ANGULAR DISTRIBUTIONS IN THE ELASTIC SCATTERING OF PROTONS BY LIGHT NUCLEI; ANALYSIS OF THE REACTIONS $Li^7(pp)Li^7$ and $Be^9(pp)Be^9$

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ABSTRACT

Use has been made of the Clebsch-Gordon series to evaluate the matrix elements involved in the one-level dispersion model for the elastic scattering of protons by light nuclei. The results have been used to describe the resonances at 440 kev and 1024 kev in the scattering of protons by lithium and the resonances at 988 and 1077 kev in beryllium. The 440 kev lithium resonance seems to be definitely due to incident p-waves forming a compound state of Be^8 with $J = 1$. The 1024 kev lithium resonance is tentatively identified as due to s-wave forming a state with $J = 1$, but more complete experimental data is necessary before a definite assignment can be made. The 988 kev beryllium resonance is ascribed to a level with $J = 2$ in B^{10} formed by incident s-wave protons and the observed γ-radiation is therefore electric dipole. The narrow resonance at 1077 has been ascribed to incident d-wave and $J = 0$.

The agreement of these assignments with other experimental data is discussed.

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I. INTRODUCTION

The scattering of protons by atomic nuclei has long been a source of information concerning the structure of the nucleus. The earliest scattering experiments were of course performed with natural alpha particle sources, but since the development of accelerators - especially the high precision electrostatic generators - proton scattering has become increasingly important.

The interest in proton scattering centers about the so called "nuclear resonance scattering" in which the proton, for certain more or less well-defined bombarding energies, is considered to enter the target nucleus and form a compound nucleus which then decomposes and re-emits the original target particle and proton. The observed elastic scattering (i.e. that for which the incident proton reappears with the same* energy) is made up of two parts. One part, the Coulomb or Rutherford scattering is of itself uninteresting since it is more or less completely understood and is described as the result of the repulsion of the classical electrostatic fields of the charges $+e$ and $+Ze$ on the proton and target nucleus. The other part is ascribed to various effect (which are not necessarily distinct) such as non-classical potentials (i.e. the break down at separations of less that 10^{-12} cm. of Coulomb's law), "nuclear scattering" in which the proton "bounces off the nucleus"

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* In the center-of-mass coordinate system.

(to give a crude classical picture), and "nuclear resonance scattering" mentioned above. As a matter of description it is perhaps easiest to say that the coulomb scattering is completely described by the well known Rutherford formula[1] and any variation from the predictions of this formula are to be ascribed to the specifically nuclear effects.

A convenient form for displaying the effects of the anomalous scattering is to give the ratio of observed scattering cross-section to the Rutherford cross-section as a function of the angle of scattering and energy of the bombarding protons. In this thesis we shall calculate this function on the basis that the scattering is due primarily to coulomb scattering and a single nuclear resonance level for various assumption as to the angular momentum of the incident protons and the compound nucleus involved in the resonance. The effects of other resonances are either neglected (as in section III) or treated as a source of general non-resonant scattering which interfers with the resonance and coulomb scattering (as in section IV). The cross-sections thus calculated are compared with the experimental data obtained in the Kellogg Radiation Laboratory (2,3,4,5) and an attempt is thus made to determine the angular momentum of the compound nucleus associated with each of the resonances.

II. THE MECHANICS OF THE PROBLEM

1. General Description of the Calculation of Cross Sections

The "cross section", σ, for a nuclear process is defined as the probability that, with an incident intensity of I bombarding particles per unit area and per unit time, the specified process will take place at the rate $I\sigma$ per unit time and per target nucleus. We may consider either a total cross section or partial cross sections, the partial cross section being essentially the probability that a more restricted type of process occur; the total cross section for the process is the sum of the partial cross sections for each of the independent subprocesses by which the less restricted reaction occurs. An ultimate example of this and the one in which we shall be particularly interested is the differential cross section for a nuclear disintegration or scattering process. The total process is the incidence of a specified particle and the subsequent emission of some other (or possibly the same) particle. The total cross section does not specify the direction in which the emitted particle is moving relative to the direction of the incident particle. The differential cross section gives the probability that the emitted particle shall have a direction of motion included in a solid angle $d\Omega$ with coordinates Θ, ϕ relative to the incident particle's direction. The total cross section is thus the integral over all angles of the differential cross section. The latter may itself be decomposed into component parts. Thus the incident particle and the target

nucleus may initially be in any one of several quantum states and for each configuration of target and incident particle states we can specify a cross section; the complete cross section is then the sum of these sub-cross sections each weighted by the probability that that configuration shall occur. There may also be more than one mechanism whereby the process under consideration can take place as is the case in the description of the scattering of protons by nuclei where the scattering by the coulomb field and the scattering produced by the existence of the compound nucleus interfere with each other.

As indicated above, the complete description of the scattering process involves the break-down of the process into its fundamental components and the subsequent reassembly of the partial cross sections for each component into the complete cross section. Specifically, in the scattering of protons by a target nucleus the angular momenta of the two particles represent a specification of the quantum states of these particles. The scattering process involves a proton with spin (total angular momentum of the proton) of $\frac{1}{2}\hbar$ and a nucleus with total angular momentum $j\hbar$ coming together with a relative orbital angular momentum $l\hbar$ and then coming apart again. The possible quantum states of the system are specified by the components of angular momentum along a given direction. We, therefore, have a proton with total angular momentum (spin) $s = \frac{1}{2}$ and component $m_s = \pm\frac{1}{2}$ and a target nucleus with total angular momentum j and component m_j' coming together with a relative orbital angular momentum l and component m' to form a compound

nucleus with total angular momentum J_c and component M. (Fig. 1). This compound nucleus then disintegrates into the original particles and since we are considering elastic scattering the particles have the same angular momentum as before. The final state is therefore specified by the quantum numbers $s, m_s ; j, m_j ; l, m$ *

We can therefore write a matrix element $f^{m_j' m_s' m'}_{m_j m_s m}$ connecting the initial and final states. The sum of the absolute square of these matrix elements gives the cross section. However, it is more convenient to separate the coulomb scattering matrix element from the nuclear matrix element and write

$$f^{m_j' m_s' m'}_{m_j m_s m} = \sigma_o^{1/2} \delta^{m_j'}_{m_j} \delta^{m_s'}_{m_s} + \mathbf{M}^{m_j' m_s' m'}_{m_j m_s m}$$

in which σ_o is the coulomb (or Rutherford) cross section and M is the specifically nuclear scattering matrix element. If all the M's vanish we then obtain pure coulomb scattering so

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* We should also allow l to be different after the collision than what it was before and this possibility must not be overlooked. However, for low bombarding energies we should expect only one value of l to be important; that one indeed which is the smallest value consistent with other selection rules. The compound nucleus will then be unable to emit particles with $l < l'$ while the probability of absorbing or emitting particles with $l > l'$ would be expected to fall off at least as fast as $(R/\chi)^{2(l-l')}$ where R = nuclear radius, $2\pi\chi$ = wave-length of the proton. For 0.5 Mev protons $\chi = 6.5 \times 10^{-13}$ cm while the nuclear radius for Li is $R = 2.7 \times 10^{-13}$cm. hence $(R/\chi)^{2l} = (0.15)^l$. Since the parity selection rule requires that the only values of l be $l', l'+2, l'+4, \ldots$ we see that the only important value is $l = l'$. For example, from the curves of Christy and Latter[6] we find that for the penetration of 0.5 Mev protons thru the coulomb barrier for lithium, the ratio of d-wave to s-wave is 0.005. The more complete quantitative discussion to be given below will include the possibility of different values of l.

that the M's contribute the anomalous scattering. The δ's are the Kronecker delta and indicate that in coulomb scattering the components of angular momentum of the particles are unaltered. The total cross section can then be written as

$$\sigma = \frac{1}{(2j+1)(2s+1)} \sum_{m_s m_j m_s' m_j'} \left| f^{m' m_j' m_s'}_{m\, m_j\, m_s} \right|^2$$

$$= \frac{1}{(2j+1)(2s+1)} \sum \left| \sigma_o^{1/2} \delta^{m_s'}_{m_s} \delta^{m_j'}_{m_j} + M^{m' m_j' m_s'}_{m\, m_j\, m_s} \right|^2 \qquad (2.01)$$

the summations being extended over all possible values of the m's, while the f's themselves may contain implicitly a summation over l. The cross section formula indicates that each configuration of the incident particles is an independent and equally probable event, therefore, since there are $2j+1$ different possible values of m_j and $2s+1$ different possible values of m_s, the a priori probability of each configuration is $1/(2j+1)(2s+1)$ justifying the factor in front of the summation. If the M's all vanish we have simply $\sigma = \sigma_o$. There are, of course, selection rules which limit the number of M's which differ from zero. We have the conservation law for component of angular momentum

$$M = m_s + m_j + m = m_s' + m_j' + m' \qquad (2.02)$$

and the "closed polygon" relationships of the vector model

$$J_c \leq j + s + l \qquad (2.03a)$$

$$J_c \geq \text{Minimum } (|j+s-l|, |j+l-s|, |l+s-j|) \qquad (2.03b)$$

These selection rules however need not be explicitly intro-

duced but arise naturally from the group-theoretical arguments used in the evaluation of the matrix elements f.

Since we are interested in resonance scattering from a more or less sharply defined level in the compound nucleus we shall carry out the evaluation of the f's in terms of the so called "one-level dispersion formula".

2. The Dispersion Formula

The dispersion formula (so-called because of the similarity in form to the well known optical dispersion formula) was first developed by Breit and Wigner[7], following the qualitative description of Bohr[8], in connection with the slow neutron capture process. Their calculation was based on second order perturbation theory and was extended to include many levels of the compound nucleus by Bethe[9] and by Bethe and Placzek[10]. Kalckar, Oppenheimer and Serber[11] also treat the dispersion formula on the basis of perturbation theory for both the case of single levels and of close, overlapping levels, and indicate that because of interference effects the cross section for the latter case is not a simple integration over single level cross sections. Objection may be validly raised to the use of ordinary perturbation treatment since the nuclear potentials can hardly be considered as contributing a small perturbation on the Hamiltonian. Kapur and Peierls[12] have recognized this objection and have derived essentially the same formula by considering perturbation on the boundary conditions without specifically consi-

dering the potentials necessary to do this. Their argument considers the unperturbed state as one for which no particles are emitted and the perturbation then changes the wave function so as to give a flux of reaction particles. Since the yields (cross sections) are small the perturbation calculation is justified.

The dispersion formula has also been derived on the basis of the S-Matrix[13,14,15] by several authors [16,17,18,19], a method which has been useful in providing a description of the cross section in the region between resonances, where the one-level formula must be abandoned in favor of the many-level formula.

The phenomenon of resonance is essentially a result of the limited range of nuclear forces and the closely bound structure of the nucleus[8] and the description is inadequate to handle the case of coulomb scattering which does not show resonances and must be separately calculated. It is important however to determine the exact phase relations between coulomb and resonance scattering matrix elements since the interference of the two must be specified. A derivation of the one-level dispersion formula is given in Appendix I. We shall here indicate how the results there given must be modified in the presence of the coulomb field.

The wave function of a particle in a coulomb field is[1]

$$\psi_c = \frac{1}{v^{1/2}}\left[\exp\{ikz + i\alpha \ln k(r-z)\} + \frac{xZe^2}{2Mv^2 \sin^2\frac{1}{2}\theta \cdot r}\exp\{ikr - i\alpha \ln 2kr - i\alpha \ln \sin^2\tfrac{1}{2}\theta + i\pi + 2i\eta_0\}\right] \tag{2.04}$$

where M is the reduced mass,

$$\alpha = z\mathcal{Z}e^2/\hbar v$$
$$k = Mv/\hbar = (2ME)^{1/2}/\hbar \qquad (2.05)$$
$$e^{2i\eta_0} = \Gamma(1+i\alpha)/\Gamma(1-i\alpha)$$

and the wave function is normalized to unit incident current. The first term represents the incident plane wave (somewhat distorted even at infinity by the long-range field) and the second term the scattered wave. Equation (2.04) is a convenient form if one is interested only in coulomb scattering. Since the coulomb field will distort the wave front at infinity it cannot be neglected in analyzing the resonance scattering, and we must use coulomb wave functions and write the scattered wave in a form in which the waves of different angular momenta are separated. Doing this for a pure coulomb potential (i.e. expanding (2.04) in spherical harmonics) we have

$$\psi_c = \frac{1}{r}\sum_l A_l f_l(r) Y_l^0(\theta,\phi) \qquad (2.06)$$

$Y_l^m(\theta,\phi)$ = normalized spherical harmonic, polar axis along the direction of propagation of the incident beam.

$$A_l = \sqrt{\frac{4\pi(2l+1)}{v}}\,\frac{1}{k}\exp\left\{i\eta_l + l\frac{\pi}{2}\right\} \qquad (2.07)$$

$f_l(r)$ is the radial partial wave function of angular momentum l and has the asymptotic form for large r

$$f_l(r) \sim \sin\left(kr + \eta_l - \alpha \ln 2kr - l\frac{\pi}{2}\right)$$

$$e^{2i\eta_l} = \Gamma(1+l+i\alpha)/\Gamma(1+l-i\alpha)$$

The wave scattered by the nucleus also may be expanded in coulomb wave functions and we have (compare appendix I, Equa-

tion (A1.16) et seq.)

$$\psi_N = \frac{1}{r}\sum_{lm} \beta_{plm}\,(g_l + i f_l)\,Y_l^m(\theta,\phi) \qquad (2.08)$$

where here f_l is the regular solution in the coulomb potential and g_l is the irregular solution. We have the asymptotic behavior (Compare A1.18)

$$g_l(r) + i f_l(r) \sim \exp i\{kr - \alpha \ln 2kr - l\tfrac{\pi}{2} + \eta_l\} \qquad (2.09)$$

The phase shift, $\eta_l - \alpha \ln 2kr$ of the wave with respect to a plane wave (Compare (A1.26)) may have a further phase shift, δ_l produced by short range nuclear potentials but these we neglect here.

The coefficients β are given by (Cf. (A1.23,34,37)

$$\beta^{m_j' m_s'}_{Plm m_j m_s} = -\pi \sum_{l'} \frac{A_{l'}\, \mathbf{H}^{CM}_{Pl'o m_j' m_s'}\, \mathbf{H}^{CM\,*}_{Plm m_j m_s}}{E - E_o + \frac{1}{2} i\gamma} \qquad (2.10)$$

where l, m are the orbital angular momentum and component of the scattered partial wave.

$m_j' m_s'$ are the components of spin of the target nucleus and incident particle before collision.

$m_j m_s$ are the same after the collision.

M is tne component of angular momentum of the compound nucleus.

E kinetic energy of the incident particle.

E_o resonance energy.

The spin quantum numbers have been included to indicate that we have a strict selection rule

$$H^{CM}_{Plmm_jm_s} = 0 \quad \text{unless } M = m + m_j + m_s \qquad (2.11)$$

The total wave function is made up of the incident beam, the coulomb scattered beam and the nuclear scattered beam distorted by the coulomb field. Therefore, adding (2.04) and (2.08) and making use of (2.07), (2.09), (2.10) we have for the partial wave $\psi^{m_j' m_s'}_{m_j m_s}$

$$\begin{aligned}
\psi^{m_j' m_s'}_{m_j m_s} &= \frac{1}{v^{1/2}} \exp i\{kz + \alpha \ln k(r-z)\}\, \delta^{m_j'}_{m_j} \delta^{m_s'}_{m_s} \\
&\quad - \frac{1}{r v^{1/2}} \Bigg[\frac{zZe^2}{2Mv^2 \sin^2\frac{1}{2}\theta} \exp i\{kr - \alpha \ln 2kr - \alpha \ln \sin^2\tfrac{1}{2}\theta + 2\eta_0\}\, \delta^{m_j'}_{m_j} \delta^{m_s'}_{m_s} \\
&\qquad + \frac{2\pi^{3/2}}{k} \sum_{lm} Y^m_l(\theta,\phi) \sum_{l'} \frac{H^{CM}_{Pl'0m_j'm_s'} H^{CM*}_{Plmm_jm_s}}{E - E_0 + \frac{1}{2} i\gamma} \\
&\qquad \times (2l'+1)^{1/2} \exp i\{kr - \alpha \ln 2kr + \eta_l + \eta_{l'} - \tfrac{\pi}{2}(l-l')\} \Bigg] \\
&= \frac{1}{v^{1/2}} \exp i\{kz + \alpha \ln k(r-z)\}\, \delta^{m_j'}_{m_j} \delta^{m_s'}_{m_s} \\
&\quad - \frac{1}{r v^{1/2}} \exp i\{kr - \alpha \ln 2kr + 2\eta_0\}\, f^{m_j' m_s'}_{m_j m_s}
\end{aligned} \qquad (2.12)$$

where

$$\begin{aligned}
f^{m_j' m_s'}_{m_j m_s} &= \frac{zZe^2}{2Mv^2 \sin^2\frac{1}{2}\theta} \exp\{-i\alpha \ln \sin^2\tfrac{1}{2}\theta\}\, \delta^{m_j'}_{m_j} \delta^{m_s'}_{m_s} \\
&\quad + 2\pi^{3/2} \chi \sum_{lm} Y^m_l(\theta,\phi) \sum_{l'} \frac{H^{CM}_{Pl'0m_j'm_s'} H^{CM*}_{Plmm_jm_s}}{E - E_0 + \frac{1}{2} i\gamma} \\
&\quad \times (2l'+1)^{1/2}\, i^{(l'-l)} \exp i(\eta_l + \eta_{l'} - 2\eta_0)
\end{aligned} \qquad (2.13)$$

$2\pi\lambda\!\!\!^{-}$ = wave length of the incident particles = $2\pi/k$

The cross section is the sum of the partial cross section for all the possible break ups of any given initial configuration averaged over the initial configurations. The cross section is therefore the absolute square of $f^{m_j' m_s'}_{m_j m_s}$ averaged over the possible values of m_j' and m_s'. Since there are $(2j+1)$ of the former and $(2s+1)$ of the latter we obtain

$$\sigma(\theta,\phi)\,d\omega = \frac{d\omega}{(2j+1)(2s+1)} \sum_{\substack{m_j' m_s' \\ m_j m_s}} \left| f^{m_j' m_s'}_{m_j m_s} \right|^2 \qquad (2.14)$$

3. The Clebsch-Gordon Series and the Evaluation of Matrix Elements.[20,21,22]

In order to calculate cross sections we must still evaluate the matrix elements $H^{CM}_{Plmm_jm_s}$. These are given by the integral

$$H^{CM}_{Plmm_jm_s} = \int \chi^*_{CJM}\, V_{AP}\, \chi_{Ajm_j}\, \chi_{Psm_s}\, F_{Plm}\, d\tau_c \qquad (2.15)$$

This is similar to equation (A1.24) except that we here specify the angular momentum quantum numbers of the nuclei and have normalized the wave functions per unit energy. We shall use group theory methods to determine the relative values of the matrix elements, $H^{CM}_{Plmm_jm_s}$ which will allow us to express these elements in terms of the experimentally determined half width of the resonance since we also have

$$\gamma_p = 2\pi \sum_{lmm_j m_s} \left| \mathbf{H}^{CM}_{plmm_j m_s} \right|^2 \qquad (2.16)$$

γ_p being the "proton half width".

We need not know the exact form of the functions χ or F but only their transformation properties. If the interaction potential V_{AP} is a scalar, then $\mathbf{H}^{CM}_{plmm_j m_s}$ has the same transformation properties as

$$\mathbf{M}^{JMl}_{mm_j m_s} = \int \chi^*_{cJM}\, \chi_{Ajm_j}\, \chi_{psm_s}\, F_{plm}\, d\tau_c \qquad (2.17)$$

that is, we can write

$$\mathbf{H}^{CM}_{plmm_j m_s} = h^c_{pl}\, \mathbf{M}^{JMl}_{mm_j m_s} \qquad (2.15a)$$

where h^c_{pl} is independent of the components $m m_j m_s$.
Now the functions $f_{JM} = \chi_{cJM}$ and $g_{mm_j m_s} = \chi_{Ajm_j}\, \chi_{psm_s}\, F_{plm}$ form two complete sets of functions and (17) is just the expansion coefficients of f_{JM} in terms of the $g_{mm_j m_s}$.

$$\begin{aligned} \gamma_p &= 2\pi \sum_l \sum_{mm_j m_s} \left| \mathbf{H}^{CM}_{plmm_j m_s} \right|^2 \\ &= 2\pi \sum_l \left| h^c_{pl} \right|^2 \sum_{mm_j m_s} \left| \mathbf{M}^{JMl}_{mm_j m_s} \right|^2 \end{aligned} \qquad (2.18)$$

We can then express the full proton width as a sum of partial widths for different angular momenta of the emitted proton

$$\gamma_p = \sum_l \gamma_{pl} \qquad h^c_{pl} = \sqrt{\frac{\gamma_{pl}}{2\pi \sum |\mathbf{M}|^2}} \qquad (2.19)$$

The coefficients $\mathbf{M}^{JMl}_{mm_j m_s}$ can be easily evaluated by making use of tne expansion coefficients derived in Appendix II.

The evaluation of $\mathbf{M}^{JMl}_{mm_j m_s}$ falls into two categories depending on the type of nuclear interaction; these have very roughly

been called LS and JJ in partial analogy to the atomic spectra notation. The LS case is one in which the spin angular momentum and orbital angular momentum are uncoupled. Physically this means that angular momentum of each kind must be separately conserved (the total angular momentum must be conserved in all cases); there are no operators in $V_{\lambda\rho}$ which couple the orbital momentum and the spin momentum. Mathematically this shows itself in the separation of the wave function into products of space and spin functions so that the integral (17) can be written as a sum of products of an integral over the coordinates of the space variables and an integral over the coordinates of the spin variables. In the case of "J J coupling" the orbital and spin momenta are not separately conserved but may be converted one into the other, only the total angular momentum J, is conserved. The integral (17) cannot, in this case, be written as the product of two integrals.

A. "L S" description

We must write out $\chi_{A j m_j} \chi_{P s m_s} F_{P l m}$ in terms of the orthogonal set χ_{CJM}. It is assumed that we know the spectroscopic notation of $\chi_{A j m_j}$; and that the total momentum j is made up of orbital momentum λ and spin momentum σ (For example χ_A may be denoted as $^2P_{3/2}$ in which case $j=3/2, \sigma=\frac{1}{2}, \lambda=1$). Similarly, the total momentum J of χ_C is assumed to be made up of a specified L and S. then

$$\chi_{A j m_j} \sim \Psi_j^{m_j} \sim \sum_{m_\sigma m_\lambda} A_{j m_j}^{\lambda \sigma m_\sigma m_\lambda} \psi_\lambda^{m_\lambda} \phi_\sigma^{m_\sigma} \qquad (2.20a)$$

In this and what follows we shall use Ψ to denote a wave function combining space and spin; ψ , a function of space only and ϕ a function of spin only. The symbol $\backsim$ is used rather than equality to indicate that only the transformation properties of the wave functions are important; we use ψ and ϕ for the space and spin parts of any wave function - - this however does not imply that the different functions are similar except in their relationship to the rotation group.

The incident particle we shall specify to be a proton and hence it has only spin, therefore we write

$$\chi_{P s m_s} \backsim \Psi_s^{m_s} \backsim \phi_s^{m_s} \tag{2.20b}$$

The relative motion wave function is of course pure orbital,

$$F_{Plm} \backsim \Psi_l^{m} \backsim \psi_l^{m} \tag{2.20c}$$

For the compound nucleus we write

$$\chi_{CJM} \backsim \Psi_J^{M} \backsim \sum_{\mu\nu} A_{JM}^{LS\mu\nu} \psi_L^{\mu} \phi_S^{\nu} \tag{2.20d}$$

then we have

$$\chi_{A_j m_j} \chi_{P s m_s} F_{Plm} \backsim \sum_{m_\sigma m_\lambda} A_{j m_j}^{\lambda\sigma m_\lambda m_\sigma} \psi_\lambda^{m_\lambda} \phi_\sigma^{m_\sigma} \phi_s^{m_s} \psi_l^{m}$$

$$\backsim \sum_{m_\sigma m_\lambda} A_{j m_j}^{\lambda\sigma m_\lambda m_\sigma} \sum_{\rho} A_{\rho, m_\lambda + m}^{\lambda l m_\lambda m} \psi_\rho^{m_\lambda + m} \sum_{\tau} A_{\tau, m_\sigma + m_s}^{\sigma s m_\sigma m_s} \phi_\tau^{m_\sigma + m_s}$$

(2.21)

combining this with (2.20d) we have

$$\mathbf{M}^{JMl}_{mm_jm_s} = \sum_{m_\sigma m_\lambda} A^{\lambda\sigma m_\lambda m_\sigma}_{jm_j} A^{\lambda l m_\lambda m}_{L,m_\lambda+m} A^{\sigma s m_\sigma m_s}_{S,m_\sigma+m_s} A^{LS,m_\lambda+m,m_\sigma+m_s}_{JM} \qquad (2.22)$$

B. "J J" description

Here we specify only the total angular momentum which gives us a more general description which includes the previous one as a special case. We write

$$\chi_{Ajm_j}\chi_{Psm_s} \sim \Psi_i^{m_j}\Psi_s^{m_s} \sim \sum_{Tm_T} A^{jsm_jm_s}_{Tm_T}\Psi_T^{m_T} \qquad (2.23)$$

and the coefficients h^c_{Pl} must now be recognized as depending on the quantum number T (In the LS description h^c_{Pl} would have been considered as depending on the LS assignment of the compound nucleus if we had wished to have a compound nucleus which was a linear combination of LS states.) The most convenient way to introduce the dependence is to write

$$h^c_{PlT} = \alpha_T h^c_{Pl} \qquad \gamma_{PlT} = 2\pi |h^c_{PlT}|^2 \sum_{mm_jm_s} |\mathbf{M}^{JMl}_{mm_jm_s}|^2 \qquad (2.24)$$

$$\gamma_{Pl} = \sum_T \gamma_{PlT}$$

then

$$\mathbf{H}^{JM}_{Plmm_jm_c} = \int \chi^*_{CJM} V_{AP} \chi_{Ajm_j}\chi_{Psm_s} F_{Plm}\, d\tau$$

$$= \sum_{Tm_T} A^{jsm_jm_s}_{Tm_T} h^c_{PlT} \int \chi^*_{CJM}\chi_{Tm_T} F_{Plm}\, d\tau \qquad (2.24)$$

$$= h^c_{Pl} \sum_{Tm_T} \alpha_T A^{jsm_jm_s}_{Tm_T} A^{Tlm_Tm}_{JM}$$

so that from (2.15a) we have

$$\mathbf{M}^{JMl}_{m m_l m_s} = \sum_{T m_T} \alpha_T A^{l s m_l m_s}_{T m_T} A^{T l m_T m}_{JM} \tag{2.25}$$

We then find (see Appendix III)

$$\sum_{m m_l m_s} |\mathbf{M}^{JMl}_{m m_l m_s}|^2 = \sum \alpha_T^2 \tag{2.26}$$

From (2.19)

$$\gamma_{Pl} = 2\pi |h^c_{Pl}|^2 \sum_{m m_l m_s} |\mathbf{M}|^2 \tag{2.19a}$$

While from (2.24)

$$\gamma_{Pl} = \sum_T 2\pi \alpha_T^2 |h^c_{Pl}|^2 \sum_{m m_l m_s} |\mathbf{M}|^2 \tag{2.24a}$$

Comparing these two expressions we see that in the JJ description $\sum_T \alpha_T^2 = 1$ and therefore

$$\sum_{m m_l m_s} |\mathbf{M}^{JMl}_{m m_l m_s}|^2 = 1 \tag{2.26a}$$

We can now write the cross section (2.14) in the form in which it will be most convenient to use by considering the ratio of the cross section to the pure coulomb cross section. We therefore write

$$\sigma_o(\theta,\phi) = \left(\frac{z Z e^2}{2 M v^2 \sin^2 \frac{1}{2}\theta}\right)^2$$

and then in place of (2.13) and (2.14)

$$\frac{\sigma}{\sigma_0} = \frac{1}{(2s+1)(2j+1)} \sum_{\substack{m_j m_s \\ m_j' m_s'}} \left| f_{0\, m_j m_s}^{m_j' m_s'} \right|^2 \tag{2.14a}$$

$$f_{0\, m_j m_s}^{m_j' m_s'} = \delta_{m_j}^{m_j'} \delta_{m_s}^{m_s'} + \frac{2\pi^{3/2} \chi}{\sigma_0^{1/2}} \sum_{lm} Y_l^m(\theta,\phi) \sum_{l'} \frac{(2l'+1)^{1/2} e^{i\Theta} \mathbf{H}^{CM}_{Pl'0m_j'm_s'} \mathbf{H}^{CM*}_{Plmm_jm_s}}{(E-E_0+\frac{1}{2}i\gamma)}$$

$$\Theta = \eta_l + \eta_{l'} - 2\eta_0 + \alpha \ln \sin^2\tfrac{1}{2}\theta - (l-l')\frac{\pi}{2} \tag{2.13a}$$

Using (2.15a), (2.19) and introducing

$$\chi = \frac{\hbar}{Mv}$$

$$\alpha = \frac{z\mathcal{Z}e^2}{\hbar v} = \frac{z\mathcal{Z}}{\text{"}137\text{"}\beta} \qquad \beta = v/c$$

we have

$$f_{0\, m_j m_s}^{m_j' m_s'} = \delta_{m_j}^{m_j'} \delta_{m_s}^{m_s'} + \frac{2\pi^{1/2} \sin^2\frac{1}{2}\theta}{\alpha} \sum_{lm} Y_l^m(\theta,\phi) \sum_{l'} \frac{(2l'+1)^{1/2} e^{i\Theta} \mathbf{M}^{JMl'}_{0m_j'm_s'} \mathbf{M}^{JMl}_{mm_jm_s} \sqrt{\gamma_{Pl}\gamma_{Pl'}}}{\sqrt{\Sigma|\mathbf{M}|^2}\sqrt{\Sigma|\mathbf{M}'|^2}\,(E-E_0+\frac{1}{2}i\gamma)} \tag{2.13b}$$

In applying these formulae to specific problems we must make some simplifying assumption in order to reduce the problem to one whose complexity is consistent with the accuracy of the experimental data. We shall assume that only one angular momentum of the incident plane wave produces the observed resonance and that therefore

$$\gamma_{Pl'} = \gamma_P \delta_l^{l'} \tag{2.27}$$

this reduces (2.13b) to the form

$$f_{o\,m_j m_s}^{m_j' m_s'} = \delta_{m_j}^{m_j'}\delta_{m_s}^{m_s'} + \frac{4\pi^{1/2}\sin^2\frac{1}{2}\theta\,(2l+1)^{1/2}e^{i\Theta_l}}{\alpha\sum_{m\mu\nu}|\mathbf{M}_{m\mu\nu}^{JMl}|^2(\chi+i)}\,\mathbf{M}_{o\,m_j' m_s'}^{JMl}\mathbf{M}_{m\,m_j m_s}^{JMl}Y_l^m(\theta,\phi) \qquad (2.13c)$$

$$m + m_j + m_s = m_j' + m_s' = M$$

where $\Theta_l = 2(\eta_l - \eta_o) + \alpha \ln \sin^2\frac{1}{2}\theta$

$$\alpha = \frac{zZ}{137\beta} \qquad \beta = \sqrt{\frac{2E}{Mc^2}}$$

$$\chi = (E-E_o)/\tfrac{1}{2}\gamma \qquad (2.28)$$

$$e^{2i\eta_l} = \frac{\Gamma(l+1+i\alpha)}{\Gamma(l+1-i\alpha)} = \frac{(l+i\alpha)(l-1+i\alpha)\cdots(1+i\alpha)}{(l-i\alpha)(l-1-i\alpha)\cdots(1-i\alpha)}\,e^{2i\eta_o}$$

From this last expression we can write

$$2(\eta_l - \eta_o) = 2\sum_{n=1}^{l}\tan^{-1}\frac{\alpha}{n} \qquad (2.28a)$$

and $$\Theta_l = 2\sum_{n=1}^{l}\tan^{-1}\frac{\alpha}{n} + \alpha\ln\sin^2\tfrac{1}{2}\theta$$

We can then write the cross section in the form

$$\frac{\sigma}{\sigma_o} = \frac{1}{(2J+1)(2S+1)}\sum_{\substack{m_j m_s\\ m_j' m_s'}}\left|f_{o\,m_j m_s}^{m_j' m_s'}\right|^2$$

$$= 1 + \frac{8\pi^{1/2}(2l+1)^{1/2}\sin^2\frac{1}{2}\theta}{(2J+1)(2S+1)\alpha\sum_{m\mu\nu}|\mathbf{M}_{m\mu\nu}^{JMl}|^2}\,\mathrm{Re}\left\{\frac{e^{i\Theta_l}}{\chi+i}\right\}\sum_{M m_j m_s}|\mathbf{M}_{o\,m_j m_s}^{JMl}|^2 Y_l^o(\theta,\phi)$$

$$+ \frac{16\pi(2l+1)\sin^4\frac{1}{2}\theta}{(2J+1)(2S+1)\alpha^2(1+\chi^2)\left(\sum|\mathbf{M}_{m\mu\nu}^{JMl}|^2\right)^2}\sum_{\substack{M m_j m_s\\ m_j' m_s'}}\left|\mathbf{M}_{o\,m_j' m_s'}^{JMl}\mathbf{M}_{m\,m_j m_s}^{JMl}Y_l^m(\theta,\phi)\right|^2 \qquad (2.29)$$

Using the values of the sums found in Appendix III we obtain our final form for the cross-section

$$\frac{\sigma}{\sigma_o} = 1+\frac{(2l+1)}{(2J+1)(2S+1)}\,\frac{4\pi\chi^2}{\sigma_o}\cdot\frac{\sum_m |Y_l^m(\theta,\phi)|^2 \sum_{MTR}\alpha_T^2\alpha_R^2 \left|A_{JM}^{TlMo}A_{JM}^{Rl,M-m,m}\right|^2}{(1+x^2)}$$

$$+ 2\,Re\left\{\frac{e^{i\Theta_l}}{x+i}\right\}\frac{2J+1}{(2J+1)(2S+1)}\,\frac{\chi}{\sigma_o^{1/2}}\,P_l(\cos\theta) \qquad (2.30$$

where $P_l(\cos\theta)$ is the ordinary Legendre polynomial in cos Θ which has been introduced in place of $Y_l^m(\theta,\phi)$ merely for reasons of convenience.

We can infer from the form of (2.30) compared to (2.29) that we can introduce the matrix elements $f_{o\,T m_T}^{T' m_T'}$ in place of $f_{o\,m_j m_s}^{m_j' m_s'}$ in which the matrix elements $\mathbf{M}_{m_T m_T}^{JMl}$ are given by

$$\mathbf{M}_{m_T m_T}^{JMl} = \alpha_T\, A_{JM}^{Tl m_T m} \qquad (2.31)$$

This amounts to describing the possible states of the initial or final system by the quantum numbers $T m_T$ instead of the numbers $m_j m_s$; it is a direct result of the possibility of writing, for the JJ case, $\Psi_j^{m_j}\Psi_s^{m_s}$ as a linear combination of $\Psi_T^{m_T}$ (Eq. (2.23)). Using the quantum numbers $m_j m_s$ we have a set of $(2j+1)(2s+1)$ functions out of which to form our system; using the quantum numbers T, m_T we obtain a set of $\sum_{T=|j-s|}^{j+s}(2T+1) = (2j+1)(2s+1)$ functions. That each set must contain the same number of functions is a necessary result of the fact that each is a complete orthonormal system of the same function space.

III. APPLICATIONS TO THE REACTION Li^7 (p_1p) Li^7

The foregoing theory shall now be applied to the elastic scattering of protons by lithium at the well known γ-ray resonance at 440 kev. It is known that the angular momentum of Li^7 in the ground state is 3/2 and the usual spectroscopic assignment is $^2P_{3/2}$. The angular momentum of the proton is 1/2. We consider as possible only incident proton waves with relative angular momentum $l = 0,1$.

The existence of the γ-ray and the non-existence of alpha particles at this resonance give us important information concerning the possible assignments of the compound nucleus Be^{8*}. The parity of Li^7 in the ground state we shall assume to be odd*; the proton of itself is even, while the relative angular momentum contributes a parity of $(-1)^l$. The parity of the compound nucleus is therefore even for $l = 1$ and odd for $l = 0$. Now the alpha particle has even parity and because the two alpha particles are identical they can come out of the compound nucleus only with even relative angular momentum ($l = 0,2,...$). Then, since the alpha particles themselves are $^1S_0$, this means that we have the reaction

$$Li^7 + p \rightarrow Be^{8*} \rightarrow He^4 + He^4$$

---

* The parity assignment has not been experimentally verified but the usual models of the nucleus indicate odd parity for Li^7.

only if the compound nucleus of Be^{8*} is even parity and even angular momentum. Since no alphas are observed we conclude that the Be^8 has either odd parity (in which case it may have any angular momentum) or even parity but odd angular momentum.

The γ-ray provides further information. It is known(23) that this 17.57 Mev γ-ray arises from a transition to the ground state of Be^8. The ground state is almost certainly $^1S_0$ and we can make use of the γ-ray selection rules (which are particularly strict if one of the states is $^1S_0$) to determine possible assignments to the excited state. The γ-ray selection rules and the states which can combine with a $^1S_0^+$ state are shown in Table III.1.

The type of radiation is given in column 1 (e.g. M_1= magnetic dipole; E_3 = electric octipole). In electric multipole radiation only L changes; in Magnetic multipole, only S. The forbidden transitions in column 4 are in addition to those not allowed in columns 2 or 3. Column 6 gives the only state which as a result of selection rules can combine with a $^1S_0^+$ state.

In the case of "JJ coupling" no separation can be made of orbit and spin and in place of ΔL or ΔS we can speak only of ΔJ. In column 6 only total spin and parity remain.

These selection rules are more general and hence less restrictive than those usually given in atomic spectroscopy, as for example by Bowen(24).

TABLE III-1

Type	Allowed		Forbidden	Parity change	$^1S_0^+$
	ΔL	ΔS			
E_1	0, ±1		0 ↔ 0	yes	$^1P_1^-$
M_1		0, ±1		no	$^3S_1^+$
E_2	0, ±1, ±2		0 ↔ 0 0 ↔ 1 1/2 ↔ 1/2		$^1D_2$
M_2		0, ±1, ±2		yes	$^5S_2^-$
E_3	0, ±1, ±2, ±3		0 ↔ 0 0 ↔ 1 0 ↔ 2		$^1F_3^-$
M_3		0, ±1, ±2, ±3	1/2 ↔ 1/2 1/2 ↔ 3/2 1 ↔ 1	no	$^7S_3^+$

Selection rules for Gamma-Radiation

1. Calculation of Cross-Sections

a. $l = 0$

For $l = 0$ we can have only J = 1,2 and with the JJ description T = J gives the only non-zero matrix elements. Since T is equal to J there are no arbitrary constants and the angular distribution is completely specified by J. The possible assignments of the compound nucleus in the "LS" description are $^1P_1^-$, $^3P_{12}^-$. Since the parity is odd we forbid alpha particles, and by reference to Table III.1 the existence of γ-rays requires $^1P_1^-$ and electric dipole radiation. If L and S are not separately conserved we can also have a compound nucleus with J = 2, but since the parity is odd this requires the radiation to be magnetic quadripole.

The angular distribution of the elastically scattered protons is then obtained directly from Equation (2.30)*.

$$\frac{\sigma}{\sigma_0} = 1 + 2\,Re\left\{\frac{e^{i\Theta_0}}{x+i}\right\}\frac{2J+1}{8}\frac{\chi}{\sigma_0^{1/2}} + \frac{(2J+1)\chi^2}{8(1+x^2)\sigma_0} \qquad (3.1)$$

$$\Theta_0 = \alpha \ln \sin^2\tfrac{1}{2}\theta$$

b. $l = 1$

For $l = 1$ we can form J = 0,1,2,3 and the possible "LS" assignments are $^1S_0$ $^1P_1$ $^1D_2$ $^3S_1$ $^3P_{012}$ $^3D_{123}$. The parity of all these states is even so that in order to forbid alpha par-

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* Inclusion of a potential phase shift as indicated in Section IV has also been investigated. The analysis involved closely follows that given in section IV and since it leads to a negative result, it will not be given here. We need only say that even with the extra freedom of arbitrary phase shifts it is not possible to ascribe this level to s-wave protons, since phase shifts can only shift the positions of maximum and minimum cross section but cannot change their values.

ticles we must take only J = 1,3. Table III.1 then shows us that the only possible γ- ray state is $^3S_1^+$ which goes to the ground state by magnetic dipole. If we do not conserve L and S separately we can include J = 3 but the radiation is then magnetic octipole.

The calculation of the elastic proton scattering cross section is somewhat more tedious than the $l = 0$ case. If we assume that the compound nucleus is $^3S_1^+$ we can calculate the matrix elements $\mathbf{M}^{JM}_{m m_l m_s}$ from equation (2.22), and these are summarized in Table III.2a. Table III.2b presents the matrix elements $f_{o\, m_l m_s}^{m_l' m_s'}$. From this, equation (2.29) gives us the cross section

$$\frac{\sigma}{\sigma_o}(^3S_1) = 1 + \frac{3}{8}\frac{\chi^2}{\sigma_o(1+\chi^2)} + \frac{3}{4}\frac{\chi}{\sigma_o^{1/2}} Re\left\{\frac{e^{i\Theta_1}}{\chi+i}\right\}\cos\theta$$

$$\Theta_1 = \alpha \ln \sin^2\tfrac{1}{2}\theta + 2\tan^{-1}\alpha \qquad (3.2)$$

If we go to the "JJ" description the possible values of J are 1 or 3. In the former case we can have T = 1 or 2, in the latter only T = 2. We use equation (2.30) and find that the angular distributions are easier to calculate than in the L-S case; only the summation in the resonance term remains to be evaluated, the arithmetic is sufficiently direct that it shall not be given here. The cross sections are:

$$\frac{\sigma}{\sigma_o}(J=1) = 1 + \frac{3}{8}\frac{\chi^2}{\sigma_o(1+\chi^2)}\left[\left(\frac{5\alpha_1^2-\alpha_2^2}{10}\right)^2(3\cos^2\theta-1)+1\right]$$

$$+\frac{3}{4} Re\left\{\frac{e^{i\Theta_1}}{\chi+i}\right\}\frac{\chi}{\sigma_o^{1/2}}\cos\theta \qquad (3.3)$$

$$\alpha_1^2 + \alpha_2^2 = 1$$

m_j m_s \ m	-1	0	1
$\frac{3}{2}$ $\frac{1}{2}$	$\sqrt{\frac{1}{3}}$		
$\frac{3}{2}$ $-\frac{1}{2}$	$\sqrt{\frac{1}{6}}$		
$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{3}\sqrt{\frac{1}{2}}$	$-\frac{1}{3}\sqrt{2}$	
$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	
$-\frac{1}{2}$ $\frac{1}{2}$		$-\frac{1}{3}$	$\frac{1}{3}$
$-\frac{1}{2}$ $-\frac{1}{2}$		$-\frac{1}{3}\sqrt{2}$	$\frac{1}{3}\sqrt{\frac{1}{2}}$
$-\frac{3}{2}$ $\frac{1}{2}$			$\sqrt{\frac{1}{6}}$
$-\frac{3}{2}$ $-\frac{1}{2}$			$\sqrt{\frac{1}{3}}$

$$M^{JMl}_{m\, m_j\, m_s}$$

$l = 1 \quad J = 1 \quad (^3S_1)$

$$\sum_{m\mu\nu} \left| M^{JMl}_{m\mu\nu} \right|^2 = \frac{2}{3}$$

TABLE III.2a

$$f_{o\ m_j' m_s'}^{\ m_j m_s}$$

$m_j' m_s'$ \ $m_j m_s$	$\frac{3}{2}\ \frac{1}{2}$	$\frac{3}{2}\ -\frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ -\frac{1}{2}$	$-\frac{1}{2}\ \frac{1}{2}$	$-\frac{1}{2}\ -\frac{1}{2}$	$-\frac{3}{2}\ \frac{1}{2}$	$-\frac{3}{2}\ -\frac{1}{2}$
$\frac{3}{2}\ \frac{1}{2}$	1							
$\frac{3}{2}\ -\frac{1}{2}$		1						
$\frac{1}{2}\ \frac{1}{2}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}AY_1^{-1}$		$1+\frac{2}{9}AY_1^{0}$		$-\frac{\sqrt{2}}{9}AY_1^{1}$			
$\frac{1}{2}\ -\frac{1}{2}$		$-\frac{1}{3\sqrt{6}}AY_1^{-1}$	$-\frac{1}{9\sqrt{2}}AY_1^{-1}$	$1+\frac{1}{9}AY_1^{0}$	$\frac{1}{9}AY_1^{0}$	$-\frac{1}{9\sqrt{2}}AY_1^{1}$	$-\frac{1}{3\sqrt{6}}AY_1^{1}$	
$-\frac{1}{2}\ \frac{1}{2}$		$-\frac{1}{3\sqrt{6}}AY_1^{-1}$	$-\frac{1}{9\sqrt{2}}AY_1^{-1}$	$\frac{1}{9}AY_1^{0}$	$1+\frac{1}{9}AY_1^{0}$	$-\frac{1}{9\sqrt{2}}AY_1^{1}$	$-\frac{1}{3\sqrt{6}}AY_1^{1}$	
$-\frac{1}{2}\ -\frac{1}{2}$						$1+\frac{2}{9}AY_1^{0}$		$-\frac{1}{3}\sqrt{\frac{2}{3}}AY_1^{1}$
$-\frac{3}{2}\ \frac{1}{2}$							1	
$-\frac{3}{2}\ -\frac{1}{2}$								1

$$A = \frac{4\pi^{1/2}(2l+1)^{1/2} e^{i\Theta_1} \sin^{2l}\frac{1}{2}\vartheta}{\alpha\,(x+i)\sum_{m\mu\nu} \left| M_{m\mu\nu}^{JMl} \right|^2} = \frac{3\pi^{1/2}\sqrt{3}\, e^{i\Theta_1}}{(x+i)} \frac{\chi}{\sigma_o^{1/2}}$$

TABLE III.2b

$$\frac{\sigma}{\sigma_0}(J=3) = 1+\frac{7}{8}\frac{\chi^2}{\sigma_0(1+\chi^2)}\left[1+\frac{6}{25}(3\cos^2\theta-1)\right]$$

$$+\frac{7}{4}\,\mathrm{Re}\left\{\frac{e^{i\Theta_1}}{\chi+i}\right\}\frac{\chi}{\sigma_0^{1/2}}\cos\theta \qquad (3.4)$$

2. Comparison with experimental results

The excitation curve (cross section as a function of incident proton energy) has been measured[25] at two different angles, approximately $\Theta = 90^\circ$ and $\Theta = 144^\circ$ in the center of mass coordinates. At 90° the curve is almost symmetric with a maximum ratio $\sigma/\sigma_0 = 1.61$, at 144° the curve is definitely asymmetric with the low energy side higher than the high energy side (see Fig. 2), the maximum ratio is $\sigma/\sigma_0 = 2.38$. These then are the dominant features which must be used to choose between the various assignments of (3.1)-(3.4). We tabulate this data in Table III.3 in which we give, for each assignment of the compound nucleus, the position and value of the maximum and minimum of the ratio σ/σ_0.

TABLE III.3

Assignment	89.2°				144.4°			
	x_{max}	$\frac{\sigma}{\sigma_0}$ max	x_{min}	$\frac{\sigma}{\sigma_0}$ min	x_{max}	$\frac{\sigma}{\sigma_0}$ max	x_{min}	$\frac{\sigma}{\sigma_0}$ min
$l=0$ $J=1$	0.78	1.60	-1.29	0.64	0.36	3.59	-2.75	0.66
$l=0$ $J=2$		2.00		0.30		5.32		0.33
$l=1\ ^3S_1$	0	1.72	--	1.00	0	2.08	--	1.00
$l=1$ $J=1$	0	1.72	--	1.00	-0.28	2.06	3.6	0.91
		1.55			-0.18	2.63	5.4	0.94
$l=1$ $J=3$	0	2.30	--	1.00	-0.19	4.68	5.3	0.87
Experiment	0	1.61	--	1.00	-0.25	2.38	?	0.95

The double entry in the row $l = 1$ $J = 1$ is a reflection of the freedom in the choice of the quantum number T and of the possibility of having an arbitrary mixture of states with T = 1 and T = 2. The two figures given are the extremes possible with different choices of the mixture of the two states.

It is evident that the assignment $l = 1$, $J = 1$ best represents the experimental data and we shall now more closely examine this case. It must be pointed out that the experimental excitation curve can not be compared with the theoretical curves until a correction has been made for the effects of resolution. There are, as always in experimental measurements, various sourves of "blurring" which tend to "smear out" the "true" excitation curve and thereby reduce the maximum. Therefore the experimental maximum values of σ/σ_o must be increased before they can be adequately compared with the predicted values. A complete discussion of the problem and an evaluation of the various contributions are presented in Appendix IV; we shall merely state here that the maximum of σ/σ_o should be raised to 1.64 at $\theta = 89.3^\circ$ and to 2.41 at $\theta = 137.8^\circ$. Considering now equation (3.3) we can plot $\frac{\sigma}{\sigma_o}\Big|_{\text{maximum}}$ as a function of α_1^2 for various scattering angles θ. This is done in Fig. 3 and it is seen that the data indicate a value of $\alpha_1^2 = 0.80 \pm 0.02$. The purely nuclear cross section is therefore found to be

$$\sigma_{\text{nuclear}}\, d\omega = \frac{3\chi^2}{8(1+\chi^2)}\left[0.14(3\cos^2\theta - 1) + 1\right] d\omega$$

$$= \frac{3\chi^2}{8(1+\chi^2)}\left[0.86(1 + 0.49\cos^2\theta)\right] d\omega. \qquad (3.3a)$$

3. Equivalence of "JJ" and "LS" description

We have concluded from the observed data that the compound nucleus of Be^8 at 17.57 Mev above the ground state which is formed by the bombardment of Li^7 with 440 kev protons is a state with total angular momentum J = 1 and with 80% T = 1 and 20% T = 2. We wish now to compare this description of the state with the more conventional description in terms of LS states. This can most easily be done by expressing the incident and compound nuclei in each of the two ways and then equating matrix elements in the two descriptions.

The incident system can therefore be written in two ways;

$$\begin{aligned}\Psi_{inc}^{m m_j m_s} &= \sum_{LSJ}' \mathbf{M}_{m m_j m_s}^{JMl}(LS)\,\Psi_J^M(LS) \\ &= \sum_{TJ}' \mathbf{M}_{m m_j m_s}^{JMl}(jT)\,\Psi_J^M(jT)\end{aligned} \tag{3.5}$$

In the second description we do not sum over j because we know definitely the angular momentum of the lithium nucleus; the situation is somewhat more specialized than a completely general case in which only the component orbit and spin momenta are given. The compound system is

$$\Psi_{comp}^M = \sum_{L'S'} \beta_{L'S'} \Psi_J^M(L'S') = \sum_{j'T'} \alpha_{j'T'} \Psi_J^M(j'T') \tag{3.6}$$

We now form the scalar product

$$\left(\Psi_{inc}^{m m_j m_s}, \Psi_{comp}^M\right) = \sum_{LS} \mathbf{M}_{m m_j m_s}^{JMl}(LS)\,\beta_{LS} = \sum_{T} \mathbf{M}_{m m_j m_s}^{JMl}(jT)\,\alpha_{jT} \tag{3.7}$$

In forming these products we have used the same mode of description for the incident and compound systems; we can however also mix our descriptions:

$$\left(\Psi_{inc}^{m m_j m_s}, \Psi_{comp}^{M}\right) = \sum_{LS} \sum_{jT} M(LS)\alpha_{jT}\left(\Psi_J^M(LS), \Psi_J^M(jT)\right) \qquad (3.7a)$$

where we have written $M(LS)$ for brevity in place of $\mathbf{M}_{m m_j m_s}^{JM\tau}(LS)$

Writing $\left(\Psi_J^M(LS), \Psi_J^M(jT)\right) = \gamma_{LS}^{jT}$ we have

$$\sum_{LS} M(LS)\beta_{LS} = \sum_{T} M(jT)\alpha_{jT} - \sum_{LS}\sum_{jT} M(LS)\alpha_{jT}\gamma_{LS}^{jT} \qquad (3.8)$$

from this we can write

$$\beta_{LS} = \sum_{jT} \alpha_{jT}\gamma_{LS}^{jT} \qquad (3.9a)$$

$$M(jT) = \sum_{LS} M(LS)\,\gamma_{LS}^{jT} \qquad (3.9b)$$

We get an equation involving the γ_{LS}^{jT} for each choice of $m m_j m_s$ and solving these equations we obtain the following matrix for the γ_{LS}^{jT}, hence the transformation matrix between "LS states" and "jT states".

jT \ LS	10	01	11	21
$\frac{1}{2}$ 0	0	$\frac{1}{3}$	$-\sqrt{\frac{1}{3}}$	$\frac{\sqrt{5}}{3}$
$\frac{1}{2}$ 1	$-\sqrt{\frac{1}{3}}$	$\frac{\sqrt{2}}{3}$	$-\sqrt{\frac{1}{6}}$	$-\frac{1}{3}\sqrt{\frac{5}{2}}$
$\frac{3}{2}$ 1	$\sqrt{\frac{2}{3}}$	$\frac{1}{3}$	$-\sqrt{\frac{1}{12}}$	$-\frac{\sqrt{5}}{6}$
$\frac{3}{2}$ 2	0	$\frac{\sqrt{5}}{3}$	$\sqrt{\frac{5}{12}}$	$\frac{1}{6}$

We can by no means determine the compound state completely; we can only determine, from these scattering experiments, the ratio $\alpha_{3/2,1}/\alpha_{3/2,2}$. In terms of this more complete description the quantities α_1 and α_2 of the previous section have the values

$$\alpha_1 = \frac{\alpha_{3/2,1}}{\sqrt{\alpha_{3/2,1}^2 + \alpha_{3/2,2}^2}}$$

$$\alpha_2 = \frac{\alpha_{3/2,2}}{\sqrt{\alpha_{3/2,1}^2 + \alpha_{3/2,2}^2}} \qquad (3.10)$$

The coefficient $\left(\frac{5\alpha_1^2-\alpha_2^2}{10}\right)$ of equation (3.3) can therefore be written as

$$\frac{5\alpha_1^2-\alpha_2^2}{10} = 0.38 =$$

$$\frac{1}{5}\cdot\frac{30\beta_{10}^2+6\beta_{21}^2+10\sqrt{6}\,\beta_{10}\beta_{01}-15\sqrt{2}\,\beta_{10}\beta_{11}-5\sqrt{30}\,\beta_{10}\beta_{21}-10\sqrt{3}\,\beta_{01}\beta_{11}-4\sqrt{5}\,\beta_{01}\beta_{21}+2\sqrt{15}\,\beta_{11}\beta_{21}}{12\beta_{10}^2+12\beta_{01}^2+9\beta_{11}^2+3\beta_{21}^2+4\sqrt{6}\,\beta_{10}\beta_{01}-6\sqrt{2}\,\beta_{10}\beta_{11}-2\sqrt{30}\,\beta_{10}\beta_{21}+8\sqrt{3}\,\beta_{01}\beta_{11}+2\sqrt{15}\,\beta_{11}\beta_{21}} \qquad (3.11)$$

This expression is not very useful since we do not have enough data to evaluate the constants β_{LS} ; in addition to (3.11)

all that we have is

$$\sum_{LS} \beta_{LS} = 1 \qquad (3.11a)$$

Conceivably, similar analyses of angular distributions for other reactions which involve the same excited level of Be^8 could yield additional relationships from which to determine the β_{LS} . In this regard the angular distribution of the γ-rays will, unfortunately, tell us nothing. According to calculations carried out by Judd[26] the distribution of the γ-rays are determined solely by the parameter $(5\alpha_1^2 - \alpha_2^2)$ and therefore, although the angular distribution is different, we can obtain from it no further relationships between the β_{LS} .

IV. INCLUSION OF POTENTIAL PHASE SHIFTS IN S-WAVE SCATTERING AND ITS APPLICATION TO THE LITHIUM RESONANCE AT 1 MEV.

In scattering problems in which the proton can approach close to the scattering nucleus the effect of the nuclear non-resonant potential scattering cannot be neglected. This is the case primarily for s-wave collisions, for which the relative angular momentum of the proton is zero. In order to include the additional scattering term we rewrite equation (3.1) in a form which explicitly displays the scattering matrix elements involved.

$$\sigma = \frac{2J+1}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o} + \frac{\chi}{\chi+i}\right|^2 + \frac{1-2J}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o}\right|^2 \qquad (4.1)$$

We shall introduce the effect of the potential scattering by means of phase shifts. The resonance denominator can be written in terms of its phase shift

$$\frac{1}{\chi+i} = -\sin\phi e^{i\phi} = \frac{i}{2}(e^{2i\phi}-1) \qquad -\cot\phi = \chi \qquad (4.2)$$

and the resonance cross section $4\pi\chi^2/(1+\chi^2)$ becomes equivalent to the phase shift cross section[27] $4\pi\chi^2\sin^2\phi$. We can rewrite (4.1) in the form

$$\sigma = \frac{2J+1}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o} - \chi\sin\phi e^{i\phi}\right|^2 + \frac{1-2J}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o}\right|^2 \qquad (4.1a)$$

which is exactly the form of the cross section due to a phase shift ϕ in the state of angular momentum J but with no inter-

action other than coulomb scattering in the state with the other orientation of spins (this has of course been specialized to the case at hand for which the possible values of J are 1 and 2). There is therefore a strong suggestion (which is indeed borne out by the calculations of Wigner[17] and of Wigner and Eisenbud[18] that if there is present nuclear potential scattering which can be represented by phase shifts δ_J, the cross section should then be of the form

$$\sigma = \frac{3}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o} - \chi \sin\psi_1 e^{i\psi_1}\right|^2 + \frac{5}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o} - \chi \sin\psi_2 e^{i\psi_2}\right|^2 \qquad (4.3)$$

where $$\psi_J = \begin{cases} \delta_J - \cot^{-1} x \\ \delta_J \end{cases}$$

depending upon whether the resonance is or is not associated with the state J.

We can fit the various parameters involved by comparing such characteristic data as the values and positions of the maximum and minimum cross section. Let us now consider only that part of σ which is resonant; therefore we write (with rather obvious notation)

$$\sigma_J = \frac{2J+1}{8}\left|\sigma_o^{1/2} e^{-i\Theta_o} - \chi \sin\psi_J e^{i\psi_J}\right|^2 \qquad (4.4)$$

$$\psi_J = \delta_J - \cot^{-1} x \qquad x = \frac{E-E_o}{\frac{1}{2}\gamma}$$

$$\sigma_J = \frac{2J+1}{8}\left\{\sigma_o + \chi^2 \sin^2\psi_J - 2\chi\sigma_o^{1/2}\sin\psi_J \cos(\psi_J + \Theta_o)\right\}$$

$$= \frac{2J+1}{8}\sigma_o\left\{A^2 + \frac{u^2}{4} - Au\cos(2\psi_J + \epsilon)\right\} \qquad (4.5)$$

where

$$u = \chi/\sigma_o^{1/2} \qquad A^2 = 1 + u\sin\Theta_o + \frac{u^2}{4}$$

$$A\sin\epsilon = -\cos\Theta_o$$

hence we have immediately that

$$\sigma_{J\ \substack{\text{Max}\\\text{min}}} = \frac{2J+1}{8}\sigma_o\left[A \pm \tfrac{1}{2}u\right]^2 \qquad \cos(2\psi_J+\epsilon)\Big|_{\substack{\text{Max}\\\text{min}}} = \mp 1$$

and

$$\sigma_{Max} - \sigma_{min} = \frac{2J+1}{8}\cdot 2Au\sigma_o \qquad (4.6)$$

This is the most convenient procedure for determining the angular momentum from the observed excitation curve assuming that the resonance is due to s-wave protons. We note that the change in cross section is independent of the phase shift δ_J, but that this latter quantity can be determined by the position of the maximum and minimum cross section. Also from the value of the cross section at maximum or minimum we can determine (but, in general, not unambiguously) the phase shift of the non-resonant component.

The position of the maximum is

$$x_{Max} = \cot\left(\delta_J + \frac{\epsilon}{2} + \frac{\pi}{2}\right)$$

and the minimum

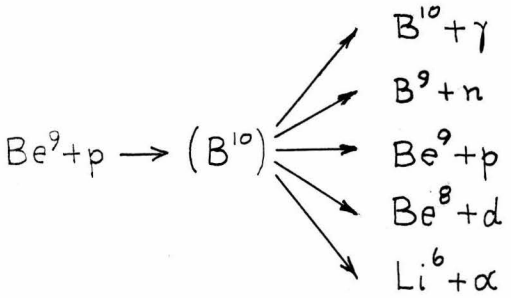
$$x_{min} = \cot\left(\delta_J + \frac{\epsilon}{2}\right)$$

The lithium resonance at 1.0 Mev is shown in Fig. 4. The fit is only qualitative but the data definitely indicates that the resonance is produced by a state with J = 1. The presence of the broad resonances above 1 Mev. which we have not included

in our analysis can easily be burdened with the responsibility for the lack of fit. We could of course have included these terms but until more data is obtained, in the form either of excitation curves at other angles or of angular distributions at various energies near the resonance, such a procedure would have little significance since agreement with one excitation curve can certainly be obtained by a proper choice of non-resonant background, particularly so if it is allowed to have a dependence on energy. Another effect is that described in the next section, i.e. the influence of competing reactions. The curve for $J = 1$ can be brought into agreement with the observed amplitude of the resonance by introducing $\gamma_p/\gamma \approx 0.7$.

V. APPLICATION TO $Be^9(p,p)Be^9$

The compound nucleus B^{10} formed by the bombardment of Be^9 by protons is known to have several modes of decay



(5.1)

The γ-ray yield shows a broad resonance at a proton bombarding energy of 988 kev. with a half-width of 94 kev and a narrow resonance at 1077 kev with a half width of 4 kev.[23] The protons respond to these resonances as well as to very broad resonances below 700 kev. and above 1300 kev.[28] The deuterons and alphas show typical resonance behaviour at 988 kev. but nothing at 1077 kev.[29]

The ground state of Be^9 is taken as $^2P_{3/2}^-$ and since this is the same as for Li^7 the cross-sections from single resonances will have the same form as those given in the previous sections. The broad resonance can properly be treated as single but the narrow resonance is not unaffected by the other upon which it is superimposed. We therefore shall first investigate the resonance at 988 kev. and then, having determined its nature, allow it to interfere with possible assignments of the narrow resonance at 1077.

1. Broad Resonance at 988 kev.

The cross section can be written in the form

$$\frac{\sigma}{\sigma_0} = 1 + 2\,\frac{2J+1}{8}\,\frac{\chi}{\sigma_0^{1/2}}\,\mathrm{Re}\left\{\frac{e^{i\Theta_l}}{\chi+i}\right\} P_l(\cos\theta) + \frac{2J+1}{8}\cdot\frac{\chi^2}{\sigma_0}\cdot\frac{Q_{Jl}(\cos\theta)}{1+\chi^2} \qquad (5.2)$$

Where $Q_{Jl}(\cos\theta)$ is given by

		$Q_{Jl}(\cos\theta)$
$l = 0$	$J = 1$	1
	$J = 2$	1
$l = 1$	$J = 0$	1
	$J = 1$	$1 + \frac{1}{2}\left(\frac{5\alpha_1^2 - \alpha_2^2}{10}\right)^2 P_2(\cos\theta)$
	$J = 2$	$1 + \frac{7}{10}\left(\alpha_1^2 - \alpha_2^2\right)^2 P_2(\cos\theta)$
	$J = 3$	$1 + \frac{12}{25} P_2(\cos\theta)$
$l = 2$	$J = 0$	1
	$J = 1$	$1 + \frac{1}{2}\left(\alpha_1^2 - \alpha_2^2\right)^2 P_2(\cos\theta)$
	$J = 2$	$1 + \frac{2}{7}\left(\frac{7\alpha_1^2 - 3\alpha_2^2}{14}\right)^2 \left\{5 P_2(\cos\theta) + 16 P_4(\cos\theta)\right\}$
	$J = 3$	$1 + 3\left(\frac{4\alpha_1^2 + \alpha_2^2}{7}\right)^2 P_2(\cos\theta) + \frac{11}{2}\left(\frac{2\alpha_1^2 - 3\alpha_2^2}{7}\right)^2 P_4(\cos\theta)$
	$J = 4$	$1 + \frac{11}{686}\left\{50 P_2(\cos\theta) + 13 P_4(\cos\theta)\right\}$

$$P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2} \qquad P_4(\cos\theta) = \frac{35\cos^4\theta - 30\cos^2\theta + 3}{8}$$

2. Comparison with Experimental Data

The differential scattering cross section for protons emerging at an angle of $142^{\circ}55'$ in the center of mass coordinates has been carefully explored by Thomas, Fowler, and Lauritsen[29]

from 300 kev to 1300 kev bombarding energy; less reliable data were also taken at approximately 105° (center of mass). The yield shows a minimum in the ratio of scattering cross section to coulomb cross section at 910 kev. ($\frac{\sigma}{\sigma_o}(\text{min}) = 0.79$) and a maximum at 1038 ($\frac{\sigma}{\sigma_o}(\text{Max}) = 5.135$). We shall compare this data with the expected changes in cross section computed from equation (5.2) and tabulated in Table V.1. In calculating these values we have modified equations (5.2) in order to take into consideration the effect of the other modes of decay. In section III we could safely omit these corrections because the γ-ray which was the only other competing reaction was of the order of 10^{-3} weaker. Here, however, the alphas and deuterons offer appreciable competition to the proton in the decay of B^{10}. Therefore γ_p in the resonance denominator of Eq. (2.13b) must be replaced by

$$\gamma = \gamma_p + \gamma_\alpha + \gamma_d$$

which introduces the factor γ_p/γ into the second term of (2.13c). The net effect can therefore be most easily described by replacing χ with $\frac{\gamma_p}{\gamma}\chi$ in equation (5.2). In computing Table V.1 we assumed the values

$$\frac{\gamma_p \chi}{\gamma \sigma_o^{1/2}} = 2.500 \quad \text{at 988 kev}$$

$$\theta = 142°55' \text{ (center of mass)}$$

$$\frac{\gamma_p}{\gamma} = 0.87 \qquad (5.3)$$

This corresponds roughly to a deuteron-plus-alpha yield which is 15% of the proton yield.

TABLE V.1

Relative change in cross section: $\frac{\sigma_{Max}-\sigma_{min}}{\sigma_o}$, $\theta = 142^{\circ}55'$ (where two figures are given they represent extreme values possible for various assignments of α_T)

J =	0	1	2	3	4
$l = 0$		2.898	4.830		
1	0.432	1.750 1.296	2.160 3.250	4.05	
2	1.064	3.73 3.19	1.040 5.320	4.91 8.94	11.46
Experimental: 4.345					

From this alone we see that possible assignments are $l = 0$, $J = 2$; $l = 1$, $J = 3$; $l = 2$, $J = 2,3$. We can eliminate the last two on the basis of the shape of the curve, which shows a cross section less than coulomb below the resonance and a cross section much larger than coulomb above resonance; on the other hand the assignment $l = 1$, $J = 3$ shows a maximum below the resonance energy while $l = 2$, $J = 2$ would require a fairly symmetric excitation curve with very little evidence of interference effects and a maximum almost exactly at resonance.

As a further verification we may compute the predicted values of $\frac{\sigma_{Max}-\sigma_{min}}{\sigma_o}$ at $\theta = 90^{\circ}$ and compare these with the experimental value (the data at 90° is not as accurate as that at 143° and therefore we shall use it sparingly and only as substantiating rather than primary evidence).

TABLE V.2

Relative change in cross section $\frac{\sigma_{Max}-\sigma_{min}}{\sigma_o}$, $\theta = 90^o$

(Where two figures are given they represent the extreme values values possible for various assignments of α_T)

J =	0	1	2	3	4
$l = 0$		0.991	1.651		
1	0.242	0.545 0.727	1.105 1.211	1.289	
2	0.089	0.151 0.268	0.446 0.716	0.338 1.313	0.996
Experimental = 1.53					

We cannot place too much importance on the experimental result since it is accurate only to about ± 20%. The form of the excitation curve, however, is much more definitely established. For all $l = 1$ curves the interference term vanishes since the Legendre polynomial $P_1(\cos\theta)$ is zero at $\theta = 90^o$. Hence we would find the simple resonance curve superimposed on the coulomb background with its maximum exactly on resonance. For all $l = 2$ curves the maximum cross section occurs at an energy less than the resonance energy and the minimum cross section occurs above the resonances. The observed curve shows a minimum below, and a maximum above, the resonance energy; which is the expected behaviour for an incident s-wave.

We therefore feel happy in choosing $l = 0$, $J = 2$ to describe the scattering. However, a more detailed analysis of the data is possible. Inspection of the excitation curve shows that a

single resonance is not an adequate description of the scattering. We shall try to include non-resonant scattering following the theory developed in section IV. We introduce two phase shifts δ_1 and δ_2 corresponding to the states J = 1 and J = 2 respectively and write our cross section in the form (see equation 4.5)

$$\frac{\sigma}{\sigma_0} = \sum_{J=1}^{2} \frac{2J+1}{8} \left\{ A^2 + \frac{u^2}{4} - Au\sin(2\psi_J + \epsilon) \right\}$$

$$u = \frac{\gamma_p}{\gamma} \frac{\lambda}{\sigma_0^{1/2}} = \frac{\gamma_p}{\gamma} \cdot \frac{1-\cos\theta}{\alpha} \qquad \alpha = \frac{4}{\text{"137"}} \sqrt{\frac{M_p c^2}{2E}}$$

$$A^2 = 1 + \frac{u^2}{4} + u\sin\Theta_0$$

$$A\sin\epsilon = -\cos\Theta_0 \qquad (5.4)$$

where $\psi_1 = \delta_1$

$$\psi_2 = \delta_2 - \cot^{-1} x \qquad (5.5)$$

We can if we wish go back and make a more accurate estimate of γ_p/γ from our data by using equation (4.6)

$$\sigma_{Max} - \sigma_{min} = \frac{2J+1}{4} Au\sigma_0 \qquad (4.6)$$

to determine u.

Table V.1 gives us $\frac{\sigma_{Max} - \sigma_{min}}{\sigma_0} = 4.345 \pm 0.10$. At $\theta = 142^\circ 55'$ and $E_p = 0.988$ Mev, $\sin\Theta_0 = -0.0669$, so that we find

$$u = 2.312 \pm 0.10$$

which implies the value

$$\gamma_p/\gamma = 0.82 \pm 0.03 \qquad (5.6)$$

If we use this value we find, by fitting the experimental values of the cross section at the maximum and minimum*

$$\delta_1 = \begin{cases} +0.915 \pm 0.03 \\ -0.176 \pm 0.03 \end{cases} \qquad (5.7)$$

where the probable error is based on the somewhat arbitrary assignment of a probable error in the cross section as plotted in Fig. 5 of approximately 5%. The two values arise from the ambiguity of determining an angle from its cosine (Equation 5.4). It has been assumed that δ_1 does not change over the resonance.

From the position of the maximum and minimum with respect to the γ-ray resonance we can deduce the half width γ and the nuclear potential phase shift δ_2.

$$\gamma = 125 \pm 15 \text{ kev} \qquad \delta_2 = s\pi - 0.317 \pm 0.07 \qquad (5.8)$$

(s is an interger)

Using the values computed here for δ_1, δ_2, γ_p/γ and γ we obtain the curve indicated as "theoretical" in Fig. 5.

3. The Narrow Resonance at 1077 kev.

To investigate the narrow resonance we must essentially repeat what has been done already for the broad resonance, but with the important difference that the background against which

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* There is also an indeterminancy of a multiple of 2π in our phase shifts which is beyond our power to determine from only a single resonance. We know that $\delta \to o$ for low energies and increases by π for each resonance. Hence only if we knew the entire excitation curve from zero energy up could we hope to determine δ unambiguously.

the resonance interferes is not simply the coulomb cross section but the entire broad resonance. This leads to the cross section formula

$$\frac{\sigma}{\sigma_0} = 988\ \text{Resonance} + \frac{2J+1}{8}\cdot\frac{\chi^2}{\sigma_0}\cdot\frac{Q_{Jl}(\cos\theta)}{1+\chi^2}$$

$$+\frac{2J+1}{8}\,\frac{\chi}{\sigma_0^{1/2}}\sum_{T=1}^{2}\alpha_T^2\,\mathrm{Re}\left\{\frac{e^{i\Theta_l}-u\sin\psi_T\exp i[\psi_T+\Theta_l-\Theta_0]}{\chi+i}\right\}P_l(\cos\theta)$$

(5.9)

where "988 Resonance" indicates the part already computed, $Q_{Jl}(\cos\theta)$ has been previously tabulated but is here applied to the narrow resonance, u and ψ_T refer to those quantities of the broad resonance and are evaluated at a specific energy. They are considered to be constant over the narrow resonance. χ refers to the narrow resonance and is of course in units of its half-width.

The cross sections predicted from this have been compared with the observed resonance (Fig. 5). The comparison has been made for $l = 0,1,2$ and for each of the possible choices of δ_1 and δ_2 for the broad resonance. The uncertainties, however, are such that no really definite assignment can be made. However, some general conclusion can be drawn. The magnitude of the resonance is small* indicating that, unless the effect can be produced as a result of a fortuitous cancellation by the interference terms, the total angular momentum of the compound

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* The effects of resolution (Appendix IV) tend to blur out the resonance, but correcting for this only increases $\sigma/\sigma_0|_{Max}$ by approximately 20%.

nucleus is zero. Hence data at other angles should be able to resolve the question. At 90° the excitation curve shows a small resonance whose phase is inverted with respect to that at 142°. This implies that the resonance is J = 0 and that furthermore $l = 2$. This last assignment results from the fact that for $l = 1$ all interference terms vanish at 90° while for $l = 2$ the sign of the interference is reversed from that at 142° (at 90° $P_2(\cos\theta) = -0.5$; at 142° $P_2(\cos\theta) = 0.5$).

4. Comparison of Beryllium assignments with other experiments

An energy level diagram is shown in Fig. 6 for the compound nucleus B^{10}. The angular momentum and parity are indicated for each level shown. Some of these assignments are however conjectural. The ground state of Be^9 is fairly certainly 3/2 (odd), and the level of B^{10} produced by the 988 kev resonance we are attempting to ascribe to 2(odd). The level would then decay to $Li^6 + \alpha$ thru a p wave and to $Be^8 + d$ also by p wave. With a 2(odd) state we can get electric dipole radiation to the ground state which has recently been shown[30] to be 3(even). The ground state is supposedly one level of a triplet and γ-radiation is equally permitted to the other two members.

The narrow resonance we have indicated as going to a state 0(odd) which could then radiate electric dipole only to the 1 (even) level of the ground state triplet, but would require electric octipole to reach the ground state. This level could

also decay to $Li^6 + \alpha$ or to $Be^8 + d$, the fact that such a decay is not observed implies that a further selection rule is involved. On the other hand we could forbid such decay if we ascribed the narrow resonance to incident p-wave which would then lead to O(even) for the compound nucleus. Such an assignment is, however, quite inconsistent with the existence of interference effects at 90^o. It is therefore evident that more experiment work is required.

Appendix I

The Dispersion Formula

The dispersion formula, as indicated in § II.2 has been adequately treated by several authors and the only purpose for presenting it here is one of completeness. The proof used is that given by Bethe[9].

We shall consider for simplicity only a single compound nucleus which has only two modes of decay. P shall designate the incident particle and A the target nucleus. Q and B are respectively the emitted particle and residual nucleus other than the reemission of P; C is the compound nucleus. Hence the reaction can be symbolically expressed as

$$A + P \rightarrow C \begin{matrix} \nearrow A + P \\ \searrow B + Q \end{matrix} \qquad \text{(A1.01)}$$

We shall use $\chi_A, \chi_B, \chi_C, \chi_P, \chi_Q$ to represent the normalized wave functions of the nuclei A, B, C, P, and Q, while ψ_P and ψ_Q are the wave functions of the relative motions of the A, P or B, Q systems each with respect to their centers of mass; these functions are as yet unnormalized. ψ_P is specified to the extent that it shall be composed of an incident plane wave and an outgoing spherical wave while ψ_Q is solely an outgoing wave.

The total Hamiltonian may be split in two ways

$$\begin{aligned} H &= H_A + T_P + U_P + V_{AP} \\ &= H_B + T_Q + U_Q + V_{BQ} \end{aligned} \qquad \text{(A1.02)}$$

H_A, H_B, H_C are the Hamiltonians of the internal structure of the nuclei A, B, C... T_P and T_Q are the kinetic energies of the respective particles relative to the center of mass of the corresponding systems. U_P, U_Q are the potential energies of P and Q in the field of A and B respectively and are functions only of the distance between A and P or B and Q. At large distances this is the coulomb field if P or Q is a proton. V_{AP} and V_{BQ} are interaction energies between particles which depend on coordinates other than the separation distance and include the internal coordinates of A and B.

The wave functions χ satisfy the equations

$$\begin{aligned} H_A\chi_A &= W_A\chi_A \\ H_B\chi_B &= W_B\chi_B \\ H\chi_C &= W_C\chi_C \end{aligned} \qquad \text{(A1.03)}$$

except that we shall assume χ_C to vanish outside the "nuclear radius". The wave function of the complete system is approximated as

$$\Psi = \chi_A\chi_P\psi_P + \chi_B\chi_Q\psi_Q + a\chi_c \qquad \text{(A1.04)}$$

The constant a is the only one we need since ψ_P and ψ_Q are not yet normalized. We now try to satisfy the Schrodinger equation as accurately as possible with this choice

$$H\Psi = W\Psi \qquad \text{(A1.05)}$$

where the energy W is given by

$$W = W_A + E_P \qquad \text{(A1.06)}$$

E_P being the kinetic energy of the relative motion of A and P.

We impose the following conditions,

$$\int \chi_c^* (H-W) \Psi \, d\tau_c = 0$$

$$\int \chi_A^* \chi_P^* (H-W) \Psi \, d\tau_A \, d\tau_P = 0 \qquad \text{(A1.07)}$$

$$\int \chi_B^* \chi_Q^* (H-W) \Psi \, d\tau_B \, d\tau_Q = 0$$

the integration in the first case is over all coordinates but in the second two cases only over the internal coordinates (including summation over spins for P or Q.) and hence is a condition for all of the relative coordinates.

From (.02) and (.03) we find (since the χ's are normalized)

$$a(W-W_c) = \int \chi_c^* (V_{AP} - L_p) \chi_A \chi_p \psi_p \, d\tau_c + \int \chi_c^* (V_{BQ} - L_Q) \chi_B \chi_Q \psi_Q \, d\tau_c \qquad \text{(A1.08)}$$

$$H - W = H_A + V_{AP} - L_p$$
$$L_p = E_p - T_p - U_p = E_p + \frac{\hbar^2}{2M} \nabla_p^2 - U_p \qquad \text{(A1.09)}$$

and

$$L_p \psi_p = a \int \chi_A^* \chi_P^* (V_{AP} - L_p) \chi_c \, d\tau_A \, d\tau_P + \int \chi_A^* \chi_P^* V_{AP} \chi_A \chi_P \psi_P \, d\tau_A \, d\tau_P + \int \chi_A^* \chi_P^* (V_{AP} - L_p) \chi_B \chi_Q \psi_Q \, d\tau_A \, d\tau_P \qquad \text{(A1.10)}$$

plus an equation similar to (.10) for $L_Q \psi_Q$. The second term in (.10) gives a contribution to the nuclear potential scattering and the third term represents a direct transition without

formation of an intermediate state from AP to BQ. Both of these possibilities we shall assume are small since we are primarily interested in the resonance with the compound nucleus.

Therefore (.10) reduces to

$$L_p \psi_p \simeq a \int \chi_A^* \chi_p^* (V_{AP} - L_p) \chi_c \, d\tau_A \, d\tau_p \qquad \text{(A1.11)}$$

L_p is a spherically symmetric operator so that we can write (.11) in spherical polar coordinates. We therefore expand ψ_p in spherical harmonics.

$$\psi_p = \frac{1}{r_p} \sum_{lm} \psi_{plm}(r_p) Y_l^m(\theta_p, \phi_p) \qquad \text{(A1.12)}$$

By multiplying (.11) by Y_l^{m*} and integrating over the solid angle ω_p of the positional coordinates of particle P and making use of equation (.09) we find that $\psi_{plm}(r_p)$ satisfies the differential equation

$$\frac{\hbar^2}{2M} \frac{d^2}{dr^2} \psi_{plm} + \left[E_p - U_p - \frac{\hbar^2 l(l+1)}{2Mr^2} \right] \psi_{plm} = a \, C_{plm}(r) \qquad \text{(A1.13)}$$

in which $C_{plm}(r)$ is defined by the integral

$$C_{plm}(r_p) = \int r_p Y_l^{m*}(\theta_p, \phi_p) \chi_A^* \chi_p^* (V_{AP} - L_p) \chi_c \, d\tau_A \, d\tau_p \, d\omega_p \qquad \text{(A1.14)}$$

in which the integration extends over all the coordinates of all the particles except the distance r_p between A and P.

Since χ_c vanishes for r_p larger than the nuclear radius (i.e. the compound nucleus exists only if all the particles are close together), for large r_p (.13) becomes the homogeneous equation,

$$\frac{\hbar^2}{2M}\frac{d^2}{dr^2}\psi_{plm} + \left[E_p - U_p - \frac{\hbar^2 l(l+1)}{2Mr^2}\right]\psi_{plm} = 0 \qquad \text{(A1.15)}$$

and the solution, which may contain both the regular solution f and the irregular solution g, is

$$\psi_{plm} \rightarrow \alpha_{plm} f_{pl}(r_p) + \beta_{plm} g_{pl}(r_p) \qquad (r_p \rightarrow \infty) \qquad \text{(A1.16)}$$

where for small r

$$f_{pl}(r) \approx r^{l+1} \qquad g_{pl}(r) \approx r^{-l} \qquad (r \rightarrow 0) \qquad \text{(A1.17)}$$

For large r we can write the asymptotic form

$$g(r) + i f(r) = \exp i\left\{kr - l\frac{\pi}{2} + \delta_l\right\} \qquad (r \rightarrow \infty) \qquad \text{(A1.18)}$$

$$k = M_p v/\hbar$$

From (.15) we find, writing the equation for f and g, multiplying respectively by g and f, subtracting and integrating

$$g\frac{df}{dr} - f\frac{dg}{dr} = \text{constant} = k \qquad \text{(A1.19)}$$

where the value of the constant has been obtained from the asymptotic forms (.18). In a similar manner multiplying equation (.13) for ψ by f and equation (.15) for f by ψ, subtracting and integrating from 0 to large radius we find (since f vanishes at r = 0)

$$\lim_{r\rightarrow\infty}\left(f\frac{d\psi}{dr} - \psi\frac{df}{dr}\right) = \frac{2M}{\hbar^2} d \int_0^\infty f_{pl} C_{plm}\, dr \qquad \text{(A1.20)}$$

(.20) now holds for large r and making use of (.16) and (.19)

we find

$$\beta_{Plm} = -\frac{2M}{\hbar^2 k} a \int_0^\infty f_{Pl} C_{Plm} dr$$

$$= -\frac{2}{\hbar v} a \int_0^\infty F^*_{Plm}(r_P \theta_P \phi_P) \chi_A^* \chi_P^* (V_{AP} - L_P) \chi_c d\tau_A d\tau_P d\Omega_P \qquad (A1.21)$$

$$F_{Plm} = \frac{1}{r} f_{Pl}(r) Y_l^m(\theta, \phi) \qquad (A1.22)$$

Now $L_P F_{Plm} = o$ and since L_P is a self-adjoint operator it can be seen that it contributes nothing to the integral in (.21) so that we obtain

$$\beta_{Plm} = \frac{-2}{\hbar v} a V^{C*}_{Plm} \qquad (A1.23)$$

where the matrix element V^C_{Plm} is defined by

$$V^C_{Plm} = \int \chi_c^* V_{AP} \chi_A \chi_P F_{Plm} d\tau_c \qquad (A1.24)$$

the integration being carried out over all the coordinates of the system.

We must now fix the coefficients α_{Plm}. These are determined by the asymptotic behaviour of the partial waves. Since the only incident waves are the ones with subscript Plo these must have an incoming part such as to give an incident plane wave while all the others must be purely outgoing. Since we have the asymptotic expansion[31] of the incident wave

$$\psi_P^{inc} = \frac{1}{v^{1/2}} e^{ikz}$$

$$= \sqrt{\frac{\pi}{v}} \frac{2}{kr} \sum_l (2l+1)^{1/2} Y_l^o(\theta, \phi) \sin(kr - l\tfrac{\pi}{2}) \qquad (A1.25)$$

we must write ψ_{Plo} as

$$\psi_{Plo} = \frac{i}{k}\sqrt{\frac{\pi(2l+1)}{v}}\, e^{i\frac{\pi}{2}l} e^{-ikr} + \eta e^{ikr} \qquad \text{(A1.26)}$$

and for the other waves

$$\psi_{Plm} = \text{const.}\, e^{ikr_P}$$

$$\psi_{Qlm} = \text{const.}\, e^{ik'r_Q} \qquad \text{(A1.27)}$$

Equations(.16), (.18), (.27) and (.28) lead immediately to

$$\alpha_{Plm} = i\,\beta_{Plm} \qquad m \neq o$$

$$\alpha_{Plo} = i\,\beta_{Plo} + \frac{2}{k}\sqrt{\frac{\pi(2l+1)}{v}}\, e^{i\delta_l} \qquad \text{(A1.28)}$$

$$\alpha_{Qlm} = i\beta_{Qlm}$$

Now (.16) is valid only for distances at which the inhomogeneous term on the right of (.13) is negligible and we must therefore modify it so as to represent the correct form of ψ_{Plm} at small distances. To do this we write

$$\psi_{Plm}(r) = \alpha_{Plm} f_{Pl}(r) + \beta_{Plm} h_{Pl}(r) \qquad \text{(A1.29)}$$

where $h_{Pl}(r)$ is a solution of the inhomogeneous equation (.13) such that $h_{Pl}(r) \longrightarrow g_{Pl}(r)$ at large distances. In analogy to (.22) we define

$$G_{Plm} = \frac{1}{r_P} h_{Plm}(r_P)\, Y_l^m(\theta_P, \phi_P) \qquad \text{(A1.30)}$$

and

$$\psi_P = \sum_{lm} (\alpha_{Plm} F_{Plm} + \beta_{Plm} G_{Plm})$$

$$= \frac{2}{k}\sqrt{\frac{\pi}{v}} \sum_l (2l+1)^{1/2} e^{i\delta_l} F_{Plm}$$

$$- \frac{2d}{\hbar v} \sum_{lm} V_{Plm}^{C*} (G_{Plm} + i F_{Plm}) \qquad \text{(A1.31)}$$

We must still determine the value of the constant a. This we can do by making use of equation (.08) which has not yet entered our discussion. Inserting (.31) and similar expressions in BQ into (.08) we obtain

$$a(W-W_c) = \sum_{lm} \left[\frac{2}{k}\sqrt{\frac{\pi(2l+1)}{v}}\, e^{i\delta_l} \delta_m^o - i\frac{2d}{\hbar v} V_{Plm}^{C*}\right]$$

$$\times \int \chi_c^* (V_{AP} - L_p) \chi_A \chi_p F_{Plm}\, d\tau$$

$$- \frac{2d}{\hbar v} \sum_{lm} V_{Plm}^{C*} \int \chi_c^* (V_{AP} - L_p) \chi_A \chi_p G_{Plm}\, d\tau_c \qquad \text{(A1.32)}$$

$$+ \text{similar terms in BQ}$$

The first integral is just V_{Plm}^{C}; the second integral is difficult to evaluate but appears only as a shift in the apparent resonance energy of the reaction. The integral may be complex but will have the same phase as V_{Plm}^{C} since G and F have the same phase. The summation is therefore real so that if we define it to be $\frac{1}{2}\hbar v\, \epsilon_{cp}$, we have, including a similar definition for the BQ integral

$$a(W-W_c)=\sum_l \frac{2}{k}\sqrt{\frac{\pi(2l+1)}{v}}\, e^{i\delta_l} V^c_{Plm} \; -\frac{2ai}{\hbar v_P}\sum_{lm}|V^c_{Plm}|^2$$
$$-\frac{2ai}{\hbar v_Q}\sum_{lm}|V^c_{Qlm}|^2 - a(\epsilon_{cP}-\epsilon_{cQ}) \qquad \text{(A1.33)}$$

or

$$a = \frac{2}{k}\sqrt{\frac{\pi}{v_P}}\;\frac{\sum_{l'}(2l'+1)^{1/2}e^{i\delta_{l'}}V^c_{Pl'o}}{W-W_o+\frac{1}{2}i\gamma_c} \qquad \text{(A1.34)}$$

$$W_o = W_c-\epsilon_{cP}-\epsilon_{cQ}$$

$$\gamma_c = \frac{4}{\hbar v_P}\sum_{lm}|V^c_{Plm}|^2+\frac{4}{\hbar v_Q}\sum_{lm}|V^c_{Qlm}|^2$$

Thus the resonance occurs when the energy of the system is W_o rather than W_c. Thus the observed resonance energy for a given compound nucleus depends slightly on the mode of formation. That the effect should be small, at least in respect to the difference between γ ray emission and elastic scattering of protons has been discussed by Breit[32] who indicates the shifts to be of the order of γ_c^2/E_P .

Now the probability of emission of particle Q per unit incident flux of P, i.e., the cross section for the emission of Q, is

$$\sigma^P_{Qlm} = v_Q|\beta_{Qlm}|^2 \qquad \text{(A1.35)}$$

Since this is just the flux of particles Q^{lm}. Therefore putting (.34) and (.23) into (.35) we get

$$\sigma^P_{Qlm} = \frac{4v_Q}{(\hbar v_Q)^2}\,\frac{4\pi}{v_P k_P^2}\left|\sum_{l'}(2l'+1)^{1/2}e^{i\delta_l}\,\frac{V^c_{Pl'o}\,V^{c*}_{Qlm}}{W-W_o+\frac{1}{2}i\gamma_c}\right|^2 \qquad \text{(A1.36)}$$

The matrix elements have been calculated with the particle wave function normalized to unit flux. If we normalize per unit energy we have (Ref. 9, p.105)

$$H^{c}_{Plm} = \left(\frac{2}{\pi\hbar v_p}\right)^{1/2} V^{c}_{Plm} \qquad \text{(A1.37)}$$

where H is the same matrix element as V but with energy rather than flux normalization. This gives us the final result

$$\sigma^{P}_{Qlm} = 4\pi^3 \lambda_P^2 \left| \sum_{l'} (2l'+1)^{1/2} e^{i\delta_{l'}} \frac{H^{c}_{Pl'0}\, H^{c*}_{Qlm}}{W - W_0 + \frac{1}{2} i\gamma_c} \right|^2$$

$$\gamma_c = 2\pi \sum_{R=PQ} \sum_{lm} \left| H^{c}_{Rlm} \right|^2$$

Appendix II

The Clebsch-Gordon Series

The Clebsch-Gordon series is the decomposition of a direct product of representations of the rotation group into a sum of representations. Since the $2J+1$ wave functionsof angular momentum J induce a representation of the rotation group, the series allows us to decompose the product of wave function into a sum of wave functions and hence becomes the mathematical basis for the vector model of the atom.(20,21,22)

A wave function ψ_j^m of a Hamiltonian which is invariant to rotation of coordinates can be represented in its transformation properties by the monomial

$$\frac{(2j)!\,(-1)^{j-m}\,\xi^{j+m}\,\eta^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \tag{A2.1}$$

where $(\xi,\eta$ $\frac{a^{j-m}b^{j+m}}{\sqrt{(j+m)!\,(j-m)!}}$) is a spinor. This is just the coefficient of in the spinor-invariant

$$(-a\eta+b\xi)^{2j} \tag{A2.2}$$

where (a,b) is a constant spinor. We are interested in evaluating the coefficients in the expansion

$$\psi_l^{\mu}\psi_s^{\nu}=\sum_{jm} A_{jm}^{ls\mu\nu}\psi_j^m \tag{A2.3}$$

or, since as we shall see, the transformation can be taken to be real and unitary

$$\psi_j^{\ m} = \sum_{\mu\nu} A_{jm}^{is\mu\nu} \psi_i^{\ \mu} \psi_s^{\ \nu} \qquad \text{(A2.4)}$$

We form an invariant

$$\Phi = (-AY + BX)^{2j} \qquad \text{(A2.5)}$$

where (A,B) is a constant spinor and (X,Y) an arbitrary spinor. The terms $X^{j+m} Y^{j-m}$ of the expansion of Φ transform like the wave functions $\psi_j^{\ m}$

We also write Φ in the form

$$\Phi = (-A\eta + B\xi)^{\alpha} (-A\eta' + B\xi')^{\beta} (-\xi\eta' + \eta\xi')^{\gamma} \qquad \text{(A2.6)}$$

The spinor (ξ, η) shall be associated with the wave functions $\psi_i^{\ \mu}$ and the spinor $(\xi'\eta')$ with the wave functions $\psi_s^{\ \nu}$.

We must obviously have $\alpha+\beta=2j$ since in the form (.5) Φ is a homogeneous polynomial of degree $2j$ in A and B, while in the form (.6) it is a homogeneous polynomial of degree $\alpha+\beta$. We shall not yet specify the values of α, β, γ any more completely.

Expanding the two expressions for Φ and equating them we have

$$\sum_{m=-j}^{j} \frac{(-1)^{j-m}(2j)!\, X^{j+m} Y^{j-m}}{(j+m)!\,(j-m)!} A^{j-m} B^{j+m}$$

$$= \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} \sum_{r=0}^{\gamma} \frac{\alpha!\,\beta!\,\gamma!\,(-1)^{p+q+r}\, \eta^{p+\gamma-r} \xi^{\alpha-p+r} \eta'^{q+r} \xi'^{\beta-q+\gamma-r}}{p!\,(\alpha-p)!\,q!\,(\beta-q)!\,r!\,(\gamma-r)!} A^{p+q} B^{\alpha-p+\beta-q}$$

(A2.7)

Identifying coefficients of like powers of $A^{j-m} B^{j+m}$ we have

$$p+q=j-m \qquad \alpha+\beta=2j \tag{A2.8}$$

and since we want to identify the monomial $\eta^{p+\gamma-r} \xi^{\alpha-p+r}$ with the wave function ψ_l^{μ} and the monomial $\eta'^{q+r} \xi'^{\beta-q+\gamma-r}$ with the wave function ψ_s^{ν} we also have

$$\begin{aligned} \alpha+\gamma &= 2l & \alpha-\gamma-2(p-r) &= 2\mu \\ \beta+\gamma &= 2s & \beta+\gamma-2(q+r) &= 2\nu \end{aligned} \tag{A2.9}$$

Since (.5) and (.6) must be finite polynomials, α, β, γ must be integers and l, s, j can be only integer or half integer;

hence

$$\begin{aligned} \alpha &= j+l-s & q &= s-\nu+r \\ \beta &= j+s-l & m &= \mu+\nu \\ \gamma &= l+s-j & p &= j-\mu-s+r \end{aligned} \tag{A2.10}$$

We see from this that l, s, j must each be integral, or one integral and the two others half-integral.

This gives us

$$\frac{(-1)^{j-m}(2j)!\, X^{j+m} Y^{j-m}}{(j+m)!\,(j-m)!}$$

$$= \sum_{\nu=l-j-r}^{s-r} \sum_{r=0}^{l+s-j} \frac{(j+l-s)!\,(j+s-l)!\,(l+s-j)!\,(-1)^{j-m+r}\, \xi^{l+\mu} \eta^{l-\mu} \xi'^{s+\nu} \eta'^{s-\nu}}{(j-\mu-s+r)!\,(l+\mu-r)!\,(s-\nu-r)!\,(j-l+\nu+r)!\,r!\,(l+s-j-r)!}$$

$$\mu+\nu=m$$

(A2.11)

The implicit condition on the indices is that the factors in the denominator of (11) must be factorials of non-negative num-

bers. (We may allow the indices to run over all values if we recognize that the factorial of a negative number is infinite; hence terms containing them in the denominator vanish). This condition gives us other important relationships

$$\begin{aligned} &j+l-s \geq 0 \\ &j+s-l \geq 0 \qquad\qquad -s \leq \nu \leq s \\ &l+s-j \geq 0 \end{aligned} \tag{A2.12}$$

If we now replace the monomials in (11) by wave functions according to (.1) we have

$$\psi_j^{\,m} = C_{lsj} \sqrt{(j+m)!(j-m)!}\ \frac{(j+l-s)!(j+s-l)!(l+s-j)!}{(2l)!(2s)!}$$

$$\times \sum_{\nu=-s}^{s} \sum_{r} \frac{\sqrt{(l+\mu)!(l-\mu)!(s+\nu)!(s-\nu)!}\,(-1)^{j-l-s+r}\,\psi_l^{\,m-\nu}\psi_s^{\,\nu}}{(j-m+\nu-s+r)!(l-m-\nu-r)!(s-\nu-r)!(j-l+\nu+r)!\,r!(l+s-j-r)!} \tag{A2.13}$$

where the constant C_{lsj} has been introduced because the set of functions $\psi_j^{\,m}$ and the monomials $\dfrac{(-1)^{j-m}(2j)!\,\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!(j-m)!}}$ are not equal but only have the same transformation properties. That C_{lsj} does not depend on the magnetic quantum numbers $m\mu\nu$ can be seen if the identification of monomials with wave functions is made in equation (.7) which is a function only of lsj.

We therefore have

$$A_{jm}^{ls\mu\nu} = C_{lsj} \sum_{r} \frac{\sqrt{(j+m)!(j-m)!(l+\mu)!(l-\mu)!(s+\nu)!(s-\nu)!}\ (j+l-s)!(j+s-l)!(l+s-j)!(-1)^{j-l-s+r}}{(2l)!(2s)!(j-m+\nu-s+r)!(l+m-\nu-r)!(s-\nu-r)!(j-l+\nu+r)!\,r!(l+s-j-r)!}$$

$$A^{l s \mu\nu}_{jm} = B_{lsj} \sum_{r} \frac{(-1)^r \sqrt{(j+m)!(j-m)!(l+\mu)!(l-\mu)!(s+\nu)!(s-\nu)!}}{(j-\mu-s+r)!(j+\nu-l+r)!(l+\mu-r)!(s-\nu-r)!r!(l+s-j-r)!} \qquad \mu+\nu=m \qquad \text{(A2.14)}$$

the summation over r being taken over all values which do not make the denominator infinite.

We must now evaluate the coefficient B_{lsj}. To do this we impose the condition that the transformation $A^{l s \mu\nu}_{jm}$ shall be unitary. Since the A's are real this condition becomes

$$\sum_{\mu\nu} \left| A^{l s \mu\nu}_{jm} \right|^2 = 1 \qquad \text{(A2.15)}$$

This is most easily carried out for the case m = j since then the sum (14) reduces to a single term. Since B_{lsj} is independent of m such a choice will not affect the applicability of the solution to cases where $m \neq j$. For m = j we have $\mu = j-\nu$, $r = s-\nu$ and

$$A^{l s, j-\nu, \nu}_{jj} = B_{lsj} \frac{(-1)^{s-\nu}\sqrt{(2j)!(l+j-\nu)!(l-j+\nu)!(s+\nu)!(s-\nu)!}}{(l+j-s)!(j-l+s)!(s-\nu)!(l-j+\nu)!}$$

$$= \frac{B_{lsj}\sqrt{(2j)!}\,(-1)^{s-\nu}}{(l+j-s)!(j+s-l)!}\sqrt{\frac{(l+j-\nu)!(s+\nu)!}{(l-j+\nu)!(s-\nu)!}} \qquad \text{(A2.16)}$$

Then, (15) becomes

$$\frac{B^2_{lsj}\,(2j)!}{[(l+j-s)!(j+s-l)!]^2} \sum_{\nu=j-l}^{s} \frac{(l+j-\nu)!(s+\nu)!}{(l-j+\nu)!(s-\nu)!} = 1 \qquad \text{(A2.17)}$$

we write this as

$$\frac{B^2_{lsj}\,(2j)!}{(l+j-s)!(s+j-l)!} \sum_{\nu=l-j}^{s} \binom{l+j-\nu}{s-\nu}\binom{s+\nu}{l-j+\nu} = 1 \qquad \text{(A2.18)}$$

where $\binom{\alpha}{\beta}$ is the coefficient of x^{β} in the expansion of $(1+x)^{\alpha}$

$$\binom{\alpha}{\beta} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-\beta+1)}{\beta!} = (-1)^{\beta}\frac{(\beta-\alpha-1)(\beta-\alpha-2)\ldots(1-\alpha)(-\alpha)}{\beta!} = (-1)^{\beta}\binom{\beta-\alpha-1}{\beta} \tag{A2.19}$$

Then the sum in (18) becomes

$$\sum_{\nu=l-j}^{s}\binom{l+j-\nu}{s-\nu}\binom{s+\nu}{l-j+\nu} = \sum_{\nu=l-j}^{s}(-1)^{s+l-j}\binom{s-l-j-1}{s-\nu}\binom{l-j-s-1}{s-\nu}$$

$$= (-1)^{s+l-j}\binom{-2j-2}{l+s-j} = \binom{l+s+j+1}{l+s-j} \tag{A2.20}$$

The summation of the binomial coefficients is verified by identifying the coefficient of x^{l+s-j} on each side of the identity

$$(1+x)^{s-l-j-1}(1+x)^{l-j-s-1} = (1+x)^{-2j-2} \tag{A2.21}$$

Solving (18) for B^{2}_{lsj} we then have

$$B^{2}_{lsj} = \frac{(l+j-s)!\,(s+j-l)!\,(l+s-j)!\,(2j+1)}{(l+s+j+1)!} \tag{A2.22}$$

hence, we now have from (14)

$$A^{lsj}_{\mu\nu m} = \sqrt{\frac{(l+s-j)!\,(l+j-s)!\,(s+j-l)!\,(2j+1)}{(l+s+j+1)!}}\,\delta^{\mu+\nu}_{m}$$

$$\times\sum_{r}\frac{(-1)^{r}\sqrt{(j+m)!\,(j-m)!\,(l+\mu)!\,(l-\mu)!\,(s+\nu)!\,(s-\nu)!}}{(j-\mu-s+r)!\,(j-l+\nu+r)!\,(l+\mu-r)!\,(s-\nu-r)!\,r!\,(l+s-j-r)!} \tag{A2.23}$$

The coefficients $A^{ls\mu\nu}_{jm}$ are tabulated in the following pages. Since the wave functions are representations (Darstellungen) of the rotation group we have labeled each table with the heading $D_l \times D_s$. The symbols on the left stand for the product of any two wave functions with l and s given by the subscripts, μ and ν given by the superscripts. The symbols along the top of the table again stand for wave functions with subscript j and superscript m. The element $A^{ls\mu\nu}_{jm}$ then stands at the intersection of the row labeled $U^{\mu}_{l} V^{\nu}_{s}$ and the column labeled W^{m}_{j}. All empty intersections are zero. Each sub-matrix is then an explicit example of the general group theoretical formula

$$\mathfrak{D}_l \times \mathfrak{D}_s = \mathfrak{D}_{l+s} + \mathfrak{D}_{l+s-1} + \cdots + \mathfrak{D}_{|l-s|} \qquad \text{(A2.24)}$$

The coefficients A may also be thought of as a transformation of coordinate axes in function space from one set of orthogonal "base vectors" $e_{\mu\nu} = \psi^{\mu}_{l} \psi^{\nu}_{s}$ to another orthogonal set $\varepsilon_{jm} = \psi^{m}_{j}$.

	W_{l+1}^{m}	W_{l}^{m}	W_{l-1}^{m}
$U_l^{m+1} V_1^{-1}$	$\sqrt{\frac{(l-m+1)(l-m)}{(2l+1)(2l+2)}}$	$\sqrt{\frac{(l-m)(l+m+1)}{l(2l+2)}}$	$\sqrt{\frac{(l+m)(l+m+1)}{(2l)(2l+1)}}$
$U_l^{m} V_1^{0}$	$\sqrt{\frac{2(l+m+1)(l-m+1)}{(2l+1)(2l+2)}}$	$\frac{m}{\sqrt{l(l+1)}}$	$-\sqrt{\frac{2(l-m)(l+m)}{(2l)(2l+1)}}$
$U_l^{m-1} V_1^{1}$	$\sqrt{\frac{(l+m+1)(l+m)}{(2l+1)(2l+2)}}$	$-\sqrt{\frac{(l+m)(l-m+1)}{l(2l+2)}}$	$\sqrt{\frac{(l-m)(l-m+1)}{(2l)(2l+1)}}$

$\mathfrak{D}_1 \times \mathfrak{D}_1$

	$W^m_{l+3/2}$	$W^m_{l+1/2}$	$W^m_{l-1/2}$	$W^m_{l-3/2}$
$U_l^{m+3/2} V_{3/2}^{-3/2}$	$\sqrt{\frac{(l-m-1/2)(l-m+1/2)(l-m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$-\sqrt{\frac{3(l-m-1/2)(l-m+1/2)(l+m+3/2)}{(2l+3)(2l+1)(2l)}}$	$\sqrt{\frac{3(l-m+1/2)(l-m+3/2)(l+m-1/2)}{(2l-1)(2l+1)(2l+2)}}$	$-\sqrt{\frac{(l+m-1/2)(l+m+1/2)(l+m+3/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m+1/2} V_{3/2}^{-1/2}$	$\sqrt{\frac{3(l+m+3/2)(l-m+1/2)(l-m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$-(l+3m+3/2)\sqrt{\frac{(l-m+1/2)}{(2l+3)(2l+1)(2l)}}$	$-(l-3m-1/2)\sqrt{\frac{(l+m+1/2}{(2l-1)(2l+1)(2l+2)}}$	$\sqrt{\frac{3(l-m-1/2)(l+m-1/2)(l+m+1/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m-1/2} V_{3/2}^{1/2}$	$\sqrt{\frac{3(l+m+3/2)(l+m+1/2)(l-m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$(l-3m+3/2)\sqrt{\frac{(l+m+1/2)}{(2l+3)(2l+1)(2l)}}$	$-(l+3m-1/2)\sqrt{\frac{(l-m+1/2)}{(2l-1)(2l+1)(2l+2)}}$	$-\sqrt{\frac{3(l-m-1/2)(l-m+1/2)(l+m-1/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m-3/2} V_{3/2}^{3/2}$	$\sqrt{\frac{(l+m+3/2)(l+m+1/2)(l+m-1/2)}{(2l+1)(2l+2)(2l+3)}}$	$\sqrt{\frac{3(l+m-1/2)(l+m+1/2)(l-m+3/2)}{(2l+3)(2l+1)(2l)}}$	$\sqrt{\frac{3(l-m+1/2)(l-m+3/2)(l+m-1/2)}{(2l-1)(2l+1)(2l+2)}}$	$\sqrt{\frac{(l-m-1/2)(l-m+1/2)(l-m+3/2)}{(2l-1)(2l)(2l+1)}}$

$$D_l \times D_{3/2}$$

	W_1^1	W_1^0 W_0^0	W_1^{-1}
$U_{1/2}^{1/2}$ $V_{1/2}^{1/2}$	1		
$U_{1/2}^{1/2}$ $V_{1/2}^{-1/2}$		$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$	
$U_{-1/2}^{-1/2}$ $V_{1/2}^{1/2}$		$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$	
$U_{-1/2}^{-1/2}$ $V_{1/2}^{-1/2}$			1

$$\mathfrak{D}_{1/2} \times \mathfrak{D}_{1/2}$$

	$W^{3/2}_{3/2}$	$W^{1/2}_{3/2}$ $W^{1/2}_{1/2}$	$W^{-1/2}_{3/2}$ $W^{-1/2}_{1/2}$	$W^{-3/2}_{3/2}$
U^{1}_{1} $V^{1/2}_{1/2}$	1			
U^{1}_{1} $V^{-1/2}_{1/2}$		$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$		
U^{0}_{1} $V^{1/2}_{1/2}$		$\sqrt{\frac{2}{3}}$ $-\sqrt{\frac{1}{3}}$		
U^{0}_{1} $V^{-1/2}_{1/2}$			$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$	
U^{-1}_{1} $V^{1/2}_{1/2}$			$\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{2}{3}}$	
U^{-1}_{1} $V^{-1/2}_{1/2}$				1

$$D_1 \times D_{1/2}$$

	W_2^2	$W_2^1 W_1^1$	$W_2^0 W_1^0$	$W_2^{-1} W_1^{-1}$	W_2^{-2}
$U_{3/2}^{3/2} V_{1/2}^{1/2}$	1				
$U_{3/2}^{3/2} V_{1/2}^{-1/2}$		$\sqrt{\frac{1}{4}}\ \sqrt{\frac{3}{4}}$			
$U_{3/2}^{1/2} V_{1/2}^{1/2}$		$\sqrt{\frac{3}{4}}\ -\sqrt{\frac{1}{4}}$			
$U_{3/2}^{1/2} V_{1/2}^{-1/2}$			$\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}}$		
$U_{3/2}^{-1/2} V_{1/2}^{1/2}$			$\sqrt{\frac{1}{2}}\ -\sqrt{\frac{1}{2}}$		
$U_{3/2}^{-1/2} V_{1/2}^{-1/2}$				$\sqrt{\frac{3}{4}}\ \sqrt{\frac{1}{4}}$	
$U_{3/2}^{-3/2} V_{1/2}^{1/2}$				$\sqrt{\frac{1}{4}}\ -\sqrt{\frac{3}{4}}$	
$U_{3/2}^{-3/2} V_{1/2}^{-1/2}$					1

$$\mathfrak{D}_{3/2} \times \mathfrak{D}_{1/2}$$

	W_2^2	W_2^1 W_1^1	W_2^0 W_1^0 W_0^0	W_2^{-1} W_1^{-1}	W_2^{-2}
U_1^1 V_1^1	1				
U_1^1 V_1^0		$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$			
U_1^0 V_1^1		$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$			
U_1^1 V_1^{-1}			$\sqrt{\frac{1}{6}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{3}}$		
U_1^0 V_1^0			$\sqrt{\frac{2}{3}}$ 0 $-\sqrt{\frac{1}{3}}$		
U_1^{-1} V_1^1			$\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{3}}$		
U_1^0 V_1^{-1}				$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$	
U_1^{-1} V_1^0				$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$	
U_1^{-1} V_1^{-1}					1

$\mathfrak{D}_1 \times \mathfrak{D}_1$

	$W^{5/2}_{5/2}$	$W^{3/2}_{5/2}$ $W^{3/2}_{3/2}$	$W^{1/2}_{5/2}$ $W^{1/2}_{3/2}$	$W^{-1/2}_{5/2}$ $W^{-1/2}_{3/2}$	$W^{-3/2}_{5/2}$ $W^{-3/2}_{3/2}$	$W^{-5/2}_{5/2}$
$U^{2}_{2}\ V^{1/2}_{1/2}$	1					
$U^{2}_{2}\ V^{-1/2}_{1/2}$		$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{4}{5}}$				
$U^{1}_{2}\ V^{1/2}_{1/2}$		$\sqrt{\frac{4}{5}}$ $-\sqrt{\frac{1}{5}}$				
$U^{1}_{2}\ V^{-1/2}_{1/2}$			$\sqrt{\frac{2}{5}}$ $\sqrt{\frac{3}{5}}$			
$U^{0}_{2}\ V^{1/2}_{1/2}$			$\sqrt{\frac{3}{5}}$ $-\sqrt{\frac{2}{5}}$			
$U^{0}_{2}\ V^{-1/2}_{1/2}$				$\sqrt{\frac{3}{5}}$ $\sqrt{\frac{2}{5}}$		
$U^{-1}_{2}\ V^{1/2}_{1/2}$				$\sqrt{\frac{2}{5}}$ $-\sqrt{\frac{3}{5}}$		
$U^{-1}_{2}\ V^{-1/2}_{1/2}$					$\sqrt{\frac{4}{5}}$ $\sqrt{\frac{1}{5}}$	
$U^{-2}_{2}\ V^{1/2}_{1/2}$					$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{4}{5}}$	
$U^{-2}_{2}\ V^{-1/2}_{1/2}$						1

$$\mathfrak{D}_2 \times \mathfrak{D}_{1/2}$$

	$W^{5/2}_{5/2}$	$W^{3/2}_{5/2}$	$W^{3/2}_{3/2}$	$W^{1/2}_{5/2}$	$W^{1/2}_{3/2}$	$W^{1/2}_{1/2}$	$W^{-1/2}_{5/2}$	$W^{-1/2}_{3/2}$	$W^{-1/2}_{1/2}$	$W^{-3/2}_{5/2}$	$W^{-3/2}_{3/2}$	$W^{-5/2}_{5/2}$
$U^{3/2}_{3/2}\ V^{1}_{1}$	1											
$U^{3/2}_{3/2}\ V^{0}_{1}$		$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$									
$U^{1/2}_{3/2}\ V^{1}_{1}$		$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$									
$U^{3/2}_{3/2}\ V^{-1}_{1}$				$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{2}}$						
$U^{1/2}_{3/2}\ V^{0}_{1}$				$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{3}}$						
$U^{-1/2}_{3/2}\ V^{1}_{1}$				$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$						
$U^{1/2}_{3/2}\ V^{-1}_{1}$							$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$			
$U^{-1/2}_{3/2}\ V^{0}_{1}$							$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{3}}$			
$U^{-3/2}_{3/2}\ V^{1}_{1}$							$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{2}}$			
$U^{-1/2}_{3/2}\ V^{-1}_{1}$										$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$	
$U^{-3/2}_{3/2}\ V^{0}_{1}$										$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{5}}$	
$U^{-3/2}_{3/2}\ V^{-1}_{1}$												1

$$\mathfrak{D}_{3/2} \times \mathfrak{D}_{1}$$

	W_3^3	W_3^2 W_2^2	W_3^1 W_2^1	W_3^0 W_2^0	W_3^{-1} W_2^{-1}	W_3^{-2} W_2^{-2}	W_3^{-3}
$U_{5/2}^{5/2}\ V_{1/2}^{1/2}$	1						
$U_{5/2}^{5/2}\ V_{1/2}^{-1/2}$		$\sqrt{\frac{1}{6}}$ $\sqrt{\frac{5}{6}}$					
$U_{5/2}^{3/2}\ V_{1/2}^{1/2}$		$\sqrt{\frac{5}{6}}$ $-\sqrt{\frac{1}{6}}$					
$U_{5/2}^{3/2}\ V_{1/2}^{-1/2}$			$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$				
$U_{5/2}^{1/2}\ V_{1/2}^{1/2}$			$\sqrt{\frac{2}{3}}$ $-\sqrt{\frac{1}{3}}$				
$U_{5/2}^{1/2}\ V_{1/2}^{-1/2}$				$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$			
$U_{5/2}^{-1/2}\ V_{1/2}^{1/2}$				$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$			
$U_{5/2}^{-1/2}\ V_{1/2}^{-1/2}$					$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$		
$U_{5/2}^{-3/2}\ V_{1/2}^{1/2}$					$\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{2}{3}}$		
$U_{5/2}^{-3/2}\ V_{1/2}^{-1/2}$						$\sqrt{\frac{5}{6}}$ $\sqrt{\frac{1}{6}}$	
$U_{5/2}^{-5/2}\ V_{1/2}^{1/2}$						$\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{5}{6}}$	
$U_{5/2}^{-5/2}\ V_{1/2}^{-1/2}$							1

$$\mathfrak{D}_{5/2} \times \mathfrak{D}_{1/2}$$

	W_3^3	W_3^2 W_2^2	W_3^1 W_2^1 W_1^1	W_3^0 W_2^0 W_1^0	W_3^{-1} W_2^{-1} W_1^{-1}	W_3^{-2} W_2^{-2}	W_3^{-3}
$U_2^2 V_1^1$	1						
$U_2^2 V_1^0$		$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$					
$U_2^1 V_1^1$		$\sqrt{\frac{2}{3}}$ $-\sqrt{\frac{1}{3}}$					
$U_2^2 V_1^{-1}$			$\sqrt{\frac{1}{15}}$ $\sqrt{\frac{1}{3}}$ $\sqrt{\frac{3}{5}}$				
$U_2^1 V_1^0$			$\sqrt{\frac{8}{15}}$ $\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{3}{10}}$				
$U_2^0 V_1^1$			$\sqrt{\frac{6}{15}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{10}}$				
$U_2^1 V_1^{-1}$				$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$			
$U_2^0 V_1^0$				$\sqrt{\frac{3}{5}}$ 0 $-\sqrt{\frac{2}{5}}$			
$U_2^{-1} V_1^1$				$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$			
$U_2^0 V_1^{-1}$					$\sqrt{\frac{6}{15}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{10}}$		
$U_2^{-1} V_1^0$					$\sqrt{\frac{8}{15}}$ $-\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{3}{10}}$		
$U_2^{-2} V_1^1$					$\sqrt{\frac{1}{15}}$ $-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{3}{5}}$		
$U_2^{-1} V_1^{-1}$						$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$	
$U_2^{-2} V_1^0$						$\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{2}{3}}$	
$U_2^{-2} V_1^{-1}$							1

$\mathfrak{D}_2 \times \mathfrak{D}_1$

	W_3^3	W_3^2 W_2^2	W_3^1 W_2^1 W_1^1	W_3^0 W_2^0 W_1^0 W_0^0	W_3^{-1} W_2^{-1} W_1^{-1}	W_3^{-2} W_2^{-2}	W_3^{-3}
$U_{3/2}^{3/2} V_{3/2}^{3/2}$	1						
$U_{3/2}^{3/2} V_{3/2}^{1/2}$		$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$					
$U_{3/2}^{1/2} V_{3/2}^{3/2}$		$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$					
$U_{3/2}^{3/2} V_{3/2}^{-1/2}$			$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$				
$U_{3/2}^{3/2} V_{3/2}^{1/2}$			$\sqrt{\frac{3}{5}}$ 0 $-\sqrt{\frac{2}{5}}$				
$U_{3/2}^{-1/2} V_{3/2}^{3/2}$			$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$				
$U_{3/2}^{3/2} V_{3/2}^{-3/2}$				$\sqrt{\frac{1}{20}}$ $\sqrt{\frac{1}{4}}$ $\sqrt{\frac{9}{20}}$ $\sqrt{\frac{1}{4}}$			
$U_{3/2}^{1/2} V_{3/2}^{-1/2}$				$\sqrt{\frac{9}{20}}$ $\sqrt{\frac{1}{4}}$ $-\sqrt{\frac{1}{20}}$ $-\sqrt{\frac{1}{4}}$			
$U_{3/2}^{-1/2} V_{3/2}^{1/2}$				$\sqrt{\frac{9}{20}}$ $-\sqrt{\frac{1}{4}}$ $-\sqrt{\frac{1}{20}}$ $\sqrt{\frac{1}{4}}$			
$U_{3/2}^{-3/2} V_{3/2}^{3/2}$				$\sqrt{\frac{1}{20}}$ $-\sqrt{\frac{1}{4}}$ $\sqrt{\frac{9}{20}}$ $-\sqrt{\frac{1}{4}}$			
$U_{3/2}^{1/2} V_{3/2}^{-3/2}$					$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$		
$U_{3/2}^{-1/2} V_{3/2}^{-1/2}$					$\sqrt{\frac{3}{5}}$ 0 $-\sqrt{\frac{2}{5}}$		
$U_{3/2}^{-3/2} V_{3/2}^{1/2}$					$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$		
$U_{3/2}^{-1/2} V_{3/2}^{-3/2}$						$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$	
$U_{3/2}^{-3/2} V_{3/2}^{-1/2}$						$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$	
$U_{3/2}^{-3/2} V_{3/2}^{-3/2}$							1

$$\mathfrak{D}_{3/2} \times \mathfrak{D}_{3/2}$$

	$W^{7/2}_{7/2}$	$W^{5/2}_{7/2}$	$W^{5/2}_{5/2}$	$W^{3/2}_{7/2}$	$W^{3/2}_{5/2}$	$W^{1/2}_{7/2}$	$W^{1/2}_{5/2}$	$W^{-1/2}_{7/2}$	$W^{-1/2}_{5/2}$	$W^{-3/2}_{7/2}$	$W^{-3/2}_{5/2}$	$W^{-5/2}_{7/2}$	$W^{-5/2}_{5/2}$	$W^{-7/2}_{7/2}$
$U_3^3\ V_{1/2}^{1/2}$	1													
$U_3^3\ V_{1/2}^{-1/2}$		$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{6}{7}}$											
$U_3^2\ V_{1/2}^{1/2}$		$\sqrt{\frac{6}{7}}$	$-\sqrt{\frac{1}{7}}$											
$U_3^2\ V_{1/2}^{-1/2}$				$\sqrt{\frac{2}{7}}$	$\sqrt{\frac{5}{7}}$									
$U_3^1\ V_{1/2}^{1/2}$				$\sqrt{\frac{5}{7}}$	$-\sqrt{\frac{2}{7}}$									
$U_3^1\ V_{1/2}^{-1/2}$						$\sqrt{\frac{3}{7}}$	$\sqrt{\frac{4}{7}}$							
$U_3^0\ V_{1/2}^{1/2}$						$\sqrt{\frac{4}{7}}$	$-\sqrt{\frac{3}{7}}$							
$U_3^0\ V_{1/2}^{-1/2}$								$\sqrt{\frac{4}{7}}$	$\sqrt{\frac{3}{7}}$					
$U_3^{-1}\ V_{1/2}^{1/2}$								$\sqrt{\frac{3}{7}}$	$-\sqrt{\frac{4}{7}}$					
$U_3^{-1}\ V_{1/2}^{-1/2}$										$\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$			
$U_3^{-2}\ V_{1/2}^{1/2}$										$\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{5}{7}}$			
$U_3^{-2}\ V_{1/2}^{-1/2}$												$\sqrt{\frac{6}{7}}$	$\sqrt{\frac{1}{7}}$	
$U_3^{-2}\ V_{1/2}^{1/2}$												$\sqrt{\frac{1}{7}}$	$-\sqrt{\frac{6}{7}}$	
$U_3^{-3}\ V_{1/2}^{-1/2}$														1

$$\mathfrak{D}_3 \times \mathfrak{D}_{1/2}$$

	$W^{7/2}_{7/2}$	$W^{5/2}_{7/2}$	$W^{5/2}_{-7/2}$	$W^{3/2}_{7/2}$	$W^{3/2}_{5/2}$	$W^{3/2}_{-3/2}$	$W^{1/2}_{7/2}$	$W^{1/2}_{5/2}$	$W^{1/2}_{-3/2}$	$W^{-1/2}_{7/2}$	$W^{-1/2}_{5/2}$	$W^{-1/2}_{-3/2}$	$W^{-3/2}_{7/2}$	$W^{-3/2}_{5/2}$	$W^{-3/2}_{-3/2}$	$W^{-5/2}_{7/2}$	$W^{-5/2}_{-5/2}$	$W^{-7/2}_{-7/2}$
$U^{5/2}_{5/2}\ V^{1}_{1}$	1																	
$U^{5/2}_{5/2}\ V^{0}_{1}$		$\sqrt{\frac{2}{7}}$	$\sqrt{\frac{5}{7}}$															
$U^{3/2}_{5/2}\ V^{1}_{1}$		$\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$															
$U^{5/2}_{5/2}\ V^{-1}_{1}$				$\sqrt{\frac{1}{21}}$	$\sqrt{\frac{2}{7}}$	$\sqrt{\frac{2}{3}}$												
$U^{3/2}_{5/2}\ V^{0}_{1}$				$\sqrt{\frac{10}{21}}$	$\sqrt{\frac{9}{35}}$	$-\sqrt{\frac{4}{15}}$												
$U^{1/2}_{5/2}\ V^{1}_{1}$				$\sqrt{\frac{10}{21}}$	$-\sqrt{\frac{16}{35}}$	$\sqrt{\frac{1}{15}}$												
$U^{3/2}_{5/2}\ V^{-1}_{1}$							$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{16}{35}}$	$\sqrt{\frac{2}{5}}$									
$U^{1/2}_{5/2}\ V^{0}_{1}$							$\sqrt{\frac{4}{7}}$	$\sqrt{\frac{1}{35}}$	$-\sqrt{\frac{2}{5}}$									
$U^{-1/2}_{5/2}\ V^{1}_{1}$							$\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{18}{35}}$	$\sqrt{\frac{1}{5}}$									
$U^{1/2}_{5/2}\ V^{-1}_{1}$										$\sqrt{\frac{2}{7}}$	$\sqrt{\frac{18}{35}}$	$\sqrt{\frac{1}{5}}$						
$U^{-1/2}_{5/2}\ V^{0}_{1}$										$\sqrt{\frac{4}{7}}$	$-\sqrt{\frac{1}{35}}$	$-\sqrt{\frac{2}{5}}$						
$U^{-3/2}_{5/2}\ V^{1}_{1}$										$\sqrt{\frac{1}{7}}$	$-\sqrt{\frac{16}{35}}$	$\sqrt{\frac{2}{5}}$						
$U^{-1/2}_{5/2}\ V^{-1}_{1}$													$\sqrt{\frac{10}{21}}$	$\sqrt{\frac{16}{35}}$	$\sqrt{\frac{1}{15}}$			
$U^{-3/2}_{5/2}\ V^{0}_{1}$													$\sqrt{\frac{10}{21}}$	$-\sqrt{\frac{9}{35}}$	$-\sqrt{\frac{4}{15}}$			
$U^{-5/2}_{5/2}\ V^{1}_{1}$													$\sqrt{\frac{1}{21}}$	$-\sqrt{\frac{2}{7}}$	$\sqrt{\frac{2}{3}}$			
$U^{-3/2}_{5/2}\ V^{-1}_{1}$																$\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$	
$U^{-5/2}_{5/2}\ V^{0}_{1}$																$\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{5}{7}}$	
$U^{-5/2}_{5/2}\ V^{-1}_{1}$																		1

$$\mathfrak{D}_{5/2} \times \mathfrak{D}_{1}$$

	$W^{7/2}_{7/2}$	$W^{5/2}_{7/2}$ $W^{5/2}_{5/2}$	$W^{3/2}_{7/2}$ $W^{3/2}_{5/2}$ $W^{3/2}_{3/2}$	$W^{1/2}_{7/2}$ $W^{1/2}_{5/2}$ $W^{1/2}_{3/2}$ $W^{1/2}_{1/2}$	$W^{-1/2}_{7/2}$ $W^{-1/2}_{5/2}$ $W^{-1/2}_{3/2}$ $W^{-1/2}_{1/2}$	$W^{-3/2}_{7/2}$ $W^{-3/2}_{5/2}$ $W^{-3/2}_{3/2}$	$W^{-5/2}_{7/2}$ $W^{-5/2}_{5/2}$	$W^{-7/2}_{7/2}$
$U_2^2\ V_{3/2}^{3/2}$	1							
$U_2^2\ V_{3/2}^{1/2}$		$\sqrt{\frac{3}{7}}\ \sqrt{\frac{4}{7}}$						
$U_2^1\ V_{3/2}^{3/2}$		$\sqrt{\frac{4}{7}}\ -\sqrt{\frac{3}{7}}$						
$U_2^2\ V_{3/2}^{-1/2}$			$\sqrt{\frac{1}{7}}\ \sqrt{\frac{16}{35}}\ \sqrt{\frac{2}{5}}$					
$U_2^1\ V_{3/2}^{1/2}$			$\sqrt{\frac{4}{7}}\ \sqrt{\frac{18}{35}}\ -\sqrt{\frac{2}{5}}$					
$U_2^0\ V_{3/2}^{3/2}$			$\sqrt{\frac{2}{7}}\ -\sqrt{\frac{18}{35}}\ \sqrt{\frac{1}{5}}$					
$U_2^2\ V_{3/2}^{-3/2}$				$\sqrt{\frac{1}{35}}\ \sqrt{\frac{6}{35}}\ \sqrt{\frac{2}{5}}\ \sqrt{\frac{2}{5}}$				
$U_2^1\ V_{3/2}^{-1/2}$				$\sqrt{\frac{12}{35}}\ \sqrt{\frac{5}{14}}\ 0\ -\sqrt{\frac{3}{10}}$				
$U_2^0\ V_{3/2}^{1/2}$				$\sqrt{\frac{18}{35}}\ -\sqrt{\frac{3}{35}}\ -\sqrt{\frac{1}{5}}\ \sqrt{\frac{1}{5}}$				
$U_2^{-1}\ V_{3/2}^{3/2}$				$\sqrt{\frac{4}{35}}\ -\sqrt{\frac{27}{70}}\ \sqrt{\frac{2}{5}}\ -\sqrt{\frac{1}{10}}$				
$U_2^1\ V_{3/2}^{-3/2}$					$\sqrt{\frac{4}{35}}\ \sqrt{\frac{27}{70}}\ \sqrt{\frac{2}{5}}\ \sqrt{\frac{1}{10}}$			
$U_2^0\ V_{3/2}^{-1/2}$					$\sqrt{\frac{18}{35}}\ \sqrt{\frac{3}{35}}\ -\sqrt{\frac{1}{5}}\ -\sqrt{\frac{1}{5}}$			
$U_2^{-1}\ V_{3/2}^{1/2}$					$\sqrt{\frac{12}{35}}\ -\sqrt{\frac{5}{14}}\ 0\ \sqrt{\frac{3}{10}}$			
$U_2^{-2}\ V_{3/2}^{3/2}$					$\sqrt{\frac{1}{35}}\ -\sqrt{\frac{6}{35}}\ \sqrt{\frac{2}{5}}\ -\sqrt{\frac{2}{5}}$			
$U_2^0\ V_{3/2}^{-3/2}$						$\sqrt{\frac{2}{7}}\ \sqrt{\frac{18}{35}}\ \sqrt{\frac{1}{5}}$		
$U_2^{-1}\ V_{3/2}^{-1/2}$						$\sqrt{\frac{4}{7}}\ -\sqrt{\frac{1}{35}}\ -\sqrt{\frac{2}{5}}$		
$U_2^{-2}\ V_{3/2}^{1/2}$						$\sqrt{\frac{1}{7}}\ -\sqrt{\frac{16}{35}}\ \sqrt{\frac{2}{5}}$		
$U_2^{-1}\ V_{3/2}^{-3/2}$							$\sqrt{\frac{4}{7}}\ \sqrt{\frac{3}{7}}$	
$U_2^{-2}\ V_{3/2}^{-1/2}$							$\sqrt{\frac{3}{7}}\ -\sqrt{\frac{4}{7}}$	
$U_2^{-2}\ V_{3/2}^{-3/2}$								1

$D_2 \times D_{3/2}$

	W_4^4	W_4^3 W_3^3	W_4^2 W_3^2	W_4^1 W_3^1	W_4^0 W_3^0	W_4^{-1} W_3^{-1}	W_4^{-2} W_3^{-2}	W_4^{-3} W_3^{-3}	W_4^{-4}
$U^{7/2}_{7/2} V^{1/2}_{1/2}$	1								
$U^{7/2}_{7/2} V^{-1/2}_{1/2}$		$\sqrt{\frac{1}{8}}$ $\sqrt{\frac{7}{8}}$							
$U^{5/2}_{7/2} V^{1/2}_{1/2}$		$\sqrt{\frac{7}{8}}$ $-\sqrt{\frac{1}{8}}$							
$U^{5/2}_{7/2} V^{-1/2}_{1/2}$			$\sqrt{\frac{1}{4}}$ $\sqrt{\frac{3}{4}}$						
$U^{3/2}_{7/2} V^{1/2}_{1/2}$			$\sqrt{\frac{3}{4}}$ $-\sqrt{\frac{1}{4}}$						
$U^{3/2}_{7/2} V^{-1/2}_{1/2}$				$\sqrt{\frac{3}{8}}$ $\sqrt{\frac{5}{8}}$					
$U^{1/2}_{7/2} V^{1/2}_{1/2}$				$\sqrt{\frac{5}{8}}$ $-\sqrt{\frac{3}{8}}$					
$U^{1/2}_{7/2} V^{-1/2}_{1/2}$					$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$				
$U^{-1/2}_{7/2} V^{1/2}_{1/2}$					$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$				
$U^{-1/2}_{7/2} V^{-1/2}_{1/2}$						$\sqrt{\frac{5}{8}}$ $\sqrt{\frac{3}{8}}$			
$U^{-3/2}_{7/2} V^{1/2}_{1/2}$						$\sqrt{\frac{3}{8}}$ $-\sqrt{\frac{5}{8}}$			
$U^{-3/2}_{7/2} V^{-1/2}_{1/2}$							$\sqrt{\frac{3}{4}}$ $\sqrt{\frac{1}{4}}$		
$U^{-5/2}_{7/2} V^{1/2}_{1/2}$							$\sqrt{\frac{1}{4}}$ $-\sqrt{\frac{3}{4}}$		
$U^{-5/2}_{7/2} V^{-1/2}_{1/2}$								$\sqrt{\frac{7}{8}}$ $\sqrt{\frac{1}{8}}$	
$U^{-7/2}_{7/2} V^{1/2}_{1/2}$								$\sqrt{\frac{1}{8}}$ $-\sqrt{\frac{7}{8}}$	
$U^{-7/2}_{7/2} V^{-1/2}_{1/2}$									1

$$\mathfrak{D}_{7/2} \times \mathfrak{D}_{1/2}$$

	W_4^4	W_4^3 W_3^3	W_4^2 W_3^2 W_2^2	W_4^1 W_3^1 W_2^1	W_4^0 W_3^0 W_2^0	W_4^{-1} W_3^{-1} W_2^{-1}	W_4^{-2} W_3^{-2} W_2^{-2}	W_4^{-3} W_3^{-3}	W_4^{-4}
$U_3^3\ V_1^1$	1								
$U_3^3\ V_1^0$		$\sqrt{\frac{1}{4}}\ \sqrt{\frac{3}{4}}$							
$U_3^2\ V_1^1$		$\sqrt{\frac{3}{4}}\ -\sqrt{\frac{1}{4}}$							
$U_3^3\ V_1^{-1}$			$\sqrt{\frac{1}{28}}\ \sqrt{\frac{1}{4}}\ \sqrt{\frac{5}{7}}$						
$U_3^2\ V_1^0$			$\sqrt{\frac{3}{7}}\ \sqrt{\frac{1}{3}}\ -\sqrt{\frac{5}{21}}$						
$U_3^1\ V_1^1$			$\sqrt{\frac{15}{28}}\ -\sqrt{\frac{5}{12}}\ \sqrt{\frac{1}{21}}$						
$U_3^2\ V^{-1}$				$\sqrt{\frac{3}{28}}\ \sqrt{\frac{5}{12}}\ \sqrt{\frac{10}{21}}$					
$U_3^1\ V_1^0$				$\sqrt{\frac{15}{28}}\ \sqrt{\frac{1}{12}}\ -\sqrt{\frac{8}{21}}$					
$U_3^0\ V_1^1$				$\sqrt{\frac{10}{28}}\ -\sqrt{\frac{6}{12}}\ \sqrt{\frac{3}{21}}$					
$U_3^1\ V_1^{-1}$					$\sqrt{\frac{3}{14}}\ \sqrt{\frac{1}{2}}\ \sqrt{\frac{2}{7}}$				
$U_3^0\ V_1^0$					$\sqrt{\frac{8}{14}}\ 0\ -\sqrt{\frac{3}{7}}$				
$U_3^{-1}\ V_1^1$					$\sqrt{\frac{3}{14}}\ -\sqrt{\frac{1}{2}}\ \sqrt{\frac{2}{7}}$				
$U_3^0\ V_1^{-1}$						$\sqrt{\frac{10}{28}}\ \sqrt{\frac{6}{12}}\ \sqrt{\frac{3}{21}}$			
$U_3^{-1}\ V_1^0$						$\sqrt{\frac{15}{28}}\ -\sqrt{\frac{1}{12}}\ -\sqrt{\frac{8}{21}}$			
$U_3^{-2}\ V_1^1$						$\sqrt{\frac{3}{28}}\ -\sqrt{\frac{5}{12}}\ \sqrt{\frac{10}{21}}$			
$U_3^{-1}\ V_1^{-1}$							$\sqrt{\frac{15}{28}}\ \sqrt{\frac{5}{12}}\ \sqrt{\frac{1}{21}}$		
$U_3^{-2}\ V_1^0$							$\sqrt{\frac{3}{7}}\ -\sqrt{\frac{1}{3}}\ -\sqrt{\frac{5}{21}}$		
$U_3^{-3}\ V_1^1$							$\sqrt{\frac{1}{28}}\ -\sqrt{\frac{1}{4}}\ \sqrt{\frac{5}{7}}$		
$U_3^{-2}\ V_1^{-1}$								$\sqrt{\frac{3}{4}}\ \sqrt{\frac{1}{4}}$	
$U_3^{-3}\ V_1^0$								$\sqrt{\frac{1}{4}}\ -\sqrt{\frac{3}{4}}$	
$U_3^{-3}\ V_1^{-1}$									1

$\mathfrak{D}_3 \times \mathfrak{D}_1$

	W_4^4	W_4^3 W_3^3	W_4^2 W_3^2 W_2^2	W_4^1 W_3^1 W_2^1 W_1^1	W_4^0 W_3^0 W_2^0 W_1^0	W_4^{-1} W_3^{-1} W_2^{-1} W_1^{-1}	W_4^{-2} W_3^{-2} W_2^{-2}	W_4^{-3} W_4^{-3}	W_4^{-4}
$U_{5/2}^{5/2}\ V_{3/2}^{3/2}$	1								
$U_{5/2}^{5/2}\ V_{3/2}^{1/2}$		$\sqrt{\frac{3}{8}}\ \sqrt{\frac{5}{8}}$							
$U_{5/2}^{3/2}\ V_{3/2}^{3/2}$		$\sqrt{\frac{5}{8}}\ -\sqrt{\frac{3}{8}}$							
$U_{5/2}^{5/2}\ V_{3/2}^{-1/2}$			$\sqrt{\frac{3}{28}}\ \sqrt{\frac{5}{12}}\ \sqrt{\frac{10}{21}}$						
$U_{5/2}^{3/2}\ V_{3/2}^{1/2}$			$\sqrt{\frac{15}{28}}\ \sqrt{\frac{1}{12}}\ -\sqrt{\frac{8}{21}}$						
$U_{5/2}^{1/2}\ V_{3/2}^{3/2}$			$\sqrt{\frac{10}{28}}\ -\sqrt{\frac{6}{12}}\ \sqrt{\frac{3}{21}}$						
$U_{5/2}^{5/2}\ V_{3/2}^{-3/2}$				$\sqrt{\frac{1}{56}}\ \sqrt{\frac{1}{8}}\ \sqrt{\frac{5}{14}}\ \sqrt{\frac{1}{2}}$					
$U_{5/2}^{3/2}\ V_{3/2}^{-1/2}$				$\sqrt{\frac{15}{56}}\ \sqrt{\frac{49}{120}}\ \sqrt{\frac{1}{42}}\ -\sqrt{\frac{3}{10}}$					
$U_{5/2}^{1/2}\ V_{3/2}^{1/2}$				$\sqrt{\frac{30}{56}}\ -\sqrt{\frac{1}{60}}\ -\sqrt{\frac{25}{84}}\ \sqrt{\frac{3}{20}}$					
$U_{5/2}^{-1/2}\ V_{3/2}^{3/2}$				$\sqrt{\frac{10}{56}}\ -\sqrt{\frac{9}{20}}\ \sqrt{\frac{27}{84}}\ -\sqrt{\frac{1}{20}}$					
$U_{5/2}^{3/2}\ V_{3/2}^{-3/2}$					$\sqrt{\frac{1}{14}}\ \sqrt{\frac{3}{10}}\ \sqrt{\frac{3}{7}}\ \sqrt{\frac{1}{5}}$				
$U_{5/2}^{1/2}\ V_{3/2}^{-1/2}$					$\sqrt{\frac{3}{7}}\ \sqrt{\frac{1}{5}}\ -\sqrt{\frac{1}{14}}\ -\sqrt{\frac{3}{10}}$				
$U_{5/2}^{-1/2}\ V_{3/2}^{1/2}$					$\sqrt{\frac{3}{7}}\ -\sqrt{\frac{1}{5}}\ -\sqrt{\frac{1}{14}}\ \sqrt{\frac{3}{10}}$				
$U_{5/2}^{-3/2}\ V_{3/2}^{3/2}$					$\sqrt{\frac{1}{14}}\ -\sqrt{\frac{3}{10}}\ \sqrt{\frac{3}{7}}\ -\sqrt{\frac{1}{5}}$				
$U_{5/2}^{1/2}\ V_{3/2}^{-3/2}$						$\sqrt{\frac{10}{56}}\ \sqrt{\frac{9}{20}}\ \sqrt{\frac{9}{28}}\ \sqrt{\frac{1}{20}}$			
$U_{5/2}^{-1/2}\ V_{3/2}^{-1/2}$						$\sqrt{\frac{30}{56}}\ \sqrt{\frac{1}{60}}\ -\sqrt{\frac{25}{84}}\ -\sqrt{\frac{3}{20}}$			
$U_{5/2}^{-3/2}\ V_{3/2}^{1/2}$						$\sqrt{\frac{15}{56}}\ -\sqrt{\frac{49}{120}}\ \sqrt{\frac{1}{42}}\ \sqrt{\frac{3}{10}}$			
$U_{5/2}^{-5/2}\ V_{3/2}^{3/2}$						$\sqrt{\frac{1}{56}}\ -\sqrt{\frac{1}{8}}\ \sqrt{\frac{5}{14}}\ -\sqrt{\frac{1}{2}}$			
$U_{5/2}^{-1/2}\ V_{3/2}^{-3/2}$							$\sqrt{\frac{10}{28}}\ \sqrt{\frac{6}{12}}\ \sqrt{\frac{1}{7}}$		
$U_{5/2}^{-3/2}\ V_{3/2}^{-1/2}$							$\sqrt{\frac{15}{28}}\ -\sqrt{\frac{1}{12}}\ -\sqrt{\frac{8}{21}}$		
$U_{5/2}^{-5/2}\ V_{3/2}^{1/2}$							$\sqrt{\frac{3}{28}}\ -\sqrt{\frac{5}{12}}\ \sqrt{\frac{10}{21}}$		
$U_{5/2}^{-3/2}\ V_{3/2}^{-3/2}$								$\sqrt{\frac{5}{8}}\ \sqrt{\frac{3}{8}}$	
$U_{5/2}^{-5/2}\ V_{3/2}^{-1/2}$								$\sqrt{\frac{3}{8}}\ -\sqrt{\frac{5}{8}}$	
$U_{5/2}^{-5/2}\ V_{3/2}^{-3/2}$									1

$\mathfrak{D}_{5/2} \times \mathfrak{D}_{3/2}$

	W_4^4	W_4^3 W_3^3	W_4^2 W_3^2 W_2^2	W_4^1 W_3^1 W_2^1 W_1^1	W_4^0 W_3^0 W_2^0 W_1^0 W_0^0	W_4^{-1} W_3^{-1} W_2^{-1} W_1^{-1}	W_4^{-2} W_3^{-2} W_2^{-2}	W_4^{-3} W_3^{-3}	W_4^{-4}
$U_2^2 V_2^2$	1								
$U_2^2 V_2^1$		$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$							
$U_2^1 V_2^2$		$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$							
$U_2^2 V_2^0$			$\sqrt{\frac{3}{14}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{2}{7}}$						
$U_2^1 V_2^1$			$\sqrt{\frac{8}{14}}$ 0 $\sqrt{\frac{3}{7}}$						
$U_2^0 V_2^2$			$\sqrt{\frac{3}{14}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{2}{7}}$						
$U_2^2 V_2^{-1}$				$\sqrt{\frac{1}{14}}$ $\sqrt{\frac{3}{10}}$ $\sqrt{\frac{6}{14}}$ $\sqrt{\frac{2}{10}}$					
$U_2^1 V_2^0$				$\sqrt{\frac{6}{14}}$ $\sqrt{\frac{2}{10}}$ $-\sqrt{\frac{1}{14}}$ $-\sqrt{\frac{3}{10}}$					
$U_2^0 V_2^1$				$\sqrt{\frac{6}{14}}$ $-\sqrt{\frac{2}{10}}$ $-\sqrt{\frac{1}{14}}$ $\sqrt{\frac{3}{10}}$					
$U_2^{-1} V_2^2$				$\sqrt{\frac{1}{14}}$ $-\sqrt{\frac{3}{10}}$ $\sqrt{\frac{6}{14}}$ $-\sqrt{\frac{2}{10}}$					
$U_2^2 V_2^{-2}$					$\sqrt{\frac{1}{70}}$ $\sqrt{\frac{1}{10}}$ $\sqrt{\frac{4}{14}}$ $\sqrt{\frac{4}{10}}$ $\sqrt{\frac{1}{5}}$				
$U_2^1 V_2^{-1}$					$\sqrt{\frac{16}{70}}$ $\sqrt{\frac{4}{10}}$ $\sqrt{\frac{1}{14}}$ $-\sqrt{\frac{1}{10}}$ $-\sqrt{\frac{1}{5}}$				
$U_2^0 V_2^0$					$\sqrt{\frac{36}{70}}$ 0 $-\sqrt{\frac{4}{14}}$ 0 $\sqrt{\frac{1}{5}}$				
$U_2^{-1} V_2^1$					$\sqrt{\frac{16}{70}}$ $-\sqrt{\frac{4}{10}}$ $\sqrt{\frac{1}{14}}$ $\sqrt{\frac{1}{10}}$ $-\sqrt{\frac{1}{5}}$				
$U_2^{-2} V_2^2$					$\sqrt{\frac{1}{70}}$ $-\sqrt{\frac{1}{10}}$ $\sqrt{\frac{4}{14}}$ $-\sqrt{\frac{4}{10}}$ $\sqrt{\frac{1}{5}}$				
$U_2^1 V_2^{-2}$						$\sqrt{\frac{1}{14}}$ $\sqrt{\frac{3}{10}}$ $\sqrt{\frac{6}{14}}$ $\sqrt{\frac{2}{10}}$			
$U_2^0 V_2^{-1}$						$\sqrt{\frac{6}{14}}$ $\sqrt{\frac{2}{10}}$ $-\sqrt{\frac{1}{14}}$ $-\sqrt{\frac{3}{10}}$			
$U_2^{-1} V_2^0$						$\sqrt{\frac{6}{14}}$ $-\sqrt{\frac{2}{10}}$ $-\sqrt{\frac{1}{14}}$ $\sqrt{\frac{3}{10}}$			
$U_2^{-2} V_2^1$						$\sqrt{\frac{1}{14}}$ $-\sqrt{\frac{3}{10}}$ $\sqrt{\frac{6}{14}}$ $-\sqrt{\frac{2}{10}}$			
$U_2^0 V_2^{-2}$							$\sqrt{\frac{3}{14}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{2}{7}}$		
$U_2^{-1} V_2^{-1}$							$\sqrt{\frac{8}{14}}$ 0 $-\sqrt{\frac{3}{7}}$		
$U_2^{-2} V_2^0$							$\sqrt{\frac{3}{14}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{2}{7}}$		
$U_2^{-1} V_2^{-2}$								$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$	
$U_2^{-2} V_2^{-1}$								$\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$	
$U_2^{-2} V_2^{-2}$									1

$D_2 \times D_2$

Appendix III

Summation Formulae Involving Clebsch-Gordon Coefficients

In section II certain sums of matrix elements appear in the equations for the scattering cross section. We have specifically sums of the type

$$\sum M^{JMl}_{mm_j m_s} M^{JMl}_{m'm'_j m'_s}$$

with various conditions on the summed indices. The matrix elements M can be written as

$$M^{JMl}_{mm_j m_s} = \sum_{Tm_T} \alpha_T A^{jsm_j m_s}_{Tm_T} A^{Tlm_T m}_{JM}$$

The matrices A^{ls} are unitary and hence

$$\sum_{\mu\nu} A^{ls\mu\nu}_{jm} A^{ls\mu\nu}_{j'm'} = \delta_{jj'} \delta_{mm'} \qquad \text{(A3.1)}$$

$$\sum_{jm} A^{ls\mu\nu}_{jm} A^{ls\mu'\nu'}_{jm} = \delta_{\mu\mu'} \delta_{\nu\nu'} \qquad \text{(A3.2)}$$

which follow directly from the condition that A^{ls} is the transformation matrix from one orthonormal set of base vectors to another.

We need also the relationship*

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* The evaluation of this sum has been given by Breit and Darling, Phys. Rev. 71, 402 (1947).

$$\sum_{\nu m} A_{jm}^{l s \mu \nu} A_{jm}^{l s \mu' \nu} = X_{\mu\mu'} = \frac{2j+1}{2l+1} \delta_{\mu\mu'} \qquad \text{(A3.3)}$$

To prove this we make use of Schurr's lemma that if the matrix X commutes with every matrix of an irreducible representation then X is a multiple of the unit matrix.

We first show that $A_{jm}^{l s \mu \nu}$ is a scalar (i.e. that $P_R A_{jm}^{l s \mu \nu} = A_{jm}^{l s \mu \nu}$ where P_R is the operator which subjects the coordinates of the wave function to a rotation R.)

$$A_{jm}^{l s \mu \nu} = (\psi_l^{\mu} \psi_s^{\nu}, \psi_j^{m})$$

$$P_R A_{jm}^{l s \mu \nu} = (P_R \psi_l^{m} \psi_s^{\nu}, P_R \psi_j^{m})$$

$$= \sum_{\mu' \nu' m'} D_{\mu' \mu}^{(l)*} D_{\nu' \nu}^{(s)*} (\psi_l^{\mu'} \psi_s^{\nu'}, \psi_j^{m'}) D_{m' m}^{(j)}$$

$$= \sum_{\mu' \nu' m'} \sum_{j' \sigma \sigma'} A_{j' \sigma}^{l s \mu' \nu'} D_{\sigma \sigma'}^{(j)*} A_{j' \sigma'}^{l s \mu \nu} A_{j m'}^{l s \mu' \nu'} D_{m' m}^{(j)}$$

$$= \sum_{j' m' \sigma \sigma'} \delta_{j j'} \delta_{m' \sigma} D_{\sigma \sigma'}^{(j')*} A_{j' \sigma'}^{l s \mu \nu} D_{m' m}^{(j)}$$

$$= \sum_{\sigma \sigma'} D_{\sigma \sigma'}^{(j)*} D_{\sigma m}^{(j)} A_{j \sigma'}^{l s \mu \nu} = A_{jm}^{l s \mu \nu}$$

Therefore

$$P_R X_{\mu\mu'} = X_{\mu\mu'} = P_R \sum_{m \nu} (\psi_l^{\mu} \psi_s^{\nu}, \psi_j^{m}) (\psi_l^{m}, \psi_l^{\mu'} \psi_s^{\nu})$$

$$= \sum_{m \nu} \sum_{\substack{\mu'' \mu''' \\ \nu' \nu'' \\ m' m''}} D_{\mu'' \mu}^{(l)*} D_{\nu' \nu}^{(s)*} A_{j m'}^{l s \mu'' \nu'} D_{m' m}^{(j)} D_{m'' m}^{(j)} A_{j m''}^{l s \mu''' \nu''} D_{\mu''' \mu'}^{(l)} D_{\nu'' \nu}^{(s)}$$

$$X_{\mu\mu'} = \sum_{m'\nu'\mu''\mu'''} D^{(l)*}_{\mu''\mu} A^{ls\mu''\nu'}_{jm'} A^{ls\mu'''\nu'}_{jm'} D^{(l)}_{\mu'''\mu'}$$

$$= \sum_{\mu''\mu'''} D^{(l)*}_{\mu''\mu} X_{\mu''\mu'''} D^{(l)}_{\mu'''\mu'}$$

hence $X_{\mu\mu'}$ commutes with the irreducible matrices $D^{(l)}(R)$ for any R and therefore we can conclude that

$$X_{\mu\mu'} = X^{lsj} \delta_{\mu\mu'}$$

The evaluation of X^{lsj} is now simple. We simply sum $X_{\mu\mu}$ and obtain

$$\sum_{\mu} X_{\mu\mu} = (2l+1) X^{lsj}$$

$$= \sum_{\mu\nu m} A^{ls\mu\nu}_{jm} A^{ls\mu\nu}_{jm} = \sum_{m} 1 = 2j+1$$

hence $X^{lsj} = \frac{2j+1}{2l+1}$. This establishes equation (A3.3).

We now wish to sum the squares and products of matrix elements appearing in the cross section formula.

We have

$$\sum_{mm_j m_s} \left| M^{JMl}_{mm_j m_s} \right|^2 = \sum_{mm_j m_s} \left| \sum_{Tm_T} \alpha_T A^{jsm_j m_s}_{Tm_T} A^{Tlm_T m}_{JM} \right|^2$$

$$= \sum_{mm_j m_s} \sum_{TRm_T m_R} \alpha_T \alpha_R A^{jsm_j m_s}_{Tm_T} A^{jsm_j m_s}_{Rm_R} A^{Tlm_T m}_{JM} A^{Rlm_R m}_{JM}$$

$$\sum_{m m_j m_s} \left| M^{JMl}_{m m_j m_s} \right|^2 = \sum_{TR} \sum_{m m_T m_R} \alpha_T \alpha_R \delta_{TR} \delta_{m_T m_R} A^{Tl m_T m}_{JM} A^{Rl m_R m}_{JM}$$

$$= \sum_T \alpha_T^2 \sum_{m m_T} \left| A^{Tl m_T m}_{JM} \right|^2 = \sum_T \alpha_T^2 \qquad \text{(A3.4)}$$

Another important sum is the same matrix elements summed over a different set of indices

$$\sum_{M m_j m_s} \left| M^{JMl}_{m m_j m_s} \right|^2 = \sum_T \alpha_T^2 \sum_{M m_T} \left| A^{Tl m_T m}_{JM} \right|^2$$

$$= \frac{2J+1}{2l+1} \sum_T \alpha_T^2 \qquad \text{(A3.5)}$$

The third important sum is the squares of products of matrix elements

$$\sum_{\substack{m m_j m_s \\ M m_j' m_s'}} \left| M^{JMl}_{o m_j' m_s'} M^{JMl}_{m m_j m_s} \right|^2$$

$$= \sum_{Mm} \sum_{m_j' m_s'} \left| M^{JMl}_{o m_j' m_s'} \right|^2 \sum_{m_j m_s} \left| M^{JMl}_{m m_j m_s} \right|^2$$

$$= \sum_{Mm} \sum_{T m_T} \alpha_T^2 \left| A^{Tl m_T o}_{JM} \right|^2 \sum_{R m_R} \alpha_R^2 \left| A^{Rl m_R m}_{JM} \right|^2$$

$$= \sum_{TR} \alpha_T^2 \alpha_R^2 \sum_M \left| A^{TlMo}_{JM} A^{Rl,M-m,m}_{JM} \right|^2$$

(A3.6)

Appendix IV

Effects of Instrumental Resolution on Observed Resonance Excitation Curves

Every physical measurement involves some amount of interpretation in order to determine, from the measurement, the actual value of the quantity whose measurement was intended. Measurements with a magnetic spectrometer are certainly no exception! Ideally we are trying to measure a cross section, specifically we wish to determine the number of protons, per incident proton, scattered at a specified direction with respect to the direction of motion of the incident proton when the latter, just before scattering, has a specified energy. The actual measurement however determines the number of protons which leave the target with proper energy and direction of motion to pass thru the spectrometer when the magnetic field has a certain strength. There are five obvious points at which such a measure fails to meet the conditions required of it and therefore produces a smearing or lack of resolution in the results.

(1) Variation in beam voltage. It is manifestly impossible to have all the incident protons of the same kinetic energy. The variation here is of the order of 0.1%.

(2) Straggling in the target. The spectrometer is set to measure an energy which corresponds ideally to incident particles which have penetrated a certain distance into the target before being scattered. Because energy loss is statistical an observation of the energy of the emerging particle after it has left the target does not uniquely determine the energy at which scattering took place.

(3) Finite source size.

(4) Entrance window. A focusing spectrometer will focus to a point image all protons leaving a point source with the same energy over a wide solid angle; in the scattering of protons from light elements the energy of the scattered particle is a function of the angle of scattering and hence the point image is no longer produced.

(5) Finite exit window.

All the data were taken using a thick target. If we denote the bombarding energy by E_B, the energy for which the spectrometer is set by E_o and the energy at which scattering takes place in the target by E_p, we have $E_2 = kE_p$ as the energy of the proton just after scattering and

$$k = \left[\frac{M_1 \cos\theta + (M_o^2 - M_1^2 \sin^2\theta)^{1/2}}{M_1 + M_o} \right]^2 \qquad \text{(A4.1)}$$

is a function of the angle of scattering, θ, and the masses of the proton and the scattering nucleus, M_1 and M_o. The target is arranged so that the normal to its surface bisects the angle between the incident and scattered directions. This means that the path length of a proton going into the target is the same as the path length coming out. The energy losses coming out and going in are therefore proportional to stopping powers at the respective energies. If η is the ratio of stopping power at energy kE_p to the stopping power at energy E_p ($\eta \approx k^{-1/2}$) we have

$$\eta(E_B - E_p) = (kE_p - E_o) \qquad \text{(A4.2)}$$

$$E_p = \frac{\eta E_B + E_o}{k + \eta} \qquad \text{(A4.2a)}$$

If we consider straggling we must consider the probability distribution

$$P(E, x; E_B)\, dE = \sqrt{\frac{\alpha}{\pi x}}\, \exp\left\{ -\frac{\alpha}{x} \left(E - E_B + n\epsilon_1 x \right)^2 \right\} dE \qquad \text{(A4.3)}$$

which gives the probability of a particle which had energy E_B at x = o having energy E at a depth x. α is a straggling parameter which is approximately independent of energy, $n\epsilon_1$ is the stopping power of the target for protons of energy E_B. Similarly,

$$P_2(E_o, x; kE)\,dE_o = \sqrt{\frac{\alpha}{\pi x}}\, exp\left\{-\frac{\alpha}{x}\left(E_o - kE + n\epsilon_2 x\right)^2\right\} dE_o \qquad \text{(A4.3a)}$$

which is the probability that a proton scattered by the scattering nucleus with energy kE shall emerge from the target with energy E_o.

$n\epsilon_2 = \eta\, n\epsilon_1$ = stopping power for particles of energy E_o; the yield is then

$$Y(E_o)\,dE_o = \int_{x=o}^{\infty}\int_{E=o}^{E_B} P_1(E, x; E_B)\,\sigma(E)\,P_2(E_o, x; kE)\,dE\,dx\,dE_o \qquad \text{(A4.4)}$$

where $\sigma(E)$ is the cross section for scattering. The integration over x is extended to infinity in the approximation that we have a "thick" target. The yield therefore is not proportional to σ but is "blurred" by the straggling function $S(E, E_o)$

$$Y(E_o)\,dE_o = \int_{E=o}^{E_B} S(E, E_o)\,\sigma(E)\,dE_o$$

$$S(E, E_o) = \int_{x=o}^{\infty} P_1(E, x; E_B)\,P_2(E_o, x; kE)\,dx \qquad \text{(A4.5)}$$

Integrating this we get, using (A4.2a)

$$S(E, E_o) = \frac{2\alpha}{\pi}\, exp\left\{2n\epsilon_1\alpha\left[(E_B - E_p)(1+\eta^2) + (k\eta - 1)(E - E_p)\right]\right\} \times K_o\left(2n\epsilon_1\alpha\sqrt{(1+\eta^2)\left\{(E_B - E)^2 + (kE - E_o)^2\right\}}\right) \qquad \text{(A4.6)}$$

$K_0(\xi)$ is the solution of the modified Bessel equation of zero order which is finite at infinity:

$$K_0(\xi) \sim \sqrt{\frac{\pi}{2\xi}}\, e^{-\xi} \tag{A4.6a}$$

Using the asymptotic form and expanding the expression in the exponential as a power series in $E - E_p$ we obtain, with sufficient accuracy for our purposes

$$S(E,E_0) = \sqrt{\frac{\alpha}{\pi}}\, \frac{\exp\left\{-\frac{n\epsilon_i\alpha(k+\eta)^2(E-E_p)^2}{(1+\eta^2)(E_B-E_p)}\right\}}{\sqrt{n\epsilon_i(1+\eta^2)(E_B-E_p)}} \tag{A4.7}$$

we therefore see that the straggling function is a Gaussian distribution with dispersion

$$\overline{(E-E_p)^2} = \frac{(1+\eta^2)(E_B-E_p)}{2n\epsilon_i\alpha(k+\eta)^2} \tag{A4.8}$$

The variation in E_p as a result of straggling is

$$\delta E_{p,s} = \frac{1}{k+\eta}\left[\frac{(1+\eta^2)(E_B-E_p)}{2n\epsilon_i\alpha}\right]^{1/2} \tag{A4.8a}$$

$$2n\epsilon_i\alpha = \frac{M}{2mE_p}\ln\frac{4E_p\, m}{I\, M}$$

m/M = ratio of electron to proton mass.

I = effective ionization potential of the target.

For variations due to beam voltage we find from (A4.2a)

$$\delta E_{p,B} = \frac{\eta}{k+\eta}\,\delta E_B \tag{A4.9}$$

For the effect of finite source size we note that changing the position of the source point changes both E_o and k (since the angle of scattering changes). Again from (A4.2a) we find

$$\delta E_{p,\,source} = \left[\frac{1}{k+\eta} \frac{\partial E_o}{\partial x} - \frac{(\eta E_B + E_o)}{(k+\eta)^2} \frac{\partial k}{\partial x} \right] \delta x \qquad \text{(A4.10)}$$

where x is displacement of the source point and δx is the mean displacement resulting from finite source size. Now, if we have a displacement x in the source we produce a displacement mx in the image, m being the magnification of the instrument, so that if r is a displacement in the image space

$$\frac{\partial E_o}{\partial x} = -m \frac{\partial E_o}{\partial r}$$

From the design of the instrument* we have

$$\frac{\partial E_o}{\partial r} = \frac{E_o}{(1+m) r_o}$$

and hence

$$\frac{\partial E_o}{\partial x} = -\frac{m E_o}{(1+m) r_o} \qquad \text{(A4.11)}$$

r_o = radius of stable proton path

If l is the distance from the source to the entrance window of the spectrometer,

$$\frac{\partial k}{\partial x} = \frac{1}{l} \frac{\partial k}{\partial \theta}$$

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* For a complete discussion of the design and focusing properties of the proton spectrometer, see C. W. Snyder, Thesis, California Institute, (1948).

and from (A4.1)*

$$\frac{\partial k}{\partial \theta} = - \frac{2k \sin\theta}{\left[\frac{M_0^2}{M_1^2} - \sin^2\theta \right]^{1/2}}$$

this gives us, with $E_0 \approx kE_p$

$$\delta E_{p,\text{source}} = - \frac{kE_p}{k+\eta} \left\{ \frac{m}{r_0(1+m)} - \frac{2\sin\theta}{l} \left[\frac{M_0^2}{M_1^2} - \sin^2\theta \right]^{-1/2} \right\} \qquad \text{(A4.12)}$$

For variations due to the finite entrance angle, we hold everything constant except k and obtain

$$\delta E_{p,\theta} = - \frac{(\eta E_B + E_0)}{(k+\eta)^2} \frac{\partial k}{\partial \theta} \delta\theta \approx - \frac{E_p}{k+\eta} \frac{\partial k}{\partial \theta} \delta\theta \qquad \text{(A4.13)}$$

The finite exit window allows E_0 to have a finite spread and

$$\delta E_{p,w} = \frac{1}{k+\eta} \frac{\partial E_0}{\partial r} \delta r = \frac{kE_p}{k+\eta} \frac{\delta r}{r_0(1+m)} \qquad \text{(A4.14)}$$

The total spread in E_p is then given by

$$\delta E_p = \left[\sum_i (\delta E_{p,i})^2 \right]^{1/2}$$

For the scattering of protons by lithium, the conditions were the following:

E_p = 440 kev $\qquad\qquad$ δE_B = 300 ev

k = 0.604 $\qquad$ η = 1.432 at θ = 137.8°

k = 0.787 $\qquad$ η = 1.241 at θ = 81.1°

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* Angles here are all laboratory angles. In discussing angular distributions in Sections III and IV we used center of mass angles.

$$m = 0.8 \qquad r_o = 26.7 \text{ cm} \qquad \ell = 12.4 \text{ cm}$$

$$\delta x = \frac{\Delta x}{\sqrt{12}} = 0.04 \text{ cm}$$

$$\delta \theta = \frac{\Delta \theta}{\sqrt{20}} = 0.0142$$

$$\delta r = \frac{\Delta r}{\sqrt{12}} = 0.102 \text{ cm}$$

Δx, $\Delta\theta$, Δr are the full widths of the respective apertures; δx, $\delta\theta$, δr are, on the other hand, the root mean square deviations measured from the center of the aperture. For δx and δr the shape was taken to be rectangular; for $\delta\theta$, parabolic.

We therefore have, for the mean spread due to resolution

$$\delta E_p = 0.926 \text{ kev} \quad \text{at} \quad 137.8^\circ$$

$$= 1.320 \text{ kev} \quad \text{at} \quad 81.1^\circ$$

Having computed this, what effect will it now have on the observed excitation curves? We can, with sufficient accuracy write the formula for the excitation curve as

$$\frac{\sigma}{\sigma_o} = 1 + f(x) = 1 + \frac{a+bx}{1+x^2} \tag{A4.15}$$

by neglecting the variation with energy of slowly varying functions in (2.30). We now expand this in a Taylor series around the maximum. Then

$$f(x) = f_{max}\left\{1 - \frac{(x-x_{max})^2}{1+x_{max}^2} + \ldots\right\}$$

$$x_{max} = \frac{-a + \sqrt{a^2+b^2}}{b}$$

$$f_{max} = \frac{b}{x_{max}} \tag{A4.16}$$

We represent the resolution by a function g(x) with the properties that

$$\int_{-\infty}^{\infty} g(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} x^2 g(x)\,dx = \left(\frac{\delta E_p}{\frac{1}{2}\gamma}\right)^2 \qquad \text{(A4.17)}$$

The observed excitation curve then has the form

$$\frac{\sigma}{\sigma_0}\Big|_{obs} = 1 + F(x) = 1 + \int_{-\infty}^{\infty} g(t)\,f(x-t)\,dt \qquad \text{(A4.18)}$$

Using the expansion (A4.16) we find for $F(x_{max})$

$$F(x_{max}) = \int_{-\infty}^{\infty} g(t)\,f(x_{max}-t)\,dt$$

$$= f_{max}\int_{-\infty}^{\infty} g(t)\left\{1 - \frac{t^2}{1+x_{max}^2} + \dots\right\} dt$$

$$= f_{max}\left\{1 - \frac{(\delta E_p/\frac{1}{2}\gamma)^2}{1+x_{max}^2} + \dots\right\} \qquad \text{(A4.19)}$$

Therefore,

$$f_{max} \approx F(x_{max})\left\{1 + \frac{(\delta E_p/\frac{1}{2}\gamma)^2}{1+x_{max}^2}\right\} \qquad \text{(A4.19a)}$$

From the experimental data we find, at $\theta = 137.8^\circ$ (144.3° in the center of mass coordinates)

$$F(x_{max}) = 1.38$$

$$\tfrac{1}{2}\gamma = 6 \text{ kev}$$

$$x_{max} = -0.2 \ (= -1.2 \text{ kev})$$

$$E_p = 0.926 \text{ kev}$$

So that

$$f_{max} = 1.41 \qquad \left.\frac{\sigma}{\sigma_o}\right|_{max} = 2.41$$

At $\theta = 81.1^\circ$ (89.3° in the center of mass coordinates)

$$F(x_{max}) = 0.61$$

$$\tfrac{1}{2}\gamma = 6 \text{ kev}$$

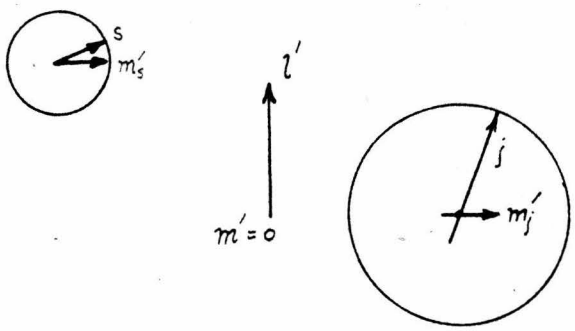
$$x_{max} = 0$$

$$E_p = 1.320 \text{ kev}$$

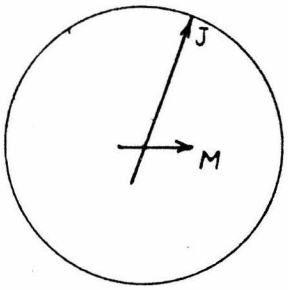
So that

$$f_{max} = 0.64 \qquad \left.\frac{\sigma}{\sigma_o}\right|_{max} = 1.64$$

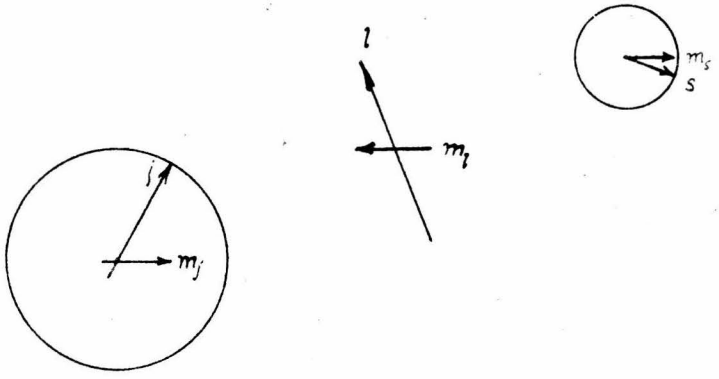
These are the values quoted in Section III.



a. Incident Configuration



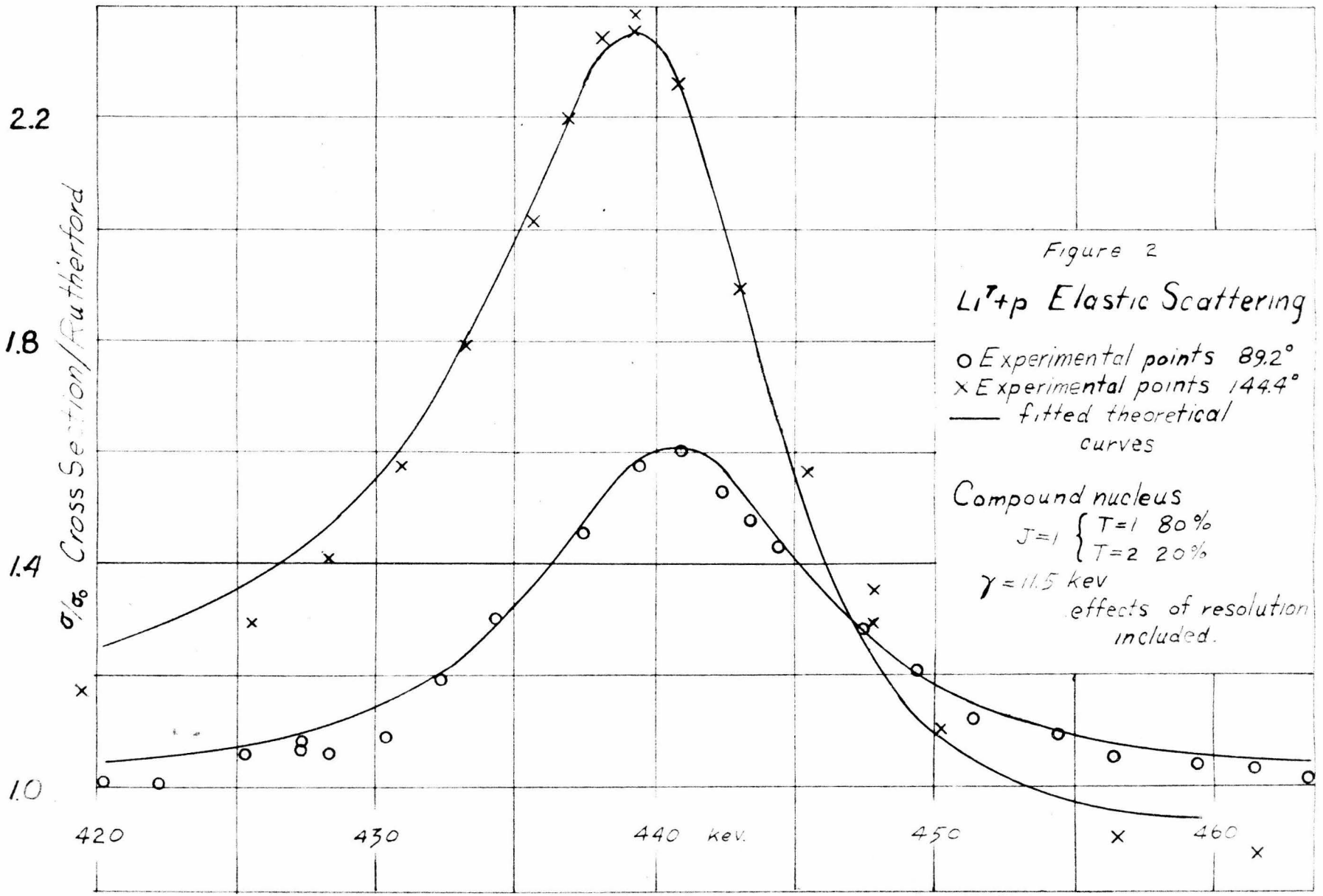
b. Compound Nucleus



c. Residual Configuration

Figure 1

Schematic Description of Momentum Vectors in a Nuclear Reaction


2.2
1.8
1.4
1.0
σ/σ₀ Cross Section/Rutherford
420
430
440 kev.
450
460
Figure 2
Li⁷+p Elastic Scattering
o Experimental points 89.2°
× Experimental points 144.4°
—— fitted theoretical curves
Compound nucleus
J=1 { T=1 80%
T=2 20%
γ = 11.5 kev
effects of resolution included.

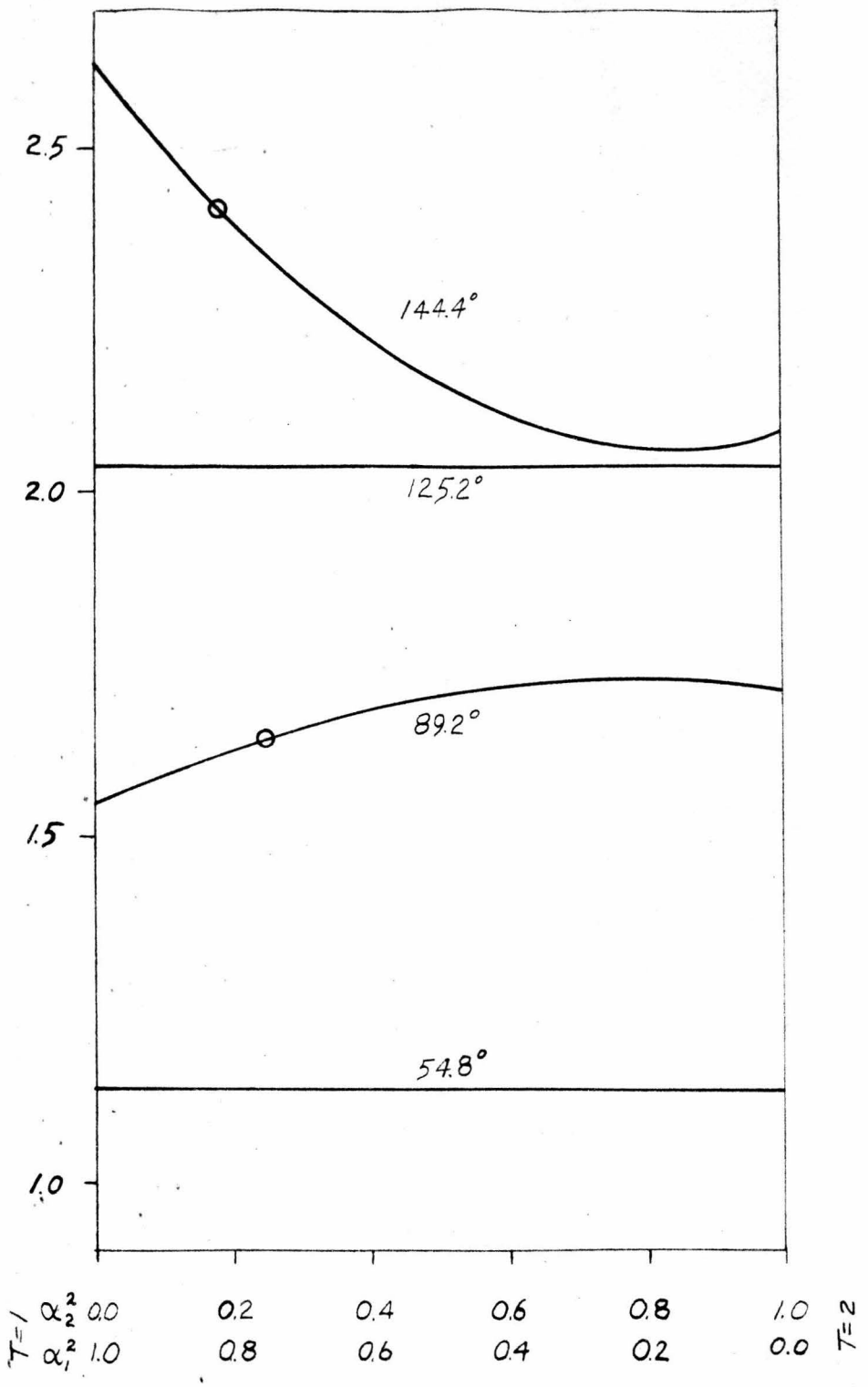


Figure 3

Ratio of Maximum Cross Section to Rutherford as a function of the mixture of T-states in the compound nucleus, for various angles of scattering in the center of mass system.

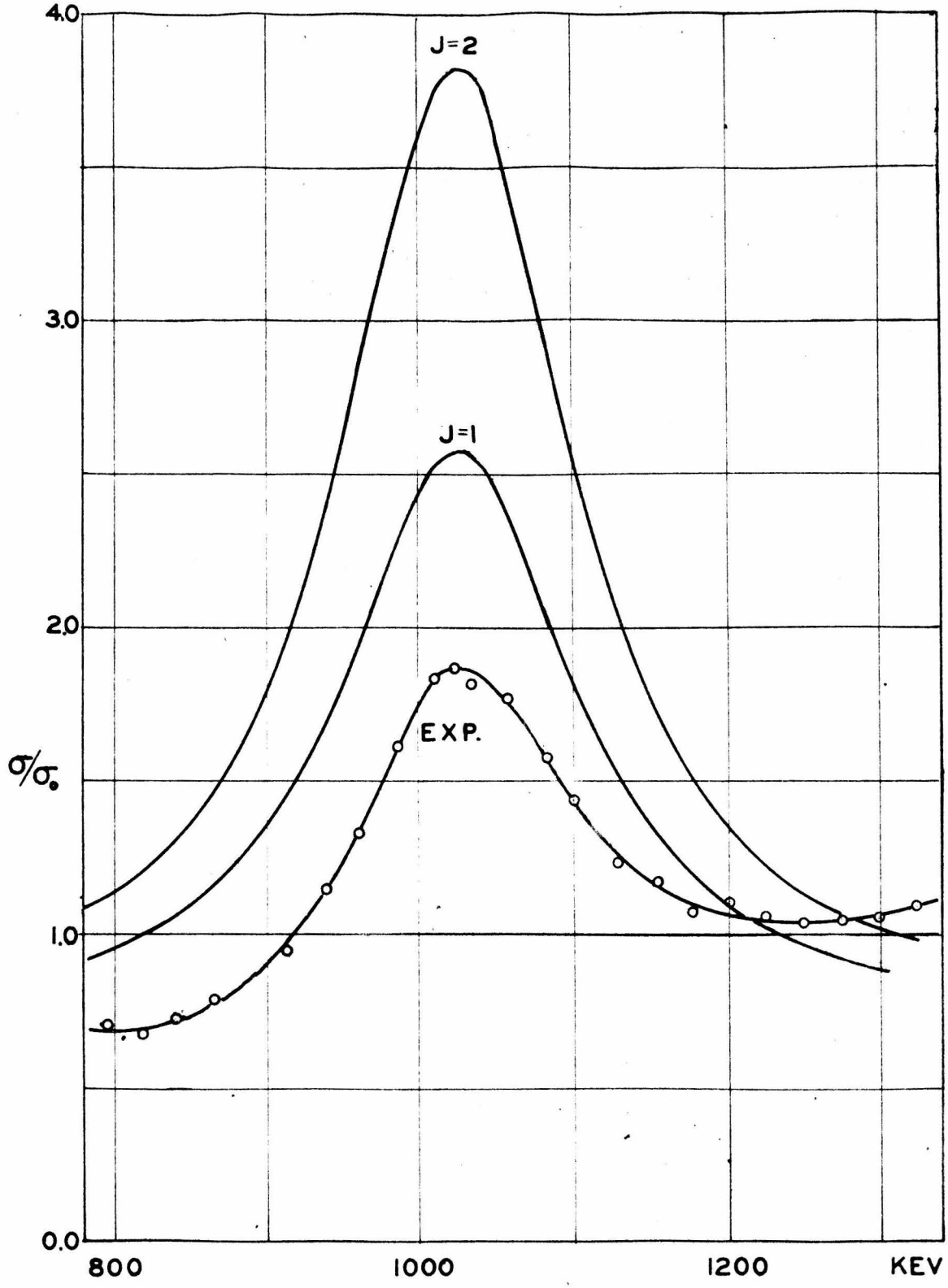


Figure 4

$Li^7 + p$ Elastic Scattering at 1.0 Mev.

E_0 = 1024 kev $\frac{1}{2}\Gamma$ = 90 kev

$J = 1$: $\delta_1 = 0.46$ $\delta_2 = 0.95$; $J = 2$: $\delta_1 = 1.11$ $\delta_2 = 0.46$

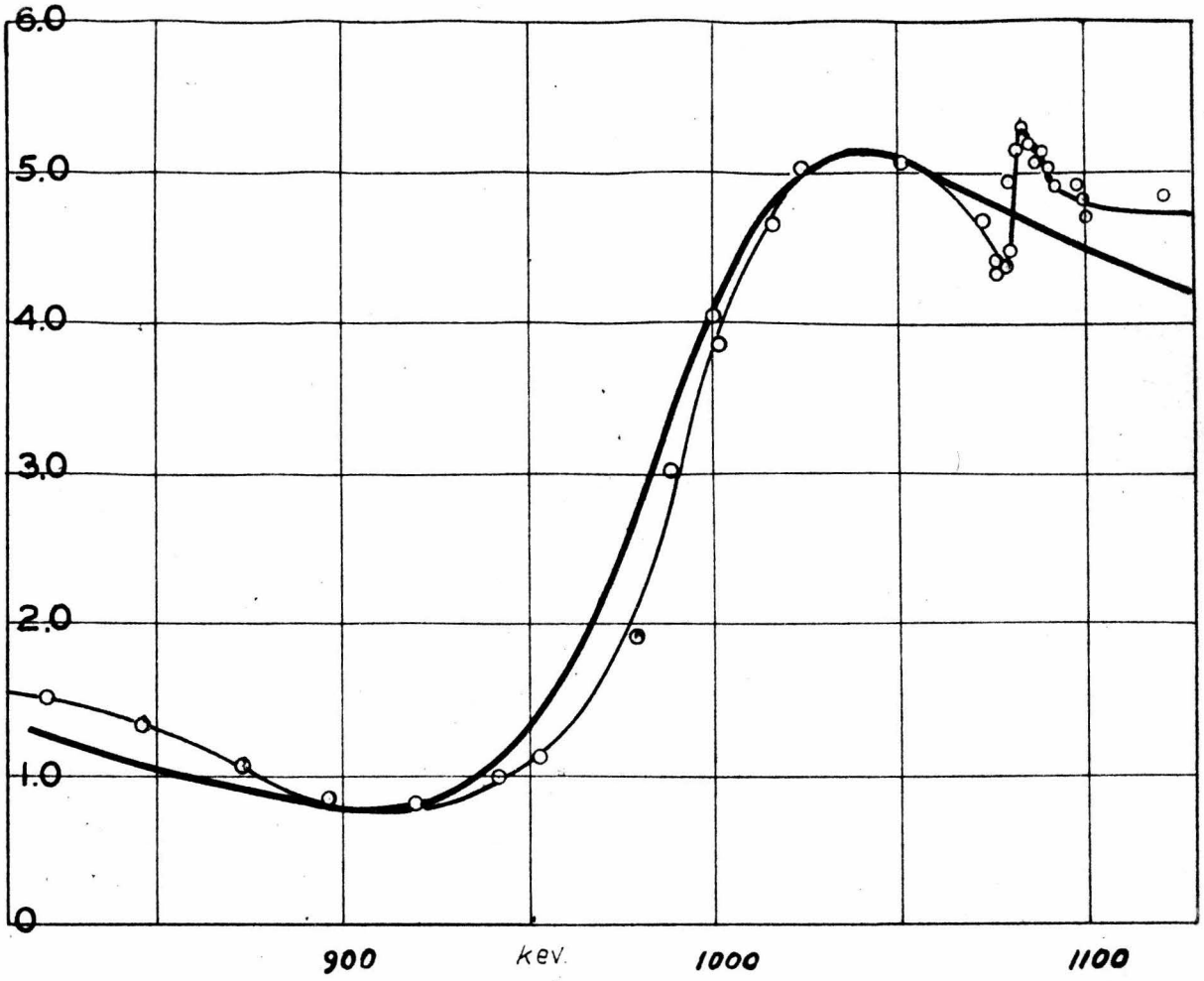


Figure 5

Be^9 + p. Resonance at 988 kev.

Ratio of Elastic Scattering to Rutherford Scattering vs. Proton Energy.

—o— Experimental Points (scattering angle 142° CM.)

——— Theoretical Curve

E_o = 988 E_{max} = 1038 E_{min} = 910

Phase shifts: $\delta_1 = \begin{cases} +0.915 \\ -0.176 \end{cases}$ $\delta_2 = 2.83 + s\pi$

γ_p/γ = 0.82 γ, (fitted from this data) = 125 kev.

γ, from gamma-ray width = 90 kev.

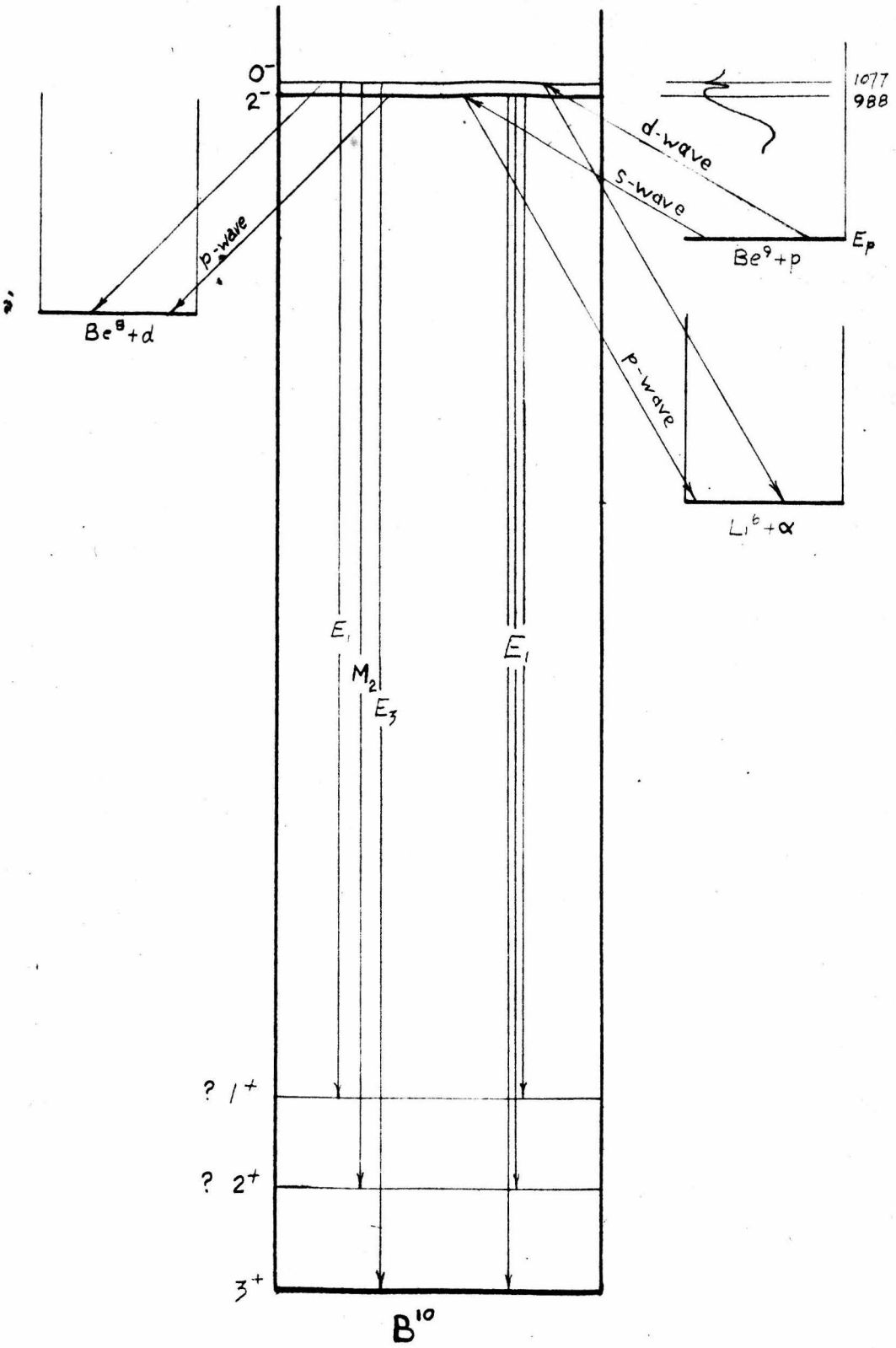


Figure 6

Energy Levels in B^{10}.

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