappa(t-\tau)}\right)$$

Substituting (3.11) and (3.12) in (3.10) and interchanging the symmetrical quantities r,θ,z with ρ,ϕ,ζ, we get as the temperature at time t at the point (r,θ,z) due to an initial heat distribution $h(r,\theta,z)$ and a boundary heat source $f(r,\theta,t)$,

$$(3.14) \qquad T(r,\theta,z,t) = T_1(r,\theta,z,t) + T_2(r,\theta,z,t)$$

where

$$T_1 = \frac{1}{8(\pi\kappa t)^{3/2}}\int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} h(\rho,\phi,\zeta)\left[\exp\left(-\frac{Q^2+(z-\zeta)^2}{4\kappa t}\right) - \exp\left(-\frac{Q^2+(z+\zeta)^2}{4\kappa t}\right)\right]d\gamma$$

and

$$T_2 = \frac{z}{8(\pi\kappa)^{3/2}}\int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{f(\rho,\phi,\tau)}{(t-\tau)^{5/2}}\exp\left(-\frac{Q^2+\zeta^2}{4\kappa(t-\tau)}\right)\rho\,d\rho\,d\phi\,d\tau$$

where $d\gamma = \rho\,d\rho\,d\phi\,d\zeta$

Since we shall restrict ourselves to the case of axial symmetry, we can take $h(r,\theta,z) = H(r,z)$

and $\qquad f(r,\theta,t) = F(r,t)$

We may then write

$$T_1 = K_1 \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} H(\rho,\zeta)\ \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right)\left[\exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) - \exp\left(-\frac{(\zeta+z)^2}{4\kappa t}\right)\right]\int_{\phi=0}^{2\pi}\exp\left(\frac{2r\rho\cos(\theta-\phi)}{4\kappa t}\right)d\phi\,\rho\,d\rho\,d\zeta$$

But $\int_0^{2\pi}\exp(\alpha\cos(\theta-\phi))\,d\phi = \int_0^{2\pi}\exp(\alpha\cos(\phi))\,d\phi = 2\pi\ I_0(\alpha)$

where $I_0(\alpha)$ is the modified Bessel function of the first kind of order zero.

Hence,

$$(3.15)\qquad T_1 = \frac{1}{4\kappa t\sqrt{(\pi\kappa t)}}\int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\rho\, H(\rho,\zeta)\ \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right)\left[\exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) - \exp\left(-\frac{(\zeta+z)^2}{4\kappa t}\right)\right] I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho\, d\zeta$$

and similarly,

$$(3.16)\qquad T_2 = \frac{z}{4\kappa\sqrt{(\pi\kappa)}}\int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\frac{F(\rho,\tau)}{(t-\tau)^{5/2}}\ \exp\left(-\frac{r^2+\rho^2+z^2}{4\kappa(t-\tau)}\right) I_0\left(\frac{2r\rho}{4\kappa(t-\tau)}\right)\rho\, d\rho\, d\tau$$

The foregoing method gives the expressions (3.15) and (3.16) as the formal solutions of the boundary value problem (2.6), (2.8). For solution of the thermal stress problem it will be convenient to write these equations in a slightly different form.

By means of the relations [22],

$$(A)\qquad \frac{1}{2\kappa t}\exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right) = \int_{b=0}^{\infty} e^{-\kappa b^2 t} J_0(br)\ J_0(b\rho)\ b\ db$$

and

$$(B)\qquad \frac{\sqrt{\pi}}{4\sqrt{(\kappa t)}}\left[\exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) - \exp\left(-\frac{(\zeta+z)^2}{4\kappa t}\right)\right] = \int_{c=0}^{\infty} e^{-\kappa c^2 t}\sin(c\zeta)\ \sin(cz)\ dc$$

(3.15) can be written in the form:

$$(3.17)\qquad T_1 = \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty}\frac{2\rho b}{\pi} H(\rho,\zeta)\ \exp\left(-\kappa(b^2+c^2)t\right)\cdot J_0(br)\ J_0(b\rho)\ \sin(c\zeta)\ \sin(cz)\ dc\,db\,d\rho\,d\zeta$$

Making the transformation $t - \tau = t'$, and using the relations (A) and

(C) $$\frac{z}{2\sqrt{(\pi\kappa t^3)}} \exp\left(-\frac{z^2}{4\kappa t}\right) = \int_{c=0}^{\infty} \frac{2\kappa c}{\pi} e^{-\kappa c^2 t} \sin(cz)\, dc$$

(3.16) can be written in the form

(3.18) $$T_2 = \frac{2\kappa}{\pi} \int_{t'=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho b\, F(\rho, t-t') \exp\left(-\kappa(b^2+c^2)t'\right) \cdot J_0(br)\, J_0(b\rho)\, c \sin(cz)\; dc\, db\, d\rho\, dt'$$

(3.17), (3.18) are the forms of the solution which lend themselves readily to the solution of the thermo-elastic equations. In order to determine what restrictions must be placed on the functions $H(r,z)$ and $F(r,t)$ in order to satisfy the boundary conditions, we can proceed as follows:

We have as the complete solution of the system (2.6), (2.8) $T = T_1 + T_2$, but T_2 vanishes identically when $t = 0$, so we must find under what conditions $\lim T_1 = H(r,z)$ as t tends to zero.

From equation (3.15) we have

$$T_1 = I_1 - I_2 \quad \text{where}$$

$$I_1 = \frac{1}{4\kappa t\sqrt{(\pi\kappa t)}} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} \rho H(\rho,\zeta) \exp\left(-\frac{r^2+\rho^2+(\zeta-z)^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho\, d\zeta$$

$$I_2 = \frac{1}{4\kappa t\sqrt{(\pi\kappa t)}} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} \rho H(\rho,\zeta) \exp\left(-\frac{r^2+\rho^2+(\zeta+z)^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho\, d\zeta$$

Since the temperature reference level can be chosen so that H is always positive, the integrals I_1 and I_2 are ≥ 0 for all values of r, z, and $t \geq 0$. If we make the transformation $\lambda^2 = \frac{(\zeta - z)^2}{4\kappa t}$ in I_1 and $\lambda^2 = \frac{(\zeta + z)^2}{4\kappa t}$ in I_2, we have

$$I_1 = \frac{1}{2\kappa t\sqrt{\pi}} \int_{\lambda=\frac{-z}{2\sqrt{\kappa t}}}^{\infty} \int_{\rho=0}^{\infty} \rho H(\rho, z+2\sqrt{\kappa t}\,\lambda) \quad \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) e^{-\lambda^2} I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho\, d\lambda$$

and

$$I_2 = \frac{1}{2\kappa t\sqrt{\pi}} \int_{\lambda=\frac{z}{2\sqrt{\kappa t}}}^{\infty} \int_{\rho=0}^{\infty} \rho H(\rho, -z+2\sqrt{\kappa t}\,\lambda) \quad \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) e^{-\lambda^2} I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho\, d\lambda$$

If we restrict H to be a bounded function, we then have

$$I_2 \leq \frac{M}{2\kappa t\sqrt{\pi}} \int_{\lambda=\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda \int_{\rho=0}^{\infty} \rho \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho$$

where M is a constant, but

$$\text{(D)} \qquad \int_{\rho=0}^{\infty} \frac{\rho}{2\kappa t} \quad \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) \quad I_0\left(\frac{2r\rho}{4\kappa t}\right) d\rho \quad = 1 \qquad [23]$$

Therefore $I_2 \leq \frac{M}{\sqrt{\pi}} \int_{\lambda=\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda$, and as t tends to zero, I_2 will tend to zero for all positive z. We also have for H bounded that

$$I_1 \leq \frac{M}{\sqrt{\pi}} \int_{\lambda=-\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda \qquad \text{by} \quad \text{(D)}$$

But this integral converges for all $t \geq 0$. Therefore I_1 is absolutely convergent and the integrations may be performed in any order. Let us then write

$$I_1 = \int_{\rho=0}^{\infty} L_t(\rho) \quad G(\rho, z, t)\, d\rho$$

where $$L_t(\rho) = \frac{\rho}{2\kappa t} \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right)$$

and $$G(\rho, z, t) = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{\zeta=0}^{\infty} H(\rho, \zeta) \quad \exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta$$

We shall need the following theorem:

Theorem I.

Given a kernel $K_t(\rho) \geq 0$ such that

(1) K is integrable and $\int_{\rho=0}^{\infty} K_t(\rho)\, d\rho = 1$

(2) Corresponding to any $\delta_1 > 0$,

$\int_{\mathcal{R}} K_t(\rho)\, d\rho$ will tend to zero as t tends to zero, where $\mathcal{R}$ is the portion of the ρ axis outside the interval $r-\delta_1 < \rho < r+\delta_1$.

And given a function $\Gamma(\rho,t)$ such that

(3) $0 \leq \Gamma \leq M$, a constant

(4) $\Gamma(\rho,t)$ is a continuous function in ρ and t, and

(5) $\Gamma(r,0) = 0$

Then

$$\int_{\rho=0}^{\infty} K_t(\rho)\; \Gamma(\rho,t)\, d\rho \text{ will tend to zero as } t \to 0.$$

Proof:

We can write the integral $\int_{\rho=0}^{\infty}$ as the sum of three integrals, $\int_{\rho=0}^{r-\delta_1} + \int_{r-\delta_1}^{r+\delta_1} + \int_{r+\delta_1}^{\infty}$

By (3) we have that

$$\int_{\rho=0}^{r-\delta_1} K_t(\rho)\; \Gamma(\rho,t)\, d\rho \;\leq\; M \int_{\rho=0}^{r-\delta_1} K_t(\rho)\, d\rho$$

and

$$\int_{\rho=r+\delta_1}^{\infty} K_t(\rho)\; \Gamma(\rho,t)\, d\rho \;\leq\; M \int_{\rho=r+\delta_1}^{\infty} K_t(\rho)\, d\rho$$

But by (2) the right members of both expressions tend to zero with t. Further, since Γ is a continuous function of ρ and t, corresponding to a given $\varepsilon > 0$, we can choose a δ_1 and a δ_2, such that

$$|\Gamma(\rho,t)| < \varepsilon \quad \text{for} \quad |\rho - r| < \delta_1 \quad \text{and} \quad |t| < \delta_2$$

Therefore

$$\int_{r-\delta_1}^{r+\delta_1} K_t(\rho)\, \Gamma(\rho,t)\, d\rho \leq \varepsilon \int_{r-\delta_1}^{r+\delta_1} K_t(\rho)\, d\rho$$

for $|t| < \delta_2$

But

$$\int_{r-\delta_1}^{r+\delta_1} K_t(\rho)\, d\rho \leq \int_0^{\infty} K_t(\rho)\, d\rho = 1$$

Hence

$$\lim_{t\to 0} \int_{\rho=0}^{\infty} K_t(\rho)\, \Gamma(\rho,t)\, d\rho = 0 \qquad \text{q.e.d.}$$

As a consequence of equation (D) the function $L_t(\rho)$ satisfies condition (1) of Theorem I. Further, the modified Bessel function $I_0(x)$ is bounded by e^x [24], so we have the inequalities,

$$\int_{\rho=0}^{r-\delta} L_t(\rho)\, d\rho \leq \int_{\rho=0}^{r-\delta} \frac{\rho}{2\kappa t} \exp\left(-\frac{(r-\rho)^2}{4\kappa t}\right) d\rho$$

and

$$\int_{\rho=r+\delta}^{\infty} L_t(\rho)\, d\rho \leq \int_{\rho=r+\delta}^{\infty} \frac{\rho}{2\kappa t} \exp\left(-\frac{(r-\rho)^2}{4\kappa t}\right) d\rho$$

Since the right members of these inequalities go to zero with t, $L_t(\rho)$ will satisfy condition (2) of Theorem I.

Next let us take

$$\Gamma(\rho,t) = |\, G(\rho,z,t) - G(r,z,0)\, |$$

If we restrict $H(\rho,\zeta)$ to be a continuous function of both variables, bounded by a constant M, then the above expression for Γ will satisfy conditions (3), (4), and (5) of Theorem I.

Therefore

$$\lim_{t\to 0} \int_{\rho=0}^{\infty} L_t(\rho)\, |\, G(\rho,z,t) - G(r,z,0)\, |\, d\rho = 0$$

Hence by (1),

$$\lim_{t\to 0} \int_{\rho=0}^{\infty} L_t(\rho)\, G(\rho,z,t)\, d\rho = G(r,z,0)$$

It remains only to evaluate $G(r,z,0)$.

Since H is a continuous function, for a given ε there will exist a δ, such that

$$| H(\rho,\zeta) - H(\rho,z) | < \frac{\varepsilon}{4} \quad \text{for} \quad |\zeta - z| < \delta$$

We can write $G(\rho,z,t) = A_1 + A_2 + A_3$

where

$$A_1 = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_0^{z-\delta} H(\rho,\zeta) \exp\left(- \frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{z}{2\sqrt{\kappa t}}}^{-\frac{\delta}{2\sqrt{\kappa t}}} e^{-\lambda^2} H(\rho, z+2\sqrt{\kappa t}\,\lambda) \, d\lambda ,$$

$$A_2 = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{z-\delta}^{z+\delta} H(\rho,\zeta) \exp\left(- \frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta,$$

and

$$A_3 = \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{z+\delta}^{\infty} H(\rho,\zeta) \exp\left(- \frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} H(\rho, z+2\sqrt{\kappa t}\,\lambda) \, d\lambda$$

Now, since H is bounded,

$$|A_1| \leq \frac{M}{\sqrt{\pi}} \left| \int_{-\frac{z}{2\sqrt{\kappa t}}}^{-\frac{\delta}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda \right|$$

and there will exist a t_1 such that

$$\left| \int_{-\frac{z}{2\sqrt{\kappa t}}}^{-\frac{\delta}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda \right| < \frac{\sqrt{\pi}}{4M}\varepsilon \qquad \text{for} \qquad 0 < t \leq t_1$$

Therefore $\quad |A_1| < \frac{\varepsilon}{4} \quad$ for $\quad 0 < t \leq t_1$

Similarly, there exists a t_3 such that

$$|A_3| < \frac{\varepsilon}{4} \quad \text{for} \quad 0 < t \leq t_3$$

Returning now to A_2, we can write it as the sum of two integrals:

$$\frac{H(\rho,z)}{2\sqrt{(\pi\kappa t)}} \int_{z-\delta}^{z+\delta} \exp\left(- \frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta + \frac{1}{2\sqrt{(\pi\kappa t)}} \int_{z-\delta}^{z+\delta} \left(H(\rho,\zeta) - H(\rho,z)\right) \exp\left(- \frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta$$

The first term of the right member is equal to

$$\frac{H(\rho,z)}{\sqrt{\pi}}\int_{-\frac{\delta}{2\sqrt{\kappa t}}}^{\frac{\delta}{2\sqrt{\kappa t}}} e^{-\lambda^2}\,d\lambda = \frac{2H(\rho,z)}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-\lambda^2}\,d\lambda - \int_{\frac{\delta}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda\right]$$

Hence,

$$\left| A_2 - H(\rho,z) \right| \leq \frac{2M}{\sqrt{\pi}}\int_{\frac{\delta}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda + \frac{1}{2\sqrt{(\pi\kappa t)}}\int_{z-\delta}^{z+\delta} \left| H(\rho,\zeta) - H(\rho,z) \right| \cdot \exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta$$

But there exists a t_2 such that

$$\int_{\frac{\delta}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda < \frac{\sqrt{\pi}}{8M}\varepsilon \qquad \text{for} \qquad 0 < t \leq t_2$$

And since H is continuous,

$$\frac{1}{2\sqrt{(\pi\kappa t)}}\int_{z-\delta}^{z+\delta} \left| H(\rho,\zeta) - H(\rho,z) \right| \exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) d\zeta < \frac{\varepsilon}{4\sqrt{\pi}}\int_{-\frac{\delta}{2\sqrt{\kappa t}}}^{\frac{\delta}{2\sqrt{\kappa t}}} e^{-\lambda^2}\,d\lambda < \frac{\varepsilon}{4}$$

We therefore have

$$\left| A_2 - H(\rho,z) \right| < \frac{\varepsilon}{2} \qquad \text{for} \qquad 0 < t \leq t_2$$

But $\quad G(\rho,z,t) - H(\rho,z) = A_1 + A_3 + A_2 - H(\rho,z) < \varepsilon$

whenever $0 < t \leq t_4$ where t_4 is the smallest of t_1, t_2, t_3.

Therefore $\quad \lim\limits_{t\to 0} G(\rho,z,t) = H(\rho,z)$ [25]

We have thus shown that

$$\lim_{t\to 0} T = H(r,z)$$

provided that $H(r,z)$ is bounded and continuous in both variables.

We must next consider the limit of T as z tends to zero. Since T vanishes identically on the boundary $z = 0$, we must find under what conditions $\lim\limits_{z\to 0} T_2 = F(r,t)$.

For the discussion of this limit it will be convenient to establish a theorem similar to the preceeding one.

Theorem II.

Given a kernel, $K_z(\rho,\lambda) \geq 0$, such that

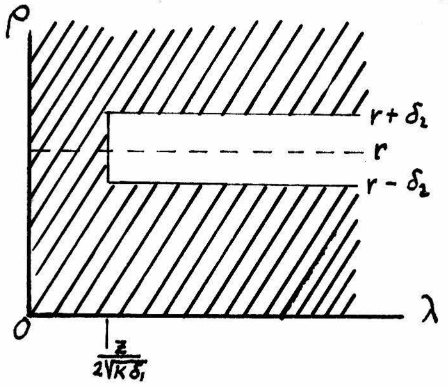
(1) K is integrable and $\int_{\rho=0}^{\infty}\int_{\lambda=0}^{\infty} K_z(\rho,\lambda)\, d\rho\, d\lambda = 1$

(2) Corresponding to any δ_1 and $\delta_2 > 0$,

$$I = \int\int_{\mathcal{R}} K_z(\rho,\lambda)\, d\rho\, d\lambda \text{ will tend to 0 as } z \to 0.$$

where $\mathcal{R}$ is the region outside the strip

$$r-\delta_2 < \rho < r+\delta_2, \qquad \frac{z}{2\sqrt{(K\delta_1)}} < \lambda$$



And given a function $G(r,t)$, such that

(3) $0 \leq G \leq M$

(4) $G(r,t)$ is continuous in both r and t, and

(5) $G(r,0) = 0$

Then,

$$II = \int_{\rho=0}^{\infty}\int_{\lambda=0}^{\infty} K_z(\rho,\lambda)\, G\left(\rho, \frac{z^2}{4K\lambda^2}\right) d\rho\, d\lambda$$

will approach zero as $z \longrightarrow 0$.

Proof:

Break up the integral II into three parts,

$$II = A + B + C$$

where

$$A = \int_{\rho=0}^{r-\delta_2}\int_{\lambda=0}^{\infty} K_z\, G\, d\rho\, d\lambda \quad + \int_{\rho=r+\delta_2}^{\infty}\int_{\lambda=0}^{\infty} K_z\, G\, d\rho\, d\lambda$$

$$B = \int_{\rho=r-\delta_2}^{r+\delta_2}\int_{\lambda=0}^{\frac{z}{2\sqrt{K\delta_1}}} K_z\, G\, d\rho\, d\lambda$$

and
$$C = \int_{\rho=r-\delta_2}^{r+\delta_2}\int_{\lambda=\frac{z}{2\sqrt{\kappa\delta_1}}}^{\infty} K_z\, G\, d\rho\, d\lambda$$

It follows, since G is bounded, that

$$A + B \leq M \iint_{\mathcal{R}} K_z(\rho,\lambda)\, d\rho\, d\lambda$$

But by (2), this integral tends to zero with z. Further, since G is a continuous function, we have by (4) and (5) that for a given ε, we can find a δ_1 and δ_2 such that $\left| G\left(\rho, \frac{z^2}{4\kappa\lambda^2}\right) \right| < \varepsilon$ for $|r-\rho| < \delta_2$ and $\frac{z^2}{4\kappa\lambda^2} < \delta_1$

but this last inequality is equivalent to $\lambda > \frac{z}{2\sqrt{(\kappa\delta_1)}}$.

Therefore in the strip $\overline{\mathcal{R}}$, it follows that

$$C < \varepsilon \iint_{\overline{\mathcal{R}}} K_z(\rho,\lambda)\, d\rho\, d\lambda \leq \varepsilon \int_{\rho=0}^{\infty}\int_{\lambda=0}^{\infty} K_z(\rho,\lambda)\, d\rho\, d\lambda$$

Hence by (1) $\qquad C < \varepsilon$

Therefore

$$\lim_{z\to 0} II = \lim_{z\to 0}(A + B + C) = 0 \qquad \text{q.e.d.}$$

If in equation (3.16), we set $\lambda^2 = \dfrac{z^2}{4\kappa(t-\tau)}$,

T_2 becomes

$$\frac{4}{\sqrt{\pi}}\int_{\lambda=\frac{z}{2\sqrt{\kappa t}}}^{\infty}\int_{\rho=0}^{\infty} F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) \frac{\lambda^2}{z^2}\, e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r^2+\rho^2)\right) I_0\left(\frac{2r\rho\lambda^2}{z^2}\right)\rho\, d\rho\, d\lambda$$

Let us define as the kernel of Theorem II,

$$K_z(\rho,\lambda) = \frac{4\rho}{\sqrt{\pi}}\,\frac{\lambda^2}{z^2}\, e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r^2+\rho^2)\right) I_0\left(\frac{2r\rho\lambda^2}{z^2}\right)$$

Then

$$(E) \quad T_2 = \int_{\lambda=0}^{\infty}\int_{\rho=0}^{\infty} F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) \quad K_z(\rho,\lambda)\ d\rho\ d\lambda \quad - \int_{\lambda=0}^{\frac{z}{2\sqrt{\kappa t}}}\int_{\rho=0}^{\infty} F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) \quad K_z(\rho,\lambda)\ d\rho\ d\lambda$$

Consider the integral of the kernel over the first quadrant,

$$\int_{\rho=0}^{\infty}\int_{\lambda=0}^{\infty} \frac{4\rho}{\sqrt{\pi}} \ \frac{\lambda^2}{z^2} \ e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r^2+\rho^2)\right) \ I_0\left(\frac{2r\rho\lambda^2}{z^2}\right)\ d\rho\ d\lambda$$

The integrand is positive and we may write the integral in the form,

$$\frac{4}{z^2\sqrt{\pi}} \int_{\lambda=0}^{\infty} \lambda^2 e^{-\lambda^2} \exp\left(-\frac{\lambda^2 r^2}{z^2}\right) \int_{\rho=0}^{\infty} \rho \exp\left(-\frac{\lambda^2\rho^2}{z^2}\right) \ I_0\left(\frac{2r\rho\lambda^2}{z^2}\right) d\rho\ d\lambda$$

But $$\int_{\rho=0}^{\infty} \rho \exp\left(-\frac{\lambda^2\rho^2}{z^2}\right) \quad I_0\left(\frac{2r\rho\lambda^2}{z^2}\right) \quad d\rho \quad = \quad \frac{z^2}{2\lambda^2} \quad \exp\left(\frac{r^2\lambda^2}{z^2}\right)$$

Therefore

$$\int_{\rho=0}^{\infty}\int_{\lambda=0}^{\infty} K_z(\rho,\lambda)\ d\lambda\ d\rho \ = \ \frac{2}{\sqrt{\pi}}\int_{\lambda=0}^{\infty} e^{-\lambda^2} d\lambda \ = 1$$

Hence condition (1) of Theorem II is satisfied by the above choice of $K(\rho,\lambda)$. Next consider

$$\lim_{z\to 0} \iint_{\mathcal{R}} \frac{4\rho}{\sqrt{\pi}} \ \frac{\lambda^2}{z^2} \ e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r^2+\rho^2)\right) \ I_0\left(\frac{2r\rho\lambda^2}{z^2}\right) \ d\rho\ d\lambda$$

Since $I_0(x) \leq e^x$, we may write

$$\iint_{\mathcal{R}} K_z(\rho,\lambda)\ d\rho\ d\lambda \ \leq \ \frac{4}{\sqrt{\pi}} \int_{\rho=0}^{r-\delta_1}\int_{\lambda=0}^{\infty} \frac{\rho\lambda^2}{z^2} \ e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r-\rho)^2\right) d\rho\ d\lambda \ +$$

$$+ \ \frac{4}{\sqrt{\pi}} \int_{\rho=r+\delta_1}^{\infty}\int_{\lambda=0}^{\infty} \frac{\rho\lambda^2}{z^2} \ e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r-\rho)^2\right) d\rho\ d\lambda \ +$$

$$+ \ \frac{4}{\sqrt{\pi}} \int_{\rho=r-\delta_1}^{r+\delta_1}\int_{\lambda=0}^{\frac{z}{2\sqrt{\kappa\delta_1}}} \frac{\rho\lambda^2}{z^2} \ e^{-\lambda^2} \exp\left(-\frac{\lambda^2}{z^2}(r-\rho)^2\right) d\rho\ d\lambda$$

If we make the transformations $\frac{\lambda}{z} = \mu$ and $r - \rho = x$,

the first integral in the right member becomes

$$\frac{4z}{\sqrt{\pi}} \int_{x=\delta_2}^{r} \int_{\mu=0}^{\infty} (r - x)\, \mu^2 \exp\left(-\,(z^2+x^2)\mu^2\right)\, d\mu\, dx$$

But upon integrating with respect to μ, it becomes

$$z \int_{x=\delta_2}^{r} \frac{(r - x)\, dx}{(z^2+x^2)^{3/2}}$$

And since this integral is bounded for all z, the product goes to zero with z.

Setting $\frac{\lambda}{z} = \mu$ and $r - \rho = -x$, the second integral in the right member becomes

$$\frac{4z}{\sqrt{\pi}} \int_{x=\delta_2}^{\infty} \int_{\mu=0}^{\infty} (r + x)\, \mu^2 \exp\left(-\,(z^2+x^2)\mu^2\right)\, d\mu\, dx$$

Which, integrating with respect to μ, becomes

$$z \int_{\delta_2}^{\infty} \frac{(r + x)\, dx}{(z^2+x^2)^{3/2}}$$

And since the integrand is $O(1/x^2)$, the integral exists and the expression goes to zero with z. Finally, the third integral becomes

$$\frac{4z}{\sqrt{\pi}} \int_{x=-\delta_2}^{+\delta_2} \int_{\mu=0}^{\frac{1}{2\sqrt{K\delta_1}}} (r - x)\, \mu^2 \exp\left(-\,(z^2+x^2)\mu^2\right)\, d\mu\, dx =$$

$$= \frac{8rz}{\sqrt{\pi}} \int_{x=0}^{\delta_2} \int_{\mu=0}^{\frac{1}{2\sqrt{K\delta_1}}} \mu^2 \exp\left(-\,(z^2+x^2)\mu^2\right)\, d\mu\, dx \;=\; 4zr \int_{\mu=0}^{\frac{1}{2\sqrt{K\delta_1}}} \mu\, e^{-\mu^2 z^2}\, d\mu$$

But the latter integral is bounded for all z, and the expression tends to zero with z. The above choice of kernel, therefore, satisfies condition (2) of Theorem II.

Let us next define the function G of Theorem II as

$$| F(\rho,t-\tau) - F(r,t) |$$

Clearly, $G(r,0) = 0$ and condition (5) is satisfied. If $F(\rho,\tau)$ is a continuous function of both variables and satisfies the relation $0 \leq F \leq M$, then all of the conditions of Theorem II are met. We then have that

$$\lim_{z\to 0} \int_{\lambda=0}^{\infty}\int_{\rho=0}^{\infty} K_z(\rho,\lambda) \left| F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) - F(r,t) \right| d\rho\, d\lambda = 0;$$

and since $\displaystyle\int_{\lambda=0}^{\infty}\int_{\rho=0}^{\infty} K_z(\rho,\lambda)\, d\rho\, d\lambda = 1,$

$$\lim_{z\to 0} \int_{\lambda=0}^{\infty}\int_{\rho=0}^{\infty} K_z(\rho,\lambda)\, F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) d\rho\, d\lambda = F(r,t)$$

Returning now to equation (E),

$$\lim_{z\to 0} T_2 = F(r,t) - \lim_{z\to 0} \int_{\lambda=0}^{\frac{z}{2\sqrt{\kappa t}}}\int_{\rho=0}^{\infty} F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) K_z(\rho,\lambda)\, d\rho\, d\lambda$$

by Theorem II. But $0 \leq F \leq M$, therefore

$$\int_{\lambda=0}^{\frac{z}{2\sqrt{\kappa t}}}\int_{\rho=0}^{\infty} F\left(\rho, t-\frac{z^2}{4\kappa\lambda^2}\right) K_z(\rho,\lambda)\, d\rho\, d\lambda \leq \frac{2M}{\sqrt{\pi}} \int_{0}^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2}\, d\lambda$$

And the limit of this last expression as $z \to 0$, is zero.

We have therefore shown that

$$\lim_{z\to 0} T = F(r,t)$$

provided that $F(r,t)$ is bounded and is continuous in both r and t.

IV. Thermal Stresses in a Semi-Infinite Solid.

It was shown in section II that the boundary value problem in thermo-elasticity for the axially symmetric case reduces to the solution of equations (2.4) subject to the boundary conditions (2.5) (page 8), in which the value of the function T is obtained from the solution of equation (2.6) with its boundary conditions (2.8) (page 9). The solution of the system (2.6), (2.8) was discussed in section III, so it remains only to solve the system (2.4), (2.5).

In attacking this system, we shall use a purely formal method and later investigate the conditions for the functions H(r,z) and F(r,t) under which the solution so obtained is valid.

First, let us write equations (2.4) in vector form. The displacement vector $\overline{V}$ (page 4) in the case of axial symmetry is equal to

$$\overline{I}_1 v_1 + \overline{I}_3 v_3$$

Hence, $\nabla^2\overline{V} = \nabla^2(\overline{I}_1 v_1) + \nabla^2(\overline{I}_3 v_3)$ where ∇^2 is the Laplacian operator in cylindrical coordinates (page 11). But

$$\nabla^2(\overline{I}_1 v_1) = \overline{I}_1(\nabla^2_{-\theta} v_1 - \frac{v_1}{r^2})$$

and

$$\nabla^2(\overline{I}_3 v_3) = \overline{I}_3(\nabla^2_{-\theta} v_3)$$

Further, $\Delta = \overline{\nabla}\cdot\overline{V}$ (page 5). Hence it is seen that the first equation of (2.4) is the $\overline{I}_1$ component, and the second equation of (2.4) is the $\overline{I}_3$ component of

(4.1) $$(\lambda + G)\overline{\nabla}(\overline{\nabla}\cdot\overline{V}) + G\nabla^2\overline{V} - \beta\overline{\nabla}T = 0$$

We shall now seek a solution of equation (4.1) for $z > 0$ which will satisfy equations (2.5) on the surface $z = 0$.

By means of the vector relation

(A) $$\nabla^2\bar{V} = \bar{\nabla}(\bar{\nabla}\cdot\bar{V}) - \bar{\nabla}\times\bar{\nabla}\times\bar{V},$$

equation (4.1) can be written in the form

$$(\lambda + 2G)\,\bar{\nabla}(\bar{\nabla}\cdot\bar{V}) - G(\bar{\nabla}\times\bar{\nabla}\times\bar{V}) - \beta\bar{\nabla}T = 0$$

Let us now consider the vector $\bar{V}$ as composed of two parts,

(4.2) $$\bar{V} = \bar{\nabla}\Phi + \bar{\nabla}\times\bar{A}$$

where Φ is a scalar quantity and $\bar{A}$ is a vector which we restrict to satisfy the relation,

(B) $$\bar{\nabla}\cdot\bar{A} = 0$$

We then have for (4.1)

$$(\lambda + 2G)\bar{\nabla}\left[(\nabla^2\Phi) + \bar{\nabla}\cdot(\bar{\nabla}\times\bar{A})\right] - G\left[\bar{\nabla}\times\bar{\nabla}\times(\bar{\nabla}\Phi + \bar{\nabla}\times\bar{A})\right] - \beta\bar{\nabla}T = 0$$

But since the curl of the gradient is zero and the divergence of the curl is zero, the above equation becomes simply,

(4.3) $$(\lambda + 2G)\bar{\nabla}(\nabla^2\Phi) - G(\bar{\nabla}\times\bar{\nabla}\times\bar{\nabla}\times\bar{A}) - \beta\bar{\nabla}T = 0$$

If we now take the curl of equation (4.3), again using the fact that curl grad = 0, we have

(4.4) $$\bar{\nabla}\times\bar{\nabla}\times\bar{\nabla}\times\bar{\nabla}\times\bar{A} = 0$$

But by means of relation (A),

$$\bar{\nabla}\times\bar{\nabla}\times(\bar{\nabla}\times\bar{\nabla}\times\bar{A}) = \bar{\nabla}[\bar{\nabla}\cdot(\bar{\nabla}\times\bar{\nabla}\times\bar{A})] - \nabla^2(\bar{\nabla}\times\bar{\nabla}\times\bar{A}) = 0$$

Again, div curl = 0, hence, by (A),

$$\nabla^2(\bar{\nabla}\times\bar{\nabla}\times\bar{A}) = \nabla^2[\bar{\nabla}(\bar{\nabla}\cdot\bar{A})] - \nabla^4\bar{A} = 0$$

But by (B), $(\bar{\nabla}\cdot\bar{A}) = 0$, therefore,

(4.5) $$\nabla^4\bar{A} = 0$$

We can find a solution of equation (4.5) by taking $\bar{A}$ such that $\nabla^2\bar{A} = \bar{B}$ where $\nabla^2\bar{B} = 0$.

Now

$$\bar{\nabla}\times\bar{V} = \bar{\nabla}\times\bar{\nabla}\times\bar{A} = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} = -\nabla^2\bar{A} = -\bar{B}$$

by (B). But in cylindrical coordinates with axial symmetry,

$$\bar{\nabla}\times\bar{V} = \frac{1}{r}\begin{vmatrix} \bar{1}_1 & \bar{1}_2 r & \bar{1}_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_1 & 0 & v_3 \end{vmatrix} = -\bar{1}_2\left(\frac{\partial v_3}{\partial r} - \frac{\partial v_1}{\partial z}\right) = -\bar{B}$$

But $$\bar{B} = \bar{1}_1\bar{B}_1 + \bar{1}_2\bar{B}_2 + \bar{1}_3\bar{B}_3 = \bar{1}_2\left(\frac{\partial v_3}{\partial r} - \frac{\partial v_1}{\partial z}\right)$$

Hence the components $B_1 = B_3 = 0$ and

(4.6) $$B_2 = \frac{\partial v_3}{\partial r} - \frac{\partial v_1}{\partial z}$$

Therefore $\nabla^2\bar{A} = \bar{1}_2 B_2$. But we can expand the left member of this equation as follows:

$$\nabla^2\bar{A} = \nabla^2(\bar{1}_1 A_1) + \nabla^2(\bar{1}_2 A_2) + \nabla^2(\bar{1}_3 A_3) =$$

$$= \bar{1}_1\nabla^2 A_1 + \frac{1}{r^2}\left[\bar{1}_1\frac{\partial^2 A_1}{\partial\theta^2} + 2\bar{1}_2\frac{\partial A_1}{\partial\theta} - \bar{1}_1 A_1\right] + \bar{1}_2\nabla^2 A_2 +$$

$$+ \frac{1}{r^2}\left[\bar{1}_2\frac{\partial^2 A_2}{\partial\theta^2} - 2\bar{1}_1\frac{\partial A_2}{\partial\theta} - \bar{1}_2 A_2\right] + \bar{1}_3\nabla^2 A_3$$

But all partials with respect to θ vanish because of axial symmetry, therefore

$$\nabla^2\bar{A} = \bar{1}_1\left[\nabla^2 A_1 - \frac{A_1}{r^2}\right] + \bar{1}_2\left[\nabla^2 A_2 - \frac{A_2}{r^2}\right] + \bar{1}_3\nabla^2 A_3 = \bar{1}_2 B_2$$

Equating components, we have

$$(4.7)\qquad \begin{cases} \nabla^2 A_1 - \dfrac{A_1}{r^2} = 0 \\ \nabla^2 A_2 - \dfrac{A_2}{r^2} = B_2 \\ \nabla^2 A_3 = 0 \end{cases}$$

But $\overline{B}$ was taken as a solution of the equation

$$\nabla^2 \overline{B} = 0$$

Hence, $\nabla^2 \overline{B} = \nabla^2(\mathbf{1}_2 B_2) = \mathbf{1}_2\left(\nabla^2 B_2 - \dfrac{B_2}{r^2}\right) = 0$

The equation $\nabla^2 B_2 - \dfrac{B_2}{r^2} = 0$ can be solved by the method of separation of variables. Setting $B_2(r,z) = R(r)\cdot Z(z)$ and dividing by RZ, we get

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} - \frac{1}{r^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Since r and z are independent variables, we may write

$$\frac{d^2Z}{dz^2} - b^2 Z = 0$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(b^2 - \frac{1}{r^2}\right)R = 0$$

The general solution is then

$$C_1 e^{-bz} J_1(br) + C_2 e^{bz} J_1(br) + C_3 e^{-bz} Y_1(br) + C_4 e^{bz} Y_1(br)$$

But we shall restrict ourselves to solutions which are finite for all positive z and r, taking

$$(4.8)\qquad B_2 = C(b)\, e^{-bz} J_1(br)$$

Finally, to evaluate the components of $\overline{\nabla}\times\overline{A}$, we write, remembering that partials with respect to θ vanish,

$$\overline{\nabla}\times\overline{A} = -\overline{1}_1 \frac{\partial A_2}{\partial z} - \overline{1}_2\left(\frac{\partial A_3}{\partial r} - \frac{\partial A_1}{\partial z}\right) + \overline{1}_3\left(\frac{\partial A_2}{\partial r} + \frac{A_2}{r}\right)$$

Now by (4.3), since $\overline{V}$ has no $\overline{1}_2$ component,

$$\frac{\partial A_3}{\partial r} = \frac{\partial A_1}{\partial z}$$

and

$$(4.9) \qquad \overline{\nabla}\times\overline{A} = -\overline{1}_1 \frac{\partial A_2}{\partial z} + \overline{1}_3\left(\frac{\partial A_2}{\partial r} + \frac{A_2}{r}\right)$$

Therefore the only component of $\overline{A}$ which is of interest is A_2, which from (4.7) and (4.8) satisfies the equation

$$(4.10) \qquad \nabla^2 A_2 - \frac{A_2}{r^2} = C(b)\, e^{-bz} J_1(br)$$

Again, we can use the method of separation of variables. Assume $A_2 = J_1(br)\, Z(z)$, then

$$\frac{1}{J_1}\frac{d^2J_1}{dr^2} + \frac{1}{rJ_1}\frac{dJ_1}{dr} - \frac{1}{r^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = C(b)\, e^{-bz}$$

But $$\frac{J_1''}{J_1} + \frac{1}{r}\frac{J_1'}{J_1} - \frac{1}{r^2} = -b^2$$

Therefore, $$Z'' - b^2 Z = C(b)\, e^{-bz}$$

And this equation has the solution

$$Z = C_1(b)\, e^{-bz} - \frac{C(b)}{2b}\, z\, e^{-bz}$$

Therefore,

$$(4.11) \qquad A_2 = J_1(br)\, e^{-bz}(C_1 + C_2 z)$$

If we set $\overline{W} = \text{curl}\,\overline{A}$, equation (4.2) becomes

$$\overline{V} = \nabla\Phi + \overline{W}$$

where by equations (4.9) and (4.11),

$$W_1 = -\frac{\partial A_2}{\partial z} = J_1(br)\, e^{-bz}\left[C_2\cdot(bz-1) + bC_1\right],$$

$$W_2 = 0,$$

and $$W_3 = \frac{\partial A_2}{\partial r} + \frac{A_2}{r} = bJ_0(br)\, e^{-bz}\left[C_1 + C_2 z\right]$$

Hence,

$$(4.12)\quad \begin{cases} v_1 = \dfrac{\partial\Phi}{\partial r} + W_1 = \dfrac{\partial\Phi}{\partial r} + J_1(br)\, e^{-bz}\left[C_2\cdot(bz-1) + bC_1\right] \\[2ex] v_3 = \dfrac{\partial\Phi}{\partial z} + W_3 = \dfrac{\partial\Phi}{\partial z} + bJ_0(br)\, e^{-bz}\left[C_1 + C_2 z\right] \end{cases}$$

In order to evaluate the scalar function Φ of equation (4.2), let us take the divergence of both sides of equation (4.3). Using the vector relation div curl = 0, we get

$$(\lambda + 2G)\nabla^4\Phi - \beta\nabla^2 T = 0$$

But by equation (2.6) we can write

$$(4.13)\qquad \nabla^4\Phi = \frac{\beta}{\kappa(\lambda + 2G)}\frac{\partial T}{\partial t}$$

Now the function T is a solution of the equation of heat conduction. Several forms of T have already been derived in section III and will be referred to later.

But for the solution of (4.13) it will be convenient to consider the fundamental solution of the heat equation which is obtained by the method of separation of variables.

Setting $T(r,z,t) = R(r)\, Z(z)\, \theta(t)$ in (2.6), we find that

$$T = D_0(b,c) \exp\left(-\kappa(b^2+c^2)t\right) b\, J_0(br) \sin(cz) \tag{C}$$

where D_0 is a function of b and c which depends on the boundary conditions.

Differentiating and substituting expression (C) in equation (4.13), we get

$$\nabla^4 \Phi = D\, b\, J_0(br) \sin(cz) \tag{4.14}$$

where $$D = -\beta(b^2+c^2) \frac{D_0(b,c)}{\lambda + 2G} \exp\left(-\kappa(b^2+c^2)t\right)$$

In order to solve equation (4.14), let us take

$$\Phi = b\, J_0(br) \cdot Z(z)$$

Then $$\nabla^2\left[\nabla^2(J_0(br)\cdot Z(z))\right] = D\, J_0(br) \sin(cz)$$

And $$\nabla^2(J_0(br)\cdot Z(z)) = Z\, J_0'' + Z \frac{J_0'}{r} + J_0\, Z'' =$$

$$= J_0\, Z \left(\frac{J_0''}{J_0} + \frac{1}{r}\frac{J_0'}{J_0} + \frac{Z''}{Z}\right)$$

But $J_0(br)$ is a solution of the differential equation

$$\frac{J_0''}{J_0} + \frac{1}{r}\frac{J_0'}{J_0} = -b^2$$

Therefore $$\nabla^2(J_0(br)\, Z(z)) = J_0(br)\,(Z'' - b^2 Z)$$

Hence,

$$\nabla^4\Phi = \nabla^2\left[J_0(br)\cdot(Z'' - b^2 Z)\right] = D\, J_0(br)\, \sin(cz)$$

Again $\quad \nabla^2\left[J_0(br)\cdot(Z'' - b^2 Z)\right] = J_0(br)\,(Z'' - b^2 Z)\cdot$

$$\cdot\left[\frac{J_0''}{J_0} + \frac{1}{r}\frac{J_0'}{J_0} + \frac{Z^{IV} - b^2 Z''}{Z'' - b^2 Z}\right] = J_0(br)\left[Z^{IV} - 2b^2 Z'' + b^4 Z\right]$$

The differential equation for $Z(z)$ thus reduces to

$$Z^{IV} - 2b^2 Z'' + b^4 Z = D \sin(cz)$$

Considering first the homogeneous equation,

$$Z^{IV} - 2b^2 Z'' + b^4 Z = 0,$$

since two pair of roots are equal, we obtain the solution,

$$Z = C_3 e^{-bz} + C_4 z\, e^{-bz}$$

which remains finite for all $z > 0$. To solve the non-homogeneous equation, we assume a solution of the form $Z(z) = C' \sin(cz)$. Hence,

$$C' \sin(cz)\left[c^4 + 2b^2c^2 + b^4\right] = D \sin(cz)$$

or $\quad$ (D) $\quad C' = \dfrac{D}{(b^2 + c^2)^2}$

Therefore the complete solution which remains finite in the solid $z > 0$, is

$$Z(z) = C_3 e^{-bz} + C_4 z\, e^{-bz} + \frac{D \sin(cz)}{(b^2 + c^2)^2}$$

Hence,

(4.15) $\quad \Phi = b\, J_0(br)\left[C_3 e^{-bz} + C_4 z\, e^{-bz} + C' \sin(cz)\right]$

But to obtain the values of the displacements in equation (4.12), we must have the components of grad(Φ).

$$\frac{\partial \Phi}{\partial r} = -b^2 J_1(br)\left[C_3 e^{-bz} + C_4 z e^{-bz} + C' \sin(cz)\right]$$

$$\frac{\partial \Phi}{\partial z} = b J_0(br)\left[e^{-bz}\left(C_4(1-bz) - b C_3\right) + cC' \cos(cz)\right]$$

Substituting these values of the components into equation (4.12), we obtain

$$v_1 = J_1(br)\left\{e^{-bz}\left[z(C_2 b - C_4 b^2) + C_1 b - C_2 - C_3 b^2\right] - C' b^2 \sin(zc)\right\}$$

$$v_3 = J_0(br)\left\{e^{-bz}\left[z(C_2 b - C_4 b^2) + C_1 b + C_4 b - C_3 b^2\right] + C' bc \cos(cz)\right\}$$

But by the transformation,

$$C_2 b - C_4 b^2 = h_1$$
$$C_1 b - C_2 - C_3 b^2 = h_2$$
$$C_1 b + C_4 b - C_3 b^2 = h_3,$$

we can reduce the unknown parameters to three. Hence,

(4.16) $$v_1 = J_1(br)\left\{e^{-bz}\left[h_1 z + h_2\right] - C' b^2 \sin(cz)\right\}$$

(4.17) $$v_3 = J_0(br)\left\{e^{-bz}\left[h_1 z + h_3\right] + C' bc \cos(cz)\right\}$$

In order to satisfy the boundary conditions and evaluate the unknowns, h_1, h_2, and h_3, we shall have need for several of the following expressions involving v_1, v_3, and their derivatives. First, from the boundary conditions (2.5), we have the equations:

(4.18) $$\frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z} = 0 \qquad \text{for} \quad z = 0$$

and

$$(4.19) \qquad \lambda\Delta + 2G\frac{\partial v_3}{\partial z} = \beta T \qquad \text{for} \quad z = 0$$

From equation (4.17)

$$\frac{\partial v_3}{\partial r} = -\,b\,J_1(br)\left\{ e^{-bz}(h_1 z + h_3) + C'bc\cos(cz)\right\}$$

and from (4.16)

$$\frac{\partial v_1}{\partial z} = J_1(br)\left\{ e^{-bz}\left[h_1(1-bz) - bh_2\right] - C'b^2c\cos(cz)\right\}$$

Hence,

$$\frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z} = J_1(br)\left\{ e^{-bz}\left[h_1(1-2bz) - b(h_2 + h_3)\right] - 2C'b^2c\cos(cz)\right\}$$

And from the boundary condition (4.18), this gives

$$(4.20) \qquad h_1 - bh_2 - bh_3 = 2C'b^2c$$

From the conditions (2.4), we have

$$(4.21) \qquad (\lambda + G)\frac{\partial\Delta}{\partial r} + G\left(\nabla^2 v_1 - \frac{v_1}{r^2}\right) - \beta\frac{\partial T}{\partial r} = 0, \quad z > 0$$

In order to obtain the expressions needed to evaluate (4.21), let us first derive a value for Δ from (4.16) and (4.17).

$$\text{Now} \qquad \Delta = \overline{\nabla}\cdot\overline{V} = \frac{\partial v_1}{\partial r} + \frac{v_1}{r} + \frac{\partial v_3}{\partial z}$$

From (4.16), we have

$$\frac{\partial v_1}{\partial r} = b\,J_1'(br)\left\{\ \text{"}\ \right\} = \left(b\,J_0(br) - \frac{J_1(br)}{r}\right)\left\{\ \text{"}\ \right\}$$

Hence,

$$\frac{\partial v_1}{\partial r} + \frac{v_1}{r} = bJ_0(br)\left\{ e^{-bz}\left[h_1 z + h_2\right] - C'b^2 \sin(cz)\right\}$$

From (4.17), we have

$$\frac{\partial v_3}{\partial z} = J_0(br)\left\{ e^{-bz}\left[h_1(1-bz) - bh_3\right] - C'bc^2 \sin(cz)\right\}$$

Therefore

$$(4.22) \quad \Delta = J_0(br)\left\{ e^{-bz}\left[h_1 + bh_2 - bh_3\right] - C'b(b^2+c^2) \sin(cz)\right\}$$

But by equation (C) on page 33

$$T = D_0(b,c) \exp\left(-\kappa(b^2+c^2)t\right) b\, J_0(br) \sin(cz)$$

Hence from (4.22)

$$(4.23) \quad (\lambda + G)\,\Delta - \beta T = J_0(br)\left\{ (\lambda+G)\, e^{-bz}\left[h_1 + b(h_2 - h_3)\right] + \right.$$

$$\left. - (\lambda+G)\, C'b(b^2+c^2) \sin(cz) + (\lambda+2G)\, C'b(b^2+c^2) \sin(cz)\right\} =$$

$$= J_0(br)\left\{ (\lambda+G)\, e^{-bz}\left[h_1 + b(h_2 - h_3)\right] + GC'b(b^2+c^2) \sin(cz)\right\}$$

We must next determine the value of the expression $\nabla^2 v_1 - \frac{v_1}{r^2}$

Now, $$\frac{\partial}{\partial r}\left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r}\right) = \frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} =$$

$$= -b^2 J_1(br)\left\{ e^{-bz}\left[h_1 z + h_2\right] - C' b^2 \sin(cz)\right\}$$

and $$\frac{\partial^2 v_1}{\partial z^2} = J_1(br)\left\{ e^{-bz}\left[-2bh_1 + b^2 h_1 z + b^2 h_2\right] + C' b^2 c^2 \sin(cz)\right\}$$

Therefore

$$(4.24) \quad \nabla^2 v_1 - \frac{v_1}{r^2} = -J_1(br)\left\{ e^{-bz}(2bh_1) \quad - \quad C' b^2 (b^2+c^2) \quad \sin(cz)\right\}$$

Differentiating (4.23) with respect to r,

$$(\lambda+G)\frac{\partial\Delta}{\partial r} - \beta\frac{\partial T}{\partial r} = -J_1(br)\left\{ (\lambda+G)e^{-bz}[bh_1 + b^2 h_2 - b^2 h_3] + \right.$$

$$\left. + \; G\, b^2 (b^2+c^2)\, C' \sin(cz)\right\}$$

Substituting this last equation and equation (4.24) in (4.21), we get

$$J_1(br)\left\{ G\, e^{-bz}(-2bh_1) + G\, C'\, b^2(b^2+c^2) \quad \sin(cz)\right\} =$$

$$= J_1(br)\left\{ (\lambda+G)\; e^{-bz}[bh_1 + b^2(h_2-h_3)] + G\, C'\, b^2(b^2+c^2) \sin(cz)\right\}$$

i.e. $\qquad - 2bh_1 G = (\lambda + G)\,[bh_1 + b^2(h_2 - h_3)]$ or

$$(4.25) \qquad (\lambda + 3G)\, h_1 + b(\lambda + G)\, h_2 - b\,(\lambda + G)\, h_3 = 0 \quad [26]$$

The remaining boundary condition (4.19) states that

$$\lambda\Delta + 2G\frac{\partial v_3}{\partial z} = \beta T \qquad \text{for} \quad z = 0$$

Substituting from (4.22) and (C) we get,

$$J_0(br)\left\{ e^{-bz}\lambda[h_1 + bh_2 - bh_3] - \lambda C'\, b\, (b^2+c^2) \sin(cz)\right\} +$$

$$+ \; J_0(br)\left\{ e^{-bz}\, 2G[h_1(1-bz) - bh_3] - 2G\, C'\, bc^2 \sin(cz)\right\} =$$

$$= J_0(br)\left\{-(\lambda + 2G)\, C'\, b(b^2+c^2) \quad \sin(cz)\right\}$$

Hence on the surface $z = 0$,

$$\lambda\,[h_1 + bh_2 - bh_3] + 2G[h_1 - bh_3] = 0, \qquad \text{or}$$

$$(4.26) \qquad (\lambda + 2G)\, h_1 + \lambda b h_2 - b(\lambda + 2G)\, h_3 = 0$$

We now have three equations in the three unknown parameters h_1, h_2, and h_3, viz.,

$$(4.20) \qquad h_1 \quad -bh_2 \quad -bh_3 = 2C'b^2c$$

$$(4.25) \qquad (\lambda + 3G)h_1 \quad +b(\lambda + G)h_2 \; -b(\lambda + G)h_3 = 0$$

$$(4.26) \qquad (\lambda + 2G)h_1 \quad +b\lambda h_2 - b(\lambda + 2G)h_3 = 0$$

This system is readily solvable by Cramer's Rule. For the determinant of coefficients we have

$$\Delta_c = \begin{vmatrix} 1 & -b & -b \\ \lambda + 3G & b(\lambda + G) & -b(\lambda + G) \\ \lambda + 2G & b\lambda & -b(\lambda + 2G) \end{vmatrix} = -4\, Gb^2(\lambda + G)$$

The determinant for h_1,

$$\Delta_{h_1} = \begin{vmatrix} 2C'b^2c & -b & -b \\ 0 & b(\lambda + G) & -b(\lambda + G) \\ 0 & b\lambda & -b(\lambda + 2G) \end{vmatrix} = -4\, C'\, b^4c\, G(\lambda+G)$$

The determinant for h_2,

$$\Delta_{h_2} = \begin{vmatrix} 1 & 2C'b^2c & -b \\ \lambda + 3G & 0 & -b(\lambda + G) \\ \lambda + 2G & 0 & -b(\lambda + 2G) \end{vmatrix} = 4\, C'\, b^3c\, G(\lambda+2G)$$

And the determinant for h_3,

$$\Delta_{h_3} = \begin{vmatrix} 1 & -b & 2C'b^2c \\ \lambda + 3G & b(\lambda + G) & 0 \\ \lambda + 2G & b\lambda & 0 \end{vmatrix} = -4C'b^3cG^2$$

Hence by Cramer's Rule

$$h_1 = \frac{\Delta_{h_1}}{\Delta_c} = C'b^2c, \qquad h_2 = \frac{\Delta_{h_2}}{\Delta_c} = -C'bc\left[\frac{\lambda+2G}{\lambda+G}\right], \qquad h_3 = \frac{\Delta_{h_3}}{\Delta_c} = C'bc\frac{G}{\lambda+G}$$

Substituting these values for h_1, h_2, and h_3 into (4.16) and (4.17), we obtain the following expressions for the displacements:

$$(4.27) \quad v_1 = C'J_1(br)\left\{bc\, e^{-bz}\left[bz - \frac{\lambda+2G}{\lambda+G}\right] - b^2 \sin(cz)\right\}$$

$$(4.28) \quad v_3 = C'J_0(br)\left\{bc\, e^{-bz}\left[bz + \frac{G}{\lambda+G}\right] + bc \cos(cz)\right\}$$

And substituting (4.22), we obtain

$$(4.29) \quad \Delta = C'J_0(br)\left\{-b^2c\, e^{-bz}\left[\frac{2G}{\lambda+G}\right] - b(b^2+c^2)\sin(cz)\right\}$$

Equations (4.27), (4.28), and (4.29) are the fundamental solutions to the boundary value problem (2.4), (2.5) which were obtained by taking for T the fundamental solution of the equation of conduction,

$$(C) \qquad T_f = D_0 \exp\left(-\kappa(b^2+c^2)t\right) b\, J_0(br)\sin(cz).$$

In order to transform the fundamental solution given by (C) into the solutions obtained in section III for the boundary value problem $T = H(r,z)$, $t = 0$; $T = F(r,t)$, $z = 0$, we shall take D_0 as the operator $D_1 + D_2$ where

$$(4.30) \quad D_1 = \frac{2}{\pi} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta)\; J_0(b\rho)\; \sin(c\zeta)\quad dc\; db\; d\rho\; d\zeta$$

and

$$(4.31) \quad D_2 = \frac{2\kappa}{\pi} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, F(\rho,\tau)\; \exp\left(\kappa(b^2+c^2)\tau\right)\; J_0(b\rho)\; c\; dc\; db\; d\rho\; d\tau$$

We then have

$$D_1\{T_f\} = T_1 \quad \text{as given by (3.17)}$$

and

$$D_2\{T_f\} = T_2 \quad \text{as given by (3.18)}$$

Similarly, in order to transform the fundamental solutions given by equations (4.27), (4.28), and (4.29) into the solutions which satisfy the boundary value problem, (2.4), (2.5) in which $T = T_1 + T_2$, as defined by equations (3.17) and (3.18), we shall take C' as the operator

$$C' = D_1\left[\frac{-\beta\exp\left(-\kappa(b^2+c^2)t\right)}{(\lambda+2G)(b^2+c^2)}\right] + D_2\left[\frac{-\beta\exp\left(-\kappa(b^2+c^2)t\right)}{(\lambda+2G)(b^2+c^2)}\right]$$

where D_1 and D_2 are given by (4.30) and (4.31). We thus obtain as the formal solution of the complete boundary value problem (2.4), (2.5), (2.6), (2.8),

$$(4.32) \quad \begin{cases} v_1 = v_1' + v_1'' \\ v_3 = v_3' + v_3'' \\ \Delta = \Delta' + \Delta'' \end{cases}$$

where

$$(4.32a) \quad v_1' = \frac{2\beta}{\pi(\lambda+2G)} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta)\; (b^2+c^2)^{-1}\; J_1(br)\; J_0(b\rho)\cdot$$

$$\exp\left(-\kappa(b^2+c^2)t\right) \sin(c\zeta) \left\{ bc\ e^{-bz}\left[\frac{\lambda+2G}{\lambda+G} - bz\right] + b^2\sin(cz)\right\} dc\,db\,d\rho\,d\zeta$$

$$(4.32b) \qquad v_1'' = \frac{2\kappa\beta}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho F(\rho,\tau)\ \frac{\exp\left(-\kappa(b^2+c^2)(t-\tau)\right)}{(b^2+c^2)}$$

$$J_1(br)\ J_0(b\rho)\ c\left\{bc\ e^{-bz}\left[\frac{\lambda+2G}{\lambda+G} - bz\right] + b^2\ \sin(cz)\right\} dc\ db\ d\rho\ d\tau$$

$$(4.32c) \qquad v_3' = \frac{-2\beta}{\pi(\lambda+2G)} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho H(\rho,\zeta)\ \frac{\exp\left(-\kappa(b^2+c^2)t\right)}{(b^2+c^2)}$$

$$J_0(br)\ J_0(b\rho)\ \sin(c\zeta)\left\{bc\ e^{-bz}\left[\frac{G}{\lambda+G} + bz\right] + bc\ \cos(cz)\right\} dc\,db\,d\rho\,d\zeta$$

$$(4.32d) \qquad v_3'' = \frac{-2\kappa\beta}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho F(\rho,\tau)\ \frac{\exp\left(-\kappa(b^2+c^2)(t-\tau)\right)}{(b^2+c^2)}$$

$$J_0(br)\ J_0(b\rho)\ c\left\{bc\ e^{-bz}\left[\frac{G}{\lambda+G} + bz\right] + bc\ \cos(cz)\right\} dc\ db\ d\rho\ d\tau$$

$$(4.32e) \qquad \Delta' = \frac{2\beta}{\pi(\lambda+2G)} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho H(\rho,\zeta)\ \frac{\exp\left(-\kappa(b^2+c^2)t\right)}{(b^2+c^2)}$$

$$J_0(br)\ J_0(b\rho)\ \sin(c\zeta)\ \left\{b^2c\ e^{-bz}\left(\frac{2G}{\lambda+G}\right) + b(b^2+c^2)\ \sin(cz)\right\} dc\,db\,d\rho\,d\zeta$$

$$(4.32f) \qquad \Delta'' = \frac{2\kappa\beta}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho F(\rho,\tau)\ \frac{\exp\left(-\kappa(b^2+c^2)(t-\tau)\right)}{(b^2+c^2)}$$

$$J_0(br)\ J_0(b\rho)\ c\left\{b^2c\ e^{-bz}\left(\frac{2G}{\lambda+G}\right) + b(b^2+c^2)\ \sin(cz)\right\} dc\ db\ d\rho\ d\tau$$

We must next verify that the expressions for the displacements given by the equations (4.32) actually do satisfy the boundary value problem (2.4), (2.5), where T is determined by (2.6), (2.8). To do this we must ascertain the conditions on the functions H and F under which the integrals converge.

We have first the equations (2.4), viz., for $z > 0$,

$$(\lambda + G)\frac{\partial\Delta}{\partial r} + G\left(\nabla^2 v_1 - \frac{v_1}{r^2}\right) - \beta\frac{\partial T}{\partial r} = 0$$

$$(\lambda + G)\frac{\partial\Delta}{\partial z} + G\nabla^2 v_2 - \beta\frac{\partial T}{\partial z} = 0$$

Two things must be shown:

(I) That the integrands formally satisfy (2.4)

(II) That the necessary differentiations under the integral signs are valid. This depends on the convergence of the germane integrals.

In order to establish (I), consider the equations (4.27), (4.28), and (4.29), the expressions for the portions of the integrands containing r and z. We shall formally differentiate these expressions and list herewith all of the equations with which we shall have occasion to deal.

From (4.27)

$$\frac{\partial v_1}{\partial r} = C' \, b \, J_1'(br)\left\{ \;\; " \;\; \right\} = \left[C' \, b \, J_0(br) - \frac{C' \, J_1(br)}{r}\right]\left\{ \;\; " \;\; \right\}$$

Hence,

$$\text{(4.33a)} \qquad \frac{\partial v_1}{\partial r} + \frac{v_1}{r} = C' \, b \, J_0(br)\left\{ bc\, e^{-bz}\left[bz - \frac{\lambda+2G}{\lambda+G}\right] - b^2 \sin(cz)\right\}$$

Differentiating (4.33a) with respect to r,

$$(4.33b)\quad \frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} = -C'\, b\, J_1(br)\cdot$$

$$\cdot\left\{bc\, e^{-bz}\left[bz - \frac{\lambda+2G}{\lambda+G}\right] - b^2 \sin(cz)\right\}$$

$$(4.33c)\quad \frac{\partial v_1}{\partial z} = C'\, J_1(br)\left\{b^2 c\, e^{-bz}\left[\frac{2\lambda+3G}{\lambda+G} - bz\right] - b^2 c\cos(cz)\right\}$$

$$(4.33d)\quad \frac{\partial^2 v_1}{\partial z^2} = C'\, J_1(br)\left\{b^3 c\, e^{-bz}\left[bz - \frac{3\lambda+4G}{\lambda+G}\right] + b^2 c^2 \sin(cz)\right\}$$

Hence, adding (4.33b) and (4.33d)

$$(4.33e)\quad \nabla^2 v_1 - \frac{v_1}{r^2} = C'\, J_1(br)\left\{b^3 c\, e^{-bz}(-2) + b^2 (b^2+c^2)\sin(cz)\right\}$$

From (4.28)

$$(4.33f)\quad \frac{\partial v_3}{\partial r} = -\,bC'\, J_1(br)\left\{bc\, e^{-bz}\left[bz + \frac{G}{\lambda+G}\right] + bc\cos(cz)\right\}$$

$$\frac{\partial^2 v_3}{\partial r^2} = -\,b^2 C'\, J_1'(br)\left\{\quad " \quad\right\} =$$

$$= \left(-b^2 C'\, J_0(br) + \frac{bC' J_1(br)}{r}\right)\left\{\quad " \quad\right\}$$

Hence,

$$(4.33g)\quad \frac{\partial^2 v_3}{\partial r^2} + \frac{1}{r}\frac{\partial v_3}{\partial r} = -\,b^2 C'\, J_0(br)\left\{bc\, e^{-bz}\left[bz + \frac{G}{\lambda+G}\right] + \right.$$

$$\left. + bc\cos(cz)\right\}$$

Again from (4.28)

$$(4.33h)\quad \frac{\partial v_3}{\partial z} = C'\, J_0(br)\left\{b^2 c\, e^{-bz}\left[\frac{\lambda}{\lambda+G} - bz\right] - bc^2 \sin(cz)\right\}$$

(4.33i) $$\frac{\partial^2 v_3}{\partial z^2} = C' J_0(br)\left\{ b^3 c\, e^{-bz}\left[bz - \frac{2\lambda+G}{\lambda+G}\right] - bc^3 \cos(cz)\right\}$$

Hence, adding (4.33g) and (4.33i),

(4.33j) $$\nabla^2 v_3 = C' J_0(br)\left\{ b^3 c\, e^{-bz}(-2) - bc(b^2+c^2)\cos(cz)\right\}$$

Now from equation (C) (page 32) and (D) (page 34),

(4.33k) $$\beta T = -C' J_0(br)\left\{ (\lambda+2G)\, b(b^2+c^2)\sin(cz)\right\}$$

Hence, from (4.29)

(4.33l) $$(\lambda+G)\Delta - \beta T = C' J_0(br)\left\{ -2Gb^2 c\, e^{-bz} + Gb(b^2+c^2)\sin(cz)\right\}$$

Differentiating with respect to r,

(4.33m) $$(\lambda+G)\frac{\partial \Delta}{\partial r} - \beta\frac{\partial T}{\partial r} = -bC' J_1(br)\left\{ -2Gb^2 c\, e^{-bz} + \right.$$

$$\left. + G\, b(b^2+c^2)\sin(cz)\right\}$$

and differentiating with respect to z,

(4.33n) $$(\lambda+G)\frac{\partial \Delta}{\partial z} - \beta\frac{\partial T}{\partial z} = C' J_0(br)\left\{ 2Gb^3 c\, e^{-bz} + Gbc(b^2+c^2)\cos(cz)\right\}$$

Adding (4.33m) to the product of G and (4.33e), we get the left member of the first equation of (2.4), viz,

$$G\, C' J_1(br)\left\{ 2b^3 c\, e^{-bz} - b^2(b^2+c^2)\sin(cz)\right\} +$$

$$+ G\, C' J_1(br)\left\{ -2b^3 c\, e^{-bz} + b^2(b^2+c^2)\sin(cz)\right\}$$

But this expression is identically zero, thus satisfying the first equation.

Adding (4.33n) to the product of G and (4.33j), we get the left member of the second equation of (2.4):

$$G\ C'\ J_0(br)\left\{2\ b^3c\ e^{-bz} + bc(b^2+c^2)\cos(cz)\right\} +$$

$$G\ C'\ J_0(br)\left\{-2b^3c\ e^{-bz} - bc(b^2+c^2)\cos(cz)\right\}$$

This expression is also identically zero. We have thus established property (I), that the integrands formally satisfy equations (2.4).

As to property (II), in order to justify the necessary differentiations under the integral signs, we must restrict H(r,z) and F(r,t) to be continuous. This assures that the integrands will be continuous. We must then find the conditions under which the integrals v', v", v', v", together with their partials $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z}$, $\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial z^2}$, $\frac{\partial^2}{\partial r \partial z}$, converge uniformly. [27].

It will not be necessary to get a separate condition for convergence for each of the above integrals, but will be sufficient to consider the integrals in which the highest powers of b and c appear. The convergence of these integrals will assure convergence of integrals involving lower powers since the appearance of the factor $\frac{bc}{b^2+c^2}$ or $\frac{b\sin(cz)}{b^2+c^2}$ in all of the expressions assures convergence near the origin.

The highest powers of b and c appearing in terms containing the function e^{-bz} occur in the expressions (4.33b), (4.33d), and (4.33i).

We must therefore consider the convergence of the integrals,

$$P_1 = \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta)\ \exp\Big(-\kappa(b^2+c^2)t\Big)\ J_i(br)\ J_0(b\rho)\cdot$$

$$\cdot \sin(c\zeta)\ \frac{b^4 c\ e^{-bz}}{b^2+c^2}\ dc\ db\ d\rho\ d\zeta$$

and

$$P_2 = \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, F(\rho,\tau)\ \exp\Big(-\kappa(b^2+c^2)(t-\tau)\Big)\ J_i(br)\ J_0(b\rho)\cdot$$

$$\cdot \frac{b^4 c^2 e^{-bz}}{b^2+c^2}\ dc\ db\ d\rho\ d\tau$$

where $i = 0$ or 1.

The highest powers of b and c appearing in terms involving the functions $\sin(cz)$ and $\cos(cz)$ occur in the expressions (4.33d), (4.33e), (4.33j), and (4.33n). We must then also consider the convergence of the integrals,

$$P_3 = \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta)\ \exp\Big(-\kappa(b^2+c^2)t\Big)\ J_i(br)\ J_0(b\rho)\cdot$$

$$\cdot \sin(c\zeta)\ \frac{b^n c^m}{b^2+c^2} \begin{Bmatrix}\sin(cz)\\ \cos(cz)\end{Bmatrix} dc\ db\ d\rho\ d\zeta$$

and

$$P_4 = \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, F(\rho,\tau)\ \exp\Big(-\kappa(b^2+c^2)(t-\tau)\Big)\ J_i(br)\ J_0(b\rho)\cdot$$

$$\cdot \frac{c\cdot b^n c^m}{b^2+c^2} \begin{Bmatrix}\sin(cz)\\ \cos(cz)\end{Bmatrix} dc\ db\ d\rho\, d\tau$$

where $i = 0$ or 1 and $n + m = 4$ according to the schedule

$$n = 1,\ 2,\ 3,\ 4$$

$$m = 3,\ 2,\ 1,\ 0$$

Now $|P_1| \leq \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\,|H|\, \exp\left(-\kappa(b^2+c^2)t\right) \frac{bc}{b^2+c^2}\; b^3 e^{-bz}.$

$\cdot\,|J_1(br)|\cdot|J_0(b\rho)|\cdot\;|\sin(c\zeta)|\quad dc\; db\; d\rho\; d\zeta$

But $\frac{bc}{b^2+c^2} \leq 1,\quad e^{-bz} \leq 1,\quad |\sin(c\zeta)| \leq 1,\quad |J_1(br)| \leq 1,$

and we can find a constant a_1, such that $|J_0(b\rho)| \leq \frac{a_1}{\sqrt{b\rho}}$

Hence,

$|P_1| \leq a_1 \int\int\int\int \sqrt{\rho}\; |H|\; \exp\left(-\kappa(b^2+c^2)t\right)\; b^{5/2}\; dc\; db\; d\rho\; d\zeta = M_1$

Since either the combination $b^m c^n$ with $n > 1$, or $b^4\sin(cz)$ appears in each of the various expressions occurring in P_3, by means of the bound $|\sin(cz)| \leq cz$, we can use the same inequalities that were used for P_1. Further because of the symmetry of the exponential function in M_1, it is immaterial whether we use b or c in the integrand. Therefore, covering all of the cases,

$$|P_3| \leq M_1$$

But
$$\int_{c=0}^{\infty} \exp\left(-\kappa c^2 t\right)\, dc = \frac{\sqrt{\pi}}{2\sqrt{\kappa t}}$$

and
$$\int_{b=0}^{\infty} b^{5/2} e^{-\kappa b^2 t}\, db = \frac{1}{2\kappa^{7/4} t^{7/4}} \int_{x=0}^{\infty} x^{3/4} e^{-x}\, dx = \frac{\Gamma(7/4)}{2\kappa^{7/4} t^{7/4}} = \frac{a_2}{t^{7/4}}$$

where a_2 is a constant.

Therefore
$$M_1 \leq \frac{a_2}{t^{9/4}} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} \sqrt{\rho}\; |H(\rho,\zeta)|\; d\rho\; d\zeta$$

Hence P_1 and P_3 converge absolutely and uniformly in r and z for all $t > 0$, provided that the integral,

$$\text{(E)} \qquad \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} \sqrt{\rho}\ |H(\rho,\zeta)|\ d\rho\ d\zeta \qquad \text{converges}$$

Returning to the integral P_2, we may write it as the sum of two integrals by means of the equality,

$$\frac{b^4 c^2 e^{-bz}}{b^2+c^2} = b^4 e^{-bz} - \frac{b^6 e^{-bz}}{b^2+c^2}$$

Namely, $\qquad P_2 = P_{2,1} - P_{2,2}$ where

$$P_{2,1} = \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\ F(\rho,\tau)\ \exp\Big(-\kappa(b^2+c^2)(t-\tau)\Big)\ J_1(br)\ J_0(b\rho)\ \cdot$$
$$\cdot\ b^4 e^{-bz}\ dc\ db\ d\rho\ d\tau$$

and

$$P_{2,2} = \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\ F(\rho,\tau)\ \exp\Big(-\kappa(b^2+c^2)(t-\tau)\Big)\ J_1(br)\ J_0(b\rho)\ \cdot$$
$$\cdot\ \frac{b^6 e^{-bz}}{b^2+c^2}\ dc\ db\ d\rho\ d\tau$$

But $$\int_{c=0}^{\infty} \exp\Big(-\kappa(t-\tau)c^2\Big)\ dc = \frac{\sqrt{\pi}}{2\sqrt{(\kappa(t-\tau))}},$$

Hence

$$P_{2,1} = \frac{\sqrt{\pi}}{2\sqrt{\kappa}} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty} \rho\ \frac{F(\rho,\tau)}{\sqrt{(t-\tau)}} \exp\Big(-\kappa b^2(t-\tau)\Big)\ b^4 e^{-bz}\ J_1(br) J_0(b\rho) db\, d\rho\, d\tau$$

and since $\Big|\exp\Big(-\kappa b^2(t-\tau)\Big)\Big| \le 1$, and $|J_1(bx)| \le 1$,

$$|P_{2,1}| \le \frac{\sqrt{\pi}}{2\sqrt{\kappa}} \left| \int_{\tau=0}^{t}\int_{\rho=0}^{\infty} \rho\ \frac{F(\rho,\tau)}{\sqrt{(t-\tau)}}\ d\rho\ d\tau \right| \cdot \left| \int_{b=0}^{\infty} b^4\ e^{-bz} db \right|$$

But $\int_0^{\infty} b^4 e^{-bz} db$ converges for all $z > 0$, therefore $P_{2,1}$ converges for $z > 0$ whenever

$$\text{(F)} \qquad \int_{\tau=0}^{t}\int_{\rho=0}^{\infty} \rho\ |F(\rho,\tau)|\ (t-\tau)^{-1/2}\ d\rho\ d\tau \qquad \text{converges.}$$

In the integral $P_{2,2}$ we may write, using the value,

$$\int_{c=0}^{\infty} \frac{\exp[-\kappa c^2(t-\tau)]}{b^2+c^2}\, dc = \frac{\sqrt{\pi}}{b} \exp[\kappa b^2(t-\tau)] \int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\mu^2} d\mu$$

$$P_{2,2} = \sqrt{\pi} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty} \rho\, F(\rho,\tau)\; J_i(br)\; J_0(b\rho)\; b^5 e^{-bz} \int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\mu^2} d\mu \;\; db\, d\rho\, d\tau$$

Hence, by means of the inequalities,

$$|J_i(bx)| \leq 1 \qquad \int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\mu^2} d\mu \leq \frac{\sqrt{\pi}}{2}$$

$$|P_{2,2}| \leq \frac{\pi}{2} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty} \rho\, |F(\rho,\tau)|\; d\rho\, d\tau \int_{b=0}^{\infty} b^5 e^{-bz}\, db$$

It follows that $P_{2,2}$ will also converge whenever condition (F) holds and z is greater than zero.

In considering the convergence of the integral P_4,

$$\int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, F(\rho,\tau)\; \exp\Big(-\kappa(b^2+c^2)(t-\tau)\Big)\; J_i(br)\; J_0(b\rho)\,\cdot$$

$$\cdot\; \frac{cb^n c^m}{b^2+c^2} \begin{Bmatrix} \sin(cz) \\ \cos(cz) \end{Bmatrix} dc\, db\, d\rho\, d\tau$$

let us break it up as follows,

$$P_4 = \int_{\tau=0}^{t-\varepsilon}\int\int\int\{\;\;"\;\;\}\, dc\, db\, d\rho\, d\tau + \int_{\tau=t-\varepsilon}^{t}\int\int\int\{\;\;"\;\;\}\, dc\, db\, d\rho\, d\tau$$

where $\varepsilon > 0$.

Now the first integral will converge absolutely provided that

$$\text{(G)} \qquad \int_{\rho=0}^{\infty} \rho\, |F(\rho,\tau)|\; d\rho \leq M \qquad \text{for all } \tau$$

where M is a constant.

For we have that,

$$\left| \int_{\tau=0}^{t-\varepsilon}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty}\{ \quad " \quad \} dc\, db\, d\rho\, d\tau \right| \leq M \int_{\tau=0}^{t-\varepsilon}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \exp\left(-\kappa(b^2+c^2)(t-\tau)\right) \cdot$$

$$\cdot |J_1(br)| \cdot |J_0(b\rho)| \frac{cb^n c^m}{b^2+c^2} \begin{Bmatrix} |\sin(cz)| \\ |\cos(cz)| \end{Bmatrix} dc\, db\, d\tau$$

$$\leq \frac{M}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} \left[\exp\left(-\kappa(b^2+c^2)\varepsilon\right) + \exp\left(-\kappa(b^2+c^2)t\right)\right] \cdot$$

$$\cdot \frac{cb^n c^m}{(b^2+c^2)^2} \begin{Bmatrix} |\sin(cz)| \\ |\cos(cz)| \end{Bmatrix} dc\, db$$

But the combination $b^4\cos(cz)$ does not occur, so by means of the inequalities,

$$|\sin(cz)| \leq cz, \qquad |\cos(cz)| \leq 1, \qquad \frac{b^2c^2}{(b^2+c^2)^2} \leq 1$$

we have,

$$\left| \int_{\tau=0}^{t-\varepsilon} \{ \quad " \quad \} d\tau \right| \leq \frac{M}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} \left[\exp\left(-\kappa(b^2+c^2)\varepsilon\right) + \right.$$

$$\left. + \exp\left(-\kappa(b^2+c^2)t\right)\right] b^{n-2} c^{m-1} dc\, db$$

which converges absolutely and uniformly with respect to r and z. If b does not appear to at least the second power, we can still avoid the difficulties of convergence near the origin by using the identity,

$$\frac{bc^n}{b^2+c^2} = bc^{n-2} - \frac{b^3c^{n-2}}{b^2+c^2}$$

Before considering the convergence of the second integral,

$$\int_{\tau=t-\varepsilon}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty}\{ \quad " \quad \} dc\, db\, d\rho\, d\tau,$$

let us list all of the cases which occur in the equations (2.4). Only the parameters that vary will be listed below and the expressions will be simplified wherever possible, in order to facilitate the initial integration with respect to c.

(1) $J_1 \dfrac{cb^2}{b^2+c^2} \sin(cz)$

(2) $J_0 \dfrac{bc^2}{b^2+c^2} \cos(cz) = J_0\, b \cos(cz) - J_0 \dfrac{b^3}{b^2+c^2} \cos(cz)$

(3) $J_0\, bc \sin(cz)$

(4) $J_0 \dfrac{b^3 c}{b^2+c^2} \sin(cz)$

(5) $J_1 \dfrac{b^2 c^2}{b^2+c^2} \cos(cz) = J_1\, b^2 \cos(cz) - J_1 \dfrac{b^4}{b^2+c^2} \cos(cz)$

(6) $J_1 \dfrac{b^2 c^3}{b^2+c^2} \sin(cz) = J_1\, b^2 c \sin(cz) - J_1 \dfrac{b^4 c}{b^2+c^2} \sin(cz)$

(7) $J_1\, b^2 c \sin(cz)$

(8) $J_0 \dfrac{b^3 c^2}{b^2+c^2} \cos(cz) = J_0\, b^3 \cos(cz) - J_0 \dfrac{b^5}{b^2+c^2} \cos(cz)$

(9) $J_0 \dfrac{bc^3}{b^2+c^2} \sin(cz) = J_0\, bc \sin(cz) - J_0 \dfrac{b^3 c}{b^2+c^2} \sin(cz)$

(10) $J_0 \dfrac{bc^4}{b^2+c^2} \cos(cz) = J_0\, bc^2 \cos(cz) - J_0\, b^3 \cos(cz) +$

$$+ J_0 \frac{b^5}{b^2+c^2} \cos(cz)$$

(11) $J_0\, bc^2 \cos(cz)$

With respect to the variable c only five different forms appear:

$$\text{(a)} \qquad \int_{c=0}^{\infty} \exp[-\kappa c^2(t-\tau)] \; \frac{\cos(cz)}{b^2+c^2} \; dc$$

$$\text{(b)} \qquad \int_{c=0}^{\infty} \exp[-\kappa c^2(t-\tau)] \; c \; \frac{\sin(cz)}{b^2+c^2} \; dc$$

$$\text{(c)} \qquad \int_{c=0}^{\infty} \exp[-\kappa c^2(t-\tau)] \; \cos(cz) \; dc$$

$$\text{(d)} \qquad \int_{c=0}^{\infty} \exp[-\kappa c^2(t-\tau)] \; c \; \sin(cz) \; dc$$

$$\text{(e)} \qquad \int_{c=0}^{\infty} \exp[-\kappa c^2(t-\tau)] \; c \; \cos(cz) \; dc$$

Integrating these expressions with respect to c, $(t-\tau)=\mu$

$$\text{(a)} \; = \; \frac{\kappa\sqrt{\pi}}{2} \int_{\lambda=0}^{\infty} e^{-\kappa b^2\lambda} \exp\left(-\frac{z^2}{4\kappa(\mu+\lambda)}\right) \; [\kappa(\mu+\lambda)]^{-\frac{1}{2}} \, d\lambda$$

$$\text{(b)} \; = \; \frac{z\kappa\sqrt{\pi}}{4} \int_{\lambda=0}^{\infty} e^{-\kappa b^2\lambda} \exp\left(-\frac{z^2}{4\kappa(\mu+\lambda)}\right) \; [\kappa(\mu+\lambda)]^{-3/2} \, d\lambda$$

$$\text{(c)} \; = \; \frac{\sqrt{\pi}}{2} \; (\kappa\mu)^{-\frac{1}{2}} \exp\left(-\frac{z^2}{4\kappa\mu}\right)$$

$$\text{(d)} \; = \; \frac{\sqrt{\pi}}{4} \; z(\kappa\mu)^{-\frac{3}{2}} \exp\left(-\frac{z^2}{4\kappa\mu}\right)$$

$$\text{(e)} \; = \; \frac{\sqrt{\pi}}{4} \; (\kappa\mu)^{-\frac{3}{2}} \exp\left(-\frac{z^2}{4\kappa\mu}\right) \; - \; \frac{\sqrt{\pi}}{8} \; z^2(\kappa\mu)^{-\frac{5}{2}} \exp\left(-\frac{z^2}{4\kappa\mu}\right)$$

Now, setting $t-\tau=\mu$,

$$\int_{\tau=t-\varepsilon}^{t} \{ \quad " \quad \} \, d\tau \; = \; \int_{\mu=0}^{\varepsilon}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho \, F(\rho, t-\mu) \; \exp\left(-\kappa(b^2+c^2)\mu\right) \; J_1(br) \; J_0(b\rho) \cdot$$

$$\cdot \; \frac{cb^n c^m}{b^2+c^2} \begin{Bmatrix} \sin(cz) \\ \cos(cz) \end{Bmatrix} dc \; db \; d\rho \; d\mu$$

Substituting the expressions (a),..., (e) in the above integral and taking absolute values, we have for the cases of the highest powers of b which occur in (2.4):

$$\text{(I)} \qquad \left| \int_0^{\varepsilon} \right| \leq \int_{\mu=0}^{\varepsilon}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty} \rho\, |F|\; b^{6} e^{-\kappa b^{2}\mu} (a)\; db\; d\rho\; d\mu$$

$$\text{(II)} \qquad \left| \int_0^{\varepsilon} \right| \leq \int_{\mu=0}^{\varepsilon}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty} \rho\, |F|\; b^{4} e^{-\kappa b^{2}\mu} (b)\; db\; d\rho\; d\mu$$

etc.

Since all of the functions appearing in the integrands are positive, we may integrate in any order. Let us first consider the integral (I).

The integral with respect to b is:

$$\int_{b=0}^{\infty} b^{6} \exp\left(-\kappa b^{2}(\mu+\lambda)\right) db$$

which is equal to $[\kappa(\mu+\lambda)]^{-3}$

Hence, assuming that condition (G) holds,

$$\text{(I)} \leq \frac{\kappa M\sqrt{\pi}}{2} \int_{\mu=0}^{\varepsilon}\int_{\lambda=0}^{\infty} [\kappa(\lambda+\mu)]^{-7/2} \exp\left(-\frac{z^{2}}{4\kappa(\lambda+\mu)}\right) d\lambda\; d\mu$$

By means of the transformation, $s^{2} = \dfrac{z^{2}}{4\kappa(\lambda+\mu)}$

$$\text{(I)} \leq \frac{2^{5}\kappa M\sqrt{\pi}}{z^{5}} \left| \int_{\mu=0}^{\varepsilon}\int_{s=0}^{\frac{z}{2\sqrt{\kappa\mu}}} s^{4} e^{-s^{2}}\, ds\, d\mu \right| =$$

$$= \frac{2^{5}\kappa M\sqrt{\pi}}{z^{5}} \left| \varepsilon \int_{0}^{\frac{z}{2\sqrt{\kappa\varepsilon}}} s^{4} e^{-s^{2}}\, ds + \frac{z^{5}}{2^{6}\kappa^{5/2}} \int_{\mu=0}^{\varepsilon} \mu^{-5/2} \exp\left(-\frac{z^{2}}{4\kappa\mu}\right) d\mu \right|$$

But $\displaystyle\int_{0}^{\frac{z}{2\sqrt{\kappa\varepsilon}}} s^{4} e^{-s^{2}}\, ds \leq \int_{0}^{\infty} s^{4} e^{-s^{2}}\, ds = \frac{3\sqrt{\pi}}{8}$ for all ε

and $\displaystyle\int_{\mu=0}^{\varepsilon} \mu^{-5/2} \exp\left(-\frac{z^2}{4\kappa\mu}\right) d\mu = \varepsilon\,(\theta\varepsilon)^{-5/2} \exp\left(-\frac{z^2}{4\kappa\theta\varepsilon}\right)$

where $0 < \theta < 1$. We therefore have that

$$\lim_{\varepsilon \to 0} (I) = 0$$

The treatment of the remaining integrals is similar to the above, and all can be shown to go to zero with ε. It follows then that P_4 will converge if conditions (F) and (G) hold. But (F) reduces to (G). For if (G) holds, (F) becomes

$$\int_{\tau=0}^{t} \frac{M}{(t-\tau)^{1/2}}\, d\tau = 2M\sqrt{t}$$

which exists for all t.

We may therefore conclude that the expressions (4.32) satisfy the differential equations (2.4) for $z > 0$, $t > 0$, whenever conditions (E) and (G) hold.

It remains to investigate the conditions under which the values of (4.32) satisfy the boundary conditions (4.18) and (4.19).

First let us consider (4.18), $\dfrac{\partial v_3}{\partial r} + \dfrac{\partial v_1}{\partial z} = 0$ for $z = 0$.

From (4.33f) and (4.33c) we have

$$\frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z} = A_1 + A_2$$

where

(4.34a) $$A_1 = \frac{4\beta}{\pi(\lambda+2G)} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta) \exp\left(-\kappa(b^2+c^2)t\right) \cdot$$

$$\cdot \frac{b^2 c}{b^2+c^2} J_1(br)\, J_0(b\rho)\, \sin(c\zeta) \left\{ e^{-bz}(1-bz) - \cos(cz) \right\} dc\, db\, d\rho\, d\zeta$$

and

$$(4.34b) \qquad A_2 = \frac{4\beta\kappa}{\pi(\lambda+2G)} \int\limits_{\tau=0}^{t}\int\limits_{\rho=0}^{\infty}\int\limits_{b=0}^{\infty}\int\limits_{c=0}^{\infty} \rho\, F(\rho,\tau) \exp\left(-\kappa(b^2+c^2)(t-\tau)\right) \cdot$$

$$\cdot \frac{b^2c^2}{b^2+c^2} J_1(br)\, J_0(b\rho) \left\{ e^{-bz}(1-bz) - \cos(cz) \right\} dc\, db\, d\rho\, d\tau$$

We must find the conditions on H and F under which

$$\lim_{z\to 0} A_1 = 0 \quad \text{and} \quad \lim_{z\to 0} A_2 = 0$$

First let us consider $\lim\limits_{z\to 0} A_1$. It will be useful to consider this limit in two parts:

$$\lim_{z\to 0} \left\{ -z\ (\text{const.}) \iiiint \rho\, H \exp\left(-\kappa(b^2+c^2)t\right) \frac{b^3c}{b^2+c^2} J_1\, J_0 \cdot \right.$$

$$\left. \cdot \sin(c\zeta)\, e^{-bz} dc\, db\, d\rho\, d\zeta \right\} +$$

$$+ \lim_{z\to 0} \left\{ (\text{const.}) \iiiint \rho\, H \exp\left(-\kappa(b^2+c^2)t\right) \frac{b^2c}{b^2+c^2} J_1\, J_0 \cdot \right.$$

$$\left. \cdot \sin(c\zeta) \left[e^{-bz} - \cos(cz) \right] dc\, db\, d\rho\, d\zeta \right\}$$

The first integral will converge when the integral P_1 converges. Therefore when condition (E) obtains, the first term will go to zero with z.

The expression $\left[e^{-bz} - \cos(cz)\right]$ occurs in the second integral.

Now, $$\left| e^{-bz} - \cos(cz) \right| \leq 2$$

and $$e^{-bz} - \cos(cz) = 1 - bz + \ldots - 1 + \frac{(cz)}{2} - \ldots = z\, f(b,c,z)$$

We may thus take $\left| e^{-bz} - \cos(cz) \right| \leq 2z$. Using this inequality, we have in a similar manner, the second integral converging when (E) holds, and the second term going to zero with z. Therefore $\lim_{z \to 0} A_1 = 0$.

We can treat the integral A_2 in the same manner as A_1. The integrals $\int_0^{t-\varepsilon} \{\cos(cz),\ \sin(cz),\ \text{and } e^{-bz}\}$ converge absolutely and uniformly with respect to r and z and

$$\lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t} \{\cos(cz) \text{ and } \sin(cz)\} = 0$$

provided that condition (G) holds. The treatment of lim A_1 can then be applied in the case of A_2 provided we can show that the integral of the exponential function also tends to zero with ε, that is

$$\lim_{\varepsilon \to 0} J = 0$$

where

$$J = \int_{\tau=t-\varepsilon}^{t} \int_{\rho=0}^{\infty} \int_{b=0}^{\infty} \int_{c=0}^{\infty} \rho\, F \exp\left(-\kappa(b^2+c^2)(t-\tau)\right) \frac{b^3 c^2}{b^2+c^2}\quad J_1(br)\ J_0(b\rho)\cdot$$

$$\cdot\, e^{-bz}\, dc\, db\, d\rho\, d\tau$$

Let us write

$$\frac{b^3 c^2}{b^2+c^2} = b^3 - \frac{b^5}{b^2+c^2}$$

Then if condition (G) holds,

$$\lim_{\varepsilon \to 0} |J| \leq \lim_{\varepsilon \to 0} M \int_{t-\varepsilon}^{t} \int_{b=0}^{\infty} \int_{c=0}^{\infty} \exp\left(-\kappa(b^2+c^2)(t-\tau)\right)\ e^{-bz} b^3 \cdot$$

$$\cdot \left| 1 - \frac{b^2}{b^2+c^2} \right| dc\, db\, d\tau$$

But $$\int_{c=0}^{\infty} \exp[-\kappa c^2 (t-\tau)]\ dc = \frac{\sqrt{\pi}}{2\sqrt{(\kappa(t-\tau))}}$$

and

$$b^2 \int_{c=0}^{\infty} \frac{\exp[-\kappa c^2(t-\tau)]}{b^2+c^2}\, dc = b\sqrt{\pi}\; \exp[\kappa b^2(t-\tau)] \int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\lambda^2} d\lambda$$

Therefore,

$$\lim_{\varepsilon\to 0} |J| \leq \lim_{\varepsilon\to 0} \sqrt{\pi}\, M \int_{t-\varepsilon}^{t}\int_{b=0}^{\infty} e^{-bz} b^3 \left[\frac{\exp[-\kappa b^2(t-\tau)]}{2\sqrt{(\kappa(t-\tau))}} + b\int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\lambda^2} d\lambda\right] db\, d\tau$$

Now $\displaystyle\int_{\tau=t-\varepsilon}^{t}\left[\frac{\exp[-\kappa b^2(t-\tau)]}{2\sqrt{(\kappa(t-\tau))}} + b\int_{b\sqrt{\kappa(t-\tau)}}^{\infty} e^{-\lambda^2} d\lambda\right] d\tau =$

$$= \int_{\mu=0}^{\varepsilon}\left[\frac{\exp(-\kappa b^2\mu)}{2\sqrt{(\kappa\mu)}} + b\int_{b\sqrt{\kappa\mu}}^{\infty} e^{-\lambda^2} d\lambda\right] d\mu =$$

$$= \frac{1}{2\sqrt{\kappa}}\int_{0}^{\varepsilon} \mu^{-1/2} e^{-\kappa b^2\mu} d\mu + \frac{\varepsilon}{2\sqrt{\kappa}}\; e^{-\kappa b^2\varepsilon} - b\varepsilon \int_{b\sqrt{\kappa\varepsilon}}^{\infty} e^{-\lambda^2} d\lambda$$

But since $|e^{-\kappa b^2\varepsilon}| \leq 1$ and $\displaystyle\int_{b\sqrt{\kappa\varepsilon}}^{\infty} e^{-\lambda^2} d\lambda \leq \frac{\sqrt{\pi}}{2}$, the limits of the last two terms as ε tends to zero are zero. We therefore need only consider

$$\lim_{\varepsilon\to 0} \int_{b=0}^{\infty} b^3\, e^{-bz} \int_{0}^{\varepsilon} \mu^{-1/2} e^{-\kappa b^2\mu}\, d\mu\, db =$$

$$= \lim_{\varepsilon\to 0} \sqrt{\varepsilon}\;(\text{const}) \int_{b=0}^{\infty} b^3 e^{-bz} \exp(-\kappa b^2\theta\varepsilon)\, db = 0$$

where $0 < \theta < 1$, by the mean value theorem.

The final boundary condition (4.19) states that

$$\lambda\Delta + 2G\frac{\partial v_3}{\partial z} - \beta T = 0 \quad \text{for } z = 0.$$

From equations (4.29), (4.33h), and (C) we have

$$\lambda\Delta + 2G\frac{\partial v_3}{\partial z} - \beta T = B_1 + B_2$$

where

$$(4.35a) \qquad B_1 = \frac{4\rho G}{\pi(\lambda+2G)} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, H(\rho,\zeta)\, \exp\left(-\kappa(b^2+c^2)t\right)$$

$$\frac{b^3}{b^2+c^2}\; J_0(br)\; J_0(b\rho)\; \sin(c\zeta) \left\{ \sin(cz) - cze^{-bz}\right\} dc\, db\, d\rho\, d\zeta$$

and

$$(4.35b) \qquad B_2 = \frac{4\rho\kappa G}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho\, F(\rho,\tau)\, \exp\left(-\kappa(b^2+c^2)(t-\tau)\right)\cdot$$

$$\frac{b^3 c}{b^2+c^2}\; J_0(br)\; J_0(b\rho) \left\{ \sin(cz) - cz\, e^{-bz}\right\} dc\, db\, d\rho\, d\tau$$

Now (H) $\left| \sin(cz) - cz\, e^{-bz} \right| = (cz) \left| \frac{\sin(cz)}{(cz)} - e^{-bz} \right| \leq 2cz$

Since no higher powers of b and c appear in B_1 and B_2 than in the integrals P_1, P_2, P_3, P_4, the treatment of the convergence is the same as for the previous boundary condition. And by a corresponding use of the inequality (H), B_1 and B_2 can be shown to go to zero with z. We thus have that boundary condition (4.19) is satisfied whenever conditions (E) and (G) hold.

To summarize the results of section IV:

The displacements v_1' and v_3' as given by (4.32) are the solutions to the boundary value problem (2.4), (2.5), (2.6), (2.8) for $t > 0$ whenever $H(r,z)$ is bounded, continuous in both variables, and the integral (E) converges. The displacements v_1'' and v_3'' as given by (4.32) satisfy equations (2.4) for $z > 0$ and the surface conditions at $z = 0$ whenever $F(r,t)$ is bounded, continuous, and the integral (G) converges and is bounded for all t.

V. Examples.

To illustrate the foregoing theory, we shall integrate the equations for some specific choices of the functions $H(r,z)$ and $F(r,t)$.

Of primary interest physically is the case in which the initial distribution of heat throughout the solid is constant,

$$H(r,z) \equiv H_0$$

Substituting in equation (3.15), we have

$$T_1 = \frac{H_0}{4\sqrt{\pi}\cdot(\kappa t)^{3/2}} \int_{\zeta=0}^{\infty}\int_{\rho=0}^{\infty} \rho \exp\left(-\frac{r^2+\rho^2}{4\kappa t}\right) I_0\left(\frac{2r\rho}{4\kappa t}\right)\left[\exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) - \exp\left(-\frac{(\zeta+z)^2}{4\kappa t}\right)\right] d\rho\, d\zeta$$

But by means of the formula,

$$\text{(A)} \qquad \int_{\zeta=0}^{\infty}\left[\exp\left(-\frac{(\zeta-z)^2}{4\kappa t}\right) - \exp\left(-\frac{(\zeta+z)^2}{4\kappa t}\right)\right] d\zeta = 4\sqrt{\kappa t}\int_0^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2}\, d\lambda$$

and formula (D) on page 17, the above expression reduces to:

$$\text{(5.1)} \qquad T_1 = \frac{2H_0}{\sqrt{\pi}} \int_0^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2}\, d\lambda$$

The values for this last integral, known as the error function, have been tabulated. See, for example, Jahnke and Emde.

If the surface temperature $F(r,t) \equiv F_0$, a constant, we have

$$T_2 = \frac{zF_0}{4\kappa\sqrt{\pi\kappa}} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty} \frac{\rho}{(t-\tau)^{5/2}} \exp\left(-\frac{r^2+\rho^2}{4\kappa(t-\tau)}\right) I_0\left(\frac{2r\rho}{4\kappa(t-\tau)}\right) \cdot \exp\left(-\frac{z^2}{4\kappa(t-\tau)}\right) d\rho\, d\tau$$

Again by formula (D) on page 17, this integral reduces to

$$(5.2) \quad T_2 = \frac{z\,F_0}{2\sqrt{(\pi\kappa)}} \int_{\tau=0}^{t} (t-\tau)^{-3/2} \exp\left(-\frac{z^2}{4\kappa(t-\tau)}\right) d\tau =$$

$$= \frac{2F_0}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda$$

Now if $H_0 = F_0 = A$, it is obvious that the temperature throughout $z \geq 0$ will always equal A. In fact,

$$T = T_1 + T_2 = \frac{2A}{\sqrt{\pi}} \left\{ \int_0^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda + \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda \right\} \equiv A$$

But in this case of constant temperature, there can be no thermal stresses. Hence for $H_0 = F_0 = A$, we have by equation (2.2) on page 7 that

$$s_1 = \lambda\Delta + 2Ge_1 - \beta T = 0$$

$$s_2 = \lambda\Delta + 2Ge_2 - \beta T = 0$$

$$s_3 = \lambda\Delta + 2Ge_3 - \beta T = 0$$

Adding these equations and using the fact that $e_1 + e_2 + e_3 = \Delta$, gives

$$(5.3) \qquad (3\lambda+2G)\Delta = 3\beta T, \quad \text{hence} \quad \Delta = 3\alpha T$$

Substituting in the above stress formulae,

$$(5.4) \qquad e_1 = e_2 = e_3 = \frac{\beta}{3\lambda+2G}\, T = \alpha T$$

Therefore $\begin{cases} v_1 = \alpha r A \\ v_3 = \alpha z A \end{cases}$

As an example of a function which represents a concentrated surface heat source of short duration and yet satisfies the conditions of integrability, continuity and boundedness, consider the function

$$F(r,t) = A\, t^{-1} \exp\left(-\frac{r^2}{4\kappa t}\right)$$

where A is a constant. Substituting this value of F in equation (3.16), we have

$$T_2 = \frac{Az}{4\kappa\sqrt{(\kappa\pi)}} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty} \tau^{-1}\exp\left(-\frac{\rho^2}{4\kappa\tau}\right) (t-\tau)^{-\frac{5}{2}} \exp\left(-\frac{r^2+\rho^2+z^2}{4\kappa(t-\tau)}\right) \cdot$$

$$\cdot\, I_0\left(\frac{2r\rho}{4\kappa(t-\tau)}\right)\rho \, d\rho \, d\tau$$

Integrating first with respect to ρ,

$$\int_{\rho=0}^{\infty} \rho \exp\left(-\frac{\rho^2 t}{4\kappa\tau(t-\tau)}\right) I_0\left(\frac{2r\rho}{4\kappa(t-\tau)}\right) d\rho =$$

$$= \frac{2\kappa\tau(t-\tau)}{t} \exp\left(\frac{r^2\tau}{4\kappa t(t-\tau)}\right)$$

Therefore

$$T = \frac{Az}{2t\sqrt{(\pi\kappa)}} \int_{\tau=0}^{t} (t-\tau)^{-3/2} \exp\left(-\frac{z^2}{4\kappa(t-\tau)}\right) \exp\left(-\frac{r^2}{4\kappa(t-\tau)}\right) \cdot$$

$$\cdot \exp\left(\frac{r^2\tau}{4\kappa t(t-\tau)}\right) d\tau =$$

$$= \frac{Az}{2t\sqrt{(\pi\kappa)}} \int_{\tau=0}^{t} (t-\tau)^{-3/2} \exp\left(-\frac{z^2}{4\kappa(t-\tau)}\right) \exp\left(-\frac{r^2(t-\tau)}{4\kappa t(t-\tau)}\right) d\tau =$$

$$= \frac{Az}{2t\sqrt{(\kappa\pi)}} \exp\left(-\frac{r^2}{4\kappa t}\right) \int_{\tau=0}^{t} (t-\tau)^{-3/2} \exp\left(-\frac{z^2}{4\kappa(t-\tau)}\right) d\tau$$

But by the transformation, $\lambda^2 = \dfrac{z^2}{4\kappa(t-\tau)}$, we obtain

$$(5.5) \qquad T_2 = \frac{2A}{\sqrt{\pi}} t^{-1} \exp\left(-\frac{r^2}{4\kappa t}\right) \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda$$

Again this last integral can be evaluated from tables. The distribution of heat throughout a semi-infinite solid as a function of r, z, and t where the initial temperature is H and the surface is heated by a source whose value is

$$A \ t^{-1} \exp\left(-\frac{r^2}{4\kappa t}\right)$$

is given by (5.1) + (5.5).

In order to determine the thermal stresses arising from the above functions, we must turn to equations (4.32). It is seen that $H(r,z) \equiv H_0$ does not satisfy the convergence condition (E). However, if the initial distribution is constant, the temperature scale can be picked so that $H_0 = 0$. Further, since no thermal stresses can arise in a solid possessing uniform temperature distribution, if we assume an initial stress-temperature equilibrium, the only stresses which can arise will be due to heat sources impressed on the surface. Hence we need only consider v_1'' and v_2''.

Now $F = A \ t^{-1} \exp\left(-\dfrac{r^2}{4\kappa t}\right)$ satisfies condition (G),

For $\dfrac{A}{t} \displaystyle\int_{\rho=0}^{\infty} \rho \exp\left(-\frac{\rho^2}{4\kappa t}\right) d\rho = 2\kappa A$ which is bounded for all t.

Substituting the function $F = A\, t^{-1} \exp\left(-\frac{r^2}{4\kappa t}\right)$

in equations (4.32) we have:

$$(5.6) \qquad v_1'' = \frac{2\kappa\beta A}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho \frac{J_0(b\rho)}{(t-\tau)} \exp\left(-\frac{\rho^2}{4\kappa(t-\tau)}\right) \cdot$$

$$\cdot J_1(br) \exp\left(-\kappa(b^2+c^2)\tau\right) \frac{c}{b^2+c^2} \left\{ e^{-bz} bc \left(\frac{\lambda+2G}{\lambda+G} - bz\right) + \right.$$

$$\left. + \; b^2 \sin(cz)\right\} dc\, db\, d\rho\, d\tau$$

$$(5.7) \qquad v_3'' = \frac{-2\kappa\beta A}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \rho \frac{J_0(b\rho)}{(t-\tau)} \exp\left(-\frac{\rho^2}{4\kappa(t-\tau)}\right) \cdot$$

$$\cdot J_0(br) \exp\left(-\kappa(b^2+c^2)\tau\right) \frac{c}{b^2+c^2} \left\{ e^{-bz} bc \left(\frac{G}{\lambda+G} + bz\right) + \right.$$

$$\left. + \; bc \cos(cz)\right\} dc\, db\, d\rho\, d\tau$$

First, we have that

$$(A) \qquad \int_{\rho=0}^{\infty} \rho\, J_0(b\rho) \exp\left(-\frac{\rho^2}{4\kappa(t-\tau)}\right) d\rho = 2\kappa(t-\tau) \exp\left(-\kappa b^2 (t-\tau)\right)$$

Hence,

$$v_1'' = \frac{4\kappa^2\beta A}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} \frac{e^{-\kappa c^2 \tau}}{b^2+c^2}\, c\, J_1(br) \left\{ e^{-bz} bc \left(\frac{\lambda+2G}{\lambda+G} - \right.\right.$$

$$\left.\left. bz\right) + b^2 \sin(cz)\right\} dc\, db\, d\tau$$

and

$$v_3'' = \frac{-4\kappa^2\beta A}{\pi(\lambda+2G)} \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} \frac{e^{-\kappa c^2 \tau}}{b^2+c^2}\, c\, J_0(br) \left\{ e^{-bz} bc \left(\frac{G}{\lambda+G} + \right.\right.$$

$$\left.\left. + \; bz\right) + bc \cos(cz)\right\} dc\, db\, d\tau$$

Next consider the integral which arises in the first two terms of both v_1'' and v_3'':

$$\text{(B)} \qquad \int_{c=0}^{\infty} (b^2+c^2)^{-1} \exp(-\kappa c^2 \tau)\; c^2 \, dc =$$

$$= \int_{c=0}^{\infty} e^{-\kappa c^2 \tau} dc \; - \; b^2 \int_{c=0}^{\infty} (b^2+c^2)^{-1} \exp(-\kappa c^2 \tau) dc =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{(\kappa\tau)}} \; - \sqrt{\pi}\; b\; e^{\kappa b^2 \tau} \int_{b\sqrt{\kappa\tau}}^{\infty} e^{-\lambda^2} d\lambda$$

We are thus led to an integral of the form

$$\text{(C)} \qquad \int_{\tau=0}^{t}\int_{b=0}^{\infty} e^{-\kappa b^2 t} b^n\, e^{-bz} J_m(br) \left\{ \frac{\sqrt{\pi}}{2\sqrt{(\kappa\tau)}} - \sqrt{\pi}\; b\; e^{\kappa b^2\tau} \int_{b\sqrt{\kappa\tau}}^{\infty} e^{-\lambda^2} d\lambda \right\} db\, d\tau$$

where m = 0,1 and n = 1, 2.

or

$$\int_{b=0}^{\infty} e^{-\kappa b^2 t} e^{-bz} b^n J_m(br) \left\{ \frac{\sqrt{\pi}}{2\sqrt{\kappa}} \int_{\tau=0}^{t} \frac{d\tau}{\sqrt{\tau}} - \sqrt{\pi}\; b \int_{\tau=0}^{t} e^{\kappa b^2 \tau} \int_{b\sqrt{\kappa\tau}}^{\infty} e^{-\lambda^2} d\lambda \; d\tau \right\} db$$

but $\displaystyle\int_{\tau=0}^{t} \frac{d\tau}{\sqrt{\tau}} = 2\sqrt{t}$ and, integrating by parts,

$$\int_{\tau=0}^{t} e^{\kappa b^2 \tau} \int_{b\sqrt{\kappa\tau}}^{\infty} e^{-\lambda^2} d\lambda \; d\tau = \left[\frac{e^{\kappa b^2 \tau}}{\kappa b^2} \int_{b\sqrt{\kappa\tau}}^{\infty} e^{-\lambda^2} d\lambda \right]_{\tau=0}^{t} + \frac{1}{(2b)\sqrt{\kappa}} \int_{0}^{t} \frac{d\tau}{\sqrt{\tau}} =$$

$$= \frac{e^{\kappa b^2 t}}{\kappa b^2} \int_{b\sqrt{\kappa t}}^{\infty} e^{-\lambda^2} d\lambda \; - \frac{\sqrt{\pi}}{2\kappa b^2} + \frac{\sqrt{t}}{b\sqrt{\kappa}}$$

Hence (C) becomes

$$\Gamma_m^n = \frac{\pi}{2\kappa} \int_{b=0}^{\infty} e^{-\kappa b^2 t} e^{-bz} b^{n-1} J_m(br)\, db \; - \frac{\sqrt{\pi}}{\kappa} \int_{b=0}^{\infty} e^{-bz} b^{n-1} J_m(br) \int_{b\sqrt{\kappa t}}^{\infty} e^{-\lambda^2} d\lambda \; db$$

We shall employ the notation

$$A_m^n(a,b,c) = \int_{x=0}^{\infty} e^{-ax^2} e^{-bx} x^n \, J_m(cx)\, dx \quad .$$

We can then write (C) as

$$\text{(C)} \quad \Gamma_m^n = \frac{\pi}{2\kappa} A_m^{n-1}(\kappa t, z, r) - \frac{\sqrt{\pi}}{2\sqrt{\kappa}} \int_{\mu=\sqrt{t}}^{\infty} A_m^n(\kappa\mu^2, z, r)\, d\mu$$

We must also evaluate the ~~the~~ following integral which occurs in the last term of v_1''.

$$\text{(D)} \quad \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} e^{-\kappa c^2 \tau} \frac{b^2 c}{b^2+c^2} J_1(br) \sin(cz)\, dc\, db\, d\tau$$

which is equal to

$$\text{(D')} \quad \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} e^{-\kappa c^2 \tau} c\, J_1(br) \sin(cz)\, dc\, db\, d\tau \; -$$

$$\text{(D'')} \quad \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} e^{-\kappa c^2 \tau} \frac{c^3}{b^2+c^2} J_1(br) \sin(cz)\, dc\, db\, d\tau$$

We can immediately write (D') as the product of two integrals:

$$\int_{b=0}^{\infty} e^{-\kappa b^2 t} J_1(br)\, db \cdot \int_{\tau=0}^{t}\int_{c=0}^{\infty} e^{-\kappa c^2 \tau} c \sin(cz)\, dc$$

The first integral is equal to $\frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{4\kappa t}\right)\right]$

And by formula (C) page 16, the second integral can be written as

$$\frac{z\sqrt{\pi}}{4\kappa^{3/2}} \int_{\tau=0}^{t} \tau^{-3/2} \exp\left(-\frac{z^2}{4\kappa\tau}\right) d\tau = \frac{\sqrt{\pi}}{\kappa} \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda$$

Integrating (D'') with respect to τ,

$$\text{(D}_1''\text{)} \quad \frac{1}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t} J_1(br)\, c\, \frac{\sin(cz)}{b^2+c^2}\, dc\, db \; -$$

$$\text{(D}_2''\text{)} \quad \frac{1}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} \exp\left(-\kappa(b^2+c^2)t\right) J_1(br)\, c\, \frac{\sin(cz)}{b^2+c^2}\, dc\, db$$

But $$\int_{c=0}^{\infty} c\,\frac{\sin(cz)}{b^2+c^2}\,dc = \frac{\pi}{2}\,e^{-bz}$$

Hence (D_1'') reduces to $\frac{\pi}{2\kappa}\,A_1^0(\kappa t,z,r)$

The integral (D_2'') can be written as

$$\int_{\tau=t}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \exp\left(-\kappa(b^2+c^2)\tau\right)\ J_1(br)\quad c\,\sin(cz)\quad dc\,db$$

Hence, using the integrals which were used for (D'),

$$(D_2'') = \int_{\tau=t}^{\infty} r^{-1}\left[1 - \exp\left(-\frac{r^2}{4\kappa\tau}\right)\right]\cdot\frac{\sqrt{\pi}}{4}\,\frac{z}{(\kappa\tau)^{3/2}}\,\exp\left(-\frac{z^2}{4\kappa\tau}\right)\,d\tau$$

$$= \frac{z\sqrt{\pi}}{4r\kappa^{3/2}}\int_{\tau=t}^{\infty}\tau^{-3/2}\exp\left(-\frac{z^2}{4\kappa\tau}\right)d\tau - \frac{z\sqrt{\pi}}{4r\kappa^{3/2}}\int_{\tau=t}^{\infty}\tau^{-3/2}\exp\left(-\frac{r^2+z^2}{4\kappa\tau}\right)d\tau$$

$$= \frac{\sqrt{\pi}}{r\kappa}\int_{0}^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2}\,d\lambda - \frac{z\sqrt{\pi}}{r\kappa\sqrt{(r^2+z^2)}}\int_{0}^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda$$

Hence,

$$(D) = \frac{1}{r}\left[1 - \exp\left(-\frac{r^2}{4\kappa t}\right)\right]\frac{\sqrt{\pi}}{\kappa}\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda - \frac{\pi}{2\kappa}\,A_1^0(\kappa t,z,r) +$$

$$+ \frac{\sqrt{\pi}}{r\kappa}\int_{0}^{\frac{z}{2\sqrt{\kappa t}}} e^{-\lambda^2}\,d\lambda - \frac{z\sqrt{\pi}}{r\kappa\sqrt{(r^2+z^2)}}\int_{0}^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda =$$

$$= \frac{\pi}{2r\kappa} - \frac{\sqrt{\pi}}{r\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda - \frac{\pi}{2\kappa}\,A_1^0(\kappa t,z,r) +$$

$$- \frac{z\sqrt{\pi}}{r\kappa\sqrt{(r^2+z^2)}}\int_{0}^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda$$

Finally, we must evaluate the integral,

$$(E)\quad \int_{\tau=0}^{t}\int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t}\,e^{-\kappa c^2\tau}\,\frac{c^2 b}{b^2+c^2}\quad J_0(br)\quad \cos(cz)\quad db\,dc\,d\tau$$

Integrating first with respect to τ, we have,

$$(E') \quad \frac{1}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} e^{-\kappa b^2 t}\, b\, J_0(br)\, \frac{\cos(cz)}{b^2+c^2}\, dc\, db \quad +$$

$$(E'') \quad -\frac{1}{\kappa} \int_{b=0}^{\infty}\int_{c=0}^{\infty} \exp\left[-\kappa(b^2+c^2)t\right]\, b\, J_0(br)\, \frac{\cos(cz)}{b^2+c^2}\, dc\, db$$

But
$$\int_{c=0}^{\infty} \frac{\cos(cz)}{b^2+c^2}\, dc = \frac{\pi}{2b}\, e^{-bz}$$

Therefore (E') becomes

$$\frac{\pi}{2\kappa} \int_{b=0}^{\infty} e^{-\kappa b^2 t} e^{-bz} J_0(br)\, db = \frac{\pi}{2\kappa}\, A_0^0(\kappa t, z, r)$$

Now, as before, we can rewrite (E") as

$$-\int_{\tau=t}^{\infty}\int_{b=0}^{\infty}\int_{c=0}^{\infty} \exp\left[-\kappa(b^2+c^2)\tau\right]\, b\, J_0(br)\, \cos(cz)\, dc\, db$$

But
$$\int_{b=0}^{\infty} b\, e^{-\kappa b^2 \tau} J_0(br)\, db = \frac{1}{2\kappa\tau} \exp\left(-\frac{r^2}{4\kappa\tau}\right)$$

and
$$\int_{c=0}^{\infty} e^{-\kappa c^2 \tau} \cos(cz)\, dc = \frac{1}{2}\, \frac{\sqrt{\pi}}{\sqrt{(\kappa\tau)}} \exp\left(-\frac{z^2}{4\kappa\tau}\right)$$

Therefore (E") becomes:

$$-\frac{\sqrt{\pi}}{4\,\kappa^{3/2}} \int_{\tau=t}^{\infty} \tau^{-3/2} \exp\left(-\frac{r^2+z^2}{4\kappa\tau}\right) d\tau = -\frac{\sqrt{\pi}}{\kappa\sqrt{(z^2+r^2)}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda$$

and

$$(E) \quad = \frac{\pi}{2\kappa}\, A_0^0(\kappa t, z, r) - \frac{\sqrt{\pi}}{\kappa\sqrt{(r^2+z^2)}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda$$

Substituting from equations (C), (D), and (E) we have,

$$(5.8) \qquad v_1'' = K\left\{\frac{\lambda+2G}{\lambda+G}\Gamma_1^1 - z\Gamma_1^2 + \frac{\pi}{2r\kappa} + \right.$$

$$\left. - \frac{\sqrt{\pi}}{r\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda - \frac{\pi}{2\kappa}A_1^0 + \frac{z\sqrt{\pi}}{r\kappa\sqrt{(r^2+z^2)}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right\}$$

and

$$(5.9) \qquad v_3'' = -K\left\{\frac{G}{\lambda+G}\Gamma_0^1 + z\Gamma_0^2 + \frac{\pi}{2\kappa}A_0^0 + \right.$$

$$\left. - \frac{\sqrt{\pi}}{\kappa\sqrt{(r^2+z^2)}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right.$$

where $\quad K = \dfrac{4\kappa^2\beta A}{\pi(\lambda+2G)}$

and $\quad \Gamma_m^n = \dfrac{\pi}{2\kappa}A_m^{n-1}(\kappa t, z, r) - \dfrac{\sqrt{\pi}}{\sqrt{\kappa}}\displaystyle\int_{\mu=\sqrt{t}}^{\infty} A_m^n(\kappa\mu^2, z, r)\,d\mu$

We have thus expressed the displacements in terms of tabulated functions and the "A" functions. It can be readily verified that equations (5.8) and (5.9) satisfy the boundary value problem (2.4), (2.5), (2.6), (2.8). [28].

We shall next obtain series representations for the functions A_m^n. In order to obtain a series for the integral,

$$A_0^1 = \int_{x=0}^{\infty} e^{-ax^2} e^{-bx} J_0(cx)\,dx$$

let us employ an interpolation formula due to Ramanujan. [30]

$$\text{If} \quad \Lambda(x) = \sum_{i=0}^{\infty} (-)^i \lambda(i)\frac{x^i}{i!} = \lambda(0) - \lambda(1)x + \lambda(2)\frac{x^2}{2!} - \dots$$

and $M(x) = \sum_{j=0}^{\infty} (-)^j \mu(j) \frac{x^j}{j!} = \mu(0) - \mu(1)x + \mu(2)\frac{x^2}{2!} - \ldots$

Then, $\int_{x=0}^{\infty} \Lambda(x)\, M(x)\, x^{s-1}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(s+n)}{n!} \lambda(-s-n)\, \mu(n) =$

$= \Gamma(s)\, \lambda(-s)\, \mu(0) - \Gamma(s+1)\, \lambda(-1-s)\, \mu(1) + \frac{\Gamma(s+2)}{2!} \lambda(-s-2)\cdot$

$\cdot\mu(2) - \frac{\Gamma(s+3)}{3!} \lambda(-3-s)\, \mu(3) + \ldots$

For $\Lambda(x)$ we shall take $e^{-ax^2} e^{-bx}$. Now the generating function of the Weber-Hermite functions of positive integral order is

$$\exp(-z^2 + 2zx) = \sum_{i=0}^{\infty} H_i(x) \frac{z^i}{i!}$$

If we set $\sqrt{a}\, y = z$ and $x = -\frac{b}{2\sqrt{a}}$ in $e^{-ay^2} e^{-by}$ we get

$$\exp(-z^2 + 2zx)$$

Therefore $e^{-ay^2} e^{-by} = \sum_{i=0}^{\infty} H_i\left(-\frac{b}{2\sqrt{a}}\right) a^{i/2} \frac{y^i}{i!}$

But $H_n(-x) = (-)^n H_n(x)$

Hence, $e^{-ay^2} e^{-by} = \sum_{i=0}^{\infty} (-)^i H_i\left(\frac{b}{2\sqrt{a}}\right) a^{i/2} \frac{y^i}{i!}$

and $\lambda(i) = a^{i/2} H_i\left(\frac{b}{2\sqrt{a}}\right)$

For $M(x)$ we shall take $J_0(cx) = \sum_{m=0}^{\infty} \frac{(-)^m c^{2m} x^{2m}}{2^{2m}\, m!\, m!}$

But $\dfrac{1}{2^{2m}\, m!\ m!} = \dfrac{1}{(2m)!!\ \ (2m)!!} = \dfrac{(2m-1)!!}{(2m)!!\ \ (2m)!} =$

$$= \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\ \ m!} \cdot \frac{1}{(2m)!}$$

Therefore $\quad J_0(cx) = \sum_{m=0}^{\infty} (-1)^m\ c^{2m}\, p(2m)\ \dfrac{x^{2m}}{(2m)!}$

where $\quad p(2m) = \dfrac{(2m-1)!!}{(2m)!!} = \dfrac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\ m!}$

Hence $\quad \mu(j) = 0 \quad$ if j is odd

$\qquad\quad \mu(j) = (\sqrt{-1})^j\ c^j\ p(j) \quad$ if j is even.

Substituting in the above formula, taking $s = 1$, we get

$$(1) \quad \int_{x=0}^{\infty} e^{-ax^2} e^{-bx} J_0(cx)\ dx = \sum_{n=0}^{\infty} (-)^n\ a^{-(n+\frac{1}{2})} H_{-(2n+1)}\left(\frac{b}{2\sqrt{a}}\right) c^{2n}\ p(2n)$$

Substituting in the above formula, taking $s = 2$, we get

$$(2) \quad \int_{0}^{\infty} e^{-ax^2} e^{-bx} J_0(cx)\ x\ dx = \sum_{n=0}^{\infty} (-1)^n\ (2n+1)\ a^{-(n+1)} \cdot$$

$$\cdot\ H_{-(2n+2)}\left(\frac{b}{2\sqrt{a}}\right) c^{2n}\ \ p(2n)$$

To evaluate the integral $\int_{x=0}^{\infty} e^{-ax^2} e^{-bx} J_1(cx)\ dx$, we take

$$M(x) = J_1(cx) = \sum_{m=0}^{\infty} (-1)^m\ \frac{x^{2m+1}}{(2m+1)!}\ \mu(2m+1) = \sum_{m=0}^{\infty} \frac{(-)^m\ x^{2m+1}\ c^{2m+1}}{2^{2m+1} m!\ (m+1)!}$$

where $\quad \mu(2m+1) = c^{2m+1}\ \dfrac{(2m+1)!!}{2^m\ m!\ (2m+2)} = c^{2m+1}\ \ p_1(2m+1)$

We thus have $\quad \mu(m) = 0 \quad$ when m is even

$\qquad$ and $\quad \mu(m) = (-)^{\frac{m-1}{2}}\ c^m\ \ p_1(m) \quad$ when m is odd.

Substituting in the interpolation formula and taking s = 1, we get

$$(3)\qquad \int_{x=0}^{\infty} e^{-ax^2} e^{-bx} J_1(cx)\,dx = \sum_{n=0}^{\infty} (-)^{n+1} a^{-(n+1)} H_{-(2n+2)}\left(\frac{b}{2\sqrt{a}}\right) c^{2n+1} p_1(2n+1)$$

If in (1) and (2) we set c = 0, we have

$$\int_0^{\infty} e^{-ax^2} e^{-bx}\,dx = \frac{1}{\sqrt{a}} \exp\left(\frac{b^2}{4a}\right) \int_{\frac{b}{2\sqrt{a}}}^{\infty} e^{-\lambda^2}\,d\lambda$$

and
$$\int_0^{\infty} e^{-ax^2} e^{-bx} x\,dx = \frac{1}{2a} - \frac{b}{2a^{3/2}} \exp\left(\frac{b^2}{4a}\right) \int_{\frac{b}{2\sqrt{a}}}^{\infty} e^{-\lambda^2}\,d\lambda$$

(1) and (2) will satisfy these equations if we take

$$H_{-1}(x) = e^{x^2} \int_x^{\infty} e^{-\lambda^2}\,d\lambda = q(x)$$

By means of the recurrence relation for the Weber-Hermite functions,

$$H_{n-1}(x) = \frac{1}{2n} H_n'(x) ,$$

we will then have

$$H_{-n}(x) = \frac{(-1)^{n-1}}{2^{n-1}\,(n-1)!} \frac{d^{(n-1)}}{dx^{(n-1)}}\left[q(x)\right]$$

or
$$H_{-n}\left(\frac{b}{2\sqrt{a}}\right) = \frac{(-1)^{n-1}\, a^{\frac{1}{2}(n-1)}}{(n-1)!} \frac{d^{(n-1)}}{db^{(n-1)}}\left[q\left(\frac{b}{2\sqrt{a}}\right)\right] \qquad [31]$$

Substituting in the above equations:

(1) becomes

$$(5.10)\qquad A_0^{\circ}(a,b,c) = \frac{1}{\sqrt{a}} \sum_{n=0}^{\infty} \frac{(-1)^n\, c^{2n}}{2^{2n}\, n!\, n!} \frac{d^{(2n)}}{db^{(2n)}}\, q\left(\frac{b}{2\sqrt{a}}\right)$$

And (3) becomes

$$(5.11) \qquad A_1^{\circ}(a,b,c) = \frac{1}{\sqrt{a}} \sum_{n=0}^{\infty} \frac{(-1)^n \, c^{2n+1}}{2^{2n+1}\, n!\,(n+1)!} \frac{d^{(2n+1)}}{db^{(2n+1)}}\left[q\left(\frac{b}{2\sqrt{a}}\right)\right]$$

By setting c = 0 and/or b = 0, and using the relations,

$$H_{-(2n+1)}(0) = \frac{\sqrt{\pi}}{2} \frac{1}{4^n \, n!}$$

$$H_{-(2n)}(0) = \frac{1}{2^n \, (2n-1)!!}$$

the above formulae can be shown to reduce to their known values on the boundary.

Since the above series are alternating, we can approximate the solution by taking the first n terms, the error will then be less than the (n+1)st term. Series for the other integrals A_m^n can be obtained from (5.10) and (5.11) by differentiating with respect to b or c. The series can also be integrated term by term to obtain the other terms in Γ_m^n.

The foregoing formulae will allow us to calculate to any desired degree of accuracy the temperature, displacements, and stresses in the solid $z > 0$, at any position and time. In order to get a physical picture of the temperature and stress distribution in the solid $z > 0$ due to the surface temperature $F = \frac{A}{t} \exp\left(-\frac{r^2}{4\kappa t}\right)$, let us first consider the behavior on the plane $z = 0$.

Now $$T = T(r,t) = t^{-1} A \exp\left(\frac{-r^2}{4\kappa t}\right)$$

At a fixed point $r = r_0$, the temperature will be initially zero, will rise to a maximum, then diminish to zero. To find the value of t at which this maximum occurs, consider

$$\frac{\partial T}{\partial t} = A \exp\left(-\frac{r_0^2}{4\kappa t}\right)\left(\frac{r_0^2}{4\kappa t^3} - \frac{1}{t^2}\right) = 0$$

Hence $t_{max} = \dfrac{r_0^2}{4\kappa}$ and $T_{max} = \dfrac{4\kappa A}{e r_0^2}$.

It is thus seen that the maximum travels out at a rate proportional to $t^{-1/2}$ and its amplitude is proportional to r^{-2}.

Next, in order to obtain an approximate picture of how a thermal stress varies, let us consider the normal radial stress s_1.

Now $$s_1 = \lambda \Delta + 2G \frac{\partial v_1''}{\partial r} - \beta T$$

Substituting from equations (5.5), (5.8), and (5.9) and their derivatives, we have:

$$(5.12) \quad s_1 = 2GK\Big\{ 2\Gamma_0^1 - z\left(\Gamma_0^3 - r^{-1}\Gamma_1^2\right) - \frac{\pi}{2r^2\kappa} + \frac{\pi}{2\kappa r} A_1^0 +$$

$$- \frac{\pi}{2\kappa} A_0^1 + \frac{\sqrt{\pi}}{\kappa r^2} \exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda - \frac{\sqrt{\pi}\, z}{\kappa}\Bigg[\frac{1}{2\sqrt{(\kappa t)}} \cdot$$

$$\cdot \frac{1}{r^2+z^2} \exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{z^2 + 2r^2}{r^2(r^2+z^2)^{3/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda\Bigg]\Big\} - \frac{8\kappa^2 \beta AG}{r\pi(\lambda+G)} \Gamma_1^1 .$$

As a first approximation, we take from equations (5.10) and (5.11),

$$A_1^\circ \doteq \frac{r}{2\sqrt{(\kappa t)}} \frac{d}{dz}\left[q\left(\frac{z}{2\sqrt{(\kappa t)}}\right)\right]$$

and
$$A_0^\circ = \frac{1}{\sqrt{(\kappa t)}}$$

Substituting in (5.12) we have,

$$(5.13) \quad s_1 = 2GK\left\{\frac{\pi}{4\kappa\sqrt{(\kappa t)}} \frac{d}{dz}\left[q\left(\frac{z}{2\sqrt{(\kappa t)}}\right)\right] - \frac{\pi}{2r^2\kappa} + \right.$$

$$+ \frac{\sqrt{\pi}}{\kappa r^2} \exp\left(-\frac{r^2}{4\kappa t}\right) \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2} d\lambda + \frac{(-)\sqrt{\pi}\ z}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}} \frac{1}{r^2+z^2}\cdot\right.$$

$$\left.\left.\cdot\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{z^2+2r^2}{r^2(r^2+z^2)^{3/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2} d\lambda\right]\right\} + \frac{(-)2\kappa\beta AG}{(\lambda+G)\sqrt{(\kappa t)}} \frac{d}{dz}\left[q\left(\frac{z}{2\sqrt{(\kappa t)}}\right)\right]$$

On the surface $z = 0$, we have approximately,

$$s_1 = 2GK\left\{\frac{\pi}{8\kappa^2 t} - \frac{\pi}{2r^2\kappa} + \frac{\pi}{2r^2\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\right\} - \frac{\beta AG}{t(\lambda+G)}$$

At a given time, $t = t_0 > 0$, s_1 will be a maximum at some point $r = r_{max} > 0$. In order to find the approximate position of this maximum, consider

$$\frac{\partial s_1}{\partial r} = \frac{\pi GK}{\kappa}\left[\frac{2}{r^3}\left(1 - \exp\left(-\frac{r^2}{4\kappa t_0}\right)\right) - \frac{1}{2r\kappa t_0}\exp\left(-\frac{r^2}{4\kappa t_0}\right)\right]$$

Equating to zero, we have

$$\exp\left(-\frac{r^2}{4\kappa t_0}\right)\left[2 + \frac{r}{2\kappa t_0}\right] - 2 = 0$$

or approximately, $$\left(1 - \frac{r}{4\kappa t_0} + \frac{r}{32\kappa^2 t_0^2}\right)\cdot\left(1 + \frac{r^2}{4\kappa t_0}\right) = 1$$

Hence, $r_{max} \doteq 2\sqrt{(\kappa t)}$. So the maximum stress occurs when the temperature is maximum and travels outward at the same rate as T_{max}. Substituting $t = \frac{r^2}{4\kappa}$ in the expression for s_1, we have

$$s_{1\,max} = \frac{\pi G K}{\kappa e r^2} = \frac{4\kappa \beta A G}{e r^2(\lambda+2G)} = \frac{\alpha E T_{max}}{2(1-2\nu)}$$

We thus have that the maximum stress also diminishes as r^{-2}.

In the case $z > 0$, at a fixed point (r_0, z_0), T will be a function of time. In order to find the maximum value we differentiate with respect to t.

$$\frac{\partial T}{\partial t} = \frac{2A}{\sqrt{\pi}}\left\{\exp\left(-\frac{r^2+z^2}{4\kappa t}\right)\frac{z}{4\sqrt{\kappa}\,t^{5/2}} + \frac{r^2}{4\kappa t^3}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-x^2}dx + \right.$$

$$\left. + \left(\frac{-1}{t^2}\right)\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-x^2}dx\right\} = 0$$

Thus t_{max} occurs when

$$\frac{z\sqrt{\kappa t}}{4\kappa t - r^2}\exp\left(-\frac{z^2}{4\kappa t}\right) = \int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda$$

Hence, $$T_{max} = \frac{2A}{\sqrt{\pi}}\frac{z\kappa}{\sqrt{(\kappa t)}}\frac{1}{(4\kappa t - r^2)}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right)$$

Again as a first approximation, we can take $s_{1\,max} = \frac{\alpha E T_{max}}{2(1-2\nu)}$

And it can be shown that the velocity of T_{max} in the z direction is also approximately proportional to $t^{-1/2}$.

In summarizing, we conclude that the maximum stress occurring at a point, as a first approximation, is proportional to the maximum temperature at the point. Since we assume a condition of temperature-stress equilibrium, the maximum stress occurs at the same time as the maximum temperature. The stress and temperature maximums move out from the origin on an advancing front which is approximately spherical. The velocity is nearly proportional to $t^{-1/2}$, the attenuation varies with direction, being proportional to r^{-1} on the surface $z = 0$, and proportional to $z\ e^{-az^2}$ along the axis $r = 0$.

NOTES

1 Sir W. Thomson, Math. and Phys. Papers, vol. 1, p. 97

2 J. Boussinesq, Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques, Paris 1885.

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16 Carslaw, loc. cit. p. 171.

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18 For Green's Theorem in cylindrical coordinates, see Webster, Dynamics p. 381.

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22 Watson, A Treatise on the Theory of Bessel Functions, Macmillan 1944. p. 395.

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24 $$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2} = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^4 \cdot 2 \cdot 2} + \frac{x^6}{2^6 \cdot 6 \cdot 6} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

Since these series are both absolutely convergent, we may compare them term by term; we have that

$$\frac{x^{2n}}{2^{2n}(n!)^2} < \frac{x^{2n}}{(2n)!}$$

since $(2n)!! > (2n-1)!!$

$$(2n)!!\,(2n)!! > (2n)!$$

$$2^{2n}(n!)^2 > (2n)!$$

Therefore $I_0(x) \leq e^x$ for all $x \geq 0$.

25 Carslaw, loc cit p. 31. The function can be shown to satisfy the boundary conditions for other hypotheses.

26 The other equation of (2.4), viz.,

$$(\lambda+G)\frac{\partial\Delta}{\partial z} + G\nabla^2 v_3 - \beta\frac{\partial T}{\partial t} = 0$$

leads to the same relation between the h's.

27 See Carslaw, Introduction to the Theory of Fourier's Series and Integrals. p. 200.

28 In order to verify that equations (5.8) and (5.9) satisfy the boundary value problem (2.4), (2.5), (2.6), (2.8), we must evaluate the following derivatives:

$$\frac{\partial v_1''}{\partial r} = K\left\{\frac{\lambda+2G}{\lambda+G}\left(\Gamma_0^2 - \frac{1}{r}\Gamma_1^1\right) - z\left(\Gamma_0^3 - \frac{1}{r}\Gamma_1^2\right) - \frac{\pi}{2r^2\kappa} + \right.$$
$$+\left(\frac{1}{2\kappa t} + \frac{1}{r^2}\right)\frac{\sqrt{\pi}}{\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda - \frac{\pi}{2\kappa}A_0^1 +$$
$$+\frac{\pi}{2\kappa r}A_1^0 - \frac{z\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\frac{1}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) + \right.$$
$$\left.\left. - \frac{z^2+2r^2}{r^2(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

$$\frac{\partial v_1''}{\partial r} + \frac{v_1''}{r} = K\left\{\frac{\lambda+2G}{\lambda+G}\Gamma_0^2 - z\Gamma_0^3 + \frac{1}{2\kappa t}\frac{\sqrt{\pi}}{\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\cdot\right.$$
$$\cdot\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda - \frac{\pi}{2\kappa}A_0^1 - \frac{z\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\frac{1}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) + \right.$$
$$\left.\left. - \frac{1}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

$$\frac{\partial v_3''}{\partial z} = -K\left\{\frac{-G}{\lambda+G}\Gamma_0^2 + \Gamma_0^2 - z\Gamma_0^3 - \frac{\pi}{2\kappa}A_0' - \frac{z\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\cdot\right.\right.$$

$$\left.\left.\frac{1}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{1}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

$$\Delta = K\left\{\frac{2G}{\lambda+G}\Gamma_0^2 + \frac{1}{2\kappa t}\frac{\sqrt{\pi}}{\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda\right\} =$$

$$= K\left\{\frac{2G}{\lambda+G}\Gamma_0^2 + \frac{\pi}{4\kappa^2}T\right\}$$

$$\frac{\partial v_1''}{\partial z} = K\left\{\frac{\lambda+2G}{\lambda+G}(-)\Gamma_1^2 + z\Gamma_1^3 - \Gamma_1^2 + \frac{\sqrt{\pi}}{2r\kappa^{3/2}\sqrt{t}}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) + \right.$$

$$\left. + \frac{\pi}{2\kappa}A_1' - \frac{\sqrt{\pi}}{r\kappa}\left[\frac{z^2}{2\sqrt{(\kappa t)}}\frac{1}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) + \frac{r^2}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

$$= K\left\{\frac{2\lambda+3G}{\lambda+G}(-)\Gamma_1^2 + z\Gamma_1^3 + \frac{\pi}{2\kappa}A_1' + \frac{\sqrt{\pi}}{r\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\frac{r^2}{r^2+z^2}\cdot\right.\right.$$

$$\left.\left.\cdot\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{r^2}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

$$\frac{\partial v_3''}{\partial r} = -K\left\{\frac{-G}{\lambda+G}\Gamma_1^2 - z\Gamma_1^3 - \frac{\pi}{2\kappa}A_1' - \frac{\sqrt{\pi}}{\kappa r}\left[\frac{1}{2\sqrt{(\kappa t)}}\frac{r^2}{r^2+z^2}\cdot\right.\right.$$

$$\left.\left.\cdot\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{r^2}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

Hence, the first boundary condition,

$$\frac{\partial v_1''}{\partial z} + \frac{\partial v_3''}{\partial r} = K\left\{-2\Gamma_1^2 + 2z\Gamma_1^3 + \frac{\pi}{\kappa}A_1' + \frac{2\sqrt{\pi}}{r\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\right.\right.$$

$$\left.\left.\frac{r^2}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{r^2}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\}$$

And $$\left[\frac{\partial v_1''}{\partial z} + \frac{\partial v_3''}{\partial r}\right]_{z=0} = K\left\{-2\left(\frac{\pi}{2\kappa} A_1' - \frac{\sqrt{\pi}}{\sqrt{\kappa}}\int_{\mu=\sqrt{t}}^{\infty} A_1^2 \, d\mu\right) + \frac{\pi}{\kappa} A_1' + \right.$$

$$\left.\frac{2\sqrt{\pi}}{\kappa r}\cdot\left[\frac{1}{2\sqrt{(\kappa t)}} \exp\left(-\frac{r^2}{4\kappa t}\right) - \frac{1}{r}\int_0^{\frac{r}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda\right]\right\} =$$

$$= K\left\{2\frac{\sqrt{\pi}}{\sqrt{\kappa}}\int_{\mu=\sqrt{t}}^{\infty} A_1^2(\kappa\mu^2, z=0, r)\, d\mu + \frac{2\sqrt{\pi}}{\kappa r}\left[\frac{1}{2\sqrt{(\kappa t)}} \exp\left(-\frac{r^2}{4\kappa t}\right) + \right.\right.$$

$$\left.\left. - \frac{1}{r}\int_0^{\frac{r}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda\right]\right\}$$

But $$A_1^2(\kappa\mu^2, z=0, r) = \int_{x=0}^{\infty} x^2 e^{-\kappa\mu^2 x^2} J_1(rx)\, dx = \frac{r}{(2\kappa\mu^2)^2} \exp\left(-\frac{r^2}{4\kappa\mu^2}\right)$$

and $$\int_{\mu=\sqrt{t}}^{\infty} \frac{r}{4\kappa^2\mu^4} \exp\left(-\frac{r^2}{4\kappa\mu^2}\right) d\mu = \frac{r}{4\kappa^2}\int_0^{\frac{r}{2\sqrt{\kappa t}}} \frac{2^4\kappa^2\lambda^4}{r^4} e^{-\lambda^2} \frac{r}{2\sqrt{\kappa}} \frac{d\lambda}{\lambda^2}$$

For if we set $\mu = \frac{r}{2\sqrt{\kappa}\,\lambda}$, then the integral becomes

$$\frac{2}{r^2\sqrt{\kappa}}\int_0^{\frac{r}{2\sqrt{\kappa t}}} \lambda^2 e^{-\lambda^2} d\lambda = \frac{2}{r^2\sqrt{\kappa}}\left\{\left[\frac{\lambda e^{-\lambda^2}}{-2}\right]_0^{\frac{r}{2\sqrt{\kappa t}}} + \frac{1}{2}\int_0^{\frac{r}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda\right\} =$$

$$= \frac{2\sqrt{\pi}}{\kappa r}\left\{\frac{2}{r}\cdot\frac{-r}{4\sqrt{(\kappa t)}} \exp\left(-\frac{r^2}{4\kappa t}\right) + \frac{1}{r}\int_0^{\frac{r}{2\sqrt{\kappa t}}} e^{-\lambda^2} d\lambda\right\}$$

Hence substituting in the above equation,

$$\left[\frac{\partial v_1''}{\partial z} + \frac{\partial v_3''}{\partial r}\right]_{z=0} = 0$$

proving the first boundary condition.

The second boundary condition states that

$$\left[\lambda\,\Delta + 2G\,\frac{\partial v_3}{\partial z} - \beta T\right]_{z=0} = 0$$

Now $\lambda\Delta + 2G\dfrac{\partial v_3}{\partial z} - \beta T = K\left\{\dfrac{2\lambda G}{\lambda+G}\Gamma_0^2 + \dfrac{\lambda}{2\kappa t}\dfrac{\sqrt{\pi}}{\kappa}\exp\left(-\dfrac{r^2}{4\kappa t}\right)\cdot\right.$

$$\left.\cdot\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda\right\} - 2GK\left\{\frac{\lambda}{\lambda+G}\Gamma_0^2 - z\Gamma_0^3 - \frac{\pi}{2\kappa}A_0^1 - \frac{z\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\cdot\right.\right.$$

$$\left.\left.\cdot\frac{1}{r^2+z^2}\exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{1}{(r^2+z^2)^{3/2}}\int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\,d\lambda\right]\right\} +$$

$$- \frac{2A\beta}{\sqrt{\pi}}\,t^{-1}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda$$

Setting $z = 0$, the expression becomes

$$K\left\{\frac{2\lambda G}{\lambda+G}\Gamma_0^2 + \frac{\lambda\pi}{4\kappa^2 t}\exp\left(-\frac{r^2}{4\kappa t}\right) - \frac{2\lambda G}{\lambda+G}\Gamma_0^2 + \frac{G\pi}{\kappa}A_0^1 +\right.$$

$$\left. - \frac{\pi(\lambda+2G)}{4\kappa^2}\,t^{-1}\exp\left(-\frac{r^2}{4\kappa t}\right)\right\} = K\left\{\frac{\pi G}{\kappa}A_0^1 - \frac{\pi G}{2\kappa^2 t}\exp\left(-\frac{r^2}{4\kappa t}\right)\right\} =$$

$$= \frac{KG\pi}{\kappa}\left\{A_0^1 - \frac{1}{2\kappa}\,t^{-1}\exp\left(-\frac{r^2}{4\kappa t}\right)\right\}$$

But $A_0^1 = \displaystyle\int_{x=0}^{\infty} e^{-\kappa x^2 t}\,x\,J_0(xr)\,dx = \frac{1}{2\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right)$

Therefore $\left[\lambda\Delta + 2G\dfrac{\partial v_3}{\partial z} - \beta T\right]_{z=0} = 0$

The equations (2.4) are also satisfied by (5.8) and (5.9).

$$\frac{\partial}{\partial r}\left[\frac{\partial v_1''}{\partial r} + \frac{v_1''}{r}\right] = \frac{\partial^2 v_1''}{\partial r^2} + \frac{1}{r}\frac{\partial v_1''}{\partial r} - \frac{v_1''}{r^2} =$$

$$= K\left\{\frac{\lambda+2G}{\lambda+G}(-)\Gamma_1^3 + z\Gamma_1^4 - \frac{r}{4\kappa^2 t^2}\frac{\sqrt{\pi}}{\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda\right. +$$

$$+ \frac{\pi}{2\kappa} A_1^2 - \frac{z\sqrt{\pi}}{\kappa}\left[- \frac{r}{2\sqrt{(\kappa t)}} \exp\left(-\frac{r^2+z^2}{4\kappa t}\right)\left(\frac{3}{(r^2+z^2)^2} + \right.\right.$$

$$\left.\left. + \frac{1}{2\kappa t(r^2+z^2)}\right) + \frac{3r}{(r^2+z^2)^{5/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda\right]\Bigg\}$$

And

$$\frac{\partial^2 v_3''}{\partial z^2} = -K\left\{ \frac{-2\lambda-G}{\lambda+G}\Gamma_0^3 + z\,\Gamma_0^4 + \frac{\pi}{2\kappa} A_0^2 - \frac{\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\,\frac{1}{r^2+z^2}\,\cdot\right.\right.$$

$$\left. \cdot \exp\left(-\frac{r^2+z^2}{4\kappa t}\right) - \frac{1}{(r^2+z^2)^{3/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda\right] - \frac{z\sqrt{\pi}}{\kappa}\left[\frac{1}{2\sqrt{(\kappa t)}}\,\cdot\right.$$

$$\left.\left. \cdot \exp\left(-\frac{r^2+z^2}{4\kappa t}\right)\left(\frac{-3z}{(r^2+z^2)^2} - \frac{z}{2\kappa t(r^2+z^2)}\right) + \frac{3z}{(r^2+z^2)^{5/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda\right]\right\}$$

Also

$$\frac{\partial\Delta}{\partial r} = K\left\{\frac{-2G}{\lambda+G}\Gamma_1^3 - \frac{r}{4\kappa^2 t^2}\,\frac{\sqrt{\pi}}{\kappa}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda\right\}$$

and

$$\beta T = K\left\{\frac{\sqrt{\pi}(\lambda+2G)}{2\kappa^2 t}\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda\right\}$$

and

$$\beta\frac{\partial T}{\partial r} = K\left\{\frac{\sqrt{\pi}(\lambda+2G)}{2\kappa^2 t}\left(\frac{-r}{2\kappa t}\right)\exp\left(-\frac{r^2}{4\kappa t}\right)\int_{\frac{z}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\, d\lambda\right\}$$

$$\frac{\partial^2 v_1''}{\partial z^2} = K\left\{\frac{2\lambda+3G}{\lambda+G}\Gamma_1^3 - z\,\Gamma_1^4 + \Gamma_1^3 - \frac{\pi}{2\kappa} A_1^2 + \frac{\sqrt{\pi}}{\kappa}\left[\frac{rz}{2\sqrt{(\kappa t)}}\,\cdot\right.\right.$$

$$\left.\left. \cdot \exp\left(-\frac{r^2+z^2}{4\kappa t}\right)\left(\frac{-3}{(r^2+z^2)^2} - \frac{1}{2\kappa t(r^2+z^2)}\right) + \frac{3rz}{(r^2+z^2)^{5/2}} \int_0^{\sqrt{\frac{r^2+z^2}{4\kappa t}}} e^{-\lambda^2}\, d\lambda\right]\right\}$$

Substituting in the first equation of system of (2.4),

$$G\left(\nabla^2 v_1 - \frac{v_1}{r^2}\right) = GK\left\{2\Gamma_1^3 - \frac{r\sqrt{\pi}}{4K^3t^2}\exp\left(-\frac{r^2}{4Kt}\right)\int_{\frac{r}{2\sqrt{Kt}}}^{\infty} e^{-\lambda^2}\,d\lambda\right\}$$

$$(\lambda+G)\frac{\partial\Delta}{\partial r} = GK(-2)\Gamma_1^3 - K(\lambda+G)\left\{\frac{r\sqrt{\pi}}{4K^3t^2}\exp\left(-\frac{r^2}{4Kt}\right)\int_{\frac{r}{2\sqrt{Kt}}}^{\infty} e^{-\lambda^2}\,d\lambda\right\}$$

$$-\beta\frac{\partial T}{\partial r} = (\lambda+2G)K\left\{\frac{r\sqrt{\pi}}{4K^3t^2}\exp\left(-\frac{r^2}{4Kt}\right)\int_{\frac{r}{2\sqrt{Kt}}}^{\infty} e^{-\lambda^2}\,d\lambda\right\}$$

Hence, adding, the right members total zero.

30 "Ramanujan" G.H.Hardy, Camb. Univ. Press 1940, p. 203.

31 Taking $H_{-1}(x) = e^{x^2}\int_x^{\infty} e^{-\lambda^2}\,d\lambda = q(x)$, we have

$\frac{dq}{dx} = 2xq - 1$ Hence, by means of the recurrence relation,

$$H_{n-1}(x) = \frac{1}{2n} H_n'(x)$$

we can obtain explicit expressions for the Weber-Hermite functions of negative integral order, viz.,

$$H_{-1}(x) = q$$

$$H_{-2}(x) = -\frac{1}{2}\left[2xq - 1\right]$$

$$H_{-3}(x) = \frac{1}{2\cdot 4}\left[(4x^2+2)q - 2x\right]$$

$$H_{-4}(x) = -\frac{1}{2\cdot 4\cdot 6}\left[(8x^3+12x)q - 4(x^2+1)\right]$$

$$H_{-5}(x) = \frac{1}{2\cdot 4\cdot 6\cdot 8}\left[(16x^4+48x^2+12)q - 4(2x^3+5x)\right]$$

$$H_{-6}(x) = -\frac{1}{2\cdot 4\cdot 6\cdot 8\cdot 10}\left[(32x^5+160x^3+120x)q - 4(4x^4+18x^2+8)\right]$$