Transient Response of Two-Dimensional Cantilevered Semi-Infinite and Finite Elastic Plates, Subjected to Base Motions

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ABSTRACT

This research is concerned with the response of a two-dimensional, isotropic, homogeneous, elastic, cantilevered plate subjected to a step transverse velocity at the base. The investigation uses a method by Miklowitz which is based on a double Laplace transform and a boundedness condition on the solution.

The case of a semi-infinite plate is solved, for long-time, to find the shear and normal stresses at the base. The solution in the interior of the plate is shown to agree with that obtained by the Bernoulli-Euler approximate theory. The solution is then extended to the case of the finite length plate, with traveling wave and vibrational forms of the solution being found for the interior of the plate.

At the base of the plate the investigation shows that the normal stress is singular at the corners while the shear stress is non-singular.

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INTRODUCTION

Analysis of semi-infinite waveguides based on the equations of motion for a linear elastic, homogeneous, isotropic medium is a subject of long standing interest. Recently a method has been developed by Miklowitz for solving nonseparable elastic waveguide problems involving nonmixed edge or end conditions. For the semi-infinite waveguide, the method uses a Laplace transform on the propagation coordinate, and a related boundedness condition on the solution which generates integral equations for the edge unknowns (displacements and strains). Solution of these integral equations determines the formal solution to a problem. The first problem solved by Miklowitz [1] was a symmetrically loaded waveguide with nonmixed displacement end conditions, i.e. a cantilevered semi-infinite plate. Further details may be found in Miklowitz [2], [3]. Sinclair and Miklowitz [4] extended the method to non-mixed symmetric stress end conditions. More recently, they have also extended the technique to antisymmetric stress end conditions [5]. They found the solution to the problem of the semi-infinite plate under a sudden end moment and zero end shear stress. Long-time information for the near and far field was obtained. References for other work on plates with non-mixed edge conditions are given by Miklowitz in [1].

In the current work, the foregoing general ideas have been extended to the finite waveguide. Here the essential differences are that a finite Laplace transform on the propagation coordinate replaces the one-sided Laplace transform for the semi-infinite waveguide, and a related entirety condition on the transformed solution replaces the above-mentioned boundedness condition.

To solve the problem of the cantilevered finite length plate, the solution of the problem for a similar semi-infinite plate is needed. So the first case solved here is the problem of a cantilevered semi-infinite plate, subjected to a step transverse velocity at the base where the normal displacement is assumed to be zero. The integral equations resulting from the boundedness condition were solved for long-time to yield the shear and normal strains at the base, with the latter becoming singular at the corners. The exact theory solution and the Euler-Bernoulli approximate theory solution are shown to agree for the long-time-near-field region away from the base.

For the finite length cantilevered plate, the solution obtained from the Euler-Bernoulli approximate theory is used to reduce the entirety condition to the same set of equations that resulted from the boundedness condition for the semi-infinite plate. The strains at the base are shown to be the strains at the base for the semi-infinite plate multiplied by a reflection function. The traveling wave and vibrational forms of the solution are found for the interior of the plate, away from the base.

I. THE SEMI-INFINITE PLATE

1. Statement of the Problem

To solve the finite cantilevered plate problem, it is first necessary to find the solution of the semi-infinite cantilevered plate. This problem is shown in Fig. 1. A homogeneous, isotropic, linear elastic plate in plane strain of width 2h is built into a rigid base, and suddenly this base is given a uniform velocity in the width direction. The problem is formulated as a standard plane strain elastodynamic boundary value problem. Displacements u and v are taken to be in the x and y directions, respectively. The governing equations are

$$c_d^2 u_{xx}(x,y,t) + \left(c_d^2 - c_s^2\right) v_{xy}(x,y,t) + c_s^2 u_{yy}(x,y,t) = u_{tt}(x,y,t) \quad ,$$

$$c_s^2 v_{xx}(x,y,t) + \left(c_d^2 - c_s^2\right) u_{xy}(x,y,t) + c_d^2 v_{yy}(x,y,t) = v_{tt}(x,y,t) \quad , \tag{1.1}$$

$$\sigma_{xx}(x,y,t) = \mu\left[k^2 u_x(x,y,t) + (k^2-2) v_y(x,y,t)\right] \quad ,$$

$$\sigma_{yy}(x,y,t) = \mu\left[(k^2-2) u_x(x,y,t) + k^2 v_y(x,y,t)\right] \quad , \tag{1.2}$$

$$\sigma_{xy}(x,y,t) = \mu\left[v_x(x,y,t) + u_y(x,y,t)\right] \quad ,$$

for $x > 0$, $-h < y < h$, $t > 0$. Here $c_d^2 = \frac{\lambda+2\mu}{\rho}$ and $c_s^2 = \frac{\mu}{\rho}$ are, respectively, the dilational and equivoluminal body wave speeds, λ and μ

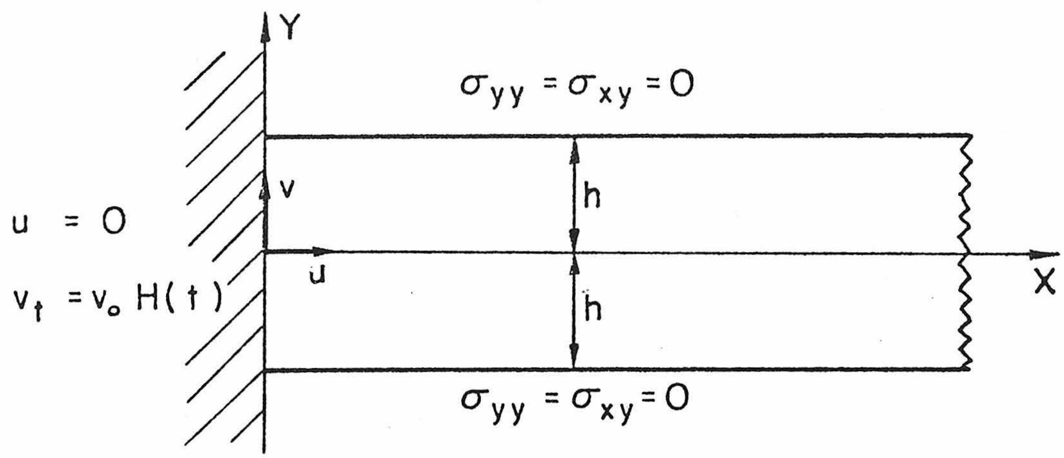

Fig. 1 Coordinates, Displacements and Boundary Conditions for the Semi-Infinite Plate in Plane Strain.

are the Lamé constants, ρ is the mass density and $k^2 = c_d^2/c_s^2$. Subscripts in this work, when associated with displacements, indicate differentiation; but when associated with stresses identify the component in the usual way.

Initial and boundary conditions are

$$u(x,y,0) = u_t(x,y,0) = v(x,y,0) = v_t(x,y,0) \quad \text{for } x>0,\ -h\le y\le h, \tag{1.3}$$

and

$$\left.\begin{aligned} u(0,y,t) &= 0 \\ v_t(0,y,t) &= v_0 H(t) \end{aligned}\right\} \text{for } -h \le y \le h,\ t \ge 0 \quad , \tag{1.4a}$$

$$\sigma_{yy}(x,\pm h,t) = \sigma_{xy}(x,\pm h,t) = 0 \quad \text{for } x>0,\ t\ge 0 \quad . \tag{1.4b}$$

The radiation conditions are

$$\lim_{x\to\infty} \left\{\begin{matrix} u, u_x, \text{ etc.} \\ v, v_x, \text{ etc.} \end{matrix}\right\} = 0 \quad \text{for } -h \le y \le h,\ t \ge 0 \quad . \tag{1.5}$$

The problem is an antisymmetric (flexural) one with respect to the midplane $y = 0$. It models a very tall building whose rigid base (ground) suddenly moves horizontally but not vertically. The problem is one of wave radiation into the plate, and no interaction with the base, except for wave reflection there.

2. Formal Solution

The problem is decomposed into a rigid body motion and a residual problem. The rigid body motion is

$$\left.\begin{aligned} u(x,y,t) &= 0 \\ v(x,y,t) &= v_0 t \end{aligned}\right\} \text{ for } x \geq 0,\ -h \leq y \leq h,\ t \geq 0 \quad . \tag{1.6}$$

The residual problem must satisfy the following initial and boundary conditions

$$\left.\begin{aligned} u(x,y,0) &= u_t(x,y,0) = v(x,y,0) = 0 \\ v_t(x,y,0) &= -v_0 \end{aligned}\right\} \text{ for } x > 0,\ -h \leq y \leq h \quad , \tag{1.7}$$

$$u(0,y,t) = v(0,y,t) = 0 \text{ for } -h \leq y \leq h,\ t \geq 0 \quad , \tag{1.8a}$$

$$\sigma_{yy}(x,\pm h,t) = \sigma_{xy}(x,\pm h,t) = 0 \text{ for } x > 0,\ t \geq 0 \quad . \tag{1.8b}$$

The radiation conditions now are

$$\left.\begin{aligned} &\lim_{x \to \infty} \left\{ v \right\} = -v_0 t \\ &\lim_{x \to \infty} \left\{ \begin{matrix} u,\ u_x,\ \text{etc.} \\ v_x,\ \text{etc.} \end{matrix} \right\} = 0 \end{aligned}\right\} \text{ for } -h \leq y \leq h,\ t \geq 0 \quad . \tag{1.9}$$

The residual problem has the same form as the ones considered in [5]. The formal solution can be obtained in exactly the same manner as was done in the first part of that paper. This gives (see Eq. (19) of [5])

$$u(x,y,t) = \frac{1}{2\pi i}\int_{Br_p} \bar{u}(x,y,p)e^{pt}dp \quad ,$$

$$v(x,y,t) = \frac{1}{2\pi i}\int_{Br_p} \bar{v}(x,y,p)e^{pt}dp \quad ,$$

$$\bar{u}(x,y,p) = \frac{1}{2\pi i}\int_{Br_s} \tilde{\bar{u}}(s,y,p)e^{sx}ds \quad , \tag{1.10}$$

$$\bar{v}(x,y,p) = \frac{1}{2\pi i}\int_{Br_s} \tilde{\bar{v}}(s,y,p)e^{sx}ds \quad ,$$

where Br_p and Br_s are the Bromwich contours in the p- and s-planes, respectively, and

$$\tilde{\bar{u}}(s,y,p) = \tilde{\bar{u}}^c(s,y,p) + \tilde{\bar{u}}^p(s,y,p) \quad ,$$
$$\tilde{\bar{v}}(s,y,p) = \tilde{\bar{v}}^c(s,y,p) + \tilde{\bar{v}}^p(s,y,p) \quad , \tag{1.11a}$$

$$\tilde{\bar{u}}^c(s,y,p) = C_1(s,p)\sinh\alpha y + C_2(s,p)\sinh\beta y \quad ,$$
$$\tilde{\bar{v}}^c(s,y,p) = \frac{\alpha}{s}C_1(s,p)\cosh\alpha y - \frac{s}{\beta}C_2(s,p)\cosh\beta y \quad , \tag{1.11b}$$

$$\tilde{\tilde{u}}^{p}(s,y,p) = \frac{1}{k_s^2}\int_0^y \left\{\left[\frac{s^2}{\alpha}\sinh\alpha(y-y')+\beta\sinh\beta(y-y')\right]g(s,y',p)\right.$$

$$\left.+k^2 s\left[\cosh\alpha(y-y')-\cosh\beta(y-y')\right]h(s,y',p)\right\}dy'$$

$$\tilde{\tilde{v}}^{p}(s,y,p) = \frac{1}{k_d^2}\int_0^y \left\{\left[\alpha\sinh\alpha(y-y')+\frac{s^2}{\beta}\sinh\beta(y-y')\right]h(s,y',p)\right. \qquad (1.11c)$$

$$\left.+\frac{s}{k^2}\left[\cosh\alpha(y-y')-\cosh\beta(y-y')\right]g(s,y',p)\right\}dy' \quad ,$$

$$C_1(s,p) = -\frac{s}{L(s,p)}\left[k^2\left(2s^2-k_s^2\right)\cosh\beta h\cdot I(s,p)+2s\beta\sinh\beta h\cdot J(s,p)\right] \quad ,$$

$$(1.11d)$$

$$C_2(s,p) = -\frac{-\beta}{L(s,p)}\left[2k^2 s\alpha\cosh\alpha h\cdot I(s,p)-\left(2s^2-k_s^2\right)\sinh\alpha h\cdot J(s,p)\right] \quad ,$$

$$I(s,p) = \frac{1}{k_s^2} \int_0^h \left\{ \frac{s}{k^2} \left[\left(2s^2 - k_s^2\right) \frac{\sinh \alpha(h-y)}{\alpha} + 2\beta \sinh \beta(h-y) \right] g(s,y,p) \right.$$

$$\left. + \left[\left(2s^2 - k_s^2\right) \cosh \alpha(h-y) - 2s^2 \cosh \beta(h-y) \right] h(s,y,p) \right\} dy$$

$$+ \left(\frac{k^2 - 2}{k^2} \right) \bar{u}(0,h,p) \quad ,$$

(1.11e)

$$J(s,p) = \frac{1}{k_s^2} \int_0^h \left\{ \left[2s^2 \cosh \alpha(h-y) - \left(2s^2 - k_s^2\right) \cosh \beta(h-y) \right] g(s,y,p) \right.$$

$$\left. + \frac{k^2 s}{\beta} \left[2\alpha\beta \sinh \alpha(h-y) + \left(2s^2 - k_s^2\right) \sinh \beta(h-y) \right] h(s,y,p) \right\} dy$$

$$- \bar{v}(0,h,p) \quad ,$$

$$g(s,y,p) = k^2 \left[s\bar{u}(0,y,p) + \bar{u}_x(0,y,p) \right] + (k^2 - 1)\bar{v}_y(0,y,p) \quad ,$$

(1.11f)

$$h(s,y,p) = \frac{v_0}{s c_d^2} + \frac{1}{k^2} \left[s\bar{v}(0,y,p) + \bar{v}_x(0,y,p) + (k^2 - 1)\bar{u}_y(0,y,p) \right] \quad ,$$

$$L(s,p) = 4s^2 \alpha\beta \cosh \alpha h \sinh \beta h + \left(2s^2 - k_s^2\right)^2 \sinh \alpha h \cosh \beta h \quad . \qquad (1.12)$$

Here $\alpha = \sqrt{k_d^2 - s^2}$, $\beta = \sqrt{k_s^2 - s^2}$, $k_d = \frac{p}{c_d}$, $k_s = \frac{p}{c_s}$ and p and s are the transform parameters for the time and x Laplace transforms, respectively. A bar over a quantity indicates that it has been Laplace transformed with respect to t while a tilda indicates that it has been Laplace transformed with respect to x. It should be noted that the first term in h(s, y, p) comes from the nonzero initial condition, which is the second of (1.7), and hence is not present in [5].

L(s, p) is the generalized Rayleigh-Lamb frequency equation for antisymmetric harmonic waves. Define $s_j(p)$ as the roots of

$$L[s_j(p), p] = 0 \quad . \tag{1.13}$$

Then, as shown in [1], [5], g(s, y, p) and h(s, y, p) must, for $\mathrm{Re}[s_j(p)] > 0$, satisfy the following boundedness condition (see pp. 8-12 in [1] and (22) in [5]).

$$0 = \begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} \left\{ \frac{1}{k_s^2} \int_0^h \left\{ \left[\left(2s_j^2 - k_s^2\right) \frac{\cosh \alpha_j y}{\cosh \alpha_j h} - 2s_j^2 \frac{\cosh \beta_j y}{\cosh \beta_j h} \right] h(s_j, y, p) \right. \right.$$

$$\left. - \frac{s_j}{k^2 \alpha_j} \left[\left(2s_j^2 - k_s^2\right) \frac{\sinh \alpha_j y}{\cosh \alpha_j h} + 2\alpha_j \beta_j \frac{\sinh \beta_j y}{\cosh \beta_j h} \right] g(s_j, y, p) \right\} dy$$

$$\left. + \left(\frac{k^2 - 2}{k^2} \right) \bar{u}(0, h, p) + Y_j(s_j, p) \bar{v}(0, h, p) \right\} , \tag{1.14}$$

$$Y_j(s_j, p) = \frac{\left(2s_j^2 - k_s^2\right)}{2k^2 s_j \alpha_j} \tanh \alpha_j h = \frac{-2s_j \beta_j}{k^2 \left(2s_j^2 - k_s^2\right)} \tanh \beta_j h \quad ,$$

$$\alpha_j = \sqrt{k_d^2 - s_j^2} \quad , \quad \beta_j = \sqrt{k_s^2 - s_j^2} \quad .$$

The first of (1. 14) are two coupled integral equations for $g(s,y,p)$ and $h(s,y,p)$. Solving these equations completes the formal solution of the problem.

3. Solution of the Boundedness Equations

Using the boundary conditions at $x = 0$, (1.11f) reduces to

$$g(s,y,p) = k^2 \bar{u}_x(0,y,p) \quad ,$$

$$h(s,y,p) = \frac{v_0}{s c_d^2} + \frac{1}{k^2} \bar{v}_x(0,y,p) \quad . \tag{1.15}$$

The unknown Laplace transformed edge strains $\bar{u}_x(0,y,p)$ and $\bar{v}_x(0,y,p)$ are found by assuming for them representations with unknown coefficients. The representations consist of a singular term which corresponds to the behavior of the strains at the corners $y = \pm h$ plus a Fourier series. If the singularity at the corner is the same as the assumed singular form, then the Fourier series only has to represent a regular function of y. The unknown coefficients in the Fourier series will decay as $1/n^2$ or faster as n, the number of the term in the series, becomes large. It should be noted that calculation of the values of the edge unknowns, to a given level of accuracy, requires only a finite number of terms in the series because of this two part representation.

In [1], Miklowitz found that a dynamically loaded elastic wave-guide that was built-in at the base had the same types of singularities at the corners as did a similar statically loaded waveguide. The types of singularities in the present problem should also be calculable from statics. Since the present problem does not have a static limit, the singularities of a static right-angled wedge with one edge built-in and the other edge free will be used.

The possible stress (and strain) singularities of a static right-angled wedge are known from the work of Knein [6], Williams [7] and Uflyand [8]. As these works show, the dominant stresses are, near the corner, proportional to r^{-q} where r is the distance from the corner and q is a real positive number. For a fixed corner angle, q depends only on Poisson's ratio. In the remainder of this work, ν will be set equal to 0.2433 which makes q very close to 1/4.

Forms for the Edge Unknowns

It is necessary to assume forms for the unknown edge strains $\bar{u}_x(0, y, p)$ and $\bar{v}_x(0, y, p)$. From the antisymmetry of the problem, $\bar{u}_x$ will be odd in y and $\bar{v}_x$ will be even. So $\bar{u}_x$ will be represented by an antisymmetrized singular term plus a Fourier sine series while $\bar{v}_x$ will be represented by a symmetrized singular term plus a Fourier cosine series.

In order that the Fourier series converge rapidly, it is necessary to choose the correct singular forms for the edge unknowns. Based on the results of the right angled wedge problem, any singularities that are present are expected to be of the $r^{-\frac{1}{4}}$ type. However, since the present problem has two corners which will interact, one or both of the strains may not be singular at the corners. Various combinations of strain singularities were tried. These were, both strains singular, both strains nonsingular, and one strain singular and one nonsingular. For each assumed form of the edge strains, $g(s, y, p)$ and $h(s, y, p)$ were calculated from (1.15), substituted into the boundedness equations (1.14) and the resulting simple integrations were performed. This gave an infinite set of algebraic equations which were then approximated for long-time. The equations were solved numerically using the method of reduction to see if the unknown coefficients could be determined. This procedure will be shown in more detail for the case that solved the boundedness equations.

The boundedness equations were solved by assuming that $\bar{u}_x$ was singular and that $\bar{v}_x$ was not. This gives the following forms for the unknown edge strains:

$$\bar{u}_x(0, y, p) = b_0(p)\left[(1-y/h)^{-\frac{1}{4}}-(1+y/h)^{-\frac{1}{4}}+2^{-\frac{1}{4}}y/h\right]+\sum_{\substack{n=2\\ n\text{ even}}}^{\infty} b_n(p)\sin\frac{n\pi y}{2h} \quad ,$$

$$\bar{v}_x(0, y, p) = a_0(p) + \sum_{\substack{n=2\\ n\text{ even}}}^{\infty} a_n(p)\cos\frac{n\pi y}{2h} \quad . \tag{1.16}$$

$a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ are the unknown coefficients, functions of the Laplace transform parameter p.

Substitute (1.16) into (1.15) to get $g(s, y, p)$ and $h(s, y, p)$ and in turn these into the boundedness equations, (1.14). The latter equations are then integrated with the aid of the following integrals:

$$\int_0^h \sinh \alpha_j y(1-y/h)^{-\frac{1}{4}} dy = S_{\alpha_j} \quad ,$$

$$\int_0^h \sinh \alpha_j y(1+y/h)^{-\frac{1}{4}} dy = T_{\alpha_j} \quad ,$$

$$(1.17)$$

$$\int_0^h \cosh \alpha_j y \cos\frac{n\pi y}{2h} dy = (-1)^{n/2} \frac{\alpha_j \sinh \alpha_j h}{(\alpha_j^2 + \theta_n^2)} \quad , \quad n \text{ even} \quad ,$$

$$\int_0^h \sinh \alpha_j y \sin\frac{n\pi y}{2h} dy = (-1)^{n/2} \frac{\theta_n \sinh \alpha_j h}{(\alpha_j^2 + \theta_n^2)} \quad , \quad n \text{ even} \quad ,$$

where $\theta_n = \frac{n\pi}{2h}$, and another analogous set of integrals with α_j replaced by β_j. Then the boundedness equations become an infinite set of algebraic equations:

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \Bigg\{ a_0(p)M_j^0(s_j, p) + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} a_n(p) M_j^n(s_j, p) + b_0(p)N_j^0(s_j, p)$$

$$(1.18)$$

$$+ \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} b_n(p)N_j^n(s_j, p) + Q_j(s_j, p)\Bigg\} = 0$$

where

$$M_j^0(s_j, p) = s_j Y_j(s_j,p)/\beta_j^2 \quad ,$$

$$M_j^n(s_j, p) = -\frac{(-1)^{n/2} s_j Y_j(s_j, p)}{\left(\beta_j^2 + \theta_n^2\right)} \left[\left(\frac{k^2-1}{k^2}\right) \frac{2\theta_n^2}{\left(\alpha_j^2 + \theta_n^2\right)} - 1\right] \quad ,$$

$$N_j^0(s_j, p) = -\frac{2s_j^2}{k_s^2} \left[\frac{s_j\left(S_{\alpha_j} - T_{\alpha_j}\right)}{\alpha_j \cosh \alpha_j h} + \frac{\beta_j\left(S_{\beta_j} - T_{\beta_j}\right)}{s_j \cosh \beta_j h}\right] + \frac{s_j\left(S_{\alpha_j} - T_{\alpha_j}\right)}{\alpha_j \cosh \alpha_j h}$$

$$+ 2^{-\frac{1}{4}}\left(\frac{k^2-2}{k^2}\right)\frac{s_j}{\alpha_j^2} + 2^{-\frac{1}{4}} \frac{k^2 Y_j(s_j, p)}{\beta_j^2 h}\left[\left(\frac{k^2-1}{k^2}\right)\frac{2s_j^2}{\alpha_j^2} + 1\right] \quad , \tag{1.19}$$

$$N_j^n(s_j, p) = \frac{(-1)^{n/2} k^2 \theta_n Y_j(s_j, p)}{\left(\beta_j^2 + \theta_j^2\right)} \left[\left(\frac{k^2-1}{k^2}\right) \frac{2s_j^2}{\left(\alpha_j^2 + \theta_n^2\right)} + 1\right] \quad ,$$

$$Q_j(s_j, p) = v_0 Y_j(s_j, p)/c_s^2 \beta_j^2 \quad ,$$

and $Y_j(s_j, p)$ is as in (1.14).

Equations (1.18) can be solved for the unknown coefficients $a_0(p), a_n(p), b_0(p)$ and $b_n(p)$. That is, for a certain number of unknowns, $a_0(p), a_2(p), a_4(p), \ldots, b_0(p), b_2(p), b_4(p), \ldots$, matching numbers of $s_j(p)$ (which are infinite in number; see [1]) are available to give a sufficient number of equations from (1.18) to solve for these unknowns.

To proceed further, a representation for the $s_j(p)$ is needed. The long-time solution will be considered here.

Long-time Approximation to the Boundedness Equations

The long-time solution can be derived from the first two of (1.10) by using Watson's Lemma. This gives (see Sec. 5.10.2.4 of [9])

$$\begin{bmatrix} u(x,y,t) \\ v(x,y,t) \end{bmatrix}_{t \gg 1} = \frac{1}{2\pi i} \int_{Br_p} \begin{bmatrix} \bar{u}(x,y,p) \\ \bar{v}(x,y,p) \end{bmatrix}_{p \ll 1} e^{pt} dp \quad . \tag{1.20}$$

For p small, the roots $s_j(p)$ of $L(s,p)$ are, for the lowest mode,

$$s_0(p) = \pm(1\pm i)\gamma \quad , \tag{1.21}$$

where $\gamma = \sqrt{\dfrac{p}{2c_p r_g}}$, $c_p = \sqrt{\dfrac{E}{\rho(1-\nu^2)}}$ is the "plate" wave speed and $r_g = \dfrac{h}{\sqrt{3}}$ is the radius of gyration for the plate section. For the higher modes, $s_j(p) = \hat{s}_j, j \geq 1$, where $\hat{s}_j$ is a complex constant satisfying

$$f(\hat{s}_j) = \sin 2\hat{s}_j h - 2\hat{s}_j h = 0 \quad . \tag{1.22}$$

Equation (1.21) is the generalized frequency-wave number relation for the Euler-Bernoulli approximate theory. $f(s)$ is a well known function in the analysis of two dimensional elastostatic layer problems, hence its occurrence here is not surprising. The zeros of $f(s)$ are an ordered infinite set, corresponding to the piercing points of $L(s,p)$ in the plane $p = 0$ (cf. [1], Fig. 8 for corresponding symmetric wave piercing points). Hillman and Salzer, [10], give the first ten roots to six decimal places.

The Euler-Bernoulli approximate theory frequency-wave number relation is known to be a good approximation to the lowest mode of the antisymmetric Rayleigh-Lamb frequency equation for a range of p small but greater than zero. Furthermore, as Sinclair and Miklowitz show in (70) of [5], for the higher modes $\lim_{p\to 0}\left\{\frac{ds_j(p)}{dp}\right\} = 0$.

It follows that the zeros of $f(s)$ are a good approximation to the $s_j(p), j \geq 1$, for p small. This shows that the long-time approximation will be valid for t large but not necessarily infinite.

It remains to approximate (1.19) for p small with $s_j(p)$ as in (1.21) and (1.22) and with $\mathrm{Re}[s_j(p)] > 0$. For the lowest mode, $s_0(p)$, this gives, for $p \to 0$

$$M_0^0(s_0, p) = -\frac{h}{k^2} + O(p) \quad ,$$

$$M_0^n(s_0, p) = -(-1)^{n/2}\left(\frac{k^2-2}{k^2}\right)\frac{ihp}{k^2 c_p r_g \theta_n^2} + O(p^{\frac{3}{2}}) \quad ,$$

$$N_0^0(s_0, p) = -(1+i)2^{-\frac{1}{4}}\,\frac{5h^2}{7}\sqrt{\frac{p}{2c_p r_g}} + O(p^{\frac{3}{2}}) \quad , \qquad (1.23)$$

$$N_0^n(s_0, p) = (-1)^{n/2}(1+i)\frac{2h^2}{n\pi}\sqrt{\frac{p}{2c_p r_g}} + O(p^{\frac{3}{2}}) \quad ,$$

$$Q_0(s_0, p) = (-1+i)\frac{v_0 h}{2c_d^2}\sqrt{\frac{2c_p r_g}{p}} + O(1) \quad .$$

Approximating (1.19) for the higher modes, $s_j(p), j \geq 1$, gives

$$M_j^0(\hat{s}_j, p) = -\tan \hat{s}_j h/k^2\hat{s}_j + O(p^2) \quad ,$$

$$M_j^n(\hat{s}_j, p) = -\frac{(-1)^{n/2}\hat{s}_j \tan \hat{s}_j h}{k^2(\theta_n^2 - \hat{s}_j^2)} \left[\left(\frac{k^2-1}{k^2}\right)\frac{2\hat{s}_j^2}{(\theta_n^2-\hat{s}_j^2)} + \left(\frac{k^2-2}{k^2}\right)\right] + O(p^2) \quad ,$$

$$N_j^0(\hat{s}_j, p) = -\frac{1}{\cos \hat{s}_j h}\left\{\left(S_{\hat{s}_j} - T_{\hat{s}_j}\right)\left[\left(\frac{k^2-1}{k^2}\right)\hat{s}_j h \tan \hat{s}_j h + \frac{1}{k^2}\right]\right. \tag{1.24}$$

$$\left. + \left(\frac{k^2-1}{k^2}\right)\hat{s}_j h\left(H_{\hat{s}_j} - R_{\hat{s}_j}\right)\right\} + 2^{-\frac{1}{4}}\left(\frac{k^2-2}{k^2}\right)\frac{(\tan \hat{s}_j h - \hat{s}_j h)}{\hat{s}_j^2 h} + O(p^2) \quad ,$$

$$N_j^n(\hat{s}_j, p) = \frac{(-1)^{n/2}\theta_n \tan \hat{s}_j h}{\left(\theta_n^2 - \hat{s}_j^2\right)}\left[\left(\frac{k^2-1}{k^2}\right)\frac{2\hat{s}_j^2}{\left(\theta_n^2-\hat{s}_j^2\right)} + 1\right] + O(p^2) \quad ,$$

$$Q_j(\hat{s}_j, p) = -v_0 \tan \hat{s}_j h/c_d^2\hat{s}_j^2 + O(p^2) \quad ,$$

where

$$H_{\hat{s}_j} = \int_0^h \frac{y}{h} \cos \hat{s}_j y\left(1-\frac{y}{h}\right)^{-\frac{1}{4}} dy \quad ,$$

$$R_{\hat{s}_j} = \int_0^h \frac{y}{h} \cos \hat{s}_j y\left(1+\frac{y}{h}\right)^{-\frac{1}{4}} dy \quad , \tag{1.25}$$

$$S_{\hat{s}_j} = \int_0^h \sin \hat{s}_j y\left(1-\frac{y}{h}\right)^{-\frac{1}{4}} dy \quad , \qquad \text{(cont.)}$$

$$T_{\hat{s}_j} = \int_0^h \sin \hat{s}_j y(1+\frac{y}{h})^{-\frac{1}{4}} dy \quad .$$

The coefficients in (1.23) and (1.24) are split into their real and imaginary parts and the order in p of each of the terms is determined. Substituting the order of each term into (1.18) gives

$$\begin{bmatrix} O(1) & O(p^{\frac{3}{2}}) & \cdots & O(p^{\frac{1}{2}}) & O(p^{\frac{1}{2}}) & \cdots \\ O(p) & O(p) & \cdots & O(p^{\frac{1}{2}}) & O(p^{\frac{1}{2}}) & \cdots \\ O(1) & O(1) & \cdots & O(1) & O(1) & \cdots \\ O(1) & O(1) & \cdots & O(1) & O(1) & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} a_0(p) \\ a_2(p) \\ \vdots \\ b_0(p) \\ b_2(p) \\ \vdots \end{bmatrix} = \begin{bmatrix} O(p^{-\frac{1}{2}}) \\ O(p^{-\frac{1}{2}}) \\ O(1) \\ O(1) \\ \vdots \end{bmatrix} \qquad (1.26)$$

Using Cramer's rule on (1.26) gives that

$$a_0(p) = O(p^{-\frac{1}{2}}) \quad ,$$

$$a_n(p) = b_0(p) = b_n(p) = O(p^{-1}) \quad . \qquad (1.27)$$

(1.27) shows that the unknown edge strains, $u_x(0,y,t)$ and $v_x(0,y,t)$, are constant for long-time. Define

$$a_0(p) = \frac{A_0}{\sqrt{p}} \ ,$$

$$a_n(p) = \frac{A_n}{p} \ , \quad b_0(p) = \frac{B_0}{p} \ , \quad b_n(p) = \frac{B_n}{p} \ , \tag{1.28}$$

where A_0, A_n, B_0 and B_n are independent of p. Substituting (1.23), (1.24) and (1.28) into (1.18), retaining only the lowest order terms in p, and simplifying gives the following set of simultaneous equations:

$$A_0 = -\frac{v_0}{c_s^2}\sqrt{2c_p r_g} \ ,$$

$$-2^{-\frac{1}{4}}\frac{5}{7}B_0 + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} (-1)^{\frac{n}{2}} \frac{2B_n}{n\pi} = -\frac{v_0}{c_d}\sqrt{\frac{2}{3k^2(1-\nu)}} \ , \tag{1.29}$$

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \left\{ \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} \left[A_n \hat{M}_j^n(z_j) + B_n \hat{N}_j^n(z_j) \right] + B_0 \hat{N}_j^0(z_j) \right\} = 0 \ ,$$

where j goes from 1 to infinity and

$$\hat{M}_j^n(z_j) = -\frac{(-1)^{n/2} 4z_j \tan z_j}{\left(n^2\pi^2 - 4z_j^2\right)} \left[\left(\frac{k^2-1}{k^2}\right) \frac{8z_j^2}{\left(n^2\pi^2-4z_j^2\right)} + \left(\frac{k^2-2}{k^2}\right) \right] \ ,$$

(cont.)

$$\hat{N}_j^0(z_j) = -\frac{1}{\cos z_j}\left\{\left(S'_j - T'_j\right)\left[\left(\frac{k^2-1}{k^2}\right) z_j \tan z_j + \frac{1}{k^2}\right] + \left(H'_j - R'_j\right)\left(\frac{k^2-1}{k^2}\right) z_j\right\}$$

$$+ 2^{-\frac{1}{4}}\left(\frac{k^2-2}{k^2}\right)\frac{1}{z_j^2}(\tan z_j - z_j) \quad ,$$

$$\hat{N}_j^n(z_j) = \frac{(-1)^{n/2} 2n\pi \tan z_j}{\left(n^2\pi^2 - 4z_j^2\right)}\left[\left(\frac{k^2-1}{k^2}\right)\frac{8z_j^2}{\left(n^2\pi^2-4z_j^2\right)} + 1\right] ,$$

$$H'_j = \int_0^1 r \cos z_j r (1-r)^{-\frac{1}{4}} dr \quad , \qquad (1.30)$$

$$R'_j = \int_0^1 r \cos z_j r (1+r)^{-\frac{1}{4}} dr \quad ,$$

$$S'_j = \int_0^1 \sin z_j r (1-r)^{-\frac{1}{4}} dr \quad ,$$

$$T'_j = \int_0^1 \sin z_j r (1+r)^{-\frac{1}{4}} dr \quad ,$$

where z_j is obtained from

$$\sin 2z_j = 2z_j \quad .$$

The first two of (1.29) have important physical meanings. The net shear force at the base is given by

$$Q(0,t) = \int_{-h}^{h} \sigma_{xy}(0,y,t)dy \quad . \tag{1.31}$$

Using (1.2), substituting for $\overline{v}_x(0,y,p)$ from (1.16) and integrating gives

$$\overline{Q}(0,p) = 2\mu h a_0(p) \quad .$$

Substituting for $a_0(p)$ from (1.28) and the first of (1.29) and inverting gives

$$Q(0,t) = -2v_0 \rho h \sqrt{\frac{2c_p r_g}{\pi t}} \quad , \tag{1.32}$$

which goes to zero for long-time. The net moment at the base is given by

$$M(0,t) = \int_{-h}^{h} \sigma_{xx}(0,y,t) y \, dy \quad . \tag{1.33}$$

Using (1.2), (1.16), (1.28) and integrating gives

$$\overline{M}(0,p) = 2\mu k^2 \left\{ 2^{-\frac{1}{4}} \frac{5}{7} \frac{B_0 h^2}{p} - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (-1)^{\frac{n}{2}} \frac{2B_n h^2}{n\pi p} \right\} \quad .$$

Substituting from the second of (1.29) and inverting gives

$$M(0,t) = 2v_0 h \rho c_p r_g H(t) \quad , \tag{1.34}$$

which is constant for long-time.

The problem can also be solved approximately using

Euler-Bernoulli approximate theory for the quantities $Q(0,t)$ and $M(0,t)$. When the shear force and moment are calculated, (1.32) and (1.34) result. So the approximate solution has the same net shear force and net moment at the base as does the exact solution for long-time. It should be emphasized however that the exact theory governs the important singularity in $\bar{u}_x(0,y,t)$. The following section assesses this important contribution to the problem.

Numerical Solution of the Boundedness Equations

Equations (1.29) were solved numerically using Fortran IV. The method of reduction was used to calculate the solution. The values of the unknowns were calculated using more and more unknowns until convergence to the final value of each unknown was reached. Convergence should be obtained for a relatively small number of unknowns since the Fourier series in (1.16) are not being called upon to represent the singularities at $y = \pm h$.

Values of the unknown coefficients for $\nu = 0.2433$ are shown in Table I. As can be seen, the coefficients converge for 30 through 38 unknowns. The coefficients also decay faster than $1/n^2$ for large n. Therefore, the coefficients in Table I are a solution to (1.29) and hence (1.16) is a solution to the boundedness condition for long-time.

A similar procedure was carried out for each of the other possible singular forms for the unknown base strains. For all of these cases, the unknown coefficients failed to converge by the time fifty unknowns were used. This indicates that the other singular terms do not

TABLE I

Coefficients A_n and B_n

(ν = 0.2433)

Coefficients	Number of Unknowns Used							
	20	24	28	30	32	34	38	42
A_2	0.5165	0.5165	0.5166	0.5166	0.5166	0.5166	0.5167	0.5167
A_4	-0.2397	-0.2398	-0.2399	-0.2399	-0.2399	-0.2399	-0.2399	-0.2400
A_6	0.1529	0.1532	0.1533	0.1533	0.1533	0.1534	0.1534	0.1534
A_8	-0.1111	-0.1118	-0.1121	-0.1122	-0.1123	-0.1123	-0.1124	-0.1124
A_{10}	0.0859	0.0875	0.0882	0.0883	0.0884	0.0885	0.0886	0.0887
A_{12}	-0.0674	-0.0710	-0.0722	-0.0725	-0.0728	-0.0729	-0.0731	-0.0732
A_{14}	0.0505	0.0586	0.0610	0.0616	0.0620	0.0623	0.0626	0.0628
A_{16}	-0.0271	-0.0470	-0.0517	-0.0527	-0.0535	-0.0539	-0.0545	-0.0548
B_0	0.7151	0.7136	0.7129	0.7127	0.7127	0.7125	0.7126	0.7128
B_2	0.2936	0.2950	0.2956	0.2958	0.2958	0.2959	0.2958	0.2956
B_4	-0.0970	-0.0978	-0.0981	-0.0982	-0.0982	-0.0983	-0.0982	-0.0981
B_6	0.0469	0.0475	0.0477	0.0478	0.0478	0.0478	0.0478	0.0477
B_8	-0.0265	-0.0271	-0.0273	-0.0274	-0.0274	-0.0274	-0.0274	-0.0273
B_{10}	0.0161	0.0168	0.0171	0.0171	0.0171	0.0172	0.0172	0.0171
B_{12}	-0.0097	-0.0108	-0.0111	-0.0112	-0.0112	-0.0113	-0.0113	-0.0112
B_{14}	0.0049	0.0069	0.0075	0.0076	0.0077	0.0077	0.0078	0.0077
B_{16}	0.0011	-0.0037	-0.0048	-0.0050	-0.0051	-0.0052	-0.0053	-0.0053

represent the base strains correctly.

The coefficients from Table I were used to calculate the strains at the base. Graphs of these strains are shown in Figs. 2 and 3. The normal strain, $u_x(0,y,t)$, becomes infinite as the corners $y=\pm h$ are approached indicating that there is restraint in the x direction. On the other hand, the shear strain, $v_x(0,y,t)$, does not become infinite for long-time, probably because the motion in the direction is restrained to a lesser degree since the motion the base of the plate wants to make in the thickness direction is the same as is given by the boundary condition. Note that the shear strain may be singular at the corners for short-time.

4. Derivation of the Formal Long-Time Solution

Once the transformed edge unknowns, $\bar{u}_x(0,y,p)$ and $\bar{v}_x(0,y,p)$, have been determined for small p, the formal long-time solution can be calculated from (1.11). $g(s,y,p)$ and $h(s,y,p)$ are found by substituting the base strains from (1.16) into (1.15). The integrands in (1.11) are now known and the indicated integrations are performed. The resulting forms for the doubly transformed solutions are

$$\tilde{\bar{u}}(s,y,p) = \sum_{j=1}^{4} \tilde{\bar{u}}_j(s,y,p) \quad ,$$

$$\tilde{\bar{v}}(s,y,p) = \sum_{j=1}^{4} \tilde{\bar{v}}_j(s,y,p) \quad , \tag{1.35}$$

where

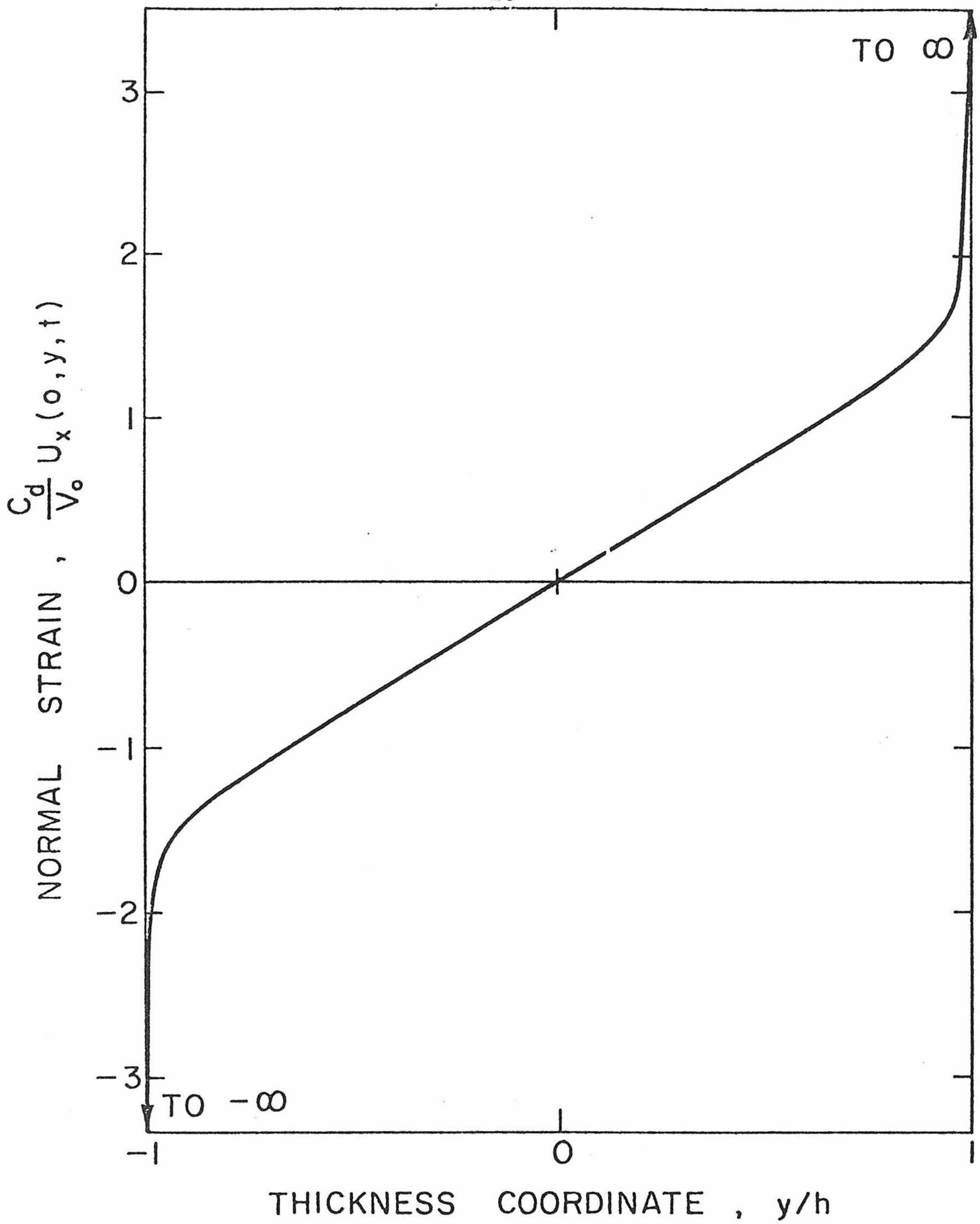

Fig. 2 Normal Strain at the Base.

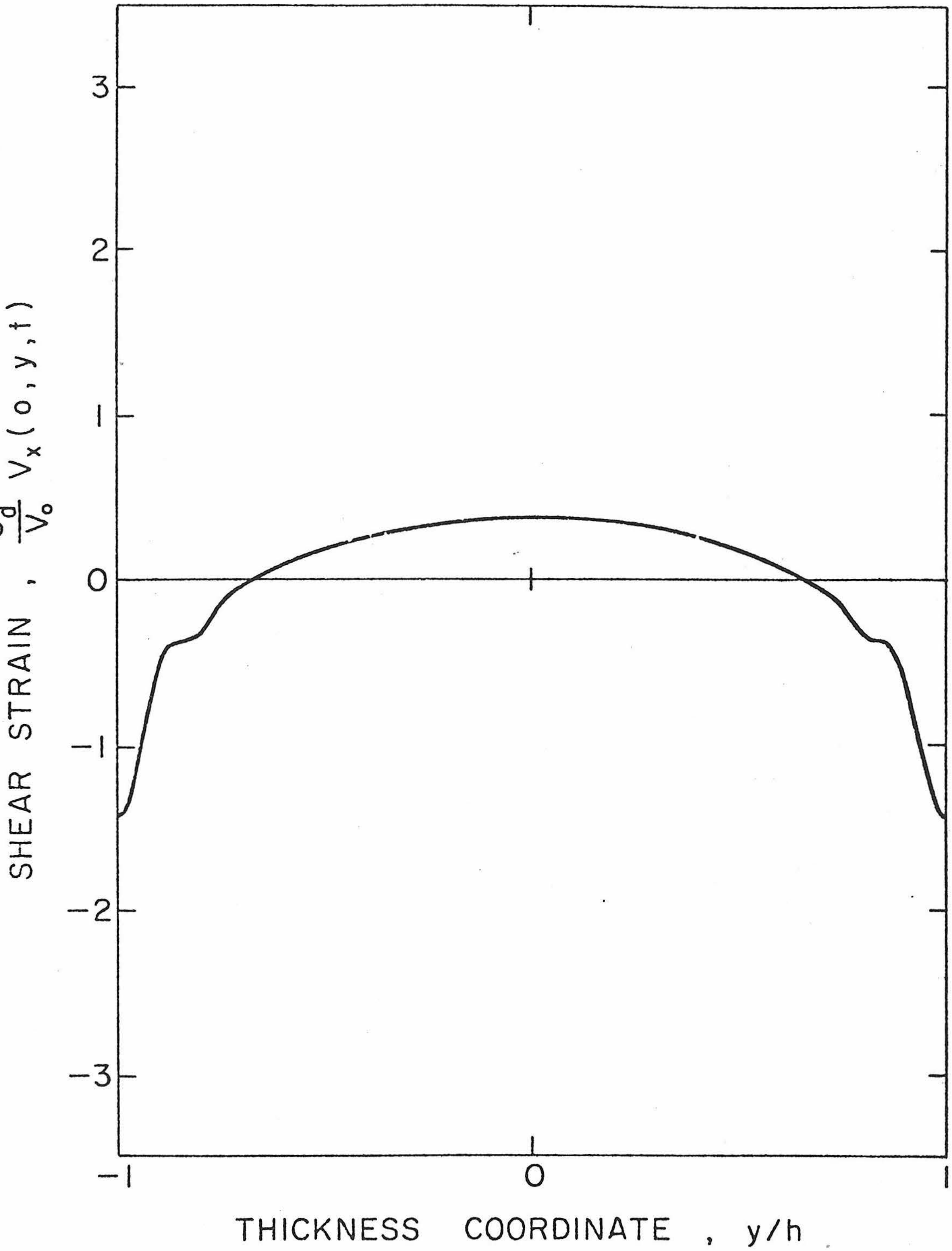

Fig. 3 Shear Strain at the Base.

$$\tilde{u}_1(s,y,p) = f_A(s,p) \cdot \frac{I_u(s,y,p)}{L(s,p)} \quad ,$$

$$\tilde{u}_2(s,y,p) = \left[f_B(s,p) + f_C(s,p) + f_D(s,p)\right] \cdot \frac{J_u(s,y,p)}{L(s,p)} \quad ,$$

$$\tilde{u}_3(s,y,p) = -\frac{k^2}{k_s^2} b_0(p)\left[\frac{s^2}{\alpha} \sinh \alpha y \,(H_\alpha - R_\alpha) + \beta \sinh \beta y \,(H_\beta - R_\beta)\right] ,$$

$$\tilde{u}_4(s,y,p) = \left[f_E(s,y,p) + f_F(s,y,p)\right] \quad , \qquad (1.36)$$

$$\tilde{v}_1(s,y,p) = f_A(s,p) \cdot \frac{I_v(s,y,p)}{L(s,p)} \quad ,$$

$$\tilde{v}_2(s,y,p) = \left[f_B(s,p) + f_C(s,p) + f_D(s,p)\right] \cdot \frac{J_v(s,y,p)}{L(s,p)} \quad ,$$

$$\tilde{v}_3(s,y,p) = -\frac{k^2{}_s}{k_s^2} b_0(p)\left[\cosh \alpha y \,(H_\alpha - R_\alpha) - \cosh \beta y \,(H_\beta - R_\beta)\right] \quad ,$$

$$\tilde{v}_4(s,y,p) = \left[f_G(s,y,p) + f_H(s,y,p) + f_K(s,y,p)\right] \quad ,$$

where

$$f_A(s,p) = -\frac{s}{k_s^2} b_0(p)\left\{\frac{\left(2s^2 - k_s^2\right)}{\alpha}\left[\cosh \alpha h \,(S_\alpha - T_\alpha) + \frac{2^{-\frac{1}{4}}}{\alpha}\right]\right.$$

$$\left. + 2\beta\left[\cosh \beta h \,(S_\beta - T_\beta) + \frac{2^{-\frac{1}{4}}}{\beta}\right]\right\} \quad , \qquad (1.37a)$$

$$f_B(s,p) = -\frac{k^2}{k_s^2} b_0(p)\Big\{2s^2\left[\sinh \alpha h(S_\alpha - T_\alpha) + \frac{2^{-\frac{1}{4}}}{\alpha^2 h}\right]$$

$$-\left(2s^2-k_s^2\right)\left[\sinh \beta h(S_\beta - T_\beta) + \frac{2^{-\frac{1}{4}}}{\beta^2 h}\right]\Big\}, \qquad (1.37b)$$

$$f_C(s,p) = -\sum_{\substack{n=2\\ n=\text{even}}}^{\infty} (-1)^{\frac{n}{2}} \Big\{\left(k^2\theta_n b_n(p) - s a_n(p)\right)\frac{1}{\left(\beta^2+\theta_n^2\right)}$$

$$\left[\left(\frac{k^2-1}{k^2}\right)\frac{2s^2}{\left(\alpha^2+\theta_n^2\right)}+1\right] + \frac{s}{k^2} a_n(p)\frac{2}{\left(\alpha^2+\theta_n^2\right)}\Big\}, \qquad (1.37c)$$

$$f_D(s,p) = -\left[\frac{v_0}{s c_d^2} + \frac{a_0(p)}{k^2}\right]\frac{k^2{}_s}{\beta^2}, \qquad (1.37d)$$

$$f_E(s,y,p) = -\frac{k^2}{k_s^2} b_0(p)\Big\{\frac{s^2}{\alpha}\left[\cosh \alpha y\left(S_\alpha^y - T_\alpha^y\right) - \sinh \alpha y\left(H_\alpha^y - R_\alpha^y\right)\right]$$

$$+\beta\left[\cosh \beta y\left(S_\beta^y - T_\beta^y\right) - \sinh \beta y\left(H_\beta^y - R_\beta^y\right)\right] + 2^{-\frac{1}{4}}\frac{y}{h}\left(\frac{s^2}{\alpha^2}+1\right)\Big\}, \qquad (1.37e)$$

$$f_F(s,y,p) = -\sum_{\substack{n=2 \\ n\ \mathrm{even}}}^{\infty} \frac{\sin\theta_n y}{\left(\beta^2+\theta_n^2\right)} \left\{ k^2 b_n(p)\left[\left(\frac{k^2-1}{k^2}\right)\frac{s^2}{\left(\alpha^2+\theta_n^2\right)} + 1\right] \right.$$

$$\left. -\left(\frac{k^2-1}{k^2}\right) a_n(p)\,\frac{s\,\theta_n}{\left(\alpha^2+\theta_n^2\right)} \right\} , \tag{1.37f}$$

$$f_G(s,y,p) = -\frac{k_s^2}{k_s^2}\, b_0(p)\left[\sinh\alpha y\left(S_\alpha^y - T_\alpha^y\right) - \cosh\alpha y\left(H_\alpha^y - R_\alpha^y\right)\right.$$

$$\left. -\sinh\beta y\left(S_\beta^y - T_\beta^y\right) + \cosh\beta y\left(H_\beta^y - R_\beta^y\right) + 2^{-\frac{1}{4}}\left(\frac{1}{\alpha^2 h} - \frac{1}{\beta^2 h}\right)\right] , \tag{1.37g}$$

$$f_H(s,y,p) = \sum_{\substack{n=2 \\ n\ \mathrm{even}}}^{\infty} \frac{\cos\theta_n y}{\left(\alpha^2+\theta_n^2\right)} \left\{ \frac{a_n(p)}{k^2}\left[\frac{(k^2-1)\,s^2}{\left(\beta^2+\theta_n^2\right)} - 1\right] - \frac{(k^2-1)b_n(p)\,s\,\theta_n}{\left(\beta^2+\theta_n^2\right)} \right\} , \tag{1.37h}$$

$$f_K(s,y,p) = -\left(\frac{v_0}{s c_d^2} + \frac{a_0(p)}{k^2}\right)\frac{k^2}{\beta^2} , \tag{1.37i}$$

$$I_u(s,y,p) = -k^2 s\left[\left(2s^2 - k_s^2\right)\cosh\beta h\,\sinh\alpha y + 2\alpha\beta\cosh\alpha h\,\sinh\beta y\right] ,$$

(cont.)

$$J_u(s,y,p) = -\beta\left[2s^2 \sinh\beta h \sinh\alpha y - \left(2s^2 - k_s^2\right)\sinh\alpha h \sinh\beta y\right] ,$$

$$I_v(s,y,p) = -k^2\alpha\left[\left(2s^2 - k_s^2\right)\cosh\beta h \cosh\alpha y - 2s^2\cosh\alpha h \cosh\beta y\right] , \tag{1.38}$$

$$J_v(s,y,p) = -s\left[2\alpha\beta \sinh\beta h \cosh\alpha y + \left(2s^2 - k_s^2\right)\sinh\alpha h \cosh\beta y\right] .$$

$L(s,p)$ is the Rayleigh-Lamb frequency equation as in (1.12), S_α and T_α are as in the first two of (1.17) with α_j replaced by α and

$$H_\alpha = \int_0^h \cosh\alpha y\,(1-\tfrac{y}{h})^{-\frac{1}{4}}\,dy ,$$

$$R_\alpha = \int_0^h \cosh\alpha y\,(1+\tfrac{y}{h})^{-\frac{1}{4}}\,dy ,$$

$$H_\alpha^y = \int_0^y \cosh\alpha y'\,(1-\tfrac{y'}{h})^{-\frac{1}{4}}\,dy ,$$

$$R_\alpha^y = \int_0^y \cosh\alpha y'\,(1+\tfrac{y'}{h})^{-\frac{1}{4}}\,dy , \tag{1.39}$$

$$S_\alpha^y = \int_0^y \sinh\alpha y'\,(1-\tfrac{y'}{h})^{-\frac{1}{4}}\,dy ,$$

$$T_\alpha^y = \int_0^y \sinh\alpha y'\,(1+\tfrac{y'}{h})^{-\frac{1}{4}}\,dy .$$

Equations (1.35) through (1.39) have resulted from the form of the edge unknowns assumed in (1.16), i.e. they are not directly dependent on the assumption of small p. For use in the long-time solution, however, they will have to be approximated for p small to be consistent with the approximations that were used to determine the unknown coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ that they contain.

The doubly transformed solution will now be inverted, using approximations that give the solution for two different regions of the plate. First, the asymptotics of the Laplace transform will be used to find the solution as $x \to 0$. In the following section the doubly transformed solution is inverted by residue theory. This solution is valid in a region away from the base, $x = 0$, but behind the body wave fronts. Since p must be small, information about the wave fronts cannot be obtained.

a. Near-field Asymptotic Solution for $x \to 0$

The near-field asymptotic solution, valid as $x \to 0$, will be obtained in the same manner as it was by Miklowitz in [1] and [2]. Applying Watson's Lemma to the present case gives (see Sec. 5.10.2.3 of [9])

$$\left\{ \begin{matrix} \bar{u}(x,y,p) \\ \bar{v}(x,y,p) \end{matrix} \right\}_{x \to 0} = \frac{1}{2\pi i} \int_{Br_s} \left\{ \begin{matrix} \tilde{\bar{u}}(s,y,p) \\ \tilde{\bar{v}}(s,y,p) \end{matrix} \right\}_{|s| \to \infty} e^{sx} \, ds \quad . \tag{1.40}$$

Since p must be small, (1.35) through (1.39) will be expanded with $|sh| >> 1$ and $|k_s^2/s^2| << 1$. This gives

$$\begin{Bmatrix} \alpha \\ \\ \beta \end{Bmatrix} = -is \left[1 - \begin{Bmatrix} k_d^2/2s^2 \\ \\ k_s^2/2s^2 \end{Bmatrix} + O\left(p^4/s^4\right) \right] . \qquad (1.41)$$

It should be noted that since large $|s|$ here is necessarily on Br_s, (1.41) shows, assuming again that p is real, that the real parts of α and β are large through the value of Im s while the imaginary parts remain constant. Note that for very large $|s|$, $\alpha = \beta \simeq \text{Im}\, s$. In the work that follows s is chosen on the upper half of Br_s. However, since $\tilde{u}$ and $\tilde{v}$ are even in α and β, it follows that the large $|sh|$ approximation holds all over Br_s in the usual way.

Because of the singular terms in (1.16), $y = h$ must be given special consideration. For the "interior" solution, $0 \le y < h$, the following approximations are valid:

$$L(s,p) \simeq \frac{1}{2} s^2 k_s^2 \left(\frac{k^2-1}{k^2}\right) e^{-2ish} ,$$

$$I_u(s,y,p) \simeq -\frac{i}{4} s^2 k_s^2 (k^2-1)(h-y)\, e^{-is(h+y)} , \qquad (1.42)$$

$$J_u(s,y,p) \simeq \frac{1}{4} s^2 k_s^2 \left(\frac{k^2-1}{k^2}\right)(h-y)\, e^{-is(h+y)} ,$$

$$I_v(s,y,p) \simeq -\frac{1}{4} s^2 k_s^2 (k^2-1)(h-y)\, e^{-is(h+y)} ,$$

(cont.)

$$J_v(s,y,p) \simeq \frac{i}{4}\, s^2 k_s^2 \left(\frac{k^2-1}{k^2}\right)(h-y)\, e^{-is(h+y)} \quad .$$

Now S_α and T_α may be written in the form

$$\left\{\begin{matrix} S_\alpha \\ \\ T_\alpha \end{matrix}\right\} = \frac{1}{2}\int_0^h \left[\frac{e^{\alpha y} - e^{-\alpha y}}{(1 \mp y/h)^{-\frac{1}{4}}}\right] dy \quad . \qquad (1.43)$$

For large α these become

$$\left\{\begin{matrix} S_\alpha \\ \\ T_\alpha \end{matrix}\right\} = \frac{1}{2}\int_0^h \frac{e^{\alpha y}\, dy}{(1 \mp y/h)} \quad .$$

Make the variable change $y = h-r$. Then

$$S_\alpha = \frac{e^{\alpha h}}{2}\int_0^h e^{-\alpha r}(h/r)^{\frac{1}{4}}\, dr \quad ,$$

$$T_\alpha = \frac{e^{\alpha h}}{2}\int_0^h e^{-\alpha r}(2-r/h)^{-\frac{1}{4}}\, dr \quad . \qquad (1.44)$$

Since α is a real, positive, large parameter in (1.44), according to (1.41), these integrals may be approximated by Watson's Lemma, with the results

$$S_\alpha = \frac{1}{2} e^{\alpha h}\left[\Gamma(\tfrac{3}{4})\, h^{\frac{1}{4}} \alpha^{-\frac{3}{4}} + O\left(\frac{1}{\alpha h}\right)\right] ,$$

$$T_\alpha = \frac{1}{2} e^{\alpha h}\left[2^{-\frac{1}{4}} \frac{1}{\alpha} + O\left(\frac{1}{\alpha^2 h^2}\right)\right] , \tag{1.45}$$

with an equivalent set for S_β and T_β. Equations (1.45) can now be used to approximate $f_A(s,p)$ and $f_B(s,p)$ in (1.37) with the results

$$f_A(s,p) = -b_0(p) \frac{is^2}{k_s^2} e^{-2ish}\left[\Gamma(\tfrac{3}{4})\, h^{\frac{1}{4}} (-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] ,$$

$$f_B(s,p) = -b_0(p) \frac{k^2 s^2}{k_s^2} e^{-2ish}\left[\Gamma(\tfrac{3}{4})\, h^{\frac{1}{4}} (-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] . \tag{1.46}$$

Similarly, the integrals H_α and R_α are written as

$$\left\{\begin{matrix} H_\alpha \\ \\ R_\alpha \end{matrix}\right\} = \frac{1}{2}\int_0^h \left[\frac{e^{\alpha y} + e^{-\alpha y}}{(1 \mp y/h)^{\frac{1}{4}}}\right] dy . \tag{1.47}$$

Approximating these integrals gives

$$H_\alpha = \frac{1}{2} e^{\alpha h}\left[\Gamma(\tfrac{3}{4})\, h^{\frac{1}{4}} \alpha^{-\frac{3}{4}} + O\left(\frac{1}{\alpha h}\right)\right] , \tag{1.48}$$

(cont.)

$$R_\alpha = \frac{1}{2} e^{\alpha h}\left[2^{-\frac{1}{4}}\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2 h^2}\right)\right] .$$

$\tilde{\tilde{u}}_3(s,y,p)$ and $\tilde{\tilde{v}}_3(s,y,p)$ in (1.36) are now approximated for $|sh|$ large with the aid of (1.48), yielding

$$\tilde{\tilde{u}}_3(s,y,p) = \frac{b_0(p)}{4k_s^2} e^{-is(h+y)} O\left(\frac{1}{s}\right) ,$$

(1.49)

$$\tilde{\tilde{v}}_3(s,y,p) = \frac{ib_0(p)}{4k_s^2} e^{-is(h+y)} O\left(\frac{1}{s}\right) .$$

Also, for $|sh|$ large

$$f_C(s,p) = O\left(\frac{1}{s}\right) ,$$

(1.50)

$$f_D(s,p) = O\left(\frac{1}{s}\right) .$$

Combining terms from (1.42) through (1.50) together gives

$$\tilde{\tilde{u}}_1(s,y,p) = -b_0(p)\frac{s^2}{4k_s^2}(h-y)e^{-is(h+y)}\left[\Gamma\left(\tfrac{3}{4}\right)h^{\frac{1}{4}}(-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{s}e^{is(h-y)}\right),$$

$$\tilde{\tilde{u}}_2(s,y,p) = b_0(p)\frac{s^2}{4k_s^2}(h-y)e^{-is(h+y)}\left[\Gamma\left(\tfrac{3}{4}\right)h^{\frac{1}{4}}(-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{s}e^{is(h-y)}\right),$$

(cont.)

$$\tilde{\tilde{u}}_3(s,y,p) = b_0(p)\frac{s^2}{4k_s^2}\, e^{-is(h+y)} O\left(\frac{1}{s^3}\right) + O\left(\frac{1}{s} e^{is(h-y)}\right) ,$$

(1.51)

$$\tilde{\tilde{v}}_1(s,y,p) = -b_0(p)\frac{is^2}{4k_s^2}(h-y)\, e^{-is(h+y)}\left[\Gamma\left(\tfrac{3}{4}\right)h^{\frac{1}{4}}(-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{s} e^{is(h-y)}\right),$$

$$\tilde{\tilde{v}}_2(s,y,p) = b_0(p)\frac{is^2}{4k_s^2}(h-y)\, e^{-is(h+y)}\left[\Gamma\left(\tfrac{3}{4}\right)h^{\frac{1}{4}}(-is)^{-\frac{3}{4}} + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{s} e^{is(h-y)}\right),$$

$$\tilde{\tilde{v}}_3(s,y,p) = b_0(p)\frac{is^2}{4k_s^2}\, e^{-is(h+y)} O\left(\frac{1}{s^3}\right) + O\left(\frac{1}{s} e^{is(h-y)}\right) .$$

Adding $\tilde{\tilde{u}}_1$, $\tilde{\tilde{u}}_2$ and $\tilde{\tilde{u}}_3$, $\tilde{\tilde{v}}_1$, $\tilde{\tilde{v}}_2$ and $\tilde{\tilde{v}}_3$, it is easily seen that the lowest order terms containing $e^{-is(h+y)}$ cancel each other. This cancellation appears to occur for all orders in s. The terms of $O\left(\frac{1}{s} e^{is(h-y)}\right)$ will be exponentially small for $y < h$. Therefore, $\tilde{\tilde{u}}_1$, $\tilde{\tilde{u}}_2$, $\tilde{\tilde{u}}_3$, $\tilde{\tilde{v}}_1$, $\tilde{\tilde{v}}_2$, and $\tilde{\tilde{v}}_3$ will not contribute to the solution as $x \to 0$.

It remains to approximate $\tilde{\tilde{u}}_4$ and $\tilde{\tilde{v}}_4$. Integrating $f_E(s,y,p)$ and $f_G(s,y,p)$ by parts and approximating yields

$$f_E(s,y,p) = b_0(p)\left\{\left[(1-y/h)^{-\frac{1}{4}} - (1+y/h)^{-\frac{1}{4}} + 2^{-\frac{1}{4}}y/h\right]\frac{1}{s^2} + O\left(\frac{1}{s^4}\right)\right\} ,$$

(1.52)

$$f_G(s,y,p) = b_0(p)\,\frac{(k^2-1)}{h}\left\{\left[\tfrac{1}{4}(1-y/h)^{-\frac{5}{4}} + \tfrac{1}{4}(1+y/h)^{-\frac{5}{4}} - 2^{-\frac{1}{4}}\right]\frac{1}{s^3} + O\left(\frac{1}{s^5}\right)\right\} .$$

The other terms can easily be approximated to give

$$f_F(s,y,p) = \sum_{\substack{n=2\\ n\text{ even}}}^{\infty} \left[b_n(p)\frac{\sin\theta_n y}{s^2} + a_n(p)\left(\frac{k^2-1}{k^2}\right)\frac{\theta_n \sin\theta_n y}{s^3}\right] + O\left(\frac{1}{s^4}\right) ,$$

$$f_H(s,y,p) = \sum_{\substack{n=2\\ n\text{ even}}}^{\infty} \left[a_n(p)\frac{\cos\theta_n y}{s^2} - b_n(p)\,\theta_n(k^2-1)\frac{\cos\theta_n y}{s^3}\right] + O\left(\frac{1}{s^4}\right) ,$$

(1.53)

$$f_K(s,y,p) = \frac{a_0(p)}{s^2} + \frac{v_0}{c_s^2 s^3} + O\left(\frac{1}{s^4}\right) .$$

From (1.51) through (1.53), it follows that

$$\tilde{u}(s,y,p) = \left\{ b_0(p)\left[(1-y/h)^{-\frac{1}{4}} - (1+y/h)^{-\frac{1}{4}} + 2^{-\frac{1}{4}}y/h\right] + \sum_{\substack{n=2\\ n\text{ even}}}^{\infty} b_n(p)\sin\theta_n y\right\}\frac{1}{s^2}$$

$$+ \sum_{\substack{n=2\\ n\text{ even}}}^{\infty} a_n(p)\left(\frac{k^2-1}{k^2}\right)\frac{\theta_n \sin\theta_n y}{s^3} + O\left(\frac{1}{s^4}\right) ,$$

(cont.)

$$\tilde{\bar{v}}(s,y,p) = \left[a_0(p) + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} a_n(p)\cos\theta_n y\right]\frac{1}{s^2} + \left\{\frac{v_0}{c_s^2}\right. \tag{1.54}$$

$$+ b_0(p)\frac{(k^2-1)}{h}\left[\frac{1}{4}(1-y/h)^{-\frac{5}{4}} + \frac{1}{4}(1+y/h)^{-\frac{5}{4}} - 2^{-\frac{1}{4}}\right]$$

$$\left.- \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} b_n(p)(k^2-1)\,\theta_n\cos\theta_n y\right\}\frac{1}{s^3} + O\left(\frac{1}{s^4}\right) \quad .$$

The s-x Laplace transform can now be easily inverted, giving

$$\bar{u}(x,y,p) = \left\{b_0(p)\left[(1-y/h)^{-\frac{1}{4}} - (1+y/h)^{-\frac{1}{4}} + 2^{-\frac{1}{4}}y/h\right] + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} b_n(p)\sin\theta_n y\right\}x$$

$$+ \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} a_n(p)\left(\frac{k^2-1}{k^2}\right)\frac{x^2\theta_n}{2}\sin\theta_n y + O(x^3) \quad ,$$

$$\tag{1.55}$$

$$\bar{v}(x,y,p) = \left[a_0(p) + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} a_n(p)\cos\theta_n y\right]x + \frac{1}{2}\left\{\frac{v_0}{c_s^2} + b_0(p)\frac{(k^2-1)}{h}\left[\frac{1}{4}(1-y/h)^{-\frac{5}{4}}\right.\right.$$

$$\left.\left.+\frac{1}{4}(1+y/h)^{-\frac{5}{4}} - 2^{-\frac{1}{4}}\right] - \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} b_n(p)(k^2-1)\,\theta_n\cos\theta_n y\right\}x^2 + O(x^3) \quad .$$

Differentiating with respect to x, (1.55) yields the same Laplace transformed strains at the base as were assumed in (1.16). The coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ are given by (1.28) and Table I. Substituting these into (1.55) and inverting gives

$$u(x,y,t) = \left\{B_0\left[(1-y/h)^{-\frac{1}{4}} - (1+y/h)^{-\frac{1}{4}} + 2^{-\frac{1}{4}} y/h\right] + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} B_n \sin\theta_n y\right\} x$$

$$+ \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} A_n \left(\frac{k^2-1}{k^2}\right) \frac{x^2\theta_n}{2} \sin\theta_n y + O(x^3) \quad ,$$

$$v(x,y,t) = \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} A_n x \cos\theta_n y + \frac{1}{2}\left\{B_0 \frac{(k^2-1)}{h}\left[\frac{1}{4}(1-y/h)^{-\frac{5}{4}} + \frac{1}{4}(1+y/h)^{-\frac{5}{4}} - 2^{-\frac{1}{4}}\right]\right. \tag{1.56}$$

$$\left. - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} B_n (k^2-1)\theta_n \cos\theta_n y\right\} x^2 + O(x^3) \quad ,$$

for x small and t large. Substituting the displacements from (1.56) into (1.1), it is easily seen that they satisfy the displacement equations of motion to lowest order in x.

The near field solution is not singular for $x > 0$ when $y = h$. This is clear from (1.37) if it is noted that $f_E(s,y,p)$ and $f_G(s,y,p)$, and hence $\tilde{u}_4$ and $\tilde{v}_4$, involve only integrable singularities. Since for $y = h$, $H_\alpha^y = H_\alpha$, $R_\alpha^y = R_\alpha$, etc., $f_E(s,y,p)$ and $f_G(s,y,p)$ can be approximated using (1.45) and (1.48), giving

$$f_E(s,h,p) = b_0(p)\, O\left(\frac{1}{s^4}\right) ,$$

$$f_G(s,h,p) = b_0(p)\, O\left(\frac{1}{s^5}\right) . \tag{1.57}$$

Also,

$$f_F(s,h,p) \equiv 0 ,$$

$$f_H(s,h,p) = \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} (-1)^{n/2} \left[\frac{a_n(p)}{s^2} - b_n(p)(k^2-1)\frac{\theta_n}{s^3}\right] + O\left(\frac{1}{s^4}\right) , \tag{1.58}$$

while $f_K(s,h,p)$ is given by the last of (1.53). From (1.57), (1.58) it follows that

$$\tilde{u}(x,h,p) = b_0(p)\, O\left(\frac{1}{s^4}\right) ,$$

$$\tilde{v}(x,h,p) = \left[a_0(p) + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} (-1)^{n/2} a_n(p)\right]\frac{1}{s^2} + \left[\frac{v_0}{c_s^2} - \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} (-1)^{n/2} b_n(p)(k^2-1)\theta_n\right]\frac{1}{s^3} \tag{1.59}$$

$$+ a_n(p)\, O\left(\frac{1}{s^4}\right) + b_0(p)\, O\left(\frac{1}{s^5}\right) .$$

Substituting for the coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ from

(1.28) and inverting yields

$$u(x,h,t) = B_0\, O(x^3) \quad ,$$

$$v(x,h,t) = \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} (-1)^{n/2} \left[A_n x - B_n \frac{(k^2-1)}{2} \theta_n x^2 \right] + O(x^3) \quad . \tag{1.60}$$

If $(1-y/h)$ gets small at the same rate as x does, the behavior of (1.60) is in agreement with that of (1.56). The singular terms in (1.56), of course, get large as $y \to h$ which is not consistent with their absence at $y = h$. As Miklowitz shows in [2], through further parts integrations of $f_E(s,y,p)$ and $f_G(s,y,p)$, a series of singular terms (valid for small x) of the form $x^n/(1-y/h)^{n-\frac{3}{4}}$ are obtained for $\tilde{\bar{u}}_4$ and $\tilde{\bar{v}}_4$. These terms alternate in sign (cf.(54) in [2]) and must cancel the singular term in (1.56). It follows then, that for the leading singular term in (1.56) to represent u, x must vanish at a faster rate than $(1-y/h)^{\frac{1}{4}}$ does, i.e., the asymptotic solution, (1.56) is limited to smaller and smaller x values as the corner at $y = h$ is approached. Otherwise the use of additional terms involving x^3, x^5, . . . , will be needed, which will still have limitations as y gets closer and closer to h.

b. <u>Inversion by Residue Theory for the Domain $h < x \ll c_s t$</u>

The doubly transformed displacements $\tilde{\bar{u}}(s,y,p)$ and $\tilde{\bar{v}}(s,y,p)$ are even functions of α and β; thus they have no branch points in the s-plane, and as a result they can be inverted solely by residue theory.

From (1.13), $s_j(p)$ is defined as the roots of

$$L[s_j(p), p] = 0 \quad .$$

Then the poles of $\tilde{\tilde{u}}(s,y,p)$ and $\tilde{\tilde{v}}(s,y,p)$ are

$$\text{and} \qquad \begin{aligned} s &= 0 \\ s &= s_j(p) \end{aligned} \tag{1.61}$$

where, as a result of the boundedness condition, (1.14), $\mathrm{Re}[s_j(p)] < 0$. This gives

$$\begin{aligned} \bar{u}_i(x,y,p) &= R^{u_i}(0) + \sum_{j=0}^{\infty} R_j^{u_i}\left(s_j(p)\right) \quad , \\ \bar{v}_i(x,y,p) &= R^{v_i}(0) + \sum_{j=0}^{\infty} R_j^{v_i}\left(s_j(p)\right) \quad , \end{aligned} \tag{1.62}$$

where $R_j^{u_i}\left(s_j(p)\right)$ and $R_j^{v_i}\left(s_j(p)\right)$ are the residues of $\tilde{\tilde{u}}_i(s,y,p)\,e^{sx}$ and $\tilde{\tilde{v}}_i(s,y,p)\,e^{sx}$, respectively, at the pole $s_j(p)$.

Now $\tilde{\tilde{u}}_3(s,y,p)$, $\tilde{\tilde{v}}_3(s,y,p)$ and $\tilde{\tilde{u}}_4(s,y,p)$ have no poles in the s-plane and the only pole of $\tilde{\tilde{v}}_4(s,y,p)$ comes from the pole of $f_\kappa(s,y,p)$ at $s = 0$. Therefore

$$\begin{aligned} \bar{u}_3(x,y,p) &= \bar{v}_3(x,y,p) = 0 \quad , \\ \bar{u}_4(x,y,p) &= 0 \quad , \\ \bar{v}_4(x,y,p) &= -\frac{v_0}{p^2} \quad . \end{aligned} \tag{1.63}$$

On the other hand, the other doubly transformed displacements do not have a pole at $s = 0$. They do, however, have an infinite number of poles at the modes $s = s_j(p)$, $\mathrm{Re}[s_j(p)] < 0$.

Lowest Mode Contribution

First consider the contribution of the lowest mode to the solution. From (1.21), for p small, the lowest mode is given by

$$s_0(p) = \pm(1 \pm i)\gamma \quad , \quad \gamma = \sqrt{\frac{p}{2c_p r_g}} \quad .$$

Of these, only the roots $s_0(p) = -(1\pm i)\gamma$ will have non-zero residues.

By definition

$$\begin{Bmatrix} R_0^{u_i}\left(s_0(p)\right) \\ R_0^{v_i}\left(s_0(p)\right) \end{Bmatrix} = \lim_{s \to s_0(p)} \left[\left(s - s_0(p)\right) \begin{Bmatrix} \tilde{\tilde{u}}_i(s, y, p) \\ \tilde{\tilde{v}}_i(s, y, p) \end{Bmatrix} e^{sx} \right] \quad . \qquad (1.64)$$

Expanding $R_0^{u_i}\left(s_0(p)\right)$ gives

$$R_0^{u_i}\left(s_0(p)\right) = f_A(s_0, p) \cdot I_u(s_0, y, p)\, e^{s_0 x} \lim_{s \to s_0(p)} \left[\left(s - s_0(p)\right)/L(s, p)\right] \quad .$$

Expand $L(s, p)$ in a Taylor series about $s = s_0(p)$. Then, since

$L(s_0, p) = 0,$

$$\underset{s \to s_0(p)}{\text{limit}} \left[\Big(s - s_0(p) \Big) / L(s,p) \right] = \left[\frac{\partial L}{\partial s} \Big|_{s = s_0(p)} \right]^{-1} \quad . \tag{1.65}$$

Along a mode of the Rayleigh-Lamb frequency equation, $L(s,p) = 0$. Therefore

$$dL = \frac{\partial L}{\partial s}\, ds + \frac{\partial L}{\partial p}\, dp = 0 \quad . \tag{1.66}$$

After a considerable amount of algebra, it can be shown that

$$\frac{\partial L}{\partial p} \Big|_{s = s_0(p)} = \frac{2ih}{c_s} k_s^3 \, s_0(p) + O(p^{\frac{9}{2}}) \quad . \tag{1.67}$$

For $s_0(p) = -(1 \pm i)\gamma$

$$\frac{dp}{ds} = \frac{2p}{s_0(p)} \quad . \tag{1.68}$$

Combining (1.66) through (1.68) gives

$$\frac{\partial L}{\partial s} \Big|_{s = s_0(p)} = -4 i k_s^4 h + O(p^5) \quad . \tag{1.69}$$

Write

$$f_A(s,p) = \hat{f}_A(s,p) + r_A(s,p) \quad ,$$

where **(cont.)**

$$\hat{f}_A(s,p) = -\frac{s}{k_s^2}\, b_0(p)\left\{\frac{\left(2s^2-k_s^2\right)}{\alpha}\cosh\alpha h\,(S_\alpha - T_\alpha) + 2\beta\cosh\beta h\,(S_\beta - T_\beta)\right\} ,$$

$$r_A(s,p) = -\frac{s}{k_s^2}\, b_0(p)\, 2^{-\frac{1}{4}}\left[\frac{\left(2s^2-k_s^2\right)}{\alpha^2} + 2\right] . \tag{1.70}$$

Then the following approximations are valid for $s = s_0(p)$ and p small

$$\left\{\begin{matrix}\cosh\alpha y\\ \cosh\beta y\end{matrix}\right\} = 1 + O(p)\,y^2 ,$$

$$\left\{\begin{matrix}\sinh\alpha y\\ \sinh\beta y\end{matrix}\right\} = i s_0(p)\,y + O(p^{\frac{3}{2}})\,y^3 , \tag{1.71}$$

$$\hat{f}_A(s_0,p) = -2^{-\frac{1}{4}}\frac{8}{21}\, b_0(p)\, s_0(p)\, h^2 + b_0(p)\, O(p^{\frac{3}{2}}) ,$$

$$I_u(s_0,y,p) = -i\,k^2 k_s^2\, s_0^2(p)\,y + O(p^4) . \tag{1.72}$$

Equations (1.69), (1.70) and (1.72) are now used to give

$$R_0^{u_1}\left(s_0(p)\right) = -2^{-\frac{1}{4}}\frac{2}{21}\, b_0(p)\frac{k^2}{k_s^2}\, s_0^3(p)\, hy\, e^{s_0(p)x} + b_0(p)\, O(p^{\frac{1}{2}})$$

$$+ r_A(s_0,p)\cdot I_u(s_0,y,p)\left[\frac{\partial L}{\partial s}\Big|_{s=s_0(p)}\right]^{-1} e^{s_0(p)\,x} . \tag{1.73}$$

$R_0^{u_2}\left(s_0(p)\right)$ is found in exactly the same manner as $R_0^{u_1}\left(s_0(p)\right)$ was. First let

$$f_B(s,p) = \hat{f}_B(s,p) + r_B(s,p)$$

where (1.74)

$$\hat{f}_B(s,p) = -\frac{k^2}{k_s^2} b_0(p)\left\{2s^2 \sinh \alpha h(S_\alpha - T_\alpha) - \left(2s^2 - k_s^2\right) \sinh \beta h(S_\beta - T_\beta)\right\} ,$$

$$r_B(s,p) = -\frac{k^2}{k_s^2} b_0(p)\frac{2^{-\frac{1}{4}}}{h}\left[\frac{2s^2}{\alpha^2} - \frac{\left(2s^2 - k_s^2\right)}{\beta^2}\right] .$$

Approximating gives

$$\hat{f}_B(s_0,p) = b_0(p)\, O(p) ,$$

$$f_C(s_0,p) = -\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty}\left[(-1)^{n/2} b_n(p)\frac{k^2}{\theta_n} + a_n(p)\, O(p^{\frac{1}{2}})\right] ,$$

(1.75)

$$f_D(s_0,p) = \frac{v_0}{c_s^2 s_0^2(p)} + \frac{a_0(p)}{s_0(p)} + O(1) + a_0(p)\, O(p^{\frac{1}{2}}) ,$$

$$J_u(s_0,y,p) = ik_s^2 s_0^3(p)\, yh + O(p^{\frac{9}{2}}) .$$

Combining all these yields

$$R_0^{u_2}\left(s_0(p)\right) = \left\{ -\frac{v_0 s_0(p)}{4p^2} - \frac{a_0(p)\, s_0^2(p)}{4k_s^2} + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{(-1)^{n/2} b_n(p)\, k^2 s_0^3(p)}{4k_s^2 \theta_n} \right\} y\, e^{s_0(p) x}$$

$$+ O(1) + a_0(p)\, O(p^{\frac{1}{2}}) + a_n(p)\, O(p^{\frac{1}{2}}) + b_0(p)\, O(p) \qquad (1.76)$$

$$+ r_B(s_0, p) \cdot J_u(s_0, y, p) \left[\frac{\partial L}{\partial s} \bigg|_{s = s_0(p)} \right]^{-1} e^{s_0(p) x} \quad .$$

The total contribution of the pole at $s_0(p)$ to $\overline{u}(x, y, p)$ can be found by adding $R_0^{u_1}\left(s_0(p)\right)$ from (1.73) to $R_0^{u_2}\left(s_0(p)\right)$ from (1.76). Set the sum of the last term in (1.73) plus the last term in (1.76) equal to $\hat{R}_0^u\left(s_0(p)\right)$, i.e.,

$$\hat{R}_0^u\left(s_0(p)\right) = \left[r_A(s_0, p) \cdot I_u(s_0, y, p) + r_B(s_0, p) \cdot J_u(s_0, y, p) \right] \left[\frac{\partial L}{\partial s} \bigg|_{s = s_0(p)} \right]^{-1} e^{s_0(p) x} \quad .$$

$$(1.77)$$

Approximating $\hat{R}_0^u\left(s_0(p)\right)$ gives, after a great deal of algebra,

$$\hat{R}_0^u\left(s_0(p)\right) = -2^{-\frac{1}{4}}\, b_0(p) \frac{k^2}{12 k_s^2}\, s_0^3(p)\, h y\, e^{s_0(p) x} + b_0(p)\, O(p^{\frac{1}{2}}) \quad . \qquad (1.78)$$

Then from (1.73), (1.76) and (1.78),

$$R_0^u\left(s_0(p)\right) = \left\{-2^{-\frac{1}{4}}\frac{5}{7}b_0(p) + \sum_{\substack{n=2\\ n\text{ even}}}^{\infty}(-1)^{n/2}b_n(p)\frac{2}{n\pi}\right\}\frac{k^2 s_0^3(p)}{4k_s^2}\, hy\, e^{s_0(p)x}$$

(1.79)

$$-\left[\frac{v_0 s_0(p)}{4p^2} + \frac{a_0(p)s_0^2(p)}{4k_s^2}\right] y\, e^{s_0(p)x} .$$

Now we substitute the coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ from (1.28) into (1.79). Then, from the boundedness equations (the first and second equations of (1.29)), the following equations hold:

$$a_0(p) = \frac{A_0}{\sqrt{p}} = -\frac{v_0}{c_s^2\gamma} ,$$

$$-2^{-\frac{1}{4}}\frac{5}{7}b_0(p) + \sum_{\substack{n=2\\ n\text{ even}}}^{\infty}(-1)^{\frac{n}{2}}\frac{2b_n(p)}{n\pi} = -\frac{v_0}{c_d p}\sqrt{\frac{2}{3k^2(1-\nu)}} .$$

Using these relations, (1.79) reduces to

$$R_0^u\left(s_0(p)\right) = \frac{v_0}{4p^2}\left[-s_0(p) + \frac{s_0^2(p)}{\gamma} - \frac{s_0^3(p)}{2\gamma^2}\right] y\, e^{s_0(p)x} + O(p^{-\frac{1}{2}}) . \quad (1.80)$$

Define the contribution to $\bar{u}(x,y,p)$ by the lowest mode by $\bar{u}^L(x,y,p)$. Then $\bar{u}^L(x,y,p) = R_0^u[(-1+i)\gamma] + R_0^u[-(1+i)\gamma]$. Evaluating this expression gives

$$\bar{u}^{L}(x,y,p) = \frac{2v_0}{p^2}\gamma y \sin\gamma x\, e^{-\gamma x} + O(p^{-\frac{1}{2}}) . \qquad (1.81)$$

The lowest mode contribution to $\bar{v}(x,y,p)$ is calculated the same way. Starting with the second of (1.64), $R_0^{v_i}\big(s_0(p)\big)$ is expanded using the same approximations as were used to calculate $R_0^{u_i}\big(s_0(p)\big)$. The only new approximations needed are

$$I_v(s_0,y,p) = -ik^2 k_s^2 s_0(p) + O(p^{\frac{7}{2}}) ,$$

$$J_v(s_0,y,p) = ik_s^2 s_0^2(p)\, h + O(p^4) . \qquad (1.82)$$

The lowest mode contribution, $\bar{v}^{L}(x,y,p)$, is

$$\bar{v}^{L}(x,y,p) = \frac{v_0}{p^2}(\cos\gamma x + \sin\gamma x)\, e^{-\gamma x} + O(p) . \qquad (1.83)$$

Contribution of the Complex Segments of the Higher Branches

For the higher modes, $s_j(p) = \hat{s}_j$, $j \geq 1$ where $\hat{s}_j$ is a complex constant satisfying (1.22). For $\hat{s}_j$ to be a pole, $\mathrm{Re}(\hat{s}_j)$ must be negative. Let

$$\hat{s}_j = -\hat{s}_j^R \pm \hat{s}_j^I \qquad (1.84)$$

where $\hat{s}_j^R$ and $\hat{s}_j^I$ are real, positive numbers. It can easily be shown that

$$\left\{ \begin{array}{l} R_j^u i\,(\hat{s}_j) \\ \\ R_j^v i\,(\hat{s}_j) \end{array} \right\} = \frac{K_j(y)}{p}\, e^{\pm i \hat{s}_j^I x}\, e^{-\hat{s}_j^R x} \qquad (1.85)$$

where, for a given mode $\hat{s}_j$, $K_j(y)$ is only a function of y. So the higher modes give edge waves that decay exponentially with x. From Hillman and Salzer, [10], the smallest $\hat{s}_j^R$ is $\hat{s}_1^R = 3.7488/h$. So the decay of the higher modes contribution is quite rapid and they may be neglected for x/h greater than about one. Their contribution is quite important, however, for x small where they supply the difference between the near field asymptotic solution and the lowest mode contribution.

Since the higher modes will not contribute for $x/h > 1$, in this region, from (1.63), (1.81) and (1.83) we have that

$$\begin{aligned} \bar{u}(x,y,p) &= \frac{2v_0}{p^2}\gamma y \sin\gamma x\, e^{-\gamma x} + O(p^{-\frac{1}{2}}) \,, \\ \bar{v}(x,y,p) &= \frac{v_0}{p^2}\left[(\cos\gamma x + \sin\gamma x)\, e^{-\gamma x} - 1\right] + O(p) \,. \end{aligned} \qquad (1.86)$$

From the second of (1.2),

$$\bar{\sigma}_{yy}(x,y,p) = \mu\left[(k^2-2)\,\bar{u}_x(x,y,p) + k^2\,\bar{v}_y(x,y,p)\right] .$$

Differentiating (1.86) gives, to lowest order in p,

$$\bar{u}_x(x,y,p) = \frac{2v_0}{p^2}\gamma^2 y\,(\cos\gamma x - \sin\gamma x)\,e^{-\gamma x} \quad ,$$

$$\bar{v}_y(x,y,p) = \frac{\partial}{\partial y}\left\{O(p) \text{ terms in } \bar{v}(x,y,p) \text{ in } (1.86)\right\} \quad . \tag{1.87}$$

Now the $O(p)$ terms in $\bar{v}(x,y,p)$ are of two types. First there are the terms that come from retaining more terms in the approximations that were used for $f_A(s_0,p)$, $f_B(s_0,p)$, $f_C(s_0,p)$, $f_D(s_0,p)$ and $\left[\left.\frac{\partial L}{\partial s}\right|_{s=s_0(p)}\right]^{-1}$. These terms will be independent of y. Secondly, there are the terms that come from retaining additional terms in the approximations for $I_v(s_0,y,p)$ and $J_v(s_0,y,p)$. Some of these terms will come from keeping the second term in the expansions for $\cosh\alpha y$ and $\cosh\beta y$. The first of (1.71) shows that these terms will be proportional to y^2. Therefore

$$\bar{v}_y(x,y,p) = y\,O(p) \quad .$$

Now, from the boundary conditions $\bar{\sigma}_{yy}(x,\pm h,p) = 0$. Since $\bar{\sigma}_{yy}(x,y,p)$ is linear in y, it must be zero everywhere. Therefore, using (1.87), it can be shown that

$$\bar{v}_y(x,y,p) = -\left(\frac{k^2-2}{k^2}\right)\bar{u}_x(x,y,p) \tag{1.88}$$

to lowest order in p, i.e., $O\left(\frac{1}{p}\right)$. Keeping the next highest order terms gives, for p small,

$$\bar{\sigma}_{yy}(x, y, p) = O(1) \quad . \tag{1.89}$$

From the first of (1.2)

$$\bar{\sigma}_{xx}(x, y, p) = \mu\left[k^2 \bar{u}_x(x, y, p) + (k^2 - 2)\bar{v}_y(x, y, p)\right] \quad .$$

Substituting from (1.87) and (1.88) gives

$$\bar{\sigma}_{xx}(x, y, p) = \frac{8v_0\gamma^2}{p^2}\mu\left(\frac{k^2 - 1}{k^2}\right) y (\cos\gamma x - \sin\gamma x) e^{-\gamma x} + O(1) \quad . \tag{1.90}$$

Similarly

$$\bar{\sigma}_{xy}(x, y, p) = \mu\left[\bar{u}_y(x, y, p) + \bar{v}_x(x, y, p)\right] \quad .$$

From (1.86), to lowest order in p, $\bar{u}_y(x, y, p) = -\bar{v}_x(x, y, p)$. So the $O(p^{-\frac{3}{2}})$ terms cancel. Keeping the next order terms gives

$$\bar{\sigma}_{xy}(x, y, p) = O(p^{-\frac{1}{2}}) \quad . \tag{1.91}$$

As (1.89) through (1.91) show, $\sigma_{xx}(x, y, t)$ will be the dominant stress for long-time provided $\frac{x}{h} > 1$.

Inversion of the Time Transform

Behind the body wave fronts, but for $\frac{x}{h} > 1$, the time Laplace transformed displacements for the residual problem are given by (1.86) and the transformed stresses by (1.89) through (1.91). To obtain the solution to the original problem, the term $\frac{v_0}{p^2}$ must be added to $\overline{v}(x,y,p)$ in (1.86). The displacements and stresses are then inverted using the tables in Abromowitz and Stegun, [11]. This gives, for t large

$$u(x,y,t) = 4v_0 y\sqrt{\frac{t}{2\pi c_p r_g}}\left[\sin\Delta + \frac{\pi}{2}\eta\left(1-2C_2(\Delta)\right)\right] + O\left(\frac{1}{\sqrt{t}}\right) \quad .$$

$$v(x,y,t) = -2v_0 t\left\{S_2(\Delta) + \eta\sin\Delta + \Delta\left[1-2C_2(\Delta)\right]\right\} + O(1) \quad , \tag{1.92}$$

$$\sigma_{xx}(x,y,t) = \frac{2v_0\mu y}{c_p r_g(1-\nu)}\left[1-2C_2(\Delta)\right] \quad ,$$

$$\sigma_{yy}(x,y,t) = 0 \quad , \tag{1.93}$$

$$\sigma_{xy}(x,y,t) = O\left(\frac{1}{\sqrt{t}}\right) \quad .$$

Here $C_2(\Delta)$ and $S_2(\Delta)$ are the Frenel integrals defined by

$$C_2(\Delta) = \frac{1}{\sqrt{2\pi}}\int_0^{\Delta}\frac{\cos z}{\sqrt{z}}\,dz \quad ,$$

(cont.)

$$S_2(\Delta) = \frac{1}{\sqrt{2\pi}}\int_0^{\Delta} \frac{\sin z}{\sqrt{z}}\, dz \quad , \tag{1.94}$$

and

$$\Delta = \frac{x^2}{4c_p r_g t} \quad ,$$

$$\eta = \frac{x}{\sqrt{2\pi c_p r_g t}} \tag{1.95}$$

The displacement at the centerline, $\frac{c_d}{v_0 h}\, v(x, 0, t)$, and the stress at $y = h$, $\frac{c_d}{v_0 u}\, \sigma_{xx}(x, \frac{h}{2}, t)$, were calculated for two times $t = 100\, t_h$ and $t = 400\, t_h$ where $t_h = h/c_d$. The displacements are graphed in Fig. 4 and the stresses in Fig. 5. Observing the stresses in Fig. 5, it is seen that the shortest wavelength (highest frequency) waves lead in the disturbance, progressively becoming longer and longer as x decreases for a given time. But since the solution is valid only for low frequency, the large x response in these curves is inadmissable. For $t = 100\, t_h$, the region $x > \xi_{100}$ has arbitrarily been ruled out. Since (1.92) and (1.93) are not valid for $x < h$, the solutions graphed in Figs. 4 and 5 only hold in the region $h < x < \xi_{100}$ for $t = 100\, t_h$. Similarly, for $t = 400\, t_h$, the solutions hold in the region $h < x < x_{400}$. It is clear, therefore, that as time increases the solution is valid for larger and larger x.

As was mentioned earlier (see discussion following (1.34)), the problem can be solved approximately using Euler-Bernoulli approximate theory. When this is done, (1.92) results for the displacements

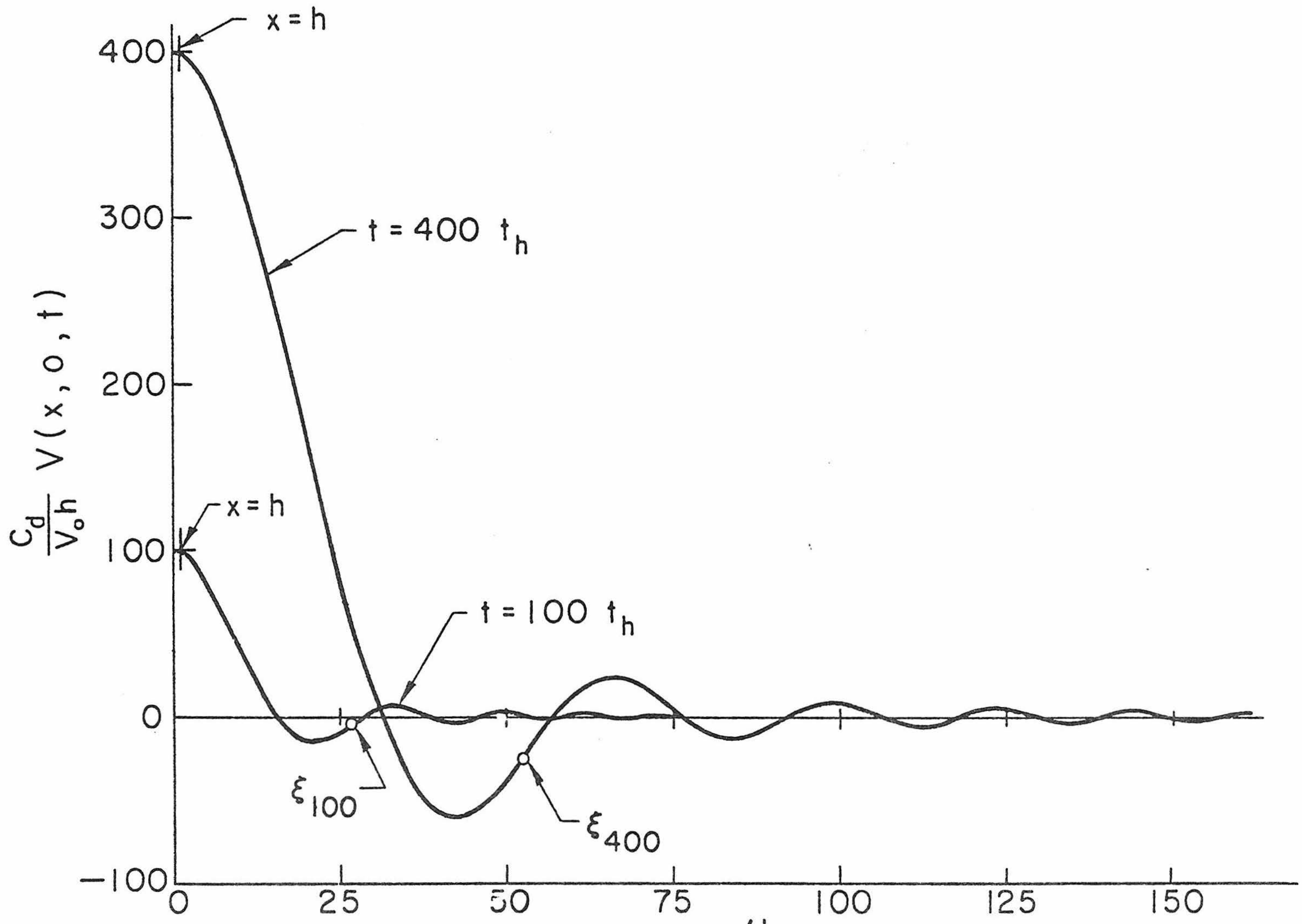

Fig. 4 Centerline Displacement, $v(x, 0, t)$, for the Semi-Infinite Plate.

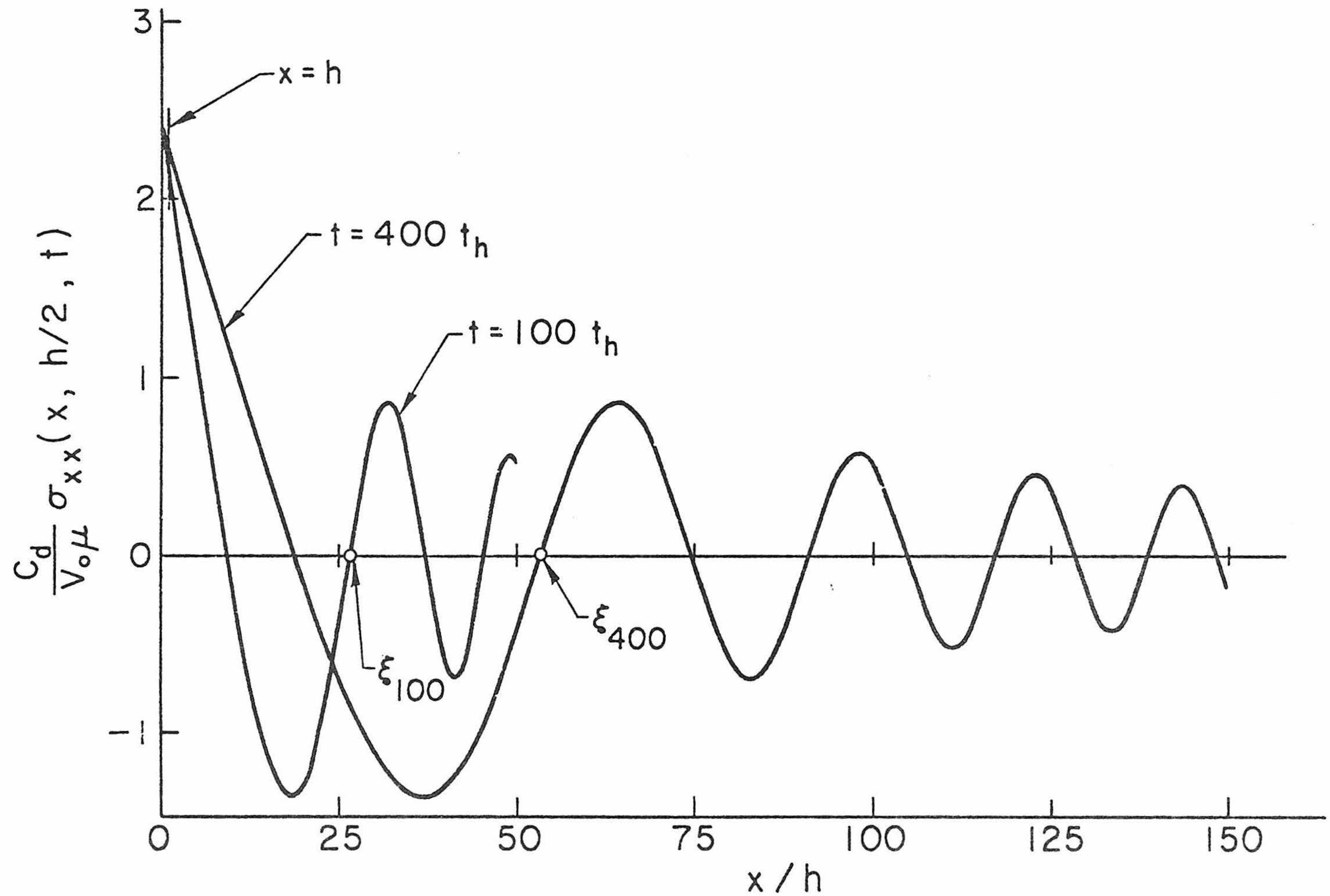

Fig. 5 Stress $\sigma_{xx}(x, y, t)$ at $y = h/2$ for the Semi-Infinite Plate.

and (1.93) for the stresses except that the order terms are missing. So for long-time in the region $x > h$, the Euler-Bernoulli approximate theory gives the dominant terms in the solution. In the region $0 \leq x < h$, the exact solution differs from the approximate solution by a series of terms that decay exponentially with x. At the base, $x = 0$, the Euler-Bernoulli approximate theory gives the total moment and the net shear force to lowest order in t. However, it must be emphasized that it gives no information at all about the important singularities at the corners $y = \pm h$, $x = 0$.

II. THE FINITE PLATE

1. Formulation, Formal Solution and Entirety Condition

Once the semi-infinite cantilevered plate problem has been solved, the problem of a similar finite cantilevered plate, built-in at the base $(x = 0)$ and stress free at the other end $(x = \ell)$, can be solved for the long-time response. The plate is depicted in Fig. 6. The problem is formulated in exactly the same way that the semi-infinite plate problem was, i.e., with the displacement equations of motion, (1.1), the stress-strain relations, (1.2), the initial conditions, (1.3), and the same boundary conditions at the base $(x = 0)$ and on the plate faces $(y = \pm h)$, (1.4). The only changes are that the radiation conditions, (1.5), are replaced by

$$\sigma_{xx}(\ell, y, t) = \sigma_{xy}(\ell, y, t) = 0 \quad \text{for} \quad -h \le y \le h, \ t \le 0 \quad , \qquad (2.1)$$

and that

$$u(x, y, t) = v(x, y, t) = 0 \quad \text{for} \quad x > \ell, \ -h \le y \le h, \ t \ge 0 \quad . \qquad (2.2)$$

In the work that follows, it will be assumed that the length of the plate, ℓ, is greater than the width, $2h$.

As before, the problem is decomposed into a rigid body motion and a residual problem. The residual problem will still satisfy the initial and boundary conditions, (1.7) and (1.8), but now the radiation condition, (1.9), is replaced by (2.1).

To derive the formal solution, first use the one-sided Laplace transform with respect to t, parameter p. Transforming the

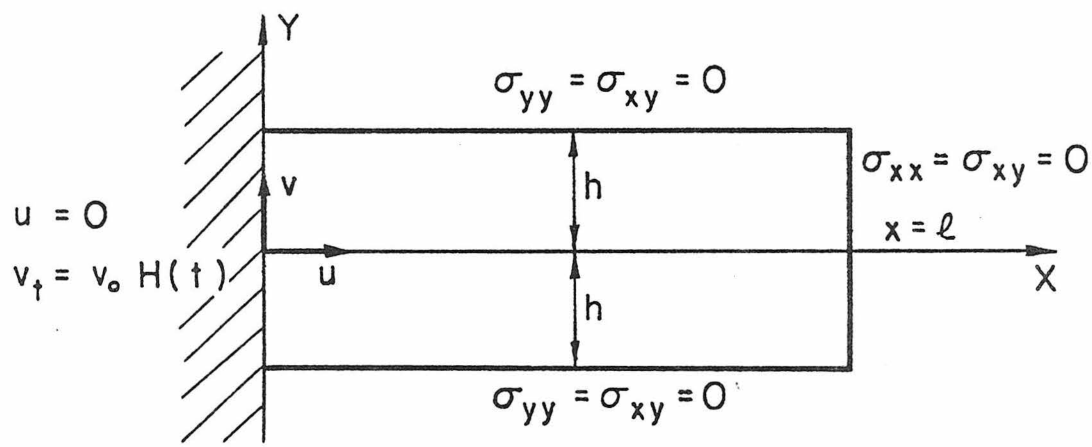

Fig. 6 Geometry and Boundary Conditions for the Finite Plate.

displacement equations of motion, (1.1), gives

$$k^2 \bar{u}_{xx}(x,y,p) + (k^2-1)\bar{v}_{xy}(x,y,p) + \bar{u}_{yy}(x,y,p) = k_s^2 \bar{u}(x,y,p) \quad ,$$

$$\bar{v}_{xx}(x,y,p) + (k^2-1)\bar{u}(x,y,p) + k^2 \bar{v}_{yy}(x,y,p) = k_s^2 \bar{v}(x,y,p) + \frac{v_0}{c_s^2} \quad , \qquad (2.3)$$

where the $\frac{v_0}{c_s^2}$ term comes from the non-zero initial condition in (1.7).

Noting (2.2), introduce the one-sided finite Laplace transform with respect to x, parameter s, defined by

$$\tilde{\bar{u}}(s,y,p) = \int_0^{\ell} \bar{u}(x,y,p)\, e^{-sx} dx \quad ,$$

$$\tilde{\bar{u}}_x(s,y,p) = s\tilde{\bar{u}}(s,y,p) - \left[\bar{u}(0,y,p) - \bar{u}(\ell,y,p)\, e^{-s\ell}\right], \text{ etc.} \qquad (2.4)$$

Transforming (2.3) gives

$$\frac{d^2\tilde{\bar{u}}}{dy^2}(s,y,p) + (k^2-1)\, s\, \frac{d\tilde{\bar{v}}}{dy}(s,y,p) + \left(k^2 s^2 - k_s^2\right)\tilde{\bar{u}}(s,y,p) = g(s,y,p) \quad ,$$

$$\frac{d^2\tilde{\bar{v}}}{dy^2}(s,y,p) + \left(\frac{k^2-1}{k^2}\right) s\, \frac{d\tilde{\bar{u}}}{dy}(s,y,p) + \left(\frac{s^2-k_s^2}{k^2}\right)\tilde{\bar{v}}(s,y,p) = h(s,y,p) \quad , \qquad (2.5)$$

where

$$
\begin{aligned}
g(s,y,p) = {} & k^2\left[s\bar{u}(0,y,p) + \bar{u}_x(0,y,p)\right] + (k^2-1)\bar{v}_y(0,y,p) \\
& - k^2\left[s\bar{u}(\ell,y,p) + \bar{u}_x(\ell,y,p)\right]e^{-s\ell} - (k^2-1)\bar{v}_y(\ell,y,p)\,e^{-s\ell} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
h(s,y,p) = {} & \frac{1}{k^2}\left[s\bar{v}(0,y,p) + \bar{v}_x(0,y,p) + (k^2-1)\bar{u}_y(0,y,p) + \frac{v_0}{sc_s^2}\right] \\
& - \frac{1}{k^2}\left[s\bar{v}(\ell,y,p) + \bar{v}_x(\ell,y,p) + (k^2-1)\bar{u}_y(\ell,y,p) + \frac{v_0}{sc_s^2}\right]e^{-s\ell} \quad .
\end{aligned}
\tag{2.6}
$$

Equation (2.5) has exactly the same form as (9) in [5]. Solving this equation once again yields (1.10) through (1.12) for the formal solution where $g(s,y,p)$ and $h(s,y,p)$ are now given by (2.6).

In [12], Widder proves that the function

$$
f(s) = \int_a^b e^{-st}\,d\alpha(t) \quad , \quad 0 \le a \le b < \infty
$$

is entire. Comparing with the first of (2.4) shows that $\tilde{\bar{u}}(s,y,p)$ and $\tilde{\bar{v}}(s,y,p)$ can have no poles in the s- plane.

For the semi-infinite waveguide it was only necessary to rule out the poles in the right half s- plane, i.e. for $\mathrm{Re}\left[s_j(p)\right] > 0$ where $s_j(p)$ was defined by (1.13). Now if $s_j(p)$ satisfies (1.13), so do $\bar{s}_j(p)$, $-s_j(p)$ and $-\bar{s}_j(p)$. Two of these will have real parts greater than zero and two will have negative real parts. For the finite waveguide it is necessary to set the residue of all four of these roots equal to zero. This generates an entirety condition which finally will determine the transformed edge unknowns. So (1.14) now must hold for $s_j(p)$ and for $-s_j(p)$ and their conjugates. This gives four coupled integral equations for the edge unknowns at $x = 0$ and $x = \ell$. These allow four of the edge unknowns, two at each end, to be calculated. The boundary conditions on the ends of the plate can now be used to completely determine the rest of the transformed edge unknowns and hence the formal solution.

2. Forms for the Edge Unknowns from Euler-Bernoulli Approximate Theory

In the theme of the earlier work by Miklowitz and Sinclair, [4] and [5], representations will be set down here for the transformed edge unknowns. They will be found from the Euler-Bernoulli theory since then they would be expected to be at least a part of the total representations for the edge unknowns for long-time.

To solve the entirety equations, it is necessary to assume forms for the edge unknowns at $x = 0$ and at $x = \ell$. The same representations for the unknown strains at the base will be used as were used for the semi-infinite plate problem. Sinclair and Miklowitz showed in [5] that, for an antisymmetric plate with both stresses prescribed on the end, the edge unknowns agreed to lowest order in time with the forms found for them using Euler-Bernoulli approximate theory. It seems reasonable to expect similar behavior for the edge unknowns at $x = \ell$ for the present problem.

The Euler-Bernoulli approximation to the present problem is formulated in the usual way. The governing equations are

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{1}{c_p^2 r_g^2} \frac{\partial^2 v(x,t)}{\partial t^2} = 0 \quad ,$$

$$u(x,y,t) = -y \frac{\partial v(x,t)}{\partial x} \quad ,$$

$$M(x,t) = -2h\, c_p^2 r_g^2 \frac{\partial^2 v(x,t)}{\partial x^2} \quad , \tag{2.7}$$

$$Q(x,t) = -2h\, c_p^2 r_g^2 \frac{\partial^3 v(x,t)}{\partial x^3} \quad ,$$

where $M(x,t)$ and $Q(x,t)$ are the net moment and the net shear force at x, respectively (see (1.31) and (1.33)). Initial and boundary conditions are

$$v(x,0) = v_t(x,0) = 0 \quad \text{for} \quad x > 0 \quad , \tag{2.8}$$

and

$$\left.\begin{aligned} v_t(0,t) &= v_0 H(t) \quad , \\ v_x(0,t) &= 0 \quad , \\ M(\ell,t) &= Q(\ell,t) = 0 \quad , \end{aligned}\right\} \text{for} \quad t \geq 0 \quad . \tag{2.9}$$

After Laplace transforming with respect to time and using the initial conditions, (2.8), the first of (2.7) becomes

$$\frac{d^4\bar{v}(x,p)}{dx^4} + \frac{p^2}{c_p^2 r_g^2}\bar{v}(x,p) = 0 \quad . \tag{2.10}$$

Next, after finite Laplace transforming with respect to x as in (2.4), (2.10) becomes

$$\left[s^4 + \frac{p^2}{c_p^2 r_g^2}\right]\tilde{\bar{v}}(s,p) = \bar{v}_{xxx}(0,p) + s\bar{v}_{xx}(0,p) + s^2\bar{v}_x(0,p) + s^3\bar{v}(0,p)$$

$$- \left[\bar{v}_{xxx}(\ell,p) + s\bar{v}_{xx}(\ell,p) + s^2\bar{v}_x(\ell,p) + s^3\bar{v}(\ell,p)\right]e^{-s\ell} \quad . \tag{2.11}$$

Using the boundary conditions and the last two of (2.7) reduces (2.11) to

$$\tilde{\bar{v}}(s,p) = \left[s^4 + \frac{p^2}{c_p^2 r_g^2}\right]^{-1}\left\{\bar{v}_{xxx}(0,p) + s\bar{v}_{xx}(0,p) + \frac{v_0 s^3}{p^2} - \left[s^2\bar{v}_x(\ell,p) + s^3\bar{v}(\ell,p)\right]e^{-s\ell}\right\} \quad . \tag{2.12}$$

To obtain the doubly transformed solution, the edge unknowns, $\bar{v}_{xxx}(0,p)$, $\bar{v}_{xx}(0,p)$, $\bar{v}_x(\ell,p)$ and $\bar{v}(\ell,p)$, must be found. Now, as was explained earlier, $\tilde{\bar{v}}(s,p)$ must be an entire function of s. Therefore the numerator of (2.12) must be zero whenever the denominator equals zero. Now the denominator of (2.12) has zeros at $s = \pm(1\pm i)\gamma$,

$\gamma = \sqrt{\frac{p}{2c_p r_g}}$. Substituting into the numerator gives the following four equations for the edge unknowns:

$$\bar{v}_{xxx}(0,p) + (1+i)\gamma \bar{v}_{xx}(0,p) + \left[-2i\ \gamma^2 \bar{v}_x(\ell,p) + 2(1-i)\gamma^3 \bar{v}(\ell,p)\right] e^{-(1+i)\gamma\ell} = 2(1-i)\frac{v_0\gamma^3}{p^2} \ ,$$

$$\bar{v}_{xxx}(0,p) + (1-i)\gamma \bar{v}_{xx}(0,p) + \left[2i\ \gamma^2 \bar{v}_x(\ell,p) + 2(1+i)\gamma^3 \bar{v}(\ell,p)\right] e^{-(1-i)\gamma\ell} = 2(1+i)\frac{v_0\gamma^3}{p^2} \ ,$$

(2.13)

$$\bar{v}_{xxx}(0,p) - (1+i)\gamma \bar{v}_{xx}(0,p) + \left[-2i\ \gamma^2 \bar{v}_x(\ell,p) - 2(1-i)\gamma^3 \bar{v}(\ell,p)\right] e^{(1+i)\gamma\ell} = -2(1-i)\frac{v_0\gamma^3}{p^2} \ ,$$

$$\bar{v}_{xxx}(0,p) - (1-i)\gamma \bar{v}_{xx}(0,p) + \left[2i\ \gamma^2 \bar{v}_x(\ell,p) - 2(1+i)\gamma^3 \bar{v}(\ell,p)\right] e^{(1-i)\gamma\ell} = -2(1+i)\frac{v_0\gamma^3}{p^2} \ .$$

Solving (2.13) for the edge unknowns yields

$$\bar{v}_{xxx}(0,p) = \frac{v_0}{c_p^2 r_g^2 \gamma}\,\frac{E(p)}{D(p)} \ ,$$

(cont.)

$$\bar{v}_{xx}(0,p) = -\frac{v_0}{c_p r_g p} \frac{F(p)}{D(p)} \quad , \tag{2.14}$$

$$\bar{v}_x(\ell,p) = \frac{v_0}{c_p r_g p \gamma} \frac{G(p)}{D(p)} \quad ,$$

(2.15)

$$\bar{v}(\ell,p) = \frac{2v_0}{p^2} \frac{H(p)}{D(p)} \quad ,$$

where

$$D(p) = \cosh^2 \gamma\ell + \cos^2 \gamma\ell \quad ,$$

$$E(p) = \cosh \gamma\ell \sinh \gamma\ell + \cos \gamma\ell \sin \gamma\ell \quad ,$$

$$F(p) = \cosh^2 \gamma\ell - \cos^2 \gamma\ell \quad , \tag{2.16}$$

$$G(p) = \sinh \gamma\ell \cos \gamma\ell - \cosh \gamma\ell \sin \gamma\ell \quad ,$$

$$H(p) = \cosh \gamma\ell \cos \gamma\ell \quad .$$

The edge unknowns obtained here are for the original problem where the base of the plate has a constant velocity (see (1.4a)). The residual problem is found by subtracting a rigid body motion $v(x,t) = v_0 t$ from the original problem. So to obtain the edge unknowns for the residual problem v_0/p^2 must be subtracted from $v(\ell, p)$. This gives

$$\bar{v}(\ell,p) = \frac{v_0}{p^2}\left[\frac{2H(p)}{D(p)} - 1\right] \quad . \tag{2.17}$$

Using the first of (2.15) and the second of (2.7) shows that

$$\bar{u}(\ell, y, p) = -\frac{v_0 y}{c_p r_g p \gamma} \frac{G(p)}{D(p)} \quad . \tag{2.18}$$

3. Solution of the Entirety Equations

The estimates for $\bar{u}(\ell, y, p)$ and $\bar{v}(\ell, y, p)$ in (2.17) and (2.18) will be called $\bar{u}^{EB}(\ell, y, p)$ and $\bar{v}^{EB}(\ell, y, p)$ where the EB superscripts show how the terms were found. To these estimates will be added a supplementary set of edge unknowns — distinguished by the superscript A — to account for the difference between the exact and Euler-Bernoulli theories. The exact theory forms for the edge unknowns are

$$\begin{aligned} \bar{u}(\ell, y, p) &= \bar{u}^{EB}(\ell, y, p) + \bar{u}^{A}(\ell, y, p) \quad , \\ \bar{v}(\ell, y, p) &= \bar{v}^{EB}(\ell, y, p) + \bar{v}^{A}(\ell, y, p) \quad , \end{aligned} \tag{2.19}$$

where $\bar{u}^{A}(\ell, y, p)$ and $\bar{v}^{A}(\ell, y, p)$ are two additional unknowns functions of y and p. Each will be represented by a Fourier series in y with the p dependence incorporated into the series coefficients. Now from the symmetry of the problem $\bar{u}^{A}(\ell, y, p)$ will be odd in y and hence will be represented by a Fourier sine series while $\bar{v}^{A}(\ell, y, p)$ will be even and represented by a cosine series. Quarter range Fourier series, similar to those used for the semi-infinite plate (see (1.16)), will be used. Thus

$$\bar{u}^{A}(\ell, y, p) = \sum_{n=1}^{\infty} c_n(p) \sin \frac{n\pi y}{2h} \quad ,$$

(cont.)

$$\bar{v}^{A}(\ell, y, p) = d_0(p) + \sum_{n=1}^{\infty} d_n(p) \cos \frac{n\pi y}{2h} \quad . \tag{2.20}$$

Later it will be shown that only the n even terms are needed to represent the edge unknowns.

Differentiating $\bar{u}(\ell, y, p)$ and $\bar{v}(\ell, y, p)$ with respect to y gives

$$\bar{u}_y(\ell, y, p) = -\frac{V_0}{c_p r_g p \gamma} \frac{G(p)}{D(p)} + \sum_{n=1}^{\infty} \frac{n\pi}{2h} c_n(p) \cos \frac{n\pi y}{2h} \quad ,$$

$$\bar{v}_y(\ell, y, p) = -\sum_{n=1}^{\infty} \frac{n\pi}{2h} d_n(p) \sin \frac{n\pi y}{2h} \quad . \tag{2.21}$$

Now use is made of the boundary conditions at $x = \ell$. From (1.2) and (2.1)

$$\bar{\sigma}_{xx}(\ell, y, p) = 0 = \mu\left[k^2 \bar{u}_x(\ell, y, p) + (k^2 - 2)\bar{v}_y(\ell, y, p)\right] \quad . \tag{2.22}$$

Therefore

$$\bar{u}_x(\ell, y, p) = \left(\frac{k^2 - 2}{k^2}\right) \sum_{n=1}^{\infty} \frac{n\pi}{2h} d_n(p) \sin \frac{n\pi y}{2h} \quad . \tag{2.23}$$

Similarly, the other boundary condition,

$$\bar{\sigma}_{xy}(\ell, y, p) = 0 = \mu\left[\bar{u}_y(\ell, y, p) + \bar{v}_x(\ell, y, p)\right] \quad , \tag{2.24}$$

implies that

$$\bar{v}_x(\ell, y, p) = \frac{v_0}{c_p r_g p \gamma} \frac{G(p)}{D(p)} - \sum_{n=1}^{\infty} \frac{n\pi}{2h} c_n(p) \cos \frac{n\pi y}{2h} \quad . \tag{2.25}$$

The boundary conditions at $y = h$, from (1.8b) and (1.2), are

$$\bar{\sigma}_{yy}(x, h, p) = 0 = \mu\left[(k^2-2)\bar{u}_x(x, h, p) + k^2 \bar{v}_y(x, h, p)\right] \quad . \tag{2.26}$$

Letting $x \to \ell$ and combining (2.22) and (2.26) shows that

$$\bar{v}_y(\ell, h, p) = \bar{u}_x(\ell, h, p) = 0 \quad . \tag{2.27}$$

This corner condition suggests that only the n even terms be used for the series in (2.23) and in the second of (2.20) and (2.21). The other boundary condition at $y = h$ does not restrict $\bar{u}_y(\ell, h, p)$ or $\bar{v}_x(\ell, h, p)$. Therefore, the series may contribute at $y = h$ and so the n even terms should be used in the Fourier series for these edge unknowns.

In summary, the following forms will be assumed for the edge unknowns

$$\bar{u}(0, y, p) = \bar{u}_y(0, y, p) = \bar{v}(0, y, p) = \bar{v}_y(0, y, p) = 0 \quad ,$$

$$\bar{u}_x(0, y, p) = b_0(p)\left[(1-y/h)^{-\frac{1}{4}} - (1+y/h)^{-\frac{1}{4}} + 2^{-\frac{1}{4}} y/h\right] + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} b_n(p) \sin \frac{n\pi y}{2h} \quad ,$$

(cont.)

$$\bar{v}_x(0,y,p) = a_0(p) + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} a_n(p) \cos \frac{n\pi y}{2h} \quad ,$$

$$\bar{u}(\ell,y,p) = -\frac{v_0 y}{c_p r_g p \gamma} \frac{G(p)}{D(p)} + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} c_n(p) \sin \frac{n\pi y}{2h} \quad ,$$

$$\bar{u}_y(\ell,y,p) = -\frac{v_0}{c_p r_g p \gamma} \frac{G(p)}{D(p)} + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{n\pi}{2h} c_n(p) \cos \frac{n\pi y}{2h} \quad ,$$

$$\bar{u}_x(\ell,y,p) = \left(\frac{k^2-2}{k^2}\right) \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{n\pi}{2h} d_n(p) \sin \frac{n\pi y}{2h} \quad , \qquad (2.28)$$

$$\bar{v}(\ell,y,p) = \frac{v_0}{p^2} \left[\frac{2H(p)}{D(p)} - 1\right] + d_0(p) + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} d_n(p) \cos \frac{n\pi y}{2h} \quad ,$$

$$\bar{v}_y(\ell,y,p) = -\sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{n\pi}{2h} d_n(p) \sin \frac{n\pi y}{2h} \quad ,$$

$$\bar{v}_x(\ell,y,p) = \frac{v_0}{c_p r_g p \gamma} \frac{G(p)}{D(p)} - \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} \frac{n\pi}{2h} c_n(p) \cos \frac{n\pi y}{2h} \quad .$$

The unknown coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ here will differ from the values found for them for the semi-infinite plate case.

The edge unknowns from (2.28) are substituted into (2.6) to give

$g(s,y,p)$ and $h(s,y,p)$. Substituting these into the entirety equations, (1.14), and integrating using (1.17) yields the following infinite set of algebraic equations:

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \Bigg\{ a_0(p)\, M_j^0(s_j,p) + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} a_n(p)\, M_j^n(s_j,p) + b_0(p)\, N_j^0(s_j,p)$$

$$+ \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} b_n(p)\, N_j^n(s_j,p) + d_0(p)\, P_j^0(s_j,p)\, e^{-s_j \ell} + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} \Big[c_n(p)\, O_j^n(s_j,p) \tag{2.29}$$

$$+ d_n(p)\, P_j^n(s_j,p) \Big] e^{-s_j \ell} + Q_j(s_j,p) + R_j(s_j,p)\, e^{-s_j \ell} \Bigg\} = 0$$

plus a similar set with $s_j(p)$ replaced by $-s_j(p)$. In (2.29), $M_j^0(s_j,p)$, $M_j^n(s_j,p)$, $N_j^0(s_j,p)$, $N_j^n(s_j,p)$ and $Q_j(s_j,p)$ are exactly the same as in (1.19) while

$$O_j^n(s_j,p) = - \frac{(-1)^{n/2}\, 2\theta_n s_j Y_j(s_j,p)}{\left(\beta_j^2 + \theta_n^2\right)\left(\alpha_j^2 + \theta_n^2\right)} \left(\frac{k^2-1}{k^2}\right)\left(k_d^2 + 2\theta_n^2\right) \quad ,$$

$$P_j^0(s_j,p) = - \frac{k_s^2}{\beta_j^2}\, Y_j(s_j,p) \quad , \tag{2.30}$$

$$P_j^n(s_j,p) = \frac{(-1)^{n/2}\, Y_j(s_j,p)}{\left(\beta_j^2 + \theta_n^2\right)} \left[\left(\frac{k^2-1}{k^2}\right) \frac{4 s_j^2 \theta_n^2}{\left(\alpha_j^2 + \theta_n^2\right)} - k_s^2 \right] \quad , \qquad \text{(cont.)}$$

$$R_j(s_j, p) = \frac{2v_0 \gamma}{c_d^2 \alpha_j^2} \frac{G(p)}{D(p)} \left[\frac{2(k^2-1)s_j}{\beta_j^2} Y_j(s_j, p) \right.$$

$$\left. + \frac{(k^2-2)h}{k^2} \right] - \frac{2v_0}{c_s^2 \beta_j^2} \frac{Y_j(s_j, p) H(p)}{D(p)} ,$$

where $Y_j(s_j, p)$ is as in (1.14).

Equations (2.30) are approximated for p small, just as was done for the semi-infinite plate. For the lowest mode, $s_0(p)$ is given by (1.21), i.e., $s_0(p) = \pm(1 \pm i)\gamma$, $\gamma = \sqrt{\frac{p}{2c_p r_g}}$. Approximating (1.19) and (2.30) for $s = s_0(p)$ and p small yields (1.23) for $M_0^0(s_0, p)$, $M_0^n(s_0, p)$, $N_0^0(s_0, p)$, $N_0^n(s_0, p)$ and $Q_0(s_0, p)$ while for the other terms the following results:

$$O_0^n(s_0, p) = -(-1)^{n/2} \left(\frac{k^2-1}{k^2} \right) \frac{4s_0^2 h}{k^2 \theta_n} + O(p^2) ,$$

$$P_0^0(s_0, p) = \frac{4}{3} \left(\frac{k^2-1}{k^2} \right) \frac{s_0^3 h^3}{k^2} + O(p^{\frac{5}{2}}) ,$$

(cont.)

$$P_0^n(s_0, p) = (-1)^{n/2}\left(\frac{k^2-1}{k^2}\right)\frac{4s_0^3 h}{\theta_n^2} + O(p^{\frac{5}{2}}) \quad ,$$

(2.31)

$$R_0(s_0, p) = \frac{2v_0 h}{c_d^2 D(p)}\left[\frac{\gamma}{s_0^2}\, G(p) + \frac{1}{s_0}\, H(p)\right] + O(1) \quad .$$

For the higher modes $s_j(p) = \hat{s}_j$, $j \geq 1$, where $\hat{s}_j$ satisfies (1.22). Approximating now gives (1.24) for $M_j^0(\hat{s}_j, p)$, $M_j^n(\hat{s}_j, p)$, $N_j^0(\hat{s}_j, p)$, $N_j^n(\hat{s}_j, p)$ and $Q_j(\hat{s}_j, p)$ while for the other terms the following results:

$$O_j^n(\hat{s}_j, p) = -\frac{(-1)^{n/2}(k^2-1)\,\theta_n \hat{s}_j \tan \hat{s}_j h}{k^4\left(\theta_n^2 - \hat{s}_j^2\right)^2} + O(p^2) \quad ,$$

$$P_j^0(\hat{s}_j, p) = \frac{p^2 \tan \hat{s}\, h}{c_d^2 \hat{s}_j^2} + O(p^4) \quad ,$$

$$P_j^n(\hat{s}_j, p) = \frac{(-1)^{n/2}(k^2-1)\, 4\hat{s}_j^2\, \theta_n^2 \tan \hat{s}_j h}{k^4\left(\theta_n^2 - \hat{s}_j^2\right)^2} + O(p^2) \quad ,$$

(cont.)

$$R_j(\hat{s}_j, p) = \frac{2v_0\gamma}{c_d^2\hat{s}_j^2}\frac{G(p)}{D(p)}\left[\left(\frac{k^2-1}{k^2}\right)\frac{2\tan\hat{s}_jh}{\hat{s}_j} - \frac{(k^2-2)h}{k^2}\right]$$

$$+\frac{2v_0\tan\hat{s}_jh}{c_s^2}\frac{H(p)}{D(p)} + O(p^2) \quad . \tag{2.32}$$

To solve this infinite set of algebraic equations, assume that the unknown coefficients are of the form

$$a_0(p) = \frac{A_0}{\sqrt{p}}\frac{E(p)}{D(p)} \quad ,$$

$$a_n(p) = \frac{A_n}{p}\frac{F(p)}{D(p)} \quad ,$$

$$b_0(p) = \frac{B_0}{p}\frac{F(p)}{D(p)} \quad ,$$

$$b_n(p) = \frac{B_n}{p}\frac{F(p)}{D(p)} \quad , \tag{2.33}$$

$$c_n(p) = \frac{C_n}{p}\frac{F(p)}{D(p)} \quad ,$$

$$d_0(p) = \frac{D_0}{p}\frac{F(p)}{D(p)} \quad ,$$

$$d_n(p) = \frac{D_n}{p}\frac{F(p)}{D(p)} \quad ,$$

where A_0, A_n, B_0, B_n, C_n, D_0 and D_n are independent of p and D(p), E(p) and F(p) are defined in (2.16). Substituting into the four equations obtained from $s = s_0(p) = \pm(1\pm i)\gamma$ and neglecting all but the lowest order terms in p yields

$$\frac{A_0}{\sqrt{p}}\frac{E(p)}{D(p)}\left(-\frac{h}{k^2}\right) + \frac{B_0}{p}\frac{F(p)}{D(p)}\left(-2^{-\frac{1}{4}}\frac{5}{7}s_0h^2\right) + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\left[\frac{B_n}{p}\frac{F(p)}{D(p)}(-1)^{\frac{n}{2}}\frac{s_0h}{\theta_n}\right] = \tag{2.34}$$

$$\frac{v_0h}{c_d^2}\left[\frac{1}{s_0} - \left(\frac{2\gamma}{s_0^2}\frac{G(p)}{D(p)} + \frac{2}{s_0}\frac{H(p)}{D(p)}\right)e^{-s_0\ell}\right] .$$

Substituting the four roots $s_0(p)$ into (2.34) and multiplying both sides of the resulting equations by $c_d^2/hc_p^2r_g^2$ gives the following equations:

$$\left(-\delta\frac{A_0}{\sqrt{p}}\frac{E(p)}{D(p)}\right) + (1+i)\gamma\left(Mk^2\delta\frac{h}{p}\frac{F(p)}{D(p)}\right) + \left[-2i\gamma^2\left(\frac{2\gamma v_0}{p^2}\frac{G(p)}{D(p)}\right)\right.$$

$$\left. + 2(1-i)\gamma^3\left(\frac{2v_0}{p^2}\frac{H(p)}{D(p)}\right)\right]\exp[-(1+i)\gamma\ell] = 2(1-i)\frac{v_0\gamma^3}{p^2} ,$$

$$\left(-\delta\frac{A_0}{\sqrt{p}}\frac{E(p)}{D(p)}\right) + (1-i)\gamma\left(Mk^2\delta\frac{h}{p}\frac{F(p)}{D(p)}\right) + \left[2i\gamma^2\left(\frac{2\gamma v_0}{p^2}\frac{G(p)}{D(p)}\right)\right.$$

$$\left. + 2(1+i)\gamma^3\left(\frac{2v_0}{p^2}\frac{H(p)}{D(p)}\right)\right]\exp[-(1-i)\gamma\ell] = 2(1+i)\frac{v_0\gamma^3}{p^2} ,$$

(cont.)

$$\left(-\delta \frac{A_0}{\sqrt{p}} \frac{E(p)}{D(p)}\right) - (1+i)\gamma \left(Mk^2 \delta \frac{h}{p} \frac{F(p)}{D(p)}\right) + \left[-2i\gamma^2\left(\frac{2\gamma v_0}{p^2} \frac{G(p)}{D(p)}\right)\right.$$

$$\left. -2(1-i)\gamma^3\left(\frac{2v_0}{p^2} \frac{H(p)}{D(p)}\right)\right] \exp\left[(1+i)\gamma \ell\right] = -2(1-i)\frac{v_0\gamma^3}{p^2} \quad , \tag{2.35}$$

$$\left(-\delta \frac{A_0}{\sqrt{p}} \frac{E(p)}{D(p)}\right) - (1-i)\gamma \left(Mk^2 \delta \frac{h}{p} \frac{F(p)}{D(p)}\right) + \left[2i\gamma^2\left(\frac{2\gamma v_0}{p^2} \frac{G(p)}{D(p)}\right)\right.$$

$$\left. -2(1+i)\gamma^3\left(\frac{2v_0}{p^2} \frac{H(p)}{D(p)}\right)\right] \exp\left[(1-i)\gamma \ell\right] = -2(1+i)\frac{v_0\gamma^3}{p^2} \quad ,$$

where

$$M = -2^{-\frac{1}{4}} \frac{5}{7} B_0 + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} (-1)^{\frac{n}{2}} \frac{2B_n}{n\pi} \quad , \text{ and}$$

$$\delta = \frac{c_s^2}{c_p^2 r_g^2} \quad .$$

Comparing (2.35) with (2.13) shows that they are identical provided

$$-\delta \frac{A_0}{\sqrt{p}} \frac{E(p)}{D(p)} = \overline{v}_{xxx}(0, p) \quad ,$$

(cont.)

$$Mk^2 \delta \frac{h}{p} \frac{F(p)}{D(p)} = \bar{v}_{xx}(0, p) \quad ,$$

$$\frac{2\gamma v_0}{p^2} \frac{G(p)}{D(p)} = \bar{v}_x(\ell, p) \quad , \tag{2.36}$$

$$\frac{2v_0}{p^2} \frac{H(p)}{D(p)} = \bar{v}(\ell, p) \quad .$$

Substituting the solution of (2.13) for $\bar{v}_x(\ell, p)$ and $\bar{v}(\ell, p)$ from (2.15) shows that the last two of (2.36) are satisfied identically. This shows that, for long-time, the dominant terms in the edge unknowns at $x = \ell$ will be in agreement with those given by the Euler-Bernoulli approximate theory. The first two of (2.36) are not satisfied identically because of the singularities at the corners. Substituting (2.14) into the first two of (2.36) gives

$$A_0 = -\frac{v_0\sqrt{p}}{c_s^2 \gamma} = -\frac{v_0}{c_s^2}\sqrt{2c_p r_g}$$

$$-2^{-\frac{1}{4}} \frac{5}{7} B_0 + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (-1)^{\frac{n}{2}} \frac{2B_n}{n\pi} = -\frac{v_0}{c_d}\sqrt{\frac{2}{\sqrt{3k^2(1-\nu)}}} \quad .$$

So the four equations coming from the lowest mode have been reduced to two, which it should be noted are identical to the first two boundedness equations for the semi-infinite plate, (1.29).

For the higher modes, $s = s_j(p) = \hat{s}_j$, $j \geq 1$, where $\hat{s}_j$ is a constant which satisfies (1.22). Now, if $\hat{s}_j$ is a root, then $\bar{\hat{s}}_j$, $-\hat{s}_j$ and $-\bar{\hat{s}}_j$ will also be roots where $\bar{\hat{s}}_j$ is the complex conjugate of $\hat{s}_j$. Since we are setting both the real and imaginary parts of (2.29) equal to zero, it is only necessary to use two of these roots, $\hat{s}_j$ and $-\hat{s}_j$. Consider a root $\hat{s}_j$ in the first quadrant, i.e.

$$\hat{s}_j = \hat{s}_j^+ = \hat{s}_j^R + i\,\hat{s}_j^I \quad , \tag{2.38}$$

where $\hat{s}_j^R$ and $\hat{s}_j^I$ are real, positive numbers. Then, for this root, using (2.33) for the unknown coefficients and retaining only the lowest order terms in p, the entirety equations become the following set of algebraic equations:

$$\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} \Bigg\{ B_0 N_j^0(\hat{s}_j^+, p) + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left[A_n M_j^n(\hat{s}_j^+, p) + B_n N_j^n(\hat{s}_j^+, p) \right] + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left[C_n O_j^n(\hat{s}_j^+, p) + D_n P_j^n(\hat{s}_j^+, p) \right] e^{-i\hat{s}_j^I \ell} e^{-\hat{s}_j^R \ell} \Bigg\} = 0 \quad , \tag{2.39}$$

where $M_j^n(\hat{s}_j^+, p)$, $N_j^0(\hat{s}_j^+, p)$ and $N_j^n(\hat{s}_j^+, p)$ are as in (1.24) and $O_j^n(\hat{s}_j^+, p)$ and $P_j^n(\hat{s}_j^+, p)$ are given by (2.32).

Next use $s_j(p) = -\hat{s}_j^+$ in the entirety equations. This will generate another set of algebraic equations. Using (2.33) for the unknown coefficients, retaining only the lowest order terms in p, and multiplying the resulting equations by $e^{-\hat{s}_j^+ \ell}$ gives the following set of algebraic equations:

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \left\{ \left[-B_0 N_j^0(\hat{s}_j^+, p) + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left(A_n M_j^n(\hat{s}_j^+, p) - B_n N_j^n(\hat{s}_j^+, p) \right) \right] e^{-i \hat{s}_j^I \ell} e^{-\hat{s}_j^R \ell} \right. \tag{2.40}$$

$$\left. + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left[C_n O_j^n(\hat{s}_j^+, p) - D_n P_j^n(\hat{s}_j^+, p) \right] \right\} = 0 \quad .$$

As was mentioned earlier, the smallest $\hat{s}_j^R$ is $\hat{s}_1^R = 3.7488\, h$. Since $\ell/h \geq 2$, $e^{-\hat{s}_j^R \ell}$ will be small. Neglecting the terms in (2.39) that are multiplied by $e^{-\hat{s}_j^R \ell}$ shows that (2.39) reduces to the last of (1.29). Therefore, the coefficients A_n, B_0 and B_n are given by Table I. Equations (2.40) can now be reduced to

$$\begin{Bmatrix} Re \\ Im \end{Bmatrix} \left\{ \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left[C_n O_j^n(\hat{s}_j^+, p) - D_n P_j^n(\hat{s}_j^+, p) \right] = \right. \tag{2.41}$$

$$\left. - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} 2 A_n M_j^n(s_j^+, p)\, e^{-i \hat{s}_j^I \ell} e^{-\hat{s}_j^R \ell} \right\}$$

where the right hand side is known. The coefficients C_n and D_n can be determined from (2.41). Since the right hand side becomes small as

$e^{-\hat{s}_j^R \ell}$, so will C_n and D_n. So the Fourier series parts of the edge unknowns at $x = \ell$ are small corrections to the Euler-Bernoulli terms.

Once the edge unknowns have been found, $g(s,y,p)$ and $h(s,y,p)$ can be calculated from (2.6). Any term that is multiplied by the shift operator, $e^{-s\ell}$, will not contribute to the solution for $x<\ell$. So, for $0 \le x < \ell$, the doubly transformed displacements are once again given by (1.35) through (1.39) where the coefficients are now given by (2.33).

Since the terms multiplied by $e^{-s\ell}$ will not contribute for $0 \le x < \ell$, the doubly transformed displacements will still have poles for x in this region. So the s-x Laplace transform can still be inverted by residue theory. The contribution of the pole at $s = 0$ is unchanged from (1.63). The contribution of the lowest mode is calculated in exactly the same way as it was in Chapter I giving (1.81) for $R_0^u(s_0(p))$. Then using (2.33) and (2.35) gives, for a root $s_0(p)$,

$$R_0^u\left(s_0(p)\right) = \frac{v_0}{4p^2}\left[-s_0(p) + \frac{s_0^2(p)}{\gamma}\frac{E(p)}{D(p)} - \frac{s_0^3(p)}{2\gamma^3}\frac{F(p)}{D(p)}\right] y\, e^{s_0(p)x} + O(p^{-\frac{1}{2}}) \quad .$$

All four of the lowest mode roots will contribute to the solution. Summing the residues for each of the $s_0(p)$ gives

$$\overline{u}^L(x,y,p) = \frac{v_0\gamma y}{2p^2}\left\{\left[\left(1+\frac{2E(p)+F(p)}{D(p)}\right)\sin\gamma x + \left(1-\frac{F(p)}{D(p)}\right)\cos\gamma x\right]e^{-\gamma x}\right.$$

$$\left. + \left[\left(1-\frac{2E(p)-F(p)}{D(p)}\right)\sin\gamma x - \left(1-\frac{F(p)}{D(p)}\right)\cos\gamma x\right]e^{\gamma x}\right\} + O(p^{-\frac{1}{2}}). \qquad (2.43)$$

Similarly, the lowest mode contribution to $\bar{v}(x,y,p)$ and to the stresses is given by

$$\bar{v}^{L}(x,y,p) = \frac{v_0}{2p^2}\left\{\left[\left(1+\frac{E(p)}{D(p)}\right)\cos\gamma x + \left(\frac{E(p)+F(p)}{D(p)}\right)\sin\gamma x\right]e^{-\gamma x}\right.$$

$$\left. + \left[\left(1-\frac{E(p)}{D(p)}\right)\cos\gamma x + \left(\frac{E(p)-F(p)}{D(p)}\right)\sin\gamma x\right]e^{\gamma x} - 2\right\} + O(p^{-1}) \quad , \tag{2.44}$$

$$\bar{\sigma}^{L}_{xx}(x,y,p) = \frac{v_0 \mu y}{c_p r_g p(1-\nu)}\left\{\left[\left(\frac{E(p)+F(p)}{D(p)}\right)\cos\gamma x - \left(1+\frac{E(p)}{D(p)}\right)\sin\gamma x\right]e^{-\gamma x}\right.$$

$$\left. - \left[\left(\frac{E(p)-F(p)}{D(p)}\right)\cos\gamma x - \left(1-\frac{E(p)}{D(p)}\right)\sin\gamma x\right]e^{\gamma x}\right\} + O(1) \quad , \tag{2.45}$$

$$\bar{\sigma}^{L}_{yy}(x,y,p) = O(1) \quad ,$$

$$\bar{\sigma}^{L}_{xy}(x,y,p) = O(p^{-\frac{1}{2}}) \quad .$$

The higher mode roots with $Re(\hat{s}_j) < 0$ will once again give edge waves of the same form as (1.85). On the other hand, for the roots with $Re(\hat{s}_j) > 0$, exponentially increasing waves will result. Calculating the residues for these roots shows that

$$\begin{Bmatrix} R^u_j(s^+_j) \\ R^v_j(\hat{s}^+_j) \end{Bmatrix} = \frac{F(p)}{pD(p)}\begin{Bmatrix} \hat{K}^u_j(y) \\ \hat{K}^v_j(y) \end{Bmatrix}\left\{B_0 N^0_j(\hat{s}^+_j, p) + \sum_{\substack{n=2 \\ n\,\text{even}}}^{\infty}\left[A_n M^n_j(\hat{s}^+_j, p) + B_n N^n_j(\hat{s}^+_j, p)\right]\right\} e^{\hat{s}^I_j x}\, e^{\hat{s}^R_j x} + O(1) \quad ,$$

where $\hat{K}_j^u(y)$ and $\hat{K}_j^v(y)$ are, for a given j, a function only of y. Using (2.39) we have

$$\begin{Bmatrix} R_j^u(\hat{s}_j^+) \\ R_j^v(\hat{s}_j^+) \end{Bmatrix} = -\frac{F(p)}{pD(p)} \begin{Bmatrix} \hat{K}_j^u(y) \\ \hat{K}_j^v(y) \end{Bmatrix} \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \Big[C_n O_j^n(\hat{s}_j^+, p) \tag{2.46}$$

$$+ D_n P_j^n(\hat{s}_j^+, p) \Big] e^{-i\,\hat{s}_j^I(\ell - x)} e^{-\hat{s}_j^R(\ell - x)} + O(1) \quad .$$

So the contribution of these terms will decay exponentially away from $x = \ell$. At $x = \ell$, summing up all of the residues should just give the Fourier series terms in $\overline{u}(\ell, y, p)$ and $\overline{v}(\ell, y, p)$ in (2.28). Since C_n and D_n are small (see discussion after (2.41)) these terms will be neglected.

So, for $\frac{x}{h} > 1$, the transformed displacements and stresses are given by (2.43) through (2.45). In the next two sections, these will be inverted in two different ways. The first method will show the solution as traveling waves while the second will bring out the vibrational form.

4. Inversion of the Time Transform-Traveling Wave Form

For the region away from the base but behind the body wavefronts the transformed displacements and stresses are given by (2.43) through (2.45). Of these, only $\overline{v}(x, y, p)$ and the dominant stress $\overline{\sigma}_{xx}(x, y, p)$ will be inverted here. $\overline{u}(x, y, p)$ can be inverted in the same way that $\overline{v}(x, y, p)$ will be while the other two stresses vanish for long-time.

The traveling wave form of the solution is found by using the binomial series expansion

$$\frac{1}{1+\Delta} = \sum_{m=0}^{\infty} (-1)^m \Delta^m \quad , \quad |\Delta| < 1 \quad , \tag{2.47}$$

to expand the denominator of the transformed solution until a form is obtained that can be inverted directly. Observing (2.44) and (2.45) we see that $\bar{v}(x, y, p)$ has $p^2 D(p)$ in its denominator while $\bar{\sigma}_{xx}(x, y, p)$ has $pD(p)$. So it is necessary to expand $\frac{1}{D(p)}$ which, from the first of (2.15), is

$$\frac{1}{D(p)} = \frac{1}{\cosh^2 \gamma \ell + \cos^2 \gamma \ell} \quad .$$

Now by Lerch's Theorem, p can be required to be real and positive. Then $\gamma \ell$ will also be real and positive; hence $\frac{\cos^2 \gamma \ell}{\cosh^2 \gamma \ell} < 1$. Using (2.47) to expand $\frac{1}{D(p)}$ with $\Delta = \frac{\cos^2 \gamma \ell}{\cosh^2 \gamma \ell}$ yields

$$\frac{1}{D(p)} = \sum_{m=0}^{\infty} (-1)^m \frac{(\cos^2 \gamma \ell)^m}{(\cosh^2 \gamma \ell)^{m+1}} \quad . \tag{2.48}$$

Using the binomial series expansion, (2.47), two more times gives

$$\frac{1}{\cosh^2 \gamma \ell} = 2 \sum_{k=0}^{\infty} (-1)^k \left[2 e^{-2\gamma \ell} \sum_{j=0}^{\infty} (-1)^j \left(e^{-4\gamma \ell} \right)^j \right]^{k+1} .$$

This series can be reduced to

$$\frac{1}{\cosh^2 \gamma \ell} = 4 \sum_{k=1}^{\infty} (-1)^{(k-1)} k\, e^{-2k\gamma \ell} . \tag{2.49}$$

Substituting this into (2.48) yields

$$\frac{1}{D(p)} = \sum_{m=0}^{\infty} \left[(-1)^m (\cos^2 \gamma \ell)^m \left(4 \sum_{k=1}^{\infty} (-1)^{(k-1)} k\, e^{-2k\gamma \ell} \right)^{m+1} \right] . \tag{2.50}$$

The terms in (2.50) decay as $e^{-2k(m+1)\gamma \ell}$ so for any given value of $\gamma \ell$ only a finite number of terms will be needed to calculate the solution to a given level of accuracy. Note that as $\gamma \ell$ gets smaller and smaller, corresponding to longer and longer time, more terms will be needed.

Evaluating the first few terms in (2.50) gives

$$\frac{1}{D(p)} = 4 \Big\{ e^{-2\gamma \ell} - (4 + 2\cos 2\gamma \ell) e^{-4\gamma \ell} + (17 + 16 \cos 2\gamma \ell + 2 \cos 4\gamma \ell) e^{-6\gamma \ell}$$

$$- (80 + 96 \cos 2\gamma \ell + 24 \cos 4\gamma \ell + 2 \cos 6\gamma \ell) e^{-8\gamma \ell} \Big\} + O(e^{-10\gamma \ell}) . \tag{2.51}$$

Using (2.51) in (2.44) shows that $\overline{v}(x, y, p)$ has the form

$$\overline{v}(x, y, p) = v_0 \sum_{j=0}^{\infty} \overline{T}_j(x, y, p) \quad ,$$

where

$$\overline{T}_0(x, y, p) = \frac{1}{p^2}\left\{[\cos \gamma x + \sin \gamma x]\, e^{-\gamma x} - 1\right\} \quad ,$$

$$\overline{T}_1(x, y, p) = \frac{1}{p^2}\,[2 \cos \gamma x + \cos \gamma (2\ell - x) - \sin \gamma (2\ell - x)]\, e^{-\gamma (2\ell - x)} \quad ,$$

$$\overline{T}_2(x, y, p) = -\frac{1}{p^2}\,[2 \cos \gamma x + 4 \sin \gamma x - 2 \sin \gamma (2\ell - x) + \cos \gamma (2\ell + x) + \sin \gamma (2\ell + x)]\, e^{-\gamma (2\ell + x)} \quad , \tag{2.52}$$

$$\overline{T}_3(x, y, p) = -\frac{1}{p^2}\,[8 \cos \gamma x + 2 \sin \gamma x + 6 \cos \gamma (2\ell - x) - 4 \sin \gamma (2\ell - x) + 2 \cos \gamma (2\ell + x) + \cos \gamma (4\ell - x) - \sin \gamma (4\ell - x)]\, e^{-\gamma (4\ell - x)} \quad ,$$

$$\overline{T}_4(x, y, p) = O\left(e^{-\gamma (4\ell + x)}\right) \quad , \quad \text{etc.}$$

Inspection of (2.52) shows that $\overline{T}_0(x, y, p)$ is exactly the same as the second of (1.89), i.e., $\overline{T}_0$ is just $\overline{v}$ for the semi-infinite plate. So $\overline{T}_0$ will be a wave traveling in the positive x-direction.

$\overline{T}_1(x, y, p)$ is proportional to $e^{-\gamma (2\ell - x)}$. So $\overline{T}_1$ represents a wave traveling in the negative x-direction. This wave is the reflection of $\overline{T}_0$ from the boundary at $x = \ell$. Similarly $\overline{T}_2$ is the reflection of $\overline{T}_1$ from the boundary $x = 0$ and is a wave traveling in the positive x-direction. The other $\overline{T}_j$'s represent further reflections from the ends of the plate.

At the base of the plate we have that $\overline{T}_0(0, y, p) = 0$ and $\overline{T}_1(0, y, p) = -\overline{T}_2(0, y, p)$, $\overline{T}_3(0, y, p) = -\overline{T}_4(0, y, p)$, etc. So the displacement changes sign when it reflects from the built-in end which is, of course, required by the boundary condition.

The transformed normal stress, $\overline{\sigma}_{xx}(x, y, p)$, can also be written as a series of traveling waves. Using (2.51) in the first of (2.45) yields

$$\overline{\sigma}_{xx}(x, y, p) = v_0 \sum_{j=0}^{\infty} \overline{U}_j(x, y, p) \quad , \tag{2.53}$$

where

$$\overline{U}_0(x, y, p) = \frac{2\mu y}{(1-\nu)\, c_p r_g p} \left[\cos \gamma x - \sin \gamma x\right] e^{-\gamma x} \quad ,$$

$$\overline{U}_1(x, y, p) = -\frac{2\mu y}{(1-\nu)\, c_p r_g p} \left[-2 \sin \nu x + \cos \gamma (2\ell - x)\right.$$

$$\left. + \sin \gamma (2\ell - x)\right] e^{-\gamma (2\ell - x)} \quad ,$$

$$\overline{U}_2(x, y, p) = \frac{2\mu y}{(1-\nu)\, c_p r_g p} \left[-4 \cos \gamma x + 2 \sin \gamma x - 2 \cos \nu (2\ell - x)\right.$$

$$\left. - \cos \nu (2\ell + x) + \sin \gamma (2\ell + x)\right] e^{-\gamma (2\ell + x)} \quad , \qquad \text{(cont.)}$$

$$\overline{U}_3(x,y,p) = O\left(e^{-\gamma(4\ell - x)}\right) \quad , \quad \text{etc.}$$

Note that $\overline{U}_0(x,y,p)$ is, once again, just the normal stress for the semi-infinite plate, (1.90). Now we have that at the end $x = \ell$, $\overline{U}_0(\ell,y,p) = -\overline{U}_1(\ell,y,p)$, $\overline{U}_2(\ell,y,p) = -\overline{U}_3(\ell,y,p)$, etc. which gives zero normal stress at the end as required by the boundary condition.

Equations (2.52) and (2.53) can be inverted by using the tables in Abramowitz and Stegun, [11]. This gives

$$v(x,y,t) = v_0 \sum_{j=0}^{\infty} T_j(x,y,t) \quad ,$$

$$\sigma_{xx}(x,y,t) = v_0 \sum_{j=0}^{\infty} U_j(x,y,t) \quad , \tag{2.54}$$

where the outgoing wave is given by

$$T_0(x,y,t) = -2t\left[S_2(\Delta_0) + \frac{x}{\sqrt{2\pi c_p r_g t}} \sin \Delta_0 + \Delta_0\left(1 - 2C_2(\Delta_0)\right)\right] \quad ,$$

$$U_0(x,y,t) = \frac{2\mu y}{(1-\nu) c_p r_g}\left[1 - C_2(\Delta_0)\right] \quad , \tag{2.55}$$

$$\Delta_0 = \frac{x^2}{4 c_p r_g t} \quad .$$

Here $C_2(\Delta_0)$ and $S_2(\Delta_0)$ are the Frenel integrals which were defined in (1.94). Inverting $\overline{T}_1(x,y,p)$ and $\overline{U}_1(x,y,p)$ shows that the first reflected wave is given by

$$T_1(x,y,t) = -2t\left[C_2(\Delta_1) + \frac{(2\ell-x)}{\sqrt{2\pi c_p r_g t}}\cos\Delta_1 - \frac{1}{2} - \Delta_1\left(1-2S_2(\Delta_1)\right)\right]$$

$$+ 2\,\mathrm{Re}\left\{\frac{1}{4c_p r_g}\left[(2\ell-x)^2 - x^2 - 2ix(2\ell-x)\right]\mathrm{erfc}\, z_1\right.$$

$$\left. - \frac{2t}{\sqrt{\pi}} z_1 \exp\left[-\frac{\ell(\ell-x)}{2c_p r_g t}\right]\exp\left[\frac{ix(2\ell-x)}{4c_p r_g t}\right] + t\,\mathrm{erfc}\, z_1\right\} ,$$

$$U_1(x,y,t) = -\frac{2\mu y}{(1-\nu)c_p r_g}\left[1 - 2S_2(\Delta_1) - 2\,\mathrm{Im}(\mathrm{erfc}\, z_1)\right] , \qquad (2.56)$$

where

$$\Delta_1 = \frac{(2\ell-x)^2}{4c_p r_g t} ,$$

$$z_1 = \frac{(2\ell-x)-ix}{\sqrt{8c_p r_g t}} ,$$

and erfc is the complementary error function. Inverting the other $\overline{T}_j(x,y,p)$ and $\overline{U}_j(x,y,p)$ yields terms that are similar to $T_1(x,y,t)$ and $U_1(x,y,t)$.

$v(x,0,t)$ is shown in Fig. 7 for the case $\frac{\ell}{h} = 20$ and $t = 700\,t_h$ where $t_h = \frac{t}{c_d}$. The outgoing wave and the first four reflections are shown. As can be seen, the fourth reflection is quite small so the fifth

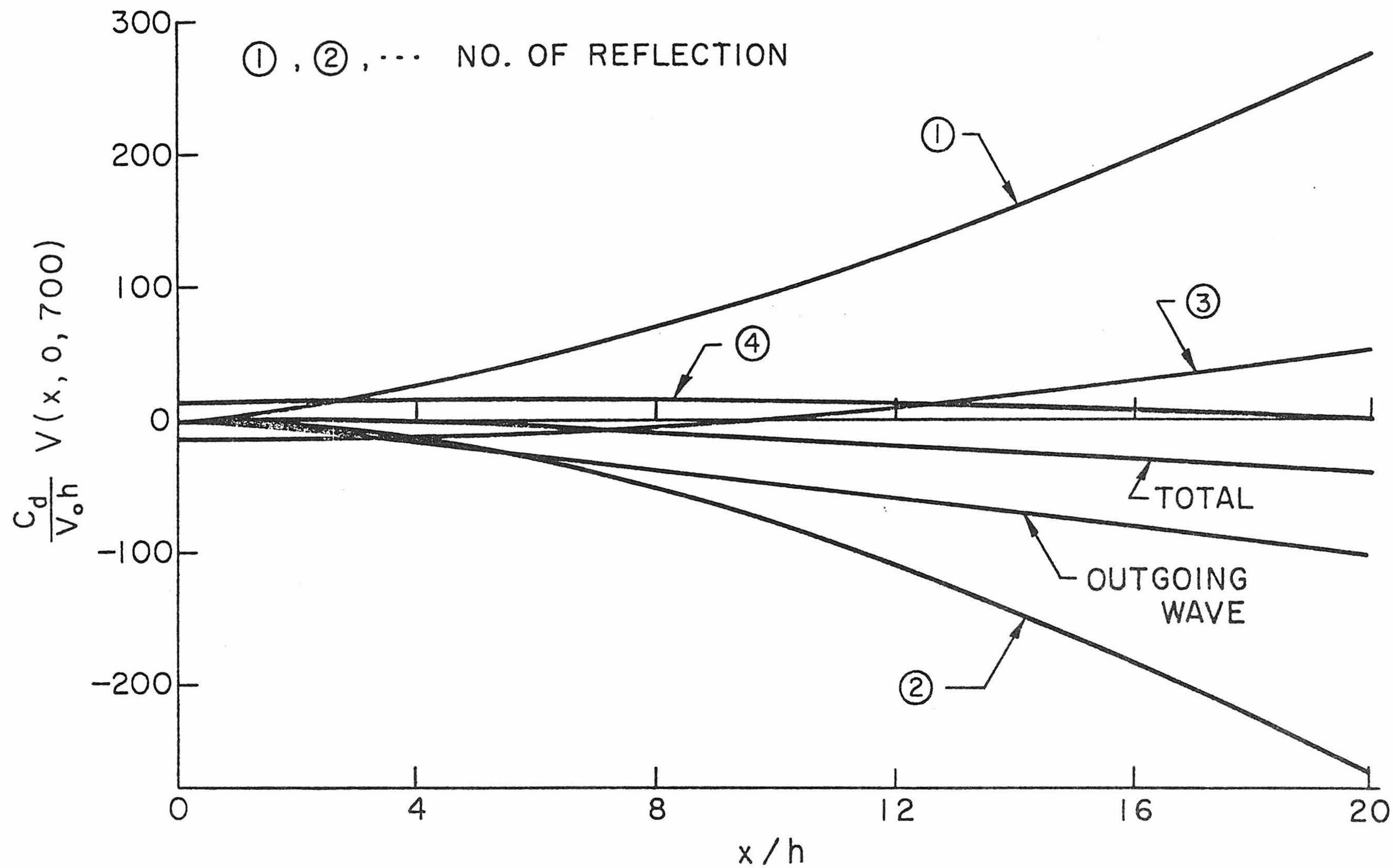

Fig. 7 Displacement v(x, 0, 700) for the Finite Plate with $\frac{\ell}{h} = 20$.

and succeeding reflections will not contribute very much to the solution. The total displacement of the plate, which was obtained by adding up the outgoing wave and the four reflections is also shown. The stress, $\sigma_{xx}(x, \frac{h}{2}, t)$, is graphed in Fig. 8. The outgoing wave, the first three reflections and the total stress are shown.

5. Inversion of the Time Transform-Vibrational Form

The Laplace transformed displacements and stresses, given by (2.43) through (2.45) for the region $h < x < \ell$, can also be inverted by means of a contour integration and residue theory. As shown by Miklowitz in [9], for example, the long-time behavior of the solution is determined by the singularities of the transformed solution closest to the Bromwich contour.

Observing (2.43) through (2.45), we see that the transformed solution has a branch point at $p = 0$ and poles wherever

$$D(p) = \cosh^2 \gamma \ell + \cos^2 \gamma \ell = 0 \quad . \tag{2.57}$$

Using l'Hopital's rule shows that $p = 0$ is not a pole of (2.43) through (2.45). A branch cut is made along the negative real axis and the branch is chosen so that $\sqrt{p}$ will be real and positive when p is.

It can be shown that (2.57) does not have any roots in the half plane $\mathrm{Re}\, p > 0$. Singularities in the half plane $\mathrm{Re}\, p < 0$ will decay with time and can be neglected for long-time. Integrating over a small circle around the branch point at $p = 0$ shows that it also does not contribute to the solution. The contributions of the integrals along the

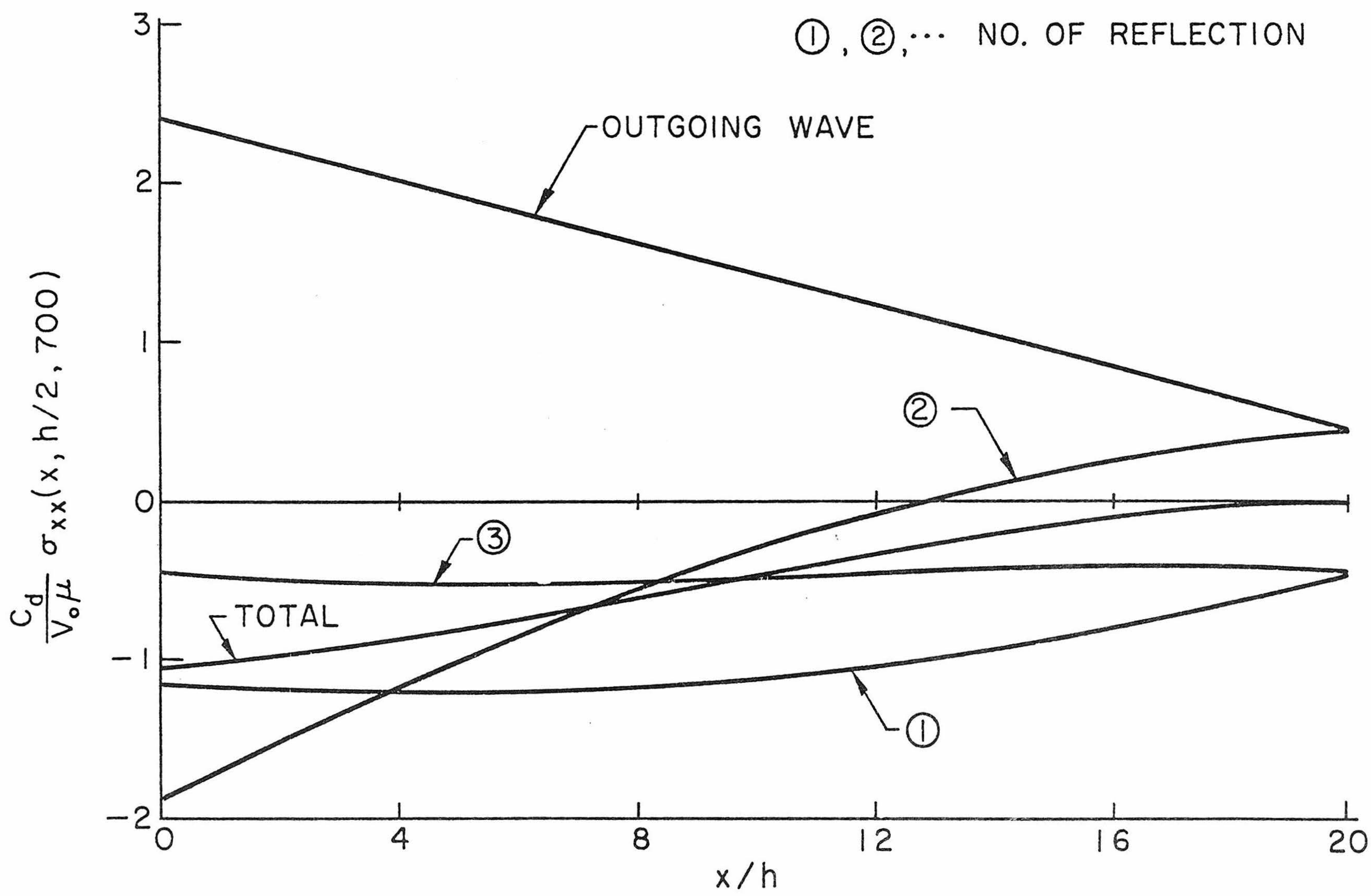

Fig. 8 Stress $\sigma_{xx}(x, \frac{h}{2}, 700)$ for the Finite Plate with $\frac{\ell}{h} = 20$.

branch cut will also decay with time. So it is only necessary to consider the roots of (2.57) on the imaginary axis.

Setting $p = i\omega$ and substituting into (2.57) gives

$$D(i\omega) = \cosh \ell\sqrt{\frac{\omega}{c_p r_g}} \cos \ell\sqrt{\frac{\omega}{c_p r_g}} + 1 = 0 \quad . \tag{2.58}$$

Equation (2.58) has an infinite number of roots corresponding to the natural frequencies of vibration of the plate. Note that if $\hat{\omega}$ is a root of (2.58), $-\hat{\omega}$ will be also.

The first ten roots of (2.58) were calculated numerically. The frequency was nondimensionalized by dividing by $\omega_s = \frac{\pi c_s}{2h}$. The resulting values for $\Omega = \frac{\omega}{\omega_s}$ are given in Table 2.

The solution given by (2.43) through (2.45) is only valid for p small. Comparing the exact Rayleigh-Lamb frequency spectrum with the small p approximations, (1.21) and (1.22), that we are using, shows that the latter are valid for at least $0 \le \Omega \le 0.10$. Selecting $\Omega = 0.10$ as the highest admissible frequency, we see from Table 2 that for a given $\frac{\ell}{h}$ ratio only a limited number of roots can be used. Note that the number of allowable roots increases as $\frac{\ell}{h}$ gets larger.

The contribution of a pole at $p = i\hat{\omega}$ to $v(x, y, t)$ and $\sigma_{xx}(x, y, t)$ is given by

$$\begin{Bmatrix} R^v(i\hat{\omega}) \\ \\ R^\sigma(i\hat{\omega}) \end{Bmatrix} = \lim_{p \to i\hat{\omega}} \left[(p - i\hat{\omega}) \begin{Bmatrix} \bar{v}(x, y, p) \\ \\ \bar{\sigma}_{xx}(x, y, p) \end{Bmatrix} e^{pt} \right] \quad , \tag{2.59}$$

TABLE 2

Natural Frequencies of Vibration for the Plate

$$\Omega = \frac{\omega}{\omega_s} \quad , \quad \omega_s = \frac{\pi c_s}{2h}$$

	$\frac{\ell}{h} = 5$	$\frac{\ell}{h} = 10$	$\frac{\ell}{h} = 20$
Ω_1	0.084	0.021	0.005
Ω_2	0.526	0.131	0.032
Ω_3	1.474	0.368	0.092
Ω_4	2.889	0.722	0.180
Ω_5	4.777	1.194	0.298
Ω_6	7.136	1.784	0.446
Ω_7	9.966	2.491	0.622
Ω_8	13.269	3.317	0.829
Ω_9	17.043	4.260	1.065
Ω_{10}	21.290	5.322	1.330

where $\overline{v}(x,y,p)$ and $\overline{\sigma}_{xx}(x,y,p)$ are given by (2.44) and (2.45) respectively. Substituting $p = i\hat{\omega}$ into (2.16) gives

$$E(i\hat{\omega}) = \frac{(1+i)}{2}\left[\sinh 2r \cos 2r + \cosh 2r \sin 2r\right] ,$$

$$F(i\hat{\omega}) = i \sinh 2r \sin 2r , \tag{2.60}$$

$$-\left.\frac{\partial D(p)}{\partial p}\right|_{p=i\hat{\omega}} = -\frac{ir}{\hat{\omega}}\left[\sinh 2r \cos 2r - \cosh 2r \sin 2r\right] ,$$

where

$$r = \frac{\ell}{2}\sqrt{\frac{\hat{\omega}}{c_p r_g}} .$$

Using (2.59) and (2.60) to calculate the residue yields

$$R^v(i\hat{\omega}) = \frac{v_0}{4\omega^2}\left[\left.\frac{\partial D}{\partial p}\right|_{p=i\hat{\omega}}\right]^{-1}\left\{iF(i\hat{\omega})\left(\cosh\frac{2rx}{\ell} - \cos\frac{2rx}{\ell}\right)\right.$$

$$\left. -(1-i)\,E(i\hat{\omega})\left(\sinh\frac{2rx}{\ell} - \sin\frac{2rx}{\ell}\right)\right\}e^{i\hat{\omega}t} . \tag{2.61}$$

Combining this with the residue from the pole at $p = -i\hat{\omega}$ shows that the part of the displacement associated with the frequency $\hat{\omega}$ can be written as

$$v^{\hat{\omega}}(x,y,t) = V(x,y,\hat{\omega}) \sin \hat{\omega} t .$$

Here $V(x,y,\hat{\omega})$ is the mode shape which is given by

$$V(x, y, \hat{\omega}) = \frac{v_0}{\hat{\omega} r d(r)} \left[f(r) \left(\cosh \frac{2rx}{\ell} - \cos \frac{2rx}{\ell} \right) - e(r) \left(\sinh \frac{2rx}{\ell} - \sin \frac{2rx}{\ell} \right) \right] ,$$

where (2.62)

$$d(r) = \sinh 2r \cos 2r - \cosh 2r \sin 2r ,$$

$$e(r) = \sinh 2r \cos 2r + \cosh 2r \sin 2r ,$$

$$f(r) = \sinh 2r \sin 2r .$$

Similarly

$$\sigma_{xx}^{\hat{\omega}}(x, y, t) = \Sigma(x, y, \hat{\omega}) \sin \hat{\omega} t , \qquad (2.63)$$

where

$$\Sigma(x, y, \hat{\omega}) = - \frac{4 v_0 \mu h}{(1-\nu) \ell^2 \hat{\omega} r d(r)} \left[f(r) \left(\cosh \frac{2rx}{\ell} + \cos \frac{2rx}{\ell} \right) \right.$$

$$\left. - e(r) \left(\sinh \frac{2rx}{\ell} + \sin \frac{2rx}{\ell} \right) \right] .$$

The lowest mode shape for the displacement, $V(x, 0, \omega_1)$, is shown in Fig. 9 for the the case $\frac{\ell}{h} = 5$ while $\Sigma(x, \frac{h}{2}, \omega_1)$ is shown in Fig. 10. These mode shapes agree with those found by Den Hartog in [13] using Euler-Bernoulli approximate theory.

The near-field asymptotic solution, valid as $x \to 0$, is obtained exactly as it was in Chapter I with the transformed displacements being given by (1.55), i.e.,

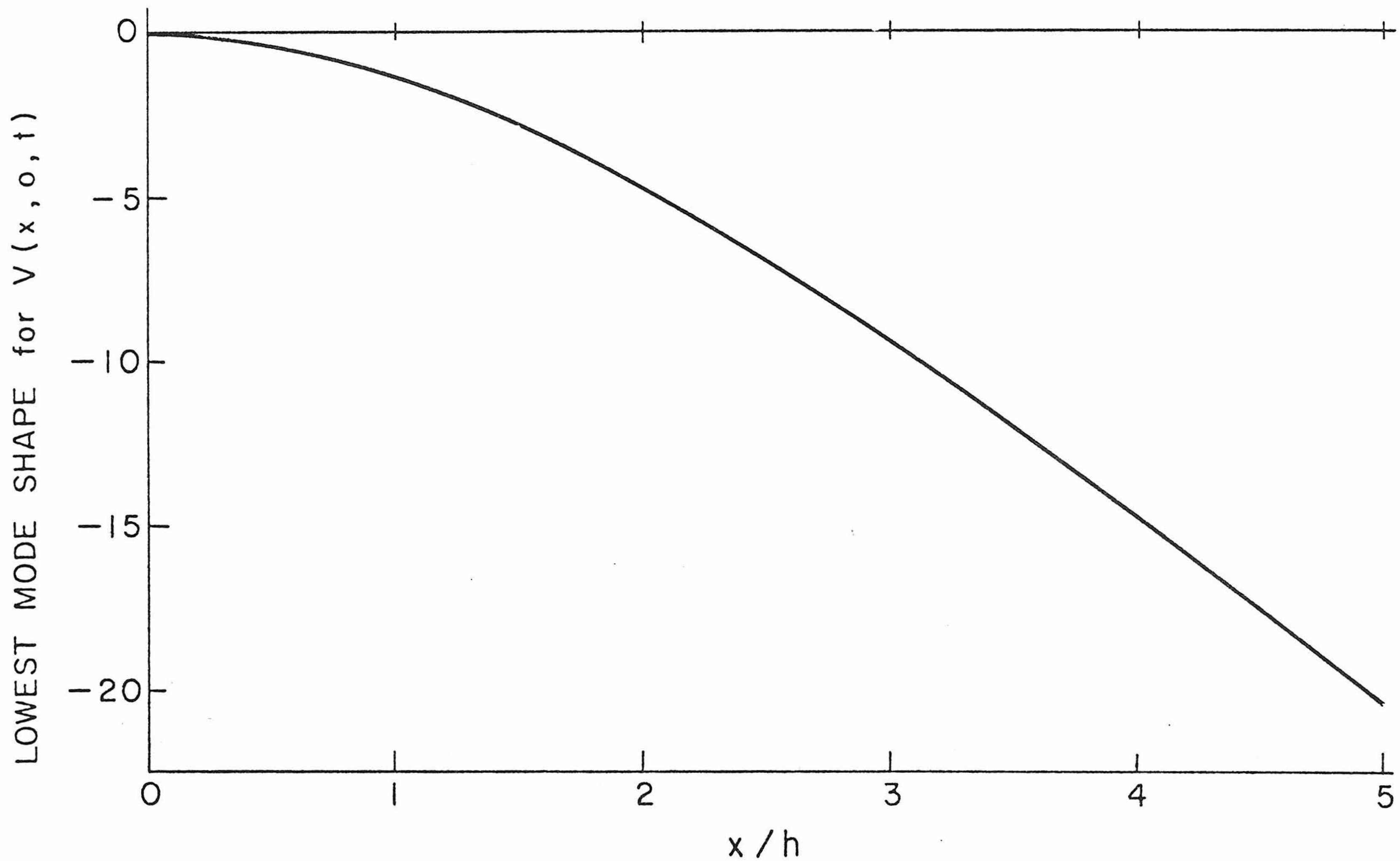

Fig. 9 Lowest Mode Shape for $v(x, 0, t)$ for $\frac{\ell}{h} = 5$.

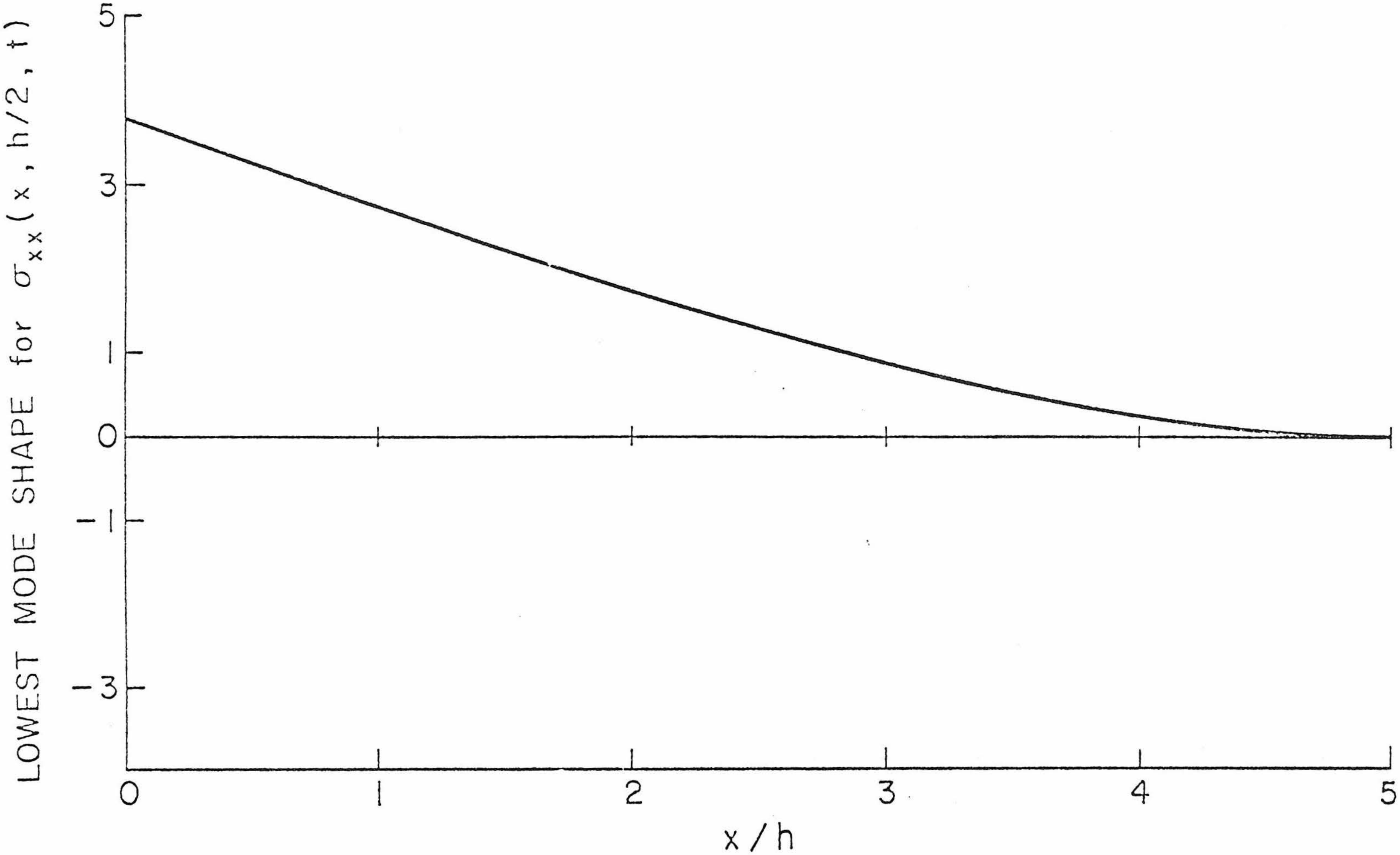

Fig. 10 Lowest Mode Shape for $\sigma_{xx}(x, \frac{h}{2}, t)$ for $\frac{\ell}{h} = 5$.

$$\bar{u}(x,y,p) = \left\{ b_0(p) \left[\left(1-\frac{y}{h}\right)^{-\frac{1}{4}} - \left(1+\frac{y}{h}\right)^{-\frac{1}{4}} + 2^{-\frac{1}{4}}\frac{y}{h} \right] + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} b_n(p) \sin\theta_n y \right\} x$$

$$+ \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} a_n(p) \left(\frac{k^2-1}{k^2}\right) \frac{x^2\theta_n}{2} \sin\theta_n y + O(x^3) \quad ,$$

$$\bar{v}(x,y,p) = \left[a_0(p) + \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} a_n(p) \cos\theta_n y \right] x + \frac{1}{2} \left\{ \frac{v_0}{c_s^2} \right.$$

$$+ b_0(p) \frac{(k^2-1)}{h} \left[\frac{1}{4}\left(1-\frac{y}{h}\right)^{-\frac{5}{4}} + \frac{1}{4}\left(1+\frac{y}{h}\right)^{-\frac{5}{4}} - 2^{-\frac{1}{4}} \right]$$

$$\left. - \sum_{\substack{n=2 \\ n\ \text{even}}}^{\infty} b_n(p)(k^2-1)\,\theta_n \cos\theta_n y \right\} x^2 + O(x^3) \quad .$$

The coefficients $a_0(p)$, $a_n(p)$, $b_0(p)$ and $b_n(p)$ are now given by (2.32). Calculating the strains at the base of the plate shows that they are the base strains for the semi-infinite plate multiplied by a reflection function, i.e.,

$$\bar{u}_x(0,y,p) = u_x^{SI}(y)\,\frac{F(p)}{p\,D(p)} \quad ,$$

$$\bar{v}_x(0,y,p) = v_x^{SI}(y)\,\frac{F(p)}{p\,D(p)} \quad , \tag{2.64}$$

where $u_x^{sI}(y)$ and $v_x^{sI}(y)$ represent the y dependence of the strains for the semi-infinite plate and are shown in Figs. 2 and 3 respectively.

The transformed strains, (2.64), can be inverted by using the same methods as in the last two sections. Formally, we have that

$$\begin{Bmatrix} u_x(0,y,t) \\ \\ v_x(0,y,t) \end{Bmatrix} = \begin{Bmatrix} u_x^{sI}(y) \\ \\ v_x^{sI}(y) \end{Bmatrix} \frac{1}{2\pi i} \int_{Br_p} \left[\frac{\cosh^2 \ell \sqrt{\frac{p}{2c_p r_g}} - \cos^2 \ell \sqrt{\frac{p}{2c_p r_g}}}{\cosh^2 \ell \sqrt{\frac{p}{2c_p r_g}} + \cos^2 \ell \sqrt{\frac{p}{2c_p r_g}}} \right] \frac{e^{pt} dp}{p} \quad . \tag{2.65}$$

As (2.65) shows, the strains at the base, including the singular term, will be time dependent for the finite plate instead of constant as was the case for the semi-infinite plate.

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