SOME APPLICATIONS OF FUNCTIONAL ANALYSIS TO PARALLEL DISPLACEMENT IN RIEMANNIAN GEOMETRY

Thesis by

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Summary

This thesis is a study of finite parallel displacement of a contravariant vector according to Levi-Civita's differential definition. The interesting features of such displacement are shown to depend upon the matrizant function, $\Omega_a^b[-\Gamma_k \frac{dx^k}{ds}]$. The precise dependence of the matrizant, and hence the displaced vectors, upon the coefficients of connection of the space and upon the directrix of displacement is studied by means of the Fréchet differentials of the matrizant. This is done both by warping the space, and by varying the directrix.

Since parallel displacement is a geometric phenomenon it is not surprising that the matrizant turns out to be a two point tensor. However, the Fréchet differentials of the matrizant are two point tensors only under rather special conditions. A second surprising result is that only for flat spaces do the Fréchet differentials of any particular order produced by varying the directrix vanish for arbitrary directrices and variations.

The interesting case of a closed directrix is discussed in some detail. In particular the "fixed" vectors are examined, where "fixed" vectors are those which return to their original direction after displacement about the directrix.

The theory is shown to generalize immediately to tensors of any order and of any type.

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CONTENTS

I. Introduction

The problem of parallelism has fascinated geometers throughout the history of mathematics. Euclid faced it in his famous fifth postulate, and Euclidean parallelism remained the standard until the nineteenth century. With the development of Riemannian geometry as a generalization of Euclidean geometry it soon became evident that no generalization of the notion of parallelism could be devised which had all the properties of Euclidean parallelism[1]. This thesis will be concerned with that generalization proposed by T. Levi-Civita.

Levi-Civita[2] suggested a differential definition of a generalized parallelism with the property that if a vector is displaced parallel to itself along a geodesic of the space, it makes a constant angle with the geodesic. However, such vectors do not have constant components. The change in the components depends in general upon the coefficients of connection (Christoffel symbols) of the space, and on the directrix curve followed in displacing the vector. We shall examine this dependence of the displaced vector on the connection of the space and on the directrix in some detail by means of the theory of Fréchet differentials in Banach spaces.

A substantial amount of study has already been made of Levi-Civita parallelism. Pérès[3] found the change produced by displacing

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1. For a discussion of several types of generalized parallelism, see Whittaker (1)
2. Levi-Civita (1) or see any text on Riemannian geometry
3. Pérès, J. (1)

a vector about an infinitesimal closed curve. Dienes[4] has integrated the differential equations essentially by the Liouville-Neumann[5] method of successive substitutions to obtain the change produced in a vector by displacement over a finite distance. He determines the change produced in the displaced vector by changing the directrix an infinitesimal amount. Applying his result to a closed curve, he obtains the Pérès formula. Appel[6] quotes Dienes and determines the Pérès formula on taking the first terms of the Liouville-Neumann series. Eisenhart[7] and Thomas[8] find the change produced by displacement about an infinitesimal parallelogram by considering infinitesimal vectors for the sides of the parallelogram. Duscheck-Mayer[9] also integrate the differential equations by the Liouville-Neumann series.

The contributions of these authors can be summarized in two statements. First they have indicated the solution to the equations of parallel displacement. Second they have displayed the local dependence of the displaced vector on the space by integrating the equations about an infinitesimal closed directrix. They have displayed the local dependence upon the directrix by making infinitesimal changes in the directrix. The difficulty of treating a function expressed as a system of series of iterated integrals has probably inhibited more thorough treatment of the vector displaced over a finite distance.

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4. Dienes, Paul (1)
5. Whittaker & Watson (1)
6. Appel, Paul (1)
7. Eisenhart, L.P. (1) pp. 65-67
8. Thomas, T.Y. (1) pp. 38-42
9. Duschek-Mayer (1)

Much of this difficulty can be simplified by regarding the system of differential equations of parallel displacement as a single matric equation. The solution can then be expressed by means of the matrizant function[10]. This technique is used in chapter III. Some of the earlier results are developed and a tensor expression for the displaced vector is given.

A discussion of the Fréchet differential properties of the matrizant as a function over a Banach ring is given in chapter IV. By means of these properties of the matrizant, the change in the displaced vector as the underlying space is warped is discussed with examples in chapter V.

A similar change as the directrix curve is varied, the space remaining the same, is discussed in chapter VI. The existence of the Fréchet differentials of the matrizant as a function of the directrix with the change in the directrix as increment is proven, and their generating formula is derived.

The tensor nature of the matrizant and its differentials is discussed in chapter VII.

In chapter VIII spaces are classified according to their degree, where degree is defined in a manner comparable to the degree of a plane curve. The surprising result is obtained that spaces are of zero or infinite degree as they are flat or not.

Vectors which return to their original position after parallel displacement about a closed directrix are discussed in chapter IX. The theory is extended to mixed tensors of any order in chapter X.

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10. Frazer, Duncan and Collar (1)

II. Notations, Conventions, and Elementary Theorems.

In this chapter are listed the general conventions which are followed in this thesis, and the elementary theorems on Fréchet differentials which will be used.

II.A. Coordinate Systems.

$x^i \quad i=1,\cdots,n$ are generic coordinates in the n dimensional Riemannian space. Where no ambiguity results, the index i will frequently be omitted, as in "the coordinates x" or "A, a point function, is a function of x".

II.B. Summation Convention.

When an index appears twice in the same term, once as a superscript and once as a subscript, that term represents the sum of n terms obtained by giving the index successively the values 1, 2, ..., n. Parameter values, on the other hand, are not summed. It is usually clear from the context when a symbol is an index and when it is a parameter value. A note is made wherever there is danger of confusion.

II.C. Tensor-Matric Notation.

The matric notation is used to reduce the number of explicitely expressed indeces. The quantity with n^2 components, B^r_c may be represented by the matric symbol B . In the matrix the component B^r_c appears in the rth row and in the cth column. The product of two matric terms A,B is the matric product, ie $(AB)^r_c = A^r_m B^m_c$. The contravariant vector λ^i is represented by the column matrix λ . Similarly if a mixed quantity has more than n^2 components it is represented by a matric term with indeces, for example, the coefficients of connection Γ^{n}_{ck} are written Γ_{k}. Where ambiguity may result from the matric notation, the indeces are inserted. The term is then said to be expressed in "component" notation.

II.D. Fundamental Theorems on Fréchet Differentials.

An ordinary differential exists for numerically valued functions of a numerical variable. The Fréchet differential is a generalization of the ordinary differential to functions with values and arguments in normed linear spaces. The Fréchet differential is defined as follows.

Definition 1. Fréchet Differential.

Let $F[A]$ be a function on the normed linear space B_1 to a normed linear space B_2. Then if for $A = A_0$ a function $\delta' F[A_0; D]$ exists which is linear in D and if for D in some neighborhood of the origin of B_1, ~~and if~~

$$(2.1) \qquad F[A_0 + D] - F[A_0] - \delta' F[A_0; D] = \|D\| \cdot \varepsilon[A_0; D]$$

and if $\|\varepsilon[A_0; D]\| \to 0$ as $\|D\| \to 0$ then $\delta' F[A_0; D]$ is the first Fréchet differential of $F[A]$ at A_0 with increment D. Higher order Fréchet differentials are successive first Fréchet differentials.

Theorem 2.1. If the Fréchet differential of F exists, it is given by the Gateaux differential

$$(2.2) \qquad \mathcal{L} F[A_0; D] = \lim_{\lambda \to 0} \left\{ \frac{F[A_0 + \lambda D] - F[A_0]}{\lambda} \right\}$$

Formula (2.2) is frequently used to compute $\delta' F[A_0; D]$. The existence of the Fréchet differential is often proved by showing that the Gateaux differential exists linear in D and satisfies (2.1) for D in some neighborhood of the origin of B_1. This proof, of course, also furnishes the formula for the Fréchet differential $\delta' F = \mathcal{L} F$.

The following elementary theorems on the Fréchet differentials are listed without proof [11].

<u>Theorem 2.2</u> <u>If the Fréchet differential of a function</u> $F[A]$ <u>exists, it is unique</u>.

<u>Theorem 2.3</u> <u>A finite linear combination of functions Fréchet differentiable at</u> A_0 <u>is itself Fréchet differentiable at</u> A_0.

<u>Theorem 2.4</u> <u>Chain Rule</u>. <u>If</u> B_1, B_2, B_3 <u>are three normed linear spaces</u>, <u>if</u> S_1, S_2 <u>are open sets in</u> B_1, B_2, <u>if</u> $F[A]$ <u>is a Fréchet differentiable function on</u> S_1 <u>to</u> S_2 <u>with differential</u> $\delta' F[A; D]$ <u>and if</u> $\mathfrak{B}[T]$ <u>is a Fréchet differentiable function on</u> S_2 <u>to</u> B_3 <u>with Fréchet differential</u> $\delta' \mathfrak{B}[T; S]$, <u>then</u> $\Phi[A]$ <u>on</u> B_1 <u>to</u> B_3 <u>defined by</u> $\Phi[A] = \mathfrak{B}[F[A]]$ <u>is Fréchet differentiable with differential</u>

(2.3) $\qquad \delta' \Phi[A; D] = \delta' \mathfrak{B}[F[A]; \delta' F[A; D]]$

<u>Definition 2</u>. <u>Multilinear Function</u>.

A function $F[A_1, \cdots, A_p]$ of p variables $A_1, \cdots, A_p$ on $B_1, \ldots, B_p$ to B where $B_1, \ldots, B_p$, B are normed linear spaces is called a multilinear function if F is additive in each variable and if a real number M exists such that $\| F[A_1, \cdots, A_p] \| \leq M \cdot \|A_1\| \cdot \|A_2\| \cdots \|A_p\|$

<u>Theorem 2.5</u> <u>A multilinear function</u> $F[A_1, \cdots, A_p]$ <u>possesses a Fréchet differential with increments</u> $D_1, \cdots, D_p$ <u>given by</u>

(2.4) $$\delta' F[A_1, \cdots, A_p; D_1, \cdots, D_p] = F[D_1, A_2, \cdots, A_p] + F[A_1, D_2, \cdots, A_p] + \cdots + F[A_1, A_2, \cdots, D_p].$$

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11. Michal, A.D. (1)

Notation for the Frechet Differential.

The first Frechet differential of a function F of a variable A with increment D, which depends parametrically on the parameters ξ, η is written $\delta^1 F[A; D | \xi, \eta]$, The ith Frechet differential with i increments D_1, D_2, ... D_i is written $\delta^i F[A; D_1; D_2; \dots; D_i | \xi, \eta]$. When all the increments are equal to D, the ith differential may be written $\delta^i F[A; D; \xi \eta]$. When no ambiguity results, the parameters ξ, η and sometimes even A and D will be omitted, writing the ith differential simply as $\delta^i F$.

III. Integration of the Equations of Levi-Civita Parallel Displacement.

III.A. The Equations of Parallel Displacement of a Contravariant Vector as a System of Ordinary Differential Equations.

Let us consider a general Riemannian space with generic coordinates x. Let a new system of generic coordinates $\bar{x}$ be defined by the equations

$$\bar{x} = \bar{x}(x)$$

where the functions $\bar{x}(x)$ possess first order derivatives and the inverse functions $x = x(\bar{x})$ also possess first order derivatives. A set of n point functions $\lambda^i(x), i=1,\cdots,n$ are said to define a contravariant vector at the point x if the functions in the two coordinate systems are related by

$$\bar{\lambda}^i(\bar{x}) = \lambda^j(x)\frac{\partial \bar{x}^i(x)}{\partial x^j}$$

The functions $\lambda^i(x)$ may be defined at a point only, along a curve or over any other suitable domain of definition.

A set of such vectors defined one at each point of a directrix curve $x^i = x^i(\xi)$ are said to be Levi-Civita parallel if their components, $\lambda^i(x)$, satisfy the system of differential equations

$$\frac{d\lambda^i}{d\xi} = -\Gamma^i_{jk}\lambda^j\frac{dx^k}{d\xi} \tag{3.1}$$

where

$\lambda^i[x(\xi)]$ are the components of the vector at the point x (ξ)

$x(\xi)$ is the point with coordinates $x^i(\xi)$

$x^j = x^j(\xi)$ are the parametric equations of the directrix with real numerical parameter, ξ.

$\frac{dx^j}{d\xi}$ are the derivatives of the parametric equations, $j=1,\cdots,n$

$\Gamma^i_{jk}[x(\xi)]$ are the Christoffel symbols of the space. They are also called the "coefficients of connection" of the space, and are symmetric in j, k.

In a general, non-Riemannian, space, the coefficients of connection, Γ^i_{jk}, need not be symmetric in k, j. In such cases two different definitions of Levi-Civita parallelism are possible as one writes $\Gamma^i_{jk}\lambda^j \frac{dx^k}{d\xi}$ or $\Gamma^i_{jk}\lambda^k \frac{dx^j}{d\xi}$ in (3.1). The theories of the two definitions are completely similar, and when Γ^i_{jk} is symmetric they are identical. We shall consider only the one definition, (3.1), in what follows, and shall assume symmetry. Many of the theorems generalize immediately to the more general non-symmetric case. However, since we do not pursue the non-symmetric theory, there will be no distinction made between those theorems which are valid only for the symmetric case and those which hold more generally.

Γ^i_{jk} and λ^i are functions of the parameter ξ by virtue of the directrix equations. For example

$$\frac{d\lambda^i}{d\xi} = \frac{\partial \lambda^i}{\partial x^j}\frac{dx^j}{d\xi}$$

It is sometimes convenient to think of a set of parallel vectors as being generated by an initial vector which moves parallel to itself along the directrix, rather than of the set of vectors as a whole. Such a vector is said to be displaced parallel to itself. The set as a whole is said to be generated by parallel displacement of the initial vector. This viewpoint will be generally followed in this discussion. We seek now to integrate equations (3.1) so that, given an initial vector, $\lambda[x(a)]$, we can determine the vectors, $\lambda[x(t)]$, which are parallel to it at different points $x(t)$ along the directrix.

The solution of equations (3.1) has been given by many authors[12]. In order that this discussion may be self contained we also give the Liouville-Neumann series solution to (3.1).

Theorem 3.1. Let a contravariant vector, λ, be displaced parallel to itself along the directrix $x^i = x^i(\xi)$. Let $\lambda^i(a)$ be a set of predetermined constants, the initial values of the components of λ at $x^i(a)$. Then the values of $\lambda^i(t)$, the components of λ at $x(t)$ are given in the region of convergence by

$$\lambda^i(t) = \lambda^i(a) - \int_a^t \Gamma^i_{jk}[x(\xi)] \frac{dx^k}{d\xi} d\xi\, \lambda^j(a) + \int_a^t \Gamma^i_{\ell k}[x(\xi)] \frac{dx^k}{d\xi} d\xi \int_a^\xi \Gamma^\ell_{jm}[x(\eta)] \frac{dx^m}{d\eta} d\eta\, \lambda^j(a) - \cdots \tag{3:2}$$

where $\Gamma^i_{jk}[x(\xi)]$ are the coefficients of connection of the space evaluated along the directrix.

Proof: (3.2) is the Liouville-Neumann series solution for the defining equations (3.1)

III.B. The Equations of Parallel Displacement of a Contravariant Vector as a Single Matric Equation.

We may consider (3.1) as a single matric valued equation

$$\frac{d\lambda}{d\xi} = -\Gamma_k \frac{dx^k}{d\xi} \lambda \tag{3.3}$$

where λ is the column matrix of the components of the vector being displaced.

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12. See introduction.

Γ_k is the square n x n matrix of the components Γ^n_{ck}. The suppressed lower index indicates the column position and the suppressed upper index indicates the row position of the term Γ^n_{ck} in the matrix Γ_k. Note that it takes n matrices, k = 1,..., n to list all the coefficients of connection.

$\Gamma_k \frac{dx^k}{d\xi}$ is the sum of n matrices, each of which is the square matric product of a square matrix, Γ_k, by a scalar point function, $\frac{dx^k}{d\xi}$. On the other hand, $\Gamma_k \lambda$ is the column matric product of a square matrix, Γ_k, multiplied on the right by a column matrix, λ. We note that it makes no difference whether λ or $\frac{dx^k}{d\xi}$ is first multiplied by Γ_k. That is

(3.4) $$\left(\Gamma_k \frac{dx^k}{d\xi}\right)\lambda = (\Gamma_k \lambda)\frac{dx^k}{d\xi}$$

The solution to (3.3) leads to

Theorem 3.2. Let a contravariant vector λ be displaced parallel to itself along the directrix $x^i = x^i(\xi)$. Let $\lambda(a)$ be a column matrix of predetermined constants, the initial values of the components of λ at $x(a)$. Then the column matrix of components of λ, $\lambda(t)$, at $x(t)$ are given in the region of convergence by

(3.5) $$\lambda(t) = \Omega^t_a \left[-\Gamma_k \frac{dx^k}{d\xi} \,|\, \xi\right] \lambda(a)$$

where the matrizant function Ω^t_a of an n x n matrix $A(\xi)$ itself a function of the parameter ξ is given by

(3.6) $$\Omega^t_a [A|\xi] = I + \int_a^t A(\xi)\,d\xi + \int_a^t A(\xi)\,d\xi \int_a^\xi A(\eta)\,d\eta + \cdots$$

where I is the unit matrix with one's on the main diagonal and zeros elsewhere.

Proof: (3.5) is the rewriting of (3.2) in matric form, and hence is the solution of (3.3). The theorem follows from theorem 3.1.

III.C. Illustration. Parallel Displacement on a Sphere in Ordinary Three Dimensional Euclidean Space.

III.C.1. Displacement Along a Parallel of Longitude.

Theorem 3.3. Let coordinates x^1, x^2 be chosen on a sphere in three dimensional Euclidean space with x^1 the colatitude and x^2 the longitude. Let a vector λ be displaced by Levi-Civita parallel displacement along $x^2 =$ constant from $x^1 = a$ to $x^1 = t$. Then $\lambda(a)$ and $\lambda(t)$ the values of λ at $x^1 = a, t$ respectively, are related by

(3.7)
$$\lambda(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin a}{\sin t} \end{pmatrix} \lambda(a)$$

Proof. If r is the radius of the sphere, the metric on the sphere is

(3.8)
$$ds^2 = r^2\, dx^{1\,2} + r^2 \sin^2 x^1\, dx^{2\,2}$$

By direct computation, the two matrices of Christoffel symbols[13] are

$$\Gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\cos x^1}{\cos x^2} \end{pmatrix} \qquad \Gamma_2 = \begin{pmatrix} 0 & \frac{\cos x^1}{\sin x^1} \\ -\sin x^1 \cos x^1 & 0 \end{pmatrix}$$

We may take x^1 to be the parameter ξ along the directrix. Then

(3.10) $\quad \frac{dx^k}{d\xi} = \frac{dx^k}{dx^1} = \delta^k_i$ where

δ^k_i is Kronecker's delta $= 1,\ k = 1;\ = 0\ k \neq 1$

Computing the matrizant from (3.6)

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13. For formulas for computing the Christoffel symbols when the coordinate lines are orthogonal, see Michal,(2) page 98 or Eisenhart,(1) page 44.

$$(3.11)\quad \Omega_a^t\left[-\Gamma_R \frac{dx^R}{dx'}\,|\,x'\right] = I - \lambda\!\int_a^t \Gamma_1(x')\,dx' + \int_a^t \Gamma_1(x')\,dx' \int_a^{x'} \Gamma_1(\xi)\,d\xi + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \ln \frac{\sin t}{\sin a} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2!}\left(\ln \frac{\sin t}{\sin a}\right)^2 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \exp \ln \frac{\sin a}{\sin t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin a}{\sin t} \end{pmatrix}$$

Substituting (3.11) in (3.5) gives the theorem.

In component notation, (3.7) is written

$$(3.12)\qquad \lambda^1(t) = \lambda^1(a) \qquad\qquad \lambda^2(t) = \frac{\sin a}{\sin t}\,\lambda^2(a)$$

The computation can be checked by means of the theorems[14] that parallel displacement along a geodesic changes neither the length of the vector nor the angle the vector makes with the geodesic. If $|\lambda|$ is the length of the vector, and if θ is the angle it makes with the geodesic, then since a parallel of longitude is a geodesic on a sphere

$$(3.13)\qquad |\lambda(a)|^2 = r^2\left\{\lambda^{1^2}(a) + \sin^2 a\,\lambda^{2^2}(a)\right\}$$

$$|\lambda(t)|^2 = r^2\left\{\lambda^{1^2}(t) + \sin^2 t\,\lambda^{2^2}(t)\right\}$$

$$= r^2\left\{\lambda^{1^2}(a) + \sin^2 t\,\frac{\sin^2 a}{\sin^2 t}\,\lambda^{2^2}(a)\right\} = |\lambda(a)|^2$$

$$(3.14)\qquad \cos\theta(a) = \frac{\lambda^1(a)}{|\lambda(a)|}$$

$$\cos\theta(t) = \frac{\lambda^1(t)}{|\lambda(t)|} = \frac{\lambda^1(a)}{|\lambda(a)|} = \cos\theta(a)$$

III. C.2. <u>Displacement Along a Parallel of Latitude.</u>

<u>Theorem 3.4.</u> <u>Let coordinates</u> x^1, x^2 <u>be chosen on a sphere in three dimensional Euclidean space with</u> x^1 <u>the colatitude and</u> x^2

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14. see Eisenhart, (1) cit., page 64.

the longitude. Let a vector λ be displaced by Levi-Civita parallel displacement along $x^1 =$ constant from $x^2 = a$ to $x^2 = t$. Then $\lambda(a)$ and $\lambda(t)$ the values of λ at $x^2 = a, t$ respectively, are related by

(3.15) $$\lambda(t) = \begin{pmatrix} \cos[(t-a)\cos x^1] & -\sin x^1 \sin[(t-a)\cos x^1] \\ \dfrac{\sin[(t-a)\cos x^1]}{\sin x^1} & \cos[(t-a)\cos x^1] \end{pmatrix}$$

Proof: We may take x^2 to be the parameter along the directrix. Then

(3.16) $$\frac{dx^k}{d\xi} = \frac{dx^k}{dx^2} = \delta^k_2$$

Computing the matrizant, using (3.9) and (3.6)

(3.17) $$\Omega^t_a\left[-\Gamma_k \frac{dx^k}{dx^2} \mid x^2\right] = I - \begin{pmatrix} 0 & -\sin x^1\,(t-a)\cos x^1 \\ \dfrac{(t-a)\cos x^1}{\sin x^1} & 0 \end{pmatrix} + \begin{pmatrix} -\dfrac{(t-a)^2\cos^2 x^1}{2!} & 0 \\ 0 & -\dfrac{(t-a)^2\cos^2 x^1}{2!} \end{pmatrix} - \cdots$$
$$= \begin{pmatrix} \cos[(t-a)\cos x^1] & -\sin x^1 \sin[(t-a)\cos x^1] \\ \dfrac{\sin[(t-a)\cos x^1]}{\sin x^1} & \cos[(t-a)\cos x^1] \end{pmatrix}$$

Substituting (3.17) in (3.5) gives the theorem.

III.D. A Tensor Expression for the Integral of Equations (3.3)

Parallel displacement is essentially a geometric phenomenon. Therefore the results should be expressible in tensor form. We shall do this by means of normal coordinate theory[15] for spaces

15. For normal coordinates see Thomas (1), chapter 5, or Veblen (1) chapter 6.

for which Γ_k can be expanded in a Taylor series about $x(a)$.

Theorem 3.5. If a contravariant vector, indicated by the column matrix λ is displaced by Levi-Civita parallel displacement along a directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = t$, if the coefficients of connection Γ_k can be expanded about $x(a)$ in a Taylor series, and if $\lambda(a)$, $\lambda(t)$ are the values of λ at $x(a)$, $x(t)$ respectively, then

$$(3.18)\quad \begin{aligned}\lambda(t) = \Big\{ I &- \int_a^t [A_{kj}\, y^j[x] + A_{kjl}\, y^j[x]\, y^l[x] + \cdots] \frac{\partial y^k}{\partial x^p}\frac{dx^p}{d\xi}\, d\xi \\ &+ \int_a^t [A_{kj}\, y^j[x] + \cdots] \frac{\partial y^k}{\partial x^p}\frac{dx^p}{d\xi}\, d\xi \int_a^\xi [A_{mn}\, y^n[x] + \cdots] \frac{\partial y^m}{\partial x^q}\frac{dx^q}{d\eta}\, d\eta \\ &- \cdots \Big\}\, \lambda(a)\end{aligned}$$

where I is the unit matrix, A_{kj}, A_{kjl} , are the matric expressions for the normal tensors[16] of the space evaluated at $x(a)$ and $y^j[x]$ are the functions of x defining the transformation $y^j = y^j[x]$ from generic to normal coordinates with origin $x(a)$. $y^j[x]$ are functions of ξ by reason of the directrix equations $x^i = x^i(\xi)$.

Proof: From (3.5) and (3.6)

$$(3.19)\quad \lambda(t) = \Big\{ I - \int_a^t \Gamma_k \frac{dx^k}{d\xi} d\xi + \int_a^t \Gamma_k \frac{dx^k}{d\xi} d\xi \int_a^\xi \Gamma_m \frac{dx^m}{d\tau} d\tau - \cdots \Big\}\, \lambda(a)$$

We may now transform to normal coordinates y^i with center at $x(a)$. We place a prime $'$ over terms which are evaluated in the normal coordinates. ξ , of course, remains unchanged. (3.19) becomes

$$(3.20)\quad \lambda'(t) = \Big\{ I - \int_a^t \Gamma'_k \frac{dy^k}{d\xi} d\xi + \int_a^t \Gamma'_k \frac{dy^k}{d\xi} d\xi \int_a^\xi \Gamma'_m \frac{dy^m}{d\tau} d\tau - \cdots \Big\}\, \lambda(a)$$

16. see Thomas, (1) pp. 102-105.

Now we assume that the coefficients of connection, Γ'_k , can be expanded in a series about the origin of the normal coordinates

$$(3.21) \qquad \Gamma'_k = \Gamma'_k(0) + \frac{\partial \Gamma'_k(0)}{\partial y^j} y^j + \frac{1}{2!} \frac{\partial^2 \Gamma'_k(0)}{\partial y^j \partial y^l} y^l y^j + \cdots$$

where $\Gamma'_k(0)$ and the different derivatives are all evaluated at the origin. But at the origin of normal coordinates,

$$(3.22) \qquad \begin{aligned} &\Gamma'_k(0) = 0 \\ &\frac{\partial \Gamma'_k}{\partial y^j} = A'_{kj} \\ &\frac{\partial^2 \Gamma'_k}{\partial y^j \partial y^l} = A'_{kjl} \\ &\vdots \end{aligned}$$

where A_{kj} , A_{kjl} , ... are the normal tensors of the space written in matric form. Hence we may write (3.20)

$$(3.23) \qquad \begin{aligned} \lambda'(t) = \Big\{ I &- \int_a^t [A'_{kj} y^j + A'_{klj} y^l y^j + \cdots] \frac{dy^k}{d\xi} d\xi \\ &+ \int_a^t [A'_{kj} y^j + A'_{klj} y^l y^j + \cdots] \frac{dy^k}{d\xi} d\xi \\ &\quad \int_a^\xi [A'_{mn} y^n + A'_{mnp} y^n y^p + \cdots] \frac{dy^m}{d\tau} d\tau \\ &- \cdots \qquad \Big\} \lambda(a) \end{aligned}$$

The law of formation of terms is readily seen. If in (3.23) we represent $\int_a^t A'_{\cdots} y^{\cdot} \cdots y^{\cdot} \frac{dy^{\cdot}}{d\xi} d\xi$ where $A'_{\cdots}$ has m indeces by the number m , and integral composition by indicated product, (3.23) may be written

$$(3.24) \qquad \begin{aligned} \lambda'(t) = \Big\{ I &- [2+3+4+\cdots] + [2+3+4+\cdots][2+3+4+\cdots] + \\ &- [2+3+4+\cdots][2+3+4+\cdots][2+3+4+\cdots] + \\ &+ [2+3+\cdots][2+3+\cdots][2+3+\cdots][2+3+\cdots] - \cdots \Big\} \lambda(a) \end{aligned}$$

On changing back to generic coordinates, (3.23) becomes the conclusion of the theorem

$$\lambda(t) = \Big\{ I - \int_a^t [A_{kj}\, y^j[x] + A_{klj}\, y^l[x]\, y^j[x] + \cdots] \frac{\partial y^k}{\partial x^p} \frac{dx^p}{d\xi} d\xi$$

(3.18)
$$+ \int_a^t [A_{kj}\, y^j[x] + \cdots] \frac{\partial y^k}{\partial x^p} \frac{dx^p}{d\xi} d\xi \int_a^\xi [A_{mn}\, y^n[x] + \cdots] \frac{\partial y^m}{\partial x^q} \frac{dx^q}{d\tau} d\tau$$

$$- \cdots \Big\} \lambda(a)$$

where the y^i are given as functions of x by the equations transforming the generic into normal coordinates. We note that the normal tensors are constants with respect to the integrations and could be written outside the integral sign if desired.
Further, since $\frac{\partial y^j}{\partial x^k} = \delta^j_k$ at $x^i(a)$, $A'_{kj} = A_{kj}$.

In general the functions $\bar{x}(x)$ in a transformation of coordinates are scalar functions of x . However under a transformation $\bar{x} = \bar{x}(x)$, the normal coordinate functions $y(x)$ undergo a linear transformation $\bar{y}^i = y^j \frac{\partial \bar{x}^i}{\partial x^j}(a)$ where the partial derivatives $\frac{\partial \bar{x}^i}{\partial x^j}(a)$ are evaluated at $x(a)$. Hence the integrands of (3.18) are mixed tensors of rank two evaluated at $x(a)$, the origin of the normal coordinates. The tensor indeces, of course, are suppressed in the matric notation.

Theorem 3.5 shows that the tensor expression for $\lambda(t)$ may be quite complicated. Hence it is worthwhile to seek some rearrangement of the result (3.18) which may make a more usable expression for $\lambda(t)$. Such a rearrangement will now be given as

<u>Theorem 3.6</u>. <u>If a contravariant vector, indicated by the column matrix</u> λ <u>is displaced by Levi-Civita parallel displacement along a directrix</u> $x^i = x^i(\xi)$ <u>from</u> $\xi = a$ <u>to</u> $\xi = t$, <u>if</u> $\Gamma_k[x(\xi)]$ <u>can be expanded in a Taylor series about</u> $x(a)$ <u>and if</u> $\lambda(a)$ <u>and</u> $\lambda(t)$

are the values of λ at $\xi = a$, and $\xi = t$ respectively, then

(3.25) $$\lambda(t) = \left\{ \sum_{r=0}^{\infty} T_r \right\} \lambda(a)$$

where the matric term T_r is homogeneous of degree r in the functions $y^i[x]$. T_r are given by the table

(3.26)
$$T_0 = I$$
$$T_1 = 0$$
$$T_2 = -A_{kj} \int_a^t y^j[x] \frac{\partial y^k}{\partial x^p}[x] \frac{dx^p}{d\xi} d\xi$$
$$T_3 = -A_{klj} \int_a^t y^j[x]\, y^l[x] \frac{\partial y^k}{\partial x^p}[x] \frac{dx^p}{d\xi} d\xi$$
$$T_4 = -A_{kljh} \int_a^t y^j[x]\, y^l[x]\, y^h[x] \frac{\partial y^k[x]}{\partial x^p} \frac{dx^p}{d\xi} d\xi$$
$$+ A_{kj} A_{lh} \int_a^t y^j[x] \frac{\partial y^k}{\partial x^p} \frac{dx^p}{d\xi_1} d\xi_1 \int_a^{\xi_1} y^h[x] \frac{\partial y^l}{\partial x^m} \frac{dx^m}{d\xi_2} d\xi_2$$
$$\vdots$$

T_r "homogeneous of degree r in the functions $y^i[x]$ " means that multiplying each $y^i[x]$ in T_r by a constant α is equivalent to multiplying T_r by α^r ; ie,

$$T_r[\alpha y^i[x]] = \alpha^r T_r[y^i[x]]$$

As in (3.24), the law of formation of the terms may be expressed

(3.27)
$$T_2 = -2 \qquad\qquad T_5 = -5 + 2\cdot 3 + 3\cdot 2$$
$$T_3 = -3 \qquad\qquad T_6 = -6 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 - 2\cdot 2\cdot 2$$
$$T_4 = -4 + 2\cdot 2 \qquad\qquad \vdots$$

and in general T_n = sum of all possible permutations of all integers ≥ 2 such that the sum of the numbers in each permutation is n, and each permutation is multiplied by $(-1)^{\sigma}$, where σ is the number of numbers in the permutation.

The numbers, m, in the right members of (3.27) stand for terms of the mth degree in $y^i[x]$ as follows

(3.28) $$m = -A_{kl\cdots j}\int_a^t y^j[x]\cdots y^l[x]\,\frac{\partial y^k[x]}{\partial x^p}\,\frac{dx^p}{d\xi}\,d\xi$$

where there are m indeces $k, l, \cdots, j$. Multiplication in the right member of (3.27) is integral composition. For example

(3.29) $$2\cdot 3 = A_{kl}A_{mhr}\int_a^t y^l[x]\,\frac{\partial y^k}{\partial x^p}\,\frac{dx^p}{d\xi_1}\,d\xi_1\int_a^{\xi_1} y^h[x]\,y^r[x]\,\frac{\partial y^m}{\partial x^q}\,\frac{dx^q}{d\xi_2}\,d\xi_2$$

where the quantities in the first integrand are functions of parameter ξ_1 and those in the second are functions of parameter ξ_2.

Proof of theorem 3.6: From the absolute convergence of the Taylor series and the matrizant function (3.6), (3.18) is absolutely convergent. Hence (3.18) may be rearranged term by term to give Theorem 3.6.

The results of theorems 3.5 and 3.6 show that the tensor representation of $\lambda(t)$ can be quite complicated. A different approach to the problem will be taken in chapter VII which will supplement these results.

III.E. <u>The Special Case of a Closed Directrix</u>.

Of all directrices, the closed directrix is the most interesting. Of the known theorems on Levi-Civita parallel displacement, one of the earliest and most fundamental is that of Pérès, giving the results on displacing a vector about an infinitesimal closed curve. It will now be shown that this formula is but a special approximation to (3.25).

Theorem 3.7. If a contravariant vector, indicated by the column matrix λ is displaced by Levi-Civita parallel displacement completely about a closed directrix $x^i = x^i(\xi)$, where ξ goes from $\xi = a$ to $\xi = b$ in one circuit, if $\Gamma_k[x]$ can be expanded in a Taylor series, and if $\lambda(a)$, and $\lambda(b)$ are the values of λ before and after the circuit, then to terms of second degree in $y[x]$

(3.30) $$\lambda(b) = \left\{ I + \frac{1}{2} R_{kj} \int_a^b y^j[x] \frac{dy^k}{dx^p} \frac{dx^p}{d\xi} d\xi \right\} \lambda(a)$$

where I is the identity matrix, R_{kj} is the matrix of the Riemann[17] curvature tensor evaluated at $x(a)$, $y^j[x]$ is the function of x which transforms to normal ~~coordination~~ coordinates y^j in transforming to normal coordinates with center at $x(a)$.

Proof: From theorem 3.6, to terms of second degree in $y[x]$,

(3.31) $$\lambda(b) = \left\{ I - A_{kj} \int_a^b y^j[x] \frac{dy^k}{dx^p} \frac{dx^p}{d\xi} d\xi \right\} \lambda(a)$$

Integrating the integral in (3.31) by parts, since $\frac{dy^k}{dx^p}\frac{dx^p}{d\xi} = \frac{dy^k}{d\xi}$

(3.32) $$\int_a^b y^j \frac{dy^k}{d\xi} d\xi = y^j y^k \Big]_a^b - \int_a^b y^k \frac{dy^j}{d\xi} d\xi$$

Since the curve is closed, $y^j[x(a)] = y^j[x(b)]$, so the first term on the right in (3.32) vanishes. Taking half of (3.32), interchanging the dummy indeces gives for (3.31)

(3.33) $$\lambda(b) = \left\{ I + (A_{jk} - A_{kj}) \frac{1}{2} \int_a^b y^j \frac{dy^k}{dx^p} \frac{dx^p}{d\xi} d\xi \right\} \lambda(a)$$

Noting that

(3.34) $$A_{jk} - A_{kj} = R_{kj}$$

completes the proof of the theorem.

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17. See Eisenhart (1) pp. 19-22, or any text on Riemannian geometry. Some authors differ in sign from the definition as given by Eisenhart.

We note that R_{kj} = matrix $(R^{r}_{c\,kj})$, where $R^{r}_{c\,kj}$ are the components of the Riemann curvature tensor.

<u>Corollary 3.7.1.</u> <u>Pérès formula.</u> <u>To terms of the second degree in</u> $x(\xi)$, <u>the generic coordinates of the curve,</u>

$$(3.35) \qquad \lambda(b) = \left\{ I + \tfrac{1}{2} R_{kj} \int_a^b x^j \frac{dx^k}{d\xi}\, d\xi \right\} \lambda(a)$$

Proof: The transformation equations from generic to normal coordinates are

(3.36) $y^j(\xi) = x^j(\xi) - x^j(a) +$ terms of degree two or more in x.

Hence

$$\frac{\partial y^k}{\partial x^p} \frac{dx^p}{d\xi} = \delta^k_p \frac{dx^p}{d\xi} = \frac{dx^k}{d\xi}$$

to first degree terms in x. Further $\int_a^b x^j(a) \frac{dx^k}{d\xi} d\xi = 0$ since the directrix is closed. Hence to second degree terms in x,

$$(3.37) \qquad \int_a^b y^j \frac{dy^k}{d\xi} d\xi = \int_a^b \{x^j(\xi) - x^j(a)\} \frac{dx^k}{d\xi} d\xi = \int_a^b x^j(\xi) \frac{dx^k}{d\xi} d\xi$$

Substituting (3.37) into (3.30) gives corollary (3.7.1.)

It is easy to show that if the directrix is a parallelogram (3.35) reduces to the forms given by Eisenhart and Thomas in their discussions of displacement about a particular infinitesimal parallelogram. For the full significance of this result we need the theory of the matrizant as an analytic function on a Banach space.

IV. The Matrizant as a Function on a Banach Ring.

IV.A. Definition of a Banach Ring. The Ring of Square Matrices.[18]

A Banach ring is a complete normed linear space, or Banach space, with a multiplication defined. There may or may not be a multiplicative unit.

A set of elements y form a normed linear space N if the following conditions are satisfied.

(1) A function of two variables called addition exists on NN to N, and is written + . In symbols, if $y_1 \in N$ and $y_2 \in N$ then $y_1 + y_2 = y_3$, $y_3 \in N$

(2) Multiplying any element $y_1 \in N$ on the left by a real number α yields another element $\alpha y_1 = y_2 \in N$.

(3) A function, called, norm, on N to the real numbers exists and is written $\|y\|$.

(4) $y_1 + (y_2 + y_3) = (y_1 + y_2) + y_3$

(5) $y_1 + y_2 = y_2 + y_1$

(6) $\alpha(y_1 + y_2) = \alpha y_1 + \alpha y_2$,

(7) $(\alpha_1 + \alpha_2) y = \alpha_1 y + \alpha_2 y$,

(8) $y + z = w$; $y, w \in N$ has a unique solution $z \in N$

(9) $\|\alpha y\| = |\alpha| \cdot \|y\|$ α any real number

(10) $\|y\| \geq 0$ for all $y \in N$

(11) $\|y\| = 0$ if and only if $y = 0$

(12) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$ (triangle law) for all $y_1, y_2 \in N$

18. Banach, S. (1)

A normed linear space is a Banach space, B, if in addition to the above 12 conditions, the Cauchy convergence criteria is satisfied in the norm. That is, every sequence $\{x_i\}$ of elements of B converges if for any ε there exists an N such that for all $m, \ell \geq N$, $\| x_m - x_\ell \| < \varepsilon$

A multiplication is a function of two variables on BB to B which is

(4.1) (a) associative $(xy)z = x(yz)$

(b) distributive with respect to addition

$$(x+y)z = xz + yz \qquad z(x+y) = zx + zy$$

(c) modular $\|xy\| \leq M \cdot \|x\| \cdot \|y\|$ where M is a fixed constant independent of x and y. The least such constant is called the "modulus" of the product xy.

As an illustration, consider the set of all square matrices of order n whose elements are real numbers. These matrices form a normed linear space if the norm is taken as the maximum of the absolute value of any element in the matrix[19]. In symbols, if A is a matrix whose element in the rth row and cth column is A^r_c

(4.2) $\|A\| = \max |A^r_c| \quad r, c = 1, \cdots, n$

Further, these matrices form a ring with ordinary matric multiplication as the ring multiplication. For, in component notation

19. Such matrices form a normed linear space with other norms also, such as the square root of the sum of the square of the elements. However we shall consider only the one norm, since it is easy to work with, and is used many places in this thesis.

$$(AB)^r_c = A^r_m B^m_c$$

$$(AB)C = A(BC) = (ABC)^r_c = A^r_m B^m_i C^i_c$$

$$(A+B)C = (A^r_m + B^r_m) C^m_c = A^r_m C^m_c + B^r_m C^m_c = AC + BC$$

$$C(A+B) = C^r_m (A^m_c + B^m_c) = C^r_m A^m_c + C^r_m B^m_c = CA + CB$$

$$\| AB \| = \| A^r_m B^m_c \| \leq n \|A\| \cdot \|B\|$$

Clearly matric multiplication satisfies conditions (4.1) with modulus $M = n$, the order of the matrices.

Now let us consider two auxiliary Banach spaces with values in the Banach ring of square matrices. Let B_1 be the set of all square matrices of order n whose elements are continuous, real, numerical, functions of a single real numerical variable ξ in the range $a \leq \xi \leq b$. Let B_2 be the set of all square matrices of order n whose elements are continuous real, numerical functions of two real numerical variables ξ, η in the range $a \leq \xi, \eta \leq b$. Then if $\| \ \|_N$ be the norm (4.2), and if $z(\xi)$ be an element in B_1 and $y(\xi,\eta)$ an element in B_2, we may take as the norms

$$\| z(\xi) \| = max \| z(\xi) \|_N \quad a \leq \xi \leq b$$

(4.3) $$\| y(\xi,\eta) \| = max \| y(\xi,\eta) \|_N \quad a \leq \xi, \eta \leq b$$

With the norms defined in (4.3) the sets B_1 and B_2 are easily seen to form Banach spaces.

Now the matrizant

(4.4) $$\Omega^t_s [A|\xi] = I + \int_s^t A(\xi)\, d\xi + \cdots$$

considered as a functional apart from its geometric role is a function of two real variables, s, t. It is clear that $\Omega^t_s [A]$ is a functional on B_1 to B_2.

IV. B. The Differential Properties of the Matrizant.

Professor A.D. Michal[20] has proved the following theorems about the matrizant, considered as a function of the matrix A in B_1 to the Banach space B_2.

Theorem 4.1. The matrizant function (4.4) on the Banach space B_1 to the Banach space B_2 satisfies the differential system in Fréchet differentials

$$(4.5)\qquad \begin{aligned} \delta' \Omega_{r}^{t}[A;\delta A] &= \int_{r}^{t} \Omega_{\xi}^{t}\, \delta A(\xi)\, \Omega_{r}^{\xi}\, d\xi \\ \Omega_{r}^{t}[0] &= I \end{aligned}$$

where $\delta' \Omega_{r}^{t}[A;\delta A]$ is the first Fréchet differential of $\Omega_{r}^{t}[A]$ with increment δA, $\Omega_{r}^{t}[0]$ is the matrizant of the zero element of B_1, I is the unit of the ring, in this case the unit matrix.

Theorem 4.2. The higher order Fréchet differentials of the matrizant with equal increments δA are given by

$$(4.6)\qquad \delta^{k} \Omega_{r}^{t}[A;\delta A] = k!\int_{r}^{t} \Omega_{\xi_1}^{t}[A]\, \delta A(\xi_1)\, d\xi_1 \int_{r}^{\xi_1} \Omega_{\xi_2}^{\xi_1}\, \delta A(\xi_2)\, d\xi_2 \cdots \int_{r}^{\xi_{k-1}} \Omega_{\xi_k}^{\xi_{k-1}}[A]\, \delta A(\xi_k)\, \Omega_{r}^{\xi_k}\, d\xi_k$$

Theorem 4.3.[21] The entire analytic[22] series (4.4) giving the matrizant function can be differentiated termwise to give

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20. Michal, A.D. (3)
21. Michal, A.D. (4)
22. An entire analytic series in a Banach space is defined as follows. A homogeneous polynomial of degree n on a Banach space possesses the modular property, $\|p_n(x)\| \le M_n \|x\|^n$. M_n is the modulus of the homogeneous polynomial, $p_n(x)$. A series of homogeneous polynomials $\Sigma p_i(x)$ is said to be an entire analytic series if the real numerical series $\Sigma M_i \lambda^i$ converges for all values of λ. While it requires more than the modular property for a function to be a homogeneous polynomial, it is easy to show that the series (4.4) is an entire analytic series in the sense of this footnote. For a more extended discussion of polynomials, see Martin (1).

(4.7) $\delta^1 \Omega_r^t [A;\delta A] = \int_r^t \delta A(\xi)\, d\xi +$

$$+ \sum_{i=2}^{\infty} \Big[\int_r^t \delta A(\xi_1)\, d\xi_1 \int_r^{\xi_1} A(\xi_2)\, d\xi_2 \cdots \int_r^{\xi_{i-1}} A(\xi_i)\, d\xi_i +$$

$$\int_r^t A(\xi_1)\, d\xi_1 \int_r^{\xi_1} \delta A(\xi_2) d\xi_2 \cdots \int_r^{\xi_{i-1}} A(\xi_i)\, d\xi_i + \cdots +$$

$$\int_r^t A(\xi_1)\, d\xi_1 \int_r^{\xi_1} A(\xi_2)\, d\xi_2 \cdots \int_r^{\xi_{i-1}} \delta A(\xi_i)\, d\xi_i$$

We note that (4.7) can also be obtained by substituting (4.4) in (4.5)

<u>Theorem 4.4.</u> <u>The following generalized Taylor Series Expansion holds for the matrizant for all continuous matrices</u> $A(\xi)$ <u>and</u> $\delta A(\xi)$.

(4.8) $$\Omega_r^t [A+\delta A \mid \xi] = \Omega_r^t [A] + \sum_{i=1}^{\infty} \frac{1}{i!} \delta^i \Omega_r^t [A;\delta A]$$

$$= \Omega_r^t [A] + \int_r^t \Omega_{\xi_1}^t [A]\, \delta A(\xi_1)\, \Omega_r^{\xi_1} [A]\, d\xi_1 +$$

$$+ \sum_{i=2}^{\infty} \int_r^t \Omega_{\xi_1}^t [A]\, \delta A(\xi_1)\, d\xi_1$$

$$\int_r^{\xi_1} \Omega_{\xi_2}^{\xi_1} [A]\, \delta A(\xi_2)\, d\xi_2 \cdots$$

$$\int_r^{\xi_{i-1}} \Omega_{\xi_i}^{\xi_{i-1}} [A]\, \delta A(\xi_i)\, \Omega_r^{\xi_i}\, d\xi_i$$

<u>Theorem 4.5.</u> <u>If</u> $A(\xi) \cdot A(\eta) = A(\eta) \cdot A(\xi)$ <u>for all real</u> ξ, η <u>in</u> $a \leq \xi, \eta \leq b$, <u>then</u>

(4.9) $$\Omega_r^t [A] = e^{\int_r^t A(\xi)\, d\xi}$$

<u>where</u> $a \leq r \leq t \leq b$ <u>and where</u>

(4.10) $e^{\int_r^t A(\xi) d\xi}$ <u>is the matric exponential of</u> $\int_r^t A(\xi)\, d\xi$

<u>The matric exponential of a matrix</u> T <u>is defined as the series</u>

(4.11) $$e^T = I + T + \frac{1}{2!} T^2 + \cdots + \frac{1}{i!} T^i + \cdots$$

<u>where</u> T^i <u>is the matric product of</u> i <u>equal matrices</u> T .

The above theorems are assumed without proof in this thesis.

V. The Change in a Set of Parallel Vectors Produced by Warping the Space.

In this chapter we discuss the effects of warping the space on a set of parallel vectors defined along a directrix. By a warping we mean such a transformation that the coordinates of all points remain unchanged, but the coefficients of connection are changed. Alternatively, a warping may be regarded as changing the coefficients, g_{ij}, of the metric of the space, with the resulting change in Γ_k.

Under a warping, the equations of curves, and in particular the equations of the directrix curve, remain unchanged. The set of parallel vectors, however, changes in general. Since any vector of the set is given by assigning a particular value to t in theorem 3.2, it suffices to consider the change in $\lambda(t)$ given by (3.5). The result is

Theorem 5.1. Let a contravariant vector, indicated by the column matrix λ be displaced by Levi-Civita parallel displacement along the directrix curve $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = t$. Let $\lambda(a)$ and $\lambda(t)$ be the values of λ at $\xi = a$ and $\xi = t$ respectively. If the space is warped in such a way that the coefficients of connection $\Gamma_k[x(\xi)]$ are given an increment $\delta\Gamma_k[x(\xi)]$, if λ is displaced as before with the same initial value $\lambda(a)$ and if now $\lambda = \bar{\lambda}(t)$ at $\xi = t$ then

$$\begin{aligned}\bar{\lambda}(t) - \lambda(t) = \Big\{ &- \int_a^t \Omega_{s_1}^{t}\Big[-\Gamma_k[x(\xi)]\frac{dx^k}{d\xi}\Big]\,\delta\Gamma_k[x(s_1)]\,\frac{dx^k}{ds_1} \\ &\quad \Omega_a^{s_1}\Big[-\Gamma_k[x(\xi)]\frac{dx^k}{d\xi}\Big]\,ds_1 \, + \\ &+ \int_a^t \Omega_{s_1}^{t}\Big[-\Gamma_k\frac{dx^k}{d\xi}\Big]\,\delta\Gamma_k(s_1)\,\frac{dx^k}{ds_1}\,ds_1 \int_a^{s_1}\Omega_{s_2}^{s_1}\Big[-\Gamma_p\frac{dx^p}{d\xi}\Big] \\ &\quad \delta\Gamma_l(s_2)\,\frac{dx^l}{ds_2}\,\Omega_a^{s_2}\Big[-\Gamma_n\frac{dx^n}{d\xi}\Big]\,ds_2 - \cdots \Big\}\,\lambda(a)\end{aligned} \tag{5.1}$$

Proof: From (3.5)

(5.2) $$\lambda(t) = \Omega_a^t\left[-\Gamma_k \frac{dx^k}{d\xi}\right]\lambda(a)$$

$$\bar{\lambda}(t) = \Omega_a^t\left[-\Gamma_k \frac{dx^k}{d\xi} - \delta\Gamma_k \frac{dx^k}{d\xi}\right]\lambda(a)$$

Forming the difference, using (4.8) gives the theorem on identifying $-\Gamma_k \frac{dx^k}{d\xi}$ with A and $-\delta\Gamma_k \frac{dx^k}{d\xi}$ with δA in (4.8).

(5.1) is obtained by regarding Ω_a^t as a function of $-\Gamma_k \frac{dx^k}{d\xi}$. Since in warping only Γ_k changes, our interpretation of (5.1) can be sharpened on considering Ω_a^t as a function of Γ_k alone. In this context

(5.3) $$\Omega_a^t[\Gamma_k] = \Omega_a^t[A(\Gamma_k)] = \Omega_a^t\left[-\Gamma_k \frac{dx^k}{d\xi}\right]$$

Now $A(\Gamma_k) = -\Gamma_k \frac{dx^k}{d\xi}$ is a linear function of Γ_k. Hence it possesses Fréchet differentials of all orders with increments given by

(5.4) $$\delta^1 A[\Gamma_k;\delta\Gamma_k] = -\delta\Gamma_k \frac{dx^k}{d\xi} \quad ; \quad \delta^i A[\Gamma_k;\delta\Gamma_k] = 0 \quad i > 1$$

Using the composition theorem for Fréchet differentials of functions of <u>linear</u> functions, (5.3), (5.4)

(5.5) $$\delta^i\Omega_a^t[\Gamma_k;\delta\Gamma_k] = \delta^i\Omega_a^t[A;\delta^1 A(\Gamma_k)]$$

$$= \delta^i\Omega_a^t\left[-\Gamma_k \frac{dx^k}{d\xi} ; -\delta\Gamma_k \frac{dx^k}{d\xi}\right]$$

Using (5.5), (4.8) we have

<u>Theorem 5.2.</u> <u>Under the conditions of theorem 5,1,</u> (5.1)<u>can be written</u>

(5.6) $$\bar{\lambda}(t) - \lambda(t) = \sum_{i=1}^{\infty} \frac{1}{i!}\,\delta^i\Omega_a^t[\Gamma_k;\delta\Gamma_k]\lambda(a)$$

V. B. Example. Warping the Plane into a Sphere.

As an example of a warping, consider the plane with polar coordinates ρ, θ. Regarding it as the equatorial plane of a unit sphere with center at the origin, we map it onto the unit sphere by projection from the south pole. Hence the center of the plane maps into the north pole, the unit circle maps into the equator, etc. We assume that the points retain their old coordinates, ρ, θ, under the mapping.

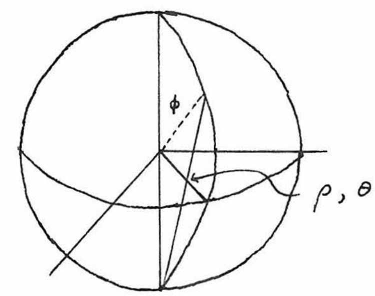

Figure 1.

We see from the geometry that the colatitude, ϕ, is related to the coordinate ρ by

(5.7) $\phi = 2\tan^{-1}\rho$

Using this relation, and the fact that on the unit sphere in spherical polar coordinates the metric is

(5.8) $ds^2 = d\phi^2 + \sin^2\phi \, d\theta^2$

We obtain

(5.9) on the plane $ds^2 = d\rho^2 + \rho^2 d\theta^2$

on the sphere $ds^2 = \frac{4}{(1+\rho^2)^2} d\rho^2 + \frac{4\rho^2}{(1+\rho^2)^2} d\theta^2$

By direct computation of the Christoffel symbols[23],

(5.10) on the plane $\Gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}$ $\Gamma_2 = \begin{pmatrix} 0 & -\rho \\ \frac{1}{\rho} & 0 \end{pmatrix}$

(5.11) on the sphere $\Gamma_1 = \begin{pmatrix} \frac{-2\rho}{1+\rho^2} & 0 \\ 0 & \frac{1-\rho^2}{(1+\rho^2)\rho} \end{pmatrix}$ $\Gamma_2 = \begin{pmatrix} 0 & -\frac{\rho(1-\rho^2)}{1+\rho^2} \\ \frac{1-\rho^2}{\rho(1+\rho^2)} & 0 \end{pmatrix}$

23. See note 13 on page 12.

where ρ, θ are considered the x^1, x^2 coordinates respectively. Hence

(5.12)
$$\delta\Gamma_1 = \begin{pmatrix} -\frac{2\rho}{1+\rho^2} & 0 \\ 0 & -\frac{2\rho}{1+\rho^2} \end{pmatrix} \qquad \delta\Gamma_2 = \begin{pmatrix} 0 & \frac{2\rho^3}{1+\rho^2} \\ -\frac{2\rho}{1+\rho^2} & 0 \end{pmatrix}$$

To fix our ideas, let a contravariant vector λ be displaced along the particular directrix with equations

(5.13)
$$\begin{array}{lll} \rho = 1 & a = 0 & dx^1/d\xi = 0 \\ \theta = \xi & t = \pi & dx^2/d\xi = 1 \end{array}$$

This directrix is half the unit circle on the plane, and half the equator on the sphere. For this directrix, from (5.10), (5.12)

(5.14)
$$\begin{array}{lll} \text{on the plane} & \Gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \Gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{on the sphere} & \Gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} & \Gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & \delta\Gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \delta\Gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}$$

Hence, in the plane, using (4.9)

(5.15)
$$\Omega_r^t\left[-\Gamma_k \frac{dx^k}{d\xi}\right] = e^{-\int_r^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\xi} = e^{-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(t-r)}$$

Then

(5.16)
$$\Omega_0^\pi = e^{-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\pi} = \begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the successive Fréchet differentials with increments

(5.17)
$$\delta\Gamma_k \frac{dx^k}{d\xi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are from (4.6)

$$(5.18)\quad \delta^1\Omega_0^{\pi} = \int_0^{\pi} e^{-\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\pi-\xi)} \begin{pmatrix}0&1\\-1&0\end{pmatrix} e^{-\begin{pmatrix}0&-1\\1&0\end{pmatrix}\xi}\, d\xi = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\pi$$

$$(5.19)\quad \delta^2\Omega_0^{\pi} = 2\int_0^{\pi} e^{-\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\pi-\xi)} \begin{pmatrix}0&1\\-1&0\end{pmatrix} d\xi \int_0^{\xi} e^{-\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\xi-\eta)} \begin{pmatrix}0&1\\-1&0\end{pmatrix} e^{-\begin{pmatrix}0&-1\\1&0\end{pmatrix}\eta}\, d\eta$$

$$= \pi^2 \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\vdots$$

$$\delta^i\Omega_0^{\pi} = \pi^i \begin{pmatrix}\varepsilon^{i+1} & \varepsilon^{i}\\ \varepsilon^{i+2} & \varepsilon^{i+1}\end{pmatrix}$$

where

$$\varepsilon^1 = -1, \quad \varepsilon^2 = 0, \quad \varepsilon^3 = +1, \quad \varepsilon^4 = 0, \quad \varepsilon^i = \varepsilon^{i-4} \quad i > 4$$

Hence the generalized Taylor series expansion for $\Omega_0^{\pi}[-\Gamma_k \frac{dx^k}{d\xi} - \delta\Gamma_k \frac{dx^k}{d\xi}]$ is

(5.20)

$$\Omega_0^{\pi}\left[-\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right] = \begin{pmatrix}-1&0\\0&-1\end{pmatrix} + \pi\begin{pmatrix}0&-1\\1&0\end{pmatrix} + \frac{\pi^2}{2!}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$+ \frac{\pi^3}{3!}\begin{pmatrix}0&1\\-1&0\end{pmatrix} + \frac{\pi^4}{4!}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \cdots$$

$$= \begin{pmatrix}-\cos\pi & -\sin\pi\\ \sin\pi & -\cos\pi\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

This result checks exactly with the result by direct computation, which is easier in this instance.

$$(5.21)\quad \Omega_0^{\pi}\left[-\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right] = \Omega_0^{\pi}\left[\begin{pmatrix}0&0\\0&0\end{pmatrix}\right] = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Hence from (5.2), (5.16), (5.20)

(5.22) on the plane $\lambda(\pi) = \Omega_0^{\pi}\lambda(0) = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\lambda(0)$

on the sphere $\lambda(\pi) = \Omega_0^{\pi}\lambda(0) = \begin{pmatrix}1&0\\0&1\end{pmatrix}\lambda(0)$

This is checked by direct readings. In plane polar coordinates the components of two parallel vectors at opposite ends of a diameter through the origin are exact negatives of each other. Further, the equator is a geodesic on the sphere, hence parallel displacement along it does not change the components of λ .

VI. The Change in Parallel Vectors Produced by Changing the Directrix.

In this chapter we consider two parallel vectors whose base points are connected by the directrix defining the parallelism. We discuss what happens to the second vector if the first vector is held fixed and the directrix is changed. The space, of course, is left unchanged. We shall assume that the new directrix is subject only to the condition that its end points be the same as the end points of the original directrix, the base points of the two vectors, and that the matrizant function $\Omega[-\Gamma_k \frac{dx^k}{d\xi}]$ exist computed along the new directrix.

We first give a tensor matric result related to (3.18) of theorem (3.5).

Theorem 6.1. *Let the contravariant vector indicated by the column matrix* λ *be displaced by Levi-Civita parallel displacement along a directrix* $x^i = x^i(\xi)$ *from* $\xi = a$ *to* $\xi = b$ *with* $\lambda(a)$, $\lambda(b)$ *the values of* λ *at* $\xi = a$ *and* $\xi = b$ *respectively. If* λ *is displaced along a new directrix with the same end points as the old,*

$$x^i = x^{i*}(\xi) = x^i(\xi) + z^i(\xi) \;, \quad z^i(\xi) = 0 \quad \xi = a, b$$

and if $\lambda^*(b)$ *is the value of* λ *after displacement along* $x^{i*}(\xi)$ *then* (6.1)

$$\lambda^*(b) - \lambda(b) = \Big\{ \int_a^b \Big(I - \int_\zeta^b [A_{kj}\, y^j[x] + A_{klj}\, y^l[x]\, y^j[x] + \cdots]\, \frac{\partial y^k[x]}{\partial x^p} \frac{dx^p}{d\xi} d\xi + \cdots \Big)$$

$$(6.1) \qquad \Big(A_{mn} \Big\{ y^n[x|\zeta]\, \frac{dy^m}{d\zeta}[x] - y^{*n}[x^*|\zeta]\, \frac{dy^{*m}}{d\zeta}[x^*] \Big\} + $$

$$+ A_{mnh} \Big\{ y^n y^h \frac{dy^m}{d\zeta} - y^{*n} y^{*h} \frac{dy^{*m}}{d\zeta} \Big\} + \cdots \Big)$$

$$\Big(I - \int_a^\zeta [A_{fg}\, y^g + A_{fgi}\, y^g y^i + \cdots]\, \frac{\partial y^f}{\partial x^q} \frac{dx^q}{d\eta} d\eta + \cdots \Big) d\zeta \Big\} \lambda(a) +$$

$$+ \cdots$$

$$(6.1)\quad \lambda^*(b) - \lambda(b) = \text{(term on page 33)} +$$

$$\Big\{ \int_a^b \Big(I - \int_3^b [A_{kj} y^j + A_{klj} y^j y^l + \dots] \frac{dy^k}{dx^p} \frac{dx^p}{d\xi} d\xi + \dots \Big)$$

$$\Big(A_{mn} \{ y^n(3) \frac{dy^m}{d3} - y^{*n} \frac{dy^{m*}}{d3} \} + A_{mnh} \{ y^n y^h \frac{dy^m}{d3} - y^{n*} y^{h*} \frac{dy^{m*}}{d3} \} + \dots \Big) d3$$

$$\int_a^3 \Big(I - \int_\tau^3 [A_{gu} y^u(\eta) + \dots] \frac{dy^g}{d\eta} + \dots \Big) \Big(A_{df} \{ y^f \frac{dy^d}{d\tau} - y^{f*} \frac{dy^{d*}}{d\tau} \} + \dots \Big)$$

$$\Big(I - \int_a^\tau [A_{gi} y^i(\phi) + \dots] \frac{dy^g}{d\phi} d\phi + \dots \Big) d\tau \Big\} \lambda(a)$$

$$+ \dots$$

where $A_{jk \dots l}$ are the normal tensors of the space evaluated at $x^i(a)$, $y^k[x]$ are the transformation functions giving the normal coordinates with center at $x^i(a)$ in terms of the generic coordinates.

Proof: From (3.5)

$$(6.2)\quad \lambda^*(b) - \lambda(b) = \{ \Omega_a^{*b} [-\Gamma_k^* \frac{dx^{*k}}{d\xi}] - \Omega_a^b [-\Gamma_k \frac{dx^k}{d\xi}] \} \lambda(a)$$

where Ω^{*b}_a is the matrizant computed along the varied directrix $x^{*i}(\xi)$. From (4.8)

$$(6.3)\quad \Omega^* - \Omega = \sum_{i=0}^{\infty} \frac{1}{i!} \delta^i \Omega [-\Gamma_k \frac{dx^k}{d\xi} ; -\delta \Gamma_k \frac{dx^k}{d\xi}]$$

where $-\delta\Gamma_k \frac{dx^k}{d\xi} = \Gamma_k \frac{dx^k}{d\xi} - \Gamma_k^* \frac{dx^{*k}}{d\xi}$ the increment in the kernel of the matrizant Ω to yield Ω^*. For clarity we write out the first two terms.

$$(6.4)\quad \delta^1 \Omega_a^b = \int_a^b \Omega_3^b [-\Gamma_l \frac{dx^l}{d\xi} | \xi] \{ \Gamma_k(3) \frac{dx^k}{d3} - \Gamma_k^*(3) \frac{dx^{*k}}{d3} \} \Omega_a^3 [-\Gamma_m \frac{dx^m}{d\tau} | \tau] d3$$

$$(6.5)\quad \frac{1}{2!} \delta^2 \Omega_a^b = \int_a^b \Omega_3^b [-\Gamma_l \frac{dx^l}{d\xi}] \{ \Gamma_k(3) \frac{dx^k}{d3} - \Gamma_k^*(3) \frac{dx^{*k}}{d3} \} d3$$

$$\int_a^3 \Omega_\tau^3 [-\Gamma_m \frac{dx^m}{d\eta}] \{ \Gamma_n \frac{dx^n}{d\tau} - \Gamma_n^* \frac{dx^{*n}}{d\tau} \} \Omega_a^\tau [-\Gamma_p \frac{dx^p}{dv}]$$

As in theorem 3.5, we transform to normal coordinates with origin at $x^i(a)$. We expand Γ_k about the origin and transform back into generic coordinates. Using (3.21) and (3.22)

$$(6.6)\quad \Gamma_k \frac{dx^k}{ds} - \Gamma^*_k \frac{dx^{*k}}{ds} = A_{kj}\left(y^j[x(\xi)]\,\frac{dy^k[x(\xi)]}{d\xi} - y^{*j}[x^*(\xi)]\,\frac{dy^{*k}[x^*(\xi)]}{d\xi}\right) + \frac{1}{2!}A_{kjl}\left(y^j y^l \frac{dy^k}{d\xi} - y^{*j} y^{*l}\frac{dy^{*k}}{d\xi}\right)$$

where $A_{kj\ldots l}$ are the normal tensors evaluated at $x^i(a)$, $y^j = y^j[x(\xi)]$ are the parametric equations of the original directrix and $y^j = y^{*j}[x^*(\xi)]$ are the equations of the varied directrix in normal coordinates transformed to generic coordinates. Using (3.6), (3.21) and (3.22)

$$(6.7)\quad \Omega_r^t\left[-\Gamma_k \frac{dx^k}{ds}\right] = I - \int_r^t \left[A_{kj}\, y^j[x] + A_{kjl}\, y^j[x]\, y^l[x] + \cdots\right]\frac{\partial y^k[x]}{\partial x^p}\frac{dx^p}{ds}\,ds + \cdots$$

Hence using (6.6) and (6.7) in (6.3) and (6.2) proves theorem 6.1.

VI.B. <u>Two Lemmas for the Existence of Fréchet Differentials</u> $\delta^i \Omega_a^b\left[-\Gamma_k \frac{dx^k}{d\xi}\,;\, z\,|\,\xi\right]$

The theorems of chapter IV permit us to compute the Fréchet differentials of the matrizant $\delta^i\Omega[A\,;\delta A|\xi]$ with increments δA . Usually, however, we do not think of a displaced vector as a function of $-\Gamma_k \frac{dx^k}{d\xi}$ but as a function of the space, or of the directrix. In chapter V we discussed the Fréchet differentials with increments $\delta\Gamma_k$ produced by warping the space. We shall now prove two theorems which will guarantee the existence of the differentials with increment $z^i(\xi)$ produced by changing the directrix.

<u>Theorem 6.2</u>. <u>If the function</u> $x^i(\xi)$ <u>is a function on the real</u>

numbers $a \le \xi \le b$ to the vector space E_1 of sets of n continuous functions differentiable almost everywhere, and if $\frac{dx^i}{d\xi}$ is regarded as a function of x on E_1 to E_2, a second vector space, and if the norms of E_1 and E_2 are respectively

(6.8) $$\| x^i(\xi) \|_{E_1} = \max \left\{ |x^i(\xi)|, \left|\frac{dx^i(\xi)}{d\xi}\right| \right\} \quad a \le \xi \le b ; i = 1, \cdots, n$$

$$\| y^i(\xi) \|_{E_2} = \max \left\{ |y^i(\xi)| \right\} \quad a \le \xi \le b ; i = 1, \cdots, n$$

then $\frac{dx^i(\xi)}{d\xi}$ possesses a first Fréchet differential with increment $z(\xi)$ for any x and z in E_1 given by

(6.9) $$\delta' \frac{dx^i}{d\xi} [x^i ; z^i] = \frac{d z^i(\xi)}{d\xi}$$

Proof: The Fréchet differential is given by the Gateaux differential provided the Gateaux differential is additive and continuous in z and provides the principal part of the first difference. The Gateaux differential is

(6.10) $$\mathcal{L} \frac{dx^i}{d\xi} = \lim_{\lambda \to 0} \frac{\frac{d(x + \lambda z)}{d\xi} - \frac{dx}{d\xi}}{\lambda} = \frac{dz}{d\xi}$$

which is obviously additive in z. Since the first difference $\Delta \frac{dx}{d\xi} = \frac{dz}{d\xi}$, the differential is obviously the whole of the first difference. Since it is additive it will be continuous in z if it is continuous in z near the origin of E_1.

But from (6.8), if $\|z\| < \varepsilon$ where ε is arbitrarily small, then $\| \frac{dz}{d\xi} \| < \varepsilon$. Hence $\frac{dz}{d\xi}$ is continuous in z and the Fréchet differential exists.

It is necessary to include $\left|\frac{dx^i}{d\xi}\right|$ in (6.8) to allow for pathological curves $z(\xi)$ for which $|z|$ is small but $\left|\frac{dz}{d\xi}\right|$ is very large. This guarantees the continuity of $\frac{dz}{d\xi}$ in z, which otherwise would not exist. For if $\|x\|_{E_1}$ were just max $|x^i(\xi)|$, then given any M no matter how large and ε no matter how small it would be easy to find a curve $x^i(\xi)$ such that $\|x^i\| < \varepsilon$ but $\|dx^i/d\xi\| > M$ and $\frac{dx^i}{d\xi}$ would not be continuous in x.

Theorem 6.3. The function $\Gamma_k \frac{dx^k}{d\xi}$ on the n Banach spaces B_1 of square matrices of order n and the n vector spaces E_2 to the Banach space B_1 possesses a first Fréchet differential with increment z in E_2 given by

(6.11) $$\delta' \Gamma_k \frac{dx^k}{d\xi}[x;z] = \frac{\partial \Gamma_k}{\partial x^l} z^l \frac{dx^k}{d\xi} + \Gamma_k \frac{dz^k}{d\xi}$$

Proof: Since Γ_k is an analytic function of the coordinates x $\delta'\Gamma_k[x;z]$ exists and is the ordinary differential, $\frac{\partial \Gamma_k}{\partial x^l} z^l$. By theorem 6.2 $\delta' \frac{dx^k}{d\xi}[x;z]$ exists and is given by $\frac{dz^k}{d\xi}$. Hence since $\Gamma_k \frac{dx^k}{d\xi}$ is a sum of multilinear functions the theorem follows from the formula for Fréchet differentials of multilinear functions.

VI.C. The First Fréchet Differential $\delta'\lambda[x;z|b]$

We now use theorem 6.3 to prove

Theorem 6.4. Let a contravariant vector, indicated by the column matrix λ be displaced by Levi-Civita parallel displacement along the directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$. Let $\lambda(a)$, $\lambda(b)$ be the values of λ at $\xi = a, b$ respectively, and related by $\lambda(b) = \Omega_a^b \lambda(a)$ where Ω_a^b is the matrizant function. If the directrix is changed to a new directrix with the same end points with equations $x^i = x^{*i}(\xi) = x^i(\xi) + z^i(\xi)$ then $\lambda(b)$ possesses a first Fréchet differential with increment z given by

$$(6.12)\qquad \delta'\lambda[x;z|b] = \int_a^b \Omega_s^b[-\Gamma_k \frac{dx^k}{d\xi}|\xi]\; R_{\alpha_1\alpha_2}\, z^{\alpha_2}(s)\, \frac{dx^{\alpha_1}}{ds}\; \Omega_a^s[-\Gamma_l \frac{dx^l}{d\xi}]\, ds$$

Proof: From theorems 4.1 and 6.2 we know that the Frechet differential of $\Omega_a^b[-\Gamma_k \frac{dx^k}{d\xi}]$ with increment $\Delta\{-\Gamma_k \frac{dx^k}{d\xi}\}$ exists, and that $\delta'[\Gamma_k \frac{dx^k}{d\xi}; z]$ exists. We may regard Ω_a^b as a function on E_1 to B_2, that is, a function of x defined by $\Omega_a^b[x] = \Omega_a^b[A[x]]$ Then by the composition theorem on Fréchet differentials, $\delta'\Omega_a^b[x;z]$ exists and is given by

$$(6.13)\qquad \begin{aligned} \delta'\Omega_a^b[x;z] &= \delta'\Omega_a^b[A;\delta' A[x;z]] \\ &= \delta'\Omega_a^b[-\Gamma_k \frac{dx^k}{d\xi};\, -\delta'\Gamma_l \frac{dx^l}{d\xi}] \\ &= \delta'\Omega_a^b[-\Gamma_k \frac{dx^k}{d\xi};\, -\frac{\partial \Gamma_l}{\partial x^m} z^m \frac{dx^l}{d\xi} - \Gamma_l \frac{dz^l}{d\xi}] \end{aligned}$$

By theorem 6.1, (6.13) is

$$(6.14)\qquad \delta'\Omega_a^b[x;z] = \int_a^b \Omega_s^b[-\Gamma_k \frac{dx^k}{d\xi}|\xi]\left(-\frac{\partial \Gamma_l}{\partial x^m} z^m \frac{dx^l}{ds} - \Gamma_l \frac{dz^l}{ds}\right) \Omega_a^s[-\Gamma_p \frac{dx^p}{d\eta}|\eta]\, ds$$

The right member of (6.14) can be broken into two parts by distributing the middle factor. Taking the second term and integrating by parts,

$$(6.15)\qquad \begin{aligned} -\int_a^b \Omega_s^b\, \Gamma_l \frac{dz^l}{ds}\, \Omega_a^s\, ds &= -\left\{\Omega_s^b\, \Gamma_l\, z^l\, \Omega_a^s\right\}_a^b \\ &+ \int_a^b \frac{d\Omega_s^b}{ds}\, \Gamma_l\, z^l\, \Omega_a^s\, ds \\ &+ \int_a^b \Omega_s^b \frac{\partial \Gamma_l}{\partial x^m} \frac{dx^m}{d\xi}\, z^l\, \Omega_a^s\, ds \\ &+ \int_a^b \Omega_s^b\, \Gamma_l\, z^l \frac{d\Omega_a^s}{ds}\, ds \end{aligned}$$

Now since the varied directrix has the same initial and end points as the initial directrix, $z(a) = z(b) = 0$. Hence the non-integral term on the right of (6.15) vanishes. Further

(6.16) $$\frac{d\Omega_s^b}{ds} = \frac{d}{ds}\left[1 - \int_s^b \Gamma_k \frac{dx^k}{d\xi} d\xi + \int_s^b \Gamma_k \frac{dx^k}{d\xi_1} d\xi_1 \int_s^{\xi_1} \Gamma_\ell \frac{dx^\ell}{d\xi_2} d\xi_2 - \cdots\right]$$

$$= \Gamma_k \frac{dx^k}{ds} - \int_s^b \Gamma_k(\xi) \frac{dx^k}{d\xi} d\xi\, \Gamma_\ell(s) \frac{dx^\ell}{ds} + \cdots$$

$$= \Omega_s^b\left[-\Gamma_\ell \frac{dx^\ell}{d\xi} \Big| \xi\right] \Gamma_k(s) \frac{dx^k(s)}{ds}$$

and

(6.17) $$\frac{d\Omega_a^s}{ds} = \frac{d}{ds}\left[1 - \int_a^s \Gamma_k \frac{dx^k}{d\xi} d\xi + \int_a^s \Gamma_k \frac{dx^k}{d\xi_1} d\xi_1 \int_a^{\xi_1} \Gamma_\ell \frac{dx^\ell}{d\xi_2} d\xi_2 - \cdots\right]$$

$$= -\Gamma_k(s) \frac{dx^k(s)}{ds} + \Gamma_k(s) \frac{dx^k(s)}{ds} \int_a^s \Gamma_\ell \frac{dx^\ell}{d\xi_2} d\xi_2 - \cdots$$

$$= -\Gamma_k(s) \frac{dx^k(s)}{ds}\, \Omega_a^s\left[-\Gamma_\ell \frac{dx^\ell}{d\xi} \Big| \xi\right]$$

Substituting in (6.15) gives

(6.18) $$-\int_a^b \Omega_s^b\, \Gamma_k \frac{dz^k}{ds}\, \Omega_a^s\, ds =$$

$$\int_a^b \Omega_s^b \left\{ \frac{\partial \Gamma_k}{\partial x^\ell} \frac{dx^\ell}{ds} z^k + \Gamma_\ell \frac{dx^\ell}{ds} \Gamma_k z^k - \Gamma_k z^k \Gamma_\ell \frac{dx^\ell}{ds} \right\} \Omega_a^s\, ds$$

Using (6.18) in (6.14)

(6.19) $$\delta' \Omega_a^b[x;z] = \int_a^b \Omega_s^b \left\{ \frac{\partial \Gamma_k}{\partial x^\ell} - \frac{\partial \Gamma_\ell}{\partial x^k} + \Gamma_\ell \Gamma_k - \Gamma_k \Gamma_\ell \right\} z^k \frac{dx^\ell}{ds}\, \Omega_a^s\, ds$$

But the matrix form of the Riemann curvature tensor with one contravariant and three covariant indeces in generic coordinates is

(6.20) $$R_{\ell k} = \frac{\partial \Gamma_k}{\partial x^\ell} - \frac{\partial \Gamma_\ell}{\partial x^k} + \Gamma_\ell \Gamma_k - \Gamma_k \Gamma_\ell$$

Using (6.20) and (6.19) gives the theorem, since

(6.21) $\delta' \lambda(b) = \delta' \Omega_a^b \lambda(a)$

because $\delta' \lambda(a) = 0$

VI.D. The Second Fréchet Differential $\delta^2 \lambda [x; z \mid b]$

Theorem 6.4 has given us the expression for the first Fréchet differential of λ . We now seek the second Fréchet differential of λ with equal increments z . We begin with a lemma.

Theorem 6.5. Under the conditions of theorem 6.4

$$(6.22) \quad \delta' \Omega_\eta^\tau [x; z] = \int_\eta^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_\eta^s \, ds + \Omega_\eta^\tau \Gamma_k(\eta) z^k(\eta) - \Gamma_k(\tau) z^k(\tau) \Omega_\eta^\tau$$

Proof: Following exactly the same line of reasoning as in theorem 6.4 we have

$$(6.23) \quad \delta' \Omega_\eta^\tau [x; z] = \int_\eta^\tau \Omega_s^\tau \left(- \frac{\partial \Gamma_k}{\partial x^l} z^l \frac{dx^k}{ds} - \Gamma_k \frac{dz^k}{ds} \right) \Omega_\eta^s \, ds$$

Integrating the second term of the right member of (6.23) by parts

$$(6.24) \quad - \int_\eta^\tau \Omega_s^\tau \Gamma_k \frac{dz^k}{ds} \Omega_\eta^s \, ds = - \left[\Omega_s^\tau \Gamma_k z^k \Omega_\eta^s \right]_\eta^\tau + \int_\eta^\tau \frac{d}{ds} \Omega_s^\tau \Gamma_k z^k \Omega_\eta^s \, ds + \int_\eta^\tau \frac{\partial \Gamma_k}{\partial x^l} z^k \frac{dx^l}{ds} \Omega_\eta^s \, ds + \int_\eta^\tau \Omega_s^\tau \Gamma_k z^k \frac{d}{ds} \Omega_\eta^s \, ds$$

Evaluating the non-integral terms on the right in (6.24)

$$(6.25) \quad - \left[\Omega_s^\tau \Gamma_k z^k \Omega_\eta^s \right]_\eta^\tau = \Omega_\eta^\tau \Gamma_k(\eta) z^k(\eta) - \Gamma_k(\tau) z^k(\tau) \Omega_\eta^\tau$$

since $\Omega_s^\tau = I$, the unit matrix, when $s = \tau$.

Using (6.20), (6.24) and (6.25) in (6.23) gives theorem 6.5

Using theorem 6.5 we now consider

Theorem 6.6. Under the conditions of theorem 6.4, $\lambda(b)$ possesses a second Fréchet differential with equal increments z given by

(6.26)

$$\delta^2\lambda[x;z;z] = \Big\{ \int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dz^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds +$$

$$+ 2! \int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \, ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1\beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} \, ds_2$$

$$+ \int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3} \Omega_a^s \, ds$$

where

(6.27) $$P_{\alpha_1\alpha_2\alpha_3} = \frac{\partial R_{\alpha_1\alpha_2}}{\partial x^{\alpha_3}} + \Gamma_{\alpha_3} R_{\alpha_1\alpha_2} - R_{\alpha_1\alpha_2} \Gamma_{\alpha_3}$$

Proof: Since $\lambda(a)$ is independent of the directrix,

(6.28) $$\delta^2\lambda[x;z;z] = \delta^2\{\Omega_a^b \lambda(a)\} = \delta^2\Omega_a^b[x;z;z]\lambda(a)$$

Further

(6.29) $$\delta^2\Omega_a^b = \delta'\{\delta'\Omega_a^b\} = \delta' \int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds$$

Now Ω_s^b, $R_{\alpha_1\alpha_2}$ are elements in the Banach spaces B_2, B_1 of square matrices. $\frac{dx^\alpha}{ds}$ and z^α are elements in the vector spaces E_2 and E_1 respectively. Clearly $\delta'\Omega_a^b$ is a sum of multilinear functions on B_2 B_1 E_2 E_1 B_2 to B_2. Hence if the various factors have Fréchet differentials with increment z, then the Fréchet differential of $\delta'\Omega_a^b$ with increment z exists.

But theorems 6.5, and 6.2 guarantee the existence of $\delta'\Omega_s^b[x;z]$ and $\delta'\frac{dx^\alpha}{ds}[x;z]$. $\delta' z = 0$ and $\delta' R_{\alpha_1\alpha_2} = \frac{\partial R_{\alpha_1\alpha_2}}{\partial x^{\alpha_3}} z^{\alpha_3}$ since $R_{\alpha_1\alpha_2}$ is an analytic point function whose Fréchet differential is the ordinary differential. Hence by the theorem on the Fréchet differential of a multilinear function

$$(6.30)\quad \delta^2\Omega_a^b[x;z] = \int_a^b \delta^1\Omega_s^b[x;z]\, R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\Omega_a^s\, ds + \qquad (1)$$

$$\int_a^b \Omega_s^b \frac{\partial R_{\alpha_1\alpha_2}}{\partial x^{\alpha_3}}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3}\Omega_a^s\, ds + \qquad (2)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds} z^{\alpha_2}\Omega_a^s\, ds + \qquad (3)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\delta^1\Omega_a^s[x;z]\, ds \qquad (4)$$

For convenience, the four terms on the right in (6.30) are numbered (1) - (4)

Using theorem 6.5, terms (1) and (4) together become

$$(6.31)\quad \int_a^b\int_s^b \Omega_t^b R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{dt} z^{\beta_2}\Omega_s^t\, dt\, R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\Omega_a^s\, ds + \qquad (i)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\, ds \int_a^s \Omega_t^s R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{dt} z^{\beta_2}\Omega_a^t\, dt + \qquad (ii)$$

$$\int_a^b \Omega_s^b \Gamma_{\alpha_3} z^{\alpha_3} R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\Omega_a^s\, ds + \qquad (iii)$$

$$-\int_a^b \Omega_s^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds} z^{\alpha_2}\Gamma_{\alpha_3} z^{\alpha_3}\Omega_a^s\, ds \qquad (iiii)$$

where the terms in (6.31) are numbered (i) - (iiii) for convenience. The terms with α indeces are functions with parameter s. Those with β indeces have parameter t.

Interchanging the order of integration and certain dummy indeces in (i) shows (i) = (ii). Combining (iii), (iiii) with (2) of (6.30), we may write (6.30) as

$$(6.32)\quad \delta^2\Omega_a^b[x;z] =$$

$$2!\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2}\, ds_1 \int_a^{s_1}\Omega_{s_2}^{s_1} R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2} z^{\beta_2}\Omega_a^{s_2}\, ds_2 +$$

$$+\int_a^b \Omega_s^b\left[\frac{\partial R_{\alpha_1\alpha_2}}{\partial x^{\alpha_3}} + \Gamma_{\alpha_3}R_{\alpha_1\alpha_2} - R_{\alpha_1\alpha_2}\Gamma_{\alpha_3}\right]\frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3}\Omega_a^s\, ds +$$

$$+\int_a^b \Omega_s^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds} z^{\alpha_2}\Omega_a^s\, ds$$

Hence on using (6.32), (6.28) and (6.27), we have theorem 6.6.

VI.E. Dagger Differentiation.

On comparing the expressions for $\delta^1 \Omega_a^b[x;z]$ and $\delta^2 \Omega_a^b[x;z]$ we see that $\delta^2 \Omega_a^b$ is almost but not quite obtained by formally differentiating the expression for $\delta^1 \Omega_a^b$ with respect to an arbitrary parameter. We shall now define a formal notation, called "dagger differentiation", and symbolized by $\swarrow$ whose rules will give $\delta^2 \Omega$ from $\delta^1 \Omega$ by formal operations.

Theorem 6.7. Dagger Differentiation. If the operation called "Dagger Differentiation" denoted by the symbol " $\swarrow$ " has the following properties

$$(6.33) \quad \Omega_n^{r\swarrow} = \int_n^r \Omega_s^r R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_n^s \, ds$$

$$R^{\swarrow}_{\alpha_1\alpha_2} = P_{\alpha_1\alpha_2\alpha_3} z^{\alpha_3}$$

$$\frac{dx^{k\swarrow}}{ds} = \frac{dz^k}{ds}$$

$$z^{\swarrow} = 0$$

and if the ordinary formulas for the derivatives of products and sums hold for dagger differentiation, then

$$\delta^1 \Omega_a^b[x;z] = \Omega_a^{b\swarrow}$$

$$\delta^2 \Omega_a^b[x;z] = \Omega_a^{b\swarrow\swarrow}$$

Proof: Setting $\eta = a$ $r = b$ in (1) of (6.33) gives (6.12). By (6.33)

$$(6.34) \quad \left\{\int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds\right\}^{\swarrow} =$$

$$\int_a^b \Omega_s^{b\swarrow} R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds + \qquad (1)$$

$$\int_a^b \Omega_s^b R^{\swarrow}_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s ds + \qquad (2)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1 \swarrow}}{ds} z^{\alpha_2} \Omega_a^s ds + \qquad (3)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2 \swarrow} \Omega_a^s ds + \qquad (4)$$

$$\int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^{s \swarrow} ds \qquad (5)$$

whose terms are numbered (1) - (5) for convenience. Numbers (1) and (5) from (6.34) are

$$(6.35)\quad \int_a^b \int_{s_1}^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \Omega_{s_2}^{s_1} ds_1 R_{\beta_1\beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} ds_2$$

$$\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1\beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} ds_2$$

Note that terms with α indeces are functions of parameter s_1 and those with β indeces are functions of parameter s_2. On interchanging the order of integration in the first of (6.35), terms (1) and (5) add together to give

$$(6.36)\quad 2! \int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1\beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} ds_2$$

Further, term (4) vanishes. We identify (6.36) with the first term in (6.32); terms (2) and (3) with the second and third terms of (6.32) to give the theorem.

VI.F. <u>The m^th Fréchet Differential</u> $\delta^m \lambda [x; z; z; \cdots; z \,|\, b]$

Having found the first and second Fréchet differentials we seek a recurrence formula for the m^{th} Fréchet differentials $\delta^m \lambda [x; z \,/\, b]$ with equal increments z. As a preliminary we define the sequence of P functions

$$(6.37)\qquad P_{\alpha_1\alpha_2\alpha_3} = \frac{\partial R_{\alpha_1\alpha_2}}{\partial x^{\alpha_3}} + \Gamma_{\alpha_3} R_{\alpha_1\alpha_2} - R_{\alpha_1\alpha_2}\Gamma_{\alpha_3}$$

$$P_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{\partial P_{\alpha_1\alpha_2\alpha_3}}{\partial x^{\alpha_4}} + \Gamma_{\alpha_4} P_{\alpha_1\alpha_2\alpha_3} - P_{\alpha_1\alpha_2\alpha_3}\Gamma_{\alpha_4}$$

$$\vdots$$

$$P_{\alpha_1\cdots\alpha_m} = \frac{\partial P_{\alpha_1\cdots\alpha_{m-1}}}{\partial x^{\alpha_m}} + \Gamma_{\alpha_m} P_{\alpha_1\cdots\alpha_{m-1}} - P_{\alpha_1\cdots\alpha_{m-1}}\Gamma_{\alpha_m}$$

$$\vdots$$

The first in this sequence of P functions has already been defined in (6.27). We extend the properties of dagger differentiation defined in (6.33) to include

$$(6.38)\qquad P^{\swarrow}_{\alpha_1\cdots\alpha_m} = P_{\alpha_1\cdots\alpha_m\alpha_{m+1}}\, z^{\alpha_{m+1}}$$

$$\frac{dz^{\alpha\swarrow}}{da} = 0$$

With this extension we have

<u>Theorem 6.8.</u> <u>Under the conditions of theorem 6.4, the mth Frechet differential</u> $\delta^m \lambda\, [x; z\,|\,b]$ <u>with equal increments</u> z <u>exists and is given by the recurrence formula</u>

$$(6.39)\qquad \delta^m \lambda\, [x; z\,|\,b] = \Omega_a^{b\,\swarrow\cdots(m)\cdots\swarrow}\, \lambda(a)$$

<u>where</u> $\Omega_a^{b\,\swarrow\cdots(m)\cdots\swarrow}$ <u>is the m</u>th <u>dagger differential of</u> Ω_a^b <u>defined in (6.33) and (6.38)</u>

Proof: As in (6.28), since $\lambda(a)$ is independent of the directrix,

$$(6.40)\qquad \delta^m \lambda(b) = \delta^m \Omega_a^b\, [x; z]\, \lambda(a)$$

We shall now show that

$$(6.41)\qquad \delta^m \Omega_a^b = \Omega_a^{b\,\swarrow\cdots(m)\cdots\swarrow}$$

The proof of (6.41) is by induction. Theorem 6.7 shows (6.41) to be true for m = 1, 2. We assume it true for m-1 and prove it true for m.

By considering the defining rules (6.33) and (6.38) of dagger differentiation, it is clear that the formula for the m-1 dagger derivative of Ω_a^b is not a simple one. However $\Omega_a^{b\,\dagger\cdots(m-1)\cdots\dagger}$ is the sum of terms each of which is homogeneous and of degree m-1 in z. A typical term is

(6.42) $$\int_a^b \Omega_{s_1}^b P_{\alpha_1\cdots\alpha_j} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2}\cdots z^{\alpha_j} \int_a^{s_1} \Omega_{s_2}^{s_1} P_{\alpha_{j+1}}\cdots P_{\alpha_m} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}}\cdots z^{\alpha_m} \Omega_a^{s_2} ds_2$$

The term with the most integrals is

(6.43)
$$\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1\beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} ds_2 \cdots$$
$$\int_a^{s_{m-2}} \Omega_{s_{m-1}}^{s_{m-2}} R_{\gamma_1\gamma_2} \frac{dx^{\gamma_1}}{ds_{m-1}} z^{\gamma_2} \Omega_a^{s_{m-1}} ds_{m-1}$$

We shall show that the Fréchet differentials of (6.42) and (6.43) with increments z are indeed given by the dagger derivative. Further the method is general and applies to any other term in the expansion for $\Omega_a^{b\,\dagger\cdots(m-1)\cdots\dagger}$.

Consider (6.42). Ω_η^τ, $P_{\alpha_1\cdots\alpha_j}$ are elements in the Banach spaces B_2, B_1 of square matrices of order n of chapter IV. z^α is also an element of the vector space E_1 while $\frac{dx^\alpha}{ds}$ and $\frac{dz^\alpha}{ds}$ are in the vector space E_2 . Clearly (6.42) is a multilinear function on $B_2\ B_1\ E_2\ E_1\ B_2\ B_1\ E_2\ E_1\ B_2$ to B_2 . Hence the Fréchet differential of (6.42) exists if the Fréchet differential exists for each factor. Since $P_{\alpha_1\cdots\alpha_j}$ is a differentiable function of x, $\delta' P_{\alpha_1\cdots\alpha_j}[x;z]$ exists given by the ordinary differential

(6.44) $$\delta' P_{\alpha_1\cdots\alpha_j}[x;z] = \frac{\partial P_{\alpha_1\cdots\alpha_j}}{\partial x^{\alpha_{j+1}}} z^{\alpha_{j+1}}.$$

Theorems 6.5 and 6.2 give the formula for $\delta' \Omega^{\gamma}_{n} [x; z]$ and $\delta' \frac{dx^{\alpha}}{ds} [x; z]$. $\delta' z^{\alpha} = \delta' \frac{dz^{\alpha}}{ds} = 0$

Let us write $\delta'(6.42)$ for the first Fréchet differential with increment z of the expression given by equation (6.42). Then by the formula for the Fréchet differential of a multilinear function

(6.45) $\qquad \delta'(6.42) =$

$$(1) \int_a^b \delta' \Omega^{b}_{s_1} P_{\alpha_1 \dots \alpha_j} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} ds_1 \int_a^{s_1} \Omega^{s_1}_{s_2} P_{\alpha_{j+1} \dots \alpha_m} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} \Omega^{s_2}_{a} ds_2 +$$

$$(2) \int_a^b \Omega^{b}_{s_1} \frac{\partial P_{\alpha_1 \dots \alpha_j}}{\partial x^{\gamma}} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} z^{\gamma} ds_1 \int_a^{s_1} \Omega^{s_1}_{s_2} P_{\alpha_{j+1} \dots \alpha_m} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} \Omega^{s_2}_{a} ds_2 +$$

$$(3) \int_a^b \Omega^{b}_{s_1} P_{\alpha_1 \dots \alpha_j} \frac{dz^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} ds_1 \int_a^{s_1} \Omega^{s_1}_{s_2} P_{\alpha_{j+1} \dots \alpha_m} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} \Omega^{s_2}_{a} ds_2 +$$

$$(4) \int_a^b \Omega^{b}_{s_1} P_{\alpha_1 \dots \alpha_j} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} ds_1 \int_a^{s_1} \delta' \Omega^{s_1}_{s_2} P_{\alpha_{j+1} \dots \alpha_m} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} \Omega^{s_2}_{a} ds_2 +$$

$$(5) \int_a^b \Omega^{b}_{s_1} P_{\alpha_1 \dots \alpha_j} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} ds_1 \int_a^{s_1} \Omega^{s_1}_{s_2} \frac{\partial P_{\alpha_{j+1} \dots \alpha_m}}{\partial x^{\gamma}} \frac{dz^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} z^{\gamma} \Omega^{s_2}_{a} ds_2 +$$

$$(6) \int_a^b \Omega^{b}_{s_1} P_{\alpha_1 \dots \alpha_j} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} \dots z^{\alpha_j} ds_1 \int_a^{s_1} \Omega^{s_1}_{s_2} \frac{\partial P_{\alpha_{j+1} \dots \alpha_m}}{\partial x^{\gamma}} \frac{dx^{\alpha_{j+1}}}{ds_2} z^{\alpha_{j+2}} \dots z^{\alpha_m} \delta' \Omega^{s_2}_{a} ds_2$$

where the terms have been numbered (1) - (6) for convenience.

We wish to compare these results with the dagger derivatives. Hence we compute $(6.42)^{\swarrow}$ omitting all terms involving $z^{\swarrow}$ or $\frac{dz}{ds}^{\swarrow}$ since these terms vanish. The terms are numbered $(1)^{\swarrow}$ - $(6)^{\swarrow}$ to compare with terms numbered (1) - (6) in (6.45). The obvious indeces have been omitted for clarity.

(6.46) $\qquad (6.42)^{\swarrow} =$

$$(1)^{\swarrow} \qquad \int_a^b \Omega^{\swarrow} P \frac{dx}{ds} z \dots z \, ds \int_a^{s} \Omega P \frac{dz}{ds} z \dots z \, \Omega \, ds +$$

$$(2)^{\swarrow} \qquad \int_a^b \Omega P^{\swarrow} \frac{dx}{ds} z \dots z \, ds \int_a^{s} \Omega P \frac{dz}{ds} z \dots z \, \Omega \, ds +$$

(6.46) continued.

$$(3)^{\checkmark} \int \Omega\, P \frac{dz}{ds}\, z \cdots z\, ds \int \Omega\, P \frac{dz}{ds}\, z \cdots z\, \Omega\, ds +$$

$$(4)^{\checkmark} \int \Omega\, P \frac{dx}{ds}\, z \cdots z\, ds \int \Omega^{\checkmark} P \frac{dz}{ds}\, z \cdots z\, \Omega\, ds +$$

$$(5)^{\checkmark} \int \Omega\, P \frac{dx}{ds}\, z \cdots z\, ds \int \Omega\, P^{\checkmark} \frac{dz}{ds}\, z \cdots z\, \Omega\, ds +$$

$$(6)^{\checkmark} \int \Omega\, P \frac{dx}{ds}\, z \cdots z\, ds \int \Omega\, P \frac{dz}{ds}\, z \cdots z\, \Omega^{\checkmark} ds$$

On comparison, $(3) \equiv (3)^{\checkmark}$. From theorem 6.5, (1) will have two terms, one involving $\int_{s_1}^{t} \Omega_t^{t} R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{dt} z^{\alpha_2} \Omega_{s_1}^{t}\, dt$. This term is identical with $(1)^{\checkmark}$ on using (6.33). The other term of (1) involving $+\Gamma_\gamma z^\gamma$ we will combine with (2).

Further from theorem 6.5 (4) will consist of three terms. The one involving $\int_{s_2}^{s_1} \Omega_t^{s_1} R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{dt} z^{\alpha_2} \Omega_{s_2}^{t}\, dt$ is identical with $(4)^{\checkmark}$. The term involving $-\Gamma_\gamma z^\gamma$ is combined with term (2). The third term involving $+\Gamma_\gamma z^\gamma$ is combined with (5).

Similarly (6) consists of two terms, one identical with (6) and one which is combined with (5).

But the three terms (2), the original (2) plus the extra term from (1) and the extra term from (4), add together to give $(2)^{\checkmark}$. The three terms (5), the original (5) plus the extra term from (4) and the extra term from (6), add together to give $(6)^{\checkmark}$. Hence $(6.45) \equiv (6.46)$.

The method is clearly perfectly general. For example (6.43) is a sum of multilinear functions on $B_2\ B_1\ E_2\ E_1\ B_2\ B_1\ E_2\ E_1 \ldots B_2$ to B_2, with the Fréchet differential computed by the product rule. Every term involving $\delta' \frac{dx^\alpha}{ds}[x; z]$ is the same as the corresponding term involving $\frac{dx^\alpha}{ds}{}^{\swarrow}$. Each term involving $\delta' R_{\alpha_1\alpha_2}$ is combined with a term involving $\Gamma_\gamma \frac{dx^\gamma}{ds}$ from the $\delta'\Omega$, the differential of the matrizant appearing just to the left of $R_{\alpha_1\alpha_2}$ in (6.43) and with a term involving $-\Gamma_\gamma \frac{dx^\gamma}{ds}$ from the differential of the matrizant on the right to give $R^{\swarrow}_{\alpha_1\alpha_2}$. The remaining terms of the differentials of the matrizants Ω are $\Omega^{\swarrow}$.

This method can be applied to any term. We note that the equality $\delta'(6.42) \equiv (6.42)^{\swarrow}$ required a regrouping of the terms resulting from $\delta'(6.42)$, but that it required no regrouping of terms arising from any other term in $\delta^{m-1}\Omega_a^b$. This quality is always true, so for each term T of $\delta^{m-1}\Omega_a^b$ we have

$$\delta' T = T^{\swarrow} \tag{6.47}$$

Hence

$$\delta'\delta^{m-1}\Omega_a^b = (\delta^{m-1}\Omega_a^b)^{\swarrow} = (\Omega_a^{b\,\swarrow\cdots(m-1)\cdots\swarrow})^{\swarrow} = \Omega_a^{b\,\swarrow\cdots(m)\cdots\swarrow} \tag{6.48}$$

which completes the induction.

<u>Corollary</u> 6.8.1. <u>Under the conditions of theorem 6.4 the vector</u> $\lambda(b)$ <u>possesses the generalized Taylor series expansion</u>

$$\lambda(b) = \left\{ I + \sum_{i=1}^{\infty} \frac{1}{i!}\, \delta^i \Omega_a^b [x; z] \right\} \lambda(a)$$

Proof: The proof is immediate from theorems 4.4 and 6.8.

VI.G. <u>The Explicit Formulae for</u> $\delta^3\Omega_a^b[x;z]$, $\delta^4\Omega_a^b[x;z]$ <u>and</u> $\delta^2\Omega_\eta^\tau[x;z]$

It is clear from the recurrsion formula (6.39) that successive Fréchet differentials of Ω_a^b with equal increments z will be more and more complicated. However, it will be of interest to compute $\delta^3\Omega_a^b$ and $\delta^4\Omega_a^b$ for future reference. Further the recurrsion formula is simplified by the fact that the original and varied directrices have the same end points. The first Frechet differential for the more general case where the original and varied directrices have different end points was given in theorem 6.5. It will be of interest to compute the second as well. We give the first result as

<u>Theorem 6.9.</u> <u>Under the conditions of theorem</u> 6.4 <u>the third and fourth Frechet differentials of</u> $\Omega_a^b[x]$ <u>with equal increments z are given by</u>

(6.49) $$\delta^3\Omega_a^b[x;z] = 2\int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3}\frac{dz^{\alpha_1}}{ds}z^{\alpha_2}z^{\alpha_3}\Omega_a^s\,ds +$$

$$+\int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3\alpha_4}\frac{dx^{\alpha_1}}{ds}z^{\alpha_2}z^{\alpha_3}z^{\alpha_4}\Omega_a^s\,ds +$$

$$+3\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dz^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2 +$$

$$+3\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2 +$$

$$+3\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}P_{\beta_1\beta_2\beta_3}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}z^{\beta_3}\Omega_a^{s_2}ds_2$$

$$+3\int_a^b \Omega_{s_1}^b P_{\alpha_1\alpha_2\alpha_3}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}z^{\alpha_3}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2$$

$$+3!\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3$$

and

(6.50) $$\delta^4\Omega_a^b[x;z] = 3\int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3\alpha_4}\frac{dz^{\alpha_1}}{ds}z^{\alpha_2}z^{\alpha_3}z^{\alpha_4}\Omega_a^s\,ds +$$

$$6\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dz^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2 +$$

$$8\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}P_{\beta_1\beta_2\beta_3}\frac{dz^{\beta_1}}{ds_2}z^{\beta_2}z^{\beta_3}\Omega_a^{s_2}ds_2 +$$

$$8\int_a^b \Omega_{s_1}^b P_{\alpha_1\alpha_2\alpha_3}\frac{dz^{\alpha_1}}{ds_1}z^{\alpha_2}z^{\alpha_3}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2 +$$

(6.50) cont'd.

$$+\;6\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}P_{\beta_1\beta_2\beta_3}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}z^{\beta_3}\Omega_a^{s_2}ds_2\;+$$

$$6\int_a^b \Omega_{s_1}^b P_{\alpha_1\alpha_2\alpha_3}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}z^{\alpha_3}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dz^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2\;+$$

$$\int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}z^{\alpha_3}z^{\alpha_4}z^{\alpha_5}\Omega_a^s ds\;+$$

$$4\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}P_{\beta_1\beta_2\beta_3\beta_4}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}z^{\beta_3}z^{\beta_4}\Omega_a^{s_2}ds_2\;+$$

$$4\int_a^b \Omega_{s_1}^b P_{\alpha_1\alpha_2\alpha_3\alpha_4}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}z^{\alpha_3}z^{\alpha_4}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}\Omega_a^{s_2}ds_2\;+$$

$$12\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dz^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3\;+$$

$$12\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dz^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3\;+$$

$$12\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dz^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3\;+$$

$$4!\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_2}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_3}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}ds_3$$
$$\int_a^{s_3}\Omega_{s_4}^{s_3}R_{\delta_1\delta_2}\frac{dx^{\delta_1}}{ds_4}z^{\delta_2}\Omega_a^{s_4}ds_4$$

Proof: The formulas (6.49) and (6.50) follow immediately from the dagger differentiation laws.

It is noticed that many of the terms resulting from the dagger differentiation are identical. Combining these like terms gives the peculiar arrangement of coefficients in (6.50). For example we note

$$(6.52)\quad \int_a^b \Omega_{s_1}^{b\dagger} R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3 =$$

$$\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1\dagger}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3 =$$

$$\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2\dagger}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3}ds_3 =$$

$$\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2}\frac{dx^{\alpha_1}}{ds_1}z^{\alpha_2}ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}R_{\beta_1\beta_2}\frac{dx^{\beta_1}}{ds_2}z^{\beta_2}ds_2\int_a^{s_2}\Omega_{s_3}^{s_2}R_{\gamma_1\gamma_2}\frac{dx^{\gamma_1}}{ds_3}z^{\gamma_2}\Omega_a^{s_3\dagger}ds_3$$

For the first of (6.52) is (not writing the curvature tensor terms or the terms summed against it)

(6.53) $\int_a^b \int_{s_1}^b \Omega_t^b\, \Omega_{s_1}^t\, dt\, ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1}\, ds_2 \int_a^{s_2} \Omega_{s_3}^{s_2}\, \Omega_a^{s_3}\, ds_3$

Interchanging the order of integration with respect to t and with respect to s_1 gives

(6.54) $\int_a^b \Omega_t^b\, dt \int_a^t \Omega_{s_1}^t\, ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1}\, ds_2 \int_a^{s_2} \Omega_{s_3}^{s_2}\, \Omega_a^{s_3}\, ds_3$

which on renumbering the dummy parameters and inserting the curvature tensor terms gives the same result as the last of (6.52). Similarly for the other two terms of (6.52).

The second result listed at the beginning of this chapter is

Theorem 6.10. *Under the conditions of theorem* 6.4, *the second Fréchet differential* $\delta^2 \Omega_n^\tau [x; z]$ *is given by*

(6.55) $$\delta^2 \Omega_n^\tau [x; z] =$$

$$2 \int_n^\tau \Omega_{s_1}^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2}\, ds_1 \int_n^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1 \beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_n^{s_2}\, ds_2 +$$

$$\int_n^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dz^{\alpha_1}}{ds} z^{\alpha_2} \Omega_n^s\, ds +$$

$$\int_n^\tau \Omega_s^\tau P_{\alpha_1 \alpha_2 \alpha_3} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3} \Omega_n^s\, ds +$$

$$- 2\, \Gamma_\gamma(\tau)\, z^\gamma(\tau) \int_n^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_n^s\, ds +$$

$$+ 2 \int_n^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_n^s\, ds\, \Gamma_\gamma(n)\, z^\gamma(n) +$$

$$- \frac{\partial \Gamma_\alpha(\tau)}{\partial x^\beta} z^\alpha(\tau)\, z^\beta(\tau)\, \Omega_n^\tau$$

$$+ \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Gamma_\beta(\tau)\, z^\beta(\tau)\, \Omega_n^\tau$$

(6.55) cont'd.

$$- 2\, \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Omega_\eta^\tau\, \Gamma_\beta(\eta)\, z^\beta(\eta) +$$

$$+ \Omega_\eta^\tau\, \Gamma_\alpha(\eta)\, z^\alpha(\eta)\, \Gamma_\beta(\eta)\, z^\beta(\eta) +$$

$$+ \Omega_\eta^\tau\, \frac{\partial \Gamma_\alpha(\eta)}{\partial x^\beta}\, z^\alpha(\eta)\, z^\beta(\eta)$$

Proof: From theorem 6.5

$$\delta^2 \Omega_\eta^\tau [x; z] = \delta' \{ \delta' \Omega_\eta^\tau \} = \tag{6.56}$$

$$\delta' \Big\{ \int_\eta^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_\eta^s \, ds + \Omega_\eta^\tau \Gamma_\alpha(\eta) z^\alpha(\eta) +$$

$$- \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Omega_\eta^\tau \Big\}$$

Clearly $\delta' \Omega_\eta^\tau [x; z]$ is the sum of multilinear functionals on B_2 B_1 E_1 E_2 to B_2. All the factors possess Fréchet differentials, so the right member of (6.56) is given by the formula for the derivative of a sum of multilinear functions. By direct computation the right member is

(6.57)

$$2 \int_\eta^\tau \Omega_{s_1}^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_\eta^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1 \beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_\eta^{s_2} ds_2 + \tag{1}$$

$$+ \int_\eta^\tau \Omega_s^\tau P_{\alpha_1 \alpha_2 \alpha_3} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3}\, \Omega_\eta^s \, ds + \tag{2}$$

$$+ \int_\eta^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dz^{\alpha_1}}{ds} z^{\alpha_2} \Omega_\eta^s \, ds \tag{3}$$

$$- \int_\eta^\tau \Gamma_\gamma(\tau)\, z^\gamma(\tau)\, \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2}(s)\, \Omega_\eta^s \, ds \tag{4}$$

$$+ \int_\eta^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_\eta^s\, \Gamma_\gamma(\eta)\, z^\gamma(\eta)\, ds \tag{5}$$

$$+ \int_\eta^\tau \Omega_s^\tau R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_\eta^s \, ds\, \Gamma_\gamma(\eta)\, z^\gamma(\eta) \tag{6}$$

$$+ \Omega_\eta^\tau\, \Gamma_\gamma(\eta)\, z^\gamma(\eta)\, \Gamma_\beta(\eta)\, z^\beta(\eta) \tag{7}$$

$$- \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Omega_\eta^\tau\, \Gamma_\beta(\eta)\, z^\beta(\eta) \tag{8}$$

(6.57) cont'd.

$$- \Gamma_\gamma(\tau)\, z^\gamma(\tau) \int_\eta^\tau \Omega_s^\tau\, R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds}\, z^{\alpha_2}\, \Omega_\eta^s\, ds \tag{9}$$

$$+ \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Gamma_\beta(\tau)\, z^\beta(\tau)\, \Omega_\eta^\tau \tag{10}$$

$$- \Gamma_\alpha(\tau)\, z^\alpha(\tau)\, \Omega_\eta^\tau\, \Gamma_\beta(\eta)\, z^\beta(\eta) \tag{11}$$

$$+ \Omega_\eta^\tau \frac{\partial \Gamma_\alpha(\eta)}{\partial x^\beta}\, z^\alpha(\eta)\, z^\beta(\eta) \tag{12}$$

$$- \frac{\partial \Gamma_\alpha}{\partial x^\beta}\, z^\alpha(\tau)\, z^\beta(\tau)\, \Omega_\eta^\tau \tag{13}$$

where the different terms are numbered (1) - (13) for convenience. In the computation of (1), (2) and (3), the contributions due to the parameters s in Ω_s^τ and Ω_η^s are included in (2), but the contributions due to parameters τ and η are given in (4) and (5). The remaining terms are straightforward. Adding all terms (1) - (13) gives (6.55), the theorem.

VI.H. Applications to a Closed Directrix.

By means of theorems 4.4, 6.5 and 6.6 a shorter and more meaningful proof of corollary 3.7.1 is possible. We give a lemma first.

Theorem 6.11. If a vector, indicated by the column matrix λ, is displaced by Levi-Civita parallel displacement about a closed directrix, $x^i = x^i(\xi)$, if in making the circuit ξ goes from $\xi = a$ to $\xi = b$, and if $v^i = x^i(\xi) - x^i(a)$ then the first and second Fréchet differentials of $\lambda[x|b]$ with increment v exist and are given by

(6.58)
$$\delta'\lambda\,[x; v|b] = 0$$
$$\delta^2\lambda\,[x; v|b] = R_{\alpha_1\alpha_2} \int_a^b v^{\alpha_2} \frac{dv^{\alpha_1}}{d\xi}\, d\xi\ \lambda(a)$$

where $R_{\alpha_1\alpha_2}$, the Riemann curvature tensor is evaluated at the point $x^i(a)$.

Proof: Consider a closed directrix $x^i = u^i(\xi)$ through the point $x^i(a) = x^i(b)$, parameterized so that ξ goes from a to b as the parameter traverses the directrix. Then give the directrix such an increment z^i that $u^i(\xi) + z^i(\xi) = x^i(\xi)$. Then from theorems 6.5 and 6.6.

$$(6.59) \qquad \delta^1\lambda[u;z|b] = \int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{du^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds \, \lambda(a)$$

$$\delta^2\lambda[u;z|b] = \int_a^b \Omega_s^b R_{\alpha_1\alpha_2} \frac{dz^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds \, \lambda(a) +$$

$$+ \int_a^b \Omega_s^b P_{\alpha_1\alpha_2\alpha_3} \frac{du^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3} \Omega_a^s \, ds \, \lambda(a) +$$

$$+2\int_a^b \Omega_{s_1}^b R_{\alpha_1\alpha_2} \frac{du^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1\beta_2} \frac{du^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} ds_2 \, \lambda(a)$$

We now shrink the original directrix $u^i(\xi)$ to the point $u^i(a)$. This implies that $\| u^i(\xi) - u^i(a) \| \to 0$. Hence $z^i(\xi) \to v^i(\xi)$, $R_{\alpha_1\alpha_2}[x(\xi)] \to R_{\alpha_1\alpha_2}[x(a)]$. In the limit $\Omega_s^b = I$, $\frac{du^\alpha}{ds} = 0$, $z^i = v^i$. The only non-vanishing term of (6.59) is the first of $\delta^2\lambda$ which is

$$\delta^2\lambda[x;v|b] = R_{\alpha_1\alpha_2} \int_a^b \frac{dv^{\alpha_1}}{d\xi} v^{\alpha_2} d\xi$$

which proves the theorem.

Theorem 6.12. If a vector λ is displaced by Levi-Civita parallel displacement about a closed directrix $x^i = x^i(\xi)$, and if ξ goes from a to b as the parameter point traverses the directrix, and if $\lambda(a)$, $\lambda(b)$ are the values of λ at $x^i(a)$ before and after displacement, then the principal term of the generalized Taylor series expansion for the difference $\lambda(b) - \lambda(a)$ is $\frac{1}{2}\delta^2\lambda[x;v]$ and

(6.60) $\lambda(b) - \lambda(a) = \left\{ \frac{1}{2} R_{\alpha_1 \alpha_2} \int_a^b \frac{dx^{\alpha_1}}{ds} x^{\alpha_2} d\xi + \cdots \right\} \lambda(a)$

Proof: Corollary 6.8.1 gives the generalized Taylor series expansion. Since the directrix is closed, $\int_a^b v^{\alpha_2} \frac{dv^{\alpha_1}}{d\xi} d\xi = \int_a^b \frac{dx^{\alpha_1}}{d\xi} x^{\alpha_2} d\xi$. Using (6.58) gives the theorem.

Corollary 6.12.1 Pérès Formula. If the directrix is an infinitesimal curve in the sense that $\| v^i \|$ is infinitesimal, then (6.60) gives the change $\lambda(b) - \lambda(a)$. The change is the second Frechet differential of λ with increment v.

Proof: From the definition of the generalized Taylor series expansion, each term is of higher degree in z than the preceeding terms. Hence if $\|z\| = \|v\|$ is infinitesimal, the only contributing term is the first non-identically vanishing term given by (6.60). (6.60) is the equation usually discussed in texts like Eisenhart and Thomas under "Parallel Displacement About an Infinitesimal Closed Curve", although frequently a parallelogram rather than a general curve is taken. It is a matter of simple quadrature to reduce (6.60) for the parallelogram to the form given by these authors.

VII. The Tensor Character of Ω_a^b and $\delta^i\Omega_a^b$.

Parallel displacement is essentially a geometric phenomenon, and is independent of the coordinate system. If the contravariant vector λ is displaced along a given directrix, the value of λ at any point of the directrix is independent of the coordinate system. That is, the components of λ in one coordinate system are related to those in another system by the ordinary contravariant tensor transformation law

(7.1) $\quad \bar{\lambda}(\xi) = \bar{\lambda}[\bar{x}(\xi)] = B(\xi)\lambda[x(\xi)]$ or

(7.2) $\quad \lambda(\xi) = A(\xi)\bar{\lambda}(\xi)$

Where $A(\xi)$, $B(\xi)$ are the matrices of the partial derivatives

(7.3) $\quad A(\xi) = \left(\dfrac{\partial x^r}{\partial \bar{x}^c}\right) \quad ; \quad B(\xi) = \left(\dfrac{\partial \bar{x}^r}{\partial x^c}\right)$

where the direct and inverse transformations are given by

(7.4) $\quad \bar{x} = \bar{x}(x) \quad , \quad x = x(\bar{x})$

It is seen, that as written, $B(\xi) = B[x|\xi]$ and $A(\xi) = A[\bar{x}|\xi]$ are functions on the vector space E_2[23] to B_1[24]. Of course (7.1) is regarded as defining $\bar{\lambda}[\bar{x}(\xi)]$ and might more fully be expressed as

(7.5) $\quad \bar{\lambda}[\bar{x}(\xi)] = B[x[\bar{x}(\xi)]]\,\lambda[x[\bar{x}(\xi)]]$

and (7.2) as

(7.6) $\quad \lambda[x(\xi)] = A[\bar{x}[x(\xi)]]\,\bar{\lambda}[\bar{x}[x(\xi)]]$

Now since $\lambda(b)$ and $\lambda(a)$ are related by theorem 3.2 as

(7.7) $\quad \lambda(b) = \Omega_a^b\,\lambda(a)$

---

23. See theorem 6.2.

24. See section IV.A.

it is worth while to enquire into the tensor characteristics of Ω_a^b. We seek the relation between $\Omega_a^b[x] = \Omega_a^b[-\Gamma_\alpha \frac{dx^\alpha}{d\xi} | \xi]$ and $\bar{\Omega}_a^b[\bar{x}] = \Omega_a^b[-\bar{\Gamma}_\alpha \frac{d\bar{x}^\alpha}{d\xi} | \xi]$. One approach from a different point of view has already been made in theorem 3.5.

Theorem 7.1. If a vector λ is displaced by Levi-Civita parallel displacement along a directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$ and if $\lambda(a)$, $\lambda(b)$ the values of λ at $\xi = a, b$ respectively are related by the equation $\lambda(b) = \Omega_a^b \lambda(a)$ then the matrizant function Ω_a^b is a two point tensor with base points $x(a)$, $x(b)$, contravariant of order one at $x(b)$ and covariant of order one at $x(a)$.

Proof: The following equations hold in the x and $\bar{x}$ coordinate systems.

(7.8) $\lambda(b) = \Omega_a^b[x]\, \lambda(a)$

(7.9) $\bar{\lambda}(b) = \bar{\Omega}_a^b[\bar{x}]\, \bar{\lambda}(a)$

where the bared terms are the functions computed in the bar coordinates. Using (7.5) in (7.9) gives

(7.10) $B(b)\, \lambda(b) = \bar{\Omega}_a^b[\bar{x}]\; B(a)\, \lambda(a)$

On multiplying on the left by $A(b) = B^{-1}(b)$

(7.11) $\lambda(b) = A(b)\, \bar{\Omega}_a^b[\bar{x}]\, B(a)\, \lambda(a)$

Since (7.11), (7.8) must hold for arbitrary vectors $\lambda(a)$

(7.12) $\Omega_a^b[x] = A(b)\, \bar{\Omega}_a^b[\bar{x}]\, B(a)$

But (7.12) are the defining equations for a two point tensor, base points $x(a)$, $x(b)$, contravariant of order one at $x(b)$, covariant of order one at $x(a)$, which proves the theorem.

It is clear that the parameter limits may be anything in the range of definition for ξ . Hence we have by the same proof

Theorem 7.2. If a contravariant vector λ is displaced by Levi-Civita parallel displacement along a directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$ and if $\lambda(\eta)$, $\lambda(\tau)$ are the values of λ at $\xi = \eta$ and $\xi = \tau$ respectively, are related by $\lambda(\tau) = \Omega_\eta^\tau \lambda(\eta)$, then Ω_η^τ forms a two point tensor field with base points $x(\eta)$, $x(\tau)$ contravariant at $x(\tau)$, covariant at $x(\eta)$ for all η, τ such that $a \le \eta, \tau \le b$ .That is

(7.13) $\Omega_\eta^\tau [x] = A(\tau)\, \bar{\Omega}_\eta^\tau [\bar{x}]\, B(\eta)$

Note that the theorem is valid even if $\eta = \tau$ or $\eta > \tau$, since $\Omega_\eta^\eta = I$ and $\Omega_\eta^\tau\, \Omega_\tau^\eta = I$ ie $\Omega_\eta^\tau = (\Omega_\tau^\eta)^{-1}$

However, $\delta^i \Omega_\eta^\tau [x; z]$ does not have such general tensor properties, as is shown by the following theorems.

Theorem 7.3. Let a contravariant vector λ be displaced by Levi-Civita parallel displacement along $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$. Let Ω_a^b be the matrizant function relating the values of λ at $\xi = a$ and $\xi = b$ by $\lambda(b) = \Omega_a^b \lambda(a)$. Let the directrix be varied to a nex directrix with the same end points $x(a)$, $x(b)$ and with equations $x = x^*(\xi) = x(\xi) + z(\xi)$. Let $\delta^i \Omega_a^b [x; z]$ be the ith Frechet differential of Ω_a^b with equal increments z. Then $\delta^i \Omega_a^b [x; z]$ is a two point tensor with base points $x(a)$, $x(b)$ for all z in E_1 contravariant of order one at $x(b)$ and covariant of order one at $x(a)$ if and only if the transformation $\bar{x} = \bar{x}(x)$ is linear.

Proof: The theorem will be proved by induction on the order of the Fréchet differential. The right member of

$$(7.14) \quad \Omega_a^b[x] = A(b)\,\bar{\Omega}_a^b[\bar{x}]\,B(a)$$

is a trilinear function on $B_1\ B_2\ B_1$ to B_2 , where B_1 is the above defined Banach space of square matrices of order n whose elements are continuous real numerical functions of one real numerical variable, and B_2 is the Banach space of square matrices of order n whose elements are real numerical continuous functions of two real numerical variables. Hence if A , B , $\Omega_a^b[\bar{x}]$ each have Fréchet differentials, the differential $\delta'\Omega_a^b[x;z]$ will be given by the formula for the derivative of a trilinear function. But A,B are matrices whose elements are differentiable functions of $\bar{x}$ and x, hence their Fréchet differentials are the ordinary differentials. Hence $\delta' A[\bar{x};\bar{z}]$ and $\delta' B[x;z]$ exist. But since $z(a) = z(b) = 0$, because of the end point conditions on the directrices, $\delta' A[\bar{x};\bar{z}|b] = \delta' B[x;z|a] = 0$. Further since $\bar{x}(x)$ is a differentiable function of x, $\delta'\bar{x}(x)$ exists and is given by the ordinary differential, in matric notation

$$(7.15) \qquad \delta'\bar{x}^{\alpha}[x;z] = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} z^{\beta} = B z$$

Hence using the composition theorem for Fréchet differentials

$$(7.16) \qquad \delta'\Omega_a^b[x;z] = A(b)\,\delta'\bar{\Omega}_a^b[\bar{x};\delta'\bar{x}(x;z)]\,B(a)$$
$$= A(b)\,\delta'\bar{\Omega}_a^b[\bar{x};Bz]\,B(a)$$

But $Bz = \bar{z}$ if and only if the transformation is linear. For if we expand $\bar{x}(x+z)$ we obtain

(7.17) $\bar{x}(x+z) = \bar{x}(x) + Bz +$ higher order terms involving $\frac{\partial^2 \bar{x}^\alpha}{\partial x^\beta \partial x^\gamma} z^\beta z^\gamma$

$= \bar{x} + \bar{z}$ by definition.

Hence

(7.18) $\bar{z} = Bz +$ higher order terms involving $\frac{\partial^2 \bar{x}^\alpha}{\partial x^\beta \partial x^\gamma} z^\beta z^\gamma$

Hence

(7.19) $\bar{z} = Bz$

if and only if the higher order terms vanish, ie, the transformation equations be linear.

Hence, if and only if the transformation is linear, then

(7.20) $\delta^1 \Omega_a^b [x; z] = A(b)\, \delta^1 \Omega_a^b [\bar{x}; \bar{z}]\, B(a)$

Clearly, if the transformation is linear, and if we make the induction hypothesis, then in equations (7.14), (7.16), (7.20) we can replace Ω_a^b by $\delta^{i-1}\Omega_a^b$ and $\delta^1\Omega_a^b$ by $\delta^i \Omega_a^b$, and all equations will be valid. Hence $\delta^i \Omega_a^b [x; z]$ is a two point tensor. This completes the induction.

Conversely, let us assume that $\delta^2 \Omega_a^b$ is a two point tensor, without assuming that $\delta^1 \Omega_a^b$ is a two point tensor. Then we can write

(7.21) $\delta^1 \Omega_a^b [x; z] = A\, \delta^1 \bar{\Omega}_a^b [\bar{x}; \bar{z}]\, B + A\, \delta^1 \bar{\Omega}_a^b [\bar{x}; y]\, B$

where $y = \delta^1 \bar{x}[x; z] - \bar{z}$. Then

(7.22) $\delta^2 \Omega_a^b [x; z; z] = A\, \delta^2 \bar{\Omega}_a^b [\bar{x}; \bar{z}; \bar{z}]\, B + 2A\, \delta^2 \Omega_a^b [\bar{x}; z; y]\, B$

$+ A\, \delta^1 \bar{\Omega}_a^b [\bar{x}; \delta^1 \bar{z}(x; z)]\, B + A\, \delta^1 \Omega_a^b [\bar{x}; \delta y [x; z]]\, B$

$+ A\, \delta^2 \Omega_a^b [\bar{x}; y; y]\, B$

Hence since $\delta^2 \Omega_\alpha^b[x;z;z]$ is a tensor, the last four terms of (7.22) must sum to zero. Of these four terms, the second,

$$(7.23) \quad A\,\delta^1\bar{\Omega}[\bar{x};\delta^1\bar{z}[x;z]]\,B$$

is homogeneous of second degree in z, and the others are, if not identically zero, of at least third degree in z. Hence the second term is linearly independent of the other two, and must vanish independently for all z. But (7.23) vanishes only when $\delta^1\bar{z}[x;z]$ vanishes. But this implies that $\bar{z}$ be independent of x, or that the transformation be linear.

Hence, if $\delta^2\Omega_\alpha^b[x;z;z]$ is to be a tensor, the transformation $\bar{x}=\bar{x}(x)$ must be linear. An identical type of argument holds if $\delta^i\Omega_\alpha^b$ is a tensor. This completes the proof of theorem 7.3.

Similarly, if and only if the transformation $\bar{x}=\bar{x}(x)$ is linear, then $\delta^i\Omega_\eta^\tau[x;z]$ is a two point tensor. For differentiating the results of theorem 7.2

$$(7.24) \quad \delta^1\Omega_\eta^\tau[x;z] = \delta^1 A[x;z|\tau]\,\bar{\Omega}_\eta^\tau[\bar{x}]\,B(\eta) + A(\tau)\,\bar{\Omega}_\eta^\tau[\bar{x}]\,\delta^1 B[x;z|\eta] + A(\tau)\,\delta^1\bar{\Omega}_\eta^\tau[\bar{x};\delta^1\bar{x}[x;z]]\,B(\eta)$$

If $\bar{x}^i=\bar{x}^i(x)$ is linear, then $\delta^1 A=\delta^1 B=0$ since A and B are matrices of constants, and further, $\delta^i\bar{x}[x;z]=Bz=\bar{z}$. If the transformation is not linear, then $\delta^1 A(\tau)\neq 0$ unless $\tau=b$, $\delta^1 B(\eta)\neq 0$ unless $\eta=\alpha$.

In a manner similar to that used in theorem 7.3 it is then easy to show

Theorem 7.4. Under the conditions of theorem 7.3, the Fréchet differentials of the matrizant $\delta^i \Omega_n^\tau[x; z]$ are two point tensor fields if and only if the transformation of coordinates $\bar{x} = \bar{x}(x)$ is linear.

The situation is entirely different when the change in the parallel vectors is caused by warping the space as in chapter V. In this case we have

Theorem 7.5. Under the conditions of theorem 5.2 $\delta^i \Omega_a^t [\Gamma_k; \delta\Gamma_k]$ is a two point tensor field, contravariant of order one at $x(t)$ and covariant of order one at $x(a)$.

Proof: $\bar{\Gamma}_k$, the coefficients of connection in the bar co-ordinates are Fréchet differentiable functions of Γ_k with increment $\delta\Gamma_k$. For, the equations of transformation of the co-efficients of connection are, in component notation

$$(7.25) \quad \bar{\Gamma}_{j\,k}^{\,i} = \Gamma_{mn}^{\,\ell} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^\ell} + \frac{\partial^2 x^\ell}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^\ell}$$

Clearly the Fréchet differentials $\delta^i \bar{\Gamma}_k [\Gamma_\ell; \delta\Gamma_\ell]$ exist and are given in matric notation by

$$(7.26) \quad \delta' \bar{\Gamma}_k [\Gamma_\ell; \delta\Gamma_\ell] = B\, \delta\Gamma_m\, A \frac{\partial x^m}{\partial \bar{x}^k}$$

$$(7.27) \quad \delta^i \bar{\Gamma}_k [\Gamma_\ell; \delta\Gamma_\ell] = 0 \qquad \ell > 1$$

and

$$(7.28) \quad \bar{\Gamma}_k[\Gamma + \delta\Gamma] - \bar{\Gamma}_k[\Gamma] = \delta\bar{\Gamma}_k = B\delta\Gamma_m\, A \frac{\partial x^m}{\partial \bar{x}^k} = \delta' \bar{\Gamma}_k [\Gamma; \delta\Gamma]$$

Now differentiating the equation corresponding to (7.12), and using the composition theorem for Frechet differentials

$$(7.29) \quad \Omega_a^t[\Gamma] = A(t)\,\bar{\Omega}_a^t[\bar{\Gamma}]\,B(a)$$

$$(7.30) \quad \delta'\Omega_a^t[\Gamma;\delta\Gamma] = A(t)\,\delta'\bar{\Omega}_a^t[\bar{\Gamma};\delta'\bar{\Gamma}[\Gamma;\delta\Gamma]]\,B(a)$$
$$= A(t)\,\delta'\bar{\Omega}_a^t[\bar{\Gamma};\delta\bar{\Gamma}]\,B(a)$$

Since A, B are independent of Γ_R , the differentials $\delta'A$, $\delta'B$ both vanish, hence no terms involving $\delta'A$ or $\delta'B$ occur in (7.30). However, (7.30) is the defining equation of a two point tensor, contravariant of order one at $x(t)$ and covariant of order one at $x(a)$. This proves the statement of theorem 7.5 for the first Fréchet differentials. The proof for all differentials is by induction, using a step by step proof which is entirely similar to that used for the first Fréchet differential.

VIII. The Classification of Riemannian Spaces by Their Degree.

VIII.A. The Concept of the Degree of a Space.

It is a common and useful practice to classify plane curves by their degree. The degree of a curve, when it is expressed parametrically, may be defined as the order of the lowest order derivative which vanishes for each coordinate equation and each parameter value[24]. Plane curves may be of finite or infinite degree. Similarly we seek to classify Riemannian spaces by means of the Fréchet differentials of parallel displaced vectors with changes in the directrix as increment. Spaces of zero degree are spaces for which the first (and hence all higher order) Frechet differential of an arbitrary vector vanishes for any and all directrices and variations. Spaces of degree one are spaces for which the first Fréchet differential of some vector for some directrix and some variation does not vanish, but for which the second (and hence all higher order) Fréchet differential does vanish for any and all vectors, directrices and variations.

In general, a space of degree m is one for which the mth Fréchet differential does not vanish for some vector directrix and variation, and the (m + 1)st Fréchet differential vanishes for arbitrary vectors, directrices and variations. A space of infinite degree is one for which no Fréchet differential vanishes for arbitrary vectors, directrices and variations.

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24. Of course the parametrization must be in terms of an essential parameter. $y = t^3$, $x = t^3 - 5$ is only a first degree curve, even though the second and third derivatives do not vanish for this parametrization.

Because of the dependence of the displaced vector on the matrizant function, this problem is equivalent to classifying spaces according to the vanishing of the Fréchet differentials of the matrizant. We now seek these conditions.

VIII.B. The Conditions for the Vanishing of the Fréchet Differentials $\delta^i \Omega_a^b [x;z]$

Theorem 8.1. Under the conditions of theorem 6.4, the necessary and sufficient condition for the vanishing of $\delta^1 \Omega_a^b [x;z]$ for all directrices $x^i = x^i(\xi)$ and variations $z^i = z^i(\xi)$ in an open region D of the space of definition is that

$$R_{kj} = 0$$

at all points in D. $k, j = 1, \cdots, n$

Proof: From 6.19

$$(8.1)\quad \delta^1 \Omega_a^b [x;z] = \int_a^b \Omega_s^b R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds$$

Clearly, if $R_{\alpha_1 \alpha_2}$ vanishes, so does (8.1). On the other hand, if (8.1) vanishes for all directrices and variations, it vanishes in particular for the directrix and variation

(8.2) $x^i(\xi)$ arbitrary except in the neighborhood of $x(\xi_0)$

$$x^i(\xi) = x^i(\xi_0) \quad i = 1, \cdots, k-1, k+1 \cdots n \quad |\xi - \xi_0| \leq 2\varepsilon$$

$$x^k(\xi) = x^k(\xi_0) + (\xi - \xi_0) \quad k \text{ unique} \quad |\xi - \xi_0| \leq 2\varepsilon$$

$$z^i(\xi) = 0 \quad i = 1, \cdots j-1, j+1, \cdots n$$

$$\begin{aligned} z^j(\xi) &= 0 & \xi < \xi_0 - \varepsilon \text{ or } \xi > \xi_0 + 2\varepsilon \\ &= \varepsilon + \xi - \xi_0 & \xi_0 - \varepsilon \leq \xi \leq \xi_0 \\ &= \varepsilon & \xi_0 \leq \xi \leq \xi_0 + \varepsilon \\ &= \xi_0 + 2\varepsilon - \xi & \xi_0 + \varepsilon \leq \xi \leq \xi_0 + 2\varepsilon \end{aligned}$$

We illustrate (8.2) in Figure 8.1 a, b, c

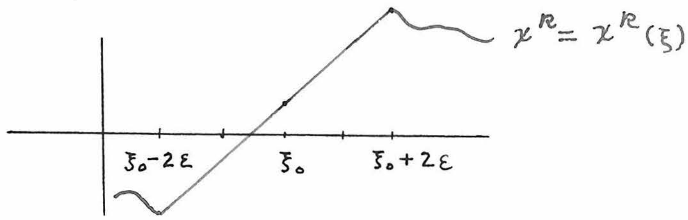


Fig. 8.1 a

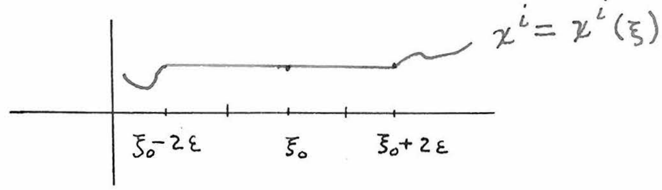


Fig. 8.1 b

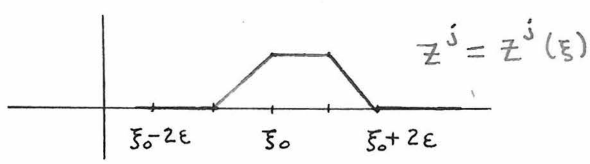


Fig. 8.1 c

With these directrices and variations,

(8.3) $$\delta' \Omega^b_a [x;z] = \int_{\xi_0-\varepsilon}^{\xi_0+2\varepsilon} \Omega^b_s \, R_{\alpha_1\alpha_2} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} \Omega^s_a \, ds$$

$$= \int_{\xi_0-\varepsilon}^{\xi_0+2\varepsilon} \Omega^b_s \, R_{kj} \, z^j \, \Omega^s_a \, ds$$

(not summed on j)

If we expand the matric terms in a Taylor series about ξ_0 we have

(8.4) $$\delta' \Omega^b_a [x;z] = \Omega^b_{\xi_0} \, R_{kj}(\xi_0) \, \Omega^{\xi_0}_a \cdot 2\varepsilon^2 + O(\varepsilon^3)$$

where $O(\varepsilon^3)$ represents terms of degree 3 and higher in ε

Since we may take ε as small as we please,

(8.5) $\Omega_{\xi_0}^{b} R_{kj}(\xi_0)\Omega_a^{\xi_0} = 0$

But $\Omega_{\xi_0}^{b}$ and $\Omega_a^{\xi_0}$ have inverses $\Omega_b^{\xi_0}$ and $\Omega_{\xi_0}^{a}$ respectively, since $\Omega_{\xi_0}^{b}\Omega_b^{\xi_0} = I$. (This result follows since $\Omega_{\xi_0}^{b}\Omega_b^{\xi_0}$ is the matrizant describing the displacement of a vector from ξ_0 to b and back to ξ_0, which leaves the vector unchanged.)

Hence,

(8.6) $R_{kj}[x|\xi_0] = 0$

But $x(\xi_0)$ is an arbitrary point. Hence R_{kj} vanishes everywhere. The openness of domain D guarantees the existence of the curves in Figure 8.1 in D for sufficiently small ε. This proves the theorem, since kj can have any values 1, ..., n.

<u>Theorem 8.2. Under the conditions of theorem 6.4, the necessary and sufficient condition for the vanishing of the second Fréchet differential,</u> $\delta^2\Omega_a^b[x;z]$ <u>for all directrices</u> $x^i(\xi)$ <u>and variations</u> $z^i(\xi)$ <u>in an open region D is that</u>

(8.7) $R_{kj} = 0$

<u>everywhere in D for</u> $k = 1,\cdots,n$; $j = 1,\cdots,n$.

Proof: If $R_{kj} = 0$ everywhere, then from theorem 8.1 $\delta^1\Omega_a^b[x;z]$ vanishes and hence so must $\delta^2\Omega_a^b[x;z]$. The proof of the necessity of the condition is similar to the proof for theorem 8.1. The two special functions for $z(\xi)$ will be sketched only in Figure 8.2 and not described analytically.

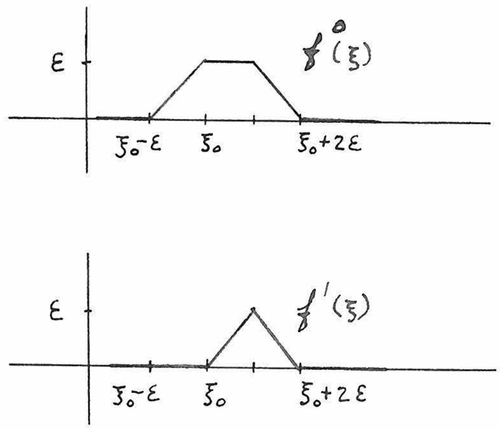


Fig. 8.2

Let the directrix be given by (8.2), and let

(8.8) $$\begin{aligned} z^1 &= f^0(\xi) \\ z^2 &= f^1(\xi) \\ z^3 &= 0 \\ &\vdots \\ z^n &= 0 \end{aligned}$$

Then from theorem 6.7

(8.9) $$\delta^2 \Omega_a^b [x; z] = \int_a^b \Omega_s^b R_{\alpha_1 \alpha_2} \frac{dz^{\alpha_1}}{ds} z^{\alpha_2} \Omega_a^s \, ds \, +$$

$$+ \int_a^b \Omega_s^b R_{\alpha_1 \alpha_2 \alpha_3} \frac{dx^{\alpha_1}}{ds} z^{\alpha_2} z^{\alpha_3} \Omega_a^s \, ds \, +$$

$$+ 2 \int_a^b \Omega_{s_1}^b R_{\alpha_1 \alpha_2} \frac{dx^{\alpha_1}}{ds_1} z^{\alpha_2} ds_1 \int_a^{s_1} \Omega_{s_2}^{s_1} R_{\beta_1 \beta_2} \frac{dx^{\beta_1}}{ds_2} z^{\beta_2} \Omega_a^{s_2} ds_2$$

We expand the matric terms in Taylor series, obtaining

(8.10) $$\delta^2 \Omega_a^b [x; z] =$$

$$\Omega_{\xi_0}^b R_{12} \Omega_a^{\xi_0} \int_{\xi_0}^{\xi_0 + \varepsilon} \left(\frac{df^0}{ds} f^1 - \frac{df^1}{ds} f^0 \right) ds + \cdots +$$

(8.10) cont'd.

$$+\Omega_{\xi_0}^{b} P_{k\alpha_2\alpha_3}\Omega_a^{\xi_0}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} f^{\alpha_2} f^{\alpha_3}\, ds+\cdots \quad +$$

$$+2!\,\Omega_{\xi_0}^{b} R_{k\alpha_2} R_{k\beta_2}\Omega_a^{\xi_0}\int_{\xi_0-\varepsilon}^{\xi_0+2\varepsilon} f^{\alpha_2}\, ds_1 \int_{\xi_0-\varepsilon}^{s_1} f^{\beta_2}\, ds_2+\cdots$$

Use is made of the skew symmetry of $R_{\alpha_1\alpha_2}$ in α_1, α_2 in the first term.

We consider the integrals of the first terms of the three series in the right member of (8.10). We are interested only in order of magnitude with respect to ε . The first integral is of order ε^2, the second of order ε^3 and the third is of order ε^4 . This can be verified by direct computation or by noting that the order is unchanged by changing the curves slightly as long as a term is not dropped or made infinite. Hence in the integrals in the second and third terms of (8.10) the functions f^0, f^1 may be replaced by the step function

$$\begin{aligned} z^i(\xi) &= 0 && \xi < \xi_0 \\ z^i(\xi) &= \varepsilon && \xi_0 \leq \xi \leq \xi_0+\varepsilon \\ z^i(\xi) &= 0 && \xi_0+\varepsilon < \xi \end{aligned}$$

The third integral of (8.10) is then of the same order as

$$(8.11) \qquad \int_{\xi_0}^{\xi_0+\varepsilon} \varepsilon\, ds_1 \int_{\xi_0}^{s_1} \varepsilon\, ds_2 = \frac{\varepsilon^4}{2}$$

and the second is of the same order as

$$(8.12) \qquad \int_{\xi_0}^{\xi_0+\varepsilon} \varepsilon^2\, ds = \varepsilon^3$$

and the first, with no approximation, is

(8.13) $\int_{\xi_0}^{\xi_0+\varepsilon} -\varepsilon\, ds = -\varepsilon^2$

All the deleted terms will be of larger degree in than the first term in the series in which they occur. Clearly then the lowest degree in ε appearing is the second. Since we may take ε as small as we choose, the coefficient of ε^2 in (8.10) must vanish, to wit,

(8.14) $\Omega^{b}_{\xi_0} R_{12}(\xi_0) \Omega^{\xi_0}_{a} = 0$

Hence as in (8.5), (8.6), since ξ_0 is arbitrary and $\Omega^{b}_{\xi_0}$, $\Omega^{\xi_0}_{a}$ have inverses,

(8.15) $R_{12} = 0$ everywhere.

But the variation could have been chosen similarly, but differently to give any permutation of the subscripts on $R_{\alpha_1\alpha_2}$. Hence

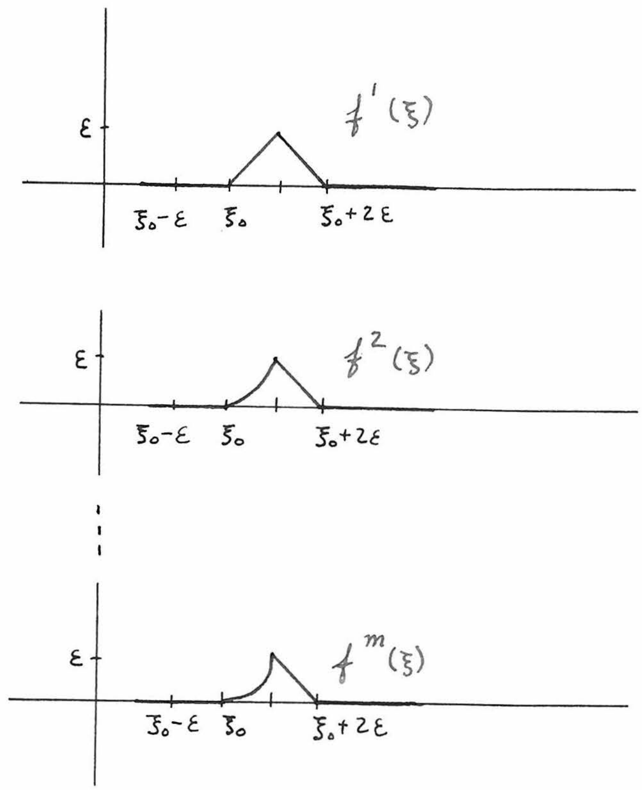
(8.16) $R_{kj} = 0$ everywhere

for all values of k, j. This completes the proof of theorem 8.2.

Theorem 8.3. The necessary and sufficient condition for the vanishing of $\delta^3\Omega^{b}_{a}$ for all directrices $x^i(\xi)$ and variations $z^i(\xi)$ in an open region D of a Riemannian space is $R_{kj} = 0$ for all k, j = 1, ..., n and all points in D.

The proof of theorem 8.3, while along the same lines as the proof for theorems 8.1 and 8.2, is rather more difficult. We define for future use as values of z the curves

(8.18)



Note that each function $f^i(\xi)$, $i=1,\cdots,m,\cdots$ is the same as every other function except in the range $\xi_0 \leq \xi \leq \xi_0+\varepsilon$ and in this range, $f^i(\xi) = \frac{(\xi-\xi_0)^i}{\varepsilon^{i-1}}$. The procedure will be to choose particular varied directrices so that $z^i(\xi) = f^k(\xi)$ or zero. The matric terms in the differential will be expanded and factored out of the integrand. We particularly note at this stage those integrals involving $\frac{d f^k(\xi)}{d\xi} f^j(\xi)$ which are summed against a term which is skew symmetric in k and j because of the skew symmetry of R_{kj} in k, j. For such integrals any contribution of $\frac{df^k(\xi)}{d\xi} f^j(\xi)$ will cancel that of $\frac{df^j(\xi)}{d\xi} f^k(\xi)$ over the region $\xi_0 + \varepsilon \leq \xi \leq \xi_0 + 2\varepsilon$. But from the dagger differentiation formulas (6.33) and (6.37) for the different Fréchet differentials, the only place $\frac{df^k}{d\xi}$ appears is when summed against the first index of R_{kj} or some term in the $P_{k\alpha\ldots\beta}$ series. The other indeces are always summed against a term z^α. Hence $\frac{dz^k}{d\xi}$ only appears in an expression $\left(\frac{dz^\alpha}{d\xi} z^\beta - \frac{dz^\beta}{d\xi} z^\alpha\right)$ since $R_{\alpha\beta}$, $P_{\alpha\beta\ldots\gamma}$ are all skew symmetric in $\alpha\beta$. If $z^\alpha = f^i(\xi)$ and $z^\beta = f^j(\xi)$, the parenthesis term vanishes over $\xi_0 + \varepsilon \leq \xi \leq \xi_0 + 2\varepsilon$ as noted above.

These functions, $f^i(\xi)$ all different, are so chosen that the degree in ε of any single or iterated integral of any combination of them can be computed at a glance. For example

$$(8.19)\quad \int_{-\varepsilon+\xi_0}^{2\varepsilon+\xi_0} f^i(\xi)\,d\xi \begin{array}{ll} = \varepsilon^2 & i = 1 \\ = \frac{5}{6}\varepsilon^2 & i = 2 \\ \vdots & \\ = \left(\frac{1}{m+1} + \frac{1}{2}\right) & i = m \\ \vdots & \end{array}$$

and hence is of order ε^2 for each i. Similarly,

$$(8.20)\quad \int_{-\varepsilon+\xi_0}^{2\varepsilon+\xi_0} \frac{df^i(\xi)}{d\xi} f^j(\xi)\,d\xi = \frac{i}{i+j}\,\varepsilon^2$$

and hence is of order ε^2 for all i, j. Further

$$(8.21)\quad \int_{-\varepsilon+\xi_0}^{2\varepsilon+\xi_0} f^i(\xi_1)\,d\xi_1 \int_{-\varepsilon+\xi_0}^{\xi_1} f^j(\xi_2)\,d\xi_2$$

is of order ε^4 as is easily seen by direct computation, or by changing the curves by letting the portion in the range $\xi_0+\varepsilon \leq \xi \leq \xi_0+2\varepsilon$ approach perpendicularity. This does not change the degree of the integral in ε, but makes it easy to integrate at sight. The rule for computing the degree in ε of an integral is given by

Lemma 8.3.1. In the proof of theorem 8.3, an iterated integral consisting of α integrations, β functions $f^i(\xi)$ defined in (8.18) and γ derived functions $\frac{df^i(\xi)}{d\xi}$ with limits a, b on the last performed integration will be of degree $\alpha+\beta$ in ε.

Proof: From the above discussion, if the integral does not involve $\frac{df^i}{d\xi}$ we may vary the $f^i(\xi)$ so the curve $f^i(\xi)$ drops perpendicularly from ε to zero at $\xi_0+\varepsilon$ and is zero for all larger ξ. This does not change the degree of the integral in ε, since the integrands are all positive. Then it is clear that each f^i introduces a term which on the final evaluation of the limits is of order ε^i. But each f^i is divided by ε^{i-1}, so the net contribution to the degree by each $f^i(\xi)$ is one, independent of i. Each integration similarly increases the degree by one.

When $\frac{df^i(\xi)}{d\xi}$ occurs in a term, the net contribution to the degree over $\xi_0 \leq \xi \leq \xi_0+\varepsilon$ will be zero, since $\frac{df^i(\xi)}{d\xi}$ is of degree one less that $f^i(\xi)$. From an above argument, in the region $\xi_0+\varepsilon \leq \xi \leq \xi_0+2\varepsilon$ the contribution will cancel that from another term. This proves lemma 8.3.1.

From (6.49) we now consider $\delta^3 \Omega_a^b[x; z]$ for the particular directrix given by (8.2) and variations z^i chosen from (8.18).

We expand the matric terms in (6.49) about $\xi = \xi_0$ obtaining

$$(8.22)\quad \Omega_{\xi}^{b}\, P_{\alpha_1\alpha_2\alpha_3}\, \Omega_{a}^{\xi} = \Omega_{\xi_0}^{b}\, P_{\alpha_1\alpha_2\alpha_3}(\xi_0)\, \Omega_{a}^{\xi_0} +$$

$$+ \Omega_{\xi_0}^{b}\, P_{\alpha_1\alpha_2\alpha_3\alpha_4}(\xi_0)\, \Omega_{a}^{\xi_0} \frac{dx^{\alpha_4}}{d\xi}(\xi-\xi_0) +$$

$$+\frac{1}{2!}\, \Omega_{\xi_0}^{b}\, P_{\alpha_1\cdots\alpha_4}(\xi_0)\, \Omega_{a}^{\xi_0} \frac{d^2x^{\alpha_4}}{d\xi^2}(\xi-\xi_0)^2 +$$

$$+\frac{1}{2!}\, \Omega_{\xi_0}^{b}\, P_{\alpha_1\cdots\alpha_5}(\xi_0)\, \Omega_{a}^{\xi_0} \frac{dx^{\alpha_4}}{d\xi}\frac{dx^{\alpha_5}}{d\xi}(\xi-\xi_0)^2 +$$

$$+\cdots$$

$$(8.23)\quad \Omega_{\xi}^{b}\, P_{\alpha_1\alpha_2\alpha_3\alpha_4}\, \Omega_{a}^{\xi} =$$

$$\Omega_{\xi_0}^{b}\, P_{\alpha_1\alpha_2\alpha_3\alpha_4}(\xi_0)\, \Omega_{a}^{\xi_0} + \cdots$$

We consider a sample term in an iterated integral in some detail, since it is more complicated. We discuss the term which may be written

$$(8.24)\quad \int_a^b \int_a^{\xi_1} \Omega_{\xi_1}^{b}\, R_{\alpha_1\alpha_2}(\xi_1)\, \Omega_{\xi_2}^{\xi_1}\, R_{\beta_1\beta_2}(\xi_2)\, \Omega_{a}^{\xi_2} \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2}(\xi_1) \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2}(\xi_2)\, d\xi_2\, d\xi_1$$

Now we expand the matric function of ξ_2 about ξ_0.

$$(8.25)\quad \Omega_{\xi_1}^{b}\, R_{\alpha_1\alpha_2}(\xi_1)\, \Omega_{\xi_2}^{\xi_1}\, R_{\beta_1\beta_2}(\xi_2)\, \Omega_{a}^{\xi_2} =$$

$$\Omega_{\xi_1}^{b}\, R_{\alpha_1\alpha_2}(\xi_1)\, \Omega_{\xi_0}^{\xi_1} \Big\{ R_{\beta_1\beta_2}(\xi_0) + P_{\beta_1\beta_2\beta_3} \frac{dx^{\beta_3}}{d\xi_2}(\xi_0)\,(\xi_2-\xi_0) +$$

$$+\frac{1}{2!} P_{\beta_1\beta_2\beta_3\beta_4} \frac{dx^{\beta_3}}{d\xi_2}\frac{dx^{\beta_4}}{d\xi_2}(\xi_2-\xi_0)^2 +$$

$$+\frac{1}{2!} P_{\beta_1\beta_2\beta_3} \frac{d^2x^{\beta_3}}{d\xi_2^2}(\xi_2-\xi_0)^2 +$$

$$+\cdots \Big\}\, \Omega_{a}^{\xi_0}$$

Similarly, expanding the matric functions of ξ_1 in (8.25) about ξ_0 we get

$$(8.26)\quad \Omega^{b}_{\xi_1} R_{\alpha_1\alpha_2}(\xi_1)\,\Omega^{\xi_1}_{\xi_2} R_{\beta_1\beta_2}(\xi_2)\,\Omega^{\xi_2}_{a} =$$

$$\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2}(\xi_0)\, R_{\beta_1\beta_2}(\xi_0)\,\Omega^{\xi_0}_{a} +$$

$$+\,\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2}(\xi_0)\, P_{\beta_1\beta_2\beta_3}(\xi_0)\,\Omega^{\xi_0}_{a}\frac{dx^{\beta_3}}{d\xi_2}(\xi_2-\xi_0) +$$

$$+\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3}(\xi_0)\, R_{\beta_1\beta_2}(\xi_0)\,\Omega^{\xi_0}_{a}\frac{dx^{\alpha_3}}{d\xi_1}(\xi_1-\xi_0) +$$

$$+\cdots$$

The matric terms are all expanded in a similar way about ξ_0, giving for the expansion of $\delta^3\Omega^{b}_{a}[x;z]$

$$(8.27)\quad \delta^3\Omega^{b}_{a}[x;z] = \Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dz^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3}\, d\xi + \quad (1)$$

$$+\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3\alpha_4}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dz^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3}\frac{dx^{\alpha_4}}{d\xi}(\xi-\xi_0)\, d\xi + \quad (2)$$

$$+\frac{1}{2!}\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_5}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dz^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3}\frac{dx^{\alpha_4}}{d\xi}\frac{dx^{\alpha_5}}{d\xi}(\xi-\xi_0)^2\, d\xi + \quad (3)$$

$$+\frac{1}{2!}\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_4}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dz^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3}\frac{d^2x^{\alpha_4}}{d\xi^2}(\xi-\xi_0)^2\, d\xi + \quad (4)$$

$$+\cdots$$

$$++\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3\alpha_4}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dx^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3} z^{\alpha_4}\, d\xi + \quad (5)$$

$$+\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_5}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dx^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3} z^{\alpha_4}\frac{dx^{\alpha_5}}{d\xi}(\xi-\xi_0)\, d\xi + \quad (6)$$

$$+\frac{1}{2!}\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_6}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dx^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3} z^{\alpha_4}\frac{dx^{\alpha_5}}{d\xi}\frac{dx^{\alpha_6}}{d\xi}(\xi-\xi_0)^2\, d\xi + \quad (7)$$

$$+\frac{1}{2!}\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_5}\,\Omega^{\xi_0}_{a}\int_a^b \frac{dx^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3} z^{\alpha_4}\frac{d^2x^{\alpha_5}}{d\xi^2}(\xi-\xi_0)^2\, d\xi + \quad (8)$$

$$+\cdots$$

(8.27) cont'd.

$$++\ 3\,\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2} R_{\beta_1\beta_2}\, \Omega^{\xi_0}_{a} \left\{ \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dz^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 + \right. \quad (9)$$

$$\left. + \int_a^b \frac{dz^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 \right\} +$$

$$+\ 3\,\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2} P_{\beta_1\beta_2\beta_3}\, \Omega^{\xi_0}_{a} \left\{ \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dz^{\beta_1}}{d\xi_2} z^{\beta_2} \frac{dx^{\beta_3}}{d\xi_2} (\xi_2 - \xi_0) d\xi_2 \right. \quad (10)$$

$$\left. + \int_a^b \frac{dz^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} \frac{dx^{\beta_3}}{d\xi_2} (\xi_2 - \xi_0) d\xi_2 \right\} +$$

$$+ 3\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3} R_{\beta_1\beta_2}\, \Omega^{\xi_0}_{a} \left\{ \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} \frac{dx^{\alpha_3}}{d\xi_1} (\xi_1 - \xi_0) d\xi_1 \int_a^{\xi_1} \frac{dz^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 + \right. \quad (11)$$

$$\left. + \int_a^b \frac{dz^{\alpha_1}}{d\xi_1} z^{\alpha_2} \frac{dx^{\alpha_3}}{d\xi_1} (\xi_1 - \xi_0) d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 \right\} +$$

$$+ \cdots$$

$$++\ 3\,\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2} P_{\beta_1\beta_2\beta_3}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} z^{\beta_3} d\xi_2 + \quad (12)$$

$$+\ 3\,\Omega^{b}_{\xi_0} R_{\alpha_1\alpha_2} P_{\beta_1\cdots\beta_4}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} z^{\beta_3} \frac{dx^{\beta_4}}{d\xi_2} (\xi_2 - \xi_0) d\xi_2 + \quad (13)$$

$$+ 3\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3} P_{\beta_1\beta_2\beta_3}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} \frac{dx^{\alpha_3}}{d\xi_1} (\xi_1 - \xi_0) d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 + \quad (14)$$

$$+ \cdots$$

$$++\ 3\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3} R_{\beta_1\beta_2}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} z^{\alpha_3} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 + \quad (15)$$

$$+ 3\,\Omega^{b}_{\xi_0} P_{\alpha_1\cdots\alpha_4} R_{\beta_1\beta_2}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} z^{\alpha_3} \frac{dx^{\alpha_4}}{d\xi_1} (\xi_1 - \xi_0) d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 + \quad (16)$$

$$+ 3\,\Omega^{b}_{\xi_0} P_{\alpha_1\alpha_2\alpha_3} P_{\beta_1\beta_2\beta_3}\, \Omega^{\xi_0}_{a} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} z^{\alpha_3} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} \frac{dx^{\beta_3}}{d\xi_2} (\xi_2 - \xi_0) d\xi_2 + \quad (17)$$

$$+ \cdots$$

(8.27) cont'd.

$$++\ 3!\,\Omega_{\xi_0}^{b} R_{\alpha_1\alpha_2} R_{\beta_1\beta_2} R_{\gamma_1\gamma_2} \Omega_a^{\xi_0} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2}\, d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 \int_a^{\xi_2} \frac{dx^{\gamma_1}}{d\xi_3} z^{\gamma_2} d\xi_3 + \quad (18)$$

$$+\ 3!\,\Omega_{\xi_0}^{b} P_{\alpha_1\alpha_2\alpha_3} R_{\beta_1\beta_2} R_{\gamma_1\gamma_2} \Omega_a^{\xi_0} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} \frac{dx^{\alpha_3}}{d\xi_1} (\xi_1 - \xi_0)\, d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 \int_a^{\xi_2} \frac{dx^{\gamma_1}}{d\xi_3} z^{\gamma_2} d\xi_3 + \quad (19)$$

$$+\ 3!\,\Omega_{\xi_0}^{b} R_{\alpha_1\alpha_2} P_{\beta_1\beta_2\beta_3} R_{\gamma_1\gamma_2} \Omega_a^{\xi_0} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} \frac{dx^{\beta_3}}{d\xi_2} (\xi_2 - \xi_0)\, d\xi_2 \int_a^{\xi_2} \frac{dx^{\gamma_1}}{d\xi_3} z^{\gamma_2} d\xi_3 + \quad (20)$$

$$+\ 3!\,\Omega_{\xi_0}^{b} R_{\alpha_1\alpha_2} R_{\beta_1\beta_2} P_{\gamma_1\gamma_2\gamma_3} \Omega_a^{\xi_0} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2} d\xi_1 \int_a^{\xi_1} \frac{dx^{\beta_1}}{d\xi_2} z^{\beta_2} d\xi_2 \int_a^{\xi_2} \frac{dx^{\gamma_1}}{d\xi_3} z^{\gamma} \frac{dx^{\gamma_3}}{d\xi_3} (\xi_3 - \xi_0)\, d\xi_3 + \quad (21)$$

$$+ \cdots$$

Note that (8.27) is the sum of several series. Each series is started with a double plus sign, and the first few explicitly written terms of lowest order in ε are written with a single plus sign.

Since $\delta^3 \Omega_a^b [x; z]$ is to vanish for all directrices and variations z^i, it vanishes for the directrix (8.2) and variations taken from (8.18). We compute all the integrals in (8.27) and collect terms of the same degree in ε. Since ε is arbitrary, the coefficient of $\varepsilon^0, \varepsilon^1, \cdots, \varepsilon^m, \cdots$ must vanish.

In particular we see that term (1) is the only term of degree three, and that no terms of lower degree exist. Hence (1) must vanish. We write (1) in detail

(8.28) $\Omega_{\xi_0}^{b} P_{\alpha_1\alpha_2\alpha_3}(\xi_0)\, \Omega_a^{\xi_0} \int_a^b \frac{dz^{\alpha_1}}{d\xi} z^{\alpha_2} z^{\alpha_3}\, d\xi$

on expanding the summations in (8.28), the terms fall naturally into groups where the indeces $\alpha_1, \alpha_2, \alpha_3$ have the values 1,2,3; 1,2,4; ...; n-2, n-1, n. There are $\binom{n}{3}$ such mutually exclusive groups. Without loss of generality we may discuss the first group only. Expanding the summation, this group in (8.28) becomes

$$(8.29)\quad \mathcal{S}\, \Omega_{\xi_0}^{b}\, P_{123}\, \Omega_a^{\xi_0} \int_a^b \frac{dz'}{d\xi} z^2 z^3 \, d\xi + \qquad \text{(A)}$$

$$\mathcal{S}\, \Omega_{\xi_0}^{b}\, P_{122}\, \Omega_a^{\xi_0} \int_a^b \frac{dz'}{d\xi} z^2 z^2 \, d\xi + \qquad \text{(B)}$$

$$\mathcal{S}\, \Omega_{\xi_0}^{b}\, P_{121}\, \Omega_a^{\xi_0} \int_a^b \frac{dz'}{d\xi} z^2 z' \, d\xi + \qquad \text{(C)}$$

$$\mathcal{S}\, \Omega_{\xi_0}^{b}\, P_{112}\, \Omega_a^{\xi_0} \int_a^b \frac{dz'}{d\xi} z' z^2 \, d\xi + \qquad \text{(D)}$$

$$\mathcal{S}\, \Omega_{\xi_0}^{b}\, P_{111}\, \Omega_a^{\xi_0} \int_a^b \frac{dz'}{d\xi} z' z' \, d\xi \qquad \text{(E)}$$

where $\mathcal{S}$ stands for the sum of all terms obtained by giving the indeces 1, 2, 3 all possible (six) permutations. The terms have been lettered A - E for convenience.

Now because of the skew symmetry of P_{abc} in ab, terms D and E vanish. If we choose variations $z^3 = 0$, terms A vanish and $\mathcal{S}$ is only over $1,2$. We choose four sets of variations $z^i, i = 1,2$ from the f^m of (8.18) as follows:

$$(8.30)\quad \begin{aligned} z^1 &= f^1,\ f^3,\ f^7,\ f^{13} \\ z^2 &= f^2,\ f^5,\ f^{11},\ f^{17} \end{aligned}$$

With these choices the statement that (1) of (8.27) or (8.29) vanishes for these four variations yields four homogeneous linear matric equations in four unknowns with non-vanishing determinant. Hence each matric term $\Omega^{b}_{\xi_0} P_{122} \Omega^{\xi_0}_{a}$, $\Omega^{b}_{\xi_0} P_{211} \Omega^{\xi_0}_{a}$, $\Omega^{b}_{\xi_0} P_{121} \Omega^{\xi_0}_{a}$, $\Omega^{b}_{\xi_0} P_{212} \Omega^{\xi_0}_{a}$ must vanish. Since Ω^{τ}_{η} has an inverse, this implies

$$P_{kjl} = 0 \tag{8.31}$$

when any two of the three indeces k, j, l are the same, for $1, 2$ might have been any pair of indeces.

We now assume $z^3 \neq 0$, but $z^i = 0$ for $i > 3$. Then since we have shown terms B, C, D, and E of (8.29) to vanish, we consider term A, which must also vanish.

We evaluate the integrals for the three choices of $z^i, i = 1, 2, 3$

$$\begin{array}{lccc} & i & ii & iii \\ z^1 = & f^1, & f^2, & f^2 \\ z^2 = & f^2, & f^3, & f^3 \\ z^3 = & f^3, & f^4, & f^5 \end{array} \tag{8.32}$$

Recalling that the integrals from $\xi_0 + \varepsilon$ to $\xi_0 + 2\varepsilon$ all cancel, the integrals in (8.29) are, for these three cases,

$$\begin{array}{lcccc} & & i & ii & iii \\ \int_a^b \frac{dz^1}{d\xi} z^2 z^3 \, d\xi & = & \varepsilon^3/6 & \varepsilon^3/3 & \varepsilon^3/3 \\ \int_a^b \frac{dz^2}{d\xi} z^1 z^3 \, d\xi & = & \varepsilon^3/3 & \varepsilon^3/2 & \varepsilon^3/2 \\ \int_a^b \frac{dz^3}{d\xi} z^1 z^2 \, d\xi & = & \varepsilon^3/2 & 2\varepsilon^3/3 & 5\varepsilon^3/6 \end{array}$$

Equating (8.29) to zero for these three variations gives three homogeneous linear matric equations in three unknowns with non-vanishing determinant. Hence each matric term vanishes. Since $\Omega^{b}_{\xi_0}$ has an inverse, we conclude that

(8.33) $\quad P_{123} + P_{132} = 0$

From (8.31), (8.33) and since we could perform the same analysis with any other triplet of components, and since P_{kjl} is skew symmetric in kj, P_{kjl} is skew symmetric in all three indeces. For adding

$$P_{123} + P_{132} = 0$$

$$P_{213} + P_{231} = 0$$

which are valid from (8.33) gives

$$P_{132} + P_{231} = 0$$

which shows skew symmetry in kl of P_{kjl}. Hence

(8.34) P_{kjl} is skew symmetric in each pair of indeces kj, kl, jl.

Now since P_{kjl} is completely skew symmetric, for all values of x, ξ in the space, all derivatives of $\Omega^{b}_{\xi} P_{\alpha_1\alpha_2\alpha_3} \Omega^{\xi}_{a}$ are completely skew symmetric in $\alpha_1\alpha_2\alpha_3$. Hence any term involving $P_{\alpha_1 \dots \alpha_n}$ which is summed against an expression $z^{\alpha_2} z^{\alpha_3}$ must vanish. Hence all the terms in the series (1), (2), (3), (4), ... of (8.27), and in the series (5), (6), (7), (8), ... vanish giving no new necessary conditions.

The term of lowest degree in ε remaining is (9), which is of degree four. (9) is the sum of two terms which involve different iterated integrals. In the first term, $\frac{dx}{d\xi}$ appears in the first of two iterated integrals, and in the second term $\frac{dx}{d\xi}$ appears in the second. By choosing the directrix as in (8.2) then choosing a similar directrix slightly differently paramaterized, (9) will be the sum of two terms with different coefficients for the two cases. Since (9) must vanish for each case, we conclude that each term in (9) vanishes separately. We write the first term in detail.

$$(8.35)\quad 3\,\Omega_{\xi_0}^{b} R_{\alpha_1\alpha_2} R_{\beta_1\beta_2} \Omega_a^{\xi_0} \int_a^b \frac{dx^{\alpha_1}}{d\xi_1} z^{\alpha_2}(\xi_1)\, d\xi_1 \int_a^{\xi_1} \frac{dz^{\beta_1}}{d\xi_2} z^{\beta_2}(\xi_2)\, d\xi_2 = 0$$

As in treating $P_{\alpha_1\alpha_2\alpha_3}$, we choose different sets of z such that, for a particular x^{α}, we obtain from (8.35) 27 linear homogeneous matric equations in 27 matric unknowns with non-vanishing determinant. Hence each matric term vanishes and

$$(8.36)\quad R_{kj} R_{lm} = 0$$

for all values of k j , lm.

Condition (8.36), however, implies that

$$(8.37)\quad R_{kj} = 0$$

For writing (8.36) in component notation for l = k, j = m, (not summed on k, j) gives

$$(8.38)\quad R^{r}_{s\,kj} R^{s}_{c\,kj} = 0$$

That is, the matrix R_{kj} must be nilpotent of order two.

Let us now choose coordinates which at $x(\xi_0)$ are orthogonal. This is always possible, since picking n mutually orthogonal directions at a point is equivalent to picking a self polar tetrahedron with respect to the quadric $g_{ij}\, y^i y^j$ in an n-1 dimensional projective space. In general, these coordinates will not be mutually orthogonal everywhere in the space.

In this coordinate system, at $x(\xi_0)$,

(8.39) $g_{ij} = g_{ii}\,\delta^i_j$ and $g^{ij} = g^{ii}\,\delta^j_i$ then

(8.40) $R_{kj} = R^r{}_{c\,kj} = g^{rr} R_{rc\,kj}$ (not summed on r.)

Since $R_{rc\,kj}$ is skew symmetric in r,c, so is $R^r{}_{c\,kj}$. Then (8.38) may be written

(8.41) $-\sum\limits_{m=1}^{n} R^m{}_{r\,kj}\, R^m{}_{c\,kj} = -\sum\limits_{m=1}^{n} (R^m{}_{r\,kj})^2 = 0$

for those terms for which c = r.

But if the sum of squares of real terms vanishes, the terms themselves vanish. Hence

(8.42) $R^m{}_{r\,kj} = 0$

at $x(\xi_0)$. But $x(\xi_0)$ can be taken as any point of D. Hence (8.42) holds at every point of D, and is a necessary condition for the vanishing of $\delta^3 \Omega^b_a [x;z]$.

But $R^m{}_{r\,kj} = 0$ for all points is the necessary and sufficient condition for the vanishing of $\delta^1 \Omega^b_a [x;z]$, which is a sufficient condition for the vanishing of $\delta^3 \Omega_a^b[x;z]$. Hence the necessary and sufficient condition for the vanishing of $\delta^3 \Omega_a^b[x;z]$ for all directrices and variations is

$$(8.43) \qquad R_{kj} = 0$$

at all points of D. This completes the proof of theorem 8.3.

In general, the same condition will hold for the mth Fréchet differential. For, from the dagger differentiation law, and the same type of argument as in theorem 8.3, at least one necessary condition for the vanishing of the mth Fréchet differential for all directrices and variations is of the form

$$(8.44) \qquad R_{kj} R_{cd} \cdots R_{ef} = 0$$

where there are $[\frac{m+1}{2}]$ factors in (8.44). $[\frac{m+1}{2}]$ is the greatest integer contained in $\frac{m+1}{2}$. In particular (8.44) holds for $k = c = \ldots = e$; $j = d = \ldots = f$ and R_{kj} is nilpotent of order $[\frac{m+1}{2}]$. Now there exists an integer p such that $2^p < [\frac{m+1}{2}] \leq 2^{p+1}$. If R_{kj} is nilpotent of order $[\frac{m+1}{2}]$ it is nilpotent of order 2^{p+1}. Then, if $q = 2^p$, let

$$(8.45) \quad B^r_{c\,kj} = R^r_{\alpha_1\,kj} R^{\alpha_1}_{\alpha_2\,kj} \cdots R^{\alpha_{q-1}}_{c\,kj} \qquad \text{and}$$

$$(8.46) \quad B^r_{\alpha\,kj} B^\alpha_{c\,kj} = 0 \qquad \text{(not summed on k, j).}$$

On choosing orthogonal coordinates at $x(\xi_0)$, $B^r_{\alpha\,kj}$ is symmetric in $r\alpha$. Hence (8.46) can be written, (8.47) $\sum_{\alpha} B^{r}_{\alpha kj} B^{c}_{\alpha kj} = 0$

For r = c (8.47) reduces to

(8.48) $\sum_{\alpha} (B^{r}_{\alpha kj})^2 = 0$

and hence

(8.49) $B^{r}_{\alpha kj} = 0$

(8.49) and (8.45) imply that R_{kj} is nilpotent of order 2^{p}. Continuing the process leads to the conclusion that

(8.50) $R_{kj} = 0$

Hence we have proved, since sufficiency is obvious from theorem 8.1,

Theorem 8.4. The necessary and sufficient condition that $\delta^{m}\Omega^{b}_{a}[x;z]$ given by (6.39) of theorem 6.8 vanish for all directrices $x^{i}(\xi)$ and variations $z^{i}(\xi)$ in an open region D of the space of definition of x, z is

(8.51) $R_{kj} = 0$

everywhere in D for all k, j = 1, ..., n and all m = 1, 2, ...

VIII.C. The Classification of Riemannian Spaces by Their Degree.

From theorem 8.4 and the fact that

(8.52) $\delta^{i}\lambda[x;z|b] = \delta^{i}\Omega^{b}_{a}[x;z]\,\lambda(a)$

we have

Theorem 8.5. Every flat Riemannian space is of degree zero and all other Riemannian spaces are of infinite degree.

IX. Applications of Matrix Theorems to Displacement About a Closed Directrix.

When the directrix is a closed curve, the displaced vector

(9.1) $\lambda(b) = \Omega_a^b \, \lambda(a)$

is located at the same point as the original. Hence Ω_a^b has the effect of a rotation of $\lambda(a)$ into $\lambda(b)$. This geometric fact leads to

Theorem 9.1. If, in a n-dimensional Riemannian space, a contravariant vector, indicated by the column matrix λ is displaced about a closed directrix, $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$, $x^i(a) = x^i(b)$ and if the values of $\lambda(a), \lambda(b)$ are related by (9.1) , then the determinant of the matrizant function is plus one. In symbols,

(9.2) $|\Omega_a^b| = +1$

Proof: Since the length of an arbitrary vector is unchanged under parallel displacement, in component notation, the square of the length is

(9.3) $$g_{ij} \lambda^i(b) \lambda^j(b) = g_{ij} \Omega_a^b{}^i_k \Omega_a^b{}^j_l \lambda^k(a) \lambda^l(a)$$

$$= g_{kl} \lambda^k(a) \lambda^l(a)$$

Hence

(9.4) $$g_{ij} \Omega_a^b{}^i_k \Omega_a^b{}^j_l = g_{kl}$$

Taking the determinant of both sides of (9.4)

(9.5) $|g| \cdot |\Omega_a^b|^2 = |g|$ and

(9.6) $|\Omega_a^b|^2 = 1$ hence

(9.7) $|\Omega_a^b| = \pm 1$

But parallel displacement does not change the angle between two displaced vectors. Hence, an orthogonal ennuple, or set of n mutually orthogonal vectors, displaces into a new set with the same orientation. Hence the plus sign only holds in (9.7) which proves theorem 9.1.

We seek next the vectors which have the same direction to within sign before and after displacement.

Theorem 9.2. Let a contravariant vector λ be displaced by parallel displacement about a closed directrix $x^i = x^i(\xi)$. Then if n, the dimension of the Riemann space, is odd, there are exactly 1,3,5,..., n-2 or n linearly independent vectors which have the same direction, to within sign, before and after displacement. If n is even, there are exactly 0, 2, ..., n-2, n such linearly independent vectors.

Proof: We regard the components λ^i as the n coordinates of a point in n-1 dimensional projective space. The matrizant Ω_a^b in (9.1) describes a linear transformation or collineation in the space. A fixed point, λ_0, will be a solution of the n linear homogeneous equations in n unknowns,

(9.8) $$\rho \lambda_0^i = \Omega_a^b{}^i_j \lambda_0^j$$

The necessary and sufficient condition for the existence of such points is that the determinant of the coefficients vanish.

(9.9) $$|\rho I - \Omega_a^b| = 0$$

where I is the unit matrix. This condition is an equation of degree n in ρ , hence has n roots, real and complex. Since complex roots occur in pairs, there are exactly 1,3,..., n-2 or 0,2,4,...,n-2, n real roots as n is odd or even. Every root, ρ_α , on substitution in (9.8) gives a vector λ_α which displaces into the vector $\rho_\alpha \lambda_\alpha$. Since parallel displacement preserves length, ρ_α must be ± 1 , or complex. Further, since angles are preserved, the elementary divisors[25] of the matrix must all be linear. Hence each real root of (9.9) yields a distinct "fixed" vector λ_α , and r distinct real roots yield r linearly independent vectors. Since complex roots occur in pairs, this completes the proof of theorem 9.2.

That linearity of the elementary divisors follows from the preservation of the angles between two displaced vectors is proved by the following contradiction. If there is an elementary divisor which is not linear, there will be fewer than n linearly independent vectors which assume their original directions. These, say n-m, vectors determine an n-m dimension space, S_{n-m} , and by subtraction determine an m dimensional space, S_m . All of the vectors of S_m are orthogonal to S_{n-m} . Since angles are preserved, each of the vectors V_m in S_m displace into a vector which is in S_m . Similarly for the vectors V_{n-m} in S_{n-m}. Hence the transformation determined by the matrizant function must be the direct sum of two transformations, one in the space S_{n-m} and the other in the orthogonal space S_m . Since every collineation has at least one fixed point, the collineation

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in S_m has at least one fixed vector. But this contradicts the statement that there were only n-m fixed vectors, and hence contradicts the assumption that at least one of the elementary divisors was not linear.

Theorem 9.3. Under the conditions of theorem 9.1, there are an even number of the vectors of theorem 9.2 which return to their original direction but opposite sense.

Proof: It is shown in theorem 9.1 that $|\Omega^b_a| = +1$. Let the coordinate system be so chosen that the real vectors of theorem 9.2 lie on some of the coordinate axes. The relation (9.1) may then be written

$$(9.10) \qquad \begin{aligned} \lambda^1(b) &= \lambda^1(a) \\ &\vdots \\ \lambda^i(b) &= \lambda^i(a) \\ \lambda^{i+1}(b) &= -\lambda^{i+1}(a) \\ &\vdots \\ \lambda^{i+j}(b) &= -\lambda^{i+j}(a) \\ \lambda^{i+j+1}(b) &= \omega^{i+j+1}_k \lambda^k(a) \qquad k = i+j+1, \cdots, n \\ &\vdots \\ \lambda^n(b) &= \omega^n_k \lambda^k(a) \end{aligned}$$

where the first i vectors are those returning to their initial direction and sense, and have components $1,0,0,\cdots,0$; $0,1,0,0,\cdots,0$; etc. The j vectors returning to their initial direction but opposite sense have components $0,\cdots,0,1,0,\cdots,0$ etc. The $n-i-j$ linearly independent real vectors which span the remaining space do not return to their original direction. However the $n-i-j$ rowed determinant $|\omega| = +1$ from the same type of reasoning as in theorem 9.1. Since

(9.11) $$|\Omega_a^b| = (-1)^j (+1)^i |\omega| = +1$$

j must be even. This completes the proof of theorem 9.3.

On using theorem 9.2 and 9.3, there follows immediately

Theorem 9.4. Let a contravariant vector λ be displaced about a closed directrix by parallel displacement in a Riemannian space. Then there are exactly 1,3,5, ..., n-2, n or 0,2,4,..., n-2, vectors which return to their original value as n is odd or even.

Clearly if two or more vectors return to their original direction and sense, any linear combination of them also does. Similarly any linear combination of vectors which return to their original direction but opposite sense also returns to its original direction and opposite sense. The two groups of vectors are quite distinct, for

Theorem 9.5. The vectors of theorem 9.2 which return to their original direction and sense are orthogonal to those which return to their original direction but opposite sense.

Proof: Let λ_1 be a vector which returns to its original direction and sense and λ_2 be a vector which returns to its original direction but opposite sense. Then, since the angle between λ_1 and λ_2 is unchanged by parallel displacement,

(9.12) $$g_{ij}\lambda_1^i(b)\lambda_2^j(b) = -g_{ij}\lambda_1^i(a)\lambda_2^j(a) = g_{ij}\lambda_1^i(a)\lambda_2^j(a)$$

But this implies

(9.13) $g_{ij}\,\lambda_1^i(a)\,\lambda_2^j(a)=0$

and hence λ_1 and λ_2 are orthogonal.

If such "fixed" vectors exist at a point of a directrix, they exist all along the directrix. In fact

Theorem 9.6. If λ is a contravariant vector which returns to its original direction either with or without the same sense after parallel displacement about a closed directrix in Riemannian space, then every vector parallel to λ on being displaced about the directrix returns to its original direction with or without its original sense, respectively.

Proof: If $\lambda(c)$ is parallel to $\lambda(a)$ at some point $x(c)$ then

(9.14) $\lambda(c) = \Omega_a^c\,\lambda(a)$

If $\lambda(c)$ is displaced once about the directrix and the curve parameter ξ goes from c to d, then

(9.15) $$\lambda(d) = \Omega_c^d\,\lambda(c) = \Omega_c^d\,\Omega_a^c\,\lambda(a) = \Omega_a^d\,\lambda(a)$$
$$= \Omega_b^d\,\Omega_a^b\,\lambda(a) = \pm\,\Omega_b^d\,\lambda(a) = \pm\,\lambda(c)$$

where the positive sign occurs if $\lambda(a)$ returns with the same sense, and the minus sign occurs if $\lambda(a)$ returns with the opposite sense after being displaced about the directrix.

Use is made of the fact that

(9.16) $\Omega_a^c = \Omega_b^d$.

X. Extension of the Theory to Tensors of Types Other Than Contravariant Vectors.

The entire theory of the previous chapters clearly extends immediately to tensors of types other than contravariant vectors. For the sake of completeness we indicate how this is done and prove some of the theorems corresponding to earlier theorems.

Theorem 10.1. If the contravariant tensor t^{ij} of rank two is displaced by Levi-Civita parallel displacement along a directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$ and if $t^{ij}(a)$ and $t^{ij}(b)$ are the values of t^{ij} at $\xi = a$ and $\xi = b$ respectively, then

$$(10.1) \quad t^{ij}(b) = \Omega_a^b{}^i{}_m \left[-\Gamma_e \frac{dx^e}{d\xi}\right] \Omega_a^b{}^j{}_\ell \left[-\Gamma_t \frac{dx^t}{d\xi}\right] t^{m\ell}(a)$$

(not summed on a,b)

where (10.1) is written in component notation, ie $\Omega_a^b{}^r{}_c \left[-\Gamma_e \frac{dx^e}{d\xi}\right]$ is the component in the rth row and cth column of the matrix $\Omega_a^b \left[-\Gamma_e \frac{dx^e}{d\xi}\right]$

Proof: The equations of parallel displacement for a contravariant tensor of rank two are

$$(10.2) \quad \frac{dt^{ij}}{d\xi} = -\Gamma^i_{mk} \frac{dx^k}{d\xi} t^{mj} - \Gamma^j_{mk} \frac{dx^k}{d\xi} t^{im}$$

We may now integrate equations (10.2) by successive approximations in exactly the same way as equations (3.1) were integrated. The verification of the result (10.1) involves interchanging orders of integration and certain theorems in combinatorial analysis. The proof, while simple, is tricky, and becomes rather complicated when extended to tensors of order greater than two.

A different proof proceeds as follows. Consider the particular tensor

(10.3) $T^{ij} = \lambda^i \mu^j$

the direct product of $\lambda^i(\xi)$, $\mu^j(\xi)$ two contravariant vectors. On displacing T^{ij} along the directrix $x^i = x^i(\xi)$we have that T^{ij} must satisfy

(10.4) $$\begin{aligned} \frac{dT^{ij}}{d\xi} &= \frac{d\lambda^i}{d\xi}\mu^j + \lambda^i \frac{d\mu^j}{d\xi} \\ &= -\Gamma^i_{mk}\frac{dx^k}{d\xi}\lambda^m\mu^j - \Gamma^j_{mk}\frac{dx^k}{d\xi}\lambda^i\mu^m \\ &= -\Gamma^i_{mk}\frac{dx^k}{d\xi}T^{mj} - \Gamma^j_{mk}\frac{dx^k}{d\xi}T^{im} \end{aligned}$$

which is the same differential equation as (10.2). But from (3.5), (10.4) has the unique solution at $\xi = b$

(10.5) $T^{ij}(b) = \Omega^{b\,i}_{a\,m}\,\Omega^{b\,j}_{a\,l}\,T^{ml}(a)$ (not summed on a, b)

Since t^{ij} satisfies the same differential equations as T^{ij} , the theorem is immediate.

In an entirely similar way,

<u>Theorem 10.2.</u> <u>If a contravariant tensor of order p,</u> $t^{ij\cdots k}$, <u>is displaced by Levi-Civita parallel displacement along a directrix</u> $x^i = x^i(\xi)$ <u>from</u> $\xi = a$ <u>to</u> $\xi = b$, <u>then</u>

(10.6) $t^{ij\cdots k}{}_{(b)} = \Omega^{b\,i}_{a\,\alpha}\,\Omega^{b\,j}_{a\,\gamma}\cdots\Omega^{b\,k}_{a\,\beta}\,t^{\alpha\gamma\cdots\beta}{}_{(a)}$ <u>(not summed on a,b)</u>

<u>where there are p indeces</u> $i, j \cdots k\,;\,\alpha, \gamma, \cdots \beta$

Now clearly, $t^{ij\cdots k}[\Omega_a^b]$ considered as a function of the matrizant Ω_a^b is a multilinear function on the Banach space B_2 of square matrices of order n whose elements are real continuous numerical functions of two real numerical variables to a new Banach space whose elements are sets of elements from B_2. Hence by the composition theorem for Fréchet differentials,

Theorem 10.3. <u>The Fréchet differentials of any order of $t^{ij\cdots k}_{(b)}$ given in theorem 10.2 exist and are given by the appropriate formulae by means of (10.6) and the theorems on Fréchet differentials of a multilinear function and the composition theorem.</u>

Theorem 10.3 is clearly valid for both the Fréchet differentials resulting from warping the space and from varying the directrix.

The details for covariant and mixed tensors are only slightly different from those for the contravariant tensors. For a covariant vector, μ , the equations of parallel displacement are

$$(10.7)\qquad \frac{d\mu_i}{d\xi} = \Gamma^k_{ji}\frac{dx^j}{d\xi}\mu_k$$

which can be written in the matrix form

$$(10.8)\qquad \frac{d\mu}{d\xi} = \mu\,\Gamma_j\frac{dx^j}{d\xi}$$

Equation (10.8) differs from (3.3) only in sign and order of matric factors. Therefore, if

$$(10.9)\qquad \mho_a{}^b = \Omega_a^b\left[\Gamma_j\frac{dx^j}{d\xi}\right] \quad \text{transposed}$$

we have

Theorem 10.4. If a covariant vector, represented by the row matrix μ is displaced by Levi-Civita parallel displacement in a Riemannian space along the directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$, and if $\mu(a)$, $\mu(b)$ are the values of μ at $\xi = a, b$ respectively, then

(10.10) $\mu(b) = \mu(a)\, \mho_a^{\,b}$

where $\mho_a^{\,b}$ is the transpose of the matrix $\Omega_a^b\left[\Gamma_k \frac{dx^k}{d\xi}\right]$. Further

Theorem 10.5. All the theorems on the Fréchet differentials of a contravariant vector under parallel displacement have their counterparts in theorems for a covariant vector μ. These theorems are obtained by interchanging $-\Gamma_k$ for Γ_k transposing all matrices in the theorems for λ and replacing λ by μ

We state the further theorem

Theorem 10.6. If a mixed tensor $t^{i\cdots j}_{k\cdots l}$ is displaced by Levi-Civita parallel displacement along the directrix $x^i = x^i(\xi)$ from $\xi = a$ to $\xi = b$, then if $t^{i\cdots j}_{k\cdots l}(a)$, $t^{i\cdots j}_{k\cdots l}(b)$, are the values at $\xi = a$ and $\xi = b$ respectively, then

(10.11) $t^{i\cdots j}_{k\cdots l}(b) = \Omega_a^{b}{}^{i}_{m} \cdots \Omega_a^{b}{}^{j}_{p}\; t^{m\cdots p}_{q\cdots r}(a)\, \mho_a^{b}{}^{q}_{k} \cdots \mho_a^{b}{}^{r}_{l}$

and all the Fréchet differentials of $t^{i\cdots j}_{k\cdots l}(b)$ exist and are given by their obvious formulas.

XI. Bibliography.

Appel, Paul

(1) "Traite de Mechanique Rationelle" Paris. (1926) vol. 5 pp 136-137.

Banach, Stephan

(1) "Theorie Des Operations Lineares" Warsaw. (1932)

Bocher, Maxime

(1) "Introduction to Higher Algebra" New York. (1938) Chapter XXI.

Dienes, Paul

(1) "Sur l'Integration Des Equations du Deplacement Parallele de M. Levi-Civita" Rendiconti Deo Circolo Matematico di Palermo t. XLVII (1923) pp 144-152.

Duscheck-Mayer

(1) "Lehrbuch der Differential-Geometrie". Band II. Riemannische Geometrie. Berlin (1930) pp 123-125.

Eisenhart, L.P.

(1) "Riemannian Geometry" Princeton (1926) pp 65-67.

Frazier, Duncan and Collar

(1) "Elementary Matrices and Some Applications to Dynamics and Differential Equations" Cambridge (1938) pp 53-56.

Levi-Civita, T.

(1) "Nozione di Parallelismo in Una Varieta Qualunque e Conseguente Specificazione Geometrica della Curvatura Riemannian" Rendiconte deo Circolo Matematico di Palermo t. XLII (1917) pp 173-205.

Martin, R. S.

(1) "Contributions to the Theory of Functionals" Ph.D. Thesis. California Institute of Technology Pasadena (1932)

Michal, Aristotle D.

(1) "First Order Differentials of Functionals With Arguments in Topological Abelian Groups" Revista de Ciencias, Lima, No 453 Ano XLVII pp 389 - 422

(2) "Matric and Tensor Calculus with Applications to Mechanics, Elasticity and Aeronautics" New York (1947)

(3) " On a Non-Linear Total Differential Equation in Normed Linear Spaces" to appear in Acta Mathematica.

(4) "The Frechet Differentials of Regular Power Series in Normed Linear Spaces" Duke Mathematical Journal vol 13(1946) pp 57-59.

Peres, J.

(1) "Le Parallelisme de Levi-Civita et la Courbure Riemannienne" Rendiconte delea R. Academia dei Lincei Serie V. vol XXVIII 1 Semestre (1919) pp 425-428.

Thomas, T.Y.

(1) "Differential Invariants of Generalized Spaces" Cambridge (1934).

Veblen, O.

(1) "Invariants of Quadratic Differential Forms" London, (1933).

Whittaker, E.T.

(1) "Parallelism and Teleparallelism in the Newer Theories of Space". Journal of the London Mathematical Society. vol.5 (1930) pp 68-80.

Whittaker and Watson

(1) "Modern Analysis" American Edition. Cambridge (1945) pp 221 - 2