DETERMINATION OF BUCKLING LOADS BY FREQUENCY MEASUREMENTS

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ABSTRACT

Analytic solutions for the relation between compressive load and natural frequency are obtained for a simple column and a circular ring under uniform radial load. The relation is of the form

$$P = P_{cr} - K \lambda_n^2$$

An implicit analytic solution for a rectangular frame under symmetrical axial load and for symmetrical buckling is also obtained. The complexity of the final transcendental equation indicates that the relation between P and λ_n will not be linear.

The relation is checked experimentally for a simple column. The extrapolated critical load is within 2% of the computed value.

The experiment on a rectangular frame shows that the relation between P and λ_n can be very closely approximated by a linear one. The extrapolated critical load is 17,500 lbs. which is about 49% higher than that for the lowest symmetrical mode and about 3% higher for the lowest unsymmetrical mode. Evidently, the measured frequency is in the latter mode. The misleading result is due to the fact that these modes have frequencies very close to each other, and the unsymmetrical mode becomes more stable because of restraints at the supports and dynamic coupling between the modes.

Some interesting points in this work are stressed in the conclusion and some suggestions for the additional experiments are made.

CONTENTS

I. INTRODUCTION

Buckling problems in elasticity have been studied for many years. Mathematical solutions have been obtained for many idealized problems, such as prismatic bars, plates, shells, etc. of different shapes and end conditions. However, such solutions are still very limited, and there remain many problems involving complicated structures, which are too cumbersome, and can not be practicably solved by analytic methods. This is especially true for a built-up structural frame in compression. In practice, a destructive experimental method is usually employed to find out the safe load.

The relations between compressive load and frequency for some simple cases have been studied analytically by Charles Massonet in his article "Les Relations entre les Modes Normaux de Vibration et la Stabilité des Systèmes Élastiques".(1)* Further work has been explored by Dr. Felix Buckens at the California Institute of Technology. However, the work is limited to theoretical discussions for cases where buckling problems have mathematical solutions.

Massonnet has shown that, for a simple prismatic column with pinned end, the relation between axial compressive load and the square of the frequency is linear, and deviates slightly from linearity for columns with other end conditions. He also has shown that the load at zero frequency is the critical load corresponding to the associated mode of vibration. Therefore, if a few points on the curve of load

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* The number in the parenthesisrefers to the number in the Table of of References following the text.

versus square of frequency could be determined, then the critical load could be extrapolated for zero frequency. If this law can be extended to structures where no analytic solution or no simple analytic solution is possible, we shall have a non-destructive experimental method for determining the critical load.

Before such a method can be applied, the accuracy of the extrapolation must first be investigated and compared with the theoretical value. It is the purpose of the present attempt, (a) to correlate buckling load and frequency for a few simple cases, (b) to check experimentally for a case where the relation is solved analytically and is known to be linear, and (c) to determine experimentally for a case where the relation is non-linear, but the critical loads can be computed, so that it may be seen how far the relation may be approximated by a linear one and how accurate the extrapolated critical load will be.

II. THEORY

In the theoretical study, the following nomenclature is used:

I	Second moment of area of a section
E	Young's modulus of elasticity
M	Moment at a section
P	Axial force
q	Radial uniformly distributed load
ρ	Mass density per unit length
l	Length of a member
r	Radius
λ_n	Frequency of the n'th mode in cycles per unit time
x	Coordinate along the direction of the axial load
y	Lateral displacement
w	Radial displacement
u	Tangential displacement
θ	Angle in polar coordinates

(1) Simple Column.

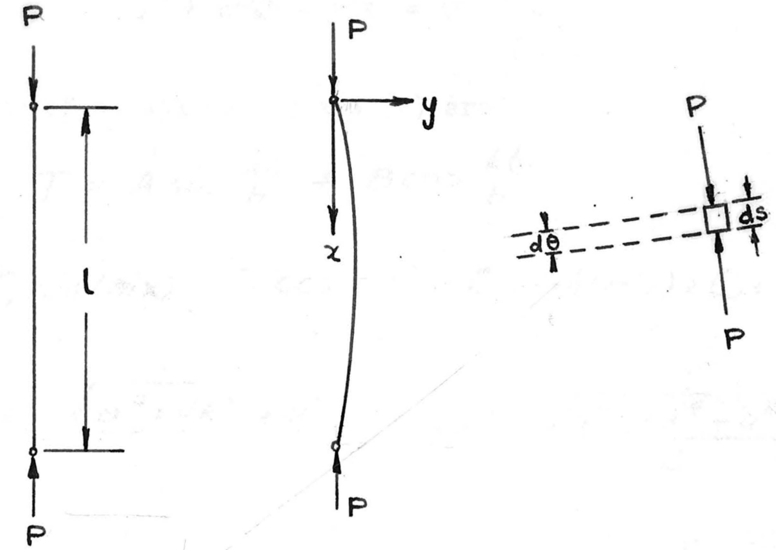


Fig. 1

When a straight prismatical column under an axial compressive load P is deformed, the moment at a section (x,y) is $-EI\frac{\partial^2 y}{\partial x^2}$

The intensity of the lateral restoring force is then $-EI\frac{\partial^4 y}{\partial x^4}$

The lateral restoring force due to the axial force P is

$$-P\frac{\partial \theta}{\partial s} = -P\frac{\partial^2 y}{\partial x^2}$$

The inertia force is $-\rho\frac{\partial^2 y}{\partial t^2}$. For equilibrium, the sum of the forces must be zero.

$$\therefore \quad EI\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial^2 y}{\partial t^2} = 0$$

Let $a^2 = \frac{P}{EI}$; and $b^2 = \frac{\rho}{EI}$; and

$$Y = X(x).T(t)$$

Then

$$\frac{X^{IV} + a^2 X''}{X} = -\frac{b^2 T''}{T} = k^2$$

where K is an arbitrary constant.

Therefore,

$$T'' + \left(\frac{k}{b}\right)^2 T = 0 \tag{1}$$

$$X^{IV} + a^2 X'' - k^2 X = 0 \tag{2}$$

The solutions of equations (1) and (2) are

$$T = A \sin\frac{kt}{b} + B\cos\frac{kt}{b} \tag{3}$$

$$X = C_1 \sin(mx) + C_2\cos(mx) + C_3 \sinh(m'x) + C_4\cosh(m'x) \tag{4}$$

where

$$m^2 = \frac{\sqrt{a^4+4k^4}+a^2}{2} \quad ; \quad m'^2 = \frac{\sqrt{a^4+4k^4}-a^2}{2}$$

To satisfy the boundary conditions:

(a) at $x = 0$, $X = 0$; (b) at $x = 0$, $X'' = 0$;

(c) at $x = l$, $X = 0$; (d) at $x = l$, $X'' = 0$;

we obtain, from equation (4), the condition

$$\sin ml = 0$$

or

$$ml = n\pi$$

where $n = 1, 2, 3, \ldots\ldots$

Hence

$$k^2 = \frac{n^4\pi^4}{l^4} - \frac{n^2\pi^2 a^2}{l^2}$$

But from equation (3), $\lambda_n = \frac{1}{2\pi}\left(\frac{k}{b}\right)$ is the natural frequency of the n'th mode. Therefore,

$$\lambda_n^2 = \frac{1}{4\pi^2}\cdot\left(\frac{k}{b}\right)^2$$

or

$$a^2 = \frac{n^2\pi^2}{l^2} - \frac{4b^2l^2}{n^2}\lambda_n^2$$

Substituting values of 'a' and 'b', we have

$$P = \frac{n^2\pi^2 EI}{l^2} - \frac{4\rho l^2}{n^2}\lambda_n^2 \tag{5}$$

From equation (5), it is seen that there is a linear relationship between λ_n^2 and P.

When $\lambda_n = 0$,

$$P = \frac{n^2 \pi^2 EI}{l^2}$$

which is Euler's buckling load.

When P = 0,

$$\lambda_n = \frac{n^2 \pi}{2 l^2} \sqrt{\frac{EI}{\rho}}$$

which is the natural frequency of the column in free vibration.

A further discussion of the equation will be found in Part IV, where the results of the experiment on a simple column are considered.

(2) Circular Ring.

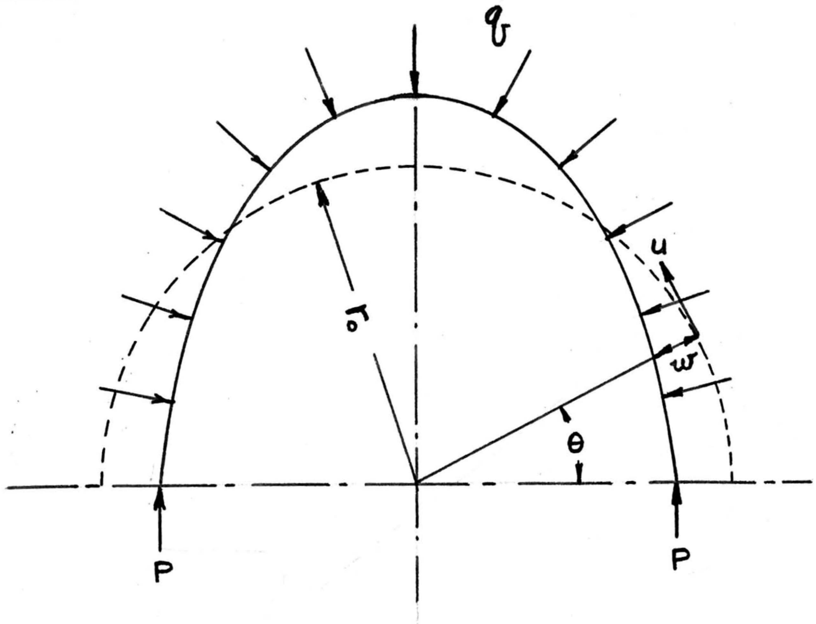


Fig. 2

The effect of a constant uniform radial load upon the frequency of a circular ring of uniform cross section, has been studied by Massonnet (Ref. 1). However, he did not give the derivation of his basic equation connecting the frequency and the radial load. A somewhat simpler equation can be obtained with the following assumptions:

(a) the axial load in the ring remains constant;

(b) the circumferential length of the ring remains constant (Ref. 2) i. e., the circumferential strain

$$\frac{du}{ds} - \frac{w}{r} = 0$$

or

$$\frac{du}{d\theta} - w = 0$$

Let $$w = \sum a_n \sin n\theta \sin \lambda_n t$$

Then $$u = \int w \, d\theta$$

$$= -\sum \frac{a_n}{n} \cos n\theta \sin \lambda_n t$$

The moment at a section of the ring due to a change in curvature (Ref. 3) is

$$M = -\frac{EI}{r_o^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)$$

Therefore, the strain energy due to this moment is

$$V_1 = \int_0^{2\pi} \frac{EI}{2 r_o^4}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)^2 r_o \, d\theta$$

$$\therefore \delta V_1 = \frac{EI}{r_o^3} \int_0^{2\pi} \left(\frac{\partial^4 w}{\partial \theta^4} + 2\frac{\partial^2 w}{\partial \theta^2} + w\right) \delta w \, d\theta$$

The effect of the axial load P is equivalent to a radially distributed load of intensity (Ref. 3)

$$\frac{P}{r_o^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)$$

Therefore, the total radial load $q + \frac{P}{r_o^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)$

The potential energy of this load with a deflection 'w' is

$$V_2 = \int_0^{2\pi} \int_0^{w} \left[q + \frac{P}{r_o^2}\left(\frac{\partial^2 w_1}{\partial \theta^2} + w_1\right) dw_1\right] r_o \, d\theta$$

$$\therefore \delta V_2 = \int_0^{2\pi} \left[q r_o + \frac{P}{r_o}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) \delta w \, d\theta\right.$$

But $$\int_0^{2\pi} q r_o \, \delta w \, d\theta = 0$$

$$\therefore \delta V_2 = \int_0^{2\pi} \frac{P}{r_o}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) \delta w \, d\theta$$

Now, $$\delta V = \delta V_1 + \delta V_2$$

$$\therefore \delta V = \frac{EI}{r_o^3}\int_0^{2\pi}\left[\frac{\partial^4 w}{\partial\theta^4} + \left(2+\frac{Pr_o^2}{EI}\right)\frac{\partial^2 w}{\partial\theta^2} + w\left(1+\frac{Pr_o^2}{EI}\right)\right]\delta w\, d\theta \tag{6}$$

The kinetic energy of the ring is

$$T = \int_0^{2\pi}\frac{\rho}{2}\left[\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] r_o\, d\theta$$

$$\therefore \delta T = \rho r_o \int_0^{2\pi}\left(\frac{\partial^2 w}{\partial t^2}\delta w + \frac{\partial^2 u}{\partial t^2}\delta u\right) d\theta \tag{7}$$

For a particular mode w_n,

$$\frac{\partial^2 w_n}{\partial t^2}\delta w_n = -a_n\lambda_n^2 \sin n\theta \sin\lambda_n t \cdot \delta a_n \sin n\theta \sin\lambda_n t$$

$$= -a_n\lambda_n^2 \sin^2 n\theta \sin^2\lambda_n t\, \delta a_n$$

$$\frac{\partial^2 u_n}{\partial t^2}\delta u_n = \frac{a_n}{n}\lambda_n^2 \cos n\theta \sin\lambda_n t \cdot \frac{-\delta a_n}{n}\cos n\theta \sin\lambda_n t$$

$$= -\frac{a_n}{n^2}\lambda_n^2 \cos^2 n\theta \sin^2\lambda_n t\, \delta a_n$$

Substituting in equation (7), we have

$$\delta T = -\rho r_o a_n \lambda_n^2 \sin^2\lambda_n t \int_0^{2\pi}\left(\sin^2 n\theta + \frac{\cos^2 n\theta}{n^2}\right)\delta a_n\, d\theta$$

$$= -a_n \rho r_o \lambda_n^2\left(1+\frac{1}{n^2}\right)\sin^2\lambda_n t\, \delta a_n \tag{8}$$

and also for the n'th mode, equation (6) gives

$$\delta V = \frac{EI}{r_o^3} a_n \sin\lambda_n t \int_0^{2\pi}\left[n^4 \sin n\theta - \left(2+\frac{Pr_o^2}{EI}\right)n^2 \sin n\theta\right.$$

$$\left. + \left(1+\frac{Pr_o^2}{EI}\right)\sin n\theta\right]\delta a_n \sin n\theta \sin\lambda_n t\, d\theta$$

$$= \frac{EI}{r_o^3} a_n\left[(n^2-1)^2 - (n^2-1)\frac{Pr_o^2}{EI}\right]\sin^2\lambda_n t\, \delta a_n \tag{9}$$

By Hamilton's Principle,

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0$$

Therefore, substituting equations (8) and (9), we have

$$\int_{t_1}^{t_2} \left[-\rho r_0 \lambda_n^2 \left(1 + \frac{1}{n^2}\right) + \frac{EI}{r_0^3}\left[(n^2-1)^2 - (n^2-1)\frac{Pr_0^2}{EI}\right] a_n \sin^2 \lambda_n t \; \delta a_n \, dt = 0$$

Since $a_n \neq 0$, and δa_n is arbitrary, therefore

$$\rho r_0 \left(1 + \frac{1}{n^2}\right) \lambda_n^2 = \frac{EI}{r_0^3}\left[(n^2-1)^2 - (n^2-1)\frac{Pr_0^2}{EI}\right]$$

$$\therefore P = \frac{(n^2-1)EI}{r_0^2} - \frac{\rho r_0^4 (n^2+1)}{n^2(n^2-1)} \lambda_n^2$$

But P is assumed to be constant,

$$P = qr_0$$

$$\therefore q = \frac{(n^2-1)EI}{r_0^3} - \frac{\rho r_0^3 (n^2+1)}{n^2(n^2-1)} \lambda_n^2 \qquad (10)$$

which gives the relation between 'q' and λ_n^2.

When $\lambda_n = 0$, the equation gives

$$q = \frac{(n^2 - 1)EI}{r_0^3}$$

which isthe critical load of the ring when it will buckle with the n'th mode (Ref. 3).

Also when $q = 0$,

$$\lambda_n = \frac{(n^2-1)n}{r_0^3}\sqrt{\frac{EI}{\rho(n^2+1)}}$$

which isthe natural frequency of a ring in free vibration (Ref. 4).

Equation (10) also shows that load and the square of frequency have a linear relationship.

(3) Rectangular Frame.

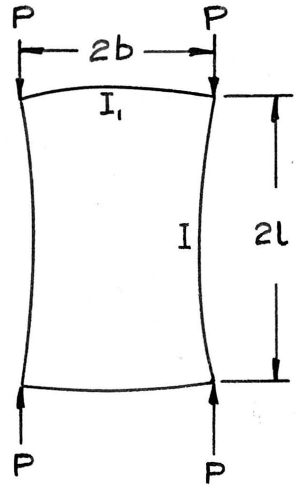


Fig. 3

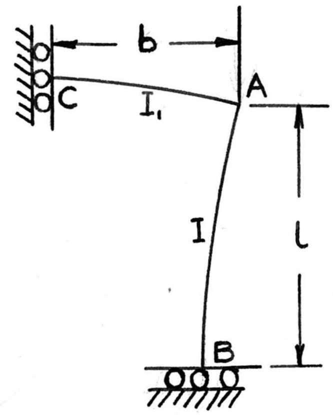


Fig. 4

When a rectangular frame of constant cross section is loaded as shown in fig. 3, the symmetrical mode of vibration may be replaced by its equivalent system CAB shown in fig. 4. Members AB and AC are now to be considered separately and the conditions at the point A are to be matched.

(i) Member AB.

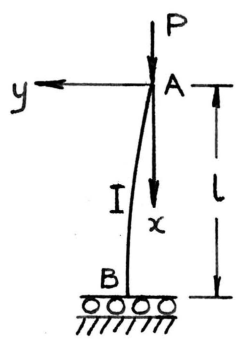


Fig. 5

The differential equation of motion is

$$EI\frac{\partial^4 y}{\partial x^4} + P\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial^2 y}{\partial t^2} = 0$$

Let

$$y = \Sigma Y_n \sin\lambda_n t$$

For a particular 'n', $\quad EI\,Y_n^{IV} + P Y_n'' - \rho\lambda_n^2 Y_n = 0$

or
$$Y_n^{IV} + a^2 Y'' - d^4 Y_n = 0$$

where $a^2 = P/EI$, and $d^4 = \rho\lambda_n^2/EI$.

The solution to the equation is

$$Y_n = C_1 \sin(kx) + C_2 \cos(kx) + C_3 \sinh(k'x) + C_4 \cosh(k'x)$$

where
$$k^2 = \frac{\sqrt{a^4 + 4d^4} + a^2}{2} \quad ; \quad k'^2 = \frac{\sqrt{a^4 + 4d^4} - a^2}{2}$$

The boundary conditions are:

(a) at $x = 0$, $Y_n = 0$; (b) at $x = 1$, $Y_n' = 0$.

Therefore, $C_2 + C_4 = 0$, or $C_4 = -C_2$;

and
$$k C_1 \cos(kl) + k' C_3 \cosh(k'l) = 0$$

or
$$C_3 = -\frac{k}{k'} \alpha C_1 \quad , \text{ where } \quad \alpha = \frac{\cos(kl)}{\cosh(k'l)}$$

$$\therefore Y_n = C_1 \left[\sin(kx) - \frac{k}{k'} \alpha \sinh(k'x)\right] + C_2 \left[\cos(kx) - \cosh(k'x)\right]$$

The slope at the point A is
$$Y_n' \Big|_{x=0} = \left[C_1 k(\cos kx - \alpha \cosh k'x) + C_2(-k \sin kx - k' \sinh k'x)\right]_{x=0}$$
$$= C_1 k(1-\alpha) \tag{11}$$

and the moment at the point A is

$$EI\, Y_n'' \Big|_{x=0} = EI\left[C_1 k(-k \sin kx - \alpha k' \sinh k'x) + C_2(-k^2 \cos kx - k'^2 \cosh k'x)\right]_{x=0}$$

$$= -C_2(k^2 + k'^2) EI \tag{12}$$

(ii) <u>Member AC</u>.

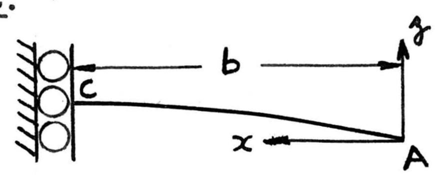


Fig. 6

The differential equation of motion is

$$EI_1 \frac{\partial^4 z}{\partial x^4} + \rho \frac{\partial^2 z}{\partial x^2} = 0$$

Let
$$z = \sum Z_n \sin \lambda_n t$$

For a particular 'n',

$$Z_n^{IV} - \frac{\rho \lambda_n^2}{EI} Z_n = 0$$

Let $q^4 = \rho \lambda_n^2 / EI$; then the solution of the equation is

$$Z_n = C_5 \sin qx + C_6 \cos qx + C_8 \cosh qx + C_7 \sinh qx$$

The boundary conditions are

(a) at $x = 0$, $Z_n = 0$; (b) at $x = b$, $Z_n' = 0$.

Therefore, $C_6 + C_8 = 0$, or $C_8 = -C_6$

and
$$C_5 q \cos qb + C_7 q \cosh qb = 0$$

or
$$C_7 = -\beta C_5 \quad , \text{ where } \beta = \frac{\cos qb}{\cosh qb}$$

$$Z_n = C_5 (\sin qx - \beta \sinh qx) + C_6 (\cos qx - \cosh qx)$$

The slope at the point A is

$$Z_n'|_{x=0} = [C_5 q (\cos qx - \beta \cosh qx) - C_6 q (\sin qx + \sinh qx)]_{x=0}$$
$$= C_5 q (1 - \beta) \qquad (13)$$

and the moment at the point A is

$$EI_1 Z_n''|_{x=0} = EI_1 [C_5 q^2 (-\sin qx - \beta \sinh qx) - C_6 q^2 (\cos qx + \cosh qx)]_{x=0}$$
$$= -2 C_6 q^2 EI_1 \qquad (14)$$

(iii) At the Point A.

The conditions to be satisfied at the point A are that the slopes and moments are to be equal. Equating ~~the~~ equations (11) and (12) to equations (13) and (14) respectively, we have

$$C_1 k(1-\alpha) - C_5 q(1-\beta) = 0 \qquad (15)$$

$$C_2(k^2+k'^2) - 2C_6 q^2 \frac{I_1}{I} = 0 \qquad (16)$$

Also we have the conditions that

$$Y_n\big|_{x=2l} = 0 \qquad Z_n\big|_{x=2b} = 0$$

which give

$$C_1\left(\sin 2kl - \frac{k\alpha}{k'}\sinh 2k'l\right) + C_2(\cos 2kl - \cosh 2k'l) = 0 \qquad (17)$$

and

$$C_5(\sin 2qb - \beta\sinh 2qb) + C_6(\cos 2qb - \cosh 2qb) = 0 \qquad (18)$$

From equations (15), (16), (17) and(18), for the constants to be consistent, the following relation must be satisfied:

$$\begin{vmatrix} k(1-\alpha) & 0 & -q(1-\beta) & 0 \\ \left(\sin 2kl - \frac{k\alpha}{k'}\sinh 2k'l\right) & (\cos 2kl - \cosh 2k'l) & 0 & 0 \\ 0 & 0 & (\sin 2qb - \beta\sinh 2qb) & (\cos 2qb - \cosh 2qb) \\ 0 & k^2+k'^2 & 0 & -2q^2\frac{I_1}{I} \end{vmatrix}$$

$$= 0$$

or
$$-2q^2\frac{I_1}{I}k(1-\alpha)(\cos zkl-\cosh zk'l)(\sin zqb-\beta\sinh zqb)$$
$$+q(1-\beta)(k^2+k'^2)(\sin zkl-\frac{k\alpha}{k'}\sinh zk'l)(\cos zqb-\cosh zqb)$$
$$=0$$

Substituting the values of k, k', q, α, and β, we will get a transcendental equation in λ_n, whose roots give the frequency of vibration of the various symmetrical modes, with the load P as a parameter. Hence it is also an equation relating P and λ_n^2. It will be evident from the complex nature of the equation that the relation between P and λ_n^2 will not be linear, and hence a graph of P vs. λ_n^2 will be a curve rather than a straight line. Due to the complexity of this relationship, we will leave to experimental investigations the problem of determining how far this curvalinear relation deviates from a straight line, and whether the mothod of frequency measurement is applicable for the determination of a critical load in this case.

III. EXPERIMENTS

(1) Simple Column.

(A) Description of the Model.

The column is made of cold rolled dural (24-ST) bar, supported on both ends on knife edges. Detailed dimensions of the column and a section of the support are shown in fig. 7 and 8. Fig. 9 shows the experimental set up of the instruments.

The ends of the column are reinforced with hardened steel anvils to serve as seats for the knife edges. The top knife edge is mounted on a differential screw and nut arrangement, so that load may be applied to the column by turning the nut. Stress in the column is measured by a pair of strain gauges at about a quarter distance from the support. The gauges form one leg of a wheatstone bridge. The unbalanced voltage of the bridge circuit is measured by a self-balancing potentiometer. The strain is then deduced by using the gauge constant supplied by the manufacturer.

The potentiometer is first calibrated for a known change of resistance in the strain gauges by shunting a known resistance across the gauges. A pair of dummy gauges mounted on a similar bar hanging freely beside the column are used as the other corresponding leg in the bridge circuit, as a means of compensating the temperature effect.

The frequency of the column is of the order of 50 cycles per second. Hence it is easily measured by a stroboscope.

(B) Calculation of the 1st Critical Load and Stress.

Length of the column	25-1/16"
Cross-section	0.750" x 0.400"
Young's Modulus (dural)	10.6×10^6 lbs./in.^{2}
Specific weight	0.100 lb./in.^{3}

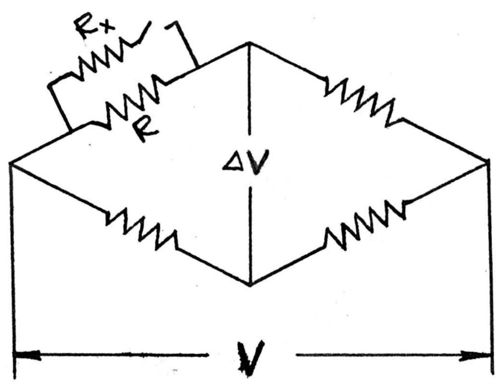
$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$= \frac{\pi^2 \times 10.6 \times 10^6 \times 0.75 \times 0.4^3}{12 \times 25.06^2}$$

$$= 665 \text{ lbs.}$$

$$\sigma_{cr} = \frac{665}{0.750 \times 0.400} = 2210 \text{ lbs./in.}^2$$

(C) Calibration of Potentiometer and Calculation of Stress.



The Calibration data are:

V	6.22 volts.
V_o	0.427 mv.
V'	1.970 mv.
R_x	245,000 ohms.
R	243 ohms.

$$\frac{\Delta V}{V} = \frac{0.001543}{6.22}$$

$$\frac{\Delta R}{R} = \frac{243}{245,000}$$

$$K = \frac{\Delta R/R}{\Delta V/V} = \frac{243 \times 6.22}{245,000 \times 0.001543}$$

$$= 3.96$$

Gauge Constant = $2.00 \pm 1\%$

$$\text{Strain (e)} = 2.00 \times \frac{\Delta R}{R}$$

$$= 2.00 \times 3.96 \times \frac{\Delta V}{V}$$

$$= \frac{7.92}{6.22} \times \Delta V$$

$$\sigma = E.e$$

$$= 10.6 \times 10^6 \times \frac{7.92}{6.22} \times \Delta V$$

$$= 13.5 \times 10^3 \times \Delta V.$$

where ΔV is measured in millivolts.

(D) Experimental Results.

The measured results are tabulated below:

No.	V mv.	ΔV	σ #/in^2	λ cps.	λ^2
1	0.427	0.000	0	52.2	2,730
2	0.448	0.021	284	48.8	2,380
3	0.470	0.043	580	44.7	2,000
4	0.490	0.063	850	41.0	1,680
5	0.506	0.079	1,065	37.2	1,380
6	0.525	0.098	1,321	33.8	1,140
7	0.540	0.113	1,522	28.9	838

The graphical representation of the above results is shown in fig. 10. It may be noted here that the extrapolated critical load is 2200 lbs./in.^{2} as compared to the calculated value of 2210 lbs./in.^{2}

(2) Rectangular Frame.

(A) Description of the Model.

The frame is made of cold rolled steel. Each member is annealed before being brazed together at the four corners. The whole frame is again annealed after completion. At the corners on the center line of each member, a shallow notch is cut to receive the knife edge for loading. Fig. 11 shows the dimensions and corner details of the frame. The knife edges are $\frac{1}{4}$ x $\frac{1}{4}$ tool bits. At the bottom side they rests on two small V-blocks. On the top, the knife edges are carried by a heavy I-beam. The load is applied through another knife edge on the beam, mid-way between the other two knife edges, so that the total load is equally divided between the two legs of the frame. The load is applied by a universal testing machine with a maxium capacity of 30,000 lbs. Fig. 12 is a close-up view of the model aligned in the testing machine.

The frequency is measured by recording the output of gauges on an oscillograph. The frame is excited by hitting one of the vertical members with a rubber mallet.

A check on the final dimension of the frame showed that the corners deviated from a right angle by an amount of about one and half degrees. This error is taken care of in the test set-up, where the bottom knife edge seats are adjusted so that the center line of the vertical member is perpendicular to the base of the testing machine.

(B) Calculation of Critical Loads.

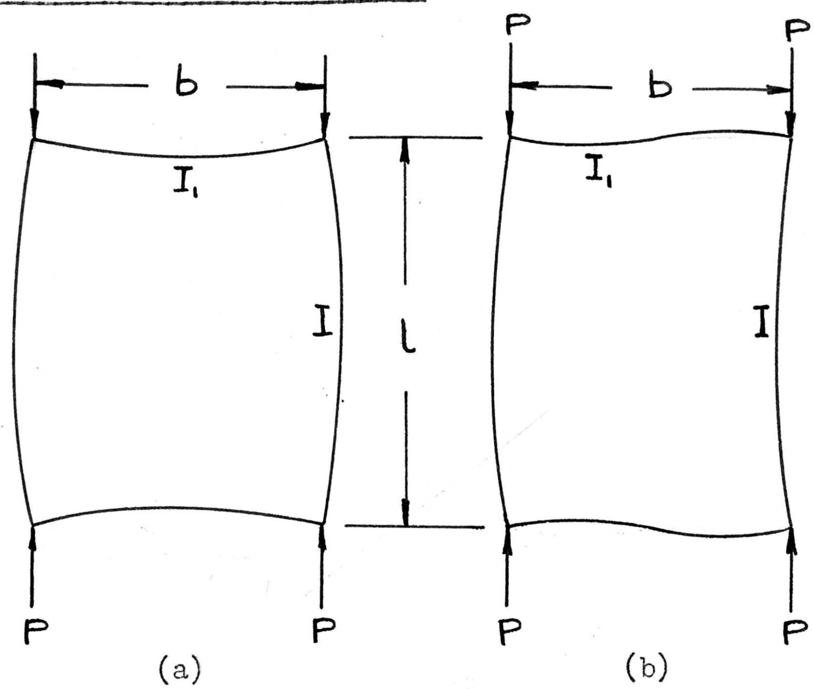


Fig. 13

The condition required for buckling in the mode shown in fig. 13a is *

$$-\frac{\tan\frac{kl}{2}}{\frac{kl}{2}} = \frac{bI}{lI_1}$$

We have $I = I_1$, $l = 14\text{-}5/8''$, and $b = 10''$,

$$\frac{b}{l} = \frac{10}{14.63} = 0.683$$

$$\frac{kl}{2} = 2.18$$

$$P_{cr} = \frac{4 \times 2.18^2 \times 30 \times 10^6 \times 0.0044}{14.63^2}$$

$$= 11{,}700 \text{ lbs.}$$

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* The derivation of the condition may be found in "Theory of Elastic Stability" by Timoshenko, page 90 and the following pages.

The condition for buckling in the mode shown in fig. 13b is

$$-\frac{\tan\frac{kl}{2}}{\frac{kl}{2}} = \frac{bI}{3I_1 l}$$

For $\frac{b}{3l} = 0.228$, $kl/2 = 2.62$.

$$P_{cr} = 11{,}700 \times \left(\frac{2.62}{2.18}\right)^2$$
$$= 17{,}000 \text{ lbs.}$$

(C) Experimental Results.

The measured results are tabulated below:

P* lbs.	λ cps.	λ^2 10^4/sec.^{2}	P* lbs.	λ cps.	λ^2 10^4/sec.^{2}
0	204	4.15	4,515	179	3.21
550	205	4.20	5,010	175	3.09
1,030	204	4.16	5,455	172	2.95
1,515	202	4.08	6,025	168	2.82
2,005	199	3.95	6,515	165	2.74
2,460	196	3.85	7,490	161	2.60
3,020	192	3.69	8,005	157	2.49
3,505	189	3.56	8,420	155	2.36
4,000	183	3.35			

A graphical representation of the above data is shown in fig. 14.

It should be noted that: (1) there is a rise in frequency in the earlier part of the loading; and (2) the experimental points lies close to a straight line, and the extrapolated buckling load is 17,500 lbs. which is about 49% higher than the calculated lowest critical load of 11,700 lbs., but is about 3% higher than the lowest critical load of 17,000 lbs. for unsymmetrical buckling.

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* Correction for the weight of the loading beam has been added.

IV. COMPARISON OF ANALYSIS AND EXPERIMENT.

(1) Simple Column.

The experimental points, being very close to a straight line, give a good check on the analysis of the simple column. The extrapolated critical stress of 2200 lbs./in.^{2} also checks well with the calculated value of 2210 lbs./in.^{2}

The biggest error in the measurement is ΔV, which is the difference of two small values. An estimate of the maximum value of the errors is about $\pm 4\%$. Hence the extrapolated result of the critical load is well within the expected experimental error. However, it must also be remembered that the calculated value is only theoretical, and it is subjected to errors in the choice of value E, and in the measured values of I and l.

It is also interesting to check the slope of the curve, K, i.e.,

$$K = \frac{2200 \times 0.3}{2730} = 0.241 \text{ lb.-sec.}^2$$

But the equation (5), for $n = 1$, gives

$$K = 4\rho l^2$$

$$= \frac{4 \times 0.1 \times 0.4 \times 0.75 \times 25.06^2}{32.2 \times 12}$$

$$= 0.195 \text{ lb.-sec.}^2$$

The experimental value is about 23% higher.

For thepurpose of better bearing of the pivots, the ends of the column are reinforced with two pieces of hardened steel. Evidently these masses lowers the frequencies of the column, for the theoretical natural frequency of the column is

$$\lambda_1 = \frac{\pi}{2 \times 25.06^2} \sqrt{\frac{10.6 \times 10^6 \times 0.4^3 \times 0.75 \times 32.2 \times 12}{12 \times 0.1 \times 0.4 \times 0.75}}$$

$$= 58.5 \text{ cps.}$$

while the measured frequency is 52.2 cps.

The correction factor for the critical load due to these reinforcements (Ref. 5) is $\left[1 + \frac{\pi^2}{6}\left(\frac{2a}{l}\right)^3\right]$, where 'a' is the length of reinforcements. For this case the factor does not have any significance.

The important result is that the linear relationship and the critical load check very well with the theory.

(2) Rectangular Frame.

The interesting result brought out by the experiment is that the curve of λ_n^2 vs. P comes out to be rather close to a straight line as far as the data go, although the theory indicates that the relation is non-linear. The fact that the extrapolated critical load of 17,500 lbs., as compared to the computed value of 17,000 lbs., for unsymmetrical buckling, indicates that the relation can be approximated with reasonable accuracy by a linear one, even though the data only extend up to about 50% of the buckling load. An increase in accuracy would be expected if the loads are carried closer to the buckling point.

The data obtained might be somewhat misleading, because the critical load extrapolated is not the lowest one, 11,700 lbs., which one would naturally expect to obtain. However, the rise in frequency at the early part of the loading indicates that the frame changes its mode of vibration when load is applied. Therefore, the critical load extrapolated will correspond to the mode in which the frequencies are measured. The records show that there is a beat frequency of the order of 10 cycles per second at the beginning of loading. Hence there are two dominant frequencies present, and they are very close to each other. It will be shown later that the supports will absorb less energy for unsymmetrical modes than for symmetrical ones. As the frequencies are so close together, it may happen that the total energy required for exciting the slightly higher mode is actually lower. The unsymmetrical mode is the next higher one, and its frequency appears very close to that of the lowest symmetrical mode. Hence it is this mode in which the frequencies are measured. However, no analysis of the natural frequency of the system is made. It is interesting to see, however, from the experimental curve, that when the straight portion of the curve is extrapolated to zero load, a frequency of 211 cps. is found. This is probably the natural frequency of the unsymmetrical mode, and is about 7 cps. higher than the frequency measured at zero load, which probably is the frequency for the symmetrical mode, the difference being of the order of the beat frequency.

The method of excitation by hitting one of the vertical members is certainly capable of introducing the two basic modes simultaneously. With no device to indicate the mode, it is difficult to tell whether one mode is completely absent or not. However, the beat frequency may give some kind of indication. The beat frequency increases with the load, as is expected, while the amplitude of the beat decreases with the load. The presence of beats is not marked when load reaches 4,000 lbs., and the beats disappear completely from the records for loads above 6,000 lbs. This indicates that one of the modes is more stable than the other. This may be clarified by considering the conditions at the supports.

It was noticed during loading, that local yielding at the knife edges took place at loads above 4,000 lbs. and the frame had to be excited several times before the load became steady. Inspection made after the test showed that the knife edges made indentations as deep as 1/32" in the notches in the frame. This indentation of the knife edge introdues friction and absorbs energy of vibration. The energy lost at the supports will be proportional to the amount of rotation of the corners. It has been shown that, for the same corner moment, the corners rotate three times as much for the symmetrical mode as for the unsymmetrical mode (the rotation being $\frac{2EIM}{b}$ and $\frac{6EIM}{b}$ respectively). So, as the knife edges dig deeper into the notches, it becomes more difficult to excite the lowest symmetrical mode, and the unsymmetrical mode becomes the only dominant one. Therefore, the measured frequency is in this mode. The check on the critical load for this mode also indicates that the frequency is in the same mode.

When two frequencies are very close together as in this case, even though the initial motion may be in the symmetrical mode, the dynamic coupling between the modes caused by non-symmetry either in dimensions, in load, in density of material, or in friction at the supports, etc., may be enough to start the other mode, initiated by a small phase difference on the two sides of the frame. Evidently, what happened in this case, is that the coupling between the modes has stabilized the unsymmetrical mode. In a full scale structure, where non-symmetry in construction and loading etc. is more likely to occur than in a model, it is quite possible to have enough such dynamic coupling to throw off the frequency from the lowest mode. Therefore, one should look out for the possibility of such misleading results.

Though the measurement indicates the structure is stable in the unsymmetrical mode of vibration, one can not, however, conclude that the structure will buckle in that mode first. Its dominance may be purely because of the method of excitation and the support conditions. To check this a further experiment will be required, where some refinement in technique may be followed.

First, a definite method of exciting the frame should be devised, such that the simplest mode will be the only dominant one. A possible method consists of stretching a wire across the middle of the vertical members to draw them toward each other, and then cutting it to start the motion. Second, the support notches should be hardened or hardened inserts should be used, in order that they may not damp out the simplest mode, and that they will not introduce too much damping force at heavy loads.

It is difficult to say that with these refinements we shall be able to record the simplest mode with the absence of the next higher mode, because they are so close together, and one might be excited by the other though possible dynamic coupling between them. If this seems likely, the mode of vibration should be measured as well as the frequency.

One method of determining the modes could be to record the traces from corresponding pairs of strain gauges on opposite members of the frame simultaneously, so that the phase differences could be compared. Another method could be to connect the gauges in the same bridge circuit in such a way that they would give double output for a symmetrical mode, but would nullify each other for an unsymmetrical mode.

V. CONCLUSIONS & SUGGESTIONS

The interesting points which represent the worthwhile results of this work are:

(1) For a simple pin-ended beam, the theory and experiment check very well.

(2) The analytic solution for a circular ring shows that there is a linear relation between the uniform radial load and the square of frequency.

(3) The analytic solution shows that for a rectangular frame, the relation of P and λ_n^2 is not linear, but the experimental result indicates that the relation is very close to a linear one.

(4) The experimental results show that if there are modes with frequencies very close to each other, the stable one may not be the lowest mode, and thus the results may be misleading, unless the modes of vibration are measured as well as the frequencies.

The structures covered in this work are of a very simple type, and therefore no generalized conclusion can be drawn. Thus far, it may be said that the relation of P and λ_n^2 for prismatic beams is very close to a linear one for all end conditions except the case with fixed ends, which may be the next step to be investigated.

Other experiments which might profitably be conducted involving structures of the following types:

(1) Compression members in a truss.

(2) Different types of elastically unstable frameworks.

(3) Columns built-up with stiffening lattice work.

(4) Curved beams and arches.

(5) Plates and shells.

(6) Webs of plate girders in bending.

When the structures become sufficiently complicated so that many modes of vibration are possible, some definite controllable means of exciting the structure and devices of detecting the modes must be provided, or the results may become meaningless. Such tests should be checked for some complicated structures by actually carrying the structures to failure.

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(1) C. Massonnet: "Les Relations entre les Modes Normaux de Vibration et la Stabilité des Systèmes Élastiques." - Bulletin des Cours et des Laboratoires d'Essais des Constructions du Génie Civil et d'Hydraulique Fluviale, Vol. 1, Nos. 1 and 2, 1940.

(2) Lord Rayleigh: Theory of Sound.

(3) Timoshenko: Elastic Stability.

(4) Timoshenko: Vibration Problems in Engineering.

(5) Karman & Biot: Mathematical Methods in Engineering.

F I G U R E S

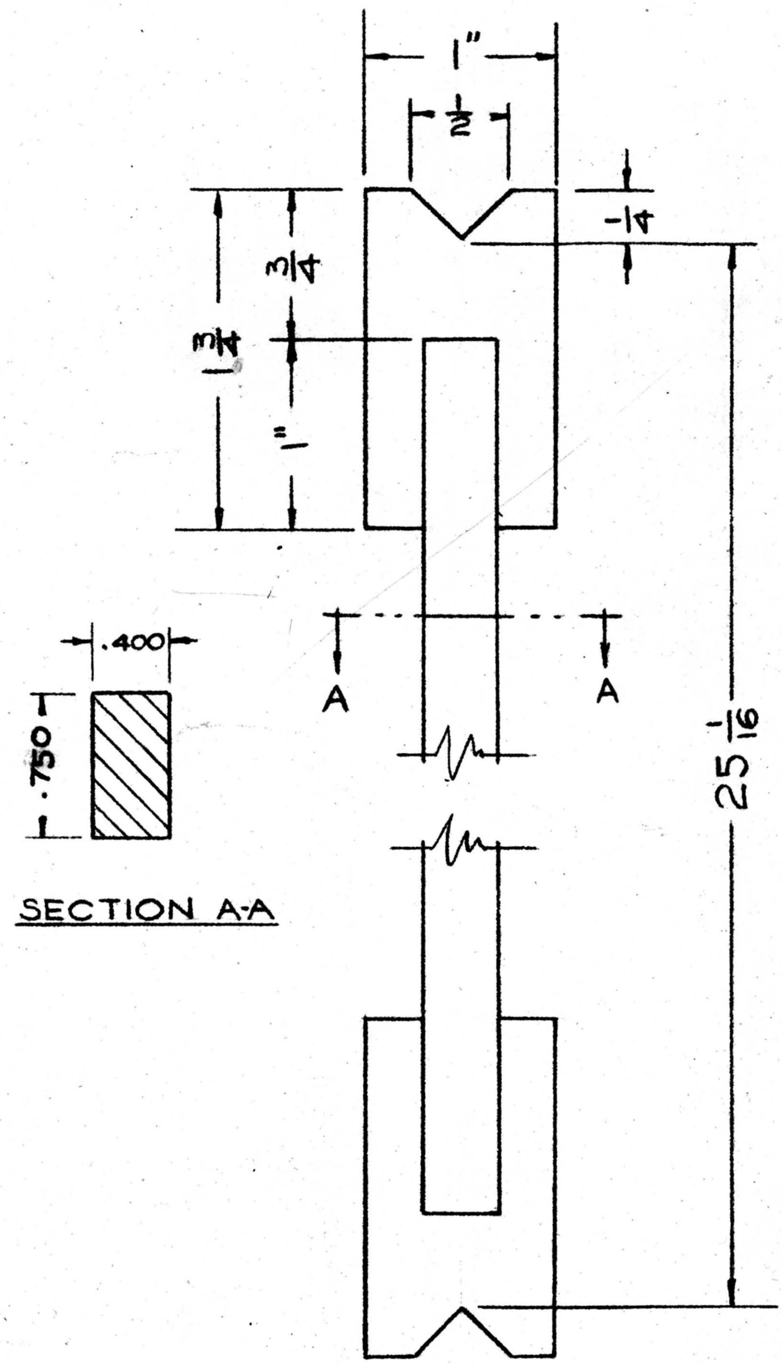

FIG. 7 - DIMENSIONS OF THE TEST MODEL OF A SIMPLE COLUMN

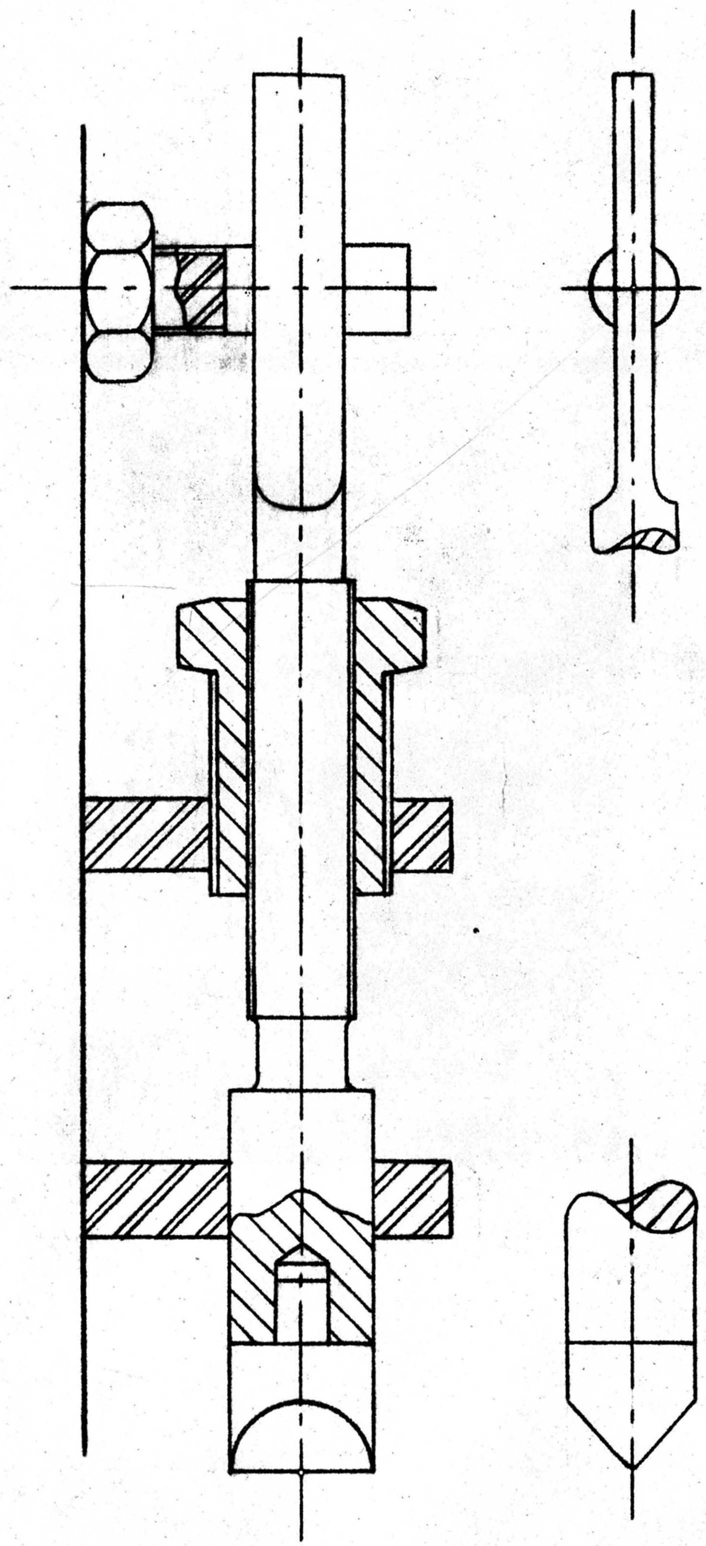

FIG. 8 . - A SECTION THROUGH THE TOP SUPPORT OF THE SIMPLE COLUMN.

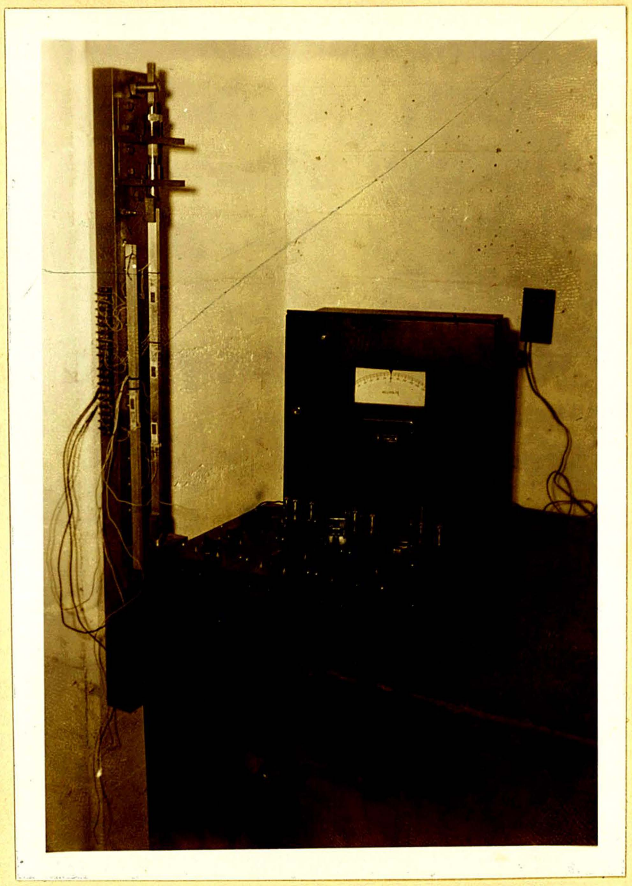

Fig. 9 - Photograph of the experimental set-up of the simple column.

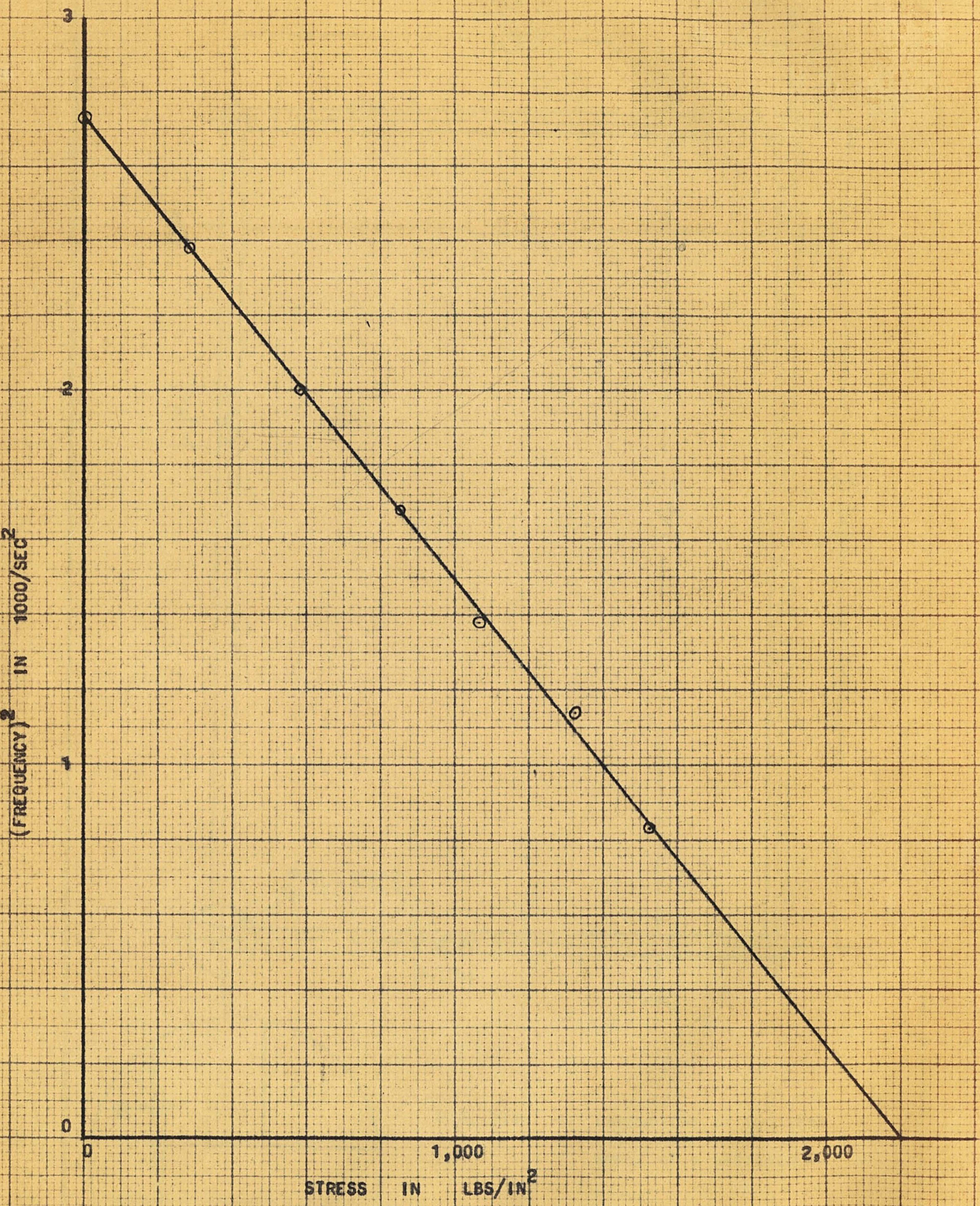

FIG. 10 - STRESS VS. (FREQUENCY)^2 RELATION OF THE SIMPLE COLUMN

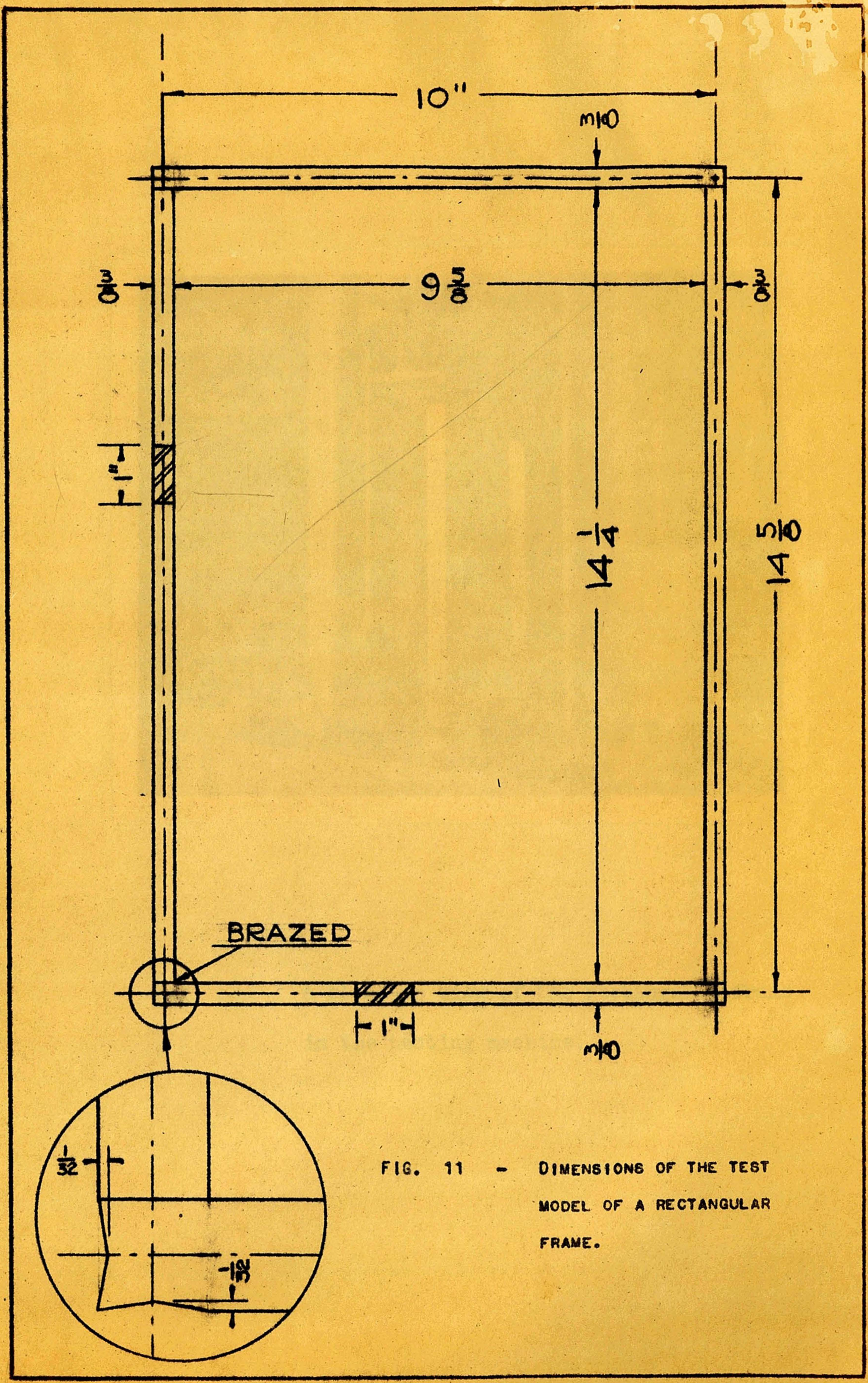

FIG. 11 - DIMENSIONS OF THE TEST MODEL OF A RECTANGULAR FRAME.

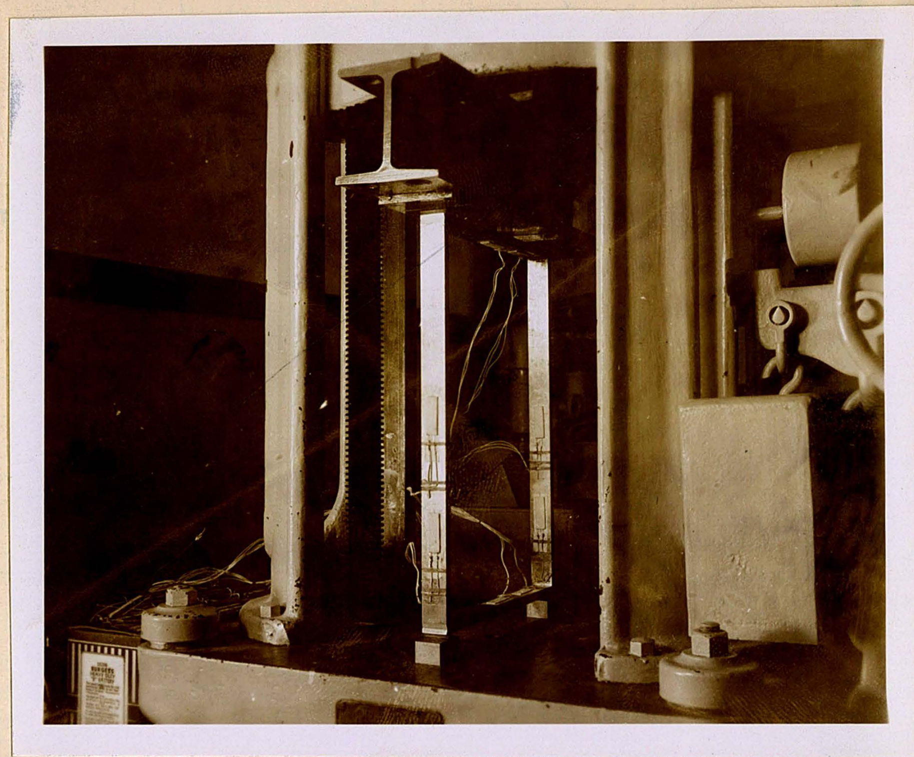

Fig. 12 - A close-up view of the rectangular frame in the testing machine.

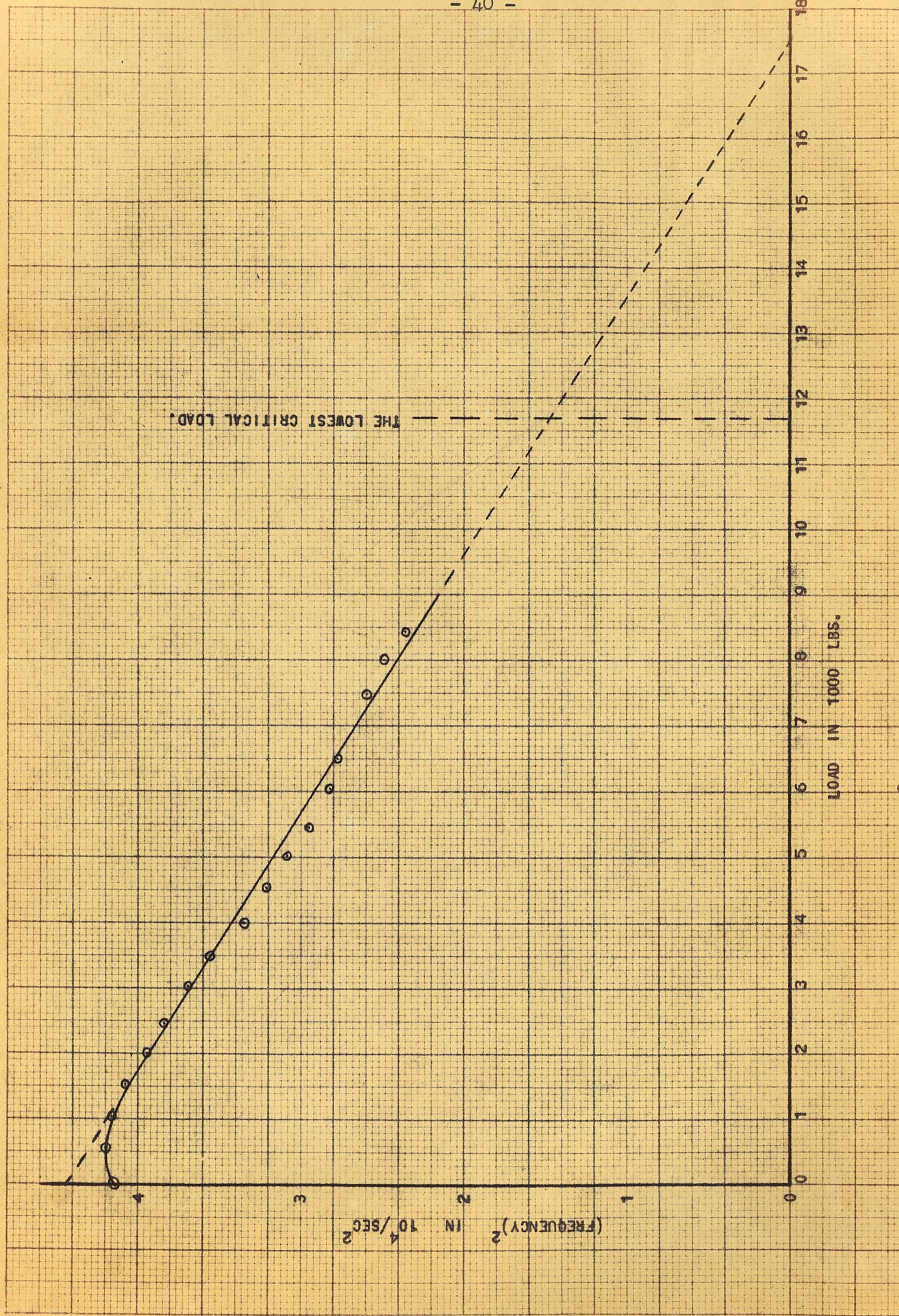

FIG. 14 - LOAD VS. (FREQUENCY)^2 RELATION OF THE RECTANGULAR FRAME - THE LONGER MEMBERS BEING LOADED.