# Two Categorifications of the Local Langlands Correspondence for the Torus

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# ABSTRACT

The stack of local Langlands parameters is a Picard stack when the relevant reductive group is a torus. We explicitly determine its Picard dual and show that the Fourier-Mukai transform gives rise to the integral categorical local Langlands correspondence for the torus. This is the categorification of the local Langlands correspondence and answers a conjecture of X. Zhu. Moreover, we establish a geometric version of this correspondence. This second categorification relates to the previous correspondence in the sense that taking the categorical trace construction allows one to reproduce the previous result.

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*Chapter 1*

# INTRODUCTION

Let $G$ be a reductive group over a non-archimedean local field $F$. X. Zhu (2021) provided an in-depth study of the stack of local and global Langlands parameters $\mathrm{Loc}_{^cG,F}$ defined over $\mathbb{Z}[1/p]$, and proposed a categorical arithmetic local Langlands correspondence (LLC). The automorphic side of this Langlands correspondence is given by the category of $\ell$-adic sheaves on the stack of isocrystals $\mathfrak{B}(G) = LG/\mathrm{Ad}_\sigma LG$, where $\sigma$ is the Frobenius on the residue field $\kappa_F$ of $F$. A central part of the conjecture is as follows.

**Conjecture 1.0.1.** (X. Zhu, 2021, Conjecture 4.6.4.) Let $(G, B, T, e)$ be a pinned quasi-split reductive group over $F$ and $\Lambda = \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. Then there is a natural equivalence of stable $\infty$-categories

$$\mathbb{L}_G : \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \to \mathrm{IndCoh}_{\hat{\mathcal{N}}_{^cG}}(\mathrm{Loc}_{^cG,F})$$

sending the Whittaker sheaf to the structural sheaf $\mathcal{O}_{\mathrm{Loc}_{^cG,F}}$. The category

$$\mathrm{IndCoh}_{\hat{\mathcal{N}}_{^cG}}(\mathrm{Loc}_{^cG}) \subset \mathrm{IndCoh}(\mathrm{Loc}_{^cG})$$

is a subcategory of ind-coherent sheaves with certain singular support condition.

In a more recent development, Zhu established the tame categorical local Langlands correspondence with $\overline{\mathbb{Q}}_\ell$-coefficient and the unipotent correspondence with $\overline{\mathbb{F}}_\ell$-coefficient (Xinwen Zhu, 2025).

1.0.2. **The First Categorification.** In this paper, we verify a more general form of the conjecture that allows $\mathbb{Z}$-coefficient for an arbitrary torus over $F$.

When $G = T$ is a torus, the stack $\mathfrak{B}(T)$ is a disjoint union of copies of the classifying stack $[*/T(F)]$ indexed by the Kottwitz set $B(T)$. It is possible to define a characteristic 0 analogue of $\mathfrak{B}(T)$ which also consists of copies of $[*/T(F)]$. In fact, we can define an algebraic stack $\mathbf{Tor}_{T,\mathrm{iso}_F}$ over $\mathbb{Z}$, such that

$$\mathrm{IndShv}\left(\mathfrak{B}(T),\ \overline{\mathbb{Z}}_\ell\right) \cong \mathrm{QCoh}\left(\mathbf{Tor}_{T,\mathrm{iso}_F} \otimes \overline{\mathbb{Z}}_\ell\right).$$

The algebraic stack $\mathbf{Tor}_{T,\mathrm{iso}_F}$ is essentially the stack of $T$-isocystals. Let $\breve{F}$ be the completion of a maximal unramified extension of $F$, and let $\sigma$ be the Frobenius action. The groupoid $\mathbf{Tor}_{T,\mathrm{iso}_F}(\mathbb{Z})$ is

$$\mathbf{Tor}_{T,\mathrm{iso}_F}(\mathbb{Z}) = [T(\breve{F})/\mathrm{Ad}_\sigma T(\breve{F})].$$

One of the main result of this paper is as follows.

**Theorem 1.0.3.** There is a canonical family of Poincaré line bundles

$$\mathcal{L}_T : \mathbf{Tor}_{T,\mathrm{iso}_F} \times \mathrm{Loc}_{^cT,F} \to \mathbb{B}\mathbb{G}_m$$

for every torus $T$ over $F$. Furthermore, these Poincaré line bundles induce isomorphism $\mathrm{Loc}^\vee_{^cT,F} \cong \mathbf{Tor}_{T,\mathrm{iso}_F}$ for all torus $T$.

We shall see in Section 2.4 that the isomorphism $\mathrm{Loc}^\vee_{^cT,F} \to \mathbf{Tor}_{T,\mathrm{iso}_F}$ is essentially taking the cup product with the fundamental class of local class field theory. This answers a conjecture of Zhu (X. Zhu, 2021).

As an application, we establish an integral local Langlands correspondence for torus.

**Theorem 1.0.4.** The Fourier-Mukai transform via the Poincaré line bundle $\mathcal{L}_T$ gives the equivalence of stable $\infty$-categories

$$\mathbb{L} : \mathrm{IndCoh}(\mathrm{Loc}_{^cT,F}) \cong \mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F}).$$

The Fourier-Mukai transform was first introduced to the Langlands program in the work of Laumon, where the transform is applied to construct the geometric Langlands correspondence for $GL(1)$. Later, in the work of Braverman-Bezrukavnikov (Bezrukavnikov and Braverman, 2007) and Chen-Zhu (Chen and X. Zhu, 2014), Fourier-Mukai is used to establish a generic version of the Langlands correspondence in positive characteristic (for any reductive group $G$).

1.0.5. **The Second Categorification.** Let $\Lambda = \overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{Z}}_\ell$ and let $L^+T$ be the positive loop group of $T$. There is an isomorphism between abelian groups of continuous characters of Serre's fundamental group $\pi_1(L^+T)$ and character sheaves on $L^+T$:

$$\mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times) = \mathrm{CS}(L^+T, \Lambda).$$

The second main result of this paper is a categorification of this isomorphism:

**Theorem 1.0.6.** There exists a fully-faithful, $t$-exact, monoidal functor

$$\mathrm{Ch} : \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) \to \mathrm{IndShv}(LT),$$

where $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}$ is the representation stack of the inertia group of $F$ in the dual group ${}^cT$.

This can be viewed as a "categorification of categorification" in the the following sense. Let $\mathrm{Shv}^{\mathrm{mon}}(LT)$ denote the essential image of Ch. It is the thick subcategory compactly generated by all character sheaves on $LT$. Let $\mathrm{Ch}^{\mathrm{mon}}$ denote the the equivalence of categories

$$\mathrm{Ch}^{\mathrm{mon}} : \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) \cong \mathrm{IndShv}^{\mathrm{mon}}(LT).$$

Note that both categories carry a Frobenius structure.

**Proposition 1.0.7.** There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}(\mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}), \sigma) & \longrightarrow & \mathrm{Tr}(\mathrm{IndShv}^{\mathrm{mon}}(LT), \sigma) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F}) & \xrightarrow{\mathbb{L}} & \mathrm{IndShv}(\mathfrak{B}(T)), \end{array} \tag{1.1}$$

where the top arrow is induced by $\mathrm{Ch}^{\mathrm{mon}}$, the bottom arrow is the functor $\mathbb{L}$ under the identification $\mathrm{IndShv}(\mathfrak{B}(T)) \cong \mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F})$. Furthermore, he two vertical arrows are canonical equivalences.

Some results in this paper are independently obtained by K. Zou (Zou, 2024), sometimes by different methods. There are two main differences between this work and Zou's. On the one hand, Zou considers integral $\ell$-adic sheaves on $\mathrm{Bun}_G$ over the Fargues-Fontaine curve, while we consider quasi-coherent sheaves on $\mathbf{Tor}_{T,\mathrm{iso}_F}$ which is defined over $\mathbb{Z}$. On the other hand, Zou uses the spectral action on the Whittaker sheaf to establish the equivalence, while we use the Fourier-Mukai transform.

The organization of the paper is as follows. Section 2.1 introduces the notations and definitions of $\mathrm{Loc}_{{}^cT,F}$ and $\mathbf{Tor}_{T,\mathrm{iso}_F}$. Section 2.3 establishes an important short exact sequence involving $\mathrm{Loc}_{{}^cT,F}$. Section 2.4 constructs the Poincaré line bundles and proves the main theorem. Section 2.5 discusses the Fourier-Mukai transform and the equivalence of categories.

*Chapter 2*

# THE FIRST CATEGORIFICATION

## 2.1 Notation

2.1.1. **Galois groups.** We first fix our notations. Let $F$ be a local field, $\kappa_F$ its residue field, and $E/F$ a Galois extension that splits the torus $T$ defined over $F$. Let $\Gamma = \mathrm{Gal}(E/F)$ and let $W_F$ and $W_E$ be the Weil groups of $F$ and $E$. Recall that the relative Weil group of $E/F$ is defined as $W_{E/F} = W_F/[W_E, W_E]$.

By local class field theory, the relative Weil group is also the unique group extension

$$1 \to E^\times \to W_{E/F} \to \Gamma \to 1$$

corresponding to the fundamental class $\alpha \in H^2(\Gamma, E^\times)$. Fix $\{w_\tau\}, \tau \in \Gamma$ to be a system of representatives of (right) cosets of $E^\times$. For any $g, w \in W_{E/F}$, there is a unique $\delta(g, w) \in E^\times$ and a unique $\tau \in \Gamma$ such that

$$gw = \delta(g, w)w_\tau.$$

The assignment $(\tau, \sigma) \mapsto \delta(w_\tau, w_\sigma)$ is a cycle and it represents the fundamental class $\alpha$.

Let $p : W_{E/F} \to \Gamma$ be the projection as above, we extend the meaning of the notation $w_\tau$ to allow elements of $W_{E/F}$ appear in the subscript, for example, $w_g = w_{p(g)}$, and $w_{g\tau} = w_{p(g)\tau}$, for all $g \in W_{E/F}, \tau \in \Gamma$.

2.1.2. **Torus and dual torus.** Let $L = X^*(T)$ and $\hat{L} = X_*(T)$ be the weight lattice and coweight lattice of $T$. They are both $\Gamma$-modules and therefore $W_{E/F}$-modules. The dual torus $\hat{T} = L \otimes \mathbb{G}_m$ is defined over $\mathbb{Z}$. The C-group of $T$ is an enhancement of the Langlands dual group

$${}^cT := \hat{T} \rtimes (\mathbb{G}_m \times \Gamma),$$

where $\Gamma$ acts naturally on $T$ and $\mathbb{G}_m$ acts trivially. The $\mathbb{G}_m$ factor in the C-group is only useful for non-abelian reductive groups, but we include it here to be consistent with notations in (X. Zhu, 2021). Let the homomorphism $\chi : W_{E/F} \to \mathbb{G}_m \times \Gamma$ be trivial on the factor $\mathbb{G}_m$ and be the canonical projection on the factor $\Gamma$.

2.1.3. **The stack of Langlands parameters.** We now define the scheme of framed Langlands parameters $\mathrm{Loc}^{\square}_{{}^cT,F}$ and the stack of Langlands parameters $\mathrm{Loc}_{{}^cT,F}$. All (Picard) stacks in this note will live over the fpqc site of $S = \operatorname{Spec}\mathbb{Z}$.

Let $M$ be an affine group scheme over $S$, and $G$ an abstract group. The functor $\mathcal{R}_{G,M}$ that sends every $\mathbb{Z}$-algebra $R$ to the set of group homomorphisms from $G$ to $M(R)$ is represented by an affine scheme. This is because, if $I \subset G$ is a set of generators, $\mathcal{R}_{G,M}$ is a closed subset of $M^I$ according to the relations between the generators. The derived scheme of representations is studied in (X. Zhu, 2021).

By local class field theory, there is the short exact sequence

$$1 \to E^\times \to W_{E/F} \to \Gamma \to 1.$$

Let $U^{(n)}$ be the $n$-th congruence subgroup of $E^\times$, so $U^{(0)} = \mathcal{O}_E^\times$ and $U^{(n)} = 1 + \mathfrak{m}_E^n, n \geq 1$. Let $W^{(n)} = W_{E/F}/U^{(n)}$. The scheme of framed Langlands parameters $\mathrm{Loc}^{\square}_{{}^cT,F}$ classifies continuous cross homomorphisms from $W_{E/F}$ to $\hat{T}$, with $\hat{T}(R)$ endowed with the discrete topology. Namely,

$$\mathrm{Loc}^{\square}_{{}^cT,F} = \varinjlim_n \mathrm{Loc}^{\square,(n)}_{{}^cT,F}, \quad \text{where} \quad \mathrm{Loc}^{\square,(n)}_{{}^cT,F} = {}^{cl}\mathcal{R}_{W^{(n)},{}^cT} \times_{{}^{cl}\mathcal{R}_{W^{(n)},\mathbb{G}_m\times\Gamma}} \{\chi\}.$$

Equivalently, $\mathrm{Loc}^{\square}_{{}^cT,F}$ is the scheme whose $R$-points are $Z^1(W_{E/F}, \hat{T}(R))$, where $Z^1$ denotes the set of *continuous* cocycles. Since $\hat{T}$ is commutative, $\mathrm{Loc}^{\square}_{{}^cT,F}$ has a canonical Picard stack structure. The stack of Langlands parameters is defined as $\mathrm{Loc}_{{}^cT,F} = \mathrm{Loc}^{\square}_{{}^cT,F}/\hat{T}$.

Recall that in (Deligne, n.d.), Deligne defined the functor $\mathrm{ch} : D^{[-1,0]}(S,\mathbb{Z}) \to \mathcal{PS}/S$ that sends a complex of abelian sheaves over $S$ whose cohomology concentrates in degree -1 and 0 to a Picard stack over $S$. It is convenient to think of $\mathrm{Loc}_{{}^cT,F}$ as

$$\mathrm{Loc}_{{}^cT,F} = \mathrm{ch}\left(\tau_{\leq 0}(C^\bullet(W_{E/F}, \hat{T})[1])\right),$$

where $C^\bullet(W_{E/F}, \hat{T})$ is the complex of abelian sheaves calculating group cohomologies $H^\bullet(W_{E/F}, \hat{T})$.

2.1.4. **The category of coherent sheaves on $\mathrm{Loc}_{{}^cT,F}$.** Recall that $\mathrm{Loc}^{\square}_{{}^cT,F} = \varinjlim \mathrm{Loc}^{\square,(n)}_{{}^cT,F}$ and $\mathrm{Loc}_{{}^cT,F} = \mathrm{Loc}^{\square}_{{}^cT,F}/\hat{T}$. As the action of $\hat{T}$ stablizes $\mathrm{Loc}^{\square,(n)}_{{}^cT,F}$ for large $n$, $\mathrm{Loc}_{{}^cT,F}$ has a natural ind-scheme structure. Accordingly, we define the category $\mathrm{Coh}(\mathrm{Loc}_{{}^cT,F})$ to be coherent sheaves supported on finitely many connected components and $\mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F})$ to be the ind-completion of it.

2.1.5. **The stack of isocrystals.** We follow the setup in (Kottwitz, 2014), and let $\hat{L} \to \mathrm{Hom}(E^\times, E^\times \otimes \hat{L}) = Z^1(E^\times, T(E))$ be the adjoint of the identity. We form the set of algebraic cycles by the following fibre product:

$$\begin{array}{ccc} Z^1_{\mathrm{alg}}(W_{E/F}, T(E)) & \longrightarrow & \hat{L} \\ \downarrow & & \downarrow \\ Z^1(W_{E/F}, T(E)) & \xrightarrow{\mathrm{res}} & Z^1(E^\times, T(E)). \end{array}$$

In other words, the algebraic cycles are those 1-cycles whose restriction to $E^\times \subset W_{E/F}$ is an algebraic character of $T$.

Recall that $U^{(n)}$ are congrunce subgroups of $E^\times$. The sets $U_n = \hat{L} \otimes U^{(n)}$ forms a basis of open subgroups of $T(E)$ that are $W_{E/F}$-invariant, and $V_n = U_n \cap T(F)$ forms a basis of open subgroups of $T(F)$.

We define the stack $\mathbf{Tor}_{T,\mathrm{iso}_F}$ as follows (non-zero terms of the chain complex are situated in degree -1 and 0)

$$\mathbf{Tor}_{T,\mathrm{iso}_F} := \varprojlim_n \mathrm{ch}\left(\cdots \to 0 \to T(E)/U_n \to Z^1_{\mathrm{alg}}(W_{E/F}, T(E)/U_n) \to 0 \to \cdots\right).$$

Since the topos of fpqc-sheaves is replete, it is shown that inverse limit of surjective homomorphisms have no higer limit (Bhatt and Scholze, 2014). Therefore the inverse limit can also be taken at each degree:

$$\mathbf{Tor}_{T,\mathrm{iso}_F} = \mathrm{ch}\left(\cdots \to 0 \to \varprojlim T(E)/U_n \to \varprojlim Z^1_{\mathrm{alg}}(W_{E/F}, T(E)/U_n) \to 0 \to \cdots\right).$$

**Remark 2.1.6.** (i) Let us consider the groupoid $\mathbf{Tor}_{T,\mathrm{iso}_F}(\mathbb{Z})$. The isomorphism class of the groupoid is the cohomology of algebraic cycles $H^1_{\mathrm{alg}}(W_{E/F}, T(E))$, and it is isomorphic to the Kottwitz set $B(T) = X^*(\hat{T}^\Gamma) = X_*(T)_\Gamma = \hat{L}_\Gamma$ which classifies $T$-isocrystals. The automorphism group of the identity object is $T(F)$.

(ii) The stack $\mathbf{Tor}_{T,\mathrm{iso}_F}$ is *not* the constant groupoid with objects $\hat{L}_\Gamma$ and with automorphism group of the identity $T(F)$. The fpqc site remembers the pro-finite topology of the automorphism group $T(F)$.

(iii) Let $\breve{F}$ be the completion of a maximal unramified extension of $F$, $\sigma$ the Frobenius. $\mathbf{Tor}_{T,\mathrm{iso}_F}(\mathbb{Z})$ is isomorphic to the groupoid of pairs $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a $T$-torsor over $\breve{F}$ and $\phi : \mathcal{E} \cong \sigma^* \mathcal{E}$ is an isomorphism of $T$-torsors. By a Tannakian formalism, this is the groupoid of exact tensor functors from $\mathrm{Rep}_T$ to the monoidal category of isocrystals over $F$. This explains the notation $\mathbf{Tor}_{T,\mathrm{iso}_F}$.

### 2.2 Quasi-Coherent Sheaves on $\mathrm{Tor}_{\mathcal{T},\mathrm{iso}_F}$

2.2.1. Recall that $V_n$ is a system of open neighbourhoods of the topological group $T(F)$. Let $T(F)_0$ be the maximal compact subgroup of $T$. We define two sheaves of abelian groups on the fpqc-site of $\mathbb{Z}$

$$\mathcal{T} = \varprojlim \underline{T(F)/V_n},$$
$$\mathcal{T}_0 = \varprojlim \underline{T(F)_0/V_n},$$

where $\underline{A}$ means the constant sheaf of an abelian group $A$.

2.2.2. We also define a sub-functor $\mathcal{T}^\circ$ of $\mathcal{T}$ so that for every unitary ring $A$,

$$\mathcal{T}^\circ(A) = \bigcup_{t\in T(F)} t\cdot \mathcal{T}_0(A).$$

In other words, $\mathcal{T}^\circ$ consists of sections which lie uniformly in one coset of $T(F)_0$. The sheafification of $\mathcal{T}^\circ$ is $\mathcal{T}$.

We take pre-stack quotients $\mathbb{B}\mathcal{T}^\circ = */\mathcal{T}^\circ$ and $\mathbb{B}\mathcal{T}_0 = */\mathcal{T}_0$. Their sheafification are the stacks $\mathbb{B}T(F)$ and $\mathbb{B}\mathcal{T}_0$. In the next proposition, we will use the pre-stack quotients because they are easier to work with. However, sheafification has no effect on the category of quasi-coherent sheaves. We quickly recall the argument for this fact in the following.

Consider the presheaf QCoh that associates a scheme $S$ with the $\infty$-category of quasi-coherent sheaves on $S$ and associates a morphism with its $*$-pullback. This presheaf is in fact a sheaf by fpqc-descent. Let $\mathcal{Y}^\flat$ be a prestack and $\mathcal{Y}$ its sheafification. A quasi-coherent sheaf on $\mathcal{Y}^\flat$ is a morphism of pre-sheaves $\mathcal{Y}^\flat \to \mathrm{QCoh}$. However, any morphism of pre-sheaf must factor through the sheafification $\mathcal{Y}^\flat \to \mathcal{Y}$. This is to say that there is a canonical equivalence $\mathrm{QCoh}(\mathcal{Y}^\flat) \cong \mathrm{QCoh}(\mathcal{Y})$.

**Proposition 2.2.3.** Let $\Lambda$ be any commutative ring. Pulling back along $* \to \mathbb{B}\mathcal{T}_0$ and $* \to \mathbb{B}T(F)$ produces the following equivalences of categories:

$$\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0 \otimes \Lambda) \cong \mathrm{Rep}_{\mathrm{sm}}(T(F)_0, \Lambda),$$
$$\mathrm{QCoh}(\mathbb{B}\mathcal{T}^\circ \otimes \Lambda) \cong \mathrm{Rep}_{\mathrm{sm}}(T(F), \Lambda).$$

*Proof.* Let $\mathbb{B}\mathcal{T}_0$ and $\mathbb{B}\mathcal{T}^\circ$ also denote their base-change over $\Lambda$. Let $p$ denote $* \to \mathbb{B}\mathcal{T}_0$. Consider the adjoint pair

$$p^* : \mathrm{QCoh}(\mathbb{B}\mathcal{T}_0) \rightleftarrows \Lambda\text{-Mod} : p_*.$$

It is clear that $p^*$ is conservative and exact. The category $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$ admits arbitrary limits. Therefore, $p^*$ exhibits $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$ as co-monadic over $\Lambda$-Mod.

Since $\mathcal{T}_0$ is pro-finite, it is represented by an affine scheme $\mathrm{Spec}(A)$. The co-monad endomorphism $p^*p_*$ is the functor $M \mapsto A \otimes M$. $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$ is hence equivalent to the category of $\Lambda$-modules with $A$-co-algebra structure, i.e., the category $\mathrm{Rep}_{\mathrm{sm}}(T(F)_0, \Lambda)$.

Let $f : \mathbb{B}\mathcal{T}_0 \to \mathbb{B}\mathcal{T}^\circ$ the map induced by the inclusion $\mathcal{T}_0 \to \mathcal{T}^\circ$. It is schematic, quasi-compact, and quasi-separated. Therefore there exists a pair of adjoint functors

$$f^* : \mathrm{QCoh}(\mathbb{B}\mathcal{T}^\circ) \rightleftarrows \mathrm{QCoh}(\mathbb{B}\mathcal{T}_0) : f_*.$$

The pullback functor $f^*$ can be identified with pre-composing a map $\mathbb{B}\mathcal{T}^\circ \to \mathrm{QCoh}$ with $f$. It is clear that $f^*$ is conservative and exact. The category $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$ admits arbitrary limits. Therefore, $f^*$ exhibits $\mathrm{QCoh}(\mathbb{B}\mathcal{T}^\circ)$ as co-monadic over $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$.

On the other hand, we have another co-monad by induction-restriction:

$$\mathrm{Res} : \mathrm{Rep}_{\mathrm{sm}}(T(F), \Lambda) \rightleftarrows \mathrm{Rep}_{\mathrm{sm}}(T(F)_0, \Lambda) : \mathrm{Ind},$$

where Res exhibit $\mathrm{Rep}_{\mathrm{sm}}(T(F))$ as monadic over $\mathrm{Rep}_{\mathrm{sm}}(T(F)_0)$.

Let $q : * \to \mathbb{B}\mathcal{T}^\circ$. By identifying $\mathrm{QCoh}(\mathbb{B}\mathcal{T}_0)$ with $\mathrm{Rep}_{\mathrm{sm}}(T(F)_0)$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathbb{B}\mathcal{T}^\circ) & \xrightarrow{\quad q^* \quad} & \mathrm{Rep}_{\mathrm{sm}}(T(F), \Lambda) \\ & {\scriptstyle f^*}\searrow \quad \swarrow{\scriptstyle \mathrm{Res}} & \\ & \mathrm{Rep}_{\mathrm{sm}}(T(F)_0, \Lambda) & \end{array}$$

This put us in the situation of the following lemma, which is a weaker version of the dual statement of Corollary 4.7.3.16 in (Lurie, 2017).

**Lemma 2.2.4.** Suppose we have a commutative diagram of $\infty$-categories

$$\begin{array}{ccc} C & \xrightarrow{\quad U \quad} & C' \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle H'} & \\ & \mathcal{D} & \end{array}$$

Assume that:

1. $H$ and $H'$ admit right adjoints $G$ and $G'$.

2. $H$ exhibit $C$ as co-monadic over $\mathcal{D}$.

3. $H'$ exhibit $C'$ as co-monadic over $\mathcal{D}$.

4. For each object $D \in \mathcal{D}$, the unit and co-unit map $UG \to G'H'UG \to G'HG \to G'$ induce equivalence $UG(D) \to G'(D)$.

Then $U$ is an equivalence of $\infty$-categories.

All conditions in the lemma are readily satisfied except 4. We notice that in our case $UG = q^* f_*$ and $G' = \mathrm{Ind}_{T(F)_0}^{T(F)}$. Using base change in the following diagram:

$$\begin{array}{ccc} \prod_{T(F)/T(F)_0} * & \longrightarrow & \mathbb{B}\mathcal{T}_0 \\ \downarrow & & \downarrow f \\ * & \xrightarrow{q} & \mathbb{B}\mathcal{T}^\circ \end{array}$$

we compute $UG(M) = \prod_{T(F)/T(F)_0} M$ for any $M \in \mathrm{Rep}_{\mathrm{sm}}(T(F)_0)$ and this is canonically isomorphic to $\mathrm{Ind}_{T(F)_0}^{T(F)} M$. This canonical isomorphism $UG \cong G'$ must agree with the morphism constructed in condition 4. □

### 2.3 A short exact sequence

The main result of this section is as follows.

**Proposition 2.3.1.** For an arbitrary torus $T$ over a local field $F$, we have a short exact sequence of Picard stacks

$$1 \to \underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m) \to \mathrm{Loc}_{{}^c T, F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m) \to 1. \qquad (2.1)$$

**Remark 2.3.2.** (i) Note the first term has the alternative form $\underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m) = B\hat{T}^\Gamma$, because $X^*(\hat{T}^\Gamma) = \hat{L}_\Gamma$.

(ii) From the proposition, we see the isomorphism classes of $\mathrm{Loc}_{{}^c T, F}$ is dual to the automorphism group of $\mathbf{Tor}_{T,\mathrm{iso}_F}$ and vice versa. This results in the duality that is treated in section 2.4.

(iii) The inner-hom $\underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)$ is defined as the colimit of inner-homs $\underline{\mathrm{Hom}}(T(F)/U, \mathbb{G}_m)$ for all open subgroups $U \subset T(F)$.

Our strategy is to first show the following short exact sequence for any $\mathbb{Z}$-algebra $R$:

$$0 \to \mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to H^1(W_{E/F}, \hat{T}(R)) \to \mathrm{Hom}_{\mathrm{cts}}(T(F), R^\times) \to 0. \tag{2.2}$$

Heuristically, this is our desired exact sequence (2.1) on the level of coarse moduli space. The group cohomology $H^1$ is interpreted as *continuous* cross homomorphisms up to conjugation. After (2.2) is shown, we will show (2.1) directly.

We let $\bar{Z}^1, \bar{H}^1$ be all cross homomorphisms and all cross homomorphisms up to conjugation respectively, removing the continuous conditions. We first show a version of the short exact sequence (2.2) without the topology constraints.

**Lemma 2.3.3.** We have the short exact sequence

$$0 \to \mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to \bar{H}^1(W_{E/F}, \hat{T}(R)) \to \mathrm{Hom}(T(F), R^\times) \to 0. \tag{2.3}$$

*Proof.* Let $C_\bullet \to \mathbb{Z} \to 0$ be the bar resolution of the trivial module $\mathbb{Z}$ by free $\mathbb{Z}[W_{E/F}]$-modules. The group cohomology $\bar{H}^1(W_{E/F}, \hat{T}(R))$ is exactly the first cohomology of the following complex (we abbreviate $W_{E/F}$ by $W$ at times):

$$\mathrm{Hom}_{\mathbb{Z}[W]}(C_\bullet, \hat{T}(R)) = \mathrm{Hom}_{\mathbb{Z}[W]}(C_\bullet, L \otimes R^\times) = \mathrm{Hom}((C_\bullet \otimes \hat{L})_W, R^\times).$$

Because the complex $(C_\bullet \otimes \hat{L})_W$ is a complex of free abelian groups, and its homology caculates the group homology of the $W_{E/F}$-module $\hat{L}$, we apply the universal coefficient theorem and get

$$0 \to \mathrm{Ext}^1(H_0(W_{E/F}, \hat{L}), R^\times) \to \bar{H}^1(W_{E/F}, \hat{T}(R)) \to \mathrm{Hom}(H_1(W_{E/F}, \hat{L}), R^\times) \to 0.$$

In the first term, $H_0(W_{E/F}, \hat{L}) = \hat{L}_W = \hat{L}_\Gamma$. In the third term, Langlands(Langlands, 1997) proved that the corestriction map $\mathrm{cores} : H_1(W_{E/F}, \hat{L}) \to H_1(E^\times, \hat{L})^\Gamma$ is an isomorphism, and therefore

$$H_1(W_{E/F}, \hat{L}) \xrightarrow{\sim} H_1(E^\times, \hat{L})^\Gamma = T(F).$$

This finishes the proof of (2.3). □

We spell out the corestriction map explicitly. We have $(C_1 \otimes \hat{L})_W = \mathbb{Z}[W_{E/F}] \otimes \hat{L}$, $(C_0 \otimes \hat{L})_W = \hat{L}$ and the differential is $\sum w \otimes x = \sum w^{-1}x - x$. The corestriction map $\mathrm{cores} : H_1(W_{E/F}, \hat{L}) \to H_1(E^\times, \hat{L})^\Gamma$ is induced by the chain map

$$\begin{aligned} \mathbb{Z}[W_{E/F}] \otimes \hat{L} &\to \mathbb{Z}[E^\times] \otimes \hat{L} \\ w \otimes x &\mapsto \sum_\tau \delta(w_\tau, w) \otimes w_\tau x. \end{aligned} \tag{2.4}$$

**Lemma 2.3.4.** We have a short exact sequence

$$0 \to \mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to H^1(W_{E/F}, \hat{T}(R)) \to \mathrm{Hom}_{\mathrm{cts}}(T(F), R^\times) \to 0.$$

*Proof.* We first show the image of $\mathrm{Ext}^1(\hat{L}_\Gamma, R^\times)$ is contained in $H^1(W_{E/F}, \hat{T}(R))$.

To show this, we work out an explicit formula for the inclusion $\mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to \bar{H}^1(W_{E/F}, \hat{T}(R))$ in (2.3). Let $\hat{L}_0 = \{gx - x | g \in W_{E/F}, x \in \hat{L}\}$, so we have

$$0 \to \hat{L}_0 \to \hat{L} \to \hat{L}_\Gamma \to 0.$$

This short exact sequence gives rise to

$$\mathrm{Hom}(\hat{L}_0, R^\times) \xrightarrow{\alpha} \mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to 0.$$

On the other hand, for each $\phi \in \mathrm{Hom}(\hat{L}_0, R^\times)$, we associate a map $W_{E/F} \times \hat{L} \to R^\times$ by

$$(g, x) \mapsto \phi(g^{-1}x - x).$$

By adjunction this defines a cross homomorphism $W_{E/F} \to L \otimes R^\times = \hat{T}(R)$. We denote this map by $\beta : \mathrm{Hom}(\hat{L}_0, R^\times) \to \bar{H}^1(W_{E/F}, \hat{T}(R))$. The inclusion map is then $\beta \circ \alpha^{-1} : \mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \to \bar{H}^1(W_{E/F}, \hat{T}(R))$, whose result is independent of the choice of a preimage of $\alpha$.

It remains to show under the surjection in (2.3), an element lies in $H^1(W_{E/F}, \hat{T}(R))$ if and only if its image is a continuous homomorphism.

Recall the bar resolution $B_\bullet \to \mathbb{Z} \to 0$ is a resolution of the trivial module $\mathbb{Z}$ by free $\mathbb{Z}[W_{E/F}]$-modules, and it begins with $\mathbb{Z}[W_{E/F}] \xrightarrow{\mathrm{tr}} \mathbb{Z}$. The first homology $H_1(W_{E/F}, \hat{L})$ is therefore represented by a cycle living in $\mathbb{Z}[W_{E/F}] \otimes \hat{L}$.

The surjection in (2.3) is induced by $\pi$ on the level of cocycle as follows:

$$\bar{Z}^1(W_{E/F}, \hat{T}(R)) \xrightarrow{\quad \pi \quad} \mathrm{Hom}(H_1(W_{E/F}, \hat{L}), R^\times)$$

$$\phi : W_{E/F} \to L \otimes R^\times \longmapsto \left(\sum w \otimes x \mapsto \sum \langle \phi(w), x \rangle\right),$$

where $\phi$ is a cocycle, $\sum w \otimes x \in \mathbb{Z}[W_{E/F}] \otimes \hat{L}$ a chosen cycle reprensenting some element in $H_1(W_{E/F}, \hat{L})$, and the pairing is by evaluating $\hat{L}$ on $L$. It remains to show that $\phi$ is continuous if and only if $\pi(\phi)$ is. This is the content of the next lemma. □

**Lemma 2.3.5.** A cocycle $\phi$ is continuous if and only if $\pi(\phi)$ is.

*Proof.* We basically follow the argument given in (Langlands, 1997). We have the corestriction map $\mathrm{cor} : H_1(E^\times, \hat{L}) \to H_1(W_{E/F}, \hat{L})$. A homomorphism $\psi : H_1(W_{E/F}, \hat{L}) \to R^\times$ is continuous if and only if $\psi \circ \mathrm{cor}$ is. On the other hand, it is clear that a cocycle is continuous if and only if its restriction to $E^\times$ is. We have the following commutative diagram:

$$\begin{array}{ccc} \bar{Z}^1(W_{E/F}, \hat{T}) & \xrightarrow{\pi} & \mathrm{Hom}(H_1(W_{E/F}, \hat{L}), R^\times) \\ \downarrow & & \downarrow \\ \bar{Z}^1(E, \hat{T}) & \xrightarrow{\tau} & \mathrm{Hom}(H_1(E, \hat{L}), R^\times), \end{array}$$

where the map $\tau$ send a cocycle $f$ to an homomorphism $\tau(f) : H_1(E, \hat{L}) \to R^\times$, $a \otimes x \mapsto \langle f(a), x\rangle$. It is then clear that $f$ is continuous if and only if $\tau(f)$ is, and the lemma follows. □

2.3.6. **Proof of proposition 2.3.1.** The functor $\mathrm{ch} : D^{[-1,0]}(S, \mathbb{Z}) \to \mathcal{PS}/S$ is an equivalence of category. Let $(-)^\flat$ be an quasi-inverse. The existence of the short exact sequence (2.1) is by definition to say there exists an exact triangle

$$\underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m)^\flat \to \mathrm{Loc}^\flat_{cT,F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)^\flat \xrightarrow{+1} .$$

We first define the homomorphisms

$$\underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m)^\flat \xrightarrow{\bar{\beta}} \mathrm{Loc}^\flat_{cT,F} \xrightarrow{\bar{\pi}} \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)^\flat$$

as follows, where each column is a 2-term complex of abelian sheaves corresponding to the Picard stack above:

$$\begin{array}{lccccc} \deg = 0 & \underline{\mathrm{Hom}}(\hat{L}_0, \mathbb{G}_m) & \xrightarrow{\beta'} & \mathrm{Loc}^\square_{cT,F} & \xrightarrow{\pi'} & \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m) \\ & \uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle d} & & \uparrow \\ \deg = -1 & \underline{\mathrm{Hom}}(\hat{L}, \mathbb{G}_m) & \xrightarrow{\sim} & \hat{T} & \longrightarrow & 0. \end{array}$$

The map $\beta'(R)$ is exactly $\beta : \mathrm{Hom}(\hat{L}_0, R^\times) \to H^1(W_{E/F}, \hat{T}(R))$ as defined in Lemma 2.3.4. The map $\pi'(R)$ is the restriction of $\pi$ to the continuous cocycles

$$Z^1(W_{E/F}, \hat{T}(R)) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(H_1(W_{E/F}, \hat{L}), R^\times)$$

$$\phi : W_{E/F} \to L \otimes R^\times \longmapsto \left(\textstyle\sum w \otimes x \mapsto \langle \phi(w), x\rangle\right).$$

It is immediate to check that the squares commute, so they define homomorphisms of complexes.

To show the above three terms forms an exact sequence of Picard stacks amounts to checking they form an exact triangle. We do so by showing a quasi-isomorphism $\mathrm{cofib}(\bar{\beta}) \cong \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)$.

Note $\mathrm{cofib}(\bar{\beta})$ is nothing other than the following complex:

$$\underline{\mathrm{Hom}}(\hat{L}, \mathbb{G}_m) \xrightarrow{(\mathrm{id}, -\mathrm{res})} \hat{T} \oplus \underline{\mathrm{Hom}}(\hat{L}_0, \mathbb{G}_m) \xrightarrow{\binom{d}{\beta'}} \mathrm{Loc}^{\square}_{^cT,F}.$$

Because the first map is injective, $H^{-2}(\mathrm{cofib}(\bar{\beta})) = 0$. Next, because $d \circ \mathrm{id} = \beta' \circ \mathrm{res}$ and $\beta'$ is injective, $H^{-1}(\mathrm{cofib}(\bar{\beta})) = 0$.

We know that $H^0(\mathrm{cofib}(\bar{\beta})) = \mathrm{Loc}^{\square}_{^cT,F}/\mathrm{Im}\binom{d}{\beta'}$. To evaluate this, we first take the quotient $\mathrm{Loc}^{\square}_{^cT,F}/\mathrm{Im}'\binom{d}{\beta'}$ as presheaf, where $\mathrm{Im}'$ is the image of maps of presheaves. We have

$$\begin{aligned} &\mathrm{Loc}^{\square}_{^cT,F}(R)/(\mathrm{Im}'(d) + \mathrm{Im}'(\beta')) \\ =\ &H^1(W_{E/F}, \hat{T}(R))/(\mathrm{Im}'(\beta')/(\mathrm{Im}'(\beta') \cap \mathrm{Im}'(d))) \\ =\ &H^1(W_{E/F}, \hat{T}(R))/\mathrm{Ext}^1(\hat{L}_\Gamma, R^\times) \\ =\ &\mathrm{Hom}_{\mathrm{cts}}(T(F), R^\times). \end{aligned}$$

The last line of the equation is just the $R$-points of $\underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)$. □

The next proposition is crutial for the duality $\mathbf{Tor}_{T,\mathrm{iso}_F} \cong \mathrm{Loc}^{\vee}_{^cT,F}$.

**Proposition 2.3.7.** The +1 map is zero for the following exact triangle we just show

$$\underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m)^\flat \to \mathrm{Loc}^\flat_{^cT,F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)^\flat \xrightarrow{+1}.$$

*Proof.* Recall that the +1 map is the inverse of the quasi-isomorphism showed above composed with a canonical projection

$$\underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m) \xleftarrow{\sim} \mathrm{cofib}(\bar{\beta}) \to B\hat{T}^\Gamma[1].$$

Due to the universal coefficient theorem, there exists a collection of maps

$$\rho : \mathrm{Hom}(H^1(W_{E/F}, \hat{L}), R^\times) \to \bar{Z}^1(W_{E/F}, \hat{T}(R))$$

that is functorial in $R$, such that after composing with the projection maps $\bar{Z}^1(W_{E/F}, \hat{T}(R)) \to \bar{H}^1(W_{E/F}, \hat{T}(R))$, give splittings of the following surjections:

$$\pi : \bar{H}^1(W_{E/F}, \hat{T}(R)) \to \mathrm{Hom}(H^1(W_{E/F}, \hat{L}), R^\times).$$

For a continuous homomorphism $f : H^1(W_{E/F}, \hat{L}) \to R^\times$, $\rho(f)$ is automatically a continuous cocycle because its orbit under action of $\hat{T}(R)$ must contain one continuous cocycle, but since all coboundries are continuous, $\rho(f)$ is continuous. Therefore we have the following maps:

$$\mathrm{Hom}_{\mathrm{cts}}(H^1(W_{E/F}, \hat{L}), R^\times) \to Z^1(W_{E/F}, \hat{T}(R)). \tag{2.5}$$

The maps (2.5) give rise to an inverse of the quasi-isomorphism $\mathrm{cofib}(\bar{\beta}) \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)$ by the composition

$$\underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m) \to \mathrm{Loc}^{\square}_{{}^cT,F} \to \mathrm{cofib}(\bar{\beta}).$$

Composing this inverse with $\mathrm{cofib}(\bar{\beta}) \to B\hat{T}^\Gamma[1]$ is nothing but the +1 map, but it is also zero by construction. □

## 2.4 Duality

The main result of this paper is the following.

**Theorem 2.4.1.** There is a unique family of Poincaré line bundles (or a family of pairings)

$$\mathcal{L}_T : \mathbf{Tor}_{T,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cT,F} \to \mathbb{B}\mathbb{G}_m$$

for every torus $T$ over $F$ that satisfies the three conditions below. Furthermore, these Poincaré line bundles induce isomorphism $\mathbf{Tor}_{T,\mathrm{iso}_F} \cong \mathrm{Loc}^{\vee}_{{}^cT,F}$ for every torus $T$.

a) **Functorality.** Let $f : S \to T$ be a map between torus. It induces $\alpha : \mathrm{Loc}_{{}^cT,F} \to \mathrm{Loc}_{{}^cS,F}$ and $\beta : \mathbf{Tor}_{S,\mathrm{iso}_F} \to \mathbf{Tor}_{T,\mathrm{iso}_F}$. The following two line bundles on $\mathbf{Tor}_{S,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cT,F}$ are canonically isomorphic

$$(\beta \times \mathrm{id})^* \mathcal{L}_T = (\mathrm{id} \times \alpha)^* \mathcal{L}_S.$$

b) **Split case.** For the split torus $T = \mathbb{G}_m$, both $\mathbf{Tor}_{T,\mathrm{iso}_F}$ and $\mathrm{Loc}_{{}^cT,F}$ canonically split as

$$\mathbf{Tor}_{T,\mathrm{iso}_F} = \mathbb{Z} \times \mathbb{B}F^\times,$$
$$\mathrm{Loc}_{{}^cT,F} = \underline{\mathrm{Hom}}_{\mathrm{cts}}(F^\times, \mathbb{G}_m) \times \mathbb{B}\mathbb{G}_m.$$

Then $\mathcal{L}_T$ is the canonical line bundle $\mathbf{Tor}_{T,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cT,F} \to \mathbb{BG}_m$ via the tautological pairings

$$\underline{\mathrm{Hom}}_{\mathrm{cts}}(F^\times, \mathbb{G}_m) \times BF^\times \to \mathbb{BG}_m$$
$$\mathbb{Z} \times \mathbb{BG}_m \to \mathbb{BG}_m.$$

c) **Induced case.** For an induced torus $S = \mathrm{Res}_{E/F}\,\mathbb{G}_m$, let $T$ be the split torus over $E$, the Shapiro isomorphism $\mathbf{Tor}_{S,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cS,F} \cong \mathrm{Tor}_{T,\mathrm{iso}_E} \times \mathrm{Loc}_{{}^cT,E}$ identifies the line bundles for $S$ and $T$.

Our strategy is to introduce an auxilary stack $\mathcal{T}_T$ for every torus $T$ in place of $\mathbf{Tor}_{T,\mathrm{iso}_F}$. When it is clear from the context, we will suppress the torus $T$ from the notation $\mathcal{T}_T$.

We will first define a family of line bundles on $\mathcal{T} \times \mathrm{Loc}_{{}^cT,F}$. Then we verify the main theorem for $\mathcal{T}$ instead of $\mathbf{Tor}_{T,\mathrm{iso}_F}$. Finally, we will show a cononical isomorphism $\mathcal{T} \cong \mathbf{Tor}_{T,\mathrm{iso}_F}$.

2.4.2. **Definition of $\mathcal{T}$.** Recall that $U^{(n)} \subset E^\times \subset W_{E/F}$ is a basis of open neighbourhoods . We define $W^{(n)} = W_{E/F}/U^{(n)}$.

Notice that $\hat{L}$ is a $W^{(n)}$-module for all $n$. Let $C_\bullet(W^{(n)}, \hat{L})$ be the chain complex that calculates the group homology of the $W^{(n)}$-module $\hat{L}$. The boundries of this chain complex are defined by

$$B_i(W^{(n)}, \hat{L}) = \mathrm{Im}\left(C_{i+1}(W^{(n)}, \hat{L}) \xrightarrow{d} C_i(W^{(n)}, \hat{L})\right).$$

We define

$$\mathcal{T}_n := \mathrm{ch}\left(\cdots \to 0 \to \frac{C_1(W^{(n)}, \hat{L})}{B_1(W^{(n)}, \hat{L})} \xrightarrow{d} C_0(W^{(n)}, \hat{L}) \to 0 \to \cdots\right),$$

and $\mathcal{T} := \varprojlim \mathcal{T}_n$. It is convenient to think of $\mathcal{T}$ as $\mathrm{ch}\left(\tau_{\geq -1}(C_\bullet(W_{E/F}, \hat{L}))\right)$ with a pro-stack structure.

The groupoid $\mathcal{T}_n(\mathbb{Z})$ calculates group homology. Hence, it has isomorphism classes $\hat{L}_\Gamma$. The following two lemma calculates its automorphism group to be $T(F)/V_n$. ($V_n$ is defined in section 2.1.5).

**Lemma 2.4.3.** The subgroups $U^{(n)}$ and $U_n \subset T(E)$ has no higer Galois cohomologies for sufficiently large $n$.

*Proof.* For sufficiently large $n$, the group $U^{(n)}$ is isomorphic to the additive group $O_E$ via the logarithm map. Since this is an induced module, its higher cohomology groups vanish. Moreover, $U_n = \hat{L} \otimes U^{(n)}$ is a direct sum of copies of $U^{(n)}$, so the same conclusion holds. □

Next we introduce a lemma in homological algebra. It generalizes a result of Langlands (Langlands, 1997). It also plays an important role in the second categorification which is the theme of the next chapter.

**Lemma 2.4.4.** Let $\Gamma$ be a finite group, $C$ be an abelian group equipped with a $\Gamma$-action. Let $\alpha \in H^2(\Gamma, C)$ represent the group extension

$$1 \to C \to G \to \Gamma \to 1.$$

Let $M$ be any $\Gamma$-module, in other words, it is a $G$-module on which $C$ acts trivially. Then we have a commutative diagram

$$\begin{array}{ccccccccccc} & & H_2(\Gamma, M) & \longrightarrow & H_1(C, M)_\Gamma & \longrightarrow & H_1(G, M) & \longrightarrow & H_1(\Gamma, M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cup\alpha} & & \| & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \cup\alpha} & & \\ 0 & \longrightarrow & \hat{H}^{-1}(\Gamma, M \otimes C) & \longrightarrow & (M \otimes C)_\Gamma & \longrightarrow & (M \otimes C)^\Gamma & \longrightarrow & \hat{H}^0(\Gamma, M \otimes C) & \longrightarrow & 0. \end{array}$$

The first row of the diagram is the long exact sequence associated to the Lyndon-Hochschild-Serre spectral sequence. The second row is the definition of the Tate cohomology. The first and last vertical map is the cup product with $\alpha$. The map res is the restriction map $H_1(G, M) \to H_1(C, M)^\Gamma$.

In particular, if $\Gamma$ and $C$ satisfies the condition of the Tate-Nakayama lemma, we would have

$$H_1(G, M) \cong (M \otimes C)^\Gamma.$$

*Proof.* See appendix A. □

**Corollary 2.4.5.** (i) Applying the lemma to the group extension

$$1 \to E^\times \to W_{E/F} \to \Gamma \to 1,$$

we get the result of Langlands which claims

$$H_1(W_{E/F}, \hat{L}) \cong (E^\times \otimes \hat{L})^\Gamma = T(F).$$

(ii) Similarly, with the group extension

$$1 \to E^\times/U^{(n)} \to W^{(n)} \to \Gamma,$$

we get

$$H_1(W^{(n)}, \hat{L}) \cong \left((E^\times/U^{(n)}) \otimes \hat{L}\right)^\Gamma = T(F)/V_n.$$

(iii) Recall the definition of the relative inertia group

$$1 \to I_E^{\mathrm{ab}} \to I_{E/F}^{\mathrm{rel}} \to I \to 1.$$

Applying the lemma to this group extension gives us

$$H_1(I_{E/F}^{\mathrm{rel}}, \hat{L}) = (I_E^{\mathrm{ab}} \otimes L)^I.$$

We now move towards proof of the main theorem.

**Proposition 2.4.6.** There is a canonical pairing $(\,\cdot\,,\,\cdot\,) : \mathcal{T} \times \mathrm{Loc}_{{}^cT,F} \to \mathbb{BG}_m$.

*Proof.* Recall that

$$\begin{aligned} \mathrm{Loc}_{{}^cT,F} &= \operatorname{colim} \mathrm{ch}\left(\cdots \to \hat{T} \xrightarrow{d} Z^1(W^{(n)}, \hat{T}) \to \cdots\right), \\ \mathcal{T} &= \ \lim\ \mathrm{ch}\left(\cdots \to \frac{C_1(W^{(n)},\hat{L})}{B_1(W^{(n)},\hat{L})} \xrightarrow{d} C_0(W_{E/F}, \hat{L}) \to \cdots\right). \end{aligned}$$

$\mathrm{Loc}_{{}^cT,F}$ has an ind-stack structure, and $\mathcal{T}$ has a corresponding pro-stack structure, so to define a pairing between them, it suffices to define it for each $\mathrm{Loc}^{(n)}_{{}^cT,F}$ and $\mathcal{T}$.

The pairing in the proposition is induced by two pairings:

$$\begin{aligned} \langle\,,\,\rangle : &\quad \hat{T} \times \hat{L} \quad \to \mathbb{G}_m, \\ [\,,\,] : &\ Z^1(W^{(n)}, \hat{T}) \times \tfrac{C_1(W^{(n)},\hat{L})}{B_1(W^{(n)},\hat{L})} \to \mathbb{G}_m. \end{aligned}$$

The first pairing is the natural pairing, since $\hat{L}$ is the character lattice of $\hat{T}$. The second pairing is given by

$$\begin{aligned} Z^1(W^{(n)}, \hat{T}) \times C_1(W^{(n)}, \hat{L}) &\to \mathbb{G}_m \\ (\phi, \psi) &\mapsto \sum_{w \in W} \langle \phi(w), \psi(w) \rangle. \end{aligned}$$

This pairing factors through $C_1(W^{(n)}, \hat{L})/B_1(W^{(n)}, \hat{L})$ because in Section 2.3, we show that the same formula defines a pairing between $Z^1(W^{(n)}, \hat{T})$ and $H_1(W^{(n)}, \hat{L}) = Z_1(W^{(n)}, \hat{L})/B_1(W^{(n)}, \hat{L})$.

Our desired pairing $\mathcal{T} \times \mathrm{Loc}_{{}^cT,F} \to \mathbb{B}\mathbb{G}_m$ is given by the following assignment. For $\psi : x \to x + d\psi$ an isomorphism in $\mathcal{T}$ and $t : \phi \to \phi + dt$ an isomorphism in $\mathrm{Loc}_{{}^cT,F}$, there is an isomorphism in the fibred groupoid $\mathrm{Loc}_{{}^cT,F} \times \mathcal{T}$

$$(t, \psi) : (\phi, x) \to (\phi + dt, x + d\psi).$$

We assign to this morphism a morphism in $\mathbb{B}\mathbb{G}_m$ which is given by

$$[\phi, \psi] \cdot \langle t, x + d\psi \rangle.$$

To show this assignment defines a homomorphism of stacks, one must check this assignment respects composition of morphisms, and a simple calculation shows it boils down to check

$$[dt, \psi] = \langle t, d\psi \rangle, \quad \forall t, \psi.$$

The verification is straightforward:

$$\begin{aligned}
[dt, \psi] &= \sum_{w \in W} \langle dt(w), \psi(w) \rangle \\
&= \sum_{w \in W} \langle wt - t, \psi(w) \rangle \\
&= \langle t, \sum_{w \in W} w^{-1}\psi(w) - \psi(w) \rangle \\
&= \langle t, d\psi \rangle.
\end{aligned}$$

□

2.4.7. **Proof of $\mathcal{T} \cong \mathrm{Loc}^\vee_{{}^cT,F}$.** We have the following short exact sequences by truncation of the $t$-structure:

$$1 \to BH_1(W^{(n)}, \hat{L}) \to \mathcal{T}_n \to \hat{L}_\Gamma \to 1.$$

The +1 map of these exact sequences are zero. This is because, in the derived category of abelian groups, every complex is quasi-isomorphic to the direct sum of its cohomologies (Keller, 1996). Therefore, the +1 map in any truncation is zero. The above exact sequences are in the image of the exact functor embedding the drived category of abelian groups to the drived category of abelian sheaves on $\mathrm{Spec}(\mathbb{Z})$, and hence its +1 map is zero.

Since in the fpqc-topology, there is no higher limit, taking derived limit on the short exact sequence gives us

$$1 \to \varprojlim \mathbb{B}H_1(W^{(n)}, \hat{L}) \to \mathcal{T} \to \hat{L}_\Gamma \to 1.$$

Note the +1 map is zero. By Lemma 2.4.4, we can rewrite the short exact sequence as

$$1 \to \varprojlim \mathbb{B}(T(F)/V_n) \to \mathcal{T} \to \hat{L}_\Gamma \to 1.$$

Since $\mathrm{Ext}^1(\underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m), \mathbb{B}\mathbb{G}_m) = 0$ (for generality on the dualality for Picard stacks, see (Brochard, 2014)), by dualizing the short exact sequence (2.1)

$$1 \to \mathbb{B}\hat{T}^\Gamma \to \mathrm{Loc}_{^cT,F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m) \to 1$$

we get

$$1 \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)^\vee \to \mathrm{Loc}^\vee_{^cT,F} \to (\mathbb{B}\hat{T}^\Gamma)^\vee \to 1.$$

The +1 map of the above exact triangle is zero because it is induced by the +1 map of (2.1), which is shown to be zero in section 2.3.7.

We form the following diagram where the first and the third vertical maps are canonical isomorphisms and the second is the map induced by our pairing:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varprojlim B(T(F)/V_n) & \longrightarrow & \mathcal{T} & \longrightarrow & \hat{L}_\Gamma & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\ 1 & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)^\vee & \longrightarrow & \mathrm{Loc}^\vee_{^cT,F} & \longrightarrow & (B\hat{T}^\Gamma)^\vee & \longrightarrow & 1. \end{array}$$

The second vertical map is an isomorphism because, first, the two visible squares commute by the construction of the pairing, and second, the square of the +1 maps commutes because the two +1 maps are shown to be zero. This concludes the proof that $\mathcal{T} \cong \mathrm{Loc}^\vee_{^cT,F}$.

2.4.8. **Proof of theorem 2.4.1.** Now we prove the main theorem with $\mathcal{T}$ in stead of $\mathbf{Tor}_{T,\mathrm{iso}_F}$. We established above that for each torus $T$ the pairing defined in proposition 2.4.6 induces an ismorphism $\mathcal{T} \cong \mathrm{Loc}^\vee_{^cT,F}$. It remains to check that they are a unique family of line bundles on $\mathcal{T} \times \mathrm{Loc}_{^cT,F}$ satisfying the three conditions in the theorem.

Condition (a) is equivalent to the following proposition.

**Proposition 2.4.9.** Let $f : S \to T$ be a map between torus. It induces $\alpha : \mathrm{Loc}_{^cT,F} \to \mathrm{Loc}_{^cS,F}$ and $\beta : \mathcal{T}_S \to \mathcal{T}_T$. Let $t : \phi \to \phi$ be an isomorphism in $\mathrm{Loc}_{^cT,F}$, and let $\psi : x \to x$ be an isomorphism in $\mathcal{T}_S$. Under the pairing, $(\alpha(\phi), \psi)$ and $(\phi, \beta(\psi))$ are isomorphisms in $\mathbb{B}\mathbb{G}_m$. We then have

$$(\alpha(\phi), \psi) = (\phi, \beta(\psi)).$$

*Proof.* Note that $\mathrm{Loc}_{{}^cT,F}$ is the truncation of the complex $H^\bullet(W_{E/F}, \hat{T})$, and $\mathcal{T}_T$ is the truncation of the complex $H_\bullet(W_{E/F}, \hat{L})$. If we let $\hat{f} : \hat{T} \to \hat{S}$, $g : \hat{L}_S \to \hat{L}_T$ be the homomorphisms induced by $f$, let $t \in \hat{T}$, and $x \in \hat{S}$, the proposition follows from the trivial fact that

$$\langle \hat{f}(t), x \rangle = \langle t, g(x) \rangle. \qquad \square$$

Condition (b) follows from the definition of the pairing.

We verify condition (c). Let $S = \mathrm{Res}_{E/F}\, \mathbb{G}_m$ an induced torus and $T = \mathbb{G}_m$ a split torus over $E$. The cocharacters of the two torus satisfy

$$\hat{L}_S = \mathrm{Ind}_*^{\mathrm{Gal}(E/F)}\, \hat{L}_T.$$

We apply Shapiro's lemma to get isomorphisms

$$\begin{aligned}\mathrm{Loc}_{{}^cS,F} &= \mathrm{ch}\left(\tau_{\leq 0}(H^\bullet(W_{E/F}, \hat{S})[1])\right) \\ &\cong \mathrm{ch}\left(\tau_{\leq 0}(H^\bullet(E^\times, \hat{T})[1])\right) = \mathrm{Loc}_{{}^cT,E},\end{aligned}$$

$$\begin{aligned}\mathbf{Tor}_{S,\mathrm{iso}_F} &= \mathrm{ch}\left(\tau_{\geq -1} H_\bullet(W_{E/F}, \hat{L}_S)\right) \\ &\cong \mathrm{ch}\left(\tau_{\geq -1} H_\bullet(E^\times, \hat{L}_T)\right) = \mathrm{Tor}_{T,\mathrm{iso}_E}.\end{aligned}$$

The cap product that is used to define the pairings clearly interwines with the Shapiro isomorphisms, and hence condition (c) is satisfied.

By condition (b) and (c), any such family of pairings is uniquely determined on all induced torus. To show such a family of pairings is unique, it remains to show the pairing for every torus $T$ is induced by the pairing for an induced torus $S$ that covers $T$.

For every torus $T$, there is always an induced torus $S$ and a cover $S \to T$ that is surjective on cocharacters $\hat{L}_S \to \hat{L}_T$. Let $t : \phi \to \phi + dt$ be an isomorphism in $\mathrm{Loc}_{{}^cT,F}$ and $\psi : x \to x + d\psi$ an isomorphism in $\mathcal{T}_T$. By surjectivity of $\hat{L}_S \to \hat{L}_T$, we can lift $\psi : x \to x + d\psi$ to $\tilde{\psi} : \tilde{x} \to \tilde{x} + d\tilde{\psi}$ in $\mathcal{T}_S$. By functorality of the pairings, we have

$$(t, \psi) = (\alpha(t), \tilde{\psi}).$$

Therefore the pairing on $T$ is predetermined by that of $S$. This concludes the proof of the main theorem.

We finish the section with

2.4.10. **Proof of $\mathcal{T} \cong \mathbf{Tor}_{T,\mathrm{iso}_F}$.** Recall that

$$\mathbf{Tor}_{T,\mathrm{iso}_F} = \lim \mathrm{ch}\left(\cdots \to 0 \to T(E)/U_n \to Z^1_{\mathrm{alg}}(W^{(n)}, T(E)) \to 0 \to \cdots\right).$$

We first define a map between groupoids $\mathcal{T}(\mathbb{Z}) \to \mathbf{Tor}_{T,\mathrm{iso}_F}(\mathbb{Z})$ by defining the two vertical maps in the following diagram such that the square commutes

$$\begin{array}{ccc} C_1(W_{E/F}, \hat{L})/B_1(W_{E/F}, \hat{L}) & \xrightarrow{\quad d \quad} & \hat{L} \\ \downarrow{\scriptstyle \mathrm{cores}} & & \downarrow{\scriptstyle c_0} \\ T(E) & \xrightarrow{\quad d \quad} & Z^1_{\mathrm{alg}}(W_{E/F}, T(E)). \end{array} \tag{2.6}$$

In the following, we make the identification $T(E) = \hat{L} \otimes E^{\times}$, with $W_{E/F}$ acting diagonally.

We define the first verticle map to be the corestriction map (2.4)

$$C_1(W_{E/F}, \hat{L})/B_1(W_{E/F}, \hat{L}) \xrightarrow{\mathrm{cores}} C_1(E^{\times}, \hat{L})/B_1(E^{\times}, \hat{L}) \cong T(E).$$

Following (2.4), it sends $w \otimes x$ to $\sum_\tau w_\tau x \otimes \delta(w_\tau, w)$. The fact that this map induces isomorphism on the cohomology of degree $-1$ is provided by Corollary 2.4.5.

The second verticle map $c_0 : \hat{L} \to Z^1_{\mathrm{alg}}(W_{E/F}, T(E))$ is defined via two maps, $\hat{L} \to \hat{L}^{\Gamma}$ and $\hat{L} \to Z^1(W_{E/F}, T(E))$, since $Z^1_{\mathrm{alg}}(W_{E/F}, T(E))$ is defined as a fibre product $\hat{L} \times_{Z^1(E^\times, T(E))} Z^1(W_{E/F}, T(E))$. We order the first map to be the norm map, and the second to be the composition

$$\hat{L} \to Z^1(E^{\times}, T(E)) \xrightarrow{\mathrm{inf}} Z^1(W_{E/F}, T(E)).$$

The inflation map on cocyles is defined as follows. Let $\phi : E^{\times} \to T(E)$ be a cocycle. The inflation of this cocycle is usually given by

$$g \mapsto \sum_\tau w_\tau^{-1} \phi(\delta(w_\tau, g)).$$

We define the inflation map by the same formula, however with a different system of representatives given by $\{\tilde{w}_\tau = w_{\tau^{-1}}^{-1}, \tau \in \Gamma\}$.

Explicitly, $c_0(x)$ is the cocycle

$$\begin{aligned} g \mapsto & \sum_\tau \tilde{w}_\tau^{-1}(x \otimes \tilde{\delta}(\tilde{w}_\tau, g)) \\ &= \sum_\tau \tilde{w}_\tau^{-1} x \otimes \tilde{w}_\tau^{-1} \tilde{\delta}(\tilde{w}_\tau, g) \\ &= \sum_\tau w_{\tau^{-1}} x \otimes \delta(g, w_{g^{-1}\tau^{-1}}). \end{aligned}$$

(For notation, see section 2.1.1.) The map $c_0$ agrees with the map $c_0$ Kottwitz assign to the Tate-Nakayama triple $(\mathbb{Z}, E^\times, \alpha)$ in (Kottwitz, 2014), and that it induce isomorphism on the cohomology of degree 0 is exactly Lemma 5.1 in *loc. cit.*

The following calculation verifies the diagram (2.6) commutes.

**Proposition 2.4.11.** We have $c_0 \circ d = d \circ \mathrm{cores}$.

*Proof.* From above, $d \circ \mathrm{cores}(w \otimes x)$ is the cocycle

$$g \mapsto \sum_{w_\tau} g w_\tau x \otimes g\delta(w_\tau, w) - \sum_{w_\tau} w_\tau x \otimes \delta(w_\tau, w).$$

We change the variable $\tau \to g^{-1}\tau$ in the first summation. The right hand side becomes

$$\begin{aligned} &\sum_\tau w_\tau x \otimes (g\delta(w_{g^{-1}\tau}, w) - \delta(w_\tau, w)) \\ =&\sum_\tau w_\tau x \otimes (\delta(g, w_{g^{-1}\tau w}) - \delta(g, w_{g^{-1}\tau})). \end{aligned} \tag{2.7}$$

On the other hand, $c_0 \circ d(w \otimes x)$ is the cocycle

$$g \mapsto \sum_\tau w_{\tau^{-1}} w^{-1} x \otimes \delta(g, w_{g^{-1}\tau^{-1}}) - \sum_\tau w_{\tau^{-1}} x \otimes \delta(g, w_{g^{-1}\tau^{-1}}).$$

We change the variable $\tau \to w^{-1}\tau$ in the first summation. The right hand side becomes

$$\sum_\tau w_{\tau^{-1}} x \otimes (\delta(g, w_{g^{-1}\tau^{-1}w}) - \delta(g, w_{g^{-1}\tau^{-1}})).$$

This equals (2.7). □

We continue with the proof that $\mathcal{T} \cong \mathbf{Tor}_{T,\mathrm{iso}_F}$. Both $\mathcal{T}$ and $\mathbf{Tor}_{T,\mathrm{iso}_F}$ has a pro-stack structure. We can use the same construction for the commuting diagram (2.6) when we replace $W_{E/F}$ by $W^{(n)}$ and $T(E)$ by $T(E)/U_n$. The definition of the algebraic cycles $Z^1_{\mathrm{alg}}(W_{E/F}, T(E))$ can be easily adapted to define $Z^1_{\mathrm{alg}}(W^{(n)}, T(E)/U_n)$. We claim that the diagram is a quasi-isomorphism for sufficiently large $n$, and therefore together they define an isomorphism $\mathcal{T} \cong \mathbf{Tor}_{T,\mathrm{iso}_F}$.

By Lemma 2.4.3, there is an isomorphism for large $n$

$$H^1_{\mathrm{alg}}(W_{E/F}, T(E)) \to H^1_{\mathrm{alg}}(W_{E/F}, T(E)/U_n) = H^1_{\mathrm{alg}}(W^{(n)}, T(E)/U_n).$$

This shows the induced map on cohomology at degree 0 is an isomorphism for large $n$. Lemma 2.4.4 says the induced map on cohomology at degree -1 is an isomorphism for large $n$. This finishes the proof.

## 2.5 Categorical LLC for the torus

In the previous section, we constructed a canonical family of Poincaré line bundles $\mathcal{L} : \mathbf{Tor}_{T,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cT,F} \to \mathbb{B}\mathbb{G}_m$. They induce isomorphisms $\mathbf{Tor}_{T,\mathrm{iso}_F} \cong \mathrm{Loc}^{\vee}_{{}^cT,F}$.

**Theorem 2.5.1.** Let $\pi_1, \pi_2$ be the projects of $\mathrm{Loc}_{{}^cT,F} \times \mathbf{Tor}_{T,\mathrm{iso}_F}$ to its factors. A Fourier-Mukai transform via the line bundle $\mathcal{L}$ establish the equivalence of $\infty$-categories that preseres t-structures:

$$\pi_{1!}(\pi_2^*(-) \otimes \mathcal{L}^{-1}) : \mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F}) \leftrightharpoons \mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F}) : \pi_{2*}(\pi_1^*(-) \otimes \mathcal{L}),$$

where the functor $\pi_{1!}$ is the right adjoint of $\pi_1^*$ and $\pi_{2*}$ is the left adjoint of $\pi_2^*$.

Before we can prove the theorem, we gather some useful facts as follows.

Recall that by the system of congrunce subgroups $U^{(n)} \subset O_E^\times$ give rise to a basis of open subgroups $V_n$ of $T(F)$. Using Corollary 2.4.5, we have the following short exact sequence:

$$0 \to \underline{\mathrm{Hom}}(\hat{L}_\Gamma, \mathbb{B}\mathbb{G}_m) \to \mathrm{Loc}^{(n)}_{{}^cT,F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F)/V_n, \mathbb{G}_m) \to 0.$$

We let $\underline{\mathrm{Loc}} = \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F), \mathbb{G}_m)$ and let $\underline{\mathrm{Loc}}^{(n)} = \underline{\mathrm{Hom}}_{\mathrm{cts}}(T(F)/V_n, \mathbb{G}_m)$. By proposition 2.3.7, $\mathrm{Loc}_{{}^cT,F}$ splits as $\mathrm{Loc}_{{}^cT,F} = \underline{\mathrm{Loc}} \times \mathbb{B}\hat{T}^\Gamma$ and similarly $\mathrm{Loc}^{(n)}_{{}^cT,F} = \underline{\mathrm{Loc}}^{(n)} \times \mathbb{B}\hat{T}^\Gamma$.

We define the regular function ring $O_{\underline{\mathrm{Loc}}}$ to be the union of all $O_{\underline{\mathrm{Loc}}^{(n)}}$, i.e., $O_{\underline{\mathrm{Loc}}}$ consists of functions supported on some $\underline{\mathrm{Loc}}^{(n)}$ for some $n$.

Let $T(F)_0$ be the maximal compact subgroup of $T(F)$ and let $\mu$ be the (left) Haar measure on $T(F)$ such that $\mu(T(F)_0) = 1$. We define the Hecke algebra of $T(F)$ to be $\mathbb{Q}$-valued compactly supported smooth function on $T(F)$ equipped with the convolution product with respect to $\mu$ and denote it by $\mathcal{H}(T(F), \mathbb{Q})$.

**Lemma 2.5.2.** There is an isomorphism $O_{\underline{\mathrm{Loc}}} \otimes \mathbb{Q} \cong \mathcal{H}(T(F), \mathbb{Q})$.

*Proof.* We choose a splitting $T(F) = T(F)_0 \oplus R$, where $R$ is a free abelian group of finite rank. It suffice to check the claim on each subgroup.

The Hecke algebra of $R$ is isomorphic to the group ring $\mathbb{Z}[R]$, and this is isomorphic to the function ring of $\underline{\mathrm{Hom}}(R, \mathbb{G}_m)$.

Let $G_n = T(F)_0/V_n$ be a tower of finite abelian groups with projections $p_n : G_{n+1} \to G_n$.

Consider the map $\mathbb{Q}[G_n] \to \mathcal{H}(G_n, \mathbb{Q})$ that sends an element $g \in G_n$ to $|G_n| \cdot \delta_g$. By direct computation, this map is an algebra isomorphism (with $\mathbb{C}$-coefficient, this is the familiar discrete Fourier transform).

The Hecke algebra $\mathcal{H}(T(F)_0, \mathbb{Q})$ is the union of all $\mathcal{H}(G_n, \mathbb{Q})$, where the inclusion map $\mathcal{H}(G_n, \mathbb{Q}) \to \mathcal{H}(G_{n+1}, \mathbb{Q})$ is the pullback of functions along projections $p_n$. Let $i_n : \mathbb{Q}[G_n] \to \mathbb{Q}[G_{n+1}]$ be the unique map such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Q}[G_n] & \xrightarrow{\sim} & \mathcal{H}(G_n, \mathbb{Q}) \\ \downarrow{\scriptstyle i_n} & & \downarrow \\ \mathbb{Q}[G_{n+1}] & \xrightarrow{\sim} & \mathcal{H}(G_{n+1}, \mathbb{Q}). \end{array}$$

It remains to identify the map $i_n$ with the inclusion $\mathcal{O}_{\underline{\mathrm{Loc}}^{(n)}} \subset \mathcal{O}_{\underline{\mathrm{Loc}}^{(n+1)}}$. This is readily manifested given that the image of 1 under $i_n$ is an idempotent

$$i_n(1) = \frac{1}{|G_{n+1}|} \sum_{g \in \ker(p_n)} g,$$

and that $i_n(1) \cdot \mathbb{Q}[G_{n+1}]$ is precisely the image of $\mathbb{Q}[G_n]$. □

**Remark 2.5.3.** This proposition amounts to saying that under our Langlands correspondence, $\mathcal{O}_{\mathrm{Loc}^c{}_{T,F}}$ is send to the Whittaker sheaf, i.e., the Hecke algebra with the usual $T(F)$-action.

2.5.4. **Decomposition of categories.** Under the splitting $\mathrm{Loc}^c{}_{T,F} = \underline{\mathrm{Loc}} \times \mathbb{B}\hat{T}^\Gamma$, the Poincaré line bundle $\mathcal{L}$ on $\mathrm{Loc}^c{}_{T,F} \times \mathbf{Tor}_{T,\mathrm{iso}_F}$ can be regarded as the structural sheaf over $\underline{\mathrm{Loc}} \times \hat{L}_\Gamma$ with both a $\hat{T}^\Gamma$ and a $T(F)$-action, such that $\hat{T}^\Gamma$ acts by the character in $x \in \hat{L}_\Gamma$ on the $x$-component, and $t \in T(F)$ acts on $\underline{\mathrm{Loc}}_n \times \hat{L}_\Gamma$ by $tV_n \in \mathbb{Z}[T(F)/V_n]$ as described in the previous lemma.

Since $\mathrm{Loc}^c{}_{T,F} = \underline{\mathrm{Loc}} \times \mathbb{B}\hat{T}^\Gamma$ is a $\hat{T}^\Gamma$-gerbe, ind-coherent sheaves on $\mathrm{Loc}^c{}_{T,F}$ decompose as

$$\mathrm{IndCoh}(\mathrm{Loc}^c{}_{T,F}) = \prod_{\alpha \in X^*(\hat{T}^\Gamma)} \mathrm{IndCoh}^\alpha(\mathrm{Loc}^c{}_{T,F}). \tag{2.8}$$

The subcategory $\mathrm{IndCoh}^\alpha(\mathrm{Loc}^c{}_{T,F})$ comprise ind-coherent sheaves on $\mathrm{Loc}^c{}_{T,F}$ that has an action of $\hat{T}^\Gamma$ via the character $\alpha$. It is equivalent to $\mathrm{IndCoh}(\underline{\mathrm{Loc}})$.

Connected components of $\mathbf{Tor}_{T,\mathrm{iso}_F}$ are indexed by the Kottwitz set $B(T) = X^*(\hat{T}^\Gamma)$. Let the corresponding component of $\beta \in X^*(\hat{T}^\Gamma)$ be $BT(F)_\beta$. We have a decomposition by connected components

$$\mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F}) = \prod_{\beta \in X^*(\hat{T}^\Gamma)} \mathrm{QCoh}(BT(F)_\beta).$$

We have an open embedding $BT(F)_\beta \times \mathrm{Loc}_{{}^cT,F} \hookrightarrow \mathbf{Tor}_{T,\mathrm{iso}_F} \times \mathrm{Loc}_{{}^cT,F}$, and we denote the restriction of the Poincaré line bundle by $\mathcal{L}_\beta$.

By abuse of notation, we let $\pi_1$ and $\pi_2$ be the two projection of $BT(F)_\beta \times \mathrm{Loc}_{{}^cT,F}$ to its factors. A simple calculation should reveal that for $\mathcal{F} \in \mathrm{Coh}^\alpha(\mathrm{Loc}_{{}^cT,F})$ and $\beta \neq -\alpha$

$$\pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{L}_\beta) = 0.$$

Therefore, showing the equivalence of category $\mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F}) \cong \mathrm{Coh}(\mathrm{Loc}_{{}^cT,F})$ is reduced to showing the equivalence of the subcategories by pull-push

$$\pi_{1!}(\pi_2^*(-) \otimes \mathcal{L}_{-\alpha}^{-1}) : \mathrm{QCoh}(BT(F)_{-\alpha}) \leftrightarrows \mathrm{IndCoh}^\alpha(\mathrm{Loc}_{{}^cT,F}) : \pi_{2*}(\pi_1^*(-) \otimes \mathcal{L}_{-\alpha}).$$

2.5.5. **The equivalence.** In this section we show the following functor is an equivalence:

$$\pi_{2*}(\pi_1^*(-) \otimes \mathcal{L}_{-\alpha}) : \mathrm{IndCoh}^\alpha(\mathrm{Loc}_{{}^cT,F}) \to \mathrm{QCoh}(BT(F)_{-\alpha}). \tag{2.9}$$

Let $\mathcal{F} \in \mathrm{Coh}^\alpha\left(\mathrm{Loc}^{(n)}_{{}^cT,F}\right)$. Consider the coherent sheaf $\pi_1^*\mathcal{F} \otimes \mathcal{L}_{-\alpha}$ on $\mathrm{Loc}_{{}^cT,F} \times \mathbb{B}T(F)_{-\alpha}$, it can be regarded as a coherent sheaf on $\underline{\mathrm{Loc}}$ with a $\hat{T}^\Gamma$ and a $T(F)$-action. By definition of $\mathcal{L}_{-\alpha}$, the $\hat{T}^\Gamma$-action of $\pi_1^*\mathcal{F} \otimes \mathcal{L}_{-\alpha}$ is trivial, and the $T(F)$-action is identified with the $\mathcal{O}_{\underline{\mathrm{Loc}}^{(n)}} = \mathbb{Z}[T(F)/V_n]$-action. The pushforward by $\pi_{2*}$ simply forgets the $\hat{T}^\Gamma$-action.

In summary, the pull-push factors as

$$\begin{array}{ccc} \mathrm{Coh}^\alpha(\mathrm{Loc}^{(n)}_{{}^cT,F}) & \xrightarrow{\pi_{2*}(\pi_1^*(-)\otimes\mathcal{L}_{-\alpha})} & \mathrm{QCoh}(BT(F)_{-\alpha}). \\ & \searrow^{\sim} \quad \nearrow & \\ & \left\{\text{finite}\, \mathbb{Z}[T(F)/V_n]\text{-Mod}\right\} & \end{array}$$

By taking colimit, this results in the equivalence of categories

$$\begin{array}{ccc} \mathrm{IndCoh}^\alpha(\mathrm{Loc}) & \xrightarrow{\pi_{2*}(\pi_1^*(-)\otimes\mathcal{L}_{-\alpha})} & \mathrm{QCoh}(BT(F)_{-\alpha}). \\ & \searrow^{\sim} \quad \nearrow^{\sim} & \\ & \mathrm{Rep}_{\mathrm{sm}}\,(T(F)) & \end{array}$$

2.5.6. **The inverse functor.** The functor $\pi_1^*$ preserves limits and is a right adjoint. We denote its left adjoint by $\pi_{1!}$. We first note that both $\pi_1^*$ and $\pi_{1!}$

respect the decomposition of $\mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F} \times \mathbb{B}T(F)_{-\alpha})$ and $\mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F})$ as $\hat{T}^{\Gamma}$-gerbes, we can restrict the following discussion to $p_1^*$ and $p_{1!}$ for the map $p_1 : \underline{\mathrm{Loc}} \times \mathbb{B}T(F)_{-\alpha} \to \underline{\mathrm{Loc}}$ and $p_2 : \underline{\mathrm{Loc}} \times \mathbb{B}T(F)_{-\alpha} \to \mathbb{B}T(F)_{-\alpha}$.

Let $X = \underline{\mathrm{Loc}} \times \mathbf{Tor}_{T,\mathrm{iso}_F}$ and $Y = \underline{\mathrm{Loc}}$. On the level of abelian categories, let $G : \mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{QCoh}(Y)^{\heartsuit}$ be the functor of taking $T(F)$-coinvariants. Then $G$ and $p_1^*$ is an adjoint pair. The derived functors $LG$ and $p_1^*$ (no need to derive) lift to $\infty$-categories and is still an adjoint pair. Therefore $LG = p_{1!}$.

Let $M \in \mathrm{QCoh}(\mathbb{B}T(F)_{-\alpha})$ such that $M$ is fixed by $V_n$. The quasi-coherent sheaf $p_2^* M \otimes \mathcal{L}_{-\alpha}^{-1} = M \otimes O_{\underline{\mathrm{Loc}}}$ restricts to $M \otimes \mathbb{Z}[T(F)/V_n]$ on $\underline{\mathrm{Loc}}^{(n)}$ and it has an $T(F)$-action that is identified with its $\mathbb{Z}[T(F)/V_n]$-module structure. Due to the fact that $\mathbb{Z}[T(F)/V_n]$ is a projective $T(F)$-module, we have

$$p_{1!}(p_2^* M \otimes \mathcal{L}_{-\alpha}^{-1})|_{\underline{\mathrm{Loc}}^{(n)}} = LG(M \otimes \mathbb{Z}[T(F)/V_n]) \cong M.$$

We emphesize again that this isomorphism endows $M$ the with structure of an $O_{\underline{\mathrm{Loc}}}$-module which is identified with its $T(F)$-module structure. In general, take $M = \operatorname{colim} M_n$ where $M_n = M^{V_n}$. As $p_{1!}$ commute with filtered colimits, the same statement holds. In other words, the functor $\pi_{1!}(\pi_2^*(-) \otimes \mathcal{L}^{-1})$ is the inverse of $\pi_{2*}(\pi_1^*(-) \otimes \mathcal{L})$.

*Chapter 3*

# THE SECOND CATEGORIFICATION

## 3.1 Notation

3.1.1. **The inertia group.** We let $I \subset \Gamma$ be the inertia group of the field extension $E/F$. Assume $\Omega_E$ and $\Omega_F$ are the maximal unramified extension of $E$ and $F$, we denote the absolute inertia group of $E$ and $F$ by $I_E = \mathrm{Gal}(\bar{E}/\Omega_E)$ and $I_F = \mathrm{Gal}(\bar{F}/\Omega_F)$. They are subgroups of $W_E$ and $W_F$ respectively. We also define the relative inertia group as $I^{\mathrm{rel}}_{E/F} := I_F/[I_E, I_E]$.

The relative inertia group is a group extension:

$$1 \to I_E^{\mathrm{ab}} \to I^{\mathrm{rel}}_{E/F} \to I \to 1.$$

3.1.2. **The loop group.** We denote by $k$ and $O$ the residue field and ring of integers of $F$. Let $\varpi$ denote a uniformizer of $O$. For a $k$-algebra $R$, we let $W(R)$ denote its ring of Witt vectors, let $W_O(R) = W(R) \otimes_{W(k)} O$ and $W_{O,n}(R) = W(R) \otimes_{W(k)} O/\varpi^n$.

Let $\mathcal{Y}$ be a finite-type $O$-scheme. According to Greenberg, the following two presheaves on the category of affine $k$-schemes:

$$L_p^+\mathcal{Y}(R) = \mathcal{Y}(W_O(R)), \quad L_p^n\mathcal{Y}(R) = \mathcal{Y}(W_{O,n}(R)),$$

are represented by schemes over $k$. We denote their perfection by

$$L^+\mathcal{Y} = (L_p^+\mathcal{Y})^{p^{-\infty}}, \quad L^n\mathcal{Y} = (L_p^n\mathcal{Y})^{p^{-\infty}},$$

and call them $p$-adic jet spaces.

Let $Y$ be an affine scheme of finite type over $F$. The $p$-adic loop space $LY$ of $Y$ is a perfect space defined by assigning a perfect $k$-algebra $R$ the set

$$LY(R) = Y(W_O(R)[1/p]).$$

$LY$ is represented by an ind-perfect scheme. Assume $\mathcal{Y}$ is an affine scheme of finite type over $O$ and $X = \mathcal{Y} \otimes_O F$, then $L^+\mathcal{Y} \subset LY$ is a closed subscheme.

Take $Y = T$. We let $L^{\geq n}T \subset L^+T$ be the congruence subgroup such that $L^{\geq n}T(\kappa_F) = V_n$.

3.1.3. **Character sheaves and Serre's fundamental group.** Let $\Lambda = \overline{\mathbb{Z}}_\ell, \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. Let $H$ be a connected pro-algebraic group over $\overline{\mathbb{F}}_\ell$, and let $m$ be the multiplication map of $H$. Recall that a character sheaf with coefficient in $\Lambda$ on $H$ is a rank one $\Lambda$-local system $\mathrm{Ch}_\xi$ on $H$ equipped with an isomorphism

$$m^* \operatorname{Ch}_\xi \cong \operatorname{Ch}_\xi \boxtimes_\Lambda \operatorname{Ch}_\xi,$$

which satisfies the usual cocycle conditions.

**Definition 3.1.4.** Let $H$ be a (not necessarily) connected pro-algebraic group, and let $H^\circ$ be its central component. We define the character sheaves on $H$ to consists of all the translations of character sheaves on $H^\circ$.

Denote by $\mathcal{P}$ the category of commutative pro-algebraic groups over $k = \overline{\mathbb{F}}_\ell$ and by $\mathcal{P}_0$ the category of abelian profinite groups. Serre defined the fundamental groups $\pi_1$ as the first derived functor of the right exact functor $\pi_0 : \mathcal{P} \to \mathcal{P}_0$, $\pi_0(G) := G/G^\circ$ (Serre, 1961).

Assume $H$ is abelian and connected. A key property of this fundamental group is that the abelian group of character sheaves on $H$, which we denote by $\mathrm{CS}(H, \Lambda)$, is isomorphic to the abelian group of continuous rank 1 representations of $\pi_1(H)$:

$$\operatorname{Hom}_{\mathrm{cts}}(\pi_1(H), \Lambda^\times) \cong \mathrm{CS}(H, \Lambda).$$

The goal of the second categorification is essentially to categorify this isomorphism into a fully faithful functor (take $H = L^+T$)

$$\mathrm{Ch} : \mathrm{Coh}\left(\underline{\mathrm{Hom}}_{\mathrm{cts}}(\pi_1(L^+T), \mathbb{G}_m)\right) \to \mathrm{Shv}(L^+T, \Lambda).$$

The following theorem, which can be called the geometric class field theory, will play a key role in the sequel.

**Theorem 3.1.5** (Serre (1961); Deshpande and Wagh (2023))**.** Let $T$ be an arbitrary torus defined over $F$ and splits over a finite Galois extension $E$ of $F$. Let $I$ be the inertia group of the field extension $F/E$ and let $I_E^{\mathrm{ab}}$ be the abelianization of the (absolute) inertia group of $E$. Then there is a canonical isomorphism

$$\pi_1(L^+T) \cong \left(\hat{L} \otimes I_E^{\mathrm{ab}}\right)^I.$$

3.1.6. **The fixed point stack.** Let $Y$ be an arbitrary stack equipped with an automorphism $\sigma_Y : Y \to Y$. We define the fix-point stack of $\sigma_Y$ to be

$$\mathcal{L}_\sigma Y = Y_{\Gamma, Y\times Y, \Delta} Y,$$

where $\Gamma = (1, \sigma_Y)$ is the graph of $\sigma_Y$ and $\Delta$ is the diagonal embedding.

**Proposition 3.1.7.** The fixed point stack $\mathcal{L}_\sigma \mathbb{B}LT = \dfrac{LT}{\mathrm{Ad}_\sigma LT}$.

*Proof.* By definition $\mathcal{L}_\sigma \mathbb{B}LT$ is the fibre product

$$\begin{array}{ccc} \mathcal{L}_\sigma \mathbb{B}LT & \longrightarrow & \mathbb{B}LT \\ \downarrow & & \downarrow \Delta \\ \mathbb{B}LT & \xrightarrow{\mathrm{id}\times\sigma} & \mathbb{B}LT \times \mathbb{B}LT. \end{array}$$

Objects of $\mathcal{L}_\sigma \mathbb{B}LT$ are represented by a tuple $(x, y) \in LT$ and morphisms $(t, s) : (x, y) \to (x', y')$ are a tuple $(t, s) \in L^+T$ such that

$$\begin{array}{ccc} * \times * & \xrightarrow{(x,y)} & * \times * \\ {\scriptstyle (t,\sigma t)}\downarrow & & \downarrow{\scriptstyle (s,s)} \\ * \times * & \xrightarrow{(x',y')} & * \times *. \end{array}$$

In other words, $(t, s) : (x, y) \to (x + s - t, y + s - \sigma t)$. It is straightforward from here to see that $\mathcal{L}_\sigma \mathbb{B}LT = \frac{LT}{\mathrm{Ad}_\sigma LT}$. □

## 3.2 Representation Stack of Inertia

**Definition 3.2.1.** The moduli space $\mathcal{R}_{I_F, {}^cT}$ of the (strongly) continuous representations of $I_F$ in ${}^cT$ is defined over $\mathbb{Z}_\ell$ as follows. Let $r = \mathrm{rank}\, \hat{T}$. For every $\mathbb{Z}_\ell$-algebra $A$, $\mathcal{R}_{I_F, {}^cT}(A)$ classifies the cross homomorphisms $\rho : I_F \to \hat{T}(A) \subset A^r$ such that for each coordinate map $\nu : A^r \to A$, the $\mathbb{Z}_\ell$ subspace $M$ generated by $\nu \circ \rho(I_F)$ is a finitely generated $\mathbb{Z}_\ell$-module, and the representation of $I_F$ on $M$ is continuous, with $M$ equipped with the usual $\ell$-adic topology.

**Definition 3.2.2.** The representation stack of inertia is $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F} := \mathcal{R}_{I_F, {}^cT}/\hat{T}$.

The Frobenius $\sigma$ acts on $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}$ via its action on $I_F$ and $\hat{T}$. Explicitly, the action sends a cross homomorphism $c : I_F \to \hat{T}$ to $\sigma c$ such that $\sigma c(g) = {}^\sigma c(\sigma^{-1} g \sigma)$.

The following proposition reveals the connection between $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}$ and $\mathrm{Loc}_{{}^cT,F}$.

**Proposition 3.2.3.** The fix-point stack $\mathcal{L}_\sigma \operatorname{Loc}^{\mathrm{geom}}_{{}^cT,F}$ is canonically isomorphic to $\operatorname{Loc}_{{}^cT,F} \otimes \mathbb{Z}_\ell$.

*Proof.* There is a natural map $\operatorname{Loc}_{{}^cT,F} \otimes \mathbb{Z}_\ell \to \mathcal{L}_\sigma \operatorname{Loc}^{\mathrm{geom}}_{{}^cT,F}$ induced by restricting a cross homomorphism $c : W_F \to \hat{T}$ to $c' : I_F \to \hat{T}$. The goal is to show that this map is surjective on objects and is a bijection on the automorphism group of an object.

Objects of $\mathcal{L}_\sigma \operatorname{Loc}^{\mathrm{geom}}_{{}^cT,F}$ can be represented by a tuple $([c], t)$ for $c \in Z^1(I_F, \hat{T})$ and $t \in \hat{T}$ such that $\sigma c = c + dt$. It is the image of $\tilde{c} \in Z^1(W_F, \hat{T})$ if $\tilde{c}$ satisfies $\tilde{c}|_{I_F} = c$ and $\tilde{c}(\sigma) = t$.

To show that such $\tilde{c}$ exists, we first need to check that $c$ factors through a finite quotient of $I_F$. The restriction of $c$ to the wild inertia $P_F$ clearly factors through a finite quotient. Therefore, it suffices to show the claim for the restriction of $c$ to $I_F^{\mathrm{tame}}$.

Let the action of $I_F^{\mathrm{tame}}$ on $\hat{T}$ factor through an open subgroup $K$. Then for all $g \in K$, we have $c(g) = {}^\sigma c(\sigma^{-1} g \sigma)$. Henceforth ${}^\sigma c(g) = c(\sigma g \sigma^{-1}) = c(g^q) = c(g)^q$. Let $N$ be a positive integer so that $\sigma^N$ acts trivially on $\hat{T}$, we have $c(g) = c(g)^{q^N}$. That is, for every $g \in K$, $c(g^{q^N - 1}) = 1$. We conclude that $c$ factors through an open subgroup of $I_F^{\mathrm{tame}}$ as $\{g^{q^{N-1}} | g \in K\}$ is open.

One define $\tilde{c}(\sigma^n)$ recursively via

$$\tilde{c}(\sigma^n) = t + {}^\sigma\tilde{c}(\sigma^{n-1})$$

for all $n \in \mathbb{N}$ and define

$$\begin{aligned}\tilde{c}(\sigma^{-n}) &= -{}^{\sigma^{-n}}\tilde{c}(\sigma^n),\\ \tilde{c}(\sigma^m g) &= \tilde{c}(\sigma^m) + {}^{\sigma^n}c(g),\end{aligned}$$

for all $n \in \mathbb{N}, m \in \mathbb{Z}$ and for all $g \in I_F$. The condition $\sigma c = c + dt$ ensures that $\tilde{c}$ is a 1-cocycle.

An automorphism of the object $([c], t)$ is represented by an element $w \in \hat{T}$ such that $dw = 0$ and the following diagram commutes:

$$\begin{array}{ccc} c & \xrightarrow{t} & \sigma c \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \sigma w} \\ c & \xrightarrow{t} & \sigma c. \end{array}$$

That is, $w \in \hat{T}^{I_F}$ and $\sigma w = w$, hence $w \in \hat{T}^{W_F}$. This automorphism group is exactly in bijection with the automorphism group of an object of $\operatorname{Loc}_{{}^cT,F}$. □

**Proposition 3.2.4.** We have a canonically split short exact sequence

$$1 \to \underline{\mathrm{Hom}}(\hat{L}_I, \mathbb{B}\mathbb{G}_m) \to \mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F} \to \underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes I_E^{\mathrm{ab}})^I, \mathbb{G}_m) \to 1. \tag{3.1}$$

*Proof.* The proof of this proposition is similar to that of proposition 2.3.1. The only difference is that we need to replace the isomorphism $H_1(W_{E/F}, \hat{L}) \cong T(F)$ with $H_1(I^{\mathrm{rel}}_{E/F}, \hat{L}) \cong (\hat{L} \otimes I_E^{\mathrm{ab}})^I$, which is supplied by Corollary 2.4.5. □

3.2.5. $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}$ **as an ind-scheme.** We want to understand the geometry of the third term in this exact sequence.

We first note that the abelian group $I_E^{\mathrm{ab}}$ splits into its tame and wild parts as an $\Gamma$-module. By class field theory, we have

$$I_E^{\mathrm{ab}} = \varprojlim_{\substack{K/E \text{ finite Galois,} \\ \text{unramified}}} \mathcal{O}_K^\times.$$

Each term $\mathcal{O}_K^\times = \mu_K \times U_K^{(1)}$ splits as a $\Gamma$-module, where $\mu_K$ is the group of roots of unity in $K$ and $U_K^{(1)} = 1 + \mathfrak{m}_K$ the principal units. This splitting clearly respects the norm map, which is the transition map in the projective limit. Therefore we have a splitting

$$I_E^{\mathrm{ab}} = \varprojlim U_K^{(1)} \times I_E^{\mathrm{tame}} = P_E^{\mathrm{ab}} \times I_E^{\mathrm{tame}}.$$

The group $\varprojlim U_K^{(1)}$ can be identified with the abelianization of the wild inertia $P_E^{\mathrm{ab}}$(Iwasawa, 1955). Although we do not rely on this fact, but we use it to simplify our notation.

With the splitting of the group $I_E^{\mathrm{ab}}$, we obtain a decomposition

$$\underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes I_E^{\mathrm{ab}})^I, \mathbb{G}_m) = \underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes I_E^{\mathrm{tame}})^I, \mathbb{G}_m) \times \underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes P_E^{\mathrm{ab}})^I, \mathbb{G}_m)$$

which is characterized by that $(\hat{L} \otimes I_E^{\mathrm{tame}})^I$ is a prime-to-$p$ profinite group and $(\hat{L} \otimes P_E^{\mathrm{ab}})^I$ is a pro-$p$ group. This gives us

$$\underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes I_E^{\mathrm{ab}})^I, \mathbb{G}_m) = \bigsqcup_{x \in \Xi} \underline{\mathrm{Hom}}_{\mathrm{cts}}(\hat{J}, \mathbb{G}_m) = \bigsqcup_{x \in \Xi} \mathcal{R}_{\hat{J},\mathbb{G}_m}. \tag{3.2}$$

where $\Xi = \underline{\mathrm{Hom}}_{\mathrm{cts}}((\hat{L} \otimes P_E^{\mathrm{ab}})^I, \mathbb{G}_m)$, and $\hat{J} = (\hat{L} \otimes I_E^{\mathrm{tame}})^I$.

The functor $\mathcal{R}_{\hat{J},\mathbb{G}_m}$ can be made explicit by embedding it into an algebraic torus. We choose $\tau$ a topological generator of $I_E^{\mathrm{tame}}$, and let the finitely generated abelian group $(\hat{L} \otimes \langle\tau\rangle)^I \subset \hat{J}$ be denoted by $J$. Denote by $\hat{H}$ the torus $\mathcal{R}_{J,\mathbb{G}_m}$. There is an inclusion $\mathcal{R}_{\hat{J},\mathbb{G}_m} \to \hat{H}$ by restricting a representation of $\hat{J}$ to $J$.

Let us denote $\hat{H}^p \subset \hat{H}$ the subfunctor which is the union of all closed subschemes $i_Z : Z \subset \hat{H}$ that are finite over $\mathbb{Z}_\ell$ and $Z(\overline{\mathbb{F}}_\ell) \subset \hat{H}(\overline{\mathbb{F}}_\ell)^p$, where $H(\overline{\mathbb{F}}_\ell)^p$ consists of all elements in $\hat{H}(\overline{\mathbb{F}}_\ell)$ of order prime to $p$. Note that by construction $\hat{H}^p$ has a natural structure of an ind-scheme.

**Proposition 3.2.6.** The inclusion $\mathcal{R}_{\hat{J},\mathbb{G}_m} \to \hat{H}$ identifies $\mathcal{R}_{\hat{J},\mathbb{G}_m}$ with $\hat{H}^p$. Consequently, $\mathcal{R}_{\hat{J},\mathbb{G}_m}$ is an ind-scheme.

*Proof.* Let $A$ be an $\mathbb{Z}_\ell$-algebra. Every $\phi \in \mathcal{R}_{\hat{J},\mathbb{G}_m}(A)$ represents a continuous cross homomorphism $\varphi : \hat{J} \to A^\times \subset A$ such that the image of $\varphi$ in $A$ spans a finitely generated $\mathbb{Z}_\ell$-module $M$, and the map $\phi : \hat{J} \to M$ is continuous.

The image of $\phi \in \hat{H}$ gives a ring homomorphism which we also denote by $\phi : \mathbb{Z}_\ell[J] \to A$. This is justified because this $\phi$ coincide with the cross homomorphism in their value in $A$. Let $\phi$ factor as $\mathbb{Z}_\ell[J] \to R \hookrightarrow A$, where Spec $R$ is the schematic image of Spec $A$. Our conditions on $\phi$ imply that $R$ is a finite $\mathbb{Z}_\ell$-algebra.

Henceforth, all $\mathbb{F}_\ell$-points of Spec $R$ can lift to $\mathbb{F}_\ell$-points of Spec $A$, their image in $\hat{H}(\mathbb{F}_\ell)$ must have finite and prime-to-$\ell$ order, because $\phi : \mathbb{Z}_\ell[J] \to A$ can be lifted to a continuous map $\mathbb{Z}_\ell[\hat{J}] \to A$. □

### 3.3 Review of the Categorical Trace

3.3.1. **Hochschiled homology.** Let us first review the general formalism of the Hochschild homology. We will mostly follow ((Hemo, 2023)).

Let $\mathcal{R}$ be a symmetric monoidal category, $A$ and $B$ be two associative algebras in $\mathcal{R}$. We denote by $A^{\mathrm{rev}}$ (resp. $B^{\mathrm{rev}}$) to be the algebra $A$ with the reversed multiplication. An $A$-$B$-bimodule can be regarded as a left $(A \otimes B^{\mathrm{rev}})$-module or a right $(B \otimes A^{\mathrm{rev}})$-module. For an $A$-$A$-bimodule $F$, the Hochschild homology of $F$, if exists, is defined as

$$\mathrm{Tr}(A, F) = A \otimes_{A \otimes A^{\mathrm{rev}}} F \in \mathcal{R}.$$

However, the Hochschild complex of $F$ always exists. It is a simplicial object in $\mathcal{R}$ given by

$$\mathrm{HH}(A, F)_\bullet = \mathrm{Bar}(A)_\bullet \otimes_{A \otimes A^{\mathrm{rev}}} F = A^{\otimes \bullet} \otimes F.$$

**Example 3.3.2.** Let $\phi$ be an endomorphism of the algebra $A$, and $F$ an $A$-bimodule. The $\phi$-twisted bimodule $F$, which we denote by ${}^\phi F$, is the bimodule whose left

$A$-action is pre-composed with $\phi$ and whose right action stays the same. In this case, we also denote the Hochschild homology of ${}^{\phi}A$ by $\mathrm{Tr}(A, \phi)$.

Roughly speaking, $\mathrm{Tr}(A, \phi)$ is determined by the universal property that a functor $\mathrm{Tr}(A, \phi) \to C$ is equivalent to a functor $G : A \to C$ equipped with equivalences

$$F(a \otimes b) \cong F(b \otimes \phi(a)), \quad a, b \in \mathrm{Ob}(A)$$

together with all the necessary higher coherence data. In particular, there is a tautological functor

$$\mathrm{Tr}_{\phi} : A \to \mathrm{Tr}(A, \phi)$$

sending an object $a$ to its universal $\phi$-twisted trace.

Now consider $\mathcal{R} = \mathrm{Lincat}_{\Lambda}$. An algebra object $A$ in $\mathrm{Lincat}_{\Lambda}$ is a presentable $\Lambda$-linear monoidal category such that the monoidal product commutes with colimits separately in each variable. Let $F$ be an $A$-bimodule category. In this case, $\mathrm{Tr}(A, F)$ always exists, and is called the categorical trace of $(A, F)$. Note that the output of categorical trace is again a presentable $\Lambda$-linear category.

We recall two key propositions which are particularly useful in the calculation of categorical traces, one in the coherent sheaf setting, and one in the $\Lambda$-sheaf setting. The gist is that the categorical trace of a sheaf theory can often be identified with, or embed in, the category of sheaves of certain fixed point object.

**Proposition 3.3.3.** Let $X$ be a smooth Artin stack with an automorphism $\sigma : X \to X$, and let $i : \mathcal{L}_{\sigma} X \to X$. There is a canonical equivalence of categories

$$H : \mathrm{Tr}(\mathrm{IndCoh}(X), \sigma) \cong \mathrm{IndCoh}(\mathcal{L}_{\sigma} X),$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\quad i^* \quad} & \mathrm{IndCoh}(\mathcal{L}_{\sigma} X) \\ & \searrow \qquad \nearrow_{H} & \\ & \mathrm{Tr}(\mathrm{IndCoh}(X), \sigma). & \end{array}$$

**Proposition 3.3.4.** Let $X$ be a placid stack such that the diagonal $\Delta_X : X \to X \times X$ is representable pro-smooth. Let $Y$ be a prestack and let $f : X \to Y$ be a ind-proper morphism such that the relative diagonal $X \to X \times_Y X$ is ind-proper. Let $\sigma_X : X \to X$ and $\sigma_Y : Y \to Y$ be endomorphisms intertwined by $f$.

Consider the following diagram:

$$\begin{array}{ccc} X \times_Y \mathcal{L}_\sigma Y & \xrightarrow{\delta_0} & X \times_{f,Y,f\circ\sigma_X} X \\ {\scriptstyle q}\downarrow & & \\ \mathcal{L}_\sigma Y, & & \end{array} \tag{3.3}$$

where $\delta_0$ is the map

$$X \times_Y (Y \times_{\Delta_Y, Y\times Y, 1\times\sigma_Y} Y) \cong X \times_{Y\times Y} Y \xrightarrow{\Delta_X} (X \times X) \times_{Y\times Y} Y \cong X \times_{f,Y,f\circ\sigma_X} X$$

and $q$ is the projection.

There is a canonical factorization

$$\begin{array}{ccc} \mathrm{Shv}(X \times_Y X) & \xrightarrow{\delta_0^*} & \mathrm{Shv}(X \times_Y \mathcal{L}_\sigma Y) \\ \downarrow & & \downarrow{\scriptstyle q_!} \\ \mathrm{Tr}(\mathrm{Shv}(X \times_Y X), \sigma) & \xrightarrow{G} & \mathrm{Shv}(\mathcal{L}_\sigma Y), \end{array}$$

where the functor $G$ is fully faithful.

### 3.4 Geometric LLC for the torus

In this section, let $\Lambda = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$. By abuse of notation, we will use $\mathrm{Loc}_{{}^cT,F}$ and $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}$ to denote the base-change $\mathrm{Loc}_{{}^cT,F} \otimes \Lambda$ and $\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F} \otimes \Lambda$ in the sequel. We will use $\mathrm{Shv}(-)$ to refer to the category of bounded constructable $\Lambda$-sheaves.

**Theorem 3.4.1.** (i) Let $T$ be an algebraic torus over $F$. There exists a fully-faithful, $t$-exact, monoidal functor

$$\mathrm{Ch} : \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) \to \mathrm{IndShv}(LT).$$

Let $\mathrm{Shv}^{\mathrm{mon}}(LT)$ denote the essential image of Ch. It is the thick subcategory compactly generated by all character sheaves on $LT$. Let $\mathrm{Ch}^{\mathrm{mon}}$ denote the equivalence of categories

$$\mathrm{Ch}^{\mathrm{mon}} : \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) \cong \mathrm{IndShv}^{\mathrm{mon}}(LT).$$

(ii) Both categories carry a Frobenius structure, and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}(\mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}), \sigma) & \xrightarrow{\mathbb{T}} & \mathrm{Tr}(\mathrm{IndShv}^{\mathrm{mon}}(LT), \sigma) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle G} \\ \mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F}) & \xrightarrow{\mathbb{L}} & \mathrm{IndShv}(\mathfrak{B}(T)), \end{array} \tag{3.4}$$

where $\mathbb{T}$ is induced by $\mathrm{Ch}^{\mathrm{mon}}$ and the functorality of the categorical trace construction, $\mathbb{L}$ is the canonical equivalence (2.9) under the identification $\mathrm{IndShv}(\mathfrak{B}(T)) \cong \mathrm{QCoh}(\mathbf{Tor}_{T,\mathrm{iso}_F})$, and two vertical arrows are the canonical maps $H$ and $G$ supplied by Proposition 3.3.3 and Proposition 3.3.4. Furthermore, $H$ and $G$ are equivalences.

3.4.2. **Construction of** Ch. We define the functor $\mathrm{Ch} : \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) \to \mathrm{IndShv}(LT)$ as follows.

Using the split short exact sequence (3.1), we have a decomposition

$$\mathrm{Coh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) = \prod_{\alpha\in\hat{L}_I} \mathrm{Coh}^{\alpha}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}).$$

On the other hand, since

$$1 \to L^+T \to LT \to \hat{L}_I \to 1,$$

we have another decomposition

$$\mathrm{Shv}(LT) = \prod_{\beta\in\hat{L}_I} \mathrm{Shv}(L^+T).$$

The index set of the two decomposition are canonically identified. However, for our purpose, the functor sends the $\alpha$ component of ind-coherent sheaves to the $-\alpha$ component of $\Lambda$-sheaves.

Given that $\mathrm{Coh}(\underline{\mathrm{Hom}}_{\mathrm{cts}}(\pi_1(L^+T), \mathbb{G}_m)) = \mathrm{Coh}^0(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F})$, the next step is to define the functor

$$\mathrm{Ch}^0 : \mathrm{Coh}(\underline{\mathrm{Hom}}_{\mathrm{cts}}(\pi_1(L^+T), \mathbb{G}_m)) \to \mathrm{Shv}(L^+T),$$

and naturally extend it to a functor between ind-completions of both sides. Notice that $L^+T = T \times L^{++}T$ with $L^{++}T$ being the first congruence subgroup.

Recall that $\Xi = \underline{\mathrm{Hom}}((\hat{L} \otimes P_E^{\mathrm{ab}})^I, \mathbb{G}_m)$, and each point $x \in \Xi$ gives a local system on $L^{++}T$ that we denote by $\mathcal{L}_x^{++}$.

By Proposition 3.2.6, with a choice of a topological generator $\tau \in I_E^{\mathrm{tame}}$, we have an inclusion $\mathcal{R}_{\hat{J},\mathbb{G}_m} \to \hat{H}$ and coherent sheaves on $\mathcal{R}_{\hat{J},\mathbb{G}_m}$ are given by

$$\mathrm{Coh}(\mathcal{R}_{\hat{J},\mathbb{G}_m}) = \operatorname*{colim}_{Z\subset\hat{H}} \mathrm{Coh}(Z).$$

Therefore we can define a functor

$$\mathrm{Ch}^{\mathrm{tame}} : \mathrm{Coh}(\mathcal{R}_{\hat{J},\mathbb{G}_m}) \cong \left\{\text{finite } \hat{J}\text{-module}\right\} \to \mathrm{Shv}(T).$$

This functor is clearly independent of the choice of the topological generator and fully faithful.

Now we can define the functor

$$\mathrm{Ch}^0 : \mathrm{Coh}(\underline{\mathrm{Hom}}_{\mathrm{cts}}(\pi_1(L^+T), \mathbb{G}_m) \to \mathrm{Shv}(L^+T) = \mathrm{Shv}(T) \otimes \mathrm{Shv}(L^{++}T)$$

by sending an ind-coherent sheaf $\mathcal{F}$ on the $x$-component to $\mathrm{Ch}^{\mathrm{tame}}(\mathcal{F}) \otimes \mathcal{L}^{++}_x$.

*of Theorem 3.4.1 (i).* It is obvious from the construction that Ch is $t$-exact.

To show Ch is fully faithful, we can decompose both categories into blocks

$$\mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) = \bigoplus_{\alpha \in \hat{L}_I} \bigoplus_{x \in \Xi} \mathrm{IndCoh}(\underline{\mathrm{Hom}}_{\mathrm{cts}}(\hat{J}, \mathbb{G}_m)$$

$$\mathrm{IndShv}(LT) = \bigoplus_{\beta \in \hat{L}_I} \bigoplus_{x \in \Xi} \mathrm{IndShv}(T),$$

where Ch respects both direct sums (with a twist $\beta = -\alpha$). On each block, Ch restricts to $\mathrm{Ch}^{\mathrm{tame}}$ and we already know the latter is fully faithful. □

3.4.3. **Decategorification.** We move on to prove part (ii) of the Theorem 3.4.1. Let $r = \operatorname{rank} T$. Consider the following diagram:

$$\begin{array}{ccccc} & \mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}) & \xrightarrow{\mathrm{Ch}^{\mathrm{mon}}} & \mathrm{IndShv}^{\mathrm{mon}}(LT) & \\ & \downarrow & & \downarrow & \\ i^* \Big( & \mathrm{Tr}(\mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}), \sigma) & \xrightarrow{\mathbb{T}} & \mathrm{Tr}(\mathrm{IndShv}^{\mathrm{mon}}(LT), \sigma) & \Big) q_* \circ \delta_0^*[2r] \\ & \downarrow H & & \downarrow G & \\ & \mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F}) & \xrightarrow{\mathbb{L}} & \mathrm{IndShv}(\mathfrak{B}(T)). & \end{array}$$

The functorality of the trace construction yields the upper square. The lower two vertical maps and the two curved maps are the canonical maps characterized in Theorem 3.3.3 and Theorem 3.3.4. They will be made explicit in the sequel.

By Proposition 3.3.3, $H$ is an equivalence of categories, and $G$ is automatically an equivalence once we established that the lower square commutes.

Instead of showing the lower square commutes directly, we first show that for a family of objects $\mathcal{F} \in \mathrm{IndCoh}(X)$ we have canonical isomorphisms

$$\mathbb{L} \circ i^*(\mathcal{F}) \cong q_* \circ \delta_0^* \circ \mathrm{Ch}^{\mathrm{mon}}(\mathcal{F})[2r]. \tag{3.5}$$

Then we show that this family is big enough to force the commutativity of the lower square.

Each element $x \in \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)$ defines a closed point of $\mathrm{Loc}^{\mathrm{geom}}_{^cT,F}$. In particular, we fix an element $x \in \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma$ and consider the skyscraper sheaf $\Lambda_x$ at this point. We show that (3.5) holds for all such $\Lambda_x$.

**Definition 3.4.4.** For a character $\chi : T(F)_0 \to \Lambda^\times$, we define $I_\chi$ to be the augmentation ideal $\{t - \chi(t) | t \in T(F)_0\}$ in the group ring $\Lambda[T(F)]$. Then we define $R_\chi = \Lambda[T(F)]/I_\chi$.

**Proposition 3.4.5.** Let $i : \mathrm{Loc}_{^cT,F} \cong \mathcal{L}_\sigma \mathrm{Loc}^{\mathrm{geom}}_{^cT,F} \to \mathrm{Loc}^{\mathrm{geom}}_{^cT,F}$, and $x \in \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma$. Let $x$ corresponds to a character $\chi$ under the isomorphism $\mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma \cong \mathrm{Hom}_{\mathrm{cts}}(T(F)_0, \Lambda^\times)$. We have

$$i^* \Lambda_x = R_\chi \otimes \bigwedge(X_*(T) \otimes \Lambda[1]).$$

*Proof.* The components of $\mathrm{Loc}_{^cT,F}$ are indexed by the set of representations $\mathrm{Hom}_{\mathrm{cts}}(T(F)_0, \Lambda^\times)$. Each component is sent to a closed subscheme in $\mathrm{Loc}^{\mathrm{geom}}_{^cT,F}$ via

$$\begin{aligned}\mathrm{Hom}_{\mathrm{cts}}(T(F)_0), \Lambda^\times) &= \mathrm{Hom}_{\mathrm{cts}}(L^+T(\kappa_F), \Lambda^\times)\\ &= \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma\\ &\hookrightarrow \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times).\end{aligned}$$

Let $U \subset \mathrm{Loc}_{^cT,F}$ be the component that is sent to $x \in \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma$. Explicitly, $U = \mathrm{Spec}(R_\chi)$. We denote the character associated to $U$ by $\chi : T(F)_0 \to \Lambda^\times$ and study $\Lambda_x = x_* \underline{\Lambda}$.

The component in which $x$ lies can be embedded in some $\mathcal{R}_{J,\mathbb{G}_m}$ as in equation (3.2). The component itself is precisely the formal neighbourhood of $x$ in that torus. Therefore,

$$i^* \Lambda_x = R_\chi \otimes^L_{\Lambda[J]} \Lambda = R_\chi \otimes^L_{\Lambda[\hat{L}]} \Lambda = R_\chi \otimes \bigwedge(X_*(T) \otimes \Lambda[1]). \qquad \square$$

Now we compute $q_! \circ \delta_0^* \circ \mathrm{Ch}^{\mathrm{mon}}(\mathcal{F})$. Let $\mathcal{L} = \mathrm{Ch}^{\mathrm{mon}}(\Lambda_x)$, and let $\chi : T(F)_0 \to \Lambda^\times$ be the character associated to $x$. There exists $n$ so that $\chi$ factors through $T(F)_0/V_n$.

In Theorem 3.3.4, we take $X = \mathbb{B}L^{\geq n}T$, $Y = \mathbb{B}LT$, $f : X \to Y$ induced by the inclusion $L^{\geq n}T \to LT$, and the Frobenius $\sigma$ induce the automorphisms of $X$ and $Y$.

The diagram (3.3) becomes

$$\begin{array}{ccc} \dfrac{LT}{\mathrm{Ad}_\sigma L^{\geq n}T} & \xrightarrow{\delta_0} & L^{\geq n}T \backslash {}^{LT}/_{L^{\geq n}T} \\ \Big\downarrow q & & \\ \dfrac{LT}{\mathrm{Ad}_\sigma LT}, & & \end{array} \tag{3.6}$$

where $\delta_0$ is given by identity of $LT$ on objects and $t \in L^{\geq n}T \mapsto (t, -\sigma t)$ on morphisms, and the map $q$ is given by identity on objects and the inclusion of $L^{\geq n}T \to LT$ on morphisms.

**Proposition 3.4.6.** Let $\mathcal{L} = \mathrm{Ch}^{\mathrm{mon}}(\Lambda_x)$, and let $\chi$ be the character associated to $x$. Let $r = \operatorname{rank} T$, we have

$$q_* \circ \delta_0^* \mathcal{L}[2r] = R_\chi \otimes \bigwedge (X_*(T) \otimes \Lambda[1]).$$

*Proof.* Let $\mathcal{L}$ be a character sheaf on the central component $L^+T/L^{\geq n}T$ such that $\sigma^* \mathcal{L} \cong \mathcal{L}$. By Lang's theorem, this character sheaf corresponds to a character $\chi : \frac{L^+T}{L^{\geq n}T}(\kappa_F) \to \Lambda^\times$.

We calculate $q_! \circ \delta_0^* \mathcal{L}$ as follows. Let $s : * \to \frac{LT}{\mathrm{Ad}_\sigma LT}$ be a point. We form the pull-back

$$\begin{array}{ccccc} \dfrac{LT}{L^{\geq n}T} & \xrightarrow{s'} & \dfrac{LT}{\mathrm{Ad}_\sigma L^{\geq n}T} & \xrightarrow{\delta_0} & L^{\geq n}T \backslash {}^{LT}/_{L^{\geq n}T} \\ \Big\downarrow q' & & \Big\downarrow q & & \\ * & \xrightarrow{s} & \dfrac{LT}{\mathrm{Ad}_\sigma LT}, & & \end{array}$$

where $s'$ is given by $t \mapsto s + t - \sigma t$ on objects and identity on morphisms.

First, we determine on which components of $\frac{LT}{\mathrm{Ad}_\sigma L^{\geq n}T}$ the pullback $s'^* \circ \delta_0^* \mathcal{L}$ is non-zero. Clearly, $\delta_0^* \mathcal{L}$ has the same support as $\mathcal{L}$. The components of $\frac{LT}{\mathrm{Ad}_\sigma L^{\geq n}T}$ are indexed by the set $X_*(T)_I$, and so is $\frac{LT}{L^{\geq n}T}$. Let $[s] \in X_*(T)_I$ be the component in which $s$ lies, then $s'^* \circ \delta_0^* \mathcal{L}$ has support on all those components $\alpha \in X_*(T)_I$ such that $[s] + \alpha - \sigma\alpha = 0$. Assume $[s] = \sigma\alpha - \alpha$, then $s'^* \circ \delta_0^* \mathcal{L}$ has support on those components indexed by $\alpha + (X_*(T)_I)^\sigma$.

Next, we determine on which components of $\frac{LT}{\mathrm{Ad}_\sigma LT}$ the sheaf $s_* \circ \delta_0^* \mathcal{L}$ is non-zero. The components of $\frac{LT}{\mathrm{Ad}_\sigma LT}$ is indexed by $B(T) = X_*(T)_\Gamma$. The fact that

$[s] = \sigma\alpha - \alpha$ in the previous analysis implies that $s$ lies in the central component of $\frac{LT}{\mathrm{Ad}_\sigma LT}$. Therefore $s_* \circ \delta_0^* \mathcal{L}$ is supported on the central component only.

To study $s_* \circ \delta_0^* \mathcal{L}$ on the central component of $\frac{LT}{\mathrm{Ad}_\sigma LT}$, it suffice to take $s = 1 \in LT$. Since we assumed that $\sigma^* \mathcal{L} \cong \mathcal{L}$, the pullback $s'^* \circ \delta_0^* \mathcal{L}$ is the trivial local system on components indexed by $(X_*(T)_I)^\sigma$. By Lang's theorem, $s'^* \circ \delta_0^* \mathcal{L}$ restricted to $\frac{L^+T}{L^{\geq n}T}$ is the trivial local system with an $(L^+T/L^{\geq n}T)(\kappa_F)$-action given by the character $\chi$.

Let $r = \operatorname{rank} T$. Since $\frac{L^+T}{L^{\geq n}T} \cong T \times \mathbb{A}^d$ as an algebraic scheme for some integer $d$, we have

$$\begin{aligned} s^* \circ q_* \circ \delta_0^* \mathcal{L}[2r] &= \bigoplus_{(X_*(T)_I)^\sigma} \left( \bigwedge (X^*(T) \otimes \Lambda[-1](-1)) \right) [2r] \\ &= \bigwedge (X_*(T) \otimes \Lambda[1](-1)). \end{aligned}$$

The sheaf $q_* \circ \delta_0^* \mathcal{L}$ is the above sheaf equipped with a $T(F)$-action. Recall that $I_\chi = \{t - \chi(t) | t \in T(F)_0\}$ is the augmentation ideal. Then as a $\Lambda[T(F)]$-module we have

$$\begin{aligned} q_* \circ \delta_0^* \mathcal{L} &= (\Lambda[T(F)]/I_\chi) \otimes \bigwedge (X_*(T) \otimes \Lambda[1](-1)) \\ &= R_\chi \otimes \bigwedge (X_*(T) \otimes \Lambda[1](-1)) \end{aligned}$$ □

*Proof of part (ii) of Theorem 3.4.1.* Let $x \in \mathrm{Hom}_{\mathrm{cts}}(\pi_1(L^+T), \Lambda^\times)^\sigma$ and consider the skyscraper sheaf $\Lambda_x$ at this point. We have shown the canonical isomorphism

$$\mathbb{L} \circ i^*(\Lambda_x) \cong q_* \circ \delta_0^* \circ \mathrm{Ch}^{\mathrm{mon}}(\Lambda_x)[2r] = R_\chi \otimes \bigwedge (X_*(T) \otimes \Lambda[1]).$$

Since $\mathrm{Ch}^{\mathrm{mon}}$ is $t$-exact, so is $\mathbb{T}$. Therefore all four functors in diagram 3.4 are $t$-exact. Let $\mathcal{F} \in \mathrm{Tr}\left(\mathrm{IndCoh}(\mathrm{Loc}^{\mathrm{geom}}_{{}^cT,F}), \sigma\right)$ be the image of $\Lambda_x$. Since

$$\mathbb{L} \circ H(\mathcal{F}) \cong G \circ \mathbb{T}(\mathcal{F}), \tag{3.7}$$

there is an isomorphism on each cohomology. The cohomology of (3.7) in degree 0 is $R_\chi$. Since the family $\{R_\chi\}$ generates the whole category of $\mathrm{IndCoh}(\mathrm{Loc}_{{}^cT,F})$, we conclude that the diagram commutes. □

*Appendix A*

# A LEMMA IN HOMOLOGICAL ALGEBRA

**Lemma A.0.1.** Let $\Gamma$ be a finite group, $A$ be an abelian group equipped with a $\Gamma$-action. Let $[\alpha] \in H^2(\Gamma, A)$ represent the group extension

$$1 \to A \to G \to \Gamma.$$

Let $M$ be any $\Gamma$-module, in other words, it is a $G$-module on which $A$ acts trivially. Then we have a commutative diagram

$$\begin{array}{ccccccccccc} & & H_2(\Gamma, M) & \xrightarrow{d_2} & H_1(A, M)_\Gamma & \longrightarrow & H_1(G, M) & \longrightarrow & H_1(\Gamma, M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cup\alpha} & & \Vert & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \cup\alpha} & & \\ 0 & \longrightarrow & \hat{H}^{-1}(\Gamma, M \otimes A) & \longrightarrow & (M \otimes A)_\Gamma & \longrightarrow & (M \otimes A)^\Gamma & \longrightarrow & \hat{H}^0(\Gamma, M \otimes A) & \longrightarrow & 0. \end{array}$$

The first row of the diagram is the long exact sequence associated to the Lyndon-Hochschild-Serre spectral sequence. The second row is the definition of the Tate cohomology. The first and last vertical map is the cup product with $\alpha$. The map res is the restriction map $H_1(G, M) \to H_1(A, M)^\Gamma$.

In particular, if $\Gamma$ and $A$ satisfies the condition of the Tate-Nakayama lemma, we would have

$$H_1(G, M) \cong (M \otimes A)^\Gamma.$$

*Proof.* The crux of the proof is the commutativity of the left-most square, which we will prove in the sequel. The commutativity of the rest of the squares are fairly standard and will be omitted.

A.0.2. **Notations.** We follow the notation of Atiyah and Wall (Cassels, 1987). The (homogeneous) bar resolution of the $G$-module $\mathbb{Z}$ is given by

$$C_n(G, \mathbb{Z}) = \bigoplus_{G^{n+1}} \mathbb{Z}(g_0, g_1, \cdots, g_n),$$

which is the free abelian group generated by the basis $G^{n+1}$ with the action of

$$g \cdot (g_0, \cdots, g_n) = (gg_0, \cdots, gg_n).$$

The differential is the usual

$$d(g_0,\cdots,g_n)=\sum_{i=0}^{n}(-1)^i(g_0,\cdots,\widehat{g_i},\cdots,g_n).$$

It is isomorphic to the more familiar inhomogeneous bar resolution via sending

$$(g_0,g_0g_1,\cdots,g_0g_1\cdots g_n)\mapsto(g_0|g_1|\cdots|g_n).$$

Let $P_\bullet$ denote a $\Gamma$-resolution of $\mathbb{Z}$ by finitely-generated free $\Gamma$-modules, and let $P^*=\mathrm{Hom}(P_\bullet,\mathbb{Z})$ be its dual, so that we have exact sequences

$$\cdots\to P_1\to P_0\to\mathbb{Z}\to 0$$
$$0\to\mathbb{Z}\to P_0^*\to P_1^*\to\cdots.$$

Let $P_{-n}=P_{n-1}^*$ and join the two sequences together we get a doubly-infinite exact sequence

$$L_\bullet:\quad\cdots\to P_1\to P_0\to P_{-1}\to P_{-2}\to\cdots$$

The Tate groups are then the cohomology groups of $\mathrm{Hom}_\Gamma(L_\bullet,M)$ for any $\Gamma$-module $M$, i.e.

$$H_T^i(\Gamma,M)=H^i(\mathrm{Hom}_\Gamma(L_\bullet,M)).$$

The cohomology in degrees $n\le -2$ agrees with the Tate groups explicitly via isomorphisms

$$P_\bullet\otimes_\Gamma M=(P_\bullet\otimes M)_\Gamma\xrightarrow{N}(P_\bullet\otimes M)^\Gamma\to(\mathrm{Hom}(P_\bullet^*,M))^\Gamma=\mathrm{Hom}_\Gamma(P_\bullet^*,M).$$

A.0.3. **Cup Product.** Let $\{w_\sigma|\sigma\in\Gamma\}$ be a system of representatives of $\Gamma$ in $G$. We define the cocycle $\delta:\Gamma\times\Gamma\to A$ via

$$w_\sigma w_\tau=\delta(\sigma,\tau)w_{\sigma\tau}.$$

Representing $\delta$ using the homogeneous bar resolution yields a cocycle $\alpha\in\mathrm{Hom}_\Gamma(C_2(\Gamma,\mathbb{Z}),A)$ which is uniquely characterized by

$$\alpha(1,c_1,c_1c_2)=\delta(c_1,c_2).$$

Let $[\dot\phi]\in H_2(\Gamma,M)$ and $\dot\phi\in\mathrm{Hom}_\Gamma(C_2^*(\Gamma,\mathbb{Z}),M)$. The cup product $\dot\phi\cup\alpha$ lives in $\mathrm{Hom}_\Gamma(C_0(\Gamma,\mathbb{Z}),M\otimes A)$ and is defined by

$$\dot\phi\cup\alpha(c^*)=\sum_{c_1,c_2}\dot\phi(c^*,c_1^*,c_2^*)\otimes\alpha(c_2,c_1,c)\tag{A.1}$$

$$=\sum_{c_1,c_2}\dot\phi(c^*,c_1^*,(c_1c_2)^*)\otimes\alpha(c_1c_2,c_1,c).\tag{A.2}$$

Its image in $(M \otimes A)_\Gamma$ is

$$\dot{\phi} \cup \alpha(1) = \sum_{c_1, c_2} \dot{\phi}(1, c_1^*, (c_1c_2)^*) \otimes \alpha(c_1c_2, c_1, 1) \tag{A.3}$$

$$= \sum_{c_1, c_2} \phi(c_1, c_2) \otimes c_1c_2 \cdot \alpha(1, c_2^{-1}, c_2^{-1}c_1^{-1}) \tag{A.4}$$

$$= \sum_{c_1, c_2} \phi(c_1, c_2) \otimes c_1c_2 \cdot \delta(c_2^{-1}, c_1^{-1}). \tag{A.5}$$

Notice that

$$\dot{\phi}(1, c_1^*, (c_1c_2)^*) = \phi(c_1, c_2)$$

is the switch from the inhomogeneous bar resolution to the homogeneous one.

A.0.4. **Spectral sequence.** For a $G$-module $M$, we set up the chain complex that produces the Lydon-Hochschild-Serre spectral sequence. Let $P_\bullet = C_\bullet(\Gamma, \mathbb{Z})$ and $Q_\bullet = C_\bullet(G, \mathbb{Z})$. The differential of the two chain complexes are denoted by $d_1$ and $d_0$ respectively. We define the double complex

$$E_{ij} = P_i \otimes Q_j.$$

The complex $\mathrm{Tot}(E_{ij})$ is a $G$-resolution of $\mathbb{Z}$, therefore the spectral sequence associated with $E_{ij} \otimes_G M$ calculates the group homology of $M$. Note that $P_\bullet$ is also a $G$-module. We have

$$E_{ij}^M := E_{ij} \otimes_G M = (P_i \otimes Q_j \otimes M)_G = \left(P_i \otimes_\Gamma (Q_j \otimes_A M)\right).$$

Since $Q_j \otimes_A (-)$ calculates $A$-homology and $P_i \otimes_\Gamma (-)$ calculates $\Gamma$-homology, the resulting spectral sequence of this double complex is the Lydon-Hochschild-Serre spectral sequence.

We need a formula that directly link the $A$-homology calculated by the complex $Q_\bullet$ with that calculated by $C_\bullet(A, \mathbb{Z})$. We choose a projection $f : G \to A$ such that

$$f(w_\gamma) = 1, \qquad \text{for all } \gamma \in \Gamma,$$
$$f(ag) = af(g), \qquad \text{for all } a \in A, g \in G.$$

This induce a map between $A$-complexes $C_\bullet(G, \mathbb{Z}) \to C_\bullet(A, \mathbb{Z})$. Let

$$Z = \ker\left(C_1(G, \mathbb{Z}) \otimes_A M \to C_0(G, \mathbb{Z}) \otimes_A M\right)$$

and consider the composition $Z \to H_1(A, M) \cong A \otimes M$. It sends a cycle

$$(g_1, g_2) \otimes m \mapsto (f(g_2) - f(g_1)) \otimes m. \tag{A.6}$$

A.0.5. **Differentials.** Recall the definition of the map $d_2 : H_2(\Gamma, M) \to H_1(A, M)_\Gamma$. Let $\phi \in E^M_{2,0}$ be a cycle and assume there exists $\psi \in E^M_{1,1}$ such that

$$d_0\,\psi + d_1\,\phi = 0,$$

then

$$d_2([\phi]) = d_1([\psi]).$$

From now on, we will use $c$, $c_i$ or Greek letters $\sigma, \tau$ to denote an element in $\Gamma$, $g, g_i$ or $h$ an element in $G$. We note that in $E^M_{i,j}$, we have

$$\begin{aligned}&(c_0, c_1, \cdots, c_i) \otimes (g_0, \cdots, g_j) \otimes m \\ = &(1, c_0^{-1}c_1, \cdots, c_0^{-1}c_i) \otimes (w_{c_0}^{-1}g_0, \cdots, w_{c_0}^{-1}g_j) \otimes c_0^{-1}m,\end{aligned}$$

and therefore $E^M_{i,j}$ is generated by by elements whose component in the $P_i$ factor starts with $c_0 = 1$. We write all cycles as a linear combination of these elements. For example, for any $\phi \in E^M_{2,0}$, we can write

$$\phi = \sum_{c_1,c_2,g} (1, c_1, c_1c_2) \otimes (g) \otimes \phi(c_1, c_2, g).$$

This expression is in general not unique. However, it is worth noting that in the special case of $\phi \in E^M_{2,0}$, the expression above is unique modulo the identity

$$(1, c_1, c_1c_2) \otimes (g) \otimes \phi(c_1, c_2, g) = (1, c_1, c_1c_2) \otimes (ag) \otimes \phi(c_1, c_2, g)$$

for any $a \in A$. That is to say, let

$$\tilde{\phi}(c_1, c_2, \sigma) = \sum_{\bar{g}=\sigma} \phi(c_1, c_2, g),$$

the following expression of $\phi$ is unique:

$$\phi = \sum_{c_1,c_2,\sigma} (1, c_1, c_1c_2) \otimes (w_\sigma) \otimes \tilde{\phi}(c_1, c_2, \sigma).$$

Similar unique expression exists for elements of $E^M_{1,0}$ with little change.

Now we compute $d_2$. The next three formulae are straightforward.

$$
\begin{aligned}
& d_1\,(1, c_1, c_1c_2) \otimes (g) \otimes m \\
&= (c_1(1, c_2) - (1, c_1c_2) + (1, c_1)) \otimes (g) \otimes m \\
&= (1, c_2) \otimes (w_{c_1}^{-1} g) \otimes c_1^{-1} m - (1, c_1, c_2) \otimes (g) \otimes m + (1, c_1) \otimes (g) \otimes m.
\end{aligned}
$$

$$
\begin{aligned}
& d_0\,(1, c) \otimes (g_0, g_1) \otimes m \\
&= (1, c) \otimes (g_0) \otimes m - (1, c) \otimes (g_1) \otimes m.
\end{aligned}
$$

$$
\begin{aligned}
& d_1\,(1, c) \otimes (g_1, g_2) \otimes m \\
&= (w_c^{-1} g_1, w_c^{-1} g_2) \otimes c^{-1} m - (g_1, g_2) \otimes m \\
&= c^{-1} m \otimes \left( f(w_c^{-1} g_2) - f(w_c^{-1} g_1) \right) - m \otimes (f(g_2) - f(g_1)) \\
&= c^{-1} m \otimes \left( \delta(c^{-1}, g_2) - \delta(c^{-1}, g_1) \right) - m \otimes (f(g_2) - f(g_1)) \,.
\end{aligned}
$$

In the second to last equality we actually send the element to $M \otimes A$ as in (A.6). From this we deduce the boundaries of a cycle as follows. For $\gamma, \sigma \in \Gamma$, we have

$$
\begin{aligned}
d_1\,\tilde{\phi}(\gamma, \sigma) &= \sum_{c_2 = \gamma} c_1^{-1} \tilde{\phi}(c_1, c_2, c_1\sigma) - \sum_{c_1c_2 = \gamma} \tilde{\phi}(c_1, c_2, \sigma) + \sum_{c_1 = \gamma} \tilde{\phi}(c_1, c_2, \sigma) \\
&= \sum_{c} c^{-1} \tilde{\phi}(c, \gamma, c\sigma) - \sum_{c} \tilde{\phi}(c, c^{-1}\gamma, \sigma) + \sum_{c} \tilde{\phi}(\gamma, c, \sigma), \qquad \text{(A.7)}
\end{aligned}
$$

$$
d_0\,\tilde{\psi}(\gamma, \sigma) = \sum_{\overline{h} = \sigma,\, g} \psi(\gamma, h, g) - \sum_{g} \psi(\gamma, g, h). \qquad \text{(A.8)}
$$

$$
\begin{aligned}
d_1\,\psi = & \sum_{c, g_1, g_2} c^{-1} \psi(c, g_1, g_2) \otimes (\delta(c^{-1}, g_2) - \delta(c^{-1}, g_1)) \\
& - \sum_{c, g_1, g_2} \psi(c, g_1, g_2) \otimes (f(g_2) - f(g_1)). \qquad \text{(A.9)}
\end{aligned}
$$

These is enough to determine the map $d_2$.

To conclude, let us assume $d_1\,\phi + d_0\,\psi = 0$. We have

$$
\begin{aligned}
d_2([\phi]) &= d_1\,\psi \\
&= \sum_{\gamma,g_1,g_2} \gamma^{-1}\psi(\gamma,g_1,g_2)\otimes(\delta(\gamma^{-1},g_2)-\delta(\gamma^{-1},g_1)) \\
&= \sum_{\gamma,g_1,g_2} \big(\gamma\cdot\psi(\gamma^{-1},g_2,g_1)-\gamma\cdot\psi(\gamma^{-1},g_1,g_2)\big)\otimes\delta(\gamma,g_1) \\
&= \sum_{\gamma,\sigma} -\gamma\cdot d_0\,\tilde{\psi}(\gamma^{-1},\sigma)\otimes\delta(\gamma,\sigma) \\
&= \sum_{\gamma,\sigma} \gamma\cdot d_1\,\tilde{\phi}(\gamma^{-1},\sigma)\otimes\delta(\gamma,\sigma) \\
&= \sum_{\gamma,\sigma,c} \big(c^{-1}\tilde{\phi}(c,\gamma^{-1},c\sigma)-\tilde{\phi}(c,c^{-1}\gamma^{-1},\sigma)+\tilde{\phi}(\gamma^{-1},c,\sigma)\big)\otimes\delta(\gamma,\sigma) \\
&= \sum_{c_1,c_2,\sigma} \tilde{\phi}(c_1,c_2,\sigma)\otimes\big(c_1c_2\cdot\delta(c_2^{-1},c_1^{-1}\sigma)-c_1c_2\cdot\delta(c_2^{-1}c_1^{-1},\sigma)+c_1\delta(c_1^{-1},\sigma)\big) \\
&= \sum_{c_1,c_2,\sigma} \tilde{\phi}(c_1,c_2,\sigma)\otimes c_1c_2\cdot\delta(c_2^{-1},c_1^{-1}) \\
&= [\phi]\cup\alpha. \qquad \square
\end{aligned}
$$

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