THE INVARIANT IMBEDDING SOLUTION FOR ELECTROMAGNETIC WAVE PROPAGATION IN PERIODIC, ALMOST HOMOGENEOUS, AND ALMOST PERIODIC MEDIA

Thesis by

Gary Bedrosian

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TO:

Professor Charles H. Papas,
who gave this thesis direction;

Gena Bedrosian,
who put up with me while I wrote it;

and

All of my Friends and Relatives,
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ABSTRACT

The technique of invariant imbedding, as introduced for the problem of electromagnetic scattering by V. A. Ambarzumian in 1943, provides a very convenient method for the solution (albeit numerical in many cases) of plane wave scattering from a one-dimensional region of inhomogeneity. In the thirty-plus years which have intervened, the usefulness of this method has been extended in the case of electromagnetic properties of the region of inhomogeneity (dielectric constant, permeability, and conductivity).

It is the purpose of this thesis to examine the invariant imbedding solution as it applies to periodic, almost periodic, and almost homogeneous media. The introduction of a complex number, Y , which is simply the reflection coefficient rotated by a fixed phase angle, is a new concept which allows the computation of the propagation constant for any periodic medium once the reflection and transmission properties for one cell are known, without any further complications such as matrix equations. The trajectory of the parameter, Y , also provides an interesting graphical representation of the properties of a periodic medium.

The concepts derived for general periodic media are then applied to the important class of media whose reflection coefficients remain small, except perhaps at special frequencies. In particular, a small reflection approximation leads to the result that for any medium which is "almost homogeneous," there will be one special frequency, for each structure constant in the cosine expansion of the index of refraction, for which the reflection gets large.

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*Mostly Non-Original Material

†Partly Non-Original Material

Symbols Used

1. Latin

a	Slab or cell length
a_T	Transformed slab length
A	Fourier coefficient of the index of refraction
B	Fourier coefficient of the derivative of the index
c	Speed of light in free space
k	Wavenumber
k_{avg}	Average wavenumber in the slab
k_{max}	Local maximum wavenumber in the slab
k_{min}	Local minimum wavenumber in the slab
k_T	Transformed wavenumber
n	Index of refraction
n_{avg}	Average index of refraction in the slab
P	Quasi-period of an orbit on the Y-plot
r	Local reflection coefficient or magnitude of R
R	Complex reflection coefficient
R_F	Fixed reflection point
t	Local transmission coefficient or magnitude of T
T	Complex transmission coefficient
V	Quasi-velocity along a trajectory on the Y-plot
X	Rotated reflection coefficient
X_F	Fixed value of X
Y	Standardized rotated reflection coefficient
Y_F	Fixed value of Y

z	Slab length coordinate
z_T	Transformed slab length coordinate
Z	$Y - Y_F$

2. Greek

α	Loss parameter
β	Propagation constant
γ	Antisymmetry loss parameter
δ	Phase shift (complex number)
Δ	Distance between interfaces
ε	Dielectric constant
η	Y-plot trajectory shape parameter
θ	Angle of incidence
κ	Structure constant for periodic and almost periodic media
μ	Permeability
ξ	Parameter (small)
σ	Conductivity
ϕ_A, ϕ_α	Loss angle parameters
ϕ_R	Phase of R
ϕ_T	Phase of T
χ	Parameter
ψ	Wave function, or phase angle
ω	Radian frequency

3. Operators

arg	Complex phase operator
exp	Exponential function

Re	Real part
Im	Imaginary part
Θ	Add one cell to Y
Φ	Add one cell to X

Chapter I

Introduction

The technique of invariant imbedding was first applied to electromagnetic scattering by Ambarzumian in 1943.[1] Since that time, there has been much work done both to extend the use of the technique in the solution of the wave-scattering problem (for instance, references 2 through 6) and to expound the theory behind the technique.[7] This thesis will be concerned with the application of Ambarzumian's technique to one-dimensional periodic and "almost periodic" materials.

As the invariant imbedding approach of Ambarzumian applies to one-dimensional, linear wave-scattering problems, it may be stated with sufficient completeness very briefly as the following quasi-algorithm: First, consider the simplest possible scattering problem, that is, the reflection from and transmission through a jump discontinuity in the properties of the medium conducting the wave, assuming uniformity on either side of the discontinuity. In optical terms, this would be transmission and reflection at an interface between media with differing indices of refraction. In quantum mechanics, a similar example would be the matching of wave solutions on two sides of a potential discontinuity. Second, assume that we know the solution for up to N such interfaces stacked together and examine what happens when we add one more interface at a distance Δ from the last. Finally, the solution of the simple case may be iteratively transformed by the derived relationship between N^{th} and $(N+1)^{st}$ solutions until we arrive at the final answer. In the case of a continuously-varying medium, the limiting process which makes the Δ's very small and the N correspondingly large yields first-order Riccati-type differential equations which, although hard to solve exactly, are very simple to approximate numerically to obtain the desired reflection and transmission properties of the inhomogeneous medium.

The alternative methods for calculating the reflection and transmission are nearly always more difficult, both conceptually and mathematically. For instance, we could start with Maxwell's equations and reduce them to the second-order linear wave equation

$$\psi'' + k^2\psi = 0$$

(when μ is constant), and solve for the proper wave solutions given $k^2(z)$, assuming that $k^2(z)$ is constant outside the slab and that the wave should look like the standard time-harmonic plane wave away from the region of inhomogeneity. Not only would this be a more difficult proposition numerically than invariant imbedding for almost all $k^2(z)$, but the physical process involved at each point along z would tend to be obscured by the sophisticated techniques used to solve the problem. Similarly, we could solve the quantum-mechanical one-dimensional "square well" problem by matching solutions at the boundaries, but this method also tends to conceal the physical process until the final solution is examined. In contradistinction, the invariant imbedding formulation remains physically understandable at every point along the path to the solution and is capable of generating the solution in a straightforward manner for every applicable case. In this spirit, we will attempt to keep the mathematical formulation relatively simple throughout.

This much of the application of the invariant imbedding technique has previously been well discussed by the authors of references 2 through 6. It would seem that, since we understand so well how to calculate the reflection and transmission coefficients for any given discrete distribution (and how to approximate accurately and easily in the continuous case), the problem is completely solved and there is no more to be done beyond, perhaps, finding some exact solutions previously unknown or solving interesting cases numerically. Indeed, the standard invariant imbedding formulation is capable of generating a numerical solution efficiently and accurately for any particular case of interest. However, the intent of this thesis is to extend the usefulness of the general invariant imbedding approach by deducing some general properties of the invariant imbedding solution and using these properties to derive new methods of exact and approximate solution of the problem based on the invariant imbedding solution. In particular, we will see how invariant imbedding leads to an interesting formulation of the problem of scattering from a periodic medium and how the Riccati equation may be reduced in many cases to a linear equation which greatly simplifies the analysis of any medium, in particular periodic and "almost periodic" media.

Chapter II is concerned with reformulation of the invariant imbedding solution and the proof of general properties to be used later. It contains a substantial amount of non-original material. The contents of Section II.2, "The Wave Equation," are common knowledge to several branches of physics and engineering. The mathematical ideas of Section II.3, "The Invariant Imbedding Solution," are available in many places, for instance Wave Propagation in Turbulent Media by R. Adams and E. Denman (2). The notation used in this report is most similar to that found in Adams et al. (2). Sections II.5, "An Alternative Invariant Imbedding Scheme," and II.8, "Variation of All Electromagnetic Parameters," contain as their basis background material (again, mainly using the notation of Adams (2)), but extra ideas have been added to the basic principles in the interest of later results. For instance, we have found it necessary to make a detailed comparison of the two schemes for invariant imbedding and to provide an explicit transformation which extends the results for normal incidence and constant permeability to the more general case. Finally, the similarity between electromagnetic and quantum mechanical scattering, as described in Section II.7, has been previously noted by Bellman et al. (3).

Chapter III uses the general results of Chapter II to look at wave propagation in periodic media in a new way. We introduce a concept called the "Y-plot" in the analysis of the propagation constant. The complex variable, Y , is equal to the complex reflection coefficient times a phase adjustment to standardize two special points in the Y-plane, called the "fixed points." The progression of the reflection coefficient as cells are added to the periodic structure is represented by a trajectory on the Y-plot. For STOP BANDS, the trajectories converge on one fixed point and diverge from the other; for PASS BANDS, the trajectories orbit a fixed point. In both cases, the propagation constant may be computed knowing only the fixed points, via very simple formulas. These results are applied to both lossless and lossy periodic media.

Chapter IV examines the class of problems for which the wavenumber (k) does not vary much as a function of position in the slab, which we call "almost homogeneous media." The very complicated (at least, difficult to solve exactly) differential equation for the reflection coefficient derived in Chapter II reduces to an almost embarrassingly simple linear approximation which possesses an equally simple Green's function. In these almost homogeneous cases, an inversion of the reflection coefficient as a function of frequency to reconstruct the index of refraction as a function of position is quite simple to perform (with the help of a digital computer). This method may even be used with modest success for cases which are not "almost homogeneous." We finally use the approximate theory to examine the STOP BANDS of periodic and almost periodic media. The conclusion is that there exists only one non-vanishing STOP BAND corresponding to each structure constant in the cosine expansion of the index of refraction for either periodic or almost periodic media.

Where applicable, all electromagnetic quantities are in MKS units.

Chapter II

The Basic Equations and Relations

II.1 Definitions

Our first task will be to reformulate the invariant imbedding equations for electromagnetic scattering from a medium with constant permeability, μ , zero conductivity, σ , and dielectric constant, ε , which is a function only of the propagation direction (and perhaps frequency.) In this case, we may think of the problem as that of a section of space between $z = -a/2$ and $z = +a/2$ where the medium may be characterized by an index of refraction

$$n(z;\omega) = c\sqrt{\varepsilon(z;\omega)\mu} \quad . \tag{II.1}$$

For now, we will drop the explicit dependence on ω . Outside the region of the "slab," $n(z)$ is constant. For reasons which will be clear later, we will let

$$n(z) = \begin{cases} n_N \; ; & z < -a/2 \\ n_0 \; ; & z > +a/2 \end{cases} \quad . \tag{II.2}$$

With this definition of $n(z)$, we may also consider the local wavenumber, $k(z)$, given by

$$k(z) = \pm(\omega/c)n(z) \quad , \tag{II.3}$$

where ω/c will often be called the "free space wavenumber" because $n = 1$ in free space.

II.2 The Wave Equation

When the permeability is constant, Maxwell's equations reduce to a simple one-dimensional wave equation for harmonic waves (we assume an $\exp(-i\omega t)$ time dependence) propagating in a direction perpendicular to both the electric and magnetic fields (which are, of course, also perpendicular to each other). The wave equation may be written in the simple form

$$\psi'' + k^2(z)\psi = 0 \quad , \tag{II.4}$$

where ψ is the (complex) length of the electric field vector. The solutions where $k^2(z)$ is constant are right- and left-hand traveling waves of the form

$$\psi = \psi_o \exp(\pm ikz) \quad , \tag{II.5}$$

where the (+,-) correspond to right (+) and left (-) traveling waves (along the z-axis). We will assume for our scattering problem that a wave of unit electric field is incident from the left, traveling to the right. We will fix the phase of the incident wave so that it has zero phase angle at the front interface $(z = -a/2)$. The scattering problem is to find the complex numbers R and T such that

$$\psi = \begin{cases} \exp[+ik_N(z+a/2)] + R\exp[-ik_N(z+a/2)]; & z \leq -a/2 \\ T\exp[+ik_o(z-a/2)] & ; \quad z \geq +a/2 \end{cases} \tag{II.6}$$

is a solution of the wave equation in the homogeneous region which matches correctly with a solution in the inhomogeneous region (see

Figure 1). At this point we are not concerned with the exact nature of the solution in the inhomogeneous region. The phase convention adopted for the transmitted wave is chosen so that the complex number T carries all of the phase information at $z = a/2$. In the trivial problem in which $\varepsilon(z)$ is constant everywhere (no reflection), $T = \exp(ik_o a)$.

We will also consider the reverse problem of finding the reflection and transmission numbers (R' and T') when a unit wave is incident from the right (traveling to the left). In this case, we want our solution to be of the form

$$\psi = \begin{cases} T' \exp[-ik_N(z+a/2)] \; ; & z \leq -a/2 \\ \exp[-ik_o(z-a/2)] + R'\exp[+ik_o(z-a/2)]; & z \geq +a/2 \; . \end{cases} \qquad (II.7)$$

We note in passing that in the trivial case of total homogeneity, $T' = \exp(ik_o a) = T$.

Although we will not find it convenient to obtain the complete solution by solving the wave equation, there are several important results which may be derived easily by integrating the wave equation. Suppose that we have two solutions of the wave equation, ψ_1 and ψ_2. Using a standard trick,

$$\psi_2\psi_1'' + k^2(z)\;\psi_1\psi_2 = 0 = \psi_1\psi_2'' + k^2(z)\;\psi_1\psi_2 \quad , \qquad (II.8)$$

which reduces to

$$\psi_2\psi_1'' - \psi_1\psi_2'' = (\psi_2\psi_1' - \psi_1\psi_2')' = 0 \quad . \qquad (II.9)$$

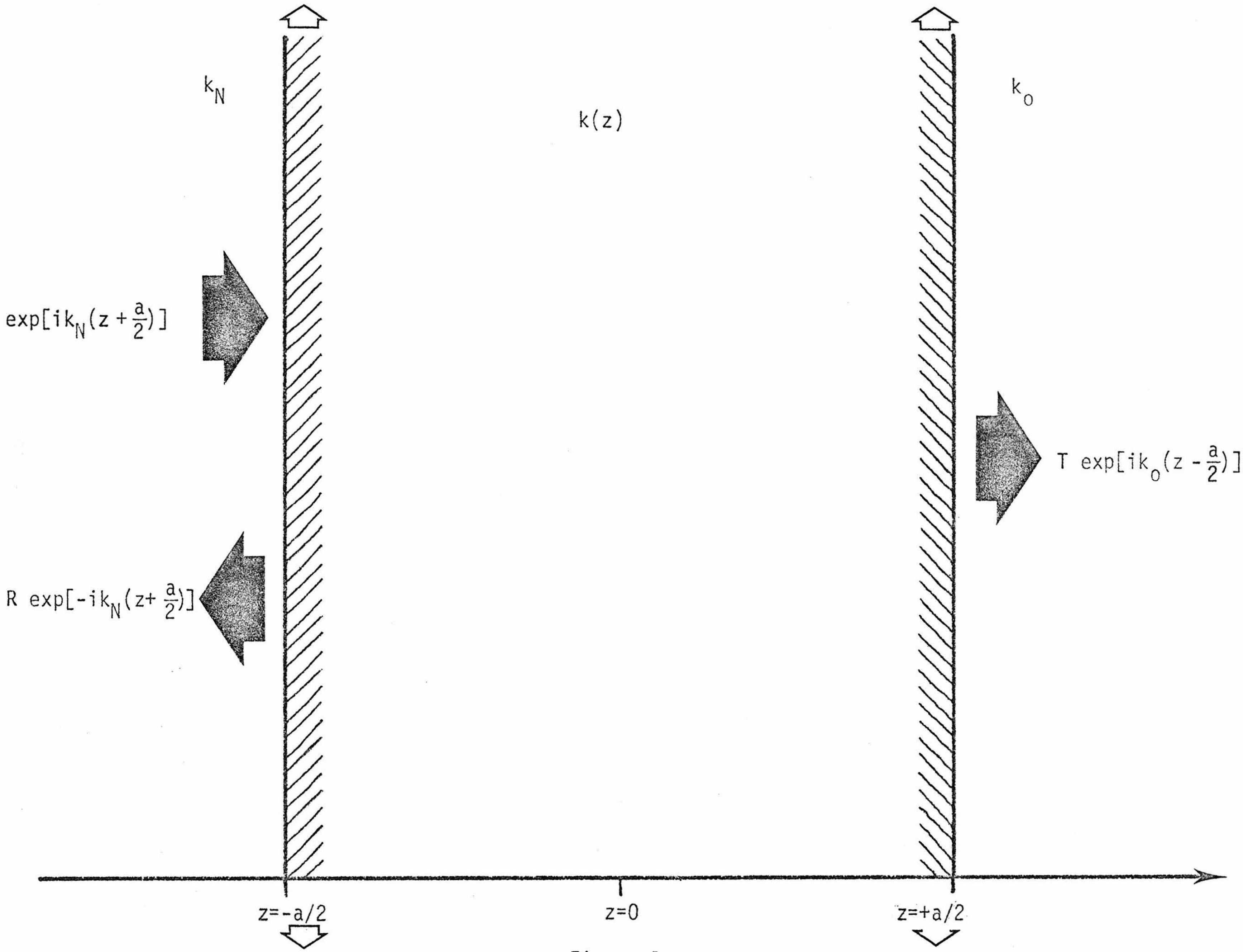

k_N
$k(z)$
k_o
$\exp[ik_N(z+\frac{a}{2})]$
$R\ \exp[-ik_N(z+\frac{a}{2})]$
$T\ \exp[ik_o(z-\frac{a}{2})]$
z=-a/2
z=0
z=+a/2

Figure 1

Integration from $z=-a/2$ to $z=+a/2$ is trivial. The result is

$$\psi_2(a/2)\psi_1'(a/2) - \psi_1(a/2)\psi_2'(a/2) =$$

$$\psi_2(-a/2)\psi_1'(-a/2) - \psi_1(-a/2)\psi_2'(-a/2) \quad . \qquad \text{(II.10)}$$

We may use this result to examine two pairs of solutions. For the first pair, consider the forward and backward scattering solutions (equations II.6 and II.7). Substitution into (II.10) gives

$$(1+R')(ik_o)(T) - (T)(ik_o)(-1+R') =$$

$$(T')(ik_N)(1-R) - (1+R)(ik_N)(-T') \quad , \qquad \text{(II.11)}$$

which reduces to

$$k_oT = k_NT' \qquad \text{or} \qquad T' = (k_o/k_N)T \quad . \qquad \text{(II.12)}$$

Equation (II.12) holds whether or not $k(z)$ has an imaginary component.

The second pair of solutions is a valid pair only when $k^2(z)$ is real and k^2 is positive in the homogeneous region. If this restriction holds, then the complex conjugate of a solution will also be a solution. The restriction $k^2 > 0$ in the homogeneous region is simply a convenience; we could as easily assume the opposite in either the right or left homogeneous regions and proceed from that assumption, but it is more usual to have $k^2(z)$ positive in the asymptotic region. If we substitute the solution (II.6) and its complex conjugate in equation (II.10), the result is

$$(T^*)(ik_o)(T) - (T)(ik_o)(-T^*) =$$

$$(1+R^*)(ik_N)(1-R) - (1+R)(ik_N)(-1+R^*) \quad . \qquad \text{(II.13)}$$

Reducing, we obtain

$$R^*R + (k_o/k_N)\ T^*\ T = 1 \quad . \qquad \text{(II.14)}$$

Had we used instead the "reverse" solution of (II.7), we would have obtained

$$(R')^*R' + (k_N/k_o)\ (T')^*T' = 1 \quad . \qquad \text{(II.15)}$$

The results given by equations (II.12), (II.14), and (II.15) are physically reasonable. The energy flux in a transverse electromagnetic wave, by Poynting's theorem, is proportional to the electric field strength times the magnetic field strength. At the same time, the magnetic field of our "reduced wave" is proportional to the wave number times the electric field (for fixed ω and μ). Therefore, the flux is proportional to the reduced wave amplitude squared times the wavenumber. Equation (II.12) implies that the same percentage of flux will get through the slab in either direction, while equations (II.14) and (II.15) imply a conservation of flux. The combination of all three equations implies

$$R^*R = (R')^*R' \quad , \qquad \text{(II.16)}$$

which implies that the same percentage of flux will also be reflected in either direction.

II.3 The Invariant Imbedding Solution

Now that we have some simple relationships (equations II.12, II.14, II.15, and II.16), we may proceed to re-derive the basic invariant imbedding solution for R, T, R', and T'. As a check of the invariant imbedding procedure, we expect to be able to prove these simple relationships from the invariant imbedding viewpoint. Although the general derivation of the invariant imbedding scheme has previously been discussed in many sources (references 2-6), we will find the derivation a useful prelude to later derivations for periodic, "almost periodic," and "almost homogeneous" media.

Consider the problem of an electromagnetic wave incident on a region where there are N abrupt discontinuities (interfaces) in $n(z)$, but $n(z)$ is constant between interfaces (Figure 2). The simplest non-trivial problem to solve is the problem of one interface. In this case, $a = 0$ and the reflection and transmission coefficients are real numbers given (respectively) by the standard Fresnel formulas:

$$R_1 = (n_1 - n_o)/(n_1 + n_o) \tag{II.17}$$

$$T_1 = 2n_1/(n_1 + n_o) \quad . \tag{II.18}$$

The derivation of these formulas is elementary, but beyond the scope of this report.[8]

Assume now that we know the solutions, R_I and T_I, for a situation with I interfaces. The index of refraction is given by

$$n(z) = \begin{cases} n_o & ; \quad z > z_1 = +a/2 \\ n_i & ; \quad z_{i+1} < z < z_i \ , \quad 1 \le i < I \\ n_I & ; \quad z < z_I = -a/2 \ , \end{cases} \tag{II.19}$$

where z_i is the position of the i^{th} interface. Suppose that we wish to add another interface, thereby increasing the slab length, a , and

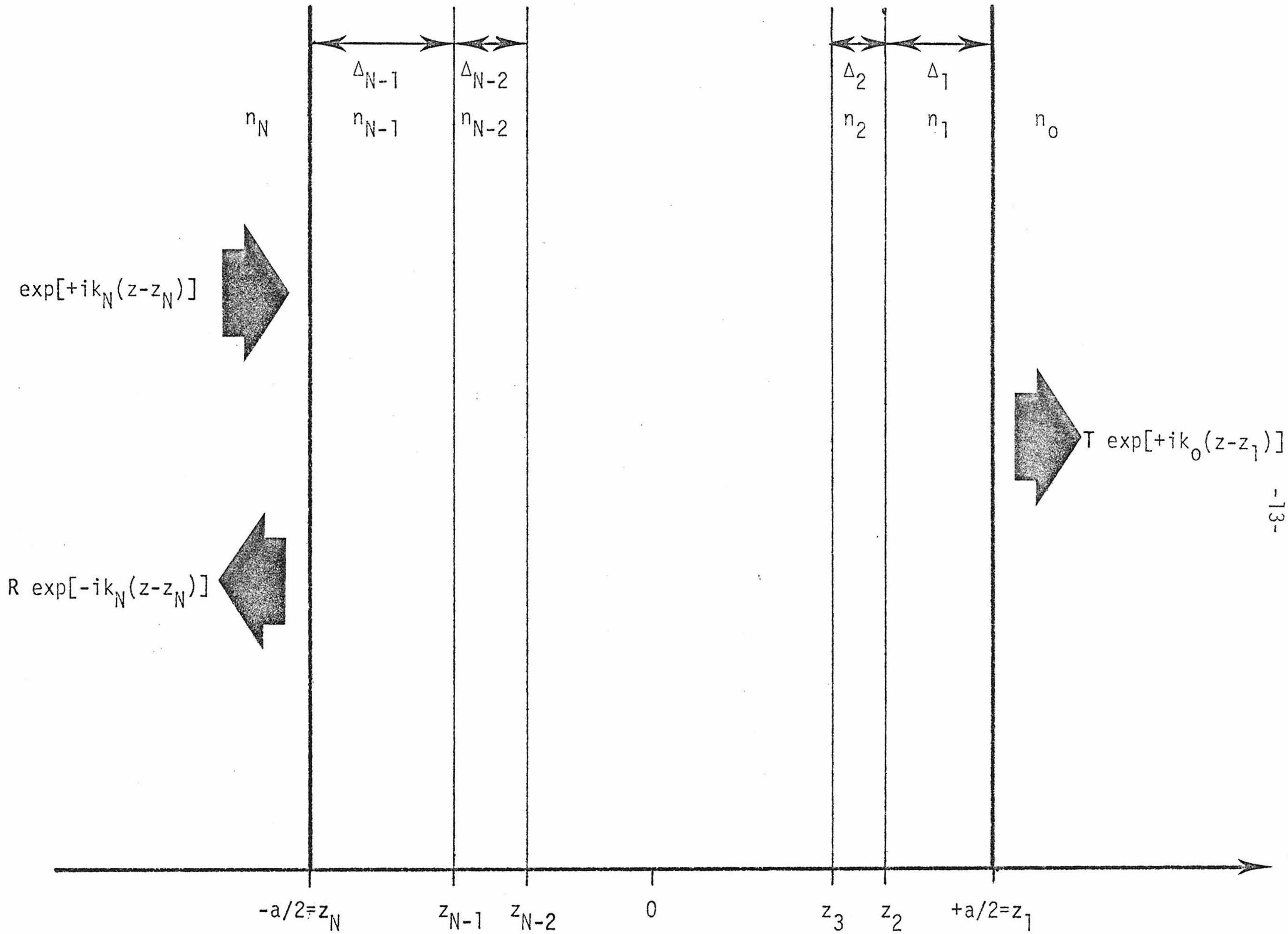

Δ_{N-1}
Δ_{N-2}
Δ_2
Δ_1
n_N
n_{N-1}
n_{N-2}
n_2
n_1
n_o
exp[+ik_N(z-z_N)]
R exp[-ik_N(z-z_N)]
T exp[+ik_o(z-z_1)]
-a/2=z_N
z_{N-1}
z_{N-2}
0
z_3
z_2
+a/2=z_1

Figure 2

changing the index of refraction in the left hand asymptotic region ($z < -a/2$). The new index of refraction function is

$$n(z) = \begin{cases} n_o & ; \quad z > z_1 = +a/2 \\ n_i & ; \quad z_{i+1} < z < z_i \,, \quad 1 \leq i < I \\ n_I & ; \quad z_{I+1} < z < z_I \\ n_{I+1} & ; \quad z < z_{I+1} = -a/2 \,, \end{cases} \tag{II.20}$$

where the new interface is added at $z = z_{I+1}$ and we have shifted the z axis to keep the slab centered about $z = 0$ (merely a matter of convenience). The new reflection and transmission coefficients will be calculated using the method of multiple reflections between interfaces I and I+1 (see Figure 3). The local reflection and transmission coefficients at the (I+1)st interface are given by the real numbers

$$r = (n_{I+1} - n_I)/(n_{I+1} + n_I) \tag{II.21}$$

$$t = 2n_{I+1}/(n_{I+1} + n_I) \quad . \tag{II.22}$$

The local reverse coefficients are given by

$$r' = (n_I - n_{I+1})/(n_I + n_{I+1}) \tag{II.23}$$

$$t' = 2n_I/(n_I + n_{I+1}) \quad . \tag{II.24}$$

These quantities are related by the Kirchhoff equations:

$$r' = -r \tag{II.25}$$

$$t = 1 + r \tag{II.26}$$

$$t' = 1 - r \quad . \tag{II.27}$$

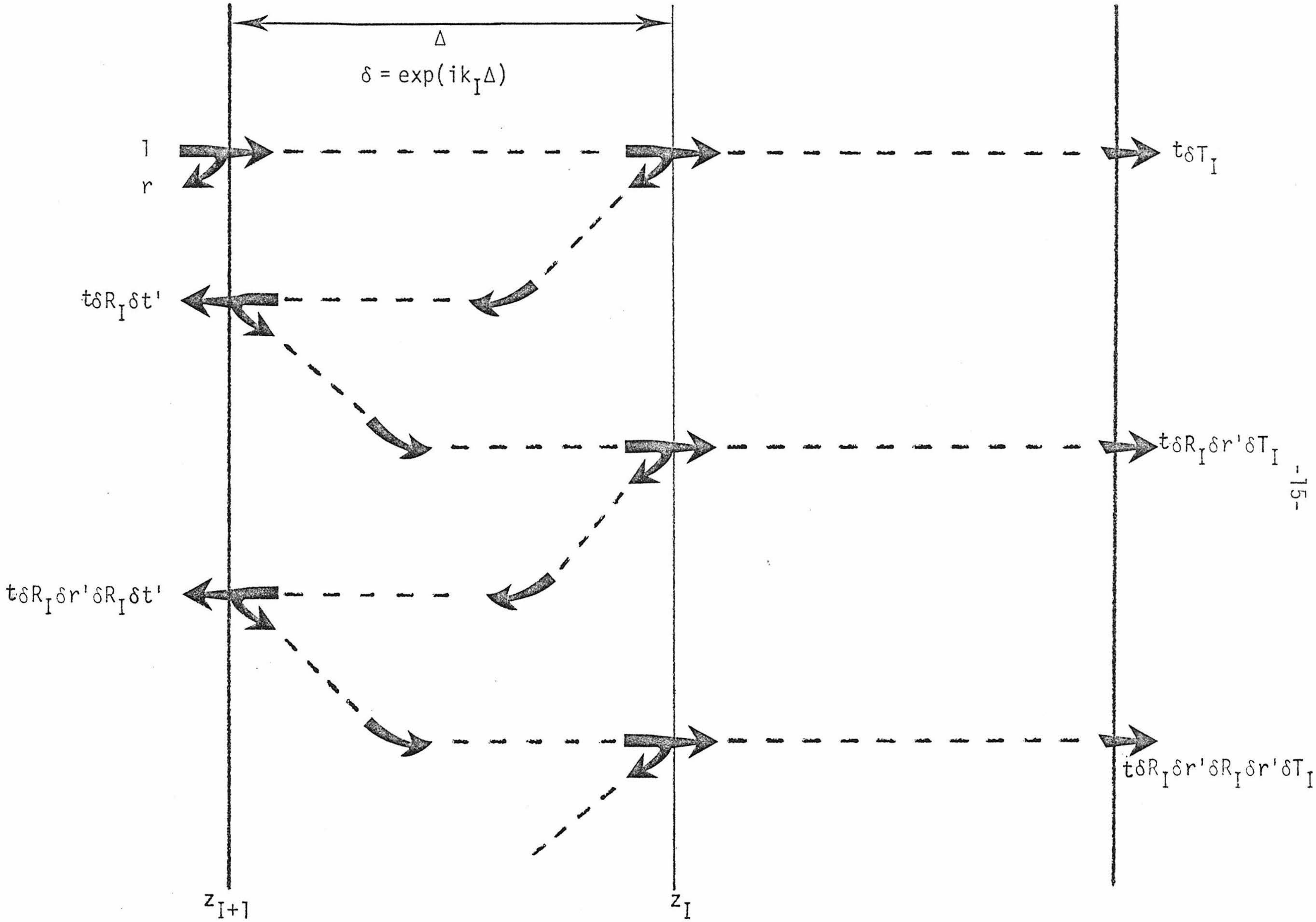


Figure 3

(N.B.: Vertical displacement is for illustration only).

The distance between interfaces, Δ , is given by

$$\Delta = z_I - z_{I+1} \quad . \tag{II.28}$$

A wave propagating between the interfaces will gain the phase factor

$$\delta = \exp(ik_I\Delta) \quad . \tag{II.29}$$

We will now consider the multiple reflections which contribute to the new reflection coefficient, R_{I+1}, and the new transmission coefficient, T_{I+1}. The unit wave is incident from the left upon the $(I+1)^{st}$ interface. The first term of the infinite series for R_{I+1} is simply the local reflection coefficient, r . The remainder of the wave, t , continues to the I^{th} interface gaining phase factor δ . At the I^{th} interface, R_I is reflected and T_I gets through to the right asymptotic region. The reflected part propagates back to the $(I+1)^{st}$ interface, again gaining phase factor δ . At the $(I+1)^{st}$ interface, t' gets through to give the second term of the series for R_{I+1} $(t\delta\, R_I\delta t')$ and r' is reflected back to become the second of the (infinite) multiple reflections. This process is illustrated in Figure 3. The series for R_{I+1} and T_{I+1} are given by

$$\begin{aligned} R_{I+1} &= r + (1+r)\delta R_I\delta(1-r) + (1+r)\delta R_I\delta(-r)\delta R_I\delta(1-r) + \cdots \\ &= r + (1-r^2)\delta^2 R_I \sum_{i=0}^{\infty} \delta^{2i} R_I^i (-r)^i \quad , \end{aligned} \tag{II.30}$$

$$\begin{aligned} T_{I+1} &= (1-r)\delta T_I + (1-r)\delta R_I\delta(-r)\delta T_I + \cdots \\ &= (1-r)\delta T_I \sum_{i=0}^{\infty} \delta^{2i} R_I^i (-r)^i \quad . \end{aligned} \tag{II.31}$$

These expressions may be summed exactly with the use of the formula

$$\sum_{i=0}^{\infty} a^i = (1-a)^{-1} \quad . \tag{II.32}$$

giving

$$\begin{aligned} R_{I+1} &= (r + \delta^2 R_I)/(1 + r\delta^2 R_I) \\ T_{I+1} &= t\delta T_I/(1 + r\delta^2 R_I) \quad . \end{aligned} \tag{II.33}$$

The recursion relationships (II.33) solve exactly any problem with discrete changes in $n(z)$. For the case of a continuous change in $n(z)$, we will derive differential equations for $R(z)$ and $T(z)$. The initial value conditions are $R(a/2) = 0$ and $T(a/2) = 1$. The qualtities $R(-a/2)$ and $T(-a/2)$ are the numbers which give the final answers for the reflection and transmission coefficients. Since the incremental interfaces are added in the negative z direction, we must be careful of our signs. If we let the Δ's get very small (and N very large) when $n(z)$ is a differentiable function, we may approximate the quantities in the discrete derivation:

$$r(z) = (n(z-dz) - n(z))/(n(z-dz) + n(z)) \tag{II.34}$$

$$n(z-dz) = n(z) - n'(z)dz \tag{II.35}$$

$$R(z-dz) = R(z) - R'(z)dz \quad . \tag{II.36}$$

In this spirit, we will expand all quantities in orders of (dz) and keep only the constant and linear terms:

$$r(z) = -n'(z)dz/2n(z) \tag{II.37}$$

$$\delta^2 = 1 + 2ik(z)dz \tag{II.38}$$

$$(1 + r\delta^2 R)^{-1} = 1 + \frac{n'(z)}{2n(z)} R(z)\, dz \quad . \tag{II.39}$$

If we make a simple substitution of these quantities in the standard recursion formula (II.33), and keep only terms of order dz, we obtain

$$R(z) - R'(z)dz = \frac{-k'(z)}{2k(z)} dz + R(z) + 2ik(z)\, R(z)\, dz$$

$$+ \frac{k'(z)}{2k(z)} R^2(z)\, dz \quad , \tag{II.40}$$

or, reducing,

$$R'(z) = \frac{1}{2}[1 - R^2(z)] \frac{k'(z)}{k(z)} - 2ik(z)\, R(z) \; . \tag{II.41}$$

Similarly, we get for $T(z)$

$$T(z) - T'(z)\, dz = T(z) + ik(z)\, T(z)\, dz$$

$$\frac{-k'(z)}{2k(z)} T(z)\, dz + \frac{k'(z)}{2k(z)} R(z)\, T(z)\, dz \tag{II.42}$$

$$T'(z) = \{\frac{1}{2}[1 - R(z)] \frac{k'(z)}{k(z)} - ik(z)\}\, T(z) \quad . \tag{II.43}$$

Equation (II.43) is very convenient to integrate once $R(z)$ is known.

$$\frac{d\, \ell n\, T}{dz} = \frac{1}{2}[1 - R(z)] \frac{d(\ell n\, k)}{dz} - ik(z) \tag{II.44}$$

$$\ell n(T(a/2)) - \ell n(T(-a/2)) =$$

$$- i \int_{-a/2}^{a/2} k(z)\, dz + \frac{1}{2} \int_{-a/2}^{a/2} [1 - R(z)] \frac{d(\ell n\, k)}{dz} dz \tag{II.45}$$

Our boundary condition on T is $T(a/2) = 1$. If we define

$$k_{avg} = 1/a \int_{-a/2}^{a/2} k(z)\, dz \quad , \tag{II.46}$$

then

$$T = T(-a/2) = \exp\left\{ ik_{avg}a - \frac{1}{2}\int_{-a/2}^{a/2} [1 - R(z)] \frac{d[\ell n\ k(z)]}{dz}\, dz \right\}. \quad (II.47)$$

II.4 Some General Properties of the Transmission Coefficient

Equation (II.47) reveals some important general properties of T. The first is that the major phase contribution when R is small comes from the $\exp(ik_{avg}a)$ term. When we actually solve a problem, we should expect the phase of T to be near $(k_{avg}a)$. The second is an interesting expression for the minimum magnitude that T could possibly have for a given k(z) (or n(z)). The magnitude of T is simply the exponential of the real part of the integral in equation (II.47) (when k is real):

$$|T| = \exp\left\{ -\frac{1}{2}\int_{-a/2}^{a/2} \mathrm{Re}[1 - R(z)] \frac{d[\ell n\ k(z)]}{dz}\, dz \right\} . \quad (II.48)$$

Since the magnitude of the reflection is constrained to be less than or equal to one, the real part of (1 - R) must lie between two and zero. Therefore, when k(z) is an increasing function, the integrand in (II.48) is positive, and when k(z) is a decreasing function, the integrand is negative. A conservative estimate of the minimum possible value of the magnitude of T results when we estimate the maximum possible value of the integral. Without solving the problem in advance, we do not know what the function R(z) looks like. However, if we let R(z) = -1 where k(z) is increasing and R(z) = +1 where k(z) is decreasing, then the integral obtains the maximum value it could possibly have for any $R(z)$:

$$|T| \geq \exp\{-\tfrac{1}{2} \cdot 2 \int_{k'(z)>0} d[\ln k(z)]\} \quad , \qquad (II.49)$$

or, integrating (over the regions where $k'(z) > 0$),

$$|T| \geq (k_{min_1}/k_{max_1}) \cdot (k_{min_2}/k_{max_2}) \cdots (k_{min_n}/k_{max_n}) \qquad (II.50)$$

where the k_{min} and k_{max} are the values of $k(z)$ at the beginning and end (respectively) of portions of $k(z)$ where $k(z)$ is an increasing function. In practice, this estimate is usually over-conservative, but it is completely reliable. For instance, suppose that $n(z)$ is given by

$$n(z) = \begin{cases} 1 & ; \; z < -a/2 \\ 1 + \chi(1 + \cos 2\pi z/a) & ; \; -a/2 \leq z \leq a/2 \\ 1 & ; \; z > a/2 \end{cases} \qquad (II.51)$$

It is relatively easy to program a computer or programmable calculator to solve the Riccati equations (II.41 and II.43) for various values of $k_0 = (\omega/c)$. We will examine some numerical solutions in detail later (Section III.9), but for the purposes of this section, we are interested in the minimum magnitude of T which we see when we solve numerically for as many k_0 as necessary. The following table lists results for three cases, $\chi = 1$, $\chi = .05$, and $\chi = .01$:

χ	$\|T\|$ at minimum	(k_{min}/k_{max})
1	0.65	0.33
.05	0.997	0.909
.01	0.99987	0.98039

If the variation in $k(z)$ is small, then

$$|T| \geq 1 - \xi \quad , \tag{II.52}$$

where ξ is a small positive number. When this is the case, then we also have a very useful estimate of the maximum magnitude of R^2 over the entire interval, not just at the end of the solution. Suppose we wish to get estimates of the magnitudes of $R(z)$ and $T(z)$ at some particular point, z, in the interval of integration of the Riccati equations. If we recognize that by integrating the equations for R and T from $+a/2$ to z, we have solved the problem of a distribution of $k(y)$ identical to the original from $+a/2$ to z and constant after that, then we have immediately from (II.14),

$$|R^2(z)| + (k_0/k(z))\ |T^2(z)| = 1 \quad . \tag{II.53}$$

Since the estimates for the "new problem" will be

$$|T(z)| \geq 1 - 0(\xi) \tag{II.54}$$

$$(k_0/k(z)) = 1 - 0(\xi) \tag{II.55}$$

due to the fact that we have the same $k(y)$ from $+a/2$ to z and a constant $k(y)$ for y less than z, we easily obtain

$$|R^2(z)| \leq 0(\xi) \quad . \tag{II.56}$$

This result is the basis of the useful approximation described in Chapter IV, in which we wish to be able to guarantee before we start that $R^2(z)$ will be "small" throughout the interval.

II.5 An Alternative Invariant Imbedding Scheme

The recursion relationships and differential equations (II.32, II.33, II.41, II.43) provide a convenient method for solving the electromagnetic scattering problem in one dimension. If we were to program a computer to solve the problem, the most convenient way would be to use the recursion relations (II.32 and II.33) directly, letting the size of the intervals become as small as practicable. We could, therefore, consider the task of solving these problems reduced to the almost trivial matter (in these days of computer abundance) of generating numerical solutions for interesting cases. However, we will obtain some very useful confirmations of the soundness of the invariant imbedding scheme (plus a new result which makes Chapter III possible) by considering an alternative invariant imbedding scheme in which we add slices from the right of the slab (rather than from the left as before). For convenience, we will call the "adding from the left method," method "A" and the "adding from the right method," method "B."

In method "B" we start at the N^{th} interface and work back to the 1^{st} interface. We may use the simple Fresnel equations to generate R_N, T_N, R'_N, T'_N :

$$R_N = (n_N - n_{N-1})/(n_N + n_{N-1}) \tag{II.57}$$

$$T_N = 1 + R_N \tag{II.58}$$

$$R'_N = -R_N \tag{II.59}$$

$$T'_N = 1 - R_N \quad . \tag{II.60}$$

Assume now that we have the solution for interfaces N, N-1, $\cdots$, I+1,I.

We wish to calculate R'_{I-1}, T'_{I-1}, R_{I-1}, and T_{I-1}. R'_{I-1} and T'_{I-1} may be found using the (suitably modified) recursion relations of method A (equations II.32 and II.33). This time, the multiple reflections are occurring at the right-hand side of the slab (see Figure 4)

As with method A, we will define local reflection and transmission coefficients and phase factor

$$r = (n_I - N_{I-1})/(n_I + n_{I-1}) \tag{II.61}$$

$$t = 1 + r \tag{II.62}$$

$$\delta = \exp[ik_{I-1}(z_{I-1} - z_I)] \tag{II.63}$$

Without further ado, the infinite series for R_{I-1} and T_{I-1} are

$$R_{I-1} = R_I + T_I\,\delta r\delta\,T'_I + T_I\,\delta r\delta\,R'_I\,\delta r\delta\,T'_I + \cdots \tag{II.64}$$

$$T_{I-1} = T_I\,\delta t + T_I\,\delta r\delta\,R'_I\,\delta t + \cdots \tag{II.65}$$

We may again sum these series with the help of equation (II.31) to give

$$R_{I-1} = \frac{R_I + r\,\delta^2(T_I T'_I - R_I R'_I)}{1 - r\,\delta^2 R'_I} \tag{II.66}$$

$$T_{I-1} = t\,\delta\,T_I/(1 - r\,\delta^2 R'_I) \quad . \tag{II.67}$$

We will not have need of the differential equation for $R(z)$ in method B. The differential equation for $T(z)$ may be found in analogy with previous arguments:

$$T_{I-1} = T(z+dz) = T(z) + (dT/dz)dz \tag{II.68}$$

$$r = -(k'(z)/2k(z))dz \tag{II.69}$$

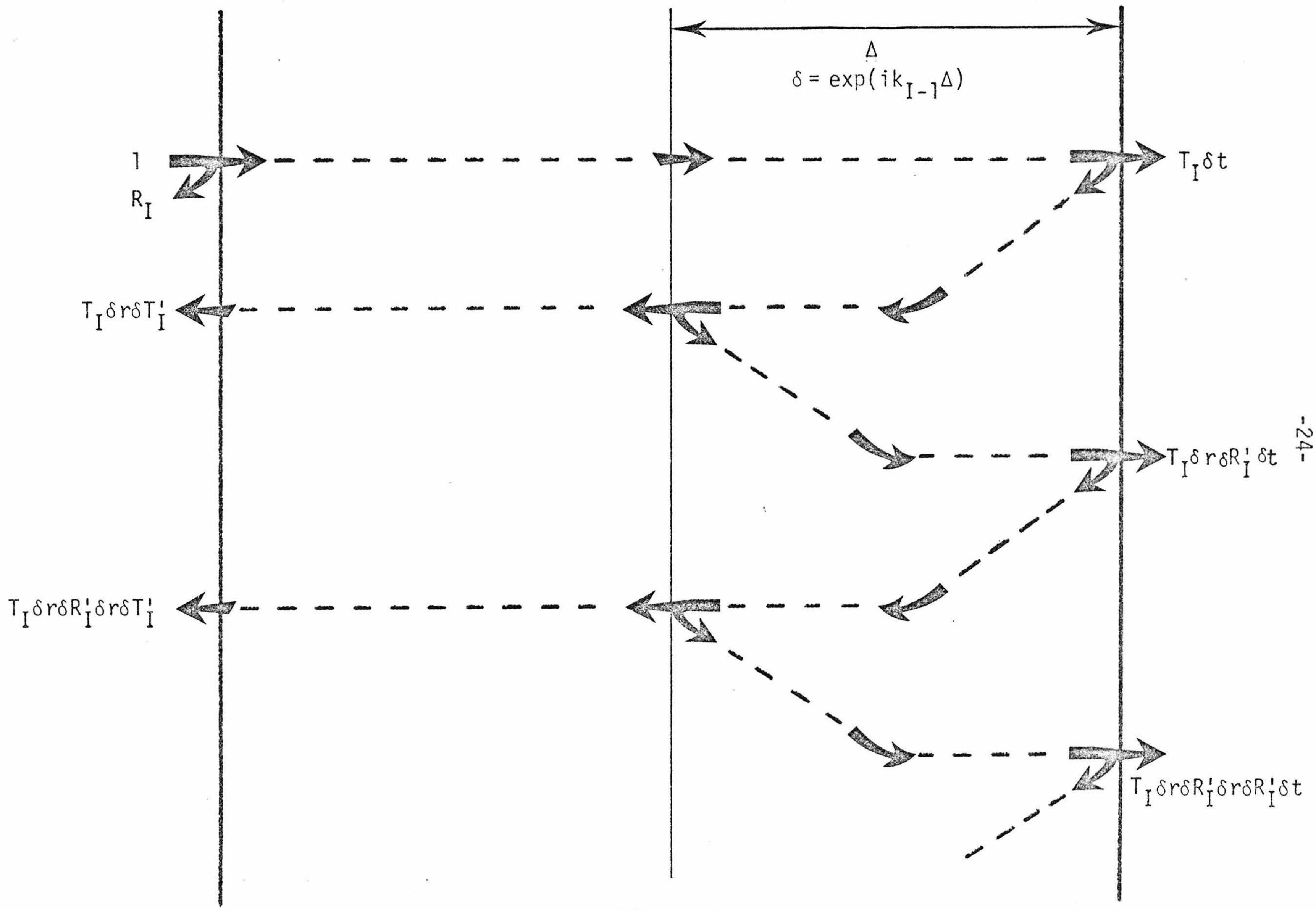


Figure 4

(N.B.: Vertical displacement is for illustration only.)

$$t = 1 + r \tag{II.70}$$

$$T(z) + (dT/dz)dz = T(z) + ik(z)\ T(z)dz - (k'(z)/2k(z))$$

$$\cdot\ R'(z)\ T(z)dz - (k'(z)/2k(z))\ T(z)dz \tag{II.71}$$

$$\frac{dT}{dz} = \{-\frac{1}{2}[1 + R'(z)]\ \frac{d[\ell n\ k(z)]}{dz} + ik(z)\}\ T(z) \quad . \tag{II.72}$$

The boundary condition is $T(-a/2) = 1$ (as opposed to method A). We easily integrate (II.72) to obtain

$$T = T(+a/2) = \exp\{ik_{avg}a - \frac{1}{2}\int_{-a/2}^{a/2} [1 + R'(z)]\ \frac{d[\ell n\ k(z)]}{dz}\ dz\} \tag{II.73}$$

Once $R'(z)$ is known, T may be calculated by integration.

Suppose we were to calculate T' by method B. The transformation $z' = -z$ gives the answer

$$T' = \exp\{ik_{avg}a - \frac{1}{2}\int_{-a/2}^{a/2} [1 + R''(z')]\ \frac{d[\ell n\ \tilde{k}(z')]}{dz'}\ dz'\} \quad . \tag{II.74}$$

$R''(z)$ is functionally the same as $R(-z)$ from method A, and $k(z) = \tilde{k}(-z)$ by the same argument. The integral expression for T' becomes

$$T' = \exp\{ik_{avg}a + \frac{1}{2}\int_{-a/2}^{a/2} [1 + R(z)]\ \frac{d[\ell n\ k(z)]}{dz}\ dz\} \quad . \tag{II.75}$$

If we compare this expression with (II.47), we easily calculate

$$T'/T = \exp[\ell n\ k(a/2) - \ell n\ k(-a/2)] = k_o/k_N \quad . \tag{II.76}$$

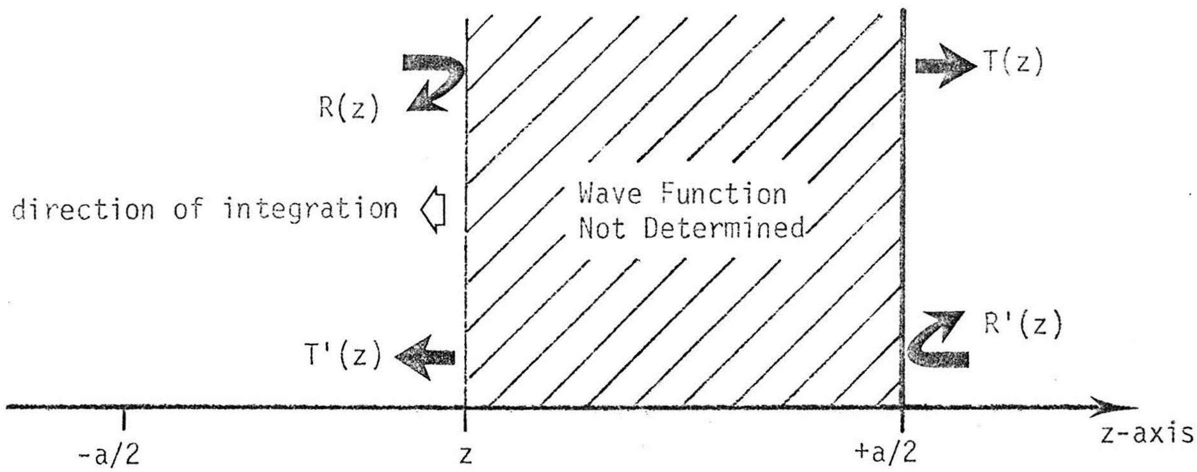
This result agrees with (II.12), which was derived directly from the reduced wave equation.

We should be careful to note the differences between the variables in the reduced wave equation and the R and T coefficients in methods A and B. This difference is illustrated in Figure 5. If we were to solve the wave equation for $\psi(z)$, we would have information about the electric (and magnetic) field at any point in the slab. We do not get the same information from $R(z)$ and $T(z)$ in the invariant imbedding method, since it is the problem itself we are changing as a function of z. If we need specific information about the electric field at any point in the slab, one method to use would be to divide the slab at z, solve each half-slab separately with invariant imbedding, then finally use the method of multiple reflections at z to give both the final reflection and transmission coefficients and the value of the electric field in the multiple reflection region. We will not find it necessary to do this. However, it will be nice to have the mathematical formulation if we should need it later.

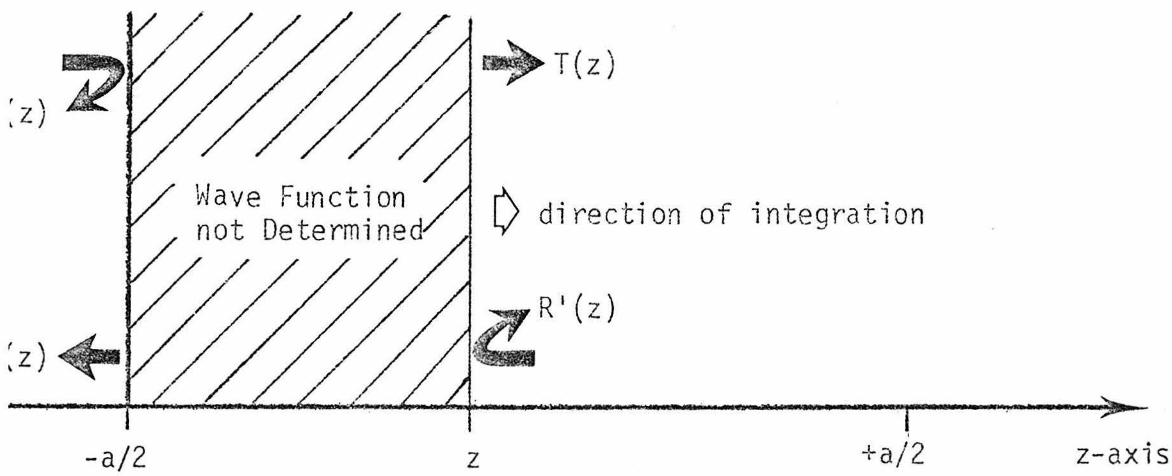
Assume that we have used invariant imbedding to get the reflection and transmission coefficients. In particular, if we use subscripts "A" and "B" to designate quantities found with methods A and B respectively, we need $T_B(z)$, $R_B'(z)$, and $R_A(z)$. $R_A(z)$ may be found by solving (II.41), and $R_B'(z)$ may be found by making the transformation $z' = -z$, solving (II.41), then making the transformation back again. Once $R_B'(z)$ is known, $T_B(z)$ is found from (II.72) or (II.73). Once these quantities are known, the series for the right- and left-going multiply-reflected waves are given (almost by inspection) by

$$\psi_+ = T_B(z) + T_B(z)R_A(z)R_B'(z) + T_B(z)R_A(z)R_B'(z)R_A(z)R_B'(z) + \cdots \qquad \text{(II.77)}$$

Invariant Imbedding - Method A



Invariant Imbedding - Method B



Solution of the Wave Equation -

Two Independent Solutions

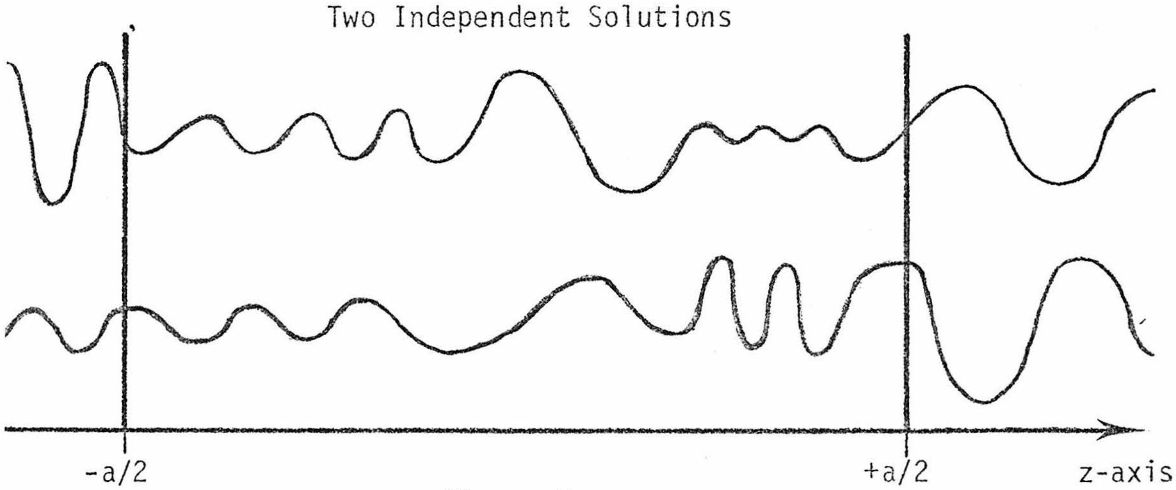


Figure 5

$$\psi_- = T_B(z)R_A(z) + T_B(z)R_A(z)R_B'(z)R_A(z)$$

$$+ T_B(z)R_A(z)R_B'(z)R_A(z)R_B'(z)R_A(z) + \cdots \quad . \tag{II.78}$$

Once again, we invoke (II.31) to sum these series to get

$$\psi_+ = T_B(z)/(1 - R_A(z)R_B'(z)) \tag{II.79}$$

$$\psi_- = T_B(z)R_A(z)/(1 - R_A(z)R_B'(z)) \tag{II.80}$$

The directions of the electric and magnetic field vectors may be found by elementary considerations (in the simple normal incidence case found here, the directions of the vectors remain constant). Their lengths (complex) may be found by

$$E = \psi_+ + \psi_- \tag{II.81}$$

$$H = n/\mu(\psi_+ - \psi_-) \tag{II.82}$$

II.6 Five Basic Relationships

We are now in a good position to derive the basic relationships among the parameters R, T, R', and T'. We will show inductively in Appendix I that the following five relationships hold. As a matter of notation, we let

$$R = r\exp(i\phi_R) \tag{II.83}$$

$$T = t\exp(i\phi_T) \tag{II.84}$$

$$R' = r'\exp(i\phi_{R'}) \tag{II.85}$$

$$T' = t'\exp(i\phi_{T'}) \quad . \tag{II.86}$$

The five relationships are

1. $r' = r$ (II.87)

2. $t' = (k_o/k_N)t$ (II.88)

3. a) $r^2 + (k_o/k_N)t^2 = 1$ (II.89)

   b) $r'^2 + (k_N/k_o)t'^2 = 1$ (II.90)

4. $\phi_{T'} = \phi_T$ (II.91)

5. $\phi_R + \phi_{R'} = \pm\pi + \phi_T + \phi_{T'}$ (II.92)

The wave equation has already given us relations 1 through 4, and we have also shown 2 and 4 with the arguments of Section II.5. Relation 5 may also be shown directly from the wave equation. Although we prove relations 1 through 5 by induction for the discrete case, there is no dependence on the number of slices or the size of the slices, so there is no problem passing to the limit which gives us the continuous case.

In the light of the five basic relationships, let us reconsider the method B recursion for R. Recall that the basic method B recursion for R (II.66) is somewhat inconvenient because it requires simultaneous calculation of T, T', and R', unlike the method A recursion (II.32). We may, however, simplify the method B recursion formula for R with the use of

$$T_I T_I' - R_I R_I' = \exp(2i\phi_T) \quad \text{(II.93)}$$

$$R_I' = -R_I \exp(2i\phi_T - 2i\phi_R) \quad , \quad \text{(II.94)}$$

which are trivial results of the five relationships. The method B solution for R becomes

$$R_{I-1} = \frac{R_I + r\,\delta^2 \exp(2i\phi_T)}{1 + rR_I\delta^2\exp(2i\phi_T - 2i\phi_R)} \quad , \qquad \text{(II.95)}$$

which eliminates the R' and T' dependences (the "r" in the equation is the "local reflection coefficient," not the magnitude of R_I). We still have an explicit dependence on T which we cannot eliminate, but we may at least eliminate the R' and T' dependences in the method B recursion for T :

$$T_{I-1} = \frac{t\,\delta\,T_I}{1 + rR_I\delta^2\exp(2i\phi_T - 2i\phi_R)} \qquad \text{(II.96)}$$

(the "t" in the equation is the local transmission coefficient and the "r" is the local reflection coefficient). Although method B is useful in proving the five relationships, it is not as convenient in practice as method A, so we will not pursue it further.

II.7 Relation to Quantum Mechanical Scattering in One Dimension

The non-relativistic one-dimensional wave equation in quantum mechanics is

$$\psi'' + 2m(E - V)/\hbar^2\psi = 0 \quad , \qquad \text{(II.97).}$$

where

$$k^2 = 2m(E - V)/\hbar^2 \qquad \text{(II.98)}$$

will put the equation in the same form as the reduced wave equation (II.4). This similarity leads us to try the invariant imbedding scheme when V is a function of z .

Suppose we have a situation where $V = 0$ for $z < 0$ and $V = V_0$ for $z > 0$. This is a simple "step" potential barrier. Assume for simplicity that $E > V_0$. Our solution for ψ is as follows. Let

$$k_0 = \sqrt{2mE/\hbar^2} \tag{II.99}$$

$$\alpha = \sqrt{2m(E - V_0)/\hbar^2} \tag{II.100}$$

$$\psi(z) = \begin{cases} \exp(ik_0z) + R\exp(-ik_0z) & ; \quad z < 0 \\ T\exp(i\alpha z) & ; \quad z > 0 \end{cases} \quad . \tag{II.101}$$

R and T are connected by the continuity of ψ and ψ' at the boundary $(z = 0)$:

$$1 + R = T \tag{II.102}$$

$$k_0(1 - R) = \alpha T \quad . \tag{II.103}$$

The solution of these equations gives us the familiar result:

$$R = (k_0 - \alpha)/(k_0 + \alpha) \tag{II.104}$$

$$T = 2k_0/(k_0 + \alpha) \quad . \tag{II.105}$$

Although $k(z)$ is calculated in two different ways for electromagnetic and quantum mechanical waves, the same formulas for the invariant imbedding scheme apply in either case once $k(z)$ is known. All of the previously derived theorems apply (with the restriction that $k(z)$ must be real). As a simple test of this idea, we will solve the one-dimensional "square well" problem.

We will let the potential barrier have height V_0 and width a, extending from $z = 0$ to $z = a$. The solution will be of the form

$$\psi = \begin{cases} \exp(ik_0 z) + R\exp(-ik_0 z) & ; \quad z < 0 \\ T\exp[ik_0(z-a)] & ; \quad z > a \quad . \end{cases} \tag{II.106}$$

There are only two interfaces to consider, so the problem is very easy. Applying (II.32) and (II.33),

$$R_1 = (\alpha - k_0)/(\alpha + k_0) \tag{II.107}$$

$$T_1 = 2\alpha/(\alpha + k_0) \tag{II.108}$$

$$R = R_2 = \frac{\frac{k_0-\alpha}{k_0+\alpha} + \frac{\alpha-k_0}{\alpha+k_0}\exp(2i\alpha a)}{1 - \left(\frac{k_0-\alpha}{k_0+\alpha}\right)^2 \exp(2i\alpha a)} \tag{II.109}$$

$$T = T_2 = \frac{\left(\frac{2\alpha}{\alpha+k_0}\right)\left(\frac{2k_0}{k_0+\alpha}\right)\exp(i\alpha a)}{1 - \left(\frac{k_0-\alpha}{k_0+\alpha}\right)^2 \exp(2i\alpha a)} \quad . \tag{II.110}$$

These expressions reduce to

$$R = \frac{(k_0^2 - \alpha^2)[1 - \exp(2i\alpha a)]}{(k_0+\alpha)^2 - (k_0-\alpha)^2 \exp(2i\alpha a)} \tag{II.111}$$

$$T = \frac{4k_0\alpha\exp(i\alpha a)}{(k_0+\alpha)^2 - (k_0-\alpha)^2 \exp(2i\alpha a)} \quad . \tag{II.112}$$

They agree with the results obtained from the usual method of solution[14], if the difference in phase convention for the outgoing wave is taken into account. If it should be the case that $E < V_0$, then α will be imaginary, but the solutions (II.111) and (II.112) will still be correct. Moreover, relations 1 through 4 (II.87 - II.91) will hold no matter how large V_0 .

II.8 Variation of All Electromagnetic Parameters

In the general case, the wavenumber (complex) of a transverse electromagnetic wave is given by

$$k = \omega\sqrt{\mu\varepsilon(1 + i\sigma/\omega\varepsilon)} \quad . \tag{II.113}$$

The positive square root is assumed, which means that exp(ikz) will "damp out" as z increases.

We now assume that we have a slab of thickness "a" (between $z = -a/2$ and $z = +a/2$) in which ε , μ , and σ are functions of z (and perhaps ω), but constant outside the slab. A plane wave with time dependence $\exp(-i\omega t)$ is incident on the left face ($z = -a/2$) at incident angle θ_N . Part is reflected at exit angle θ_N , part is absorbed in the slab, and part exits from the right face ($z = +a/2$) at exit angle θ_0 .

Let us see how the previous arguments are altered, without going through the complete derivation in detail again. For the moment, set $\sigma = 0$ ($k^2(z)$ real). There will be two polarizations to consider; the "$\perp$" case in which the electric field is perpendicular to the plane of incidence, and the "$\parallel$" case in which the electric field is parallel to the plane of incidence. As θ_N approaches zero, these two cases become degenerate.

The local reflection coefficients are given by[8]

$$r_\perp = \frac{k_I(\cos\theta_I)/\mu_I - k_{I-1}(\cos\theta_{I-1})/\mu_{I-1}}{k_I(\cos\theta_I)/\mu_I + k_{I-1}(\cos\theta_{I-1})/\mu_{I-1}} \qquad \text{(II.114)}$$

$$r_{||} = \frac{k_I/(\mu_I\cos\theta_I) - k_{I-1}/(\mu_{I-1}\cos\theta_{I-1})}{k_I/(\mu_I\cos\theta_I) + k_{I-1}/(\mu_{I-1}\cos\theta_{I-1})} \quad . \qquad \text{(II.115)}$$

The Kirchhoff relations (II.25 - II.27) still apply. There is a sign convention in common usage[8] which reverses the sign of $r_{||}$, so that at zero incidence angle,the local reflection coefficients are equal in magnitude but have opposite signs. We will not use this convention.

There remains the problem of determining the phase difference between adjacent interfaces. As a first guess, we might be tempted to say that the ray going between interfaces I and I-1 travels a distance $\Delta \sec\theta_{I-1}$ (true), so therefore the phase factor gained across the intermediate region is $\exp(ik_{I-1}\Delta\sec\theta_{I-1})$ (false). Figure 6 illustrates this problem. We must remember to refer the phases to the z-axis (the line defined by $x = 0$, $y = 0$) and that the waves are no longer traveling along the z-axis, but at some angle to it. The phase of the wave at the z-axis on the front or back of the slab is the phase of the wavefront which intersects the surface of the slab at the z-axis. Consider the reflected rays "1" and "2" of Figure 6. These two rays are originally in phase at the first contact with the slab at point P. Ray 1 is immediately reflected and travels a distance

$$d_1 = (\sin\theta_I)\,(2\Delta\tan\theta_{I-1}) \qquad \text{(II.116)}$$

in medium I to the wavefront which intersects point Q, the exit point of ray 2. We will assume that point P lies on the z-axis, so we will have to refer phases to that point. Ray 2 travels a distance

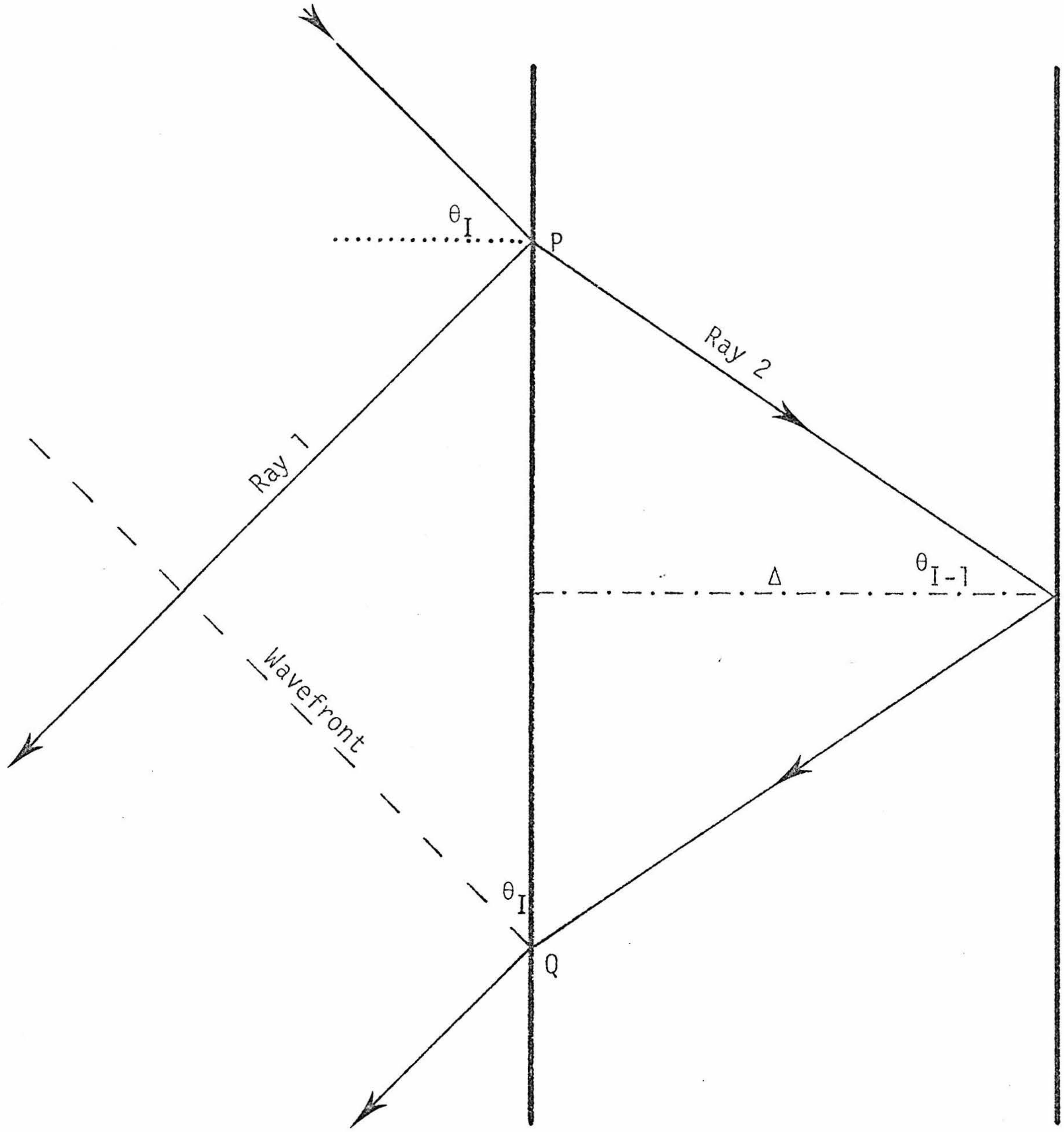


Figure 6

$$d_2 = 2\Delta \sec \theta_{I-1} \qquad \text{(II.117)}$$

in medium I-1 to the wavefront originally mentioned. The phase gained by ray 1, referred back to point P, is zero. The phase gained by ray 2, referred back to point P again, is

$$p = k_{I-1} d_2 - k_I d_1 \quad . \qquad \text{(II.118)}$$

By virtue of Snell's law, this reduces to

$$p = 2k_{I-1} \Delta(\sec \theta_{I-1} - \sin \theta_{I-1} \tan \theta_{I-1}) = 2k_{I-1} \Delta \cos\theta_{I-1} \quad . \qquad \text{(II.119)}$$

Alternatively, we could think of ray (2) as having started at such a point higher up on the slab so as to exit at point P . In any case, the phase factor gained is given by

$$\delta^2 = \exp(2ik_{I-1} \Delta \cos \theta_{I-1}) \quad . \qquad \text{(II.120)}$$

We may ask whether the phase factor gained by the transmitted wave is simply δ , as equation (II.120) would imply. Similar arguments will verify this conjecture.

The invariant imbedding analysis remains remarkably unchanged. Where before we had the expression "k_I" in differences (or derivatives), we replace it by "$k_I \cos \theta_I/\mu_I$" in the perpendicular polarization ("$\perp$") case or by "$k_I/\mu_I \cos \theta_I$" in the parallel polarization ("$\|$ ") case. Where we had the expression "$k_I \Delta_I$" (or "$k(z)dz$") in phase factors, we replace it by "$k_I \Delta_I \cos \theta_I$" in either case. This replacement leads to two interesting transformations for the two polarizations.

For $\perp$ let

$$a_T = \int_{-a/2}^{a/2} \mu(z)/\mu_0 \, dz \qquad \text{(II.121)}$$

$$z_T = -a_T/2 + \int_{-a/2}^{z} \mu(y)/\mu_0 \, dy \qquad \text{(II.122)}$$

$$k_T = k(z)\left[\cos\,\theta(z)\right]\mu_0/\mu(z) \quad (\theta(z) \text{ from Snell's Law}) \qquad \text{(II.123)}$$

The Riccati equations for R and T are given in the general case by

$$\frac{dR}{dz} = (1 - R^2(z))\,\frac{d[k(z)\,\cos\,\theta(z)/\mu(z)]}{2k(z)\,\cos\,\theta(z)/\mu(z)\,dz} - 2ik(z)\,\cos\,\theta(z)$$

$$\text{(II.124)}$$

$$\frac{dT}{dz} = (1 - R(z))\,\frac{d[k(z)\,\cos\,\theta(z)/\mu(z)]}{2k(z)\,\cos\,\theta(z)/\mu(z)\,dz} - ik(z)\,\cos\,\theta(z) \quad .$$

Since

$$\frac{dz}{dz_T} = \left(\frac{dz_T}{dz}\right)^{-1} = \frac{\mu_0}{\mu(z)} \quad , \qquad \text{(II.125)}$$

the transformed Riccati equations are

$$\frac{dR}{dz_T} = (1 - R^2(z_T))\,\frac{dk_T}{2k_T\,dz_T} - 2ik_T(z_T) \qquad \text{(II.126)}$$

$$\frac{dT}{dz_T} = (1 - R(z_T))\,\frac{dk_T}{2k_T\,dz_T} - ik_T(z_T) \quad . \qquad \text{(II.127)}$$

This somewhat surprising result implies that any problem with perpendicularly polarized waves may be transformed to a problem for which the incidence is normal and $\mu(z)$ is constant.

For $\|$ let

$$a_T = \int_{-a/2}^{a/2} \mu(z)\ \cos^2\theta(z)/\mu_0\ dz \qquad \text{(II.128)}$$

$$z_T = -a_T/2 + \int_{-a/2}^{z} \mu(y)\ \left[\cos^2\theta(y)\right]/\mu_0\ dy \qquad \text{(II.129)}$$

$$k_T = k(z)\ \mu_0/\left[\mu(z)\ \cos\ \theta(z)\right]\ . \qquad \text{(II.130)}$$

The Riccati equations are

$$\frac{dR}{dz} = (1 - R^2(z))\ \frac{d[k(z)/\cos\ \theta(z)\ \mu(z)]}{2k(z)/\cos\ \theta(z)\ \mu(z)\ dz} - 2ik(z)\cos\ \theta(z) \qquad \text{(II.131)}$$

$$\frac{dT}{dz} = (1 - R(z))\ \frac{d[k(z)/\cos\ \theta(z)\ \mu(z)]}{2k(z)/\cos\ \theta(z)\ \mu(z)\ dz} - ik(z)\ \cos\ \theta(z)\ . \qquad \text{(II.132)}$$

It is a trivial matter to verify that the transformation defined by equations (II.128) through (II.130) will reduce (II.131) and (II.132) to the standard form of (II.126) and (II.127). However, the parallel formulation encounters serious trouble when the factor, cos θ, goes to zero or becomes imaginary due to Snell's law ("total internal reflection"). The transformation to z_T becomes non-invertible and we are left with the full Riccati equations (II.131) and (II.132).

The case of "total internal reflection" ($\cos^2\theta \leq 0$) is interesting because it is an example of how relation 5 (II.92) breaks down. We may always make a piecewise transformation similar to (II.128) through (II.130). If we do that, then k_T^2 will be real even if $\cos^2\theta$ is negative. The wave equation guarantees that relations 1 through 4

(II.87 through II.91) will continue to hold (in a slightly altered form which we will see later) as long as the homogeneous regions have $\cos^2\theta > 0$. However, relationship 5 relies on the unit magnitude of the phase factor, δ , which is not the case when k_T is not real. Regardless of any restrictions on k_T , the altered version of (II.76) will always be true.

We may introduce $\sigma \neq 0$ in the previous formulations with little change in the results, except we must be prepared for any quantity to become complex as a result, thereby invalidating all relationships except the connection between transmission coefficients as derived from the integrated Riccati equations of methods A and B:

$$\perp : \quad T'/T = \mu_N k_o \cos\theta_o / \mu_o k_N \cos\theta_N \tag{II.133}$$

$$|| : \quad T'/T = \mu_N k_o \cos\theta_N / \mu_o k_N \cos\theta_o \tag{II.134}$$

II.9 Summary

The condition that $n_0 \sin\theta_o < n(z)$ applies to many cases of interest for which the plane wave originates in a region where $n(z)$ is approximately unity, propagates through a region where $n(z)$ is greater than or equal to unity, and exits into a region where $n(z)$ is again close to unity. Unless otherwise specified, we will assume that this is the case, because we may then always transform to the normal incidence, constant permeability case. If the conductivity is zero, we have the following relationships for perpendicular and parallel polarization:

$\perp$:

1. $r' = r$ (II.135)

2. $t' = (\mu_N k_o \cos\,\theta_o / \mu_o k_N \cos\,\theta_N)\; t$ (II.136)

3. $r^2 + (\mu_N k_o \cos\,\theta_o / \mu_o k_N \cos\,\theta_N)\; t^2 = 1$ (II.137)

4. $\phi_T = \phi_{T'}$ (II.138)

5. $\phi_R + \phi_{R'} = \pm\pi + \phi_T + \phi_{T'}$ (II.139)

$\|$:

1. $r' = r$ (II.140)

2. $t' = (\mu_N k_o \cos\,\theta_N / \mu_o k_N \cos\,\theta_o)\; t$ (II.141)

3. $r^2 + (\mu_N k_o \cos\,\theta_N / \mu_o k_N \cos\,\theta_o)\; t^2 = 1$ (II.142)

4. $\phi_T = \phi_{T'}$ (II.143)

5. $\phi_R + \phi_{R'} = \pm\pi + \phi_T + \phi_{T'}$ (II.144)

It is appropriate to ask the question whether there could be other relations connecting R, R', T, and T' when $k(z)$ is complex, besides (II.76) or its alternatives for oblique incidence. A simple thought experiment will show that there can be no other general relations when the slab is arbitrarily conductive. Suppose we have a slab which is reflective at the ends but highly absorptive in the middle. The attenuative middle effectively "decouples" R from R' so that we could change the parameters on one side without changing the reflection from the other side. In contrast, if there is no absorption, it is impossible to change the parameters anywhere without affecting all quantities. Of course, if the slab is only slightly absorptive at frequencies of interest, then all the relations will be approximately true; the amount by which they are not true is a measure of the lossiness. We will use this idea in our discussion of periodic media in the next chapter.

In the following chapters, we will generally restrict our attention to cases with zero conductivity and no regions of "total internal reflection." Since it is always possible, with these restrictions, to make the transformation to the constant permeability, normal incidence case, we will assume that we have done so.

As a final comment on the results of this chapter, we remark that the invariant imbedding analysis will remain correct even if we make all quantities complex, so that we should have no difficulty using invariant imbedding in more sophisticated models of interaction with matter, such as the complex permeability of a ferromagnetic material, as used for example by Sommerfeld in Optics.[9]

Chapter III

Periodic Media

III.1 The Periodic Recursion Formula

We will consider the class of problems for which the index of refraction is a periodic function within the slab (defined between $z = -aM/2$ and $z = +aM/2$), with M periods. Each "cell" of the slab has length a (see Figure 7). We might worry about whether the index of refraction function fits continuously at the junctions of the cells with each other and at the two junctions with the "background," or homogeneous index of refraction outside the slab. The junctions with the background will be taken into account when the problem of just one cell is solved, whether or not the cell fits continuously with the background. The junctions between the cells will be handled by the method of this section.

A necessary preliminary step will be to show that the index of refraction of a thin zone intermediate to two semi-infinite zones will not affect the reflection and transmission in the limit as the size of the intermediate zone goes to zero. Let the index of refraction be given by

$$n(z) = \begin{cases} n_a \; ; & z < -\Delta/2 \\ n_i \; ; & -\Delta/2 \leq z \leq +\Delta/2 \\ n_b \; ; & z > +\Delta/2 \end{cases} \qquad \text{(III.1)}$$

The standard invariant imbedding results are

$$R_2 = R = \frac{(n_b-n_i)/(n_b+n_i) + (n_i-n_a)/(n_i+n_a)\ \exp(2ik_i\Delta)}{1 + \frac{(n_b-n_i)}{(n_b+n_i)}\frac{(n_i-n_a)}{(n_i+n_a)}\exp(2ik_i\Delta)} \tag{III.2}$$

$$T_2 = T = \frac{\frac{2n_b}{(n_b+n_i)}\frac{2n_i}{(n_i+n_a)}\exp(ik_i\Delta)}{1 + \frac{(n_b-n_i)}{(n_b+n_i)}\frac{(n_i-n_a)}{(n_i+n_a)}\exp(2ik_i\Delta)} \tag{III.3}$$

As $\Delta \to 0$, these solutions become

$$R = \frac{(n_b-n_i)(n_i+n_a) + (n_i-n_a)(n_b+n_i)}{2n_bn_i + 2n_an_i} = \frac{n_b - n_a}{n_b + n_a} \tag{III.4}$$

$$T = \frac{4n_bn_i}{2n_bn_i + 2n_an_i} = \frac{2n_b}{n_b + n_a} \tag{III.5}$$

As we readily see, R and T are independent of n_i as the size of the intermediate zone goes to zero. This result enables us to link cells together without worrying what is the intermediate index of refraction in the zero distance between cells. We will find it convenient to assume that the index of refraction is the same between the cells as to the right of the first cell. When we solve the problem for L cells, we will assume that the index of refraction to the left is the same as the index to the right. If this is not so, one final application of the recursion relations (II.31) and (II.33) will give the correct final answer. In other words, we will assume that the background is identical on both sides, with the understanding that one simple final calculation will correct the result if this assumption is false. In case we wish

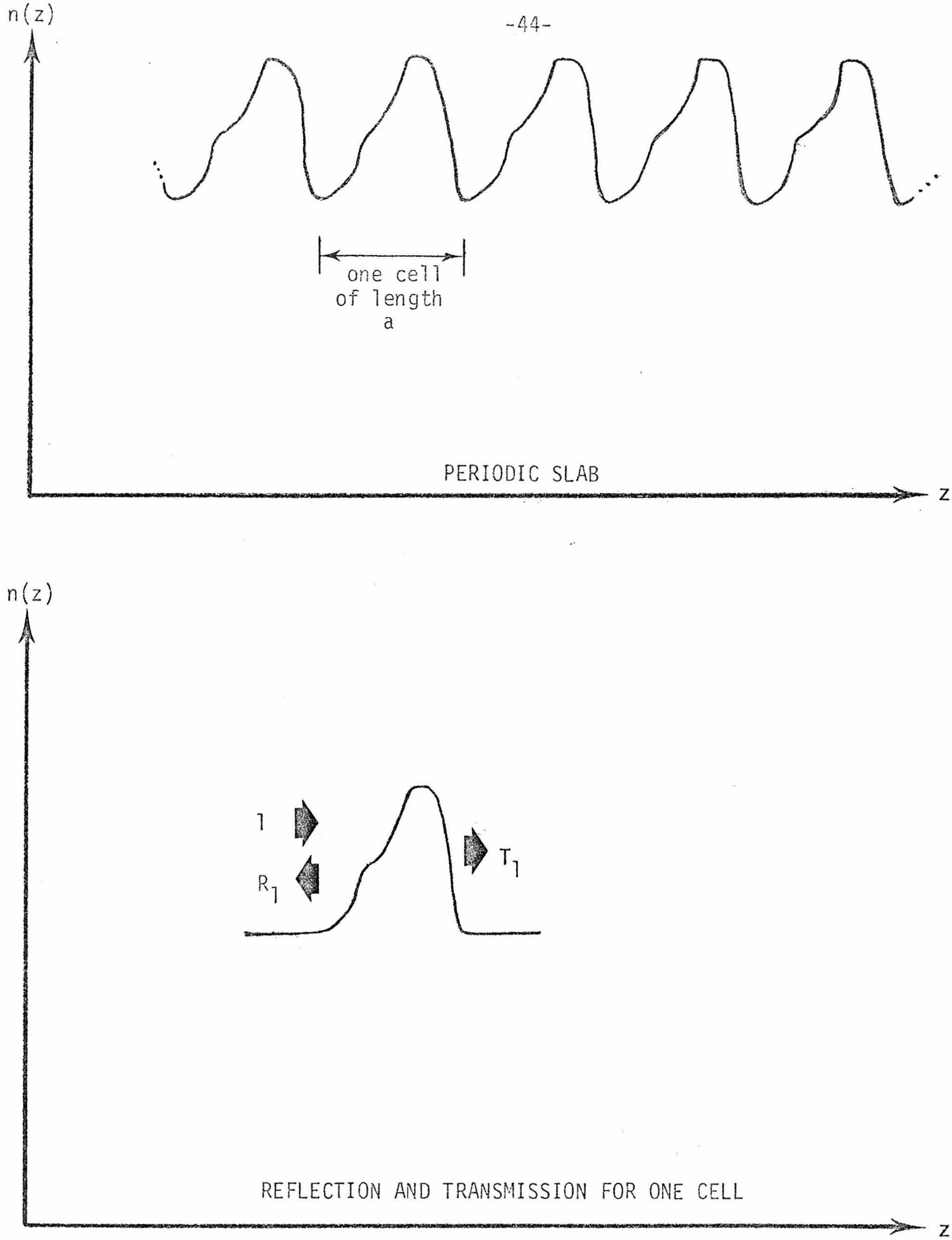

n(z)
one cell
of length
a
PERIODIC SLAB
z
n(z)
1
R_1
T_1
REFLECTION AND TRANSMISSION FOR ONE CELL
z

Figure 7

to have a non-zero distance between cells (constant to maintain periodicity), we should incorporate the extra distance into the defined cell.

We could, of course, simply solve the problem with our standard recursion formulas (II.32) and (II.33) and/or the Ricatti equations (II.41) and (II.43), but that would be inefficient and would not lend itself to a general solution. Suppose, instead, that we have used invariant imbedding (or any other method) to calculate R_1, T_1, R_1', and T_1' for one cell. The invariant imbedding method may be used to calculate R_M and T_M for M cells.

Suppose that we know the solutions, R_L and T_L, for L cells, and we wish to add one more cell (see Figure 8). The method of multiple reflections in the zero-distance region between the new cell and the L original cells is straightforward (and analogous to the previous examples of multiple reflection). The recursion formulas obtained are

$$R_{L+1} = \frac{R_1 + R_L(T_1' T_1 - R_1' R_1)}{1 - R_1' R_L} \tag{III.6}$$

$$T_{L+1} = \frac{T_1 T_L}{1 - R_1' R_L} \quad . \tag{III.7}$$

As a matter of notation, let

$$r = |R_1| \tag{III.8}$$

$$\phi_R = \arg(R_1) \tag{III.9}$$

$$t = |T_1| \tag{III.10}$$

$$\phi_T = \arg(T_1) \quad . \tag{III.11}$$

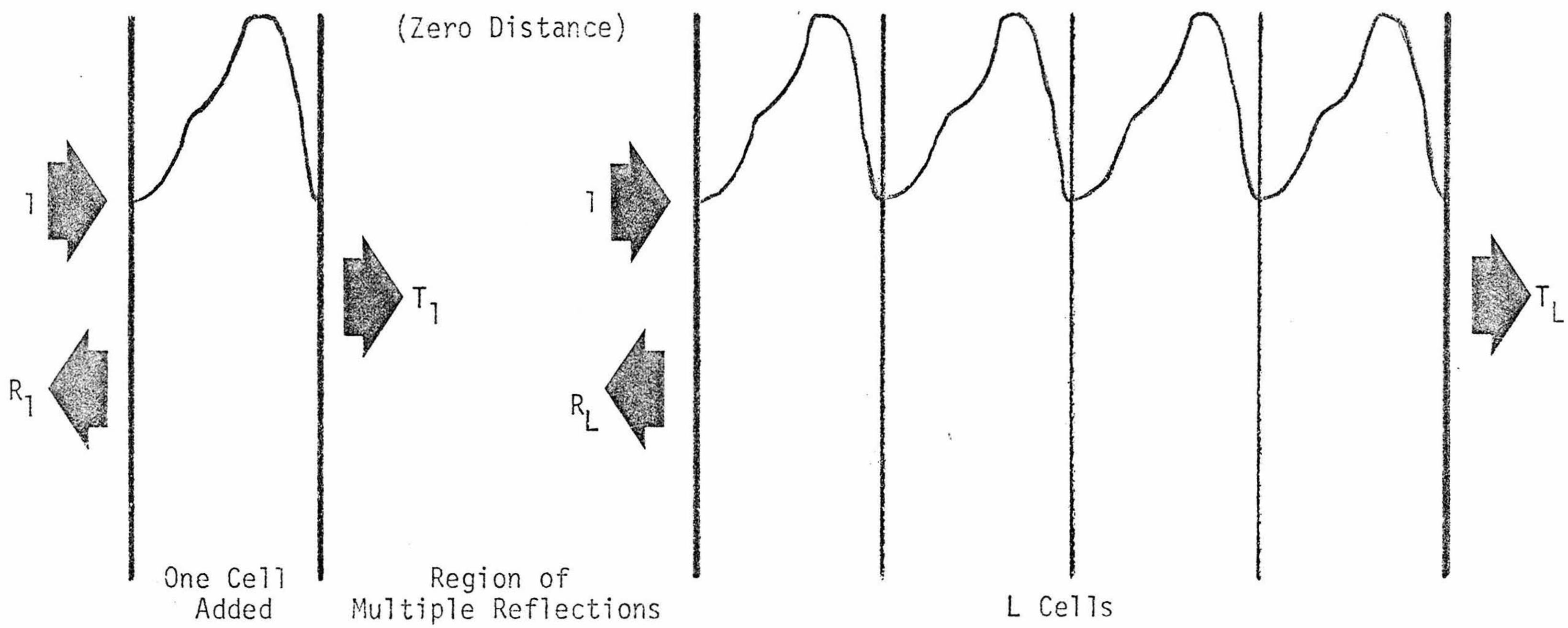

(Zero Distance)
I
R1
T1
I
RL
TL
One Cell Added
Region of Multiple Reflections
L Cells

Figure 8

Using the reciprocity relationships (II.87 - II.92), we may rewrite the periodic recursion formulas:

$$R_{L+1} = \exp(i\phi_R) \frac{r + R_L \exp(2i\phi_T - i\phi_R)}{1 + rR_L \exp(2i\phi_T - i\phi_R)} \tag{III.12}$$

$$T_{L+1} = \frac{t\ T_L \exp(i\phi_T)}{1 + rR_L \exp(2i\phi_T - i\phi_R)} \quad . \tag{III.13}$$

III.2 Fixed Points of the Recursion

In general, we should not expect the sequence $|R_L|$ to converge. If the sequence is to converge, it will converge to a fixed point of the transformation, R_F, where R_F obeys the equation

$$R_F = \exp(i\phi_R) \left[\frac{r + R_F \exp(2i\phi_T - i\phi_R)}{1 + rR_F \exp(2i\phi_T - i\phi_R)}\right] . \tag{III.14}$$

This is simply a quadratic equation whose solutions are

$$R_F = \frac{[\exp(2i\phi_T)-1] \pm \{[\exp(2i\phi_T)-1]^2 + 4r^2 \exp(2i\phi_T)\}^{1/2}}{2r \exp(2i\phi_T - i\phi_R)} . \tag{III.15}$$

We will leave it for later discussion to show that the sequence R_L converges to R_F only if $|R_F| = 1$. Let us see what happens if $|R_F| = 1$:

$$R_F = \exp(i\phi_{R_F}) \quad . \tag{III.16}$$

Substitution of (III.16) into (III.14) yields

$$\exp(i\phi_{R_F}) = \exp(i\phi_R)\ [\frac{\exp(2i\phi_T - i\phi_R + i\phi_{R_F}) + r}{1 + r\exp(2i\phi_T - i\phi_R + i\phi_{R_F})}] \quad . \qquad (III.17)$$

Let

$$\psi = \arg[1 + r\exp(-2i\phi_T + i\phi_R - i\phi_{R_F})] \quad . \qquad (III.18)$$

With a small amount of manipulation, assuming ϕ_{R_F} is real,

$$1 = \exp(2i\phi_T + 2i\psi) \quad . \qquad (III.19)$$

So far, ϕ_{R_F} has not been determined. It will be equal to whatever it must equal to make (III.19) true. Figure 9 illustrates the maximum range of angles which ψ may obtain given r. It is obvious from the diagram and elementary geometry that

$$|\psi| \leq \arcsin(r) \quad . \qquad (III.20)$$

If we convert ϕ_T to the region $\pm\pi$, then we have a very simple relationship which must hold if and only if $|R_F| = 1$: *

$$|\phi_T| \leq \arcsin(r) \quad . \qquad (III.21)$$

It is more convenient to rewrite (III.21) so that it is not necessary to convert ϕ_T back to the region $\pm\pi$:

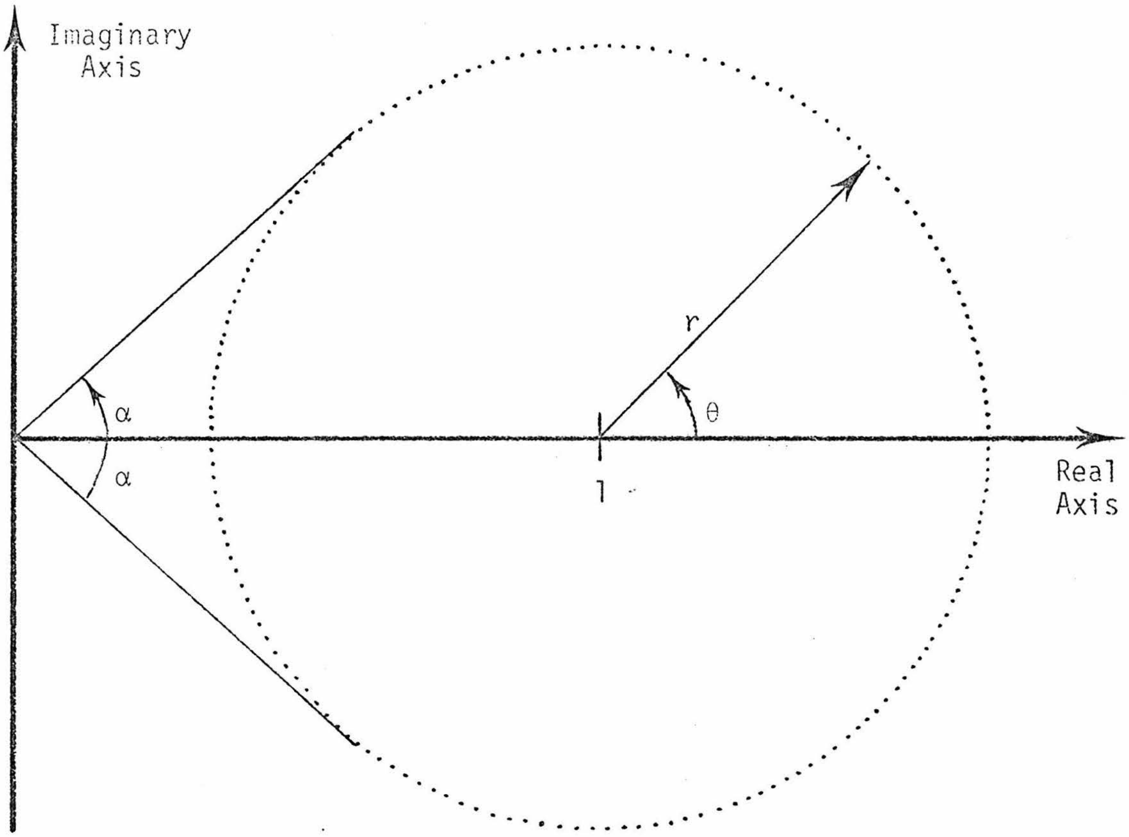
$$|\sin\phi_T| \leq r \quad . \qquad (III.22)$$

The alternative, which holds if and only if $|R_F| \neq 1$*, is

$$|\sin\phi_T| > r \quad . \qquad (III.23)$$

Since the condition that the sequence $\{|R_L|\}$ converges to unity means

*See also next section.



($\pm\alpha$) is the maximum range of the phase of ($1 + re^{i\theta}$), where θ is arbitrary. $\alpha = \arcsin(r)$.

Figure 9

that no flux will get through in the limit as we add many cells, we will call condition (III.22) the "STOP BAND" condition. The obvious corollary is to call condition (III.23) the "PASS BAND" condition. We have not given a rigorous proof yet, but conditions (III.22) and (III.23) are enough to make a PASS BAND/STOP BAND analysis of any (lossless) periodic medium. One need only find r and ϕ_T as a function of frequency and then use (III.22) and (III.23) to determine whether any particular frequency is in a PASS BAND or a STOP BAND. We will do this later for several media.

III.3 Convergence to a Fixed Point; Method One

Suppose that R_L is "very close" to R_F. We may write

$$R_L = (1 + \xi) R_F \quad , \qquad \text{(III.24)}$$

where $|\xi| << 1$. Using (III.12) to calculate R_{L+1} and expanding in terms of ξ, we obtain

$$R_{L+1} = [1 + \xi(\frac{-rR_F\exp(2i\phi_T - i\phi_R) + \exp(2i\phi_T)}{1 + rR_F\exp(2i\phi_T - i\phi_R)}) + O(\xi^2)]R_F \ . \qquad \text{(III.25)}$$

If we write

$$R_{L+1} = (1 + \xi') R_F \ , \qquad \text{(III.26)}$$

then

$$|\xi'| = |\xi| \left| \frac{1 - rR_F\exp(-i\phi_R)}{1 + rR_F\exp(2i\phi_T - i\phi_R)} \right| + O(\xi^2) \quad . \qquad \text{(III.27)}$$

The $O(\xi^2)$ term is very small when $|\xi|$ is small, but it is not zero. We define

$$P(r,\phi_T,R_F) = \frac{1 - rR_F\exp(-i\phi_R)}{1 + rR_F\exp(2i\phi_T - i\phi_R)} \quad . \tag{III.28}$$

If we examine P simultaneously with R_F , we obtain the following results:

A. $|R_F| = 1$

In this case the STOP BAND criterion (III.22) holds. There are two solutions for R_F . For one of them, $|P| > 1$. For the other, $|P| < 1$; this will be the one to which $\{R_L\}$ converges.

B. $|R_F| \neq 1$

Here the PASS BAND criterion (III.23) holds. For both solutions of R_F , $|P| = 1$. We cannot prove it with this simple analysis, but the $O(\xi^2)$ term will prevent convergence, and instead leads to orbits around R_F . The solution in which we are interested lies inside the unity circle ($|R_F| = 1$). The other solution is not interesting because it lies outside the unity circle and is therefore not possible for usual media.

III.4 Convergence to a Fixed Point; Method Two

We may think of equation (III.12) as a transformation which maps points in the complex plane to points in the complex plane. This transformation is parameterized by the numbers r , ϕ_R , and ϕ_T . Part of the transformation is a simple rotation which we may factor out by noting

$$[R_{L+1}\exp(-i\phi_R)] = \frac{r + [R_L\exp(-i\phi_R)]\ \exp(2i\phi_T)}{1 + r[R_L\exp(-i\phi_R)]\ \exp(2i\phi_T)} \quad . \tag{III.29}$$

We let

$$X_L = R_L \exp(-i\phi_R) \tag{III.30}$$

and write

$$\Phi_{\phi_T,r}(X_L) = X_{L+1} = \frac{r + X_L \exp(2i\phi_T)}{1 + rX_L \exp(2i\phi_T)} \quad . \tag{III.31}$$

We are only interested in those points which lie inside the unity circle and only those transformations for which $0 \leq r < 1$. The inverse transformation is

$$\Phi^{-1}_{\phi_T,r}(X_{L+1}) = X_L = \exp(-2i\phi_T)\left[\frac{r - X_{L+1}}{1 - rX_{L+1}}\right] \tag{III.32}$$

It is very easy to prove that both Φ and Φ^{-1} will leave points inside the unity circle. Let Z be any complex number with magnitude less than one, and let $\psi = \arg(Z)$. Then

$$r^2 + |Z|^2 < 1 + r^2|Z|^2 \tag{III.33}$$

$$r^2 + 2r|Z|\cos\psi + |Z|^2 < 1 + 2r|Z|\cos\psi + r^2|Z|^2 \tag{III.34}$$

$$\left|\frac{r + Z}{1 + rZ}\right| < 1 \quad . \tag{III.35}$$

We conclude that for any ϕ_T and $r < 1$, the transformation

$$\Phi_{\phi_T,r}(X) = \frac{r + X \exp(2i\phi_T)}{1 + rX \exp(2i\phi_T)} \tag{III.36}$$

is one-to-one and onto inside the unity circle. (This topic is also mentioned by Bellman et al. (3).) The fixed points of Φ reveal where the transformation may converge after many applications. If we call the fixed point of Φ, X_F, then X_F obeys

$$X_F = \frac{r + X_F \exp(2i\phi_T)}{1 + rX_F \exp(2i\phi_T)} \quad . \tag{III.37}$$

This is a quadratic equation, whose solutions are

$$X_F \exp(i\phi_T) = i[\frac{\sin \phi_T}{r}] \pm [1 - \frac{\sin^2\phi_T}{r^2}]^{1/2} \quad . \tag{III.38}$$

This is a slightly inconvenient result because it depends directly on ϕ_T , so we will make one last change of variable. Let

$$Y = X \exp(i\phi_T) = R \exp(i\phi_T - i\phi_R) \quad , \tag{III.39}$$

$$\Theta_{\phi_T,r}(Y) = \exp(i\phi_T)\ \Phi_{\phi_r,r}(Y \exp(-i\phi_T)) \ . \tag{III.40}$$

Evidently, Θ is one-to-one and onto in the same sense as Φ . If we wish to calculate R_L given Y_L , it is easy:

$$R_L = Y_L \exp(i\phi_R - i\phi_T) \quad . \tag{III.41}$$

The fixed points of Θ are calculated using

$$\eta = \frac{\sin \phi_T}{r} \quad , \tag{III.42}$$

giving

$$Y_F = i\eta \pm [1 - \eta^2]^{1/2} \quad . \tag{III.43}$$

The fixed points lie either on the imaginary axis ($|\eta| > 1$: PASS BAND) or on the unity circle ($|\eta| < 1$: STOP BAND).

Let us consider now the transformation (which we tentatively assume is a member of the class of transformations $\{\Theta\}$) defined by

$$\Theta_{\phi_T',r'} = [\Theta_{\phi_T,r}]^2 \quad . \tag{III.45}$$

We may calculate Y_{L+2} two ways and compare the answers:

$$Y_{L+2} = \Theta_{\phi_T',r'}(Y_L) = \frac{1 + \frac{1}{r'} Y_L \exp(i\phi_T)}{\frac{1}{r'} \exp(-i\phi_T) + Y_L} \tag{III.46}$$

$$Y_{L+2} = \Theta_{\phi_T,r}[\Theta_{\phi_T,r}(Y_L)] = \frac{1 + \dfrac{r^2\exp(2i\phi_T) + \exp(4i\phi_T)}{r\exp(i\phi_T) + r\exp(3i\phi_T)} Y_L}{\dfrac{1 + r^2\exp(2i\phi_T)}{r\exp(i\phi_T) + r\exp(3i\phi_T)} + Y_L} \quad . \tag{III.47}$$

Examination of (III.46) and (III.47) reveals the following connections between r, ϕ_T, r', and ϕ_T' :

$$r' = r \left| \frac{1 + \exp(2i\phi_T)}{1 + r^2\exp(2i\phi_T)} \right| \tag{III.48}$$

$$\frac{\sin \phi_T'}{r'} = \frac{\sin \phi_T}{r} = \eta \quad . \tag{III.49}$$

We would like to use equations (III.48) and (III.49) to find r and ϕ_T given r' and ϕ_T' so that we may decompose any transformation into two "smaller" transformations. Of course, we may always invert (III.48) and (III.49) numerically to obtain the desired results. It is not important that we actually obtain a closed-form inversion of (III.48) and (III.49), only that we recognize that we may find the solutions if necessary. When r' and ϕ_T' are small, then the approximate inversion is very simple:

$$r = \frac{1}{2} r' \tag{III.50}$$

$$\phi_T = \frac{1}{2} \phi_T' \quad . \tag{III.51}$$

The point of this argument is that we may always decompose any $\Theta_{\phi_T,r}$ into 2^M smaller transformations; in the limit, we may consider $\Theta_{\phi_T,r}$ to be infinitely-many infinitesimal transformations which preserve the fixed points (or, equivalently, η) .

An infinitesimal transformation will have the parameters ξr and $\xi\phi_T$, where ξ is a small real number. If r itself is small, then ξ will be (1/N), where N is the number of "smaller" transformations which are equivalent to the transformation which is parameterized by r and ϕ_T . The infinitesimal transformation may be written

$$Y_{L+\xi} = \left[\frac{r + Y_L \exp(i\xi\phi_T)}{1 + \xi r \, Y_L \exp(i\xi\phi_T)}\right] \exp(i\xi\phi_T) \quad . \tag{III.52}$$

Since

$$\eta = \sin(\xi\phi_T)/\xi r \simeq \phi_T/r \quad , \tag{III.53}$$

we easily expand (III.52) in powers of ξ , keeping up to linear terms, to obtain

$$Y_{L+\xi} = Y_L + \xi r[(1 - Y_L^2) + 2i\eta Y_L] \quad . \tag{III.54}$$

The form of (III.54) very closely resembles the Riccati equation for R(z) (II.41), which is not surprising. It is interesting that explicit dependence on ξ and r occurs in the expression for $(Y_{L+\xi} - Y_L)$ only as the multiplier, (ξr), which implies that the path of evolution of $Y_{L+\xi}$ is independent of ξ and r (for fixed η), but the "speed" of evolution is linearly dependent on (ξr). If we associate a quasi-time coordinate with ξ, so that adding a cell is equivalent to advancing quasi-time by some increment, then we may define a velocity function

$$V(Y_L) = \lim_{\xi \to 0} (Y_{L+\xi} - Y_L)/\xi = r[(1 - Y_L^2) + 2i\eta Y_L] \quad . \tag{III.55}$$

The value of r may vary, but the shape of the velocity field depends only on η. When r is small, one "unit" of quasi-time corresponds to one cell added. When r is close to unity, the correspondence is not so easy, but may be found using (III.48); in any event, the relation between quasi-time and adding one cell is simply a multiplicative constant. Straightforward substitution of (III.43) into (III.55) yields

$$V(Y_F) = 0 \quad . \tag{III.56}$$

We now have the necessary background for a detailed analysis of the paths of Y_L (and therefore R_L) for the PASS BAND and STOP BAND cases.

A. $|\eta| < 1$ (PASS BAND)

For simplicity, we will assume that η is positive. The fixed point which is inside the unity circle is given by (III.43) as

$$Y_F = i(\eta - \sqrt{\eta^2 - 1}) \quad . \tag{III.57}$$

Let

$$Z = Y - Y_F \quad . \tag{III.58}$$

The velocity field is

$$V(Y) = r(2i\sqrt{\eta^2 - 1}\,Z - Z^2) \quad . \tag{III.59}$$

When Z has a small magnitude (Y is close to Y_F), the linear term of (III.59) dominates and the orbit path circulates about Y_F ($Z = 0$). The quadratic term causes the orbit to depart somewhat from a purely circular form. Since Θ is continuous and invertible, no two orbits may cross. We can see from (III.59) that an orbit cannot converge to the fixed point; neither can it get outside the unity circle. For these reasons, the orbit may neither shrink nor expand each time it completes one cycle around Y_F . The orbits simply repeat themselves indefinitely. Figure 10 illustrates two orbits which were calculated using

$$Y_{L+1} = \left[\frac{r + Y_L \exp(i\phi_T)}{1 + r\, Y_L \exp(i\phi_T)}\right] \exp(i\phi_T) \tag{III.60}$$

for $r = 0.1$ and $\eta = 1.4$. Two starting values, Y_0 , were used: 0.0 and $-(0.8)i$. The first starting value corresponds to the physical process of adding cells, starting with no cells (no reflection). The ends and tips of the arrows in Figure 10 are located at the points Y_L .

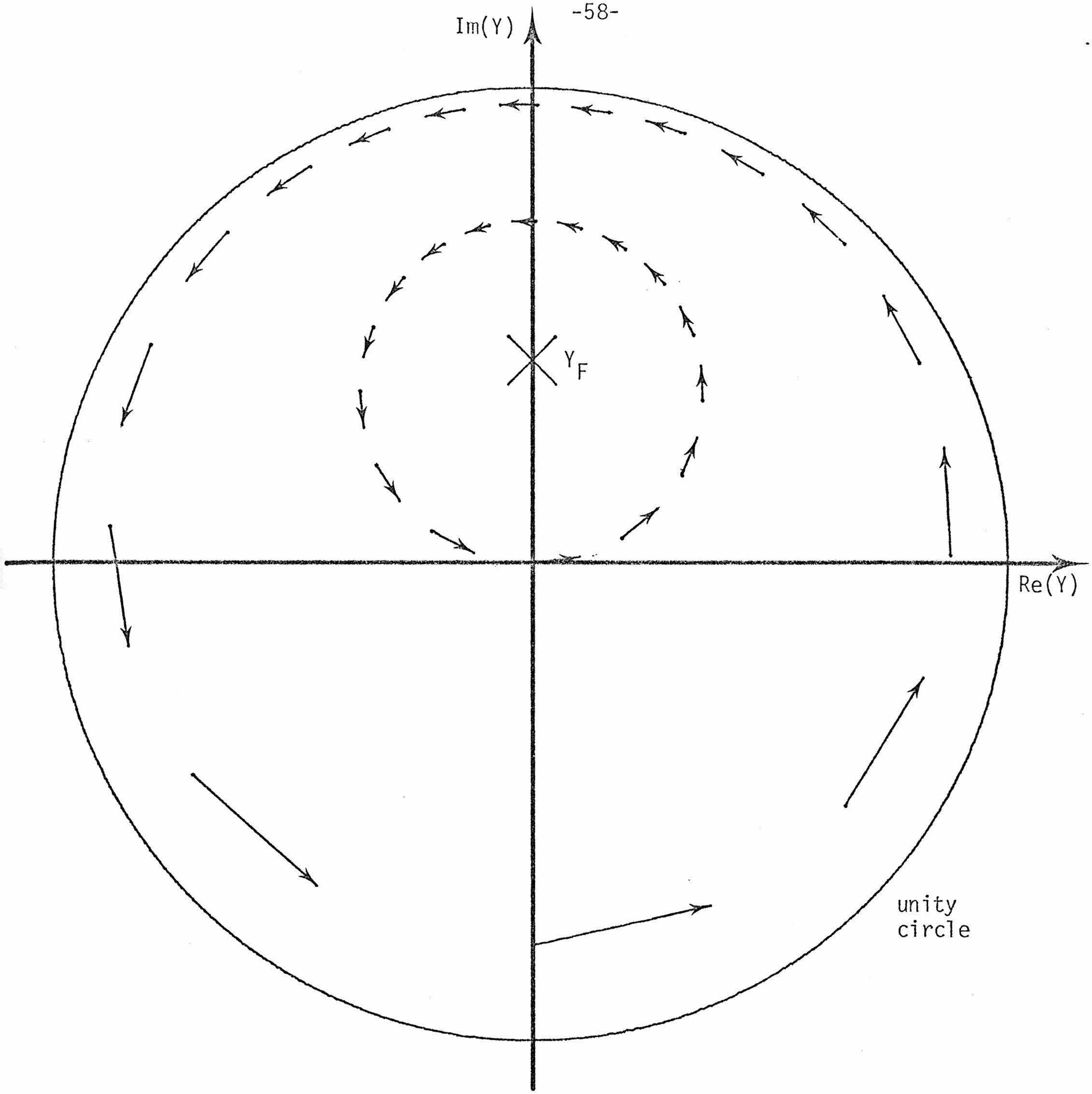
B. $|\eta| < 1$ (STOP BAND)

Again for simplicity, we assume that η is positive. There are two fixed points of concern, both on the unity circle. Let

$$Y_{F1} = i\eta + \sqrt{1 - \eta^2} \tag{III.61}$$

$$Y_{F2} = i\eta - \sqrt{1 - \eta^2} \quad . \tag{III.62}$$

For Y_{F1} let:



Orbits in the Y-plane for

r = 0.1
ϕ_T = 8° > PASS BAND
lossless medium

Figure 10

$$Z = Y - Y_F \quad . \tag{III.63}$$

The velocity field is given by

$$V(Y) = r(-2\sqrt{1-\eta^2}\, Z - Z^2) \quad . \tag{III.64}$$

Clearly, the orbits near Y_{F1} will be exponentially converging toward Y_{F1}.

For Y_{F2} let:

$$Z = Y - Y_{F2} \quad . \tag{III.66}$$

The velocity field is given by

$$V(Y) = r(+2\sqrt{1-\eta^2}\, Z - Z^2) \quad . \tag{III.67}$$

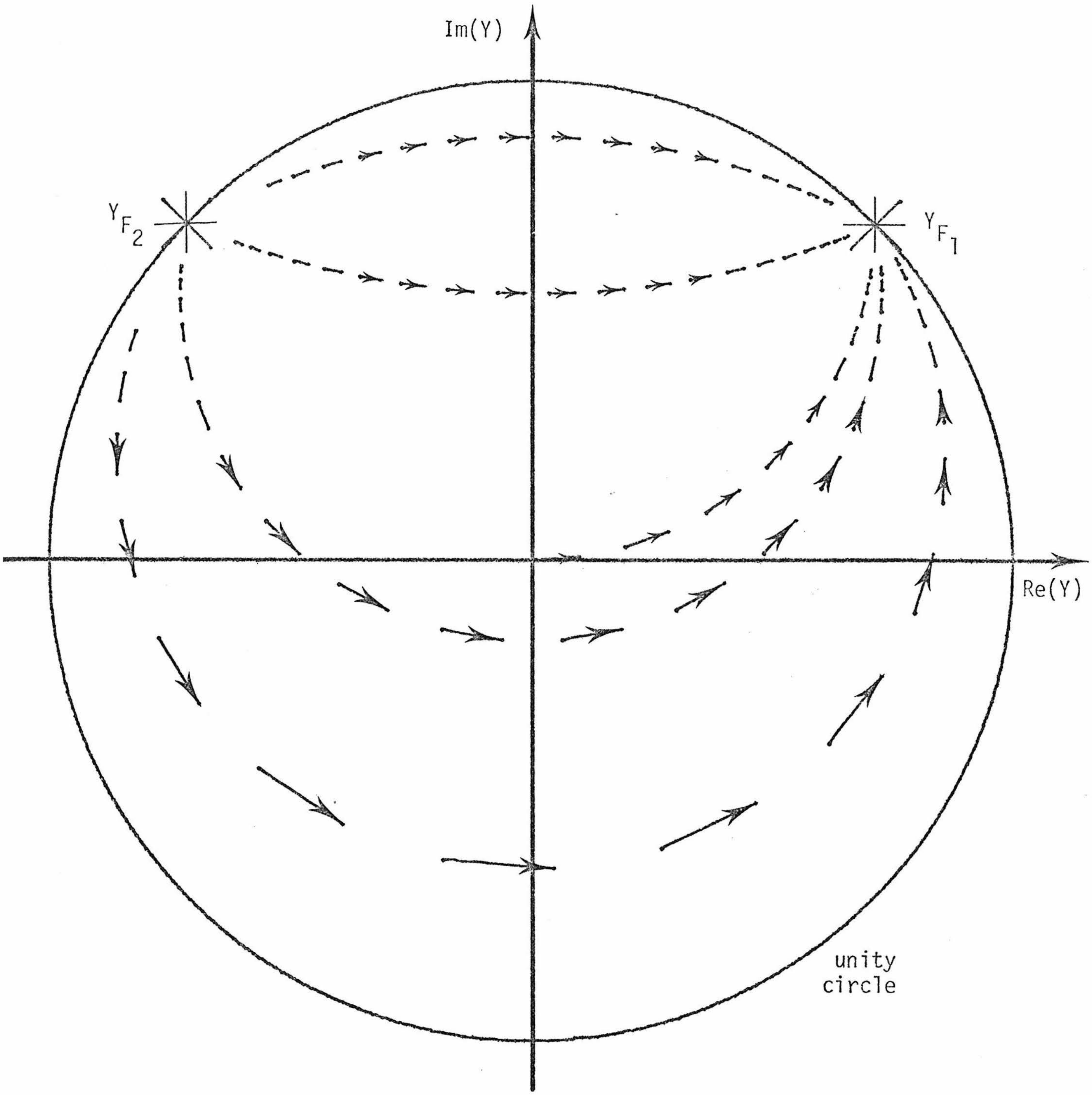
The orbits near Y_{F2} will be exponentially diverging from Y_{F2} .

It is easy to see that the paths of the orbits for the STOP BAND case originate at Y_{F2} and converge toward Y_{F1}. Figure 11 illustrates the paths as calculated with (III.60) for $r = 0.1$ and $\eta = 0.7$. One path beginning at $Y_o = 0$ was calculated (the physical case); the rest were chosen to begin near Y_{F2} . The meaning of the arrows is identical to Figure 10.

III.5 Calculation of the Propagation Constant; STOP BAND

The propagation constant, β , is a complex number which is defined by the ratio (suitably averaged) of transmission coefficients for L and L+1 cells:

$$\langle T_{L+1}/T_L \rangle = \exp(i\beta a) \quad , \tag{III.67}$$



Orbits in the Y-plane for

$r = 0.1$

$\phi_T = 4°$ > STOP BAND

lossless medium

Figure 11

$$\exp(i\beta a) = \frac{t}{\cos\phi_T \pm \sqrt{\cos^2\phi_T - t^2}} \tag{III.70}$$

In view of the fact that the transmission is expected to gain the phase ϕ_T for each cell, and ϕ_T is close to $n\pi$ (the Bragg condition) for the STOP BANDS, we may write our final solution for β in the STOP BAND as

$$\beta = i/a[\ell n(|\cos\phi_T| + \sqrt{\cos^2\phi_T - t^2}) - \ell n(t)] + n\pi/a \quad , \tag{III.71}$$

where the absolute value operator and the $(n\pi/a)$ term account for the case of $\cos\phi_T < 0$. The positive imaginary component of β represents an exponential average decrease in transmission for the STOP BAND, which is the reason it is called the "STOP BAND" ! When $|\eta| > 1$, we are in the PASS BAND and the situation is not as easy, because the reflection coefficient never converges to R_F . We must therefore develop a technique to do the proper averaging.

III.6 Calculation of the Propagation Constant; PASS BAND

From the definition of β , we must compute the following average:

$$\beta = \lim_{L\to\infty} i/aL \sum_{M=1}^{L} \ell n(T_M/T_{M+1}) = \lim_{L\to\infty} i/aL\ \ell n(T_1/T_{L+1}). \tag{III.72}$$

As L approaches infinity, R_L will make many orbits around R_F , the fixed point, as discussed in Section III.4. It is sufficient to calculate the average of $\ell n(T_M/T_{M+1})$ as one orbit is made. Since the relation between the "quasi-time" to get once around the orbit and the number of cells to get once around the orbit is simply a multiplicative constant, we need only average $\ell n(T_M/T_{M+1})$ over quasi-time once around the orbit. We must first calculate the quasi-period of one orbit. The quasi-period is simply the contour integral of $V^{-1}(Y)$ around the orbit:

$$P = \oint V^{-1}(Y)\, dY \qquad \text{(III.73)}$$

$V^{-1}(Y)$ has only one simple pole at Y_F . By Cauchy's integral theorem, using equation (III.59), we easily have

$$P = \frac{2\pi i}{2ir\sqrt{n^2 - 1}} = \frac{\pi}{r\sqrt{n^2 - 1}} \quad . \qquad \text{(III.74)}$$

This equation is supported by Figure 10. When r is small (0.1 is small enough), one unit of quasi-time corresponds to one cell added. From the figure, we see that approximately 32 cells are needed for one orbit. Equation (III.74) predicts 32.5 units of quasi-time.

If we wish to compute the average of any function, G , of R_L around one orbit, the equation to use is

$$\langle G\rangle = P^{-1} \oint G(R)\, V^{-1}(R)\, dR \quad . \qquad \text{(III.75)}$$

If $G(R)$ is a nonsingular function everywhere within the unity circle (or even just along and inside the orbit), then we have the simple and convenient result

$$\langle G\rangle = G(R_F) \quad . \qquad \text{(III.76)}$$

For the case at hand,

$$G(R) = \ln\left[\frac{1 + rR\exp(2i\phi_T - i\phi_R)}{t\exp(i\phi_T)}\right] \quad . \qquad \text{(III.77)}$$

The restriction on the magnitudes of r and R (less than one) keeps $G(R)$ "well-behaved" in the region of integration: no singularities or branch cuts. The propagation constant is given by

$$\beta = i/a \; G(R_F) \quad . \tag{III.78}$$

Using the formula for the fixed point inside the unity circle, equation (III.14), we obtain

$$\beta = i/a \; \ell n\left[\frac{\cos \phi_T - i\sqrt{t^2 - \cos^2\phi_T}}{t}\right] \quad . \tag{III.79}$$

Since we are following orbits which neither expand nor contract, it follows that β cannot have an imaginary component. Since this is the case, it makes sense to find the cosine of βa :

$$\cos(\beta a) = \mathrm{Re}[\exp(i\beta a)] = \cos \phi_T/t \quad . \tag{III.80}$$

Our final answer for the propagation constant in the PASS BAND case is

$$\beta = a^{-1} \arccos[\cos \phi_T/t] \quad , \tag{III.81}$$

where by "arccos" we mean that (βa) is to be adjusted by $2n\pi$ to be as close as possible to ϕ_T . Clearly, for the trivial case $t = 1$, it follows that $\beta a = \phi_T$. The adjustment of (βa) to the proper quadrant allows us to make a Brillouin diagram in the traditional sense.

III.7 Summary of Results for Lossless Media

If we have a periodic, lossless medium, then an analysis of one period (by any method) will yield all the necessary information to calculate the PASS and STOP BANDS for propagation in the medium, including the propagation constant. In particular, we need only calculate the fraction of flux reflected from one period (r^2) and the phase of transmission through one period (ϕ_T). If we define $t = \sqrt{1-r^2}$ we have the following formulas:

A. PASS BAND: $|\sin \phi_T|/r > 1$ or $|\cos \phi_T|/t < 1$

$$\beta = a^{-1} \arccos[\cos \phi_T/t] \quad . \tag{III.81}$$

When r is small, then the reflection from the periodic medium repeats every

$$P = \pi/\sqrt{t^2 - \cos^2 \phi_T} \tag{III.82}$$

cells. If M, the actual number of cells in the whole slab, is much less than P, then $(M\beta a)$ is not necessarily a good estimate of the phase gained in transmission through M cells.

B. STOP BAND: $|\sin \phi_T|/r < 1$ or $|\cos \phi_T|/t > 1$

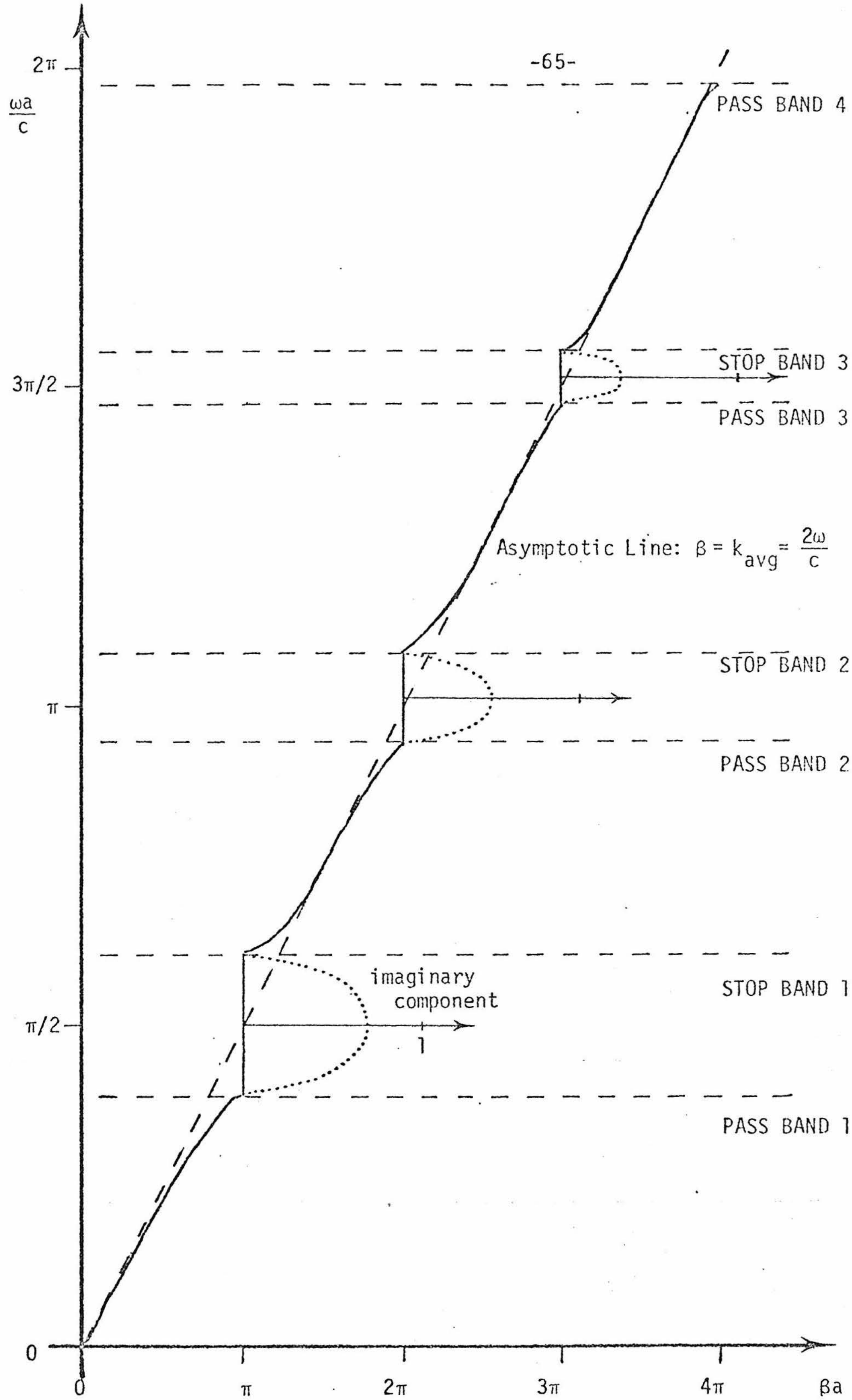
$$\beta = n\pi/a + i/a \, \ell n[|\cos \phi_T|/t + \sqrt{\cos^2\phi_T/t^2 - 1}\,]. \tag{III.83}$$

Whether r is small or is close to one, $\exp(iM\beta a)$ is a good estimate to use to calculate the transmission coefficient after M cells.

Figure 12 is a Brillouin[10,11] diagram of the first several PASS and STOP BANDS for the periodic medium with cell index of refraction given by

$$n(z) = 2 + \cos(2\pi z/a) \tag{III.84}$$

(see also Figure 15). The form of this diagram is very similar to one



Brillouin Diagram: $n(z) = 2 + \cos(2\pi z/a)$

Figure 12

done by D. Jaggard and G. Evans of the California Institute on a similar periodic medium using a different approach.[11]

III.8 Lossy Periodic Media

If the periodic medium is lossy, then the derivations of the preceding sections are no longer correct past equation (III.7). The quantity $(T_1T_1' - R_1R_1')$ is no longer of unit magnitude. We will find it convenient to define the following quantities

$$\alpha \exp(2i\phi_\alpha) = T_1'T_1 - R_1'R_1 \qquad \text{(III.85)}$$

$$\phi_\gamma = 2\phi_\alpha - \phi_R - \phi_{R'} \pm \pi \qquad \text{(III.86)}$$

$$\gamma = |R_1'/R_1| \qquad \text{(III.87)}$$

$$Y_L = R_L \exp(i\phi_\alpha - i\phi_R) \quad . \qquad \text{(III.88)}$$

The recursion relation for Y_L becomes

$$Y_{L+1} = \left[\frac{r + \exp(i\phi_\alpha)\, Y_L}{1 + \gamma r \exp(i\phi_\alpha - i\phi_\gamma) Y_L}\right] \exp(i\phi_\alpha) \quad . \qquad \text{(III.89)}$$

As the losses in the medium go to zero, the "loss parameters" approach the following limits:

$$\alpha = 1 \qquad \text{(III.90)}$$

$$\phi_\alpha = \phi_T \qquad \text{(III.91)}$$

$$\gamma = 1 \qquad \text{(III.92)}$$

$$\phi_\gamma = 0 \quad , \qquad \text{(III.93)}$$

reducing equation (III.89) to the lossless form (III.60). The fixed points of the new recursion obey the equation

$$[\gamma r \exp(-i\phi_\gamma)]Y_F^2 + [\exp(-i\phi_\alpha) - \alpha \exp(i\phi_\alpha)]Y_F + [r] = 0 \quad . \qquad \text{(III.94)}$$

The solutions may be computed trivially. As compared with the lossless fixed points, there are two general comments to make. For the PASS BAND fixed points, both will move off the imaginary axis, but the fixed points will remain inside and outside the unity circle, respectively. For the STOP BAND fixed points, the convergent fixed point will move inside the unity circle, while the divergent fixed point will move outside the unity circle. (These results would not be true if the magnitude of α were not less than or equal to one.) Once we have found Y_F (or R_F), it is a simple matter to compute the propagation constant by substitution of R_F in

$$\exp(i\beta a) = \frac{T_1}{1 + R_1' R_F} \quad . \qquad \text{(III.95)}$$

When the medium is very lossy, convergence to R_F is very rapid. When the medium is only slightly lossy, then the STOP BAND case still converges to R_F , but the PASS BAND case displays a decaying orbit behavior. There is no problem using R_F even in this event, because either the orbit will not be decaying very rapidly, in which case equation (III.76) is approximately true, or the orbit is decaying rapidly, in which case R_L converges to R_F quickly. Of course, the distinction between PASS and STOP BANDS is blurred somewhat in a lossy medium, since enough cells will eliminate the bulk of the transmitted wave in either case; the greater the "lossiness," the more the distinction between PASS and

STOP BANDS is lost. When the losses are slight, then the PASS BAND path on the Y-plane retains its general orbit feature. A good approximate criterion to use to separate the PASS and STOP BANDS is to determine whether $|\sin(\phi_\alpha)|/r$ is greater than one (PASS) or less than one (STOP); however, this should only be used to get an idea of the PASS and STOP BANDS and not to provide the fine distinction that was possible in the lossless case. Figures 13 and 14 illustrate the Y-orbits for two cases of slightly lossy, symmetric ($\gamma = 1$) cells, which are identical in the limit as the losses go to zero with the cases illustrated in Figures 10 and 11.

III.9 Multiply Periodic Media

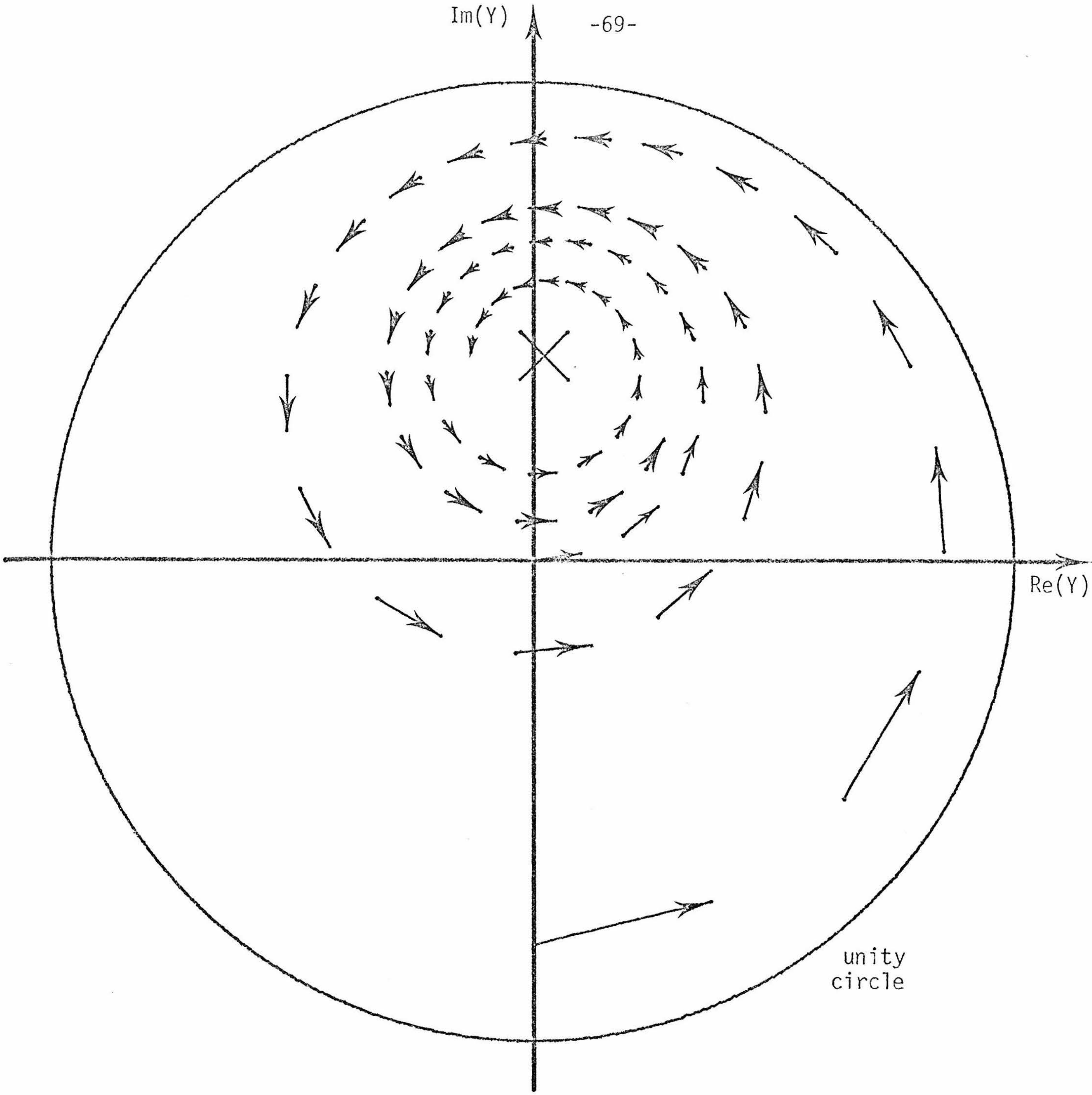
If an inhomogeneous slab has a periodic structure with cell length a, so that

$$n(z + Na) = n(z) \,, \qquad \text{(III.96)}$$

and an average index of refraction in the cell, n_{avg}, then we expect to have STOP BANDS whenever the phase gained through one cell is an integral multiple of π (but not zero):

$$k_{STOP-N} = N\pi/an_{avg} \quad . \qquad \text{(III.97)}$$

This formula helps us to understand the positions of the STOP BANDS in Figure 12. As (ω/c) gets larger, the reflection coefficient gets smaller for cases with continuous variation in $n(z)$, so we expect that the width of the STOP BANDS will get smaller as (ω/c) increases. This feature is also apparent in Figure 12. When r is small and almost constant as a function of (ω/c), the width of the STOP BAND is given



Orbits in the Y-plane for

$r = 0.1$

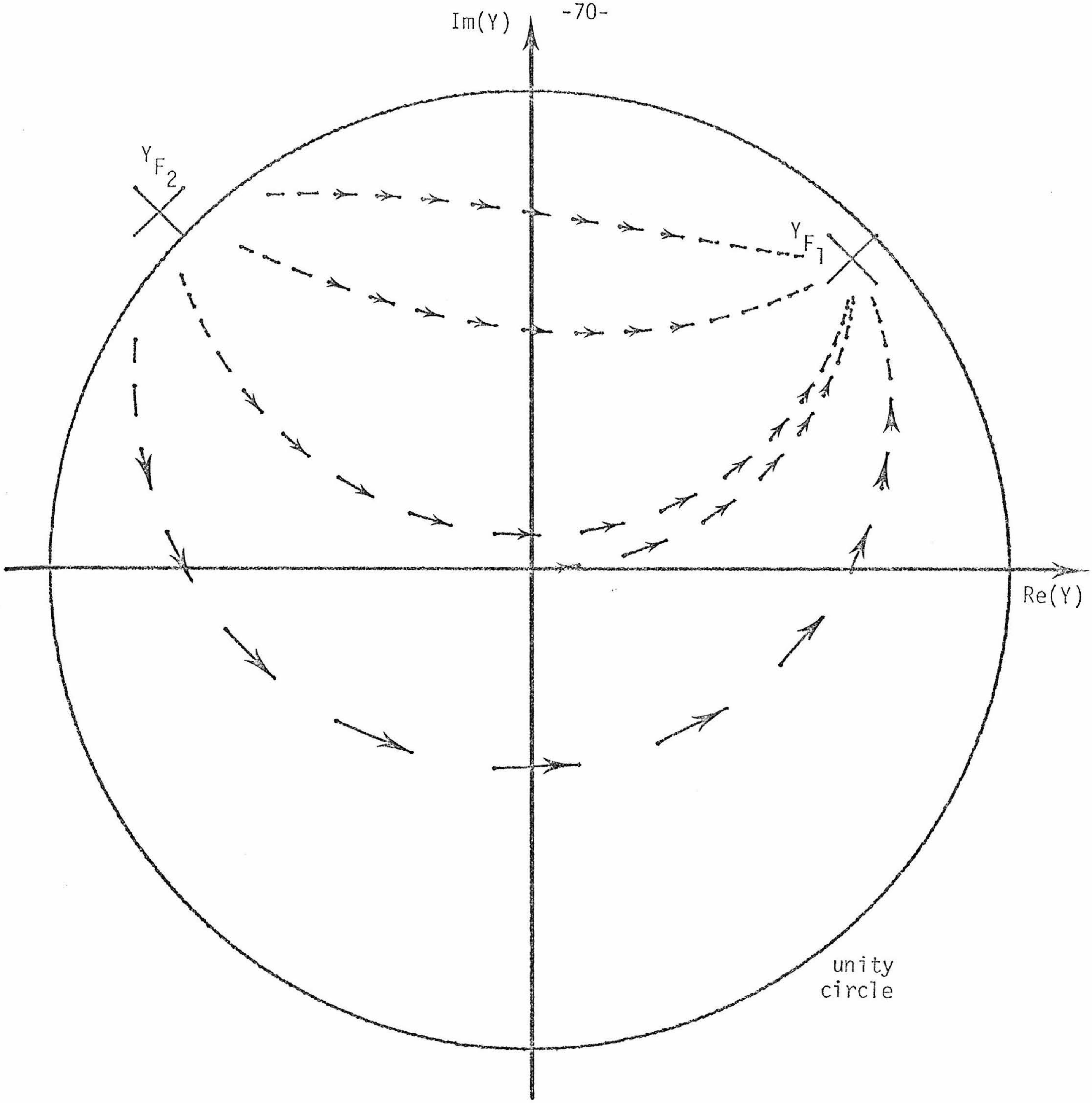
$\phi_\alpha = 8° \simeq \phi_T$

$\alpha = 0.99$ > lossy PASS BAND

$\gamma = 1$

$\phi_\gamma = 0.01°$

Figure 13



Orbits in the Y-plane for

$r = 0.1$

$\phi_\alpha = 4° \simeq \phi_T$

$\alpha = 0.99$ > lossy STOP BAND

$\phi_\gamma = 0.01°$

$\gamma = 1$

Figure 14

approximately by

$$\Delta k_{STOP-N} = 2\arcsin(r)/an_{avg} \simeq 2r/an_{avg} , \qquad \text{(III.98)}$$

because the phase for small reflection cases is very nearly

$$\phi_T = (\omega/c)an_{avg} . \qquad \text{(III.99)}$$

(Recall the arguments of Section II.4).

The index of refraction in any periodic medium with symmetric cells of length a may be written in the Fourier cosine series form:

$$n(z) = n_{avg} + \sum_{\ell=1}^{\infty} n_\ell \cos(\kappa_\ell z) , \qquad \text{(III.100)}$$

where

$$\kappa_\ell = 2\ell\pi/a . \qquad \text{(III.101)}$$

(We will generally restrict our attention to indices of refraction which always remain greater than or equal to one, but this restriction is not necessary.) Equation (III.97) may be written in terms of the lattice structure numbers, κ_ℓ , as

$$k_{STOP-\ell} = \frac{1}{2}\kappa_\ell/n_{avg} . \qquad \text{(III.102)}$$

We expect all STOP BANDS to exist for $\ell = 1$ to infinity whether or not there is a component in $n(z)$ with corresponding κ_ℓ , but we also expect that the widths (strengths) of the STOP BANDS will be strongly influenced by the presence of a non-zero component in $n(z)$ with corresponding κ_ℓ .

To illustrate this idea, consider the two cell indices of refraction given by

$$n_A(z) = 2 + \cos(\kappa_1 z) \qquad \text{(III.103)}$$

$$n_B(z) = 2 + \frac{1}{2}\cos(\kappa_1 z) + \frac{1}{2}\cos(\kappa_3 z) \quad . \qquad \text{(III.104)}$$

These two functions have identical averages ($n_{avg} = 2$), minimum values ($n_{min} = 1$), and maximum values ($n_{max} = 3$). They are shown in Figure 15 ($\chi = 1$). Consider also the index of refraction function given by

$$n_C(z) = 2 - \frac{1}{2}\cos(\kappa_1 z) - \frac{1}{2}\cos(\kappa_3 z) \qquad \text{(III.105)}$$

We have not illustrated $n_C(z)$ because it is so similar to $n_B(z)$. In fact, if we had one slab composed of many cells of type B, and another composed of many cells of type C, they would differ only at the very ends, since $n_C(z)$ is $n_B(z)$ shifted by $a/2$. We expect that the PASS and STOP BANDS should be identical for the two slabs with cells of types B and C.

Figures 16, 17, and 18 represent the computed magnitude of reflection from one cell with index of refraction given by equations (III.103), (III.104), and (III.105) respectively, where we have assumed a background index of unity. Note in particular that $|R(\omega/c)|$ is not identical for $n_B(z)$ and $n_C(z)$. We expect this because, among other reasons, $n_C(z)$ has abrupt discontinuities which prevent $|R(\omega/c)|$ from becoming smaller as (ω/c) becomes large. (In fact, the maxima of $|R(\omega/c)|$ will approach 0.8 for case C as (ω/c) becomes large.) We have not plotted $\phi_T(\omega/c)$, but of course it was also calculated so that

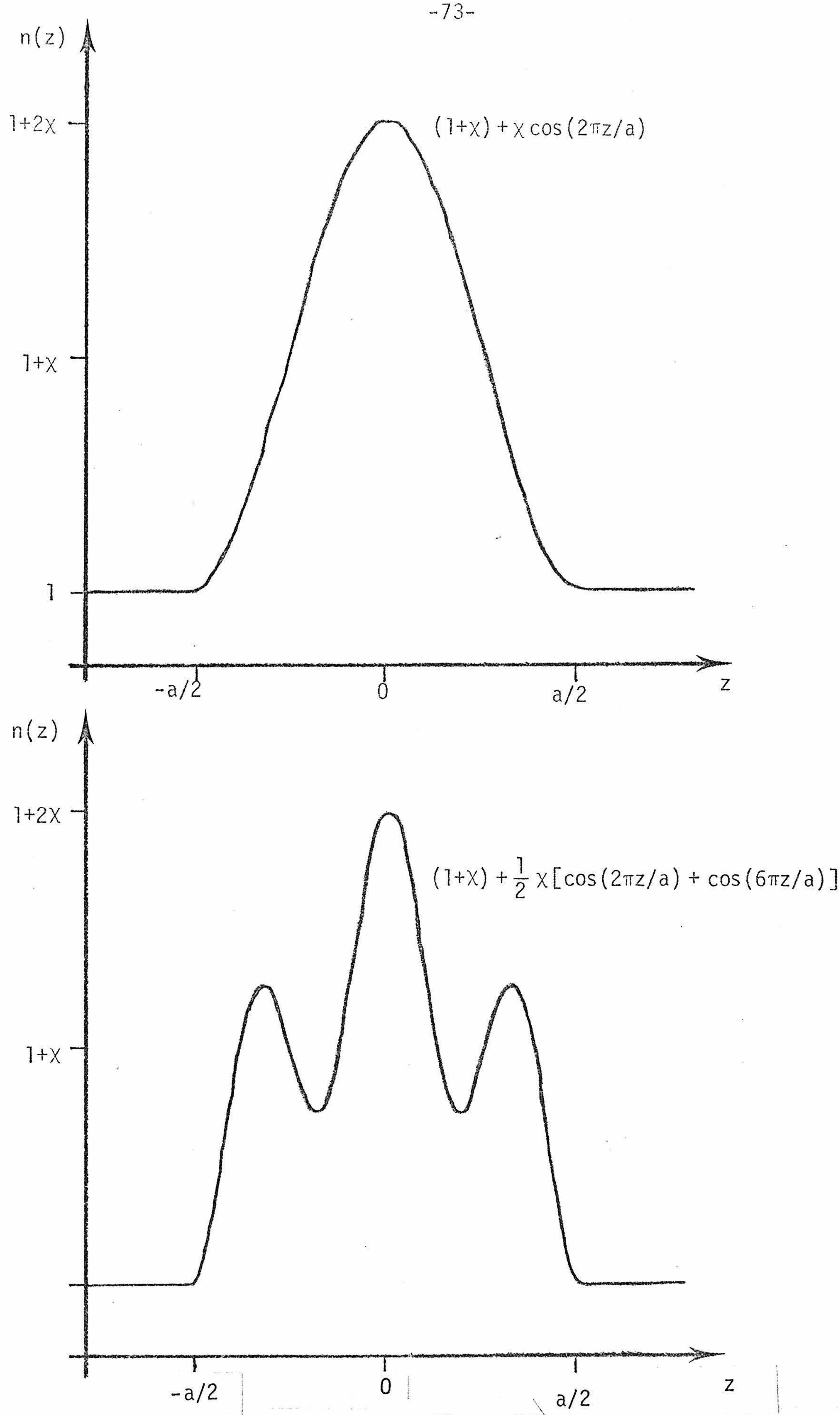
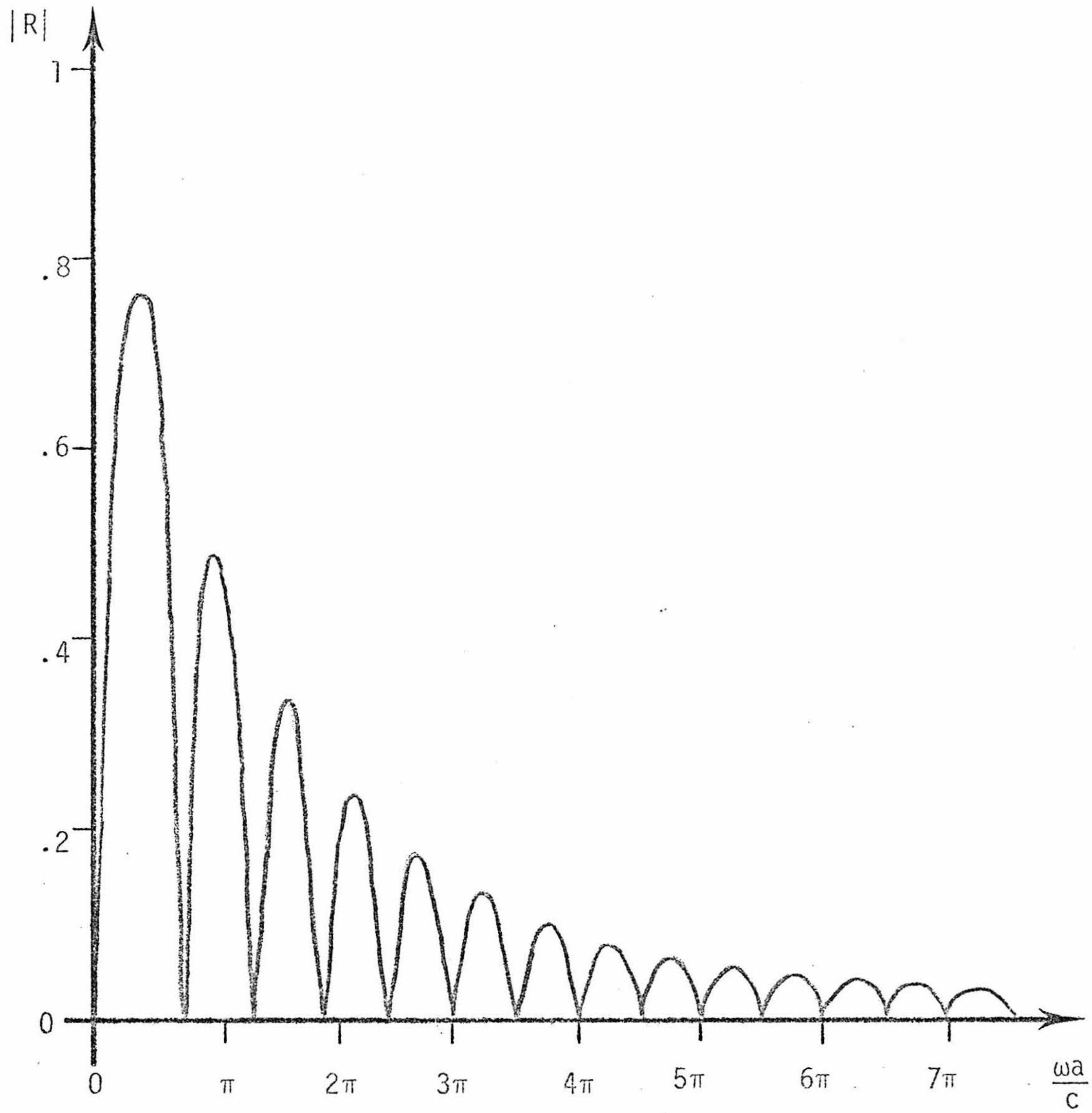
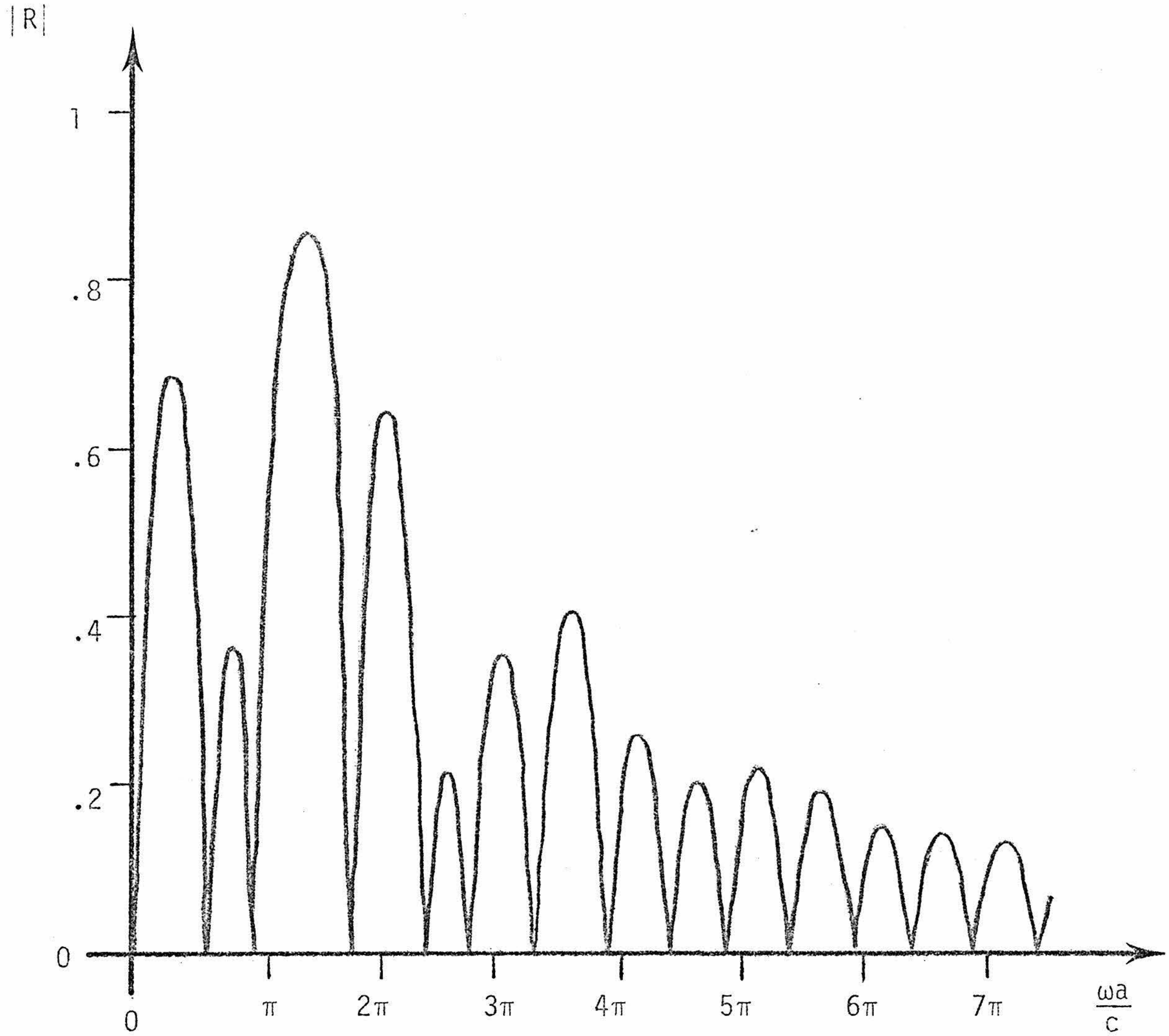

n(z)
1+2χ
(1+χ) + χ cos(2πz/a)
1+χ
1
-a/2
0
a/2
z
n(z)
1+2χ
(1+χ) + 1/2 χ[cos(2πz/a) + cos(6πz/a)]
1+χ
-a/2
0
a/2
z

Figure 15



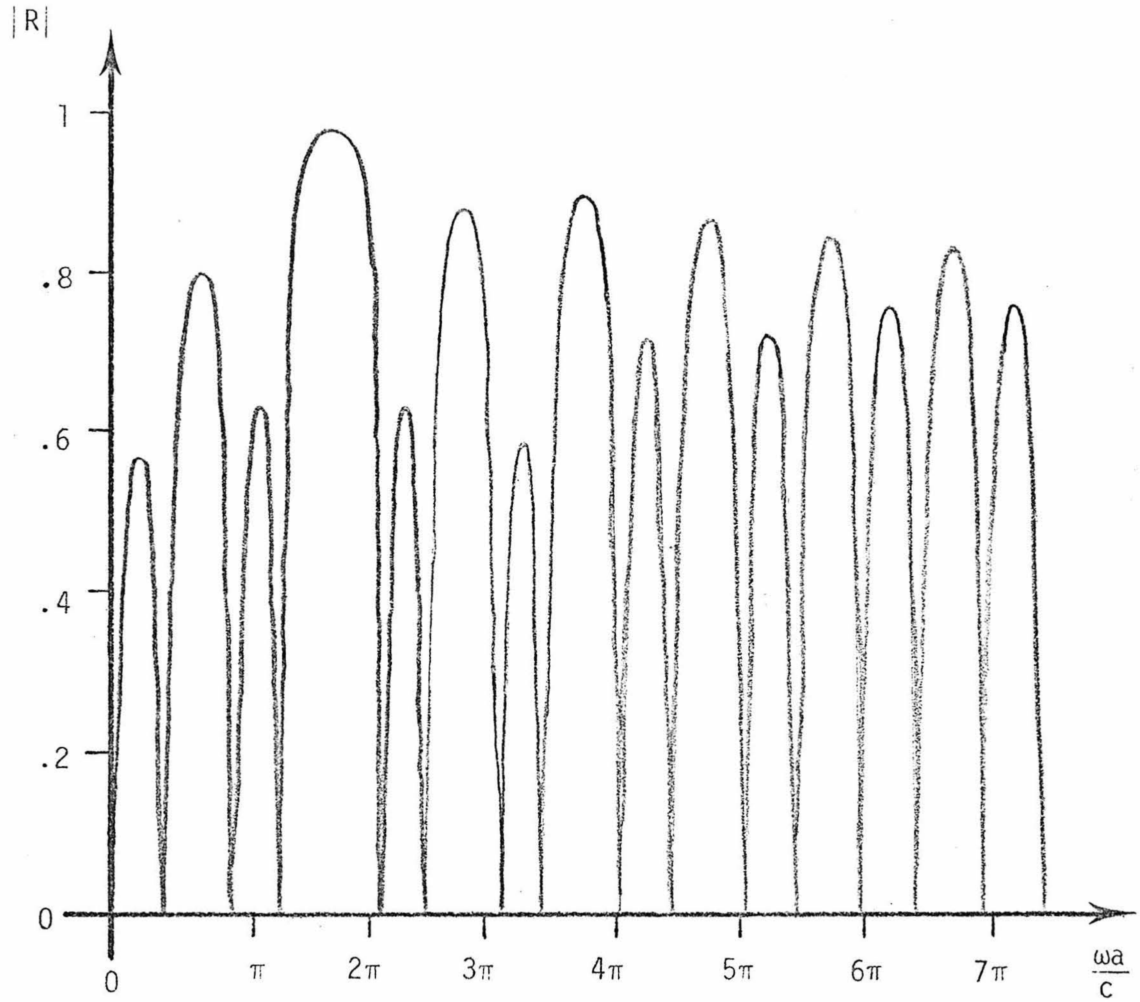
Magnitude of Reflection versus Free-Space Wavenumber Times Cell Size Index of Refraction in the cell: $2+\cos(2\pi z/a)$; $|z| < a/2$ Background Index: 1.

Figure 16



Magnitude of Reflection versus Free-Space Wavenumber Times Cell Size. Index of Refraction in the Cell: $2 + \frac{1}{2}\cos(2\pi z/a) + \frac{1}{2}\cos(6\pi z/a)$; $|z| < a/2$. Background Index: 1.

Figure 17



Magnitude of Reflection versus Free-Space Wavenumber Times Cell Size. Index of Refraction in the Cell: $2 - \frac{1}{2}\cos(2\pi z/a) - \frac{1}{2}\cos(6\pi z/a)$; $|z| < a/2$. Background Index: 1 .

Figure 18

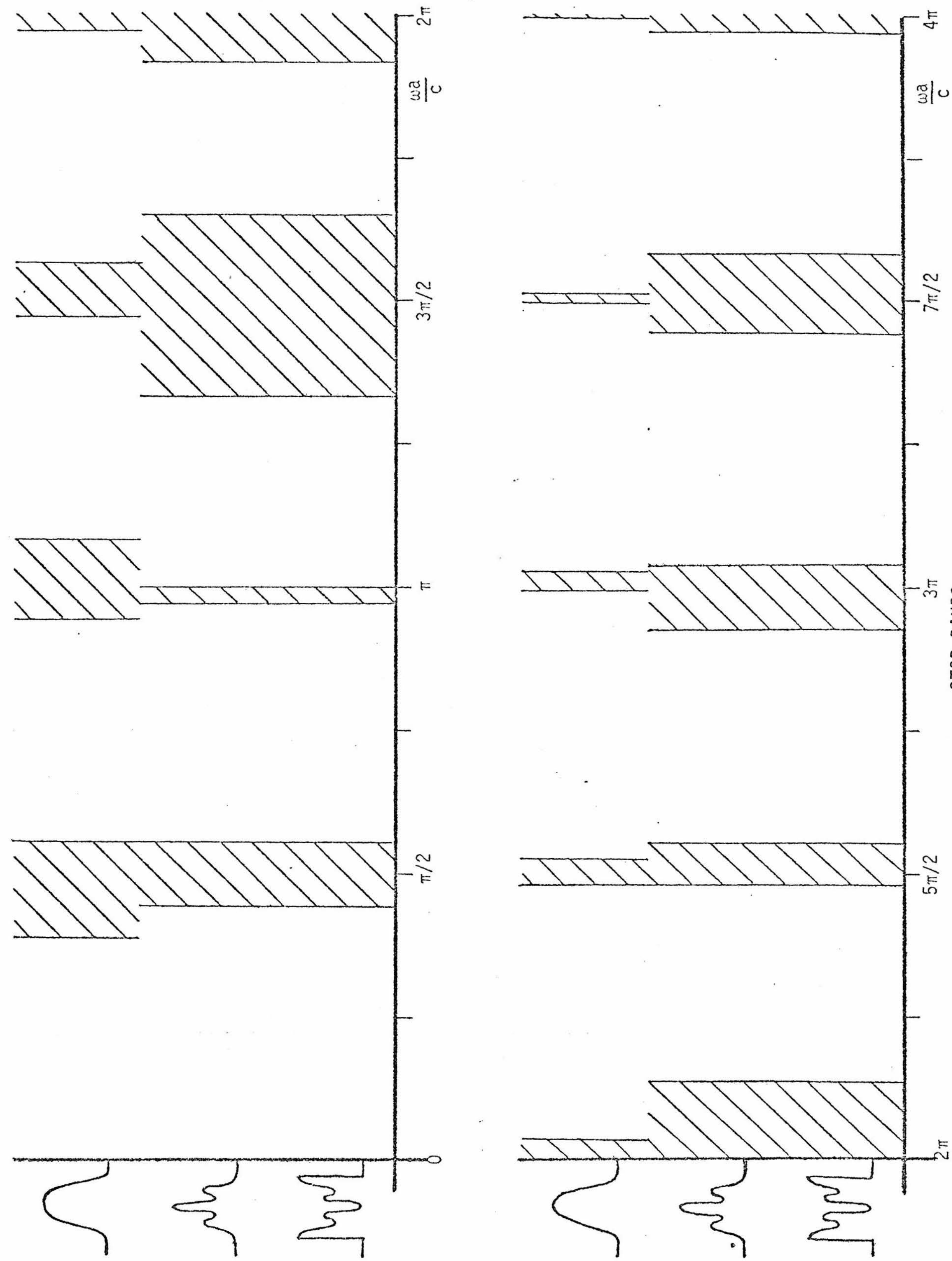
$\beta(\omega/c)$ could be computed. In all cases, $\phi_T(\omega/c)$ is close to the expected value:

$$\phi_T(\omega/c)_{expected} = \omega\, an_{avg}/c \quad , \qquad (III.106)$$

but the small differences between the expected and actual values determine the exact nature of the STOP BANDS. Figure 19 reveals the STOP BANDS (shaded areas) for the three cell types. It is interesting to note that, as predicted, the STOP BANDS for cell types B and C are identical [in fact, so is $\beta(\omega/c)$], even though both $|R(\omega/c)|$ and $\phi_T(\omega/c)$ are different for the two cases. This result gives us greater confidence in our theory of periodic media, especially the arguments of Section III.1, since the calculations for the cell type C case assumed a unit index of refraction between cells when the "actual" value is three. Figure 19 also gives instancial evidence for our claim that some STOP BANDS will be enhanced and others diminished by the addition of higher-order components in $n(z)$. The exact amount of enhancement is a matter for calculation in each particular case.

III.10 Conclusions for Periodic Media

We have seen that electromagnetic wave propagation in any periodic medium may be calculated very conveniently in terms of a propagation constant (β) once the reflection and transmission properties of one cell are determined. We have done our calculations assuming that the incidence is normal and the permeability is constant. However, the transformations of Section II.8 extend all results to the more general case in which all electromagnetic parameters are variable. Although we have



STOP BANDS
Figure 19

dropped the explicit dependence on ω , it is also possible to assume that the electromagnetic parameters are functions of ω as well as z ; in this case, the formulas for the propagation constant and the criteria for PASS and STOP BANDS will be the same, but the form of the Brillouin diagram will be altered somewhat because n_{avg} is no longer a constant, but depends on ω .

Chapter IV will deal with media which are "almost homogeneous" and, perhaps, "almost periodic." We will see that, if we are so fortunate as to have a medium which is both "almost homogeneous" and purely periodic, then the calculation of the propagation constant becomes particularly easy. Based on the material of Section III.9, we pose the interesting question of what would happen to the PASS and STOP BANDS if the index of refraction has cosine components at κ-values which are not rational multiples; in other words, the index of refraction is an "almost periodic" function. The results of this chapter are, unfortunately, not suited to answering this question, since we assumed from the outset that we were dealing with a purely periodic index of refraction. However, we will find the results of the next chapter very convenient for handling the important subset of "almost periodic" cases which are also "almost homogeneous."

Chapter IV

Almost Homogeneous Media

IV.1 The Almost Riccati Equation

The main difficulty in integrating the Riccati differential equation for $R(z)$ (equation II.41) is the R^2 term. It would be nice if we could simply ignore this term entirely. The purpose of this chapter is to consider those cases for which the R^2 term may a priori be neglected.

We will consider first exactly what we mean by an "almost homogeneous medium." We mean by "almost homogeneous" that the index of refraction is very nearly constant, or

$$n(z) = n_{avg}(1 + \xi(z)) \quad , \qquad (IV.1)$$

where $\xi(z)$ is small compared with unity. For simplicity, we will let

$$\xi = \max[|\xi(z)|] \qquad (IV.2)$$

be the measure of "almost homogeneity." We will deal later with the situation for which there are many oscillations in $n(z)$. For now, we will assume that there are only "a few" oscillations in $n(z)$ over the length of the slab. In this case, the results of Chapter II (equation II.50) give

$$|T| \geq 1 - L\xi \qquad (IV.3)$$

$$|T^2| \geq 1 - 2L\xi \qquad (IV.4)$$

$$|R^2| \leq 2L\xi \quad , \qquad (IV.5)$$

where L is the small number of oscillations, and we have dropped all terms beyond the linear terms in ξ .

With an "almost homogeneous medium," we need an "almost Riccati equation" to use to find the reflection. We may find this equation by substitution of appropriate approximations in the standard Riccati equation for reflection (II.41):

$$\frac{dR}{dz} = \frac{1}{2}\,[1 - O(\xi)]\,\frac{dn/dz}{n_{avg}[1 + O(\xi)]} - 2in_{avg}(\omega/c)\,[1 + O(\xi)]\,R(z). \qquad \text{(IV.6)}$$

The two terms in equation (IV.6) do two different things. The first term generates local reflections as $n(z)$ varies. The second term acts to change $R(z)$ at right angles to itself in the complex plane; in other words, it is a rotation. Since the two terms do two different things, there is no problem taking them to two different orders in ξ . Since R is small (of order $\xi^{1/2}$), we might want to eliminate the second term in (IV.6) completely. However, with no knowledge of the size of (dn/dz), this is inadvisable, not to mention the loss of the "rotation" effect which would result. Therefore, whatever the size of (dn/dz), we will take each term in equation (IV.6) to lowest order in ξ . This yields the "almost Riccati equation":

$$\frac{dR}{dz} = \frac{dn/dz}{2\,n_{avg}} - 2in_{avg}(\omega/c)\,R(z) \quad . \qquad \text{(IV.7)}$$

One of the wonderful things about the "almost Riccati equation" is that it is a linear equation, in the sense that if $R_1(z)$ is a solution for $n(z) = n_{avg} + n_1(z)$ and $R_2(z)$ is a solution for $n(z) = n_{avg} + n_2(z)$, then

$$R(z) = aR_1(z) + bR_2(z) \qquad \text{(IV.8)}$$

is a solution for

$$n(z) \;=\; n_{avg} + an_1(z) + bn_2(z) \quad , \qquad \text{(IV.9)}$$

as long as a and b are not large numbers (compared with unity). This may be verified by direct substitution in (IV.7).

IV.2 Green's Function for the Almost Riccati Equation

Since the almost Riccati equation is linear, it makes good sense to attempt to find a Green's function to reduce it to a simple integration over a "source function." This is actually fairly easy, so we will not needlessly obfuscate the physical process involved in finding the Green's function with unnecessary mathematical ballast.

Consider the partitioning of the function $n(z) - n_{avg}$ into many pieces which are constant except over a very small region and which are differentiable where $n(z)$ is differentiable. If we have M pieces, then

$$n(z) \;=\; n_{avg} + \sum_{L=1}^{M} n_L(z) \qquad \text{(IV.10)}$$

$$dn/dz = \sum_{L=1}^{M} dn_L/dz \quad . \qquad \text{(IV.11)}$$

The differential equation for $R_L(z)$* begins with the initial value $R_L(+a/2) = 0$, at the right hand boundary of the slab. Since (dn_L/dz) is zero until we reach z_L (the z-coordinate at the region where (dn_L/dz) is non-zero), $R_L(z_L+) = 0$. In the very small region in which (dn_L/dz) is non-zero, we get a contribution to R from the first term of the almost Riccati equation (IV.7):

---

*$R_L(z)$ is a generalization of R_1 and R_2 in equation (IV.8).

$$R_L(z_L-) = \int_{z_L+}^{z_L-} \frac{(dn_L/dz)}{2\,n_{avg}}\,dz \quad . \qquad \text{(IV.12)}$$

The "phase shifting" term has negligible effect over the small region (z_L-, z_L+) . As we continue to solve the almost Riccati equation in the negative (dz) direction, we only have the phase shifting term as a contributor, since (dn_L/dz) is zero beyond z_L-. This adds a phase factor multiplier to $R_L(z_L-)$ to give the final answer for one piece:

$$\frac{R_L(-a/2)}{R_L(z_L-)} = \exp\left[\int_{z_L-}^{-a/2} - 2in_{avg}(\omega/c)\,dz\right]$$

$$= \exp[2in_{avg}(\omega/c)(z_L- + a/2)] \quad . \qquad \text{(IV.13)}$$

Putting equations (IV.12) and (IV.13) together, we obtain

$$R_L = \exp[ian_{avg}(\omega/c)]\,\exp[2iz_L n_{avg}(\omega/c)]$$

$$\times \int_{z_L-}^{z_L+} - \frac{1}{2}\,(dn_L/n_{avg}dz)\,dz \quad . \qquad \text{(IV.14)}$$

To get the reflection for the original problem, we need only sum over all L :

$$R = \sum_{L=1}^{M} R_L = \exp[ian_{avg}\ \omega/c)] \sum_{L=1}^{M} \exp[2in_{avg}z_L(\omega/c)]$$

$$\times \int_{z_L-}^{z_L+} - \frac{1}{2}(dn_L/n_{avg}dz)\,dz \quad . \qquad \text{(IV.15)}$$

If we make the number of pieces, M , very large, then we may make the regions (z_L-, z_L+) very small, small enough so that (dn/dz) is constant over each region. Since only one (dn_L/dz) is non-zero over each region,

(dn_L/dz) is nearly constant over the small region (or is a delta function if $n(z)$ has a jump discontinuity in the region), so

$$\int_{z_L^-}^{z_L^+} -\frac{1}{2}\,(dn_L/n_{avg}dz)\;dz = -\frac{1}{2}\,\Delta z_L(dn_L/n_{avg}dz) \quad . \qquad \text{(IV.16)}$$

The Δz_L is just the factor we need to convert the sum to an integral. Over each interval, (dn/dz) is equal to (dn_L/dz). The final integrated form is

$$R = \exp[ian_{avg}(\omega/c)] \int_{-a/2}^{a/2} \exp[2in_{avg}(\omega/c)z][-\frac{1}{2}\,(dn/n_{avg}dz)]\;dz . \qquad \text{(IV.17)}$$

If we consider the function

$$S(z) = \frac{1}{2}\,(dn/n_{avg}dz) \qquad \text{(IV.18)}$$

to be the "source function" giving rise to reflection, then the Green's function is

$$G(\omega/c;z) = -\exp[ian_{avg}(\omega/c)]\;\exp[2in_{avg}(\omega/c)z] \quad , \qquad \text{(IV.19)}$$

giving us the formal solution to the problem

$$R(\omega/c) = \int_{-a/2}^{a/2} G(\omega/c;z)\;S(z)\;dz \quad . \qquad \text{(IV.20)}$$

IV.3 Almost Inversion of the Reflection

The second wonderful thing about the almost Riccati equation is that its Green's function is just a Fourier transform with slightly altered coordinates and a slightly different constant. Recall that the standard Fourier transform pairs are related by the equations

$$f(\nu) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(z) \exp(i\nu z)\, dz \qquad \text{(IV.21)}$$

$$F(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(\nu) \exp(-i\nu z)\, d\nu \quad . \qquad \text{(IV.22)}$$

If we make the identification of $S(z)$ with $F(z)$, with the additional (obvious) extension that $S(z) = 0$ outside the slab, then

$$f(2n_{avg}\omega/c) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} S(z) \exp[2in_{avg}(\omega/c)z]\, dz \qquad \text{(IV.23)}$$

and

$$R(\omega/c) = -(2\pi)^{+1/2}\ f(2n_{avg}\omega/c)\exp(ian_{avg}\omega/c) \qquad \text{(IV.24)}$$

or

$$f(2n_{avg}\omega/c) = -(2\pi)^{-1/2} \exp(-ian_{avg}\omega/c)\ R(\omega/c) \quad . \qquad \text{(IV.25)}$$

It is then a simple matter to get back the original source function by the inverse Fourier transform:

$$S(z) = -1/2\pi \int_{-\infty}^{\infty} \exp(-ian_{avg}\omega/c)\ R(\omega/c)\ \exp(-2in_{avg}z\omega/c)(2n_{avg})d(\omega/c). \qquad \text{(IV.26)}$$

Suppose that the index of refraction is represented by a Fourier series as

$$n(z) = \int_{-\infty}^{\infty} A(\kappa) \exp(-i\kappa z)\, d\kappa \quad , \qquad \text{(IV.27)}$$

where

$$A(\kappa) = 1/2\pi \int_{-\infty}^{\infty} n(z) \exp(i\kappa z)\, dz \quad . \qquad \text{(IV.28)}$$

The derivative of $n(z)$ is given by

$$dn/dz = \int_{-\infty}^{\infty} B(\kappa)\,\exp(-i\kappa z)\,d\kappa \quad , \tag{IV.29}$$

where

$$B(\kappa) = -i\kappa\,A(\kappa) \quad . \tag{IV.30}$$

$B(\kappa)$ may also be obtained from the Fourier inversion of dn/dz :

$$B(\kappa) = 1/2\pi \int_{-\infty}^{\infty} (dn/dz)\,\exp(i\kappa z)\,dz \quad . \tag{IV.31}$$

Comparing this with the integral for $R(\omega/c)$, we have

$$B(\kappa) = -\exp(-ian_{avg}\omega/c)n_{avg}/\pi\;R(\omega/c) \quad , \tag{IV.32}$$

or

$$A(\kappa) = \frac{-i\,\exp(-i\kappa a/2)}{2\pi(\omega/c)}\,R(\omega/c) \quad , \tag{IV.33}$$

where

$$\kappa = 2n_{avg}\omega/c \quad . \tag{IV.34}$$

The inversion is complete with the final integral

$$n(z) = n_o + \int_{-\infty}^{\infty} \frac{-i\,\exp(-i\kappa a/2)}{2\pi(\omega/c)}\,R(\omega/c)\,\exp(i\kappa z)\,d\kappa \quad . \tag{IV.35}$$

The additive constant, n_o , must be determined by some extra information, such as knowledge of the index of refraction in the left homogeneous region.

We have assumed that n_{avg} is somehow known before the inversion. This will probably not be the case, but no matter. In any event, n_{avg}

will not be very different from the background index of refraction because the medium is almost homogeneous. The average index was chosen as the representative index of refraction for substitution in the almost Riccati equation (over, say, the left background index) because it might tend to minimize the errors in the approximation, but mostly because of previous results relating the phase of the transmission coefficient to the average index of refraction (equation II.47). To find a better guess for n_{avg} than the background index, it is a simple matter to iterate equation (IV.35) with n_{avg} as calculated from the previous iteration. Since we are approximating by dropping $O(\xi)$ terms anyway, there is no point in doing the iteration more than once, if even once.

The other term to worry about in the inversion is $\exp(-i\kappa a/2)$, since we do not necessarily know what is the slab length before the inversion. Here we are in luck, because we recognize this term to be a simple translation operator, which shifts $n(z)$ to the left by $a/2$. Since the origin of the z-axis is arbitrary anyway, this is no problem. If we wish, we may ignore this factor in the inversion, which means that the slab will be located between zero and a after inversion, rather than between $-a/2$ and $+a/2$.

IV.4 Examples of Almost Inversion

A good question to ask at this point is how well does the inversion work? This question is best answered by way of example. We will consider three index of refraction functions. They are essentially the same as the three functions of Section III.9, except that the coefficients of the cosine terms are reduced by two orders of magnitude and the constant term is adjusted accordingly. In terms of Figure 15, the χ parameter is 1/100. The three indices of refraction are

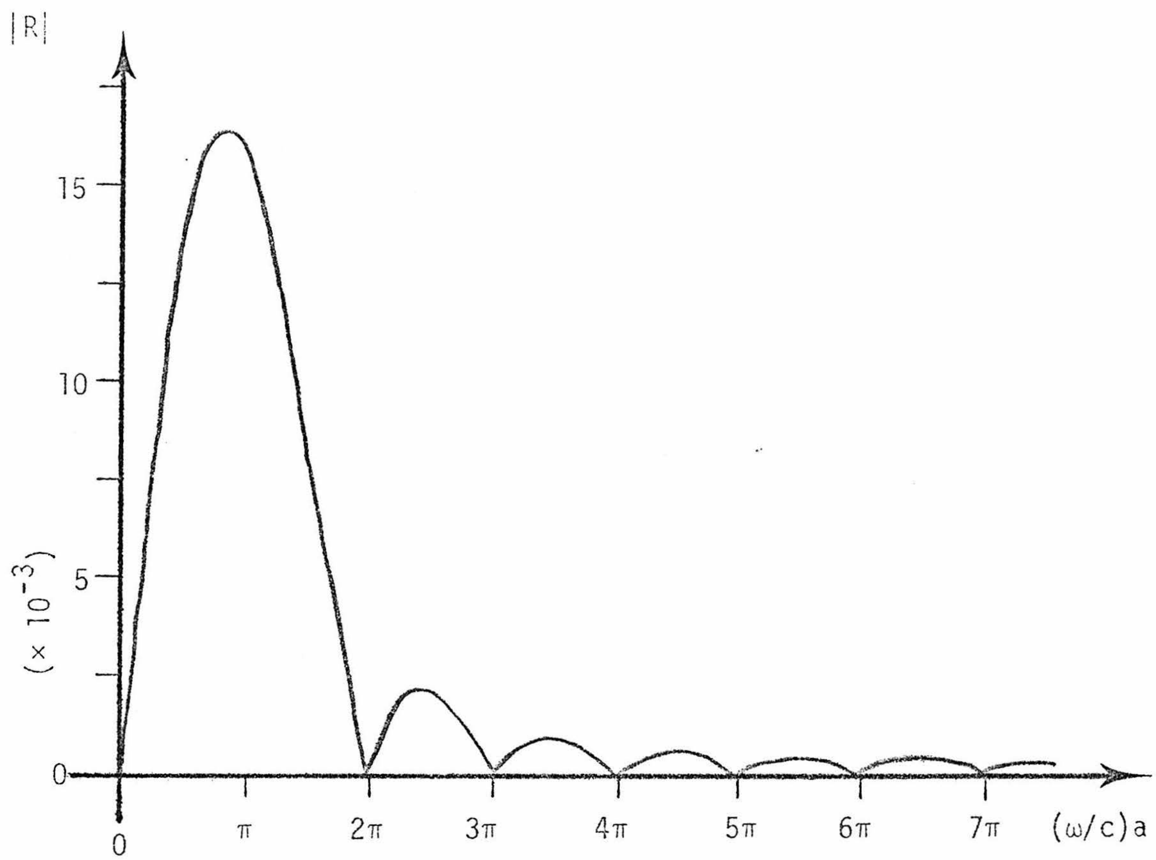
$$n_1(z) = 1.01 + 0.01 \cos(2\pi z/a) \tag{IV.36}$$

$$n_2(z) = 1.01 + 0.005[\cos(2\pi z/a) + \cos(6\pi z/a)] \tag{IV.37}$$

$$n_3(z) = 1.01 - 0.005[\cos(2\pi z/a) + \cos(6\pi z/a)] \ . \tag{IV.38}$$

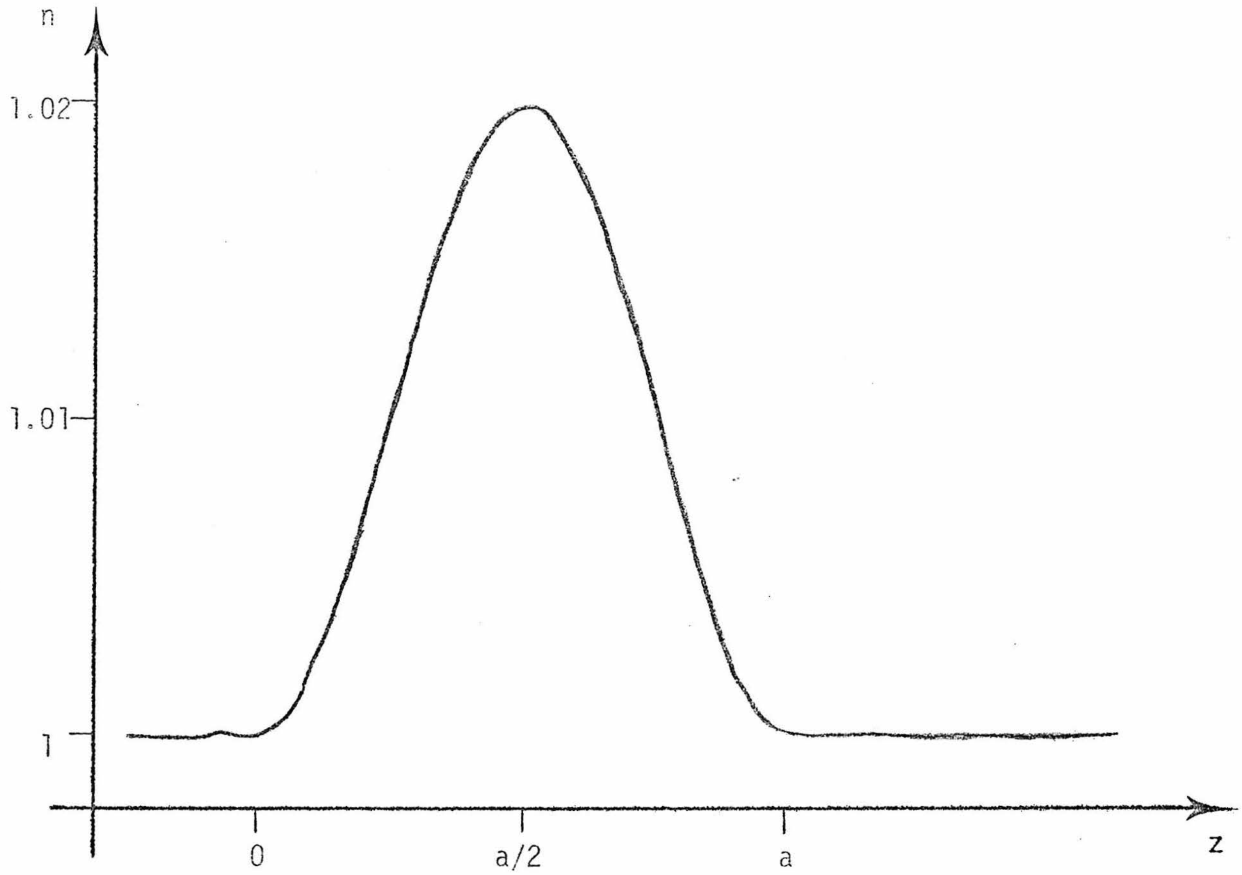
The full invariant imbedding Riccati equation (as programmed on a digital computer) was used to calculate $R(\omega/c)$ for the relevant arguments and a fast Fourier transform was calculated to reconstruct $n(z)$. The magnitude of $R(\omega/c)$ is illustrated in Figures 20, 22, and 24. The phase is not plotted, but is close to $(\omega a n_{avg}/c) \pm \pi/2$ (recall equation II.92 for symmetric $n(z)$). Figures 21, 23, and 25 illustrate the fast Fourier reconstruction of $n(z)$ for the three cases (respectively). The "high frequency wiggles" in $n(z)$ as reconstructed (which are especially noticeable in Figure 25) are due totally to the arbitrary cut-off of the frequencies used in the fast Fourier transform. Except for the non-essential "wiggles," the reconstruction of $n(z)$ is faithful for these almost homogeneous cases.

The next question to ask is how poorly does the reconstruction work for cases which are not almost homogeneous by any stretch of the imagination? Good examples of this would be the original functions of Section III.9. Since we already wrote the computer program to handle the almost homogeneous reconstruction, there is no reason not to try it out on the other cases as well. The results are illustrated in



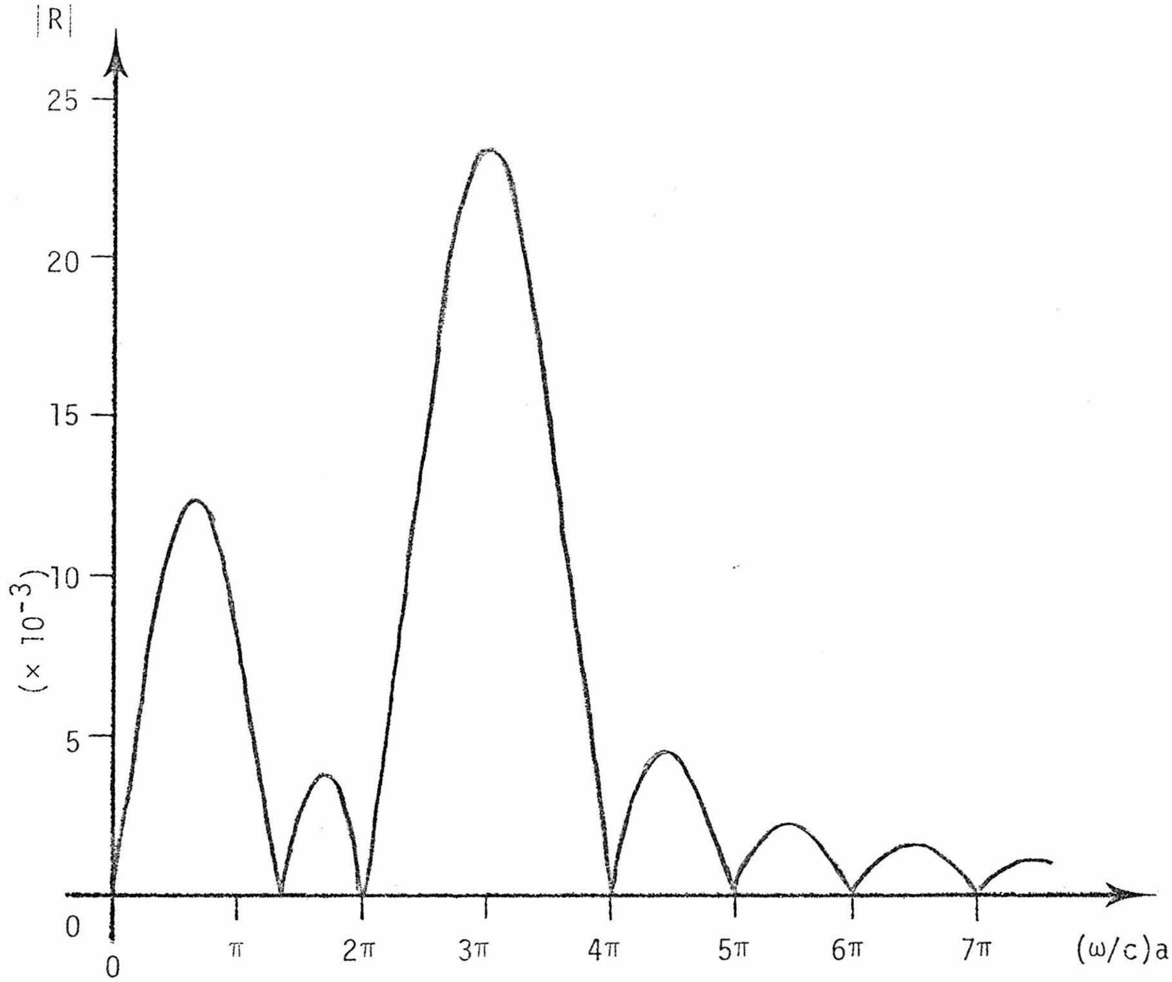
Magnitude of Reflection versus Free-Space Wavenumber (times Cell Size) for $n(z) = 1.01 + 0.01 \cos(2\pi z/a)$.

Figure 20



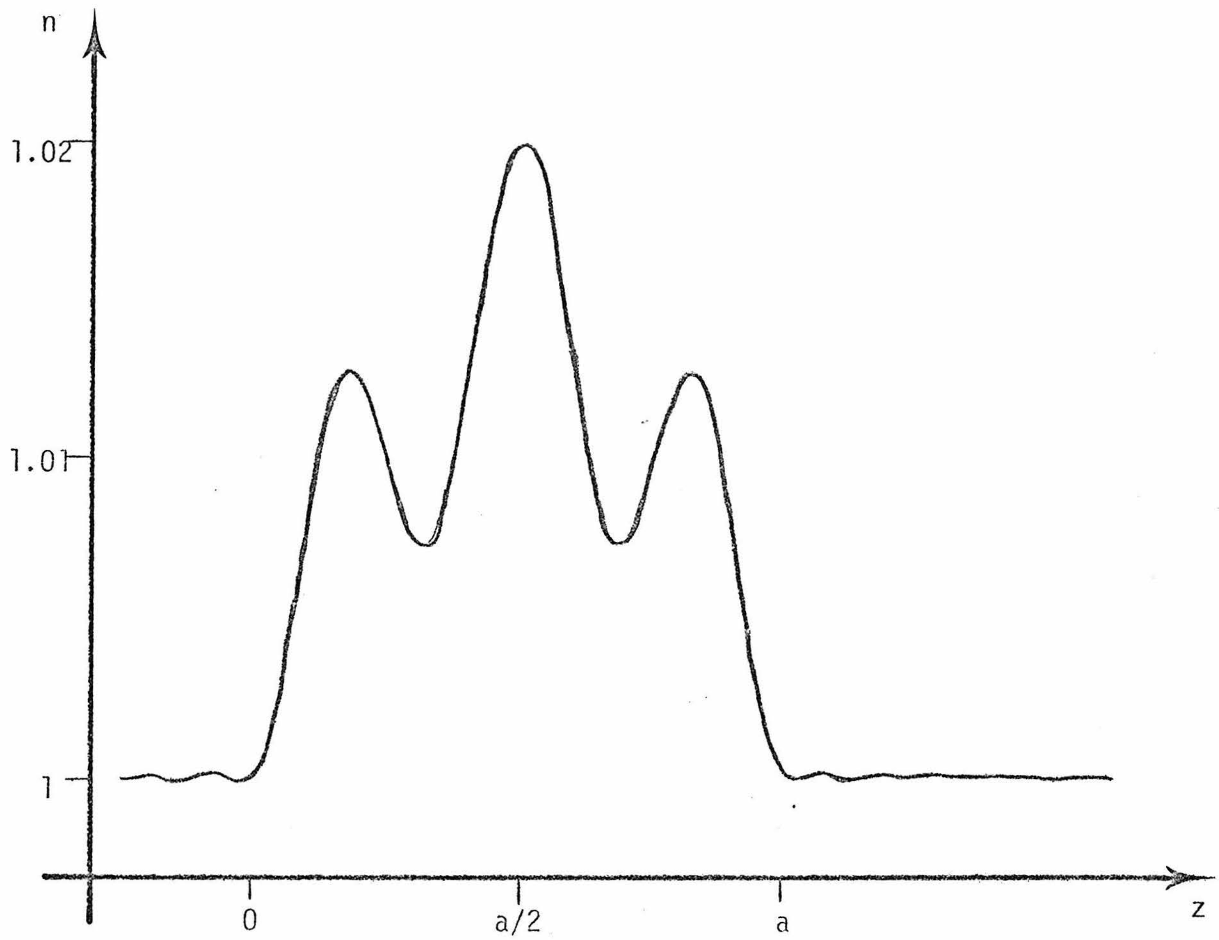
n(z) from Reflection Inversion (Reflection on Figure 20)

Figure 21



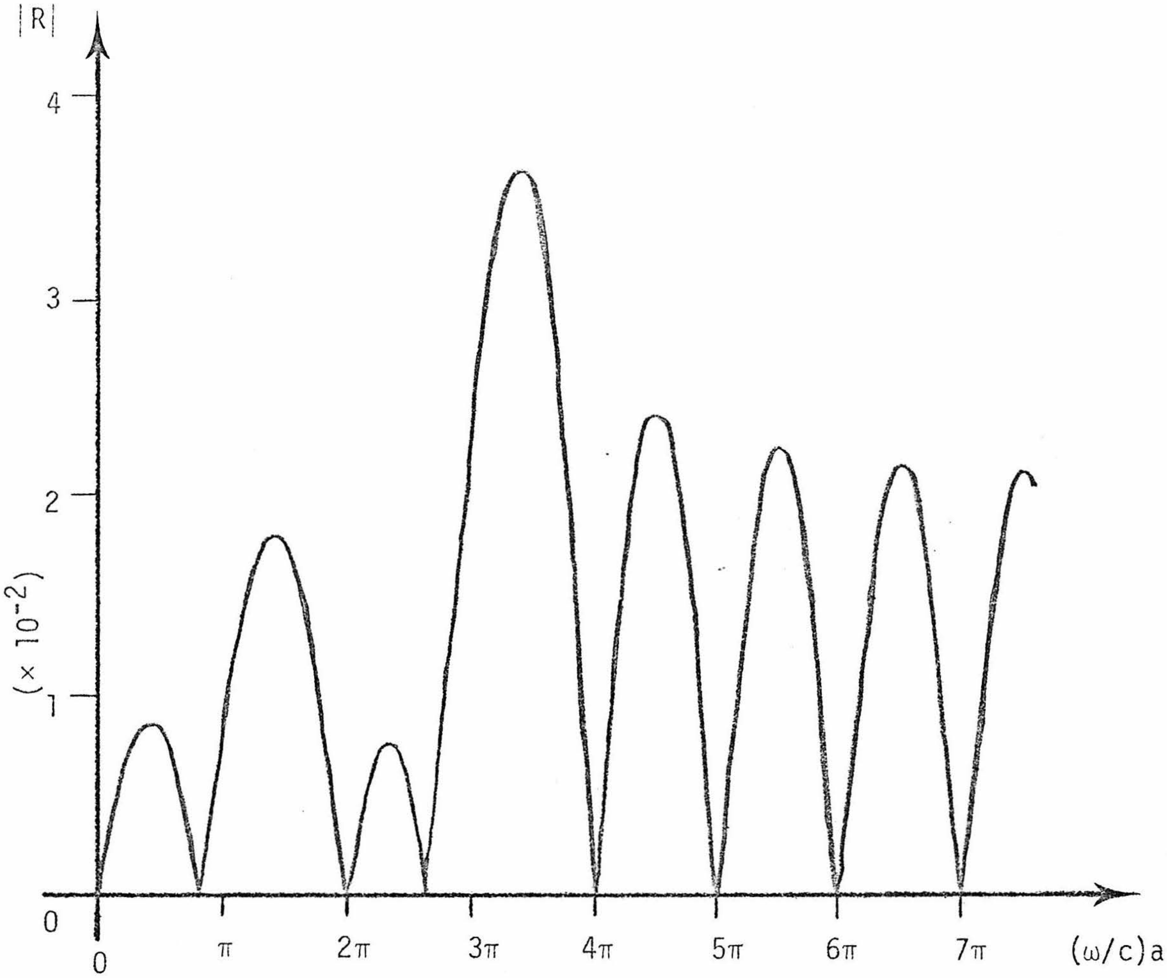
Magnitude of Reflection versus Free-Space Wavenumber (times Cell Size) for $n(z) = 1.01 + 0.005[\cos(2\pi z/a) + \cos(6\pi z/a)]$.

Figure 22



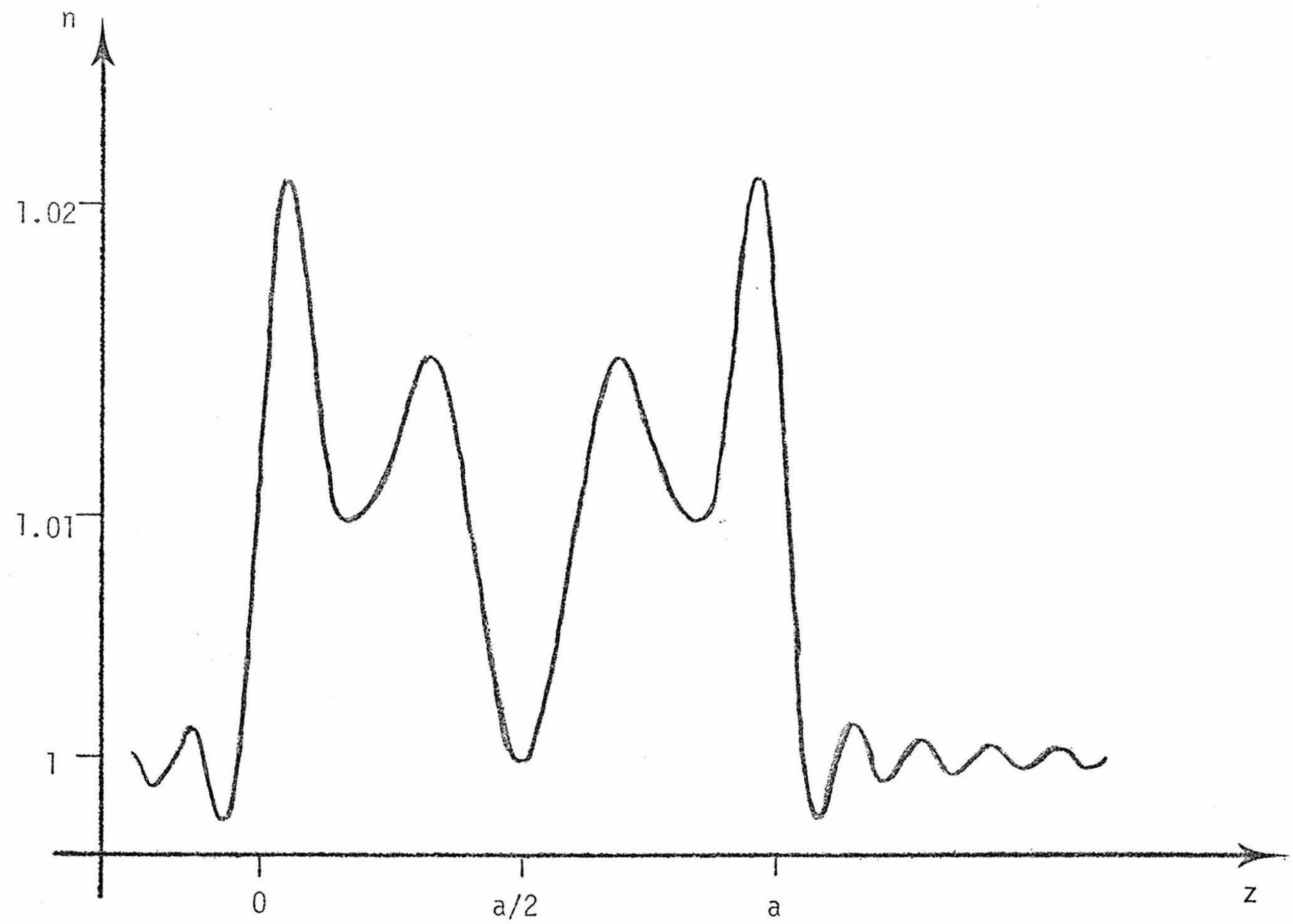
n(z) from Reflection Inversion (Reflection on Figure 22)

Figure 23



Magnitude of Reflection versus Free-Space Wavenumber (times Cell Size) for $n(z) = 1.01 - 0.005[\cos(2\pi z/a) + \cos(6\pi z/a)]$.

Figure 24



n(z) from Reflection Inversion (Reflection on Figure 24).

Figure 25

Figures 26 through 28 (respectively). As expected, they are not faithful reproductions of the original functions, but they do still show the general features. The inversion represented in Figure 28 is particularly bad because of the discontinuities in the original $n(z)$ function which keep the reflection coefficient high even as frequency is increased. An interesting feature of all three reconstructions is that the graph "jerks" near $z = 2a$, which marks the end of the slab, although the variation as reconstructed continues beyond that point. The reason that the slab as reconstructed seems to end at $z = 2a$ (rather than at $z = a$) is that the inversion assumed that $n_{avg} = 1$ when it was really twice as much. The factor of two, if inserted in the inversion, shrinks the slab size back to its original length, "a". The general similarity between the indices of refraction as reconstructed and as originally defined leads us to attempt to find a somewhat better method of inversion for the large-variation cases, but which remains the same as the old inversion for the almost homogeneous cases, which were inverted satisfactorily.

IV.5 Modified Almost Inversion

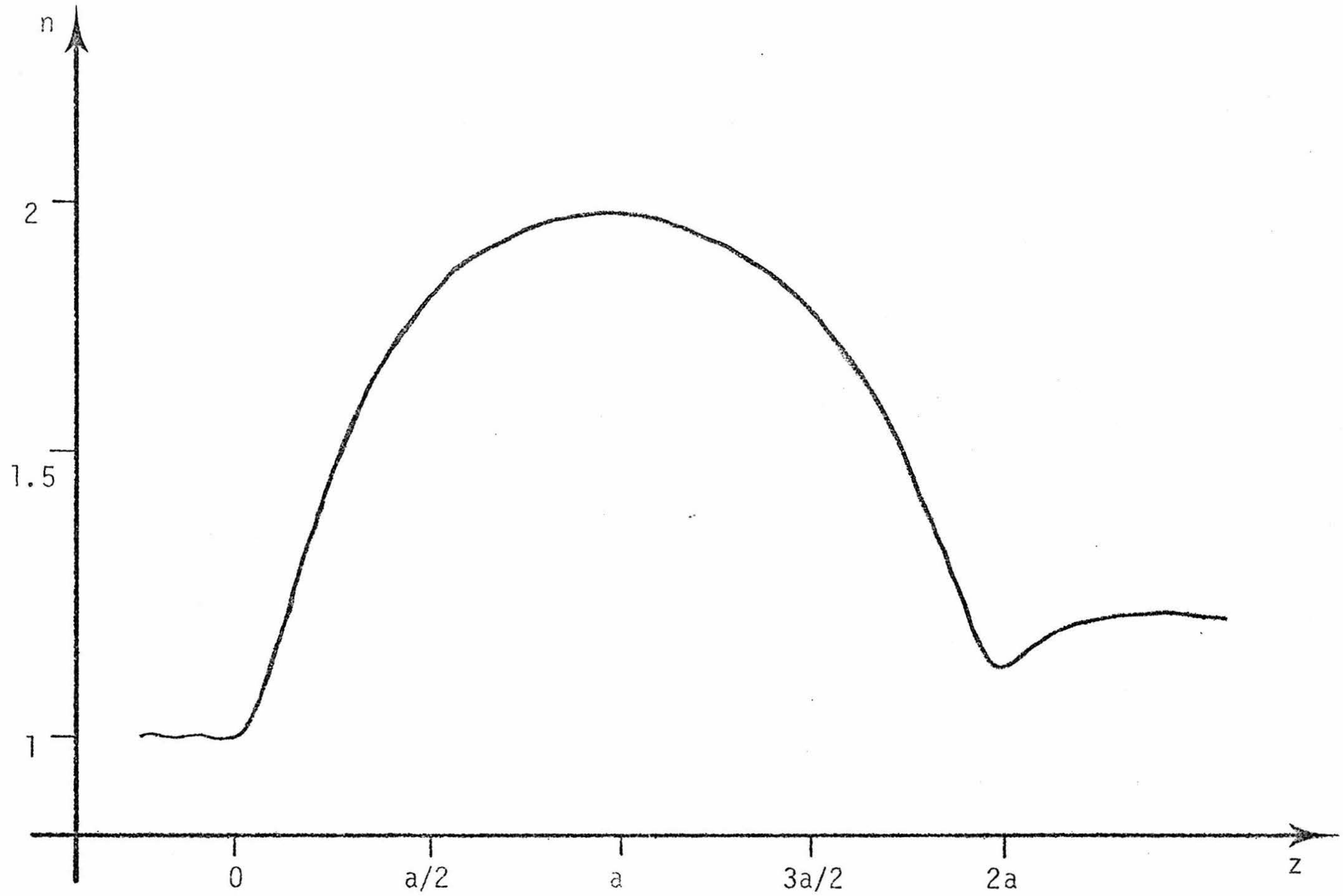
In the derivation of the almost inversion, the source function was defined to be

$$S(z) = \frac{1}{2}\,(dn/dz)/n_{avg} \quad . \qquad \text{(IV.18)}$$

This "source function" would be more accurately represented as

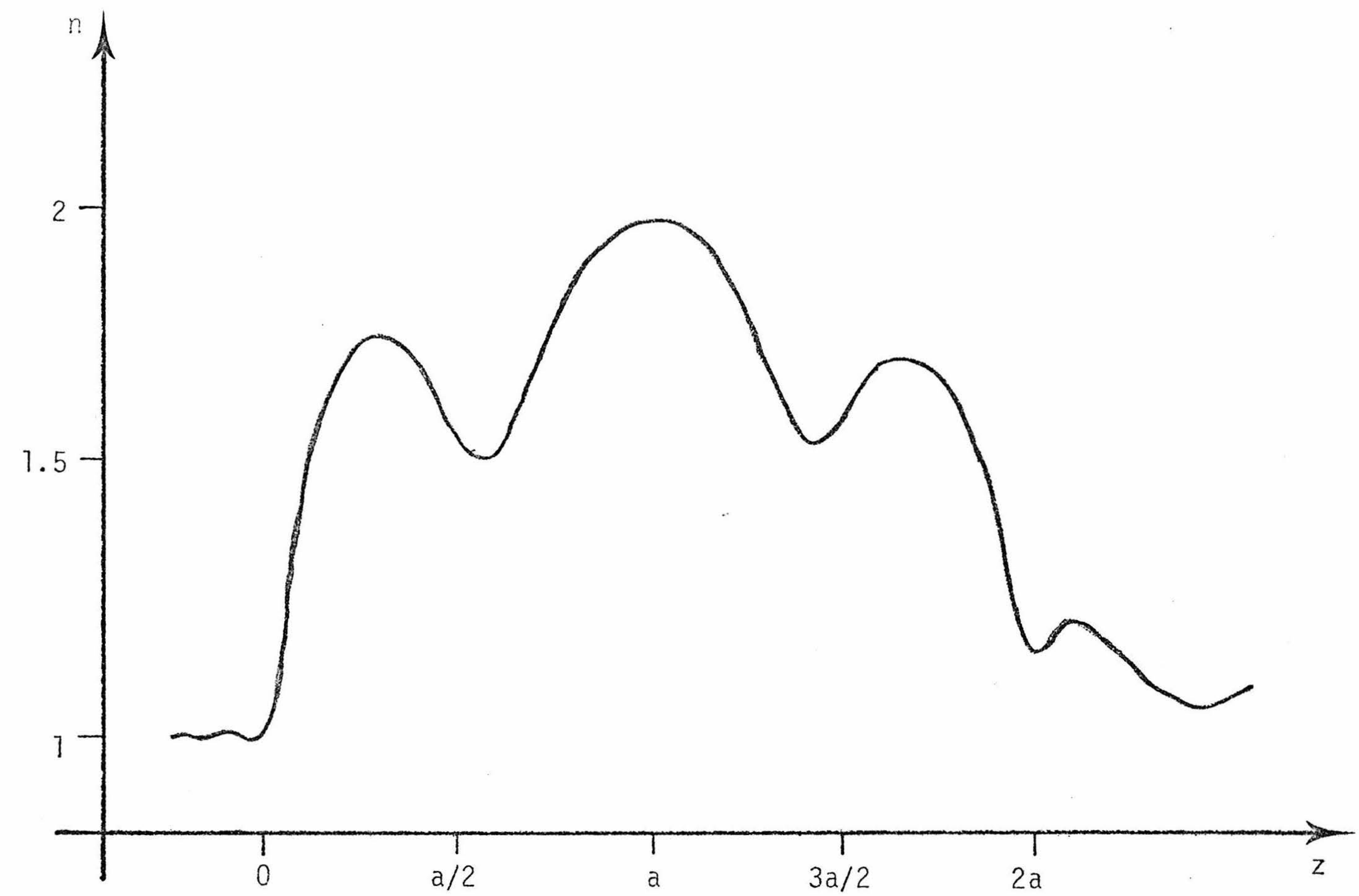
$$S(z) = \frac{1}{2}\,d[\ln(n)]/dz \qquad \text{(IV.39)}$$

for the large variation cases (but reduces to the same definition for



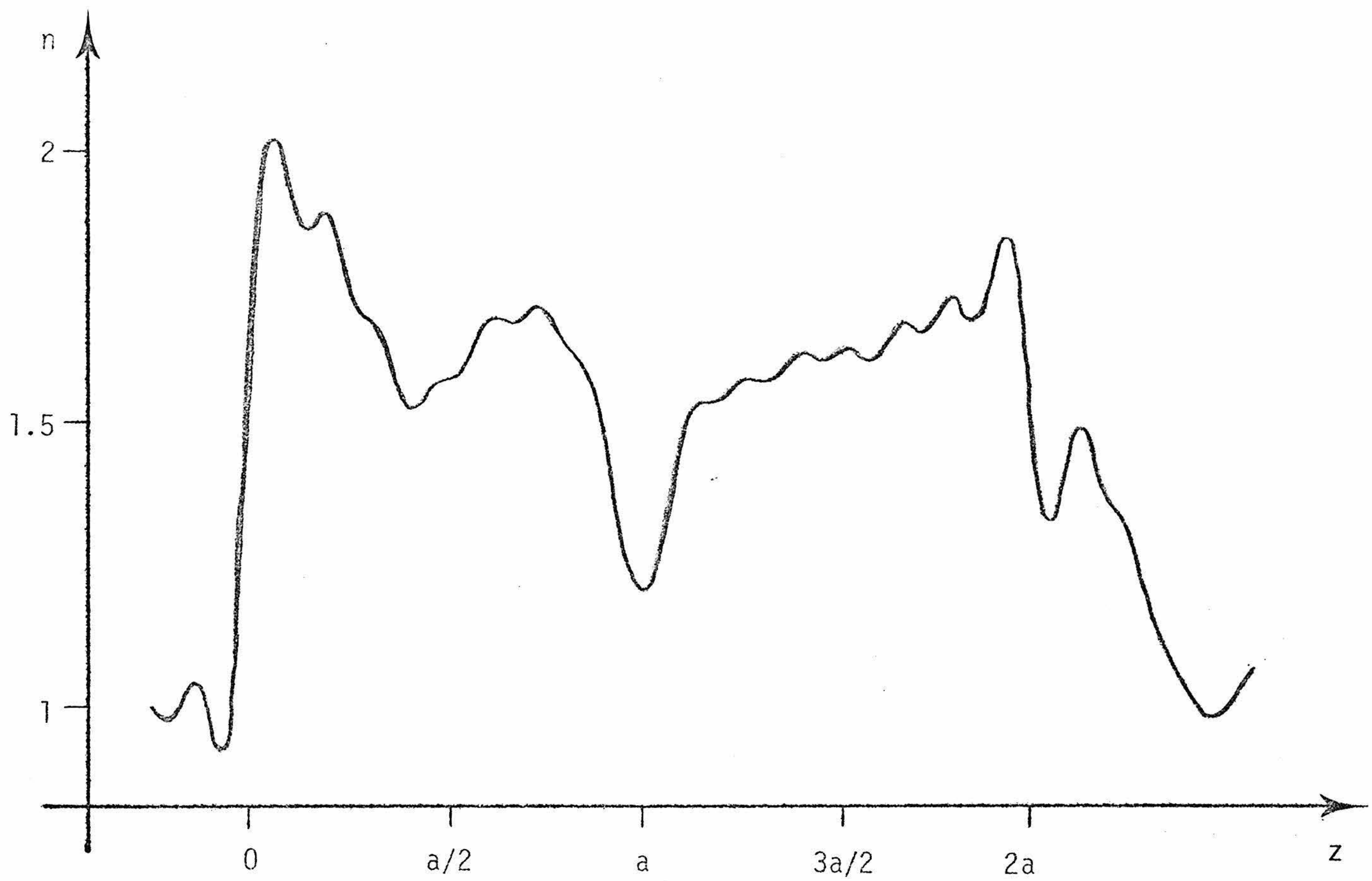
n(z) from Reflection Inversion (Reflection on Figure 16).

Figure 26



n(z) from Reflection Inversion (Reflection on Figure 17).

Figure 27



n(z) from Reflection Inversion (Reflection on Figure 18)

Figure 28

the small variation cases). If we use the second definition, then we will find $\ell n[n(z)]$ when we reconstruct. Of course, the over-all additive constant will be different (so that $\ell n(n)$ is matched in the left homogeneous region). Taking the exponential, the modified inversion becomes

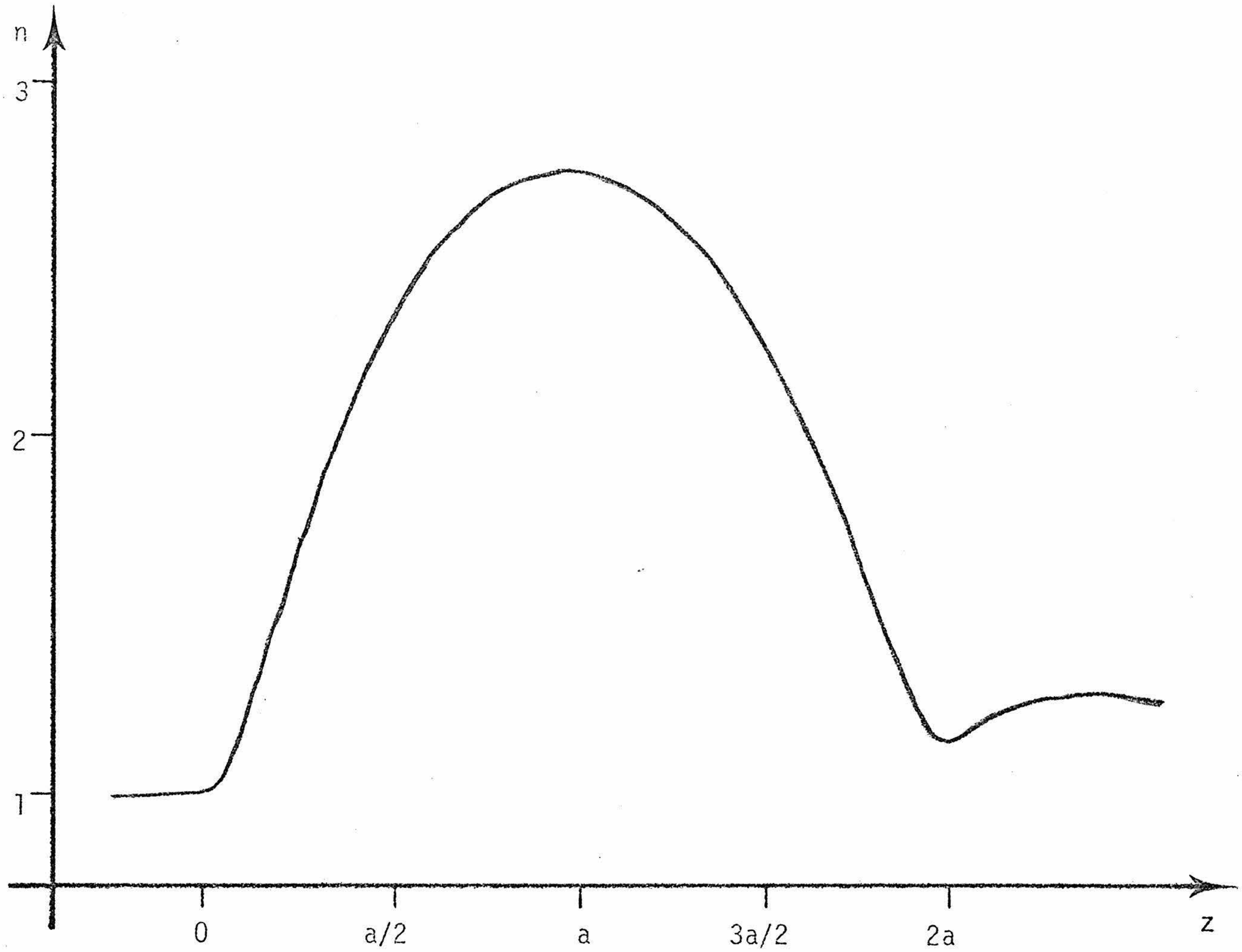
$$n(z) = n_0 \exp\left[\int_{-\infty}^{\infty} \frac{-i \exp(-i\kappa a/2)}{2\pi(\omega/c)} R(\omega/c) \exp(i\kappa z)\, d\kappa\right] , \qquad \text{(IV.40)}$$

where n_0 is a multiplicative constant which provides normalization based on $n(z)$ at a known point (presumably in the left homogeneous region where R would be measured).

The results of the modified inversion in the three cases are shown in Figures 29 through 31. The two improvements which may be seen over the original inversion are the increase in height (coming closer to the original n_{max}) and the somewhat subtler effect of shaping the peaks to conform more closely (but still not exactly) with the original functions. The factor of two in the slab length is still a problem, but this time it can more easily be solved because it is immediately apparent from the figures that the average index of refraction in the slabs is approximately two. The third case (with the discontinuities) is more or less hopeless with either method, but there is still some useful information about $n(z)$ in the reconstruction.

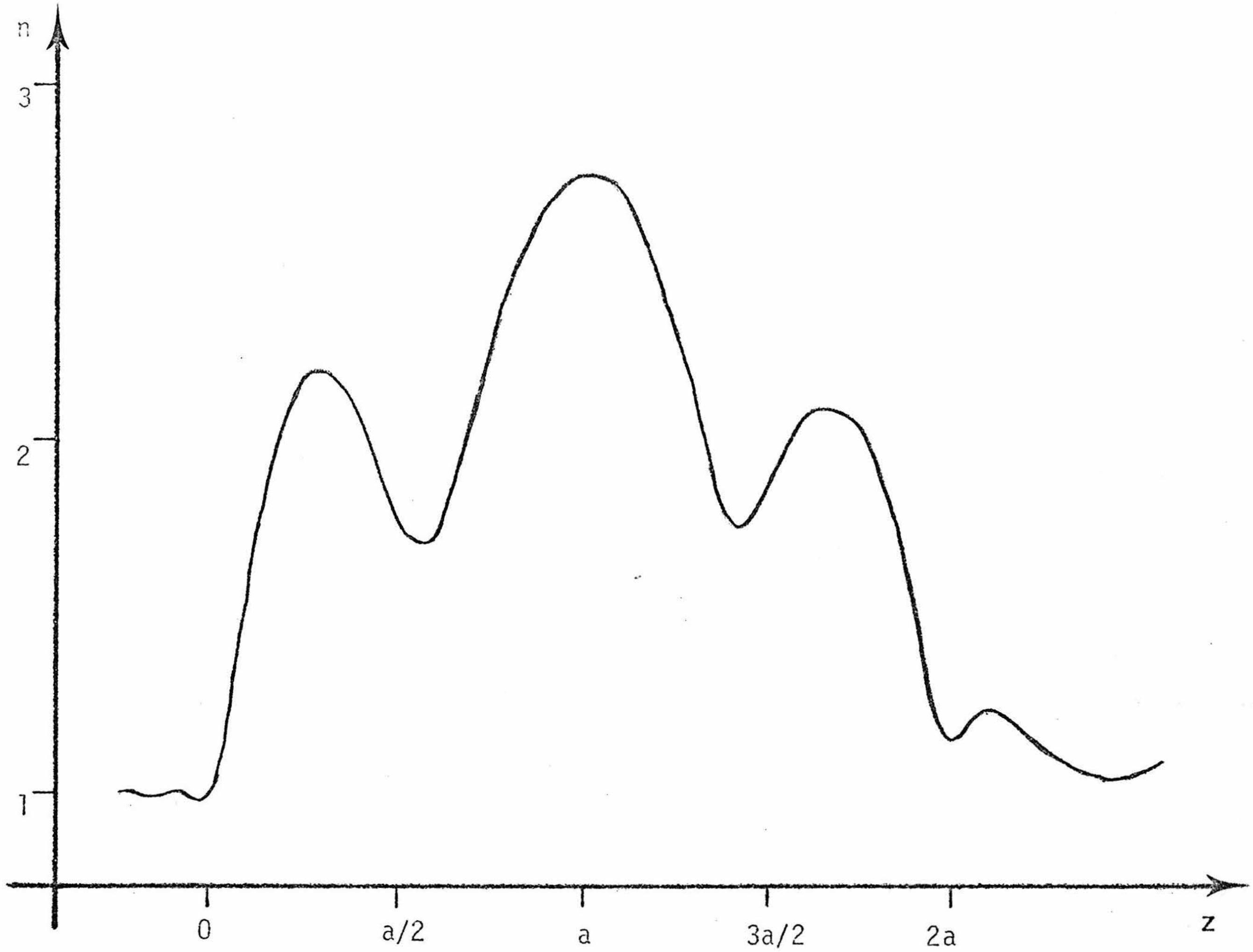
IV.6 Purely Periodic, Almost Homogeneous Media

If we are ever faced with the problem of a purely periodic, almost homogeneous medium, we are really in luck. For the purposes of this analysis, let us assume that the index of refraction in one cell is



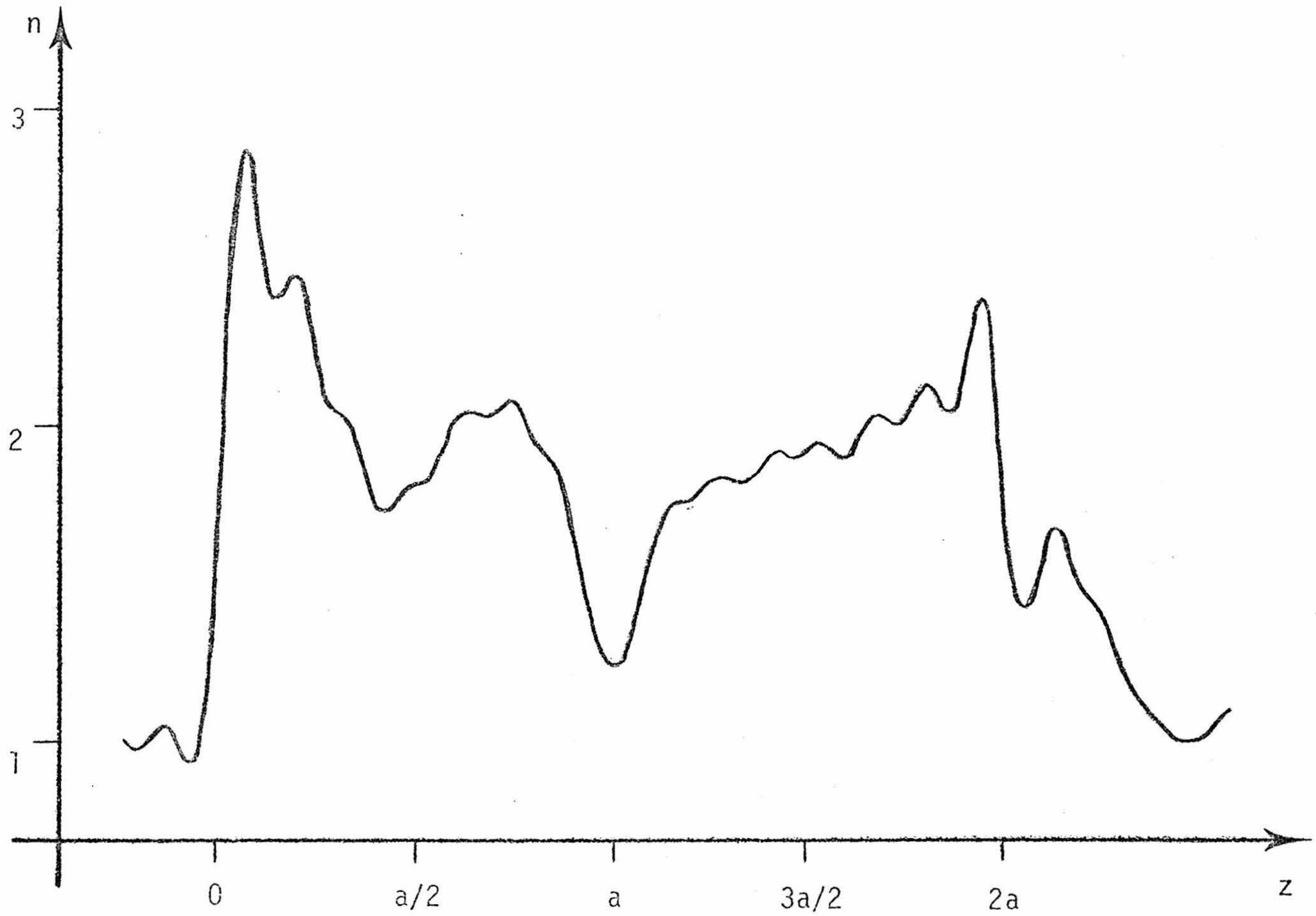
n(z) from Modified Reflection Inversion (Reflection on Figure 16)

Figure 29



n(z) from Modified Reflection Inversion (Reflection on Figure 17)

Figure 30



n(z) from Modified Reflection Inversion (Reflection on Figure 18)

Figure 31

given by

$$n(z) = n_{avg} + \chi \cos(2\pi z/a) \quad , \qquad (IV.41)$$

where "a" is now the cell size. The source function is

$$S(z) = -\frac{\pi\chi}{n_{avg}a} \sin(2\pi z/a) \quad . \qquad (IV.42)$$

Because of parity considerations, the Green's function integral reduces to

$$R(\omega/c) = \frac{2i\pi\chi}{n_{avg}a} \exp(ian_{avg}\omega/c) \int_0^{a/2} \sin(2\pi z/a) \sin(2zn_{avg}\omega/c)dz \ . \qquad (IV.43)$$

This definite integral is trivial. Evaluation of it gives

$$R(\omega/c) = \frac{i\pi\chi}{2n_{avg}} \exp(ian_{avg}\omega/c)\left[\frac{\sin(\pi - an_{avg}\omega/c)}{\pi - an_{avg}\omega/c} - \frac{\sin(\pi + an_{avg}\omega/c)}{\pi + an_{avg}\omega/c}\right] . \qquad (IV.44)$$

Since χ is small, R remains small for all values of (ω/c).

The PASS BANDS in this case will be very large and, since the reflection is very small, the propagation constant (real part) is given very closely by

$$\beta = n_{avg}(\omega/c) \qquad (IV.45)$$

everywhere. The STOP BANDS are expected to be located near the regions where $\sin(\phi_T) = \sin(\beta a) = 0$. The first STOP BAND brackets the propagation constant $\beta = \pi/a$, giving the free-space wavenumber at the first STOP BAND, $(\omega/c) = \pi/an_{avg}$. The magnitude of the reflection coefficient there is a maximum (therefore almost constant) and has the simple form

$$r = |R| = \pi\chi/2n_{avg} \quad . \tag{IV.46}$$

Based on this, the maximum imaginary component of β in the first STOP BAND is

$$\mathrm{Im}(\beta)_{max} = a^{-1} \ln\left[\frac{1}{\sqrt{1-r^2}} + \sqrt{\frac{1}{1-r^2} - 1}\right] \quad , \tag{IV.47}$$

or, expanding based on the "smallness" of r ,

$$\mathrm{Im}(\beta)_{max} = a^{-1}r = \pi\chi/2n_{avg}a \quad . \tag{IV.48}$$

The width of the first STOP BAND (from equation III.98) is

$$\Delta(\omega/c)_{STOP-1} = \pi\chi/n_{avg}^2 a \quad . \tag{IV.49}$$

A similarly straightforward analysis is not possible for STOP BANDS two and beyond, because the magnitude of the reflection coefficient is not nearly constant in the vicinity of

$$(\omega/c)_N = k_{STOP-N} = N\pi/an_{avg} \qquad (N \geq 2) \quad ; \tag{IV.50}$$

in fact, the reflection goes to zero, so we must be more careful in our investigation than for the first STOP BAND. The relevant parameter to calculate is η , given by

$$\eta = |\sin \phi_T|/r \quad , \tag{IV.51}$$

where we have added the absolute value operator for convenience. Suppose that the incident wave has a wavenumber near the STOP BAND expected wavenumber:

$$(\omega/c) = N\pi/an_{avg} + \xi/an_{avg} \qquad (N \geq 2) \quad . \tag{IV.52}$$

ξ is a small number. The magnitude of reflection is

$$r = \frac{\pi\chi}{2n_{avg}} \left[\frac{\sin[(N-1)\pi + \xi]}{(N-1)\pi + \xi} - \frac{\sin[(N+1)\pi + \xi]}{(N+1)\pi + \xi} \right] . \quad (IV.53)$$

This expression is easily expanded in terms of ξ to give (to first order in ξ):

$$r = \frac{\chi|\xi|}{n_{avg}(N^2-1)} . \quad (IV.54)$$

At the same time, the sine of the transmission phase is approximately

$$|\sin \phi_T| = |\sin(N\pi + \xi)| = |\xi| , \quad (IV.55)$$

giving the value of η near the STOP BAND expected wavenumber:

$$\eta = n_{avg}(N^2-1)/\chi . \quad (IV.56)$$

Since χ is small, η remains large (greater than one), which in turn means that the STOP BAND criterion is never satisfied for any expected STOP BANDS beyond the first. The $\sin(\phi_T) = 0$ condition is repeatedly met, but the magnitude of reflection goes to zero quickly enough for STOP BANDS two and up that they are reduced to mere ghosts, making themselves known only through very small perturbations in the (real) propagation constant near the STOP wavenumbers. Of course, as the reflection becomes non-infinitesimal, the STOP BANDS for all N will reappear. However, they will be <u>very</u> small compared with the first STOP BAND and very difficult to detect in an experiment; still mere ghosts of STOP BANDS. (This matter is treated further in Appendix II.)

This result should not be surprising, because the almost inversion should reproduce an infinite, purely sinusoidal index of refraction by way of a delta function for $R(\omega/c)$. On the other hand, it might not have been the case, since there are many cycles (instead of the few assumed in the almost Riccati equation derivation) and we have no a priori guarantee that the reflection will stay small, which we need to do a proper almost inversion.

The minimum value of η is found at the (expected) STOP BANDS. The result represented by equation (IV.56) and the theory of Y-orbits presented in Chapter III guarantee that the reflection will not become large for any values of (ω/c) except near the first STOP BAND for each sinusoidal term in $n(z)$. In general, a purely periodic index of refraction, with cell length a, may be written

$$n(z) = n_{avg} + \sum_{N=1}^{\infty} A_N \cos(2N\pi z/a + \delta_N) \quad . \tag{IV.57}$$

As long as the A_N's remain small, $n(z)$ is almost homogeneous in the cell, and we are out of the STOP BANDS for all terms, then the almost Riccati equation serves to calculate the reflection and transmission for one cell, from which the PASS BAND formula (III.81) may be used to calculate the (real) propagation constant. The propagation constant in one of the STOP BANDS may be calculated similarly with the STOP BAND formula (III.83).

IV.7 Almost Periodic Functions

Our main use of the properties of almost periodic functions will be confined to the idea that we wish to have a medium which is almost periodic (but not quite) and examine what, if anything, can be said about the propagation constant.

The first concern is with what an almost periodic function will "look like". Our prototype almost periodic function will be

$$f(x) = \cos(\kappa_\alpha x) + \cos(\kappa_\beta x) \quad , \tag{IV.58}$$

where κ_α and κ_β are not rational multiples of each other. We can see that this function cannot be purely periodic, because if we think that we have found a period, P , then

$$2 = f(0) = f(P) = \cos(\kappa_\alpha P) + \cos(\kappa_\beta P) \quad , \tag{IV.59}$$

which implies that

$$\kappa_\alpha P = 2M\pi \qquad \text{and} \qquad \kappa_\beta P = 2N\pi \quad , \tag{IV.60}$$

or

$$\kappa_\alpha/\kappa_\beta = M/N \quad , \tag{IV.61}$$

which contradicts the hypothesis about κ_α and κ_β . Of course, the "almost periodic" nature of $f(x)$ arises because we may always let

$$(\kappa_\alpha/\kappa_\beta) = (M/N)(1+\xi) \quad , \tag{IV.62}$$

where (M/N) is a rational approximation to $(\kappa_\alpha/\kappa_\beta)$ taken to any accuracy we desire, and ξ is a correspondingly small number. If this is the case, then any period near

$$P_\alpha = 2M\pi/\kappa_\alpha \qquad \text{and} \qquad P_\beta = 2N\pi/\kappa_\beta \tag{IV.63}$$

may be considered an "almost period."[12,13]

For a definite example, consider the function

$$f(x) = \cos(\pi x) + \cos(\sqrt{2}\pi x) \quad . \tag{IV.64}$$

This function is represented graphically in Figure 32 for the range of argument $x = 0$ to $x = 10$. We will let $\kappa_\beta = \pi$ and $\kappa_\alpha = \sqrt{2}\pi$. One "reasonable" rational approximation of $\kappa_\alpha/\kappa_\beta = \sqrt{2}$ is $7/5 = M/N$. Associated with this approximation are the "almost periods"

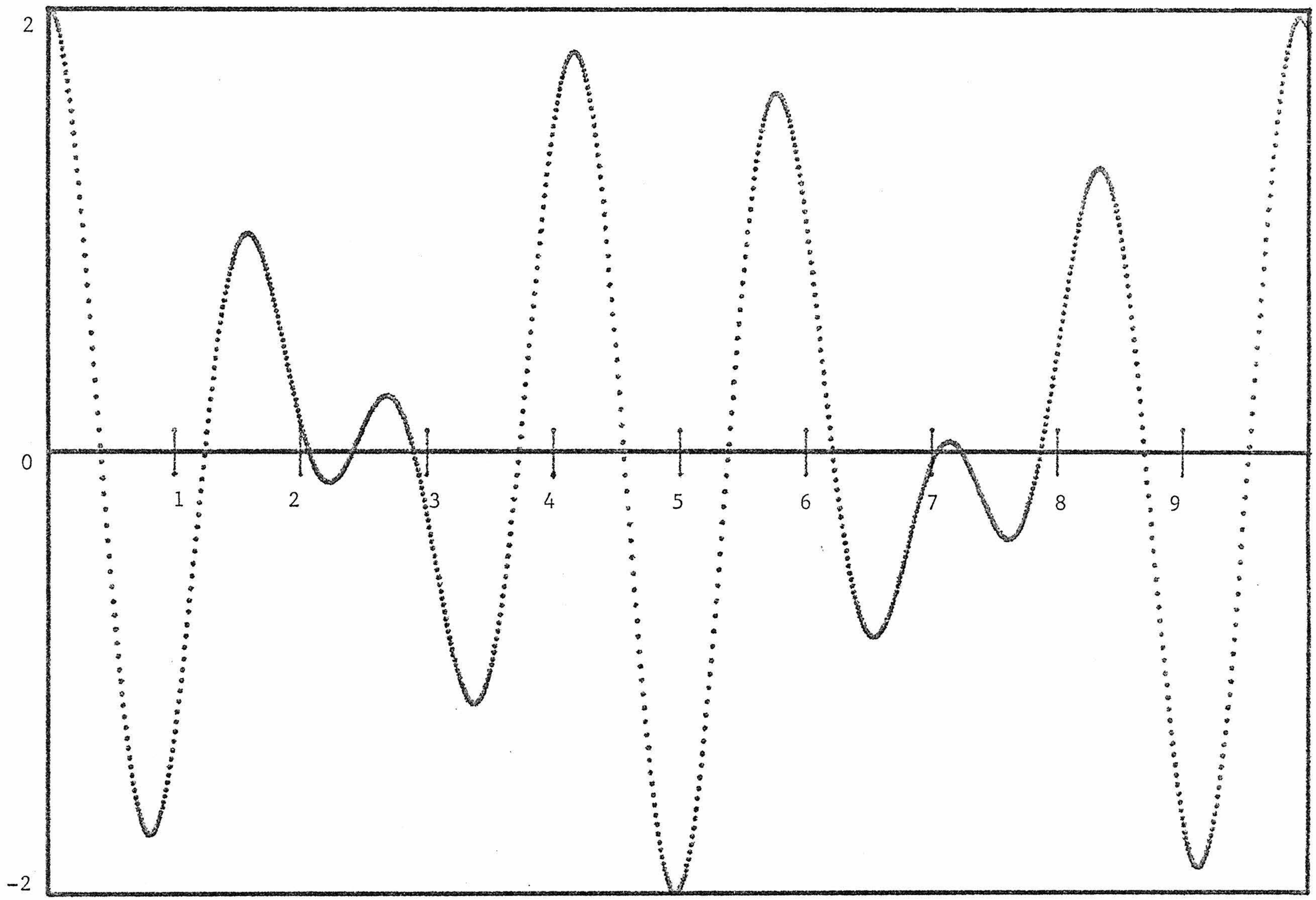
$$P_\alpha = 14\pi/\sqrt{2} = 7\sqrt{2} \quad \text{and} \quad P_\beta = 10\pi/\pi = 10 \quad . \tag{IV.65}$$

Examination of Figure 32 reveals that a number near 10 is indeed a good estimate of the first "almost period."

A better rational estimate of $\sqrt{2}$ is $(M/N) = 75/53 = 1.415 \cdots$. This approximation gives an "almost period" of $P_\beta = 106$. Figure 33 illustrates our almost periodic function in the region from $x = 100$ to $x = 110$. It is readily seen that $f(x)$ gets very close to 2 near $x = 106$. In fact, $f(x)$ reaches a maximum around $x = 106.04$, obtaining the value $f(106.04) = 1.986$. However, $f(x)$ quickly loses its similarity to $f(x-106.04)$, as may be seen by comparing $f(x)$ in the region between $x = 2$ and $x = 3$ (Figure 32) with $f(x)$ in the region between $x = 108$ and $x = 109$ (Figure 33).

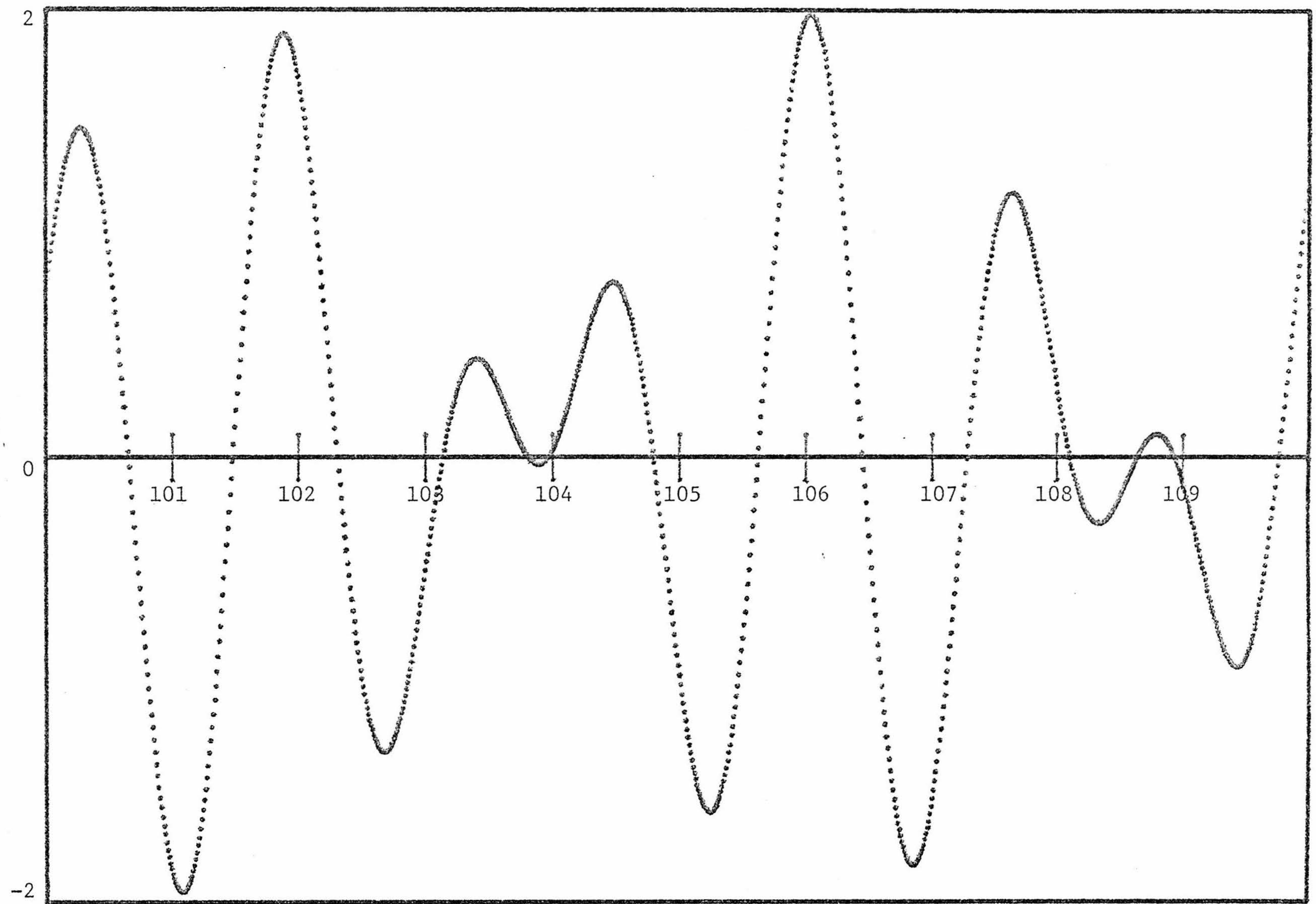
IV.8 STOP BANDS in Almost Periodic, Almost Homogeneous Media

All of the work has already been done to find the STOP BANDS in an almost periodic, almost homogeneous medium. Although the results are somewhat disappointing, since we would like to see something



$\cos(\pi x) + \cos(\sqrt{2}\pi x)$

Figure 32



$\cos(\pi x) + \cos(\sqrt{2}\pi x)$

Figure 33

"special" happen with almost periodic structures, they are comforting in the sense that it would be a big surprise if it really mattered (in a physical sense) whether the structure of a medium is periodic or only "almost periodic."

Let the index of refraction be given in the general case by

$$n(z) = n_{avg} + \sum_{\alpha=1}^{\infty} \chi_\alpha \cos(\kappa_\alpha z + \delta_\alpha) \quad , \tag{IV.66}$$

where the κ_α are "well-separated" constants (we will define this more precisely later), not necessarily rational multiples, and

$$\sum_{\alpha=1}^{\infty} \chi_\alpha << 1 \quad . \tag{IV.67}$$

There will be <u>one and only one</u> non-vanishing STOP BAND associated with each α , centered at

$$(\omega/c)_\alpha = \frac{1}{2} \kappa_\alpha / n_{avg} \quad , \tag{IV.68}$$

with width

$$\Delta(\omega/c)_\alpha = \frac{1}{2} \chi_\alpha \, \kappa_\alpha / n_{avg}^2 \quad , \tag{IV.69}$$

and maximum imaginary component of the propagation constant

$$\mathrm{Im}(\beta)_{max-\alpha} = \chi_\alpha \, \kappa_\alpha / 4n_{avg} \ . \tag{IV.70}$$

These results apply as long as condition (IV.67) holds and the STOP BANDS do not overlap; in other words

$$\mathrm{MIN}_\beta[|\kappa_\beta - \kappa_\alpha|] > \chi_\alpha \, \kappa_\alpha / n_{avg} \quad (\text{for all } \alpha) \quad . \tag{IV.71}$$

We need not worry about whether the κ's are or are not rational multiples of each other; the formulas work just as well in any event.

Chapter V

Conclusion

It has been the purpose of this thesis to make a general investigation of the properties of the invariant imbedding solution for electromagnetic wave propagation in general, periodic, almost homogeneous, and almost periodic media. While the analysis was (almost) exclusively made for electromagnetic waves, a section of the second chapter (Section II.7) extends all results to quantum mechanical waves, or in a more general sense to any wave which may be characterized by the standard wave equation (equation II.4).

Chapter II contained a reformulation of the invariant imbedding solution and the derivations of some very important general properties of the solution. These general properties, which related the reflection and transmission coefficients in two directions through the region of inhomogeneity, and related the minimum transmitted flux with the range of variation of the local wavenumber, provided the basis for Chapters III and IV.

Chapter III was a new way of looking at waves in a periodic medium. Assuming that we have already solved the problem of reflection and transmission for a single cell, an invariant imbedding recursion using the cell as the basic unit of recursion gave the propagation constant according to very simple formulas which did not involve any matrix operations. The computation of the PASS BANDS and STOP BANDS was seen to be completely trivial once the transmission and reflection coefficients for one cell are known as a function of frequency. The general treatment was extended to include absorptive media as well, although the sharp distinction between PASS and STOP BANDS was lost.

Chapter IV looked at those situations for which the reflection coefficient is small. In this case, a very simple exponential Green's function is available to transform the differential equation for the reflection coefficient into a definite integral. The reflection was also transformed back to get the original index of refraction function. It was possible simply to use this same method when the reflection was large, and we have seen that the resulting reconstructed index of refraction has the same general appearance as the original, although the fact that the reflection was not small made the shape progressively worse as the reflection failed to get smaller with increasing frequency. Finally, we used the small reflection theory to show that an almost periodic medium will not exhibit interference between structure constants to produce STOP BANDS at frequencies beyond the fundamental for each structure constant, in the limit as the variation in wavenumber with distance is small. We have not addressed ourselves to the problem of a large-variation type of almost periodic medium, since neither the methods of Chapter III nor Chapter IV are applicable to that case. It is very possible that there could be an interference effect between irrationally related structure constants to produce STOP BANDS at frequencies which are sums of integral multiples of the fundamental frequencies, when the variation in wavenumber with distance becomes large. This would be a very difficult thing to calculate, even numerically, since no finite slice of an almost periodic structure may be considered as representative of the whole, in contradistinction to a purely periodic medium.

Appendix I

Inductive Proof of Reciprocity Relations

We will prove inductively, using (II.32), (II.33), (II.66), and (II.67), the five reciprocity relationships:

1. $r' = r$ (AI.1)

2. $t' = (k_o/k_N)t$ (AI.2)

3 a) $r'^2 + (k_o/k_N)t^2 = 1$ (AI.3)

b) $r'^2 + (k_N/k_o)t'^2 = 1$ (AI.4)

4. $\phi_T = \phi_{T'}$ (AI.5)

5. $\phi_R + \phi_{R'} = \pm\pi + \phi_T + \phi_{T'}$ (AI.6)

Inductive Proof:

I. When there is only one interface (N=1), the transmission and reflection coefficients are

$$R = (k_1-k_o)/(k_1+k_o) \quad \text{(AI.7)}$$

$$T = 2k_1/(k_1+k_o) \quad \text{(AI.8)}$$

$$R' = (k_o-k_1)/(k_1+k_o) \quad \text{(AI.9)}$$

$$T' = 2k_o(k_1+k_o) \quad . \quad \text{(AI.10)}$$

All of the relations follow trivially.

II. Assume that we have shown (AI.1-6) for I interfaces. We wish to verify that they will also hold for I+1 interfaces. It will be convenient to use Method A (II.32 and II.33) to calculate R_{I+1} and T_{I+1}, and Method B (II.66 and II.67) to calculate R'_{I+1} and T'_{I+1}. We should note a change of notation (of subscripts) for method B due to the fact that we will be adding interfaces in the usual direction (that is, 1 to N) when calculating the reverse coefficients (R' and T').

Let

$$r_I = (k_{I+1}-k_I)/(k_{I+1}+ k_I) \qquad \text{(AI.11)}$$

$$t_I = 1 + r_I \qquad \text{(AI.12)}$$

$$r'_I = -r_I \qquad \text{(AI.13)}$$

$$t'_I = 1 + r'_I = 1 - r_I \qquad \text{(AI.14)}$$

$$\delta = \exp[ik_I(z_I-z_{I+1})] \qquad \text{(AI.15)}$$

The invariant imbedding recursions give

$$R_{I+1} = \frac{r_I + \delta^2 R_I}{1 + r_I\delta^2 R_I} \qquad \text{(AI.16)}$$

$$T_{I+1} = \frac{t_I\, \delta\, T_I}{1 + r_2\delta^2 R_I} \qquad \text{(AI.17)}$$

$$R'_{I+1} = \frac{R'_I + r'_I\delta^2(T'_I T_I - R'_I R_I)}{1 - r'_I\delta^2 R_I} \qquad \text{(AI.18)}$$

$$T'_{I+1} = \frac{t'_I\, \delta\, T'_I}{1 - r'_I\delta^2 R_I} \quad . \qquad \text{(AI.19)}$$

Straightforward calculation will verify the relationships.

1:

$$r = |R_{I+1}| = |r_I + \delta^2 R_I|/|1 + r_I\delta^2 R_I| \qquad \text{(AI.20)}$$

$$r' = R_{I+1} = \frac{|R'_I + r'_I\delta^2(T'_I T_I - R'_I R_I)|}{|1 + r_I\delta^2 R_I|} \quad . \qquad \text{(AI.21)}$$

We need only compare numerators as denominators are equal. From the previous step in the recursion, we have

$$T_I' = (k_o/k_I)\, T_I \qquad \text{(AI.22)}$$

$$R_I' R_I = -\exp(2i\phi_{T_I})|R_I|^2 \qquad \text{(AI.23)}$$

$$T_I' T_I - R_I' R_I = \exp(2i\phi_{T_I}) \quad . \qquad \text{(AI.24)}$$

Substitution of (AI.24) in (AI.21) and division by (AI.20) gives

$$r'/r = \frac{|R_I' - r_I \delta^2 \exp(2i\phi_{T_I})|}{|r_I + \delta^2 R_I|} \qquad \text{(AI.25)}$$

A little manipulation and the use of (AI.6) gives

$$r'/r = \frac{|-\delta^2 \exp(2i\phi_{T_I})| \cdot |r_I + \delta^{-2}|R_I| \exp(-i\phi_{R_I})|}{r_I + \delta^2\ |R_I|\ \exp(i\phi_{R_I})|} \quad . \qquad \text{(AI.26)}$$

Since $|\delta| = 1$ and $|x^*| = |x|$ for any complex number x ,

$$r'/r = \frac{1 \cdot |(r_I + \delta^2\ |R_I|\ \exp(i\phi_{R_I}))^*|}{|(r_I + \delta^2\ |R_I|\ \exp(i\phi_{R_I}))|} = 1 \quad . \qquad \text{(AI.27)}$$

(This would not work if k_I had a complex nature.)

2 and 4:

$$\frac{T'_{I+1}}{T_{I+1}} = \frac{(1 - r_I)\ \delta T_I'}{(1 + r_I)\ \delta T_I} = \frac{2k_I/(k_{I+1} + k_I)}{2k_{I+1}/(k_{I+1} + k_I)} \frac{k_o}{k_I} = \frac{k_o}{k_{I+1}} \quad . \qquad \text{(AI.28)}$$

(This works even if k_I is complex, but if it is, 2 and 4 must be combined.)

3a:

$$|R_{I+1}|^2 + (k_o/k_{I+1})|T_{I+1}|^2 = R^*_{I+1}R_{I+1} + (k_o/k_{I+1})T^*_{I+1}T_{I+1} =$$

$$\{\frac{(k_{I+1}-k_I)^2}{(k_{I+1}+k_I)^2} + \delta^2 R_I \frac{(k_{I+1}-k_I)}{(k_{I+1}+k_I)} + \delta^{-2}R^*_I \frac{(k_{I+1}-k_I)}{(k_{I+1}+k_I)} + R^*_I R_I$$

$$+ (\frac{k_o}{k_{I+1}}) \frac{4k^2_{I+1}}{(k_{I+1}+k_I)^2} (1 - R^*_I R_I)(\frac{k_I}{k_o})\} \div$$

$$\{1 + (\frac{k_{I+1}-k_I}{k_{I+1}+k_I})(\delta^{-2}R^*_I + \delta^2 R_I) + (\frac{k_{I+1}-k_I}{k_{I+1}+k_I})^2 R^*_I R_I\} =$$

$$\{\frac{(k_{I+1}-k_I)^2}{(k_{I+1}+k_I)^2} + \frac{4k_{I+1}k_I}{(k_{I+1}+k_I)^2} + \frac{(k_{I+1}-k_I)}{(k_{I+1}+k_I)} (\delta^2 R_I + \delta^{-2}R^*_I)$$

$$+ [\frac{(k_{I+1}+k_I)^2}{(k_{I+1}+k_I)^2} - \frac{4k_{I+1}k_I}{(k_{I+1}+k_I)^2}] R^*_I R_I\} \div$$

$$\{1 + \frac{(k_{I+1}-k_I)}{(k_{I+1}+k_I)} (\delta^2 R_I + \delta^{-2}R^*_I) + \frac{(k_{I+1}-k_I)^2}{(k_{I+1}+k_I)^2} R^*_I R_I\} = 1 . \quad (AI.29)$$

3b:

Follows immediately from 1, 2, and 3a.

5:

Let $-\theta_D = \arg(1 + r_I \delta^2 R_I)$ (AI.30)

$\psi = \arg(\delta)$. (AI.31)

Then

$$\phi_{T_{I+1}} = \psi + \phi_{T_I} + \theta_D \tag{AI.32}$$

$$\phi_{T'_{I+1}} = \psi + \phi_{T'_I} + \theta_D \qquad (= \phi_{T_{I+1}}) \quad . \tag{AI.33}$$

From the first step of the iteration, we have

$$\phi_{R_I} + \phi_{R'_I} = \pm\pi + \phi_{T_I} + \phi_{T'_I} \quad . \tag{AI.34}$$

With this, equation (AI.18) reduces to

$$R'_{I+1} = \frac{R'_I - r_I\delta^2\exp(2i\phi_{T_I})}{1 + r_I\delta^2\, R_I} \tag{AI.35}$$

Let

$$\theta_N = \arg(r + \delta^2 R_I) \quad . \tag{AI.36}$$

Then

$$\phi_{R_{I+1}} = \theta_N + \theta_D \quad . \tag{AI.37}$$

If we examine the numerator in (AI.35), we can manipulate it with the help of 1, 4, and (AI.34) to read:

$$R'_I - r_I\delta^2\exp(2i\phi_{T_I}) = -\delta^2\exp(2i\phi_{T_I})[|R_I|\delta^{-2}\exp(-i\phi_{R_I}) + r_I]. \tag{AI.38}$$

Then

$$\phi_{R'_{I+1}} = \pm\pi + 2\psi + 2\phi_{T_I} + [-\theta_N] + \theta_D \quad . \tag{AI.39}$$

We finally obtain the desired result:

$$\phi_{R'_{I+1}} + \phi_{R_{I+1}} = \theta_N + \theta_D + 2\phi_{T_I} \pm\pi - \theta_N + \theta_D + 2\psi$$

$$= \pm\pi + \phi_{T_{I+1}} + \phi_{T'_{I+1}} \quad . \qquad \text{(AI.40)}$$

(This proof depends heavily on the fact that $|\delta| = 1$, or k_I real.)

This completes the inductive proof.

Appendix II

Careful Analysis of Higher Order Stop Bands

We have seen in Section IV.6 that the magnitude of reflection for wavenumber

$$(\omega/c) = N\pi/an_{avg} + \xi/an_{avg} \qquad (N \geq 2) \quad , \tag{IV.52}$$

where ξ is a small number, is approximately

$$r = \chi|\xi|/n_{avg}(N^2-1) \quad . \tag{IV.54}$$

In that section we assumed that the phase of transmission was

$$\phi_T = N\pi + \xi \quad , \tag{AII.1}$$

which is the basic approximation from equation (II.47). However, since the reflection vanishes where we expect the STOP BANDS, we should compute a correction based on equation (II.47) for $n(z)$ real:

$$\phi_T = N\pi + \xi + \frac{1}{2}\int_{-a/2}^{a/2} \mathrm{Im}[R(z)]\,\frac{(dn/dz)}{n_{avg}}\,dz \quad . \tag{AII.2}$$

The function $R(\omega/c,z)$ is a simple matter to calculate with the almost Riccati equation. Recall that $R(\omega/c,a/2) = 0$ and $R(\omega/c,-a/z) = R(\omega/c)$. The integrated form of $R(\omega/c,z)$ is

$$R(\omega/c,z) = \frac{\pi\chi}{n_{avg}a}\exp[-2izn_{avg}\omega/c]\int_z^{a/2} \sin(2\pi z'/a)\exp[2iz'n_{avg}\omega/c]dz' . \tag{AII.3}$$

For convenience, we let

$$\alpha = n_{avg} a(\omega/c) = N\pi + \xi \quad . \qquad \text{(AII.4)}$$

The imaginary part of $R(\omega/c,z)$ is easily integrated to become

$$\mathrm{Im}[R(\omega/c,z)] = \frac{\pi\chi}{4n_{avg}} [\cos(2\alpha z/a)(\frac{\sin(\pi-\alpha)}{\pi-\alpha} - \frac{\sin(\pi+\alpha)}{\pi+\alpha}$$

$$- \frac{\sin[2(\pi-\alpha)\frac{z}{a}]}{\pi-\alpha} + \frac{\sin[2(\pi+\alpha)\frac{z}{a}}{\pi+\alpha}$$

$$- \sin(2\alpha z/a)(\frac{-\cos(\pi-\alpha)}{\pi-\alpha} - \frac{\cos(\pi+\alpha)}{\pi+\alpha}$$

$$+ \frac{\cos[2(\pi-\alpha)\frac{z}{a}]}{\pi-\alpha} + \frac{\cos[2(\pi+\alpha)\frac{z}{a}]}{\pi+\alpha})]. \qquad \text{(AII.5)}$$

One more integral will give us the desired final answer. The (dn/dz) term is a sine, so some of the terms in the integral will drop out by parity considerations, others will cancel. The result is

$$\phi_T = N\pi + \xi + \frac{\pi^2\chi^2}{4n^2_{avg}} [\frac{2}{\pi+\alpha} - \frac{2}{\pi-\alpha} + (\frac{\cos(\pi-\alpha)}{\pi-\alpha} + \frac{\cos(\pi+\alpha)}{\pi+\alpha})$$

$$\times (\frac{\sin(\pi-\alpha)}{\pi-\alpha} - \frac{\sin(\pi+\alpha)}{\pi+\alpha})] \quad . \qquad \text{(AII.6)}$$

Expanding in terms of ξ and retaining only low-order terms in ξ and χ^2, we obtain

$$\phi_T = N\pi + \xi + \frac{N\pi\chi^2}{(N^2-1)n^2_{avg}} + \cdots \quad . \qquad \text{(AII.7)}$$

Equation (AII.7) implies that the STOP BANDS are somewhat shifted from $N\pi/an_{avg}$.* They will be extremely small, so r will be approximately

*If we let $\xi = -(N\pi\chi^2)/[(N^2-1)n^2_{avg}]$, we will be right in the middle of the STOP BAND.

constant, and equal to

$$r = \frac{n\pi\chi^3}{(N^2-1)^2 n_{avg}^3} \quad . \tag{AII.8}$$

The maximum imaginary component of β is

$$Im(\beta)_{max} = \frac{N\pi\chi^3}{(N^2-1)^2 a n_{avg}^3} \tag{AII.9}$$

and the width of the STOP BAND is approximately

$$\Delta(\omega/c)_{STOP-N} = \frac{2N\pi\chi^3}{(N^2-1)^2 a n_{avg}^4} \quad . \tag{AII.10}$$

Compared with the first STOP BAND, since χ is a small number, these are "ghosts" indeed.

References

1. Ambarzumian, V. A., "Diffuse Reflection of Light by a Foggy Medium," Comptes Rendus (Doklady) de L'Academy des Sciences de l'URSS 38, 229-232 (1943).
2. Adams, R. N. and E. D. Denman, Wave Propagation and Turbulent Media (New York: American Elsevier, 1966).
3. Bellman, R. and R. Kalaba, "Invariant Imbedding and Wave Propagation in Stochastic Media," in Electromagnetic Wave Propagation, Desirant and Michaels (eds.) (New York: Academic Press, 1960), pp. 243-253.
4. Papas, C. H., "Plane Inhomogeneous Dielectric Slab," Caltech Antenna Laboratory Note, March 1954.
5. Kritikos, H. N., K.S.H. Lee, and C. H. Papas, "Electromagnetic Reflectivity of Non-Uniform Jet Streams, Caltech Antenna Laboratory Report, August 1966.
6. Carlisle, G. W., "Reflection from and Transmission through a Plane Inhomogeneous Medium," Douglas Paper No. 4148, October 1966.
7. Bellman, R. and G. Wing, An Introduction to Invariant Imbedding (New York: Wiley, 1975).
8. Klein, M. V., Optics (New York: Wiley, 1970).
9. Sommerfeld, A., Optics (New York: Academic Press, 1964).
10. Brillouin, L., Wave Propagation in Periodic Structures (New York: Dover, 1953).
11. Jaggard, D. L. and G. A. Evans, "Coupled Waves and Floquet Approach to Periodic Structures," Caltech Antenna Laboratory Report, August 1975.

12. Bohr, H., Almost Periodic Functions (New York: Chelsea, 1951).

13. Besicovitch, A. S., Almost Periodic Functions (Cambridge: University Press, 1932).

14. Schiff, L.I., Quantum Mechanics (New York: McGraw-Hill, 1968).