A STUDY OF THE INJECTION PROCESS

IN BETATRONS AND SYNCHROTRONS

Thesis by

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ABSTRACT

The effects of azimuthal inhomogeneities in the magnetic guiding fields of betatrons and synchrotrons, and of non-linear terms in the equations of motion of the particles have been investigated in detail, with particular emphasis on their influence during injection. The principal objective has been to determine whether these effects may provide an explanation of the success of the injection process, and whether by appropriate adjustments they may be used to enhance its effectiveness, by inducing a resonant behavior of a favorable nature. It is concluded that these effects alone cannot explain the observed injection phenomena, and that it is very difficult to use magnetic inhomogeneities to produce favorable effects. An explanation has been found for the satisfactory performance of some accelerators in which unfavorable resonant effects have been predicted by other workers.

An additional study of the electromagnetic interactions of the electrons has yielded a more satisfactory description of the injection process. These results, which may be discussed in terms of space charge and self-inductance effects, extend and clarify the recent suggestions of Kerst, and provide quantitative estimates of the effects responsible for the success of injection.

I

INTRODUCTION

This thesis constitutes a study of the process of injecting electrons into betatrons[1,2,3] and synchrotrons[4,5,6,7]. These recently invented accelerators have already become important tools for exploring the nature of high-energy interactions among fundamental particles. One of the most important and interesting tasks facing both experimental and theoretical physicists today is that of investigating and understanding these interactions, and therefore a considerable number of such accelerators have been designed and built in rapid succession, each with a greater output energy than that of its predecessors. Electrons are now being accelerated to an energy of 100 million electron volts by betatrons and to 300 Mev. by synchrotrons; accelerators now being designed will have output energies in the billion electron volt range. Almost all aspects of these machines are now well understood theoretically, the principal exception being the nature of the injection process.

In this section we shall first explain the basic principles on which these accelerators operate. We shall then discuss the importance of understanding the injection process, and formulate the general problems to be solved. The objectives set for the present study are next motivated and described, and the results are summarized.

The research to be described separates naturally into two parts. In Section II we deal with the problem of determining the motion of a single electron under the assumption that the other electrons simultaneously present have a negligibly small influence on its behavior, while Section III contains an account of the interaction effects neglected in Section II.

To make the text of these sections more readable, we have placed all lengthy mathematical derivations in an appendix, which therefore constitutes an important part of this thesis.

1. Basic Principles of the Betatron and Synchrotron

The betatron, or induction electron accelerator, was invented by Kerst[1] in 1941. It is unique among particle accelerators in that the electric field providing the accelerating force is produced by the change of magnetic flux through the circular electron orbit, in accordance with Faraday's law of electromagnetic induction. The electrons are held in their circular path by the magnetic field whose changing flux accelerates them. The field at the orbit and the flux through it increase together during the acceleration. It can easily be shown that if

$$d\Phi/dt = 2\pi R^2 \, dB_o/dt$$

where Φ is the flux through the orbit and B_o is the magnetic induction at the orbit, then the orbit radius R will remain constant independent of the electron mass and velocity even in the range of extremely relativistic energies. The integrated form of this equation,

$$\Phi(t) = 2\pi R^2 B_o(t)$$

is known as the betatron flux condition; it expresses the fact that the space average of the magnetic field over the orbit area must be twice the field at the orbit at all times during the acceleration if the orbit is not to shrink or expand. The electron orbits are made stable against small displacements by magnetic focusing forces[8] due to the space dependence of the magnetic field in the neighborhood of the orbit;

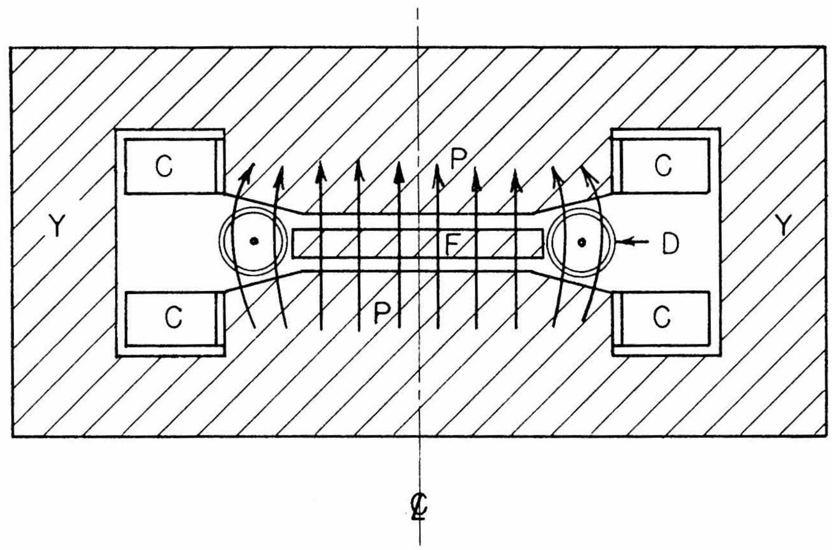
these forces will be considered in detail further on.

The electrons circulate in an evacuated toroidal accelerating chamber (usually referred to as a doughnut) containing an injection gun to supply electrons. The doughnut is placed in a magnetic field produced by an iron core electromagnet having pole pieces carefully shaped to provide magnetic focusing and central flux bars adjusted to satisfy the flux condition. These components are indicated in Fig. 1. The magnet is resonated with a capacitor bank at a low frequency (usually sixty cycles per second), the losses being supplied by an external power source. A group of electrons are accelerated during a portion of each cycle, producing a pulsed high-energy output beam. During each trip around the doughnut an electron gains an amount of energy in electron volts equal to the instantaneous voltage which would be induced by the changing flux in a single turn of wire placed at the orbit position. The electron beam itself thus constitutes the secondary coil of a transformer.

The sequence of events in each cycle is shown in Fig. 2a. The electron gun is pulsed with a voltage of the order of 50 kilovolts for about two microseconds, at a time in the cycle when the magnetic field is weak but increasing. The electrons are injected nearly tangentially, at approximately constant energy. If the injection energy is exactly constant, the first ones injected will enter too weak a magnetic field and strike the outer wall, but as the field rises there is a finite but short interval during which they are accepted into orbits lying within the doughnut. After this interval, the field will curve newly injected electrons too rapidly and they will strike the inner wall. An attempt is usually made to shape the injection pulse so that it lies within

FIGURE I

BETATRON COMPONENTS



C Energizing coils	G Injecting gun
D Glass accelerating chamber	P Magnet pole piece
F Central flux bars	Y Magnet yoke for flux return

Lines with arrows denote magnetic flux

Shaded areas are laminated iron

Black dot denotes equilibrium orbit

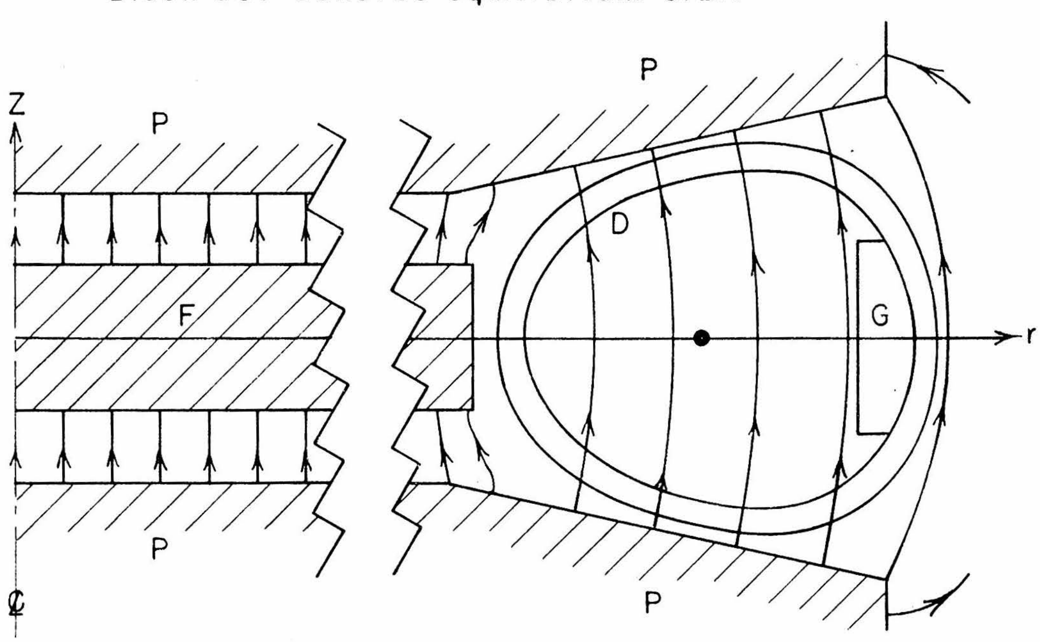


FIGURE 2a

BETATRON OPERATING CYCLES

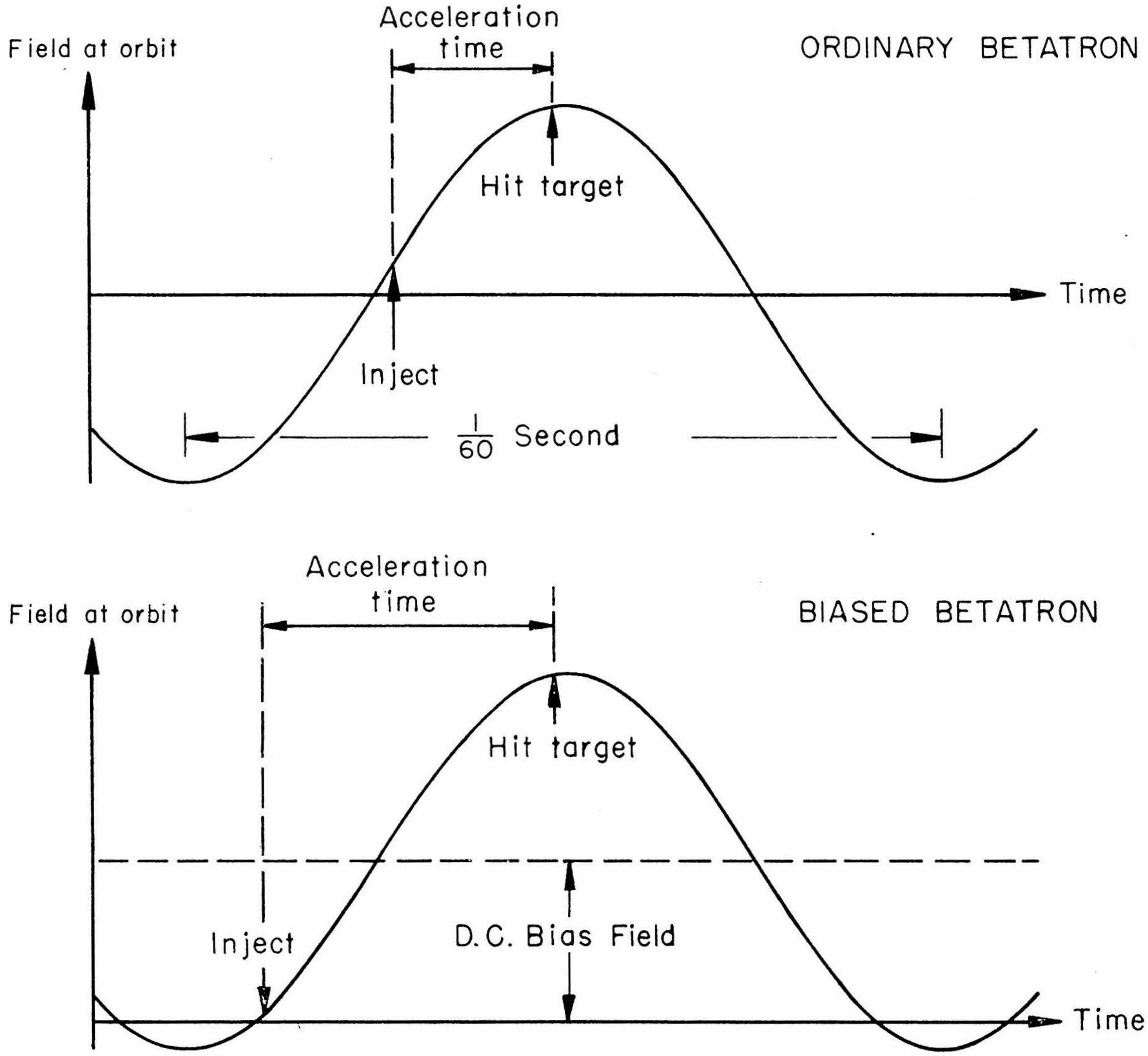
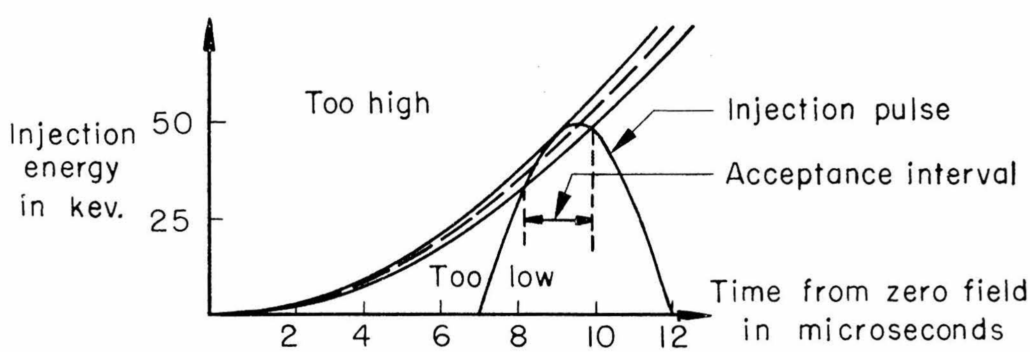


FIGURE 2b

LIMITS OF ACCEPTANCE



the limits of acceptance for a longer time; this is indicated in Fig. 2b. The accepted electrons are accelerated continuously as long as the field increases and the betatron flux condition is maintained. When they have attained the desired energy, the orbit may be deflected inward into a metal target, or outward into the rear of the gun which acts as a target, by one of a variety of methods which all depend on invalidating the flux condition at the proper time. If the orbit is not deflected, the field will reach its maximum and decrease again, and the electrons will be decelerated by the reverse effect.

Since the electrons are undergoing a centripetal acceleration, they will radiate away some of the energy imparted to them. This radiation loss is large enough to be important only for electron energies in excess of 100 Mev., beyond which its effect becomes equivalent to a failure of the flux condition, since the net energy gain per turn is no longer sufficient to maintain a constant orbit radius. The orbit will then shrink into the inner doughnut wall. Since the radiation loss increases as the cube of the electron energy, this effect sets an upper limit to the energy attainable in a betatron.

Acceleration to higher energies is made possible by the principle of phase stability, which was discovered independently by McMillan(4) and Veksler(5) in 1945. Consider a charged particle moving in a circle in a magnetic field, and passing once each revolution through an accelerating gap across which is impressed an alternating voltage. The particle gains or loses an amount of energy dependent on the phase of this voltage at the time of each crossing. If the period of the gap voltage is equal to the time of one revolution, one possible motion is that in which the particle always crosses when the gap voltage is passing through zero.

McMillan and Veksler showed that the motion with crossings at this phase is stable; that is, particles started off with slightly different energies will tend to lose energy at the gap when their energy is too great, and vice versa. This comes about because of the interdependence of radius, velocity, and time of rotation. In consequence, these particles execute slow oscillations, both in phase and in radius, about the stable values. Such particles are said to be locked in synchronism with the alternating frequency of the gap.

If this frequency, or the magnetic field strength, is slowly varied, the particles will tend to do whatever is needed to maintain this synchronism. In the synchrotron the magnetic field is slowly increased, while the accelerating frequency is held constant. The electrons then find it necessary to gain energy from the gap, on the average, in order to remain synchronized. If the magnetic field is fixed, the oscillation about the phase of zero voltage is analogous to the motion of a simple pendulum about its rest point; the effect of slowly changing the field is analogous to the introduction of a constant torque so that the pendulum's stable position is displaced. The energy lost to radiation does not upset this scheme but merely introduces an additional shift in the stable phase, just sufficient to provide for the compensating energy gain.

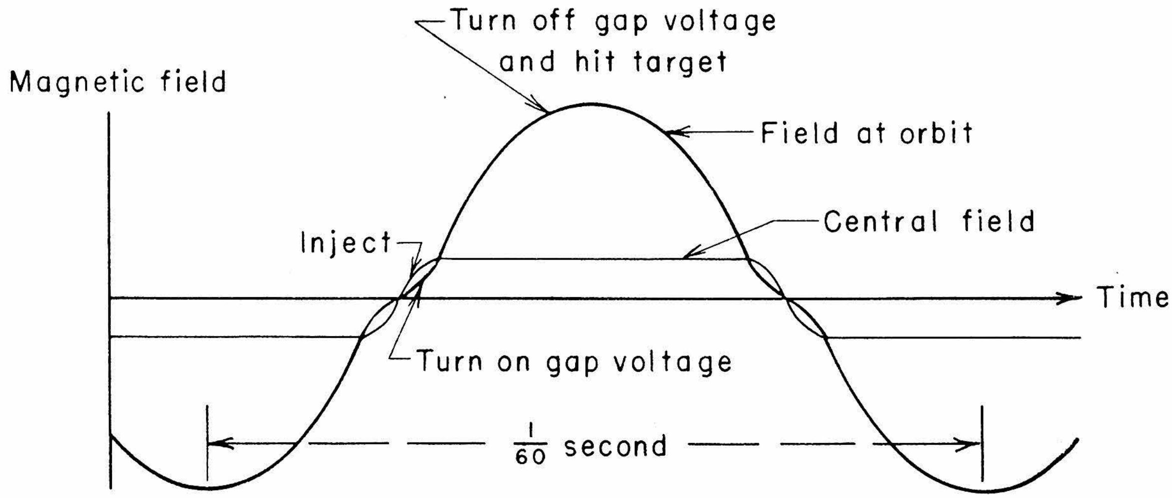
The frequency of the oscillating electric field in a synchrotron is $v/2\pi r$, where v is the electron's velocity and r the radius of the circle in which it moves; thus if the velocity is to increase considerably during acceleration, this frequency must be appropriately modulated or provision must be made to allow for the corresponding increase in radius. As it is desirable to avoid both of these alternatives, synchrotron acceleration must start at an electron energy of over one Mev., since above this energy an electron's velocity is nearly that of light. For this reason, following the suggestion of Pollock[9], it has become common practice to operate a synchrotron as a betatron in the first stage of acceleration. The central flux bars carrying the betatron flux are constructed so as to saturate early in the cycle, when the electron energy is around one Mev.; no further increase in central flux occurs although the field at the orbit continues to rise. At this time the oscillating electric field is turned on, and accomplishes the rest of the acceleration while operating at constant frequency as described above. The beam may be brought inward by shutting off the accelerating voltage when the desired energy is reached. This sequence of events is shown in Fig. 3, together with a diagram indicating the arrangement of the principal components of a synchrotron.

2. Statement of the Problem

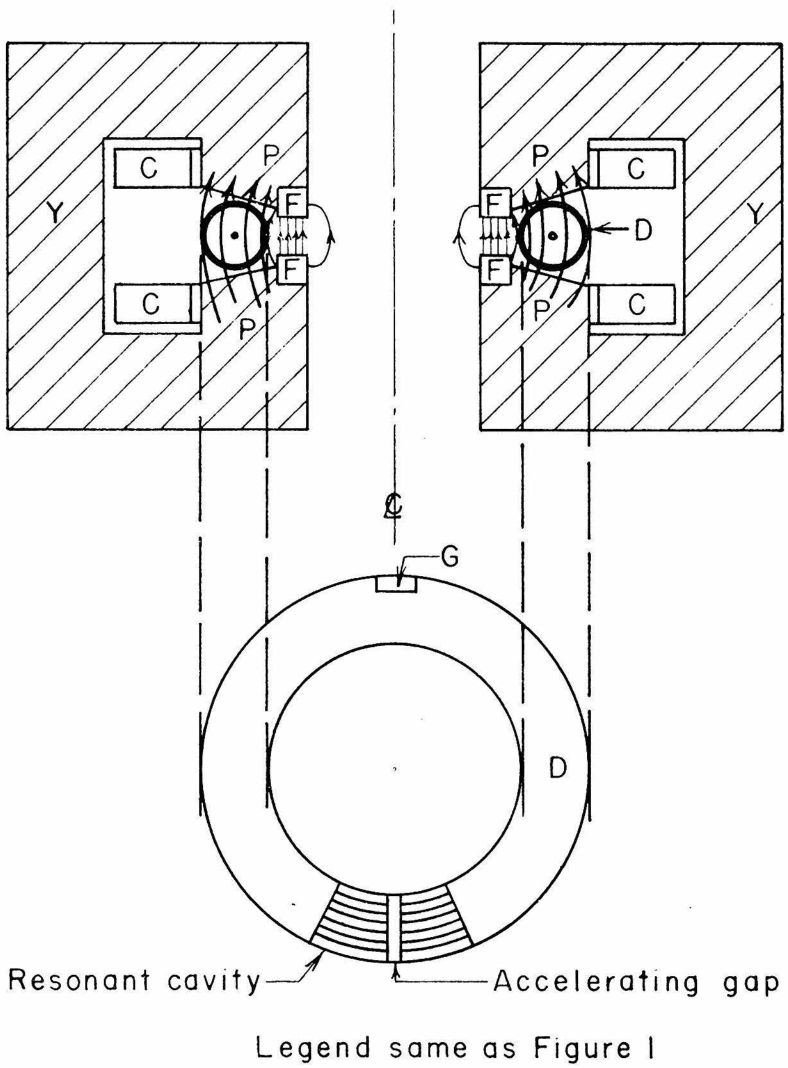
The basic principles of operation described above have been well verified by many experiments, and no betatron or synchrotron thus far constructed has failed to function, in spite of the fact that the various models differ considerably in design and construction. However, when a new machine is first operated, its high energy output is usually zero. The operators vary all possible gun parameters (such as gun position and orientation, and pulse shape, height and timing) and adjust the magnetic field shape by shimming the central flux bars and by varying the currents induced by the changing field in various coils attached to the pole pieces, until a beam is first detected. This procedure may require weeks or months of continual experimentation of a more or less random sort, although it has recently been found possible to shorten it

FIGURE 3

SYNCHROTRON OPERATING CYCLE



SYNCHROTRON COMPONENTS



Legend same as Figure I

by operating the magnet with reduced excitation during the search for a beam. When once located, the beam is maximized by repeated adjustment of all parameters; these adjustments greatly increase the original output.

The difficulties in adjusting a new machine are centered about the initial establishment of a stable circulating current of electrons, since once a current is well established there is nothing to hinder its acceleration. The injection process consists, in the broadest sense, of the establishment of this current. A complete understanding of the process would be valuable for several reasons. First, it might lead to methods for more quickly and efficiently adjusting a machine for maximum output. Second, and more important, it might suggest methods for greatly increasing the intensity of the high-energy output beam above that now attainable by empirical adjustments. Third, and perhaps most important of all, such an understanding might provide assurance that a new machine can be made to work; the crucial requirements for successful injection are not known, and at present there is no guarantee that they will be satisfied in future accelerators merely because they can be met in existing ones.

In these accelerators an electron must follow an approximately circular path for a very large number of revolutions, and the fundamental question immediately arises as to how it may avoid a collision with the rear of the injecting structure at the end of its first revolution. The original theory describing the orbits followed by electrons in a betatron, developed by Kerst and Serber(8), provided an explanation of how this mishap could be postponed for several revolutions. Their conclusions were based on the equations of motion of a single electron in an axially symmetric field. By the usual technique of linearizing the equations for study of small oscillations about steady motion, they showed that small radial and vertical oscillations, about an instantaneous circle* of radius appropriate to the momentum of the electron, were each sinusoidal, with frequencies respectively $(1-n)^{\frac{1}{2}}$ and $n^{\frac{1}{2}}$ times the rotation frequency. Here n is the exponent in the expression for the vertical component of the magnetic field as a function of radial distance in the neighborhood of the orbit ($B_z \propto r^{-n}$); it must lie in the range $0 < n < 1$ if the orbit is to be stable against both vertical and radial displacements. The effects of these oscillations may be visualized by plotting the vertical displacement from the instantaneous circle against the radial displacement, with the azimuthal angle as the parameter of the curve. The resulting Lissajous figure, of course, does not show the motion around the instantaneous circle; this motion can be indicated by placing dots along the curve at the successive positions where the electron passes the gun as it traverses the curve. This is shown in Fig. 4a, drawn for $n = 2/3$ with arbitrarily chosen initial conditions. In this example the electron will make seven revolutions before encountering the gun.

Kerst and Serber also showed the existence of two damping effects, due to the time variation of the magnetic field. The oscillations about the instantaneous circle are damped, and the instantaneous circle slowly approaches the equilibrium orbit defined by the betatron

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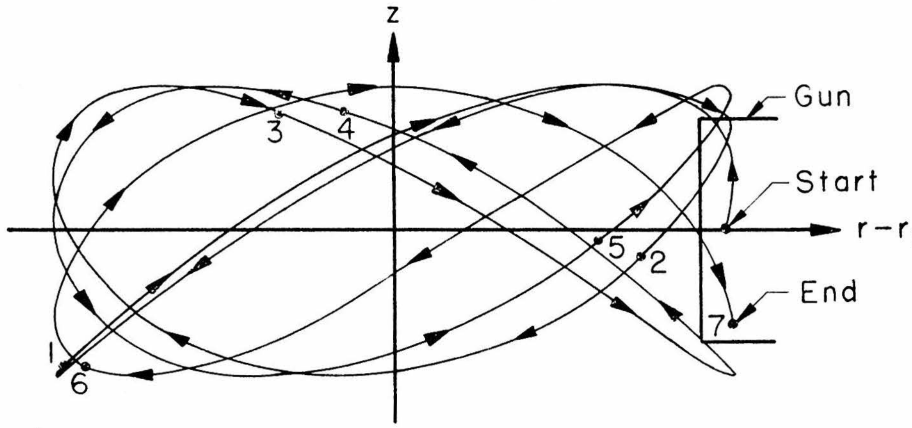
* In discussing the linearized problem we shall adhere to the terminology of Kerst and Serber. They define the radius r_i of the instantaneous circle to be that satisfying $mv = er_i\, B_z(r_i)$; it will be different for electrons of different momenta in the same field. The equilibrium orbit, of radius R, is defined by the field alone; it is the circle for which the betatron flux condition is satisfied.

FIGURE 4a

LISSAJOUS FIGURE SHOWING KERST-SERBER OSCILLATIONS

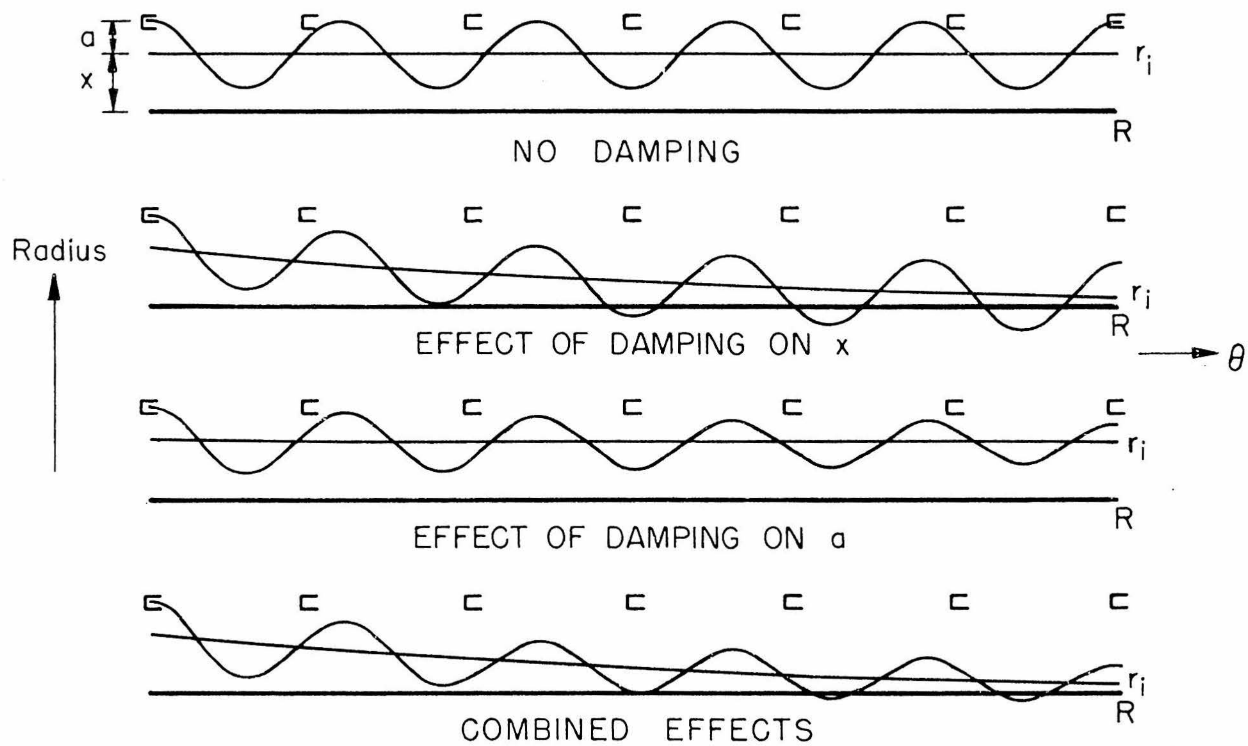
DRAWN FOR $n = \frac{2}{3}$



Numbers indicate position at successive passings of the gun

FIGURE 4b

EFFECTS OF ADIABATIC DAMPING



r_i = radius of instantaneous circle, R = radius of equilibrium orbit

Damping greatly exaggerated

flux condition. Quantitatively, they proved that

$$\Delta a/a = -(\tfrac{1}{4})\,\Delta E/E \qquad \text{and} \qquad \Delta x/x = -\tfrac{1}{2}\,\Delta E/E$$

in the non-relativistic energy region; here a denotes the amplitude of either oscillation, x the difference $r_1 - R$ between the radii of the instantaneous and equilibrium circles, and E the electron's kinetic energy. The effects of this damping in various cases are shown in Fig. 4b, where the damping is greatly exaggerated. Its effect on the Lissajous figure is to slowly translate it toward the equilibrium circle while shrinking its linear dimensions at half the rate of translation.

If the damping is negligible, it is clear that after several revolutions the electron will return to the immediate vicinity of the injection point, collide with the rear of the gun, and be lost. This is made even more certain by the fact that it will spend a larger fraction of its time near the crests of its radial oscillations, where the gun obstructs its path, than near its instantaneous circle. Kerst's original betatron(1) was designed to avoid this difficulty by utilizing the damping, which was made large by choosing low injection energy ($\sim$ 200 e.v.) and high energy gain per turn ($\sim$ 25 e.v.). It was calculated that this damping would be sufficient to avoid collisions with the rear of the gun, and that the injection energy would still be high enough to prevent excessive loss of electrons by scattering from the residual gas in the doughnut. However, experiments with this machine showed that the yield increased rapidly with increase of injection energy over the available range (to 600 e.v.); this is in direct contradiction to the theory described above, since the per cent energy gain per turn, and hence the damping, decreases with increasing injection energy. In subsequently constructed accelerators(2,3,6,7) it has been found that still greater injection energies further improve the yield. Present machines inject at the highest energies their guns will stand without arcing over; these are in the vicinity of 50 Kev., where the Kerst-Serber damping is completely inadequate to explain how some electrons do in fact manage to avoid collisions with the gun for hundreds of revolutions. No entirely satisfactory explanation of this injection phenomenon has thus far appeared.

The fraction of injected electrons which escape such collisions is not large, even in the best-adjusted machines. In a typical accelerator, about one electron in fifty to one hundred manages to successfully complete two thirds of its first circuit, the rest being lost by collision with the doughnut walls, due to the rather large divergence of the beam emitted by the injection gun (which must be compact so as to offer as small an obstructing cross-sectional area as possible). Of those which survive the first revolution, all but one in twenty or so are soon lost, many of them presumably to the rear of the gun, when a machine is finally adjusted for optimum output. However, it takes months of manipulation to raise this proportion from an initial value of perhaps one in 10^5*. The initial cost of these machines is so high that it would be worth considerable study to raise it to one in five, but it is probably more important to gain assurance that it can be improved from one in 10^5 to at least one in 100, say, in a new machine; if this could not be done, the entire initial cost might be wasted.

The amount of circulating current which can be established

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* These data were supplied by Dr. R. V. Langmuir.

probably depends on a great number of factors. Among these are the injection energy and current; the shape, duration, and timing of the injection pulse; the space and time dependence of the magnetic field, including eddy current and hysteresis effects and azimuthal inhomogeneities; the position, orientation, and design of the gun; the conductivity of the coating on the inside of the doughnut, and the resistance between it and ground. A complete theory of the injection process would include an evaluation of the influence of each of these factors on output; the most important requirement of a successful theory is that it should explain the mechanism which allows even a few electrons to clear the gun.

3. Motivation, Description, and Summary of the Present Study

The adjustment of the azimuthal variation of the magnetic field has always seemed to be an essential part of the procedure for obtaining and maximizing the output of those accelerators[7,10]. For this reason it was felt that a detailed investigation of the effect of azimuthal inhomogeneities in the magnetic field would be worth while, since only the first-order effects of such deviations of the field from axial symmetry have been considered by other workers. These deviations may not be small relative to the main field at the orbit at injection time. The main field is approximately 1.07 ($E_{Kev.}^{\frac{1}{2}}$ / R meter) gauss for non-relativistic injection energies; at 50 Kev., this is about 25 gauss for R = 0.3 m., and still less for larger accelerators. The deviations have been measured on various magnets[10] and have been found to be as large as six gauss. They are due to eddy current and hysteresis effects and to asymmetries in the magnet structure (yokes, cavities for instrumentation, etc.). They will play a much smaller role later in the cycle, as the main field gradually rises to several thousand gauss.

An investigation of the effects of such inhomogeneities on those electron orbits which lie in the plane of symmetry was initiated by Davis and Langmuir(11). They invented a magnetic field with azimuthal inhomogeneities* which at first sight showed possibility of greatly aiding the electrons in missing the gun. It was arranged to make the time of one radial oscillation twice that of one revolution, at such a phase relative to the inhomogeneities that the amplitude of the free oscillation was halved after every two revolutions. However, closer examination indicated that this phase was unstable and tended to shift to a stable value at which the amplitude was doubled after every two revolutions.

Section II of the present study forms a continuation and extension of their investigation, in which the principal objective has been to analyze the possibility of avoiding this difficulty, in order to make use of such inhomogeneities to increase the injection efficiency. After carefully considering a variety of mechanisms, the conclusion was finally reached that this difficulty is inherent in the situation, and that it is impractical to rely on adjustment of the inhomogeneities to achieve this objective. The same conclusion applies to attempts to utilize the effects of non-linear terms in the equations of motion for the same purpose. In establishing these results, the concept of the flow of particle trajectories in a phase space and a study of its divergence has played an important part. In particular, it has been proved

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* This field had a constant large value of n everywhere inside the instantaneous circle and 5/7 of the way around outside (measured from the gun in the direction of motion of the electrons), and a constant small value outside this circle over the remaining 2/7 of the range in azimuth.

that this divergence is always zero, at least to the degree of approximation justified by the mathematical methods used; this state of affairs is inconsistent with attainment of the desired result.

Another objective of this part of the research was to ascertain whether or not the success of the injection process in existing machines could be related to effects due to the presence of axial inhomogeneities. The results indicate that the inhomogeneities probably never play a favorable role in the process, but that they may hinder it considerably by producing forced oscillations and resonant effects, unless they are rather small and properly adjusted. It seems likely that all of the empirical field adjustments, which have been found necessary in obtaining and increasing output, only serve to decrease or eliminate these hindering effects. This research uncovered no reason why a machine whose field has inhomogeneities should function better than one with an axially symmetric field.

In spite of these negative conclusions, the investigation has cast light on the general behavior of accelerators in which the periods of free and forced oscillations are commensurable. Theoretical studies by previous workers[12,13] have led them to the conclusion that the oscillations in such machines would be divergent; the present research has provided an explanation of the observed fact that these accelerators may function satisfactorily. It has also led to conclusions concerning which inhomogeneities, and in particular which phases of these inhomogeneities with respect to the injection gun, are harmful to satisfactory injection in systems where such commensurabilities exist.

These conclusions are all based on an investigation of the orbits of single electrons in a specified magnetic field without taking into account the interactions among the large number of electrons simultaneously moving in these orbits. It was originally intended to analyze these interactions only to the extent of discovering in what ways they would modify the parameters entering into the single-electron study. In view of the conclusions stated above, however, it seemed that these effects must be of decisive importance in the injection process, as was first suggested by Kerst[14]. His recently proposed explanation of injection is based on some of the effects due to these interactions. Section III of the present work consists of some calculations concerning these effects, which show their importance in an analysis of the injection problem. Except for the work of Kerst[1,14] and Blewett[15], nothing concerning them has thus far been published.

The study of interaction effects described in Section III is divided into two parts, dealing with time-independent effects and time-dependent effects, respectively. The first part treats the effects of electrostatic and electromagnetic actions of a circulating beam of electrons on itself. Some qualitative conclusions are drawn concerning the probable distribution of current and charge within the beam, and expressions are derived giving the maximum amount of charge and current which can be held by the magnetic focusing forces against its own self-repulsion, as a function of injection energy and other machine parameters. There is also indicated a way of including the effects of an assumed space charge distribution in the equations of motion of an individual electron.

The second part contains a discussion of the probable magnitude of the circulating current as a function of time during the injection period, and the effects on orbit locations and oscillation amplitudes of changes in this magnitude. It is found that the effects of space charge repulsion and self-inductance of the beam are not of greatly different orders of magnitude, and that both are enhanced by locating the gun near a point of radial instability and by filling the doughnut as full of electrons as possible. Finally, means are discussed by which these effects may result in trapping a fraction of the beam in stable orbits, for a variety of assumptions as to gun location and relative magnitudes of the effects. These are shown to be consistent with various experimental discoveries concerning the injection process.

The principal contribution of this part is to extend and clarify the remarks of Kerst[14], to obtain somewhat more quantitative estimates of the magnitudes of the effects described by him, and to discuss in greater detail the importance of these interaction effects in accomplishing the injection process.

II

SINGLE ELECTRON CALCULATIONS

1. Equations of Motion

Certain simplifications are possible in investigating the motion of an electron in a betatron or synchrotron near the time of injection. As all existing synchrotrons use betatron starting, we have not considered the oscillating electric field in analyzing the injection process; the injection problems in both types of accelerators are the same and may be studied together. The transition from betatron to synchrotron action has been studied by several workers[16,17,18,19] and has been shown to be quite efficient. Studies of injection with the electric field turned on have been published only for proton synchrotrons[20,21] but are probably applicable to electron accelerators with appropriate modifications.

The magnetic field is of course increasing with time, but the characteristic period fixing the time-scale of the motion of the electron in its orbit, and of the oscillations about the stable orbit, is the time of one rotation around the doughnut; the fractional increase of the magnetic field in this time is extremely small. It is a well-known principle of mechanics that if a parameter of a system undergoes such a slow (adiabatic) change, the effects of the change may be easily predicted, provided that the motion for no change is known. In fact, if angle and action variables are employed, the effects are just those which leave the action variable invariant to the adiabatic change. For this reason, the time variation of the magnetic field and the presence of the concomitant electric field supplying the betatron accelerating force may be neglected in studying the orbits. Their effects can be included at a later stage, since Kerst and Serber[8] have already shown that they merely provide damping for the free oscillations, and cause the instantaneous circles about which these oscillations occur to converge toward the equilibrium orbit. As was mentioned above, the inhomogeneities will be adiabatically decreasing with respect to the slowly rising main field, and this relative time variation may also be neglected. The orbit problem is then reduced to the study of the motion of an electron in a specified time-independent magnetic field, provided the effects due to the presence and interaction of many electrons may be neglected.

We shall describe the motion by reference to a cylindrical polar coordinate system (r, θ, z) in which the ideal orbit satisfying the betatron flux condition, called the equilibrium orbit, is defined by $z = 0$, $r = R$ (a constant). The equation of motion in vector form is $\ddot{\vec{r}} = -(e/m)\vec{v} \times \vec{B}$ in M.K.S. rational units, where the electron has rest mass m_0, mass $m = m_0/(1 - v^2/c^2)^{\frac{1}{2}}$, velocity $\vec{v}$, charge $-e = -|e|$, and position vector $\vec{r}$; $\vec{B}$ is the magnetic induction and c is the velocity of light. Since the acceleration is normal to the velocity, the magnitude of the velocity is a constant of the motion. This can be conveniently used to eliminate the time from the differential equations of motion, replacing it by the azimuthal angle as the independent variable, while simultaneously reducing the number of equations from three to two. In order to study small oscillations about steady motion, we shall express r and z in terms of the new variables $\rho \equiv (r - R)/R$, $\zeta \equiv z/R$, since these dimensionless coordinates are small with respect to unity for all points within the doughnut. If B_z has the value B_0 at every point on the equilibrium circle, then $V = B_0 eR/m$ is the velocity a particle must have to undergo steady motion in this orbit. (We choose $\dot{\theta}$ to be positive: thus B_z is positive for electrons and would be negative if protons were considered). We define $\Delta_v \equiv (v - V)/v$; the electrons will immediately strike the doughnut walls unless v is close to V, and hence Δ_v will be very small with respect to unity. We shall express B_z as $B_0 \beta_z$, and similarly for the other field components; $B_r \equiv B_0 \beta_r$, and $B_\theta \equiv B_0 \beta_\theta$. In the appendix it is shown that, when the equations of motion are written out in cylindrical coordinates, the time eliminated, and the new notations introduced, the following equations result:

$$\rho'' = (1+\rho) + (1-\Delta_v)\sqrt{X}\left[(1+\rho)\zeta'\beta_\theta + \rho'\zeta'\beta_r - \{\rho'^2 + (1+\rho)^2\}\beta_z\right] + 2\rho'^2/(1+\rho);$$

$$\zeta'' = (1-\Delta_v)\sqrt{X}\left[-(1+\rho)\rho'\beta_\theta - \rho'\zeta'\beta_z + \{\zeta'^2 + (1+\rho)^2\}\beta_r\right] + 2\rho'\zeta'/(1+\rho).$$

Here primes denote differentiations with respect to θ and

$$X \equiv 1 + (\rho'^2 + \zeta'^2)/(1+\rho)^2.$$

These equations are rigorously correct, and are seen to be non-linear for even the simplest magnetic fields. Their complexity is greatly increased by the azimuthal variation of the β's, the specification of which we now consider.

2. Description of the Magnetic field

The desired magnetic field is approximately of the form $B_z \propto r^{-n}$; Kerst and Serber[(8)] have shown that unless $0 < n < 1$ the orbits are unstable in the linearized equations of first approximation. An inspection of the general equations of motion above will show that if $B_r = B_\theta = 0$ in the plane $z = 0$, it is possible to have a motion confined to this plane, as a special case. In discussing this case, we shall expand B_z in this plane about its value on the equilibrium orbit as a power series in ρ , with coefficients which are Fourier series in θ , as follows:

$$B_z = B_o \beta_z = B_o \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{jk} \rho^k \cos(j\theta + \alpha_{jk}).$$

Here the A's and α 's are real constants, with $A_{oo} = 1$, $A_{01} = -n$, and $\alpha_{ok} = 0$, in order to obtain the correct form of field in first order. We shall show later that the high frequency* components of the azimuthal inhomogeneities (those with large j) play a minor role, and may often be neglected. On physical grounds they should be quite small unless the magnet structure varies abruptly at certain azimuths. From qualitative considerations, we may assume that no component of the azimuthal inhomogeneities will approach the total field in magnitude; hence all of the A_{jk} with $j \neq 0$ will be somewhat less than unity.

In the more general case, we must specify all three components of the field. We set

$$B_z = B_o \beta_z = B_o \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} A_{jk\ell}\, e^{ij\theta} \rho^k \zeta^\ell ;$$

similar expressions will hold for B_r and B_θ, with coefficients $C_{jk\ell}$ and $D_{jk\ell}$, respectively. Here j, k, and ℓ are integers lying in the ranges indicated on the summation signs: $i^2 = -1$. The A's, C's, and D's are complex constants; the β 's will be real if we require that $A_{jk\ell}$ and $A_{-jk\ell}$ be complex conjugates, and similarly for the C's and D's.

Here we obtain a field having the correct form in first order by setting $A_{ooo} = 1$, $A_{010} = -n$, $C_{ooo} = D_{ooo} = 0$. Maxwell's equations for a static field require that the divergence of $\vec{B}$ must vanish, and

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* Henceforth we write "frequency" for "angular frequency with respect to θ".

that the curl of $\vec{B}$ must also vanish if we neglect the magnetic field due to the circulating electrons. These two conditions impose interrelationships among the three components of the magnetic field. The simplest such relation requires that, if $B_z = B_o\,(1-n\rho)$ to first order in ρ and ζ, then $B_r = B_o(-n\zeta)$ to the same order; thus we must have $A_{010} = C_{001}$. We shall have little occasion to deal with the other requirements imposed on the constant coefficients by Maxwell's equations; they are developed in the appendix. Here again the terms with large $|j|$ are less important, and we may also assume that the constant coefficients whose $j \neq 0$ will be somewhat less than $\frac{1}{2}$.

A determination of the order of magnitude of those A and C coefficients whose $j = 0$ is not so easy to make. These terms depend on the shape of the axially symmetric part of the magnetic field as a function of ρ and ζ at considerable distances from the equilibrium circle, where the field is changing more rapidly than in its vicinity. Even at these distances ρ and ζ are still much less than unity, and therefore the coefficients of rather high powers of them will be very large if these changes are at all sudden. As long as the oscillations are confined to a region in which the field follows the law $B_z \propto r^{-n}$ fairly closely, the necessity for consideration of such terms does not arise. Fig. 5a shows a typical graph of log B against log r in the plane z = 0, as measured experimentally[22] on a half-scale model; Fig. 5b is the corresponding graph of B_z as a function of ρ. It is apparent that if the oscillations are confined to the region AA', the field may be quite closely approximated by the second-degree curve shown on Fig. 5b in dotted lines, which coincides with the exact solid-line curve in this region. However, to get a reasonably good fit everywhere within the

FIGURE 5a

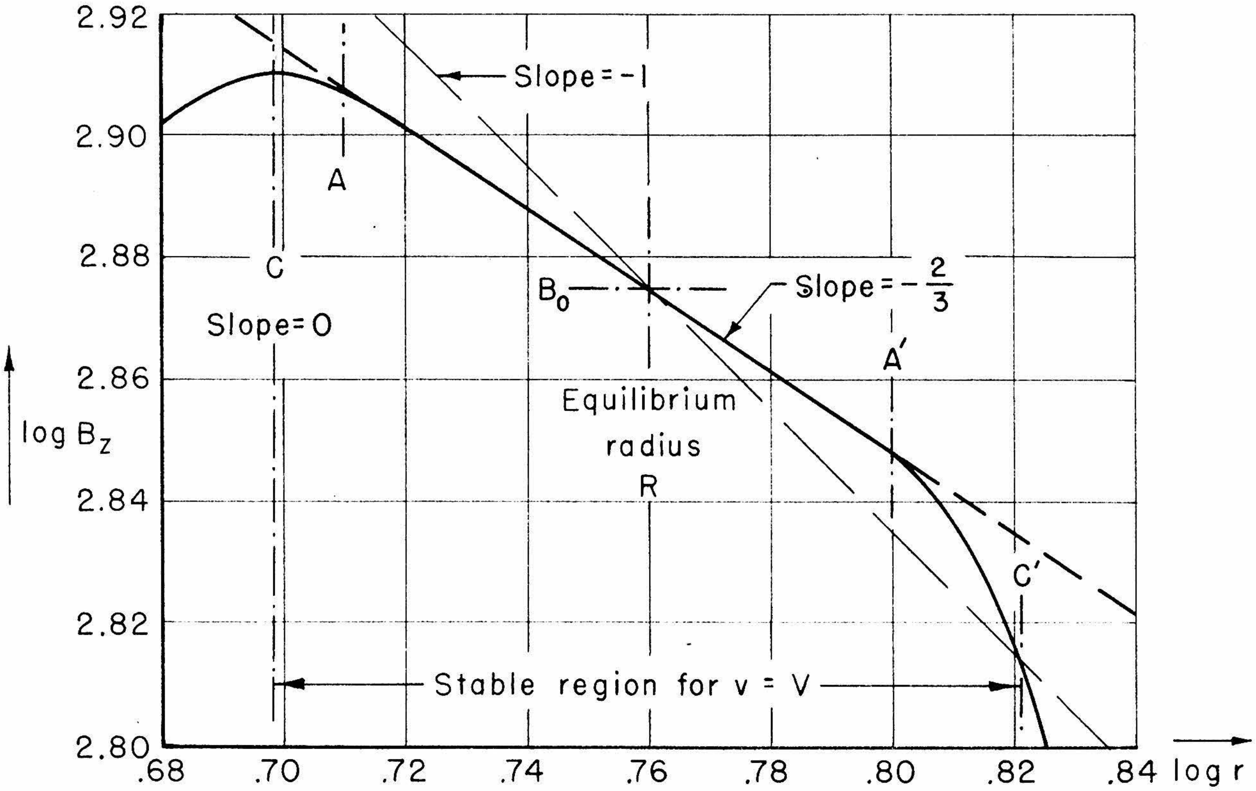
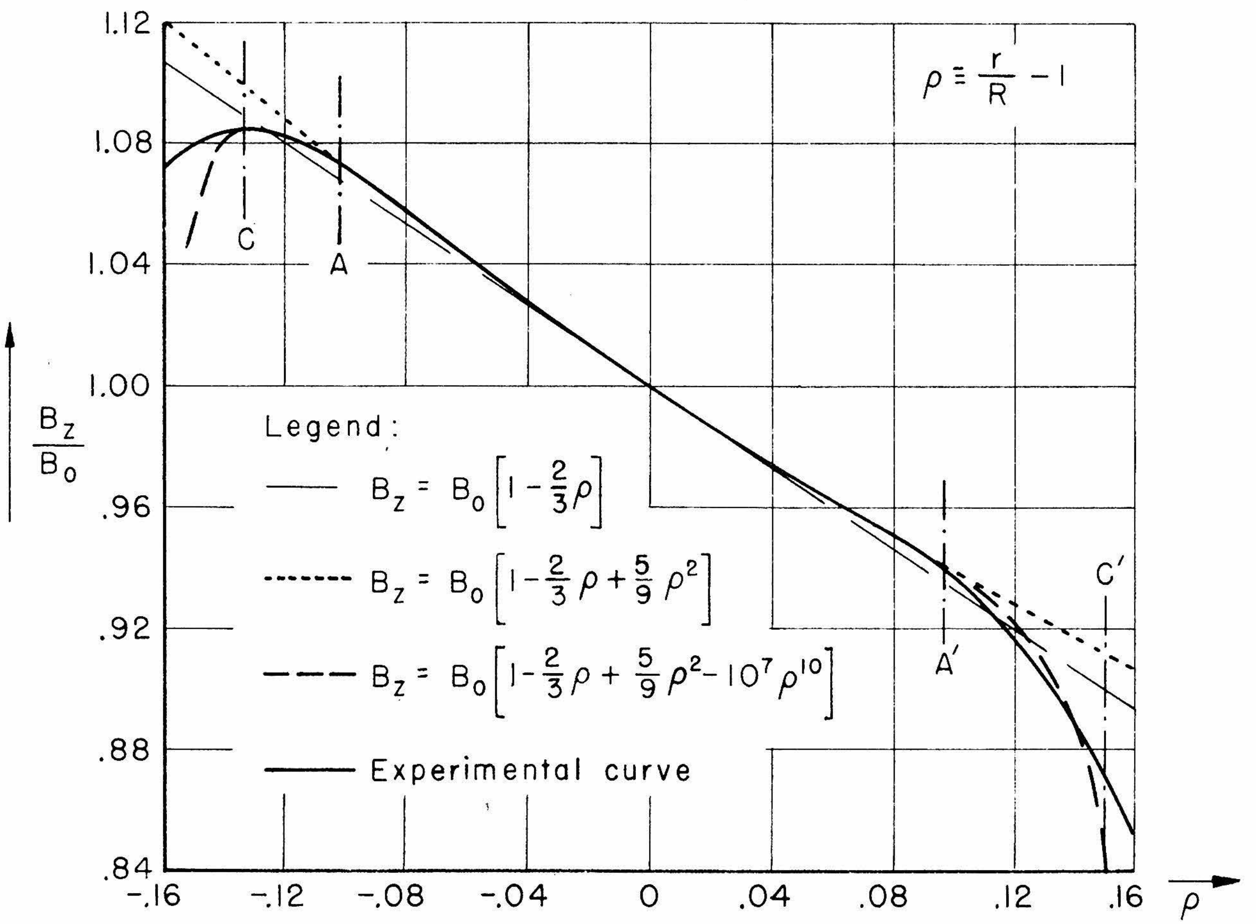


FIGURE 5b



limits of stability CC', one must use an expression containing very high powers of ρ, such as the one given on the figure having a term in ρ^{10} and shown in dashed lines. The determination of the limits of stability is discussed in the following paragraphs. In our approximation procedures we shall treat higher-order coefficients on the assumption that the terms in which they appear are not greater than those of lowest order, and in many cases are considerably smaller.

3. Motion in the Plane of Symmetry

There are two important reasons for treating the two-dimensional motion in the plane $z = 0$, which was mentioned above. From a mathematical point of view, a single differential equation is much easier to discuss than a pair of coupled equations; physically, the problem of missing the gun is primarily that of properly controlling the radial oscillations, since the best position for the gun seems to be near the outer edge of the doughnut in its plane of symmetry (although machines can be made to work with the gun in almost any position[23]).

3a. Axially Symmetric Field

i. Hamiltonian Solution

The simplest general motion is that in an axially symmetric field, as in this case, the solution can be reduced to quadratures. This is most simply done by putting the equations of motion in Hamiltonian form and noting that the canonical momentum p_θ conjugate to the azimuthal coordinate θ is a constant of the motion. We will give only the two-dimensional solution here, but we will generalize it to three-dimensional motion later in this section, and extend it still further in Section III. In the appendix it is shown that the two-dimensional solution is

$$v(t-t_I) = \int_{r_I}^{r} \left\{ 1-y^{-2}\left[r_I \cos \phi_I + (V/vR) \int_{r_I}^{y} x \, \beta_z(x)dx \right]^2 \right\}^{-\frac{1}{2}} dy .$$

Here r_I is the injection radius and ϕ_I the angle between the direction of injection and the tangent to the circle $r = r_I$ at the injection point; v is the constant velocity of the electron, and t_I the time of injection; t is the time at which the radius is r; x and y are integration variables. This integral is analogous to that giving rise to the inverse sine function;

$$a\,\omega\,(t-t_I) = \int_{r_I}^{r} (1-x^2/a^2)^{-\frac{1}{2}}dx .$$

In fact, if the integral be expressed in terms of ρ , and only constant and first power terms in ρ retained in β_z, it reduces to an integral of this form. This result may also be written

$$\dot{r} = \pm\, v\left\{ 1-r^{-2}\left[r_I \cos \phi_I + (V/vR)\int_{r_I}^{r} x\ \beta_z(x)dx \right]^2 \right\}^{\frac{1}{2}} .$$

The radial velocity is zero at the radii r_{max} and r_{min} which satisfy

$$r = r_I \cos \phi_I + (V/vR)\int_{r_I}^{r} x\ \beta_z(x)dx .$$

If no such values of r exist, the electron will monotonically depart from the equilibrium radius and be lost. If the initial conditions are such that r_{max} and r_{min} exist, then the electron will oscillate between these extremes if $\ddot{r} < 0$ at r_{max} and $\ddot{r} > 0$ at r_{min}. As shown in the appendix, these conditions become

$$\beta_z(r_{max}) > vR/Vr_{max}; \quad \beta_z(r_{min}) < vR/Vr_{min} .$$

For an electron with v = V, the limits of radial stability are the points at which $rB_z(r)/RB_z(R) = 1$; this quantity is greater than unity for $R < r < r_{max}$ and less than unity for $r_{min} < r < R$. This result may also be derived from elementary arguments about centripetal acceleration.

Taking logarithms, $\log B_z(r) - \log B_z(R) = -(\log r - \log R)$, giving rise to the straight line of slope -1 on the logarithmic graph of Fig. 5a; this critical line makes clear the geometric significance of this condition in locating the outer limit of radial stability C'. In discussing the two-dimensional problem it must be borne in mind that, in the example chosen, the condition of vertical stability will impose a more stringent inner limit than that given by this condition; the vertical oscillations are unstable inside the radius at which $dB_z/dr = 0$, denoted by C on the figure. If the field continued to rise with decreasing r, however, the inner limit would be given by the inner intersection of the straight line and the actual field curve. For an electron with $v \neq V$, the critical line will still have slope -1 but will pass through the field curve at the radius r_i corresponding to the instantaneous circle appropriate to this velocity, defined by $r_i B_z(r_i)/RB_z(R) = v/V$.

Since the radial velocity differs only in sign for outward and inward motions, the oscillation will be symmetric about an extreme point and can be represented by a Fourier cosine series about such a point; this is characteristic of all conservative oscillating systems. By a conservative system we mean here one whose coordinate acceleration (in this case, $\ddot{r}$) is a function of the coordinate only, so that a time-independent potential function for the motion of this coordinate can be defined. Here, this potential function is

$$U(r) = (mV^2/2R^2r^2)\left[(r_I Rv/V)\cos\phi_I + \int_{r_I}^{r} x\beta z(x)dx\right]^2 ;$$

the equation for radial motion is then

$$d(m\dot{r})/dt = -dU/dr ,$$

and the energy conservation equation is

$$m\dot{r}^2/2 + U = mv^2/2 = \text{constant},$$

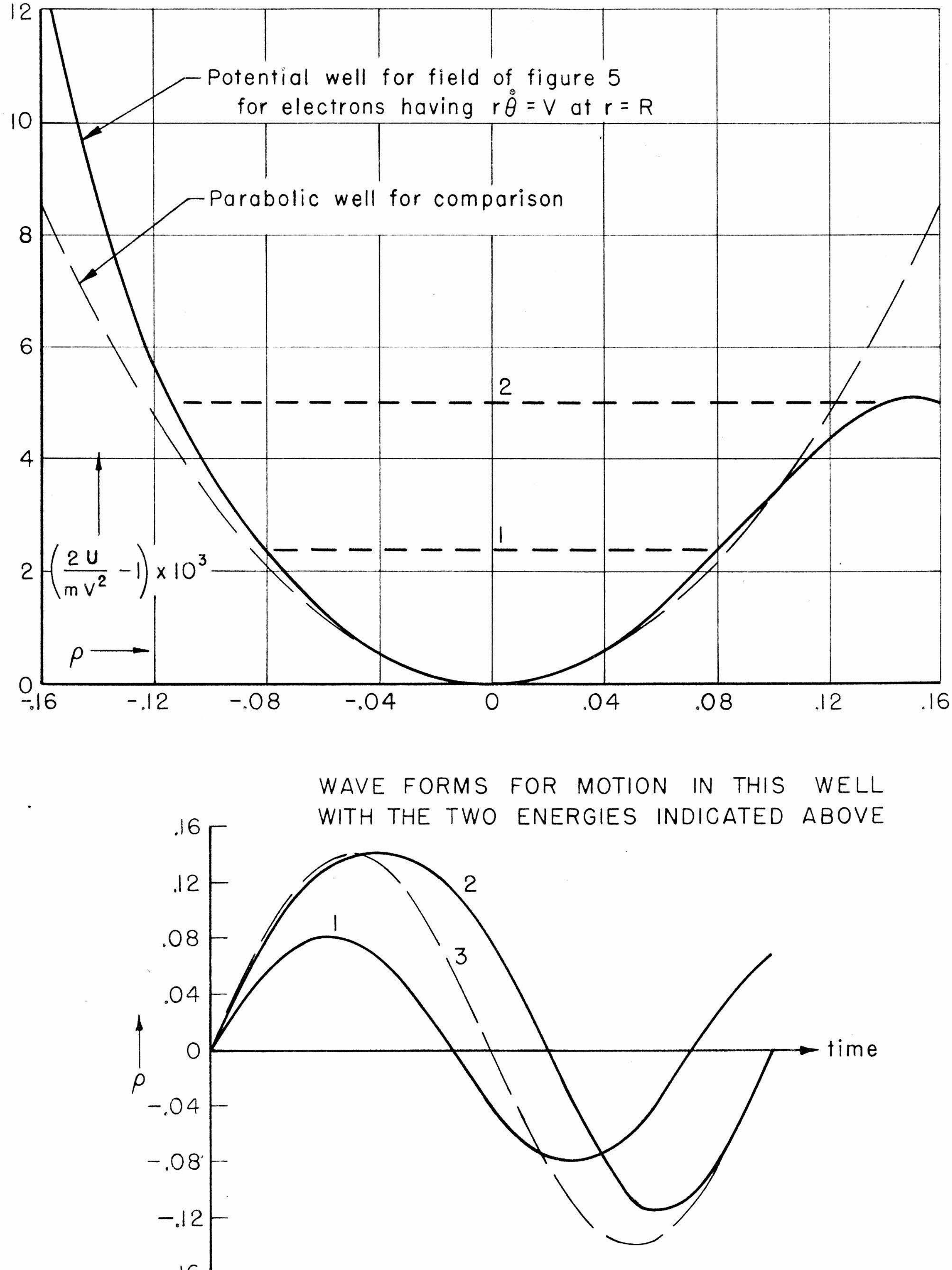
as can easily be verified by differentiation. The radial motion is that of a particle in a potential well whose shape and extent depend on its injection radius, velocity, and direction, as well as on the form of the magnetic field. The too-rapid decrease of field strength at the outer field edge corresponds to a lowering of the wall of the potential well, allowing a particle with large amplitude to spend a longer time in this region, thus flattening the crests of its waveform and lengthening its period of oscillation. These effects are shown in Fig. 6 which is drawn to correspond to the field illustrated in Fig. 5. The well shape is a somewhat complicated function of the parameters r_I, ϕ_I, and v. Its bottom need not be at $r = R$, even if $v = V$; it is located at the radius which satisfies

$$(vr_I/VR)\cos\phi_I + R^{-2}\int_{r_I}^{r} x\,\beta_z(x)dx = (r/R)^2\,\beta_z(r).$$

This can be satisfied for $r = r_I = R$, $v \cos \phi_I = V$; therefore we have chosen to illustrate the well for this condition, since an electron can be pictured as being injected at any point along its path. The group of electrons to which this well corresponds consists of those whose azimuthal velocity component is V at the time they cross the circle $r = R$.

Various applications of this analysis can be made. For example, if only constant and linear terms in $\beta_z(\rho)$ are of importance, it is shown in the appendix that the extremes of motion for any given injection conditions are given by

FIGURE 6



3 is a sine wave with same wavelength as 2, for comparison

$$\rho_{\substack{max.\\min.}} = (1-n)^{-1}\left\{\Delta_v \pm \left[(1-n)^2\rho_I^2 - 2(1-n)\Delta_v\rho_I + \Delta_v^2 + (1-n)\phi_I^2\right]^{1/2}\right\}.$$

Here $\beta_z = 1 - n\rho$, and $\cos\phi_I$ has been set equal to $1-\phi_I^2/2$. To include the effect of the next highest order term in β_z one must solve a cubic equation, or resort to graphical or numerical methods. It is also possible to write an integral for θ as a function of r. However, the principal value of this whole treatment lies in the establishment of the conditions under which the radial motion is oscillatory, and of the fact that it is conservative in the sense defined above. It may be thought at first sight that, since ρ' and ζ' appear in the general differential equations derived earlier, they might provide damping of the oscillations. In general, though, they enter in such a way that their roles are similar to those played by the coordinates. In the present case, where we deal only with ρ', it always occurs to an even power, whereas damping can only arise from terms odd in ρ' or of the form $\rho'\cdot|\rho'|$.

ii. <u>Successive Approximation Solution</u>

Numerical integration is required to determine the quantitative effects of the higher order terms in β_z and the nonlinear terms in the differential equation from the expressions given above. Therefore, we shall now obtain a solution by successive approximations, which will converge fairly rapidly if these higher order terms are small enough. Setting $\zeta = \zeta' = \beta_\theta = \beta_r = 0$, setting $A_{jk} = 0$ for $j \neq 0$, and expanding the right side of the differential equation for ρ as a function of θ in powers of ρ and ρ', we obtain

$$\rho'' + [1-n-(2-n)\Delta_v]\rho = \Delta_v + A_1\rho^2 + A_2\rho'^2 + B_1\rho^3 + B_2\rho\rho'^2 + C_1\rho^4 + C_2\rho^2\rho'^2 + C_3\rho'^4 + \text{Terms of order } \rho^5 \text{ and higher, where}$$

$$A_1 = (1-\Delta_v)(2n-1-A_{02}) , \quad A_2 = \tfrac{1}{2}(1+3\Delta_v) ;$$

$$B_1 = (1-\Delta_v)(n-2A_{02}-A_{03}) , \quad B_2 = \tfrac{1}{2}\left[3n(1-\Delta_v)-4\right] ;$$

$$C_1 = -(1-\Delta_v)(A_{02}+2A_{03}+A_{04}) , C_2 = \tfrac{1}{2}\left[4-3A_{02}(1-\Delta_v)\right] , C_3 = -\tfrac{3}{8}(1-\Delta_v) .$$

If we neglect the non-linear terms, we will have the Kerst-Serber sinusoidal oscillations taking place with a frequency of approximately $(1-n)^{\frac{1}{2}}$. The dependence of the frequency and equilibrium radius on Δ_v is also exhibited. We see that, if $|\rho| < 0.1$ inside the doughnut, we must have $\Delta_v < 0.02$ for n = 3/4 and < 0.03 for n = 2/3, for electrons which are to avoid the walls. We may evaluate the effects of the non-linear terms by the method described on page 219 of Minorsky's text[24]. The calculations are given in the appendix. The results, to third order in the small quantities a and ϵ, are:

$$\rho = \rho_c + a \cos\tau - b \cos 2\tau - c \cos 3\tau + - - - - ,$$

where $\tau \equiv \Omega\theta$,

$$\Omega^2 = \omega_r^2 - 2\epsilon A_1 - \epsilon^2\left(\frac{2A_1^2}{\omega_r^2}+3B_1\right) + \frac{a^2}{12}\left[\frac{10A_1^2}{\omega_r^2} + \left(\frac{14}{\omega_r^2}-4\right)A_1A_2 + 4\omega_r^2A_2^2 + 3\omega_r^2 B_2\right] + 3^{rd} \text{ order in } a \text{ and } \epsilon ;$$

$$\omega_r^2 \equiv 1-n-(2-n)\Delta_v , \quad \text{and} \quad \epsilon \equiv \Delta_v/\omega_r^2 ;$$

$$b = \frac{a^2}{6}\left(\frac{A_1}{\omega_r^2}-A_2\right) + \frac{\epsilon a^2}{6\omega_r^2}\left(\frac{2A_1^2}{\omega_r^2}+3B_1\right) + 4^{th} \text{ order} ;$$

$$c = \frac{a^3}{48\omega_r^2}\left[-\frac{2}{\omega_r^2}(A_1-2\omega_r^2A_2)(A_1-\omega_r^2A_2) + 3(B_1-\omega_r^2B_2)\right] + 4^{th} \text{ order} ;$$

$$\rho_c = \epsilon + \frac{a^2}{2}\left(\frac{A_1}{\omega_r^2}+A_2\right) + \frac{\epsilon^2}{\omega_r^2}A_1 + \frac{\epsilon}{\omega_r^2}\left[\frac{a^2}{2}\left(\frac{A_1}{\omega_r^2}-A_2+3B_1\right) + \epsilon^2 B_1\right] + 4^{th} \text{ order}.$$

The calculation may be carried to any order desired.

The amplitude of the n^{th} harmonic of the waveform is thus shown to be of order α^n. The frequency is shifted by an amount proportional to α^2 even if $v = V$ (when $\epsilon = 0$); it is shifted to first order in the difference $v - V$. As was noted earlier, if the oscillations extend into a region where the field is changing rapidly, some of the A_{ok} for large k may be very large; the higher harmonics need not be small in this case, and the frequency shift may be large also. Although the waveform for ρ as a function of θ will not be the same as that for ρ as a function of time, it will be very similar; the radial velocity is always such a small part of the total velocity that θ increases nearly linearly with time. The strong higher harmonics and the large frequency shift distort the waveform and lengthen its period as illustrated in Fig. 6.

3b. Linearized Treatment of a General Field

If the magnetic field is perpendicular to the plane of the orbit but not axially symmetric, Hamilton's canonical equations no longer lead to a simple solution, for two reasons. First, the canonical momentum p_θ is no longer a constant of the motion; second, the vector potential of such a field will have a non-vanishing radial component, so that the canonical momentum p_r is no longer simply $m\dot{r}$ but involves A_r as well. The canonical equations reduce to the ordinary equations of motion; therefore we shall treat this motion by use of our differential equation for ρ as a function of θ.

Expanding this equation in powers of ρ and ρ', and inserting the power series expression for β_z developed earlier, we have

$$\rho'' + \omega^2\rho = \Delta_v + A_1\rho^2 + A_2\rho'^2 + B_1\rho^3 + B_2\rho\rho'^2 + ----$$
$$- (1-\Delta_v)(1 + 2\rho + \rho^2 + \tfrac{3}{2}\rho'^2 + \tfrac{3}{8}\rho'^4 + \cdots)\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} A_{jk}\,\rho^k \cos(j\theta + \alpha_{jk}),$$

where $\omega^2 = 1 - n - (2-n)\Delta_v$; A_1, A_2, B_1, B_2, etc., were defined in the preceding paragraphs. We have mentioned earlier that in the physical situations of interest, the quantities ρ, Δ_v, and $A_{jk} (j \neq 0)$ will be considerably less than unity. We shall first see what can be learned by linearizing the equation. Keeping only first order terms in these small quantities, we have

$$\rho'' + \omega^2\rho = \Delta_v - \sum_{j=1}^{\infty} A_{j0} \cos(j\theta + \alpha_{j0}),$$

whose solution is

$$\rho = \frac{\Delta_v}{\omega^2} + \sum_{j=1}^{\infty} \frac{A_{j0}\cos(j\theta + \alpha_{j0})}{j^2 - \omega^2} + a\cos(\omega\theta + \delta),$$

where a and δ are integration constants.

i. Forced Oscillations and the Distorted Equilibrium Orbit

This result is of considerable importance, since it shows that the first-order effect of the inhomogeneities is to superimpose forced oscillations on the free oscillations. Bohm and Foldy[18] have obtained a similar result; they point out that it can be used to obtain an estimate of the maximum allowable inhomogeneities if the orbits are not to be forced into the doughnut walls, and further note that no simple resonances can occur, since $\omega^2 < 1$. Goward[25] has also obtained and discussed this result and its bearing on the injection problem. However, he did not seem to recognize that the simplest interpretation is to regard the forced oscillations as defining a new equilibrium orbit, distorted from circular shape and fixed with respect to the pole pieces, about which the free oscillations take place. When viewed from this standpoint, it is clear that the existence of the forced oscillations contributes nothing essentially new to the injection situation in the first-order equations since the problem of hitting the gun is the same whether the free oscillations take place about a circular orbit or a distorted one. However, if the A_{j0} coefficients are under sufficiently accurate control, they afford a means of dodging obstacles other than the gun, or otherwise modifying the position of the equilibrium orbit.

The effect of the resonance denominators is to decrease the influence of the forcing terms of higher frequency, as was stated earlier. Even at this stage, it seems plausible that a major part of the empirical adjustment of the magnetic field required to make a machine function may consist of altering the coefficients A_{j0} sufficiently so that the new distorted equilibrium orbit lies reasonably near the center of the doughnut's aperture at all azimuths. Reference was made earlier to data showing that the inhomogeneities may be as much as a third of the total field at injection, corresponding to values of the A_{j0} up to 0.3. We may take as typical values $n \approx 3/4$, $\omega^2 \approx \frac{1}{2}$, and assume that the aperture extends radially for distances of the order $|\rho| = 0.1$. In this case, even if the phase angles of the inhomogeneities are most favorably disposed, we must have $A_{10} < 0.025$, $A_{20} < 0.17$, and $A_{30} < 0.42$, in order that the equilibrium orbit will be confined to the center half of the doughnut's radial aperture. Furthermore, if the phases are such that this orbit passes the gun at a considerable distance, the free oscillations will have an excessive amplitude and will be apt to cause collisions with the wall if the equilibrium orbit approaches it at some other azimuth. This is borne out by the empirical fact that the output of a machine may be improved by adjusting the azimuth of the injecting gun with respect to the pole pieces.

ii. Mathieu-Hill Equations; Resonance and Divergence of Oscillations

If we retain also the terms of next lowest order and bring terms linear in ρ to the left, we have

$$\rho'' + \left[\omega^2 + \sum_{j=1}^{\infty}\left\{2A_{j0}\cos(j\theta+\alpha_{j0}) + A_{j1}\cos(j\theta+\alpha_{j1})\right\}\right]\rho = \Delta_\nu - \sum_{j=1}^{\infty} A_{j0}\cos(j\theta+\alpha_{j0}) .$$

This has the form of a Hill equation*, with an inhomogeneous term. By introducing the displacement x from the distorted equilibrium orbit, defined by

$$x \equiv \rho - \frac{\Delta_\nu}{\omega^2} - \sum_{j=1}^{\infty} \frac{A_{j0}\cos(j\theta+\alpha_{j0})}{j^2-\omega^2} ,$$

we obtain $x'' + \left[\omega^2 + \sum_{j=1}^{\infty}\left\{2A_{j0}\cos(j\theta+\alpha_{j0}) + A_{j1}\cos(j\theta+\alpha_{j1})\right\}\right]x =$

$$-\left[\frac{\Delta_\nu}{\omega^2} + \sum_{j=1}^{\infty}\frac{A_{j0}\cos(j\theta+\alpha_{j0})}{j^2-\omega^2}\right]\sum_{j=1}^{\infty}\left\{2A_{j0}\cos(j\theta+\alpha_{j0}) + A_{j1}\cos(j\theta+\alpha_{j1})\right\} ,$$

which is still an inhomogeneous Hill equation; however, the inhomogeneous term is now of second order in the A_{j0} and Δ_ν, rather than first order as above. If we neglect this term and consider the effect of a single frequency component only by setting $A_{j'k} = 0$ for $j' \neq j$, we obtain

$$x'' + \left[\omega^2 + \left\{2A_{j0}\cos(j\theta+\alpha_{j0}) + A_{j1}\cos(j\theta+\alpha_{j1})\right\}\right]x \approx 0 .$$

This is a Mathieu equation[26], whose form may be simplified by introducing new parameters a and q, and a new variable s, through the

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* See, for example, Reference 26, Chapter VI.

definitions

$$a \equiv 4\omega^2/j^2 , \qquad z \equiv j(\theta - \theta_0)/2 ,$$

$$2A_{j0} \cos \alpha_{j0} + A_{j1} \cos \alpha_{j1} \equiv -\tfrac{1}{2} j^2 q \cos j\theta_0 ,$$

$$2A_{j0} \sin \alpha_{j0} + A_{j1} \sin \alpha_{j1} \equiv \tfrac{1}{2} j^2 q \sin j\theta_0 .$$

Inserting these, we obtain

$$\frac{d^2x}{dz^2} + (a - 2q \cos 2z)\, x = 0 ,$$

which McLachnan[26] takes as the canonical form of the Mathieu equation.

Since the Mathieu equation is a linear second order differential equation, its general solution is an arbitrary linear combination of two linearly independent solutions. It is well known that it is possible to choose these two fundamental solutions of a Mathieu equation in such a way that each solution consists of a product of two factors, one of which is an exponential function of z and the other a periodic function of z; the periodic factors have the same period as the coefficient, while the exponential factors of the two solutions differ only in the sign of the exponent. For certain ranges of the parameters a and q, the exponents are imaginary and the solutions are neither positively nor negatively damped. For other ranges, the exponents are complex, so that one solution contains an exponentially decreasing factor and the other a rising factor tending exponentially to infinity; the solution of the equation is then unstable or divergent, except for very special initial conditions. The regions of stability and instability in the a-q plane have been computed, plotted, and discussed by several writers[24,26,27].

Since $0 < \omega^2 < 1$, the range of a in our problem is $0 < a < 4$; also, we may assume $|q| < \frac{1}{2}$ for reasonable values of the A_{jk}. An examination of the stability plot* shows that the solution will be stable unless a is near unity, which can occur only for $j = 1$ and $\omega^2 \approx \frac{1}{4}$, corresponding to $n \approx 3/4$. For these values the oscillations are unstable if a lies within the limits given by**

$$a = 1 \pm q - \frac{1}{8}q^2 \mp \frac{1}{64}q^3 - \frac{1}{1536}q^4 \pm --- .$$

If q is small enough so that we may retain only the linear term, the oscillations are unstable if $|3/4 - n| < |q/4|$. We may interpret this geometrically, using the definition of q. If we regard A_{jk} and α_{jk} as specifying the length and azimuth angle of a vector $\vec{A}_{jk}$ in plane polar coordinates, then $|q/4| = \frac{1}{2}|2\vec{A}_{10} + \vec{A}_{11}|$, and the motion is unstable in this approximation only if n differs from 3/4 by less than this amount. The instability occurs because the ratio of rotation frequency to oscillation frequency is nearly two to one if $\omega \approx \frac{1}{2}$, giving rise to the phenomenon of subharmonic resonance, which we shall discuss in detail farther on.

It may be mentioned that the behavior of electrons in the invented field of Davis and Langmuir[11], cited earlier, may be understood on the basis of the Mathieu equation. Rapidly decreasing amplitudes occur when the injection initial conditions are such that only the exponentially damped solution is involved. If the initial conditions are slightly altered, a component of the rising solution will be introduced; it will eventually dominate the motion. If the parameters a and q are such that there is no exponentially rising factor in one solution, there will be no falling term in the other one and hence no

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* Reference 26, Fig. 8(A), p. 40.
** Ibid., p. 16, Eqs, (2) and (3).

possibility of aiding injection in this way.

The instability which may occur here represents a special case of the general conclusions of Dennison and Berlin[12]. They pointed out that when commensurabilities occur between any two of the three characteristic frequencies (of rotation, radial oscillation, and vertical oscillation), secular terms will appear in a successive approximations treatment of the equations of motion which will result in a continual increase in the amplitude of oscillation. This type of behavior has long been known in celestial mechanics, and has been studied extensively in recent years by use of the new methods of non-linear mechanics. Minorsky[24] classifies the present example as external subharmonic resonance of order one-half; the period of the externally applied forcing oscillation is half that of the free radial oscillation, since the electron makes two rotations in the time required for one free oscillation if $\omega = \frac{1}{2}$. Dennison and Berlin have also noted that, for increasingly higher-order subharmonic resonances (corresponding to larger commensurability integers r and s, where $\omega = (1-n)^{\frac{1}{2}} \approx r/s$) the rate of build-up of the amplitude of oscillation is progressively slower.

E. Courant[13] has discussed the $n = 3/4$ resonance at considerable length, by different although equivalent methods, based on the linearized equations. His inclusion of the effect of the synchrotron oscillations due to the oscillating electric accelerating field complicates the mathematics but does not alter his conclusion, which is that machines with $n \approx 3/4$ will not operate successfully unless $A_{10} < 0.001$. Dennison and Berlin also conclude that this and other critical values of n should be avoided in order to prevent a divergence of the oscillations due to the secular terms. These conclusions conflict with the experimental fact that several machines have been designed to have n exactly 3/4; these machines are operating satisfactorily, and the problems of getting them into operation have not seemed to be qualitatively different from those with machines having noncritical n values. In fact, Kerst has constructed a betatron containing auxilliary coils by means of which he is able to vary n continuously over almost the entire stable range[28], and has found no peculiarities of behavior in the vicinity of the critical values*. We shall offer an explanation of these observed facts, based on the investigations described below.

3c. Investigation of the General Non-Linear Equation

Thus far we have only investigated linearized equations of the motion in an axially varying field. It is a general characteristic of such equations that if a solution is initially divergent its amplitude will continue to grow beyond any bound. In a physical problem, however, the non-linear terms neglected in linearizing the equations begin to play appreciable roles after the amplitude has increased sufficiently. These terms may act to modify the character of the motion considerably, and, in particular, may perhaps prevent the amplitude from rising beyond a certain point. We have already seen that the only subharmonic resonance appearing in the linear theory is that of order one-half; higher-order fractional (subharmonic) resonances are nonlinear phenomena in our problem.

To make further progress, then, we must consider a nonlinear differential equation with periodic coefficients. In discussing the two-dimensional motion in an axially symmetric field we were able to

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* Private communication from Dr. R.V. Langmuir.

solve a non-linear equation with constant coefficients by successive approximations, and to feel confident of the convergence of our procedure because of the establishment of the stability and conservative nature of the motion by use of Hamilton's canonical equations. Even in the absence of such assurance, an inspection of the result would make its convergence seem plausible if none of the high-order coefficients were exceptionally large (although Poincare* has shown by an example that such approximations need not always converge). Also, we were able to use the results of many earlier workers pertaining to the solutions of linear equations with periodic coefficients (Mathieu-Hill type) which, while complicated, are well-understood; the successive approximation methods used in discussing them have been proved convergent. However, there exists no general theory of non-linear equations with periodic coefficients**, and the convergence, time of validity, or size of error involved in the existing methods of approximate solution of such equations are impossible to establish rigorously and even difficult to determine approximately in special cases. The best we can do with existing analytic techniques is to ascertain some of the typical features of such systems and to calculate their magnitudes approximately; the details of the motion are usually too complicated to follow accurately.

i. <u>Description of the Kryloff-Bogoliuboff Method</u>

Several techniques for ascertaining the approximate effects of the non-linear periodic terms have been investigated. The most satisfactory one seems to be that developed by the Russian mathematicians Kryloff and Bogoliuboff. This method is extensively discussed by Minorsky

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* See Reference 24, p. 210, Ch. XIX.
** Loc. cit., Ch. XIX.

in his text[24]. It may be applied to differential equations of the type

$$\rho'' + \omega_r^2 \rho = f(\rho, \rho', \theta)$$

where f is periodic in the independent variable θ and small compared to the terms on the left. In first approximation, neglecting f, the solution is

$$\rho = a \cos(\omega_r \theta + \delta) ;$$

in this approximation we also have

$$\rho' = -\omega_r a \sin(\omega_r \theta + \delta) ,$$

where a and δ are arbitrary constants dependent on the intitial conditions. We take these as the generating solutions from which a solution of the complete equation is to be obtained by allowing a and δ to vary appropriately with θ. Replacing ω_r by an arbitrary ω for greater generality, and carrying through (in the appendix) the standard operations of the method of variation of parameters, we obtain the following two first order differential equations for a and δ which are rigorously equivalent to the single original second order equation, in a way similar to that in which the Hamiltonian equations may replace the Lagrangian equations in analytical mechanics:

$$a'/a = \frac{\omega_r^2 - \omega^2}{\omega} \sin\phi \cos\phi - \frac{1}{a\omega} f(a\cos\phi, -\omega a \sin\phi, \theta) \sin\phi$$

$$\delta' = \frac{\omega_r^2 - \omega^2}{\omega} \cos^2\phi - \frac{1}{a\omega} f(a\cos\phi, -\omega a \sin\phi, \theta) \cos\phi$$

where $\phi \equiv \omega\theta + \delta$.

It is impossible to solve these equations rigorously in general;

however, we note that, if ω is chosen very near to ω_r, the rates of change of a and δ with respect to θ will be very small, and, because of the trigonometric dependence on θ, will be rapidly oscillating. The approximation procedure consists of assuming that a and δ will be nearly constant over a period, replacing them by their average values $\bar{a}$ and $\bar{\delta}$ on the right sides, and averaging these terms over a complete period in θ, to obtain $\bar{a}'$ and $\bar{\delta}'$, the rates of change of the average values. These rates of change will be functions of $\bar{a}$, $\bar{\delta}$, and constant parameters in f, and will be periodic in $\bar{\delta}$ since $\bar{\delta}$ enters only through trigonometric functions of ϕ. We shall postpone a detailed treatment of the effects of this averaging approximation, first discussing some of the principal consequences of applying this method to our equation:

$$\rho'' + \omega_r^2 \rho = \Delta_v + A_1 \rho^2 + A_2 \rho'^2 + B_1 \rho^3 + B_2 \rho \rho'^2 + \cdots$$
$$- (1-\Delta_v)\left(1 + 2\rho + \rho^2 + \frac{3}{2}\rho'^2 + \frac{3}{8}\rho'^4 + \cdots\right) \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} A_{jk} \rho^k \cos(j\theta + \alpha_{jk}).$$

We see that the terms in the first row, which are independent of θ, will not contribute to $\bar{a}'/\bar{a}$, since, after substituting $a \cos \phi$ for ρ, $-\omega a \sin \phi$ for ρ', and multiplying by $\sin \phi$, they will be of the form $\cos^{\mu} \phi \sin^{\nu} \phi$ with ν always odd (because ρ' always occurs raised to an even power). Such terms have an average value of zero. However, those containing ρ to an odd power will contribute to $\bar{\delta}'$, because the substitutions, and multiplication by $\cos \phi$, bring them to the form $\cos^{\mu} \phi \sin^{\nu} \phi$, with both μ and ν even. Since these terms are everywhere positive, their averages cannot vanish. Such a term does not involve $\bar{\delta}$ but does contain $(\mu + \nu - 2)$ powers of $\bar{a}$ (because every term is multiplied by $-1/(\bar{a}\omega)$). The lowest value of $(\mu + \nu - 2)$ which occurs here is 2. Therefore, if we consider only these first-row terms, corresponding to an axially symmetric field, we obtain a shift in frequency proportional to the square of the amplitude, but no change in the amplitude, in agreement with our earlier findings. However, we do not obtain any information about the higher harmonics of the oscillation by this technique.

The second-row terms depending explicitly on θ, will lead to expressions of the form $\cos^{\mu}\phi \sin^{\nu}\phi \cos\left[\frac{j}{\omega}(\phi-\delta)+\alpha\right]$, where θ has been replaced by $\frac{1}{\omega}(\phi-\delta)$; μ , ν , and j are integers. If ω is not a rational number, the periods of the different trigonometric factors are incommensurable, and the long-time average of the term will be zero. However, we have made provision for an arbitrary choice of ω, restricted only in that it must be quite near to ω_r, the natural frequency if f = 0. There will always be many rational numbers fulfilling this condition, and we will now indicate the considerations determining our choice.

The typical term above will be multiplied by $(\mu+\nu-2)$ powers of $\bar{a}$ and by an A_{jk}. If we assume that $\omega = r/s$, a rational proper fraction in lowest terms, then we may decompose the term into single trigonometric terms of the form

$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\mu' \pm \nu' \pm \frac{js}{r}\right)\phi,$$

where

$$\mu' \leq \mu , \quad \nu' \leq \nu ,$$

and where μ' and ν' have the same parity as μ and ν, respectively. The average values of such terms are always zero except when the argument of the cosine is zero; this requires that (js/r) be an integer, and hence that j be a multiple of r. Now if we assume j = mr, where m is a positive integer, we have a non-vanishing average only if $\mu' \pm \nu' \pm ms = 0$.

For given m and s, the terms of lowest order in $\bar{a}$ for which this can be satisfied are those in which $\mu' = \mu$, $\nu' = \nu$, and $\mu + \nu = ms$. Since m is at least unity, the lowest power of $\bar{a}$ in the expressions for $\bar{a}'/\bar{a}$ and $\bar{\delta}'$ arising from such terms will be the (s-2) power. We therefore conclude that we should choose for ω a rational number near ω_r which is the ratio of small integers r and s. If we try to approximate a given ω_r more and more closely by rational fractions, both r and s increase together; the contributions become very small due to the high power of the small quantity $\bar{a}$, and are due only to high-frequency components of the field (with respect to θ) because of the large value of r. If we do not take ω near ω_r, the first terms in a' and δ' will be large, and the original assumption on which the averaging process was based will become invalid. Although we have not derived a unique criterion for selecting r and s, we will postpone a discussion of nearby resonances and merely state here that no two resonances of practical interest overlap, so that for any given ω_r it is always clear which resonance, if any, is of importance.

We shall now show that the lowest-order terms in $\bar{a}'$ and $\bar{\delta}'$ arising from terms in the second line of our f always tend to produce an ultimately divergent behavior similar to that in the unstable region of the Mathieu equation. These lowest-order terms are proportional to the averages of $\cos^{\mu}\phi \sin^{\nu}\phi \cos(s\phi - s\bar{\delta} + \alpha)$ and $\cos^{\mu+1}\phi \sin^{\nu-1}\phi \cos(s\phi - s\bar{\delta} + \alpha)$, respectively, where $\mu + \nu = s$, and ν is always odd. The average values of these are $(-1)^{\frac{\nu-1}{2}} 2^{-s} \sin(s\bar{\delta} - \alpha)$ and $(-1)^{\frac{\nu-1}{2}} 2^{-s} \cos(s\bar{\delta} - \alpha)$, respectively. If we neglect all terms but these, we see that $\bar{\delta}' = 0$ and $\frac{\partial \bar{\delta}'}{\partial \bar{\delta}} < 0$ when $\bar{\delta} = \frac{1}{s}\left(\alpha \pm \frac{\pi}{2}\right)$ for $\frac{\nu-1}{2}$ $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$, indicating that this is a stable equilibrium value of $\bar{\delta}$; but at this value, $\bar{a}'$ is positive, and the oscillations will diverge. If we start with $\bar{\delta} = \frac{1}{s}(\alpha \mp \frac{\pi}{2})$, we will have a decreasing $\bar{a}$, but this value of $\bar{\delta}$ is unstable, and the slightest shift will start it moving toward the range in which $\bar{a}$ increases. This behavior is just that of the unstable Mathieu solutions described above.

It is important to note that the other terms in $\bar{a}'$ and $\bar{\delta}'$, which are independent of $\bar{\delta}$, may alter this situation, and that higher order terms than the ones here considered must be taken into account to learn what, if anything, limits the motion which is unstable on the basis of these terms alone. It is difficult to visualize the effects of these various terms from the mathematical formulas, so we now describe a geometrical representation of the solutions which enables one to obtain an overall view of all possible motions in any particular case.

ii. <u>Phase Diagrams and their Properties</u>

Since the averaging approximation yields expressions for $\bar{a}'$ and $\bar{\delta}'$ (as functions of $\bar{a}$, $\bar{\delta}$, and the constant parameters) which are periodic in $\bar{\delta}$, it is possible to represent the approximate behavior of the system by means of trajectories drawn on a cylindrical surface constituting a two-dimensional phase space. The representative point of the system moves along some trajectory in this space, which constitutes a plot of $\bar{a}$ (measured parallel to the axis of the cylinder) against $\bar{\delta}$ (measured around the cylinder); the components of the point's velocity are $\bar{a}'$ and $\bar{\delta}'$. Such a representation is approximate in that the high-frequency fluctuations of a' and δ', having been averaged out, do not appear*. By

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* The exact solution could be represented in the same way in an $a - \delta$ surface, of course; however, the representative point would not have a unique velocity at a given point of the space because of the explicit dependence of a' and δ' on θ.

inspecting the family of all possible trajectories one can obtain a picture of the various types of behavior which may result from all possible sets of initial conditions.

A number of general properties of such $\bar{\alpha} - \bar{\delta}$ phase plots may be easily deduced. No two trajectories may cross except at a point at which the velocity of the representative point is zero, since the slope of a trajectory is uniquely determined everywhere except at these singular points. Closed trajectories, representing periodic motion, may be separated into two classes, those which do not wind completely around the cylinder (periodic trajectories of the first kind) and those which do, passing through all values of $\bar{\delta}$ from 0 to 2π (periodic trajectories of the second kind). Those of the first kind correspond to motions in which the free oscillation is locked in synchronism with the forcing frequency, while those of the second kind represent free-running oscillations which are out of synchronism. Non-closed trajectories of various sorts are also possible. We define critical trajectories as those separating regions in which the trajectories exhibit qualitatively different behavior.

The topology of trajectories in the phase plane depends strongly on the location and nature of the singular points, the simplest of which may be classified as vortex points, saddle points, stable and unstable focal points, and stable and unstable nodal points. The behavior of trajectories near these singular points is extensively discussed by Minorsky[24]; we shall not repeat the discussion here. It has been helpful in this work to regard the trajectories as representing streamlines of two-dimensional fluid flow. From this point of view, stable focal and nodal points become sinks with and without vorticity, respectively, while unstable ones become sources. The simplest saddle points behave like points of stagnation, while others may represent the confluence of a simple saddle point and a nodal point. A vortex point has vorticity but no source or sink.

Strictly speaking, a and δ are not the proper coordinates for a phase space, since they are not canonically conjugate; if the angle variable δ is used, its conjugate momentum is the action A, which is proportional to a^2. However, the use of a is preferable here because of the symmetry of the resulting formulas for a' and δ'; in obtaining a picture of the trajectories, it does not matter what power of a is plotted linearly. Nevertheless, we shall derive an interesting result by introducing $A = a^2$ (the value of the proportionality constant is immaterial in what follows). The components of the velocity vector of a representative point moving in an $A - \delta$ surface are A' and δ'. The source strength per unit area for the flow of trajectories is then the divergence $\frac{\partial A'}{\partial A} + \frac{\partial \delta'}{\partial \delta}$ of this vector. It is proved in the appendix that this divergence is rigorously equal to $\frac{\partial f}{\partial \rho'}$. This shows at once that, if the function f is independent of ρ', the flow is divergenceless and contains neither sources or sinks*. This is equivalent to saying that the flow is that of an incompressible fluid. No focal or nodal points may occur, and all trajectories will be either closed and periodic, or asymptotic (coming from and going to infinity), as would be expected in conservative systems; a saddle point represents a point of stagnation. If f is independent of θ and contains only even powers of ρ' (as in two-dimensional motion in an axially symmetric field) then every term of $\frac{\partial f}{\partial \rho'}$ will take the form $\cos^{\mu} \phi \sin^{\nu} \phi$

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* The velocity, being assumed small, cannot become infinite as at a point source with finite flow.

after substituting, where μ and ν are integers, with ν always odd; the average value of such a term is zero. Since $\frac{\partial \bar{A}'}{\partial \bar{A}} + \frac{\partial \bar{\delta}'}{\partial \bar{\delta}} \equiv \overline{\frac{\partial A'}{\partial A} + \frac{\partial \delta'}{\partial \delta}} \equiv \overline{\frac{\partial f}{\partial \rho'}}$, the divergence in the $\bar{A} - \bar{\delta}$ plane will vanish and the above conclusions still apply, in agreement with our earlier treatment of this case. If f contains trigonometric terms in θ, and if the radial oscillation frequency ω is commensurable with the rotation frequency, it may happen that $\overline{\frac{\partial f}{\partial \rho'}}$ does not vanish everywhere; here the above conclusions fail to apply, but we will show that terms contributing a non-vanishing divergence are of higher order than that to which our approximation procedure is valid. The significance of divergenceless flow will be discussed further on, where we will apply this concept to the problem of dodging the gun.

iii. Discussion and Interpretation of Phase Diagrams

The results obtained so far may be summarised by the equations

$$\bar{a}'/\bar{a} = K_1 \bar{a}^{s-2} \sin(s\bar{\delta} - \alpha) \quad + \text{ higher order terms in } \bar{a} \text{ from 2nd line of } f,$$

$$\bar{\delta}' = K_1 \bar{a}^{s-2} \cos(s\bar{\delta} - \alpha) \quad + \text{ higher order terms in } \bar{a} \text{ from 2nd line of } f,$$

$$+ K_0 + K_2 \bar{a}^2 \quad + \text{ higher order terms in } \bar{a} \text{ from 1st line of } f.$$

Here K_1 is a certain linear combination of the $A_{r,k}$ in which $k = 0, 1, \cdots (s-1)$; K_0 is proportional to $\omega_r^2 - r^2/s^2$, K_2 is a certain linear combination of B_1 and B_2, and α is a determined constant. Many of the essential features of the situation may be learned from a study of the phase plots of these equations, using only the terms given explicitly above.

We now wish to indicate the desirability of applying this method to a somewhat different equation. The accuracy of the method depends on the smallness of the function f with respect to ρ, but the f given above contains terms of first order in the A_{jo} and Δ_v which are independent of ρ and ρ', so that the method becomes invalid for oscillations of very small amplitude. By introducing as a new variable the radial displacement from the distorted equilibrium orbit (as was done earlier to obtain a smaller inhomogeneous term in the Mathieu-Hill equation), we may eliminate from f those first-order forcing terms which are independent of both ρ and ρ'. The resulting equation is similar to that discussed above, but is more complex, in that on the right side ρ is replaced by a more involved expression, consisting of the new variable plus a Fourier series in θ, and similarly for ρ'. However, it can be expanded and put into the same form as the simpler equation. Similar equations for $\bar{a}'$ and $\bar{\delta}'$ will result, differing only in that the K's and α will be more complicated functions of the parameters. The evaluation of the K's is discussed in the appendix.

In treating these equations, we set $s\bar{\delta} - \alpha \equiv x$ for brevity. In this discussion, we shall neglect the terms not written explicitly. If $K_1 = K_2 = 0$, the trajectories are straight lines parallel to the x or $\bar{\delta}$ axis, winding around the cylinder with constant velocity K_o independent of the amplitude; the system executes isochronous simple harmonic motion with angular frequency $(r/s) + K_o$. If $K_1 = 0$, but $K_2 \neq 0$, the system is still simple harmonic but no longer isochronous; the velocity along a straight-line trajectory depends on the amplitude of oscillation, and the angular frequency becomes $(r/s) + K_o + K_2 \bar{a}^2$. These cases are simple and easily understood.

If $K_1 \neq 0$, we have a more complicated situation. The differential equation of the trajectory becomes

$$\frac{\bar{a}}{s}\frac{dx}{d\bar{a}} = \frac{1}{\sin x}\cdot\left[(K_o/K_1)\bar{a}^{2-s} + (K_2/K_1)\bar{a}^{4-s} + \cos x\right] ;$$

the general solution of this differential equation is

$$\cos x = -\frac{s}{2}\frac{K_0}{K_1}\bar{a}^{2-s} - \frac{s}{4}\frac{K_2}{K_1}\bar{a}^{4-s} + \frac{C}{\bar{a}^s} \quad ,$$

where C is the arbitrary constant of integration. If $K_0 = K_2 = 0$, we obtain $\bar{a}^s = \bar{a}^s_{min} \cdot \sec x$, which leads to the trajectories of Fig. 7, describing the eventually unstable motion characteristic of subharmonic resonance*. It is difficult to determine the modifications introduced by the terms in K_0 and K_1 directly from the integrated equation of the trajectory, but a qualitative picture can be gained in the following manner. We first determine the approximate positions of the lines along which $\bar{a}' = 0$ and those along which $\bar{\delta}' = 0$; their intersections are the singular points. The signs of $\bar{a}'$ and $\bar{\delta}'$ are next determined in each of the regions into which these lines divide the surface, and the trajectories may then be sketched in, with the aid of the method of isoclines if necessary. The nature of each singularity is usually evident from this procedure; it can be easily determined by use of the theorem of Liapounoff** in most doubtful cases. We can find the magnitude of the velocity at various points and insert arrows whose size is proportional to the velocity, as an aid in visualizing the motion of the representative point along a trajectory.

If $s = 2$, corresponding to $n \approx 3/4$, and if $K_2 = 0$, we find that as $|K_0|$ increases from zero, the diagram of Fig. 7 shifts to the form of Fig. 8 until $|K_0| = |K_1|$, beyond

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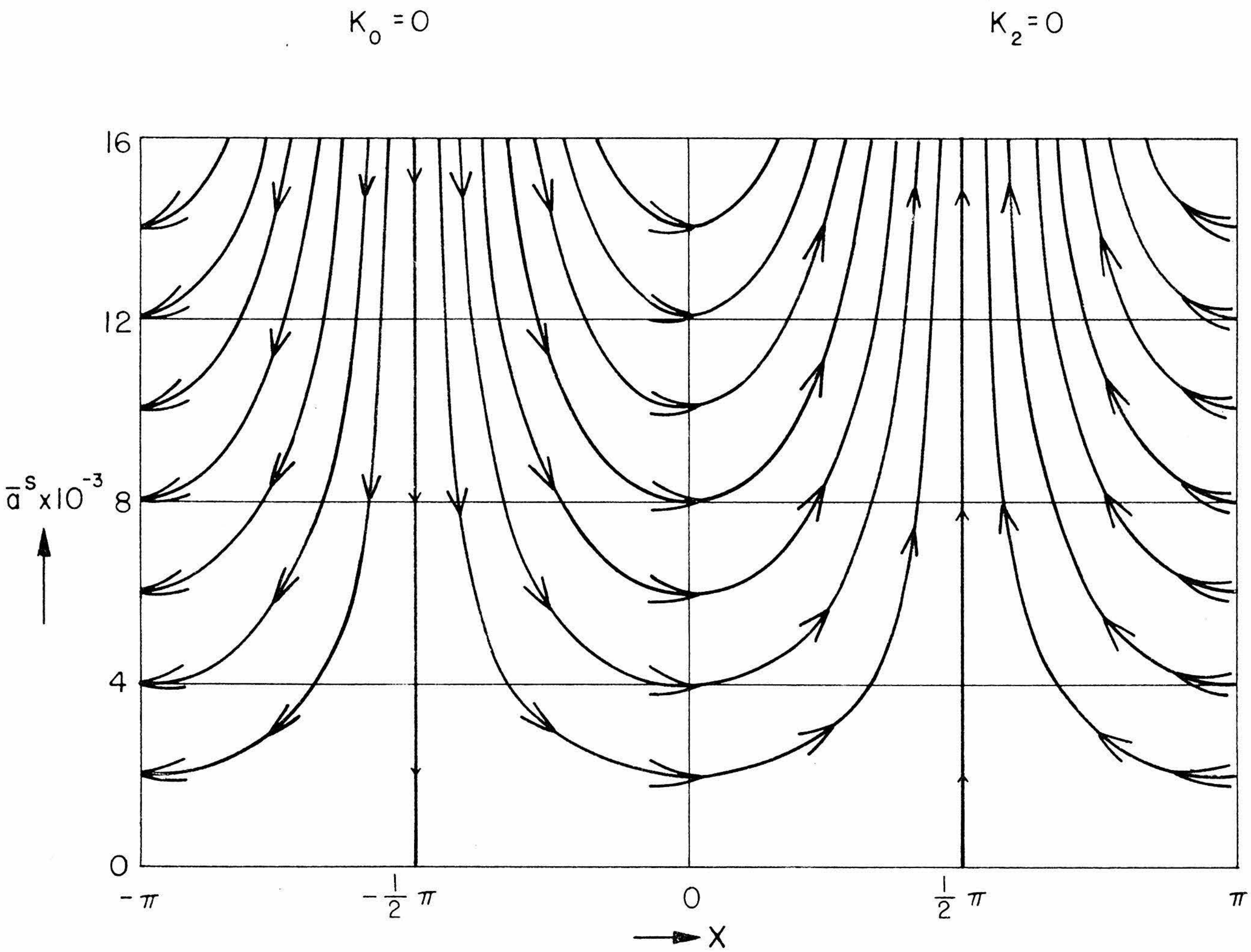
* The higher-order terms in $\bar{a}'$ may ultimately limit this motion even if $K_0 = K_2 = 0$, although this occurs at too large an amplitude to be physically useful in all cases which have been investigated; the validity of higher-order effects in this approximation is doubtful in any case.

** Reference 24, p. 51.

FIGURE 7



$$\frac{\bar{a}'}{\bar{a}} = K_1 \bar{a}^{s-2} \sin X \; ; \quad X' = sK_1 \bar{a}^{s-2} \cos X ; \quad \bar{a}^s = \bar{a}^s_{min} \sec X$$

Arrow size $\propto$ velocity for $s = 2$

FIGURE 8

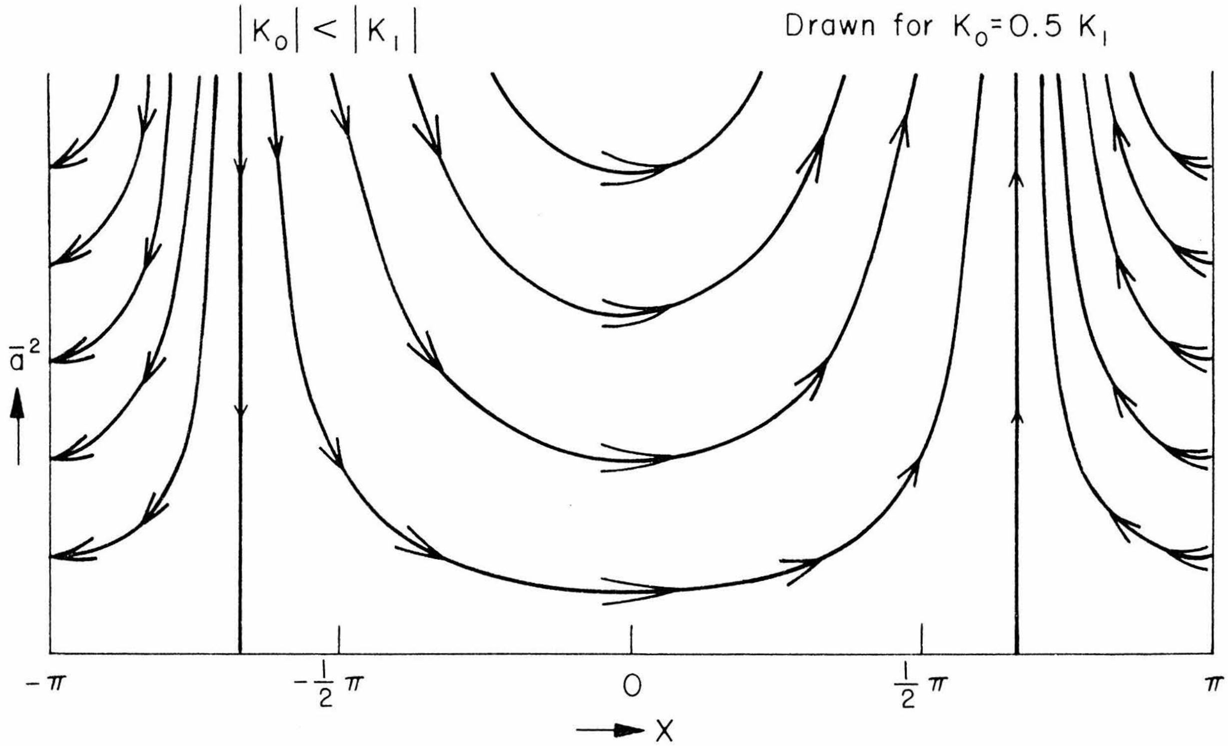
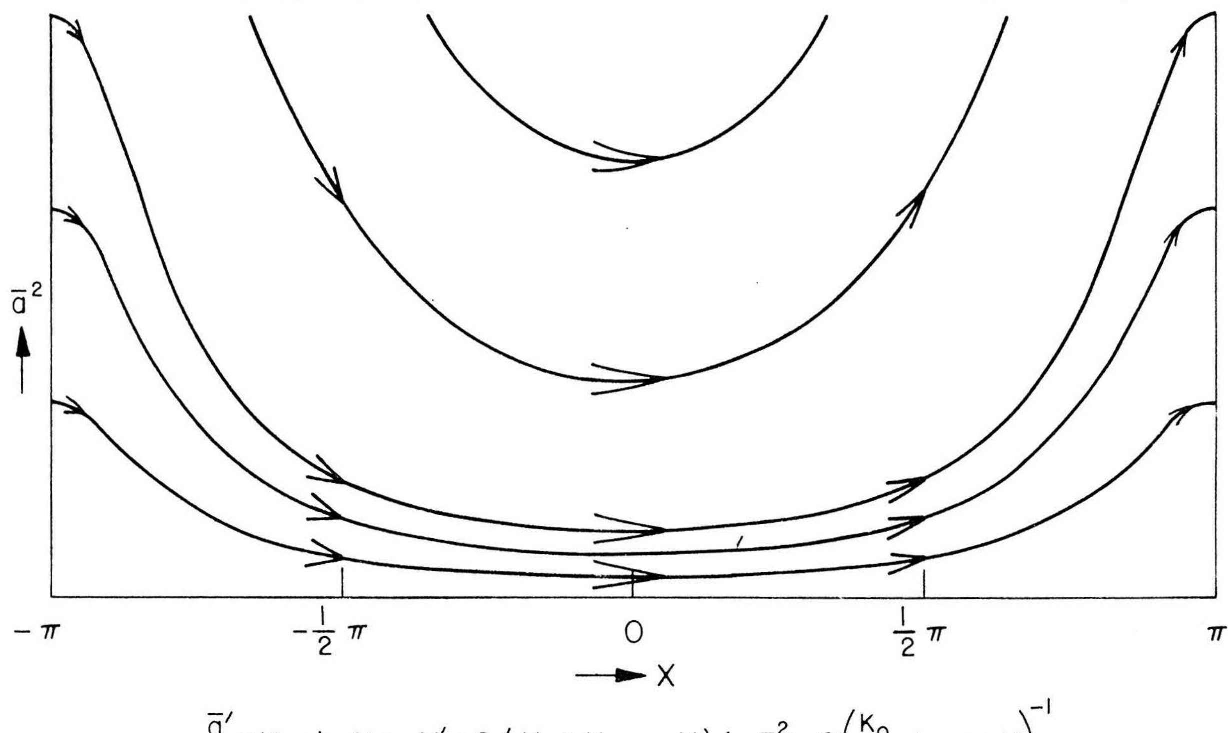


FIGURE 9



$$\frac{\bar{a}'}{\bar{a}} = K_1 \sin X\,;\quad X' = 2\,(K_0 + K_1 \cos X)\,;\quad \bar{a}^2 = C\left(\frac{K_0}{K_1} + \cos X\right)^{-1}$$

which point the asymptotic trajectories disappear and we obtain the periodic trajectories of the second kind shown in Fig. 9. We obtain from this the width of the frequency range in which the free oscillation is locked in synchronism with the forcing oscillation; for frequencies outside this range the oscillation has slipped out of synchronism, and the divergent resonant effect disappears. If $K_0 = 0$ but $K_2 \neq 0$, we may solve for $\bar{a}^2$, obtaining

$$\bar{a}^2 = \frac{K_1}{K_2}\left[-\cos X \pm \sqrt{\cos^2 X + C'}\right],$$

where C' is an arbitrary constant; this is plotted in Fig. 10. We see that, while $\bar{a}$ may become very large if K_2 is small, it will never become infinite; the existence of the frequency shift is proportional to $\bar{a}^2$ is sufficient to destroy the resonance eventually, even if the system is precisely in the center of the resonant range for small amplitudes. Of course, if K_2 is small, this may not occur until $\bar{a}$ is so large as to invalidate the approximations used. In the general case we may have a variety of types of behavior, depending on the relative magnitudes of the K's; all of these are similar to those already discussed except when K_0 and K_2 are of opposite sign and $|K_0| > 2|K_1|$, in which case we may get trajectories like those of Fig. 11.

If s = 3, and $K_2 = 0$, we find that $\bar{\delta}'$ is of invariable sign unless $|a| > |K_0/K_1|$; this means that the resonant effect will only occur for oscillations of amplitude greater than this critical amplitude, which is equivalent to saying that the width of the resonance range is proportional to the amplitude, instead of being constant as it is for s = 2. This situation is shown by the phase diagram in Fig. 12. We can now see why the only subharmonic resonance appearing in the linear theory was that

FIGURE 10

$K_0 = 0$ Drawn for $K_1 = -5 \times 10^{-3} K_2$

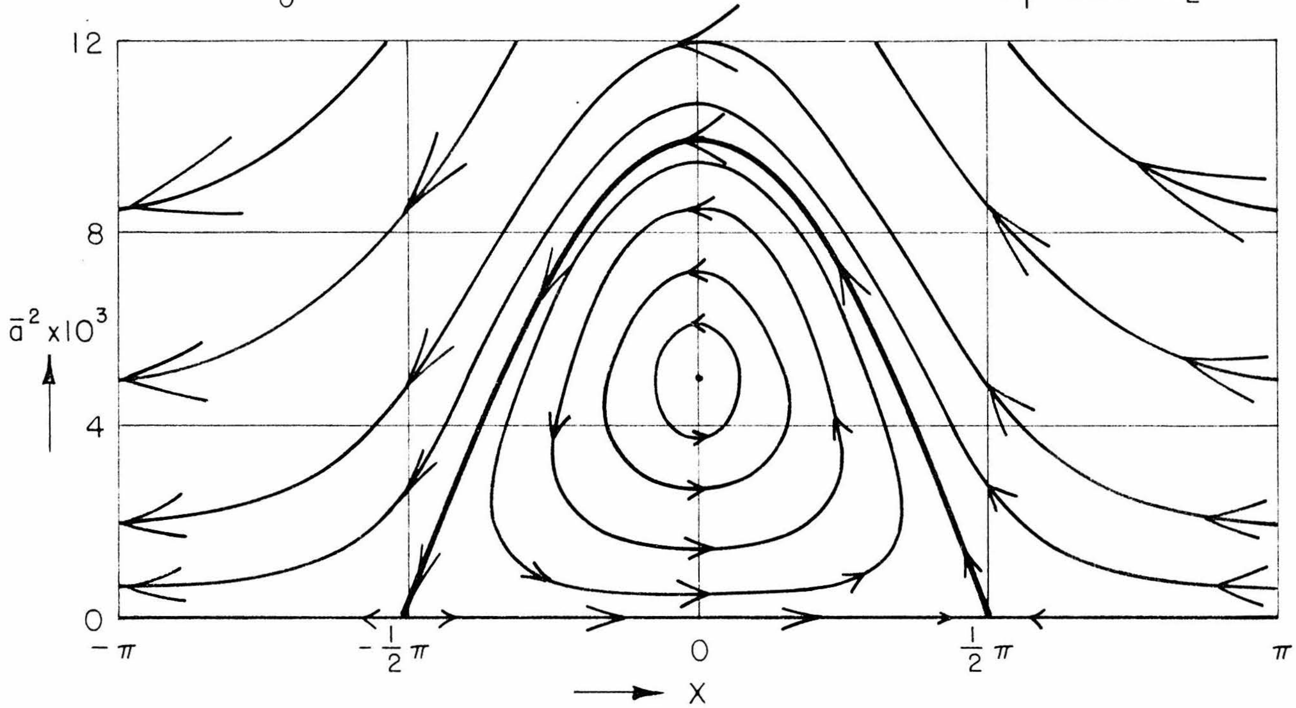
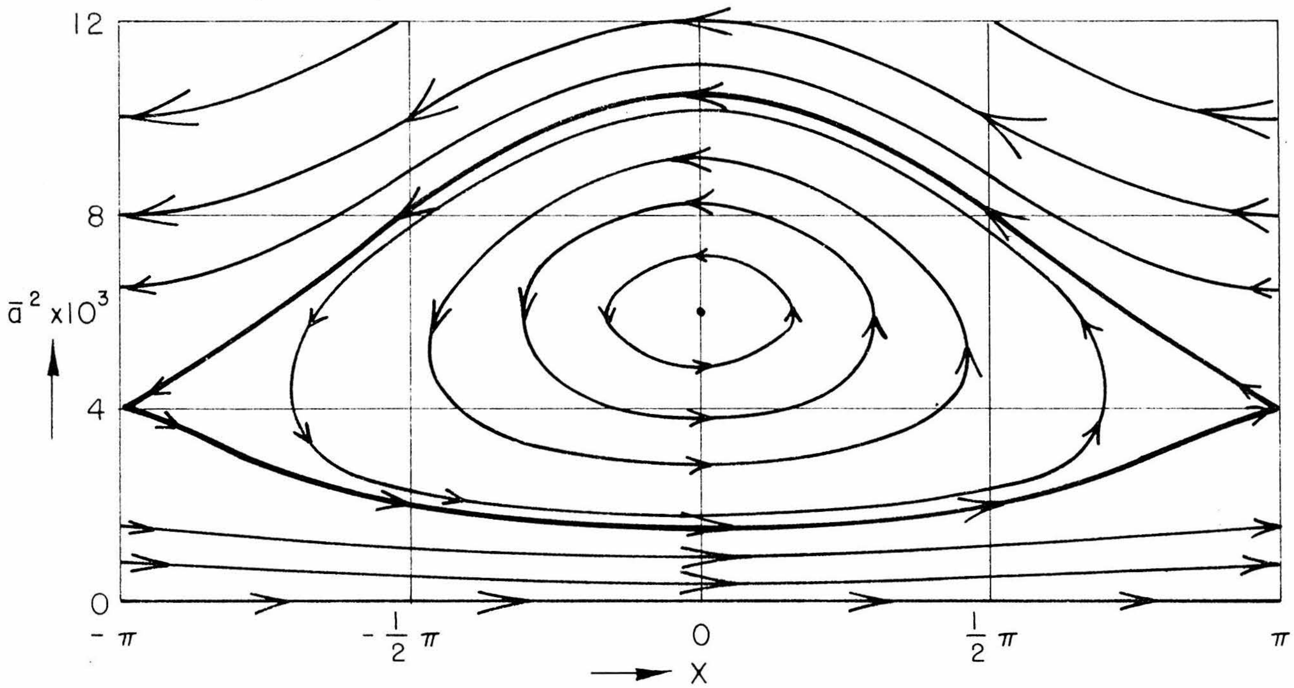


FIGURE 11

$\left|\frac{K_0}{K_1}\right| > 2, \frac{K_2}{K_1} < 0$ Drawn for $K_0 = 5K_1$, $K_2 = -10^3 K_1$



$$\frac{\bar{a}'}{\bar{a}} = K \sin X \qquad X' = 2\left(K_0 + K_1 \cos X + K_2 \bar{a}^2\right)$$

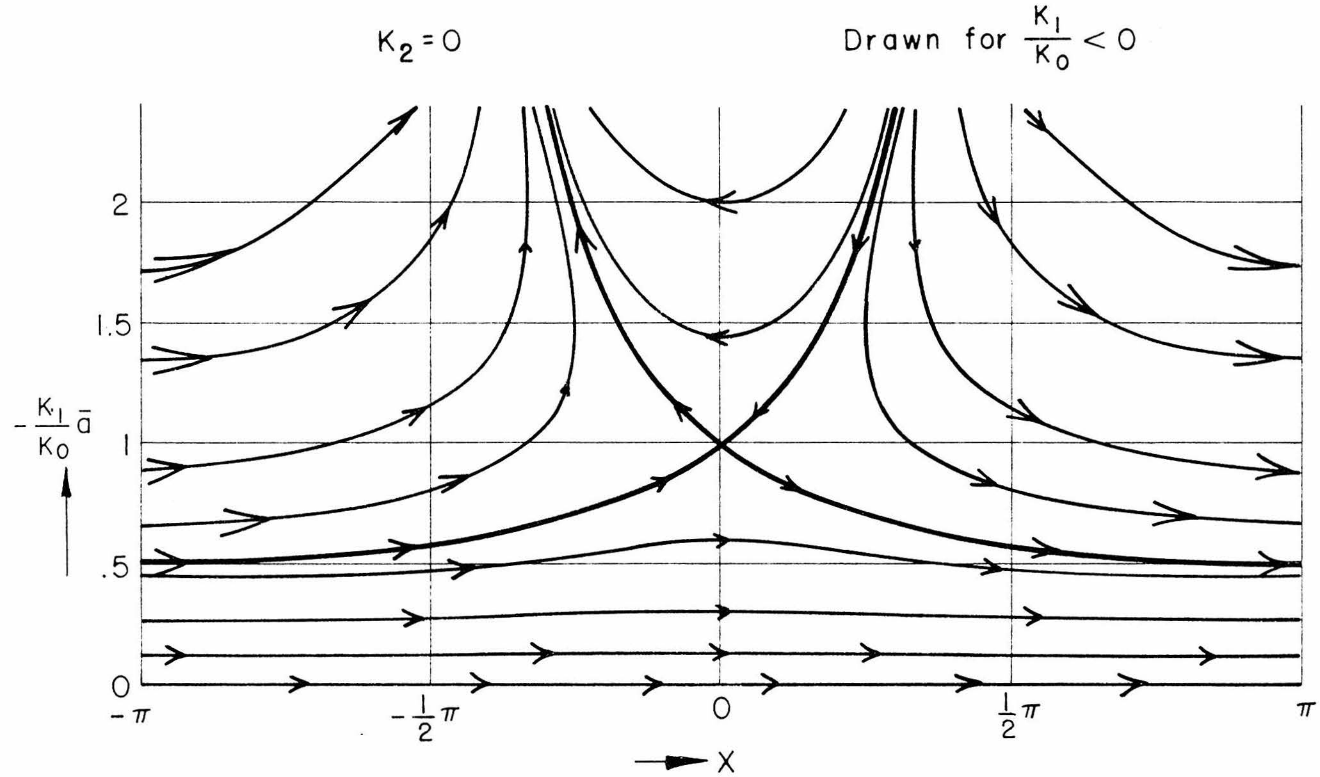
$$\bar{a}^2 = K_2^{-1}\left\{K_0 + K_1 \cos X \pm \left[\left(K_0 + K_1 \cos X\right)^2 + C'\right]^{\frac{1}{2}}\right\}$$

of order one-half; in higher fractional orders, the resonance effect vanishes as the amplitude approaches zero. In general, the width of the resonance range will be proportional to $\bar{a}^{s-2}$, as is indicated in Fig. 13. From this figure it is clear that, for moderately large inhomogeneities and for the small $\bar{a}$'s of physical interest, the widths of the resonant ranges decrease very rapidly with increasing values of s. Even if $K_o = 0$, the frequency shift proportional to $\bar{a}^2$ may become large enough to destroy the resonance.

3d. <u>General Features of the Motion; Satisfactory Operation in Spite of Resonances</u>

We are now in a position to describe qualitatively the general behavior of such a system. We have seen that the non-linear terms present even with an axially symmetric field are responsible for two effects: the introduction of higher harmonics of the fundamental frequency (which are progressively smaller if the coefficients of high order terms are not too large), and the shifting of this frequency with increasing amplitude of the principal oscillation. From the Mathieu-Hill equation we found that the terms with periodic coefficients caused the appearance of divergent oscillations when the frequencies of forced and free oscillations were nearly commensurable. We have found both types of behavior in our more general equation; some multiple of the fundamental free frequency will resonate with some forcing term, due to a near-commensurability, and thus will give rise to a continual change in its amplitude. This will cause a change in the free frequency and all its harmonics, until the resonance responsible for the increase in amplitude is destroyed. Since the high-order resonances are very narrow and exert a very weak influence on the amplitude, they are relatively unimportant,

FIGURE 12

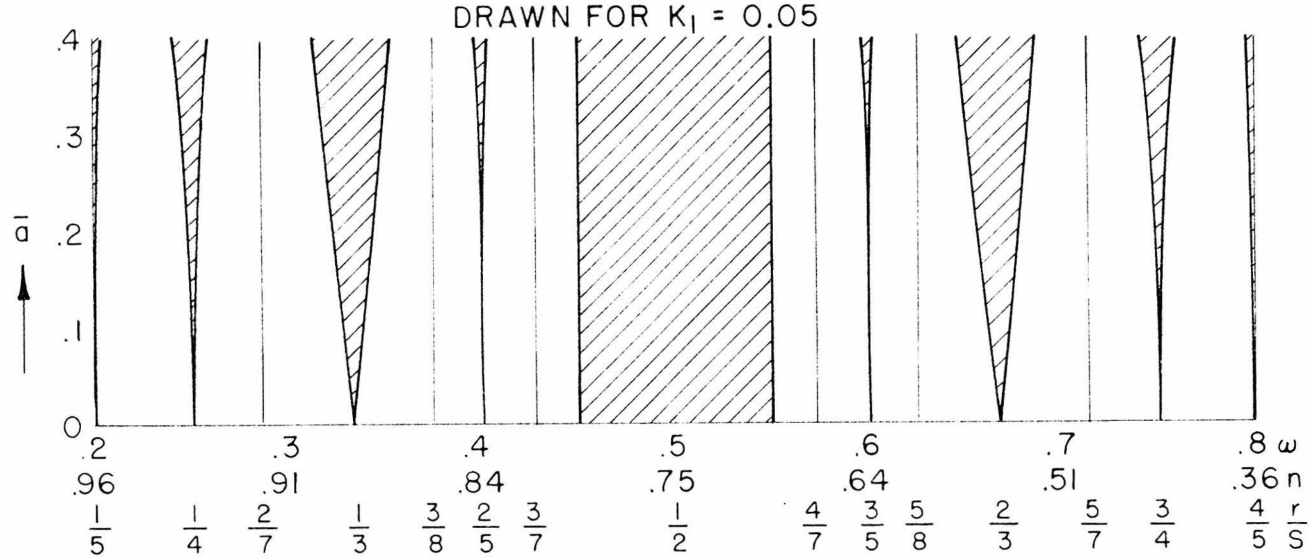


$$\frac{\bar{a}'}{\bar{a}} = K_1 \sin X \; ; \; X' = 3 \; (K_0 + K_1 \; \bar{a} \cos X) ; \quad \cos X = C \bar{a}^{-3} - \frac{3K_0}{2K_1 \bar{a}}$$

FIGURE 13

DEPENDENCE OF RESONANCE WIDTH ON AMPLITUDE

DRAWN FOR $K_1 = 0.05$



Half-width of shaded region is frequency shift required to obtain non-divergent motion.

as is clear from an inspection of Fig. 13.

One of the important results of this study is the theoretical explanation, on these grounds, of the experimental observation that machines may function well in spite of commensurabilities. We have seen that the frequency may vary quite strongly with amplitude if the field deviates considerably from a linear dependence on ρ. A resonance which is harmful at one amplitude may easily disappear at another, especially if the deviations from axial symmetry are not large. We must remember that, in addition to the effects described above, the Kerst-Serber damping and the adiabatic decrease of the inhomogeneities relative to the total field both work against any divergence of the oscillations. Also, the circulating electron current produces electric and magnetic fields which change the effective value of n. All of these considerations make it seem plausible that resonance effects will interfere with proper operation only in particular situations which have a low probability of being realised by empirical adjustments. The only potentially dangerous resonance is that of order one half; even here it seems likely that the trajectories may be similar to those of Figs. 9, 10, or 11 rather than to the unfavorable ones of Figs. 7 and 8. It may be, however, that the observed escape from stable orbits of some of the electrons throughout the acceleration cycle* of the General Electric Company's 70 Mev. synchrotron(7), in which $n = 3/4$, is due to this resonance effect, as first suggested by Courant(13).

It does not seem worth while to continue a study of the equation governing motion in the plane of symmetry in its full generality, since

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* Private communication from Dr. R. V. Langmuir.

the approximation procedures to be followed and the terms to be singled out for further study depend on the magnitudes of the various constants, which can only be determined by extensive measurements on a particular machine. We shall be content at this point with the general description above, and now turn to a discussion of the possibility of choosing certain inhomogeneities in such a way that the chance of avoiding collisions with the gun is greatly increased.

3e. The Problem of Using the Inhomogeneities to Aid Injection

To aid in increasing the injection efficiency, the inhomogeneities must reduce the amplitude of free oscillation by an amount sufficient to miss the gun within a few revolutions, and hold it below this value for several hundred revolutions, until the Kerst-Serber damping has had enough time to take over this function. In terms of the phase plots, this means that a cluster of representative points initially in a region of the phase plane having finite area, corresponding to electrons injected under similar initial conditions, must all follow trajectories which rapidly drop down toward smaller $\bar{a}$ values and remain below their initial values for a rather long time. It is clear that periodic trajectories of the second kind (corresponding to a lack of synchronisation between radial oscillation and rotation) are unsuitable for this purpose, since they will return to their original amplitudes too rapidly, in general. Therefore, it is necessary to consider only situations in which synchronism exists. We have both experimental and analytical evidence that it is difficult to maintain this condition, due to the dependence of the free frequency on amplitude. We will therefore consider only the strongest resonance, when $\omega_r \approx \frac{1}{3}$; this should be the most easily maintained.

A desirable state of affairs would be that in which all of the trajectories passing through the area corresponding to the injection conditions would flow downward toward a sink of some sort at smaller amplitude, from whose neighborhood they would never depart. However, such a sink can occur only if the divergence of the flow of trajectories is negative in this region. If it is zero, the flow is that of an incompressible fluid, and representative points may not accumulate anywhere, but must flow in and out at equal rates; all trajectories must be either closed or asymptotic (coming from and going to infinity). The critical trajectories leading to a saddle point (which is a point of stagnation in this case) are exceptions, but even infinitesimally neighboring trajectories will eventually lead away again. Therefore a necessary condition that a group of trajectories shall drop down to small values and never return to their original amplitudes is that the flow shall not be everywhere divergenceless.

That this is not a sufficient condition may be seen as follows. Suppose that the trajectories, or lines offlow, be drawn with such a spacing that the flow per unit time across a line-element of the phase plane is given by the number of lines passing through it times the component of the velocity at that point normal to the line-element. Then the existence of a non-vanishing divergence means that the lines of flow are not conserved; in a representation containing a discrete number of lines, certain lines would have to terminate or start abruptly. However, any particular representative point will continue to move onward along its trajectory through the region (except at a singular point where the velocity is zero) regardless of whether neighboring fictitious lines are required to terminate or start; the number of real particles will remain constant. A sufficient condition for occurrence of the desired situation is that the flow be inward at every point of a closed curve surrounding the region within which we wish the trajectory to remain; this requires that the area integral of the divergence over this region be negative and that the region contain at least one stable focal or nodal point*.

It is easy to show that the flow due to the terms thus far considered will always be divergenceless. Expressing our results in terms of $\bar{A} = \bar{a}^2$, we have

$$\bar{A}' = 2K_1 \bar{A}^{\frac{1}{2}s} \sin(s\bar{\delta} - \alpha) + \text{higher order},$$

$$\bar{\delta}' = K_1 \bar{A}^{\frac{s}{2}-1} \cos(s\bar{\delta} - \alpha) + K_0 + K_1 \bar{a}^2 + \text{higher order},$$

and the divergence $\dfrac{\partial \bar{A}'}{\partial \bar{A}} + \dfrac{\partial \bar{\delta}'}{\partial \bar{\delta}} = 0$ to the order of terms written explicitly. Thus we must bring in higher order terms to obtain a non-vanishing divergence, for any order resonance. We will show later that our approximation procedure will not properly account for the influence of these higher order terms; however, it will be instructive to investigate their effects under the assumption that the procedure is valid. It can be easily shown that the next terms in the equations will contain two more powers of $\bar{a}$, and are therefore much smaller. For $s = 2$, the lowest-order term in f leading to a non-vanishing divergence is $\rho\rho'^2 \cos(\theta + \alpha')$; its contribution to $\bar{a}'/\bar{a}$ will be of the form $K_3 \bar{a}^2 \sin(2\bar{\delta} - \alpha')$, where K_3 and α' are determined functions of the coefficients; there is no corresponding contribution to $\bar{\delta}$. If we neglect the terms in K_0 and K_2 which produce the frequency shift, we are led

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* Compare with Bendixson's First Theorem (the "negative criterion"), Reference 24, page 77.

to consider the equations

$$\bar{a}'/\bar{a} = K_1 \sin(2\bar{\delta} - \alpha) + K_3 \bar{a}^2 \sin(2\bar{\delta} - \alpha') \; ; \quad \bar{\delta}' = K_1 \cos(2\bar{\delta} - \alpha) .$$

Setting $2\bar{\delta} - \alpha = x$ and $\alpha - \alpha' = \epsilon$, we have $\bar{\delta}' = 0$ along the lines $x_0 = \pm \frac{\pi}{2}$, and $\bar{a}' = 0$ along the lines $K_1 \sin x = -K_3 \bar{a}^2 \sin(x + \epsilon)$. The singular points where these lines intersect are at $\bar{a}_0^2 = -\frac{K_1}{K_3 \cos \epsilon}$. If we set $\bar{a} = \bar{a}_0 + \xi$, $x = x_0 + \eta$, and expand about the singular points, keeping only first powers, we obtain

$$\begin{aligned} \xi' &= \mp 2K_1 \xi \mp (K_3 \bar{a}_0^2 \sin \epsilon)\eta \equiv a\xi + b\eta \; ; \\ \eta' &= \mp 2K_1 \eta \equiv c\xi + d\eta . \end{aligned}$$

Using the theorem of Liapounoff*, we find that the roots S_1 and S_2 of the characteristic equation $S^2 - (a + d)S + (ad - bc) = 0$ are $S_1 = S_2 = \mp 2K_1$; since we have real roots of the same sign, the singular points are both nodal points, one being stable and the other unstable. If a stable nodal point could be produced at a value of $\bar{a}$ less than that at injection, the desired results might be achieved. However, to do this the quantity $K_1/(K_3 \cos \epsilon)$ must be made very small, say less than 10^{-3}. For reasonable values of the coefficients this quantity turns out to correspond to values of $\bar{a}_0$ which are much too large. This is because K_3 is a coefficient of a higher order term than K_1, and would not be expected to be more than a thousand times as great as K_1. If it were, our approximation procedure would have little chance of yielding meaningful results. If K_1 is too small, the electrons will not move away from the gun fast enough at the start. The other terms not yet considered would make it even more difficult to obtain such a nodal point in the correct region; the effects of vertical oscillation and interaction of many electrons would further complicate the situation. Therefore it

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* Reference 24, p. 51.

seems thoroughly impractical to depend on such an effect for satisfactory injection.

Thus far we have not taken the Kerst-Serber adiabatic damping into account explicitly. It is easy to show that its cumulative effect on the amplitude would not become decisive until after several hundred revolutions. Since $da/a = -\frac{1}{4}\, dE/E$ in the non-relativistic energy range, we have $\frac{a'}{a} = -\frac{1}{4}\left(\frac{2\pi E_0}{\epsilon} + \theta\right)^{-1}$, where ϵ is the energy gain per turn, and $E = E_0 + \frac{\epsilon\theta}{2\pi}$. If gun clearance requires $a/a_0 = 0.95$, say, we must rely on other means of attaining it until $\theta/2\pi \approx 0.22E_0/\epsilon$. If $E_0 \sim 50$ Kev. and $\epsilon \sim 50$ v. per turn, this is about 220 revolutions. However, we may get more detailed information about the effects of this damping on the behavior at a resonance by including it directly in the equation for $\bar{a}'/\bar{a}$. The term K_d to be subtracted may be approximated, over the range of interest, by $\epsilon/(8\pi E_0) \approx 4 \times 10^{-5}$ for the values given above. It can be increased by raising the energy gain per turn, but will probably never exceed 10^{-4} for injection energies of the order of 50 Kev.

We shall briefly consider the effects of this term on some of the cases discussed earlier. Its general effect is to introduce a constant negative divergence into the flow pattern, thereby allowing nodal or focal points to exist where none were previously allowed. Since the divergence is very small, the rate of approach to these points is too slow to be helpful in avoiding the gun. If $K_2 = 0$, we have

$$\bar{a}'/\bar{a} = K_1 \sin x - K_d \;;$$

$$\tfrac{1}{2}x' = K_1 \cos x + K_0 \;.$$

If $|K_0| > |K_1|$, we have a continuous decrease of amplitude superimposed on the plot of Fig. 9; the average damping is the same as if there were no resonance. If $|K_o| < |K_1|$, the trajectories of Fig. 7 are qualitatively unaltered as long as $K_1^2 > K_o^2 + K_d^2$. If the inequality is reversed, we find a stable nodal point at $\bar{a} = 0$, $x = \cos^{-1}(-K_o/K_1)$; however, to obtain $|K_o| < |K_1|$ and $K_1^2 < K_o^2 + K_d^2$ for $K_d \sim 10^{-4}$ requires an extremely delicate adjustment of the K's which would be impossible in practice, and the rate of decrease of $\bar{a}$ is still too small to help in missing the gun. If $K_2 \neq 0$, we find that the vortex points of Figs. 10 and 11 become stable focal points toward which the trajectories spiral slowly inward. The approach is very slow since K_d is so small. The higher order terms discussed earlier in connection with divergence may alter the position and strength of such a sink, while higher order terms in the frequency shift may be of sufficient importance to move the stable point to an unfavorable position or destroy it. In any event, the damping is too small to be of use in explaining or aiding injection.

These difficulties may be displayed more explicitly in analytical form. Let us consider the situation illustrated in Fig. 7, with s = 2. If we accept the conclusion that the flow of trajectories is divergenceless in the lowest approximation, which can be calculated and experimentally controlled with reasonable accuracy, we have

$$\bar{a}^2 = (\text{const.}) \sec (2\bar{\delta} - \alpha),$$

if we neglect the adiabatic damping. It is shown in the appendix that by substituting this in the differential equations for $\bar{a}'$ and $\bar{\delta}'$ and integrating with respect to θ one obtains

$$(\bar{a}/\bar{a}_o)^2 = \sin^2\delta_o \, e^{2K_1\theta} + \cos^2\delta_o \, e^{-2K_1\theta} .$$

This holds if $\alpha = \pi/2$, corresponding to injection at the most favorable phase, at which all trajectories are dropping down. Here $\bar{a}_o$ is the initial amplitude of oscillation, given by $\bar{a}_o = \rho_I \sec \delta_o$, and $\tan \delta_o = -2 \tan \phi_I/\rho_I$, where ρ_I is the injection radius and ϕ_I is the angle between the actual direction of injection and that for a tangentially injected electron. It is also shown that inclusion of the adiabatic damping leads to the following solution:

$$(\bar{a}/\bar{a}_o)^2 = \left[\sin^2\delta_o\, e^{2K_1\theta} + \cos^2\delta_o\, e^{-2K_1\theta}\right] e^{-2K_d\theta} .$$

This result shows that all the oscillations will diverge unless $K_1 < K_d$, or unless the resonance is destroyed before they strike the gun or the doughnut wall. The latter possibility, requiring time-dependent inhomogeneities, will be discussed later. Let us now investigate the consequences of having K_1 less than about 10^{-4}.

First, even the most strongly damped electrons (those with $\delta_o = 0$) will require N_c revolutions to clear the gun, where N_c is given by

$$N_c = \frac{\ln|\rho_I/\rho_c|}{2\pi(K_1+K_d)} > \frac{\ln|\rho_I/\rho_c|}{4\pi K_d} ,$$

the inequality holding if $K_1 < K_d$ as is here assumed: ρ_c is the value of ρ at the inside edge of the gun. For $\rho_c/\rho_I = 0.95$, and $K_d = 10^{-4}$, we obtain $N_c > 40$ revolutions. This is much too great to help injection, even if allowance be made for some assistance from the vertical oscillations as indicated in Fig. 4a. ρ_c/ρ_I cannot be pushed much nearer to unity than the value given; for a machine with R = 30 cm. This allows only 1.5 millimeter between the injection point and the inner edge of the gun. Second, the width of the resonant range of radial oscillation frequencies is directly proportional to K_1, so that for small K_1 an extremely minute change in this frequency will remove it from the resonant range. In fact, if $K_1 = 10^{-4}$, a shift of only 0.02% will destroy the resonance. It seems impractical to rely on an effect which is so sensitively dependant on the parameters involved.

The changes introduced by a smaller frequency shift, which convert the trajectories of Fig. 7 into those of Fig. 8 through 11, do not essentially modify the situation. If hopes for satisfactory injection are pinned on any particular configuration of trajectories it would be very difficult to control the sizes of the various K's and the values of the α 's precisely enough to assure that the desired configuration is realised, and the same inconsistency between rapid departure from and lack of eventual return to the original amplitude will arise.

An important conclusion of this study is, therefore, that the difficulties described above, together with the additional complications introduced by the vertical oscillations and by interaction effects, make it impractical to rely on magnetic inhomogeneities which are relatively time-independent to aid in the injection of electrons into a machine whose parameters are in the ranges assumed in this discussion. The lowest-order effects which have been calculated here are universally unfavorable; the modifications produced by the higher-order terms can only be calculated accurately by extensive numerical computation. It is not certain that they could ever be made to produce the desired result; if they could, many parameters would require critical adjustment which might prove to be experimentally impossible.

There remains the possibility that the inhomogeneities might be maintained by appropriate auxilliary coils and rapidly eliminated when the trajectories had fallen to adequately low values of $\bar{a}$. If this could be done, there would be no need to restrict K_1 to such small values. An upper limit on K_1 is imposed by the condition that the most poorly damped first-injected electrons must not return to the gun until the last-injected electrons have cleared it. A lower limit on K_1 is imposed by the condition that an appreciable fraction of all electrons must clear the gun within a few revolutions. In examining these conditions we shall neglect the adiabatic damping.

We shall again consider the situation illustrated in Fig. 7, with s = 2 and $\alpha = \pi/2$. It is shown in the appendix that an electron will return to its original amplitude after N_r revolutions, where

$$N_r = (2\pi K_1)^{-1} \log \left| \tfrac{1}{2} \rho_I \cot \phi_I \right| .$$

Of course, when the initial amplitude is regained, the wave crests will not be near the azimuth of the gun, but rather about $\pi/2$ radians away from it. However, their amplitudes are rising here as rapidly as they were falling near the gun, and the electrons will soon strike the dough-nut walls at this azimuth. It is also shown that an electron will be clear of the gun after about N_c revolutions, where

$$N_c = (2\pi K_1)^{-1} \log \left| \rho_I / \rho_c \right|$$

for electrons having $\phi_I = 0$; N_c is slightly larger for electrons injected not quite tangentially.

If we rely entirely on control of the radial oscillation to escape collisions with the gun, we must have N_c less than two revolutions, because of the synchronism. However, it can be shown that N_c may be increased to as much as ten revolutions for electrons with the largest vertical amplitudes if the gun occupies less than one third of the available vertical clearance. Since we wish to attain a considerable improvement in injection efficiency we shall require that N_c be less than four revolutions, in order not to de-pend too heavily on the vertical oscillation. We must also require that N_r be greater than a hundred revolutions, since one revolution takes place in about 10^{-8} sec. and injection proceeds for a time of the order of 10^{-6} sec. From the equations above we may then determine the radial divergence of those electrons which could successfully miss the gun by this mechanism:

$$\tan \phi_{I_{max}} = \tfrac{1}{2} \rho_I \left(\frac{\rho_c}{\rho_I}\right)^{N_r/N_c} .$$

Taking $\rho_I = 0.1$, $\rho_c/\rho_I = 0.95$, $N_r = 150$, and $N_c = 4$, we have $\phi_{Imax} \simeq 25'$, which is a very small angular spread. Further, it would be necessary to reduce the inhomogeneity from maximum to nearly zero in (150-100) = 50 revolutions, or one half micro-second. In addition, the value of K_1 specified by the choice of values above is 2×10^{-3}, and the radial oscillation frequency must be adjusted to within 0.4% in order for the effect to exist at all. Therefore this procedure seems equally impractical*, and we conclude that an explanation of injection phenomena and a method for improving injection efficiency must be sought along other lines. We will show in Section III that the effects due to the interaction of many electrons have an important bearing on both.

4. Three-Dimensional Motion

If the magnetic field is axially symmetric, it is possible to generalize the treatment given earlier for motion in the plane of symmetry, obtaining a potential function $U(r,z)$ such that $d(m\dot{r})/dt = - \partial U/\partial r$ and $d(m\dot{z})/dt = - \partial U/\partial z$. After obtaining this result by the simple method presented below, the writer discovered that a special case of it had been derived earlier by Gans[29], who used a more complicated procedure.

The Hamiltonian for an electron with charge - e moving in an axially symmetric magnetic field is

$$\mathcal{H} = \frac{1}{2m}\left[p_r^2 + (r^{-1} p_\theta + eA_\theta)^2 + p_z^2 \right],$$

where $A_\theta = A_\theta(r,z)$ is the azimuthal component of the vector potential.

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* See, however, the note in the appendix on time-varying inhomogeneities.

The Hamiltonian equations defining the canonical momenta are

$$\frac{\partial \mathcal{H}}{\partial p_r} = \dot{r} = p_r/m \; ;$$

$$\frac{\partial \mathcal{H}}{\partial p_\theta} = \dot{\theta} = \left(\frac{p_\theta}{r} + eA_\theta\right)/m \; ;$$

$$\frac{\partial \mathcal{H}}{\partial p_z} = \dot{z} = p_z/m \; .$$

From these we obtain

$$p_r = m\dot{r} \; ; \; p_\theta = mr^2\dot{\theta} - erA_\theta \; ; \; p_z = m\dot{z} \; .$$

The other Hamiltonian equations yield

$$-\frac{\partial \mathcal{H}}{\partial r} = \dot{p}_r = \frac{d}{dt}(m\dot{r}) = -\frac{\partial}{\partial r}\left[\frac{1}{2m}\left(\frac{p_\theta}{r} + eA_\theta\right)^2\right] = -\frac{\partial U}{\partial r} \; ;$$

$$-\frac{\partial \mathcal{H}}{\partial z} = \dot{p}_z = \frac{d}{dt}(m\dot{z}) = -\frac{\partial}{\partial z}\left[\frac{1}{2m}\left(\frac{p_\theta}{r} + eA_\theta\right)^2\right] = -\frac{\partial U}{\partial z} \; ;$$

$$-\frac{\partial \mathcal{H}}{\partial \theta} = \dot{p}_\theta = 0 \; ; \; p_\theta = constant = mr_I^2\dot{\theta}_I - er_I A_\theta(r_I, z_I) \; ;$$

here subscripts I denote initial or injection values. We define $r_I\dot{\theta}_I = v \cos \phi_I$, where v is the velocity of the electron. It is easy to show that the vector potential $A_\theta(r,z) = \Phi(r,z)/2\pi r$, where $\Phi(r,z)$ is the total flux of magnetic induction passing upward through a surface bounded by the circle (r,z). (The flux through any surface having this boundary is the same, since the divergence of the induction is zero.) The central flux Φ_c through the surface bounded by the circle (R,0) is equal to $2\pi R^2 B_0$, since this is the betatron flux condition which defines R. By inserting these values, and remembering that $mV = + B_0 eR$, we obtain our potential function in the following form:

$$U(r,z) = (m/2)\left[V/2\pi B_0 Rr\right]^2 \left[k\,\Phi_c + \Phi(r,z) - \Phi(r_I,z_I)\right]^2 ,$$

where $k = (r_I v \cos \phi_I)/RV)$. In the special case discussed by Gans, k = 1, $r_I = R$, $z_I = 0$, and the potential function becomes

$$U(r,z) = (m/2)\left[V/2\pi B_o R\right]^2\left[\Phi(r,z)/r\right]^2.$$

An interesting feature of this potential is that it is valid even if the magnetic field is changing in time. The potential function is then no longer time-independent, but the equations for radial and vertical motion are still given correctly by the simple formulas above, as shown by Gans. The dependence of θ on time is given by the equation p_θ = constant. The equations are even valid if the particle's energy is in the relativistic region where its mass is varying with time; the mass variation can be found from the equation

$$\dot{m} = +\,(er\dot{\theta}/c^2)(\partial A_\theta/\partial t)\ ,$$

as shown by Dennison and Berlin(12).

This result allows one to determine the region of stability in a general way. The motion is stable everywhere within the region where U is increasing with increasing departure from its minimum at the instantaneous circle. This rigorously correct method, based on the work of Gans, does not seem to be well known. His paper appeared in the first volume of the new publication Zeitschrift für Naturforschung in 1946, but Goward and Wilkins(30) in 1948 were still using the criterion that $(r/B_z)(dB_z/dr)$ should lie between zero and unity in the region within which both vertical and radial oscillations are stable. Their plotted results do not appear to be entirely consistent with the method above, especially near the edges of the stable region.

In principle, this result enables one to determine the effects of all nonlinearities if the field is axially symmetric. We are able to visualize the radial and vertical motion in terms of a mass-point moving without friction on a surface whose height above an r-z plane is U(r,z), under the influence of a constant force acting downward toward this plane.

In the linear approximation, this potential surface is an elliptical parabaloid. The gradual growth of the potential function in time provides the adiabatic damping mentioned earlier. If the damping is negligibly small, we see that an electron will eventually return to the gun, as shown in Fig. 4a. However, if its motion extends into a region where the shape of the potential well is no longer parabaloidal, its radial and vertical displacements need not be sinusoidal, and their periods may differ from those calculated from the linearized theory; the task of determining how many revolutions are required to return to the gun under these conditions is very difficult indeed.

To avoid tedious repetition, we will not discuss in detail the general motion in the presence of axial inhomogeneities, but will only point out the important features and indicate the techniques by which the calculations can be perfored. This procedure is suggested by the results obtained above, which showed that in the two-dimensional problem no useful results can be obtained by utilizing the inhomogeneities, and by the conclusions to be presented later concerning interaction effects, which cast doubt on the validity of the single-particle approximation in practical accelerators.

If the various parameters are small enough to justify approximations of the type used above, it is easy to show that the appearance of interesting deviations from the simple sinusoidal behavior of the first approximation depends on the existence of near-commensurabilities between at least two of the three characteristic frequencies (of radial oscillation, vertical oscillation, and rotation). A commensurability may be represented in the form

$$r\,\omega_r + s\,\omega_z + t\,\omega_\theta = 0\,,$$

where r, s, and t are positive or negative integers or zero. If these integers are not small, the resulting resonances are weak, slow to act, and easily destroyed by small frequency shifts due to the dependence of the oscillation frequencies on amplitude. Consequently we shall consider only the simplest cases, where none of these integers exceeds two. We may write our condition as one on n; $r\sqrt{1-n} + s\sqrt{n} + t = 0$.

The resulting resonances may be tabulated as follows:

n	$\frac{1}{2}(1-\frac{1}{4}\sqrt{7})$	1/5	$\frac{1}{4}$	9/25		$\frac{1}{2}$	16/25		3/4	4/5	$\frac{1}{2}(1+\frac{1}{4}\sqrt{7})$
r	2	1	0	2	1	1	-1	2	2	2	-2
s	-2	-2	2	-1	2	-1	2	1	0	-1	2
t	-1	0	-1	-1	-2	0	-1	-2	-1	0	-1

Since betatrons and synchrotrons are usually designed with $\frac{1}{2} < n < 4/5$, we shall consider only the four resonances in this range and the resonance at $n = 0.2$, which is similar to that at $n = 0.8$ and is of interest in cyclotrons. The resonance at $n = 3/4$ does not involve the vertical oscillation and has already been discussed above. Those at $n = 1/5$, $\frac{1}{2}$, and $4/5$ do not involve the rotation frequency, and in these cases the effects of axial inhomogeneities will cancel out over a few revolutions and thus may be neglected in studying the interplay of the two oscillations. At $n = 16/25$, there are important interactions only with the first and second Fourier components of the field; we may neglect the others.

Considering first an axially symmetric field, we expand the general equations of motion obtained earlier, keeping terms through second order in ρ and ζ and their first derivatives. We may take $D_{ooo} = 0$, since no current loops through and around the equilibrium circle. Also, $C_{ooo} = 0$, since it is the first term in the expansion of the field

$B = B_r \propto r^{-1}$ about $r = R$, and no field of this form will be present. We may always make A_{001} equal to zero by proper choice of the plane $z = 0$. Setting $\Delta_r = 0$ and making use of the interrelations between the coefficients derived in the appendix to express the C's and D's in terms of the A's, we obtain

$$\rho'' + (1-n)\rho = -A_{011}\rho\zeta + (2n-1-A_{020})\rho^2 + \left(-\tfrac{n}{2} + A_{020}\right)\zeta^2 + \tfrac{1}{2}\rho'^2 - \tfrac{1}{2}\zeta'^2 + \text{higher order};$$

$$\zeta'' + n\zeta = -\tfrac{1}{2}A_{011}\rho^2 + \tfrac{1}{2}A_{011}\zeta^2 + 2(A_{020} - n)\rho\zeta + \rho'\zeta' + \text{higher order}.$$

These equations are of the type studied extensively by Beth[31]. He has published a lengthy and thorough discussion of the behavior of the solution of such equations in a wide variety of cases. In first approximation, the equations are equivalent to those of two simple harmonic oscillators, with angular frequencies $(1-n)^{\frac{1}{2}}$ and $n^{\frac{1}{2}}$, respectively. The non-linear terms on the right represent perturbations and couplings between them. Each oscillator has a kinetic and a potential energy, whose sum would be a constant of its motion in the absence of the non-linear terms. Beth finds that, if the frequencies are commensurable, the principal effect of the non-linear terms is to cause an interchange of these total energies between the two oscillators, the sum of the total energies being conserved. In general the interchange is cyclic, so that the energy given up by one vibration to the other will ultimately be returned to it; however, for certain critical values of the parameters there may be an asymptotic approach to a particular partition of the energy among the different vibrations. He notes two general types of behavior, which he calls "libration" and "general motion"; these correspond to what we have called periodicities of the first and second kinds, respectively.

If the amplitudes of radial and vertical oscillations are denoted by $\bar{a}$ and $\bar{b}$, respectively, the energy conservation equation becomes

$$(1-n)\bar{a}^2 + n\bar{b}^2 = c^2 \text{, a constant}$$

to a good approximation, since the spring constants of the equivalent oscillators are proportional to (1-n) and n, respectively. If the energy transfer from one mode to the other and back were complete, the beam would have an elliptical cross-section in the long time average, with the ratio of radial to vertical semi-axes being $[n/(1-n)]^{\frac{1}{2}}$. Incidentally, Kerst and Serber[8] have pointed out that this ratio would eventually be attained even if the beam were originally confined to the equilibrium orbit, because of random scattering from the residual gas molecules in the doughnut which tends to produce an equipartition of energy between the two oscillations.

The method of Kryloff and Bogoliuboff, generalized to two coupled equations, has been applied to these equations by the writer, with results which confirm those of Beth and have the additional advantage of allowing a representation in a two-dimensional phase space like that used above with a single equation. This is possible because the energy conservation equation determines one amplitude for any value of the other, and the differential phase equations all depend on trigonometric functions whose arguments are the same linear combination of the two phase angles, as will be seen below.

We will only summarize the analysis here, since the generalization to two equations is quite obvious. As before, we set $\rho = a \cos(\omega_r\theta + \delta_r)$, $\zeta = b \cos(\omega_z\theta + \delta_z)$, and by the same procedure obtain four first-order equations, for $\bar{a}'$, $\bar{b}'$, $\bar{\delta}_r'$, and $\bar{\delta}_z'$. For $n \simeq 4/5$, these are

$$\bar{a}'/\bar{a} = K_1\bar{b}\,\sin(2\bar{\delta}_r - \bar{\delta}_z)\;;$$

$$\bar{b}'/\bar{b} = -\tfrac{1}{4}K_1(\bar{a}^2/\bar{b})\sin(2\bar{\delta}_r - \bar{\delta}_z)\;;$$

$$\bar{\delta}_r' = K_1\bar{b}\,\cos(2\bar{\delta}_r - \bar{\delta}_z) - (\sqrt{5}/2)(n - 4/5)\;;$$

$$\bar{\delta}_z' = \tfrac{1}{4}K_1(\bar{a}^2/\bar{b})\cos(2\bar{\delta}_r - \bar{\delta}_z) + (\sqrt{5}/4)(n - 4/5)\;;$$

where $K_1 = \frac{\sqrt{5}}{4} A_{011}$. Taking the quotient of the first two equations, we get $\bar{a}\,d\bar{a} + 4\bar{b}d\bar{b} = 0$; integrating, $\bar{a}^2 + 4\bar{b}^2 = c^2$, a constant, in agreement with our energy conservation equation above. By introducing $\bar{A} = \bar{a}^2$ and $\bar{B} = \bar{b}^2$, it may easily be verified that the divergence

$$\frac{\partial \bar{A}'}{\partial \bar{A}} + \frac{\partial \bar{B}'}{\partial \bar{B}} + \frac{\partial \bar{\delta}_r'}{\partial \bar{\delta}_r} + \frac{\partial \bar{\delta}_z'}{\partial \bar{\delta}_z} = 0 .$$

Setting $\bar{a}^2 = c^2 - 4\bar{b}^2$, and $2\bar{\delta}_r - \bar{\delta}_z = \phi$, we have

$$\bar{b}' = -\tfrac{1}{4} K_1 (c^2 - 4\bar{b}^2) \sin\phi ;$$

$$\phi' = \tfrac{1}{4} K_1 \left(12\bar{b} - \frac{c^2}{\bar{b}}\right) \cos\phi - \frac{5\sqrt{5}}{4} (n - 4/5).$$

These equations may be used to obtain a phase plot of $\bar{b}$ against ϕ, from which the essential features of the motion can be determined. The differential equation of the trajectories is

$$\frac{d\phi}{dy} = \frac{k - (3y - y^{-1}) \cos\phi}{(1-y^2) \sin\phi} ,$$

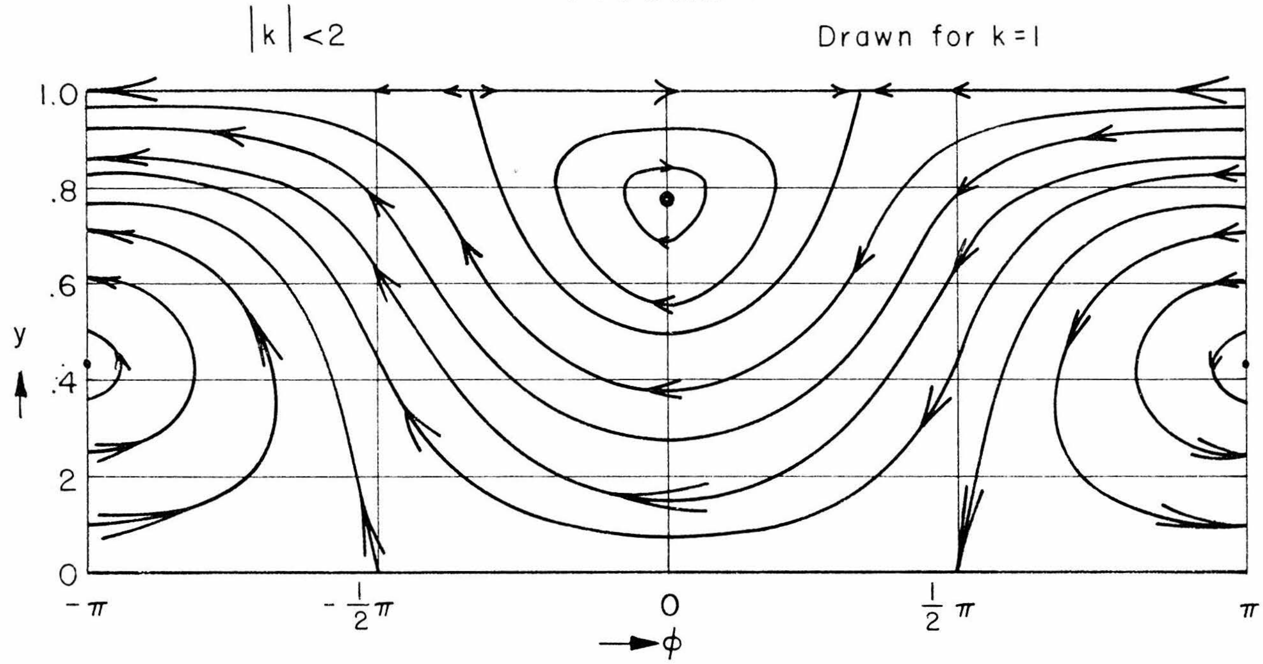
where $y = 2\bar{b}/c$ and $k = 2(5n - 4)/A_{011}c$. The general solution of this differential equation is

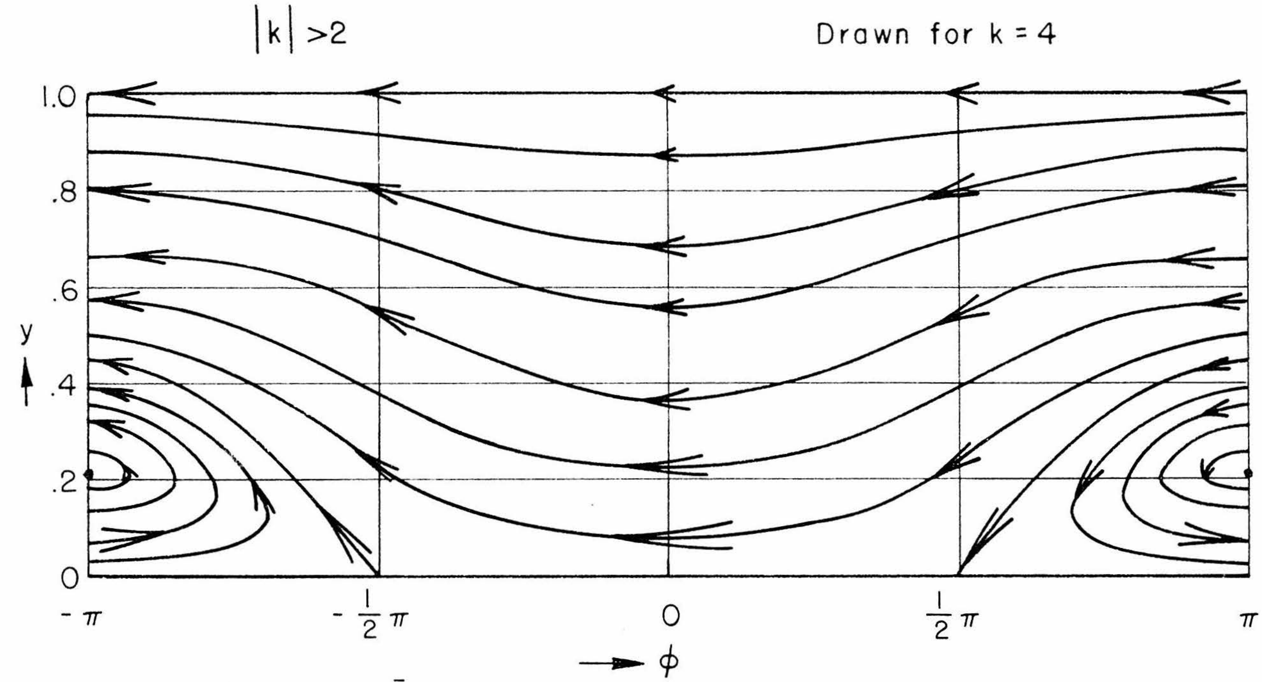
$$\cos\phi = \frac{k}{2y} + \frac{D}{y(1-y^2)} ,$$

in which D is the arbitrary constant of integration. The trajectories are of the form shown in Fig. 14. There are two qualitatively different cases, according to whether $|k| \lessgtr 2$, as indicated on the figure. The central vortex point moves up from $y = 3^{-\frac{1}{2}}$ to $y = 1$ as $|k|$ increases from 0 to 2, and is not present for larger $|k|$. The outer vortex point at $\phi = \pi$ moves down from $y = 3^{-\frac{1}{2}}$ to $y = 0$ as $|k|$ increases from zero to infinity. The analysis becomes invalid very near $y = 0$, since ϕ' grows without bound as y diminishes.

As before, the apparent possibility of quickly transforming the radial oscillation energy into that of vertical oscillation and holding

FIGURE 14





For $n = \frac{4}{5}$: $y = -2\,\frac{\bar{b}}{C}$, $\bar{a}^2 + 4\,\bar{b}^2 = C^2$, $k = 2\,\frac{5n-4}{A_{011}C}$.

$$y' = -\frac{5^{\frac{1}{2}}}{8}\,A_{011}\;C\left(1-y^2\right)\sin\phi;\;\; \phi' = \frac{5^{\frac{1}{2}}}{8}\,A_{011}\;C\left[\left(3y-y^{-1}\right)\cos\phi - k\right];$$

$$\cos\phi = \frac{k}{2y} + D\left[y\left(1-y^2\right)\right]^{-1}$$

For $n = \frac{1}{5}$; replace A_{011} by $\frac{1}{2}\,A_{020}$, $(5n-4)$ by $(5n-1)$,

and interchange $\bar{a}$ and $\bar{b}$.

it there to avoid the gun is circumvented by the divergenceless character of the plot, which brings $\bar{a}$ back to its original value after a short time.

For $n \approx 1/5$, we obtain

$$\bar{a}'/\bar{a} = - K_1 \frac{\bar{b}^2}{\bar{a}} \sin(\bar{\delta}_r - 2\bar{\delta}_z)$$
$$\bar{b}'/\bar{b} = 4 K_1 \bar{a} \sin(\bar{\delta}_r - 2\bar{\delta}_z)$$
$$\bar{\delta}_r' = - K_1 \frac{\bar{b}^2}{\bar{a}} \cos(\bar{\delta}_r - 2\bar{\delta}_z) - \frac{\sqrt{5}}{4}(n - \tfrac{1}{5})$$
$$\bar{\delta}_z' = -4 K_1 \bar{a} \cos(\bar{\delta}_r - 2\bar{\delta}_z) + \frac{\sqrt{5}}{2}(n - \tfrac{1}{5}),$$

where $K_1 = \frac{\sqrt{5}}{8} A_{020}$. Taking the quotient of the first two equations, we get $4\bar{a}d\bar{a} + \bar{b}d\bar{b} = 0$; integrating, $4\bar{a}^2 + \bar{b}^2 = c^2$, a constant, again agreeing with the energy conservation equation. The divergence vanishes as before. Setting $\bar{\delta}_r - 2\bar{\delta}_z = \phi$ and $\bar{b}^2 = c^2 - 4\bar{a}^2$, we have

$$\bar{a}' = - K_1 (c^2 - 4\bar{a}^2) \sin\phi \;;$$
$$\phi' = K_1 \left(12\bar{a} - \frac{c^2}{\bar{a}}\right) \cos\phi - \frac{5\sqrt{5}}{4}(n - \tfrac{1}{5}) .$$

These equations become identical with those for $n = 4/5$ if one replaces $\bar{a}$ by $\bar{b}$, A_{020} by A_{011}, and $(n - 1/5)$ by $(n - 4/5)$. The same phase plots of Fig. 14 therefore represent this motion when properly interpreted.

It is interesting to note that the results of this method, to the order of terms explicitly written above, depend on different terms in the differential equations for different resonances. In particular, if $n = 1/5$ corresponding to $\omega_r/\omega_z = 2$, only the terms in ζ^2 and ζ'^2 in the first equation and those in $\rho\zeta$ and $\rho'\zeta'$ in the second contribute; but if $n = 4/5$, for $\omega_r/\omega_z = \frac{1}{2}$, only the term in $\rho\zeta$ in the first equation and that in ρ^2 in the second do so. None of these terms contribute to the other resonances, which all depend on higher order terms. However, in our problem, the coefficient A_{011} of the terms affecting the $n = 4/5$ resonance will be very small since it depends on the asymmetry about the plane $z = 0$, while the coefficient A_{020} affecting the $n = 1/5$ resonance may be quite large. This resonance is of no interest in betatrons but is the only serious resonance to be encountered in a cyclotron, and is thought to be responsible for the disappearance of its beam[32] at the radius where this value of n is reached. In fact, the phase plot shows that the majority of trajectories do pass through small values of $\bar{a}$, corresponding to large vertical amplitudes, which may well cause the observed effect because of the small vertical dee clearance in cyclotrons.

We have now only to discuss the $n = 16/25$ resonance. The first-order equations will be of the form

$$\bar{a}'/\bar{a} = K_a \frac{\bar{b}^2}{\bar{a}} \sin(2\bar{\delta}_z - \bar{\delta}_r - \alpha_r) + L_a \bar{b} \sin(2\bar{\delta}_r + \bar{\delta}_z - \beta_r)$$
$$\bar{b}'/\bar{b} = K_b \bar{b} \sin(2\bar{\delta}_z - \bar{\delta}_r - \alpha_z) + L_b \frac{\bar{a}^2}{\bar{b}} \sin(2\bar{\delta}_r + \bar{\delta}_z - \beta_z)$$
$$\bar{\delta}_r' = -K_a \frac{\bar{b}^2}{\bar{a}} \cos(2\bar{\delta}_z - \bar{\delta}_r - \alpha_r) + L_a \bar{b} \cos(2\bar{\delta}_r + \bar{\delta}_z - \beta_r) + c_1\left(n - \frac{16}{25}\right)$$
$$\bar{\delta}_z' = K_b \bar{b} \cos(2\bar{\delta}_z - \bar{\delta}_r - \alpha_z) + L_b \frac{\bar{a}^2}{\bar{b}} \cos(2\bar{\delta}_r + \bar{\delta}_z - \beta_z) + c_2\left(n - \frac{16}{25}\right),$$

where K_a, K_b, α_r, and α_z depend on the amplitudes and phases of those inhomogeneities with $|j| = 1$, while L_a, L_b, β_r and β_z depend on those with $|j| = 2$; $c_1 = -5/6$ and $c_2 = 5/8$. The situation here is very complicated since there are two essentially distinct resonant effects acting simultaneously, each due to a different Fourier component of the inhomogeneities. Even if one assumes that only a single component exists, the simplifications of the above paragraph disappear and one must deal with a four-dimensional phase space. There is no longer a conservation of energy in the oscillations since the magnetic field may supply energy to or take it from either vibration. The differential phase equations no longer depend on trigonometric functions of the same argument as they do above, because of the arbitrary phases of the inhomogeneities. However, the divergence is zero even in this case. In principle, these equations could be investigated further if they were of sufficient interest to justify the labor involved.

We shall close this discussion of the motion in three dimensions with a few general remarks. First, the ζ equation contains ordinary trigonometric forcing terms in the C_{joo} like those in the A_{joo} in the ρ equation, whose first-order effect is to distort the orbit out of the plane $z = 0$. As before, the low-frequency components are the most effective because of the resonance denominators in the forced solution. If these coefficients are too large, the distorted equilibrium orbit will intersect the top or bottom of the doughnut aperture. Strictly, these terms in the first-order solution should be removed from the right side before applying the approximation method, as was done before for the radial forcing terms, by defining a new coordinate in the vertical direction measured from the distorted orbit. The right side may then be rearranged into the same form as that discussed above, but with more complicated coefficients, and the method then applied. The results will not be of a qualitatively different character.

Second, the method used above is essentially an approximation of first order in the small quantities ρ and ζ, in which we have considered only the lowest-order terms giving non-vanishing contributions. While the method will also give the contributions of higher-order terms if carried out as a prescribed routine, it is not clear that these terms are correctly taken into account without including the corrected low-order solution in some way in such a higher approximation. Since the objective of the present study has been to determine whether the inhomogeneities and effects of non-linearities could be used to aid in injection, we shall not investigate the problems of a next approximation in detail; our conclusion is that the lowest-order terms lead to unfavorable effects, and it seems unreasonable from the standpoint of feasibility to depend on higher-order modifications to reverse this general trend. There is even some doubt about the quantitative validity of the first approximation used here if some of the high-order coefficients with $j = 0$ are large enough to introduce strong higher harmonics of the free oscillations. These matters are considered in greater detail in the following paragraphs.

Finally, we may point out that all of the resonances discussed in this section are non-linear phenomena, and tend to vanish as the amplitudes of oscillation diminish. The adiabatic damping will probably compete successfully with all but the strongest of them in the circular accelerators considered here. In two- and four-section racetrack accelerators[33], however, the higher Fourier components of the field will be much stronger and these resonances may be more serious[34,35,36].

5. Discussion of the Approximation Method Used

It has been pointed out above that the lack of a theory of non-linear equations whose coefficients are periodic in the independent variable makes it difficult to discuss the size of error, convergence, and time of validity of their approximate solutions, no matter by what means they are obtained. To be sure, these difficulties appear even with non-linear equations having constant coefficients, such as we obtained for the motion in an axially symmetric field. However, successive approximation procedures for such equations based on the Hamiltonian equations as a starting point, have been developed by several writers[37,38]; these cannot always be proved to converge in general*, but give results in agreement with experiment and are clearly convergent in particular cases to

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* See Reference 37, § 198, and Reference 38, p. 584.

which they are applied. There is an extensive literature* in this field, motivated for many years largely by the problems of celestial mechanics. In contrast, approximate analytical methods for non-linear equations with periodic coefficients remained practically unexplored until comparatively recently, and in general, do not provide a systematic regime for obtaining approximations of higher order than the first.

It may be asked why we have replaced time by azimuthal angle as the independent variable, when by retaining it we could have made use of the Hamiltonian formalism, and at the same time could have avoided the problem of coefficients periodic in the independent variable. There are several reasons for our choice. First, and foremost, we were interested in the problem of dodging the injection gun, which is located at a fixed value of θ. Even if we had obtained all three coordinates as functions of time, a complicated substitution would have been required to determine the values of r and z each time the electron was at the azimuth of the gun. Second, the Hamiltonian method requires the specification of the vector potential, rather than the magnetic field strengths which are directly measurable. The determination of the vector potentials corresponding to the complicated fields of interest is not a simple matter. Third, the available constant of the motion is much more efficiently used in our method, by reducing the number of differential equations from three to two; the constancy of the Hamiltonian function does not lead easily to a similarly essential simplification in the canonical equations of motion, owing to its involved dependence on the coordinates.

Our equations are rigorously correct up to the point at which the characteristic averaging approximations of the Kryloff-Bogoliuboff method

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* A comprehensive bibliography is given in Reference 37.

are made. We may obtain some information about its accuracy by examining the effects of this averaging. For simplicity, we will consider only the solution of a single second-order equation. First, let us assume that all terms to be averaged depend explicitly on θ; we shall identify these as terms of type I. Each is then of the form

$$\cos^{\beta}(\omega\theta+\delta)\sin^{\gamma}(\omega\theta+\delta)\cos(j\theta+\alpha)$$

and is of order $\beta+\gamma-2$ in the amplitude a. The oscillating parts of such terms can be decomposed into single trigonometric functions of the form $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left[(\mu\pm\nu)\omega\pm j\right]\theta$, where $\mu\leq\beta$, $\nu\leq\gamma$, and μ and ν have the same parity as β and γ, respectively. Recalling that these terms contribute to the rates of change of log a and δ, we now wish to examine their periods of oscillation. Those with periods small with respect to the time required for a to change appreciably will be correctly taken into account in a first-order approximation by averaging, as their averages are zero, whereas those requiring much longer than the time of interest in our problem (which we may take to be several hundred revolutions) to oscillate once had best be accounted for by taking their initial values throughout. Those with intermediate periods cannot be correctly included by either method. The number of revolutions in a period is $\left[(\mu\pm\nu)\omega\pm j\right]^{-1}$; hence, if this bracket is very small, we must take the initial value, while if it is very large, we must average. This point of view makes clear the significance of selecting the proper ω near the unperturbed ω_r. By choosing it so that the very small brackets become zero, the initial values of these terms become identical with their average values, and the averaging approximation may be applied to all terms indiscriminately, leading to the routine prescription given earlier. The errors involved will depend on the terms whose periods are then of large or intermediate magnitude, corresponding to nearby resonances.

To consider the possibility of overlapping resonances, assume n is exactly that required to produce a resonance of order r/s, i.e., $\sqrt{1-n} = r/s$. Then the resonant terms arise from the average of the trigonometric functions $\cos\left[(\mu \pm \nu)\frac{r}{s} - j\right]\theta$ in which $(\mu \pm \nu) = ks$, and $j = kr$, where k is a positive integer, giving cos 0, whose average is unity. A nearby resonance is one with different integers $(\mu' \pm \nu') = s'$ and $j' = r'$ for which r'/s' is not a integral multiple of r/s, but for which $r'/s' \approx r/s$ so that the frequency of the corresponding trigonometric terms is very small. We can easily prove that the smallest possible value of this frequency is 1/s. Its frequency will be $(s'r/s) - r' = (s'r-r's)/s$. The smallest possible value of the numerator (other than zero, which has been ruled out) is unity; therefore the result is proved. It shows that the largest possible number of revolutions required for any other term to undergo one oscillation is s. Since our interest is confined to resonances of small s, we conclude that there will never be any terms with intermediate periods, and the justification of the averaging procedure is complete, provided that the perturbing terms are indeed small as originally assumed. Between every two resonances of interest there is a frequency interval in which the resonant effects dissappear.

It is hard to determine, except by numerical methods in special cases, what errors are introduced by the rapidly varying terms which are averaged to zero in the first approximation, but we can show that they will be at least one order of magnitude smaller than the effects predicted from the lowest order non-vanishing terms. Suppose that in the first approximation a is decreasing by a small fraction p per revolution; $a_1 \approx a_0\left[1 - \frac{p\theta}{2\pi}\right]$. Then the largest possible averaged-out term in a'/a

is that of lowest (zero) order in a, say $q \cos \Omega \theta$, where q is one of the coefficients assumed small of first order with respect to unity, and $\Omega > s^{-1}$. Substituting the first-order value of a, we have as a second approximation $da_2 = a_0\left[1 - \frac{p\theta}{2\pi}\right] q \cos \Omega\theta \, d\theta$, which integrates to $a_2 = a_0\left[1 + \frac{q}{\Omega}\sin \Omega\theta - \frac{pq}{2\pi}(\theta \sin \Omega\theta + 1 - \cos \Omega\theta)\right]$, and $a \approx a_1 + a_2$. Considering only the secular term, we see that it contributes a fractional error of $\frac{pq}{\Omega} <$ pqs per revolution, which is small of first order compared with the original rate of change, for the small s values of interest. Therefore it will only affect the higher order terms on which we have placed little reliance in this work. The next order perturbing term will contain another factor a which will make its effects small by another order of magnitude. We have shown here, however, that our method is really only an approximation of first order in the small quantities involved.

Let us now assume that none of the non-linear terms contain θ explicitly; we shall identify these as terms of type II. We find that our method will give only a frequency shift, and will supply no information about the higher harmonics of the free oscillation. This brings out again the fact that the method is essentially an approximation of the first order in the fundamental amplitude a; we have already discovered by the method used earlier that the amplitude of the nth harmonic is of order a^n. Therefore our method is not appropriate in problems where the higher harmonics are of considerable importance. Whether this is the case or not may be ascertained by this other method. If they are not small, the calculation of resonance effects will become inaccurate, for the following reason. To have a resonance of order r/s, we must have a term of type I in f of order s-1 in ρ and ρ', according to our formalism, to produce the requisite sth harmonic of the unperturbed free oscillation frequency. However, if the terms of type II are large enough to produce this harmonic with appreciable strength in absence of type I terms, they will resonate with type I terms of order lower than the (s-1)st in general, and our calculation will overlook these contributions. This can most easily be seen by regarding the type I terms as producing a small perturbation on the known solution of an equation having only type II terms. For example, a resonance of order 1/3 arises from a type I term in $\rho^2 \cos\theta$, which becomes $\overline{-\frac{3}{a}(a\cos\frac{\theta}{3})^2\cos\frac{\theta}{3}\cos\theta} = -\frac{3\bar{a}}{8}\overline{\cos(3\frac{\theta}{3}-\theta)} = -\frac{3\bar{a}}{8}$; however, if $\rho = a\cos\frac{\theta}{3} + ca^2\cos\frac{2\theta}{3}$, we may also get a contribution from a type I term in $\rho\cos\theta$, which becomes

$$\overline{-\frac{3}{a}\left(a\cos\frac{\theta}{3} + ca^2\cos\frac{2\theta}{3}\right)\cos\frac{\theta}{3}\cos\theta} = -\frac{3c\bar{a}}{4}\overline{\cos\left(3\frac{\theta}{3}-\theta\right)} = -\frac{3c\bar{a}}{4},$$

and if c is of order unity, this contribution is of equal importance. It would not have been included had we replaced ρ by $a\cos\frac{\theta}{3}$, as our general method requires. c will only be of order unity if the coefficients of the type II terms are not small.

We may sum up by saying that if we have only type II terms, an iteration method can be used for higher order approximations, while if we have only type I terms no such general method exists and each case must be specially investigated; if we have terms of both types, the situation is even more complicated. We have been satisfied with the lowest approximation since it has given us a qualitative description of the phenomena to be expected. The problem of developing general methods of higher approximation would seem to be sufficiently difficult to justify a thesis in mathematics, and has not been attempted here.

III

INTERACTION EFFECTS

The effects due to interactions among the many electrons simultaneously occupying the doughnut may be conveniently divided into two parts, the time-independent effects due to the mere presence of the electrons, and those effects produced by a change in their number with time. The discussion of the time-independent effects will serve to orient our thinking and to indicate the orders of magnitude which are important in the problem. It will also lead us naturally into the consideration of time-dependent effects which, as we shall see, turn out to be very important in understanding the injection process.

In this section we shall assume the magnetic field to be axially symmetric so as to simplify as much as possible the treatment of the interaction effects, which are very difficult to evaluate accurately even with this simplification. However, we shall not confine ourselves to a linearized treatment throughout, as some of the effects due to deviations of the magnetic field from the desired linear space dependence at the edges of the orbit region will be shown to provide an explanation of certain empirical facts which have not hitherto been satisfactorily accounted for.

1. Time-Independent Effects

Let us suppose that a large number of electrons have been successfully injected into an axially symmetric betatron field, and have in some manner avoided the gun, which has been removed, leaving them circulating without obstacles in their path. Let us further suppose that the magnetic field has been fixed in time, so as to produce no electric accelerating field, and that a perfect vacuum has been attained so that gas scattering can be neglected. This hypothetical situation should then be a stable one, in which all the electrons would continue to circulate indefinitely. How is the motion of an individual electron in this situation influenced by the others, and how will the electrons ultimately distribute themselves?

Any individual electron will of course be influenced by the electric and magnetic fields of the others, as well as by the external field. However, the electric and magnetic fields which determine the trajectories are determined by the distribution of current and charge, that is, by the trajectories. Thus we have a situation similar to that treated by the self-consistent field method in atomic structure problems. Here we find it necessary to introduce several simplifying assumptions in order to obtain a qualitative picture of what can be expected.

An estimate of this sort was first obtained by Kerst[1]. His object was to determine the maximum amount of charge which could be held within a doughnut of specified aperture by the magnetic focusing forces against the effects of space charge repulsion. We shall use our notation in describing his work. He assumed that the external magnetic field in the plane of symmetry followed the relation $B_z = B_0(1 - n\rho)$, and neglected terms depending on higher powers of ρ. He also used what we shall refer to as the infinite straight wire approximation. This consists of assuming that the electrostatic field of the electrons can be calculated as a function of distance from the center of the beam by regarding the beam as an infinite circular cylinder of radius $R\rho_b$ whose total charge per unit length is the same as that in the real toroidal beam, and whose charge distribution is uniform throughout the cylinder. The center of the beam was assumed to be at $r = R$, the equilibrium orbit. ρ_b is called the relative radial aperture.

The electric field inside an infinite uniformly charged cylinder is proportional to the distance from its center, while the field outside varies inversely with distance, as is known from elementary electrostatics. If the total charge in the beam is Q, the charge per unit length of our cylinder is $Q/(2\pi R)$. The field inside is then given by $Q\rho/(4\pi^2\epsilon_o R^2 \rho_b^2)$, while that outside is $Q/(4\pi^2\epsilon_o R^2\rho)$, in M.K.S. rational units; $\epsilon_o = 8.85 \times 10^{-12}$ farad per meter. The radial magnetic focusing force on an electron can easily be shown to be given by $mV^2(1 - n)\rho/R$, in the linear approximation, while the vertical focusing force is $mV^2 n \zeta/R$. Since the radial focusing force is weaker than the vertical force for $n > \frac{1}{2}$, Kerst equated the space charge repulsive force to the magnetic restoring force on an electron at the outer edge of the beam in the plane of symmetry, to find out how much charge was required to neutralize the focusing force in this plane. By carrying this out, we obtain

$$Q = -4\pi^2\epsilon_o mRV^2(1-n)\rho_b^2/e\ .$$

Since $mV = B_o eR$, this may be written

$$Q = -4\pi^2\epsilon_o B_o R^2 V(1-n)\rho_b^2$$

The corresponding circulating current $I = -QV/(2\pi R)$, if we define it to be positive, and is given by

$$I = 2\pi(1-n)\frac{B_o R}{\mu_o}\frac{V^2}{c^2}\rho_b^2\ ,$$

since $\epsilon_o\mu_o = c^{-2}$; $\mu_o = 4\ \ \times 10^{-7}$ henry per meter. Kerst also pointed out that the magnetic field due to the circulating current produces a self-focusing action on the beam which tends to counteract the electrostatic defocusing. This force is $(V/c)^2$ times as great as the electrostatic force; as he was interested in injection energies in the non-relativistic region, he neglected the magnetic self-focusing force. His further analysis involved reference to the adiabatic damping mechanism which was believed at that time to be responsible for the success of the injection process. Since this no longer seems likely, we shall not include these results here. His statement that the space charge force will be less than the focusing force everywhere inside the beam if the two are equal and opposite at the edge is incorrect under the assumptions made; since both are directly proportional to the distance from the center, they will be equal in magnitude inside, in the plane of symmetry, if they balance at the edge. Outside the beam, of course, the focusing force continues to increase while the space charge force falls off.

A considerably more refined treatment of this problem was given by Blewett[15], who sought to find that electron distribution for which both radial and vertical forces on an electron anywhere within the beam would vanish. This situation would be expected to occur in practice only after the adiabatic damping had acted for a long enough time to compress the beam, both radially and vertically, into the smallest cross-sectional area consistent with the amount of charge present. Blewett's calculations were intended to apply to this case, and were made in connection with a discussion of radiation loss, which is appreciable only at energies above about 100 Mev., where the damping is essentially complete. He used his results to determine the actual area and shape of the beam for a given total charge or current, but also noticed that his result could be used, like that of Kerst above, to obtain an upper limit on the amount of charge which could be held in the doughnut at injection time.

Blewett carried out his calculations for a particular value of n, namely, n = 3/4. We shall outline them here, having generalized them to arbitrary n and expressed them in our units. We first show that the magnetic self-focusing forces are less than the electrostatic forces by a factor $(V/c)^2$. Applying Maxwell's equations in the vicinity of the equilibrium orbit by regarding ρ, ζ, and θ as cartesian coordinates, we have

$$\frac{1}{R}\left(\frac{\partial B_z'}{\partial \rho} - \frac{\partial B_r'}{\partial \zeta}\right) = \mu_0 \sigma V \, ; \quad \frac{\partial B_r'}{\partial \rho} + \frac{\partial B_z'}{\partial \zeta} = 0 \, ;$$

$$\frac{1}{R}\left(\frac{\partial \mathcal{E}_r'}{\partial \rho} + \frac{\partial \mathcal{E}_z'}{\partial \zeta}\right) = \frac{\sigma}{\epsilon_0} \, ; \quad \frac{\partial \mathcal{E}_r'}{\partial \zeta} - \frac{\partial \mathcal{E}_z'}{\partial \rho} = 0 .$$

Here we have chosen to represent the charge per unit volume in coulombs per cubic meter by σ, having already denoted our dimensionless radial coordinate by ρ. The field strengths with primes are those due to the charge and current in the beam. These equations imply that

$$B_r' = -\mu_0 \epsilon_0 V \mathcal{E}_z' \, , \quad B_z' = \mu_0 \epsilon_0 V \mathcal{E}_r' .$$

The components of force are

$$F_r = e(\mathcal{E}_r' - VB_z') \, , \quad F_z = e(\mathcal{E}_z' \quad VB_r') \, ;$$

in view of the relations above, these become

$$F_r = e\mathcal{E}_r'(1 - V^2/c^2) \, ; \quad F_z = e\mathcal{E}_z'(1 - V^2/c^2).$$

We now assume a linear space dependence of the magnetic field as above and equate the forces;

$$\frac{mV^2}{R}(1-n)\rho = -e\mathcal{E}_r'(1-V^2/c^2) \, ;$$

$$\frac{mV^2}{R} n \zeta = -e\mathcal{E}_z'(1-V^2/c^2) .$$

The electrostatic fields inside the beam are therefore given by

$$\mathcal{E}_r' = -\frac{mV^2}{eR}\frac{(1-n)\rho}{(1-V^2/c^2)} = -B_oV\frac{(1-n)\rho}{(1-V^2/c^2)} \;;$$

$$\mathcal{E}_z' = -\frac{mV^2}{eR}\frac{n\varsigma}{(1-V^2/c^2)} = -B_oV\frac{n\varsigma}{(1-V^2/c^2)} \;.$$

By the same simplifications Poisson's equation in the vicinity of the equilibrium orbit becomes

$$\frac{1}{R}\left(\frac{\partial \mathcal{E}_r'}{\partial \rho} + \frac{\partial \mathcal{E}_z'}{\partial \varsigma}\right) = \frac{\sigma}{\epsilon_o} = -\frac{B_oV}{R}\frac{1}{(1-V^2/c^2)} \;.$$

The application of Maxwell's equations and of Poisson's equation in this way is equivalent to neglecting the curvature of the beam, and constitutes a generalization of the infinite straight wire approximation to wires of elliptical cross-section. The charge density turns out to be uniform over the beam, with the value

$$\sigma = -\frac{\epsilon_o B_o V}{R(1-V^2/c^2)} \;.$$

The boundary of the beam will be an equipotential, and therefore has the equation

$$(1-n)\rho^2 + n\varsigma^2 = \text{constant.}$$

The beam will have an elliptical cross-section, the ratio of vertical to horizontal axes being $[(1-n)/n]^{\frac{1}{2}}$. This result is pleasing, since circular beams would not fit into most doughnuts, whose vertical clearance is less than the radial clearance by about this factor.

The cross-sectional area of the beam is $\pi[(1-n)/n]^{\frac{1}{2}}\rho_b^2$, where ρ_b is the relative radial aperture defined earlier. The total charge will be given by

$$Q = \sigma(2\pi R)\pi R^2\left(\frac{1-n}{n}\right)^{\frac{1}{2}}\rho_b^2 = -2\pi^2\epsilon_o B_o R^2 V\left(\frac{1-n}{n}\right)^{\frac{1}{2}}(1-V^2/c^2)^{-1}\rho_b^2 \;.$$

The corresponding circulating current, defined as a positive quantity, is then

$$I = -\frac{QV}{2\pi R} = \pi \frac{B_o R}{\mu_o}\left(\frac{1-n}{n}\right)^{1/2} \frac{V^2/c^2}{1-V^2/c^2}\,\rho_b^2 \,.$$

It is possible to express this maximum circulating current as a function of the injection voltage E, the relative radial aperture ρ_b, and the magnetic fall-off parameter n. By a straightforward use of the usual relativistic relations between mass, velocity, and energy, we obtain

$$I = \frac{\pi m_o c}{\mu_o e}\left[\frac{E}{E_o}\left(2 + \frac{E}{E_o}\right)\right]^{3/2}\left(\frac{1-n}{n}\right)^{1/2} \rho_b^2 \,,$$

where $E_o = m_o c^2 = 511$ Kev. Relative apertures range from 0.07 for large machines(3) to 0.22 for small ones(39). For n = 3/4, we obtain the following upper limits on the circulating current;

E (Kev.)	I (Amp.)	
	$\rho_b = 0.08$	$\rho_b = 0.20$
1	0.0038	0.024
10	0.12	0.77
30	0.66	4.1
50	1.5	9.2
70	2.4	15.
500	78.	490.
1000	340.	2100.

Of greater interest, perhaps, is the estimated upper limit on the total charge Q, which by the relativistic relations can be expressed as

$$Q = -\frac{2\pi^2 m_0}{\mu_0 e}\frac{E}{E_0}\left(1+\frac{E}{E_0}\right)\left(2+\frac{E}{E_0}\right)\left(\frac{1-n}{n}\right)^{1/2} R\rho_b^2 .$$

The following table shows that this upper limit is the same, for a given injection energy, within a factor of three for a wide variety of existing accelerators.

MACHINE	Final Energy (Mev.)	R (meters)	n	ρ_b	Q(coul.) x 10^8 for E = 40 Kev.
Kerst's First Betatron(1)	2.2	0.075	2/3	0.28	6.5
Kerst's Second Betatron(2)	20	0.19	3/4	0.13	2.9
Big. G.E. Betatron(3)	100	0.84	3/4	0.07	3.7
British Synchrotron(6)	30	0.10	7/10	0.205	4.4
G.E. Synchrotron(7)	70	0.293	3/4	0.10	2.6
Model(28) for 300 Mev.	70	0.26	1/2	0.07	2.0
Illinois Betatron					
G.E.Biased Betatron(39)	50	0.292	3/4	0.096	2.4
G.E.Industrial Betatron(39)	10	0.133	3/4	0.22	5.8

These accelerators could never hold more than about 10^{11} electrons of 40 Kev. energy, even under the most ideal conditions.

It is hard to determine accurately the actual number of electrons which are accelerated to high energies; the only published measurements* are those of Bess and Hanson(40), made on the 22 Mev. betatron(2) at

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* Estimates(41) based on measurement of thermal radiation from the beam of the G.E. synchrotron indicated that about 10^8 electrons were accelerated per cycle in this machine.

Illinois. They found that, under favorable conditions, as many as 5.6×10^9 electrons struck the target per cycle of operation, with an error of less than ten per cent. The upper limit for this machine at its injection energy[2] of 20 Kev. is 1.37×10^{-8} coulombs, or 8.6×10^{10} electrons, according to the formula above; thus this accelerator seems to be operating at about seven per cent of its ideal capacity at this injection energy. It would probably be very hard to increase its performance greatly by any means whatever, if the injection energy remains unchanged.

It is easy to see that a space-charge saturated beam with the elliptical cross-section and uniform density derived above is stable. If the charge density is infinitesimally increased in one part of the beam and correspondingly decreased in another so as to keep the total charge constant, the forces thereby created are always directed so as to restore the uniform distribution of charge. The effect of this uniform distribution on the potential $U(r,z)$ in which an individual particle moves is to make it exactly constant over the entire area of the beam, by subtracting off an elliptic paraboloid potential identical to the magnetic restoring potential in this region. This makes the potential well flat-bottomed, and all the electrons are assumed to have been brought to rest in it, so as to lie evenly distributed on the bottom.

The details of the behavior of such a beam might form an interesting subject for further study. The beam will have a variety of normal modes of oscillation, each with a characteristic frequency and a particular effect on the beam's shape. We shall illustrate this by demonstrating the existence of a normal mode in which the density remains uniform in space but oscillates sinusoidally in time with a frequency equal to that of rotation around the doughnut. This mode can be discussed easily because it involves no analysis of the electrostatic effects of a non-uniform charge density. If the charge density is uniform, the radial and vertical forces on an electron are given by

$$F_r = \frac{mV^2}{R}\left(\frac{\sigma}{\sigma_0} - 1\right)(1-n)\,\rho, \quad F_z = \frac{mV^2}{R}\left(\frac{\sigma}{\sigma_0} - 1\right)n\zeta,$$

where σ_0 is the charge density defined above, for which both forces vanish. The radial and vertical motion of the electrons can be treated as that of a two-dimensional non-viscous fluid with the equation of state $p = 0$, subject to the body forces given above. The hydrodynamical equation of motion, in Euler's form, can be reduced to

$$\frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_n \cdot \vec{\nabla})\vec{v}_n = \vec{F}/m,$$

where $\vec{v}_n = (R\dot{\rho}, R\dot{\zeta})$ is the velocity in a plane through the axis of symmetry of the system, and the components of $\vec{F}$ are given above. The charge density and velocity are also subject to the equation of continuity, which can be written as

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla}\cdot(\sigma \vec{v}_n) = 0.$$

We assume that $\sigma = \sigma_0(1 + \epsilon \sin\omega t)$, where ϵ is the infinitesimal magnitude of the relative density change, and ω its angular frequency. Substituting in the equation of continuity, and neglecting terms in ϵ^2, we have

$$\sigma_0\,\omega\epsilon\cos\omega t + \sigma_0\left(\frac{\partial\dot{\rho}}{\partial\rho} + \frac{\partial\dot{\zeta}}{\partial\zeta}\right) = 0.$$

Hence $\dot{\rho} = -\omega\epsilon\cos\omega t\, f_1(\rho,\zeta)$, $\dot{\zeta} = -\omega\epsilon\cos\omega t\, f_2(\rho,\zeta)$, and

$$\partial f_1/\partial\rho + \partial f_2/\partial\zeta = 1.$$

Substituting in the equations of motion, and again neglecting terms of order ϵ^2, we obtain

$$\omega^2(\epsilon \sin \omega t) f_1 = (V^2/R^2)(\epsilon \sin \omega t)(1-n)\rho ;$$

$$\omega^2(\epsilon \sin \omega t) f_2 = (V^2/R^2)(\epsilon \sin \omega t) n \zeta .$$

We find that the three equations above can be satisfied if and only if

$$f_1 = (1-n)\rho , \quad f_2 = n\zeta , \quad \text{and } \omega^2 = V^2/R^2 ,$$

thus proving the statement made above. To find the direction of the vector $\vec{v}_n$ for any electron, draw through it an ellipse concentric with the elliptical boundary of the beam and having the same eccentricity. The normal to this ellipse at the point in question has the direction of $\vec{v}_n$. This can be shown as follows. The slope of $\vec{v}_n$ in the $\rho - \zeta$ plane is $n\zeta/(1-n)\rho$. The equation of the ellipse is $(1-n)\rho^2 + n\zeta^2 =$ constant; its slope is $d\zeta/d\rho = -(1-n)\rho/n\zeta$, which is the negative reciprocal of the slope of $\vec{v}_n$.

To consider other modes, a more general expression for the force, applicable to non-uniform charge distributions, is needed. To study larger oscillations, terms in ϵ^2 must be included. We have not investigated this problem further, as it has little bearing on the injection problem.

We have found it useful to develop an expression for the motion of an electron in an axially symmetric magnetic field, which includes the forces due to other electrons, and is more general than those of Kerst and Blewett. It assumes only that the charge distribution is also axially symmetric. The Hamiltonian function for the electron is

$$\mathcal{H} = \frac{1}{2m}\left[p_r^2 + p_z^2 + (p_\theta r^{-1} + eA_\theta)^2 \right] + (-e)\mathcal{V}(1 - v^2/c^2),$$

where A_θ is the vector potential of the external magnetic field, and $\mathcal{V}$ is the electrostatic scalar potential of the space charge. The factor $(1 - V^2/c^2)$ accounts for the magnetic self-focusing action of the beam, as shown above. If $\mathcal{V}$ and A_θ are independent of θ, p_θ is still a constant of the motion. By a repetition of the operations performed earlier, it is easy to show that the radial and vertical motion has a potential function, such that

$$d(m\dot{r})/dt = -\partial U(r,z)/\partial r \; ;$$

$$d(m\dot{z})/dt = -\partial U(r,z)/\partial z \; ;$$

$$U(r,z) = U_0(r,z) - e\mathcal{V}(r,z)(1 - V^2/c^2),$$

and U_0 is the potential function derived earlier for the motion of a single electron alone. We shall make use of this result later on.

The greatest difficulty in using this equation is the determination of the actual charge distribution which produces the potential $\mathcal{V}$. If the number of electrons is less than that required to fill the beam, the potential well will no longer be flat-bottomed, and the distribution of electrons will depend on the initial conditions. If there are so few of them that their interactions may be neglected, and if we assume a linear external field variation as we did above, each electron will execute a Lissajous figure, like that shown in Fig. 4a, in the potential U_0. Its contribution to the charge density $\sigma(\rho, \zeta)$ in an element $d\rho\, d\zeta$ will be inversely proportional to its velocity $v_n(\rho, \zeta)$ in the $\rho - \zeta$ plane and directly proportional to the area-element $d\rho\, d\zeta$ since on the average it will cross all elements of equal area with equal frequency. (This is true because a non-commensurable Lissajous figure will cover its rectangle with "uniform density".) If we assume that all electrons have as their instantaneous circle the equilibrium orbit of radius R, the resulting charge density will be symmetric in ρ and in ζ, and can be represented as follows:

$$\sigma(\rho,\zeta) \propto \int_{\rho}^{\rho_b}\int_{\zeta}^{\zeta_b} \frac{N(\rho_m, \zeta_m)\, d\rho_m\, d\zeta_m}{\left[(1-n)(\rho_m^2-\rho^2)+n(\zeta_m^2-\zeta^2)\right]^{1/2}} ,$$

where $N(\rho_m, \zeta_m)d\rho_m d\zeta_m$ is the number of electrons injected in such a way that their radial and vertical amplitudes are between ρ_m and $\rho_m + d\rho_m$, and between ζ_m and $\zeta_m + d\zeta_m$, respectively. If some of the electrons are oscillating about other instantaneous circles, a more general expression is required. These integrals are so complicated that the results of an attempt to evaluate them for specified injection conditions would not be worth the labor involved. In particular the simple and logical assumption that N is constant over a rectangular region of the $\rho - \zeta$ plane and zero elsewhere makes the second integration quite involved.

It will be noticed that the integrand contains integrable singularities at the four corners of the Lissajous rectangle. These will be smoothed somewhat by any reasonable assumption about N. If injection is from outside the equilibrium orbit in the plane of symmetry, the two singularities on the inside will be further smoothed if there is a distribution over various instantaneous circles. However, a qualitative consideration of these singularities, supplemented by some rough numerical calculations, has led us to believe that the electron density may tend to be somewhat higher near the gun than elsewhere, under normal injection conditions. The density over the remainder of the beam can probably be taken as uniform to first approximation.

If the injection conditions are such as to produce a uniform distribution, this analysis will remain valid as more electrons are injected, since the electrostatic potential function will be an elliptic paraboloid like that of the magnetic restoring force, but of opposite sign, and smaller. The net potential is their algebraic sum, and will also be paraboloidal, so that the radial and vertical oscillations will still be sinusoidal, giving rise to the Lissajous motion assumed above, but with different periods. If the injection conditions are continually adjusted for uniform distribution as the charge gradually increases, the doughtnut will fill up uniformly to the limiting density derived earlier.

If the injection conditions are such as to produce a non-uniform charge distribution, as will generally be the case, the net potential will gradually become distorted as the total charge increases, thereby altering the distribution which was assumed in calculating it. As stated above, it would be very difficult to determine the self-consistent distribution of charge for given initial conditions. From qualitative considerations, one can see that as the total charge approaches the upper limit, electrons will be accepted only into those regions where the charge density is lagging behind, since after the slopes of the two potential functions become equal and opposite in any region, the net restoring force vanishes there. If the density is lower near the center at this stage, it may be difficult to raise it by injecting from a gun at the outside.

In the linear approximation, the net potential function for uniform charge distribution is of the form

$$(1 - \sigma/\sigma_0)\left[(1-n)\rho^2 + n\zeta^2\right] ,$$

where σ_0 is the upper limit on charge density derived earlier. From this we see that the frequencies of both radial and vertical oscillations are reduced by the presence of other electrons, in the ratio $(1-\sigma/\sigma_0)^{\frac{1}{2}}$. We mentioned above that the charge density at high energies has been observed to be as much as seven per cent of the theoretical upper limit; it is believed to be as much as ten times as great as this during injection[14]. Therefore this frequency shift can be very large, and provides a convincing argument against any attempt to obtain very large circulating currents by relying on the resonant phenomena discussed in Section II. It also affords a satisfying explanation of the failure to observe poor performance in accelerators having values of n which would produce a resonance if this effect were absent. A resonance would tend to reduce the charge density by spreading the beam out, and would thereby shift the frequency of oscillation out of the resonant range.

We shall close this discussion of time-independent effects by pointing out an advantage of using a large injection energy. From the expression derived earlier for Q, we found the maximum charge to be proportional to

$$\frac{E}{E_0}\left(1+\frac{E}{E_0}\right)\left(2+\frac{E}{E_0}\right) = f(E)$$

for a given accelerator, where $E_0 = 511$ Kev. If the mechanism which allows some electrons to miss the gun saves an equal fraction of them at any injection energy, then the number of electrons successfully accelerated to high energy should vary directly with the injection energy if it is in the non-relativistic range, provided a plentiful supply of electrons is available. It is observed[39] that the output varies as the 1.25 power of the injection energy under these conditions, which indicates that the efficiency of the mechanism is also enhanced by increasing the component of the injection velocity, which determines the instantaneous circle. Since the electrons injected at a given voltage have varying azimuthal velocity components, they will not all oscillate about the same instantaneous circle. Also, the radius of an instantaneous circle corresponding to a particular azimuthal velocity is a rapidly varying function of time during the injection period. Thus the effectiveness of injection will depend quite sensitively on the degree to which the shape of the voltage pulse to the gun, as a function of time, matches the acceptance band of energies shown in Fig. 2b.

Schwartz[42] has made an analysis of one aspect of this matching. He assumed that the injection energy is constant, corresponding to a square-wave voltage pulse, and that the gun was located directly above the equilibrium orbit. He considered only motion in the plane of symmetry and neglected interaction effects and vertical oscillations. Under these assumptions he calculated, tabulated, and plotted the time of acceptance, and the fraction of injected electrons which are accepted as a function of time during the acceptance interval, for different values of n and E, and for both biased and ordinary betatrons. He concluded that injection would be most efficient for low n, high E, and a biased betatron. These conclusions are so obvious from qualitative reasoning that the calculations seem somewhat unnecessary. As n is decreased, the difference between the radii of the instantaneous circle and the equilibrium orbit for a given velocity defect is also decreased; we showed earlier that $r_i - R = (v - V)/[(1 - n)v]$ to first order. As E is increased, the acceptance interval also increases, as is obvious from an inspection of Fig. 2b. The field in a biased betatron is increasing more slowly at injection than that in an ordinary betatron, as is made clear in Fig. 2a, and therefore the acceptance interval will be correspondingly wider. The advantage to be gained by a properly shaped voltage pulse seems to be much greater than any gain one might make by changing n or by changing to a biased betatron. The gain of this sort resulting from a large increase in injection energy may perhaps be offset by the greater technical difficulties of properly shaping the pulse. Although the appropriate injection energy varies parabolically with time for non-relativistic energies, it becomes a linear function in the relativistic range, so that the possibility of further increase in injection interval disappears.

If there is no gun-avoiding mechanism, the lifetime of a successfully injected electron will depend on the number of oscillations it makes before hitting the gun. In the example illustrated in Fig. 4a, this requires about four radial oscillations, corresponding to seven revolutions. One can only estimate the mean number of revolutions M for an average electron. It depends principally on the phase of the radial oscillation at injection and on its frequency, which is a function of n, the space-change density, and the amplitude of oscillation. (The dependence on amplitude is illustrated in Fig. 6.) It also depends on the vertical oscillation frequency, if there is the possibility of going over or under the gun. In the linear approximation, neglecting space charge effects, M will be 2 if $n = 3/4$, and if the vertical oscillations are unable to help in increasing it. For any other n, M will be considerably larger. We have shown that the effects of too rapid field fall-off at the edge and of space charge both slow down the radial oscillations, thereby tending to further increase M in general.

Let us assume that the gun successfully injects a constant current i into acceptable orbits during the injection interval. The circulating current I will then rise to a value of about Mi in the time required for M revolutions, on these simple assumptions. This is because each injected electron will be counted M times before it is lost. The current will then remain constant until injection stops, after which it will drop to zero after another M revolutions if no gun-avoiding effect acts. It is known that, in any case, there is present for a short time a considerably larger circulating current than that finally trapped in stable orbits and accelerated to high energy. We thus have to consider the effects of a large rise of circulating current and a subsequent decrease of the same order of magnitude. Now it may be that a rising current produces effects, not as yet considered, which act to raise the M of electrons already present to a larger value or even to very great values, effectively saving them indefinitely. If the reverse effects of a falling current fail to completely counteract this advantage, these effects may then provide an explanation of the success of the injection process. On the other hand, the effects of a rising current may be unfavorable, but not sufficiently so to unduly decrease M. In this case, the opposite effects of the subsequent decrease in current may provide just the saving influence required to trap part of the beam in stable orbits.

These effects were first considered by Kerst[14]. It is qualitatively clear that the circulating current loop has self-inductance, and that an increase in current must produce a back electromotive force tending to slow down the electrons, while a decrease will produce an accelerating force. It is also clear that changing the number of electrons in the beam will change the magnetic and electrostatic actions of the beam on itself. Kerst discussed the change in orbit radius in terms of the electrostatic and magnetic energy associated with the beam, and also mentioned the change in electrostatic self-repulsion, without indicating how to calculate it. In the remainder of this paragraph we shall briefly summarize his conclusions. He asserts that the electromagnetic and electrostatic energy of the beam is rising during the early stage of injection. It is formed at the expense of the kinetic energy of the electrons, thus slowing them down so that they spiral inwards, thereby avoiding the gun if it is outside the orbital radius. However, if the gun is inside, he invokes the reverse effect, starting with maximum circulating current. As the current decreases, its electromagnetic energy is returned to the beam, speeding up the electrons so that they spiral outwards to avoid the gun. For either position of the gun, the decreasing space charge repulsion associated with a falling current is stated to provide damping of the oscillations.

These considerations led Kerst to propose the "orbit contractor" as a device for shrinking the orbits quickly inward so as to avoid an external gun. This device consists of a single turn of wire above and one below the orbit, outside the doughnut, through which a rising pulse of current is passed during injection. The effect of the magnetic field of these currents is similar to that of the beam on itself, and has been found to be beneficial in raising the output of badly performing accelerators, although it has proved to be of lesser value in increasing the output of well-adjusted machines. The increase in yield resulting from use of this arrangement depends strongly on the injector current and on the condition of the accelerator. Adams[43] states that "the circuit should certainly be disconnected when attempting to diagnose the ills of a betatron".

There are three objections which may be raised against Kerst's treatment of these effects. First, he computes the change in electromagnetic energy in terms of the inductance of a single turn of wire placed at the electron orbit, but arrives at this value by actually inserting a wire and measuring its inductance. The result of such a measurement will no doubt be very sensitive to the frequency at which it is made, since the degree of penetration of magnetic flux through the conducting coating of the doughnut and into the iron pole pieces will be very different at different frequencies. The relevant periods are comparable to the time of current rise or fall, perhaps half a microsecond or so, while most inductance bridges operate at a few hundred or thousand cycles per second. Therefore there may be some doubt as to the correctness of the numerical values he uses.

Second, it would be more satisfactory to exhibit in detail the forces acting on the electrons; the energy arguments as stated by Kerst are difficult to visualize in terms of these forces.

Third, and most serious, is the incompleteness of an explanation which can be made to provide effects of either sign at will. For injection from outside, the reverse effect of spiralling outward during a reduction in circulating current is not considered, while for inside injection, which is empirically known to work, the initial effect of spiralling in due to a rising current is ignored. Finally, the undamping effect of increasing space change repulsion at the start is not mentioned, although the reverse effect is stated to be of considerable importance.

We shall now describe some investigations undertaken to elucidate some of these points. They are rather idealized and incomplete but serve to indicate orders of magnitude which are important. We shall find it convenient to make the following assumptions in this rough analysis. First, we shall consider only radial displacements in the plane $z = 0$. Second, we shall assume that the circulating charge occupies a toroidal region of circular cross-section with uniform density; we shall take the radius of this cross-section to be small compared to the orbit radius. Finally, we will neglect the influence of the conducting doughnut coating and of the iron pole pieces on the distribution of flux due to the circulating current. Some of these assumptions are not justifiable, but the resulting calculations are intended merely to be illustrative and not quantitatively accurate. Certain other assumptions are discussed as they enter.

We start with the expression, developed earlier in this section, for the motion of an electron in an axially symmetric magnetic field in a region containing an axially symmetric distribution of similarly moving charges. For motion in the plane z = 0, we have

$$\frac{d}{dt}(m\dot{r}) = -\frac{dU}{dr} \;;$$

$$\tfrac{1}{2} m\dot{r}^2 + U(r) = \text{constant};$$

$$U(r) = \frac{mV^2}{2R^2r^2}\left[\frac{r_I R v}{V}\cos\phi_I + \int_{r_I}^{r} x\,\beta_z(x)\,dx\right]^2 - e\left(1 - V^2/c^2\right)\mathcal{V}(r) \;;$$

$$\mathcal{V}(r) = -\int^{r} \mathcal{E}_r(r)\,dr \;.$$

Here $\mathcal{E}_r$ is the radial electrostatic field of the circulating electrons, while r_I and ϕ_I are values at injection or at some other arbitrary point. We shall choose this point to be at an extreme value r_m (where m may stand for either maximum or minimum). At such a point $\phi = 0$ and $\dot{r} = 0$, so that the constant may be evaluated to give

$$\dot{r}^2 = v^2\left[1 - \frac{2e(1 - V^2/c^2)}{mv^2}\int_{r_m}^{r}\mathcal{E}_r(r)\,dr - \left\{\frac{r_m}{r} + \frac{V}{vRr}\int_{r_m}^{r} x\,\beta_z(x)\,dx\right\}^2\right].$$

If in this expression we substitute the new variable $\rho = (r-R)/R$, we obtain

$$\dot{\rho}^2 = \left(\frac{v}{R}\right)^2\left[1 - \frac{2e(1-V^2/c^2)}{mv^2}\int_{\rho_m}^{\rho}\mathcal{E}_r(y)\,R\,dy - \left\{\frac{1+\rho_m}{1+\rho} + \frac{(V/v)}{1+\rho}\int_{\rho_m}^{\rho}(1+y)\,\beta_z(y)\,dy\right\}^2\right];$$

here β_z and $\mathcal{E}_r$ are regarded as functions of ρ, represented by y as a new variable of integration. Assuming the charge per unit volume σ to be a function of ρ alone, and using the infinite straight wire approximation, we have

$$\mathcal{E}_r(y) = \frac{1}{\epsilon_0 R y}\int_0^y R\eta\,\sigma(\eta)\,R\,d\eta,$$

where η is a new variable of integration. Finally

$$\dot{\rho}^2 = \left(\frac{v}{R}\right)^2\left[1 - \frac{2e(1-V^2/c^2)R^2}{\epsilon_0 m v^2}\int_{\rho_m}^{\rho}\frac{dy}{y}\int_0^y \eta\,\sigma(\eta)\,d\eta - \left\{\frac{1+\rho_m}{1+\rho} + \frac{V/v}{1+\rho}\int_{\rho_m}^{\rho}(1+y)\,\beta_z(y)\,dy\right\}^2\right].$$

We may use this expression to determine the change in the maximum or minimum relative radial excursion ρ_m resulting from changes in the parameters which determine it. A change in charge density σ will effect ρ_m directly through its explicit appearance in the above expression, and also indirectly through electromagnetic induction, the effect of which will be to change the particle velocity v in this expression. These two changes produce different effects on the radial oscillation. The direct effect of changing σ is to alter the effective "spring constant" of the radial oscillation, while a change of v moves the whole region of oscillation radially inward or outward, by translating the position of the entire potential well (with consequent changes in its shape if non-linear effects are considered).

Ideally, one would compute the direct effect of a change in σ by calculating the action, given by $A = \oint |\dot{\rho}| d\rho = A(\rho_m, \sigma)$, and set its variation equal to zero for adiabatic changes, thus obtaining the connection between $\Delta\sigma$ and $\Delta\rho_m$. Actually, this is not feasible analytically unless σ is constant and β_z is a linear function of ρ. Since we wish to show qualitatively the effect of a nonlinearity in β_z, we shall compute the result of suddenly changing this "spring constant" instead of doing so adiabatically. If the motion were simple harmonic, this error would be properly corrected for by reducing the resulting relative amplitude change by one-half. We shall assume that the same factor is approximately applicable here.

The same remarks apply in some measure to a variation in v, but, due to its different effect on the motion, the factor one-half will not be needed in this case. If the potential well of a simple harmonic oscillator be suddenly translated a small amount at the moment the oscillating particle is at the bottom of the well, the resulting effect on the extremes of oscillation is the same as if the well had been adiabatically translated by the same amount. As we shall perform our sudden change at this point, no correction factor is required.

Suppose that, as an electron passes through $\rho = 0$, the amount of space charge is suddenly changed from $\sigma(\eta)$ to $\sigma(\eta)(1 + \Delta I/I)$, where I denotes the amount of circulating current, while v is suddenly changed from the value V to $V + \Delta v$. The instantaneous velocity $\dot{\rho}$ will remain the same, but the maximum (or minimum) radial excursion will change from ρ_m to $\rho_m + \Delta\rho_m$. It is established in the appendix that this change is given by

$$\left[\frac{2e(1-v^2/c^2)R^2}{\epsilon_0 m v^2}\int_0^{\rho_m}\eta\sigma(\eta)\,d\eta - 2\rho_m\left\{1-(1+\rho_m)\beta_2(\rho_m)\right\}\right]\frac{\Delta\rho_m}{\rho_m} =$$

$$-2\left[1-\left\{1+\rho_m-\int_0^{\rho_m}(1+y)\beta_2(y)\,dy\right\}^2-\int_0^{\rho_m}(1+y)\beta_2(y)\,dy\right]\frac{\Delta v}{v}$$

$$-\left[\frac{2e(1-v^2/c^2)R^2}{\epsilon_0 m v^2}\int_0^{\rho_m}\frac{dy}{y}\int_0^{y}\eta\sigma(\eta)\,d\eta\right]\frac{\Delta I}{I} \quad .$$

For the circular charge distribution assumed,

$$\int_0^y \eta\sigma(\eta)\,d\eta = \frac{1}{2\pi R^2}\,\frac{Q(y)}{2\pi R} \quad ,$$

where Q(y) is the charge inside a toroidal region of cross-sectional radius Ry. Since this charge is moving with velocity v, we have

$$I(y) = -\frac{v\,Q(y)}{2\pi R} \quad ,$$

where I is defined to be a positive current, and

$$\int_0^y \eta\sigma(\eta)\,d\eta = -\frac{I(y)}{2\pi R^2 v} \quad .$$

If σ is independent of η and has the constant value σ_0 out to $\rho = \rho_m$, as we shall assume to simplify the computation, we have

$$-\sigma_0\int_0^y \eta\,d\eta = \frac{I(y)}{2\pi R^2 v} = -\frac{\sigma_0 y^2}{2} \quad ,$$

and

$$-\sigma_0\int_0^{\rho_m}\frac{dy}{y}\int_0^y \eta\,d\eta = -\frac{\sigma_0}{4}\rho_m^2 = \frac{I(\rho_m)}{4\pi R^2 v} \quad .$$

Thus the coefficient of $\Delta I/I$ becomes

$$\frac{e(1-v^2/c^2)\, I(\rho_m)}{2\pi\epsilon_0 m v^3} = \frac{\mu_0}{2\pi}\frac{e}{mv} I(\rho_m)\frac{1-v^2/c^2}{v^2/c^2} = \frac{\mu_0 I(\rho_m)}{2\pi B_0 R}\frac{1-v^2/c^2}{v^2/c^2} ,$$

since $\epsilon_0^{-1} = \mu_0 c^2$ and $mv = B_0 eR$. We define

$$\rho_c^2 \equiv \frac{\mu_0 I(\rho_m)}{\pi B_0 R}\frac{1-v^2/c^2}{v^2/c^2} ;$$

the implication of this definition will be examined later.

It can easily be shown that, to first order in ρ, the coefficient of $\Delta v/v$ is equal to $2\rho_m$, independent of β_z. We now define $\beta_z(\rho_m) \equiv 1 - n_e\rho_m$; this determines n_e as the "effective n" at $\rho = \rho_m$, and allows us to consider non-linear functions $\beta_z(\rho)$. It should be noted that the value of n_e will in general be different for $\rho_m = \rho_{max}$ than for $\rho_m = \rho_{min}$. When all these substitutions have been introduced, and $\Delta I/I$ replaced by $\frac{1}{2}\Delta I/I$ to convert to the adiabatic result, we obtain

$$\frac{\Delta\rho_m}{\rho_m} = \frac{\frac{1}{4}\rho_c^2\frac{\Delta I}{I} + 2\rho_m\frac{\Delta v}{v}}{\left[2(1-n_e)\rho_m^2 - \rho_c^2\right]} .$$

To understand the meaning of the denominator of the right side, and the definition of ρ_c, we recall that, for a beam of circular cross-section (which implied $n = \frac{1}{2}$) we found that the maximum amount of current which could be held inside a radius $\rho = \rho_m$ was

$$I_{max} = \frac{\pi B_0 R}{\mu_0}\frac{v^2/c^2}{1-v^2/c^2}\rho_m^2 ;$$

hence $\rho_c^2/\rho_m^2 = I/I_{max}$, and ρ_c is the smallest relative radial aperture into which a current I could be compressed. For $n = \frac{1}{2}$, our denominator becomes $\rho_m^2(1 - \rho_c^2/\rho_m^2)$, which shows that this denominator is just proportional to the difference between the current actually present and that which could be present in the same doughnut. We may expect this result to hold approximately in our present more general case also, even with non-constant $\sigma(\rho)$ and non-linear $\beta_z(\rho)$. We shall denote this fractional filling factor by f, so that in general we may write the denominator as $2(1 - n_e)(1 - f)\rho_m^2$. We then have

$$\frac{\Delta\rho_m}{\rho_m} = \frac{\frac{1}{4}\rho_c^2\frac{\Delta I}{I} + 2\rho_m\frac{\Delta v}{v}}{2(1-n_e)(1-f)\rho_m^2} \quad ;$$

the first part of our analysis is now complete, and we turn to an approximate determination of $\Delta v/v$, the fractional change in particle velocity produced by electromagnetic induction when the amount of beam current is changed.

To compute the relation between velocity change and current change, we first consider two circular current elements of radii $R(1 + \rho)$ and $R(1 + \rho + x)$, coaxial and lying in parallel planes separated by a distance Ry. We assume ρ, x, and y to be small compared to unity. The flux through one due to a negative current of magnitude I in the other is given by $\Phi = -IM$, where M is their mutual inductance. This can be worked out approximately by use of the formulas given in reference 44, chapters 7 and 8. In M.K.S. rational units, we have

$$M \simeq \frac{2\mu_0}{k} R\left[(1 - \tfrac{1}{2}k^2)K(k) - E(k)\right] \quad ,$$

where K and E are complete elliptic integrals, and

$$k^2 \simeq 1 - \tfrac{1}{4}(x^2 + y^2).$$

For $k \simeq 1$, we have $K \simeq \log\left(8/\sqrt{x^2 + y^2}\right)$ and $E \simeq 1$, so that

$$\Phi \simeq -\mu_0 RI\left[\log \frac{8}{\sqrt{x^2+y^2}} - 2\right].$$

The electromotive force around one filament due to an increase ΔI of the magnitude of the current in the other in a time Δt is given by

$$\text{e.m.f.} = -\Delta\Phi / \Delta t,$$

and the corresponding tangential electric field $\mathcal{E}_T$ is given by

$$\mathcal{E}_T = (\text{e.m.f.})/(2\pi R).$$

The force F_T is obtained from

$$F_T = (-e)\mathcal{E}_T = \Delta(mv)/\Delta t,$$

so that $\Delta v \simeq -\frac{\mu_0}{2\pi}\frac{e}{m}\Delta I\left[\log\frac{1}{\sqrt{x^2+y^2}} + (\log 8 - 2)\right].$

Also, $(\log 8 - 2) \simeq 0.08$, which is negligible compared with the first term for small x and y.

We now need to find the total effect on any particular current element due to all the elements in the beam. If the beam has uniform current density over an area A in the $\rho - \zeta$ plane, we then have

$$\Delta v \simeq -\frac{\mu_0}{2\pi}\frac{e}{m}\Delta I \frac{\iint_A \log\frac{1}{\sqrt{x^2+y^2}}\,dx\,dy}{\iint_A dx\,dy},$$

where ΔI is now the total change of current and the x - y coordinate system has its origin at the element at which Δv is being evaluated.

It should be noted that this computation gives only that effect on the beam which is due to electromagnetic induction; the effects due to the existence of magnetic fields of the various elements (as opposed to those due to their change) are included in the earlier computation of electrostatic defocusing and associated magnetic self-focusing of the beam.

The integral above is difficult to evaluate in general, but for a beam of circular cross-section of radius $R\rho_b$, two simple cases are easily worked out. First consider an element at the center. By a simple integration, we find that

$$\Delta v_{center} = -\frac{\mu_0}{2\pi}\frac{e}{m}\Delta I\left(\log\frac{1}{\rho_b}+\frac{1}{2}\right).$$

We have also evaluated the integral for an element at the outside edge of the circular beam. After several transformations and an integration by parts, we obtain

$$\Delta v_{edge} = -\frac{\mu_0}{2\pi}\frac{e}{m}\Delta I\left(\log\frac{1}{2\rho_b}+\frac{2}{\pi}\int_0^1\frac{\sin^{-1}y}{y}\,dy\right).$$

The definite integral is found(45) to have the value $\frac{\pi}{2}\log 2$; hence

$$\Delta v_{edge} = -\frac{\mu_0}{2\pi}\frac{e}{m}\Delta I\log\frac{1}{\rho_b}.$$

As these two cases are extremes, and differ by only 20 per cent for $\rho_b = 0.1$, we shall use the latter value for all elements, since each particle spends the greatest fraction of its time near the outside. It seems probable that this value will be approximately correct for beams of other shapes as well. Since the electrons of interest are those with the largest amplitudes, we shall set $\rho_b = |\rho_m|$.

If we replace $\rho_c^{\,2}$ by its value in terms of I, replace v by $B_o eR/m$, and insert our result for Δv in the expression for $\Delta\rho_m/\rho_m$ derived above, we obtain

$$\frac{\Delta\rho_m}{\rho_m} \simeq \frac{\frac{1-v^2/c^2}{4v^2/c^2} - \rho_m \log\frac{1}{|\rho_m|}}{2(1-n_e)(1-f)\,\rho_m^2}\;\frac{\mu_o \Delta I}{\pi B_o R}\;.$$

The remainder of this section will be devoted to a discussion of this equation and its implications.

First, it should be pointed out that, due to the several assumptions made in deriving this result, each term in the numerator should be multiplied by some correction factor. The first term represents the effect on the radial "spring constant" of changing the amount of circulating charge and current; its numerical coefficient depends on the actual beam shape and on the charge distribution within it, which were taken as circular and constant, respectively, in this derivation. The second term represents the effect on the radial equilibrium point of a change in particle velocity due to the electromagnetic self-inductance of the beam; its numerical coefficient depends on the influence of pole pieces and doughnut walls on the distribution of flux due to the beam, as well as on the approximations made in the derivation above.

Next we shall examine the signs of the two terms in each possible case. If we are interested in injecting from outside the equilibrium orbit, ρ_m will stand for the maximum value of ρ, and will be positive; conversely, for inside injection, ρ_m will mean the minimum value of ρ, and will be negative. In either case we will use the value of n_e which applies to the vicinity of the gun. For outside injection, a positive $\Delta\rho_m$ would be unfavorable, moving the extreme out toward the gun, while for inside injection a positive $\Delta\rho_m$ would assist in clearing the gun.

It will be noted that the first term, representing a damping effect, will move ρ_m away from the center of the doughnut toward the gun for either sign of ρ_m (that is, for either outside or inside injection) for a rising current, while the second term, representing a translation of the entire oscillation, will move ρ_m inward toward the center of the orbit for either sign of ρ_m and rising current. If the current is falling, both effects are reversed. We recall that I was defined as a positive quantity, representing the magnitude of the circulating current.

Since the two terms have opposite signs for injection from outside (the most usual arrangement), it is of interest to estimate their relative magnitudes. The first term in the numerator is very large at low injection energies and falls as injection energy increases; it has the values 1.5 at 40 Kev., and 0.75 at 80 Kev. The second term is somewhat smaller, being 0.23 for $\rho_m = 0.1$ and 0.15 for $\rho_m = 0.05$. However, as pointed out above, it is entirely possible that the numerical corrections mentioned above may be such as to reverse the inequality, or to bring about a cancellation at a particular injection energy. At least we may conclude that these two effects are not of entirely different orders of magnitude for parameters in the ranges considered here.

Next, we shall consider the denominator. The reason for dependence on the factor $1 - n_e$ is fairly evident, as this factor is proportional to the slope of the potential surface at the injection point. If this slope is very flat, corresponding to a value of n_e near to unity, a very small change in energy will produce a large change in amplitude. An inspection of Fig. 6 near $\rho = 0.15$, where $n_e = 1$, may aid in visualizing this situation. This feature is well-confirmed experimentally; a number of workers (see, for example, reference 7) have found that accelerator output increased continuously as the gun was withdrawn toward the point where

$n_s = 1$.

The importance of the factor $(1 - f)$ will doubtless vary greatly from one accelerator to another. One would expect f to increase as the available gun current increases, and to thereby enhance the success of injection. This conclusion is also borne out by experience[(39)]. Also it is known[(43)] that if the doughnut is being well-filled, Kerst's orbit contractor is needed less than if f is small. An important consequence of the presence of this term is that the change of ρ_m per unit change of current is not independent of the value of the current; this circumstance will play a part in our discussion of the non-cancellation of rising and falling current effects.

We have mentioned earlier that large currents exist for a short time during injection, so that f may approach fairly near to unity. There is abundant experimental evidence indicating that best performance is attained when the doughnut is well-filled by circulating electrons during injection.

The presence of the factor B_oR, which is equal to mv/e, in the denominator indicates that the amount of current required to produce a given effect on the orbits is directly proportional to the momentum at injection, except for the explicitly written dependence in the first term in the numerator. However, the amount of charge required to produce a given effect is independent of injection momentum, except for this term, and varies only as the particle mass, so that it is essentially constant at non-relativistic injection energies. Thus the fraction of total retainable charge required to produce a given effect through the first term is essentially independent of injection energy at low energies, while the fraction required for a second-term effect decreases linearly with increasing injection energy. We may conclude that, as higher and higher injection energies are used, effects due to electromagnetic induction will become increasingly preponderant over space charge effects.

In addition to the effects included in the equation under discussion, we shall mention again the adiabatic damping discovered by Kerst and Serber[8]. Although this damping is too small to be of much help if the factor $(1 - n_e)(1 - f)$ is reasonably large, it can assume considerable importance if either n_e or f is near unity. This effect differs from those depending on a change in circulating current in that it is always of a favorable nature, for any gun location. Thus it also assists in explaining the lack of cancellation of rising and falling current effects.

We now turn to a qualitative discussion of possible injection mechanisms, for outside and inside guns, based on the effects described above. Although a more analytical approach would be possible, it is felt that the following treatment adequately indicates the various possibilities.

First we shall assume that the gun is outside the orbit, and that the first term predominates over the second. As electrons are injected, the current rises, causing the oscillations to grow. This will reduce the average lifetime of the electrons, but may not reduce it rapidly enough to compensate for those continually being injected until rather late in the injection pulse. Thus the current will be rising at a decreasing rate during this period. The adiabatic damping may become more important as f approaches unity, thus tending to save some of the later injected electrons which would otherwise have been lost. Finally, the injection voltage no longer matches the acceptance voltage, and effective injection ceases. There will then be a rapid drop in current, as the small lifetime of electrons whose amplitude has been increased suddenly, begins to take its toll. At this time the favorable reverse effect is at its greatest magnitude, and is being aided by the adiabatic damping most effectively. By the time a certain fraction of the total current has been lost, all remaining electrons are damped down into stable orbits which are permanently clear of the gun.

The same explanation applies to injection from inside the orbit. However, in this case, the two terms add, tending to give a bigger effect; on the other hand, n_s is not apt to be as near to unity, tending to give a smaller effect. The general features of the explanation remain valid in either case.

We shall next consider a case which seems somewhat less likely on the basis of the numerical values of the terms stated above, but which cannot be ruled out, on account of the unknown correction factors multiplying each. Assume that injection is from outside, and that the second term is larger than the first. Then as injection proceeds, each electron will be shifted inward away from the gun. Most of those injected at an early stage will be progressively aided by this mechanism until they attain orbits entirely clear of the gun, especially since the mechanism becomes more effective as the circulating current increases. Finally, the injection of electrons with proper velocities ceases. Some of the last-injected electrons will be lost, and the reverse mechanism, being most effective at this stage, will bring out others to be lost. The cumulative build-up of this unfavorable tendency is however opposed by the adiabatic damping and therefore is not able to completely reverse the situation; the earliest injected electrons, with the smallest amplitudes, are never brought all the way out to the gun. In this case there is a special reason for trying to lengthen the time of effective injection by appropriately shaping the voltage pulse to the gun. It is clear that lengthening the time available for the slowly growing field to produce damping will increase the difference between the initial favorable effect and the final unfavorable one.

This account contains one interesting feature, namely, an explanation of why the final output of an accelerator might increase faster than linearly with increasing injection voltage, as has been observed(39) on accelerators with outside injectors. It is not known whether a similar effect is observed with an inside injector, as this type of injection is less frequently used and has been less thoroughly studied. We have seen that a linear increase (in the non-relativistic range of injection energies) might be expected due to the number of electrons required to fill the doughnut. (Actually this is increasing as the 1.1 power at 40 Kev.). In the case just described, the effectiveness of the injection mechanism depends on the difference between the two terms discussed above, the smaller one of which is a decreasing function of injection energy so that the net effect increases as injection energy increases. On the other hand, this change may be small compared with others, as for example, the influence of greater currents or higher voltages on the electron optics of the injection gun.

Finally, we will consider injection from the inside, with the second term again predominating. In this case the two terms again add and the effect is unfavorable as the current rises. As in the first case discussed, however, the doughnut may become fairly full before the break-even point is reached, and from that time on, the loss of part of the current will help to save the rest. The mechanisms of injection discussed above are consistent with various experimental observations concerning the influence of constricted apertures and lowered injection current on output, all of which seem to indicate that, in the absence of orbit contractor coils, an accelerator operates best when conditions are favorable to completely filling the doughnut with charge during some part of the injection period. Thus it is concluded that, although the original purpose of the investigation described in this thesis was the study of the effects of azimuthal inhomogeneities in the magnetic field, the result has been an increased understanding of the effects which play an important part in the injection process.

APPENDIX

1. Equations of Motion

Using the notation explained on page 21, the ponderomotive equation

$$\vec{r} = -(e/m)\,\vec{v} \times \vec{B}$$

becomes

$$\ddot{r} - r\dot{\theta}^2 = -(e/m)(r\dot{\theta}B_z - \dot{z}B_\theta)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -(e/m)(\dot{z}B_r - \dot{r}B_z)$$

$$\ddot{z} = -(e/m)(\dot{r}B_\theta - r\dot{\theta}B_r)$$

in cylindrical polar coordinates, where differentiations with respect to time are denoted by dots. The negative sign occurs because of the electron's negative charge $-e = -|e|$. Since the force is normal to the velocity, $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 =$ constant. Introducing ρ and ζ through the definitions $r \equiv R(1+\rho)$, $z \equiv R\zeta$, where R is a constant length, we obtain

$$\ddot{\rho} - (1+\rho)\dot{\theta}^2 = -(e/m)\left[(1+\rho)\dot{\theta}B_z - \dot{\zeta}B_\theta\right]$$

$$(1+\rho)\ddot{\theta} + 2\dot{\rho}\dot{\theta} = -(e/m)\left[\dot{\zeta}B_r - \dot{\rho}B_z\right]$$

$$\ddot{\zeta} = -(e/m)\left[\dot{\rho}B_\theta - (1+\rho)\dot{\theta}B_r\right]$$

$$(v/R)^2 = \dot{\rho}^2 + (1+\rho)^2\dot{\theta}^2 + \dot{\zeta}^2 = \text{constant.}$$

We define $V \equiv B_0 eR/m$, thus determining the velocity V appropriate to circular motion with radius R in a constant field B_0. Regarding ρ and ζ as functions of θ, we have $\rho = \rho[\theta(t)]$, $\zeta = \zeta[\theta(t)]$; hence $\dot{\rho} = \rho'\dot{\theta}$, $\dot{\zeta} = \zeta'\dot{\theta}$, $\ddot{\rho} = \rho''\dot{\theta}^2 + \rho'\ddot{\theta}$, and $\ddot{\zeta} = \zeta''\dot{\theta}^2 + \zeta'\ddot{\theta}$, where primes denote differentiations with respect to θ. Inserting these into the four equations above, we obtain

$$\rho''\dot\theta^2 + \rho'\ddot\theta - (1+\rho)\dot\theta^2 = -(V/R)\left[(1+\rho)\beta_z - \zeta'\beta_\theta\right]\dot\theta$$

$$(1+\rho)\ddot\theta + 2\rho'\dot\theta^2 = -(V/R)\left[\zeta'\beta_r - \rho'\beta_z\right]\dot\theta$$

$$\zeta''\dot\theta^2 + \zeta'\ddot\theta = -(V/R)\left[\rho'\beta_\theta - (1+\rho)\beta_r\right]\dot\theta$$

$$(v/R)^2 = \left[\rho'^2 + (1+\rho)^2 + \zeta'^2\right]\dot\theta^2 = \text{constant},$$

where $\beta_r = B_r/B_0$, and similarly for β_θ and β_z. Defining $\Delta_v \equiv (v-V)/v$, we have $V/R = (1-\Delta_v)(v/R)$. Inserting this value in the first three equations, substituting $v/R = +\left[\rho'^2 + (1+\rho)^2 + \zeta'^2\right]^{\frac{1}{2}}\dot\theta$ (abbreviated as $(1+\rho)X^{\frac{1}{2}}\dot\theta$, where $X = 1 + (\rho'^2 + \zeta'^2)/(1+\rho)^2$) from the fourth, and dividing by $\dot\theta^2$, we have

$$\rho'' + \rho'\ddot\theta/\dot\theta^2 - (1+\rho) = -(1-\Delta_v)(1+\rho)X^{\frac{1}{2}}\left[(1+\rho)\beta_z - \zeta'\beta_\theta\right]$$

$$(1+\rho)\ddot\theta/\dot\theta^2 + 2\rho' = -(1-\Delta_v)(1+\rho)X^{\frac{1}{2}}\left[\zeta'\beta_r - \rho'\beta_z\right]$$

$$\zeta'' + \zeta'\ddot\theta/\dot\theta^2 = -(1-\Delta_v)(1+\rho)X^{\frac{1}{2}}\left[\rho'\beta_\theta - (1+\rho)\beta_r\right].$$

Solving the second of these equations for $\ddot\theta/\dot\theta^2$, substituting in the other two, and rearranging, we obtain

$$\rho'' = 1 + \rho + (1-\Delta_v) X^{1/2} \left[(1+\rho)\zeta'\beta_\theta + \rho'\zeta'\beta_r - \{\rho'^2 + (1+\rho)^2\}\beta_z \right] + 2\rho'^2/(1+\rho)$$

$$\zeta'' = (1-\Delta_v) X^{1/2} \left[-(1+\rho)\rho'\beta_\theta - \rho'\zeta'\beta_z + \{\zeta'^2 + (1+\rho)^2\}\beta_r \right] + 2\rho'\zeta'/(1+\rho)$$

as given on page 22.

2. Relations Among Coefficients Determined by Maxwell's Equations

Maxwell's equations for a static magnetic field require that $\vec{\nabla} \cdot \vec{B} = 0$ and also that $\vec{\nabla} \times \vec{B} = 0$ if we neglect the magnetic field due to the circulating electrons, as mentioned on pages 23 and 24. In cylindrical coordinates, these equations are:

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0$$

Introducing ρ and ζ by the definitions $r \equiv R(1 + \rho)$ and $z \equiv R\zeta$, where R is constant, these become:

$$B_r + \frac{\partial B_\theta}{\partial \theta} + (1+\rho)\left(\frac{\partial B_r}{\partial \rho} + \frac{\partial B_z}{\partial \zeta}\right) = 0$$

$$\frac{\partial B_z}{\partial \theta} - (1+\rho)\frac{\partial B_\theta}{\partial \zeta} = 0$$

$$\frac{\partial B_r}{\partial \zeta} - \frac{\partial B_z}{\partial \rho} = 0$$

$$B_\theta - \frac{\partial B_r}{\partial \theta} + (1+\rho)\frac{\partial B_\theta}{\partial \rho} = 0$$

We represent the field components by

$$B_z = B_0 \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} A_{jk\ell}\, e^{ij\theta} \rho^k \zeta^\ell$$

$$B_r = B_0 \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} C_{jk\ell}\, e^{ij\theta} \rho^k \zeta^\ell$$

$$B_\theta = B_0 \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} D_{jk\ell}\, e^{ij\theta} \rho^k \zeta^\ell$$

where $A_{jk\ell} = A^*_{-jk\ell}$ and similarly for $C_{jk\ell}$ and $D_{jk\ell}$; $A_{000} = 1$, $i^2 = -1$, and j, k, and ℓ are integers. We shall denote the triple sum by $\sum_3'$, and shall use the convention that $A_{jk\ell} = C_{jk\ell} = D_{jk\ell} = 0$ for $k < 0$, $\ell < 0$, or both. Inserting these into the equations above, and relabeling so as to obtain the same powers of ρ and ζ in each term, we obtain

$$\sum_3' \left[C_{jk\ell} + ij D_{jk\ell} + (k+1) C_{j,k+1,\ell} + k C_{jk\ell} + (\ell+1)(A_{j,k,\ell+1} + A_{j,k-1,\ell+1}) \right] e^{ij\theta} \rho^k \zeta^\ell = 0$$

$$\sum_3' \left[ij A_{jk\ell} - (\ell+1)(D_{j,k,\ell+1} + D_{j,k-1,\ell+1}) \right] e^{ij\theta} \rho^k \zeta^\ell = 0$$

$$\sum_3' \left[(\ell+1) C_{j,k,\ell+1} - (k+1) A_{j,k+1,\ell} \right] e^{ij\theta} \rho^k \zeta^\ell = 0$$

$$\sum_3' \left[D_{jk\ell} - ij C_{jk\ell} + (k+1) D_{j,k+1,\ell} + k D_{jk\ell} \right] e^{ij\theta} \rho^k \zeta^\ell = 0 \ .$$

Since these equations must hold for all values of ρ , ζ , and θ, each square bracket must vanish separately. We therefore have the following scheme:

$$\begin{aligned}
(k+1)(C_{jk\ell} + C_{j,k+1,\ell}) + (\ell+1)(A_{j,k-1,\ell+1} + A_{j,k,\ell+1}) + ij D_{jk\ell} &= 0 \\
(\ell+1)(D_{j,k-1,\ell+1} + D_{j,k,\ell+1}) - ij A_{jk\ell} &= 0 \\
(k+1)(\quad A_{j,k+1,\ell}) - (\ell+1)(\quad C_{j,k,\ell+1}) &= 0 \\
(k+1)(D_{jk\ell} + D_{j,k+1,\ell}) \qquad - ij C_{jk\ell} &= 0
\end{aligned}$$

By setting $k = k + 1$ and $\ell = \ell - 1$ in the second equation, substituting the result into the fourth, replacing ℓ by $\ell + 1$, and multiplying this expression by ℓ/ij, we obtain the third equation, thus showing that any one of the last three is redundant and may be omitted.

There is no interrelation of coefficients with different j.

If we assume that the seven coefficients A_{j00}, C_{j00}, D_{j00}, A_{j01}, A_{j10}, and A_{j11}, and A_{j20} are given, we thereby determine all the other eleven coefficients with this j for which $k + \ell \leq 2$, as follows:

$$D_{j10} = - D_{j00} + ij\, C_{j00}$$

$$D_{j01} = ij\, A_{j00}$$

$$C_{j10} = - C_{j00} - A_{j01} - ij\, D_{j00}$$

$$C_{j01} = A_{j10}$$

$$D_{j11} = ij(A_{j10} - C_{j00}) + D_{j00}$$

$$D_{j20} = -\tfrac{1}{2} ij\, (3C_{j00} + A_{j01}) + (1 + \tfrac{1}{2}j^2)D_{j00}$$

$$D_{j02} = \tfrac{1}{2} ij\, A_{j01}$$

$$C_{j11} = 2A_{j20}$$

$$C_{j20} = \tfrac{3}{2}\, ij\, D_{j00} + \tfrac{1}{2}(A_{j01} - A_{j11}) + (1 + \tfrac{1}{2}j^2)C_{j00}$$

$$C_{j02} = \tfrac{1}{2} A_{j11}$$

$$A_{j02} = - \tfrac{1}{2} A_{j10} - A_{j20} + \tfrac{1}{2} j^2 A_{j00}$$

For the terms with $j = 0$, we have assumed $A_{000} = 1$, $A_{010} = -n$. We can always make $A_{001} = 0$ by proper choice of the plane $z = 0$. We may take

$D_{000} = 0$ because none of the current causing the field will loop through the circular electron path; $C_{000} = 0$ since it represents the first term in the expansion of a field $B_r \propto r^{-1}$ about $r = R$, and no such field will be present. We then have only two remaining arbitrary constants, A_{011} and A_{020}, determining the axially symmetric field to second order in ρ and ζ. The above equations then yield

$$D_{010} = D_{001} = D_{011} = D_{020} = D_{002} = C_{010} = 0$$

$$C_{001} = -n$$

$$C_{011} = 2A_{020}$$

$$C_{020} = -\tfrac{1}{2}A_{011}$$

$$C_{002} = \tfrac{1}{2}A_{011}$$

$$A_{002} = \tfrac{1}{2}n - A_{020}$$

These results are used in deriving the second-order equations for motion in three dimensions in an axially symmetric field, given on page 73.

3. Motion in Two Dimensions from the Hamiltonian Equations

The Hamiltonian function for an electron with charge $-e$ moving in a time-independent magnetic field is $\mathcal{H} = \frac{1}{2m}\left|\vec{p} + e\vec{A}\right|^2$, in M.K.S. rational units, where $\vec{p}$ is the canonical momentum vector and $\vec{A}$ is the vector potential. If $B_r = B_\theta = 0$ in the plane $z = 0$, a particle whose initial velocity is in this plane will not leave it. If the magnetic field is axially symmetric, we have

$$B_r = B_\theta = 0\ ; \quad B_z = B_z(r)\ ;$$

$$B_z = (\vec{\nabla} \times \vec{A})_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r A_\theta(r) \right]\ ;$$

$$A_r = A_z = 0\ ; \quad A_\theta(r) = \frac{1}{r}\int^r x\, B_z(x)\, dx$$

in this plane; x is an integration variable. In plane polar coordinates,

$$\mathcal{H} = \frac{1}{2m}\left[p_r^{\ 2} + (r^{-1}p_\theta + eA_\theta)^2 \right] = \tfrac{1}{2} m v^2 = \text{constant},$$

$$p_r = m\dot{r}\ , \quad p_\theta = mr^2\dot{\theta} - erA_\theta.$$

The Hamiltonian equations of motion are

$$\frac{\partial \mathcal{H}}{\partial \theta} = -\dot{p}_\theta\ , \quad \frac{\partial \mathcal{H}}{\partial r} = -\dot{p}_r\ , \quad \frac{\partial \mathcal{H}}{\partial p_\theta} = \dot{\theta}\ , \quad \frac{\partial \mathcal{H}}{\partial p_r} = \dot{r}\ .$$

The last two merely serve to define p_r and p_θ given above; since H is independent of θ, p_θ is a constant of the motion.

$$p_\theta = \text{constant} = P_\theta = mr_I^{\ 2}\dot{\theta}_I - er_I A_\theta\,(r_I),$$

where subscripts I denote values at injection. From this, $\dot{\theta} = (P_\theta + erA_\theta)/mr^2$; from the conservation of energy, $\dot{r}^2 + r^2\dot{\theta}^2 = v^2 =$ constant. Combining these, $\dot{r}^2 = v^2 - (P_\theta + erA_\theta)^2/m^2r^2$.

Now

$$P_\theta + erA_\theta = mr_I^{\ 2}\dot{\theta}_I - e\left[r_I A_\theta(r_I) - rA_\theta(r) \right]$$

$$= mr_I^{\ 2}\dot{\theta}_I + e\int_{r_I}^{r} x\, B_z(x)dx\ ;$$

hence

$$\dot{r} = \pm\left[v^2 - \left\{ \frac{r_I\dot{\theta}_I}{r} + \frac{eB_0}{mr}\int_{r_I}^{r} x\beta_z(x)dx \right\}^2 \right]^{1/2}.$$

Since $V/R = eB_0/m$ and $r_I\dot{\theta}_I = v\cos\phi_I$, where ϕ_I is the angle between the direction of injection and the tangent to the circle $r = r_I$ at the injection point, this may be written

$$\dot{r} = \pm v\left[1 - \left\{\frac{r_I\cos\phi_I}{r} + \frac{V}{vRr}\int_{r_I}^{r} x\beta_z(x)\,dx\right\}^2\right]^{1/2};$$

integrating and setting $r = r_I$ at $t = t_I$,

$$v(t - t_I) = \int_r^{r_I}\left[1 - y^{-2}\left\{r_I\cos\phi_I + \frac{V}{vR}\int_{r_I}^{y} x\beta_z(x)\,dx\right\}^2\right]^{-1/2} dy,$$

the result given on page 26; y is another variable of integration.

To find the conditions under which oscillations will occur, we first find whether radii r_{max} and r_{min} exist at which $\dot{r} = 0$, as indicated in the text. Then, differentiating the expression for $\dot{r}$ and setting $\dot{r} = 0$, we obtain the result

$$\ddot{r} = (v^2/r)\left[1 - Vr\,\beta_z(r)/(vR)\right],$$

which is valid at $r = r_{max}$ or r_{min}. It then follows that there will be a restoring force if

$$\beta_z(r_{max}) > vR/(Vr_{max}) \text{ and } \beta_z(r_{min}) < vR/(Vr_{min}),$$

as stated on page 27.

To find the limits of motion in the case where $\beta_z(r)$ may be approximated as $\beta_z = 1 - n(r - R)/R = 1 - n\rho$, we set $\dot{r} = 0$, obtaining

$$r_{\substack{max\\min}} = r_I\cos\phi_I + \frac{V}{vR}\int_{r_I}^{r_{\substack{max\\min}}} x\beta_z(x)\,dx;$$

expressing this in terms of ρ, we have

$$1 + \rho_{\substack{max\\min}} = (1+\rho_I)(1 - \tfrac{1}{2}\phi_I^2) + \frac{V}{v}\int_{\rho_I}^{\rho_{\substack{max\\min}}} (1+y)(1-ny)dy,$$

where we assume cos ϕ_I is near unity; y is an integration variable. We have defined $V/v \equiv 1 - \Delta_v$; integrating, expanding, dropping third order terms in the small quantities ρ, ρ_I, ϕ_I and Δ_v, and simplifying, we obtain

$$\rho^2_{\substack{max\\min}} - \frac{2\Delta_v}{1-n}\rho_{\substack{max\\min}} - \rho_I^2 + \frac{2\Delta_v\rho_I - \phi_I^2}{1-n} = 0 .$$

Solving this quadratic equation, the final result is

$$\rho_{\substack{max\\min}} = \frac{1}{1-n}\left\{\Delta_v \pm\left[(1-n)^2\rho_I^2 - 2(1-n)\Delta_v\rho_I + \Delta_v^2 + (1-n)\phi_I^2\right]^{1/2}\right\},$$

as stated on page 31.

To obtain an integral for θ as a function of r, we note that

$$r' = \dot{r}/\dot{\theta} = (v^2 - r^2\dot{\theta}^2)^{\frac{1}{2}}/\dot{\theta} .$$

Since we have found $\dot{\theta}$ as a function of r, we may write at once that

$$\theta - \theta_I = \int_{r_I}^{r} \frac{\left\{\frac{r_I\cos\phi_I}{y} + \frac{V}{vRy}\int_{r_I}^{y} x\beta_z(x)\,dx\right\}\frac{dy}{y}}{\left[1 - \left\{\frac{r_I\cos\phi_I}{y} + \frac{V}{vRy}\int_{r_I}^{y} x\beta_z(x)\,dx\right\}^2\right]^{1/2}} .$$

4. <u>Method of Successive Approximations for Radial Motion in an Axially Symmetric Field</u>

This method is described in Reference 24, § 70. We shall only summarize it here. The equation we wish to treat has the form

$$\rho'' + \left[1 - n - (2 - n)\Delta_v\right]\rho = \Delta_v + A_1\rho^2 + A_2\rho'^2 + B_1\rho^3$$

$$+ B_2\rho\rho'^2 + C_1\rho^4 + C_2\rho^2\rho'^2 + C_3\rho'^4 \quad -----;$$

the coefficients A_1, A_2, B_1, B_2, ----- are given on page 32. The method is applicable to any equation of this form in which ρ' appears only to even powers, as long as the right side is small with respect to either term on the left. This can be arranged in our problem by first introducing a new independent variable

$$x \equiv \rho - \Delta_v / \left[1 - n - (2 - n)\Delta_v\right] \equiv \rho - \epsilon$$

to eliminate the constant term in Δ_v; we now find that every term on the right of the equation for x is of second or higher order in x and ϵ, whose ranges in the physical problem at hand extend from zero to values small compared with unity. Hence we can introduce δ, a parameter of smallness, by setting $x \equiv \delta\xi$, $\epsilon \equiv \delta\eta$, where ξ and η are assumed to be of order unity or less. By introducing the abbreviation $\omega_r^2 \equiv 1 - n - (2 - n)\Delta_v$, and cancelling out one factor δ, our equation becomes

$$\xi'' + \omega_r^2\xi = \delta\Big\{\left[A_1(\xi+\eta)^2 + A_2\xi'^2\right] + \delta\left[B_1(\xi+\eta)^3 + B_2(\xi+\eta)\xi'^2\right]$$

$$+ \delta^2\left[C_1(\xi+\eta)^4 + C_2(\xi+\eta)^2\xi'^2 + C_3\xi'^4\right] + ----\Big\},$$

which is of the type desired if the A's are of order unity, the B's of order δ^{-1}, and so forth. However, in the procedure to be followed in the present instance it is assumed that all of these coefficients are of order unity or less.

Since we have previously determined that the solution of our equation is a periodic oscillation symmetric about an extreme point, and therefore representable by a Fourier cosine series, we postulate here that the solution is $\xi(\mathcal{S}\theta + \phi) = z(\tau)$, with angular frequency $\mathcal{S}$ to be determined, and arbitrary phase ϕ. Changing variables, $\xi'' = \mathcal{S}^2 \ddot{z}$ and $\xi'^2 = \mathcal{S}^2 \dot{z}^2$, where dots denote differentiations with respect to τ. We then have

$$\mathcal{S}^2 \ddot{z} + \omega_r^2 z = \delta\left\{\left[A_1(z+\eta)^2 + \mathcal{S}^2 A_2 \dot{z}^2\right] + \delta\left[B_1(z+\eta)^3 + \mathcal{S}^2 B_2 (z+\eta)\dot{z}^2\right] + \cdots\right\}.$$

We now set $z(\tau) = \sum_{n=0}^{\infty} \delta^n z_n(\tau)$ and $\mathcal{S}^2 = \sum_{n=0}^{\infty} \delta^n \mathcal{S}_n^2$, and require that $z_n(\tau + 2\pi) = z_n(\tau)$ and that $\dot{z}_n(0) = 0$ for all $n \geq 0$. In solving for the z_n by successive approximation we will find that secular terms (increasing linearly with τ) will appear unless we require that z_n shall contain no term in $\cos \tau$ for $n \geq 1$.

Zero Order Equation:

$$\mathcal{S}_0^2 \ddot{z}_0 + \omega_r^2 z_0 = 0;$$

$$z_0 = a \cos\tau; \quad \mathcal{S}_0^2 = \omega_r^2.$$

First Order Equation:

$$\omega_r^2(\ddot{z}_1 + z_1) = -\mathcal{S}_1^2 \ddot{z}_0 + A_1(z_0+\eta)^2 + \omega_r^2 A_2 \dot{z}_0^2$$

$$= a_1 \mathcal{S}_1^2 \cos\tau + A_1 a^2 \cos^2\tau + 2\eta a A_1 \cos\tau + \eta^2 A_1 + \omega_r^2 a^2 A_2 \sin^2\tau$$

$$= F_0 + F_1 \cos\tau + F_2 \cos 2\tau,$$

where

$$F_0 = \frac{1}{2} a^2 (A_1 + \omega_r^2 A_2) + \eta^2 A_1 \;;$$

$$F_1 = a \delta_1^2 + 2\eta a A_1 \;; \quad F_2 = \frac{1}{2} a^2 (A_1 - \omega_r^2 A_2).$$

By setting $F_1 = 0$, we find that $\delta_1^2 = -2\eta A_1$. z_1 is then determined by solving the remaining equation;

$$z_1 = \frac{F_0}{\omega_r^2} - \frac{F_2}{3\omega_r^2} \cos 2\tau .$$

Second Order Equation:

$$\omega_r^2 (\ddot{z}_2 + z_2) = -\delta_2^2 \ddot{z}_0 - \delta_1^2 \ddot{z}_1 + 2A_1 (z_0 + \eta) z_1 + 2\omega_r^2 A_2 \dot{z}_0 \dot{z}_1$$

$$+ \delta_1^2 A_2 \dot{z}_0^2 + B_1 (z_0 + \eta)^3 + \omega_r^2 B_2 (z_0 + \eta) \dot{z}_0^2 .$$

By substituting for z_0 and z_1 and collecting terms, the right side may be written in the form $G_0 + G_1 \cos\tau + G_2 \cos 2\tau + G_3 \cos 3\tau$.

Setting $G_1 = 0$, we obtain

$$\delta_2^2 = -\frac{2A_1 F_0}{\omega_r^2} + \frac{A_1 F_2}{3\omega_r^2} + \frac{2}{3} A_2 F_2 - \frac{3}{4} a^2 B_1 - 3\eta^2 B_1 - \frac{a^2}{4} \omega_r^2 B_2 .$$

The solution for z_2 may then be written as

$$z_2 = \frac{G_0}{\omega_r^2} - \frac{G_2}{3\omega_r^2} \cos 2\tau - \frac{G_3}{8\omega_r^2} \cos 3\tau ,$$

where

$$G_0 = \eta \frac{A_1 F_2}{\omega_r^2} + 2\eta \frac{A_1 F_0}{\omega_r^2} - \eta a^2 A_1 A_2 + \eta (\eta^2 + \frac{3}{2} a^2) B_1 \;;$$

$$G_2 = 2\eta \frac{A_1 F_1}{\omega_r^2} + \eta a^2 A_1 A_2 + \frac{3}{2} \eta a^2 B_1 \;;$$

$$G_3 = -\frac{1}{3} a \frac{A_1 F_2}{\omega_r^2} + \frac{2}{3} a A_2 F_2 + \frac{1}{4} a^3 (B_1 - \omega_r^2 B_2).$$

By combining our results thus far, setting $\delta = 1$, and returning to the original notation, we obtain the results given on page 32. The convergence of these series now depends on the smallness of the constants a and ϵ.

5. Derivation of Formulas for a' and δ'

The solution of the differential equation

$$\rho'' + \omega_r^2 \rho = f(\rho, \rho', \theta) ,$$

in which $f(\theta)$ has the properties stated on page 42, may be represented by the expression

$$\rho = a(\theta) \cos[\omega\theta + \delta(\theta)] \equiv a(\theta) \cos\phi(\theta) .$$

In fact, an additional constraint on the relation between a and δ may be imposed, such that

$$\rho' = -\omega a(\theta) \sin\phi(\theta) .$$

By differentiating the expression for ρ and equating this to the assumed expression for ρ', we see that this constraint implies that

$$a' \cos\phi = \delta' a \sin\phi .$$

By differentiating the assumed expression for ρ' again, and inserting this on the left side of the differential equation, we obtain

$$(\omega_r^2 - \omega^2 - \omega\delta') a \cos\phi - \omega a' \sin\phi = f(\rho, \rho', \theta) .$$

If we substitute for δ' in terms of a' on the left, replace ρ and ρ' by their assumed expressions on the right, and solve for a'/a, we obtain

$$\frac{a'}{a} = \frac{\omega_r^2 - \omega^2}{\omega} \sin\phi \cos\phi - \frac{1}{a\omega} f(a\cos\phi, -\omega a \sin\phi, \theta) \sin\phi .$$

On the other hand, if we substitute for a' in terms of δ', we obtain

$$\delta' = \frac{\omega_r^2-\omega^2}{\omega}\cos^2\phi - \frac{1}{a\omega} f(a\cos\phi, -\omega a \sin\phi, \theta)\cos\phi .$$

These two first order equations are now equivalent to, and may replace the original second order equation, as indicated on page 42.

6. Proof of the Divergence Theorem

The theorem to be proved may be stated as follows. If the solution to the differential equation $\rho'' + \omega_r^2 \rho = f(\rho, \rho', \theta)$ be represented in the form $\rho = \sqrt{A(\theta)}\cos[\omega\theta + \delta(\theta)]$, and if it also be required that $A(\theta)$ and $\delta(\theta)$ be such that $\rho' = -\omega\sqrt{A(\theta)}\sin[\omega\theta + \delta(\theta)]$, then $\partial A'/\partial A + \partial\delta'/\partial\delta \equiv \partial f/\partial\rho'$.

We have obtained formulas for a' and δ' in the preceding section of the appendix. By differentiating them, we obtain

$$\frac{\partial a'}{\partial a} = \frac{\omega_r^2-\omega^2}{\omega}\sin\phi\cos\phi - \frac{1}{\omega}\left[\frac{\partial f}{\partial\rho}\cos\phi - \omega\frac{\partial f}{\partial\rho'}\sin\phi\right]\sin\phi ;$$

$$\frac{\partial\delta'}{\partial\delta} = -2\frac{\omega_r^2-\omega^2}{\omega}\sin\phi\cos\phi - \frac{1}{a\omega}\left[-\frac{\partial f}{\partial\rho}a\sin\phi - \frac{\partial f}{\partial\rho'}a\omega\cos\phi\right]\cos\phi + \frac{f}{a\omega}\sin\phi .$$

Since $A = a^2$, $A' = 2aa' = 2A^{\frac{1}{2}} a'$; therefore

$$\frac{\partial A'}{\partial A} = \frac{a'}{A^{1/2}} + 2A^{1/2}\frac{\partial a'}{\partial a}\frac{\partial a}{\partial A}$$

$$= \frac{a'}{A^{1/2}} + \frac{\partial a'}{\partial a} = \frac{a'}{a} + \frac{\partial a'}{\partial a} .$$

Thus

$$\frac{\partial A'}{\partial A} = 2\frac{\omega_r^2-\omega^2}{\omega}\sin\phi\cos\phi - \frac{1}{\omega}\left[\frac{\partial f}{\partial\rho}\cos\phi - \omega\frac{\partial f}{\partial\rho'}\sin\phi\right]\sin\phi - \frac{f}{a\omega}\sin\phi ,$$

and

$$\frac{\partial A'}{\partial A} + \frac{\partial\delta'}{\partial\delta} = \frac{\partial f}{\partial\rho'} ,$$

as stated on page 48, which proves the theorem.

7. Evaluation of the K's

In this section we wish to indicate how to determine the constants K_0, K_1, and K_2 introduced on page 49 which result from the application of the equations for α' and δ' introduced on page 42 to the differential equation on page 43, using the averaging procedure described on that page. We shall work out the results in detail for the approach in which the method is applied directly, for the two cases $s = 2$ and $s = 3$ (where $\omega = r/s$), and then indicate the modifications which arise if the equation to be solved is that for motion about the distorted equilibrium orbit, as mentioned on pages 34, 36, and 50. In the course of the investigation described in this thesis, a large number of very general calculations of such coefficients have been made. However, there seems to be no point in including them here, since the computations are straightforward but very tedious, and the resulting expressions are extremely unwieldly because of their generality. Also, in the application of these techniques to a situation of practical interest, it would be fairly easy to ascertain which of the A_{jk} coefficients were important; it would then be as simple to carry out the derivations afresh, keeping only these terms, as it would be to specialize general results to this case. In fact, the best technique in a particular situation might differ from that discussed here if certain higher order coefficients were known to be large, as mentioned on page 84.

For direct application to the equation on page 43, we will consider the terms in the first and second rows separately as was done in the text. The first row terms with odd powers of ρ contribute only to $\bar{\delta}'$; all other first row terms average out, as pointed out on page 43. This contribution to $\bar{\delta}'$, through second order in $\bar{a}$, is

$$- \frac{1}{\bar{a}\omega} \left\{ \left[B_1(\bar{a} \cos \phi)^3 + B_2(\bar{a} \cos \phi)(-\omega \bar{a} \sin \phi)^2 \right] \cos \phi \right\}_{Ave}$$

$$= - \frac{1}{\omega} \left(\frac{3}{8} B_1 + \frac{\omega^2}{8} B_2 \right) \bar{a}^2 ,$$

since $(\cos^4 \phi)_{Ave} = 3/8$ and $(\sin^2 \phi \cos^2 \phi)_{Ave} = 1/8$. Therefore

$$K_2 = - \frac{1}{8\omega} (3B_1 + \omega^2 B_2) .$$

The values of B_1 and B_2 are given on page 32.

We may evaluate K_0 immediately from the formula for δ' on page 42.

$$K_0 = \frac{\omega_r^2 - \omega^2}{\omega} (\cos^2 \phi)_{Ave} = \frac{1}{2\omega} (\omega_r^2 - \omega^2).$$

The corresponding term in the formula for a' averages to zero. In evaluating the contributions of terms in the second row, we need to know the value of the integer s. For s = 2, we see from the general formulas on page 49 that the contributions of lowest order are independent of $\bar{a}$. By expanding the terms in this row, we find that these are given by the average values of

$$2 \frac{1 - \Delta_v}{\bar{a}} \sum_{j=1}^{\infty} \bar{a} \left[2A_{j0} \cos(j\theta + \alpha_{j0}) + A_{j1} \cos(j\theta + \alpha_{j1}) \right] \cos \phi \ {\sin \atop \cos} \phi,$$

where the sine is taken for $\bar{a}'/\bar{a}$ and the cosine for $\bar{\delta}'$. Also, $\theta = \omega^{-1}(\phi - \bar{\delta})$, which gives $2(\phi - \bar{\delta})$ for s = 2. Inserting this, expanding the trigonometric functions, neglecting Δ_v compared with unity since the A_{jk} are assumed to be small of first order, and averaging out certain terms by inspection, we obtain

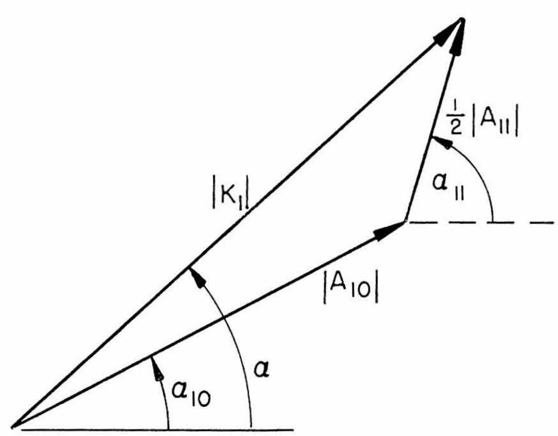
$$2\sum_{j=1}^{\infty}\left\{A_{j0}\left[\cos 2j\phi \, {}^{\sin}_{\cos} 2\phi\cos(2j\bar{\delta} - \alpha_{j0}) + \sin 2j\phi \, {}^{\sin}_{\cos} 2\phi\sin(2j\bar{\delta} - \alpha_{j0})\right]\right.$$

$$\left. + \tfrac{1}{2}A_{j1}\left[\cos 2j\phi \, {}^{\sin}_{\cos} 2\phi\cos(2j\bar{\delta} - \alpha_{j1}) + \sin 2j\phi \, {}^{\sin}_{\cos} 2\phi\sin(2j\bar{\delta} - \alpha_{j1})\right]\right\}.$$

We see that all terms with $j \neq 1$ disappear when averaged with respect to ϕ, leaving

$$A_{10} \, {}^{\sin}_{\cos} (2\bar{\delta} - \alpha_{10}) + \tfrac{1}{2}A_{11} \, {}^{\sin}_{\cos} (2\bar{\delta} - \alpha_{11}) = K_1 \, {}^{\sin}_{\cos} (2\bar{\delta} - \alpha)$$

as the final result. K_1 and α are most simply evaluated by the graphical construction which follows.



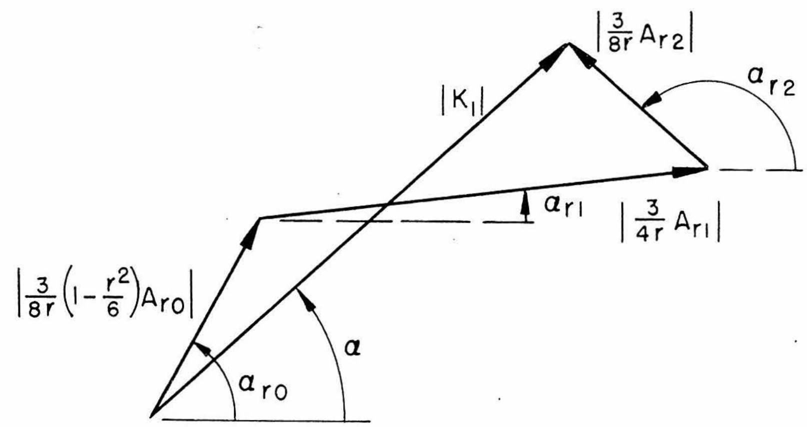
This result should be compared with that on page 38 obtained from the Mathieu equation. It will be seen that $|K_1|$ is the same as the quantity $|q/4|$ derived there.

If $s = 3$, r may be either 1 or 2, and the lowest order contributions of second-row terms contain one power of $\bar{a}$. By similar manipulations, we find that only terms with $j = r$ contribute; the result is

$$\frac{3}{8}\frac{\bar{a}}{r}\Big[(1 - r^2/6)A_{ro}{}^{\sin}_{\cos}(3\bar{\delta} - \alpha_{ro}) + 2A_{r1}{}^{\sin}_{\cos}(3\bar{\delta} - \alpha_{r1})$$

$$+ A_{r2}{}^{\sin}_{\cos}(3\bar{\delta} - \alpha_{r2})\Big] = K_1\bar{a}{}^{\sin}_{\cos}(3\bar{\delta} - \alpha).$$

K_1 and α can again be evaluated by a geometrical construction.



For other values of r and s, similar results can be obtained. The higher-order terms which are merely indicated on page 49 may also be evaluated, although it is pointed out on page 84 that these terms are not necessarily taken into account correctly by the formalism adopted here. However, if some of the A_{jk} coefficients are large, different procedures can easily be devised for obtaining a first approximation to their influence on the motion.

One such procedure which should be applied, if any of the A_{j0} with small j are appreciable, is to eliminate the lowest order effect of these terms by solving for the motion about a distorted equilibrium orbit. The reasons for doing this are discussed on pages 36 and 50. The differential equation may be written in the form

$$\rho'' + \omega_r^2 \rho = \Delta_v - \sum_{j=1}^{\infty} A_{j0} \cos(j\theta + \alpha_{j0})$$
$$+ A_1 \rho^2 + A_2 \rho'^2 + B_1 \rho^3 + B_2 \rho \rho'^2 + \cdots$$
$$- (1 + 2\rho + \cdots) \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{jk} \rho^k \cos(j\theta + \alpha_{jk}),$$

if we neglect Δ_v compared with unity. The second and third lines of this expression are now of order ρ^2, $A_{jk}\rho$, and higher orders. Neglecting them for the moment, the solution for the distorted orbit ρ_0 is

$$\rho_0(\theta) = \frac{\Delta_v}{\omega_r^2} + \sum_{j=1}^{\infty} \frac{A_{j0}}{j^2 - \omega_r^2} \cos(j\theta + \alpha_{j0}) .$$

If we denote the displacement from this orbit by x, we have

$$\rho = x + \rho_0 ,$$

and the differential equation satisfied by x is

$$x'' + \omega_r^2 x = A_1 (x + \rho_0)^2 + A_2 (x' + \rho_0')^2 + B_1 (x + \rho_0)^3 + B_2 (x + \rho_0)(x' + \rho_0')^2 + \cdots$$
$$- \left[1 + 2(x + \rho_0) + \cdots\right] \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{jk} (x + \rho_0)^k \cos(j\theta + \alpha_{jk}).$$

We now proceed to apply the method under discussion to this equation. It is clear that, in principle, the right side can be multiplied out and rearranged to bring it into the form of the original equation. However, the coefficients will then be much more complicated functions of the original parameters, and it seems unnecessary to evaluate them in detail here. We will merely give the lowest-order results for the case $r/s = \frac{1}{2}$, $\Delta_v = 0$, for comparison with those of the simpler case.

$$\frac{\bar{a}'}{\bar{a}} = A_{10}\left[1 - \left(\tfrac{4}{3}A_1 + \tfrac{1}{3}\right)\right]\sin(2\bar{\delta} - \alpha_{10}) + \tfrac{1}{2}A_{11}\sin(2\bar{\delta} - \alpha_{11})\ ;$$

$$\bar{\delta}' = A_{10}\left[1 - \left(\tfrac{4}{3}A_1 + \tfrac{1}{3}\right)\right]\cos(2\bar{\delta} - \alpha_{10}) + \tfrac{1}{2}A_{11}\cos(2\bar{\delta} - \alpha_{11})$$

$$+ (\omega_r^2 - \tfrac{1}{4}) - (\tfrac{3}{4}B_1 + \tfrac{1}{16}B_2)\bar{a}^2 - \tfrac{3}{2}B_1\sum_{j=1}^{\infty}\frac{A_{j0}^2}{(j^2-\frac{1}{4})^2}$$

$$- \tfrac{1}{2}B_2\sum_{j=1}^{\infty}\frac{j^2A_{j0}^2}{(j^2-\frac{1}{4})^2} + \sum_{j=1}^{\infty}\frac{1}{(j^2-\frac{1}{4})}\Big[A_{j0}^2 + 2A_{j0}A_{j1}\cos(\alpha_{j0}-\alpha_{j1})$$

$$+ A_{j0}A_{j2}\cos(\alpha_{j0}-\alpha_{j2})\Big].$$

The value of A_1 is given on page 32. We see that K_2 is the same as before but additional terms appear in K_0 and K_1. The complexity of these expressions increases rapidly for larger s and for higher-order terms.

8. Dependence of Oscillation Amplitude on θ

For the situation discussed on page 64, we have

$$\bar{a}^2 = (\text{const.})\sec(2\bar{\delta} - \alpha),$$

and from the results on page 49 we obtain the differential equations satisfied by $\bar{a}$ and $\bar{\delta}$ as functions of θ in this case (as illustrated in Fig. 7), in which $K_0 = K_2 = 0$:

$$\bar{a}'/\bar{a} = K_1\sin(2\bar{\delta} - \alpha)\ ;$$

$$\bar{\delta}' = K_1\cos(2\bar{\delta} - \alpha)\ .$$

The second of these can be integrated at once, to give

$$\sin(2\bar{\delta} - \alpha) = \frac{Ce^{4K_1\theta} - 1}{Ce^{4K_1\theta} + 1}\ ;$$

the value of the integration constant C is determined by the condition that $\bar{\delta} = \delta_0$ at $\theta = 0$ to be

$$C = \frac{1 + \sin(2\delta_0 - \alpha)}{1 - \sin(2\delta_0 - \alpha)} .$$

From this we obtain

$$\bar{a}^2 = (\text{const.}) \sec(2\bar{\delta} - \alpha) = (\text{const.})\left[1 - \left(\frac{Ce^{4K_1\theta} - 1}{Ce^{4K_1\theta} + 1}\right)^2\right]^{-1/2}$$

$$= (\text{const.})\, \tfrac{1}{2}\, (C^{\frac{1}{2}} e^{2K_1\theta} + C^{-\frac{1}{2}} e^{-2K_1\theta}) .$$

In the situation of interest, $\alpha = \pi/2$ and $C^{\frac{1}{2}} = \tan \delta_0$. In this case, if we require that $\bar{a} = \bar{a}_0$ at $\theta = 0$, the constant must have the value $2 \sin \delta_0 \cos \delta_0$; we then obtain

$$(\bar{a}/\bar{a}_0)^2 = \sin^2 \delta_0 e^{2K_1\theta} + \cos^2 \delta_0 e^{-2K_1\theta} ,$$

as stated on page 64.

If the adiabatic damping term discussed on page 63 is included, we have

$$\bar{a}'/\bar{a} = K_1 \sin (2\bar{\delta} - \alpha) - K_d ;$$

$$\bar{\delta}' = K_1 \cos (2\bar{\delta} - \alpha) .$$

If we make the substitution $\bar{a}(\theta) = f(\theta)e^{-K_d\theta}$, we find that

$$f'/f = K_1 \sin (2\bar{\delta} - \alpha) ,$$

so that $f(\theta)$ is the same as $\bar{a}(\theta)$ for the case without damping; therefore when damping is present,

$$(\bar{a}/\bar{a}_0)^2 = \left[\sin^2 \delta_0 e^{2K_1\theta} + \cos^2 \delta_0 e^{-2K_1\theta}\right] e^{-2K_d\theta} ,$$

as stated on page 65.

To determine the number of revolutions required for an electron to regain its original amplitude, with $K_d = 0$, we note from Fig. 7 that the curves are symmetric about their minima, determine the number of revolutions to the minimum, and double it. Carrying this out,

$$d(\bar{a}/\bar{a}_\circ)^2/d\theta = 2K_1\left[\sin^2\delta_o e^{2K_1\theta_m} - \cos^2\delta_o e^{-2K_1\theta_m}\right] = 0 ,$$

where θ_m denotes the value of θ at which $\bar{a}$ reaches a minimum. Solving for θ_m, we obtain

$$\theta_m = (2K_1)^{-1} \log|\cot\delta_o| .$$

The number of revolutions N_r required to return to the original amplitude is given by

$$N_r = 2\theta_m/(2\pi),$$

and $|\cot\delta_o|$ is found from page 65 to have the value $|\frac{1}{2}\rho_I \cot\phi_I|$; hence

$$N_r = (2\pi K_1)^{-1} \log|\tfrac{1}{2}\rho_I \cot\phi_I| ,$$

as stated on page 67.

If $\phi_I = 0$, then $\delta_o = 0$, $\bar{a}_o = \rho_I$, and $\bar{a} = \rho_I e^{-K_1\theta}$.

Therefore $\bar{a} = \rho_c$ after N_c revolutions, where

$$N_c = \theta_c/(2\pi) = (2\pi K_1)^{-1} \log|\rho_I/\rho_c| ,$$

the other result quoted on page 67.

9. Sudden Approximation for Change in Oscillation Amplitude

It was shown on page 108 that the radial velocity of an electron could be expressed by the equation

$$\dot{\rho}^2 = \left(\frac{v}{R}\right)^2\left[1 - \frac{2e(1-V^2/c^2)R^2}{\epsilon_0 m v^2}\int_{\rho_m}^{\rho}\frac{dy}{y}\int_0^y \eta\sigma(\eta)\,d\eta - \left\{\frac{1+\rho_m}{1+\rho} + \frac{V/v}{1+\rho}\int_{\rho_m}^{\rho}(1+y)\,\beta_z(y)\,dy\right\}^2\right].$$

This expression gives $\dot{\rho}$ as a function of ρ, ρ_m, σ, and v. Our approximation consists of equating $\dot{\rho}\left[0, \rho_m, \sigma, V\right]$ and $\dot{\rho}\left[0, \rho_m + \Delta\rho_m, \sigma(1 + \Delta I/I), V + \Delta v\right.$, expanding, retaining only first powers of $\Delta\rho_m$, ΔI, and Δv, and determining the relation connecting these three changes.

Carrying this out in detail, we first obtain

$$\frac{V^2}{R^2} + \frac{2e(1-V^2/c^2)}{\epsilon_0 m}\int_0^{\rho_m}\frac{dy}{y}\int_0^y \eta\sigma(\eta)\,d\eta - \frac{V^2}{R^2}\left\{1+\rho_m - \int_0^{\rho_m}(1+y)\,\beta_z(y)\,dy\right\}^2$$

$$\simeq \frac{V^2}{R^2}\left(1+\frac{2\Delta v}{v}\right) + \frac{2e(1-V^2/c^2)}{\epsilon_0 m}\left[\int_0^{\rho_m}\frac{dy}{y}\int_0^y \eta\sigma(\eta)\,d\eta + \int_{\rho_m}^{\rho_m+\Delta\rho_m}\frac{dy}{y}\int_0^y \eta\sigma(\eta)\,d\eta\right.$$

$$\left.+ \frac{\Delta I}{I}\int_0^{\rho_m}\frac{dy}{y}\int_0^y \eta\sigma(\eta)\,d\eta\right] - \frac{V^2}{R^2}\left(1+\frac{2\Delta v}{v}\right)\left\{1+\rho_m + \Delta\rho_m - \int_0^{\rho_m}(1+y)\,\beta_z(y)\,dy\right.$$

$$\left. - \int_{\rho_m}^{\rho_m+\Delta\rho_m}(1+y)\,\beta_z(y)\,dy + \frac{\Delta v}{v}\int_0^{\rho_m}(1+y)\,\beta_z(y)\,dy\right\}^2.$$

By evaluating the integrals over infinitesimal ranges in the usual way, cancelling where possible, and expanding, we obtain

$$0 \simeq \frac{V^2}{R^2} 2 \frac{\Delta v}{v} + \frac{2e(1-V^2/c^2)}{\epsilon_0 m}\left[\frac{\Delta \rho_m}{\rho_m}\int_0^{\rho_m} \eta\sigma(\eta)\,d\eta + \frac{\Delta I}{I}\int_0^{\rho_m}\frac{dy}{y}\int_0^{y}\eta\sigma(\eta)\,d\eta\right]$$

$$-\frac{V^2}{R^2}\cdot 2\frac{\Delta v}{v}\left\{1+\rho_m-\int_0^{\rho_m}(1+y)\beta_z(y)\,dy\right\}^2 - \frac{2V^2}{R^2}\left[\left\{1-(1+\rho_m)\beta_z(\rho_m)\right\}\Delta\rho_m\right.$$

$$\left. + \frac{\Delta v}{v}\int_0^{\rho_m}(1+y)\beta_z(y)\,dy\right] .$$

Rearranging and multiplying by R^2/V^2, this becomes

$$\left[\frac{2e(1-V^2/c^2)R^2}{\epsilon_0 m V^2}\int_0^{\rho_m}\eta\sigma(\eta)\,d\eta - 2\rho_m\left\{1-(1+\rho_m)\beta_z(\rho_m)\right\}\right]\frac{\Delta\rho_m}{\rho_m}$$

$$= -2\left[1-\left\{1+\rho_m-\int_0^{\rho_m}(1+y)\beta_z(y)\,dy\right\}^2 - \int_0^{\rho_m}(1+y)\beta_z(y)\,dy\right]\frac{\Delta v}{v}$$

$$-\left[\frac{2e(1-V^2/c^2)R^2}{\epsilon_0 m V^2}\int_0^{\rho_m}\frac{dy}{y}\int_0^{y}\eta\sigma(\eta)\,d\eta\right]\frac{\Delta I}{I} ,$$

as stated on page 110.

10. Note on Time-Varying Inhomogeneities

After the text was prepared in final form, an article(46) appeared describing the successful application of a rapidly varying inhomogeneity to the process of injection, in precisely the way which was considered on pages 66 to 68 of this thesis. However, the accelerator used had an air core magnet, rather than an iron core one as was considered in the present work. It is probably easier to change quickly the form of an air core field than that of an iron core one.

These experimental results seem to agree with the analysis. In particular, it was found that (1) the switching-off time of the inhomogeneity is critical to about a microsecond; (2) a very rapid fall-off is desirable (a decay time of 5×10^{-8} second was used); (3) the effect is very sensitive to a change of the value of n by as little as 0.01; (4) the azimuthal inhomogeneity must be completely removed; and (5) the scheme is successful for a variety of gun positions and for small relative apertures. All of these features are predicted by our analysis of this problem.

There is as yet no indication of the relative merits of this injection scheme as compared with the more conventional one, because the accelerator in question has properties which have made it thus far impossible to obtain an accelerated beam by the usual technique. We still believe that it will be very difficult, if not impossible, to increase the output of a well adjusted accelerator by changing over to a scheme of this kind.

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