SHARP BOUNDS FOR THE OPERATOR NORM
OF THE MEHLER KERNEL OPERATOR

Thesis by

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To my father, Ralph Deutsch

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ABSTRACT

In this thesis the norms of the Mehler kernel operators are calculated. In particular, Babenko's conjecture about the norms of these operators with purely imaginary parameter is settled. The proof is inspired by Wiener's approach to Fourier theory and his proof of Plancherel's theorem. An account is given of Wiener's approach, as well as of the important tools and theorems, particularly those of Beckner, needed to prove the main result. Applications to Kober operators and smoothing operators are given.

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NOTATION

Throughout this thesis we will use the following definitions.

(1) $\mathbb{R}$ is the set of real numbers.

(2) $\mathbb{C}$ is the set of complex numbers.

(3) If $1 \leq p < \infty$ then

$$L^p = L^p(\mathbb{R}) = \{\text{Lebesgue measurable } f:\mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)|^p dx < \infty\}$$

and, if $f \in L^p$, then

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} .$$

(4) $L^\infty = \left\{ \text{Lebesgue measurable } f:\mathbb{R} \to \mathbb{C} \mid \text{essential } \sup_x |f(x)| < \infty \right\}$

and, if $f \in L^\infty$, then

$$\|f\|_\infty = \text{essential } \sup_x |f(x)| .$$

(5) Let A be a linear mapping of L^p into L^q $(1 \leq p,q \leq \infty)$.

We write

$$\|A\|_{p,q} = \sup_{f \in L^p, f \neq 0} \|Af\|_q / \|f\|_p$$

and call this the p,q operator norm of A.

Let $d\mu$ be a measure other than Lebesgue measure.

(6) If $1 \leq p < \infty$ then

$$L^p(d\mu) = \left\{ \mu\text{-measurable } f:\mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)|^p d\mu(x) < \infty \right\}$$

and

$$L^{\infty}(d\mu) = \left\{ \mu\text{-measurable } f:\mathbb{R} \to \mathbb{C} \ \middle|\ \text{essential } \sup_{x} |f(x)| < \infty \right\} .$$

The corresponding norms , $\|f\|_{\mu;p}$, $\|f\|_{\mu;\infty}$, and $\|A\|_{\mu;p,q}$ have the obvious definitions.

INTRODUCTION

This thesis deals with integral operators associated with the kernels

$$N_\omega(x,y) = \frac{1}{\sqrt{\pi(1-\omega^2)}} \exp\left\{\frac{4xy\omega - (x^2+y^2)(1+\omega^2)}{2(1-\omega^2)}\right\} ,$$

where x and y are real and ω is a complex number with $|\omega| \leq 1$ and $\omega \neq \pm 1$. Our main results consist of formulae for the values of the norms of these integral operators as mappings of L^p into $L^q (1 \leq p \leq 2$ and $1/p + 1/q = 1)$. This generalizes the work of Babenko where the case that ω is purely imaginary and q is an even integer was considered. By taking $|\omega| = 1$, $\omega \neq \pm 1$, we get expressions for the norms of what are called the Kober operators, and by taking $0 \leq \omega < 1$ we get expressions for the norms of the smoothing operators which were considered by De Bruijn.

The kernel N_ω was introduced by F. G. Mehler [9] in 1866 in connection with the Laplace equation. He proved the identity

$$N_\omega(x,y) = \sum_{n=0}^{\infty} \omega^n \varphi_n(x)\varphi_n(y),$$

where φ_n denotes the n^{th} Hermite function. This identity, which plays a very important role in this thesis, was reestablished as an identity for the corresponding integral operators by Myller and Lebedeff in 1907 [10].

For sufficiently well behaved functions f we define

$$(N_\omega f)(x) = \int_{-\infty}^{\infty} N_\omega(x,y)f(y)dy \qquad (x \in \mathbb{R}) .$$

If we take $\omega = -i$ then $N_\omega f$ becomes $\mathfrak{F}f$, the Fourier transform of f. If $1 \leq p \leq 2$ and $1/p + 1/q = 1$ then it is well known that $\mathfrak{F}$ is a bounded linear mapping of L^p into L^q. In fact, we have the classical Hausdorff-Young inequality

$$\|\mathfrak{F}\|_{p,q} \leq (2\pi)^{1/q-1/2} .$$

However, $\mathfrak{F}$ is not a compact operator. This sometimes makes the study of $\mathfrak{F}$ difficult. The Mehler kernel operator N_ω, for $|\omega| < 1$, is compact and it may be used to approximate $\mathfrak{F}$. For this reason N_ω has become an important tool in Fourier transform theory.

In 1933, N. Wiener [15] used the Mehler kernel operators to give a proof of Plancherel's theorem which states that $\mathfrak{F}$ is an isometry of $L^2(\mathbb{R})$. Wiener's approach was used in 1961 by K. I. Babenko [2] to find the norm of the Fourier transform as a mapping of L^p into L^q in the case q is an even integer. This was done by calculating the norms of N_ω as a mapping of L^p into L^q for ω purely imaginary and q an even integer. Babenko conjectured that the formula he found for this norm (with q even) holds for all $q \geq 2$. One of the purposes of this thesis is to settle this conjecture.

In the proof of Babenko's conjecture we shall use a result of W. Beckner [4] which was generalized by F. B. Weissler [16]. This result, which was proven using a method of E. Nelson [11] concerns the norms of Mehler-type operators defined on $L^p(d\mu)$, where $1 \leq p \leq 2$ and $d\mu$ denotes the Gaussian measure on the real line. Beckner was able, by this method, to find the exact value of $\|\mathfrak{F}\|_{p,q}$.

We now briefly mention some areas in mathematics where operators of the Mehler type occur.

In a paper of 1939, H. Kober [8] considered the operators N_ω with $|\omega| = 1$ as roots of the Fourier operator $\mathfrak{F}$. At about the same time (1937), E. U. Condon [5] considered the same operators to construct a continuous group of transformations that includes the cyclic group of transformations generated by $\mathfrak{F}$.

In 1961, V. Bargmann [3] constructed a Hilbert space of entire functions on which Fock's operator solution to a commutation relation is realized (Bargmann-Fock representation). The transition from the usual Hilbert space $L^2(\mathbb{R})$ to this space is given as follows. The mapping B defined by

$$(Bf)(x) = \pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} f(y)\exp\{-\tfrac{1}{2}(x^2+y^2) + \sqrt{2}\, xy\}dy$$

maps $L^2(\mathbb{R})$ onto the space of all entire functions, g, of order ≤ 2 and type $\leq \frac{1}{2}$ such that

$$\int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} dz < \infty .$$

The latter space is a Hilbert space if we take the obvious inner product. Then B is an isometry that maps the n^{th} Hermite function onto $z^n/\sqrt{n!}$. If, now, ω is a complex number of modulus less than or equal to one, then

$$(BN_\omega f)(z) = (Bf)(\omega z) ,$$

(where we take $(N_\omega f)(x) = f(\omega x)$ if $\omega = \pm 1$).

N. G. De Bruijn, in a paper of 1973 [6] presented, among other things, a method for studying generalized functions by means of smoothing operators. For $\alpha > 0$, the smoothing operator S_α is essentially the Mehler operator with $\omega = e^{-\alpha}$ (see proposition 5.5). He also noted that the set of smoothing operators (and so the set of Mehler operators) forms a semigroup under composition. To sketch De Bruijn's approach to generalized functions, we let S be the set of all functions of the form $S_\alpha f$ with $\alpha > 0$ and $f \in L^2(\mathbb{R})$. Thus S can be thought of as the test function space on which the theory is built. A generalized function is then defined as a trace in the space S, i.e. a mapping F of $(0,\infty)$ into S such that $S_\alpha F(\beta) = F(\alpha+\beta)$ for all $0<\alpha,\beta<\infty$.

Both De Bruijn and Bargmann note that if H is the Hermite operator

$$H = \left(\frac{\partial^2}{\partial x^2} - x^2\right)$$

(harmonic oscillator), then

$$N_\omega = \omega^{-1/2} e^{-(\frac{1}{2}\log\omega)H} .$$

In particular, if we put $U(t) = e^{it/2} N_{e^{it}}$ for $t \in \mathbb{R}$, then U is the solution to the Schrödinger equation $U_t = -\frac{1}{2}iHU$ with $U(0) = I$.

We finally give a survey of the chapters of this thesis.

In chapter one of this thesis, the basic properties of N_ω are developed and Plancherel's theorem is proven by the method of Wiener. Chapter two deals with the Hausdorff-Young inequality for $\mathfrak{F}$ and the result and conjecture of Babenko. The Gaussian measure form of the Mehler operator is treated in chapter three. Chapter four contains the main result, i.e. the calculation of $\|N_\omega\|_{p,q}$. The norm is expressed as a function of a parameter γ which is given implicitly in terms of the roots of a certain fourth degree polynomial. In chapter five we calculate γ and hence $\|N_\omega\|_{p,q}$ in the special cases for Kober and smoothing operators. We also complete the Wiener-style proof of the sharp Hausdorff-Young inequality for the Fourier transform.

Chapter 1

L^2 THEORY AND THE FOURIER TRANSFORM

We begin by defining the Fourier transform. For functions $f \in L^1(\mathbb{R})$ we may define, for $x \in \mathbb{R}$,

$$(\mathfrak{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-ixy}dy \ . \qquad (1.1)$$

Norbert Wiener, in his book *The Fourier Integral and Certain of Its Applications* [15], showed by using Hermite expansions that $\mathfrak{F}$ is also definable for $f \in L^2(\mathbb{R})$ and that $\mathfrak{F}$ is in fact an isometry of $L^2(\mathbb{R})$. This result is usually referred to as Plancherel's theorem. Because Wiener's methods are similar to those we will use in a more general form later in this thesis, it is instructive to study them in some detail.

We begin with some elementary properties of $\mathfrak{F}$.

PROPOSITION 1.1(i) $\mathfrak{F}$ *is a linear operator from* L^1 *to* L^∞.

(ii) *If* A *is the operator defined by* $(Af)(x) = ixf(x)$ *then*

$$\mathfrak{F}f' = A\mathfrak{F}f \quad \textit{and} \quad \mathfrak{F}Af = -(\mathfrak{F}f)' \ .$$

(iii) *If* B *is the operator defined by* $(Bf)(x) = f''(x) - x^2f(x)$ *then*

$$\mathfrak{F}Bf = B\mathfrak{F}f \ .$$

Here we assume f is sufficiently well behaved for these statements to make sense. The validity of (i) is clear, (ii) follows from direct calculation, and (iii) is a consequence of (ii).

From (iii) one might expect the eigenfunctions of the operator B to be of some importance in the study of the Fourier transform. Hence we fix a complex number λ and consider the differential equation

$$y'' - x^2 y = \lambda y. \tag{1.2}$$

We let $h_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + 2^n x^n$, where each a_i is complex, and substitute $g(x) = e^{-x^2/2}h_n(x)$ into this equation. This yields the formulae

$$a_{k-2} = \frac{k(k-1)}{2k+\lambda-3}\, a_k \qquad (k = 2, 3, \ldots, n),$$

$$a_{n-1}(2n+\lambda-1) = a_n(2n+\lambda+1) = 0,$$

from which we find $\lambda = -(2n+1)$ and

$$h_n(x) = 2^n\left(x^n - \frac{n(n-1)}{4}\, x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{32}\, x^{n-4} - \ldots\right), \tag{1.3}$$

where the last term in the series involves x or 1 according to whether n is even or odd. It is clear that $e^{-x^2/2}h_n(x)$ is the only solution of (1.2)(aside from constant multiples) of the form $e^{-x^2/2}P(x)$, where P is a polynomial of degree n, and $\lambda = -(2n+1)$.

DEFINITION 1.2. *The polynomial* h_n *defined in* (1.3) *for a nonnegative integer* n *is called the* n^{th} *Hermite polynomial.*

PROPOSITION 1.3. $h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.

PROOF. Let $g(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. Then one may verify that g is a polynomial of degree n and $e^{-x^2/2}g(x)$ is a solution of (1.2) with $\lambda = -(2n+1)$. Furthermore, its leading coefficient is 2^n whence $g = h_n$ by uniqueness.

DEFINITION 1.4. *We define the* n^{th} *Hermite function by*

$$\varphi_n(x) = e^{-x^2/2}h_n(x)/\|e^{-x^2/2}h_n(x)\|_2 \qquad (n = 0, 1, 2, \ldots).$$

We notice that $\varphi_n \in L^p$ for each $p \geq 1$ as h_n decays exponentially to zero at $+\infty$ and $-\infty$.

Some important properties of the Hermite functions are contained in the following theorem.

THEOREM 1.5. *Let* n *be a nonnegative integer. Then the following are true.*

(i) $\|e^{-x^2/2}h_n(x)\|_2^2 = 2^n\, n!\, \sqrt{\pi}$.

(ii) φ_n *is a solution of* (1.2) with $\lambda = -(2n+1)$.

(iii) $\mathfrak{F}\varphi_n = (-i)^n \varphi_n$.

(iv) $\{\varphi_k\}_{k=0}^{\infty}$ *is an orthonormal system in* L^2.

(v) $e^{-x^2/2+2x\lambda-\lambda^2} = \sum_{k=0}^{\infty} \lambda^k \left[\frac{2^k\sqrt{\pi}}{k!}\right]^{\frac{1}{2}} \varphi_k(x) \qquad (\lambda \in \mathbb{C}).$

PROOF. Using proposition 1.3 and integration by parts we get the recursion relation

$$\int_{-\infty}^{\infty} \left[e^{-x^2/2}h_{n+1}(x)\right]^2 dx = 2(n+1) \int_{-\infty}^{\infty} \left[e^{-x^2/2}h_n(x)\right]^2 dx.$$

Also, $h_0 = 1$ so

$$\int_{-\infty}^{\infty} \left[e^{-x^2/2}h_0(x)\right]^2 dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

and (i) follows. (ii) is clearly true. To prove (iv) let n and $m \neq n$ be nonnegative integers. Then from (ii) we know

$$\varphi_m'' \varphi_n - \varphi_n'' \varphi_m = 2(n-m)\varphi_m \varphi_n .$$

We integrate this equation by parts twice to get

$$\int_{-\infty}^{\infty} \varphi_m(x)\varphi_n(x)dx = 0 ,$$

whence $\{\varphi_n\}$ is an orthonormal system in L^2.

To prove (v) let $g_n(x) = e^{-x^2/2}h_n(x)$. Then expanding the function $\exp(-x^2/2 + 2x\lambda-\lambda^2) = \exp(x^2/2)\exp(-(x-\lambda)^2)$ about $\lambda = 0$ we get

$$g_n(x) = \left(\frac{d^n}{d\lambda^n} \exp(-x^2/2 + 2x\lambda - \lambda^2)\right)(0)$$

so (v) follows from Taylor's theorem and (i).

We may now prove (iii) by calculating

$$\sum_{k=0}^{\infty} \lambda^k \left[\frac{2^k\sqrt{\pi}}{k!}\right]^{\frac{1}{2}} (\mathcal{F}\varphi_k)(y) = \mathcal{F}(e^{-x^2/2+2x\lambda-\lambda^2})$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2 + x(2\lambda-iy) - \lambda^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(- [x/\sqrt{2} - \frac{(2\lambda-iy)}{\sqrt{2}}]^2$$

$$+ [(2\lambda-iy)^2/2 - \lambda^2])dx$$

$$= e^{-y^2/2-2iy\lambda+\lambda^2}$$

$$= \sum_{k=0}^{\infty} \lambda^k \left[\frac{2^k\sqrt{\pi}}{k!}\right]^{\frac{1}{2}} (-i)^k \varphi_k(y) \qquad (\lambda \in \mathbb{C}).$$

(iii) now follows by comparing coefficients of λ^k.

We are now ready to introduce the Mehler kernel.

DEFINITION 1.6. *Let* $|\omega| \leq 1$ *but* $\omega \neq \pm 1$ *and let* ($\sqrt{\ }$ *is principal value)*

$$N_\omega(x,y) = \frac{1}{\sqrt{\pi(1-\omega^2)}} \exp\left\{\frac{4xy\omega - (x^2+y^2)(1+\omega^2)}{2(1-\omega^2)}\right\}.$$

$N_\omega(x,y)$ *is called the Mehler kernel of parameter* ω.

THEOREM 1.7. *Let* $|\omega| < 1$. *Then*

$$N_\omega(x,y) = \sum_{n=0}^{\infty} \omega^n \varphi_n(x)\varphi_n(y) .$$

PROOF. Fix $|\omega| < 1$. Direct computation yields

$$\frac{\partial^2}{\partial x^2} N_\omega - x^2 N_\omega = \frac{\partial^2}{\partial y^2} N_\omega - y^2 N_\omega = - 2\omega \frac{\partial}{\partial \omega} N_\omega - N_\omega .$$

It may be shown by induction that there exist polynomials $B_n(x,y)$ in x and y with coefficients dependent on ω such that

$$\frac{\partial^n N_\omega}{\partial \omega^n} = B_n(x,y)N_\omega(x,y) .$$

Taylor's theorem then implies

$$N_\omega(x,y) = \sum_{n=0}^{\infty} P_n(x,y)\omega^n e^{-(x^2+y^2)/2},$$

where the P_n are polynomials in x and y that are independent of ω and so

$$- 2\omega \frac{\partial N_\omega}{\partial \omega} - N_\omega = \sum_{n=0}^{\infty} - (2n+1)P_n(x,y)\omega^n e^{-(x^2+y^2)/2} . \qquad (1.5)$$

We may write

$$N_\omega(x,y) = \sum_{n=0}^{\infty} \omega^n Q_n(x,y); \; Q_n(x,y) = \frac{1}{2\pi i} \oint \frac{N_Z(x,y)}{Z^{n+1}} \, dZ \, ,$$

where the contour of integration lies inside $|Z| < 1$. From this one may show that for all positive A and B, $r > 1$, and nonnegative integers k and ℓ we have

$$\frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} Q_n(x,y) = O(r^n)$$

uniformly in $|x| \leq A$, $|y| \leq B$. Hence the sum in (1.5) converges uniformly on compact sets and hence we may differentiate it with respect to x and y freely. Doing this we obtain

$$\frac{\partial^2 N_\omega}{\partial x^2} - x^2 N_\omega = \sum_{n=0}^{\infty} \omega^n \left(\frac{\partial^2}{\partial x^2} - x^2 \right) \left(P_n(x,y) e^{-(x^2+y^2)/2} \right)$$

and, of course, a similar equation with x and y interchanged. Then (1.5) implies

$$\left(\frac{\partial^2}{\partial x^2} - x^2 \right) \left(P_n(x,y) e^{-(x^2+y^2)/2} \right) = \left(\frac{\partial^2}{\partial y^2} - y^2 \right) \left(P_n(x,y) e^{-(x^2+y^2)/2} \right)$$

$$= - (2n+1) P_n(x,y) e^{-(x^2+y^2)/2}$$

whence P_n satisfies the differential equations

$$\frac{\partial^2}{\partial x^2} P_n - 2x \frac{\partial}{\partial x} P_n + 2nP_n = \frac{\partial^2}{\partial y^2} P_n - 2yP_n + 2nP_n = 0 \, .$$

This has the same form as the differential equation for h_n (compare with (1.2) and following paragraph), therefore there exist functions

F and G such that

$$P_n(x,y) = F(y)h_n(x) = G(x)h_n(y) .$$

Now $G(x)/h_n(x) = F(y)/h_n(y)$ whence F and G are constants. We have obtained

$$P_n(x,y) = c_n h_n(x)h_n(y) ,$$

for constants c_n. Hence

$$N_\omega (x,y) = \sum_{n=0}^{\infty} d_n \omega^n \varphi_n(x)\varphi_n(y) ,$$

for some constants d_n that depend only upon n. We now have

$$N_\omega (x,x) = \frac{1}{\sqrt{\pi(1-\omega^2)}} \exp \left\{- x^2 \frac{1-\omega}{1+\omega}\right\} = \sum_{n=0}^{\infty} d_n \omega^n (\varphi_n(x))^2 , \quad (1.6)$$

and, in particular, for $x = 0$,

$$[\pi(1-\omega^2)]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} d_n \omega^n (\varphi_n(0))^2 .$$

From theorem 1.5 (v) we know that for n even,

$$(\varphi_n(0))^2 = \frac{n!}{2^n \sqrt{\pi}[(n/2)!]^2} ,$$

whence, expanding $(1-\omega^2)^{-\frac{1}{2}}$ in its power series, we find $d_n = 1$ for n even. Furthermore

$$\frac{\partial^2 N_\omega}{\partial x \partial y}(0,0) = \sum_{n=0}^{\infty} d_n \omega^n (\varphi_n'(0))^2 = \frac{2\omega}{1-\omega^2} N_\omega (0,0) ,$$

and a similar, though more involved calculation of $\varphi_n'(0)$ implies $d_n = 1$ for n odd also. The theorem is now proven.

DEFINITION 1.8. *For* $|\omega| < 1$, *we define the Mehler kernel operator,* N_ω , *with parameter* ω *by*

$$(N_\omega f)(x) = \int_{-\infty}^{\infty} f(y) N_\omega (x,y) dy .$$

For now we will consider the domain of N_ω to be L^2 although later we will consider other domains. Some important properties of N_ω are listed in the following proposition.

PROPOSITION 1.9. *Let* $|\omega| < 1$. *Then the following are true.*

(i) $N_\omega \varphi_n = \omega^n \varphi_n$ *for* $n = 0, 1, 2, \ldots$.

(ii) N_ω *is a bounded operator of* L^2.

(iii) N_ω *is a compact operator of* L^2.

PROOF (i) follows directly from theorem 1.7. To prove (ii) we let $f \in L^2$ and calculate,

$$\|N_\omega f\|_2^2 = \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} f(y)N_\omega(x,y)dy\right|^2 dx$$

$$\leq \|f\|_2^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_\omega(x,y)|^2 dydx .$$

However, we also know

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_\omega(x,y)|^2 dydx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|\sum_{n=0}^{\infty} \omega^n \varphi_n(x)\varphi_n(y)\right|^2 dydx$$

$$\leq \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} |\omega^n \varphi_n(x)|^2 dx$$

$$= \frac{1}{1 - |\omega|^2} .$$

Finally, N_ω is a bounded integral operator with a kernel that is in $L^2(\mathbb{R}^2)$, hence it is compact and we have (ii) and (iii) (see [17] page 319).

We have mentioned that $\varphi_n \in L^p$ for each $1 \leq p \leq \infty$. The following theorem gives bounds on the L^p norm of φ_n.

THEOREM 1.10. *Let* $1 \leq p \leq \infty$, $0 < a < 1$. *Then there exists a function* $c(a,p)$ *depending only on* a *and* p *such that* $\|\varphi_n\|_p \leq a^{-n}c(a,p)$, $(n = 0, 1, 2, \ldots)$.

PROOF. Let $1 \leq p \leq \infty$. From proposition 1.9 (i) we have

$$a^n \|\varphi_n\|_p = \|N_a \varphi_n\|_p$$

$$= \| \int_{-\infty}^{\infty} \varphi_n(y) N_a(x,y)dy \|_p$$

$$\leq \|(\|\varphi_n\|_2 \|N_a(x,y)\|_2)\|_p$$

$$= \left\| \left(\frac{1}{\pi(1-a^4)} \right)^{\frac{1}{4}} \exp \left\{ \frac{-x^2}{2} \left(\frac{1-a^2}{1+a^2} \right) \right\} \right\|_p$$

$$= c(a,p).$$

THEOREM 1.11. *Let* $|\omega| < 1$. *Then*

$$\|N_\omega\|_{2,2} = 1.$$

PROOF. Let $f \in L^2$. Then

$$N_\omega f = \sum_{n=0}^{\infty} \omega^n (f,\varphi_n)\varphi_n ,$$

so

$$\|N_\omega f\|_2^2 = \sum_{n=0}^{\infty} |\omega|^{2n} \, |(f,\varphi_n)|^2$$

$$\leq \sum_{n=0}^{\infty} |(f,\varphi_n)|^2$$

$$= \|f\|_2^2 .$$

Also, we clearly have equality if and only if f is a constant multiple of φ_0.

THEOREM 1.12. (i) *Let* $f \in L^p$, $1 \le p < \infty$. *Then*

$$\lim_{a\to 1} \|N_a f - f\|_p = 0 .$$

(ii) $\{\varphi_n\}_{n=0}^{\infty}$ *is a complete orthonormal system in* L^2.

PROOF. Let $0 < a < 1$. We calculate

$$\int_{-\infty}^{\infty} N_a(x,y)dy = \frac{1}{\sqrt{\pi(1-a^2)}} \int_{-\infty}^{\infty} \exp\left\{ \frac{4axy - (x^2+y^2)(1+a^2)}{2(1-a^2)} \right\} dy$$

$$= \frac{1}{\sqrt{\pi(1-a^2)}} \exp\left\{\left(\frac{2a^2}{1-a^4} - \frac{1+a^2}{2(1-a^2)}\right) x^2 \right\}$$

$$\cdot \int_{-\infty}^{\infty} \exp\left\{ - \left(y\sqrt{\frac{1+a^2}{2(1-a^2)}} - x\sqrt{\frac{2a^2}{1-a^4}}\right)^2 \right\} dy$$

$$= \left(\frac{2}{1+a^2}\right)^{\frac{1}{2}} \exp\left\{ - x^2\left(\frac{1-a^2}{2(1+a^2)}\right)\right\}$$

$$\le \sqrt{2} .$$

Let $1/p + 1/q = 1$. We will now prove $\|N_a\|_{p,p} \le \sqrt{2}$. The case $p = 1$ may be easily verified so we will assume $p > 1$. Then if $f \in L^p$,

$$\|N_a f\|_p^p = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) N_a(x,y)dy \right|^p dx$$

$$\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(y)|^p N_a(x,y)dy \right) \left(\int_{-\infty}^{\infty} N_a(x,y)dy \right)^{p/q} dx$$

$$\leq (\sqrt{2})^{\frac{p}{q}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)|^p N_a(x,y)dy\ dx.$$

We use Fubini's theorem to interchange the order of integration and in so doing we conclude

$$\|N_a\|_{p,p} \leq \sqrt{2}.$$

We now allow the value $p = 1$. Let $\epsilon > 0$ and let g be a step function with a finite number of jumps such that $\|f-g\|_p < \epsilon/4$. Then it may be verified that there exist positive numbers M and A such that

$$|(N_a g)(x)| \leq Me^{-Ax^2}$$

for all $0 < a < 1$ and all real x. Also, $N_a g \to g$ pointwise except possibly at the points where g is not continuous. It now follows from Lebesgue's dominated convergence theorem that

$$\lim_{a \to 1} \|N_a g - g\|_p = 0 .$$

Let a be so close to one that $\|N_a g - g\|_p < \epsilon/4$.
Then

$$\|N_a f - f\|_p \leq \|N_a(f-g)\|_p + \|N_a g - g\|_p + \|g - f\|_p < \epsilon$$

and we have proven (i).

Now let $f \in L^2$. Let $\varepsilon > 0$ and let a be so close to one that $\|N_a f - f\|_2 < \varepsilon/2$. Define for each positive integer m

$$(P_m f)(x) = \sum_{n=0}^{m} a^n (f,\varphi_n)\varphi_n(x) .$$

Then

$$\|N_a f - P_m f\|_2 = \|\sum_{n=m}^{\infty} a^n (f,\varphi_n)\varphi_n\|_2$$

$$\leq \|f\|_2 \sum_{n=m}^{\infty} a^n \|\varphi_n\|_2$$

$$= \| f\|_2 \sum_{n=m}^{\infty} a^n .$$

Let m be so large that $\|N_a f - P_m f\|_2 < \varepsilon/2$. Then

$$\|f - P_m f\|_2 \leq \|f - N_a f\|_2 + \|N_a f - P_m f\| < \varepsilon .$$

This completes the proof of (ii).

We may now prove Plancherel's theorem by Wiener's method.

THEOREM 1.13. *For all* $f \in L^1 \cap L^2$ *we have*

$$\|\mathfrak{F} f\|_2 = \| f\|_2 ,$$

whence $\mathfrak{F}$ *may be extended to an isometry of* L^2.

PROOF. Let $0 < t < 1$, and $f \in L^1 \cap L^2$. Then

$$|f(y)N_{-it}(x,y)| \leq |f(y)| \exp\left\{-\frac{x^2}{2}\left(\frac{1-t^2}{1+t^2}\right)\right\}$$

$$\leq |f(y)| \in L^1 .$$

Also

$$\lim_{t \to 1} N_{-it}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-ixy} .$$

It then follows from Lebesgue's dominated convergence theorem that

$$\lim_{t \to 1} (N_{-it}f)(x) = (\mathfrak{F}f)(x)$$

pointwise.

Fatou's lemma and theorem 1.11 imply

$$\|\mathfrak{F}f\|_2 \leq \liminf_{t \to 1} \|N_{-it}f\|_2 \leq \|f\|_2 .$$

Since any function in L^2 may be approximated in L^2 by functions in $L^1 \cap L^2$, we have shown that $\mathfrak{F}$ can be extended to a bounded linear operator of L^2. We now have, by Theorem 1.12, for $f \in L^2$,

$$\|\mathfrak{F}f\|_2 = \|\mathfrak{F} \sum_{n=0}^{\infty} (f,\varphi_n)\varphi_n\|_2$$

$$= \|\sum_{n=0}^{\infty} (-i)^n (f,\varphi_n)\varphi_n\|_2$$

$$= \left(\sum_{n=0}^{\infty} (f, \varphi_n)^2 \right)^{\frac{1}{2}}$$

$$= \|f\|_2 \quad .$$

Chapter 2

THE FOURIER TRANSFORM IN L^p

In this chapter we will examine the development of the classical Hausdorff-Young inequality. Throughout this chapter p will be a real number in the range $1 \leq p \leq 2$, and q will be defined by the conjugacy relation $1/p + 1/q = 1$. We will allow the value $q = +\infty$. The first inequality we will look at is due originally to Titchmarsh and its complete proof may be found in his book *Introduction to the Theory of Fourier Integrals* [14].

THEOREM 2.1. *For* $f \in L^p \cap L^1$ *we have*

$$\|\mathfrak{F}f\|_q \leq (2\pi)^{1/q-1/2} \|f\|_p,$$

so $\mathfrak{F}$ *may be extended to a continuous linear operator of* L^p *into* L^q *such that the above inequality holds.*

PROOF. The case $p = 1$ is treated separately. If $f \in L^1$ then

$$\|\mathfrak{F}f\|_\infty = \sup_x \left\{ \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ixy} dy \right| \right\}$$

$$\leq \sup_x \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(y)| \, |e^{-ixy}| dy \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \|f\|_1 .$$

Notice that if we take $g(x) = e^{-x^2/2}$ then

$$\| \mathfrak{F}g \|_\infty = \frac{1}{\sqrt{2\pi}} \| g \|_1 ,$$

whence $\| \mathfrak{F} \|_{1,\infty} = {}^1/\sqrt{2\pi}$. In the remainder of this proof we will assume $p > 1$. The proof of theorem 2.1 makes no use of the Mehler kernel operator. Titchmarsh first treats the case of q an even integer. For such q, $\mathfrak{F}f$ may be written as a q-fold convolution and so the norm of $\mathfrak{F}f$ may be easily calculated. To do this, several lemmas are needed.

LEMMA 2.2. (YOUNG'S INEQUALITY) *If* $f \in L^{1/(1-\lambda)}$ *and* $g \in L^{1/(1-\mu)}$ *where* $\lambda, \mu > 0$ *and* $\lambda + \mu < 1$, *then*

$$\left| \int_{-\infty}^{\infty} fg\,dx \right| \le \left(\int_{-\infty}^{\infty} |f|^{1/(1-\lambda)} |g|^{1/(1-\mu)} dx \right)^{1-\lambda-\mu} \left(\int_{-\infty}^{\infty} |f|^{1/(1-\lambda)} dx \right)^{\mu} \cdot \left(\int_{-\infty}^{\infty} |g|^{1/(1-\mu)} dx \right)^{\lambda} .$$

LEMMA 2.3. *Let* λ, μ, f, g *be as in lemma* 2.2, *and let*

$$c(x) = f * g\,(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt.$$

Then

$$\| c \|_{1/(1-\lambda-\mu)} \le \| f \|_{1/(1-\lambda)} \| g \|_{1/(1-\mu)} .$$

We now assume f to be a continuous function of compact support.

LEMMA 2.4. *Let* k *be an integer. Then the* k*-fold convolution*

$$c_k(x) = \int_{-\infty}^{\infty} f(u_{k-1}) \ldots \int_{-\infty}^{\infty} f(u_1)f(x-u_1-u_2\ldots - u_{k-1})du_1 \ldots du_{k-1}$$

belongs to L^2 *and*

$$\|c_k\|_2^2 \le \left(\int_{-\infty}^{\infty} |f(x)|^{2k/(2k-1)}dx\right)^{2k-1} .$$

Lemma 2.2 follows directly from Holder's inequality for three functions. Using lemma 2.2 and c as in lemma 2.3 we see

$$|c(x)| \le \left(\int_{-\infty}^{\infty} |f(t)|^{1/(1-\lambda)} |g(x-t)|^{1/(1-\mu)}dt\right)^{1-\lambda-\mu} \|f\|_{1/(1-\lambda)}^{\mu/(1-\lambda)} \cdot \|g\|_{1/(1-\mu)}^{\lambda/(1-\mu)} .$$

Whence

$$\int_{-\infty}^{\infty} |c(x)|^{1/(1-\lambda-\mu)}dx \le \left\{\|f\|_{1/(1-\lambda)}\right\}^{\frac{1}{1-\lambda}(1+\frac{\mu}{1-\lambda-\mu})} \left\{\|g\|_{1/(1-\mu)}\right\}^{\frac{1}{1-\mu}(1+\frac{\lambda}{1-\lambda-\mu})},$$

and lemma 2.3 follows. To prove lemma 2.4 we notice

$$c_k = f * c_{k-1},$$

so upon several applications of lemma 2.3 we have

$$\|c_k\|_{1/(1-k/2k)} \le \|f\|_{1/(1-1/2k)}^k ,$$

and hence the desired result.

Now let c_k be as in lemma 2.4. Then by writing the iterated integral as a multiple one and performing a change of variables,

$$(\mathfrak{F}f)^k = \mathfrak{F}\left((2\pi)^{-k/2+1/2}c_k\right),$$

and we may use theorem 1.10 to get

$$\int_{-\infty}^{\infty} |\mathfrak{F}f|^{2k}\,dx = (2\pi)^{1-k}\int_{-\infty}^{\infty} |c_k|^2 dx \le (2\pi)^{1-k}\left(\int_{-\infty}^{\infty}|f|^{2k/(2k-1)}dx\right)^{2k-1}.$$

By our usual extension procedure, the above inequality holds for all $f \in L^p$ and q an even integer.

Titchmarsh then attacks the problem of general q by using the following lemma of Hausdorff and Young [14].

LEMMA 2.5. *For any finite set of real numbers* $\{d_m\}_{m=-n}^{n}$ *we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{m=-n}^{n} d_m e^{imx}\right|^q dx \le \left(\sum_{m=-n}^{n}|d_m|^p\right)^{1/(p-1)}.$$

The proof of this lemma is long and cumbersome;it will not be presented here.

Let f be a continuous function of compact support, $\lambda > 0$, $b > 0$, and define

$$a_v = \int_{v/\lambda}^{(v+1)/\lambda} f(x)dx$$

for integers v, and

$$g_\lambda(x) = \sum_{v=-n}^{n} a_v e^{-ivx/\lambda}$$

where $n = [\lambda b] - 1$. So

$$\lim_{\lambda\to\infty} g_\lambda(x) = \int_{-b}^{b} f(y)e^{-ixy}dy ,$$

uniformly on compact sets. Furthermore, by lemma 2.5

$$\int_{-\pi\lambda}^{\pi\lambda} |g_n(x)|^q dx \le 2\pi\lambda \left(\sum_{v=-n}^{n} |a_v|^p\right)^{1/(p-1)}$$

$$\le 2\pi\left(\int_{-b}^{b} |f(x)|^p dx\right)^{1/(p-1)}$$

$$\le 2\pi \; \|f\|_p^{p/(p-1)} \quad .$$

Hence, by Fatou's lemma

$$\int_{-\infty}^{\infty} |(\mathfrak{F} f)(x)|^q \, dx = \int_{-\infty}^{\infty} \liminf_{\lambda\to\infty} |g_\lambda(x)|^q \chi_{(-\pi\lambda,\pi\lambda)} dx$$

$$\le 2\pi \|f\|_p^{p/(p-1)} .$$

We again approximate a general $f \in L^p$ by continuous functions of compact support to finally prove the theorem.

There is another, very quick proof of theorem 2.1 using M. Riesz's convexity theorem (see [12] chapter 5).

THEOREM 2.5 (M. RIESZ) *Let* T *be a linear operator of norm* K_i *from* L^{p_i} *to* L^{q_i} (i=1,2) *(the* p_i *and* q_i *are not necessarily conjugate)*. *Then* T *has norm* $\leq K_1^{1-t}K_2^t$ *from* L^p *to* L^q *where*

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1-t}{q_1} + \frac{1}{q_2} \quad .$$

The proof of this theorem is not important to this thesis. However, if we use the already proven facts that $\mathfrak{F}$ is an isometry on L^2 and that it has norm equal to $1/\sqrt{2\pi}$ from L^1 to L^∞ we get immediately

$$\|\mathfrak{F}\|_{p,q} \leq \left(\frac{1}{\sqrt{2\pi}}\right)^{1-2(1-\frac{1}{p})} = (2\pi)^{1/q-1/2} ,$$

which is exactly theorem 2.1. It is interesting to note that even this extremely powerful theorem gives no better result than theorem 2.1.

In order to improve on theorem 2.1 we return to the study of Mehler kernel operators, - this time on L^p. We will need a few more properties of the Hermite functions and of the operator N_ω in order to proceed.

THEOREM 2.6. *Let* $1 \leq a \leq \infty$, $1 \leq b \leq \infty$, $|\omega| < 1$, *and* $f \in L^a$. *Then*

$$N_\omega f = \sum_{n=0}^{\infty} \omega^n (f,\varphi_n)\varphi_n \quad ,$$

where the sum converges in L^b. *Moreover,* N_ω *is a compact operator of* L^a *into* L^b *and* $N_{\omega_1\omega_2} = N_{\omega_1} N_{\omega_2}$ *for* $|\omega_1| < 1$, $|\omega_2| < 1$ (*i.e.* $\{N_\omega\}$ *forms a semigroup*).

PROOF. From theorem 1.7 we know that, for fixed x,

$$N_\omega(x,y) = \sum_{n=0}^{\infty} \omega^n \varphi_n(x)\varphi_n(y) ,$$

and from theorem 1.10 this sum converges in L^b sense (as $\|\varphi_n\|_c = O(t^{-n})$ for all $0 < t < 1$, $1 \le c \le \infty$). Hence

$$(N_\omega f)(x) = \int_{-\infty}^{\infty} N_\omega (x,y)f(y)dy$$

$$= \sum_{n=0}^{\infty} \omega^n(f,\varphi_n)\varphi_n(x)$$

for all x. The convergence of the latter series is surely in L^b sense, again because of theorem 1.10. Also,

$$\|N_\omega f - \sum_{n=0}^{m-1} \omega^n(f,\varphi_n)\varphi_n\|_b \le \sum_{n=m}^{\infty} |\omega|^n |(f,\varphi_n)| \; \|\varphi_n\|_b$$

$$\le \|f\|_a \sum_{n=m}^{\infty} |\omega|^n \; \|\varphi_n\|_b^2 .$$

This last sum approaches zero as $m \to \infty$, again by theorem 1.10. This shows that N_ω can be approximated by the finite rank operators P_m, where

$$P_m f = \sum_{n=0}^{m} \omega^n (f,\varphi_n)\varphi_n \qquad (f \in L^a) .$$

It follows that N_ω is a compact operator of L^a. The semigroup property of N_ω follows easily from the identity for $N_\omega f$ proven in this theorem.

THEOREM 2.7. *The linear span of the set* $\{\varphi_n\}_{n=0}^{\infty}$ *is dense in* L^a *for* $1 \leq a < \infty$.

PROOF. Let $f \in L^a$ and $\varepsilon > 0$. Let g be a continuous function of compact support such that $\| f-g \|_a < \varepsilon/4$. Let $t < 1$ be so close to one that $\| N_t f-f \|_a < \varepsilon/4$. Let P_m be defined by

$$(P_m g)(x) = \sum_{n=0}^{m} t^n (g,\varphi_n)\varphi_n(x).$$

Since $g \in L^b$, where $1/a + 1/b = 1$, it follows from the proof of theorem 2.6 that for m sufficiently large $\| (N_t-P_m)g \|_a < \varepsilon/4$. Hence

$$\| f-P_m g \|_a \leq \| N_t f-f \|_a + \| N_t(f-g) \|_a + \| (N_t-P_m)g \|_a < \varepsilon ,$$

and the theorem is proven.

A bit more calculation gets us our first Hausdorff-Young inequality for N_ω .

THEOREM 2.8. *Let* $|\omega| < 1, 1 < p \leq 2$, *and let* ω *be real. Then*

$$\| N_\omega \|_{p,q} \leq [\pi(1-\omega^2)]^{-1/2} (2\pi/q)^{1/q} .$$

PROOF. Let $a = (1+\omega^2)/2(1-\omega^2)$. Then a is positive and we calculate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_\omega(x,y)|^q dy\, dx = [\pi(1-\omega^2)]^{-q/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-aq(x^2+y^2) \pm 2q\sqrt{a^2-1/4}\, xy\right\} dy\, dx$$

$$= [\pi(1-\omega^2)]^{-q/2} \left(\frac{\pi}{qa}\right)^{1/2} \int_{-\infty}^{\infty} e^{-qx^2/4a} dx$$

$$= [\pi(1-\omega^2)]^{-q/2} (2\pi/q) .$$

We now have,

$$\| N_\omega f \|_q \leq [\pi(1-\omega^2)]^{-1/2} (2\pi/q)^{1/q} \| f \|_p$$

for any $f \in L^p$. The theorem follows.

Theorem 2.8 is already good enough to give us a result for the smoothing operators of De Bruijn (see page 71). We may also bound $\|N_\omega\|$ for ω purely imaginary.

THEOREM 2.9. *Let* $|\omega| < 1$, $1 < p \leq 2$, *and let* ω *be purely imaginary. Then*

$$\|N_\omega\|_{p,q} \leq [\pi(1-\omega^2)]^{-1/2} \, [2\pi(1-\omega^2)/q(1+\omega^2)]^{1/q}.$$

PROOF. We again let $a = (1+\omega^2)/2(1-\omega^2)$. Then a is positive and we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |N_\omega(x,y)|^q dydx = [\pi(1-\omega^2)]^{-q/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{qa(x^2+y^2)\}\, dydx$$

$$= [\pi(1-\omega^2)]^{-q/2} \, [\pi/qa].$$

We now have,

$$\|N_\omega f\|_q \leq [\pi(1-\omega^2)]^{-1/2} \, [\pi/qa]^{1/q} \, \|f\|_p$$

for any $f \in L^p$. The theorem follows.

We observe that the bounds of theorems 2.8 and 2.9 both approach infinity as $|\omega| \to 1$. For this reason theorem 2.9, for example, is not good enough to prove a Hausdorff-Young inequality for the Fourier transform. If a bound for $\|N_\omega\|$ is proven that remains finite as $\omega \to -i$, then such an inequality could be proven using the method of Wiener.

For special values of ω and p, K.I. Babenko was able to calculate the norm $\|N_\omega\|_{p,q}$ explicitly. We summarize his results in the following theorem. The full proof may be found in [2].

THEOREM 2.10 (BABENKO'S THEOREM) *Let* $-1 \leq t \leq 1$ *and let* q *be a positive even integer. Define*

$$\gamma = \frac{1}{4t^2}\,[-p(1-t^2) + (p^2t^4-(2p^2-16p+16)t^2 + p^2)^{1/2}] \,.$$

Then

$$\| N_{it} \|_{p,q} = \sqrt{\gamma}\left(\frac{1+\gamma^2t^2}{\pi\gamma(1+t^2)}\right)^{1/2p-1/2q} .$$

Moreover, this norm is achieved for the Gaussian function

$$k(x) = \exp\left\{-x^2\left(\frac{1+\gamma^2t^2}{p\gamma(1+t^2)}\right)\right\} .$$

If we let $t = -1$ *we get*

$$\| \mathfrak{F} \|_{p,q} = (2\pi)^{1/2q-1/2p}(p^{1/2p}/q^{1/2q}) .$$

SKETCH OF PROOF. Babenko's proof is much too long to be presented here in detail. Instead we present an outline of his proof. We approach the theorem as an extremal problem. Let $\mu_t = \| N_{-it} \|_{p,q}$ and let f be an L^p function of L^p norm one for which N_{-it} achieves its norm. Let $g = N_{-it}f$. If f exists then it can be shown that f satisfies

LEMMA 2.12. $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N_t(|g|^{q-2}g)e^{ixy}dy = \mu_t{}^q |f(x)|^{p-2}f(x).$

This lemma is proven by perturbation of f.

We now define $u(x) = |f(x)|^{p-2}f(x)$. Then it can be shown that u and g are entire functions of order two and finite type . Let

$$\beta = \frac{1+\gamma^2t^2}{q\gamma(1+t^2)}, \quad \xi(x) = e^{\beta x^2}u(x).$$

Then the following is true.

LEMMA 2.13. *For all real* x *and* y *we have*

$$\int_{-\infty}^{\infty} e^{-q\beta s^2} |\xi(s+x+iy)|^q \, ds \leq 1.$$

The proof of lemma 2.13 relies heavily on the theory of entire functions. At one point in the proof the path of integration must be changed. To do this, q must be an even integer to assure that the integrand is entire.

As a consequence of lemma 2.13, for any $h > 0$, the entire function

$$\xi_1(z) = \int_0^h e^{-q\beta s^2} \xi^q(s+z)ds$$

is bounded and hence is constant. Therefore

$$\int_0^h e^{-q\beta s^2} \xi^q(s+z)ds = \int_0^h e^{-q\beta s^2} \xi^q(s)ds,$$

and, as h is arbitrary, ξ is a constant. Since

$$\| e^{-q\beta s^2} \xi^q(s) \|_q = 1$$

we have

$$\xi(x) = \left(\frac{q\beta}{\pi}\right)^{1/2q} .$$

It follows that

$$f(x) = \left(\frac{q\beta}{\pi}\right)^{1/2p} e^{-q\beta x^2/p}$$

and the theorem follows by direct computation of $N_{-it}f$.

Chapter 3

THE MEHLER KERNEL OPERATOR FOR GAUSSIAN MEASURE

Throughout this chapter we denote by μ the Gaussian measure for $\mathbb{R}$,

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx .$$

Lebesgue measure will be denoted by λ or simply by dx if there is no confusion possible. The measure μ is finite and has been normalized so that $\mu(\mathbb{R}) = 1$. We define the modified Hermite polynomials H_n by

$$H_n(x) = h_n(x/\sqrt{2}) .$$

The role played by the Hermite functions in the previous chapters will now be played by the polynomials H_n. Some of the properties of H_n are listed in the following proposition, which is a direct consequence of theorems 1.5 and 2.7.

PROPOSITION 3.1. (i) $\{H_n\}_{n=0}^{\infty}$ *is a complete orthonormal system in* $L^2(d\mu)$.

(ii) *The linear span of* $\{H_n\}_{n=0}^{\infty}$ *is dense in* $L^a(d\mu)$ *for* $1 \leq a < \infty$.

(iii) $e^{\lambda x - \lambda^2/2} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H_k(x) \qquad (\lambda \in \mathbb{C}).$

We now define the Mehler kernel operator for Gaussian measure.

DEFINITION 3.2. *Let* $|\omega| < 1$. *We denote by*

$$K_\omega(x,y) = \frac{1}{\sqrt{1-\omega^2}} \exp\left\{\frac{2\omega xy - (x^2+y^2)\omega^2}{2(1-\omega^2)}\right\}$$

the Mehler kernel for Gaussian measure and parameter ω. *We denote by* K_ω *the operator defined for* $f \in L^p(d\mu), 1 \le p \le 2$, *by*

$$(K_\omega f)(x) = \int_{-\infty}^{\infty} f(y) K_\omega(x,y) d\mu(y),$$

and we call K_ω *the Mehler kernel operator for Gaussian measure and parameter* ω.

F. Weissler, in his paper [16], presents a more general version of the following theorem. For the case we are immediately interested in, this simplified version suffices.

THEOREM 3.3. *Let* $1 \le p \le 2$, $1/p + 1/q = 1$, *and let* ω *be a complex number of modulus less than one that satisfies the two relations*

$$\operatorname{Re} \frac{1}{1-\omega^2} > 1/p \tag{3.1}$$

and

$$\left[\operatorname{Re} \frac{1}{1-\omega^2} - \frac{1}{p}\right]^2 > \left[\operatorname{Re} \frac{\omega}{1-\omega^2}\right]^2 . \tag{3.2}$$

Then K_ω *is a compact operator of* $L^p(d\mu)$ *to* $L^q(d\mu)$.

PROOF. Let $p > 1$ and put

$$a = \text{Re}\,\frac{1}{1-\omega^2} - \frac{1}{p}$$

and

$$b = \text{Re}\,\frac{\omega}{1-\omega^2}\,.$$

Then

$$2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_\omega(x,y)|^q d\mu(y)d\mu(x) =$$

$$= |1-\omega^2|^{-q/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{qa}{2}(x^2+y^2) + qbxy\right\} dydx$$

$$= |1-\omega^2|^{-q/2} (2\pi/qa)^{1/2} \int_{-\infty}^{\infty} \exp\left\{\frac{qx^2}{2}\left(\frac{b^2}{a} - a\right)\right\} dx$$

$$= |1-\omega^2|^{-q/2} (2\pi/qa)^{1/2} (2\pi a/q(a^2-b^2))^{1/2}\,,$$

which is finite by (3.1) and (3.2). It follows that K_ω is compact (see [17] page 319). If $p = 1$ then we notice that (3.1) implies $|\omega^2 - 1/2| < 1/2$ so

$$a + b = \text{Re}\,\frac{\omega}{1-\omega} > 0$$

and

$$a - b = - \operatorname{Re} \frac{\omega}{1+\omega} < 0$$

whence $a^2 - b^2 \leq 0$ for all ω and so (3.1) and (3.2) are never satisfied. The theorem now follows trivially.

We also have the following theorem.

THEOREM 3.4. *Let* ω, ω_1 *and* ω_2 *be complex numbers of modulus less than one that satisfy* (3.1) *and* (3.2). *Then the following statements are true.*

(i) $K_\omega H_n = \omega^n H_n$ $\qquad (n = 0, 1, 2, \ldots)$.

(ii) $K_\omega(x,y) = \sum_{n=0}^{\infty} \omega^n H_n(x)H_n(y)$.

PROOF. To prove (i) we let $s = x/\sqrt{2}$, $t = y/\sqrt{2}$ and calculate

$$(K_\omega H_n)(x) = \int_{-\infty}^{\infty} H_n(y) K_\omega(x,y)d\mu(y)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} h_n(t)e^{-t^2} K_\omega(\sqrt{2}s, \sqrt{2}t)dt$$

$$= e^{s^2/2} \int_{-\infty}^{\infty} (h_n(t)e^{-t^2/2})N_\omega(s,t)dt$$

$$= e^{s^2/2}\omega^n(h_n(s)e^{-s^2/2})$$

$$= \omega^n H_n(x).$$

The kernel defined in the sum in (ii) defines a kernel operator that also maps H_n to $\omega^n H_n$, hence it must be the same operator as K_ω.

W. Beckner, in his paper [4], was the first person to precisely calculate the norm of K_ω for ω pure imaginary and satisfying (3.1) and (3.2). He was able, from this, to determine the norm of the Fourier operator $\mathfrak{F}$. The norm Beckner derived for $\mathfrak{F}$ is the same as that derived by Babenko for the special case that q is an even integer. We state and sketch a proof of Beckner's theorem below. The main idea is to prove the result first for Mehler-type operators over a discrete probability measure and then use the central limit theorem. The main tool in the proof is the "two point inequality" (see lemma 3.6).

THEOREM 3.5 (BECKNER'S THEOREM) *Let* $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $-\sqrt{p-1} \le t \le \sqrt{p-1}$.

Then

$$\| K_{it} \|_{\mu;p,q} = 1 .$$

REMARKS. Beckner states this theorem only for $t = \sqrt{p-1}$, however his proof still works if t is in the above specified range. Also, he does not consider the case $p = 1$ which is trivially proven as then $t = 0$.

SKETCH OF PROOF. The idea of Beckners proof is due to Nelson [11]. We will use the central limit theorem to obtain the Gaussian measure, $d\mu$, as a limiting probability measure of convolutions of Bernoulli measures. Let $d\nu(x)$ be the discrete probability measure with weight 1/2 at the points $x = \pm 1$. The measure $d\nu$ is referred to as Bernoulli measure. Let $d\nu_n$ be the n-fold convolution of the measure $d\nu(\sqrt{n}\, x)$ with itself. The central limit theorem says that $d\nu_n$ converges to $d\mu$ in the sense of the space $C_0(\mathbb{R})^*$ ($C_0(\mathbb{R})$ is the set of continuous functions of $\mathbb{R}$ that vanish at $\pm\infty$ with the topology induced by the L^∞ norm; $C_0(\mathbb{R})^*$ is its dual). So, if $f \in C_0(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(x)d\nu_n(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1 + \cdots + x_n)d\nu(\sqrt{n}x_1) \ldots d\nu(\sqrt{n}x_n),$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)d\nu_n(x) = \int_{-\infty}^{\infty} f(x)d\mu(x) .$$

Beckner proves that a result analogous to theorem 3.5 holds with respect to each of the measures $d\nu_n$. The inequality of the theorem then follows as a limit of inequalities with respect to these product measures.

The product measures $d\mu_n = dv(\sqrt{n}\,x_1) \cdots dv(\sqrt{n}\,x_n)$ are discrete and each x_i can assume only the values $\pm 1/\sqrt{n}$. If f is a function on the corresponding measure space, then f always has a representative that is a polynomial of degree at most one in each of the n variables. By imitating the action of K_ω on the first two modified Hermite polynomials $(H_0(x) = 1,\ H_1(x) = x)$ we define, inductively, Mehler type operators over these discrete measure spaces. On the measure space over dv we define an operator C by

$$C_\omega\,(a+bx) = a + \omega b x.$$

The following lemma may be proven by careful calculation [4].

LEMMA 3.6. *Let* $-\sqrt{p-1} \le t \le \sqrt{p-1}$. *Then* C_{it} *is a bounded operator of norm one of* $L^p(dv)$ *to* $L^q(dv)$.

Lemma 3.6 is known as a two-point inequality. We define operators $B_{\omega,n,k}$ on the set of functions defined on the measure space over $d\mu_n$ by

$$B_{\omega,n,k}\,(a+bx_k) = a + \omega b x_k \qquad (k = 1, 2, \ldots, n)\ ,$$

where a and b are functions of the remaining $n-1$ variables. Let

$$C_{\omega,n} = B_{\omega,n,1} \cdots B_{\omega,n,n}\ .$$

From lemma 3.6 each $B_{it,n,k}$ is an operator of norm one of $L^p(dv_n)$ to $L^q(dv_n)$. It follows that $C_{it,n}$ is also an operator of norm one of

$L^p(dv_n)$ to $L^q(dv_n)$.

Let X_n denote the space of functions symmetric in the n variables $x_1, \ldots x_n$ over dv_n, and let $D_{\omega,n}$ be the restriction of $C_{\omega,n}$ to X_n. Then $D_{it,n}$ may be shown to also be an operator of norm one. The symmetric functions

$$\sigma_{n,\ell} = \ell! \sum_{m_1 < \cdots < m_\ell} x_{m_1} \cdots x_{m_\ell} \quad (\ell=0,1,\ldots,n) ,$$

form an orthonormal basis in $L^2(X_n)$. We also have

$$D_{\omega,n} \; \sigma_{n,\ell} = \omega^\ell \sigma_{n,\ell} \; .$$

Let $g \in X_n$ and write

$$g(x_1, \ldots, x_n) = \sum_{\ell=0}^{n} d_\ell \sigma_{n,\ell} \; .$$

Then we have $\| D_{it,n} g \|_{v_n;q} \leq \| g \|_{v_n;p}$, or more explicitly,

$$\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \sum_{\ell=0}^{n} (it)^\ell d_\ell \, \sigma_{n,\ell} \right|^q d\, v_n \right)^{1/q} \leq \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \sum_{\ell=0}^{n} d_\ell \, \sigma_{n,\ell} \right|^p dv_n \right)^{1/p} . \quad (3.3)$$

By comparing the generating functions for H_n and $\sigma_{n,\ell}$ one may prove that if each $x_i = \pm 1/\sqrt{n}$, then

$$\sigma_{n,\ell}(x_1, \ldots, x_n) = H_\ell(x_1 + \cdots + x_n) + \frac{1}{n}\sum_{r=1}^{[\ell/2]} a_{\ell,r} \cdot H_{\ell-2r}(x_1 + \cdots + x_n),$$

where the coefficients $a_{\ell,r}$ are bounded with respect to n for fixed ℓ. It may now be shown that the inequality (3.3) implies the theorem as $n \to \infty$. A complete account of this proof may be found in [4].

It should be noted here that lemma 3.6 is still true for ω real and $1 - p \le \omega \le p - 1$, that is $C_{\omega,1}$ has norm one for these values of ω. The rest of the theorem remains unchanged and we get the following.

COROLLARY 3.7. *Let* $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $1 - p \le t \le p - 1$, *then*

$$\|K_t\|_{\mu;p,q} = 1.$$

Beckner was then able to use his theorem to prove the sharp Hausdorff-Young inequality that Babenko conjectured.

COROLLARY 3.8. *Let* $1 \le p \le 2$ *and* $1/p + 1/q = 1$. *Then*

$$\|\mathfrak{F}\|_{p,q} = (2\pi)^{1/2q-1/2p}(p^{1/2p}/q^{1/2q}).$$

PROOF. The case $p = 1$ was treated in chapter 2 (see theorem 2.1). We now assume $p > 1$. Theorem 3.5 for $t = i\sqrt{p-1}$ may be written as

$$\left\{\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} K_{-i\sqrt{p-1}}(x,y)g(y)d\mu(y)\right|^q d\mu(x)\right\}^{1/q} \le \left\{\int_{-\infty}^{\infty} |g(y)|^p d\mu(y)\right\}^{1/p},$$

for all $g \in L^p(d\mu)$. If we let $x = \sqrt{q}\ u$ and $y = \sqrt{p}\ v$ then this becomes

$$\left\{ \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{-iuv} h(v)dv \right|^q du \right\}^{1/q} \leq (2\pi)^{1/2q-1/2p}(p^{1/2p}/q^{1/2q}) \cdot \left\{ \int_{-\infty}^{\infty} |h(v)|^p dv \right\}^{1/p}$$

where

$$h(v) = g(\sqrt{p}v)e^{-v^2/2} .$$

F. Weissler was able to determine necessary and sufficient conditions on ω for K_ω to be bounded. In the case that K_ω is bounded, Weissler showed further that its norm is usually equal to one.

THEOREM 3.8. (WEISSLER'S THEOREM) *Let* $1 \leq p,\ q \leq \infty$, *but exclude the values* $2 < p \leq q < 3$ *and* $3/2 < p \leq q < 2$. *Let* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, *and let* $|\omega| \leq 1,\ \omega \neq \pm 1$ satisfy

$$\mathrm{Re}\,\frac{1}{1-\omega^2} \geq \max\{1/p,\ 1/q'\} \tag{3.4}$$

and

$$\left(\mathrm{Re}\,\frac{1}{1-\omega^2} - \frac{1}{p}\right)\left(\mathrm{Re}\,\frac{1}{1-\omega^2} - \frac{1}{q'}\right) \geq \left(\mathrm{Re}\,\frac{\omega}{1-\omega^2}\right)^2 . \tag{3.5}$$

Then

$$\| K_\omega \|_{\mu;p,q} = 1.$$

Moreover, K_ω *is bounded if and only if* ω *satisfies* (3.4) *and* (3.5).

We observe that the case $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ is covered by this theorem. In this case, (3.4) implies $|\omega| \le 1$ with equality allowed only if $p = 2$.

For some p and q the case $|\omega| = 1$ is allowed in Weissler's theorem. By this we mean that K_ω may be extended to $|\omega| = 1$, $\omega \ne \pm 1$ and that the extension is an operator of norm one.

SKETCH OF PROOF. Weissler proves the boundedness of K_ω by relating K_ω to the Gauss-Weierstrass operator

$$(e^{z\Delta}f)(x) = (4\pi z)^{-1/2} \int_{-\infty}^{\infty} f(y)\exp\left\{ -\frac{1}{4z}(x-y)^2 \right\} dy.$$

If ω satisfies (3.4) and (3.5) then the Gauss-Weierstrass operator with $z = \gamma(1-\omega^2)/4\omega$ (where $\mathrm{Re}(\gamma/\omega) \ge 0$) is bounded, from which he concludes that K_ω is bounded. It is possible, however, to prove this theorem without using the Gauss-Weierstrass operator. If ω $|\omega| < 1$ satisfies

$$\mathrm{Re}\, \frac{1}{1-\omega^2} > \max\left\{ \frac{1}{p}, \frac{1}{q'} \right\}$$

and

$$(\mathrm{Re}\,\frac{1}{1-\omega^2} - \frac{1}{p})(\mathrm{Re}\,\frac{1}{1-\omega^2} - \frac{1}{q'}) > (\mathrm{Re}\,\frac{\omega}{1-\omega^2})^2 ,$$

then calculations similar to those in theorem 3.3 imply the boundedness of K_ω. The result that $\|K_\omega\|_{\mu;p,q} = 1$ for these ω will then imply boundedness for the region (3.4) and (3.5). To prove that K_ω is bounded implies (3.4) and (3.5), Weissler exhibits explicitly the effect of K_ω on Gaussian functions and shows that for ω satisfying (3.4) and (3.5), their images are unbounded under K_ω.

To show that $\|K_\omega\| = 1$ in the prescribed cases, Weissler proves a two point inequality. It is the same one as in Beckner's proof but Weissler was more careful with his estimates and so he could prove it to hold for more general p,q, and ω. The remainder of the proof is then identical to the proof of Beckner's theorem. A detailed account of the proof of this theorem is to be found in [16].

Chapter 4

THE NORM OF THE MEHLER KERNEL OPERATOR

This chapter is devoted to proving the main result of this dissertation. The norm of N_ω as an operator from L^p to L^q is calculated explicitly for all $|\omega| \leq 1$, $\omega \neq \pm 1$. Since N_ω is unbounded for all other ω this settles the question completely. We will also show that if $p \neq 1$ then N_ω always achieves its norm for Gaussian functions of a particular form. The main result is stated in theorem 4.1.

THEOREM 4.1 (MAIN RESULT) *Let* $|\omega| \leq 1$, $\omega \neq \pm 1$, $1 < p \leq 2$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then there exists a complex number* γ *such that*

$$\mathrm{Re}\left\{ \frac{1+\omega^2}{2(1-\omega^2)} - \frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)} \right\} = 0 \tag{4.1}$$

and

$$(1-\gamma^2\omega^2)/\gamma(1-\omega^2) > 0 . \tag{4.2}$$

For any such γ *we have*

$$\|N_\omega\|_{p,q} = |\gamma|^{\frac{1}{2}} \left(\frac{1-\gamma^2\omega^2}{\pi\gamma(1-\omega^2)} \right)^{1/2p-1/2q} .$$

Moreover, this norm is achieved for the function

$$f(x) = \left(\frac{1-\gamma^2\omega^2}{\pi\gamma(1-\omega^2)} \right)^{1/2p} \exp\left\{ x^2 \left(\frac{1+\omega^2}{2(1-\omega^2)} - \frac{1}{\gamma(1-\omega^2)} \right) \right\} .$$

We remark that the operators N_ω have not yet been defined on L^p for $|\omega| = 1$, $\omega \neq \pm 1$, and for this case the theorem should be read more carefully. What we mean to say is that N_ω, defined at least on $L^1 \cap L^2$, satisfies

$$\|N_\omega f\|_q \leq C \, \|f\|_p$$

for $f \in L^1 \cap L^2$, where C is the bounding constant in the theorem. Hence, N_ω may be extended linearly to L^p in such a way that $\|N_\omega\|_{p,q} \leq C$ and the norm is assumed for the Gaussian function f of the theorem.

In the course of the proof it will be apparent that we can choose $\gamma = \gamma(\omega)$ in such a way that for $|\omega_0| = 1$, $\omega_0 \neq \pm 1$ we have

$$\lim_{\omega \to \omega_0,\, |\omega| < 1} \gamma(\omega) = \gamma(\omega_0) \quad .$$

The inequality $\|N_{\omega_0}\|_{p,q} \leq C$ is thus a consequence of the corresponding inequality for $|\omega| < 1$, for if $f \in L^1 \cap L^2$, then we have (see theorem 1.3 for method),

$$\|N_{\omega_0} f\|_{p,q}^q = \int_{-\infty}^{\infty} \liminf_{r \uparrow 1} |(N_{\omega_0 r} f)(x)|^q dx$$

$$\leq \liminf_{r \uparrow 1} C(\omega_0 r) \, \|f\|_p$$

$$= C(\omega_0) \, \|f\|_p \quad .$$

We also remark that the case $p = 1$ will be treated separately at the end of this chapter.

We begin the proof of theorem 4.1 with the following lemma.

LEMMA 4.2. *Let* $|\omega| \leq 1$, $\omega \neq \pm 1$, $1 < p \leq 2$ *and* $1/p + 1/q = 1$. *Then there exists a complex number* γ *that satisfies* (4.1) *and* (4.2).

PROOF. Let a and b be real numbers with $a > 0$. Then (4.1) and (4.2) are true if and only if we can choose a and b so that the following are true:

$$\frac{1+\omega^2}{2(1-\omega^2)} - \frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)} = ib$$

$$(1-\gamma^2\omega^2)/\gamma(1-\omega^2) = a .$$

These may be rewritten in the form

$$\frac{1+\omega^2}{2(1-\omega^2)} - \frac{p-1}{p(1-\omega^2)} \cdot \frac{1}{\gamma} - \frac{\omega^2}{p(1-\omega^2)} \cdot \gamma = ib$$

$$\frac{1}{1-\omega^2} \cdot \frac{1}{\gamma} - \frac{\omega^2}{1-\omega^2} \cdot \gamma = a,$$

which are equivalent to the following equations.

$$-\frac{p}{2} \cdot \frac{1+\omega^2}{1-\omega^2} \cdot \frac{p}{1-\omega^2} \cdot \frac{1}{\gamma} = a - ipb$$

$$\frac{p}{2(p-1)} \cdot \frac{1+\omega^2}{1-\omega^2} - \frac{p}{p-1} \cdot \frac{\omega^2}{1-\omega^2} \cdot \gamma = a + \frac{p}{p-1} ib$$

This system of equations has a solution γ if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$\left(a - ipb + \frac{p}{2} \cdot \frac{1+\omega^2}{1-\omega^2}\right)\left(\frac{p}{2} \cdot \frac{1+\omega^2}{1-\omega^2} - (p-1)a - ipb\right) = \frac{p^2\omega^2}{(1-\omega^2)^2} . \tag{4.3}$$

Hence there exists such a γ if and only if there exist real numbers a and b with $a > 0$ that satisfy (4.3). We let

$$\frac{p}{2} \cdot \frac{1+\omega^2}{1-\omega^2} = x + iy$$

where x and y are real. It follows that $x \geq 0$. Also

$$\frac{p^2\omega^2}{(1-\omega^2)^2} = \left(\frac{p}{2} \cdot \frac{1+\omega^2}{1-\omega^2}\right)^2 - \frac{p^2}{4} = (x^2 - y^2 - p^2/4) + 2ixy ,$$

whence (4.3) becomes

$$[(a+x) + i(y-pb)][(x-a(p-1)) + i(y-pb)] = (x^2-y^2-p^2/4) + 2ixy.$$

We take real and imaginary parts of the above equation to get the following two equations.

$$-(p-1)a^2 + (2-p)xa - p^2b^2 + 2by + p^2/4 = 0 , \tag{4.4}$$

$$(2-p)ya - (2-p)pab - 2xpb = 0 . \tag{4.5}$$

It will now be shown that for all ω and p satisfying the hypothesis of the lemma we may find real solutions a, b of (4.4) and (4.5) such that $a > 0$. This suffices to prove the lemma.

First suppose $p = 2$ or $x = y = 0$. Then we have the solution

$$b = 0 , \quad a = \frac{p}{2\sqrt{p-1}} .$$

Also, if $p \neq 2$ and x is equal to zero then (4.4) and (4.5) have the solution

$$b = y/p \quad , \quad a = \sqrt{\left[\left(\frac{2}{p} - 1\right) y^2 + \frac{p^2}{y}\right] / (p-1)} \quad .$$

If $p \neq 2$ then from (4.5) we obtain

$$a = \frac{2xpb}{(2-p)(y-pb)} \tag{4.6}$$

substituting this into (4.4) gives us

$$g(b) = -(p-1)4\, x^2p^2b^2 + 2(2-p)^2x^2pb(y-pb) + (2-p)^2(y-pb)^2(2pyb - p^2b^2 + \frac{p^2}{2})$$

$$= 0. \tag{4.6a}$$

Now $g(b)$ is a fourth degree polynomial in b. The coefficient of b^4 in g is $-(2-p)^2p^4$ which is always negative as $p \neq 2$. Also, $g(0) = [(2-p)py/2]^2$ which is nonnegative.

If $y = 0$ then (4.5) implies $b = 0$ whence (4.4) becomes the following quadratic equation in a.

$$-(p-1)a^2 + (2-p)xa + p^2/4 = 0.$$

We then take

$$a = \frac{(2-p) + \sqrt{(2-p)^2x^2 + p^2(p-1)}}{2(p-1)}$$

which is always positive.

Finally, if $y \neq 0$, then $g(0) > 0$ and so g has at least two real roots, one positive and the other negative. We now notice that

$$g(y/p) = -(p-1)\,4x^2y^2 < 0 ,$$

as $x \neq 0$. So g has a real root b that satisfies

$$0 < |b| < |y/p| \quad , \quad \mathrm{sign}(y) = \mathrm{sign}(b),$$

whence (4.6) yields a positive a. The lemma is now proven.

We need to analyze the root b of (4.6a) for ω near the unit circle in more detail. Let $|\omega_0| = 1$, $\omega_0 \neq \pm 1$. If we now write

$$\frac{p}{2}\left(\frac{1+\omega_0^2}{1-\omega_0^2}\right) = x_0 + iy_0 \quad ,$$

then $x_0 = 0$. Suppose $y_0 \neq 0$. From what we found above we see that we can take

$$b_0 = y_0/p \quad , \quad a_0 = \sqrt{\left[\left(\frac{2}{p}-1\right)y_0^2 + \frac{p^2}{4}\right]/(p-1)} \quad ;$$

This gives rise to a $\gamma(\omega_0)$. Let

$$f_1(b,y) = (2-p)^2(2pyb - p^2b^2 + p^2/4) ,$$

$$f_2(b) = 2(2-p)^2pb \quad ,$$

$$f_3(b) = (p-1)\,4p^2b^2 .$$

We may then rewrite (4.6a) as

$$f_1(b,y)(y-pb)^2 + x^2(y-pb)f_2(b) = x^2 f_3(b) .$$

By an appeal to the implicit function theorem one can show the above equation has two roots (one to the left and one to the right of y/p) which are close to y_0/p if $x > 0$ and $x + iy$ is close to iy_0. Taking for b the root whose modulus is less than $|y/p|$, we get a positive a and a $\gamma(\omega)$ that satisfies $\gamma(\omega) \to \gamma(\omega_0)$ if $|\omega| < 1$ and $\omega \to \omega_0$.

If $y_0 = 0$ then $\omega_0 = \pm i$. If $\omega = it$, for $-1 \leq t \leq 1$, then γ may be calculated explicitly (see theorem 5.1)

$$\gamma(it) = \frac{1}{4t^2}[-p(1-t^2) + (p^2t^4 - (2p^2-16p+16)t^2 + p^2)^{\frac{1}{2}}].$$

This γ is clearly continuous in t for t close to ± 1.

With this rather technical lemma out of the way we may proceed with the proof of the main theorem. In the following pages we will find a relation between N_ω and $K_{\gamma\omega}$. From the results of chapter 3, we know a great deal about the norm of K_ω.

In particular, we will show that for the γ of lemma 4.2 $K_{\gamma\omega}$ is in fact a bounded operator with norm equal to one from $L^p(d\mu)$ to $L^q(d\mu)$. This will be done by applying Weissler's theorem to $\gamma\omega$ rather than ω. The factor γ is just small enough to force $\gamma\omega$ to be within the domain specified by Weissler.

LEMMA 4.3. *Let* $|\omega| \leq 1$, $\omega \neq \pm 1$, $1 < p \leq 2$, $1/p + 1/q = 1$, *and let* γ *satisfy* (4.1) *and* (4.2). *Then* $K_{\gamma\omega}$ *has norm equal to one.*

PROOF. We need only show that $\gamma\omega$ satisfies

$$\mathrm{Re}\,\frac{1}{1-\gamma^2\omega^2} \geq \frac{1}{p} \tag{4.7}$$

and

$$\left(\mathrm{Re}\,\frac{1}{1-\gamma^2\omega^2} - \frac{1}{p}\right)^2 \geq \left(\mathrm{Re}\,\frac{\gamma\omega}{1-\gamma^2\omega^2}\right)^2 \quad , \tag{4.8}$$

for then Weissler's theorem (theorem 3.8) will imply that $K_{\gamma\omega}$ has norm equal to one. We let α and β be as in theorem 4.5. Then

$$\mathrm{Re}\,\frac{1}{1-\gamma^2\omega^2} - \frac{1}{p} = \mathrm{Re}\,\frac{p-1+\gamma^2\omega^2}{p(1-\gamma^2\omega^2)}$$

$$= \frac{2}{\beta}\,\mathrm{Re}\,\frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)}$$

$$= \frac{1}{\beta}\,\mathrm{Re}\,\frac{1+\omega^2}{1-\omega^2} \geq 0$$

as $\beta > 0$ and $(1+\omega^2)/(1-\omega^2)$ maps $|\omega| \leq 1$ into $\mathrm{Re}\,\omega \geq 0$. This implies (4.7). We now notice that (4.8) may be rewritten as

$$\left(\mathrm{Re}\,\frac{p-1+\gamma^2\omega^2}{p(1-\gamma^2\omega^2)}\right)^2 \geq \left(\mathrm{Re}\,\frac{\gamma\omega}{1-\gamma^2\omega^2}\right)^2 .$$

We multiply both sides by $\beta/2$ (which does not change the sign of the inequality as $\beta > 0$) to get

$$\left(\mathrm{Re}\,\frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)}\right)^2 \geq \left(\mathrm{Re}\,\frac{\omega}{1-\omega^2}\right)^2 ,$$

which is equivalent to

$$\left(\mathrm{Re}\,\frac{1+\omega^2}{2(1-\omega^2)}\right)^2 \geq \left(\mathrm{Re}\,\frac{\omega}{1-\omega^2}\right)^2 . \qquad (4.9)$$

We now notice that

$$\mathrm{Re}\,\frac{1+\omega^2}{2(1-\omega^2)} - \mathrm{Re}\,\frac{\omega}{1-\omega^2} = \frac{1}{2}\,\mathrm{Re}\,\frac{1-\omega}{1+\omega} \geq 0 \qquad (4.10)$$

and

$$\mathrm{Re}\,\frac{1+\omega^2}{2(1-\omega^2)} + \mathrm{Re}\,\frac{\omega}{1-\omega^2} = \frac{1}{2}\,\mathrm{Re}\,\frac{1+\omega}{1-\omega} \geq 0 , \qquad (4.11)$$

because the transformations $z_1 = (1+\omega)/(1-\omega)$ and $z_2 = (1-\omega)/(1+\omega)$ are both conformal mappings of $|\omega| \leq 1$ into the right half plane. Hence $\mathrm{Re} z_1 \geq 0$ and $\mathrm{Re} z_2 \geq 0$. Multiplying (4.10) and (4.11) together then yields (4.9) and our lemma is established.

In order to state the relationship between $K_{\gamma\omega}$ and N_{ω} we will need the following definitions.

DEFINITION 4.4. *Let* $a \geq 1$, *let* β *be a real number,* $\beta \neq 0$, *and let* α *be a complex number,* $\text{Re}\,\alpha \geq 0$.

(i) *Define the map* I_a *from* $L^a(d\mu)$ *to* $L^a(dx)$ *by*

$$(I_a f)(x) = \begin{cases} (2\pi)^{-1/2a}\, e^{-x^2/2a} f(x) & \text{if } a < \infty \\ f(x) & \text{if } a = \infty . \end{cases}$$

(ii) *Define the map* M_α *from* $L^a(dx)$ *to* $L^a(dx)$ *by*

$$(M_\alpha f)(x) = e^{-\alpha x^2} f(x).$$

(iii) *Define the map* T_β *from* $L^a(dx)$ *to* $L^a(dx)$ *by*

$$(T_\beta f)(x) = f(\beta x).$$

We have the following proposition.

PROPOSITION 4.5. *Let* $a \geq 1$. *Then the following are true.*

(i) I_a *is an isometry from* $L^a(d\mu)$ *to* $L^a(dx)$.

(ii) *If* $\text{Re}\,\alpha = 0$ *then* M_α *is an isometry of* $L^a(dx)$.

(iii) *For real* $\beta \neq 0$ *and* $f \in L^a(dx)$ *we have*

$$\|T_\beta f\|_a = |\beta|^{-1/a}\, \|f\|_a .$$

PROOF. The proof is by computation. Let $f \in L^a(d\mu)$. Then if $a < \infty$,

$$\|I_a f\|_a^a = \int_{-\infty}^{\infty} |(2\pi)^{-1/2a}\, e^{-x^2/2a}\, f(x)|^a\, dx = \|f\|_{\mu;a}^a .$$

Since $I_\infty f = f$, this result is trivial for $a = \infty$, hence (i) is true. Now let $f \in L^a(dx)$ and compute

$$\|M_\alpha f\|_a = \| \, |M_\alpha f| \, \|_a$$

$$= \| \, |f| \, \|_a$$

$$= \|f\|_a ,$$

and so (ii) is true. Lastly, if $a < \infty$,

$$\|T_\beta f\|_a^a = \int_{-\infty}^{\infty} |f(\beta x)|^a dx$$

$$= \left|\frac{1}{\beta}\right| \int_{-\infty}^{\infty} |f(x)|^a dx ,$$

and if $a = \infty$ then $\|T_\beta f\|_\infty = \|f\|_\infty$ trivially. This proves (iii).

We now state the identity that relates $K_{\gamma\omega}$ to N_ω.

THEOREM 4.6. *Let* $|\omega| < 1$, $1 < p \leq 2$, $1/p + 1/q = 1$, *and let* γ *satisfy* (4.1) *and* (4.2). *Then*

$$N_\omega = (2\pi)^{1/2q-1/2p} \, \gamma^{1/2} \, M_\alpha \, T_{\sqrt{\beta}} \, I_q \, K_{\gamma\omega} \, I_p^{-1} \, T_{1/\sqrt{\beta}} \, M_\alpha$$

where

$$\beta = 2(1-\gamma^2\omega^2)/\gamma(1-\omega^2)$$

and

$$\alpha = \frac{1+\omega^2}{2(1-\omega^2)} - \frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)} .$$

We remark that the identity in theorem 4.5 is also true if we remove the restriction $1/p + 1/q = 1$ and replace the leftmost M_α with M_{α_1} where

$$\alpha_1 = \frac{1+\omega^2}{2(1-\omega^2)} - \frac{(q-1)\gamma^2\omega^2+1}{q\gamma(1-\omega^2)} .$$

This will be evident from the proof.

PROOF. Let $f \in L^p$ and compute

$(I_q K_{\gamma\omega} I_p^{-1} f)(x)$

$$= (2\pi)^{-1/2q} e^{-x^2/2q} \frac{1}{\sqrt{2\pi(1-\gamma^2\omega^2)}} \int_{-\infty}^{\infty} (2\pi)^{1/2p} e^{y^2/2p} f(y) \exp\left\{ \frac{-(y-\gamma\omega x)^2}{2(1-\gamma^2\omega^2)} \right\} dy$$

$$= \frac{(2\pi)^{1/2p-1/2q}}{\sqrt{2\pi(1-\gamma^2\omega^2)}} \int_{-\infty}^{\infty} f(y) \exp\left\{ -y^2\left[\frac{p-1+\gamma^2\omega^2}{2p(1-\gamma^2\omega^2)}\right] + xy\left[\frac{\omega}{1-\gamma^2\omega^2}\right] - x^2\left[\frac{1+(q-1)\gamma^2\omega^2}{2q(1-\gamma^2\omega^2)}\right] \right\} dy .$$

Substituting $\sqrt{\beta}x$ and $\sqrt{\beta}y$ for x and y respectively in the above gives us

$(T_{\sqrt{\beta}} I_q K_{\gamma\omega} I_p^{-1} f)(x)$

$$= \frac{(2\pi)^{1/2p-1/2q}}{\sqrt{\gamma\pi(1-\omega^2)}} \int_{-\infty}^{\infty} (T_{\sqrt{\beta}} f)(y) \exp\left\{ -y^2 \left[\frac{p-1+\gamma^2\omega^2}{\gamma(1-\omega^2)}\right] + xy\left[\frac{2\omega}{1-\omega^2}\right] - x^2\left[\frac{1-(q-1)\gamma^2\omega^2}{q\gamma(1-\omega^2)}\right]\right\} dy$$

$$= \gamma^{-1/2}(2\pi)^{1/2p-1/2q} e^{\alpha_1 x^2} \int_{-\infty}^{\infty} (T_{\sqrt{\beta}} f)(y)\ N_\omega(x,y) e^{\alpha y^2} dy$$

$$= \gamma^{-1/2}(2\pi)^{1/2q-1/2p}\ (M_{-\alpha_1} N_\omega M_{-\alpha} T_{\sqrt{\beta}} f)(x),$$

whence

$$N_\omega = (2\pi)^{1/2q-1/2p} \gamma^{1/2}\ M_{\alpha_1} T_{\sqrt{\beta}} I_q K_{\gamma\omega} I_p^{-1}\ T_{1/\sqrt{\beta}}\ M_\alpha .$$

If $1/p + 1/q = 1$ then $(p-1)(q-1) = 1$ and we have, in addition

$$\alpha_1 = \frac{1+\omega^2}{2(1-\omega^2)} - \frac{(p-1)(q-1)\gamma^2\omega^2 + p-1}{(p-1)q\gamma(1-\omega^2)} = \alpha$$

and the theorem is proven.

We may now prove theorem 4.1. If we let ω and p be according to the hypothesis of the theorem and let γ satisfy (4.1) and (4.2), then it follows immediately from theorem 4.5, proposition 4.4, and lemma 4.6 that

$$\|N_\omega\|_{p,q} \le (2\pi)^{1/2q-1/2p} \; |\gamma|^{1/2} \; (\beta)^{1/2p-1/2q}$$

$$= |\gamma|^{1/2}\left(\frac{1-\gamma^2\omega^2}{\pi\gamma(1-\omega^2)}\right)^{1/2p-1/2q} .$$

It suffices now to prove that for the function

$$f(x) = \left(\frac{1-\gamma^2\omega^2}{\pi\gamma(1-\omega^2)}\right)^{1/2p} \exp\left\{x^2\left(\frac{1+\omega^2}{2(1-\omega^2)} - \frac{1}{\gamma(1-\omega^2)}\right)\right\}$$

$$= (\beta/2\pi)^{1/2p}\, e^{(\alpha-\beta/2p)x^2} ,$$

N_ω actually achieves this bound. We use the fact that α is purely imaginary and $\beta > 0$ to compute

$$\|f\|_p = (\beta/2\pi)^{1/2p}\left(\int_{-\infty}^{\infty} e^{-\beta x^2/2}dx\right)^{1/p} = 1.$$

Also

$$(N_\omega f)(x) = \frac{(\beta/2\pi)^{1/2p}}{\sqrt{\pi(1-\omega^2)}}\int_{-\infty}^{\infty} e^{(\alpha-\beta/2p)y^2} \exp\left\{\frac{4\omega xy - (x^2+y^2)(1+\omega^2)}{2(1-\omega^2)}\right\}dy$$

$$= \frac{(\beta/2\pi)^{1/2p}}{\sqrt{\pi(1-\omega^2)}} \int_{-\infty}^{\infty} \exp\left\{-y^2\left[\frac{1+\omega^2}{2(1-\omega^2)} - \alpha + \beta/2p\right] + xy\left[\frac{2\omega}{1-\omega^2}\right] - x^2\left[\frac{1+\omega^2}{2(1-\omega^2)}\right]\right\} dy$$

$$= \frac{(\beta/2\pi)^{1/2p}}{\sqrt{\pi(1-\omega^2)}} \int_{-\infty}^{\infty} \exp\left\{-y^2\left[\frac{1}{\gamma(1-\omega^2)}\right] + xy\left[\frac{2\omega}{1-\omega^2}\right] - x^2\left[\frac{1+\omega^2}{2(1-\omega^2)}\right]\right\} dy$$

$$= \frac{(\beta/2\pi)^{1/2p}}{\sqrt{\pi(1-\omega^2)}} \exp\left\{x^2\left(\frac{\omega^2\gamma}{1-\omega^2} - \frac{1+\omega^2}{2(1-\omega^2)}\right)\right\} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{y}{\sqrt{\gamma(1-\omega^2)}} - \omega x\sqrt{\frac{\gamma}{1-\omega^2}}\right)^2\right\} dy$$

$$= \gamma^{1/2}\ (\beta/2\pi)^{1/2p} \exp\left\{x^2\left(\frac{\omega^2\gamma}{1-\omega^2} - \frac{1+\omega^2}{2(1-\omega^2)}\right)\right\}.$$

We now have

$$\|N_\omega f\|_q = |\gamma|^{1/2}(\beta/2\pi)^{1/2p}\left(\int_{-\infty}^{\infty} \exp\left\{qx^2 \operatorname{Re}\left[\frac{\omega^2\gamma}{1-\omega^2} - \frac{1+\omega^2}{2(1-\omega^2)}\right]\right\} dx\right)^{1/q}$$

$$= |\gamma|^{1/2}(\beta/2\pi)^{1/2p}\left(\int_{-\infty}^{\infty} \exp\left\{qx^2 \operatorname{Re}\left[\frac{(p-1)(\gamma^2\omega^2-1)}{p\gamma(1-\omega^2)}\right]\right\} dx\right)^{1/q}$$

$$= |\gamma|^{1/2}(\beta/2\pi)^{1/2p}\left(\int_{-\infty}^{\infty} e^{-\beta x^2/2}\, dx\right)^{1/q}$$

$$= |\gamma|^{1/2}(\beta/2\pi)^{1/2p-1/2q}$$

$$= |\gamma|^{1/2}\left(\frac{1-\gamma^2\omega^2}{\pi\gamma(1-\omega^2)}\right)^{1/2p-1/2q}.$$

This completes the proof of theorem 4.1

We now treat the case $p = 1$.

THEOREM 4.7. *Let* $|\omega| \leq 1$, $\omega \neq \pm 1$. *Then*

$$\| N_\omega \|_{1,\infty} = |\pi(1-\omega^2)|^{-1/2} .$$

PROOF. We again assume at first that $|\omega| < 1$.

Let

$$a = \text{Re}\,\frac{1+\omega^2}{2(1-\omega^2)} , \qquad b = \text{Re}\,\frac{\omega}{1-\omega^2} .$$

Then $a > 0$ and

$$a + b = \text{Re}\,\frac{1+\omega}{2(1-\omega)} > 0$$

$$a - b = \text{Re}\,\frac{1-\omega}{2(1+\omega)} > 0$$

so $a^2 - b^2 > 0$. We then have

$$\sup_y |N_\omega(x,y)| = |\pi(1-\omega^2)|^{-1/2} \sup_y \exp\left\{2xyb - (x^2+y^2)a\right\}$$

$$= |\pi(1-\omega^2)|^{-1/2} \exp\left\{x^2(b^2-a^2)/a\right\} \sup_y \exp\left\{-(\sqrt{a}y - bx/\sqrt{a})^2\right\}$$

$$= |\pi(1-\omega^2)|^{-1/2} \exp\left\{x^2(b^2-a^2)/a\right\} .$$

Hence, if $f \in L^1$,

$$\| N_\omega f \|_\infty = \sup_x \left| \int_{-\infty}^{\infty} f(y) N_\omega(x,y)\,dy \right|$$

$$\leq \sup_x \| f \|_1 \sup_y |N_\omega(x,y)|$$

$$= \| f \|_1 \sup_x |\pi(1-\omega^2)|^{-1/2} \exp\left\{ x^2(b^2-a^2)/a \right\}$$

$$= \| f \|_1 |\pi(1-\omega^2)|^{-1/2} ,$$

and so

$$\| N_\omega \|_{1,\infty} \leq |\pi(1-\omega^2)|^{-1/2} .$$

To show we have equality, we consider, for $u > 0$, the function $g(x) = e^{-ux^2}$. Let

$$s = \frac{1+\omega^2}{2(1-\omega^2)} , \quad t = \frac{\omega}{1-\omega^2} ,$$

so we know $\operatorname{Re} s > 0$ and

$$\operatorname{Re}\left(\frac{t^2 - s^2 - us}{s+u} \right) = -\frac{1}{2} \operatorname{Re}\left[\frac{(1+2u) - (1-2u)\omega^2}{(1+2u) + (1-2u)\omega^2} \right] < 0.$$

We calculate

$$(N_\omega g)(x) = (\pi(1-\omega^2))^{-1/2}\int_{-\infty}^{\infty} \exp\left\{-(s+u)y^2 + 2txy - sx^2\right\} dy$$

$$= (\pi(1-\omega^2))^{-1/2} \exp\left\{x^2\left(\frac{t^2-s^2-us}{s+ u}\right)\right\}\int_{-\infty}^{\infty} \exp\left\{-\left(y\sqrt{s+u}-\frac{t}{\sqrt{s+u}} x\right)^2\right\}dy$$

$$= [(1-\omega^2)(s+a)]^{-1/2} \exp\left\{x^2\left(\frac{t^2-s^2-us}{s + u}\right)\right\} ,$$

and so

$$\frac{\|N_\omega g\|_\infty}{\|g\|_1} = |(1-\omega^2)(s+u)|^{-1/2}(\pi/u)^{-1/2}$$

$$= |\pi(1-\omega^2)|^{-1/2} |u/(s+u)|^{1/2} .$$

Since this ratio approaches $|\pi(1-\omega^2)|^{-1/2}$ as $u \to \infty$ we have completed the proof of the theorem for $|\omega| < 1$.

Now let $f \in L^1$. Let $\epsilon > 0$ and let g be a continuous function of compact support such that $\|f-g\|_1 < \epsilon$. If $|\omega| = 1$ it follows that

$$|N_{r\omega}(f-g)(x)| \leq k\epsilon$$

for all $r \leq 1$ where k depends only on ω. Integration by parts yields the result

$$(N_{r\omega}g)(x) \to 0 \qquad \text{as} \qquad |x| \to \infty,$$

uniformly in r(i.e., N_ω obeys a Riemann-Legesgue type theorem). Hence

$$\lim_{r\uparrow 1} (N_{r\omega}f)(x) = (N_\omega f)(x)$$

uniformly for $x \in \mathbb{R}$. The usual arguments may now be used to extend the result of this theorem to $|\omega| = 1$.

Chapter 5

APPLICATIONS

1. BABENKO'S CONJECTURE

In chapter 3 we mentioned a theorem of K. I. Babenko which exhibits the norm of N_ω for ω imaginary and q equal to an even integer. Based on the results of chapter 4 we can now prove this theorem for general q. The theorem is restated in this general form for convenience.

THEOREM 5.1. *Let* $-1 \le t \le 1$, $1 < p \le 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. *Define*

$$\gamma = \frac{1}{4t^2}\left[-p(1-t^2) + \left(p^2t^4 - (2p^2-16p+16)t^2 + p^2\right)^{1/2}\right].$$

Then

$$\| N_{it} \|_{p,q} = \sqrt{\gamma}\left(\frac{1+\gamma^2t^2}{\pi\gamma(1+t^2)}\right)^{1/2p-1/2q}.$$

Moreover, this norm is achieved for the Gaussian function

$$g(x) = \exp\left\{-x^2\left(\frac{1+\gamma^2t^2}{p\gamma(1+t^2)}\right)\right\}.$$

Finally $\| N_{it} \|_{1,\infty} = (\pi(1+t^2))^{-1/2}$.

PROOF. The case $p = 1$ is a direct consequence of theorem 4.7. We assume now that $p > 1$. Let

$$\alpha = [p^2t^4 - (2p^2 - 16p + 16)t^2 + p^2]^{1/2} \tag{5.1}$$

so

$$\gamma = \frac{-p(1-t^2) + \alpha}{4t^2} . \tag{5.2}$$

We now compute

$$p - 1 - \gamma^2t^2 = p - 1 - \frac{1}{16t^2}(p^2(1-t^2)^2 + \alpha^2 - 2p(1-t^2)\alpha)$$

$$= \frac{1}{8t^2}(- p^2t^4 + 2p^2t^2 - p^4 - p(1-t^2)\alpha)$$

$$= \frac{p}{8t^2}(- p(1-t^2) + \alpha)(1 - t^2) ,$$

whence

$$\frac{p - 1 - \gamma^2t^2}{p\gamma(1+t^2)} = \frac{1 - t^2}{2(1+t^2)} ,$$

which is exactly condition (4.1) for $\omega = it$ in theorem 4.1. Also

$$\frac{1 + \gamma^2t^2}{\gamma(1+t^2)} = \frac{p^2t^4 + (8p-2p^2)t^2 + p^4 - p(1-t^2)\alpha}{2(1+t^2)(-p(1-t^2)+\alpha)}$$

$$= \frac{p[(2-p)(1-t^2) + \alpha)][-p(1-t^2) + \alpha]}{4(p-1)(1+t^2)(-p(1-t^2) + \alpha)}$$

$$= \frac{q}{4(1+t^2)} [(2-p)(1-t^2) + \alpha]. \tag{5.3}$$

However, we have

$$-2p^2 \leq 2p^2 - 16p + 16 \leq 2p^2$$

whence from (5.1)

$$\alpha \geq p|1 - t| \geq 0. \tag{5.4}$$

If we now assume $t \neq 1$ then by (5.3),condition (4.2) is also satisfied for $\omega = it$ and we apply theorem 4.1 to get

$$\|N_{it}\| = \sqrt{\gamma} \left(\frac{1 + \gamma^2 t^2}{\pi\gamma(1+t^2)}\right) \quad .$$

Since $\|N_i\|_{p,q} = \lim_{t \uparrow 1} \|N_{it}\|_{p,q}$ we get by the usual argument the corresponding equation for $t = 1$ also. Finally 4.1 says that N_{it} achieves this norm for

$$g(x) = \exp\left\{x^2\left(\frac{1 - t^2}{2(1+t^2)} - \frac{1}{\gamma(1+t^2)}\right)\right\}$$

$$= \exp\left\{x^2\left(\frac{p - 1 - \gamma^2 t^2}{p\gamma(1+t^2)} - \frac{1}{\gamma(1+t^2)}\right)\right\}$$

$$= \exp\left\{-x^2\left(\frac{1 + \gamma^2 t^2}{p\gamma(1+t^2)}\right)\right\} .$$

The theorem is now proven.

We may now let $t = -1$ in theorem 5.1 to achieve the sharp Hausdorff-Young inequality for the Fourier transform $\mathfrak{F}$, first proven by Beckner (see theorem 3.5 and corollary 3.8).

COROLLARY 5.2. *Let* $1 \leq p \leq 2$ *and* $1/p + 1/q = 1$. *Then*

$$\| \mathfrak{F} \|_{p,q} = (2\pi)^{1/2q - 1/2p} \left(p^{1/2p} / q^{1/2q} \right) .$$

From this it is seen that the method used by Wiener to prove Plancherel's theorem can be generalized to yield the Hausdorff-Young inequality directly from the norm of N_ω.

Finally, we notice that we may replace t by $i \cdot t$ everywhere in the proof of theorem 5.1 and so doing we get the following theorem.

THEOREM 5.3. *Let* $-1 \leq t \leq 1$, $1 < p \leq 2$, *and* $1/p + 1/q = 1$. *Define*

$$\gamma = \frac{1}{4t^2} [p(1+t^2) - (p^2 t^4 + (2p^2 - 16p + 16)t^2 + p^2)^{1/2}] .$$

Then

$$\| N_t \|_{p,q} = \sqrt{\gamma} \left(\frac{1 - \gamma^2 t^2}{\pi \gamma (1 - t^2)} \right)^{1/2p - 1/2q} .$$

Moreover, this norm is achieved for the Gaussian function

$$g(x) = \exp \left\{ -x^2 \left(\frac{1 - \gamma^2 t^2}{p\gamma(1 - t^2)} \right) \right\} .$$

Finally, if $t \neq \pm 1$ *then*

$$\|N_t\|_{1,\infty} = (\pi(1-t^2))^{-1/2} \quad .$$

2. SMOOTHING OPERATORS

Smoothing operators, as used by De Bruijn [6] are invaluable tools in the study of generalized functions. They are defined as follows.

DEFINITION 5.4. *Let* $\alpha > 0$ *and define*

$$S_\alpha(z,u) = (\sin h\,\alpha)^{-1/2} \exp\left\{- \frac{\pi}{\sin h\,\alpha}\,((z^2+u^2)\cos h\,\alpha - 2zu)\right\}$$

for all complex z *and* u. *We define* S_α *, the smoothing operator with parameter* ω, *by*

$$(S_\alpha f)(z) = \int_{-\infty}^{\infty} S_\alpha(z,u)f(u)du \; .$$

We also have the following representation of S_α .

PROPOSITION 5.5 (MYLLER-LEBEDEFF) *Let* $\alpha > 0$. *Then*

$$S_\alpha(z,u) = \sqrt{2\pi} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\alpha}\, \varphi_n(z\sqrt{2\pi})\varphi_n(u\sqrt{2\pi}), \tag{5.5}$$

and so

$$S_\alpha = \sqrt{2\pi}\, e^{-\alpha/2}\, T_{\sqrt{2\pi}}\, N_{e^{-\alpha}}\, T_{\sqrt{2\pi}} \; . \tag{5.6}$$

PROOF. The identity in (5.5) follows from comparison with theorem 1.7. To prove (5.6) we take f to be a function in $L^p(1 \leq p \leq 2)$

$$(S_\alpha f)(z/\sqrt{2\pi}) = \int_{-\infty}^{\infty} S_\alpha(z/\sqrt{2\pi}, u) f(u) du$$

$$= \int_{-\infty}^{\infty} f(u)\sqrt{2\pi} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\alpha} \varphi_n(z)\varphi_n(u/\sqrt{2\pi}) du$$

$$= \sqrt{2\pi} \int_{-\infty}^{\infty} f(\sqrt{2\pi}\, u) \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\alpha} \varphi_n(z)\varphi_n(u) du$$

$$= e^{-\alpha/2} \sqrt{2\pi} \int_{-\infty}^{\infty} f(\sqrt{2\pi}\, u) \sum_{n=0}^{\infty} (e^{-\alpha})^n \varphi_n(z)\varphi_n(u) du ,$$

from which the result follows.

We may now calculate the norm of the smoothing operator from L^p to L^q.

THEOREM 5.6. *Let* $1 < p \leq 2$, $1/p + 1/q = 1$, *and* $\alpha > 0$. *Define*

$$\gamma = \frac{e^{2\alpha}}{4} [p(1+e^{-2\alpha}) - (p^2 e^{-4\alpha} + (2p^2-16p+16)e^{-2\alpha} + p^2)^{1/2}].$$

Then

$$\| S_\alpha \|_{p,q} = (\gamma e^{-\alpha})^{1/2} \left(\frac{1-\gamma^2 e^{-2\alpha}}{\pi\gamma(1-e^{-2\alpha})} \right)^{1/2p - 1/2q} .$$

Moreover, the norm is achieved for the Gaussian function

$$g(x) = \exp\left\{-x^2\left(\frac{1-\gamma^2 e^{-2\alpha}}{p\gamma(1-e^{-2\alpha})}\right)\right\} .$$

Finally,

$$\|S_\alpha\|_{1,\infty} = e^{-\alpha/2}(\pi(1-e^{-2\alpha}))^{-1/2} .$$

PROOF. This is an immediate consequence of proposition 5.5 and theorem 5.3 with $\omega = e^{-\alpha}$.

3. THE OPERATORS OF KOBER

In [8] and [5], H. Kober and Condon used yet another Mehler-type operator in their study of the Fourier and Hankel transforms. These operators are defined as follows.

DEFINITION 5.7. *Let* r *be real,* $-\frac{1}{2} < r < \frac{1}{2}$ *and define*

$$T_r(x,y) = C_r \exp\left\{\frac{i\,x\,y}{\sin 2\pi r} - (x^2+y^2)(\tfrac{i}{2}\cot 2\pi r)\right\} ,$$

where

$$C_r = (2\pi|\sin 2\pi r|)^{-1/2}\, e^{\frac{1}{2}i\pi(\frac{1}{2}\,\mathrm{sign}(r) - 2r)}$$

for x *and* y *real.* $T_r(x,y)$ *is called the Kober kernel of parameter* r. *For* $f \in L^1$, *we define*

$$(T_r f)(x) = \int_{-\infty}^{\infty} f(y) T_r(x,y) dy$$

and call T_r *the Kober operator of parameter* r.

The operator T_r may also be defined on $L^p(1 \le p \le 2)$ but the same care must be taken as in the case of the Mehler operator on the unit circle.

The following lemma relates the Kober operator to the Mehler kernel operator.

LEMMA 5.8. $T_r = N_{e^{2\pi i r}}$

PROOF. We let $\omega = e^{i\theta}$ for θ real and calculate

$$\frac{1+\omega^2}{2(1-\omega^2)} = \frac{1+e^{2i\theta}}{2(1-e^{2i\theta})} = \frac{i \sin 2\theta}{2 - 2\cos 2\theta}$$

$$= \frac{i}{2} \cot \theta \quad ,$$

and

$$\frac{2\omega}{1-\omega^2} = \frac{2e^{i\theta}}{1-e^{2i\theta}} = \frac{4i \sin \theta}{2 - 2\cos 2\theta}$$

$$= \frac{i}{\sin\theta} \quad .$$

Furthermore,

$$\frac{1}{\sqrt{\pi(1-\omega^2)}} = \left(\frac{1}{\pi(1-e^{2i\theta})}\right)^{1/2}$$

$$= \left(\frac{ie^{-i\theta}}{2\pi \sin \theta} \right)^{1/2}$$

$$= (2\pi|\sin \theta|)^{-1/2} e^{\frac{1}{2} i\pi(\frac{1}{2} \operatorname{sign}(r) - \frac{\theta}{\pi})}.$$

We now have

$$N_\omega(x,y) = \frac{1}{\sqrt{\pi(1-\omega^2)}} \exp\left\{ \frac{4xy\omega - (x^2+y^2)(1+\omega^2)}{2(1-\omega^2)} \right\}$$

$$= (2\pi|\sin \theta|)^{-1/2} e^{\frac{1}{2} i\pi(\frac{1}{2} - \frac{\theta}{\pi})} \exp\left\{ \frac{ixy}{\sin \theta} - (x^2+y^2)(\tfrac{i}{2}\cot \theta) \right\}$$

$$= T_{\theta/2\pi}(x,y)$$

and the lemma follows by letting $\theta = 2\pi r$.

We now know that T_r maps L^p into L^q (compare with theorem 4.1 to see how T_r may be defined for L^p, $1 \le p \le 2$) and we may calculate its norm.

THEOREM 5.9. *Let* $1 < p \le 2$, $1/p + 1/q = 1$. *Let* r *be a real number but* $-\frac{1}{2} < r < \frac{1}{2}$. *Define*

$$\gamma = \frac{i\sqrt{p-1}}{e^{i2\pi r}} \operatorname{sign}(\sin 2\pi r).$$

Then

$$\|T_r\|_{p,q} = |\gamma|^{1/2}\left(\frac{1-\gamma^2 e^{i4\pi r}}{\pi\gamma(1-e^{i4\pi r})}\right)^{1/2p^{-1}/2q} .$$

Moreover, this norm is achieved by the Gaussian function

$$f(x) = \exp\left\{x^2\left(\frac{i}{2}\cot 2\pi r - \frac{1}{\gamma(1-e^{i4\pi r})}\right)\right\} .$$

Finally

$$\|T_r\|_{1,\infty} = |\pi(1-e^{i4\pi r})|^{-1/2} .$$

PROOF. The cases $p = 1$ and $p = 2$ are clear so we will assume $p \neq 1,2$ and $2r$ is not an integer. Let $\omega = e^{i2\pi r}$. Then

$$\gamma = \frac{i\sqrt{p-1}}{\omega} \text{ sign}(\sin 2\pi r)$$

and so

$$\frac{1+\omega^2}{2(1-\omega^2)} - \frac{p-1+\gamma^2\omega^2}{p\gamma(1-\omega^2)} = \frac{1+\omega^2}{2(1-\omega^2)}$$

$$= \frac{i}{2}\cot 2\pi r \quad . \tag{5.7}$$

Also

$$\frac{1-\gamma^2\omega^2}{\gamma(1-\omega^2)} = \frac{2-p}{i\sqrt{p-1}\,(\omega^{-1}-\omega)} \text{ sign}(\sin 2\pi r)$$

$$= \frac{(2-p)\text{sign}(\sin 2\pi r)}{i\sqrt{p-1}(-2i \sin 2\pi r)}$$

$$= \frac{2-p}{2\sqrt{p-1}\,|\sin 2\pi r|} > 0 \ . \tag{5.8}$$

The theorem now follows from theorem 4.1.

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