LEVEL SETS ON SPHERES

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ABSTRACT

Let $f: S^n \times I^1 \to E^1$ be a continuous, real-valued function on $S^n \times I^1$ for $n > 1$. Then for every $t \in I^1$ there is a subset $A_t \times t$ of the n-sphere $S^n \times t$ with the following properties:

i) $f(A_t \times t) = k_t$ independent of $x \in A_t$.

ii) $A_t \times t$ is connected.

iii) $(S^n \times t) - (A_t \times t)$ has no component containing more than half the n-dimensional measure of $S^n \times t$.

iv) For any measure-preserving homeomorphism, g, of $S^n \times t$, $A_t \times t$ contains the image of at least one of its points. (e.g. $A_t \times t$ contains a pair of antipodal points of $S^n \times t$.)

v) k_t varies continuously with t.

Further, if $g: T^2 \to E^1$ is a continuous real-valued function defined on a torus, then there is a connected, non-contractible subset of T^2 on which g is constant.

LEVEL SETS ON SPHERES

<u>Definitions and notation.</u> E^n denotes Euclidean n-space, or the space of n real variables. S^n is the set $S^n = \left\{x = (x_1, x_2, \ldots, x_{n+1}) \mid x \in E^{n+1} \text{ and } |x| = \sum_{i=1}^{n+1} (x_i)^2 = 1\right\}$, the n-sphere embedded in E^{n+1}. $I^n = \left\{x \mid x \in E^n, 0 \leq x_i \leq 1, i = 1, 2, 3, \ldots, n\right\}$ is the unit cube of E^n, and the faces of I^n are given by $\left\{x \mid x \in I^n, \text{ some } x_i = 0 \text{ or } 1\right\}$. For all three of these sets n is assumed to be greater than 1 unless otherwise indicated. The measure used on S^n is the ordinary measure normalized so that the measure of S^n is 1. In the derivations to follow, $M(A)$ denotes the measure of a set A and in each case the symbol is used, the set A is either open or closed and is therefore measurable. For any set A, $F(A)$ is the boundary of A and $S - A$ the complement of A in S. The notation $f: X \to Y$ for a function f defined on a set X with image in a set Y carries also the implication that f is continuous with respect to the relevant topology, which in all cases will be that one induced by the metric. $d(a,b)$ is the distance from a to b; $d(a,A)$ is the distance from the point a to the set A; $d(A)$ is the diameter of the set A.

The space T^2, the torus, is given by the product of two circles, or one-spheres. Its universal covering space is the plane E^2, [1]. For convenience $p: E^2 \to T^2$, the projection map will map the unit square I^2 of E^2 with opposite faces identified exactly once onto T^2. All spaces considered are locally connected, and thus the components of any open set are open ([2], page 45). This last will be used freely without reference as will the fact that any open set has a denumerable number of components.

Let O be an open subset of S^n with components O_i, $(i = 1,2, \ldots, n)$ such that each O_i has complement $S^n - O_i$ containing one and, obviously, only one component T_i with $M(T_i) > 1/2$. It will be established that, under this assumption, there is a component, A, of $S^n - O$, whose complement, $S^n - A$, contains no component with measure greater than one-half. Any set containing such a component, A, will be said to possess "property P". Thus it will be established that if O is an open subset of S^n, then either O or $S^n - O$ has property P.

LEMMA 1: Let O be an open connected subset of S^n, $n > 1$. If A is a component of $S^n - O$, then

i) A is closed and connected.

ii) $S^n - A$ is open and connected.

iii) $F(A) = F(S^n - A)$ is connected.

Proof: i) A is closed and connected by virtue of being a component of a closed set.

ii) $S^n - A$ is open because A is closed. Wilder ([2], Theorem 9.11, page 21) shows that $S^n - A$ is connected.

iii) The proof of iii) is the culmination of the argument on pp. 47-60 of Wilder [2].

LEMMA 2: Let O be an open subset of S^n not possessing property P. Then corresponding to each component, O_i, of O, there is a unique component, T_i, of $S^n - O_i$ whose measure is greater than one-half. For $i \neq j$ one of the following must hold:

i) $S^n - T_i \subseteq S^n - T_j$

ii) $S^n - T_j \subseteq S^n - T_i$

iii) $(S^n - T_j) \cap (S^n - T_i) = \emptyset$

Proof: O_i and O_j are open, connected, and disjoint. By lemma 1, $S^n - T_i$ and $S^n - T_j$ are connected. O_j, being connected and disjoint from O_i, lies in a single component of $S^n - O_i$.

1) Suppose that component is T_i. Then $O_j \subseteq T_i \implies S^n - T_i \subseteq S^n - O_j$. $S^n - T_i$, being connected, lies in some component of $S^n - O_j$. Suppose this component is T_j. Then $S^n - T_i \subseteq T_j \implies (S^n - T_i) \cap (S^n - T_j) = \emptyset$. Otherwise $(S^n - T_i) \cap T_j = \emptyset$ and $S^n - T_i \subseteq S^n - T_j$.

2) The other possibility is that $O_j \cap T_i = \emptyset$. Then $T_i \subseteq S^n - O_j$. But T_i is connected and lies in a single component of $S^n - O_j$. But $M(T_i) > 1/2$ and $M(T_j) > 1/2$ indicate that component must be T_j. Hence $T_i \subseteq T_j$ and $S^n - T_j \subseteq S^n - T_i$, establishing the lemma.

Now let $O' = \bigcup_{i=1}^{\infty} S^n - T_i$. Clearly $O' \supseteq O$, since for each i, $O_i \subseteq S^n - T_i$ and the O_i are the components of O. O' is the union of open sets, $S^n - T_i$, and is therefore open. Let $O' = \bigcup_{j=1}^{\infty} X_j$ where the X_j are the components of O'. Since each $S^n - T_i$ is connected it must be entirely contained in a single X_j. Hence each X_j is the union of the $S^n - T_i$ contained in it.

LEMMA 3: Suppose $S^n - T_i$ and $S^n - T_j$ are disjoint but are both contained in the same component X_k of O'. Then there is an integer l, such that $S^n - T_l$ contains both $S^n - T_i$ and $S^n - T_j$.

Proof: Assume there is no such l. Let T be the union of all $S^n - T_m$ which contain $S^n - T_i$. Clearly none of these intersects $S^n - T_j$ by lemma 2. Let S be an arc connecting $x \in S^n - T_i$ to $y \in S^n - T_j$ such that S lies in X_k, which is open and connected and therefore arcwise connected. S must intersect $F(T)$. Let $p \in S \cap F(T)$. $p \in S^n - T_q$ for some q, as does some neighborhood of p. This neighborhood also contains a point z of T by virtue of p belonging to $F(T)$. But $z \in S^n - T_m$ for some m where $S^n - T_m \supseteq S^n - T_i$. Since $S^n - T_m$ and $S^n - T_q$ intersect, one must contain the other from lemma 2. In either case, however, $S^n - T_q \subseteq T$. But then $p \in T$. But p is a boundary point of the open set T and as such cannot belong to T. This contradiction establishes the lemma.

LEMMA 4: Each X_k can be expressed as a countable expanding union of sets $S^n - T_i$ (i.e. each subset contains all the preceding).

Proof: From the remarks preceding lemma 3, it is seen that each X_k can be expressed as the ordered union of the $S^n - T_i$ contained in it. Let $X_k = \bigcup_{i=1}^{\infty} S^n - T_i$ be such a representation. A subunion of this union is chosen as follows: Let $I_1 = S^n - T_1$, $I_k = S^n - T_{i_k}$ for $k > 1$ where i_k is the smallest number for which $S^n - T_{i_k} \supseteq S^n - T_{i_{k-1}}$. Now consider $J_m = \bigcup_{k=1}^{m} I_k$. Since $I_k \supseteq I_{k-1}$ for all k, $J_m = I_m$. For any $x \in X_k$ there is a smallest number j, for which $x \in S^n - T_j$. Suppose $S^n - T_j \not\subseteq J = \bigcup_{k=1}^{\infty} I_k$. Clearly $S^n - T_j$ must be disjoint from J_m, where m is the largest number for which $i_m < j$. Then there is a first number p, for which $S^n - T_p$ contains both $S^n - T_j$ and $J_m = S^n - T_{i_m}$. But then $S^n - T_p$ is neither contained in nor disjoint from J_r where r is the greatest number for which $i_r < p$. Hence, $S^n - T_p \supseteq J_r = I_r$ and $S^n - T_p = I_{r+1}$. Therefore, $J = \bigcup_{k=1}^{\infty} I_k = X_k$. Notice also that the above argument is valid for the case where there are only a finite number of I_k, and that in this case, in fact, X_k is equal to the last I_k. This completes the proof of lemma 4.

Now each $S^n - X_i$ is either one of the T_j or is expressible as the decreasing intersection of a countable number of T_j, which are closed and connected. By lemma 3.8 of page 80 of Wilder [2], $S^n - X_i$ is connected. Then for each i, X_i and $S^n - X_i$ are connected, which implies, by lemma 1, that $F(X_i)$ is connected.

THEOREM 1: $S^n - O' = S^n - \bigcup_{i=1}^{\infty} (S^n - T_i) = S^n - \bigcup_{k=1}^{\infty} X_k$ is connected.

<u>Proof</u>: Suppose $S^n - O'$ is not connected. Then $S^n - O' = A \cup B$ where A and B are disjoint, non-empty, and relatively closed in the closed set $S^n - O'$, and hence closed in S^n. Since each X_i has connected boundary, each X_i has its boundary entirely in A or entirely in B. Then consider $S^n = A' \cup B'$ where $A' = A \cup \bigcup_{i \in I} X_i$, $I = \{i \mid F(X_i) \subseteq A\}$ and $B' = B \cup \bigcup_{j \in J} X_j$, $J = \{j \mid F(X_j) \subseteq B\}$. A' and B' are closed. Hence S^n is not connected. This contradiction establishes the theorem.

THEOREM 2: In any partition of S^n into an open set O and its complement $S^n - O$, there is a component, A, of either O or $S^n - O$ with the property that $S^n - A$ has no component with measure exceeding one-half (i.e. either O or $S^n - O$ has property P).

Proof: Suppose O does not have property P. Lemmas 2, 3, and 4 and theorem 1 were all derived from this hypothesis. Thus O', which is open and contains O, has a connected complement $S^n - O'$ which must therefore belong to a single component A of $S^n - O$. But, $A \supseteq S^n - O' \Longrightarrow S^n - A \subseteq O'$. Hence each component of $S^n - A$ must be contained in a single component of O'. The components of O' are the X_k which are expanding unions of the I_i for each of which $M(I_i) \leq 1/2$. The expanding union of open sets with measure no greater than one-half has measure no greater than one-half [3]. Thus for each k, $M(X_k) \leq 1/2$. Hence $S^n - O$ has property P.

THEOREM 3: Let A be a closed connected subset of S^n having property P. Then for any measure-preserving homeomorphism $g: S^n \to S^n$, there is a point $x \in A$ for which $g(x) \in A$.

Proof: Suppose A and B are closed connected sets of S^n with property P. They cannot be disjoint. For suppose they are; then $A \subseteq B_1$, a component of $S^n - B$ and, by lemma 1, $S^n - B_1$ lies in A_1 a component of $S^n - A$. A is closed $\Longrightarrow S^n - A$ is open $\Longrightarrow A_1$ is open and, similarly, B_1 is open $\Longrightarrow S^n - B_1$ is closed. But $S^n - B_1 \subseteq A_1$ combined with the above implies that $M(S^n - B_1) < M(A_1)$. However, $M(B_1) \leq 1/2$ since B has property P, and $M(A_1) > M(S^n - B_1) \geq 1/2$. Thus $M(A_1) > 1/2$ which is impossible since A has property P. Hence A and B meet.

Now consider $g(A)$. It has property P because g is a measure-preserving homeomorphism. It is also closed and connected. Therefore $g(A) \cap A \neq \emptyset$. If $y \in g(A) \cap A$, then $y = g(x)$, where $x \in A$ and $g(x) \in A$.

DEFINITION: A metric space K will be called "diameter divisible" if, for every $\varepsilon > 0$, there is an open set $O \subseteq K$ such that no component of either O or $K - O$ has point set diameter greater than ε.

LEMMA 5: Euclidean 1-space, E^1, is diameter divisible.

Proof: By construction.

THEOREM 4: Let $f: S^n \rightarrow K$ be a continuous function, where K is any diameter divisible space. Then for one and only one $k \in K$, $f^{-1}(k)$ has property P.

Proof: For each n, there is an open set $O_n \subseteq K$ such that both O_n and $K - O_n$ have components of diameter less than $1/n$. $f^{-1}(O_n)$ is an open set on S^n. Either $f^{-1}(O_n)$ or its complement has property P. Thus there is a connected set A_n with $f(A_n) \subseteq O_n$ or $f(A_n) \subseteq K - O_n$ such that $S^n - A_n$ contains components of measure not greater than one-half. $f(A_n)$ is connected, by virtue of being the continuous image of a connected set, and must belong to a single component of O_n or $K - O_n$. Therefore diameter $(f(A_n)) < 1/n$. Let $x_n \in A_n$. The sequence $\{x_n\}$ has a limit point $x \in S^n$. Let $f(x) = k$, and let B_n be the component of $f^{-1}(K_n)$ containing x, where $K_n = \left\{ l \middle| d(k,l) \leq \frac{1}{n} \right\}$. Every B_n has property P because the sequence B_n is a decreasing sequence of sets and, in addition, $B_n \supseteq A_n$ for arbitrarily large n. $B = \bigcap_{n=1}^{\infty} B_n$ is closed and connected (Wilder [2], page 80, lemma 3.8). Also $f(B) = k$, for suppose $x_1 \in B$ has $f(x_1) = k_1$. Then $d(k_1, k) = p \neq 0$ and for sufficiently large m, $1/m < p \implies x_1 \notin B_m$. Furthermore, B has property P. For suppose B^1 is a component of $S^n - B$. Let $x \in B^1$. For $n > n_o$, $x \notin B_n$. Denote by C_i the component of $S^n - B_i$ containing x. Then, if $j > i$, $C_i \subseteq C_j$ or $C_i \cap C_j = \emptyset$. Furthermore, by the argument of lemma 3, $B^1 = \bigcup_{i=n_o}^{\infty} C_i$. Hence, by Halmos [3], $M(B^1) = \lim_{n\to\infty} M(C_i) \leq 1/2$. Suppose B' has property P and $f(B') = k'$ for some k'. Then from the proof of theorem 3 we see that $B \cap B' \neq \emptyset$ and for $x \in B \cap B'$, $f(x) = k' = k$.

COROLLARY 1: Let $f: S^n \to K$ be a continuous function where K is any diameter divisible space. Then there is a set $A \subsetneq S^n$ such that

i) A is connected.

ii) $f(A) = k$ for some $k \in K$.

iii) No component of $S^n - A$ has measure greater than one-half.

iv) For any $g: S^n \to S^n$ which is a measure-preserving homeomorphism, there is a point $x \in A$ such that $g(x) \in A$.

<u>Proof</u>: This combines theorems 3 and 4.

The theorem of Johnson [4] established i), ii), iv) of corollary 1 for the case where g is the antipodal map of S^n. His proof, communicated to me privately, follows the same lines as some similar theorems of Sorgenfrey [5].

COROLLARY 2: E^n is not diameter divisible for $n > 1$.

Proof: Let $f: S^n \to E^n$, $n > 1$, be given by $f(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_n)$. Then no inverse image of a point of E^n contains more than two points and hence none has property P.

COROLLARY 3: Let $f: S^n X I^1 \to E^1$ be a continuous real-valued function defined on $S^n X [0,1]$. Then for each $t \varepsilon [0,1]$, there is a connected set $A_t X t$ which has property P with respect to the n-sphere $S^n X t$ and such that $f(x,t) = r_t$ for all $x \varepsilon A_t$. Furthermore r_t is a continuous function of t.

Proof: Only the continuity of r_t is non-obvious. Let $d((x_1,t_1),(x_2,t_2)) = d(x_1,x_2) + |t_2 - t_1|$. Then f being continuous on a compact space must be uniformly continuous. Given $\varepsilon > 0$, there exists δ such that $d((x_1,t_1),(x_2,t_2)) < \delta \Longrightarrow |f(x_2,t_2) - f(x_1,t_1)| < \varepsilon$ for arbitrary (x_1,t_1). Let $|t_2 - t_1| < \delta$. Then there exists x such that $(x,t_1) \varepsilon A_{t_1} X t_1$ and $(x,t_2) \varepsilon A_{t_2} X t_2$. But then, $|r_{t_2} - r_{t_1}| = |f(x,t_1) - f(x, t_2)| < \varepsilon$ by uniform contimuity. Thus r_t is a continuous function of t.

COROLLARY 4: Let $f: S^n X I^1 \to M$ be continuous where M is any metric space. Let C be a compact closed set which separates M. If $f(S^n X 0) \subseteq C_1$ and $f(S^n X 1) \subseteq C_2$ where C_1 and C_2 are different components of $M - C$, then for some $t \varepsilon I^1$, $f(S^n X t)$ intersects C on a set with property P.

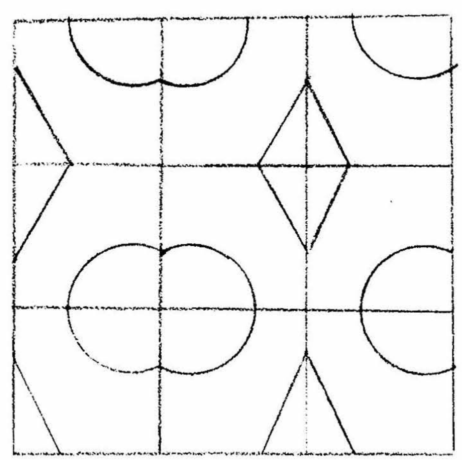
<u>Proof:</u> Define $g: M \to E^1$ by

$$g(x) = \begin{cases} 0, & x \in C \\ d(x,C), & x \in C_2 \\ -d(x,C), & x \notin C, C_2 \end{cases}$$

Then gf is continuous and satisfies the conditions of Corollary 3. But $r_0 < 0$, $r_1 > 0 \Longrightarrow$ there exists t such that $r_t = 0$.

LEMMA 6: Let O be an open set in the unit n-cube I^n of E^n, $n > 1$. Then either O or $I^n - O$ connects a pair of opposite faces of I^n.

<u>Proof:</u> It suffices to consider I^2, the unit square in E^2. All higher dimensional cases follow by taking a cross section. The lemma will be proved by showing that its falsity implies the falsity of corollary 2 above. Let O be an open set in the unit square in E^2 such that neither O nor $I^2 - O$ connects opposite faces of I^2. Reflect O about all edges of the squares determined by the integers until all its images in the square $(0,0)$, $(0,n)$, (n,n), $(n,0)$ are obtained. The accompanying figure illustrates this construction for $n = 3$. Let O and its images be denoted by O'. O' is open and its complement is closed. Three cases are possible for each component A of O and $I^2 - O$.



1) A intersects no faces of I^2. Then no reflection of A intersects a line determined by an integer.

2) A intersects one face of I^2. Then when reflected about that face an extension of A is obtained. The components which include reflections of A cannot, however, lie in more than two adjacent 1 X 1 squares in the n X n square.

3) A intersects two adjacent faces of I^2. Then the corresponding components in the n X n square lie in a 2 X 2 square.

Since each component in the n X n square must come from a single component in the unit square subdivision, the components of O' and its complement must have diameter $\leq 2\sqrt{2}$. The n X n square can now be linearly shrunk to the unit square preserving the decomposition into O' and its complement. But, in the new decomposition of the unit square the diameter of each component is bounded by $\frac{2\sqrt{2}}{n}$. The truth of this for arbitrary n contradicts corollary 2.

LEMMA 7: For any $f: I^n \to E^1$, $n > 1$, there exists $r \in E^1$ such that $f^{-1}(r)$ connects opposite faces of I^n.

<u>Proof:</u> It again suffices to consider I^2. Since E' is diameter divisible, given n, there is $O_n \subseteq E^1$ such that every component of O_n and of $E^1 - O_n$ has diameter $< \frac{1}{n}$. For every n, there is a component A_n of $f^1(O_n)$ or of $f^{-1}(E^1 - O_n)$ connecting opposite faces of I^n. As in the proof of theorem 4, $\{x_n\}$, $x_n \in A_n$ has a limit point x for which $f(x) = r$. B_n, the component of the set $\left\{y \mid r - \frac{1}{n} \leq f(y) \leq r + \frac{1}{n}\right\}$ containing x, contains A_n for arbitrarily

large x. $B = \bigcap_{n=1}^{\infty} B_n$ connects opposite faces of I^2 and $f(B) = r$ as in the proof of theorem 4.

THEOREM 5: For A, B, and A X B compact, arcwise connected topological spaces, and $f: A \to E^1$, $g: B \to E^1$ and $h: A \times B \to E^1$ having the property that the range of h is contained in both the range of f and the range of g, there exist $a \in A$ and $b \in B$ such that $f(a) = g(b) = h(a,b)$.

<u>Proof</u>: Since all spaces under consideration are compact, maxima and minima of f, g and h must exist. Let

$$h_0 = \min_{\substack{a \in A \\ b \in B}} (h(a,b)), \qquad h_1 = \max_{\substack{a \in A \\ b \in B}} (h(a,b)),$$

$$f_0 = \min_{a \in A} f(a), \qquad f_1 = \max_{a \in A} f(a),$$

$$g_0 = \min_{b \in B} g(b), \quad \text{and} \quad g_1 = \max_{b \in B} g(b) .$$

Further, let $f(a_i) = f_i$, $g(b_i) = g_i$, $i = 0,1$. There are arcs connecting, respectively a_0 and a_1 in A, and b_0 and b_1 in B. Along each of these arcs, there is a last time the value h_0 is assumed and a first subsequent time h_1 is assumed by the appropriate function. Let $F(t_1)$, $G(t_2)$, $0 \le t_i \le 1$, be parametrizations respectively of the sub-arc joining the points a_2 and a_3 of A found above for which $f(a_2) = h_0$ and $f(a_3) = h_1$, and of the sub-arc joining b_2 and b_3 for which $g(b_2) = h_0$ and $g(b_3) = h_1$. For every t_i, $0 \le t_i \le 1$,

$h_o \leq f(F(t_1)) \leq h_1$ and $h_o \leq g(G(t_1)) \leq h_1$ with the left hand equalities holding only for $t_1 = 0$ and the right, for $t_1 = 1$. Consider the function $\phi(t_1,t_2):I^2 \to E^1$, given by $\phi(t_1,t_2) = f(F(t_1)) - g(G(t_2))$. ϕ satisfies $\phi(0,0) = \phi(1,1) = 0$, $\phi(t_1,0) > 0$ for $t_1 > 0$, $\phi(t_2,1) < 0$ for $t_1 < 1$, $\phi(0,t_2) < 0$ for $t_2 > 0$, and $\phi(1,t_2) > 0$ for $t_2 < 1$. By lemma 7, there is a real number r, such that $\phi^{-1}(r)$ connects opposite sides of I^n. The above inequalities indicate that $r = 0$ is the only r which can connect, and further that $\phi^{-1}(0)$ connects $(0,0)$ to $(1,1)$. Thus there is a connected set K on which $f(F(t_1)) = g(G(t_2))$. $h(F(t_1),G(t_2))$ is defined and continuous on K. For $t_1 = t_2 = 0$, $h(F(0), G(0)) \geq f(F(0)) = h_o$ by construction, and for $t_1 = t_2 = 1$, $h(F(1), G(1)) \leq f(F(1)) = h_1$. Hence, for some $(t_1,t_2) \in K$, $h(F(t_1), G(t_2)) = f(F(t_1)) = g(G(t_2))$ for otherwise the sets for which $h > f$ and $h < f$ would disconnect $(0,0)$ from $(1,1)$ in K. Let $a = F(t_1)$, $b = G(t_2)$ and the theorem is established.

THEOREM 6: If $f:T^2 \to E^1$ is any continuous real-valued function defined on the torus T^2, and f_1 is the induced function on E^2, the universal covering space for T^2 defined by $f_1(\tilde{x}) = f(p(\tilde{x}))$ where $\tilde{x} \in E^2$ and $p:E^2 \to T^2$ is the projection function for the covering space E^2, then there is a connected set $A \subseteq E^2$ on which f_1 is constant such that A has infinite point-set diameter.

Proof: f_1 restricted to an $n \times n$ by n square of E^2 connects opposite faces by lemma 7. However, f_1 is doubly periodic with $f_1(\tilde{x}_1 + 1, \tilde{x}_2) = f(\tilde{x}_1, \tilde{x}_2) = f(\tilde{x}_1, \tilde{x}_2 + 1)$. Hence, for any n, there is a point $\tilde{x}_n$ lying on one of the two segments (0,0), (0,1) and (0,0) (1,0) such that $\tilde{x}_n$ is connected by f_1 to a point at distance n from $\tilde{x}_n$. The sequence $\{\tilde{x}_n\}$ has a limit point $\tilde{x}_o$ for which $f_1(\tilde{x}_o) = r_o$. Denote by $R_{n,\varepsilon}$ the set which is the component of $f_1^{-1}(r_o \pm \varepsilon)$ in the disk of radius n, with center $\tilde{x}_o$, which contains $\tilde{x}_o$. Then $R_n = \bigcap_{\varepsilon} R_{n,\varepsilon}$ is a connected set containing $\tilde{x}_o$ on which f_1 is constant, and has diameter at least n, as in the proof of theorem 4. Thus the component of $f_1^{-1}(r_o)$ containing $\tilde{x}_o$ has diameter at least n for every n.

THEOREM 7: For any connected, closed subset of T^2, the following are equivalent:

i) A is not contractible in T^2.

ii) Every ε-neighborhood of A contains a homotopically non-trivial cycle of T^2.

iii) $p^{-1}(A) \subseteq E^2$ contains a component of infinite point set diameter.

Proof: The proof is cyclic.

i) ==> ii): Assume ii) is false. Then there is an ε-neighborhood, A*, of A which contains no non-trivial cycles. A*, being open and connected, is arcwise connected. Then A* can be lifted to $\tilde{A}^* \subseteq E^2$ by some continuous function f, [1]. f(A*) is contractible. Let $F:\tilde{A}^* \times I^1 \to E^2$ be such a contraction. Then $G:A^* \times I^1 \to T^2$ given by $G(x,t) = p\,F(f(x),t)$ is a contraction of A*. $A \subseteq A^*$; hence A is also contractible in contradiction to i).

ii) $\Longrightarrow$ iii). This follows immediately upon application of the proof of theorem 6 to the function which is 0 on A and distance from A elsewhere, after noticing that $p^{-1}(A^*)$ is infinite if A^* contains a non-trivial cycle.

iii) $\Longrightarrow$ i). The second homotopy covering theorem is used ([6], page 54). Assume A is contractible. Let $\bar{f}: A \times I^1 \to T^2$ be a contraction of A where, for convenience, $\bar{f}(x,1) = x$ and $\bar{f}(x,0) = k$. Let $\tilde{k}$ be any point of E^2 for which $p(\tilde{k}) = k$. Then $f_o: A \to E^2$, given by $f_o(x) = \tilde{k}$ is a map such that $\bar{f}$ is a homotopy of $p\, f_o$. Then by the homotopy covering theorem, there is a homotopy $f: A \times I^1 \to E^2$ for which $p\, f(x,t) = \bar{f}(x,t)$. For $t = 1$, then, we have that $f|A \times 1$ is a homeomorphism of A. But A is compact and the components of $p^{-1}(A)$ are $f(A \times 1)$ and its translates by integers (from all of which it is distinct by virtue of $f|A \times 1$ being one to one). Thus $p^{-1}(A)$ has no infinite components, contrary to assumption.

COROLLARY 5: Given any $f: T^2 \to E^1$, there exists $r_o \in E^1$ such that $f^{-1}(r_o)$ contains a non-contractible component.

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