

- I. MICROWAVE IMPEDANCE OF SUPERCONDUCTING WEAK LINKS
- II. EFFECTS OF NONEQUILIBRIUM QUASI-PARTICLE DISTRIBUTIONS IN SUPERCONDUCTING WEAK LINKS

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ABSTRACT

Three separate but somewhat related experiments having to do with the nonequilibrium properties of proximity coupled superconducting weak links are reported.

The first experiment consisted of calibrated measurements of both the real and imaginary parts of the microwave impedance of proximity effect weak links. The impedance of a current-biased weak link was calculated analytically on the basis of several different theoretical models for the supercurrent I_s and the quasi-particle current I_{qp} , and the results were compared with the experimental measurements. For dc bias currents I_{dc} less than the critical current I_c , the measurements were consistent with the usual resistively shunted junction (RSJ) model. However, for $I_{dc} > I_c$, the Fourier component of I_s at the Josephson frequency was found to be less by a factor of $\sim 0.65 \pm .1$ than would be expected from the RSJ model, but consistent with the predictions of the phase-slip model. The phase-dependent part of the quasi-particle current, if it exists at all, was found to be no larger than 10% of the total quasi-particle current.

In the second experiment a very small tunnel junction was used to inject quasi-particles directly into the center of a proximity effect weak link. It was found that the critical current I_c of the weak link was, over a wide range of experimental conditions, proportional to $\exp(-\alpha\sqrt{P})$, where P is the injection power and α is a constant. This result was somewhat surprising since, on the basis of

simple heating or various microscopic theories, I_c would be expected to depend exponentially on the power itself. It was suggested that the order parameter inside the weak link is sufficiently small that the excitation of collective modes is more important than the excitation of quasi-particles, and that the transverse collective modes are excited by the noise voltages produced by the injected power.

The final experiment consisted of using a small tunnel junction as a probe of the nonequilibrium quasi-particle chemical potential μ_{qp} inside a proximity effect weak link. The nonequilibrium value of μ_{qp} was produced by simultaneously passing a dc bias current less than I_c and a microwave current I_μ through the weak link. For most values of I_μ the measured value of μ_{qp} was proportional to the dc supercurrent carried by the weak link, as reported in a similar experiment by Yu and Mercereau. In addition, for certain values of I_μ , the measured value of μ_{qp} became a discontinuous function of the supercurrent. A physical mechanism which may be responsible for these effects was proposed and the calculated value of μ_{qp} was found to be in qualitative agreement with the data, including the newly discovered discontinuities.

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INTRODUCTION

This thesis describes an experimental investigation of some of the properties of superconductors under nonequilibrium conditions. In particular, the properties of proximity coupled superconducting weak links, which are quantum interference devices similar to Josephson tunneling junctions, were studied under various nonequilibrium situations. Although superconductivity is reasonably well understood under equilibrium conditions, the nonequilibrium theory of superconductivity is still in an early stage of development. This is especially true for those cases in which electric fields exist inside the superconductor. An example of the need for further theoretical understanding of nonequilibrium superconductivity is the fact that, of the three experiments described in this thesis, two gave results which were unexpected on the basis of the theory or models prevalent in the literature. The remaining experiment dealt with a situation for which no detailed theory has currently been worked out, other than the model suggested in this thesis. In other words, there are many aspects of superconductivity which are not well understood, despite the great success of the BCS theory.

Although the abstract and table of contents have already outlined the topics to be considered, a brief review may be useful here. Chapter I provides an intuitive account of the fundamental properties of superconductivity, with the introductory material necessary for the understanding of the rest of the thesis. (Those readers who are familiar with superconductivity are urged to skip Chapter I.) The

remainder of the thesis describes three separate but somewhat related experiments. Chapter II describes the measurement and analytic calculation of the microwave impedance of proximity effect weak links. Since the form of the impedance is strongly dependent on quantum mechanical effects, its measurement provides a great deal of physical information which cannot be obtained in any other way. (In addition, the impedance is important in various practical applications of these devices.) Chapter III describes an experiment in which quasi-particle excitations were injected into the weak link using a small tunnel junction. This perturbs the various properties of the weak link and the information obtained is relevant to the effects of nonequilibrium conditions in superconductors. In Chapter IV, the technique described in Chapter III is reversed; a tunnel junction is used as a probe of the nonequilibrium conditions which ordinarily exist in a proximity effect weak link during its normal operation. (The tunnel junction does not perturb the weak link in the experiment described in Chapter IV.) Although each chapter contains a summary of the results from that particular experiment, Chapter IV also contains a brief review of what information has been obtained from the thesis as a whole. Finally, Appendix A provides the mathematical calculations relevant to Chapter II, while Appendix B provides the calculations needed in Chapter IV.

I. HISTORICAL BACKGROUND AND INTRODUCTORY MATERIAL

1.1 Introduction

One of the major purposes of this thesis is to present technical details which would generally not be published in the literature. Consequently, Chapters II, III, and IV, as well as the appendices, do not contain a great deal of introductory material for the benefit of nonspecialists. Since it is customary to include a certain amount of such material, this chapter has been set off from the rest of the thesis to serve as such an introduction. Those readers who are familiar with superconductivity are urged to skip this chapter.

All of the discussion in this chapter is intuitive and extremely nonrigorous. More complete and formal reviews of the theory, as well as historical accounts, can be found in any of the theses by earlier graduate students (Ref. 1-6). This chapter differs somewhat from those of the earlier theses mentioned in that the emphasis here has been placed on intuitive explanation rather than historical documentation. It was assumed that such details are of little interest to readers who are not specialists in the field, and common knowledge to those who are.

A secondary purpose of this chapter is to provide a discussion (however brief and simplified) of the nature of the so-called macroscopic wavefunction (order parameter) Ψ . As will be seen, many of the properties of a superconductor can be described by assuming that

the electrons are governed by such a wave function and then applying the methods of elementary quantum mechanics. The equations which result are extremely simple in nature; they are so simple, in fact, that they may appear to be naive or even absurd. Simple equations such as these appear throughout the thesis, making it essential to discuss the nature of the mysterious wave function Ψ .

1.2 The Macroscopic Wave Function (Order Parameter) Ψ

Superconductivity is often referred to as a macroscopic quantum state. This is because many of the equilibrium properties of a superconductor may be phenomenologically described by assuming that all of the superconducting electrons are governed by a wave function $\Psi(\vec{R})$ which is a function of position $\vec{R}$. For example, the current carried by the superconductor may be found by applying the usual quantum mechanical current density operator to Ψ :

$$\vec{j}(\vec{R}) = \frac{e^*\hbar}{2m^*i} (\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*) \quad (1.1)$$

Here e^* and m^* are, respectively, twice the charge and twice the mass of an electron. The reason for the factors of two will become apparent shortly.

The wave function Ψ is referred to as being macroscopic for several reasons. First of all, notice that Ψ is a function of only one position vector $\vec{R}$. The electrons in any metal can be described by a quantum mechanical wave function, but such wave functions are ordinarily functions of all the coordinates

of all the electrons: $\psi = \psi(\vec{r}_1, \vec{r}_2, \dots)$. The macroscopic wave function, however, is defined as a function of position inside the superconductor. More importantly, the wave function is coherent over macroscopic distances. For example, if the two ends of a superconducting wire are brought together in the right way, it is possible to observe interference effects between $\Psi(\vec{R})$ in the two ends of the wire. This is analogous to the interference effects which can be observed if two beams of microwaves (or any other coherent waves) are brought together. The electrons in a normal wire do not show such interference effects because their wave function is coherent only over distances comparable to the mean free path.

In addition to being coherent over macroscopic distances, the wave function Ψ can be seen to be quantum mechanical in nature as well. For example, if a superconductor is formed into a loop, the continuity requirements on Ψ result in the quantization of the (conjugate) angular momentum of the superconducting electrons. Furthermore, Ψ obeys a differential equation which is similar in form to the Schrodinger equation. Thus in many respects a superconductor may be considered to be a coherent, macroscopic quantum state. Much of the excitement in this area of research results from this ability to observe and manipulate quantum effects on a macroscopic scale.

Thus far some of the properties of Ψ have been described, but no reasons have been given why they are true. In fact, Ψ has not yet been properly defined nor has the question of how to calculate Ψ been discussed. As it turns out, Ψ is not really the quantum wave function

of anything. Thus the question of what Ψ is and how it is to be calculated needs further discussion.

The concept of a macroscopic wave function was first introduced (Ref. 7) in the phenomenological theory of Ginzburg and Landau (GL), who referred to Ψ as the "complex order parameter". Their treatment consisted of an assumed expansion of the free energy of a superconductor near its transition temperature in terms of the parameter Ψ . The exact physical nature of Ψ could not be understood on the basis of their phenomenological theory. (The possibility of a macroscopic quantum state had earlier been suggested (Ref. 8) by F. London.) Their treatment did, however, give an explicit equation for the spatial dependence of Ψ , known as the Ginzburg-Landau equation

$$\frac{1}{2m^*} \left(\hbar \vec{\nabla} - \frac{e^*}{c} \vec{A} \right)^2 \Psi + (\beta |\Psi|^2) \Psi = E^* \Psi \quad (1.2)$$

Here $\vec{A}$ is the electromagnetic vector potential and E^* and β are constants which can only be experimentally determined (in the context of the GL theory). Note the similarity to the Schrodinger equation if the second term is interpreted as a self-interaction potential energy. This equation was used to make several new physical predictions, including the existence of Type II superconductors (Ref. 9).

One obvious shortcoming of the GL theory was that it gave no physical interpretation to the mysterious wave function or order parameter Ψ . Since that time, the physical nature of superconductivity has been satisfactorily explained by the microscopic pairing

theory (Ref. 10) of Bardeen, Cooper, and Schrieffer (BCS). The BCS theory is based on first principles (quantum mechanics) and gives an explicit equation for the actual wave function $|\psi\rangle$ of the electrons. At absolute zero (Ref. 10)

$$|\psi\rangle = \prod_k (u_k + v_k c_{k\uparrow}^* c_{-k\downarrow}^*) |\phi_0\rangle \quad (1.3)$$

where $|\phi_0\rangle$ is the vacuum and $c_{k\uparrow}^*$ creates an electron in a Bloch state of momentum $\vec{k}$ and spin up; u_k and v_k are constants explicitly determined by the theory. Physically, at absolute zero u_k and v_k represent the probability that the Bloch states of momentum $\vec{k}$ are, respectively, unoccupied or occupied; v_k does not abruptly drop to zero at the Fermi level even at absolute zero.

A detailed understanding of the BCS theory is not necessary for the purposes of this chapter, and eq. (1.3) has been included only because it can be used to give a precise definition of the macroscopic wave function (or order parameter) Ψ . An important quantity defined in the BCS theory is the energy gap Δ_k :

$$\Delta_k = - \sum_{\ell} V_{k\ell} u_{\ell} v_{\ell} \quad (1.4)$$

where $V_{k\ell}$ is the matrix element of the attractive electron-electron interaction which gives rise to the superconducting state. By the use of Green's function methods, Gor'kov (Ref. 11) was able to show that (under certain conditions) the order parameter Ψ of the GL theory is equivalent to the energy gap parameter Δ , and to derive

eq. (1.2). (Under the appropriate approximations Δ_k is independent of k .)

Thus eq. (1.4) serves as a rigorous but nonintuitive definition of Ψ , and Gor'kov's derivation proves the properties of Ψ which have already been discussed. Clearly, Ψ is not really the quantum wave function of anything. Nevertheless, many of the properties of a superconductor can be calculated as if it were.

Before leaving the BCS theory, it should be pointed out that the magnitude of Δ can be shown (Ref. 10) to be the minimum energy of an excited state of the system; thus the term energy gap. These excited states are many-body states rather than single-particle excitations and are therefore called quasi-particles. Some effects of quasi-particles are studied in Chapters III and IV. Crudely speaking, the magnitude of the macroscopic wave function (order parameter) Ψ measures the strength of the superconductivity, while its phase measures the relative phase of u_k and v_k in eq. (1.3).

Equation (1.4) and Gor'kov's derivation have been taken as the rigorous definition of Ψ and the proof of its properties. In order to obtain some insight into why a superconductor might be described by a macroscopic wave function Ψ , a crude physical model will now be discussed. This model might be referred to as the point-particle boson model. (The author does not recall seeing this model explicitly written down anywhere, but it is undoubtedly well known.)

In the BCS theory, superconductivity is caused by an attractive interaction between electrons which causes them to form bound pairs

known historically (Ref. 12) as Cooper pairs. The electrons in a pair have opposite spin and so the net spin of a Cooper pair is zero; a Cooper pair is a boson. The binding energy of a pair turns out to be 2Δ . (The factor of 2 arises from the fact that two quasi-particles, a quasi-electron and a quasi-hole, are created when a Cooper pair is broken apart.) The point-particle boson model simplifies (grossly) the situation by assuming that the binding energy of a pair is much larger than any other energy involved in the problem. In particular, it is assumed that the binding energy is so large that the radius of a Cooper pair is zero for all practical purposes. (The radius will decrease as the binding energy increases.) Furthermore, it is assumed that the Cooper pairs do not interact with each other, and they move as free particles inside the superconductor.

From the point of view of this approximation, a superconductor consists of a box full of bosons. At low temperatures such bosons will undergo Bose-Einstein condensation into the ground state and so are all described by a single wave function $\Psi(\vec{R})$. The electrical current would be given by eq. (1.1), etc. Thus, the existence of bound Cooper pairs makes the macroscopic wave function Ψ plausible at least. Ψ can be viewed (although somewhat erroneously) as the wave function of the Cooper pair bosons.

It should be immediately pointed out, however, that the point-particle boson model is at best a gross oversimplification. It would be quite correct if Δ were infinite, but in fact Δ is so small compared to the Fermi energy ϵ_F ($\Delta \sim 2$ meV, $\epsilon_F \sim 10$ eV) that the model has

little physical reality. Because $\Delta \ll \epsilon_F$, the radius of the Cooper pairs is larger than their separation; the electrons are still almost Fermi degenerate. In addition, the model does not take into account the existence of quasi-particle excitations, which form the basis of Chapters III and IV.

It might be wondered, then, if the point-particle boson model is of any use at all. Aside from providing an intuitive (and somewhat erroneous) physical interpretation of the order parameter Ψ , it is also capable, as will be seen, of demonstrating many of the important features of the Josephson effect which otherwise are rather nonintuitive. More importantly (in the author's opinion) the point-particle boson model also demonstrates some of the main limitations of the GL theory.

The Ginzburg-Landau theory is widely used to analyze many properties of superconductors, primarily because the more rigorous Green's function methods are too cumbersome to be of much use in many situations. But, for reasons which are beyond the scope of this chapter, the GL theory implicitly includes the assumption that the radius of the Cooper pair is negligibly small. To be more precise, if the radius of a Cooper pair is not negligible compared to all other relevant dimensions, the GL theory is invalid.

For example, if an electromagnetic field is applied which varies on a scale small compared to a Cooper pair radius, then the GL theory does not give the correct results (the Gor'kov derivation

becomes invalid). The point-particle boson model helps keep these and other limitations of the GL theory in mind. This is especially important here, since the experiments in this thesis deal with situations where the GL theory is frequently invalid for various reasons.

Although too many words have probably already been devoted to the nature of Ψ , there is one more equation which is relevant to the question of the approximations involved in the point-particle boson model. At absolute zero, the BCS wave function (Ref. 13) can be put (Ref. 14) in the following (exact) form

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = A \phi(\vec{r}_1 - \vec{r}_2) \phi(\vec{r}_3 - \vec{r}_4) \dots \phi(\vec{r}_{N-1} - \vec{r}_N) \quad (1.5)$$

where $\vec{r}_i$ is the coordinate of the i th electron and spin indices have been dropped. Here A is the antisymmetrization operator and ϕ is the relative coordinate wave function of a Cooper pair, which is explicitly given by the BCS theory. Note that the same ϕ appears for each pair of coordinates. If A were ignored or had negligible effect then the wave function would be equivalent to that of Bose-condensed Cooper pairs. However, the effect of A is not at all negligible; neglecting A is in some respects equivalent to assuming that the pairs are infinitely bound.

This section would probably not be complete without some explanation of how a superconductor can carry a perpetual current. The reason has to do with the fact that Ψ is determined by the motion of all the electrons; it is a kind of macroscopic average. If a superconducting wire is joined into a loop, then the continuity of Ψ

requires that its phase change integrated once around the loop be $2\pi n$, where n is an integer. If n is nonzero, the superconductor is carrying a current.

Now suppose that an electron in this wire collides with an impurity, is scattered by a cosmic ray, or in some other way has its motion changed. Since Ψ is a macroscopic average, the value of n cannot be changed by such an event and the displaced electron will eventually relax back to its original state, thus preserving the current. This does not happen in a normal metal because the quantum phases are not coherent over macroscopic distances. Thus it is the long-range coherence of Ψ , rather than the presence of an energy gap in the excitation spectrum (as is often erroneously assumed) which gives rise to perpetual currents.

The preceding comments on the nature of Ψ might be summarized by saying that many properties of a superconductor are formally given by a macroscopic wave function Ψ , even though Ψ itself is clearly not a true quantum wave function. As will be discussed later in this chapter, Ψ has as much to do with thermodynamics as it does quantum mechanics. Before pursuing this topic further, the properties of Ψ already discussed can be put to use to give an intuitive explanation of the Josephson effect.

1.3 The Josephson Effect

The Josephson effect (Ref. 15) is the name given to several phenomena which occur in the tunneling of electrons between two

superconductors separated by a thin region of insulating material. The term is also applied to similar phenomena which occur in certain superconducting devices which do not employ tunneling as the basic means of charge transfer. For example, this thesis studies some of the properties of devices known as proximity effect weak links. (These devices will be described later in the chapter.) Proximity effect weak links are said to exhibit the Josephson effect, even though electron tunneling does not occur. In this section we will be concerned only with true tunneling phenomena, and will postpone a discussion of non-tunneling devices until the following section.

Electron tunneling between two normal metals is a fairly simple and straightforward phenomenon. One might wonder, then, what makes tunneling between two superconductors so different. First of all, the same kind of tunneling which occurs between two normal metals also occurs between two superconductors. This type of tunneling (quasi-particle current) was, in fact, studied extensively by Giaever (Ref. 16) and others before Josephson's predictions had been made. This research formed the basis for Giaever's Nobel prize. Quasi-particle tunneling is essentially the same in superconductors as in normal metals except that the excitation spectrum for a superconductor is altered by the presence of the energy gap. The Josephson effect has to do with some additional phenomena not found in normal metal tunneling and predicted by Josephson (Ref. 15) while he was still a graduate student. These predictions were initially met with great skepticism because of their unusual nature and because such effects had not been observed in

the work carried out by Giaever and others. This skepticism is a good indication of the theoretical and experimental difficulties involved in any study of the Josephson effect.

There are two features which distinguish the Josephson effect from normal metal tunneling. First of all, a finite current will flow at zero applied voltage. This tunneling supercurrent is reminiscent of the supercurrents which will flow in bulk superconductors. Secondly, the magnitude of the supercurrent goes as the square of the tunneling Hamiltonian (to be discussed), rather than as the fourth power as would be naively expected. This second property is really the crux of Josephson's contribution, since the first property is perhaps not very surprising.

An intuitive "derivation" of the Josephson effect will now be given, based on the point-particle boson model previously described. A more rigorous derivation (Ref. 15) using the correct BCS wave function shows that the essential features of the Josephson effect are not very sensitive to the exact form of the wave function. Thus our approximate model, crude as it is, will show the important qualitative features of the phenomenon.

To begin, suppose that there are two boxes (in the spirit of sophomore quantum mechanics) each capable of confining a quantum particle, such as an electron. We can imagine each box to be constructed of a three-dimensional square well, the walls of the well consisting of a large but finite potential barrier. Since the height of the potential barrier is finite, the wave function of a particle

confined in one of the boxes will have an amplitude which is constant inside the well, but dropping off exponentially like e^{-Kx} outside the well. Here x is the distance outside the well and K is a constant calculated in the way familiar from elementary quantum mechanics. Let $\psi_L(x)$ denote such a wave function for a particle confined in a particular box (the one on the left), and $\psi_R(x)$ the corresponding wave function for a particle confined in the other box.

If the two boxes are not an infinite distance apart, then it is, of course, possible for a particle to tunnel from one box to the other. The exact kind of particle to be placed inside the box has not yet been specified; in order to be specific, they can be assumed for the time being to be electrons. (Later the tunneling of Cooper pair bosons will be considered.) In order to calculate the rate of tunneling (current) from one box to the other, we must specify what wave function $\psi(x)$ describes the particle. The tunneling which occurs between two normal metals, as well as the tunneling studied by Giaever (quasi-particle tunneling) corresponds to a wave function of the form $\psi(x) = \psi_L(x)$ or $\psi(x) = \psi_R(x)$. In other words, the particle is initially located in one box or the other.

For a wave function of the form $\psi(x) = \psi_L(x)$, the transition rate to ψ_R can be readily calculated by time-dependent perturbation theory. The result, of course, depends on the matrix element between the two states which is readily shown to be proportional to e^{-KL} , where L is the separation of the two boxes. The current (transition rate) then goes as the matrix element squared, or $(e^{-KL})^2$. For real metals

it is, of course, necessary to sum over occupied states from which tunneling is possible, and over empty states into which tunneling can occur; then a net current flows only when an external voltage is applied.

The Josephson effect, on the other hand, corresponds in this simple model to initial states of the form $\psi_c(x) = \alpha[\psi_L(x) + e^{i\theta} \psi_R(x)]$ where α is a normalization constant and θ is an arbitrary phase angle. In a coherent state of this sort the particle does not initially occupy either box with certainty. The current in this type of state is most readily evaluated by directly substituting $\psi_c(x)$ into the current operator of elementary quantum mechanics and evaluating the result at some point between the two boxes. (The current density which results is independent of position.) The calculation is quite straightforward and gives the result

$$I_s = I_c \sin \theta \quad (1.6)$$

where I_s is the tunneling supercurrent and I_c is a constant proportional to e^{-KL} . The algebraic details can be found in a slightly more general calculation carried out in the GL formalism in Appendix B.

Perhaps the most important thing to note about eq. (1.6) is that the tunneling current goes as e^{-KL} , rather than the $(e^{-KL})^2$ obtained previously. Coherent tunneling of this sort always occurs with a lower power of the matrix element than does incoherent tunneling; this results from the coherence of the wave function $\psi_c(x)$ and has nothing to do with the nature of the actual particles involved. It might also

be pointed out that the derivation of eq. (1.6) does not involve the transition from one state to another, but represents the current carried in one particular state.

The Josephson effect in this model results from the tunneling of Cooper pairs. (As will be discussed shortly, the tunneling of pairs rather than single electrons is necessary to avoid the creation of quasi-particle excitations; it is still the coherent nature of superconductivity, rather than the structure of the pairs, which is important.) Let H_T be the matrix element for single electron tunneling, which is proportional to e^{-KL} . The tunneling of a pair occurs in second order in H_T since a quasi-particle excitation would be created on each side of the barrier if the two electrons making up a pair were to tunnel separately in first order. Since, in the absence of an applied voltage, there is no energy available for the production of quasi-particles, the tunneling must occur in second order with a virtual intermediate state. Thus the simplest picture of the Josephson effect is the tunneling of a Cooper pair with an effective matrix element H_T^2 . Equation (1.6) still results if e^{-KL} is replaced by the effective tunneling Hamiltonian H_T^2 . Note that the tunneling supercurrent then goes as $(e^{-KL})^2$, rather than the $(e^{-KL})^4$ which might have been expected for incoherent tunneling.

The tunneling between normal metals and the quasi-particle tunneling between superconductors are first order and go as $(e^{-KL})^2$. The supercurrent, on the other hand, is a second order effect. But since it is coherent, the supercurrent ends up going as $(e^{-KL})^2$ also.

Since e^{-KL} is an extremely small number for a real tunnel junction, the supercurrent would have been unobservably small if this were not the case. Josephson's real contribution was the discovery of this fact.

These intuitive remarks on the nature of the Josephson effect have been based on the approximation that the Cooper pairs are bound point particles. It has already been mentioned that this approximation is far from realistic. The correct results can be obtained by performing an analogous calculation using the correct wave functions provided by the BCS theory. At a temperature of absolute zero and for zero applied voltage, the results obtained by rigorous means are identical to eq. (1.6), where θ is the difference in the phase of the order parameter Ψ from one side of the oxide barrier to the other.

At finite voltages the presence of quasi-particles will contribute to the current. Since all experiments are performed at finite temperature, some quasi-particles will always be present. (The probability of a quasi-particle state being occupied is given by the Fermi-Dirac distribution, since quasi-particles behave as fermions.) Additional quasi-particles will be created if the applied voltage is large enough to break apart the Cooper pairs. The current due to quasi-particles, I_{qp} , can be shown (Ref. 15) to be given by

$$I_{qp} = v(G_1 + G_2 \cos \theta) \quad (1.7)$$

where G_1 and G_2 are both functions of temperature and of v , the applied voltage.

The first term in eq. (1.7) (incoherent quasi-particle current) is analogous to the tunneling current between normal metals, differing only in the density of states. The second term in eq. (1.7) (coherent or phase-dependent quasi-particle current) depends on both the voltage and phase and has no analogy in normal metal tunneling. It has been described by Josephson as an interference between the ordinary quasi-particle current and the supercurrent. This phase-dependent quasi-particle current has recently been the focus of considerable interest because all experimental measurements (Refs. 17-20) of G_2 (excluding this thesis) have given the opposite sign from that predicted by Josephson's theory. One of the purposes of Chapter II is to describe a new measurement of G_2 .

The time dependence of θ , the phase difference across the bridge, can be shown (Ref. 15) to be given by

$$\frac{d\theta}{dt} = \frac{2ev}{\hbar} \quad (1.8)$$

where e is the charge of the electron and $\hbar$ is Planck's constant. In either our simplified point-particle boson picture, or the rigorous calculations of Josephson, eq. (1.8) results from the fact that the phase of a quantum system precesses at a rate proportional to its energy. (The factor of 2 comes from the fact that a Cooper pair has two electronic charges.) For a nonzero DC voltage applied across the tunnel junction, eq. (1.8) can be integrated and inserted into eq. (1.6) to show that a sinusoidal supercurrent results at an angular frequency ω given by $2ev = \hbar\omega$. This is the so-called AC Josephson

effect (Ref. 15) and will be dealt with extensively in Chapter II.

Equations (1.6) and (1.7) together constitute the Josephson effect. Both of these equations were predicted in Josephson's original paper, which dealt primarily with tunnel junctions. As will be seen in the next section, similar effects occur in superconducting devices not employing tunneling, such as the proximity effect weak link.

1.4 The Proximity Effect

If a normal metal is placed in contact with a superconducting metal, then the order parameter Ψ does not immediately drop to zero at the boundary of the normal metal. Instead, Ψ drops off gradually (Ref. 21) with distance into the normal metal. For example, if a copper (normal) film is evaporated on top of a lead (superconducting) film, then various superconducting properties, such as the ability to carry a supercurrent, will exist some distance into the copper film. The distances involved may vary from $1000\text{\AA}$ or less to 1μ or more. This phenomenon is known as the proximity effect (Ref. 22).

The proximity effect is ordinarily discussed in terms of the phenomenological GL theory, primarily because the more fundamental Green's function methods are frequently too cumbersome to be of use. Consequently, in this section the GL eq. (1.2) will be applied. Consideration will be limited to the region well inside the normal metal where the magnitude of Ψ is small. Under these conditions ($|\Psi| \ll 1$) eq. (1.2) can be linearized with negligible error and yields

$$\frac{1}{2m^*} \left(\frac{\hbar}{i} \vec{\nabla} - \frac{e^*}{c} \vec{A} \right)^2 \Psi = E^* \Psi \quad (1.9)$$

Note that eq. (1.9) is identical in form to the Schrodinger equation for a single particle, and has exponentially decaying solutions which are mathematically equivalent to those of a quantum mechanical particle tunneling into a low potential barrier. For this reason the proximity effect is sometimes intuitively explained by assuming that Ψ is actually a quantum wave function and that particles (Cooper pairs) are actually tunneling into the normal metal. The appropriate barrier height would be on the order of the energy difference between a Cooper pair inside the normal metal as opposed to one inside the superconductor. An alternative "explanation" of the proximity effect is that the Cooper pairs have a lifetime inside the normal metal given by the uncertainty principle. They then "diffuse" into the normal metal a distance given by the product of this lifetime and the Fermi velocity.

The author does not find either of the above intuitive explanations to be very satisfactory. This is because of the fact that the mean free path of the electrons is much smaller (at least for the experiments in this thesis) than the decay length of Ψ . In fact, examples can be found in many branches of physics, including electromagnetism, optics, classical mechanics, and quantum mechanics, of wave motions which propagate freely in one medium but which drop off exponentially in another adjoining medium. The cut-off frequency of a waveguide, for instance, sets a frequency below which the microwaves

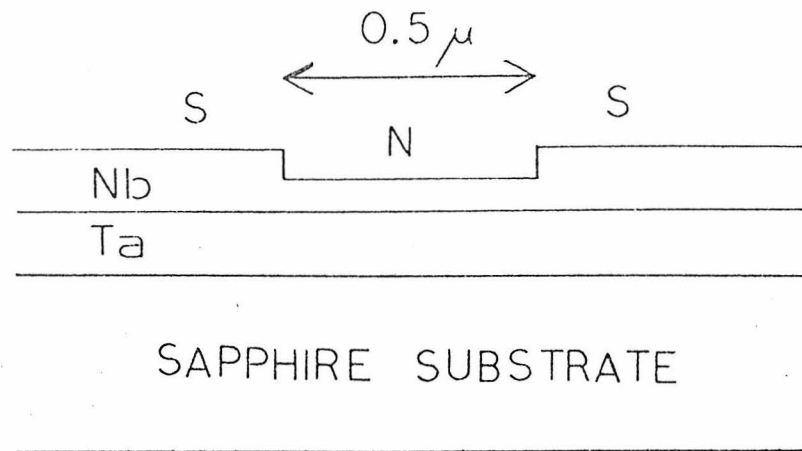
are exponentially damped rather than propagating in the waveguide. The exponential decay length of the waves will, in general, depend on how far the frequency is below the cut-off.

The fact that eq. (1.9) has the same mathematical form as quantum-mechanical tunneling does not necessarily mean, however, that the physical origin of the proximity effect is any more closely related to tunneling than it is to the corresponding phenomena in mechanics, optics, or electromagnetism. Some insight into the true nature of the proximity effect can be obtained by the following reasoning: If, for whatever reason, the electrons at some point P in a metal are in a superconducting state ($\Psi \neq 0$), then the electrons at a nearby point P' will have a tendency to be in a superconducting state as well. This is because the electrons at P' can lower their free energy to some extent by the formation of Cooper pairs in conjunction with the electrons at P . The coupling of one point P to another point P' involves the formation of mutual bound pairs, and thermodynamics plays a large role. These features are not found in true quantum mechanical tunneling. The exponential decay length is ultimately determined by a balance between the kinetic energy which would result for a rapidly changing Ψ , and the slight energy reduction which could be obtained if the metal were in the normal state ($\Psi = 0$). In any event, if the normal metal is actually a superconductor slightly above its transition temperature, then the exponential decay length becomes quite large.

The proximity effect, which has been intuitively discussed above, forms the basis for the operation of the so-called proximity effect weak link (Ref. 23). Such a weak link is formed by separating two superconducting metal films by a short length of normal metal film. By virtue of the proximity effect, the order parameter Ψ will then have an exponential tail coming into the normal region from the two superconducting regions on each side. The overlap of the two exponential tails in the center of the weak link gives rise to interference effects similar to those found in Josephson tunnel junctions. These interference effects are still referred to as the Josephson effect, and the existence of such effects not involving tunneling were in fact suggested in Josephson's original paper (Ref. 15).

Figure 1.1 shows the construction of a practical proximity effect weak link of the type actually used in the experimental work described in this thesis. A normal metal film, typically Ta, was evaporated on top of a sapphire substrate. A superconducting film, typically Nb, was then immediately evaporated on top of the first film before an oxide layer had the opportunity to form. Both films were on the order of $100\text{\AA}$ thick. At the temperature at which the device is used, which is typically around 4°K , Nb is superconducting and Ta, which has a transition temperature T_c of approximately 3.8°K , is really a superconductor slightly above its T_c . As has already been mentioned, this causes the decay length of Ψ in the Ta to be relatively large.

-24-
EDGE VIEW



TOP VIEW

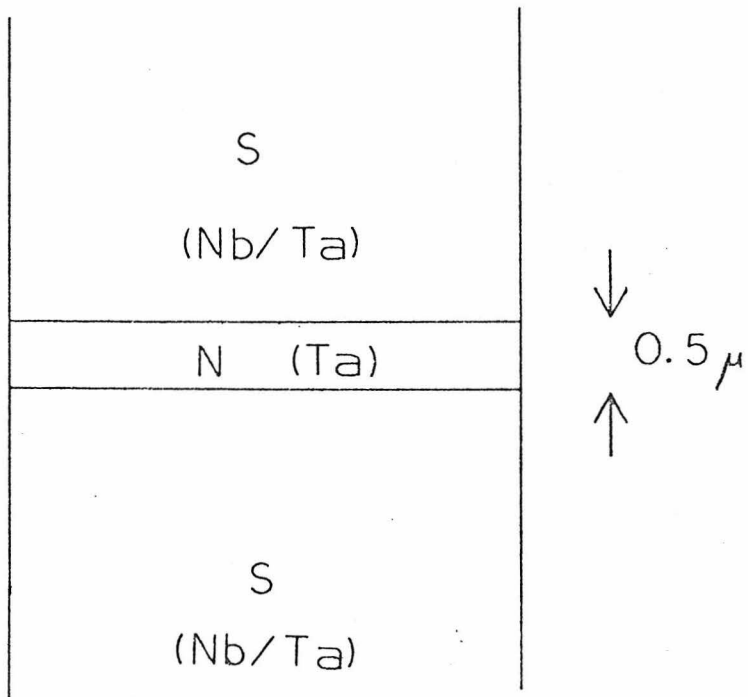


Fig. 1.1 Structure of a typical proximity effect weak link (not drawn to scale).

In order to remove the Nb over a small region as shown in the figure, a thin layer of photo-resist (PR) is first spun on top of the film using a centrifuge. PR is a photo-sensitive plastic which, after development, is removed where it has been exposed to light but remains intact elsewhere. The PR covering those regions where the Nb is to be removed are exposed and developed away using a microscope exposure technique developed by earlier graduate students (Ref. 24). Basically, the technique consists of using an optical microscope working in reverse to greatly reduce the size of masks when imaged onto the substrate. Since 0.5μ is near the limit of the resolution of an optical microscope, great care must be exercised in carrying out these procedures and the work is tedious to say the least.

Once the PR has been exposed and developed the metal which is uncovered may be etched away in several different ways. (The metal still covered by PR is protected and not affected by the etching process.) The standard procedure in the laboratory is to anodize the metal to a certain depth. Both Nb and Ta have the property that the thickness of metal converted to oxide by anodization is proportional to the anodizing voltage; this allows accurate depth control. The author, however, prefers to etch using an energetic beam of argon ions, which "sandblast" the metal away, so to speak. This process has better lateral resolution.

Proximity effect weak links such as the one shown in Fig. 1.1 exhibit many of the same properties as true tunnel junction Josephson

devices. The theory of their operation is usually formulated in terms of the GL theory. Such a calculation, although generalized to include effects beyond the scope of the GL theory, can be found in Appendix B. The results of such calculations turn out to be very similar to those for true tunnel junctions because of the mathematical similarity of eq. (1.9) to the Schrodinger equation. In particular, a supercurrent similar to eq. (1.6) results from these kinds of calculations. The quasi-particle currents, however, are experimentally found to be qualitatively different from those of a true tunnel junction. This topic is dealt with in Chapter II, where several explicit theoretical models for proximity effect weak links are discussed.

1.5 Summary

In this chapter the description of superconductors in terms of a macroscopic wave function or order parameter Ψ has been introduced. The nature of Ψ , both in terms of the GL theory and the over-simplified point-particle boson model, has been discussed. The Josephson effect has been described for non-tunneling devices, such as the proximity effect weak link, as well as for true tunnel junctions.

The remaining chapters will contain a much higher density of equations and a much lower density of words. Those readers who might be interested are reminded that a more complete and historical account of elementary topics in superconductivity, as they relate to the work

in this thesis, may be found in the introductory chapters of preceding graduate students (Refs. 1-6). The new text by Tinkham (Ref. 25) is an even better source of such information.

II. MICROWAVE IMPEDANCE OF SUPERCONDUCTING WEAK LINKS

2.1 Introduction

In this chapter calibrated measurements of both the real and imaginary parts of the microwave impedance of proximity coupled superconducting weak links are described. The measurements are compared with analytic calculations based on several different theoretical models. Information obtained in this way is of a qualitatively different nature than can be obtained from low frequency measurements. As will be seen, these measurements provide a sensitive test of various aspects of the theoretical models to be discussed.

The chapter begins with a discussion of the usefulness of information obtained from high frequency measurements and a brief history of past work in this area. The experimental problems which prevented earlier measurements of this kind are briefly mentioned. The experimental methods employed and the results obtained are described. Approximate mathematical solutions to the most prominent theoretical models used to describe proximity effect weak links are found and compared with the data. Finally, conclusions regarding the validity or accuracy of the various models are drawn.

Before going on to a discussion of the usefulness of high frequency measurements, it should probably be pointed out that most of the material in this chapter has already been published in Refs. (1) and (2). The treatment here is similar but contains details not published elsewhere. A few of the passages here are worded very nearly

the same as in the published accounts; there seemed to be no good reason to change what was already a suitable description.

Many fundamental properties of superconducting devices are not readily investigated at low frequencies for the simple reason that the impedance of superconductors (below the critical current) goes to zero in the zero frequency limit. For example, many of the basic properties of a transistor can be investigated by passing low frequency currents through its various leads and measuring the voltages which are produced. Unfortunately, there is a wide range of current (any current below the critical current) for which the analogous measurements performed on a superconductor yield no information at all; the voltage generated is zero, aside from that due to the uninteresting geometrical inductance.

One way to avoid this problem is to operate at relatively large frequencies where the impedance of the superconducting device, even below the critical current I_c , is no longer immeasurably small. For proximity effect weak links this impedance takes the form of the well-known (theoretically) Josephson inductance, L_J , (Ref. 3). Most of the measurements reported in this chapter were performed at a frequency of ~ 10 GHz, where the Josephson inductance for a critical current of $30\mu\text{A}$ corresponds to an impedance of 0.5 ohms. Since the quasi-particle conductivity of these devices typically corresponds to a resistance, R_N , of 0.2 ohms, the effects due to L_J are relatively large and can be accurately measured and compared with theory.

Although the Josephson inductance is well known (theoretically), it may be worthwhile to describe briefly its physical origin. The supercurrent is, of course, a function of the quantum phase difference ϕ across the weak link. (See, for example, eq. (2.4a) to be introduced below.) If a microwave voltage is applied across the weak link, ϕ will precess at microwave frequencies (see eq. (2.5b) below) and produce a supercurrent which also oscillates at microwave frequencies. The fact that a given voltage produces only a finite current means that the impedance of the device is nonzero. The details of this process are described briefly in Section 2.4 and calculated in detail in Appendix A. For small voltages the response is linear and can be represented by an inductance in series with the path of the supercurrent. Physically, the phenomenon is (loosely) analogous to the voltage which would be required classically to accelerate the electrons due to their inertial mass.

One physical quantity which may be directly investigated by a measurement of L_J is the sign and magnitude of the phase-dependent quasi-particle current predicted by Josephson (Ref. 3). This topic has recently been the focus of considerable attention because all previous measurements (Ref. 4-6) of this term have given a sign which is opposite to that predicted by theory. Part of the motivation for this experiment is to provide a new technique for the measurement of this term for proximity effect weak links.

The measurements of L_J which have been discussed above are relevant for dc bias currents I_1 less than I_c , in which case the dc

(time-averaged) voltage $\bar{v}$ across the weak link is zero. A second class of measurements which are qualitatively different from those already described result when $\bar{v}$ is nonzero. The supercurrent then oscillates at a frequency ω_J given by $2e\bar{v} = \hbar\omega_J$ (Ref. 3). If an external microwave current is applied at a frequency ω_μ which is near ω_J , then a strong interaction occurs between the oscillating supercurrent and the externally applied microwave current. This interaction gives rise to the well-known step structure (Ref. 7), which is discussed briefly in Section 2.4 and calculated in detail in Appendix A. The microwave impedance will show structure reflecting this interaction, and measurements of this structure give further information for comparison with the theoretical models.

2.2 Experimental Method

Before the experimental measurements can be described, the microwave impedance of a weak link must be properly defined. This is not as trivial a problem as it may seem, since the definition of microwave impedance is (ordinarily) arbitrary to a considerable extent. This difficulty can probably be most easily understood with the aid of an example.

Consider a standard rectangular X-band waveguide terminated with a short (that is, with a metal plate soldered across the end) but containing some object O located a few centimeters in front of the short. The standard definition (actually there are several) used by electrical engineers for the microwave impedance in such a case consists of a

formula involving the magnitude and phase of the reflected microwaves, relative to the incident microwaves. (This kind of formula is almost totally irrelevant to this experiment, and need not be given here.) The difficulty with this kind of definition is that, although it is precise, it may not be related in a reasonable way to the properties of object 0.

For example, the impedance Z defined in this way will depend on the exact position and orientation of 0 within the waveguide. Z depends upon the size and shape of the waveguide as well as object 0 itself. Furthermore, if 0 occupies a volume which is not small compared with the size of the waveguide itself, then the electric and magnetic fields and the current density within 0 will be non-uniform. Thus Z defined in this way will reflect some sort of average of the properties of the various regions of 0 and of the waveguide itself. Sorting out any useful information about 0 from a measurement of Z may be a formidable task, although the necessary calculations in principle can be performed.

Proximity effect weak links are necessarily mounted on a bulky dielectric substrate with various irregularly shaped areas of superconducting film, indium blobs, normal metal lead wires, and plastic deposits nearby to complicate the measurement and interpretation of Z . For reasons such as these the microwave impedance of proximity effect weak links had never before been attempted, and calibrated measurements of the type reported in this thesis had never been made on any type of weak link structure. (See Ref. (8) for what

is probably the most similar previous measurement.) In order to carry out the measurements and analysis reported here, the microwave impedance had to be defined in a way which was more closely related to the properties of the device itself.

Fortunately, the dimensions of a weak link itself are much smaller than a wavelength of the microwaves. Therefore a definite voltage exists across the device and a definite current flows through it at each instant of time. This allows the microwave impedance Z to be defined to be the (complex) ratio of the microwave voltage across the weak link to the microwave current flowing through the weak link. (Strictly speaking, the Fourier component of the current and voltage at the frequency of the externally applied microwave current drive are to be used in the definition.) This definition of Z , which is quite different from the standard electrical engineering definition, will be used throughout the remainder of the thesis. Z defined in this way is exactly what is measured by the experiment and what is calculated from the various theoretical models.

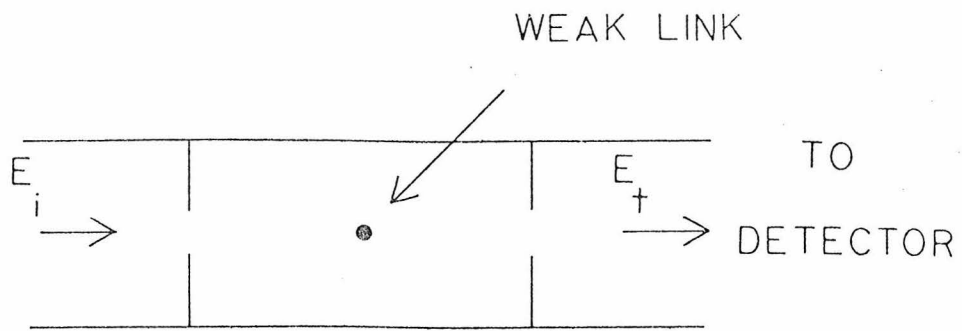
Because of this different definition and because the problem is different in other ways, the standard methods used in microwave engineering or in electron spin resonance (ESR) work were found to be of little use. Consequently, most of the familiar techniques, such as the use of Smith charts, etc., will not be discussed here.

The measurement of Z was made possible by the use of two very important techniques for simplifying the measurements and analysis. The first of these was the use of weak coupling between the weak link

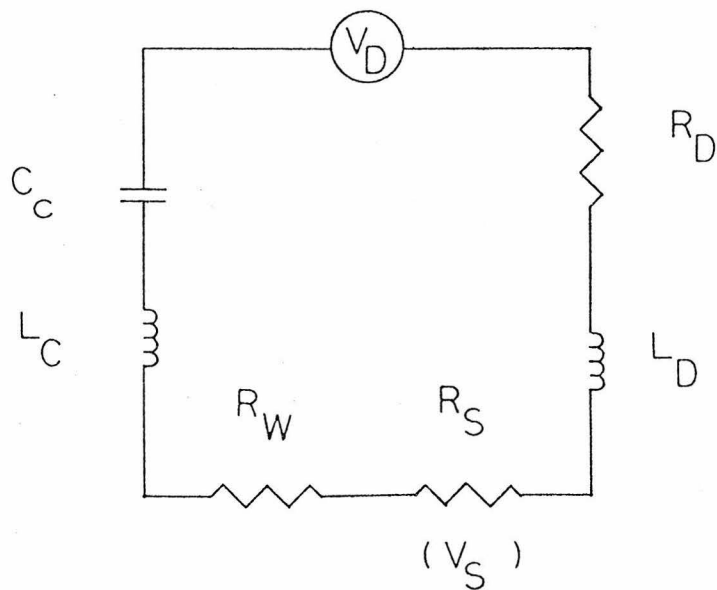
and the electromagnetic field of the microwaves. This had the effect of linearizing the equations which had to be dealt with in order to deduce Z from the signals detected. The second simplifying technique was the use of a built-in calibration method which was an intrinsic part of the thin film structure. Both of these techniques will now be discussed in detail.

The weak link was located near the center of a rectangular X-band transmission cavity as indicated schematically in Fig. 2.1a. The cavity was resonant in the TE_{102} mode at a frequency of 9.69 GHz. In this mode the cavity is one wavelength long and so the electric field is zero at the center of the cavity. The weak link was displaced slightly from the position of zero electric field and oriented in such a way that the electric field was parallel to the usual direction of current flow in the weak link. This results in a capacitive coupling to the weak link which is (deliberately) very weak. As will be seen, the exact position or orientation of the weak link does not affect the final results.

An electric field E_i was incident upon one end of the resonant cavity as shown in Fig. 2.1a, where the arrow denotes the direction of power flow, not the direction of the field itself. Some fraction of the incident power is transmitted through the cavity and gives rise to the transmitted electric field E_t . As may be expected, E_t depends to some extent on Z , the impedance of the weak link. The quantity actually measured is the (complex) ratio E_t/E_i . The method used for measuring E_t/E_i , as well as that for relating it to Z , will



(a)



(b)

Fig. 2.1 Coupling of proximity effect weak link to microwave cavity. (a) Incident and transmitted electric fields, $\vec{E}_i$ and $\vec{E}_t$. (The arrows represent the direction of power flow.) (b) One of several possible equivalent circuits.

be discussed later.

Four lead wires were connected to the weak link so that a standard four-terminal resistance measurement could be made simultaneously with the microwave measurements. These wires were made of Al (Al is not superconducting at these temperatures) and were thermomechanically bonded to the superconducting film using a commercial bonder. The more conventional method of attaching tinned leads with In contacts sometimes causes extraneous signals due to the superconducting joints, and so was avoided here. The wires were passed out of the cavity through a small hole in the cavity wall where the fields and currents are zero; this avoids large distortions of the fields due to the lead wires. In addition, an external magnetic field could be applied and a dc bias current passed through the weak link. As will be seen, the impedance of the lead wires, etc. had no effect on the measured weak link impedance.

An equivalent low frequency circuit (Ref. 9) for the microwave cavity with weak link coupled to it is shown in Fig. 2.1b. (Those readers whose intelligence is easily offended are advised not to look at Fig. 2.1b.) This particular equivalent circuit is not unique in the sense that many other circuits could just as well have been chosen. Here C_c and L_c represent, respectively, the distributed capacitance and inductance of the cavity and determine its resonant frequency (to first order). R_w represents the dissipation of energy in the walls of the cavity and R_s represents the loss of energy through the iris (hole) on the right. The energy escaping through this iris gives

rise to E_t and so the voltage v_s is proportional to the signal (electric field) measured by the detection system. The effects of the incident electric field E_i are represented by the voltage v_D . The power loss through the iris on the left is negligible, since it is made much smaller than the one on the right. (This is the advantage of using a transmission cavity; the left hand iris can be made as small as desired and need not be adjustable.) R_D and L_D represent the real and imaginary parts of Z as reflected into the tank circuit. (R_D and L_D are usually proportional, but not necessarily equal, to the real and imaginary parts of Z .)

As will be seen, there is a limit to the maximum voltage which can be applied across the weak link and still have it function properly. Consequently, the maximum signal would be obtained (by the usual impedance matching argument) if R_w were made negligible and R_s were set equal to R_D . R_w can, indeed, be made negligible, but weak coupling requires $R_D \ll R_s$. Thus the signal to noise ratio could be improved but at the expense of making the analysis nonlinear (which is the case for $R_D \sim R_s$). In these experiments R_D was kept much smaller than R_s , and so the Q of the cavity was determined by the size of the exit iris. Typically, $Q = 2000$.

Figure 2.1b is really not necessary for relating Z to E_t/E_i , and has been included mainly for the sake of completeness. (In particular, it is not necessary to know the relation between R_D , L_D and Z .) For weak coupling E_t/E_i can simply be expanded in a Taylor series in Z and higher order terms dropped to obtain

$$E_t/E_i = a + cZ \quad (2.1)$$

Here a and c are complex constants ($|a| \gg |cZ|$) which depend on the relative geometry of the cavity, the weak link, the lead wires, etc. In principle, a and c could be calculated, but in practice they cannot. At this point the second simplifying technique, the internal calibration, is put to use.

In this experiment only impedance changes were measured. For example, Z is a function of the dc bias current I_1 passing through the weak link. Thus it is possible to measure the impedance change $\Delta Z(I_1)$ defined by

$$\Delta Z(I_1) = Z(I_1) - Z(0) \quad (2.2)$$

This does not result in any real loss of information, however, because the limiting value of Z for large bias currents I_1 is the same for all the theoretical models to be considered and can be used to relate the impedance changes to the impedance itself.

The experiment, then, consisted of chopping I_1 on and off (square wave modulation) at a frequency of 10^3 to 10^4 Hz and measuring the resulting change in E_t/E_i . From eq. (2.1) ΔZ is then given by

$$\Delta Z(I_1) = \frac{1}{c} \frac{\Delta E_t}{E_i} \quad (2.3)$$

Before discussing how the calibration constant c was found, the detection system used to measure $\Delta E_t/E_i$ will be described.

Both the amplitude and phase of the transmitted electric field E_t were measured using the phase-locked heterodyne detection system

shown in Fig. 2.2. During the measurements E_i was held constant and served as the reference phase for E_t . Since this type of circuit is more or less standard, it will be discussed only briefly. Basically, the circuit mixes the original 10 GHz signal down to 60 MHz and then again to dc. The signal is mixed twice so that the $1/f$ noise can be avoided in mixer #1, where it would be a problem if the signal were mixed all the way from 10 GHz to dc. The part of the circuit drawn with dotted lines keeps track of the phase, so to speak, between the signal mixed to 60 MHz and the 60 MHz reference oscillator. This is accomplished by means of the feedback circuit. The phase shifter allows the real or imaginary part of Z to be measured, as will be described in connection with the calibration technique. The use of the circuit shown in Fig. 2.2 would have been straightforward except that it was not commercially purchased; all of the feedback loops, etc. were made in the laboratory.

The method used to determine the calibration constant c will now be discussed. From eq. (2.3) it can be seen that the value of c could be determined if a known value of impedance change could be produced and the resulting value of $\Delta E_t/E_i$ measured. A known value of ΔZ was produced in the following way: By passing a very large dc current through the bridge, the bridge and a small region of the surrounding superconducting film could be driven into the normal state. The measured dc resistance R_C of this region was made very large compared to that of the bridge alone ($R_C \gg R_N$). The assumption was that ΔZ due to chopping this large dc current was purely resistive

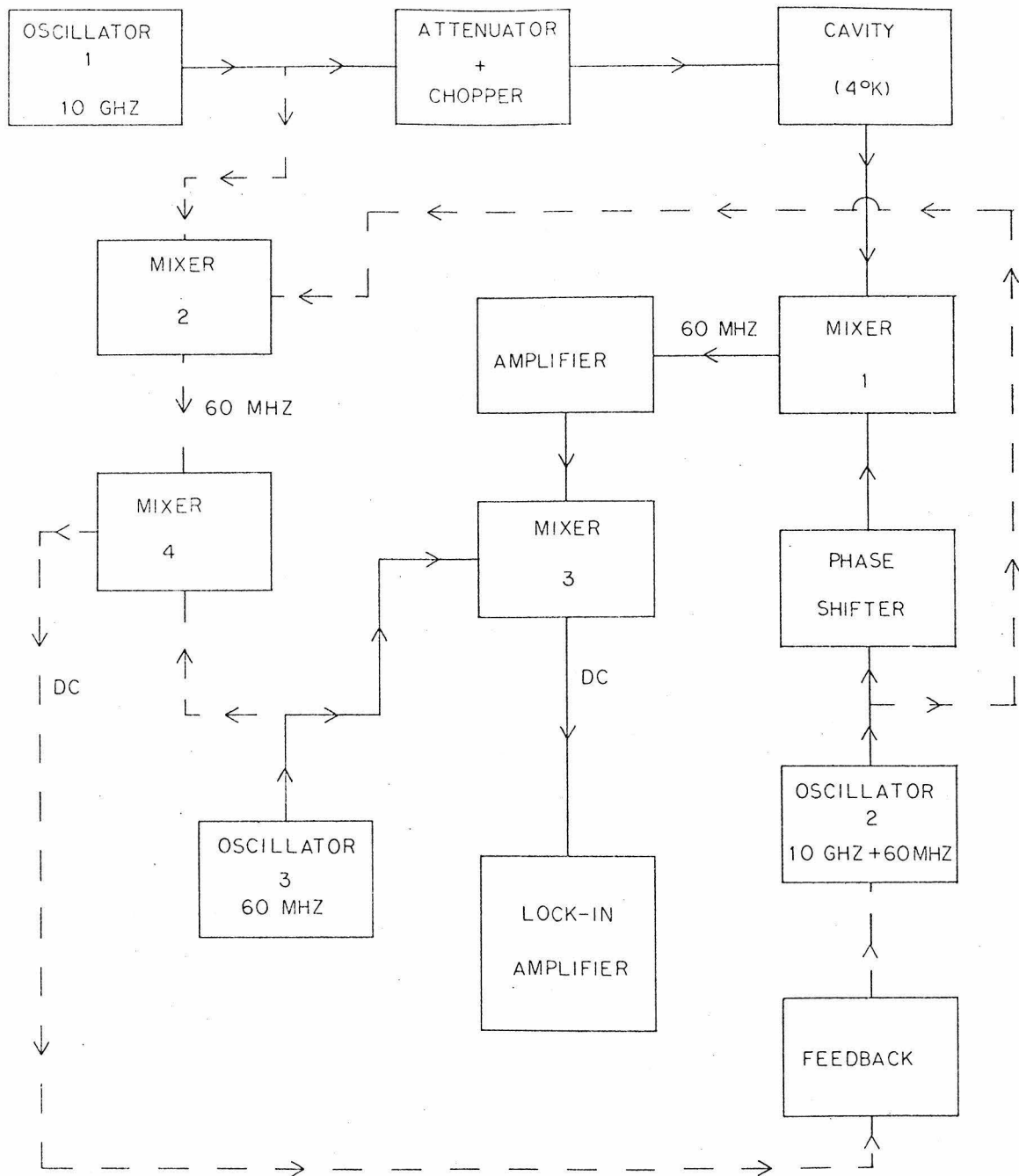


Fig. 2.2 Phase-locked heterodyne detection system

and equal to R_C . This provides a known value of Z for calibration purposes.

This assumption was justified because of the small dimensions of this normal region, because of the smallness of effects due to its stray capacitance and inductance, and because of the fact that the conductivity of thin films is independent of frequency up to frequencies beyond 10 GHz (Ref. 10). The accuracy of the calibration appears to be limited primarily by the conductance (shorting) due to the stray capacitance across R_C . Although the conductivity due to stray capacitance is negligible compared to R_N , it may not be entirely negligible compared to R_C (Ref. 11). The major effect of this is to introduce a small error into the phase of the signals.

By using the method described above, the value of c was determined to an accuracy which was estimated to be 10 percent or better. This calibrated both the amplitude and phase of the signals measured during the remainder of the experiment. In actual practice the microwave phase shifter was used in such a way that either the real or imaginary part of ΔZ could be directly recorded as a function of I_1 . The impedance of the lead wires or the position of the bridge had no effect on the calibrated results since they did not change with I_1 .

The fabrication of the proximity effect weak links used in this experiment is briefly described in Chapter I. The details of this fabrication process as well as their (well-known) dc properties have been described in earlier publications by the Caltech low-temperature group (Ref. 12). The particular weak links studied in this chapter were constructed from a layered thin film composed of Ta and W (other

materials are used in Chapters III and IV). The length of the bridges in the direction of current flow was typically about 0.5μ , the width perpendicular to the current flow about 12μ , and the resistance in the normal state, R_N , about 0.2 ohms. The weakened sections were formed by anodization. (Argon ion sputtering was used for the devices of Chapters III and IV.)

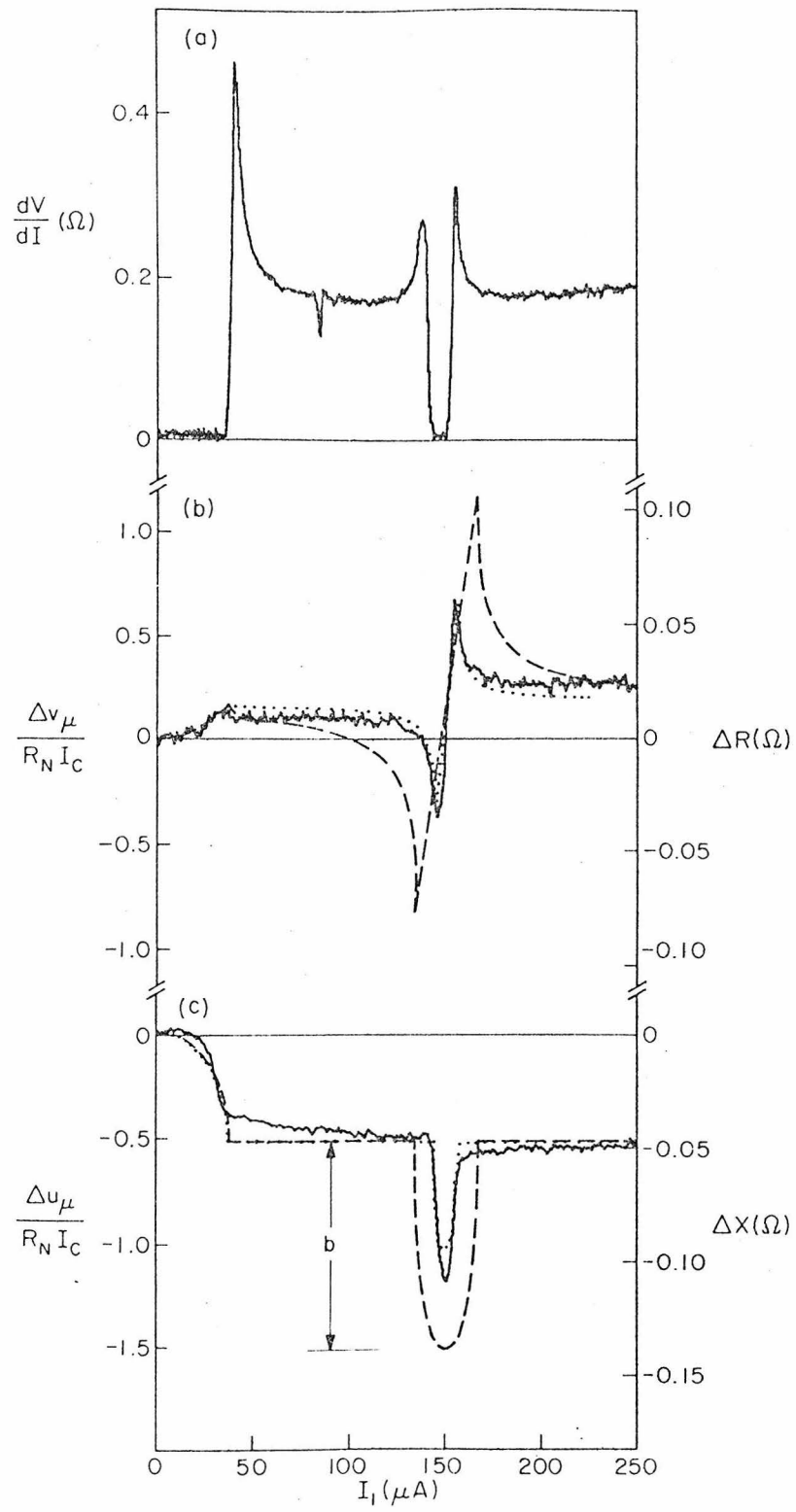
Other experimental techniques such as the measurement and regulation of the He bath temperature, etc. are well known and used throughout the field of low temperature physics. The particular techniques used by the low temperature group at Caltech are described in detail in the thesis by Palmer (Ref. 13). No special cryogenic or electromagnetic shielding techniques beyond those mentioned by Palmer were needed for the work described in this chapter. Later chapters do require some special techniques, which will be mentioned as the need arises.

2.3 Experimental Results

Measurements were carried out on several different devices fabricated at intervals of several months. The results were essentially the same for all devices tested. Consequently, only the results for one particular device are shown in the thesis. This helps avoid confusion over the exact value of the transition temperature, R_N , etc.

Some typical experimental results are shown in Figs. 2.3 and 2.4. Figure 2.3a shows the usual dc differential resistance (dV/dI) as a function of the (unchopped) dc bias current I_1 . A small microwave current I_2 was being driven through the weak link and produced

Fig. 2.3 Effects of the dc bias current I_1 on the impedance of a proximity effect weak link. (a) dc differential resistance, dV/dI . (b) Change in the real part of the microwave impedance, ΔR . (c) Change in the imaginary part of the microwave impedance, ΔX .



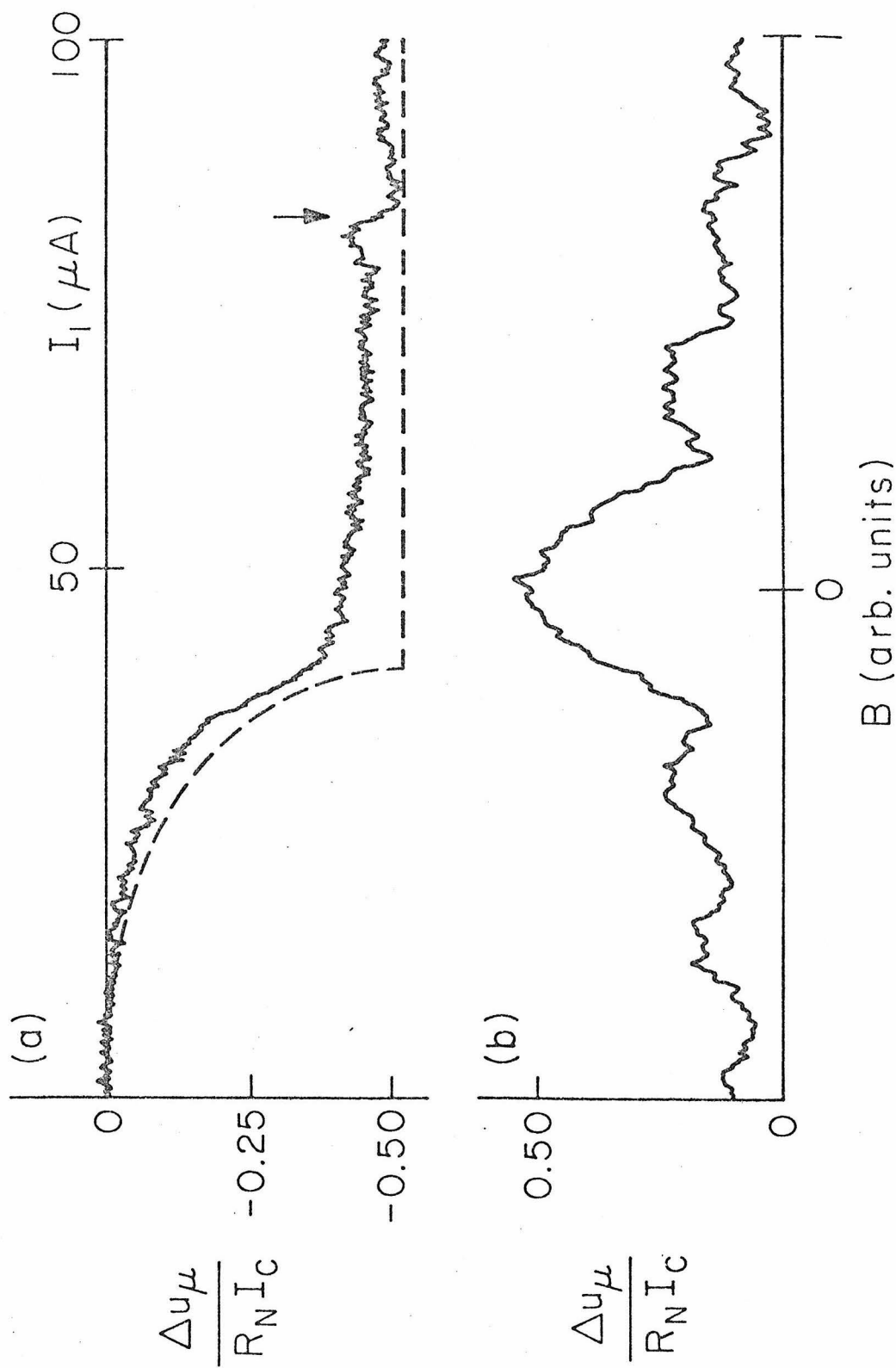


Fig. 2.4 Details of the change in the imaginary part of the microwave impedance, ΔX , as a function of the dc bias current, I_1 . (a) The Josephson inductance region. (b) Modulation of ΔX at $I_1 = 100 \mu A$ due to an external magnetic field, B .

the familiar first step. An incomplete subharmonic step can also be seen. The microwave frequency $\omega_\mu/2\pi$ was 9.69 GHz. The temperature was 2.85°K, and the critical current I_c was 38 μ A. At this value of I_c and ω_μ , the Josephson inductance, $\hbar/2eI_c$, corresponds to an impedance of 0.53 ohms, whereas R_N for this particular bridge was 0.17 ohms. The microwave current passing through the bridge has reduced the maximum zero-voltage dc current by an amount ΔI_c equal to 3.2 μ A. (The value of ΔI_c can later be used to calculate the amplitude of I_2 , once a particular theoretical model has been chosen.)

The experimental data in Fig. 2.3b (right-hand scale) show the change ΔR in the real part R of the microwave impedance as a function of I_1 for the same bridge and microwave current of Fig. 2.3a. Similarly, Fig. 2.3c (right-hand scale) shows the change ΔX in the imaginary part X of the impedance. All three curves measure different aspects of the same physical situation; Fig. 2.3b was traced immediately after tracing Fig. 2.3a, etc. (The dimensionless scales on the left-hand side of Figs. 2.3b and 2.3c will be explained later in connection with the theoretical analysis.) It should be emphasized here that ΔR and ΔX are directly determined quantities and that no theoretical assumptions have yet been made. The theoretical curves (dashed and dotted lines) should be ignored for the present time.

There are several qualitative comments which can be made about the data shown in Fig. 2.3 before discussing any of the theories. The region below the critical current ($I_1 < I_c$) is the region where the effects of the Josephson inductance L_J are expected to be seen. Note that

ΔR increases as I_1 approaches I_c , but that ΔR remains less than ~ 0.02 ohms throughout this region. This is to be compared with the value of 0.17 ohms for R_N . Thus the resistive impedance changes by only a small fraction of R_N itself. (This feature will be crucial when comparing the various theories.) Note also that there is no abrupt jump in ΔR when the critical current is reached. The reactive impedance ΔX decreases with increasing I_1 throughout this region but its magnitude also remains smaller than R_N . The magnitude of ΔX can be seen to be several times larger than the magnitude of ΔR .

Some qualitative comments can also be made about the structure seen near the first step. Near the first step ΔR shows a structure which is antisymmetric about the center of the first step, whereas ΔX is nearly symmetric. (The slight lack of symmetry seen in ΔX can probably be blamed on the small phase error involved in the calibration method, as already discussed. It can be removed by a slight phase adjustment but it is probably more honest to leave it in.) It can be shown on the basis of any of the theoretical models to be discussed below that, for a nonsynchronized bridge and $I_1 \gg I_c$, the real part of the microwave impedance approaches R_N and the imaginary part approaches zero. Thus the data could have been plotted directly in terms of R and X rather than ΔR and ΔX , if any of the models had been assumed to be correct.

Keeping this in mind, the data shown in Fig. 2.3 can be discussed in terms of the impedance itself, rather than the changes in impedance. It can be seen that R drops (and then rises) near the

first step but never actually becomes negative for this value of I_C , since the constant background in R_N corresponds to .17 ohms. Note that ΔR very nearly forms a straight line in between its relative maximum and minimum points near the step. If the microwave current is increased slightly above that used here, this line becomes even straighter. (In Fig. 2.3b this line is slightly rounded due to thermal noise effects.) Similarly, X (as opposed to ΔX) can be seen to be inductive for $I_1 < I_C$, capacitive near the first step, and approximately zero elsewhere, since the constant background in X corresponds to zero. For $I_1 > I_C$ and away from the first step, both ΔR and ΔX are approximately constant. (The slowly rising background seen in ΔR is due to a small (~ 0.01 ohm) change in R_N itself as a function of I_1 .)

All the general features noted above can be qualitatively understood in terms of the calculations to be discussed below. It should probably be noted in passing that the signal to noise ratio can be greatly improved by increasing I_C (the signals are crudely proportional to $R_N I_C$) but that the data shown here provide a better test of the various theoretical models than would be obtained for larger I_C . If desired, the signal to noise ratio can also be improved by increasing the coupling of the weak link to the fields in the cavity. Although the data shown in Fig. 2.3 are too noisy to be aesthetically pleasing, they are more than sufficient for a comparison with the theories.

2.4 Theoretical Models and Their Mathematical Solution

Several different theoretical models for proximity effect weak links have been proposed. What are probably the most commonly accepted and used theories lead to the following forms for the current-phase relationship

$$I = I_C \sin \phi + v/R_N \quad (2.4a)$$

$$I = \frac{1}{2} I_C (1 + \cos \phi) + v/R_N \quad (2.4b)$$

$$I = I_C \sin \phi + v\sigma(1 - \cos \phi) \quad (2.4c)$$

Here I is the total instantaneous current through the weak link, v is the instantaneous voltage across the weak link, and σ is a constant related to R_N . Equation 2.4a is just the current through an ideal Josephson junction shunted by a resistor, and is the model most commonly used (because of its simplicity) to describe weak links of various kinds (Refs. 14,15). Equation (2.4a) will be referred to as the resistivity shunted junction (RSJ) model. A priori, eq. (2.4a) would seem to be the most naive of the three models. Equation (2.4b) was first empirically formulated (Ref. 16) to account for the experimentally observed excess supercurrents in the dc characteristics of proximity effect weak links biased above the critical current. It was qualitatively explained (Ref. 17) on the basis of the phase slip model, and was later theoretically justified (Ref. 18) using time-dependent Ginzburg-Landau theory. Equation (2.4b) is not meant to be applied for

zero dc voltages ($I_1 < I_c$). (The physical basis of the phase-slip model will be described in more detail shortly.) Equation (2.4c) is similar to eq. (2.4a) but includes a phase-dependent quasi-particle current as predicted by Josephson (Ref. 3) in addition to the phase-independent quasi-particle current of eq. (2.4a). Here the phase-dependent quasi-particle current has been chosen to be equal and opposite to the phase-independent quasi-particle current on the basis of recent experimental results (Refs. 4-6). It would be fair to say that (at least before the publication of the present experiment) eq. (2.4c) was generally considered to be the most accurate and rigorous of the three theories or models.

A comment about the inclusion of eq. (2.4c) in the analysis of this experiment must be made here. After the results of this experiment were published (Ref. 2) the applicability of eq. (2.4c) to the analysis of this experiment was questioned for various reasons. Equation (2.4c) was included in the analysis of this experiment because it was extensively used in the literature, being applied to proximity effect weak links as well as other types of devices. In fact, one of the first measurements of the coefficient of the phase-dependent quasi-particle term (Ref. 5) was carried out using a proximity effect weak link. Furthermore, eq. (2.4c) was used with σ taken to be independent of voltage in all of the previous measurements of the phase-dependent quasi-particle term. Equation (2.4c) was included in the original publication of this experiment, and is included here, because it was used in the literature. The inclusion

of eq. (2.4c) was never meant to imply that the application of eq. (2.4c) to this experiment, that of Ref. (5), or any other experiment for that matter, is correct, justified, or reasonable. (On the contrary, the original publication of this experiment pointed out that the use of eq. (2.4c) is not very reasonable, at least for proximity effect weak links.) The appropriateness of eq. (2.4c) will be discussed in more detail in Sections 2.5 and 2.6.

The phase-slip model, eq. (2.4b), will be seen to be of special importance in this chapter, so it might be appropriate to describe its physical origin. Basically, the idea is (crudely) that a proximity effect weak link might be more similar in some respects to a short, narrow superconducting wire than to a true tunnel junction Josephson device, or at least should show some behavior in between that of a tunnel junction and a short, narrow wire. According to the simplest form of the phase-slip model, if a voltage is applied across a proximity effect weak link carrying no supercurrent, the electric field will continuously accelerate the superfluid until it reaches its critical velocity (critical current). At that point the kinetic energy of the superfluid has become so large that it is thermodynamically favorable for the superconductor to be in the normal state ($\Psi = 0$), and so within a time τ_R the electrons will relax back into the normal state. Once the kinetic energy of the superfluid has been dissipated as heat by this process, the superconducting state ($\Psi \neq 0$) is once again favorable and so the superfluid will quickly be reformed with zero superfluid velocity. The electric field will begin to

accelerate the superfluid once more, and the process is repeated over and over. This continuous acceleration, destruction, and subsequent regeneration of the superfluid is commonly referred to as a relaxation oscillation (Ref. 17). It is assumed that the destruction and regeneration of the superfluid takes place in a time period which is negligible compared with the time required for the acceleration by the electric field. Thus a plot of the supercurrent as a function of time would be approximately a triangular waveform.

The intuitive explanation given above can be made quantitative using the time-dependent GL theory (Ref. 18), which shows that variations in the amplitude of Ψ are important in addition to variations in the phase. One important feature which results is that, for given values of ϕ_L and ϕ_R , the phases on the left hand side and the right hand side, respectively, of the weak link, there may exist more than one solution for $\Psi(x)$; the value of $\nabla\phi$ integrated from one side to the other differs between these various solutions by $n2\pi$, where n is an integer. The relaxation process described above corresponds to a discontinuous change in n ; hence the term phase-slip. (Note that Ψ must go to zero at at least one point inside the weak link during the phase-slip process in order to maintain the continuity of Ψ .) If n is small (on the order of one) then these quantized phase-slips give rise to effects similar to those observed in true tunnel junctions. Since the time-average of the supercurrent for a triangular waveform is $I_C/2$, the phase-slip model predicts a nonzero dc supercurrent (referred to as the excess supercurrent) even for large

voltages. In addition, the nonsinusoidal waveform will reduce the amplitude of the Fourier component of I_s at the Josephson frequency and also generate Fourier components at other frequencies. This accounts for the factor of 1/2 in eq. (2.4b). As will be seen later, some of these predictions of the phase slip model can be tested in this experiment.

In addition to eq. (2.4), it was assumed that

$$I = I_1 + I_2 \sin \omega_\mu t \quad (2.5a)$$

$$\frac{d\phi}{dt} = \frac{2ev}{\hbar} \quad (2.5b)$$

where I_2 is the amplitude of the microwave current at angular frequency ω_μ , and e and $\hbar$ are, respectively, the charge of the electron and Planck's constant. Equation (2.5a) expresses the fact that the weak link is current biased; this assumption is physically justified at microwave frequencies by the impedance in series with the weak link due to the inductance of the current path leading to the weak link, and by the smallness of the stray capacitance shunting the bridge. Equation (2.5b) is well known from Josephson's original paper (Ref. 3).

Equations (2.5a) and (2.5b), along with the choice of either eq. (2.4a), (2.4b), or (2.4c), completely determine the behavior of the weak link. The mathematical calculation of the microwave impedance of a current biased device based on these equations is, unfortunately, made rather difficult by the nonlinearity of eq. (2.4). (The voltage biased case is trivial, since eq. (2.5b) can then be immediately

integrated.) No general analytic solution has ever been found. Digital computer calculations of the solution based on eq. (2.4a) for some specific cases have been published (Ref. 19). These digital solutions were not general enough to be of much use in the analysis of this experiment, and did not deal with eqs. (2.4b) or (2.4c). Some further computer calculations were carried out in the course of this work, but it was found that this method of analysis was prohibitively expensive. (For reasons which need not be discussed in detail, the peculiar nonlinearity of these equations causes the computer calculations to converge much more slowly than would normally be the case.)

Fortunately, an approximate solution to these equations which was suitable for the analysis of the data was eventually worked out. This approximate solution will be outlined below and is carried out in detail in Appendix A. Several approximate treatments also exist in the literature (Ref. 20-23), some of which were brought to the attention of the author after the completion of this experiment. At least one of these treatments appears to be in error, and the others do not appear to be in a form suitable for use in this experiment. Due to the length and complexity of the solutions appearing in the literature, the author has not determined to what extent their methods of solution are similar to that used here. The method used by Volkov and Nad' (Ref. 23) to calculate the dc properties of weak links based on eq. (2.4a) may be somewhat similar to the method to be described below.

A relatively simple solution to eqs. (2.4) and (2.5) can be found for the limiting case of small I_c and I_2 . The solutions for eqs. (2.4a), (2.4b), and (2.4c) are similar; in order to be precise and to avoid confusion, only the solution to eq. (2.4a) will be discussed. In particular, an approximate solution can be found if $\alpha \ll 1$ and $\beta \ll 1$, where α and β are defined by

$$\alpha = 2e R_N I_c / \hbar \omega_\mu \quad (2.6a)$$

$$\beta = 2e R_N I_2 / \hbar \omega_\mu \quad (2.6b)$$

(The inequality need not be strong.) For $I_1 < I_c$, the results can be readily obtained by the use of first-order perturbation theory and consist of the well-known Josephson inductance. The details of this calculation are included in Appendix A and need not be discussed here. For $I_1 > I_c$, however, the situation is much more difficult. Attempts have been made (Ref. 20) to use perturbation theory in this situation as well, but straightforward perturbation theory is not appropriate because the application of an arbitrarily small microwave current ΔI_2 will cause a change in phase $\Delta\phi$ which can become arbitrarily large as time progresses. Straightforward expansion in Fourier series is also not appropriate because the correct solution (see Appendix A) can be shown not to be a periodic function of time.

The difficulty in finding a solution to these equations arises from the nonlinearity and periodicity of the sine term in eq. (2.4a). A perturbation expansion would be valid if a suitable approximation

were to keep the argument of the sine term always small compared with 2π . This cannot be accomplished, however, except for the trivial case $I_1 < I_C$ already mentioned. This problem can be avoided (with a certain amount of difficulty) by making a change of variables to ϕ_R defined by

$$\phi_R = \phi - \omega_\mu t \quad (2.7)$$

Physically, ϕ_R represents the deviation of the quantum phase from the electromagnetic phase. The approximation to be described makes use of the fact that, near the first step, ϕ_R will be slowly varying (on the time scale of a microwave period τ) for $\alpha \ll 1$ and $\beta \ll 1$. Strictly speaking, the approximation is not valid at large distances from the step. But, as will be shown, the superconducting response is zero ($Z = R_N$) far from the step so that this complication is of no importance. The results obtained become exact in the limit $\alpha \rightarrow 0$ and $\beta \rightarrow 0$.

The method of solution consists of finding an approximate differential equation obeyed by ϕ_R which is more readily solved than the exact differential equation resulting from the substitution of eq. (2.7) into eqs. (2.4) and (2.5). This approximate differential equation is found by first calculating $\Delta\phi_R$, the change in ϕ_R from the beginning to end of one microwave period τ . Since $\Delta\phi_R$ is small, the differential equation governing the calculation of $\Delta\phi_R$ can be linearized and fairly easily solved to give an analytic solution for $\Delta\phi_R$. Then $d\phi_R/dt$ can be approximated by $\Delta\phi_R/\tau$. This approximation

is justified because ϕ_R is slowly varying. (As has already been mentioned, the results are exact in the limit of small α and β). The resulting approximate differential equation for ϕ_R at the first step turns out to have exactly the same form as the equation obeyed by ϕ itself at the zero step with no microwaves applied. This analogy had earlier been observed in the dc calculations of Volkov and Nad' (Ref. 23). (The author was nevertheless amazed by this analogy, since he was not aware of the work by Volkov and Nad' at the time these calculations were performed.) An analytic solution can then be found for this approximate differential equation in ϕ_R .

The details of the calculations outlined above are included in Appendix A. For convenience, the final results for the microwave impedance based on eq. (2.4a) will be summarized here. Let v_μ and u_μ be defined as the time-averaged Fourier coefficients of $v(t)$ at angular frequency ω_μ which are, respectively, in phase and out of phase with the external microwave current drive. (The time average must be taken since the solution is not a true periodic function of time and so the Fourier coefficients calculated over a few microwave periods will themselves be slowly varying functions of time. The experiment itself does measure just this time average.) The microwave impedances are then given by $R = v_\mu/I_2$ and $X = u_\mu/I_2$. For $I_1 < I_c$

$$v_\mu = \frac{I_2 R_N (\omega_\mu L_J)^2}{R_N^2 + (\omega_\mu L_J)^2} \quad (2.8a)$$

$$u_\mu = \frac{I_2 R_N^2 \omega_\mu L_J}{R_N^2 + (\omega_\mu L_J)^2} \quad (2.8b)$$

where

$$L_J = \frac{\hbar}{2eI_c [1 - (I_1/I_c)^2]^{1/2}} \quad (2.9)$$

For $I_1 > I_c$

$$\begin{aligned} v_\mu &= R_N I_c \omega_1 / \omega_2 + R_N I_2 && \text{for } |\omega_1| < \omega_2 \\ &= \pm (R_N I_c / \omega_2) [(\omega_1^2 - \omega_2^2)^{1/2} - |\omega_1|] + R_N I_2 && \text{for } |\omega_1| > \omega_2 \\ u_\mu &= -R_N I_c [1 - (\omega_1 / \omega_2)^2]^{1/2} && \text{for } |\omega_1| < \omega_2 \\ &= 0 && \text{for } |\omega_1| > \omega_2 \end{aligned} \quad (2.10)$$

where $\omega_0 = 2eR_N I_1 / \hbar$, $\omega_1 = \omega_0 - \omega_\mu$, and $\omega_2 = \frac{1}{2} \alpha \beta \omega_\mu$. It should be mentioned once again that the above results correspond to the current-phase relation of eq. (2.4a) (RSJ model), the other relations giving similar results. These solutions are in qualitative agreement with the computer calculations of Auracher and Van Duzer (Ref. 19).

Equations (2.8), (2.9), and (2.10) give the impedances themselves. The impedance changes actually measured by the experiment can be found by direct substitution. For example, $\Delta R(I_1) = I_2^{-1} [v_\mu(I_1) - v_\mu(0)]$, etc. As has already been mentioned, the measurement of impedance changes rather than the impedances themselves does not result in the loss of any physical information.

2.5 Comparison of Theory and Experiment

In Figs. 2.3b and 2.3c the results from eq. (2.4a) (RSJ model) are plotted with a dashed line. The results from eq. (2.4b) (phase-slip model) are plotted with a dotted line. (The dotted line is sometimes obscured by the data.) The results of eq. (2.4c) have not been plotted for reasons to be discussed later. The phase-slip model is meant to be applied only for nonzero dc voltages; therefore only eq. (2.4a) has been used for $I_1 < I_c$. For convenience, the left-hand scales give the in and out of phase microwave voltages, Δv_μ and Δu_μ , relative to $R_N I_c$. The only parameters in either theory are I_c , ΔI_c , R_N , and ω_μ , and all these parameters are directly measured so that the theory has not been fitted to the data in any way. (ΔI_c was used to determine I_2 as described in Appendix A; I_2 obtained in this way was then used in eqs. (2.8), (2.9), and (2.10).) The only exception is that, to aid in comparison of the theories with the data, the current axis has been normalized slightly so as to put the step at exactly the right voltage corresponding to 9.69 GHz.

First, it can be seen that either the RSJ model or the phase-slip model gives results which have the same general features as the data. The most obvious discrepancy is that at the first step, the RSJ model gives structures in both ΔR and ΔX which are too large by a factor of approximately two. The phase-slip model gives better agreement in this region. Using the methods described in Appendix A, it can be shown that the $\cos \phi$ in eq. (2.4b) (phase-slip model) can be

replaced by $\sin \phi$ with no change in the structure observed at the first step. Thus the differences in the results of these two models at the first step are primarily due to the factor of $1/2$ in eq. (2.4b). Physically, the data indicate that the Fourier component of the supercurrent at angular frequency ω_μ (AC Josephson effect) is less than I_c by a factor f whose experimental value at this temperature is 0.65 ± 0.1 . Equation (2.4a) (RSJ model) predicts a value for f of 1.0 , eq. (2.4b) (phase-slip model) gives 0.5 . (As has already been mentioned, the phase-slip model gives a reduced Fourier component of I_s at the Josephson frequency because I_s is not a sinusoidal function of time.) There are many variations of the phase-slip model; one of them (Ref. 24) happens to predict a value of 0.65 for f which is in good agreement with the data.

The exact value of f measured in this experiment is probably not of great physical significance for several reasons. First, the values of f predicted by various models, such as the phase-slip models, generally involve unrealistic approximations which cause the predicted value of f to contain a great deal of uncertainty. Thus the agreement between the measured value of f and the prediction of Ref. (24) may well be fortuitous. Secondly, in this experiment a careful study of the temperature dependence of f was not carried out. Since f may be a function of temperature (and thus critical current), the particular value of f measured here may not be of any great significance. What is of significance is the fact that f is definitely less than unity, which rules out the RSJ model in favor of the phase-

slip models. Physically, the results tend to support the idea that the supercurrent does not undergo pure sinusoidal oscillation, in agreement with the results of the phase-slip model.

There are some possible spurious experimental effects which might conceivably lower the measured value of f and therefore need to be ruled out. First, the analytic solution of the theoretical models was only approximate. Care must be taken to insure that the values of α and β are small enough so that the errors introduced are not significant. This was verified experimentally by varying the microwave current drive and the critical current to be sure that the character of the results was not altered. The size of the errors introduced in this way could also be estimated analytically and were not found to seriously affect the measurement of f . A related problem has to do with the effects of thermal noise, which set the lower limits on the magnitude of α and β for which the measurements can be made. Because of the highly nonlinear nature of the equations, the effects of thermal noise cannot be calculated in a rigorous way (except by computer). However, a crude estimate of the effects of thermal noise can be obtained by various intuitive arguments. For example, if I_c and I_2 are both much larger than the noise currents, then the effects of noise should be negligible. Once again, the best check on the effects of noise was to experimentally vary I_c and I_2 to determine if the character of the results had been altered. It was concluded that neither the neglect of higher order terms in the calculations nor of thermal noise effects could cause the observed reduction in f .

The stray capacitance of the weak link (Ref. 11) is approximately 100 times smaller than would be required to produce the measured reduction in f and is therefore negligible. The effects of heating due to power dissipation in the weak link can easily be shown to increase exponentially in the step number squared. The observed amplitude of the higher steps can therefore be used to set an upper limit on heating effects at the first step. These effects were found to be too small to significantly reduce the measured value of f .

An obvious source of possible error in the measurement of f is the method used to calibrate the impedance measurements. The nature of this calibration method and its principal limitation have already been discussed. An additional check on possible errors in the value of f due to calibration error was made by putting the theoretical and experimental results in dimensionless form. Such a procedure makes the calibration irrelevant. It was found that no set of dc parameters would give a theoretical curve based on the RSJ model (eq. (2.4a)) whose shape was the same as that observed. Yet another independent check on the experimental value of f was made by measuring the maximum width of the first step. A value of 0.62 ± 0.05 was obtained, which is in good agreement with the value already quoted. The preceding comments might be summarized by saying that no source of experimental error could be found which would account for the observed reduction of f from that predicted by the RSJ model.

In Fig. 2.4a a more detailed measurement of u_μ in the Josephson inductance region ($I_1 < I_c$) is shown. For $I_1 < I_c$ the experimental

results are in reasonable agreement with the analysis based on the RSJ model, if the uncertainty in the microwave calibration and the possible effects of thermal noise are taken into account. There is a significant discrepancy, however, just above I_c as the measured value of u_μ slowly rises to meet the theoretical calculation. Although the calculations carried out in Appendix A are only exact in the limit of small α and β , it can rigorously be shown on the basis of eq. (2.4a) or (2.4b) that u_μ should be constant (zero) for $I_1 > I_c$, provided that the weak link is not phase locked to the external microwaves or their harmonics or subharmonics. No quantitative explanation has yet been found for the fact that the experimental u_μ is not constant in this region; it does, however, seem quite possible that this effect is caused by the transition from the RSJ model to the phase-slip model as the dc voltage increases from zero at I_c . In other words, as the voltage increases the supercurrent starts to become nonsinusoidal, but the approximations made in the phase-slip model may not yet be valid. The smaller discrepancy below I_c (although within the experimental uncertainty) may also be caused by the finite size of the microwave voltages being applied and the resulting non-equilibrium situation, which for dc voltages gives rise to the phase-slip models.

Some structure at the subharmonic step is indicated by the arrow in Fig. 2.4a. An attempt was made to explain this structure by adding harmonic terms such as $\sin 2\phi$ or $\cos 2\phi$ to eq. (2.4a). (Such terms are expected, for example, on the basis of the phase-slip model.)

As reported in the original publication of this work, to lowest order in α , such terms cause u_μ to have a symmetric structure around the subharmonic step, whereas the data are antisymmetric. Subsequently, an interaction between the $\sin \phi$ term and the $\sin 2\phi$ term (which is second order in α) has been discovered; it may be large enough to explain the observed symmetry. This topic is discussed in Appendix A.

Finally, Fig. 2.4b shows the dependence of u_μ on an external magnetic field. Here I_1 was being chopped (square-wave modulated) from 0 to 100 μ A and the external field was being slowly varied. Equivalent results were obtained by chopping the external field and slowly varying I_1 . The period and extent of the modulation shows effects of magnetic shielding due to the surrounding film, but is consistent with previous results (Ref. 12) on the magnetic field modulation of these devices.

The preceding comments on the comparison of the RSJ model and the phase-slip model with the data might be summarized by saying that the RSJ model is in reasonable agreement with the data for $I_1 < I_c$ but in serious disagreement with the data for $I_1 > I_c$. The phase-slip model is in apparent good agreement. The disagreement between the RSJ model and the data cannot be explained on the basis of any obvious experimental error. These results tend to substantiate the nonsinusoidal oscillation of I_s predicted by the phase slip model. A comparison with the phase-dependent quasi-particle theory (eq. (2.4c)) will now be made.

The results calculated from the phase-dependent quasi-particle current model have not been plotted since they go off scale in Fig. 2.3.

The constant σ was determined by assuming it independent of voltage, as was done in all previous measurements of this type (Refs. 4-6) and fitting to the I-V curve for large voltages. (An alternative way of interpreting the data within the context of eq. (2.4c) will be discussed later; the same conclusions are reached in any event.) This requires $\sigma = 1/R_N$. The phase-dependent quasi-particle current then turns out to be much larger than the supercurrent for these low-resistance devices at this value of I_C , which causes the predictions of this theory to differ by an order of magnitude from the experimental results, both for $I_1 < I_C$ and at the first step. For example, if eq. (2.4c) were correct, then in going from $I_1 = 0$ to $I_1 = I_C$, ΔR would increase by 0.17 ohms instead of the ~ 0.02 ohms observed. Similarly, ΔX would change by 0.52 ohms instead of the less than 0.05 ohms observed. The details of the calculations are included in Appendix A; there seems to be little point in discussing them here in view of the gross disagreement with the measurements. If the sign of the $\cos \phi$ term is reversed (so as to agree with the BCS predictions rather than the previous experiments) the agreement is equally poor.

Thus these measurements provide a rather sensitive test of the phase-dependent quasi-particle term. The experimental results may be summarized by saying that the phase-dependent quasi-particle current (in proximity effect weak links), if it exists at all, is no larger than 10 percent of the total quasi-particle current. When the experimental results are stated in that form they become independent of any particular model. Consequently, the question of whether or

not σ should be taken to be constant or whether eq. (2.4c) should be applied at the first step, etc. become irrelevant. The phase-dependent quasi-particle current (in this experiment) is simply immeasurably small, for whatever reason. (A possible reason for this result is discussed in Section 2.6.) It should probably be immediately pointed out, however, that these results do not contradict those previous measurements made on true tunnel junctions because of the fundamental differences between the two types of devices. They do, however, seem to be inconsistent with the present interpretation of the dc measurements of Falco et al. (Ref. 5).

2.6 Conclusions

It has been shown that for $I_1 < I_c$ the RSJ model is in reasonable agreement with the data whereas the phase-dependent quasi-particle current model is grossly incorrect. Thus what might have a priori appeared to be the most naive theory turned out to be much more accurate than the theory generally regarded as most rigorous. Physically, the results indicate that the phase-dependent quasi-particle current, if it exists at all, is much smaller than the phase-independent quasi-particle current (for proximity effect weak links). An intuitive explanation for this observation will now be offered.

Since the energy gap inside the weak link (far from the superconducting boundaries) is extremely small, the properties of the material in this region should to a first approximation be the same as those of a normal material. In particular, if an electric

field is applied then a normal (quasi-particle) current should flow whose physical origin is, at least to a high degree of accuracy, exactly the same as that of a current flowing through a copper wire at room temperature. This follows from the fact that, for energies large compared to Δ , the quasi-particle excitations reduce to the usual single electron states of a normal metal. In other words, Ohm's law should prevail to a first approximation when $\Delta \ll kT$.

In addition to the Ohm's law currents there will be superimposed the effects of the small but nonzero order parameter. This will give rise to supercurrents and possibly (although it is far from obvious) phase-dependent quasi-particle currents as well. The point is that in this experiment the Ohm's law current is much larger than any phase-dependent quasi-particle current which could reasonably be expected (in the author's opinion). In fact, the disagreement between eq. (2.4c) and the data can, in principle, be made arbitrarily large by lowering the value of I_c . (The Ohm's law currents are roughly independent of temperature, whereas the critical current, and presumably any phase-dependent quasi-particle currents, decrease exponentially with temperature.)

Some further comments might be made about the applicability of eq. (2.4c) to the first step. First of all, the arguments presented in the preceding paragraphs indicate that this question is probably of no significance for proximity effect weak links and really need not be considered in this thesis. Aside from that, the

rigorous calculations of Werthamer (Ref. 25) and Harris (Ref. 26) for ideal tunnel junctions suggest that the use of eq. (2.4c) together with eq. (2.5) may not properly include the effects of photon absorption and emission at the step. The author, although not a theorist, is inclined to agree; the results of eq. (2.4c) have been included in the analysis of the step structure in Appendix A mostly for the sake of completeness. Those readers who prefer to do so can simply ignore the disagreement with the data in this region. (The analysis for $I_1 < I_c$ is not affected by these arguments.)

Before leaving this topic, the author cannot resist including an intuitive comment about the nature of the current-phase relation. Any current-phase relation, such as eqs. (2.4a), (2.4b), or (2.4c), asserts that the supercurrent can be calculated at any instant, provided that the voltage has been integrated to give ϕ at that instant. It can readily be seen that this information is not sufficient to give a complete Fourier spectrum of the electromagnetic fields (the voltage $v(t)$) involved. Thus the energy of any photons present or created in the field is unknown. If the emission or absorption of photons is an important process (which it frequently is) then it would seem that any current-phase relation should be suspected of giving incomplete results. In other words, assuming that the results are given by any current-phase relation restricts the functional form of the results in a way which may not be of sufficient generality. The above arguments are, of course, equivalent to saying that the Heisenberg uncertainty principle relating ΔE and Δt is being violated

by the form of the current-phase relations. These comments can be made more precise by simply pointing out, for example, that the rigorous results of Werthamer for true tunnel junctions (Ref. 25) do not have the form of a current-phase relation. This fact does not seem to be very generally appreciated.

The comments of the preceding paragraph do not mean, of course, that the use of a macroscopic current-phase relation will always give the wrong result. They do, however, suggest that such relations should be used with a certain amount of caution. For example, it should not a priori be assumed that the symmetry of the observed subharmonic step structure which has already been mentioned can necessarily be explained within the context of any current-phase relation.

2.7 Summary

The chapter began by describing some of the ways in which high-frequency measurements can yield information which cannot be obtained from low-frequency measurements. An experimental technique for the measurement of microwave impedances was described which employed weak coupling and self-calibration. This technique allows many of the difficulties encountered in microwave measurements to be circumvented. The experimental measurements of the microwave impedance of proximity effect weak links were then presented and discussed in a form which is independent of any theoretical assumptions.

The most prominent theoretical models for proximity effect weak links were introduced and the impedance predicted by each of

them was calculated in Appendix A using an appropriate approximate method. It was found that the resistively shunted junction model was in reasonable agreement for dc bias currents below the critical current, but showed discrepancies at the first step which could not be dismissed on the basis of experimental error. The phase-slip model, on the other hand, was in good agreement with the data at the first step. This result tends to support the prediction of the phase-slip model that the supercurrent does not oscillate in a purely sinusoidal fashion. This conclusion is significant because it shows that the behavior of proximity effect weak links is qualitatively different from that of true tunnel junctions. The theory including phase-dependent quasi-particle currents was found to be in gross disagreement with the data for all values of dc currents. Physically, the phase-dependent quasi-particle current was found to be no larger than 10 percent of the total quasi-particle current. This result does not necessarily contradict previous measurements made on true tunnel junctions and appears to be physically reasonable for proximity effect weak links. It suggests that the quasi-particle current is carried by a proximity effect weak link in approximately the same way as is the case for a normal metal.

III. EXPERIMENTAL EFFECTS OF QUASI-PARTICLE INJECTION AND EXTRACTION IN PROXIMITY EFFECT WEAK LINKS

3.1 Introduction

This chapter describes an experiment in which quasi-particle excitations were injected directly into a proximity effect weak link. The quasi-particles were injected by means of a very small tunnel junction fabricated directly on top of the normal region of the weak link. The effects of the quasi-particle injection on various properties of the weak link, such as the critical current and first step width, were then measured.

The effects of nonequilibrium quasi-particle distributions in bulk superconductors have recently been the subject of considerable experimental and theoretical interest. There are several reasons for this interest. First, most of the important superconducting parameters, such as the transition temperature, energy gap, and critical current density, are strongly dependent on the distribution of quasi-particles. For example, nonequilibrium quasi-particle distributions generated in various ways can sometimes raise the transition temperature T_c of a superconductor above its equilibrium value (Refs. 1-6), although in general they tend to lower it. These effects may be of practical importance in certain applications. The relaxation times of the quasi-particles have been experimentally measured (Refs. 7-20) and are of theoretical interest (Refs. 21-25). Since nonequilibrium quasi-particle distributions interfere with the operation of various types of weak link devices, these relaxation times are also of practical importance (Refs. 26,27). There are various dissipative processes which occur in the operation of large

superconducting magnets which limit their usefulness and impose design limitations. Once again, the nonequilibrium quasi-particles produced in these dissipative processes are of practical concern (Ref. 28). Finally, the phonons generated by the relaxation of the nonequilibrium quasi-particles provide a means of producing phonons of a given energy, and the inverse process provides a means of detecting phonons of a given energy. These methods are currently being used to study the interaction of phonons with various physical systems (Refs. 29-43). Although several practical applications have been mentioned above, the study of the effects of nonequilibrium quasi-particle distributions is, of course, interesting and important in its own right.

As can be seen, the effects of nonequilibrium quasi-particle distributions in bulk superconductors are being thoroughly studied, although a great deal of further work remains to be done. Unfortunately, most of the results for bulk superconductors are not directly applicable to proximity effect weak links, as will be discussed later. However, the work done on bulk superconductors must be discussed in order to put the present experiment in perspective, and so the topics mentioned above will be reviewed in greater detail shortly.

In contrast to the large amount of work carried out in bulk superconductors, relatively little work has been done on the effects of nonequilibrium quasi-particle distributions in weak links. This is due, at least in part, to the greater experimental and theoretical difficulty involved in studying such effects in weak links. However, even the small amount of work which has been carried out indicates that the

effects of nonequilibrium quasi-particle distributions in proximity effect weak links are of practical as well as theoretical importance.

One problem of interest has to do with the relatively low upper frequency limit on the operation of proximity effect weak links. Proximity effect weak links generally do not function well at frequencies much beyond 50 GHz (Refs. 26,44), whereas other types of weak links (Ref. 45) will function well at frequencies of several thousand GHz. This lower operating frequency of proximity effect weak links is believed to be due, at least in part (Ref. 26), to the energy dissipated in the device by the large voltages and currents required at these high operating frequencies. This energy is transferred to quasi-particle excitations which in turn relax and transfer the energy to the metal lattice and substrate. Unfortunately, the detailed effects of such nonequilibrium situations are not well understood.

As another example, the work by Palmer and Mercereau (Refs. 46, 47) on large arrays of many closely-spaced proximity effect weak links seems to indicate the existence of coupling between adjacent weak links due to the generation of quasi-particles by one weak link and their subsequent effect upon neighboring weak links. This interaction was observed to aid in the synchronization of the oscillating supercurrents and thus improved the performance of closely spaced arrays. The separation between neighboring weak links had to be less than $\sim 1\mu$ in order for this coupling to be effective. Similar work on two closely-spaced tin Dayem bridges by Jillie et al (Ref. 48), on the other hand, indicated a coupling mechanism due (presumably) to quasi-particle interactions which

tended to synchronize the two bridges out of phase with each other and so would degrade the performance of an array. (A Dayem bridge is a narrowed or constricted region of a superconducting film which shows quantum interference effects.) There is a clear need for a better understanding of the effects of nonequilibrium quasi-particle distributions in weak links.

The remainder of this chapter begins with an introductory account of the current understanding of nonequilibrium effects in bulk superconductors. As has already been mentioned, most of this information is not directly applicable to the present experiment but is needed nevertheless as background information. The chapter then continues with a discussion of the method used to fabricate the devices studied. (Most of the work which had to be carried out in order to do this experiment had to do with the problem of finding a suitable means of fabricating the devices.) The special experimental techniques needed to make the measurements are described and the experimental results are presented. Unfortunately, no detailed theories which are applicable to these measurements currently exist in the literature. Consequently a simple model which might a priori be expected (on the basis of work done on bulk superconductors) to show the main features of the data is presented. It will be seen that the experimental data are in surprising disagreement with this simple model, and suggest that the fundamental processes involved are of a completely different nature than had previously been expected. Several possible explanations for this result are considered and ruled out for various reasons. One particular explanation is suggested which appears

to be consistent with the data. The differences between this experiment and those carried out on bulk superconductors and other types of weak links will be discussed.

3.2 Background Information

A theoretical discussion of the effects of nonequilibrium quasi-particle distributions in proximity effect weak links will be postponed until Section 3.6. For the time being it should suffice to say that, if a bias voltage large compared to kT and Δ is applied across the tunnel junction, then electrons will tunnel into or out of the weak link region. (k is Boltzmann's constant and T is the temperature.) Regardless of whether an electron or a hole is created in the weak link region, the energy of the system has been raised above that of the ground state. Thus both electrons and holes are excitations and the tunneling process can be thought of as the injection of excitations in either case. Crudely speaking, the injected energy might be expected to "heat up" the weak link region and in general reduce the critical current, etc. (since the critical current decreases exponentially with increasing temperature).

The situation described above becomes more complicated when the energy of an excitation is comparable to the energy gap Δ . In that case the excitation is no longer a single-particle excited state of the type familiar from normal metals, but is instead a more complicated many-body excitation known as a quasi-particle. (As the excitation energy becomes large compared to Δ , the quasi-electrons and quasi-holes reduce to the usual single-particle excitations.) Since Δ inside the weak link region is much smaller than kT (at least far from the superconducting

boundaries), this distinction is not of any importance for many purposes and can usually be ignored in the weak link region. However, some of the devices studied used Pb (a superconductor) as the other electrode of the tunnel junction (the geometry of the devices studied will be discussed later) and so Δ is not negligible there. Although the nature of the excitations created in the Pb are usually of no interest in this experiment, the gap in the excitation spectrum of the Pb does have some important effects. As will be discussed in Section 3.6, the gap in the excitation spectrum of the Pb makes it possible, for bias voltages near Δ_{Pb}/e , to extract excitations from the weak link region. (In other words, the weak link region is "cooled off.") Most of the measurements to be discussed were made with bias voltages much larger than Δ_{Pb}/e , so that even this complication can generally be ignored in proximity effect weak links.

The conditions of this experiment are such that most of the experimental and theoretical work which has been carried out on bulk superconductors is not directly applicable here. The reasons for this will be explained at length in Section 3.6. Nevertheless, it is necessary to discuss the properties of nonequilibrium bulk superconductivity in some detail in order to provide a framework in which to describe the results of this experiment. (In addition, the discussion to be presented here will be needed for the analysis of the experiment to be described in Chapter IV.) As will be seen, the results of this experiment are quite different from what would a priori have been expected. The surprising nature of these results cannot be appreciated without a thorough discussion of nonequilibrium bulk superconductivity.

There are several review articles on both the experimental study (Refs. 49,50) and theoretical understanding (Refs. 24,25) of nonequilibrium bulk superconductors. (Here the term "bulk" is meant to include thin films or any other macroscopically large volume of superconductor which is not part of a weak link device.) The discussion to be presented here is not meant to be a detailed review of this field, but is only intended to provide an introduction to the major concepts involved for those readers who are not already familiar with them.

First, it should be pointed out that the concepts and results to be discussed in this section have been rather thoroughly investigated and therefore can be considered to be "well known". For example, non-equilibrium quasi-particle distributions in bulk superconductors have been generated by a variety of means, including direct injection by tunneling (Refs. 7-13,15), laser beam irradiation (Refs. 14,16-20), microwave irradiation (Refs. 1-3,5), phonon irradiation (Refs. 4,29-32), direct heating (Refs. 27,30), and α -particle bombardment (Ref. 51). The effects of the nonequilibrium quasi-particle distributions generated in these experiments were measured using a variety of techniques which need not be discussed here, but which generally involve tunnel junction probes (Refs. 4,7-13,15-17,19), resistance measurements (Refs. 1-3), microwave reflection properties (Ref. 18), or the spectrum of phonons emitted in the relaxation process (Refs. 29-35). (The list of references given above is far from complete.) Some of the experiments and theories referred to above will be discussed in more detail throughout this chapter. In any event, the large number of experiments and theories listed

above all appear to be consistent with the physical picture of non-equilibrium bulk superconductivity to be discussed below.

The term "quasi-particle", as has already been mentioned, is applied to the elementary excitations of the system. (It should be emphasized that quasi-particles are excited states of the system and not objects like baseballs or electrons.) Quasi-particles are described (or labeled) by a momentum $\hbar\vec{k}$ and an energy E_k which are, respectively, the change in momentum and energy of the system due to the creation of such a quasi-particle excitation. The energy E_k of a quasi-particle of momentum $\hbar\vec{k}$ can be shown to be (Ref. 52)

$$E_k = [\epsilon_k^2 + \Delta^2]^{1/2} \quad (3.1)$$

(This simple form must be generalized somewhat in the more elaborate theories of superconductivity.) Here ϵ_k is the energy of a Bloch state (unperturbed by the superconducting interaction) of momentum $\hbar\vec{k}$ measured with respect to the Fermi energy μ . For a free electron gas

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} - \mu \quad (3.2)$$

Note that E_k defined by eq. (3.1) is always positive, even for values of k below the Fermi surface. This is consistent with the convention that quasi-holes will be considered to have a positive energy. Also, note that, for large values of $|\epsilon_k|$, $E_k \rightarrow |\epsilon_k|$, so that the energy of a quasi-particle is the same as that of the corresponding Bloch state. The same is also true for the quasi-particle wave function. (The details of the wave function need not be discussed here.) This fact forms

the basis for the statement already made that quasi-particles with $E_k \gg \Delta$ reduce to the usual electrons and holes of a normal metal.

Creation and annihilation operators for the quasi-particles can be formed (Ref. 53), and their properties are formally very similar to the usual creation and annihilation operators for electrons and holes in a normal metal. The excited states of the system can then be formed by operating on the ground state with these quasi-particle operators. The similarity of these operators to those of true electrons (particles) in normal metals forms the basis for the term quasi-particle. The thermal distribution of the quasi-particles can be shown (Ref. 52) to obey Fermi-Dirac statistics, in the spirit of the Landau Fermi liquid theory (Ref. 54). To be more precise, the probability $f(k)$ that a quasi-particle state of momentum $\hbar k$ is "occupied" (that the corresponding creation operator has been applied to the ground state) is, under equilibrium conditions, given by

$$f(k) = \frac{1}{e^{E_k/kT} + 1} \quad (3.3)$$

The chemical potential μ does not appear in the equation above because all energies have been measured with respect to μ .

Under nonequilibrium conditions it may be the case that $f(k)$ cannot be described by any simple equation like eq. (3.3). However, under certain special situations to be discussed below, it may be the case that eq. (3.3) is still approximately correct if T or μ are replaced by some effective values, T^* or μ^* . In order to discuss this possibility there are several concepts which need to be introduced.

First, quasi-particles with a given energy E can be divided into two groups, known as branches of the excitation spectrum (Ref. 55). Those with momentum $\hbar k$ larger than the Fermi momentum $\hbar k_F$ form one group and are said to lie on the upper branch. Those with $\hbar k < \hbar k_F$ are said to lie on the lower branch. (Note that the absolute value of momentum, not its direction, is relevant here.) It is sometimes said that the situation in superconductors is totally different from that in normal metals, in the sense that a given energy E only corresponds to one (absolute value of) momentum in a normal metal, whereas a superconductor has two different kinds of excitations (the two branches) for each value of E . This is not completely correct, however, because even in a normal metal an electron with $k > k_F$ can have the same excitation energy as an appropriate hole with $k < k_F$, provided that all excitation energies are chosen to be positive as has been done here. The point is, as has already been emphasized, that for $E \gg \Delta$ the complicated quasi-electrons and quasi-holes reduce to the ordinary electrons and holes familiar from normal metals. It is only for $E \sim \Delta$ that the quasi-particles are linear combinations of single-particle states, and so the nature of the excitations and their related properties becomes somewhat confusing. (A great deal of confusion in this connection arises because of the fact that the usual BCS theory does not (formally) conserve the total number of electrons.) Whenever the concepts to be discussed below, such as branch imbalance, become confusing, the author recommends that the limit $\Delta \rightarrow 0$ be considered, in which case the results should reduce to those of a normal metal.

Under certain special conditions (to be discussed shortly) the quasi-particle distribution may be described by a generalization of Eq. (3.3), in which the quasi-particles on the upper branch are (very nearly) in equilibrium with each other at the equilibrium temperature T and a chemical potential $\mu_>$. That is,

$$f(k) = \frac{1}{e^{(E_k - \mu_>)/kT} + 1} \quad (k > k_F) \quad (3.4)$$

Similarly, the quasi-particles in the lower branch are governed by

$$f(k) = \frac{1}{e^{(E_k - \mu_<)/kT} + 1} \quad (k < k_F) \quad (3.5)$$

In these equations $\mu_>$ need not equal $\mu_<$. Since all energies were defined in eqs. (3.1) and (3.2) with respect to the Fermi level, $\mu_>$ and $\mu_<$ are measured from the Fermi level. As has already been pointed out, the justification of eqs. (3.4) and (3.5) for certain physical situations will be discussed shortly.

If it should happen to be the case that $\mu_> = -\mu_<$, then it is convenient to define μ_{qp} by

$$\mu_{qp} = \mu_> = -\mu_< \quad (3.6)$$

The value of μ_{qp} under these conditions will be referred to as the quasi-particle chemical potential because of the following analogy: In an intrinsic semiconductor the Fermi level μ will generally lie in the center of the energy gap. If, however, the semiconductor is doped

in an appropriate way, then the position of μ can be raised above the center of the gap. This causes an increased number of electrons and a decreased number of holes. (Charge neutrality is maintained by the impurity ions.) If the distributions of the electrons and holes are to be described by eqs. (3.4) and (3.5), then clearly $\mu_>$ will be positive and $\mu_<$ negative (in order that the number of holes decreases), if holes are taken to have positive energy. In addition, it will be expected that $\mu_> = -\mu_<$. As has already been mentioned, the quasi-particles in a superconductor reduce to the usual electrons and holes for large energies, so that this analogy to a semiconductor suggests that μ_{qp} , as defined by eq. (3.6), does indeed correspond to what would ordinarily be considered the chemical potential of the quasi-particles. (The minus sign in eq. (3.6) arises from the convention that the energies of holes are positive.)

If $\mu_> \neq -\mu_<$, it is possible to define μ_{qp} in a more general way; for example, the chemical potential of a normal tunnel junction probe in equilibrium with the superconductor might provide such a definition. In this chapter μ_{qp} will only be considered to be defined if $\mu_> = -\mu_<$. Once again, the physical conditions under which this occurs will be discussed shortly.

On the other hand, if it should happen to be the case that $\mu_> = \mu_<$, then it is convenient to define μ^* by

$$\mu^* = \mu_< = \mu_> \quad (3.7)$$

The variable μ^* is sometimes also referred to as the quasi-particle chemical potential, which causes a certain amount of confusion between

μ_{qp} and μ^* . In this thesis, the term quasi-particle chemical potential will always refer to μ_{qp} ; μ^* will be referred to by symbol only. (Note that μ_{qp} and μ^* are only defined under certain special conditions.) A positive value of μ^* , for example, corresponds to a larger number of both quasi-electrons and quasi-holes than would be the case under equilibrium. Once again, such conditions are believed to occur under certain physical situations to be described shortly.

Finally, there are situations in which both branches of the quasi-particles are (approximately) populated according to

$$f(k) = \frac{1}{e^{E_k/kT^*} + 1} \quad (3.8)$$

Here T^* is an effective temperature which is different from the temperature of the substrate or metal lattice (phonons).

An important concept in this chapter is the branch imbalance, Q , which can simply be taken to be the total number of excitations in the upper branch minus the total number in the lower branch (Ref. 55). In other words, Q is positive if there are more quasi-electrons than quasi-holes. A moment's reflection will show that Q is related to μ_{qp} defined above, since raising μ_{qp} will increase the number of quasi-electrons and decrease the number of quasi-holes. At this point it might be appropriate to point out that some authors prefer to use μ_{qp} , whereas others prefer to use Q . Still others feel that none of these concepts are very appropriate, and prefer to work entirely with $f(k)$ itself. In particular, the concept of branch imbalance has been criticized on theoretical grounds (Ref. 56), since the arbitrary division of the quasi-particle excitations

into two groups appears to have little physical basis. (In a superconductor, even at absolute zero, there is no discontinuous change in the physics at the Fermi surface, as there is in a normal metal.) This thesis will use the concepts introduced above, since they are currently being used by nearly all experimentalists and perhaps half of the theorists. (The remaining theorists are primarily concerned with Green's function methods.) The physical situations under which the various approximate forms for $f(k)$ described above are valid will now be discussed, along with an account of the various kinds of relaxation times which are important.

First, consider a nonequilibrium situation in which μ^* has a positive value, so that there are more quasi-particles on both branches than would be the case under equilibrium conditions. Such a situation might be created by phonon or photon irradiation, which tends to populate both branches equally. (Incident phonons or photons with an energy greater than 2Δ can break a Cooper pair to form two quasi-particles.) The excess quasi-particles can then recombine to form pairs with a relaxation time τ_R known as the recombination time. In a semiconductor, the analogous recombination process consists of an electron above the energy gap dropping down to fill an empty state (hole) below the gap. In a superconductor, the end result can be thought of as the annihilation of two quasi-particles to form a Cooper pair. A phonon with a minimum energy of 2Δ is emitted in the process. The value of τ_R is fairly independent of temperature near the transition temperature (Ref. 49), where it has the value of ~ 0.15 nsec in Nb and ~ 440 nsec in Al, for example

(Ref. 24). The value of τ_R is (approximately) inversely proportional to the density of quasi-particles. This is because, crudely speaking, two quasi-particles must "collide" in order for them to recombine to form a pair. The rate at which this occurs depends on the density of quasi-particles. Because of the presence of an energy gap, the number of thermally excited quasi-particles decreases exponentially with decreasing temperature (at sufficiently low temperatures), and τ_R can become quite large at low temperatures.

At very low temperatures τ_R becomes much larger than τ_T , the time required for the quasi-particles to come into thermal (temperature) equilibrium with their environment (Refs. 49,57). (τ_T is essentially the quasi-particle inelastic scattering time.) Under these conditions the quasi-particles survive long enough to be thermalized before they recombine and are well described by the equilibrium temperature T and some value of the parameter μ^* . Under static conditions where quasi-particles are being created at a fixed rate, the value of μ^* will depend on the value of the recombination time τ_R . A theory of this sort was developed by Owen and Scalapino (Ref. 22), and is known, appropriately enough, as the μ^* model. It might be pointed out that the energy gap Δ and the other parameters of the superconductor can be calculated easily, at least on a computer, once $f(k)$ has been determined. The results of the μ^* theory were found to be in good agreement with the results of various experiments, for example the laser experiment of Ref. (16).

If, under the low temperature conditions described above, a high-energy quasi-particle is created in the superconductor by some means,

such as phonon irradiation, then there is a definite sequence of events which takes place, leading to the relaxation of the superconductor toward its equilibrium state. The correctness of the sequence about to be discussed has been established by a large number of experiments (Refs. 4,5,7-19). First, the quasi-particle relaxes to the energy gap edge by phonon emission. There is some evidence (Ref. 58) that this is essentially a one-step process, with the emission of a single, high-energy phonon. The high energy phonon (or phonons) created in this process decay primarily by the breaking of pairs, that is, the creation of two quasi-particles, which requires a minimum energy of 2Δ . This assumes that the superconducting film is thick enough that the phonon does not simply escape into the substrate or He bath before decaying. The secondary quasi-particles created by this phonon decay process subsequently relax to the gap edge, just as the original quasi-particle had done. This process is repeated until most of the energy of the original quasi-particle has been transferred to a large number of secondary quasi-particles, all with an energy near Δ . The total number of quasi-particles created is then proportional to the initial energy of the original quasi-particle. This entire process is known as a cascade and is quite similar to the cosmic ray showers of high energy physics.

The quasi-particles which have relaxed to the gap edge then recombine after a relatively long time τ_R to form pairs. If the original high energy quasi-particles are continuously being generated by some process (such as phonon irradiation), then a static condition will be reached in which there is an excess number of quasi-particles, δN , above the equilibrium value, N_0 . From the discussion given above, δN is

proportional to P , the power being dissipated in the superconductor. This is because the ultimate number of secondary quasi-particles with energy near Δ created from each primary quasi-particle of energy E is proportional to E/Δ . In the μ^* model, for low temperatures and small δN , the change in the energy gap due to δN is proportional to δN and thus to the power dissipated. This fact will be crucial in the analysis of the present experiment given in Section 3.6.

There are several situations in which the results described above need to be modified. First, at very low temperatures it is possible for δN to be made larger than N_0 , a condition referred to as over-injection. Since τ_R depends on the density of quasi-particles, its value then becomes a function of P and so the linear dependence of the energy gap change on P is lost. A second complication arises if the energy distribution of the phonons is taken into account. Since the phonon distribution, as well as the quasi-particle distribution, may be altered by the nonequilibrium conditions, the quasi-particles tend to thermally relax to the effective temperature T^* of the phonons rather than the equilibrium temperature T . A theory incorporating these effects has been developed by Parker (Ref. 23) and is known as the T^* model. At low temperatures the T^* model also gives a linear dependence of the energy gap change on P (Ref. 49). Finally, at temperatures near T_c the values of τ_R and τ_T become comparable, so that the quasi-particle distribution cannot be simply described as a quasi-equilibrium state with some well-defined parameter like μ^* or T^* . Under these conditions it is necessary to work with $f(k)$ itself and the

calculations become much more involved. Such calculations have recently been carried out, for example, by Chang and Scalapino (Ref. 24).

The discussion presented above is applicable to the case in which $\mu_> = \mu_<$, a situation created by phonon or photon irradiation. A different situation results if the initial quasi-particles which are created lie primarily only on one branch. This results, for example, if the initial quasi-particles are injected by a tunnel junction biased at a voltage much larger than Δ/e . The branch imbalance Q will then relax in a characteristic time τ_Q known as the branch mixing time. (τ_Q is roughly the amount of time required for a quasi-particle on one branch to be scattered to the other branch.) In an isotropic, uniform superconductor branch mixing is forbidden for elastic process by the coherence factors (superconducting matrix elements). In addition, for inelastic processes the probability of a quasi-particle with a given energy being scattered to the opposite branch is roughly proportional to the energy gap Δ (Ref. 55). Because of this, the value of τ_Q calculated by Tinkham and Clarke (Ref. 55) diverges near T_c as $\Delta(0)/\Delta(T)$. ($\Delta(T)$ goes to zero at T_c .) This result should not be surprising, since branch mixing is forbidden in the normal state ($\Delta = 0$) by charge conservation. In a normal metal, branch mixing corresponds to the conversion of a hole into an electron, or vice versa. This is not the case in a superconductor for quasi-particle energies near Δ , where the quasi-particles are linear combinations of electron and hole states.

Near T_c , τ_T approaches a constant value (Ref. 49), which in Ta (the material of the weak link) is ~ 1.8 nsec (Ref. 24). Since τ_Q

diverges near T_c , the quasi-particles on each branch will therefore be thermalized before undergoing branch mixing. Under these conditions, the quasi-particle distribution can, indeed, be described by $\mu_>$ and $\mu_<$ as in eqs. (3.4) and (3.5). In an elegant experiment by Clarke (Ref. 15) and theoretical analysis by Tinkham et al (Ref. 55), τ_Q was measured in Sn and found to diverge as $1/\Delta(T)$ as expected. To the author's knowledge, this is the only direct measurement of τ_Q . (It is a difficult experiment.) Some related information about τ_Q was indirectly obtained by Skocpol, Beasley, and Tinkham (Ref. 59) from an experiment which involved a measurement of the resistance of a "hot spot" in a tin strip. In this experiment, a current larger than the critical current of some localized region was passed through the tin strip, creating a localized nonequilibrium region. The relaxation time of the quasi-particles determined in this experiment did not diverge near T_c but remained fairly constant. This was somewhat surprising, since the most important delay in the relaxation of the quasi-particles might have been expected to be due to the large value of τ_Q . The reason for this result is perhaps not clear (Ref. 59), but in any event the experimental relaxation time for a quasi-particle distribution with a nonzero value of Q does not diverge near T_c if the experimental conditions are spatially inhomogeneous. This will be an important point in the analysis of the present experiment given in Section 3.6, where it will be argued that τ_Q and the other quasi-particle lifetimes remain finite near T_c . In the experiment to be described in this chapter, it seems clear that the excess quasi-particles can diffuse away to a region with a relatively large value of Δ , so

that the effective value of τ_Q does not diverge, even if the local value of Δ is very small.

This concludes the introductory remarks having to do with non-equilibrium quasi-particles in bulk superconductors. There are, however, some additional kinds of excitations in superconductors which are known as collective modes of the order parameter (or pair field) (Refs. 20, 56, 60-65). Crudely speaking, the order parameter Ψ governs the motion of the "superfluid", while the existence of quasi-particles gives rise to a "normal fluid" which behaves in a way which is similar to that of the electron gas in a normal metal. This so-called two-fluid model was extensively used before the advent of the BCS theory, and is still used to some extent in connection with the phenomenological Ginzburg-Landau theory. From the point of view of the two-fluid model, a quasi-particle excitation can be thought of, very crudely, as the transfer of a "particle" from the superconducting phase to the normal phase. Although this interpretation is extremely crude, in any event the quasi-particles refer to the motion of a single "particle." The collective modes, on the other hand, correspond crudely to a bulk motion of the superfluid, somewhat similar to the waves on the surface of a large body of water. The collective modes generally involve non-equilibrium variations in the order parameter over characteristic lengths which are large compared to the atomic spacing. The intuitive discussion given above is rather naive and so the collective modes will be described in more detail below.

The collective modes can only be adequately dealt with by the use of Green's function methods, which are beyond the scope of this

thesis. Furthermore, according to a leading expert in the field of nonequilibrium superconductivity, no Green's function method has yet been devised which treats this topic in a general way (Ref. 66). And finally, it would be fair to say that the existence of collective modes in superconductors has been more or less completely ignored by all but a very few authors. Only very recently has this topic received some interest in the literature. For all these reasons, no rigorous account of the collective modes can be given here.

The discussion in this thesis of the collective modes will be restricted to the situation near T_c , where the lifetime τ_{GL} of these modes diverges like $1/\Delta(T)$ (Refs. 20,56), or in some cases like $1/\Delta(T)^2$ (Refs. 67-69). As has already been mentioned, the quasi-particle relaxation times remain fairly constant near T_c . (τ_Q can be shown to be related to or included in τ_{GL} (Ref. 56), and will be ignored here.) Consequently, τ_{GL} is much larger than the relaxation time of the quasi-particles near T_c , so that the collective modes might be thought of as variations in the magnitude of Δ with the quasi-particles in equilibrium with the local value of Δ . There are other modes in which the phase of Δ (or Ψ) varies rather than its magnitude. Variations in the magnitude of Δ are called longitudinal modes, and variations in the phase are called transverse modes (Ref. 56).

As shown by Schmid and Schön (Ref. 56), the quasi-particle distribution is described, under certain restricted conditions, by a Boltzmann-like equation, while the pair field (order parameter Ψ) is governed by a time-dependent generalization of the Ginzburg-Landau equation. Since

these two equations are coupled, it is perhaps more appropriate to speak in terms of the total relaxation time of the entire system under a given circumstance, rather than the separate relaxation times for the quasi-particles and Ψ . Under the simplest conditions, the equation governing Ψ is (Ref. 68):

$$\frac{\partial \Psi}{\partial t} = \frac{1}{\tau_{GL}} \frac{1}{2\alpha''} \left[\frac{(\hbar \vec{\nabla})^2}{2m} + \alpha'' - \beta'' |\Psi|^2 \right] \Psi \quad (3.9)$$

where α'' and β'' are the phenomenological constants which ordinarily appear in the usual Ginzburg-Landau theory. Under equilibrium conditions the term in the brackets is zero (from the usual Ginzburg-Landau equation) and so Ψ relaxes back to its equilibrium value with a relaxation time τ_{GL} . The value of τ_{GL} is $\sim \hbar/\Delta$ (Ref. 68) at low temperatures, which is on the order of 10^{-12} sec, but diverges near T_c . The idea that Ψ should relax in such a way appears to have first been suggested by Tinkham (Ref. 67), who pointed out that τ_{GL} should be approximately equal to the time required for an electron to diffuse a distance of one coherence length. (The coherence length ξ is roughly the smallest distance over which the phase of Ψ can vary by 2π without causing the increased kinetic energy to give the superconducting state a higher free energy than the normal state. At low temperatures ξ is also roughly equal to the radius of a Cooper pair (Ref. 70).) The value of τ_{GL} can also be thought of as being due to the relaxation of the order parameter due to a generalized force equal to the derivative of the superconducting-normal free energy difference with respect to Ψ . This generalized force goes to zero at T_c .

The collective modes will be seen to be important in attempting to understand the results of this experiment and will be discussed further in Section 3.6. This concludes the introductory remarks on nonequilibrium effects in bulk superconductors. Before going on to discuss the details of this experiment, it is appropriate to review those investigations of nonequilibrium quasi-particle distributions in superconducting weak links which have already been performed. To the author's knowledge, no measurements of the effects of quasi-particle injection in proximity effect weak links have previously been carried out. The experiment which superficially appears to be the most similar to the one described here is that of Wong, Yeh, and Langenberg (Ref. 71). In the experiment of Wong et al a superconducting tin strip was locally heated by various means, including a tunneling current. Under certain conditions this heating resulted in a "weak link" showing quantum interference effects similar to those observed in tin Dayem bridges. (A Dayem bridge is a constricted or narrowed region of a superconducting film which shows quantum interference effects.) As will be discussed in Section 3.6, the experiment of Wong et al actually has very little physical similarity to the present experiment, and in any event does not provide a quantitative measurement of the parameters of interest in this chapter. In a similar experiment by Janik et al (Ref. 72) a "SQUID" was irradiated by a laser beam. (The reference cited does not describe the details of the device studied, except to say that it was a "SQUID", which is an acronym for superconducting quantum interference device.) Once again, there were no quantitative results, other than the observation that the critical current was reduced to some extent.

Perhaps the most thorough investigation of nonequilibrium effects in weak links is an experiment by Octavio, Skocpol, and Tinkham (Ref. 45) and its theoretical analysis by Tinkham et al (Ref. 27). The maximum frequency at which a variable-thickness Dayem bridge would show the AC Josephson effect was analyzed successfully using a simple heating model (to be discussed further in Section 3.6)). Once again, this experiment was physically very different from the present experiment. For example, the nonequilibrium region in the experiment by Octavio et al was very small compared to a coherence length, whereas just the opposite was true in the present experiment; these and other differences will be discussed in Section 3.6. Finally, the experiment which is perhaps physically the most closely related to the experiment described in this chapter was a study of the effects of laser irradiation on a superconducting Al film recently carried out by Schuller and Gray (Ref. 20). Although this experiment was performed on a bulk superconductor rather than a weak link, it was a careful study of the relaxation effects very near T_c . As will be discussed in Section 3.6, these measurements have some relevance to the analysis of the present experiment.

Although it may be obvious, it should perhaps be pointed out that this chapter is concerned with the effects of quasi-particle injection on the proximity effect. As a result, most of the experiments or theories described above are not directly applicable to the present experiment.

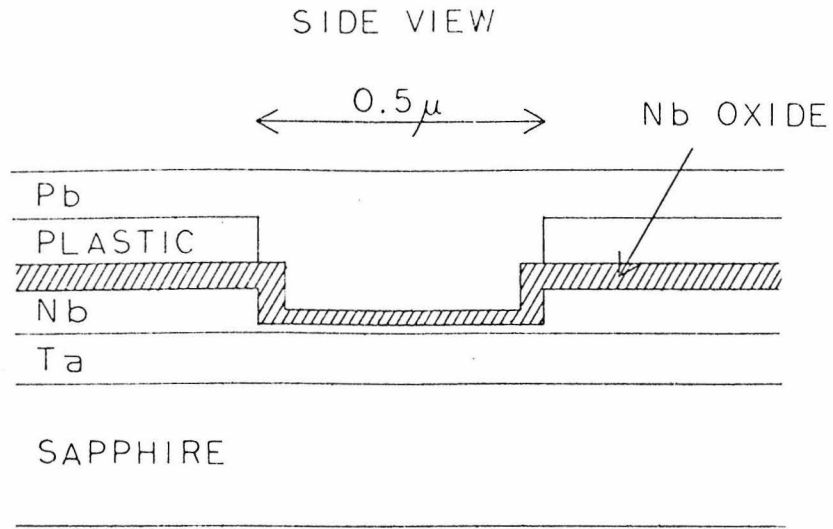
3.3 Fabrication Techniques

It has already been pointed out that the search for a suitable means of fabricating the devices to be studied was the major difficulty

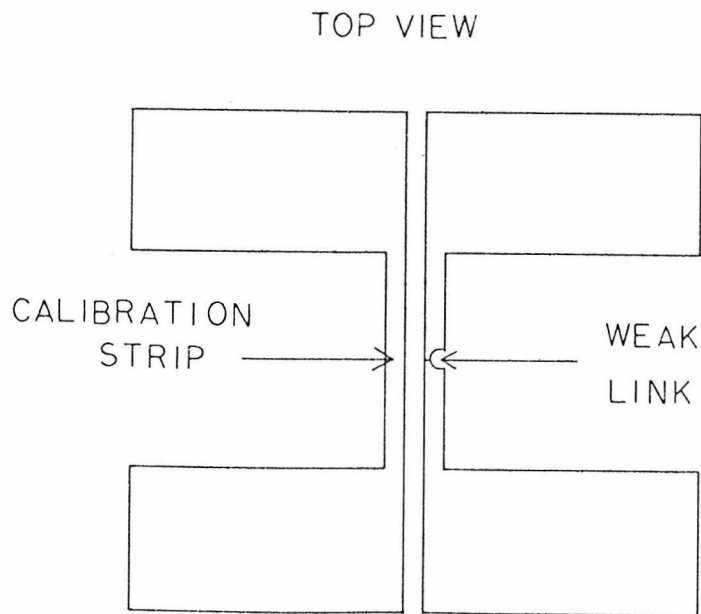
encountered in carrying out the work described in this chapter. Many different approaches to the problem were tried out over a considerable length of time. Testing each of these possible methods required the development of techniques and the construction of equipment which were essentially of no use in any of the other possible methods. After spending a certain amount of time and effort on any particular method, it was invariably found that a dead end had been reached, and that it was necessary to start over with a completely different method. Eventually a method was found which was reasonably successful; it will be described in detail below. The unsuccessful methods will not be discussed at all, since they appear to be of no use.

The structure of the devices fabricated and studied in this chapter is shown in Fig. 3.1, which is not drawn to scale. Basically, the device consisted of a Nb/Ta proximity effect weak link (of the type discussed in Chapters I and II) separated by a thin layer of Nb oxide from a Pb electrode. The oxide formed an insulating barrier which allowed tunneling between the Pb and the Nb. The plastic layer restricted the tunneling to the weak link region itself. The thicknesses and other properties of the various layers shown in Fig. 3.1 will be specified below as the fabrication process is chronologically described.

The fabrication process began with a $\sim 100\text{\AA}$ Nb film on top of a $\sim 200\text{\AA}$ Ta film on a polished sapphire substrate. The two films were successively evaporated in an ultra-high vacuum in such a way that no oxide layer had the opportunity to form between the two films. The evaporation technique for these films has been reported elsewhere (Ref. 26). The films ultimately used in this experiment were no different from those



(a)



(b)

Fig. 3.1 Structure of proximity effect weak link with tunnel junction (not drawn to scale). (a) Edge view. (b) Top view.

ordinarily used in our laboratory.

The Nb/Ta film was first etched into the large scale pattern shown in Fig. 3.1b. The weak link had not yet been formed, even though it is shown in the figure. The purpose of the calibration strip running parallel to the area where the weak link was to be fabricated will be discussed later. The areas to be protected during the etching procedure were covered with photoresistive plastic (PR) as described in Chapter I. The PR was removed to form the pattern shown by a combination of contact masking and microscope exposure. The etching process itself was first attempted using argon ion sputtering, but it was found that the superconducting properties of the films were destroyed by the heat generated during this process. Instead, the etching could successfully be accomplished using an RF-induced plasma of fluorocarbon-14 gas. The etching itself was due to the fluorine ions present in the discharge. The plasma was generated using a commercially available apparatus (Ref. 73) for that purpose.

In order to obtain complete removal of the metal films it was found necessary to use a high-power plasma discharge which tended to bake the PR onto the remaining film. This burned-on PR could be loosened by ultrasonic cleaning in acetone and then removed by gentle brushing with cotton blown free of abrasive grit.

Once the metal had been etched into the large scale pattern shown, the entire film was covered with a thin layer of PR spun on with a centrifuge. No equipment existed in the laboratory for the exact measurement of the PR thickness, but it could be estimated at $\sim 2000\text{\AA}$. The oxide

layer shown between the Nb and the PR was the natural oxide layer existing on any Nb which has been exposed to the atmosphere. The PR was then exposed and developed as described in Chapter I to leave a line whose width was approximately 0.5μ . The width of the Nb/Ta strip left after the previous etching was $\sim 30\mu$, so that a region of bare metal approximately 0.5μ by 30μ was left uncovered by the PR. This region formed the normal region of the proximity effect weak link after the thickness of the Nb had been reduced as shown in Fig. 3.1a; it also formed one electrode of the tunnel junction.

The thickness of the Nb film was reduced by etching in an energetic beam of argon ions. As has already been mentioned, this process tended to overheat the metal film; this was avoided by thermally anchoring the sapphire substrate to a large Cu heat sink using vacuum grease. (The process normally used in the laboratory to form the weak link was anodization of the Nb into an oxide. This process could not be used in this experiment because the oxide layer left behind was much too thick to allow the formation of a useful tunnel junction. Argon ion sputtering, however, does not cause the formation of such an oxide layer.) The argon ion sputtering was done using an apparatus constructed by D. Palmer, an earlier graduate student (Ref. 26). Unfortunately, the etching rate of this apparatus is not reproducible and varies greatly with position with respect to the center of the beam. It was thus necessary to monitor the amount of Nb metal which had been removed as the etching proceeded. This was accomplished by measuring the resistance of the calibration strip shown in Fig. 3.1b, whose center was located near the weak link and which was also being etched at the same time as the weak link. It was

empirically found that the desired amount of Nb had been removed when the four-terminal resistance of the calibration strip had increased by a factor of 2.5. This method was found to be even more reproducible than anodization; the transition temperature of the weak links formed in this way could be controlled to within a fraction of a degree.

After etching the weak link area, a thin oxide layer was formed by immediately admitting high-purity dry O_2 gas to the same vacuum chamber where the etching had been done. Tunnel junctions with the desired properties would be formed if the oxidation was allowed to continue for approximately 10 min, followed immediately by the evaporation of $\sim 1000\text{\AA}$ of Pb as shown in Fig. 3.1a. Unfortunately, the evaporation of the Pb had to be carried out in a second vacuum chamber. Thus it was necessary to transfer the substrate as quickly as possible from one vacuum chamber to the other in order to minimize the length of exposure to the atmosphere. Since the atmosphere in Pasadena (the location of the laboratory) is highly polluted, this step in the fabrication process may (depending on the atmospheric conditions) greatly affect the resistance and other properties of the tunnel junction. It would, of course, be highly desirable to perform the argon ion sputtering and the Pb evaporation in the same vacuum chamber, but this was not feasible using the equipment currently available in the laboratory. (Incidentally, the Pb electrode was actually formed using an alloy of 95% Pb/5% In, which is much more resistant to atmospheric corrosion than pure Pb, but which has superconducting properties very similar to pure Pb.)

3.4 Experimental Methods

The experimental measurements consisted of passing an injection current I_T through the tunnel junction and into the weak link; the critical current and maximum width of the first step were then measured as a function of I_T . In the event that the resistance R_T of the tunnel junction was a function of I_T , then the voltage V_T developed across the tunnel junction was also measured as a function of I_T . The current bias circuit indicated schematically in Fig. 3.2b was used to inject I_T into the weak link region and symmetrically extract half of it from each side of the weak link. The circles containing letters in Fig. 3.2 represent electrical contacts to the device. Voltage source v was actually a ~ 1.5 volt battery floating off ground; avoiding the use of an electronic power supply minimizes ground loop problems and RF pick-up. R_1 and R_2 actually consisted of a large number of precision metal film resistors and appropriate switches arranged so that $R_1 = R_2$. Both R_1 and R_2 were always much larger than R_T .

The four-terminal differential resistance, dV_W/dI_W , of the weak link was measured in the usual way by passing a 10^3 Hz current I_W through the leads marked I and G in Fig. 3.2a, and measuring the voltage V_W developed across the leads marked V and V'. An electronic feedback circuit was built which automatically measured the full width of the zero step, w_0 (which is twice the critical current), and the maximum full width of the first step, w_1 . (Throughout this chapter w_0 and w_1 will be used, rather than the critical current I_c and the half-width of the first step. This avoids any possible confusion or errors

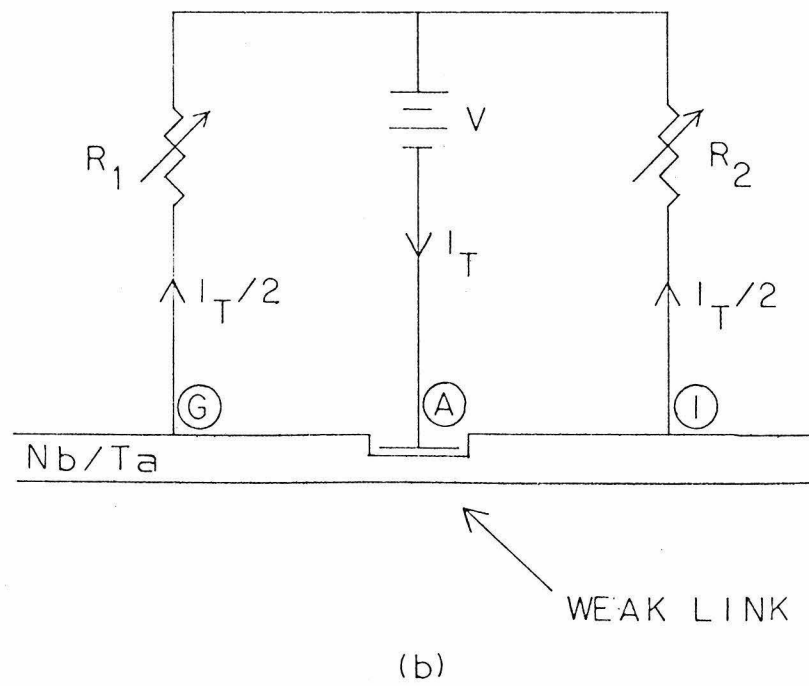
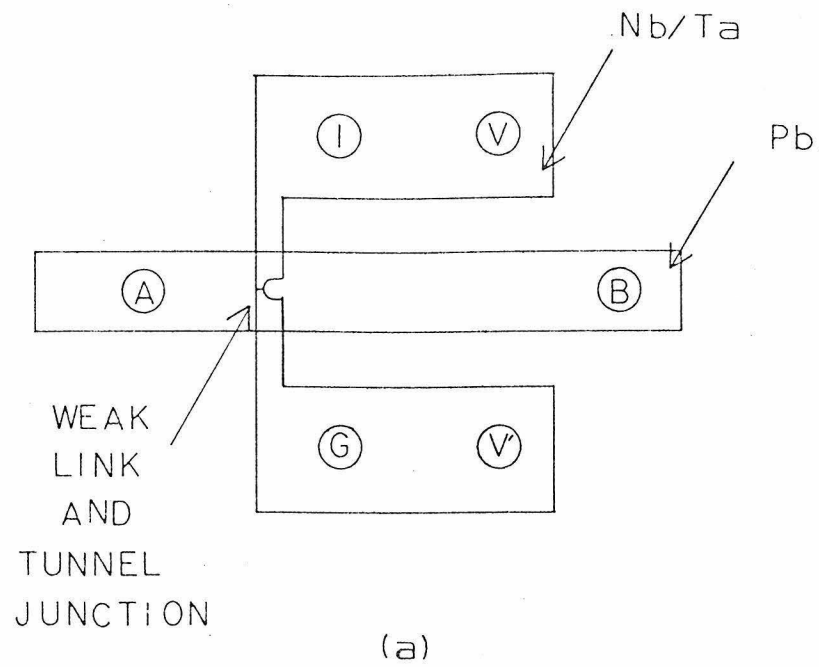


Fig. 3.2 Electrical connections to the weak link and tunnel junction. (a) Position of the various contacts. (b) Idealized injection current bias circuit.

involving factors of 2, which have sometimes appeared in the literature.) The values of w_0 and w_1 could be read on a digital voltmeter with an uncertainty of $\sim 0.5 \mu\text{A}$ or $\sim 1\%$, whichever is larger. The errors in w_1 were sometimes larger than 1% , since the microwave drive which maximized w_1 had to be determined manually. The Pb ground plane above the weak link also served to shield out any residual magnetic fields inside the dewar (the standard Pb bucket and μ -metal shields were, of course, used) to such an extent that critical currents stable to within 1 part in 300 for several hours were observed. The primary source of errors in the measurements was temperature variations in the He bath, which in turn were due to temperature variations in the laboratory room temperature. The room temperature would sometimes vary by over 25°F in less than one hour.

The first attempts at measuring w_0 and w_1 were made impossible by the trapping of fluxoids in the Pb film above the weak link. (Superconductors are capable of maintaining quantized units of magnetic flux by means of circulating supercurrents.) Such flux trapping is caused by transient voltages due to electrostatic discharges, switching of chart recorder scales, etc. In addition, it was necessary to somehow eliminate all RF currents which might be picked up by the various lead wires and passed through the weak link. Although a screened room was available, it could not eliminate any RF or other noise generated by the experimental equipment itself and was therefore not used. The RF filters already in place at the bottom of the dewar were not sufficient to eliminate either of these problems, and so an elaborate filter system

was constructed for use in this experiment. This filter system will now be described.

First, the proximity effect weak link was completely enclosed in a metal "can" which extended all the way from the bottom of the dewar to the top of the probe. The top portion of the probe (that part extending outside of the dewar) consisted of a large aluminum block into which had been machined many smaller compartments. Each of the seven lead wires going down to the weak link first passed through three separate stages of RF filters, each of which was contained in a separate compartment of the Al block. (Actually, there were 5 stages of attenuation altogether, due to the inductance of the lead wires themselves.) Each stage of each RF filter consisted of an LC π filter containing three different RF chokes, since no single choke can be effective at all frequencies. The effectiveness of this RF filter could only be judged by the response of the weak link itself, since the weak link was by far the most sensitive detector of radiation available in the laboratory. The filter appeared to be quite effective, since w_0 was quite stable and no step structure could be seen. The filter also completely eliminated all flux trapping; after installation of the new filter, not a single fluxoid was ever trapped.

All other experimental techniques used in this experiment are the same as those described in the thesis by Palmer (Ref. 26).

3.5 Experimental Results

It has already been pointed out that no detailed theory exists in the literature which is applicable to the results of this experiment. In

addition, the results will be seen to be in disagreement with a simple model which would have been expected to be appropriate (on the basis of the work carried out on bulk superconductors). Therefore, this chapter must to some extent simply present the experimental results without a detailed comparison with any theory. Hopefully, a suitable theory will be available at some time in the future.

First, it is necessary to describe the experimental characteristics of the tunnel junctions which were fabricated. It is well known experimentally that Nb oxide tunnel junctions show several features not found in ideal tunnel junctions; see, for example, Ref. (74). First, the tunneling resistance R_T generally increases greatly as the temperature is lowered from room temperature, whereas the resistance of an ideal tunnel junction increases by only a few percent. This suggests that the Nb oxide is similar to a semiconductor and that true tunneling only occurs when the temperature is low enough to "freeze-out" the charge carriers in the oxide. R_T for the tunnel junctions used in this experiment generally increased by a factor of ~ 2.5 from room temperature to 77°K, and then only slightly from 77°K to 4°K. Second, and more important, the tunneling I-V curves for Nb oxide junctions generally exhibit characteristics indicative of inelastic tunneling. These features include discontinuities in d^2V/dI^2 ("kinks" in dV/dI) at voltages corresponding to the excitation energies of impurities included in the oxide (Ref. 75), and a general smearing out of the sharp features which normally are present due to the superconducting energy gap. The tunnel junctions used in this experiment generally showed several

"kinks" in dV/dI which are presumably due to inelastic tunneling. The inelastic tunneling in Nb oxide tunnel junctions is not very well understood theoretically, but may be due to surface states of the metal as well as oxide impurity states (Ref. 74). The interpretation of these characteristics is made difficult by the fact that Nb can form several different types of oxide.

The tunnel junctions used in this experiment showed varying degrees of what is presumably inelastic tunneling. At low temperatures ($\Delta \gg kT$) ideal superconducting-normal tunnel junctions show a much larger resistance for zero bias voltage than they do for voltages above the gap. This is because a minimum energy Δ is required to create a quasi-particle in the superconductor. The low temperature ratio of unbiased to biased resistance varied from ~ 100 to ~ 1 for the tunnel junctions used in this experiment, which indicates varying amounts of inelastic tunneling. (The possibility of metallic shorts will be ruled out below.) The essential features of the results of this experiment were not affected by the inelasticity, but there were some fine points which were altered. Consequently, when the experimental results are presented it will be necessary to specify whether the results were for tunneling which was mostly elastic or highly inelastic.

It is also necessary to discuss to what extent metallic shorts may have been present in the tunnel junctions used. Those junctions having a high ratio of unbiased to biased resistance clearly had negligible shorting, but those with a ratio near unity might be suspected of being shorted. First of all, if the oxidation time was increased, R_T

invariably increased greatly, which suggests that shorts were not present. Secondly, any supercurrents observed were always less than $1 \mu\text{A}$, which is not usually the case if metallic shorts are present. In addition, those junctions which might have been suspected of being shorted showed a tunneling resistance which was either constant or decreasing with increasing bias current, whereas shorted junctions would be expected to show an increasing resistance as the shorted region heats up. It has already been mentioned that R_T always increased as T was lowered from room temperature, whereas a distinguishing feature of shorts is a decreasing resistance. (It should probably be pointed out that the author has, unfortunately, had considerable experience with shorts in other experiments.) In addition, all of the experimental data were essentially the same from one junction to another, which indicates that if any shorts were present they did not have an appreciable effect on the results. Very few shorts would be expected a priori in this experiment because of the very small surface area of the junctions ($\sim 15 \times 10^{-12} \text{ m}^2$), the lack of defects in the films, and the physical strength of Nb oxide. Because of the reasons discussed above, it will be assumed throughout the rest of this chapter that no metallic shorts were present in any of the tunnel junctions.

Because of the difficulty in reproducibly fabricating these devices (most of which was due to air pollution), extensive measurements were made on only four separate devices out of the many which were actually fabricated and tested. Some typical experimental results are shown in Fig. 3.3, which is a plot of w_0 (twice the critical current)

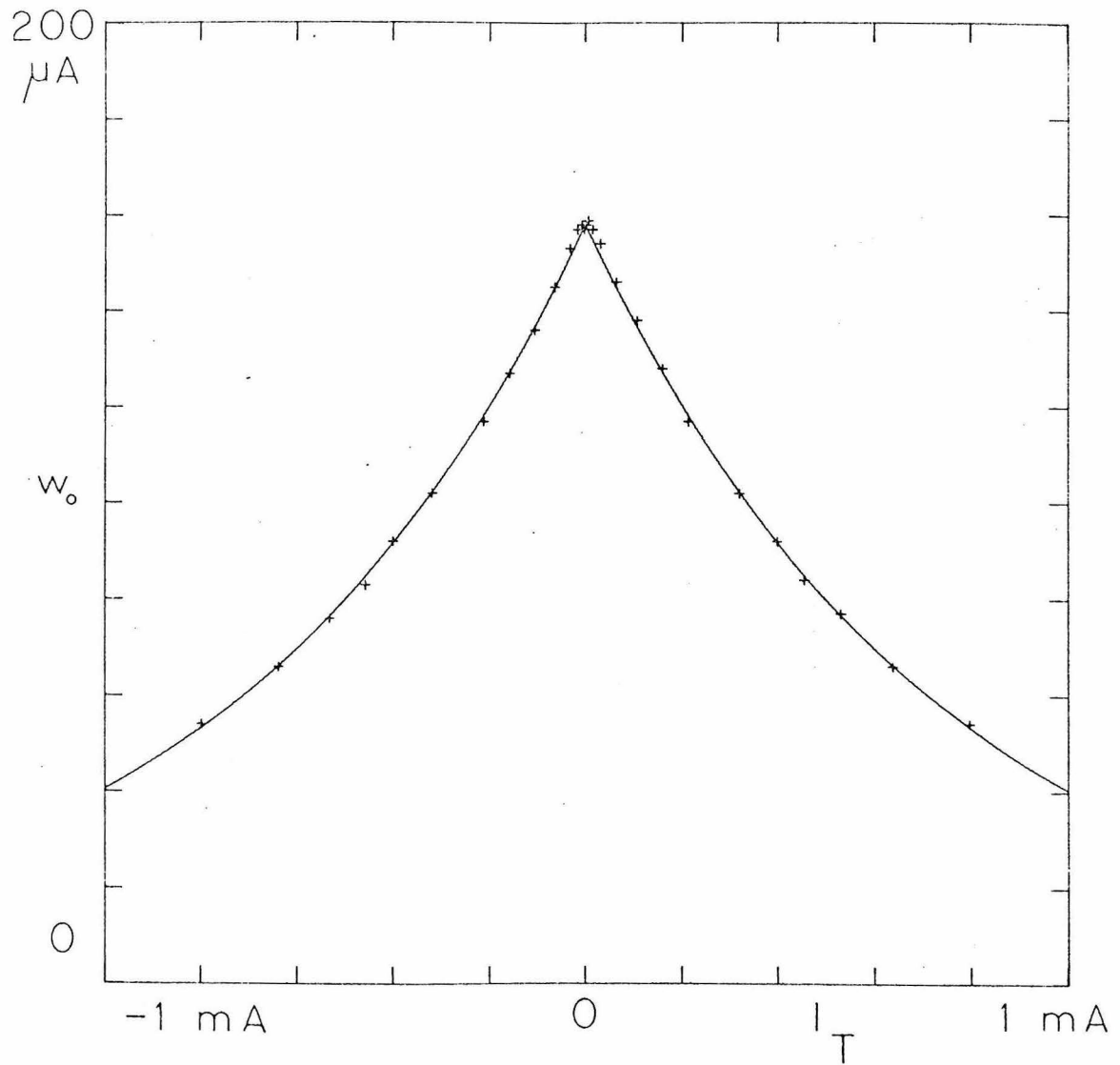


Fig. 3.3 Full width of the zero step, w_0 (twice the critical current), as a function of the injection current I_T . The temperature is fixed.

as a function of the quasi-particle injection current I_T (at a fixed temperature). The bias voltage for this particular device was sufficiently high, and the tunneling sufficiently inelastic, that R_T was essentially constant over the range of I_T shown in Fig. 3.3. The power dissipated by the tunnel junction, which will be of interest later, is therefore proportional to I_T^2 . The theoretical curve drawn through the data should be ignored for the present time, since it is desirable to first discuss the data independently of any theoretical assumptions. The most noticeable feature in Fig. 3.3 is the sharp peak or cusp at zero injection current. The sharpness of this cusp seems to be limited by the experimental uncertainty rather than by the phenomenon itself. Some of the devices tested showed some deviation from the shape of the data in Fig. 3.3. These deviations appear to have reasonable explanations based on the parameters of those devices and will be discussed later. Figure 3.3 is, however, rather typical. Although the polarity of the injection current I_T is evidently of no importance here, it was defined to be positive when the tunnel junction electrode (Pb) was at a positive electrostatic potential with respect to the weak link structure (Nb/Ta).

Much more information can be obtained by studying the temperature dependence of the kind of data shown in Fig. 3.3. It was found that if w_0 was normalized to its value w_0^* at $I_T = 0$, then the data for critical currents varying by nearly two orders of magnitude would all lie (approximately) on a universal curve as shown in Fig. 3.4. It should be pointed out that the deviations from this universal curve, although

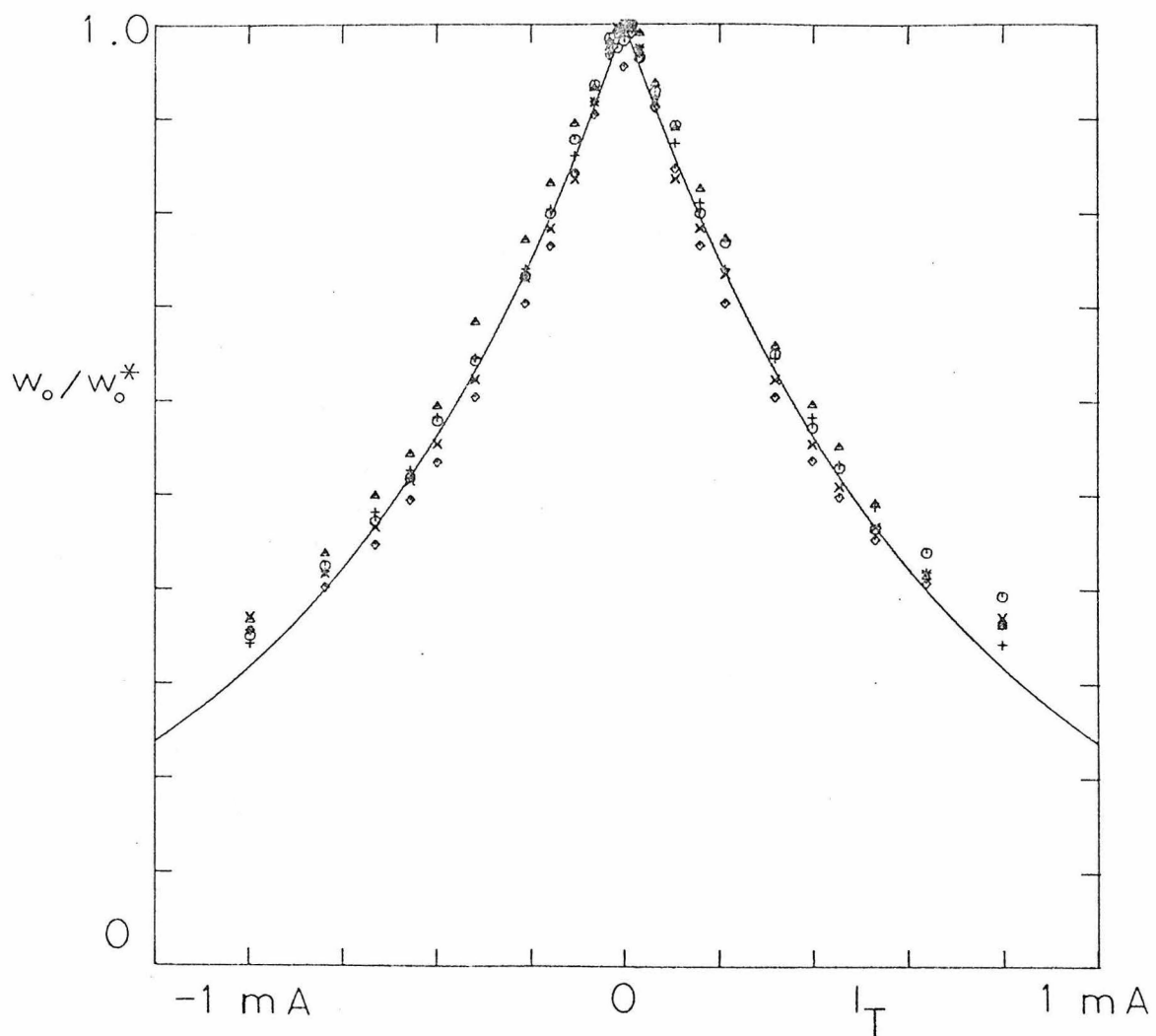


Fig. 3.4 Full width of the zero step, w_0 , normalized to its value w_0^* for zero injection current I_T as a function of I_T at several different temperatures. The circles, triangles, plus signs, x's, and diamonds correspond to values of w_0^* of 28, 61, 158, 305, and 573 μA , respectively.

relatively small, are larger than the experimental uncertainties. The data shown in Fig. 3.4 correspond to values of w_0^* of 28, 61, 158, 305, and 573 μA . All the data points shown in this figure were taken on a single device, for which R_T was essentially constant. The dependence of this kind of data on R_T will be discussed later. The fact that the normalized values of w_0 all roughly lie on a universal curve is significant and will be discussed in Section 3.6.

The data shown in Figs. 3.3 and 3.4, along with their dependence on R_T (to be discussed shortly), form the most important experimental results of this chapter. There are some additional experimental results which are, in some sense, only fine points and not as important or useful as the results already presented. (The reason for the lesser importance of the remaining experimental results will be discussed in the theoretical analysis of Section 3.6.) Although the remaining experimental results are interesting, it may be useful to keep in mind that they are of lesser importance.

The maximum full width of the first step, w_1 , is plotted in Fig. 3.5 for the same five temperatures (and the same device) giving the data shown in the previous figure. Once again, the data points have been normalized to w_1^* , the value of w_1 for $I_T = 0$, and lie roughly on a universal curve. The step was produced by microwave radiation at a frequency of ~ 9.7 GHz (see Chapter II). Although it may not be immediately obvious, the value of w_1 generally falls off somewhat more slowly than the corresponding value of w_0 . Also, the values of w_1 lie along their universal curve with somewhat less scatter than w_0 (even though the experimental uncertainties in w_0 are less than in w_1).

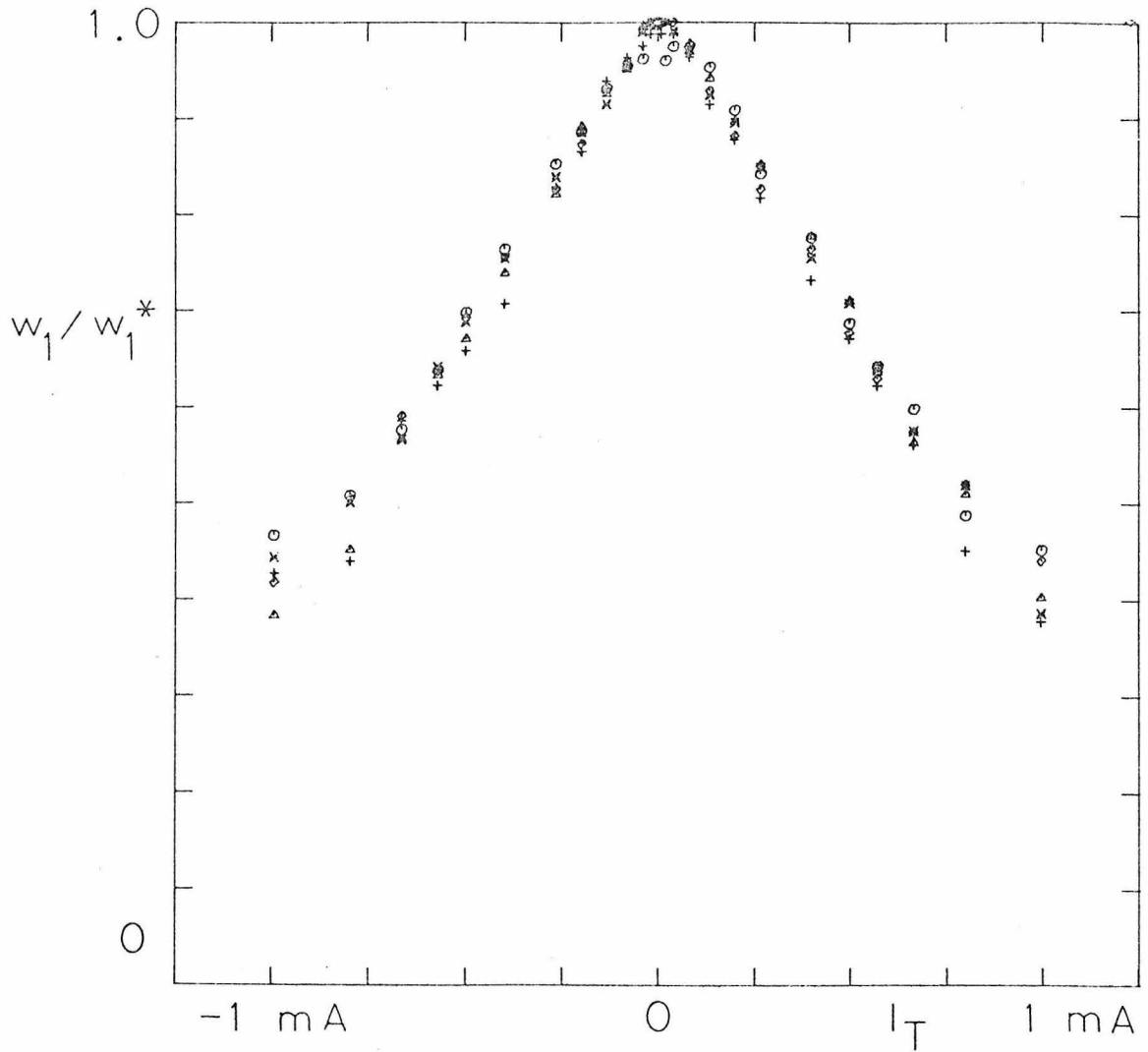


Fig. 3.5 Full width of the first step, w_1 , normalized to its value w_1^* for zero injection current I_T as a function of I_T at several different temperatures. The circles, triangles, plus signs, x's, and diamonds correspond to values of w_1^* of 45, 58, 82, 106, and 127 μA , respectively.

Figure 3.6 shows L_0 , the position of the left edge of the zero step, and R_0 , the position of the right edge of the zero step, as functions of I_T . The zero step consists of the range of dc bias currents through the weak link for which its differential resistance is zero. The choice of left and right is arbitrary, since the weak link is (presumably) symmetric about its center. Thus the lead wires to the device could have been reversed, with a subsequent reversal of the chart recorder tracing. Consequently, L_0 and R_0 can be defined more precisely as the two values of the dc bias current through the weak link at which the differential resistance of the weak link first becomes nonzero. The choice of which is labeled L_0 and which R_0 is arbitrary. Some readers may find it useful to rotate Fig. 3.6 by 90° . In principle, Fig. 3.6 contains all the information contained in plots of w_0 , such as Fig. 3.3, since $w_0 = R_0 - L_0$. Figure 3.6 does not, however, correspond to the same temperature as Fig. 3.3. Because of the steep slopes in Fig. 3.6, it is not easy to visualize w_0 from a plot of L_0 and R_0 ; thus w_0 has already been plotted separately. (In addition, w_0 was actually measured separately, since that is more accurate than subtracting L_0 from R_0 .) On the other hand, plots of L_0 and R_0 contain some information not contained in w_0 , such as the apparent discontinuity in slope at $I_T = \pm 160 \mu\text{A}$. It is somewhat surprising that this change in slope is not reflected very strongly in w_0 itself.

The values of L_0 and R_0 taken at different temperatures do not lie on any universal curve; at lower temperatures (larger I_c) the discontinuities in slope as well as the asymmetry for $I_T \rightarrow -I_T$ tend to become smaller and smaller until they are no

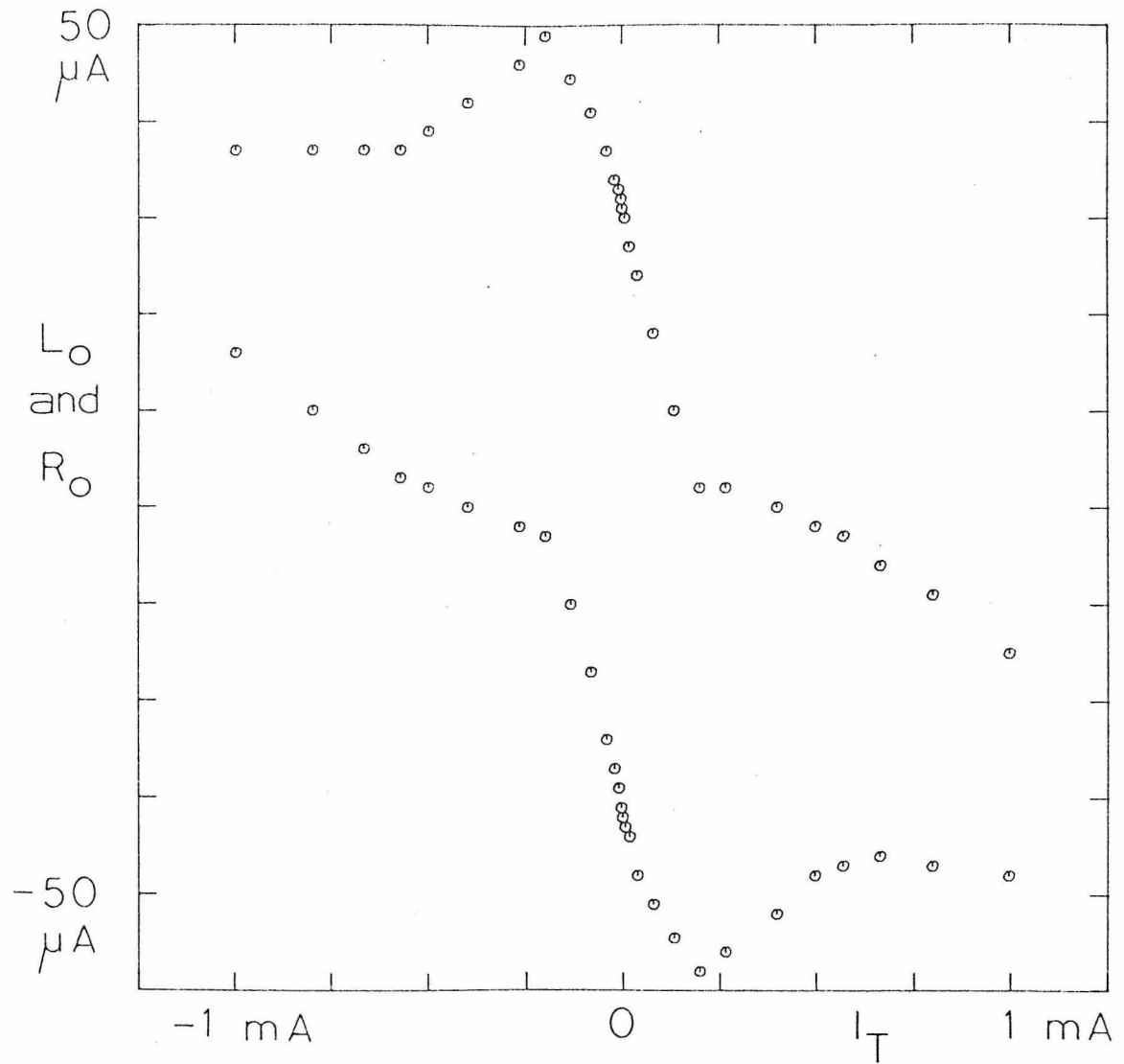


Fig. 3.6 Left edge and right edge of the zero step, L_0 and R_0 , as functions of the injection current I_T . The temperature is fixed.

longer noticeable for the lowest temperatures. The curves for smaller I_C than that shown in Fig. 3.6, on the other hand, have features which are even more pronounced. The region between the slope discontinuities very nearly becomes a straight line, intersected by a slowly curving line at an angle of approximately 90° . (Unfortunately, there is no room in the thesis for many more figures.) It may not be apparent from Fig. 3.6 and should be pointed out, that L_0 and R_0 obey exactly, to within the experimental uncertainty, the symmetry $L_0(-I_T) = -R_0(I_T)$. As will be shown using a symmetry argument in Section 3.6, the features seen in Fig. 3.6 must be due to a slight asymmetry in the tunnel junction injection with respect to the functional center of the weak link; that is, somewhat more than half of the quasi-particle current is injected on one particular side of the functional center of the weak link. It should be emphasized that, for reasons to be discussed in Section 3.6, the measurements of L_0 and R_0 are not as meaningful or useful as those of w_0 ; if they are confusing, they should perhaps be ignored.

The final quantity measured in this experiment was L_1 and R_1 , defined, respectively, as the left edge and right edge of the maximized first step. (The choice of L_1 and R_1 is arbitrary in much the same way as L_0 and R_0 .) A typical set of measurements of L_1 and R_1 is shown in Fig. 3.7. Probably the most interesting feature in Fig. 3.7 is the barely noticeable "hump" at about $200 \mu A$ of injection current. Note that this feature is not symmetric under $I_T \rightarrow -I_T$, probably because the dc bias current through the weak link serves to define a preferred

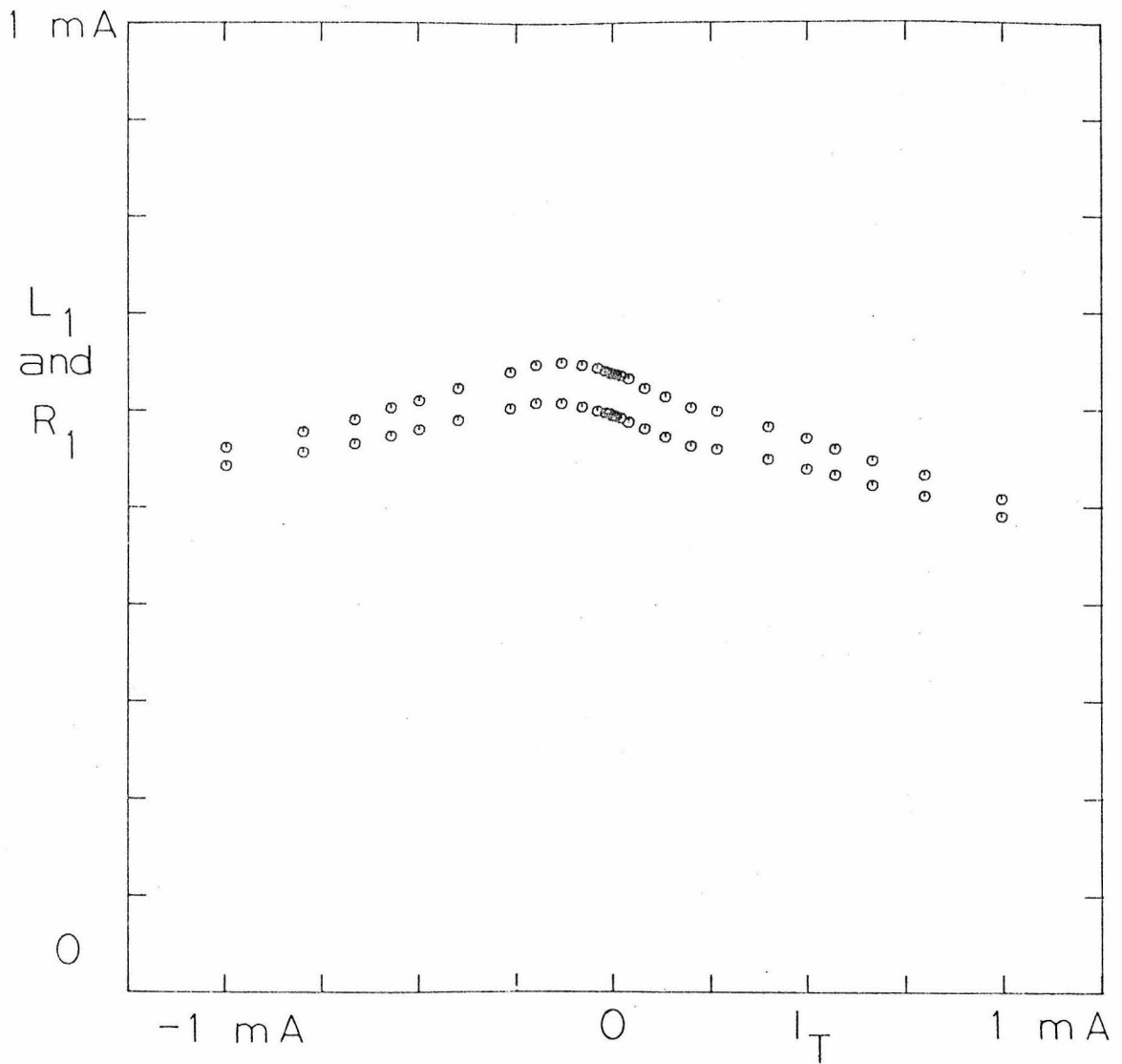


Fig. 3.7 Left edge and right edge of the first step, L_1 and R_1 , as functions of the injection current I_T . The temperature is fixed.

direction; that is, it breaks the symmetry. This kind of asymmetry was not seen in L_0 and R_0 . This asymmetrical feature disappears as I_c becomes much larger than that corresponding to Fig. 3.7, but the general curvature of Fig. 3.7 remains.

As has already been mentioned, the measurements of w_1 , L_0 , R_0 , L_1 , and R_1 are not as important or useful as the measurements of w_0 . The dependence of the reduction in w_0 on the value of R_T will be, however, quite important in the theoretical interpretation of the experimental results. For the purpose of discussing the dependence of the data on the value of R_T , it will be convenient first to discuss the temperature dependence of I_c under equilibrium conditions ($I_T = 0$). (This temperature dependence is the same as that of proximity effect weak links with no tunnel junction present.) For sufficiently small values of I_c , I_c is found empirically to vary exponentially with temperature:

$$I_c = I_{c0} \exp(-\beta T) \quad (3.10)$$

Here β is an empirical constant which has the value of $11.0^\circ\text{K}^{-1} \pm 10\%$ for all the devices studied in this chapter. (The ten percent variation is from device to device.) A plot of the experimental value of $\ln I_c$ as a function of T for one of the devices is shown in Fig. 3.8; the deviations from a straight line are due to experimental uncertainty in the temperature. Equation (3.10) can be understood theoretically on the basis of the Ginzburg-Landau theory; see, for example, the derivation of I_c given in Appendix B in terms of the parameter K . The temperature dependence of K can be found in Ref. (76). At larger values of I_c , the

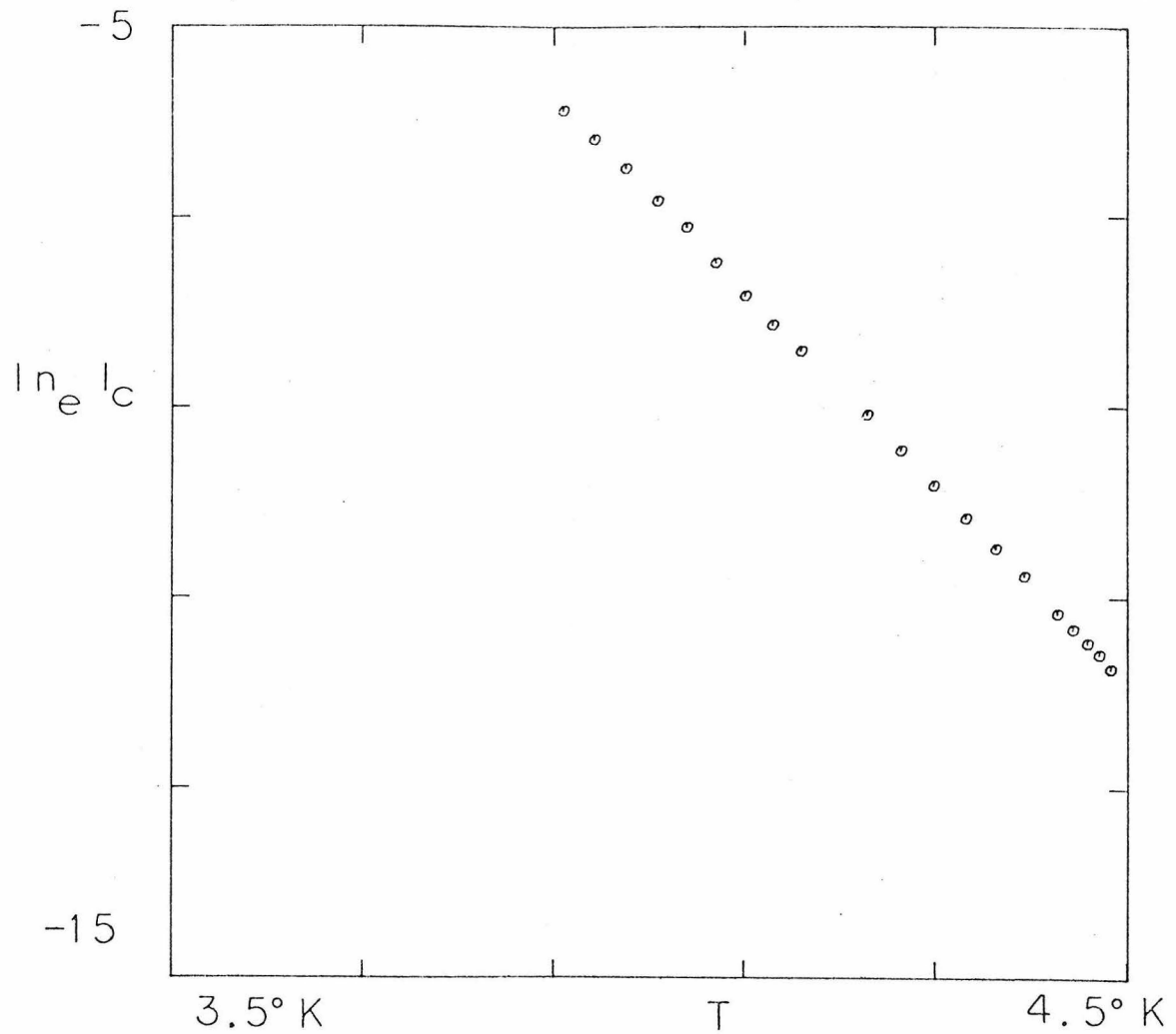


Fig. 3.8 Experimental value of the natural logarithm of the critical current I_c as a function of temperature T . The injection current is zero.

exponential dependence on T gradually changes to a power law which corresponds to the temperature dependence of a bulk superconducting film. This implies that the "normal" region of the weak link has become superconducting in its own right, so that the proximity effect is no longer the dominant effect. The measurements in this chapter were restricted to the temperature range where I_c depends exponentially on temperature.

The dependence of the data on R_T can probably be most easily described by simply pointing out that w_0 is given reasonably well by the empirical relation

$$w_0 = w_0^* \exp[-\alpha\beta \sqrt{I_T^2 R_T}] \quad (3.11)$$

Here α is a constant which is the same for all values of temperature, I_T , and R_T , and for all those devices studied which showed reasonably elastic tunneling. (One particular device showed highly inelastic tunneling and its value of α was somewhat smaller than that of the other devices studied, as will be discussed shortly.) The empirical value of α is $99^\circ\text{K/watt}^{1/2} \pm 15\%$. The inclusion of β in eq. (3.11) is somewhat arbitrary, since its value was the same to within 10% for all of the devices studied; its inclusion was suggested by the analysis to be presented in Section 3.6, and does improve the agreement with the data to some extent.

The dimensions of α are such that α itself is not a very convenient parameter. From eq. (3.11) it is evident that w_0 depends exponentially on the square root of the power P dissipated by the tunneling process. (This fact will be very important in the interpretation of the results of this experiment given in Section 3.6.) It is convenient to define

$P_{1/2}$ as the value of P which reduces w_0 by a factor of two. The value of $P_{1/2}$ was roughly (the factor of β is not included here) $0.4 \mu W$ for those devices showing reasonably elastic tunneling, and somewhat higher for inelastic tunneling, as might be expected. The value of $\beta\sqrt{P_{1/2}}$ was .0067, .0067, and .0073(MKS units) for the three devices showing reasonably elastic tunneling, and .034 for the device with highly inelastic tunneling. Since highly inelastic tunneling results in only some fraction of the total power being dissipated in the weak link itself, it seems reasonable (or at least consistent) for $\beta\sqrt{P_{1/2}}$ to have a larger value in such a device. (Even for completely elastic tunneling, half the power is dissipated in the weak link and the other half in the Pb electrode.) Unfortunately, the tunneling in this particular device was so inelastic that the dV/dI curve could not be used to determine quantitatively the amount of inelasticity present. The values of R_T at $P = P_{1/2}$ for the four devices studied were 12K, 2K, 250 and 50 ohms. This represents a reasonably large range of values of R_T for which eq. (3.11) is valid. The solid line drawn through the data in Fig. 3.3 is a plot of eq. (3.11), where the value of α in this particular instance was chosen using a least squares fit to the data in that figure.

One very important experimental fact has to do with the relation between $P_{1/2}$, as measured in this experiment, and the reduction of the critical current by various other mechanisms which occur in the ordinary use of proximity effect weak links without tunnel junctions. For example, as discussed in Chapter II and calculated in Appendix A, the

maximum dc supercurrent, $\tilde{I}_c$, in the presence of microwave radiation is given by

$$\tilde{I}_c = I_{c0} J_0(2 e v_{\mu 0} / \hbar \omega_{\mu}) \quad (3.12)$$

Here I_{c0} is the value of I_c for no microwave voltage, e is the charge of the electron, $v_{\mu 0}$ is the amplitude of the microwave voltage across the weak link, $\hbar$ is Planck's constant, $\omega_{\mu}/2\pi$ is the frequency of the microwaves, and J_0 is the zero order Bessel function. Experimentally, I_c is found to agree reasonably well with eq. (3.12) for small values of $v_{\mu 0}$, but drops off rapidly at larger values. The value of $v_{\mu 0}$ at which this drop-off occurs corresponds to a microwave power dissipation in the weak link which is roughly equal to $P_{1/2}$, as measured in this experiment. The comparison can only be approximate, since the Wyatt effect tends to cause the "constant" I_{c0} to be a weak function of $v_{\mu 0}$. (The Wyatt effect (Refs. 1-6) is the enhancement of the superconducting energy gap in the presence of microwave radiation, as briefly mentioned in Section 3.1.) Similar "heating" effects limit the maximum number of steps which can be observed in the presence of microwave radiation (see Chapter II). Once again, the power dissipation at which the superconducting effects begin to fall off is roughly equal to $P_{1/2}$. The fact that $P_{1/2}$ is quite comparable to the power dissipation which reduces the critical current in these other situations suggests that the same physical mechanism is responsible for the results in all these cases.

It has already been mentioned that some of the devices studied showed some deviation from the results for w_0 plotted in Fig. 3.3.

These deviations are not considered important (for reasons to be discussed shortly) and should not be allowed to cause any confusion. (These departures from the type of data shown in Fig. 3.3 will be discussed primarily for the sake of completeness.) The most common departure from the kind of data shown in Fig. 3.3 occurred for large values of I_T , where w_0 sometimes fell off somewhat more slowly than would be expected from eq. (3.11). The sharp cusp feature at $I_T = 0$, however, was unaltered in these cases. One possible experimental cause of this effect has to do with the possibility that the tunnel junction oxide may not be completely uniform over the entire surface area of the weak link. This would result in a distortion of the curve, especially for large values of I_T . Another possibility is that this distortion is due to the presence of higher order terms in the theoretical analysis to be presented in Section 3.6. Since this possibility is theoretical rather than experimental, it seems more appropriate to defer any further discussion until Section 3.6.

One particular device showed a plot of w_0 which was qualitatively different from that of the other devices. This is shown in Fig. 3.9. Although the data seem to show a cusp at $I_T = 0$, it is not as sharp as that of Fig. 3.3. The general shape of the curve is also not quite the same. The other properties of this device were also anomalous in the sense that its tunneling characteristics were reasonably elastic and its resistance was sufficiently low that the voltage across the tunnel junction was sometimes comparable to Δ_{Pb}/e , where Δ_{Pb} is the energy gap in the Pb electrode. As will be discussed in Section 3.6,

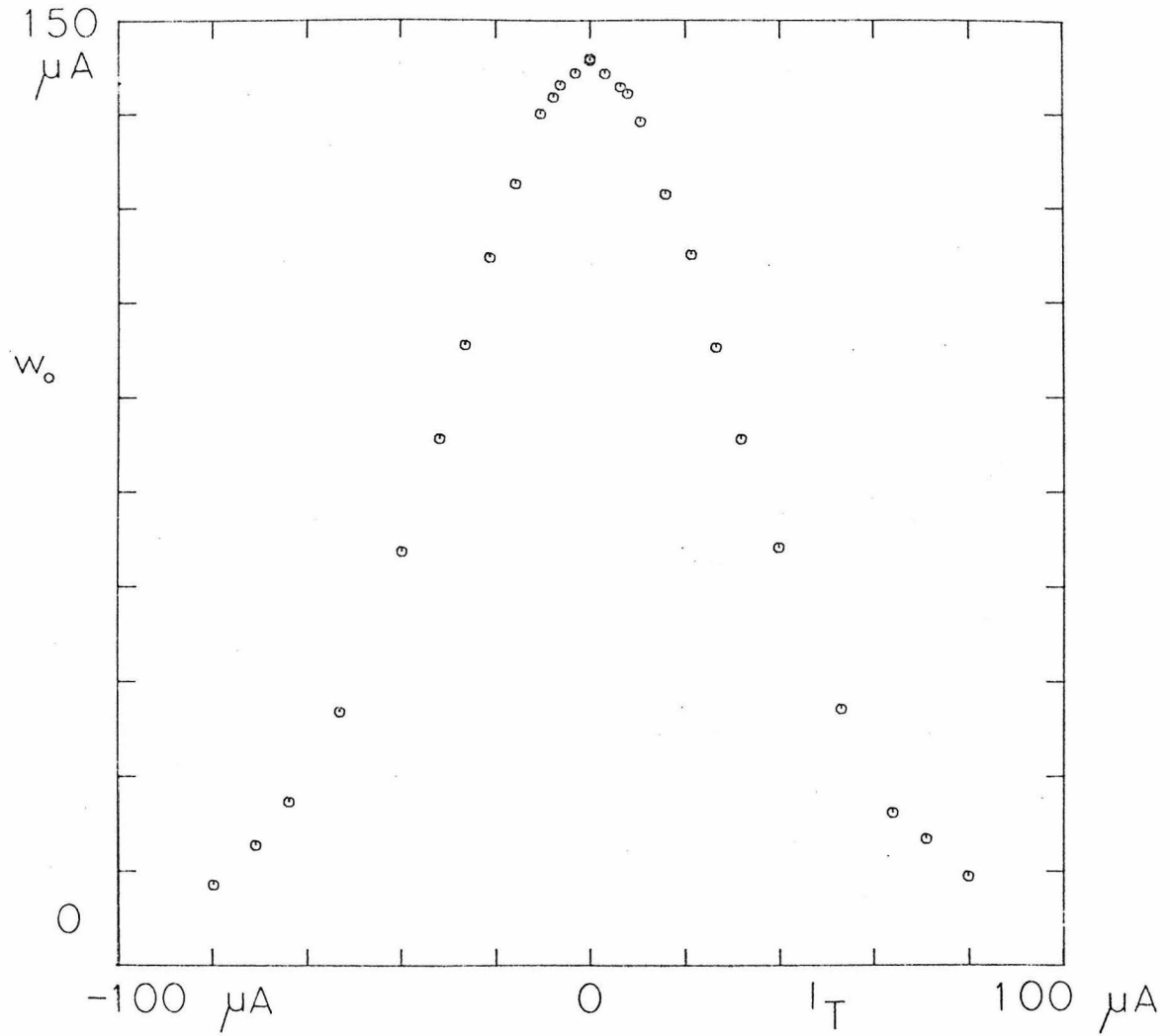


Fig. 3.9 The full width of the zero step, w_0 , as a function of the injection current I_T for a tunnel junction with low resistance (250 ohms) and elastic tunneling.

the combination of these two features can reasonably be expected to alter the shape of the data in a way which is consistent with that observed.

Before going on to a theoretical discussion of the meaning of these results, it may be appropriate to summarize the most important experimental conclusions. Actually, the important experimental results are already summarized fairly well by eq. (3.11), which says that the reduction in the critical current is exponential in the square root of the injection power. (Note that this statement already contains the nontrivial experimental result that $w_0(-I_T) = w_0(I_T)$.) In addition, the power required to reduce I_c by a factor of two by tunneling injection is approximately the same as that required to reduce I_c by other means, such as power dissipation due to the passage of dc or microwave currents through the weak link itself. There are also some additional experimental results, such as the structure in L_0 and R_0 , which will be seen to be of secondary importance and so are deliberately not summarized here.

3.6 Theoretical Discussion

It has already been pointed out that no detailed theory applicable to the results of this experiment currently exists in the literature. This is partly because the proximity effect is inherently more complicated than bulk superconductivity, but there are other reasons as well (as will be discussed shortly). Although a simple quantitative model can be formulated which might be expected (on the basis of the theoretical and experimental work done on bulk superconductors) to give the general features of the data, it will be seen that the experimental

results are completely different from what would have been expected on the basis of that model. The cause of this disagreement appears to be more fundamental than just the mathematical complexity of the proximity effect. In any event, for the reasons given above, the discussion in this section is necessarily of a descriptive, rather than a quantitative nature.

The role of the experimentalist, in some sense, is to gather data and to compare it with all the available theories. If no appropriate theory exists, or if the data are found to disagree with those theories which do exist, then the experimentalist should attempt to find an empirical relation which describes the data. Since such an empirical relation has already been found, this chapter should perhaps end at this point. However, there are some "obvious" explanations of the results of this experiment which can be ruled out by various arguments. It seems appropriate to discuss these plausible but unsuccessful explanations here. In addition, there is one rather novel explanation of these results which does not appear to be ruled out by any simple arguments. This explanation is so novel that it has (apparently) not been thoroughly investigated by the theorists. As a result, it cannot be presented in a quantitative fashion but instead is only suggested as a possible explanation. There is a tendency among graduate students not to include any "intuitive" or "suggestive" interpretations of experimental results in theses, since they would prefer not to defend any imprecise statements. Despite these considerations, this novel interpretation of these results will be presented here, since, having specifically labeled them as a

speculative suggestion, the author does not feel compelled to defend them in a rigorous, quantitative way. As will be seen, the ideas to be presented are at least consistent with common sense.

The primary goal of this section will be to understand the physical mechanism which causes the dependence of w_0 on I_T to be given by eq. (3.11). The other experimental results of Section 3.5, such as the structure in L_0 and R_0 , will only be considered once w_0 itself is understood. The reasons for taking this approach will be explained shortly. This section is rather long and deals with a relatively large number of somewhat unrelated topics. Consequently, the section has been divided up into a number of subdivisions. The section begins by considering the reasons why the polarity of the injection current had no effect on the measurements of w_0 . The mechanism responsible for the removal of the injected charge is a related topic which is discussed along with the injection polarity. The local nature of the exponential decay length of the order parameter is then considered. Because the decay length is determined locally, the results of the experiment described in this chapter can be expected to have a simpler form than would otherwise be the case. An attempt will then be made to interpret the results of this experiment on the basis of a simple heating model and various microscopic theories. It will be seen that the experimental results are fundamentally different than would be expected on the basis of these models or theories. It will then be suggested that the experimental results may be due to the increased importance of the excitation of collective modes or fluctuations, as

compared to the excitation of quasi-particles, due to the small magnitude of the order parameter in the center of the weak link. Although this suggestion is not quantitative, it appears to be in agreement with the form of the data. The possibility of "cooling" rather than "heating" the weak link under appropriate conditions is then considered. Finally, the degree of geometrical asymmetry of the devices studied and the conversion process of quasi-particles to pairs will be discussed; these effects cause the measurements of other parameters, such as L_0 and R_0 , to be less useful or important than measurements of w_0 .

3.6a Injection Polarity and the Charge Removal Mechanism

First of all, the fact that $w_0(I_T) = w_0(-I_T)$ to within the experimental uncertainty should have perhaps been expected, since the theory of superconductivity is symmetric about the Fermi surface. In other words, quasi-electrons and quasi-holes play the same role in the BCS theory, so that it is not surprising that the results are the same for either polarity of the injection current. (This would appear to be true regardless of the actual mechanism responsible for the reduction in w_0 .) This argument breaks down if the weak link is not initially in equilibrium (even for $I_T = 0$) and has an excess of quasi-electrons as compared with quasi-holes (a condition of nonzero branch imbalance) or vice-versa; in this case the polarity of the injection current may be important, since the injection current will tend to add or subtract from the quasi-particle imbalance already present. (Incidentally, a possible example of this is the asymmetric "bump" in

w_1 barely visible in Fig. 3.7 and already discussed in Section 3.5.

This feature may be the result of a branch imbalance caused by the dc bias current through the weak link.) Since the weak link is in equilibrium for $I_T = 0$ in the measurements of w_0 , $w_0(I_T) = w_0(-I_T)$ is to be expected.

Next, it is crucial to note that I_T does not simply subtract some fraction of itself from the maximum supercurrent which can be carried by the weak link. To be more precise, the data does not have the form $w_0 = w_0^* - f(I_T)$, where f is some arbitrary function. In other words, the reduction in w_0 is not due to the electrical current which necessarily flows in the weak link in order to carry the injected electrons away. Instead, the reduction in w_0 depends on R_T as well. In fact, the reduction in critical current is frequently much larger than the injection current which causes it. Thus I_T modulates w_0 rather than subtracting from it. In addition, the function f might reasonably be expected to have the form $f = cI_T$, where c is a constant; this form does not have a discontinuity in slope, as does the data. One fascinating but essentially irrelevant question (by experimental design) has to do with the exact mechanism by which the injected (or extracted) charge leaves (or enters) the weak link region. For example, under static (independent of time) situations, the injected charge might build up an electric field which removes the injected charge by Ohm's law. Or the injected charge might eventually be converted to Cooper pairs and leave the weak link region as a supercurrent. Regardless of how the charge leaves the area, the details of this process can be made irrelevant by

performing the experiment in the limit of large R_T . In this limit whatever effects might be due to the charge removal mechanism become insignificant compared to the mechanism which depends on the injection power (or its square root). In other words, the form of eq. (3.11) shows experimentally that the method of charge removal (and its related current) has no significant effect on w_0 for the experimental conditions which were studied. (This is not true for the measurements of L_0 and R_0 , and is one reason why their measurement is not as useful as that of w_0 ; this topic will be discussed in some detail later in this section.)

3.6b Local Nature of the Exponential Decay Length

Before considering any specific models or theories, there is one topic of a general nature which needs to be considered. Theorists who are not very familiar with proximity effect weak links have sometimes expressed the following opinion about the results of this experiment; the proximity effect weak link is spatially inhomogeneous, so that its properties, including the order parameter, quasi-particle distribution, etc. may be strong functions of position. Because of the nonlocal nature of superconductivity (superconducting properties are, crudely speaking, always averaged over a distance of one coherence length) it may appear to be very difficult to analyze theoretically the results of this experiment. Because of this, it is sometimes said that the results of this experiment would not be very surprising regardless of what form they might have, since it is (evidently) difficult or impossible to predict what the results should be. In other words, the experimental conditions are sufficiently complicated that

the results may not be very useful. There are several reasons why this opinion is not correct, and these reasons will now be discussed.

As discussed in Chapter I (and explicitly calculated in Appendix B for a slightly more general case), the critical current I_c can be understood as being due to the overlap of two exponential "tails" of the order parameter, each dropping off into the normal metal from one edge of the weak link. When Ψ is small enough for the Ginzburg-Landau equations to be linearized, Ψ varies as e^{-Kx} , where x is the distance into the normal metal and K is a constant. The value of K depends on the properties of the normal metal (such as its temperature or quasi-particle distribution) but is independent of position as long as the properties of the normal metal are uniform. Proximity effect weak links, as opposed to other kinds of weak links, have a very short electron mean free path and the length of the weak link ($\sim 0.5\mu$) is much larger than the thickness of the metal film ($\sim 200\text{\AA}$). In addition, the thermal contact to the sapphire substrate is very good (Ref. 26). As a result, nearly all the heat transfer is into the substrate rather than along the metal film, so that, to a first approximation, the temperature, or quasi-particle distribution, should be independent of position. The short mean free path (compared to the length of the weak link) also causes the value of K to be determined locally. (In the limit of a small mean free path, the theory of superconductivity becomes local, in the same sense that Maxwell's equations are local. The fact that the value of Ψ in the center of the weak link is determined by its value in the superconducting regions adjacent to the weak link does not mean that the theory is nonlocal; for example, the fact that electric

fields depend on distant charge distributions does not mean that the theory of electromagnetism is nonlocal.) For these reasons, this experiment can be thought of as measuring the effects on the constant K of the quasi-particle injection process. Since K and the quasi-particle distribution are spatially uniform and determined locally, the conditions of this experiment are actually quite homogeneous and well defined. In this respect experiments on weakly coupled (small critical current) proximity effect weak links are more controlled and subject to theoretical analysis than are experiments on other types of weak links. This is precisely the reason the results of this experiment, such as eq. (3.11), have such a simple form and are more quantitative than "similar" experiments in other weak links (for example, that of Ref. (71)).

In contrast to the local limit discussed above, the successful analysis of heating effects in variable-thickness tin Dayem bridges by Tinkham et al (Ref. 27) assumes the extreme nonlocal case, which is quite appropriate to that experiment. In the experiment analyzed by Tinkham et al, the region being "heated" is very small compared with a mean free path, whereas in the present experiment just the opposite is true. In either limiting case the analysis is simplified and the results are meaningful. In the experiment already discussed by Wong, Yeh, and Langenberg (Ref. 71) in which a superconducting tin strip was heated by various means, it is not clear that either of the simplifying limits described above are appropriate. (The experiment by Wong et al is probably, for other reasons, more similar to the experiment analyzed

by Tinkham than to the present experiment.) Because of this, the experiment by Wong et al did not produce simple, quantitative results of the kind given by this experiment (for example, eq. (3.11)). In addition to the arguments given above, which have to do with the local and spatially independent nature of K, further arguments will be given shortly which indicate that certain types of results for this experiment would be "surprising" even if the situation were inherently spatially inhomogeneous. These arguments have to do with the fact that the injection power is sufficiently low that the physical properties of the materials involved, such as their thermal conductivity, cannot change appreciably with increasing injection power. This simplifying feature is also limited to experiments on proximity effect weak links.

3.6c Simple Heating Model and Microscopic Theories

As a first attempt at understanding the experimental results, it might be worthwhile to simply assume that the effect of the quasi-particle injection is to heat (in a classical sense) the weak link and thereby raise its temperature. (The idea is to tentatively make this assumption and determine its consequences, since this kind of assumption has frequently been made in other experiments, as will be discussed shortly.) From a purely classical point of view, the temperature rise δT would be related to the power dissipated in the weak link region by

$$\delta T = \alpha' P \quad (3.13)$$

Here α' is a constant related to the thermal conductivity into the substrate or He bath, etc. The power dissipated is just

$$P = I_T^2 R_T \quad (3.14)$$

This approach, which might be called the simple heating model (SHM), has several variations which have appeared in the literature. (The term "simple heating" is used by some authors to mean that the phonons as well as the electrons are raised to a new effective temperature. In this chapter, the term "heating" is used in a more general sense without specifying the exact details of the electron and phonon distributions.) For example, it has already been mentioned that Tinkham et al (Ref. 27) have used a model similar to the SHM to successfully calculate the effects of heating in tin Dayem bridges. Although the heating calculations of Tinkham et al were purely classical, various microscopic approaches (such as the μ^* and T^* models already discussed) give final results which are, for the purposes of this experiment, similar to the results of the SHM. These microscopic theories will be discussed shortly, after the consequences of the SHM for the conditions of this experiment have been examined.

Given the temperature rise δT of eq. (3.13), it is necessary to determine what effect that will have on the value of the critical current. Once again, the simplest assumption is that the critical current will have the value that it would have under equilibrium conditions at the new temperature. An equivalent assumption was also made in the calculations of Tinkham et al (Ref. 27). As pointed out in Section 3.5, the critical current varies exponentially with temperature as given by eq. (3.10). Combining eqs. (3.10), (3.13), and (3.14) gives

$$w_0 = w_0^* \exp(-\alpha' \beta I_T^2 R_T) \quad (3.15)$$

Note that eq. (3.15) is a Gaussian in I_T , which has the familiar bell-shaped curve. The data in Fig. 3.3, however, have a sharp cusp which is quite different from the results of the SHM. From eq. (3.11), it can be seen that w_0 depends exponentially on the square root of the injection power, rather than on the power itself as predicted by eq. (3.15).

Although the SHM model may appear to be rather naive, it does give basically the same kind of results for bulk superconductors as the microscopic theories already described in Section 3.2. For example, both the μ^* model and the T^* model, as well as the SHM, give an energy gap change which for small values of P is proportional to P . (Here the quasi-particle relaxation times discussed earlier have been assumed to be independent of P ; the validity of this assumption will be discussed shortly.) It is this linear dependence on P which puts the SHM, the μ^* and T^* models, and essentially any other microscopic theory in disagreement with the square-root dependence on P (for small values of P) observed in this experiment. In fact, the details of any of these theories become unimportant compared to this difference as to the power dependence. The fact that the SHM and the μ^* and T^* theories all give a linear dependence on P forms the basis for the earlier statement that all these models give results which are essentially the same for the purposes of this experiment.

The linear dependence on P of all these models is a direct result of the cascade process described in Section 3.2. In this process, the energy of a primary quasi-particle is ultimately divided up among many

secondaries, so that the total number of quasi-particles created per unit time is proportional to P . Under static conditions, this gives rise to an excess number δN of quasi-particles above the equilibrium value N_0 , where δN is proportional to P . This is a rather fundamental result which is fairly independent of the details of any particular model. The problem now becomes one of understanding why the results of this experiment are so different from what appears to be an essential feature of nonequilibrium bulk superconductivity. To begin, there are several apparent explanations which can be ruled out immediately by various arguments. These unsuccessful explanations will now be discussed.

At first it might appear that the discrepancy between the SHM and the data could be easily understood by simply postulating that, for some reason, I_c really depends on the square root of the temperature in the center of the weak link, rather than on the temperature itself. This might be the case, for example, if part of the temperature dependence seen in Fig. 3.8 were due to the properties of the central part of the weak link, and the rest of the temperature dependence were due to the boundaries of the weak link where the superconducting metal joins the normal metal. Since the injection power primarily "heats up" the center of the weak link, it is conceivable that w might therefore go as $w_0 \exp(-\alpha\beta\sqrt{T})$. However, any argument such as this can be seen to be totally incorrect for the following simple and fundamental reason: The change in temperature δT is small compared with the original temperature T_0 (δT is at most a small fraction of a degree, judging from the observed reduction in

w_0) and so the square root can be expanded in Taylor series to give $\sqrt{T_0 + \delta T} = \sqrt{T_0}(1 + \delta T/2T_0)$. Thus one obtains a linear dependence on δT , even if a square-root dependence on T is assumed. This gives the Gaussian curve, eq. (3.15), just as before.

It should perhaps be apparent by now that the sort of argument presented above is of a rather general nature and rules out a great many explanations which might otherwise be plausible. To be more specific, suppose that w_0 is some function f of the number of quasi-particles N ; $w_0 = f(N)$. Here f can be any reasonable (analytic) function whatsoever, including the square root. As long as the injection process changes N_0 , the initial number of quasi-particles, by a small amount δN (which is evidently the case at these temperatures) then $w_0(N_0 + \delta N) = f(N_0) + f'(N_0)\delta N$ to a high degree of accuracy. In other words, w_0 depends linearly on δN , regardless of the form of f . For lack of a better term, the preceding argument will be referred to as the "linearization argument" throughout the remainder of this chapter.

The difficulty in explaining the results of this experiment have to do with the fact that the data shown in Fig. 3.3 are not analytic at $I_T = 0$ (the cusp feature); this nonanalyticity comes from the square root function in eq. (3.11), which has a branch cut starting at zero. (Physically, of course, the data must eventually be rounded off due to thermal noise, etc., but that is beside the point.) As the linearization argument shows, it is not easy to find plausible explanations which avoid this difficulty. This is because most (and perhaps all) physically reasonable theories do not contain singularities in w_0 as a function of N at a nonzero value of N . This difficulty could be overcome by a

theory in which w_0 is an analytic function of N , but N is a non-analytic function of P . For example, the μ^* or T^* models might be modified in such a way that w_0 is linear in δN for small values of δN , but that δN somehow depends on the square root of the injection power. Three classes of explanations which (unsuccessfully) attempt to explain the data in this way will now be discussed.

The first class of explanations tries to interpret the results by assuming that somehow the cooling rate depends on I_T or δN ; thus, α' in eq. (3.15) would become a function of I_T . One example of such a possibility has to do with the fact that the relaxation time τ_R for the recombination of quasi-particles to pairs depends on the density of quasi-particles, as discussed in Section 3.2. This effect is routinely observed in injection experiments on bulk superconductors at low temperatures, where the number of injected quasi-particles can be much larger than the number present in equilibrium (and is therefore known as over-injection). Since in this experiment $\Delta \sim \kappa T$ (even in the bulk superconducting film outside the weak link), over-injection is physically impossible and the linearization argument can immediately be seen to rule out this explanation. There are several other possible mechanisms in this class which give a removal rate of quasi-particles which depends nonlinearly on the number present. Since they all appear to be ruled out by the linearization argument, they will not be mentioned here.

The second class of explanations attempts to interpret the results by assuming that the volume of material being "heated up" by the

injection power expands as I_T increases; thus the increased power may be partially offset by the increased volume to be heated. First of all, the linearization argument rules out any significant change in the properties of the materials themselves due to δN . Any change occurring must therefore be due to a change in the properties of the excess, nonequilibrium quasi-particles or phonons as their energy rises with the rising injection voltage. For example, the mean free path of the phonons might (although it is really physically unreasonable) be proportional to their energy, so that they would travel further into the sapphire substrate before depositing most of their energy. This is not ruled out by the linearization argument, but can be shown to give the result that δN is proportional to $I_T^{-1}(I_T R_T)^\epsilon$. More generally, the last term can be replaced by an arbitrary power series expansion with no change in the results. The first factor I_T is unavoidable, since the energy deposited is always proportional to the number of quasi-particles injected, regardless of how far they or their secondary phonons travel. The second factor, $(I_T R_T)^\epsilon$, includes any dependence on the injection voltage. The value of ϵ depends on the details of the mechanism involved. But note that this result does not give δN proportional to $\sqrt{P}$ for any choice of ϵ , which appears to rule out all explanations of this sort. In other words, one cannot obtain the correct dependence on both I_T and R_T in this manner. Additionally, such dependences of the mean free path on energy seem physically unreasonable.

The third class of possible explanations argues that most of the energy given off in the decay of a quasi-particle immediately escapes from the weak link without generating additional quasi-particles. For example, the phonons created might have a high probability of propagating into the sapphire substrate and essentially leaving the area, since the phonon mean free path in sapphire is very long. In this case the cascade process would be greatly altered. If p is the average number of secondary quasi-particles produced in a quasi-particle decay, and $1/q$ is the average fraction of the primary energy received by each secondary quasi-particle ($p/q \leq 1$), then it can readily be shown that the ultimate number of quasi-particles generated in the weak link by the cascade process is proportional to $(I_T)^1 (I_T R_T)^\gamma$, where $\gamma = \ln(p)/\ln(q)$. Once again, the first factor of I_T is unavoidable and no choice of γ gives δN proportional to $\sqrt{P}$.

The three classes of possible explanations discussed above all seem to have the following characteristics in common: Since the injection current produces a value of δN which is small compared to N_0 , the properties of the material itself cannot change appreciably. Consequently, if any property (such as the thermal conductivity) changes, it must be a function of the energy of the injected quasi-particles, or the product $(I_T R_T)$. Since δN must also be proportional to I_T (the injection rate), the final result must have the form $\delta N = I_T f(I_T R_T)$. Here f is some function whose form is not important, since δN cannot have the correct dependence on both I_T and R_T regardless of the choice of f . (This can be seen by power series expansion.) It thus seems unlikely that δN could be proportional to the square root of the

injection power. This would appear to be the case, even though the three classes of explanations discussed above obviously do not include every conceivable mechanism.

It thus appears unlikely that the SHM or the μ^* or T^* theories can be brought into agreement with the data. This is because they all appear to require that w_0 be linear in δN , for small values of δN , whereas it does not seem likely that δN could be proportional to $\sqrt{P}$. Before going on to introduce a completely different theoretical approach to the problem, there is an additional experimental possibility which should be discussed. If the oxide thickness varies greatly as a function of position, then the injection power density will also be a non-uniform function of position within the weak link. If this is the case, then a plot of w_0 as a function of I_T would have a different shape than it would for a uniform power density. This might conceivably account for the fact that the data in Fig. 3.3 does not have the shape of a Gaussian curve. This possibility seems rather remote for the following reasons: First, even though it is difficult or impossible to experimentally determine the degree of uniformity of the oxide, the experimental fact that the shape of the results is the same for all oxide thicknesses (values of R_T) suggests that variations in the oxide are not important. (This is because such problems with oxide nonuniformity generally are much worse for thinner oxide layers than thicker ones.) In addition, although oxide nonuniformity would distort the results, it does not seem likely that the distortion would produce data in such close agreement with eq. (3.11) as is evident in Fig. 3.3. (In other words, the data in Fig. 3.3 suggest that the form of eq. (3.11) is not

just coincidental.) Finally, if the discrepancy between these data and the SHM model were to be dismissed on the basis of possible oxide nonuniformity, then all the previous quasi-particle injection experiments on bulk superconductors should also be dismissed for the same reason.

An analogous situation would arise if the weak link itself were spatially inhomogeneous. For example, the mechanical uncertainties in the fabrication process might cause the constant β to vary as a function of the position along the weak link. This might conceivably distort the data from a Gaussian to the sharply peaked function observed experimentally. Variations in β over distances much larger than the resolution of an optical microscope ($\sim 4000\text{\AA}$) are ruled out by the fact that the weak links appear uniform on that scale. Furthermore, the value of β is reproducible from one weak link to the next, which suggests that β is not much affected by any mechanical uncertainty in the fabrication process. It is much more difficult, however, to rule out the possibility that there are small "patches" in the film which are too small to be seen under the microscope, and which have different values of β and so might alter the shape of the data. (The average value of β over the area of a weak link need not vary from one weak link to the next, so the uniformity of the measured value of β does not rule out this possibility.)

First of all, if such patches did exist, then the temperature dependence of I_c would probably also be distorted from the pure exponential behavior seen in Fig. 3.8. The fact that these data form such a straight line suggests that such patches do not exist. In addition,

the nonlocality of the theory of superconductivity causes the properties of the superconducting material to be averaged over a distance of roughly one coherence length. Since the coherence length is a strong function of temperature near the transition temperature, the extent to which the effects of patches are important should vary with temperature. Because of this, the fact that eq. (3.11) agrees with the data over at least two orders of magnitude of critical current suggests that patches are not important. And, just as in the case of oxide nonuniformity, the form of the data plotted in Fig. 3.3 does not appear to be accidental in nature. Finally, if patches are a source of error in this experiment, then they should also be a source of error in the experiments carried out in bulk superconducting films. Consequently, this chapter will proceed on the assumption that spatial inhomogeneity in the oxide or the weak link itself is not the source of the discrepancy between the data and what would be expected on the basis of the SHM or other theories of nonequilibrium bulk superconductors.

3.6d Collective Modes and Fluctuations

For the reasons mentioned above, it does not appear likely that the SHM or other theories of nonequilibrium quasi-particle distributions in bulk superconductors are capable of explaining the experimental data. Neither does it seem likely that this discrepancy could be due to spatial inhomogeneity of the oxide or weak link. At this point a totally different interpretation of the results will be suggested, in which it is assumed that the effects of δN itself are insignificantly

small. Instead, it will be proposed that the injection process induces a certain amount of noise (random voltages) in the weak link region, which tends to excite the collective modes of the order parameter discussed briefly in Section 3.2. In particular, note that the noise introduced into the system will vary as the square root of the power, rather than the power itself. These ideas will be explained in greater detail shortly. As pointed out in Section 3.2, the collective modes have been generally ignored by most theorists and experimentalists. Consequently, the ideas about to be presented cannot be put in a quantitative form. It should therefore be emphasized that these ideas are meant to be a suggestion of a possible mechanism, and not a complete or rigorous theory.

Only recently have the collective modes become of interest in the literature (Refs. 20,56,60-65). Actually, there is a very good reason why the collective modes have generally been ignored in most of the work on bulk superconductors. At relatively low temperatures, the characteristic relaxation time of the collective modes, τ_{GL} , is typically on the order of 10^{-12} sec, whereas τ_R varies from ~ 1 nsec to ~ 500 nsec at the transition temperature T_c (depending on the material) and increases exponentially with decreasing temperature (see Section 3.2). Because $\tau_{GL} \ll \tau_R$, the departure from equilibrium due to the excitation of collective modes is ordinarily completely negligible compared with the excitation of quasi-particles. In other words, the order parameter is well described by its equilibrium value. Near T_c , however, the inequality $\tau_{GL} \ll \tau_R$ will be reversed, since τ_{GL} diverges

at T_c whereas τ_R remains approximately constant in the vicinity of T_c . The transition from domination of the relaxation process by τ_R to domination by τ_{GL} is very abrupt (due to the very rapid increase in τ_{GL}) and has been observed in a very recent experiment by Schuller and Gray (Ref. 20). Their experiment, which consisted of a careful study of the relaxation time of a superconducting film irradiated by a short laser pulse, showed that τ_{GL} begins to dominate the relaxation process at temperatures within about 1% (for the particular samples used in their experiment) of T_c . In the experiment described in this chapter, the energy gap in the weak link region is very small and corresponds to temperatures in a bulk superconductor which are very near T_c . Consequently, the excitation of collective modes might be expected to be of importance in this experiment.

Very near T_c the thermal excitation of the collective modes actually dominates the properties of the superconductor. These effects are known as fluctuations, and are familiar in many physical systems other than superconductors. A recent (and readable) review of fluctuation effects in superconductors is given in Ref. (77). (The literature on superconducting fluctuations generally does not use the term collective mode. This appears to be due to the fact that the theory of fluctuations was developed phenomenologically before the theory of collective modes in its present form became of interest.) Crudely speaking, these fluctuation effects arise because the free energy difference between the superconducting state and the normal state becomes quite small compared with kT for temperatures near T_c . Thus, if the metal is thought of as

being divided up into small regions about a coherence length on a side, any particular region can be expected to have a reasonable probability of being in either the superconducting state or the normal state.

It may be useful to discuss the magnitude of these thermally excited fluctuations, since they are actually much more important than is ordinarily realized. For example, the difference in free energy, F_0 , at 0°K between the superconducting and normal states for a block of dirty Ta (the weak link material) one coherence length on a side is roughly 0.5 eV. Here the coherence length has been taken to be $300\text{\AA}$, which is fairly typical for the films used in this experiment (Ref. 78). A coherence length is the relevant dimension here because of the fact that, if the value of Ψ is perturbed locally, the perturbation will drop off over a distance of one coherence length (Ref. 79). (This minimizes the free energy and represents a compromise between kinetic energy and binding energy.) Thus the lowest energy fluctuations involve a volume of superconductor roughly one coherence length on a side. The probability of Ψ being zero (normal state) in such a volume is roughly $\exp(-F_0/kT)$. At 3°K, for example, kT is roughly equal to 0.3 meV, so that this probability is approximately 10^{-720} . (The transition temperature of Ta is $\sim 4^\circ\text{K}$.) F will be reduced somewhat at 3°K from its value at 0°K, but the difference is not important here. There are some fluctuation effects, such as the nonzero resistance of superconducting wires very near T_c , whose existence does require that Ψ go all the way to zero. (This is necessary for the phase of Ψ to "slip" by 2π .) Such effects are small unless the temperature is very near T_c ; this situation gives the

erroneous impression that all fluctuation effects are negligible except very near T_c .

There are other fluctuation effects (for example, the thermal excitation of the collective modes of interest in this experiment) for which it is not necessary for Ψ to go all the way to zero. It will be instructive to use the value of F_0 given above to calculate the expected root-mean-square deviation, $\delta\Psi_{rms}$, in Ψ . The value of $\delta\Psi_{rms}$ can be estimated by setting the free energy change δF due to $\delta\Psi_{rms}$ equal to κT . The free energy change is most easily calculated by expanding the free energy in a Taylor series in $\delta\Psi$ (since $\delta\Psi \ll \Psi$), and gives the result that δF is proportional to $\delta\Psi^2$. (There is no term linear in $\delta\Psi$ because the equilibrium value of Ψ , by definition, causes F to be an extremum.) The result of such a simple calculation is (Ref. 80) equivalent to

$$\frac{\delta\Psi_{rms}^2}{\Psi^2} \approx \frac{\kappa T}{F} \quad (3.16)$$

Here F is the free energy difference between the superconducting and normal states at the temperature of interest. (This derivation is based on the Ginzburg-Landau theory and so is only valid near T_c or in dirty superconductors where the Ginzburg-Landau theory is applicable.) Solving for $\delta\Psi_{rms}$, using $T = 3^\circ K$, and approximating F by its value at zero degrees gives

$$\delta\Psi_{rms} \approx .025\Psi \quad (3.17)$$

In other words, the value of Ψ can be expected to vary by several percent even for temperatures several degrees below T_c . (This fact does

not seem to be very generally appreciated.) Near T_c , the free energy difference between the normal and superconducting states becomes significantly less than F_0 (it goes to zero at T_c), and so the value of $\delta\Psi_{\text{rms}}/\Psi$ becomes even larger until the inequality $\delta\Psi_{\text{rms}} \ll \Psi$ is no longer valid. These effects have been observed in several experiments, the most dramatic of which is a measurement of the magnetic susceptibility of superconducting aluminum powders carried out by Buhrman and Halperin (Ref. 81). For aluminum particles roughly $250\text{\AA}$ in radius, fluctuation effects dominated the susceptibility ($\delta\Psi_{\text{rms}}^2/\Psi^2$ was not small) over a temperature range of about 10% of T_c . The purpose of this discussion is to show that fluctuation effects are not necessarily small (as might have been suspected from numbers like 10^{-720}) in the experiment described in this chapter. That is, the collective modes are likely to be thermally excited to a significant extent, even under equilibrium conditions.

A physical mechanism by which the quasi-particle injection process can excite the collective modes will now be discussed. If the voltage $v(t)$ is measured between any two points in a metal, it will be found that, even under conditions of thermal equilibrium, $v(t)$ will be non-zero. The value of $v(t)$ will be random in nature, have some root-mean-square value v_{rms} , and is generally known as noise. The injection of the quasi-particles almost certainly increases the value of v_{rms} to some extent. (For example, the injection process itself might produce shot noise, and additional noise might be produced by the relaxation of the high-energy quasi-particles to states of lower energy.) The

transverse collective modes, which represent deviations in the phase of Ψ , are known theoretically to be coupled to voltages in the metal (Ref. 56). This seems quite reasonable, since voltages precess the phase of Ψ , as evidenced by the AC Josephson effect. (See, for example, the discussion of the Josephson inductance in Chapter II.) Consequently, the increase in v_{rms} due to the injection of quasi-particles can be expected to excite the collective modes to a larger extent than would be the case under equilibrium conditions. The quasi-particle injection might also conceivably excite the collective modes by mechanisms other than the noise voltage, a possibility which will not be discussed here.

The typical (or rms) noise voltage δv due to the quasi-particle injection might be expected to be proportional to the square root of P , the injection power (at least under certain conditions). This can be seen by the following argument. Suppose that the i^{th} quasi-particle created by the cascade process (discussed in Section 3.2) produces a voltage δv_i between any two points. Provided that the δv_i are random (independent), the total voltage change due to the injection process, given by $\delta v = \sum \delta v_i$, will not be $\delta N \delta v_i$, where δN is the total number of quasi-particles created by the cascade process. Instead, the random voltages will tend to cancel each other out so that δv is proportional to $\sqrt{\delta N}$, just as in an elementary random walk problem. This gives δv proportional to the square root of P . (It should perhaps be pointed out that P should include all the forms of energy dissipation which create high-energy quasi-particles in the weak link. In particular,

the recombination phonons created in the Pb electrode may propagate into the weak link region and create additional quasi-particles.)

There is one further reason why the excitation of collective modes can be expected to be more important in this experiment than in experiments on bulk superconductors. This has to do with the fact that the low impedance of a superconductor (zero impedance at zero frequency) tends to "short out" any noise voltage v_{rms} . For equilibrium conditions this is a rigorous consequence of Nyquist's theorem (Ref. 82); for noise voltages due to nonequilibrium effects, this result seems intuitively obvious. When the order parameter becomes very small (as it is in the weak link, for example) this "shorting" process becomes less effective at any finite frequency. For example, Chapter II showed that at microwave frequencies the impedance of the weak link is nearly equal to its value in the normal state. Thus, at least at microwave frequencies, the noise voltages will be shorted to a much larger extent in a bulk superconductor than in the present experiment.

Several reasons have been given which suggest that the excitation of collective modes may be much more important in this experiment than in bulk superconductors at low temperatures (where they are ignored). First of all, τ_{GL} becomes large compared to τ_{R} , so that the density of collective modes will increase, even if the rate of excitation of quasi-particles and collective modes stays the same as T_{c} is approached. Next, the free energy difference between the superconducting and normal state decreases as T approaches T_{c} , so that less energy

is required to excite the collective modes (as evidenced by fluctuation effects). And finally, the noise voltages are not so effectively "shorted out" near T_c , so that the rate of excitation of the collective modes might be expected to increase. Sufficiently close to T_c the excitation of collective modes might therefore reasonably be expected to dominate over the excitation of quasi-particles in the present experiment. These intuitive remarks are not quantitative and as a result it is not possible to say with certainty that the excitation of the collective modes is the dominant effect. In this connection it might be useful to point out that $P_{1/2}$ corresponds to an incident power density roughly 50 times more intense than ordinary sunlight, but that the temperature of the weak link rises at most a small fraction of a degree. Such a situation might be expected to be very "noisy". Since these ideas are not quantitative, it will simply be suggested that this experiment is performed near enough to T_c that the excitation of collective modes by the introduction of noise is the dominant effect.

At this point the assumption will (tentatively) be made that the excitation of the collective modes is the dominant effect in this experiment. The consequences of this assumption will then be determined and compared to the experimental results. As discussed in Chapter I (and explicitly calculated in Appendix B for other purposes), under appropriate conditions I_c is given by

$$I_c = I_{c0} e^{-KL} \quad (3.17)$$

Here I_{c0} is some constant, L is the length of the weak link parallel to

the direction of current flow ($\sim 0.5\mu$), and K is a constant which depends on the temperature. (The constant K was discussed earlier in this section.) K^{-1} is the exponential decay length of the order parameter which, under appropriate conditions, drops off as e^{-Kx} at the interface between a normal and a superconducting material. This is known as the proximity effect; x is the distance into the normal material. As discussed intuitively in Chapter I, K can be considered to be a local function of the free energy difference F between the normal and superconducting states (as well as other parameters, such as the mean free path, which are not of interest here) in the "normal" region of the weak link.

Although K can (presumably) be calculated theoretically (Ref. 76) it is sufficient here to simply assume that K is some unknown but analytic function of F , $K(F)$. (The theoretical calculations of K will not be considered here because, to the author's knowledge, none of these theories includes the effects of the collective modes.) The idea is that the excitation of collective modes should reduce F , just as the excitation of quasi-particles reduces F . If the change δF in F due to the noise voltage δv is small, $K(F + \delta F)$ can be approximated by

$$K(F+\delta F) \doteq K(F) + K' \delta F \quad (3.18)$$

Here higher order terms have been neglected and K' is $dK(F)/dF$. (This expansion is almost certainly valid since $K(F)$ is expected to be analytic at any nonzero value of F .) For sufficiently small values of δv , the phase of Ψ is affected by δv in a linear way (see Chapter II). It therefore seems reasonable (although far from certain) to assume that

the excitation rate of the collective modes is linear in δv , and that δF is also linear in δv . If this assumption is made, then for small values of δv , K becomes a function of δv :

$$K = K_0 + \epsilon \delta v \quad (3.19)$$

Here ϵ is some constant and higher order terms in δv have been dropped. It should be emphasized that eq. (3.19) is an expansion which seems reasonable but whose validity has certainly not been proven. Using the fact that δv is proportional to the square root of the injection power, and combining eqs. (3.17) and (3.19) yields

$$I_c = I'_{co} \exp(-\epsilon' \sqrt{I_T^2 R_T}) \quad (3.20)$$

Here I'_{co} is a constant which includes the factor $e^{-K_0 L}$, and ϵ' is a constant which depends on ϵ and the unknown constant of proportionality between δv and $\sqrt{P}$. Note that eq. (3.20) is identical in form to the empirical relation of eq. (3.11). Thus the assumptions made above, which have to do with the dominance of the excitation of collective modes and their effect on the free energy, are in qualitative agreement with the experimental data.

It might be wondered why the linearization argument does not apply to the noise produced by the injection process, since a considerable amount of noise is present even under equilibrium conditions ($I_T = 0$). The reason would have to be that the noise introduced can probably be characterized by a typical frequency ω and a spatial dependence λ (Fourier components) which are different from that characterizing the

equilibrium noise. For example, if the injection voltage is much larger than κT (which it is), then $\hbar\omega$ may be much larger than κT ; there is essentially no thermal noise at such a frequency and so the linearization argument does not apply. It might also be pointed out that the increase in noise is not necessarily equivalent to just adding a few more photons to the system. This is because the most common forms of quantum electrodynamics deal with photons which describe radiation fields, and not static or quasi-static fields. If static fields (such as that from a charged metal sphere) are to be described by photons, then these photons evidently do not obey Bose-Einstein statistics. In other words, static fields resulting from charge and current distributions are calculated from Maxwell's equations, not thermodynamics. In any event, the voltage noise δv introduced by the injection process does not necessarily correspond to simply a change in the distribution of photons from its equilibrium value.

It may be useful to summarize the intuitive ideas which have been suggested above. Several reasons were given which explain why it is possible for the excitation of collective modes to become more important near T_c than the excitation of quasi-particles. These reasons include the divergence of the lifetime of the collective modes, the decrease in the free energy difference between the superconducting and normal states, and the decrease in the ability of the superfluid to "short out" any noise voltages. The excitation of collective modes might be expected to "weaken" the superconducting state, just as is the case for the excitation of quasi-particles. Since the transverse collective modes are coupled to voltages, it was further suggested that the excitation

of the collective modes is primarily due to the noise voltages produced by the quasi-particle injection. The dependence of the noise voltage on the square root of the injection power leads to a reduction in critical current which is in qualitative agreement with the experimental results. It should be pointed out that these ideas are expected to apply only to proximity effect weak links, where the short mean path of the electrons causes the superconducting parameters to be determined locally. This is in contrast to the situation in variable-thickness tin Dayem bridges analyzed by Tinkham et al (Ref. 27), where the situation is highly nonlocal. In fact, in Tinkham's analysis the properties of the weak link are primarily determined by the superconducting region roughly one-half a coherence length away. The properties of this region are similar to those of a bulk superconductor well below its T_c (the collective modes are not important), even though the temperature at the center of the weak link was calculated to be as high as 70°K. Thus the physical properties of proximity effect weak links are quite different from those of bulk superconductors or even other types of weak links, which is the probable reason for the surprising results of this experiment. It might also be pointed out that the ideas presented here also suggest that the temperature dependence of the critical current in proximity effect weak links under equilibrium conditions may also be dominated by fluctuation effects (rather than pair breaking), an idea which has apparently been ignored by the theorists. This concludes the discussion of a suggested mechanism for the reduction of I_c ; the remainder of this section discusses some further points which are independent of the nature of the mechanism

involved.

3.6e The Possibility of Quasi-Particle Extraction

Regardless of what causes it, it has been experimentally established that w_0 depends exponentially on the square root of the injection power. A brief discussion will now be given of some of the "fine details" mentioned in Section 3.4, starting with the anomalous shape of Fig. 3.9. The discussion of these details will be independent of the physical mechanism responsible for the dependence on the square root of the injected power. Experimentally, the device which produced these data was somewhat anomalous itself, in the sense that its tunneling characteristics were relatively elastic and, in addition, R_T was much smaller than that of the other junctions showing reasonably elastic tunneling. When R_T is "small" and the tunneling is elastic, then the tunneling process can, under the appropriate bias conditions, remove excitations from the weak link, rather than inject them. This is illustrated schematically in Fig. 3.10 for the case of $\Delta_{pb} \gg kT$.

Figure 3.10 shows the situation when the bias voltage is equal to Δ_{pb}/e , where Δ_{pb} is the gap in the Pb, not the weak link. (The gap in the weak link has been assumed to be small compared to kT and is ignored.) Then the Fermi level μ_w of the weak link lines up against the upper edge of the gap in the Pb excitation spectrum, as shown. (This drawing makes use of the so-called modified semiconductor model; if the true nature of the quasi-particle states in the Pb is included, the picture is more complicated but the results are the same.) Since

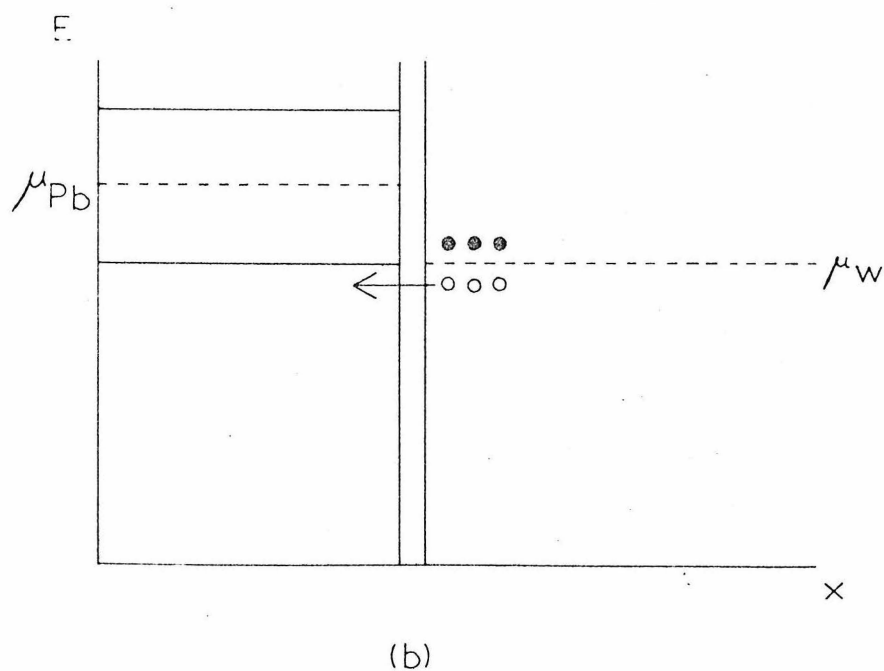
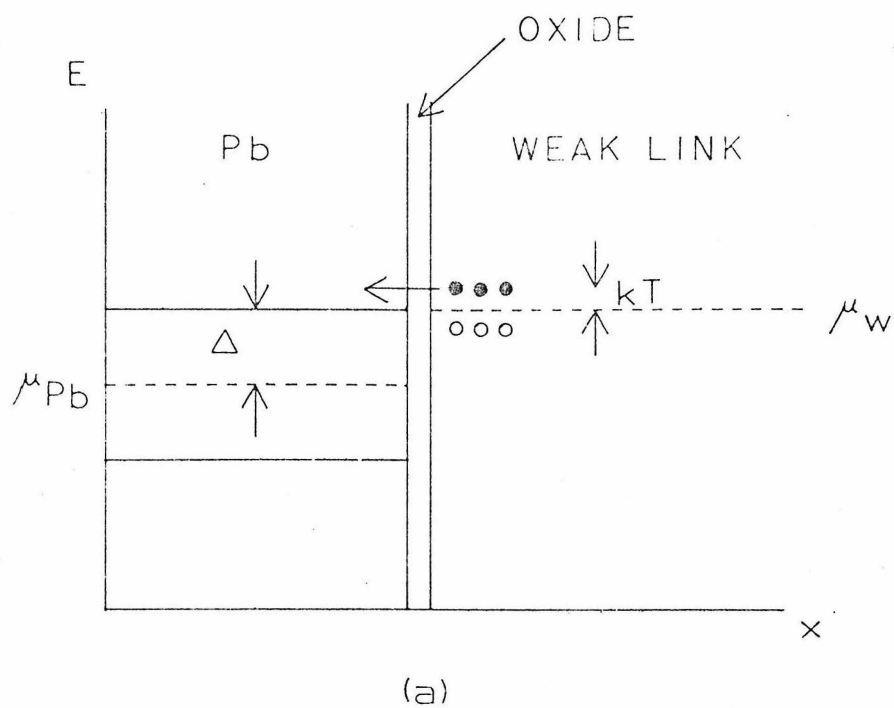


Fig. 3.10 Semiconductor model of the tunneling between the Pb electrode and the weak link region, under conditions in which the weak link region may be "cooled" rather than "heated". (a) Extraction of electrons. (b) Extraction of holes.

$\Delta_{pb} \gg kT$ (by assumption), the states above the band gap in the Pb are essentially empty, whereas the states up to approximately a distance kT above μ_w are partially occupied in the weak link. Thus a net current flows from the weak link into the empty states in the Pb. Since the particles removed from the weak link are excitations, the total energy of the weak link is lowered. A "cooling" effect then results as electrons (not shown) come into the weak link region from the surrounding metal and absorb energy from the lattice as they take the place of the excitations which had been removed. A similar effect takes place when quasi-hole excitations tunnel out of the weak link as shown in Fig. 3.10b. Mechanisms similar to this have previously been suggested by Parmenter (Ref. 83) and elaborated upon by others (Ref. 84).

The process becomes more complicated when kT is not very small compared with Δ_{pb} , when the bias voltage does not exactly equal Δ_{pb}/e , or when the inelastic tunneling effectively "smears out" the energy gap of the Pb. Detailed calculations are not included here mostly because the results strongly depend on the details of the inelasticity in the tunneling. In particular, the inelasticity can reduce the cooling effect to such an extent that the effect becomes a reduction in the heating (as compared to what would be expected for $\Delta_{pb} = 0$), rather than a net cooling. In addition, recombination phonons from the Pb can propagate into the weak link and further reduce the cooling effect. Thus it is not known quantitatively what effect this mechanism should have on the curve shown in Fig. 3.9, but in any event

the curve should be raised above the value expected from eq. (3.11). The maximum cooling should occur at about $I_T = 7 \mu\text{A}$ (this corresponds to a bias voltage equal to Δ_{PB}/e), which is roughly the position of the worst anomaly. For larger values of I_T , the curve would be expected to be raised up by a more or less constant amount. This mechanism seems to provide a reasonable explanation for the shape of Fig. 3.9, although its correctness cannot be firmly established without more quantitative analysis. This effect is not noticeable in the other devices tested because, in all cases, either the tunneling is much more inelastic, or the higher values of R_T cause the bias voltage to be much larger than Δ_{PB}/e for a reasonable injection power.

3.6f Geometrical Symmetry and the Conversion of Quasi-Particles to Pairs

Some of the experimental data presented in Section 3.5, such as the structure seen in L_0 and R_0 , have been considered unimportant or unnecessarily confusing and have been consistently ignored throughout this section. The reasons for ignoring these data will now be explained. The fact that $L_0(I_T) \neq R_0(I_T)$ means that, in a plot of the differential resistance of the weak link as a function of the dc bias current through the weak link, the position of the center of the zero step has been shifted in one direction. In other words, the critical current is not the same for the two directions of current flow through the weak link. This result would be impossible if the weak link and tunnel junction were actually symmetric about the center of the weak link. Therefore, such symmetry is at best approximate. Note

that, if the injection process were symmetric, then the current required to remove the injected electric charge would be exactly zero at the center of the weak link, which is its weakest part, so to speak. (The order parameter is the smallest there.) The lack of symmetry produces a nonzero current of this type at the center of the weak link and is an unnecessary (and probably undesirable) complication. Chapter IV describes some even stronger experimental evidence that the tunnel junction is not exactly symmetric.

As already mentioned in Section 3.5, the structure in L_0 and R_0 disappeared at large values of the critical current. This fact suggests that the structure in L_0 and R_0 may be due, at least in part, to a conversion of some fraction of the injected charge into pairs (superfluid) at positions relatively far from the center of the weak link. The charge could then be carried away by a supercurrent, part of which would pass through the center of the weak link, since the tunnel junction is evidently not symmetric. At large values of I_c , this supercurrent used to carry away the injected charge would become negligible compared to I_c ; this would explain the disappearance of the structure in L_0 and R_0 for large values of I_c . If the center of the weak link can be thought of as being able to carry a fixed maximum amount of supercurrent, regardless of its origin, then the extra supercurrent needed to carry away the injected charge should shift the position of the zero step without affecting its width. In other words, this extra supercurrent due to the charge injection should add to I_c for one direction of current flow, and subtract an equal amount

for the other direction of current flow. This is also in agreement with the data; w_0 does not show the structure seen in L_0 and R_0 and the reduction in w_0 , as has already been discussed, is experimentally found to be independent of whatever mechanism removes the injected charge.

The ideas just presented can be summarized by saying that both the apparent lack of symmetry of the data and the disappearance of the structure in L_0 and R_0 at large values of I_c suggest that this structure is due to a partial conversion of the injected charge to pairs. (This idea has not, of course, been proven.) Some fraction of the supercurrent which carries the injected charge away appears to pass through the center of the weak link in the same way as any other supercurrent, and offsets the position of the zero step. These ideas also explain the symmetry $L_0(-I_T) = -R_0(I_T)$. If this interpretation is correct, then the structures observed in L_0 and R_0 have more to do with the conversion process of injected charge to pairs than it does with the properties of the weak link itself. On this basis, these measurements of L_0 and R_0 are not as important to the understanding of the operation of the weak link as are the measurements of w_0 . The structure seen in L_0 and R_0 , in particular the discontinuities of slope, are still of interest. (These measurements may provide some information about the conversion process of quasi-particles to pairs under dynamic situations, or the interaction between the quasi-particle current and the gradient of the phase of the order parameter to be suggested in Chapter IV.) However, in designing an experiment it is

desirable to eliminate as many variables as possible; this chapter has chosen to study w_0 under conditions in which the removal of the injected charge has no significant effect. Although the structure observed in L_0 and R_0 is interesting in its own right, it is a topic which has been left for further experiments and analysis.

3.7 Summary and Conclusions

This chapter began by describing an experimental method by which quasi-particles could be directly injected into a proximity effect weak link using a very small tunnel junction biased with a voltage large compared with Δ_{pb}/e . Under appropriate conditions, there is some evidence that a net quasi-particle extraction can be accomplished as well (for a bias voltage comparable to Δ_{pb}/e). For tunnel junction bias voltages large compared to Δ_{pb}/e , it was found experimentally that the critical current reduction was exponential in the square root of the injection power. This result is valid over a relatively large range of (unperturbed) critical currents and tunnel junction resistances. The injection power required to reduce the critical current by a factor of two was approximately 0.4 μW for those devices showing reasonably elastic tunneling (and somewhat higher for those showing highly inelastic tunneling). This value of injection power is quite comparable to the power dissipation which significantly reduces the critical current in the usual applications of proximity effect weak links, in which dc or microwave voltages are applied directly across the weak link itself. This suggests that the same mechanism is responsible for the critical current reduction in all these cases. Several other experimental param-

eters were also measured but were considered to be less important or useful than the reduction in critical current.

It was pointed out that a simple heating model or a microscopic theory applicable to the usual nonequilibrium experiments in bulk superconductors (such as the μ^* or T^* theories) would be expected to give a dependence of the critical current which is exponential in the injected power itself, rather than its square root. This is a consequence of the fact that, in those theories, the effects on the superconducting properties are due to the production of an excess number of quasi-particles by the cascade process described in Section 3.2. The cascade process gives an excess number of quasi-particles which is proportional to the injected power. Several possible mechanisms which might be capable of altering the results of these theories in such a way as to bring them into agreement with the experimental results were considered. All of these explanations were found to be unacceptable for various reasons. Although it is obviously impossible to consider every conceivable mechanism of this sort, arguments were given which suggested that the discrepancy between the data and the conventional theories being applied to bulk superconductors is too fundamental to be explained by a simple modification of these theories.

It was then suggested (not proven) that the effects observed experimentally are due to the excitation of the collective modes of the order parameter, rather than the excitation of quasi-particles. Very near the transition temperature, T_c , there are several reasons why the effects of the excitation of collective modes might be expected to

dominate the effects due to the excitation of quasi-particles. First, the relaxation time of the collective modes becomes larger than that of the quasi-particles near T_c . In addition, the free energy difference between the normal and superconducting states goes to zero at T_c , so that less energy is required to excite the collective modes. And finally, the noise voltages to which the collective modes are expected to be coupled are not as efficiently shorted out by the superfluid near T_c . For these reasons, it was suggested that the noise voltages created by the injection process excite the collective modes, and that this excitation reduces the critical current. Since the noise voltage should be proportional to the square root of the injection power, this assumption leads to results which are in qualitative agreement with the data. It was also pointed out that this interpretation suggests (although it certainly does not prove) that the effects of fluctuations may dominate the behavior of proximity effect weak links even under equilibrium conditions. Several fundamental differences between this experiment and those performed on bulk superconductors or other types of weak links were discussed.

There is one conclusion of practical importance which perhaps can be drawn: If the interpretation given above is correct, it suggests that the high frequency response of proximity effect weak links may not necessarily be improved by additional means of heat transfer between the weak link and its surroundings. Instead, the high frequency response may be limited by the generation of noise at the high currents and voltages involved. This may or may not be improved by better heat transfer,

since such noise may exist even in the limit of infinite thermal conductivity. In any event, such an improvement in the high frequency response might be expected to go only as the square root of the increase in heat transfer.

IV. MEASUREMENT OF QUASI-PARTICLE NONEQUILIBRIUM IN PROXIMITY EFFECT WEAK LINKS USING A TUNNEL JUNCTION PROBE

4.1 Introduction

In Chapter III a tunnel junction was used to perturb a proximity effect weak link (by injecting quasi-particles) and the resulting changes in some of the properties of the weak link were measured. In this chapter an experiment is described in which a tunnel junction was used as a probe of some of the nonequilibrium properties of a proximity effect weak link which are intrinsically present whenever a finite voltage is applied across the weak link itself; the tunnel junction in these measurements constituted a negligible perturbation on the weak link. In particular, the change in chemical potential of the quasi-particles inside the weak link caused by the application of a microwave voltage to the weak link was measured under situations where the dc voltage was zero.

Similar measurements made by Yu and Mercereau (Ref. 1) have recently been reported, in which a direct ohmic contact to one edge of the weak link area was used as a probe. Since such an ohmic contact might be expected to greatly perturb the properties of the weak link, the interpretation of those experiments is perhaps not clear. One purpose of this experiment was to repeat the measurements of Yu et al (Ref. 1) using a tunnel junction probe which does not perturb the weak link. A second purpose of this chapter is to suggest a physical mechanism which might be responsible for the change in quasi-particle chemical potential observed, and to compare the results of calculations based on this

mechanism with the measurements. It was found that the experimental results obtained using a tunnel junction probe were similar to those observed by Yu et al, but contained additional fine structure not observed in the previous work. The experimental measurements, including the new fine structure, were found to be in qualitative agreement with the calculations.

The chapter begins with a brief review of related work in this area and of some of the properties of nonequilibrium quasi-particle distributions already discussed in Chapter III. The experimental methods employed and some possible sources of spurious results are discussed and ruled out. The experimental results are presented, followed by a discussion of a physical mechanism which might be responsible for the results obtained. Calculations based on this mechanism are carried out and compared with the experimental results. The calculations and the measurements will be seen to be in reasonably good agreement. Finally, the physical significance of the results will be considered.

The basic properties of nonequilibrium quasi-particle distributions were described in some detail in Chapter III. Many of the concepts introduced there will also be needed in this chapter. Those readers who are not familiar with branch mixing or the various relaxation times of the quasi-particles are referred to Section 3.2 for a review, as well as a list of references to relevant theoretical and experimental work. There are a few additional topics which were not mentioned in Chapter III and will now be considered. A more detailed theoretical discussion will be postponed until Section 4.3.

If quasi-particles are injected into a certain volume of superconductor and pairs are extracted, a nonequilibrium condition is created because of the nonzero recombination time τ_R required to convert quasi-particles to pairs or vice versa. If a branch imbalance is created, the nonzero value of the branch mixing time τ_Q will also contribute to the departure from equilibrium. The latter phenomenon was first observed in the experiments of Yu and Mercereau (Refs. 2-4), in which a small normal probe (ohmic contact) was placed at various positions near one of the superconducting-normal boundaries of a proximity effect weak link. At finite voltages, where a supercurrent flows up to the boundary and is converted into a normal (quasi-particle) current inside the weak link, it was found that the chemical potential μ_{qp} of the quasi-particles does not equal the chemical potential μ_p of the Cooper pairs for distances up to a few microns from the boundary. Crudely speaking, pairs tend to accumulate in that region while the number of quasi-particles is depleted because the finite value of τ_R (and to some extent τ_Q and the Ginzburg-Landau time τ_{GL}) slows down the conversion process of pairs into quasi-particles. Using time-dependent Ginzburg-Landau theory (Ref. 5,6), it was shown that

$$\mu_{qp} = a' \vec{\nabla} \cdot \vec{j}_s \quad (4.1)$$

where $\vec{j}_s$ is the supercurrent density and a' is a constant proportional to the appropriate relaxation time.

The essential correctness of this result was later confirmed in an experiment by Clarke (Ref. 7) in which quasi-particles were injected

into a bulk superconducting film using a tunnel junction. In this case the voltage measured could be shown (Ref. 8) to be proportional to the branch imbalance Q (which is related to μ_{qp}) maintained by the injection process. The two experiments were not completely equivalent in that different relaxation processes were involved.

After this brief introduction, the experiment which forms the basis for this chapter can now be described. The experiment made use of the same kind of proximity effect weak links with tunnel junctions fabricated directly on top as were used in the measurements described in Chapter III. A thorough description of the properties of these devices was given in Chapter III and need not be repeated here. The only difference was that in this experiment it was, as will be discussed later, crucial to have the tunneling resistance R_T as low as possible (no larger than ~ 50 ohms). If no current was passed through the tunnel junction its presence did not significantly affect the weak link. If, however, the tunnel junction was connected to one input of a high impedance amplifier the voltage V_T measured would be equal to μ_{qp} inside the weak link, since quasi-particle tunneling would bring the two electrodes into equilibrium with each other. (Pair tunneling did not occur because the junction was too small to maintain coherence.) The second input to the amplifier (differential mode) was connected to the bulk superconducting film, which was in thermodynamic equilibrium. Thus the amplifier measured the difference between the value of μ_{qp} in the weak link and its value in the bulk film. Since μ_{qp} was defined with respect to the Fermi level (see Chapter III), its value in the

bulk film was zero. Consequently, the signal measured by the differential amplifier was equal to μ_{qp} itself in the weak link region. As will be explained shortly, the experiment consisted of measuring the change in μ_{qp} due to various physical processes.

Since μ_{qp} may, in general, vary as a function of position inside the weak link, the measured value of V_T was actually a spatial average of μ_{qp} over the area covered by the tunnel junction probe. (As described in Chapter III, tunneling was restricted to the "normal" region of the weak link.) For example, if a large dc voltage $\bar{v}$ was applied across the weak link, μ_{qp} would vary (this is known theoretically) from zero at one end of the weak link to $\bar{v}$ at the other. In that case V_T would be expected to be approximately equal to $\bar{v}/2$. It was found experimentally that in such situations V_T would differ from $\bar{v}/2$ by anywhere from 1% to 15% (from device to device), indicating that the functional center of the tunnel junction need not coincide exactly with the functional center of the weak link. Consequently, the spatial average of μ_{qp} was weighted by a function of position within the weak link, where the weighting function was evidently not completely uniform over the area of the weak link. It might be pointed out that a 1% asymmetry corresponded to a displacement, of one center with respect to the other, of $\sim 50\text{\AA}$ ⁰; it could not be determined whether the tunnel junction, the weak link, or both were actually responsible for the asymmetry observed. As it turns out, this geometrical asymmetry was quite useful because some of the nonequilibrium quasi-particle distributions studied were expected (see Section 4.3) to be antisymmetric about the center of the weak link, so that the true spatial average of μ_{qp} would have been zero. But due to the geometrical

asymmetry of the tunnel junction probe, it was still possible to measure a nonzero value of V_T which was proportional to the magnitude of the quasi-particle imbalance. (Because the exact form of the tunnel junction asymmetry could not be determined, the constant of proportionality was not experimentally known.) Since the signals measured were always on the order of a few nanovolts, a reasonably large amount of asymmetry was not only useful but absolutely essential. (The analysis of Section 4.3 suggests that the previous experiment of Yu et al (Ref. 1) would have given a null result if their normal metal probe did not in fact contain some degree of asymmetry.) It should also be pointed out that if μ_{qp} varied in time, say at microwave frequencies, then V_T as measured was actually the time-average and should perhaps be denoted $\bar{V}_T$.

Although there are a variety of situations which can produce $\mu_{qp} \neq 0$, the measurements reported in this chapter will be restricted to one particular source of quasi-particle nonequilibrium: A microwave current I_μ was driven through the weak link while a dc bias current I_{dc} was simultaneously passed through it, where $|I_{dc}|$ was less than the critical current I_c . The frequency $\omega/2\pi$ of the microwaves was in all cases ~ 9.7 GHz. In order that a lock-in detector could be used to measure $\bar{V}_T$, I_{dc} was chopped (square-wave modulated) at a frequency of 5 to 10 Hz. The value of $\bar{V}_T$ measured was thus proportional to the change in the space and time averaged value of μ_{qp} , due to changing the dc bias current from zero to I_{dc} . Such low modulation frequencies are very inconvenient because of $1/f$ noise and various other experimental

reasons. The necessity of this low-frequency modulation will be discussed after the experimental results are presented in Section 4.2, since the possible sources of spurious results can be better understood (and ruled out) after the nature of the results obtained have been discussed. For the present it will suffice to say that the measurements were far from trivial, and that the more sophisticated aspects of the experimental method will be discussed later in Section 4.2.

The measurements were made using the probe and RF filters described in Chapter III; all other experimental techniques were the same as those described in the thesis by Palmer (Ref. 9).

4.2 Experimental Measurements

Before presenting the experimental results, it might be useful to mention briefly why it is that $\bar{V}_T$ can be nonzero, even if $|I_\mu|$ and $|I_{dc}|$ are both less than I_c . (In the actual measurements $|I_\mu|$ is frequently, but not always, less than I_c .) First of all, it was established in Chapter II that the microwave impedance Z of the weak link is nonzero even for $|I_\mu| < I_c$. Therefore, a microwave voltage will exist across the weak link and can be expected to produce some sort of nonequilibrium value of μ_{qp} . What is surprising is that the time average of μ_{qp} , and thus $\bar{V}_T$, is not zero even though the time average of the microwave voltage which produces it is zero. As will be shown in Section 4.3, there exists a rectification mechanism, dependent on the value of I_{dc} , which can give a time-averaged nonequilibrium in μ_{qp} . As will be discussed later, a dc current is necessary to break the

left-right symmetry of the device.

A typical measurement of the tunnel junction voltage $\bar{V}_T$ as a function of the dc bias current through the weak link, I_{dc} , is shown in Fig. 4.1. (The microwave current was held at a fixed value when taking the data shown here.) These data were typical in the sense that a curve of this form would almost always result if the microwave current $I_{\mu 0}$ was chosen more or less at random. ($I_{\mu 0}$ will be used to denote the amplitude of $I_{\mu}(t)$.) The data reported by Yu et al (Ref. 1) were essentially the same as that shown here; the additional fine structure observed in this experiment only occurred at very specific values of $I_{\mu 0}$ and will be discussed shortly. The fine structure does not appear in Fig. 4.1. There are some very sharp noise spikes in Fig. 4.1 which were caused by heavy electrical equipment operating somewhere in the vicinity of the laboratory. These spikes can be readily distinguished (by their shape) from the true signal or other sources of noise, and should be ignored. (The spikes disappeared late at night, on Sunday morning, etc.)

Next, the position of the critical current can be readily distinguished by the fact that, when $|I_{dc}|$ becomes larger than I_c , the usual voltage drop across the weak link causes $\bar{V}_T$ to go rapidly off scale. The values of $\bar{V}_T$ for $|I_{dc}| > I_c$ are of no interest in this experiment, and should also be ignored. It should perhaps be pointed out that the measured dc voltage across the entire weak link, that is, between the two superconducting regions on each side, actually was zero for $|I_{dc}| \leq I_c$, even though the voltage at the tunnel junction

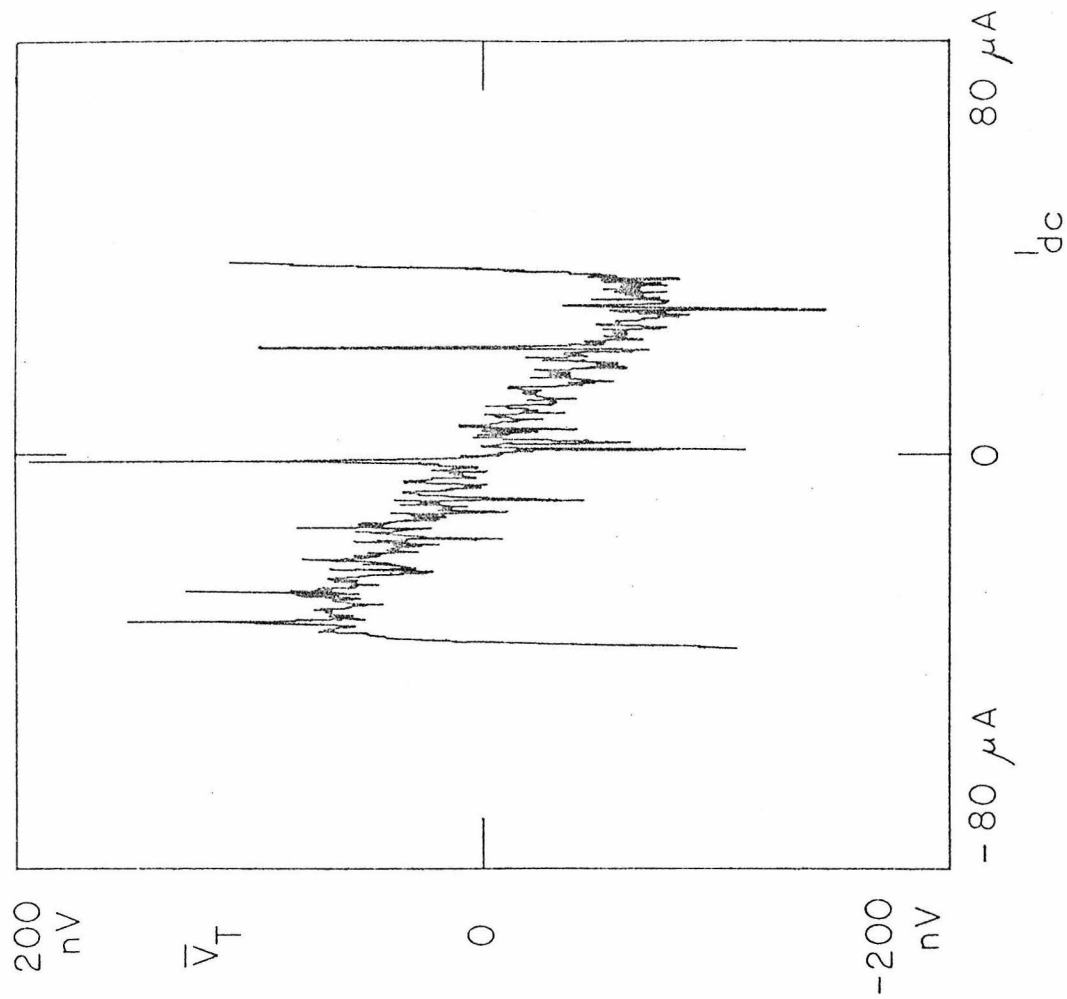


Fig. 4.1 A typical measurement of the time-averaged potential of the tunnel junction probe, $\bar{V}_T$, as a function of the dc bias current through the weak link, I_{dc} . The amplitude of the microwave bias current is fixed.

probe was nonzero, as shown in Fig. 4.1. Note that the polarity of $\bar{V}_T$ for the data shown here is just opposite what would be expected for an ohmic drop ($|I_{dc}| > I_c$); both polarities were observed, however, as will be discussed shortly. When the irrelevant features are ignored, the data shown in Fig. 4.1 form a reasonably straight line, with perhaps some curvature near the ends. Some data were taken which formed even straighter lines than those shown in Fig. 4.1, whereas still other data formed lines which were more curved. Figure 4.1 is, however, rather typical. The dependence of the data on the value of $I_{\mu 0}$ will be discussed later.

If the value of $I_{\mu 0}$ was chosen at random, then occasionally a curve of the type shown in Fig. 4.2 would be obtained. It was determined empirically that such curves resulted when the value of $I_{\mu 0}$ was in a narrow range near that value which reduced the critical current to its first minimum. (The critical current is expected to vary as the zero order Bessel function of a constant times $I_{\mu 0}$; see eq. (4.9), below.) It was also observed that the critical current did not go to zero for any value of $I_{\mu 0}$, but had nonzero minima where the usual Bessel function relation would have been expected to give a zero. The data shown in Fig. 4.2 seem to consist of three straight line segments separated by two discontinuities. Small changes in $I_{\mu 0}$ would cause the position of these discontinuities to move a relatively large distance closer together or farther apart. Note that the position of the two discontinuities is approximately symmetric about $I_{dc} = 0$. The peculiar small-scale step-like structure seen in Fig. 4.2 is due to the use of a digital circuit to fix the value of I_{dc} and should be ignored.

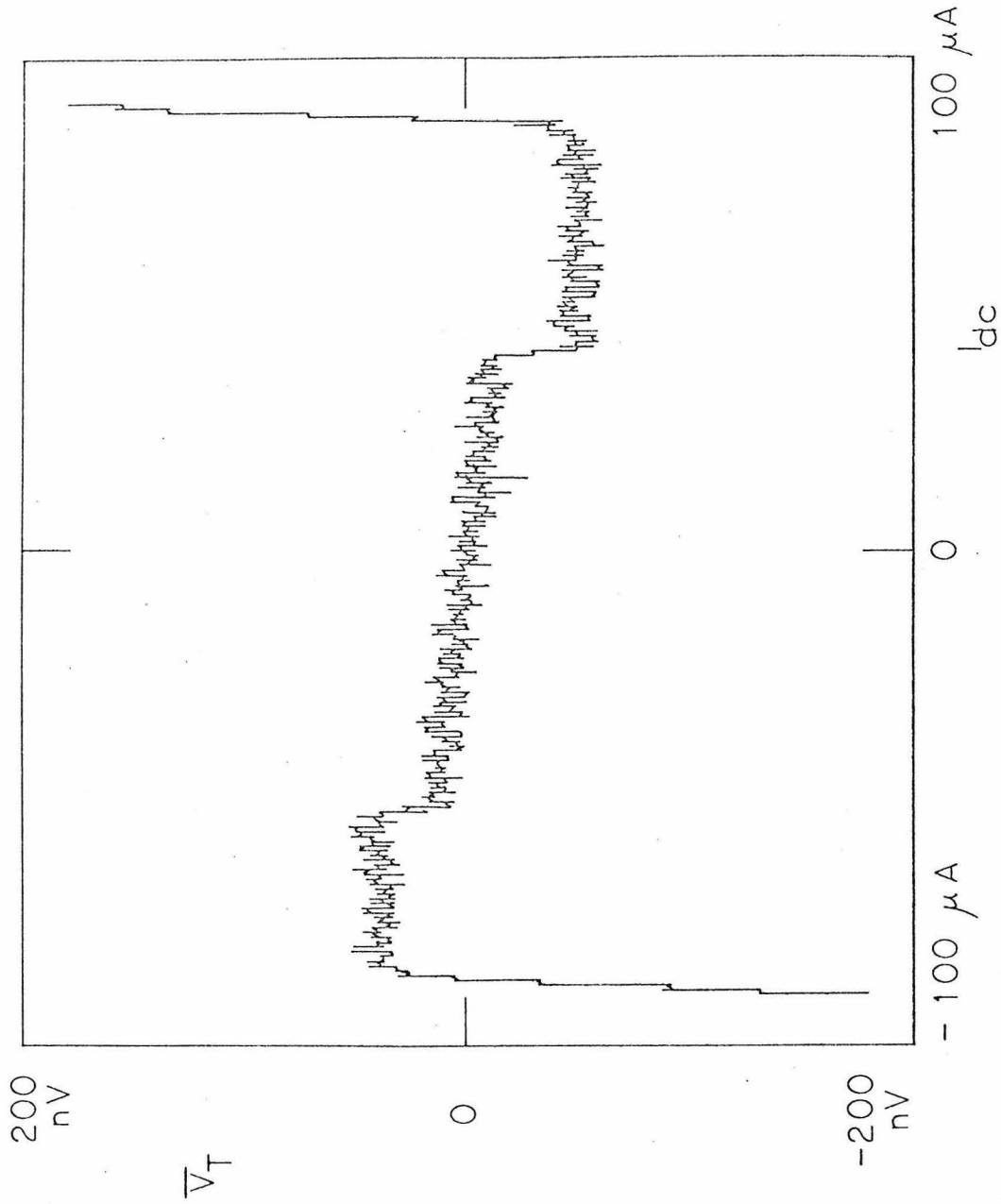


Fig. 4.2 A measurement of the time-averaged potential of the tunnel junction probe, $\bar{V}_T$, as a function of the dc bias current through the weak link, I_{dc} . Curves of this sort resulted when the microwave bias current nearly reduced the critical current to zero.

The kind of data shown in Fig. 4.3 was obtained even more rarely than that shown in Fig. 4.2, if the value of $I_{\mu 0}$ was chosen more or less at random. Figure 4.3 is a degenerate case of Fig. 4.2, and resulted if the value of $I_{\mu 0}$ was adjusted in such a way that the two discontinuities of Fig. 4.2 just came together at $I_{dc} = 0$. Then the data basically consisted of two straight line segments separated by a single discontinuity. (The slowly curving regions near each end of the curve correspond to dc bias currents larger than I_c , and should be ignored.) In addition to the three kinds of curves discussed so far, measurements of $\bar{V}_T$ also yielded slightly curved line segments separated by zero, one, or two discontinuities. For reasons which will become apparent in the theoretical discussion of Section 4.3, these discontinuities will be referred to as "phase-jumps" throughout the remainder of this chapter.

Figure 4.4 shows the dependence of the maximum value of $\bar{V}_T$ ($\bar{V}_T$ at $I_{dc} = I_c$) on the value of $I_{\mu 0}$. $I_{\mu 0}$ itself could not be directly measured. Instead, the microwave power incident on the weak link could be varied using a precision attenuator, and $I_{\mu 0}$ could later be related to the attenuator setting by observing the position of the minima in I_c (see Appendix A). The most outstanding feature in Fig. 4.4 is an apparent discontinuity at $I_{\mu 0} = 1.1$ mA, which is near the first minimum in I_c .

Now that the nature of the experimental results has been discussed, there are some further experimental details which need to be considered. (This discussion has to do with the possibility of errors

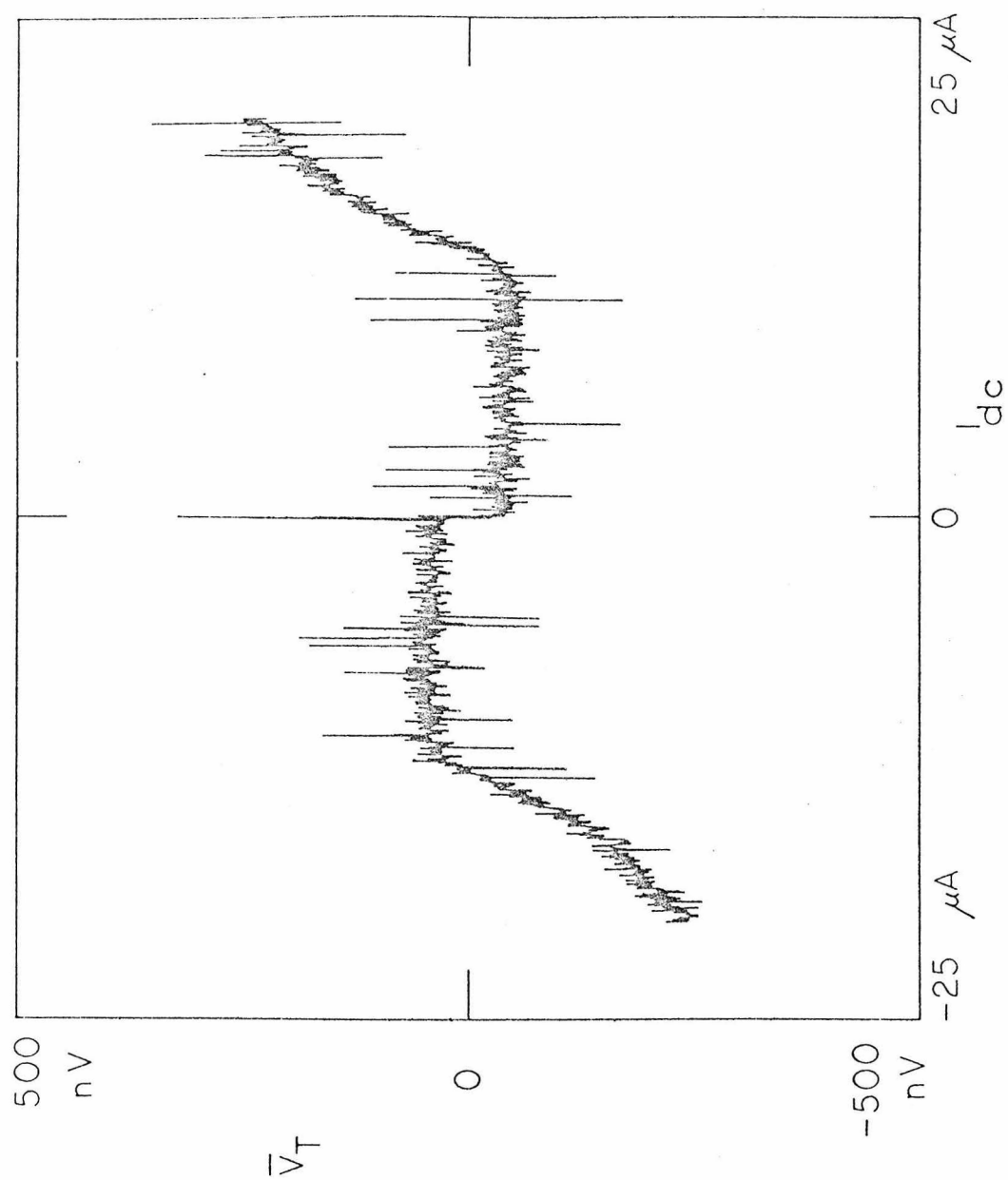


Fig. 4.3 A measurement of the time-averaged potential of the tunnel junction probe, $\bar{V}_T$, as a function of the dc bias current through the weak link, I_{dc} . This is a degenerate case of the kind of data shown in the previous figure.

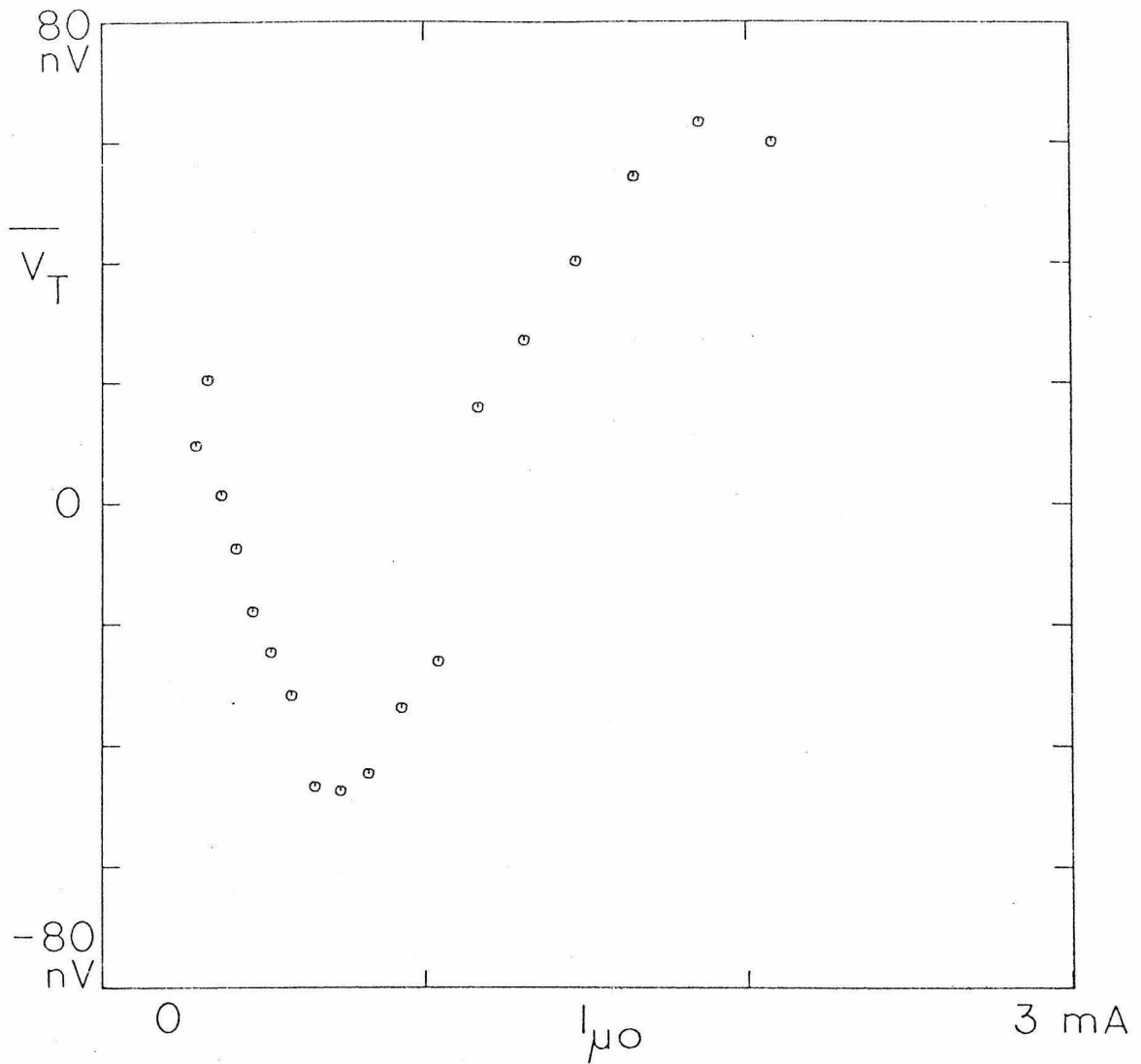


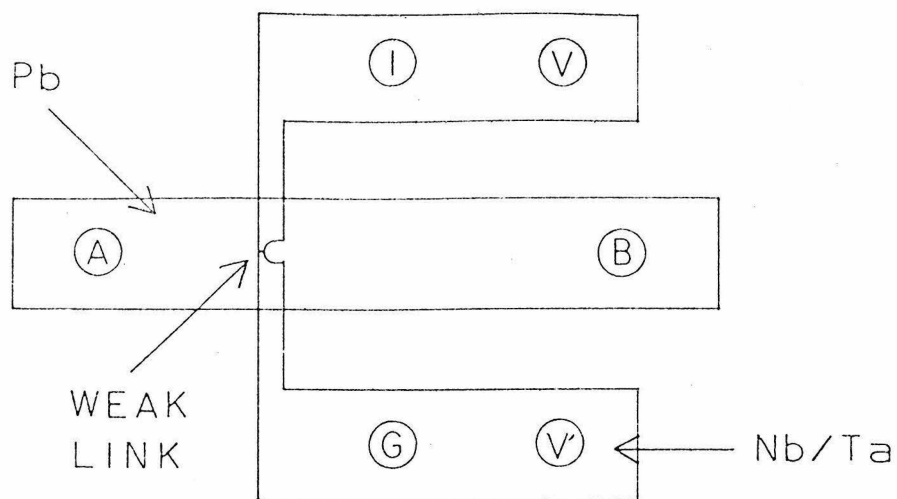
Fig. 4.4 Experimental value of the maximum time-averaged potential of the tunnel junction probe, $\bar{V}_T$, as a function of the amplitude of the microwave current in the weak link, $I_{\mu 0}$. The temperature is fixed.

in the measurements, and does not introduce any new data; those readers not interested in such details are advised to go on to Section 4.3.) Although the data presented above are quite reproducible and clearly represent some sort of physical phenomenon, this thesis has not yet established that this phenomenon actually occurs in the weak link itself. For example, a small scratch going from one side of the superconducting film to the other might conceivably have produced discontinuities in $\bar{V}_T$ if the scratch had a critical current near that of the weak link. Such possibilities need to be ruled out.

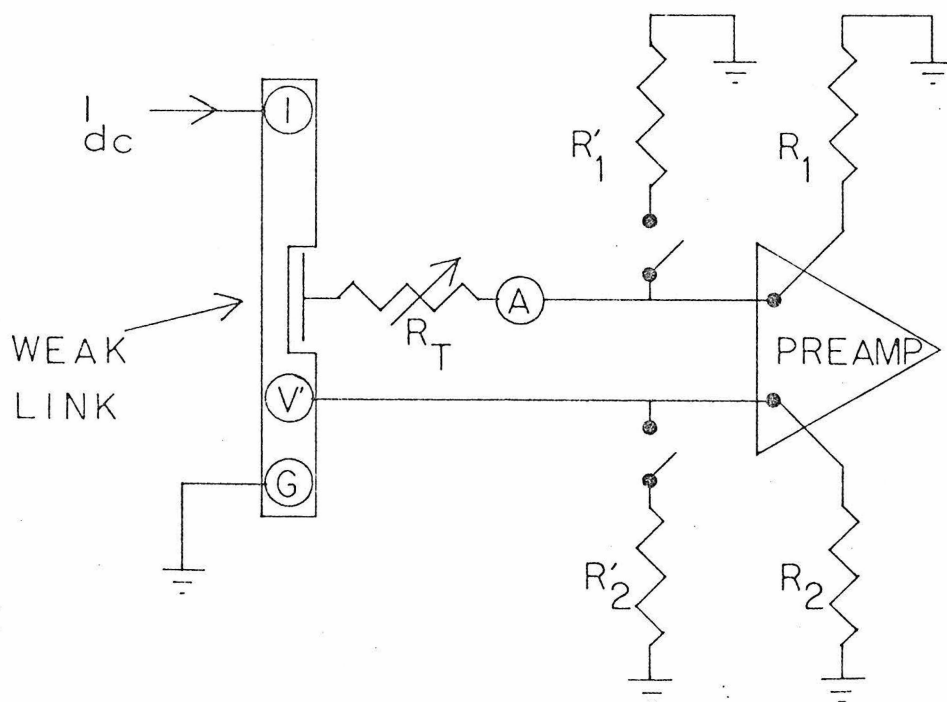
This problem is made more acute by the fact that, of all the devices fabricated and tested, only one particular device was found to be suitable for the measurements described in this chapter. The difficulties encountered in fabricating these devices have already been discussed in detail in Chapter III and this discussion need not be repeated here. For reasons to be explained shortly, it was necessary for the purposes of this experiment (but not that of Chapter III) that R_T be as small as possible and also that R_T be relatively independent of bias voltage (inelastic tunneling was preferred). Because of these restrictions only one device out of the large number fabricated was found to be acceptable. It would, of course, be desirable to repeat these measurements with a second device. However, it did not seem feasible to continue fabricating large numbers of devices using the equipment presently available in the laboratory. Consequently, an attempt will now be made to show that the phenomenon observed did indeed occur in the weak link itself and not in a scratch or the tunnel junction, etc. Once this has been established it will be argued

that the weak links (as opposed to the tunnel junctions) are highly reproducible and all have essentially the same properties, so that the measurements reported here can be assumed to be of a general nature, and not a property of this particular device. This assumption is strengthened by the fact that similar results were obtained by Yu et al (Ref. 1) and by the fact that the measurements, as will be seen, show an unmistakable similarity to the results of the calculations to be discussed in Section 4.3.

The discussion of possible spurious signals will make use of the schematic diagrams shown in Fig. 4.5. The geometrical location of the various electrical connections to the device, which were made using tinned copper wire held in place with pressed In contacts, is shown in Fig. 4.5a. (The electrical contacts are represented by circles containing letters.) Note that each bulk superconducting region (for example, the Pb electrode) had two lead wire connections. Figure 4.5b illustrates schematically the connections between the device, the current drive, and the first-stage amplifier. Here R_T represents the resistance of the tunnel junction, which might conceivably vary. R_1 and R_2 represent the internal impedance (~ 50 K ohms each) of the amplifier. R'_1 and R'_2 represent external resistors (also 50 K ohms) which could be switched into or out of the circuit so as to change the effective input impedance of the amplifier. (The reason for this will be explained later.) Note that the second input to the amplifier was connected to contact V' and so avoids the voltage drop due to I_{dc} which was developed across the wire and contacts of connection G. The material between contact V' and the weak link is super-



(a)



(b)

Fig. 4.5 Electrical connections to the weak link and tunnel junction. (a) Position of the various contacts. (b) Connections to the current drive and amplifier.

conducting and so there is no voltage drop between those two points; this is the reason for attaching two lead wires to each bulk superconducting region.

The most likely source of spurious signals is related to the fact that the joints between the In contacts and the tinned lead wires or superconducting film may have been superconducting and so have had a resistance which changed whenever their critical current was exceeded. Such a changing resistance ΔR_C in series with contact G would give rise to a voltage $I_{dc} \Delta R_C$ which constitutes a common-mode signal to the amplifier. (A common-mode signal is one in which both inputs rise by equal amounts; a differential amplifier is designed to eliminate such signals.) Although this amplifier had high common-mode rejection, such signals would not be entirely eliminated and might conceivably have been responsible for the phase-jump structures observed. The problem was complicated by the fact that the common-mode voltage to one channel of the amplifier would be attenuated by a factor of $(R_1 - R_T)/R_1$, which reduces the common-mode rejection. This would have been especially bothersome if R_T were not constant in value, in addition to not being negligible compared to R_1 .

The possibility of spurious signals due to the In contacts was completely ruled out by passing a large dc current I' through each pair of lead wires connected to the same bulk superconducting region. For example, I' was injected into contact G and extracted through contact V'. This had no effect on the weak link, since I' did not pass through the weak link, but would have drastically altered ΔR_C . This

procedure was applied to all three pairs of connections and was observed to have no effect on the measurements of $\bar{V}_T$. Note that this ruled out the possibility of any spurious signals occurring in the tinned lead wires, their external solder connections, etc., in addition to the In contacts themselves. To be more precise, the signals observed depended on I_{dc} (they showed discontinuities at a particular value of I_{dc}) but not I' , which proved that the signals originated somewhere in the superconducting film or in the device itself. Although this test was itself conclusive, any effects due to the In contacts were further ruled out by the procedures to be described below (which were carried out for other purposes).

A second possible source of error considered was that due to inductive signals picked up on the amplifier lead wires. That is, modulating I_{dc} produced changing magnetic fields which in turn generated voltages in the amplifier lead wires. First of all, $\bar{V}_T$ became zero if $I_{\mu 0}$ was set to zero. This immediately rules out inductive pick-up as the sole source of the signals measured. Inductive pick-up (which is proportional to the modulation frequency) was further ruled out by reducing the modulation frequency of I_{dc} by a factor of two, which was observed to have no effect. (The measurements were carried out at a modulation frequency of 10 Hz for the purpose of eliminating inductive pick-up.) The possible complications in these arguments which might have arisen if R_T were changing (modulated by I_{dc}) and not completely negligible compared to R_T were ruled out by a procedure to be discussed below.

A very unlikely source of spurious signals had to be considered because of the fact that the measurements were made on only one device. This was the possibility that a scratch in the Nb/Ta film functioned as a kind of weak link with a definite critical current. This possibility was immediately ruled out by the fact that any signals due to a scratch would have had to appear at contact V as well as at the tunnel junction. It is crucial in this regard to note that the hypothetical "scratch" (which the author has never observed) cannot be ruled out in the usual four-terminal measurements; only the presence of the tunnel junction probe allowed it to be ruled out here.

It has been shown so far that the signals observed did not originate in the In contacts or a "scratch" in the superconducting film, etc. The only remaining possibility is that they originated in the weak link or the tunnel junction. This does not yet rule out the possibility of spurious signals, however, for the following reason: Suppose, however unlikely it may seem, that R_T was modulated by I_{dc} or $I_{\mu 0}$. Suppose further that the entire Nb/Ta film was raised to some potential V_C , either by the voltage drop across contact G or by inductive pick-up (either of which is quite reasonable). In that case the common-mode signal V_C to one input of the amplifier would have been reduced by a factor of $(R_1 - R_T)/R_1$, whereas the second input to the amplifier would be very nearly V_C . Since R_T (hypothetically) depended on the value of I_{dc} or $I_{\mu 0}$, the voltage difference between the two amplifier inputs, which was $V_C R_T/R_1$, would also have varied. Such a voltage would have been accepted by the lock-in amplifier and could

have given rise to a spurious signal which would have been a function of I_{dc} or $I_{\mu 0}$. A moment's reflection will show that this signal ($V_C R_T / R_1$) corresponds to the voltage drop across R_T due to the nonzero current $I_T = V_C / R_1$ which flows through the tunnel junction when R_1 is finite. Thus this kind of hypothetical, spurious signal corresponds to a voltage drop developed across R_T , as opposed to the quasi-particle potential which the tunnel junction was designed to probe. In any event, a spurious signal of this kind would have been inversely proportional to R_1 , the amplifier input impedance. By switching in resistors R_1' and R_2' , the effective input impedance of the amplifier was decreased by a factor of two, which would have increased the size of such spurious signals by a factor of two. The fact that switching in R_1' and R_2' had no effect on the measurements proves that the values of $\bar{V}_T$ measured were not due to spurious signals of this sort. It also indicates that the nonzero value of R_T / R_1 does not complicate the discussion of other possible spurious effects already given above.

Several of the possible sources of spurious effects discussed above depended upon the possibility of R_T being modulated in some way. These possibilities were further ruled out by the fact that the data were taken with a device in which the tunneling was highly inelastic (see Chapter III) and consequently R_T was quite independent of bias voltage. As a further check on the various possible spurious effects discussed above, the amplifier, which had a common-mode rejection adjustment, was deliberately unbalanced so as to increase any common-mode errors. This also had no effect on the data.

It has now been established that the structure in $\bar{V}_T$ originated in the weak link and not the tunnel junction, In contacts, scratches, etc. All the likely sources of spurious signals were ruled out by several independent arguments. The weak links, as opposed to the tunnel junctions, could be fabricated in a highly reproducible manner and so the data taken on this device can be assumed to be typical of all weak links. In addition, the data are similar to those reported by Yu et al (Ref. 1), except for the fine structure observed here which could easily have been overlooked (since it occurs only in a very narrow range of $I_{\mu 0}$) or destroyed (by the strong coupling to the ohmic contact) in their experiment. Furthermore, it will be seen that there are unmistakable similarities between the fine structure observed here and the calculations to be carried out in Section 4.3. For all of these reasons, it was decided that these measurements should be reported in this chapter, even though it would obviously be desirable to fabricate and test more devices if the proper equipment were available. Those readers who feel that these data should be ignored can certainly do so; the calculations to be presented in the next section can then be taken to be a theoretical prediction rather than a confirmation.

4.3 Theory and Calculations

First of all, there is some information about μ_{qp} which can be obtained from the symmetry (or assumed symmetry) of the weak link which is independent of any particular theoretical model. This argument can perhaps be somewhat confusing because I_{dc} breaks the symmetry, and so it

will be presented here in somewhat unnecessary detail. The argument shows that any change $\Delta\mu_{qp}$ in μ_{qp} due to changing I_{dc} from zero must be antisymmetric about the center of the weak link (and therefore zero at the center). (Recall that $\bar{V}_T$ measures only changes in μ_{qp} .)

Suppose that an observer is looking at the weak link from the side with his eye in the plane of the substrate and film. Let the distance along the direction of current flow be z , with $z = 0$ at the center of the weak link. Suppose that a current I_{dc} is flowing from left to right (in addition to the microwave current I_μ) and causes a $\Delta\mu_{qp}$ which is symmetric; $\Delta\mu_{qp}(-z) = \Delta\mu_{qp}(z)$. It is experimentally known that, if the direction of I_{dc} were to be reversed, $\Delta\mu_{qp}$ would change sign. Now suppose that the observer changes his position so that he is looking at the weak link from the other side. (Alternatively, the observer could remain at the same position and turn himself upside down, assuming that he could continue to observe in that orientation.) The situation will be the same as before, except that the direction of current flow appears to have been reversed. This is because the properties of the weak link itself are symmetric. The observer can, if he likes, define a new coordinate system z' as before, where $z' = -z$.

The point is that the direction of current flow appears to have been reversed so that $\Delta\mu_{qp}$ must change sign. Since $\Delta\mu_{qp}$ was assumed to be symmetric, it makes no difference whether its argument is z or z' . Consequently, the sign reversal gives $\Delta\mu_{qp}(z) = -\Delta\mu_{qp}(z)$, or $\Delta\mu_{qp}(z) = 0$ for all z . If a general form had been assumed for $\Delta\mu_{qp}(z)$,

it follows that the symmetric part would be zero and so $\Delta\mu_{qp}$ must be antisymmetric about the center of the weak link. This result forms the basis for the comment already made that the observed value of $\bar{V}_T$ must be dependent upon the tunnel junction being slightly off center. Note that this argument makes use of the experimental fact that $\Delta\mu_{qp}$ reverses sign when I_{dc} is reversed; thus it depends on more than just the geometrical symmetry of the weak link.

Before introducing the particular model which has been investigated in detail, it might be useful to discuss what general features any prospective theory would have to have in order to explain the data. First of all, the phenomenon involves nonequilibrium quasiparticles, which makes the standard Ginzburg-Landau theory more or less useless. From eq. (4.1) it can be seen that a nonzero value of μ_{qp} requires that the supercurrent density have a nonzero divergence. Under equilibrium conditions the supercurrent has zero divergence; thus the data indicate that the superfluid as well as the quasiparticles must be in nonequilibrium. In particular, the Ginzburg-Landau theory, as discussed in Chapter I, predicts that under equilibrium conditions the amplitude of ψ drops off like e^{-Kz} into the weak link region from the strongly superconducting regions on each side of the weak link. (This form for ψ is only valid once the value of ψ has become small enough to linearize the Ginzburg-Landau equation.) Here K is a real constant discussed at some length in Chapter III and z is the distance into the normal region. In addition, the phase of these "exponential tails" is independent of position. It can readily be

shown (see Appendix B) that such a form for Ψ gives a supercurrent in the weak link region which has zero divergence (and so $\mu_{qp} = 0$). (This argument does not apply to the superconducting-normal boundary itself.) It might therefore be suspected that the form of the exponential tail of Ψ must be altered in order to account for the phenomena observed.

The particular model to be discussed here will suggest that the exponential tail coming in from the left-hand side of the weak link should have the form

$$\Psi_L(z) = \Psi_{L0} e^{-Kz} e^{i(\nabla\theta)z} \quad (4.2)$$

where $\Psi_L(z)$ is, more precisely, the order parameter induced in the weak link by the superconducting region on the left. (This form is only valid far enough away from the superconducting-normal boundary that the Ginzburg-Landau equation has become linear.) Here Ψ_{L0} is the order parameter in the bulk superconductor on the left (a complex constant), K is a real constant which determines the exponential decay length, z is the distance into the normal region, and $\nabla\theta$ is a parameter which is also a real constant. An analogous equation gives $\Psi_R(z)$, the order parameter induced in the weak link by the superconducting region on the right. The total order parameter $\Psi(z)$ in the weak link region is then just the sum $\Psi(z) = \Psi_R(z) + \Psi_L(z)$. Under equilibrium conditions, the standard Ginzburg-Landau theory gives the parameter $\nabla\theta$ equal to zero. (In other words, the gradient in the phase of the exponential tails Ψ_L or Ψ_R is ordinarily zero.) As will be discussed in more detail shortly,

the supercurrent calculated from this form for $\Psi(z)$ in Appendix B gives a supercurrent with a nonzero divergence as required, provided that the parameter $\nabla\theta$ is itself a weak function of z .

Although $\nabla\theta$ is obviously the gradient in the phase of the exponential tail Ψ_L (and also Ψ_R), it is not the gradient in the phase of Ψ itself. To be more precise, suppose that Ψ is written in the form $\Psi = \rho(z) e^{i\phi(z)}$, where ρ and ϕ are real. Then $d\phi/dz$ will, in general, not be zero, even if $\nabla\theta = 0$. This is because the phases of the complex constants Ψ_{L0} and Ψ_{R0} need not be equal. It is important to keep this distinction in mind, since, as was discussed in Chapter I, supercurrents are usually associated with nonzero gradients in the phase of Ψ . Thus, even under equilibrium conditions, the gradient in the phase of Ψ is not zero. (Recall that a supercurrent flowing in a superconductor does not constitute a nonequilibrium condition, even though a current in a normal metal always implies nonequilibrium.) The parameter $\nabla\theta$ introduced here is not the same as the gradient in the phase of Ψ . Because of the fact that $\nabla\theta = 0$ under the usual equilibrium conditions, the idea that the gradient in the phase of Ψ_L or Ψ_R (as opposed to Ψ itself) is nonzero is a rather novel suggestion. It will be convenient to introduce $\Delta\phi$ as the phase difference in Ψ between one side of the weak link and the other. (ϕ is essentially constant in the strongly superconducting regions on either side of the weak link.) In other words, $\Delta\phi$ is the difference in phase of the two complex constants Ψ_{R0} and Ψ_{L0} . Note that $\Delta\phi$ need not be zero, even if $\nabla\theta$ is zero; $\Delta\phi$ and $\nabla\theta$ are two completely independent variables, both of which affect the form of Ψ .

The idea that $\Psi_L(z)$, the exponential tail, might contain a gradient in its phase was first proposed by Yu et al (Refs. 3,10,11) and used in an attempt to explain the Wyatt effect. The author first became interested in such a possibility when it occurred to him that eq. (4.2) might be justified using a generalized form of the BCS theory. (This was before the present experiment had begun.) In the model suggested by Yu et al, the parameter $\nabla\theta$ was proportional to the time integral of the voltage, as might be expected from the phenomenological London equations. (In the London theory, the superfluid was considered to be a classical fluid with no resistance to flow, so that the effect of an electric field would be to accelerate the superfluid; as a result, the superfluid velocity, and thus $\nabla\theta$, would be 90° out of phase with the applied voltage.) The mechanism to be described here suggests that $\nabla\theta$ is proportional to the voltage itself; thus the physics is quite different. Before attempting to motivate eq. (4.2), it should be pointed out that the fact that eq. (4.2) will be seen to give results which are in agreement with this experiment does not necessarily mean that other explanations might not do just as well. In fact, the author can think of several other mechanisms which might be appropriate. The decision was made to analyze the results of this experiment using eq. (4.2) mostly because the calculations which result are reasonably tractable. The question of other possible explanations will be discussed further at the end of this section. In particular, the model suggested by Yu et al can be shown to give results for this experiment which are the same as the mechanism to be described here, to within a constant.

The calculation of $\bar{V}_T$ based on the mechanism about to be discussed is, unfortunately, rather long and complicated. It appears to be impossible to take a difficult and complex analysis and make it look simple or transparent. For this reason, the remainder of this section can probably be read only with a certain amount of difficulty, even by an expert in this field. This is perhaps the case, even though the algebraic details are given only in Appendix B. The calculations consist of a large number of essentially unrelated steps. In order to make the calculations as understandable as possible, an outline of all the various steps and the purpose of each will be given shortly. In addition, the text will explicitly point out when one step has been finished and the next is about to begin. This appears to be all that can be done in order to make the calculations as understandable as possible. Those readers who are not interested in the calculations can, of course, skip all but the final results.

The analysis will begin with the assumption that, to a first approximation, the effect of the microwave voltage across the weak link is to displace the entire electron distribution in momentum space in accordance with Ohm's law. An intuitive argument will then be presented which suggests that such a displacement of the electrons causes a nonzero gradient in the phase of the exponential tails ψ_L and ψ_R (the parameter $\nabla\theta$) which would otherwise not be present. This nonzero value of $\nabla\theta$ will then be used to calculate a new relationship between the phase difference $\Delta\phi$ across the weak link (not to be confused with the parameter $\nabla\theta$) and the supercurrent. This current-phase relationship will then be solved

to give $\bar{V}_T$ as a function of the time-average of $\Delta\phi$, ϕ_0 , and the microwave voltage across the weak link. (Although ϕ_0 might logically be written as $\Delta\phi_0$, the Δ will not be included; this is in accord with the standard notation used in this field.) The calculations would end at that point, except for the fact that the weak link is current biased, rather than voltage biased. Thus it is necessary to solve for ϕ_0 and the microwave voltage in terms of the dc and microwave currents.

Discontinuities in ϕ_0 as a function of the dc bias current will then be shown to exist. This is due to a complex series of interactions based on the fact that the microwave impedance Z (discussed in Chapter II) is a function of ϕ_0 . This situation is described by a set of nonlinear, coupled equations which must be systematically solved. These discontinuities in ϕ_0 , combined with the expression for $\bar{V}_T$, will be seen to give results which are in reasonable agreement with the data. The nature of the various steps in the analysis outlined above should become clearer as the calculations proceed.

The first step in the analysis is the suggestion of a possible physical mechanism which might be expected to produce a nonzero value of the parameter $\nabla\theta$. In order to put the following discussion in perspective, it should be pointed out that the mechanism about to be suggested is a new idea in the sense that such effects are not contained in the usual BCS theory or time-dependent Ginzburg-Landau theory. This mechanism is expected to be important only at the boundary between superconducting and normal metals under nonequilibrium conditions. Consequently, the ideas about to be discussed are not very closely

related to the usual properties of bulk, equilibrium superconductivity. In addition, it turns out that the results of the calculation of $\bar{V}_T$ will depend on the form of Ψ given in eq. (4.2), but are fairly insensitive to the physical mechanism responsible for the value of $\nabla\theta$. For this reason the mechanism about to be discussed is not essential to the analysis of the experiment; the agreement between the calculations and the measurements primarily indicates that the form of eq. (4.2) is correct, regardless of the mechanism responsible for $\nabla\theta$. Consequently, the mechanism is not meant to be a unique explanation of the experimental results, and in some sense its discussion is not necessary for the remainder of the chapter. It is discussed here primarily because it is a novel idea which (to the author's knowledge) has not previously been considered in the literature.

An attempt will now be made to show that eq. (4.2) is at least plausible, based on the nonequilibrium electron distribution shown schematically in Fig. 4.6. The solid circle shown in Fig. 4.6 is meant to represent the Fermi surface in momentum space under equilibrium conditions. The dashed circle represents the electron distribution displaced a distance $\vec{d}$ from its equilibrium position by the effect of an electric field $\vec{E}$. (In this experiment, $\vec{E}$ is produced by the microwave voltage across the weak link.) As discussed in Chapter II, the measurements of the microwave impedance strongly suggest that, under appropriate conditions, the quasi-particle conductivity at microwave frequencies in the center of a proximity effect weak link is, to a first approximation, given by Ohm's law; any effects due to superconductivity appear to be superimposed on top of the ohmic current as a

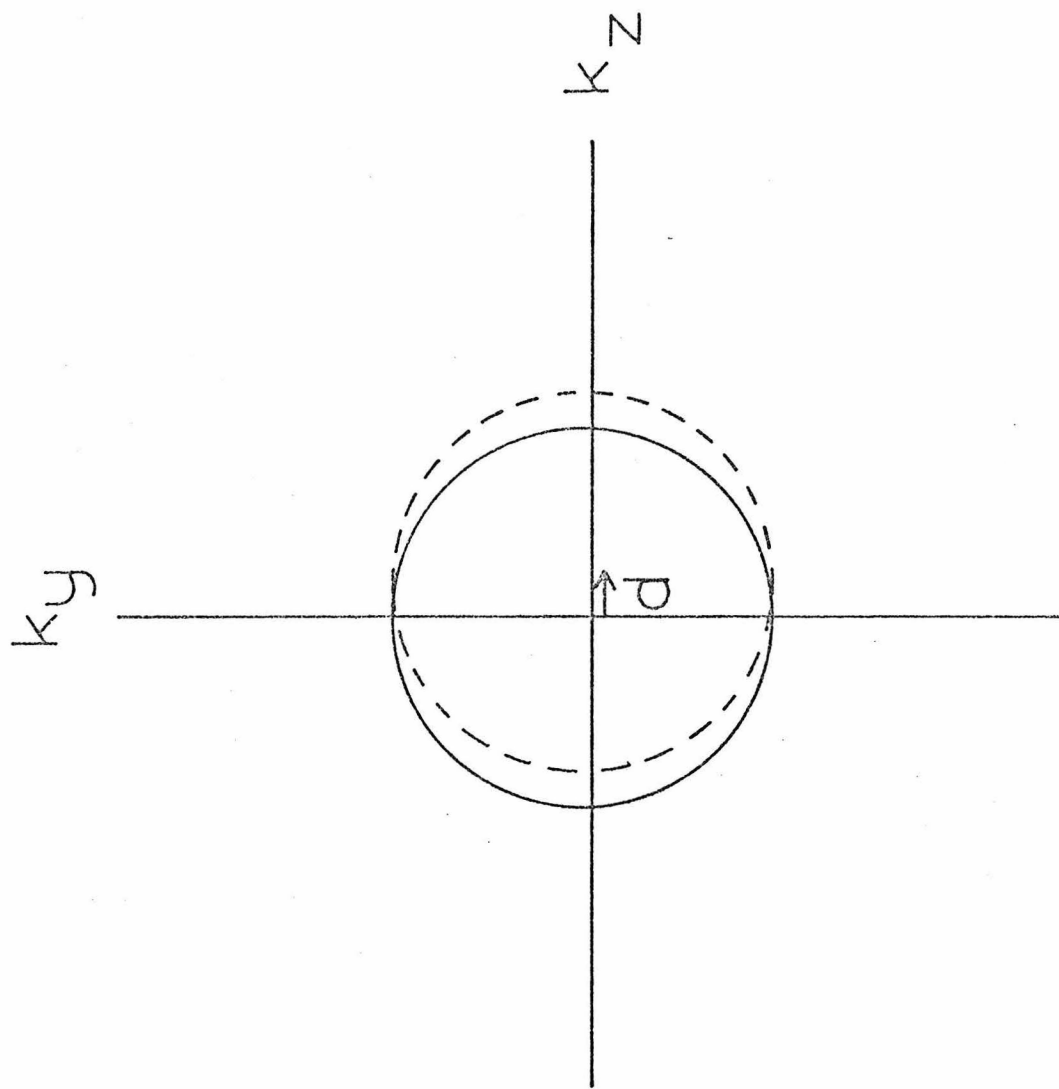


Fig. 4.6 Displacement of the overall electron distribution in momentum space by an electric field.

small perturbation. Thus the value of the displacement $\vec{d}$ is related by Ohm's law to the applied voltage. The basic concept behind the physical mechanism about to be suggested is that because the electron distribution is displaced in momentum space as shown in Fig. 4.6, the superconducting state has a tendency to form about this displaced distribution rather than the position of the equilibrium Fermi surface. This idea will now be discussed in some detail.

Under equilibrium conditions, the superconducting wavefunction is built up out of momentum states chosen symmetrically about the Fermi surface. In fact, all the electron-electron interaction which is responsible for the formation of Cooper pairs occurs within a distance of the Fermi surface which corresponds roughly to an energy of Δ (the energy gap parameter) above or below the Fermi energy. If the overall electron distribution is displaced, as in Fig. 4.6, it seems likely that the superconducting state would then have a tendency to be centered around the displaced Fermi surface in order to maintain the pairing interaction (since that is the only region of momentum space where such pairing can occur). Since $\Delta \ll kT$ in the center of the weak link, the formation of Cooper pairs necessary for the construction of the superconducting state should constitute only a small perturbation on the distribution shown in Fig. 4.6, so that the overall distribution of electrons is still approximately given by Ohm's law. By constructing the superconducting state centered around the displaced Fermi surface, the binding energy due to the formation of pairs can be increased (that is, maintained at its equilibrium value), but at the expense of the kinetic energy

which is made larger than would be the case if the superconducting state were centered around the position of the original equilibrium Fermi surface. Thus the free energy of the system might be expected to be minimized by a displacement of the superconducting state which represents a compromise between kinetic and binding energies. In other words, there may exist a self-consistent wavefunction which minimizes the free energy, given the fact that the overall electron distribution has been displaced; such a wavefunction may correspond to a nonzero value of $\nabla\theta$, since, as will be seen, $\nabla\theta \neq 0$ displaces the superconducting state in momentum space. An attempt to make these ideas quantitative will be described shortly.

A superconducting state which has been displaced in momentum space is equivalent, in a uniform, bulk superconductor, to the usual BCS wavefunction with a nonzero gradient in the phase, $\nabla\theta$. It will be convenient to denote the usual BCS wavefunction corresponding to a constant gradient in the phase by $\psi_{\text{BCS}}(\nabla\theta)$. (At absolute zero, there are no quasi-particle excitations and a wavefunction with $\nabla\theta \neq 0$ simply corresponds to shifting the entire electron system over in momentum space, thus producing the usual supercurrent. At finite temperatures the effect of $\nabla\theta \neq 0$ is similar, although somewhat more complicated due to the fact that the quasi-particle distribution is also altered.) In the usual BCS theory the free energy is minimized by $\psi_{\text{BCS}}(0)$, and nonzero values of $\nabla\theta$ only arise as a result of the application of boundary conditions, such as is the case for a closed superconducting loop. The idea is that, for the electron distribution shown

in Fig. 4.6, the free energy may well be minimized by a nonzero value of $\nabla\theta$, even in the absence of boundary conditions (or, more specifically, when the boundary conditions on the wavefunction are specified at a single plane, such as the superconducting-normal interface at one edge of the weak link). This would account for the extra phase gradient in eq. (4.2), as desired. (It should immediately be pointed out that the value of $\nabla\theta$ in a bulk superconductor is ordinarily fixed by the boundary conditions, so that these ideas do not directly apply to bulk superconductors. This will be discussed in more detail later.) Once again, an attempt will be made shortly to make these ideas more quantitative.

The mechanism proposed here assumes that the exponential tail of the order parameter (eq. (4.2), for example) will have that form which minimizes the free energy. The ideas discussed above suggest that this occurs for $\nabla\theta \neq 0$, due to the displacement of the Fermi surface in accordance with Ohm's law. One apparent problem which needs to be discussed has to do with the fact that the electrostatic potential (when a voltage has been applied) varies as a function of position inside the exponential tail. This might be expected to cause the quantum phase to precess at different rates at different positions in accordance with Josephson's equation for the time dependence of the quantum phases (see eq. (4.17), below), thus continuously changing the value of $\nabla\theta$ away from that value which minimizes the free energy. This Josephson equation was meant to be applied to bulk superconducting regions in which the electrostatic potential is essentially constant throughout that

region. On the contrary, it is assumed here that the relaxation time of the superconducting state is so short that the actual superconducting state over the short distances involved in the weak link region is always (at least approximately) the one which corresponds to the minimum free energy; as has already been suggested, this corresponds to a certain value of $\nabla\theta \neq 0$ and the phase is not expected to precess at different rates at different positions. This question will be discussed in greater detail when a comparison is made to the model proposed by Yu et al (Refs. 3,10,11). As will be seen, the model proposed by Yu et al corresponds to the opposite limit, where departures from equilibrium on the part of the superfluid give rise to the important effects. As has already been mentioned, the mechanism suggested here also depends on the fact that the boundary conditions on Ψ_L or Ψ_R leave $\nabla\theta$ a free parameter, which is not the case in a bulk superconductor. These topics will be discussed further at the end of this section.

A physical mechanism has just been suggested which might be expected to give a nonzero value of the parameter $\nabla\theta$ in eq. (4.2). It was assumed that an electric field has the effect of displacing the electron distribution in momentum space, and that the superconducting state which has the lowest free energy consistent with this displacement corresponds to a nonzero value of the parameter $\nabla\theta$. In addition, it was assumed that the actual form of Ψ will be very nearly equal to that which minimizes the free energy. The author is currently involved in an attempt to make the preceding intuitive ideas quantitative by performing a modified BCS variational calculation, using the appropriate

microscopic wavefunctions. The approach is based upon the assumption that the quasi-particle distribution is given by

$$f(\vec{k}) = f_0(\vec{k} - \vec{d}) \quad (4.3)$$

where $f_0(\vec{k})$ is the equilibrium (Fermi-Dirac) distribution; this is required by Ohm's law. The variational form for the wavefunction is chosen to be $\psi_{\text{BCS}}(\nabla\theta)$, with $\nabla\theta$ considered a free parameter; the correct value of $\nabla\theta$ is found by minimizing the free energy which results. (Once again, the usual BCS treatment using the equilibrium distribution $f_0(\vec{k})$ gives $\nabla\theta = 0$ for the minimum free energy.) The calculation is made algebraically feasible by assuming that d is small and linearizing. By use of an appropriate gauge transformation, the calculations can be cast in a form which is similar to the usual BCS calculation of the temperature-dependent penetration depth. Calculations of this sort are analogous to the theories described in Chapter III, in which non-equilibrium quasi-particle distributions were found to affect the value of the energy gap Δ . In this particular instance, however, the non-equilibrium quasi-particle distribution is expected to change the value of the parameter $\nabla\theta$. This generalized BCS calculation has not yet been finished, primarily because of some difficulties having to do with the fact that $f(\vec{k})$ governs only the quasi-particles and not all the electrons, as it does in a normal metal. There appear to be several ways of avoiding this difficulty, and it is hoped that this sort of calculation can eventually be completed. For the present time, however, the intuitive discussion presented above can only be considered to show the

plausibility of eq. (4.2) for the conditions of this experiment.

For small values of d , the value of $\nabla\theta$ produced by the mechanism suggested above can be expected to be proportional to d , which in turn is proportional to $\vec{E}$ or the voltage across the weak link. Since the ideas presented above are not quantitative, the exact dependence of the parameter $\nabla\theta$ on position within the weak link cannot be determined. In order that the necessary calculations can proceed, it was assumed that $\nabla\theta$ is approximately independent of position (a slowly varying function of z). It should be emphasized that this assumption may be rather crude and has only been made for the sake of expediency. (The nature of this assumption is discussed more thoroughly in Appendix B.) For this reason, as well as the fact that d was assumed to be small enough that $\nabla\theta$ is linear in d , the results of the calculations below should be expected to show only qualitative agreement with the data, especially for large microwave currents. (It was assumed by the author that a qualitative comparison with the data would be better than no comparison at all.) In any event, a physical mechanism which might produce a nonzero value of $\nabla\theta$ in eq. (4.2) has now been suggested. The calculations given in Appendix B show that the supercurrent under these conditions has a nonzero divergence, as required. This completes the first step in the analysis. Although the form of Ψ given by eq. (4.2) has not been derived or proven in any rigorous way, the calculation of $\bar{V}_T$ will continue, based on eq. (4.2). The idea is to determine the consequences of such a form for Ψ and to compare them with the data. As has been mentioned, the results of the calculation are fairly independent of the physical

mechanism responsible for the nonzero value of $\nabla\theta$.

The next step in the analysis is to use the form of the order parameter shown in eq. (4.2) to calculate the dependence of the supercurrent through the weak link on the phase difference $\Delta\phi$ across the weak link. (Recall that $\Delta\phi$ and $\nabla\theta$ are completely independent variables.) The nonzero value of $\nabla\theta$ gives rise to a modified current-phase relation. This calculation is straightforward and the details are given in Appendix B. The dominant (largest) term was found to give the result

$$I_S = I_C \sin(\phi_R - \phi_L - \nabla\theta L) \quad (4.4)$$

Here I_S is the instantaneous supercurrent, I_C is the usual critical current (a constant), ϕ_R and ϕ_L are the phases of the order parameter in the bulk superconducting regions on the right and left sides of the weak link, and L is the length of the weak link ($\sim 0.5\mu$); ($\phi_R - \phi_L = \Delta\phi$.) Once again, $\nabla\theta$ is (approximately) independent of position but proportional to v , the instantaneous voltage across the weak link.

Equation (4.4) has been interpreted as being a modification of the dependence of the supercurrent on the value of $\Delta\phi$ (the current-phase relation). An alternative interpretation is to view $\nabla\theta$ as changing the relation between the applied voltage and the "effective" phase, while leaving the dependence of the supercurrent on the phase unaltered. Physically, what happens is that the gradient in the phase of the exponential tails ψ_L and ψ_R (that is, $\nabla\theta$) simply changes the phase difference between the wavefunctions at the point (or points) where they interfere from their phase difference at the edges of the weak link. The basic

interference effect which produces the supercurrent is unaltered. From this point of view it is the voltage-phase relation which has been altered, rather than the current-phase relation. However, $\Delta\phi$ is defined as the phase difference between the superconducting regions at the edges of the weak link, not at the point where ψ_L and ψ_R interfere. (Actually, they interfere throughout the entire weak link region.) Thus $\Delta\phi$ is not altered by $\nabla\theta$ and so, strictly speaking, the voltage-phase relation is not altered, since the word "phase" here means $\Delta\phi$. Viewing the supercurrent as a function of the phase difference across the weak link, it is thus the current-phase relation which has been altered, although either interpretation has its advantages.

The calculation (Appendix B) of the current-phase relation, eq. (4.4), completes this step in the analysis. Before going on to the next step it might be appropriate to comment on the nature of the results which would have been obtained if the model proposed by Yu et al (Refs. 3,10,11) had been used, rather than the mechanism suggested here. (Although Yu et al (Ref. 1) recently published the results of an experiment very similar to the one described in this chapter, they did not analyze the results using their previously published (Refs. 3,10,11) model.) Recall that the equations proposed by Yu et al require that the parameter $\nabla\theta$ be proportional to the time integral of the voltage $v(t)$ across the weak link, whereas the mechanism discussed in this thesis gives $\nabla\theta$ proportional to v itself. Although the physical mechanism responsible for the final results in the two cases is totally different, the calculated form of $\bar{V}_T$ for this experiment can be shown

to be the same to within a constant. (Since the calculations using the model by Yu et al are very similar to the calculations using the mechanism proposed here, only the latter will be carried out in detail below. The method is, however, very easily generalized to the other case.) This is unfortunate, since this experiment is therefore unable to distinguish between these two mechanisms. The physical differences between the two mechanisms would allow them to be distinguished by other experiments, at least in principle, however. The physical differences between these two mechanisms, as well as the conditions under which one or the other might be expected to dominate, will be discussed in some detail at the end of this section. As has been pointed out, the results of the calculation of $\bar{V}_T$ are fairly independent of the mechanism responsible for the nonzero value of $\nabla\theta$, so that the experimental results should be viewed as a test of the form of eq. (4.2), rather than of any particular mechanism for $\nabla\theta$.

The next step in the analysis is the use of eq. (4.4) to calculate the value of $\bar{V}_T$ to be expected for the conditions of this experiment. It is convenient to define $\bar{I}_S(\nabla\theta)$ as the value of the time-average of I_S as a function of the parameter $\nabla\theta$. As shown in Appendix B, the time-averaged divergence of the supercurrent (which is proportional to $\bar{V}_T$) is proportional to $[\bar{I}_S(\nabla\theta) - \bar{I}_S(0)]$. A lengthy calculation of $\bar{I}_S$ as a function of $\nabla\theta$ gives the following result for $\bar{V}_T$:

$$\begin{aligned} \bar{V}_T = \gamma I_c \sin(\phi_0) [& (J_0(\epsilon' v_0) - 1) J_0\left(\frac{2ev_0}{\hbar\omega}\right) \\ & + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(\epsilon' v_0) J_{2k}\left(\frac{2ev_0}{\hbar\omega}\right)] \end{aligned} \quad (4.5)$$

Here ϵ' is defined by the relation $L\dot{\nabla}\theta = \epsilon'v$, v_0 is the amplitude of the microwave voltage, ϕ_0 is the time-averaged value of $\phi_R - \phi_L$ (which is determined by the dc bias current), γ is a constant relating V_T to μ_{qp} (the value of γ depends on the asymmetry of the tunnel junction), ω is the frequency of the microwave voltage, and J_N are Bessel functions of integer order. (ϕ_0 should not be confused with the flux quantum $h/2e$, which is sometimes denoted by ϕ_0 .) Note that $\bar{V}_T$ is now given as a function of ϕ_0 and v_0 .

It will be convenient to define x and y by

$$x = 2ev_0/\hbar\omega \quad (4.6)$$

$$y = \epsilon'v_0 \quad (4.7)$$

(x and y are both dimensionless.) For $y \ll 1$, eq. (4.5) can be expanded in power series (see Appendix B) to give

$$\bar{V}_T = -\gamma I_C \frac{y^2}{4} \sin(\phi_0) [J_0(x) + J_2(x)] \quad (4.8)$$

where terms of order y^4 or higher have been dropped. Equation (4.8) probably looks rather peculiar (to an expert in this field) because J_2 appears to have been kept, whereas higher order Bessel functions in x appear to have been dropped. It must be emphasized that eq. (4.8) is not an expansion in x ; only these two Bessel functions remain in the calculations at this point, and the results are valid even for $x \gg 1$. This is important to keep in mind, since x is frequently larger than one in the experiments performed. In addition, not only is J_2 not necessarily small compared to J_0 under such conditions, but most of the

interesting results will be seen to come from the J_2 term rather than the J_0 term. Equation (4.8) has been left in its present form, even though the recurrence relations for Bessel functions give $J_0(x) + J_2(x) = 2J_1(x)/x$; this was done to emphasize the fact that the two terms have different origins and that, because of the approximations made in the theory, their coefficients may differ from unity. The present form will be seen to be useful for other reasons as well. Equation (4.8) gives $\bar{V}_T$ in terms of the time-averaged phase difference, ϕ_0 , and the amplitude of the microwave voltage, v_0 . (Given v_0 , x and y are both immediately determined.) This concludes the present step in the analysis.

If the weak link were voltage biased rather than current biased, the analysis would now be finished. This is because v_0 would then be a free parameter (its value could be experimentally fixed) and the value of ϕ_0 could be determined in a trivial way from v_0 and the dc bias current I_{dc} , which is also a free parameter. (ϕ_0 could be determined directly from eq. (4.9), as will be discussed below.) However, the weak link is very nearly current biased at microwave frequencies, as was discussed in Chapter II. As a result, the independent parameters are I_{dc} and $I_{\mu 0}$, rather than I_{dc} and v_0 . Thus, in order to evaluate eq. (4.8) for $\bar{V}_T$, it is necessary to solve for v_0 and ϕ_0 in terms of I_{dc} and $I_{\mu 0}$. This process, which turns out to be equivalent to solving a set of nonlinear coupled equations, will form the next step in the analysis.

Before beginning the next step in the analysis, it may be useful to consider first those special conditions under which ϕ_0 and v_0 can

immediately be determined from I_{dc} and $I_{\mu 0}$ without having to solve the full set of nonlinear coupled equations. In fact, the only situation (for reasons to be discussed shortly) in which ϕ_0 and v_0 cannot be immediately determined occurs when v_0 causes x to be near one of the zeros of $J_0(x)$. (In other words, if $J_0(2ev_0/\hbar\omega) \approx 0$.) In order to determine v_0 from $I_{\mu 0}$, it is obviously necessary to know the value of the microwave impedance Z of the weak link. It was shown in Chapter II that, under appropriate conditions, Z is very nearly equal to R_N , the resistance of the weak link in the normal state, for bias currents less than the critical current. (This is true only under certain conditions, such as when the critical current is sufficiently small; these conditions are well satisfied in the present experiment.) R_N is a constant and can readily be determined from dc measurements. If x is not too near one of the zeros of J_0 , then it is an excellent approximation to simply take $v_0 = I_{\mu 0} R_N$. Having determined v_0 , the value of ϕ_0 must now be determined. In order to determine ϕ_0 , it will be necessary to find the relation between the time-averaged supercurrent and ϕ_0 when a microwave voltage is applied across the weak link. Under these conditions it is well-known (and explicitly calculated in Appendix B) that the maximum dc supercurrent is limited by an effective critical current $\tilde{I}_c$ given by $\tilde{I}_c = I_c J_0(x)$. In other words, the time-averaged supercurrent $\bar{I}_s$ is then given by

$$\bar{I}_s = \tilde{I}_c \sin(\phi_0) = I_c \sin(\phi_0) J_0(x) \quad (4.9)$$

The experiments in this chapter were performed with zero dc voltage across the weak link. This is only possible for $|I_{dc}| \leq \tilde{I}_c$, and further

requires that $I_{dc} = \bar{I}_s$ (see Appendix B). Equation (4.9) can then be solved for ϕ_0 , with the result that $\phi_0 = \sin^{-1}[I_{dc}/I_c J_0(x)]$. Here x , which is proportional to v_0 , is known, since v_0 has been approximated by $R_N I_{\mu 0}$. Both v_0 and ϕ_0 have now been determined from the independent parameters I_{dc} and $I_{\mu 0}$ for the special case of x not too near a zero of $J_0(x)$.

The values of v_0 and ϕ_0 determined above for x not too near a zero of J_0 can now be used to evaluate $\bar{V}_T$ from eq. (4.8). Note that, from eqs. (4.8) and (4.9), $\bar{I}_s$ and $\bar{V}_T$ are both proportional to $\sin(\phi_0)$, and so are proportional to each other if v_0 and thus x and y do not vary "significantly" as a function of ϕ_0 . (This will be seen to be true only if x is not too near a zero of J_0 .) Thus a plot of $\bar{V}_T$ as a function of $\bar{I}_s$ should be a straight line. The data shown in Fig. (4.1) are indeed a straight line, as was the vast majority of the data observed in this experiment, and all of the data observed in the previous experiment of Yu et al (Ref. 1). Consequently, the shape of most of the data observed can be seen to be consistent with the model used in the calculations, without having to solve the coupled nonlinear equations. This simple situation, however, breaks down if x is too near a zero of $J_0(x)$, as might be suspected from the result obtained above, that $\phi_0 = \sin^{-1}[I_{dc}/I_c J_0(x)]$. (This expression is not well defined if $J_0(x) = 0$.)

When x is near one of the zeros of J_0 , it is no longer sufficiently accurate to assume that $Z = R_N$. The reason for this is as follows: Suppose that the approximation $v_0 = R_N I_{\mu 0}$ causes a certain

error δx in the value of x , which in turn causes an error δJ_0 in J_0 . If x is far from any zero of J_0 , then the fractional error $\delta J_0/J_0$ in J_0 turns out to be sufficiently small to allow ϕ_0 to be accurately calculated from $\phi_0 = \sin^{-1}[I_{dc}/I_c J_0(x)]$. The fractional errors in all the quantities of interest will be small compared to the quantities themselves and so the simple analysis outlined above is valid. However, as x approaches one of the zeros of J_0 , the fractional error $\delta J_0/J_0$ in J_0 may become larger than unity if the approximation $v_0 = I_{\mu 0} R_N$ is used. (The value of δJ_0 remains fairly constant as J_0 passes through zero.) Then the estimate of ϕ_0 and all the other parameters becomes highly inaccurate. Under these conditions it is necessary to use a more accurate expression for Z which includes the dependence of Z on ϕ_0 . This causes the equations to become very nonlinear, as ϕ_0 becomes a function of itself, so to speak. The strong dependence of ϕ_0 on $1/J_0$, and the dependence of J_0 upon ϕ_0 , leads to instabilities in the equations and multivalued solutions near the zeros of J_0 . These instabilities manifest themselves as the "phase-jumps" observed in Figs. 4.2 and 4.3, which are perhaps the most interesting experimental results. The explanation of these additional fine structures is the tour de force of the calculations presented here. (Incidentally, the calculations will show that x is never exactly equal to a zero of $J_0(x)$ for any value of I_{dc} and $I_{\mu 0}$; as a result, the equations never "diverge" in the mathematical sense.)

The next step (or several steps) in the analysis has to do with the problem of calculating v_0 and ϕ_0 as functions of I_{dc} and $I_{\mu 0}$ when x

is reasonably close to a zero of $J_0(x)$. The analysis continues at this point by calculating the small but important deviation of Z from R_N (which is quite straightforward). In Chapter II the Josephson inductance was discussed at length and calculated for small microwave currents. In Appendix B this calculation is generalized to arbitrarily large values of $I_{\mu 0}$ and gives the approximate result:

$$L_J = \frac{R_N I_{\mu 0}}{2\omega J_1(x) I_c \cos(\phi_0)} \quad (4.10)$$

As explained in detail in Appendix B, the remaining calculations in this chapter were carried out to lowest order in $(R_N/\omega L_J)$, which was assumed to be small (and is small for the conditions of this experiment). Consequently, all of the results presented here, for example eq. (4.10), retain only those terms in $(R_N/\omega L_J)$ which are needed to give a final result accurate to lowest order. The important feature of eq. (4.10) is the $\cos(\phi_0)$ term, which causes Z to be a (weak) function of ϕ_0 . The impedance itself is equivalent to an inductance L_J in parallel with a resistance R_N . As shown in Appendix B, the magnitude of Z (which is all that is needed here) is

$$|Z| = \frac{R_N |\omega L_J|}{(R_N^2 + \omega^2 L_J^2)^{1/2}} \quad (4.11)$$

Since L_J is a function of ϕ_0 , so is Z . This concludes the calculation of the impedance in terms of the other parameters, so that Z can now be thought of as having been eliminated from the problem in favor of the other variables.

The next step in the analysis is the use of the calculated value of $|Z|$ to solve for the microwave voltage in terms of $I_{\mu 0}$, thus eliminating v_0 (or, equivalently, x , since x is proportional to v_0) from the problem. The remaining equations will involve only ϕ_0 and the independent parameters I_{dc} and $I_{\mu 0}$. The microwave voltage is eliminated using the equation

$$v_0 = I_{\mu 0} |Z| \quad (4.12)$$

and also eq. (4.11) for $|Z|$. Because most of the equations, such as eq. (4.9) for $\bar{I}_S$, involve x rather than v_0 , it is more convenient to solve for x rather than v_0 . Since $|Z|$ is nearly equal to R_N , x is nearly equal to

$$x_0 = \frac{2eR_N I_{\mu 0}}{\hbar\omega} \quad (4.13)$$

Equation (4.13) serves to define the quantity x_0 , which can be taken to be an independent parameter, replacing $I_{\mu 0}$. (That is, I_{dc} and x_0 can just as well be thought of as being experimentally fixed.) It will also be convenient to define

$$L_0 = R_N I_{\mu 0} / 2\omega J_1(x_0) I_c \quad (4.14)$$

$$\epsilon = R_N / \omega L_0 \quad (4.15)$$

Here ϵ should not be confused with the ϵ' already defined; ϵ and ϵ' are two unrelated parameters. (Recall that $\epsilon \ll 1$ has already been assumed.) Since x_0 , L_0 , and ϵ depend only on $I_{\mu 0}$, they can be considered to be "known." As shown in Appendix B, the dependence of Z on ϕ_0 leads to

the result

$$x = x_0 \left(1 - \frac{1}{2} \epsilon^2 \cos^2 \phi_0 \right) \quad (4.16)$$

Once again, this result is only accurate to lowest order in ϵ .

Equation (4.16) can now be used to eliminate x (and v_0) from the problem, so that all the remaining equations involve only ϕ_0 and the independent variables I_{dc} and x_0 (or $I_{\mu 0}$).

At this point x (and v_0) has been eliminated, leaving equations in which ϕ_0 is the only unknown. Before solving these equations for ϕ_0 and evaluating $\bar{V}_T$ (the equations of interest are generally cubic) it may be useful to consider the physical nature of these equations involving ϕ_0 . This can best be accomplished by simply plotting the value of $\bar{I}_S$ calculated in Appendix B as a function of ϕ_0 . (That is, eq. (4.16) for x can be substituted into eq. (4.9) for $\bar{I}_S$, giving $\bar{I}_S$ as a function of ϕ_0 .) This is shown in Fig. 4.7, which consists of a sequence of plots of $\bar{I}_S$ as a function of ϕ_0 for various values of x_0 . (Recall that x_0 is proportional to $I_{\mu 0}$, and so can be considered to be experimentally fixed.) All the values of x_0 are near z_1 , the first zero of J_0 , and increase monotonically from below z_1 to above. These calculations were performed for the case of $I_c = 300 \mu A$, $R_N = .044$ ohms, and $\omega = 6.04 \times 10^{10}$ rad/sec.

Figure 4.7a shows the situation when x_0 is far enough below z_1 ($z_1 = 2.4048$) that the change in Z (as ϕ_0 varies) has only a small percentage effect on the value of $J_0(x)$ in eq. (4.9); in other words, $\bar{I}_S$ varies roughly as $\sin(\phi_0)$, although the curve in Fig. 4.7a is already distorted somewhat from a true sine. In Fig. 4.7b, x_0 has

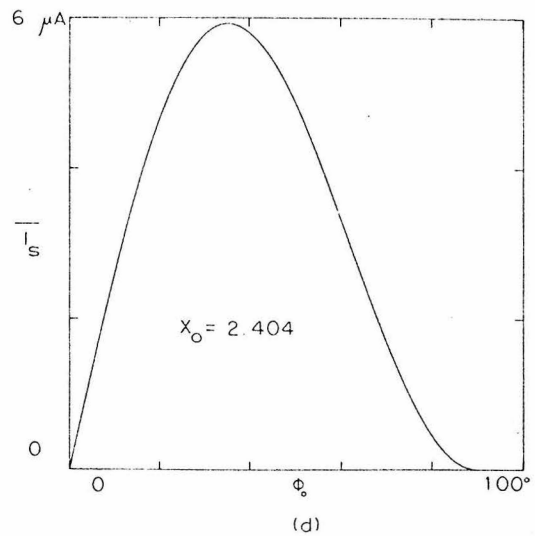
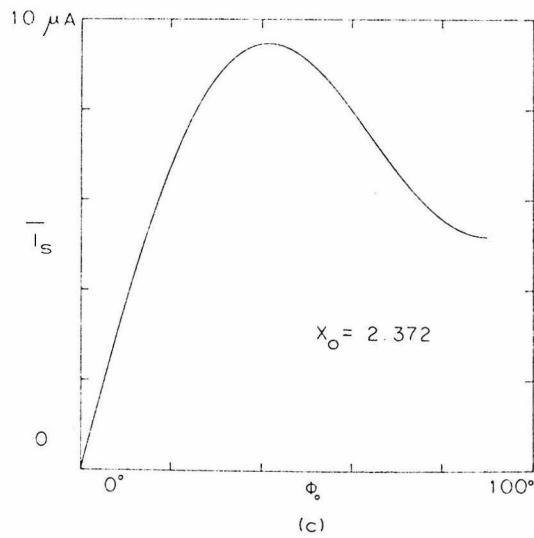
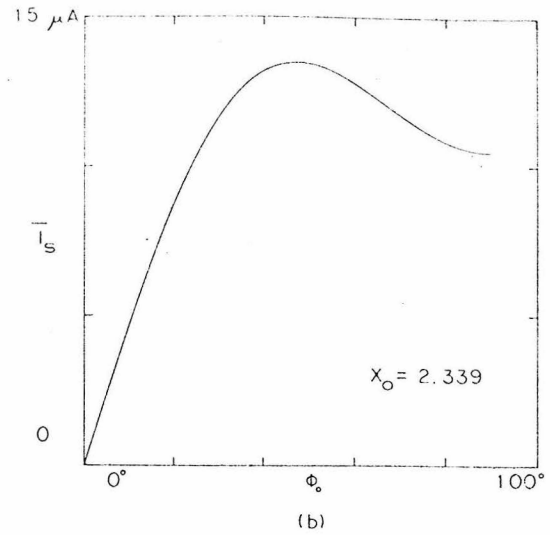
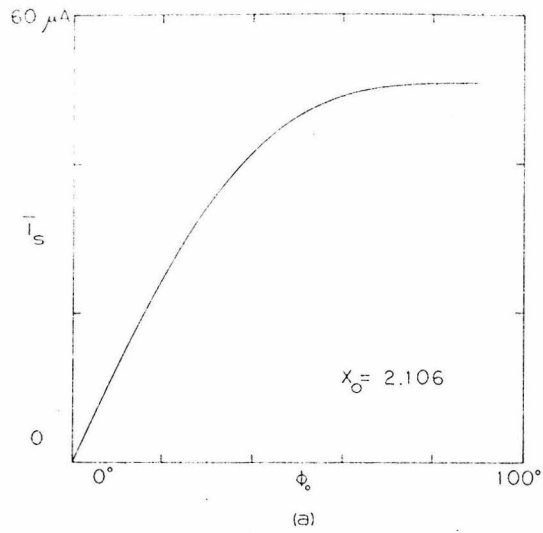


Fig. 4.7 (part I) Calculated value of the time-averaged supercurrent $\bar{I}_s$, as a function of the time-averaged quantum phase across the weak link, ϕ_0 . The results are plotted for various values of the amplitude of the microwave current. (Continued on next page.)

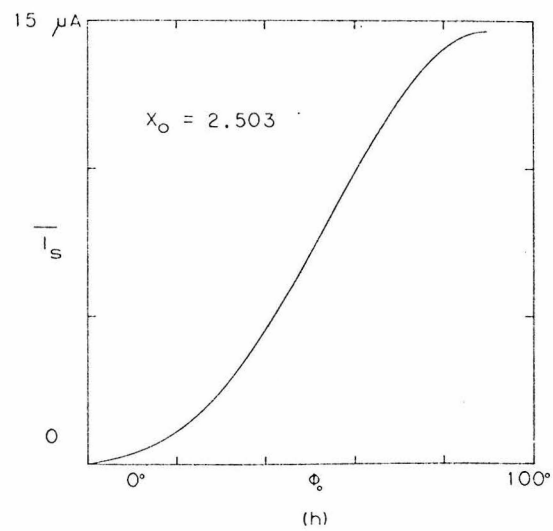
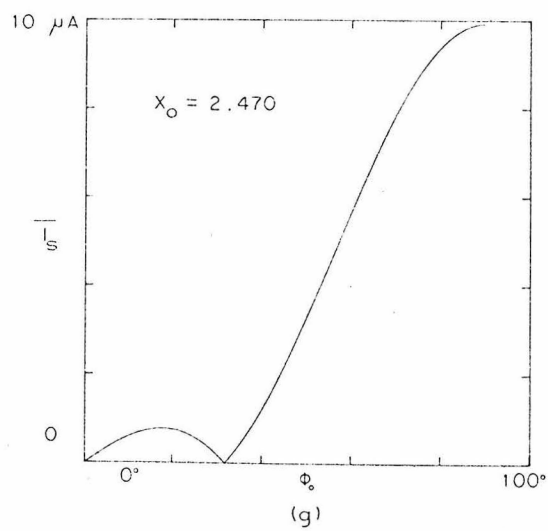
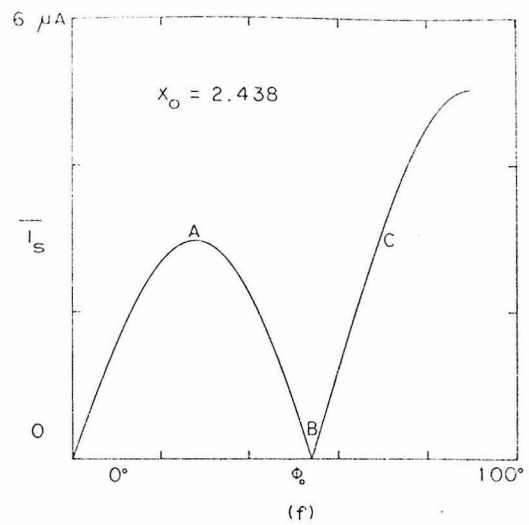
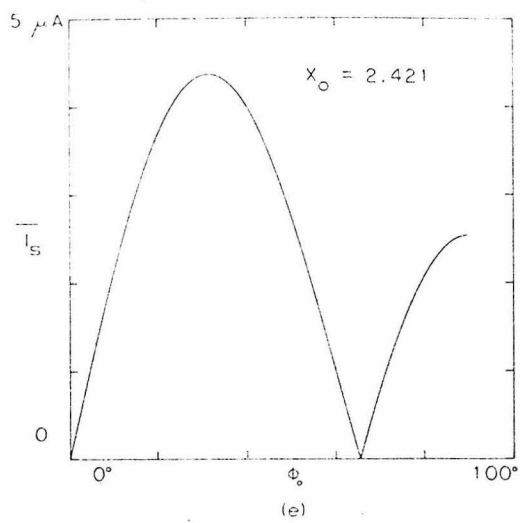


Fig. 4.7 (part II) Continued from previous page.

gotten close enough to z_1 that the variation in the microwave impedance as a function of ϕ_0 brings x (which is proportional to $|Z|$) significantly closer to z_1 for $\phi_0 = 90^\circ$ than it was for $\phi_0 = 0^\circ$. Since $\bar{I}_s$ is proportional to $J_0(x)$, this displacement of x towards one of the zeros of $J_0(x)$ reduces the value of $\bar{I}_s$. (Note that the vertical scale here is less than in the previous figure due to the closeness to z_1 .) This effect has become large enough to depress $\bar{I}_s$ at $\phi_0 = 90^\circ$ and cause the maximum supercurrent, $\tilde{I}_c$, to occur at some value of ϕ_0 not equal to 90° . Note also that, although $\bar{I}_s$ is a single-valued function of ϕ_0 , ϕ_0 is no longer a single-valued function of $\bar{I}_s$. (In other words, $\bar{I}_s = I_{dc}$ and is fixed by the experimental bias conditions; for a given value of $\bar{I}_s$, there is now more than one possible value of ϕ_0 .) The stability of these additional values of ϕ_0 for a given $\bar{I}_s$ will be discussed later.

What is happening is that, as x_0 gets closer to z_1 , the percentage change in J_0 as ϕ_0 varies becomes larger and larger. As a result, the relatively small variation in $|Z|$ as a function of ϕ_0 becomes more and more important. In Fig. 4.7c the reduction of $\bar{I}_s$ near $\phi_0 = 90^\circ$ increases, until $\bar{I}_s$ goes to zero at 90° in Fig. 4.7d, where $x_0 = z_1$. (That is, $J_0(x)$ in eq. (4.9) for $\bar{I}_s$ is zero at this value of ϕ_0 .) In Fig. 4.7e x_0 is now above z_1 , so that, as ϕ_0 varies from 0° to 90° , x passes through z_1 from below. In all the parts of Fig. 4.7, what has actually been plotted is the absolute value of $\bar{I}_s$; for example, J_0 and $\bar{I}_s$ are actually negative at $\phi_0 = 90^\circ$ in Fig. 4.7e. The absolute value has been plotted because a positive value of $\bar{I}_s$ will be produced by a corresponding value of ϕ_0 in some other quadrant;

it was desirable to avoid plotting all four quadrants of ϕ_0 here. (The fact that ϕ_0 jumps from one quadrant to another in order to keep $\bar{I}_S$ positive is a nuisance dealt with in Appendix B but which is best ignored here.) Note that $\tilde{I}_C$ (maximum $\bar{I}_S$) still occurs at some value of $\phi_0 \neq 90^\circ$, but that $\phi_0 = 90^\circ$ is always an extremum, here a local maximum.

In Fig. 4.7f, x_0 is now sufficiently large that $\tilde{I}_C$ (the maximum value of $\bar{I}_S$) once again occurs at exactly 90° . Note that $\bar{I}_S$ still has a relative maximum at some value of $\phi_0 \neq 90^\circ$. As x_0 increases further, this relative maximum becomes smaller and smaller (Fig. 4.7g) until it has disappeared altogether in Fig. 4.7h. In Fig. 4.7h, x_0 is so large that x never passes through z_1 for the entire range of ϕ_0 . Note, however, that the curve is still distorted from a true sine.

The reader has probably surmised by now that the discontinuities or phase-jumps observed in the data are caused by the multivaluedness of ϕ_0 as a function of $\bar{I}_S$. Note the (superficial) similarity between Fig. 4.7f and the V-I characteristics of an Isaki tunnel diode, a semiconductor device used for switching applications. Imagine that, for the conditions shown in Fig. 4.7f, $\bar{I}_S$ and ϕ_0 are both initially zero. If the value of $\bar{I}_S$ is increased (remember that $\bar{I}_S = I_{dc}$, the external dc bias) eventually ϕ_0 will find itself in the position of having to jump over to the other branch in order to maintain the required value of $\bar{I}_S$. Physically, what happens is that, in the region between the points marked A and B in Fig. 4.7f, $\bar{I}_S$ will not equal I_{dc} and so a voltage v will exist across the device. This voltage will precess ϕ_0 according to the Josephson equation (Ref. 12),

$$\frac{d\phi_0}{dt} = \frac{2ev}{\hbar} \quad (4.17)$$

This precession of ϕ_0 will continue until $\bar{I}_S$ again equals I_{dc} at point C. (Actually, ϕ_0 must jump all the way to a new quadrant, a complication which will be ignored here.) Similar reasoning (explained in detail in Appendix B) shows that the region between A and B, where $d\bar{I}_S/d\phi_0 < 0$, is physically unstable. If the value of $\bar{I}_S$ is now lowered, ϕ_0 will follow the curve from C to B. (Once again, ϕ_0 will actually be in another quadrant; the interested reader is encouraged to sketch all four quadrants for himself.) It should be pointed out that, although $d\bar{I}_S/d\phi_0$ is negative between A and B, there exists a corresponding value of ϕ_0 in another quadrant which is stable. Thus there are values of $\bar{I}_S$ which correspond to three stable values of ϕ_0 . The long-term stability of ϕ_0 on any particular branch will need to be discussed later. It is possible that ϕ_0 might spontaneously jump from one "stable" position to another due to noise.

Before going on to discuss the effects on $\bar{V}_T$ of these discontinuities in ϕ_0 (as a function of I_{dc}), there are some more general topics which should be discussed. First of all, it might be wondered why these discontinuities had not been discovered in the past, since their occurrence is not limited to experiments whose purpose is to measure $\bar{V}_T$. (The discontinuous nature of ϕ_0 is due to the properties of the microwave impedance, and does not depend on quasi-particle nonequilibrium for its existence.) This is partly because, in the usual dc measurements, ϕ_0 is not directly observable. When ϕ_0 jumps

from one value to another, the value of $\bar{I}_S$ remains unchanged. (The phase-jumps are observable in this experiment only because of the J_2 term in eq. (4.8); the term $\sin(\phi_0)J_0(x)$, which is all that appears in eq. (4.9) for $\bar{I}_S$, is unchanged in a phase-jump.) In addition, the author was perhaps in a unique position to think of these arguments, since he had just finished the study of the microwave impedance described in Chapter II before beginning this experiment. In any event, the discussion presented here does have some implications for measurements of other weak link properties besides $\bar{V}_T$. In particular, the maximum supercurrent $\tilde{I}_C$ for a current biased weak link no longer periodically passes through zero as the microwave drive is varied, as might have been expected from the simple Bessel function relation, eq. (4.9). This is shown in Fig. 4.8, where $\tilde{I}_C$ is plotted as a function of $I_{\mu 0}$, both for the current biased results obtained in Appendix B and for the usual Bessel function relation (voltage-biased or constant Z). Here I_C was chosen to be 700 μA so as to increase the magnitude of the effects (which increase greatly as I_C increases.) $\tilde{I}_C$ no longer goes to zero because, crudely speaking, ϕ_0 can (and will) adjust itself so as to avoid having $x = z_1$, regardless of the value of x_0 . The mechanism responsible for this can be seen graphically in the sequence of diagrams in Fig. 4.7. Although the details of the calculation of $\tilde{I}_C$ have been relegated to Appendix B, there are some final results which should probably be summarized here. The first minimum of $\tilde{I}_C$, which will be designated $\tilde{I}_{cm}$, occurs at a value of x_0 , designated x_{0m} , which can be found by finding the roots of a certain

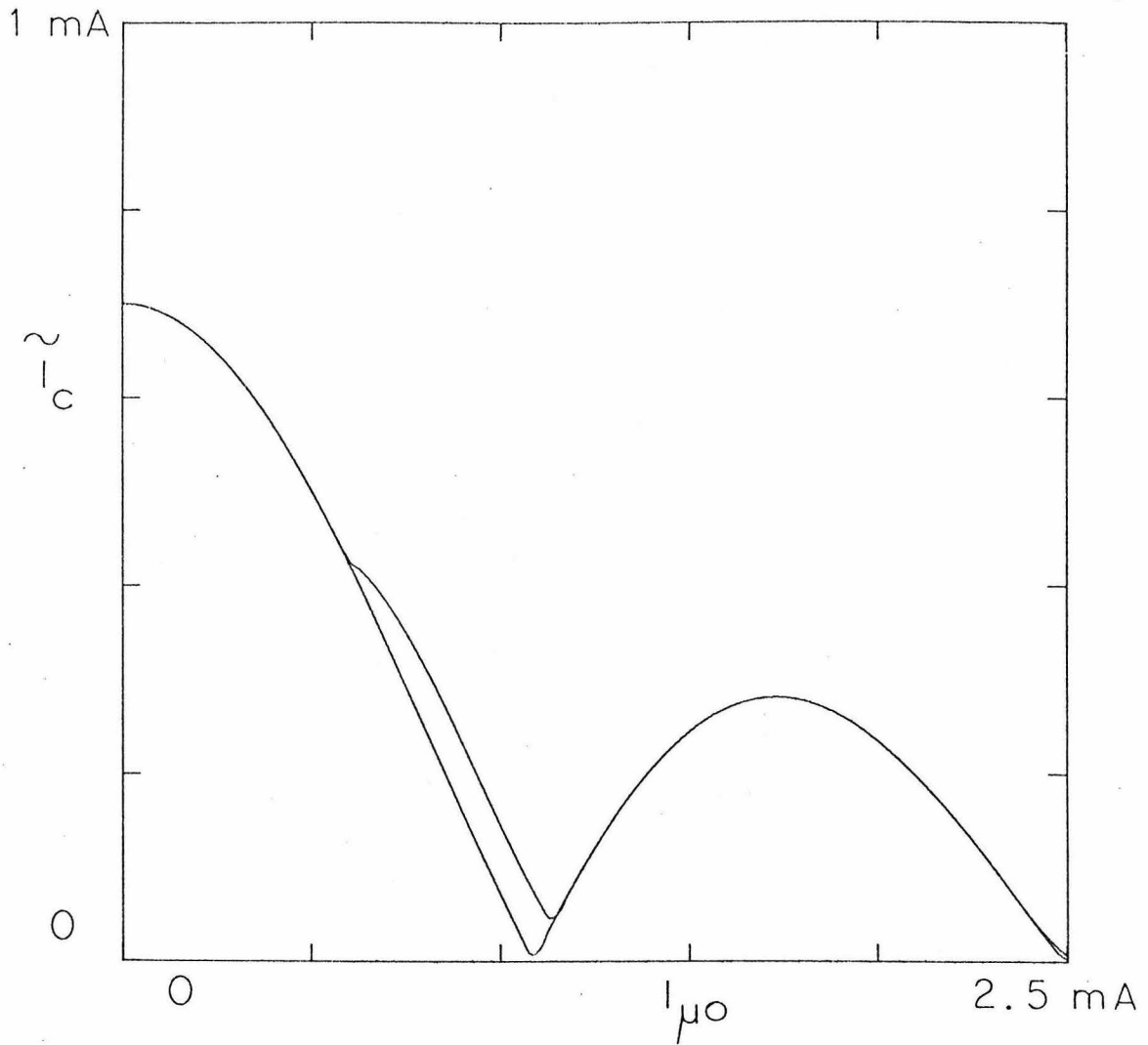


Fig. 4.8 Calculated effective dc critical current, $\tilde{I}_c$, as a function of the amplitude of the microwave current, $I_{\mu 0}$. Results are shown for the case of a voltage-biased, as well as a current-biased, proximity effect weak link with a critical current of 700 μ A in the absence of radiation.

cubic equation. The results of this calculation have the surprisingly simple form

$$x_{0m} = z_1(1 + \epsilon^2/8) \quad (4.18)$$

$$\tilde{I}_{cm} = I_c z_1 \epsilon^2 J_1(z_1) / 8 \quad (4.19)$$

Similar results can be derived for the other zeros of J_0 , but this chapter will restrict itself to the structure near z_1 . Unfortunately, the existence of the Wyatt effect (which is the enhancement of the constant I_c due to microwave radiation; see Chapter III) makes a quantitative comparison of the value of $\tilde{I}_{cm}$ given by eq. (4.19) with the experimental data impossible. However, as has already been mentioned, $\tilde{I}_c$ did, in fact, never go to zero as a function of the microwave drive. Having paused to discuss the results for $\tilde{I}_c$, the calculation of $\bar{V}_T$ will now be resumed.

The next step in the analysis is the calculation of the range of values of x_0 over which phase-jumps occur. This was done by setting $d\bar{I}_s/d\phi_0 = 0$, which is the requirement for the existence of a relative maximum (or minimum) at $\phi_0 \neq 90^\circ$, such as the one shown at point A in Fig. 4.7f. (No phase-jumps can occur without the existence of such an extremum at some value of ϕ_0 other than 90° .) It was found that $\phi_0 = 90^\circ$ is always an extremum. In addition, an extremum will occur at some other value of ϕ_0 , which will be designated ϕ_0^* , if

$$z_1(1 - \epsilon^2) \leq x_0 \leq z_1(1 + \epsilon^2/2) \quad (4.20)$$

The value of ϕ_0^* is given by

$$\phi_0^* = \sin^{-1} \left[\pm \left[\frac{1 - \frac{2}{\epsilon^2} \frac{(x_0 - z_1)}{x_0}}{3} \right]^{1/2} \right] \quad (4.21)$$

$\bar{I}_s$ has the value $\tilde{I}_c^*$ at ϕ_0^* , given by

$$\tilde{I}_c^* = \epsilon^2 x_0 I_c J_1(z_1) \left[\frac{1 - \frac{2}{\epsilon^2} \frac{(x_0 - z_1)}{x_0}}{3} \right]^{3/2} \quad (4.22)$$

Incidentally, the value of $\tilde{I}_c$ shown in Fig. 4.8 was calculated by taking the larger of $\tilde{I}_c^*$ or $I_c J_0(x_0)$, which is the value of $\bar{I}_s$ at the other extremum, $\phi_0 = 90^\circ$.

Having determined the range of x_0 over which phase jumps can occur, the next step in the analysis is the determination of the number of solutions for ϕ_0 and the value of each for a given value of I_{dc} and x_0 (or $I_{\mu 0}$). There may be zero, one, two, or three solutions for ϕ_0 ; they are given by the roots of the following cubic equation

$$\sin^3(\phi_0) - \left[1 - \frac{2}{\epsilon^2} \frac{(x_0 - z_1)}{x_0} \right] \sin(\phi_0) + \frac{2I_{dc}}{I_c J_1(x_0) x_0 \epsilon^2} = 0 \quad (4.23)$$

The solution to this cubic equation will be presented here, since there are so many equations in Appendix B that the results should be summarized in the text; those readers who are not interested in even the final result are advised to skip these equations, as they do not give any physical understanding of the situation. It is convenient to define

$$a = - \left[1 - \frac{2(x_0 - z_1)}{\epsilon^2 x_0} \right] \quad (4.24)$$

$$b = \frac{2I_{dc}}{I_c J_1(x_0) x_0 \epsilon^2} \quad (4.25)$$

$$g = \frac{b^2}{4} + \frac{a^3}{27} \quad (4.26)$$

If $g > 0$, then there is only one real root:

$$\phi_0 = \sin^{-1}(A + B) \quad (4.27)$$

where A and B are defined by

$$A = (-b/2 + g^{1/2})^{1/3} \quad (4.28)$$

$$B = (-b/2 - g^{1/2})^{1/3} \quad (4.29)$$

For $g < 0$, there are three real roots given by

$$\phi_{01} = \sin^{-1}[2 \sqrt{-a/3} \cos(\rho'/3)] \quad (4.30)$$

$$\phi_{02} = \sin^{-1}[2 \sqrt{-a/3} \cos(\rho'/3 + 120^\circ)] \quad (4.31)$$

$$\phi_{03} = \sin^{-1}[2 \sqrt{-a/3} \cos(\rho'/3 + 240^\circ)] \quad (4.32)$$

where ρ' is defined by

$$\rho' = \cos^{-1}[-\frac{b}{2} / \sqrt{-a^3/27}] \quad (4.33)$$

In the above equations both branches of the arcsin function must be used, since the two branches are not physically equivalent. Thus up to six different values of ϕ_0 are found; however, half of them are

physically unstable and should be discarded. (The physically unstable solutions can be determined by evaluating $d\bar{I}_s/d\phi_0$ for each solution; see Appendix B for the relevant formulas.) In addition, if the magnitude of the argument of the arcsin is greater than one, those solutions must also be discarded. As a result, the number of physically acceptable solutions may be zero, one, two, or three.

Having found all the solutions for ϕ_0 corresponding to a given value of I_{dc} , there still remains the question of which one of the solutions actually determines the value of $\bar{V}_T$; or perhaps ϕ_0 jumps from one solution to another as I_{dc} is varied. (In other words, the discussion presented in connection with points A, B, and C in Fig. 4.7f suggests that $\bar{V}_T$ may also be a multivalued function of I_{dc} , a condition referred to in the laboratory as hysteresis.) Fortunately, from the point of view of the analysis, it was found experimentally that very little hysteresis would result if the modulation frequency of I_{dc} was sufficiently low (~ 10 Hz). If, on the other hand, the modulation frequency was sufficiently high ($\sim 10^4$ Hz), then a noticeable amount of hysteresis was present. This indicates that the various "stable" solutions for ϕ_0 are actually quasi-stable, and that transitions from one solution to another occur at a rate of somewhere between 10^1 to 10^4 times per second. These transitions are almost certainly due to amplitude-modulated (AM) noise in the microwave drive, or some other form of external noise, although a quantitative analysis of this topic is made prohibitively difficult by the nonlinearity of the equations involved. (Thermal noise is much too small to cause transitions

between the various solutions for ϕ_0 .) Because of the transition rate mentioned above, it seemed reasonable to assume that all the possible values of ϕ_0 occur with equal likelihood. Therefore, the calculated value of $\bar{V}_T$ was taken to be the average corresponding to all the possible "stable" solutions for ϕ_0 . This assumption is also expected to be of limited validity, and provides yet another reason to expect only qualitative agreement between the data and the calculations. The correctness of this assumption is substantiated by the resulting agreement with the data. All of the calculations necessary for the evaluation of $\bar{V}_T$ exist in a closed, analytic form in Appendix B. Numerical evaluation of the resulting formulas is, as is probably evident, sufficiently tedious that it is best done by computer. The numerical evaluation of these formulas by computer is the final step in the calculation of $\bar{V}_T$.

Figure 4.9 shows the calculated results for $\bar{V}_T$ as a function of I_{dc} for the same sequence of values of x_0 as was used in Fig. 4.7. In Fig. 4.9a, x_0 is sufficiently far from z_1 that $\bar{V}_T$ is nearly a straight line. Such straight lines formed the bulk of the data. The calculations were done for the case $I_c = 300 \mu A$, $\omega = 6.04 \times 10^{10}$ rad/sec, and $R_N = .044$ ohms, which is (roughly) the same as that which produced the data shown in Figs. 4.1, 4.2, and 4.3, and also the calculations in Fig. 4.7. Once again, the Wyatt effect makes the value of I_c uncertain; unfortunately, the calculated results are strongly dependent on I_c .

Since this chapter is already longer than was intended, the progression of the various plots shown in Fig. 4.9 will not be described.

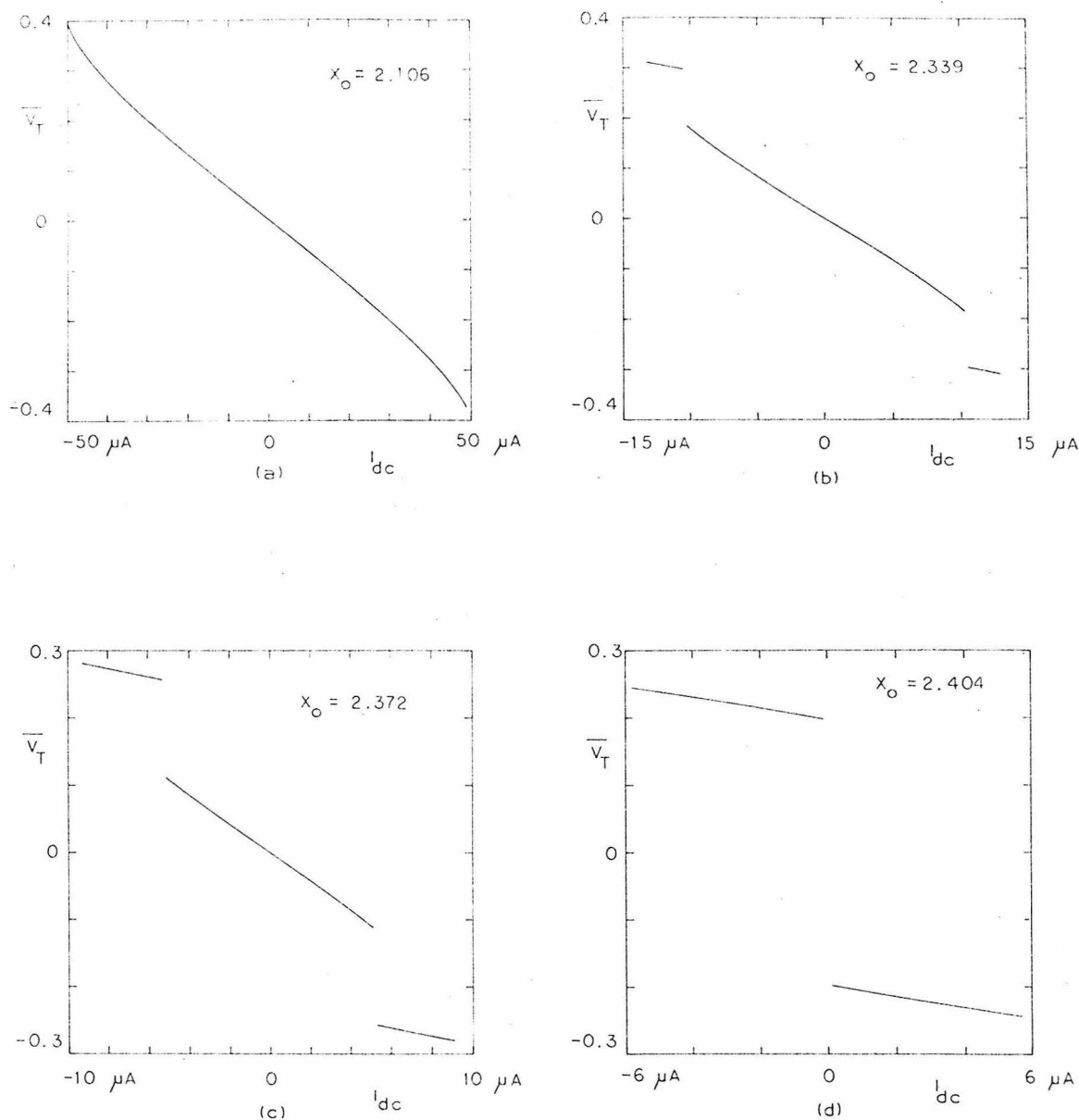


Fig. 4.9 (part I) Calculated value of the time-averaged potential of the tunnel junction probe, $\bar{V}_T$, as a function of the dc bias current through the weak link, I_{dc} . ($\bar{V}_T$ is normalized by an arbitrary constant.) The results are plotted for various values of the parameter x_0 , which is proportional to the amplitude of the microwave current. (Continued on next page.)

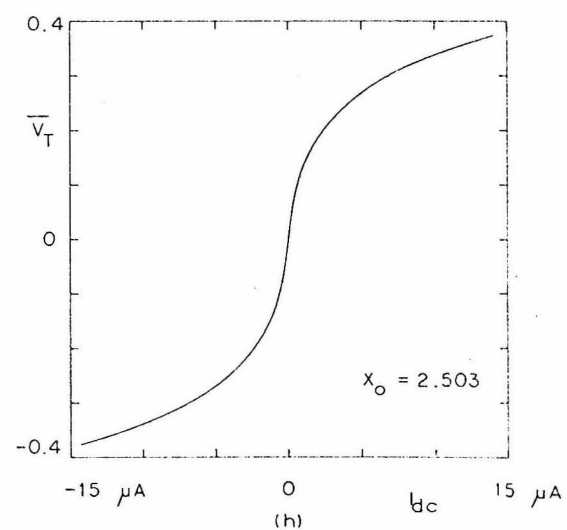
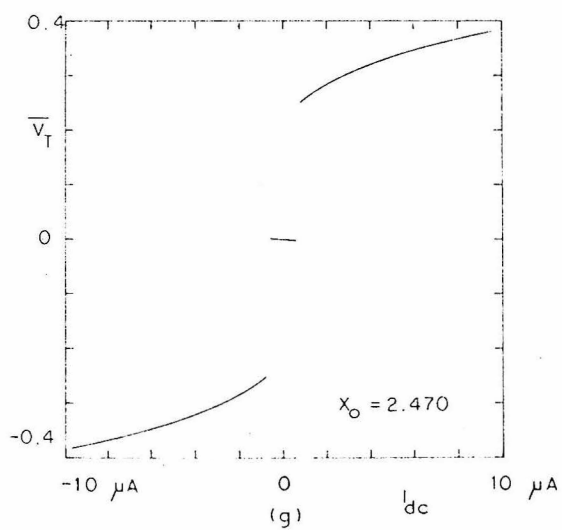
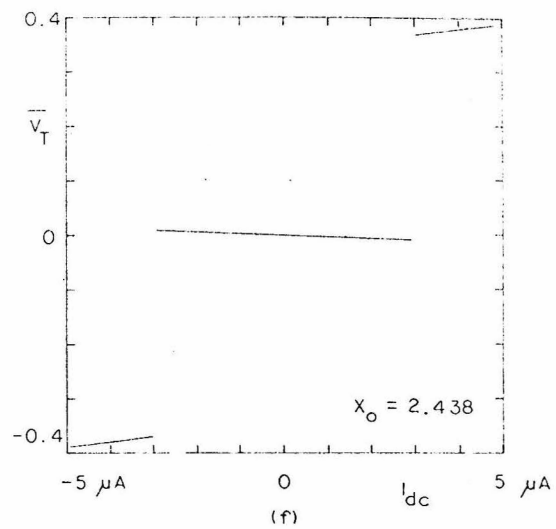
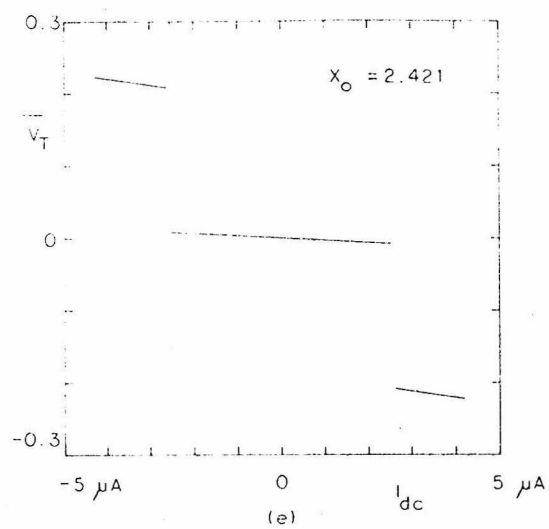


Fig. 4.9 (part II) Continued from previous page.

The interested reader can establish for himself the causal relationship between each plot in Fig. 4.7 and the resulting phase-jumps in the corresponding plot of Fig. 4.9. Mathematically, the phase-jumps correspond to a change in the number of real roots of the cubic equation, eq. (4.23). The similarity between the data in Fig. 4.2 and the calculations plotted in Fig. 4.9c should be pointed out, as well as the similarity between Figs. 4.3 and 4.9d. The horizontal axes should not be expected to agree because the calculations are proportional to I_C^2 (through ϵ) and the Wyatt effect increases I_C by a factor of anywhere from 1.5 to 4.0 for the device studied here. (The Wyatt effect enhancement near the zeros of J_0 cannot be readily determined.) Similarly, the scale of the vertical axis in the calculated results cannot be determined because of an undetermined constant in the analysis; $\bar{V}_T$ has therefore been normalized to a convenient constant. This does not result in any loss of information, since the experimental measurements are also multiplied by an unknown constant related to the degree of asymmetry of the tunnel junction. In any event, the shapes of the theory and data plots are surprisingly similar, considering the number of approximations made. Experimental results similar to each of the plots in Fig. 4.9 were obtained.

The calculated dependence of the maximum value of $\bar{V}_T$ ($\bar{V}_T$ at $\tilde{I}_C$) on the microwave drive $I_{\mu 0}$ is shown in Fig. 4.10. The most noticeable features in Fig. 4.10 are the discontinuity in slope near $I_{\mu 0} = 1$ ma, followed by a discontinuity in $\bar{V}_T$ itself. The discontinuity in slope occurs when the maximum supercurrent begins to

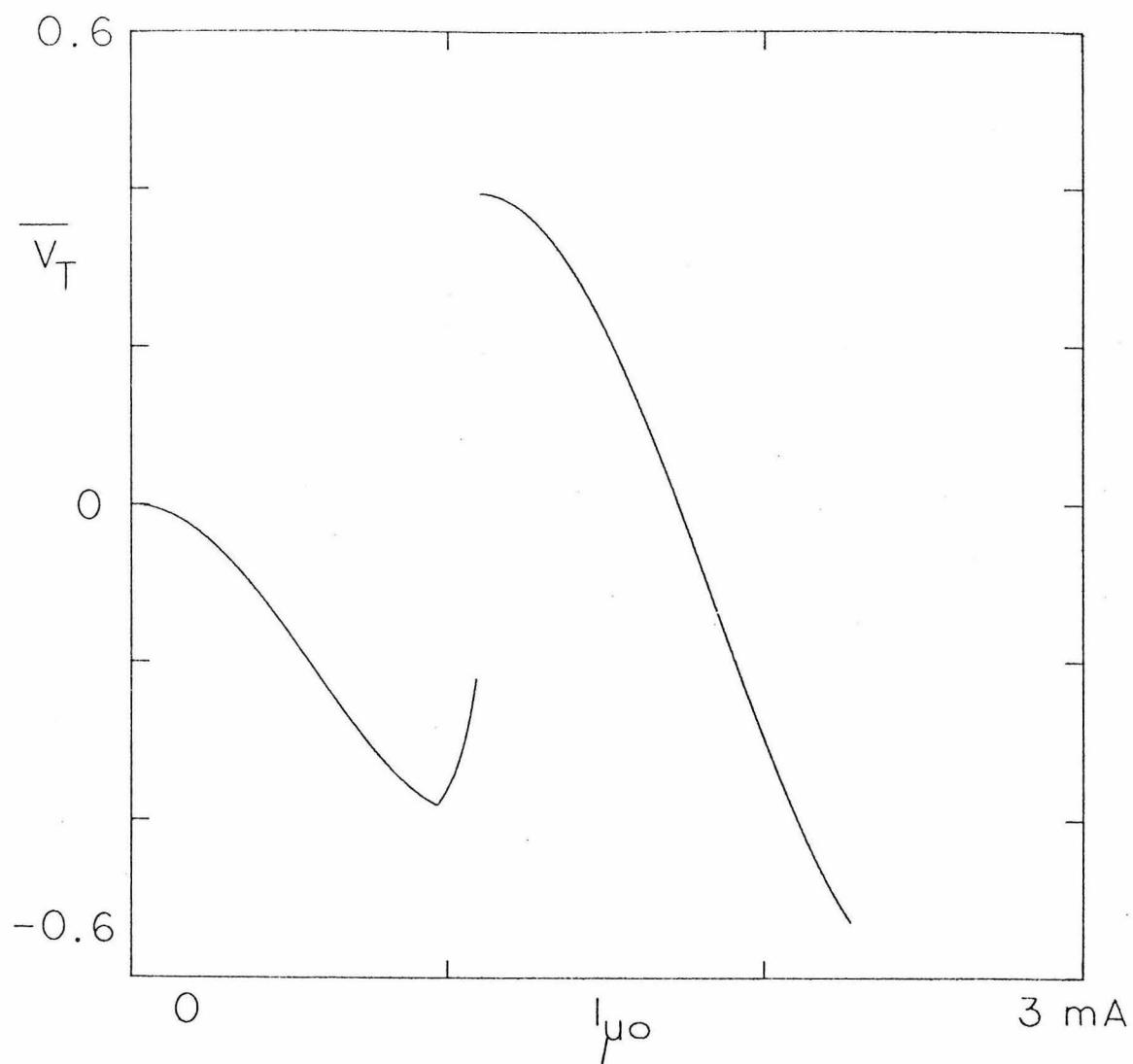


Fig. 4.10 Calculated value of the maximum time-averaged potential of the tunnel junction probe $\bar{V}_T$ as a function of the amplitude of the microwave current drive, $I_{\mu 0}$.

correspond to a phase ϕ_0^* which is not 90° . (ϕ_0^* decreases continuously from 90° , but its "velocity" $d\phi_0^*/dx_0$ changes discontinuously.) The discontinuity in $\bar{V}_T$ itself occurs when ϕ_0^* abruptly jumps back to 90° again. This occurs when the two relative maxima shown in Fig. 4.7e become just equal in magnitude. The data shown in Fig. 4.4 contain a great deal of experimental uncertainty, but the discontinuity in $\bar{V}_T$ can clearly be seen, as well as the "fish-hook" structure just below it. It should be pointed out that the horizontal scales of these two figures were not normalized to each other. (The horizontal scale in Fig. 4.4 was determined from the location of the minima in $\tilde{I}_c$.) The fact that the discontinuity in $\bar{V}_T$ occurs at the correct position is thus a genuine verification of the model used. (It is probably irrelevant but interesting nonetheless that all the calculations were in existence before the data curve, Fig. 4.4, so that the model actually was capable of predicting this aspect of the data in advance.) Aside from these two features, the two curves differ considerably, especially at large values of $I_{\mu 0}$. So many approximations have been introduced into the calculation by this point that it is probably not worthwhile to speculate on the cause of these discrepancies. In fact, it is somewhat surprising that the agreement is as good as it is.

The lengthy calculation of $\bar{V}_T$ just concluded shows that the suggested mechanism for producing a nonzero value of the parameter $\nabla\theta$ gives results which are in qualitative agreement with the data. At this point it is appropriate to consider the extent to which this agreement depends on the details of the mechanism suggested, and to what extent physical

mechanisms other than the one suggested here might equally well account for the data. In particular, the physical differences between the mechanism suggested here (which gives $\nabla\theta$ proportional to v) and that proposed by Yu et al (which gives $\nabla\theta$ proportional to the time integral of v) should be discussed. As has already been pointed out, the equations proposed by Yu et al can be shown to give results for $\bar{V}_T$ which are identical to those already obtained, to within a constant. Consequently, this experiment is totally incapable of distinguishing between these two models.

There are several reasons why the mechanism suggested here was used in the analysis of this experiment, rather than the equations proposed by Yu et al. Yu et al proposed these equations several years ago (Refs. 3,10,11) so that the mechanism responsible for such effects is (or should be) already well known in the literature. (Although the mechanism responsible for the equations proposed by Yu et al should be well known, the consequences of these equations, as just calculated, represent a new result.) On the contrary, the mechanism proposed here has not (to the author's knowledge) been described elsewhere. In addition, the equations proposed by Yu et al, as will be discussed shortly, can be understood on the basis of a time-dependent generalization of the Ginzburg-Landau theory (Refs. 13,14) (see also Chapter III). The results of their equations represent a departure from equilibrium (on the part of the superfluid) and so is largest when the Ginzburg-Landau relaxation time τ_{GL} is large compared with the other times in the problem. The author had intuitively felt for some time that proximity effect

weak links were actually very nearly in equilibrium (at least for small applied voltages) and so had deliberately searched for effects not included in the time-dependent Ginzburg-Landau (TDGL) equation. As has been pointed out, the mechanism suggested here is due to a strong adherence, on the part of the superfluid, to quasi-equilibrium and is largest when τ_{GL} is effectively zero. Thus these two mechanisms represent, in some sense, different limiting cases of the value of τ_{GL} . The physical conditions under which one or the other of these mechanisms might be expected to dominate will now be discussed.

The new mechanism suggested in this chapter, which has to do with the displacement of the electron distribution in accordance with Ohm's law, has already been explained in some detail. At this point it may be useful to discuss the physical origin of the equations proposed by Yu et al. This can best be understood in terms of the TDGL equation, which can be written in the form (Refs. 5,6)

$$\left(\frac{\partial}{\partial t} + \frac{2i\mu}{\hbar}\right)\Psi = \frac{1}{\tau_{GL}} [1 - \eta|\Psi|^2 + \xi^2(\vec{\nabla} - \frac{2ie}{\hbar c} \vec{A})^2]\Psi \quad (4.34)$$

Here η and ξ^2 were phenomenological constants in the Ginzburg-Landau theory which can now be explicitly evaluated using the BCS theory. (Their value is not important here.) If the electron density is essentially uniform as a function of position, then μ can be taken to be equal to ev , where v is the voltage. Inside the weak link where Ψ is small, the $\eta|\Psi|^2$ term can be dropped. In addition, $\vec{A} = 0$ in the problem of interest here. Under equilibrium conditions the term inside the

the brackets is zero (from the usual Ginzburg-Landau equations). If this term is assumed to be zero, or if τ_{GL} is so large that the right-hand side of eq. (4.34) is essentially zero, then the time dependence of Ψ can readily be found as a function of v . (In a bulk superconductor, there may be some complications in this argument having to do with the fact that the temperature dependence of ξ^2 can sometimes cancel the temperature dependence of $1/\tau_{GL}$, so that it may not be reasonable to assume that the right-hand side of eq. (4.34) becomes negligibly small as τ_{GL} becomes very large. In the proximity effect this is not the case, however, as both ξ^2 and $1/\tau_{GL}$ decrease with increasing temperature.) The result is that the local phase ϕ of Ψ precesses according to the familiar (from Chapter II) Josephson relation

$$\frac{\partial}{\partial t} \phi(z,t) = \frac{2ev(z,t)}{\hbar} \quad (4.35)$$

This is equivalent to eq. (4.17) except that here $\phi(z,t)$ is the local phase of Ψ . In the experiment described in this chapter, the microwave voltage is a function of position within the weak link. Thus the phase, according to eq. (4.35) should precess at different rates at different positions along the exponential tail $\psi_L(z)$. If the electric field $\vec{E}$ is uniform, then the parameter $\nabla\theta$ would also be independent of position and proportional to the time-integral of v . This is consistent with the form proposed for Ψ by Yu et al, and can be seen to result in those cases where τ_{GL} is larger than any other times in the problem. (Although the situation described above gives rise to a value of $\nabla\theta$ which is proportional to the time-integral of the voltage, it is

not entirely equivalent to the situation envisaged by Yu et al, whose original idea was that the $(1 - \xi^2 \nabla^2)\Psi$ term in eq. (4.34) should equal the $2i\mu/\hbar$ term.)

Suppose, on the other hand, that τ_{GL} is very small compared to any other time in the problem. In particular, suppose that $1/\tau_{GL}$ is so large that the term $2i\mu/\hbar$ becomes negligible compared to the right hand side of eq. (4.34) for even a small departure from equilibrium. Then the applied voltage will be unsuccessful in appreciably precessing the phase of Ψ , since any departure from equilibrium will be reflected in the right-hand side of the equation and will dominate the effects of v . In other words, the smaller the value of τ_{GL} , the less effective v will be in changing Ψ from its equilibrium value. (Once again, in bulk superconductors, but not in the proximity effect, there may be some difficulties in this argument having to do with the temperature dependence of ξ^2 cancelling that of $1/\tau_{GL}$.) In this limit the mechanism in which $\nabla\theta$ is proportional to the time-integral of v goes to zero and Ψ strongly adheres to its equilibrium value, as assumed in the new mechanism for $\nabla\theta$ suggested in this chapter. The mechanism suggested in this chapter represents a change in the quasi-equilibrium state to which Ψ relaxes, as a result of the quasi-particle or normal current. This effect is not included in eq. (4.34) but corresponds to a coupling between the normal current and the supercurrent. In any event, it can be seen that the mechanism in which $\nabla\theta$ is proportional to the time-integral of v should dominate as $\tau_{GL} \rightarrow \infty$, whereas the new mechanism suggested here should dominate as $\tau_{GL} \rightarrow 0$. It would be of interest to estimate which of these mechanisms should be the larger under the conditions of this

experiment, despite the fact that the experiment cannot distinguish between the two mechanisms. Unfortunately, this is not possible because of the fact that an appropriate value of τ_{GL} for the proximity effect is not, in the author's opinion, theoretically known. In addition, quantitative calculations of the new mechanism suggested here have not yet been finished.

It has already been pointed out that the mechanism proposed in this chapter is not applicable to bulk superconductors, where the value of $\nabla\theta$ is essentially fixed by the boundary conditions on Ψ . As a result, in bulk superconductors the value of $\nabla\theta$ is generally not equal to the value which would minimize the free energy in the absence of the boundary conditions. (That is, the free energy in bulk superconductors is minimized subject to the constraint of the applied boundary conditions.) The proximity effect differs from this behavior in that the boundary conditions on either of the exponential tails, ψ_L and ψ_R , are applied only at one plane, so that $\nabla\theta$ is free to vary. It is also quite possible that the rate at which Ψ relaxes to its quasi-equilibrium value depends on the distances involved. (Even if the ideas suggested in this chapter were made quantitative (which is currently being attempted) they would not give the rate of relaxation.) In other words, the shorter the distances involved, the more likely it is that Ψ will be well described by that form which minimized the free energy. Thus the mechanism described in this chapter is expected to apply to situations in which the relaxation time is short, the distances involved are small, and the boundary conditions are applied at a single plane. Otherwise, the

equations proposed by Yu et al would appear to be more appropriate. As has been pointed out, both of these mechanisms predict essentially the same results for this experiment; these results are in qualitative agreement with the data.

In view of the fact that these two mechanisms cannot be experimentally distinguished, it is appropriate to consider what, if any, specific conclusions can be drawn from this experiment. First, the agreement between either of these two models and the data provides some evidence that divergences of the supercurrent do exist inside the weak link and are dependent on the phase difference across the weak link. The exact nature of the mechanism responsible for these divergences cannot be determined from this experiment. On the other hand, the existence and explanation of the phase-jumps (discontinuous changes in ϕ_0) appear to be well established. As an examination of the calculations outlined above will show, the phase-jumps result from the properties of the microwave impedance. Although they (presumably) occur even under equilibrium conditions, they can be observed, at least in this experiment, only by using the nonequilibrium quasi-particle chemical potential as a probe.

4.4 Summary and Conclusions

An experimental method was described in which a very small tunnel junction was used as a probe of the nonequilibrium chemical potential μ_{qp} of the quasi-particles inside a proximity effect weak link. The probe was used to measure the effects upon μ_{qp} when a dc bias current

less than the critical current was passed through the weak link at the same time that an external microwave current was being applied. The bulk of the experimental data showed that the value of μ_{qp} was approximately proportional to the dc bias current (supercurrent) carried by the weak link, as previously reported by Yu et al (Ref. 1). In addition, it was found that, over a narrow range of microwave current drive, μ_{qp} became a discontinuous function of the dc bias current. These discontinuities in μ_{qp} , referred to as "phase-jumps", occurred for values of the microwave current near those values which minimized the critical current. The critical current was never completely reduced to zero by the applied microwave currents.

A physical mechanism was suggested which assumes that, to a first approximation, the effect of the microwave voltage was to shift the entire electron distribution in accordance with Ohm's law. The microwave impedance measurements of Chapter II indicated that this is a reasonable assumption and that the superconducting effects can be considered to be a small perturbation on this behavior. It was further suggested that such a displacement of the electron distribution might be expected to cause a gradient in the phase ($\nabla\theta$) of the exponential tail of the order parameter which would otherwise not be present. The value of $\nabla\theta$ is expected to be proportional to the voltage for sufficiently small voltages. This nonzero value of $\nabla\theta$ represents the state of lowest free energy, consistent with the displacement of the electron distribution as a whole. This condition is expected to exist in a proximity effect weak link, in which the distances involved are

comparable to a coherence length and the boundary conditions on the exponential "tail" of the order parameter are specified at only one plane. Otherwise, this gradient in the phase would not be a free parameter and could not be chosen so as to minimize the free energy. In a bulk superconductor the boundary conditions on the order parameter fix the gradient in the phase, so that this situation does not occur. (That is, the free energy in a bulk superconductor is minimized subject to the constraints introduced by the boundary conditions, which fix the value of $\nabla\theta$.) In addition, the order parameter will not have that form which minimizes the free energy unless the relaxation time of the superfluid is negligibly small. The opposite limit, in which the relaxation time is very long, gives rise to a value of $\nabla\theta$ which is proportional to the time integral of the voltage. This possibility was first suggested by Yu et al (Refs. 3,10,11) and can be shown to give results which cannot be distinguished (on the basis of this experiment) from those of the mechanism suggested here. The mechanism suggested in this chapter corresponds to a coupling or interaction between the normal (or quasi-particle) current and the supercurrent.

The mechanism described above was used to calculate the experimentally measured voltage at the tunnel junction probe, $\bar{V}_T$, which was proportional to the space and time average of μ_{qp} . The calculation began by deriving a modified current-phase relation which includes the effects of a nonzero value of $\nabla\theta$. This current-phase relation was, in turn, used to calculate $\bar{V}_T$ as a function of the time-averaged phase difference ϕ_0 across the weak link and the amplitude of the microwave

voltage v_0 . Because the weak link is current biased rather than voltage biased, the independent (experimentally fixed) parameters in the calculation are the dc bias current I_{dc} and the amplitude of the microwave bias current, $I_{\mu 0}$. Determining ϕ_0 and v_0 in terms of I_{dc} and $I_{\mu 0}$ is somewhat complicated by the fact that the microwave impedance Z is a function of ϕ_0 . This gives rise to a set of nonlinear coupled equations. The solution of these equations shows that ϕ_0 , and thus $\bar{V}_T$, is a discontinuous function of I_{dc} . For this reason the discontinuities experimentally observed in $\bar{V}_T$ were referred to as "phase-jumps."

The results of the calculations described above were found to be in qualitative agreement with the data. The agreement was surprisingly good, especially with regard to the structure of the phase-jumps, considering the number of approximations made in the calculations. This agreement does not necessarily mean that either of these two mechanisms is physically correct, however, since there may very well exist other mechanisms which agree just as well with the data. This is because both the calculations and the experiment give results which are undetermined to within a constant. In addition, the structure of the phase-jumps is primarily determined by the properties of the microwave impedance and is somewhat (although not entirely) insensitive to the mechanism responsible for the quasi-particle nonequilibrium. The results of the experiment clearly indicate, however, the existence of a divergence of the supercurrent (by whatever mechanism) which is dependent on the phase difference across the weak link. The ideas presented in this chapter, if not the measurements, suggest that a coupling or interaction exists between the supercurrent and the quasi-particle (or normal) current under

appropriate conditions.

In addition to the conclusions regarding the nonequilibrium effects discussed above, the observed structure of the phase-jumps clearly indicates the existence of some effects due to the properties of the microwave impedance which have not previously been appreciated. First of all, there are convincing experimental and theoretical reasons to believe that the interdependence of the microwave impedance and the quantum phase angle causes the time-averaged supercurrent not to have a purely sinusoidal dependence on the quantum phase angle, as is usually considered to be the case. The maximum time-averaged supercurrent does not always occur when the quantum phase difference is 90° , and does not go to zero anywhere as a function of the microwave current drive. In addition, there may be more than one value of the quantum phase which produces a given value of the time-averaged supercurrent. Discontinuous transitions or "phase-jumps" can occur between these different solutions for the quantum phase. All of these unusual and surprising properties result from the fact that the weak link is actually current biased, rather than voltage biased, as is usually assumed for the sake of simplicity. The results mentioned in this paragraph are basically equilibrium (or quasi-equilibrium) properties of proximity effect weak links and are manifestations of the Josephson inductance. Such phase-jumps presumably occur in the ordinary operation of proximity effect weak links but are generally not directly observable. The nonequilibrium quasi-particle distribution measured in this experiment serves as a probe of such effects.

Since this is the last chapter in the thesis, it might be appropriate to include a discussion of what, in the author's opinion, is known

and what is not known about proximity effect weak links. Chapter II has shown that those properties of proximity effect weak links, such as the Josephson inductance and microwave impedance, which occur under near equilibrium conditions and which are not directly related to nonequilibrium phenomena, appear to be well described by a current-phase relation such as the phase-slip model. On the other hand, phenomena such as those described in Chapters III and IV, which are fundamentally nonequilibrium in nature, do not appear to be very well understood and further theoretical and experimental work is called for.

In this regard it is probably relevant to point out that the theory of bulk, equilibrium superconductivity has historically progressed through three stages of development: the phenomenological Ginzburg-Landau theory, the microscopic BCS theory, and finally the Green's function approach, which is a generalization of the original BCS theory capable, among other things, of dealing with situations in which spatial variations are important. Most of the work being carried out on proximity effect weak links (and most other types of weak links) has, in some respects, not progressed beyond the first stage of development, the Ginzburg-Landau phenomenological approach. (Although Green's function methods have been attempted, they have not, to the author's knowledge, produced many significant results and in any event are not generally applied.)

In fact, it is probably fair to say that the general state of weak link research has not yet reached even the Ginzburg-Landau stage of development, since many research groups are currently engaged in studying the mathematical implications of the resistively shunted junction

(RSJ) model when various capacitors, resistors, etc. are added to the equivalent circuit. As shown by the last three chapters of this thesis, there are physical processes occurring in proximity effect weak links which are not described by the RSJ model. Many of the attempts which have been made to use a more realistic theoretical model than the RSJ model have been based on the phase-dependent quasi-particle current model, which Chapter II showed to be fundamentally incorrect for proximity effect weak links. Most of the remaining theoretical effort has been applied to the phase-slip model (or other nonlinear applications of the Ginzburg-Landau theory); unfortunately, the phase-slip model is phenomenological, does not properly include any effects due to quasi-particle nonequilibrium, and (for reasons to be discussed in the next paragraph) is perhaps not applicable to proximity effect weak links in the weak coupling limit. In fact, it might be concluded that many aspects of proximity effect weak links are not well understood in any real sense. The specific reasons for this conclusion, based on the experimental results of this thesis, will now be described.

First of all, Chapter II showed that the RSJ model and the phase-dependent quasi-particle current model were incorrect for proximity effect weak links, but that the phase-slip model was in agreement with the data. In particular, the data showed that the amplitude of the oscillating supercurrent is less than the critical current in the AC Josephson effect, as predicted by the phase-slip model. Although this tends to suggest that the phase-slip model is correct, the author would like to emphasize that there remains the possibility that some other

theory (which may be undeveloped at the present time) might also agree with those measurements. The author emphasizes this possibility for the following reason: The phase-slip model results from the (assumed) nonlinearity of the Ginzburg-Landau theory, and the catastrophic phase-slips would not occur without this nonlinearity. For example, the calculations of the current-phase relation in Appendix B are identical to the Ginzburg-Landau theory (if the parameter $\nabla\theta$ is set to zero) for the case of weak coupling, where the Ginzburg-Landau theory becomes linearized in the center of the weak link. If the calculations in Appendix B are examined, it will be found that the phase difference in Ψ (not to be confused with $\nabla\theta$) does periodically change by 2π , and that Ψ indeed must go to zero at some point in the weak link in order for this to happen. (That is, the algebraic sum of the exponential tails Ψ_L and Ψ_R becomes zero, whereas ψ_L and ψ_R themselves do not change except in phase.) It will also be found that, in the calculations of Appendix B, this "phase-slip" is not a catastrophic process involving the evolution of heat, does not require the "destruction" of the order parameter, and does not make the current-phase relation nonsinusoidal. The point is that all the salient features of the phase-slip model depend on the nonlinearity of the equations involved, and it is not obvious that a weakly coupled proximity effect weak link should, under all conditions, be inherently nonlinear. (For example, the calculations in Appendix B of the current-phase relation are quite linear.) On the contrary, the many orders of magnitude over which the critical current can vary exponentially strongly suggest that the equations governing the

weak link are indeed linear. (This tends to indicate that the linear calculations in Appendix B are appropriate.) Thus it is perhaps not obvious that the phase-slip model is always applicable to proximity effect weak links. It should, of course, be pointed out that there may exist some mechanism, of which the author is not aware, which causes the behavior of proximity effect weak links to always be non-linear (in contrast to Appendix B). Furthermore, the phase-slip model does, in any event, appear to be appropriate for strong coupling devices such as Dayem bridges, proximity effect weak links with large critical currents, etc.

Because of these theoretical considerations, it might be suspected that the reduction of the oscillating supercurrent in the AC Josephson effect observed in Chapter II might actually be caused by an interaction mechanism between the quasi-particle current and the supercurrent similar to the one outlined in this chapter, or by the unexpected importance of fluctuations and noise suggested by the results of Chapter III. Or perhaps none of these mechanisms is truly responsible for the results observed. In any event, further work is probably needed before these questions can be resolved.

It might be useful to outline an approach that might be successful in resolving the theoretical difficulties mentioned above. First of all, note that Chapters III and IV have concentrated on the region away from the superconducting-normal boundaries, and on the weak coupling limit; this linearizes the equations involved and gives $\Delta \ll kT$. The purpose of this approach is to make the elaborate Green's function

methods unnecessary (by removing the strong spatial variations), so that the problem can perhaps be treated through a simple generalization of the original BCS method as was attempted in the mechanism suggested in this chapter. If the theorists were to consider this limit, it might be possible for them to deduce more about the physical processes occurring in the weak link. The interaction mechanism between the quasiparticle current and the supercurrent suggested in this chapter was intended to be a first step in this direction. It is the author's feeling that this approach might lead to a detailed understanding of the excess supercurrent phenomenon, the reduction of the oscillating supercurrent in the AC Josephson effect, and any aspects of the Wyatt effect which may not be explained by the Eliashberg theory (Ref. 15).

There are several difficulties involved with the approach outlined above. First, the results of Chapter III suggest that fluctuation effects or the excitation of collective modes might be more important than previously realized, and the inclusion of such effects may be a very difficult task. Second, the approach completely ignores the region near the superconducting-normal interface, where interesting and important phenomena are known to occur. And finally, at high frequencies (large voltages), relaxation time effects may become important. The approach outlined above may be useful despite all these difficulties, since the same difficulties will occur in any other approach as well; the approach outlined here has the advantage of eliminating as many complications as possible. Finally, the real difficulty which will have the biggest effect on the development of the necessary theory is the

fact that proximity effect weak links are extensively studied by only a few groups and have consequently been ignored to some extent by other workers in this field. This is especially unfortunate because of the fact that the approach outlined above is not applicable to Dayem bridges, etc. which are intrinsically nonlinear and not as reproducibly fabricated and tested as proximity effect weak links. This situation may improve in the future as the number of practical applications of proximity effect weak links increases.

APPENDIX A MICROWAVE IMPEDANCE CALCULATIONS

A.1 Introduction

The purpose of this appendix is to calculate the microwave impedance Z of a current-biased weak link, as well as a few other related properties, as outlined in Chapter II. First of all, the calculations will be done for the case of small microwave currents and small critical currents. In addition, most of the calculations will be carried out using the resistively shunted junction (RSJ) model; the only exception consists of one section which has been devoted to generalizing the results obtained for the RSJ model to the other models described in Chapter II. The notation used here will be the same as that used in Chapter II, and the definition of variables already given in the text will sometimes not be repeated here. (It should be pointed out that the notation used in each chapter and its corresponding appendix, if there is one, may not be the same as that of other chapters; there are so many variables that it is difficult to avoid duplicating the use of some symbols.) Any physical arguments or discussions given in the text will generally not be repeated here.

The appendix begins with a calculation of Z for the case of zero dc voltage. As pointed out in the text, these calculations are straightforward since ordinary perturbation theory can be used. Next, Z will be calculated for nonzero dc voltages using the approximation method described in the text. The results obtained for the RSJ model will then be generalized to include the other models dealt with in Chapter

II. The appendix concludes with a calculation of the subharmonic step structure.

A.2 Impedance Calculation for Zero DC Voltage

As discussed in Chapter II, the RSJ model for a current biased weak link gives rise to the following equations:

$$I = I_c \sin(\phi) + v/R_N \quad (A.1)$$

$$I = I_1 + I_2 \sin(\omega_\mu t) \quad (A.2)$$

$$\frac{d\phi}{dt} = \frac{2ev}{\hbar} \quad (A.3)$$

where all the variables have already been defined in the text. All the calculations in this appendix, unless specifically stated otherwise, are done for the approximate case of $\alpha \ll 1$ and $\beta \ll 1$, where

$$\alpha = \frac{2eR_N I_c}{\hbar \omega_\mu} \quad (A.4)$$

$$\beta = \frac{2eR_N I_2}{\hbar \omega_\mu} \quad (A.5)$$

In addition, this section assumes that $|I_1| \leq I_c$, so that the dc (time-averaged) voltage across the weak link is zero.

There are several methods which could be used to solve this set of nonlinear equations. An attempt will be made throughout this appendix to use that method which gives the greatest physical insight into the problem. Frequently, as is the case here, the method of successive iteration (Ref. 1) appears to be most suitable. This method will now be used to calculate the Josephson inductance, L_J .

Physically, the requirement $\alpha \ll 1$ and $\beta \ll 1$ will be seen to cause the voltage across the weak link to be very nearly sinusoidal at frequency ω_μ . (This is only true for $|I_1| \leq I_c$, the case being considered in this section.) Thus the iteration can start with the approximation

$$v = v_0 \sin(\omega_\mu t + \theta) \quad (\text{A.6})$$

where v_0 and θ are constants. This can be inserted back into eq. (A.3) and integrated to obtain

$$\phi = \phi_0 - \frac{2ev_0}{\hbar \omega_\mu} \cos(\omega_\mu t + \theta) \quad (\text{A.7})$$

where ϕ_0 is the constant of integration. (ϕ_0 is actually determined by the value of I_1 .) With this value of ϕ , eq. (A.1) becomes

$$I = I_c \sin\left[\phi_0 - \frac{2ev_0}{\hbar \omega_\mu} \cos(\omega_\mu t + \theta)\right] + v/R_N \quad (\text{A.8})$$

This can be expanded in Taylor series to give

$$I = I_c \sin(\phi_0) - I_c \cos(\phi_0) \frac{2ev_0}{\hbar \omega_\mu} \cos(\omega_\mu t + \theta) + v/R_N \quad (\text{A.9})$$

This expansion, in which terms of order $(2ev_0/\hbar\omega_\mu)^2$ or higher have been dropped, is valid because, as will be shown later, $2ev_0/\hbar\omega_\mu \leq \beta$. Physically, $\beta \ll 1$ implies that the microwave voltage precesses ϕ by an amount small compared to 2π , so that eq. (A.1) can be linearized. (In other words, if the change in ϕ from its dc value ϕ_0 is small, then $\sin(\phi)$ can reasonably be expanded in Taylor series.)

If desired, eq. (A.9) could now be used to find the next approximation to v , and the whole process repeated until the required accuracy is obtained. However, the linear approximation already obtained is accurate enough for the purposes of this section. Note that eq. (A.9) contains two currents at frequency ω_μ ; the v/R_N term which is resistive, and the $I_c \cos(\phi_0)$ term which is 90° out of phase with the voltage and corresponds to an inductive impedance $\omega_\mu L_J$ given by (from the ratio of the voltage to current)

$$\omega_\mu L_J = \frac{\hbar \omega_\mu}{2e I_c \cos(\phi_0)} \quad (\text{A.10})$$

In other words, the current in eq. (A.9) is equivalent to that which would pass through a resistor R_N in parallel with an inductor L_J . The existence of this linear equivalent circuit is a direct consequence of the fact that, when the deviation of ϕ from ϕ_0 is small, all the equations can be linearized.

Two assertions having to do with the effects of $\alpha \ll 1$ and $\beta \ll 1$ were made in the analysis above, and they must now be shown to be consistent with the results obtained. (Alternatively, the solution obtained could simply be inserted back into the original equations to show that it does indeed satisfy the equations to a certain order in α and β .) Since the (approximate) results were equivalent to a resistor in parallel with an inductor, the passage of a microwave current I_2 will generate a voltage which is, indeed, very nearly sinusoidal at frequency ω_μ , in agreement with eq. (A.6). And since $|Z| \leq R_N$ (although this is probably apparent, it is explicitly proven

in Appendix B), it follows that $2eV_0/\hbar\omega_\mu \leq \beta$, as mentioned earlier. Physically, $\alpha \ll 1$ and $\beta \ll 1$ imply that voltages of $I_C R_N$ or $I_2 R_N$ precess ϕ by an amount small compared with 2π , so that the microwave response of the device is approximately linear (for $|I_1| \leq I_C$ only).

The Josephson inductance L_J will now be expressed in terms of the experimentally observable quantity I_1 , rather than ϕ_0 . As will be shown shortly, the time-averaged supercurrent, $\bar{I}_S$, is just $I_C \sin(\phi_0)$, where terms of order β^2 or higher have again been dropped. This must equal I_1 if $\bar{V}$ is zero, as can be seen from the time average of eqs. (A.1) and (A.2). Thus $\sin(\phi_0) = I_1/I_C$, and eq. (A.10) can be rewritten as

$$L_J = \frac{\hbar}{2eI_C [1 - (I_1/I_C)^2]^{1/2}} \quad (\text{A.11})$$

where the identity $\cos^2(\phi_0) = 1 - \sin^2(\phi_0)$ has been used. The impedance itself is that of a resistor R_N in parallel with inductor L_J , which gives

$$Z = \left[\frac{1}{R_N} + \frac{1}{j\omega L_J} \right]^{-1} \quad (\text{A.12})$$

This reduces to

$$Z = \frac{R_N (\omega_\mu L_J)^2}{R_N^2 + (\omega_\mu L_J)^2} + j \frac{R_N^2 \omega_\mu L_J}{R_N^2 + (\omega_\mu L_J)^2} \quad (\text{A.13})$$

This concludes the calculation of the microwave impedance for the case $|I_1| \leq I_C$.

This is a convenient point in the analysis to calculate the time-averaged supercurrent, $\bar{I}_s$ (also for the case $|I_1| \leq I_c$). (The result was also used in the preceding paragraph.) This is most easily accomplished by rewriting eq. (A.8) using the trigonometric identity for the sine of the sum of two terms, with the result that

$$\begin{aligned} I_s = & I_c \sin(\phi_0) \cos[x \cos(\omega_\mu t + \theta)] \\ & - I_c \cos(\phi_0) \sin[x \cos(\omega_\mu t + \theta)] \end{aligned} \quad (\text{A.14})$$

Here x has been defined by

$$x = \frac{2ev_0}{\hbar \omega_\mu} \quad (\text{A.15})$$

The trigonometric functions whose arguments are themselves trigonometric functions can be expanded using the generating function and related series for Bessel functions (Ref. 2) and give :

$$\begin{aligned} I_s = & I_c \sin(\phi_0) [J_0(x) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(x) \cos(2k(\omega_\mu t + \theta))] \\ & - I_c \cos(\phi_0) [2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x) \cos((2k+1)(\omega_\mu t + \theta))] \end{aligned} \quad (\text{A.16})$$

The time-average of eq. (A.16) is, by inspection,

$$\bar{I}_s = I_c \sin(\phi_0) J_0(x) \quad (\text{A.17})$$

Since $J_0(x)$ has a maximum value of 1 for $x = 0$, nonzero microwave voltages reduce the effective critical current. Because $x \leq \beta \ll 1$, J_0 can be expanded in Taylor series (Ref. 3) to give

$$J_0(x) = 1 - x^2/4 \quad (A.18)$$

where terms of order x^4 or higher have been dropped. Thus the maximum zero-voltage dc current is reduced by an amount δI_c which is proportional to v_0^2 . (Recall that x_0 is proportional to v_0 .) As pointed out in the text, the measured value of $\delta I_c/I_c$ was used to determine the value of I_2 .

This concludes the calculation of the weak link properties of interest for the case where $|I_1| \leq I_c$. These calculations were made straightforward (the equations were linearized) by the fact that ϕ is always very nearly equal to ϕ_0 , a constant. The next section goes on to consider the inherently nonlinear situation which results for $|I_1| > I_c$.

A.3 Impedance Calculation for Nonzero DC Voltage

A lengthy outline and intuitive explanation of the calculations in this section were given in the text and need not be repeated here. The goal of the calculation is to derive Z from eqs. (A.1) through (A.5), but in this section the time-averaged voltage is nonzero so that the deviation of ϕ from ϕ_0 is generally not small (as it was in the previous section.) The calculation begins by defining

$$\omega_0 = \frac{2eR_N I_1}{\hbar} \quad (A.19)$$

and making the change of variables from $\phi(t)$ to $\gamma(t)$ given by

$$\phi(t) = \phi_{R0} + \omega_0 t + \gamma(t) \quad (A.20)$$

where ϕ_{R0} is a constant chosen to make $\gamma(t) = 0$ at $t = 0$. Physically, ω_0 is the Josephson frequency of oscillation of the supercurrent in the limit of $I_c \rightarrow 0$ and $I_2 \rightarrow 0$. For small values of α and β , $\gamma(t)$ will be small compared with 2π for one microwave period (or perhaps for a few, but not indefinitely). This allows the equations in γ to be linearized. (These linearized equations will be an excellent approximation for a few microwave periods only; the usefulness of this approach will be seen shortly.)

From eqs. (A.3) and (A.20)

$$\dot{\phi} = \omega_0 + \dot{\gamma} = \frac{2ev}{\hbar} \quad (A.21)$$

where dots will be used to denote time derivatives. This equation can be solved for v as a function of $\dot{\gamma}$ and inserted, along with eq. (A.20), into eqs. (A.1) and (A.2) to give

$$I_c \sin(\phi_{R0} + \omega_0 t + \gamma) + \frac{(\omega_0 + \dot{\gamma})\hbar}{2eR_N} = I_1 + I_2 \sin \omega_\mu t \quad (A.22)$$

This equation can be expanded in Taylor series to give, to first order in γ ,

$$I_c \sin(\phi_{R0} + \omega_0 t) + \gamma I_c \cos(\phi_{R0} + \omega_0 t) + \frac{\omega_0 \hbar}{2eR_N} + \frac{\dot{\gamma} \hbar}{2eR_N} = I_1 + I_2 \sin \omega_\mu t \quad (A.23)$$

Note that, from the definition of ω_0 , $\omega_0 \hbar / 2eR_N = I_1$, so that the two largest terms in eq. (A.23) cancel out. (This was the purpose of introducing γ .) Solving for $\dot{\gamma}$ and using the definition of α and β yields

$$\begin{aligned}\dot{\gamma} = & \beta \omega_{\mu} \sin \omega_{\mu} t - \alpha \omega_{\mu} \sin(\phi_{R0} + \omega_0 t) \\ & - \alpha \omega_{\mu} \gamma \cos(\phi_{R0} + \omega_0 t)\end{aligned}\quad (A.24)$$

Note the dependence of $\dot{\gamma}$ on γ itself, which will later be seen to be responsible for the phase-locking which creates the first step. (This section will restrict itself to the structure near the first step.)

As outlined in Chapter II, the plan is to solve this approximate differential equation in $\dot{\gamma}$ for $\gamma(\tau)$, the value of $\gamma(t)$ after one microwave period. ($\tau = 2\pi/\omega_{\mu}$.) The usefulness of this has been explained in the text and will be apparent here shortly. Equation (A.24) is first order linear and therefore possesses an integrating factor λ (Ref. 4). Although λ can be (and was) found in a straightforward manner, it is much easier to solve eq. (A.24) by successive iteration. Since α , β , and γ are all small, the last term in eq. (A.24), which is of order $\alpha\gamma$, is smaller than the other two terms. Thus the zero order approximation for γ , which will be denoted γ_0 , can be found by (temporarily) dropping the last term in eq. (A.24) and integrating

$$\gamma_0(t) = -\beta \cos(\omega_{\mu} t) + \alpha \cos(\phi_{R0} + \omega_0 t) \quad (A.25)$$

This result for γ can now be used in the last term of eq. (A.24) to obtain the next order approximation for γ , denoted $\gamma_1(t)$:

$$\begin{aligned}\dot{\gamma}_1(t) = & \beta \omega_{\mu} \sin(\omega_{\mu} t) - \alpha \omega_{\mu} \sin(\phi_{R0} + \omega_0 t) \\ & - \alpha \omega_{\mu} \cos(\phi_{R0} + \omega_0 t)[- \beta \cos(\omega_{\mu} t) + \alpha \cos(\phi_{R0} + \omega_0 t)]\end{aligned}\quad (A.26)$$

Although the iteration could be continued to obtain arbitrary accuracy (higher order terms would also have to be kept in eq. (A.23) and below), this result is accurate enough for the purposes of this appendix. Fortunately, eq. (A.26) need not be solved for γ as a function of time, since all that is required is $\gamma(\tau)$; this can be obtained by integrating eq. (A.26) from 0 to τ . The result is

$$\begin{aligned} \gamma(\tau) = & - \int_0^{\tau} \alpha^2 \omega_{\mu} \cos^2(\phi_{R0} + \omega_0 t) dt \\ & + \int_0^{\tau} \omega_{\mu} \alpha \beta \cos(\phi_{R0} + \omega_0 t) \cos(\omega_{\mu} t) dt \end{aligned} \quad (\text{A.27})$$

where the fact that $\gamma(0) = 0$ has been used. The first integral in eq. (A.27) is independent of the microwave current and does not affect the structure of the step (it gives the usual shape of the I-V curve for $I_2 = 0$); this term will be ignored here. The $\cos(\phi_{R0} + \omega_0 t)$ term in the second integral can be expanded using trigonometric identities and the integral carried out. (Recall that ω_0 is very nearly equal to ω_{μ} near the first step for $\alpha \ll 1$ and $\beta \ll 1$.) The orthonormality of the sine and cosine reduce the result to

$$\gamma(\tau) = \frac{1}{2} \alpha \beta \omega_{\mu} \tau \cos(\phi_{R0}) \quad (\text{A.28})$$

As pointed out in the text, the variable of interest is not γ , but ϕ_R , defined by

$$\phi_R = \phi - \omega_{\mu} t \quad (\text{A.29})$$

Physically, ϕ_R is the difference between the quantum phase and the phase of the external microwave current drive. For $\alpha \ll 1$ and $\beta \ll 1$, ϕ_R is slowly varying near the first step (since the voltages corresponding to $I_1 R_N$ and $I_2 R_N$ can change ϕ during one microwave period by only an amount which is small compared to 2π). Thus $\dot{\phi}_R$ can be approximated by $\Delta\phi_R/\tau$, where $\Delta\phi_R$ is the change in ϕ_R over one period τ . From the definitions of ϕ_R and γ

$$\phi_R = \phi_{R0} + (\omega_0 - \omega_\mu)t + \gamma \quad (\text{A.30})$$

Since ϕ_{R0} is a constant, differentiation gives the change in ϕ_R over one period τ as

$$\Delta\phi_R = (\omega_0 - \omega_\mu)\tau + \Delta\gamma \quad (\text{A.31})$$

The time derivative of ϕ_R is then approximately

$$\dot{\phi}_R = \frac{\Delta\phi_R}{\tau} = \omega_0 - \omega_\mu + \frac{1}{2} \alpha \beta \omega_\mu \cos(\phi_R) \quad (\text{A.32})$$

where eq. (A.28) for $\Delta\gamma$ has been used with the subscript on ϕ_R dropped (since $\dot{\phi}_R \doteq \dot{\phi}_{R0}$ for one period). Strictly speaking this approximation, which is sometimes known as an adiabatic approximation, is not valid for values of I_1 which differ greatly from the value of I_1 at the center of the step, but this is of no importance since the superconducting response is zero away from the step ($Z \doteq R_N$). Equation (A.32) would appear to be exact in the limit $\alpha \rightarrow 0$, $\beta \rightarrow 0$.

Equation (A.32) can be cast in a somewhat more useful form by defining

$$\omega_1 = \omega_0 - \omega_\mu \quad (\text{A.33})$$

$$\omega_2 = \frac{\alpha\beta}{2} \omega_0 \quad (\text{A.34})$$

$$\phi_P = \phi_R - 90^\circ \quad (\text{A.35})$$

Then the approximate differential equation for ϕ_P is (from eq. (A.32)):

$$\dot{\phi}_P = \omega_1 - \omega_2 \sin \phi_P \quad (\text{A.36})$$

This differential equation has exactly the same form as that for ϕ itself at the zero step (critical current) with no microwave current present. This can be seen by solving eqs. (A.1) and (A.2) for v and inserting into eq. (A.3) to give

$$\dot{\phi} = \left(\frac{2eR_N I_1}{\hbar} \right) - \left(\frac{2eR_N I_c}{\hbar} \right) \sin \phi \quad (\text{A.37})$$

where, in the equation above, I_2 was set to zero. Both the terms in parentheses in eq. (A.37) have the dimensions of frequency, so that eqs. (A.36) and (A.37) have exactly the same form. This analogy between the first step and the zero step was (evidently) first pointed out in the dc calculations of Volkov and Nad' (Ref. 5). (The author was not aware of the work by Volkov and Nad' at the time these calculations were done.) The solution to eq. (A.37) is well known in the literature (Ref. 6), since it deals with the dc properties of weak links. Only an outline and not the details of its solution will be given here, since the algebra would occupy almost as much space as the

rest of this section. There are two different cases which will now be considered separately.

First, there is a static solution, $\dot{\phi}_p = 0$, which corresponds to the condition of complete phase-lock between the oscillating super-current and the microwave current drive. (This occurs in the central region of the step where $d\bar{v}/d\bar{I} = 0$.) Equation (A.36) then immediately gives

$$\phi_p = \sin^{-1}(\omega_1/\omega_2) \quad (\text{A.38})$$

It is convenient to define v_μ and u_μ as the time-averaged Fourier components of $v(t)$ at frequency ω_μ which are, respectively, in phase or out of phase with the external microwave current drive. (The time-average is required because $v(t)$ is not, in general, a periodic function of time, so that its Fourier components are slowly varying functions of time.) That is,

$$v_\mu = \frac{2}{N\tau} \int_0^{N\tau} v(t) \sin(\omega_\mu t) dt \quad (\text{A.39})$$

$$u_\mu = \frac{2}{N\tau} \int_0^{N\tau} v(t) \cos(\omega_\mu t) dt \quad (\text{A.40})$$

Here N is a large integer.

Equations (A.1) and (A.2) can be solved for $v(t)$ and yield

$$v(t) = -R_N I_C \sin(\phi) + R_N I_2 \sin(\omega_\mu t) + I_1 R_N \quad (\text{A.41})$$

The definitions of ϕ_R and ϕ_p can be used to eliminate ϕ in the equation above. The results reduce to

$$v(t) = -R_N I_C \cos(\phi_p) \cos(\omega_\mu t) + R_N I_C \sin(\phi_p) \sin(\omega_\mu t) + R_N I_2 \sin(\omega_\mu t) + I_1 R_N \quad (A.42)$$

Using the results for $\sin(\phi_p)$ in eq. (A.38), and the relation $\cos^2(\phi_p) = 1 - \sin^2(\phi_p)$, the Fourier components are

$$v_\mu = (\omega_1/\omega_2) R_N I_C + R_N I_2 \quad (A.43)$$

$$u_\mu = -R_N I_C [1 - (\omega_1/\omega_2)^2]^{1/2} \quad (A.44)$$

From the definition given in the text (in terms of the ratio of voltage to current), the real, R , and imaginary, X , parts of Z are just

$R = v_\mu/I_2$ and $X = u_\mu/I_2$. Consequently,

$$R = (\omega_1/\omega_2)(I_C/I_2) R_N + R_N \quad (A.45)$$

$$X = -R_N (I_C/I_2) [1 - (\omega_1/\omega_2)^2]^{1/2} \quad (A.46)$$

Once again, these solutions are for the case of $\dot{\phi}_R = 0$, which (from eq. (A.38)) occurs only for $|\omega_1| < \omega_2$, which corresponds to the phase-locked central part of the step.

The second case, where $|\omega_1| > \omega_2$ and, as a result, $\dot{\phi}_R \neq 0$, is much more complicated. Although the calculation will not be given here, eq. (A.36) can be solved for ϕ_p (see, for example, Ref.6) by making the change of variables $D = \dot{\phi}_p$ and putting eq. (A.36) into the form

$$\int_0^t D \, dt = \sin^{-1} \left(\frac{\omega_1 - D}{\omega_2} \right) \quad (\text{A.47})$$

Differentiating both sides of the equation and rearranging terms gives

$$\omega_2 dt = - \frac{dD}{D \left[1 - \left(\frac{\omega_1 - D}{\omega_2} \right)^2 \right]^{1/2}} \quad (\text{A.48})$$

This reduces the problem to an indefinite integral. As can be seen from eq. (A.42), the time-averaged values of $\sin(\phi_p)$ and $\cos(\phi_p)$ are what is needed to calculate v_μ and u_μ . After a considerable amount of algebra, it can be shown that (Ref. 6)

$$\overline{\dot{\phi}_p} = \pm (\omega_1^2 - \omega_2^2)^{1/2} \quad (\text{A.49})$$

where the bar denotes the time average and the choice of $\pm$ sign depends on the sign of ω_1 . From the original differential equation, eq. (A.36):

$$\overline{\sin(\phi_p)} = \frac{\omega_1 - \overline{\dot{\phi}_p}}{\omega_2} \quad (\text{A.50})$$

Combining eqs. (A.49) and (A.50) gives

$$\overline{\sin(\phi_p)} = \overline{\cos(\phi_R)} = \frac{\omega_1 \mp (\omega_1^2 - \omega_2^2)^{1/2}}{\omega_2} \quad (\text{A.51})$$

From the symmetry of eq. (A.36)

$$\overline{\cos(\phi_p)} = \overline{\sin(\phi_R)} = 0 \quad (\text{A.52})$$

This can be seen from the fact that the value of $\dot{\phi}_R$ is unaltered by the transformation $\phi_R \rightarrow -\phi_R$, which reverses the sign of $\sin(\phi_R)$; thus the time-average of $\sin(\phi_R)$ is zero.

Using eqs. (A.51), (A.52), and (A.42), the values of v_μ and u_μ can now be evaluated:

$$v_\mu = \pm \left(\frac{R_N I_C}{\omega_2} \right) [(\omega_1^2 - \omega_2^2)^{1/2} - |\omega_1|] + R_N I_2, \quad (A.53)$$

$$u_\mu = 0 \quad (A.54)$$

Here the plus (minus) sign is to be used for negative (positive) values of ω_1 . Once again, this solution is for the case of $|\omega_1| > \omega_2$, which corresponds to incomplete phase-locking between the oscillating super-current and the external microwave current drive. As before, $R = v_\mu / I_2$ and $X = u_\mu / I_2$. It should also be emphasized once again that these results are for the RSJ model, eq. (A.1).

A.4 Generalization of Results to Other Models

In order to avoid confusion, the calculations presented above were limited to the RSJ model, eq. (A.1). The purpose of this section is to show that the results can easily be generalized to the phase-slip model or the phase-dependent quasi-particle current model, usually without having to perform any additional calculations. The phase-slip model will be treated first, followed by the phase-dependent quasi-particle current model.

In the phase-slip model, eq. (A.1) is replaced by

$$I = \frac{I_C}{2} (1 + \cos \phi) + v/R_N \quad (A.55)$$

As pointed out in the text, this equation is only meant to be applied for nonzero dc voltages, so that a calculation of L_J similar to that of Section A.2 is not appropriate. Only the step structure remains to be calculated, as was done in Section A.3 for the RSJ model. It should be apparent that the first term on the right-hand side of eq. (A.55), the constant $I_C/2$, cannot give rise to microwave currents or voltages and can therefore be ignored for the purposes of this appendix. This term does affect the location of the step as a function of I_J , but even this effect goes to zero in the limit $\alpha \ll 1$. $I_C/2$ could be subtracted from I_J , but this will not be done here in order to be consistent with the fact that the rest of the calculations are lowest order in α . The same comments can be made about the α^2 term in eq. (A.27), which was dropped for the same reason, even though it has the effect of slightly reducing the value of the dc voltage across the weak link and changing the location of the step.

Having dropped the constant $I_C/2$ term, the calculation of Z can be carried out using the same kinds of techniques as were used in Section A.4. The results turn out to be exactly the same, with I_C replaced by $I_C/2$. Thus replacing $\sin(\phi)$ by $\cos(\phi)$ has no effect on the structure at the step, and the only difference between the RSJ model and the phase-slip model at the first step is the factor of $1/2$ in eq. (A.55). This calculation need not be worked out here (the author will admit that he actually did the calculation) because the results can be immediately obtained by making the change of variables $\phi^* = \phi + 90^\circ$, $I_C^* = I_C/2$. This reduces eq. (A.55) (with the constant

$I_C/2$ term dropped) to the same form as eq. (A.1), which proves that the result for Z is the same, with I_C replaced by $I_C^* = I_C/2$. This concludes the discussion of the results for the phase-slip model.

In the phase-dependent quasi-particle current model, eq. (A.1) is replaced with

$$I = I_C \sin \phi + v\sigma[1 - \cos(\phi)] \quad (\text{A.56})$$

where $\sigma = 1/R_N$, as described in the text. Although this equation, in its present form, should perhaps not be applied at the step (see the text for a discussion), the structure in Z at the step will now be calculated. (The predictions of this equation can be compared with the data regardless of the theory's validity.) At the step, the value of v in eq. (A.56) is approximately given by

$$v = R_N I_1 + 0\alpha + 0\beta \quad (\text{A.57})$$

If the terms linear in α and β are dropped (the validity of this will be examined shortly) then eq. (A.56) can be rewritten as

$$I = (I_C \sin \phi - I_1 \cos \phi) + v/R_N \quad (\text{A.58})$$

where $\sigma = 1/R_N$ has been used.

The first two terms in eq. (A.58) have been grouped together because their effects are the same, as can be seen by using the following change of variables:

$$\phi^* = \tan^{-1}(-I_1/I_C) + \phi \quad (\text{A.59})$$

$$I_C^* = (I_C^2 + I_1^2)^{1/2} \quad (\text{A.60})$$

This puts eq. (A.58) in the form

$$I = I_C^* \sin \phi^* + v/R_N \quad (\text{A.61})$$

which is identical to the form of eq. (A.1). Then the step structure would be the same as in the RSJ model and the formulas for Z would be correct if I_C were replaced by I_C^* . However, if the term linear in β is included in eq. (A.57), it can readily be shown that the phase-dependent quasi-particle current does not contribute to the phase-lock mechanism. Consequently, for the values of I_C and I_1 shown in the text, this gives a step structure whose width is approximately the same but whose height is approximately ten times larger than was observed experimentally. (In addition, the symmetry of v_μ and u_μ would be reversed.) Although this analysis shows that eq. (A.56) is grossly incorrect at the first step, there are some possible theoretical reasons (discussed in the text) to believe that it should not be applied to this situation.

The final task of this section is to calculate L_J using eq. (A.56) instead of eq. (A.1). Now the calculations can be linearized, as in Section A.2, so that they are straightforward. From the arguments presented in Section A.2, it is known that, for $\alpha \ll 1$ and $\beta \ll 1$, ϕ is not precessed very much from its time-average, ϕ_0 . (This can be shown for eq. (A.56) in the same way that it was shown for eq. (A.1).) Thus ϕ can be written as $\phi = \phi_0 + \delta\phi(t)$, where $\delta\phi(t) \ll 2\pi$.

Dropping higher order terms, eq. (A.56) becomes

$$I = I_c \sin(\phi_0 + \delta\phi) + v\sigma[1 - \cos(\phi_0)] \quad (\text{A.62})$$

Here the $v \cos(\phi_0)$ term differs from $v \cos(\phi)$ only by an amount which is higher order in v . (For small voltages, $v\delta\phi$ is second order in v .) Since $\sigma = 1/R_N$ and $1 - \cos(\phi_0)$ is independent of $\delta\phi$ or v , the following change of variables

$$R_N^* = R_N / [1 - \cos(\phi_0)] \quad (\text{A.63})$$

reduces eq. (A.62) to

$$I = I_c \sin(\phi) + v/R_N^* \quad (\text{A.64})$$

Once again, this result is identical in form to eq. (A.1), so that the formulas for Z or L_J are still correct if R_N is replaced by R_N^* . Note that R_N^* is sometimes very different from R_N ; in particular, $R_N^* \rightarrow \infty$ as $\phi_0 \rightarrow 0$. This result, although mathematically correct, strikes the author as being physically unreasonable. It should be pointed out that this peculiar result can not be blamed on Josephson or any other theorist, since eq. (A.56) was empirically formulated by some experimentalists (Ref. 7).

The expression for R_N^* given in eq. (A.63) can be used to evaluate the formula for L_J given in Section A.2. As discussed in the text, these results also show order of magnitude discrepancies with the data.

A.5 Calculation of Subharmonic Step Structure

As indicated in the text, the structure at the subharmonic (where the Josephson frequency due to the dc voltage is equal to half the microwave frequency ω_{μ}) can be calculated if harmonic terms are present in the current-phase relation. Such terms are expected on the basis of the phase-slip model, for example. Thus eq. (A.1) should perhaps be replaced with

$$I = I_C \sin(\phi) + I'_C \sin(2\phi) + v/R_N \quad (\text{A.65})$$

The case of $I'_C \ll I_C$ will be considered here (primarily for mathematical convenience). As has already been pointed out in the text, the results for Z to lowest order in α have just the opposite symmetry as the data. Such a calculation is completely analogous to that of Section A.3 and will not be carried out here. Instead, a higher order calculation will be presented here which does give the correct symmetry. As will be seen, this higher order calculation can, in fact, give the largest contribution to Z if I'_C/I_C is sufficiently small compared to α . Although there is no way of knowing whether or not this condition is satisfied in the experiment, the calculations will proceed by assuming that it is. The idea is to present a possible explanation for the symmetry observed at the subharmonic, even though it is not clear if the necessary conditions are actually met.

In order to simplify the algebra, only the central portion of the step, where the phase-locking is complete, will be considered, although the results can be generalized to the rest of the step as well. Once again, the method of successive iteration will be used,

starting with the zero-order approximation for ϕ :

$$\phi = \omega_0' t \quad (\text{A.66})$$

where ω_0' is defined by $\omega_0' = 2eR_N I_1 / \hbar$ ($\omega_0' \doteq \omega_\mu / 2$). Here t has been chosen so that $\phi = 0$ at $t = 0$. This necessitates rewriting the current drive as

$$I = I_1 + I_2 \sin(\omega_\mu t + \theta_0) \quad (\text{A.67})$$

which replaces eq. (A.2). The reason for putting the initial phase in the microwave current, rather than ϕ , will become apparent shortly. Equations (A.65) and (A.67) can be solved for $v(t)$:

$$\begin{aligned} v = & R_N I_1 + R_N I_2 \sin(\omega_\mu t + \theta_0) \\ & - R_N I_c \sin(\omega_0' t) - R_N I_c' \sin(2\omega_0' t) \end{aligned} \quad (\text{A.68})$$

Approximating ω_μ by $2\omega_0'$, dropping the I_c' term (which is higher order than needed at this point), and integrating eq. (A.3) (with v given by eq. (A.68)) gives the next approximation for ϕ :

$$\phi = \omega_0' t - \beta \cos(2\omega_0' t + \theta_0) + 2\alpha \cos(\omega_0' t) \quad (\text{A.69})$$

The current can now be calculated by inserting this value of ϕ into eq. (A.65) and expanding the trigonometric functions to yield

$$\begin{aligned} I = & I_c \sin(\omega_0' t) \\ & - \beta I_c \cos(\omega_0' t) \cos(2\omega_0' t) \cos(\theta_0) \\ & + \beta I_c \cos(\omega_0' t) \sin(2\omega_0' t) \sin(\theta_0) \end{aligned}$$

$$\begin{aligned}
 & + 2\alpha I'_c \cos(\omega'_0 t) \cos(\omega'_0 t) \\
 & + I'_c \sin(2\omega'_0 t) \\
 & - 2\beta I'_c \cos(2\omega'_0 t) \cos(2\omega'_0 t) \cos(\theta_0) \\
 & + 2\beta I'_c \cos(2\omega'_0 t) \sin(2\omega'_0 t) \sin(\theta_0) \\
 & + 4\alpha I'_c \cos(2\omega'_0 t) \cos(\omega'_0 t) \\
 & + v/R_N
 \end{aligned} \tag{A.70}$$

As seen in Section A.3, the phase-locking is maintained by those terms whose time-average is nonzero and dependent upon the value of θ_0 . The only such term in eq. (A.70) is the sixth, whose time-average is

$$\bar{I}_S = -\beta I'_c \cos(\theta_0) \tag{A.71}$$

It will be convenient to define

$$\omega'_1 = \omega'_0 - \omega_\mu/2 \tag{A.72}$$

$$\alpha' = \frac{2eR_N I'_c}{\hbar \omega'_0} \tag{A.73}$$

and

$$\omega'_2 = \alpha' \beta \omega'_0 \tag{A.74}$$

The primes are used only because similar symbols have already been used in Section A.3. Equation (A.65) can be written

$$I = I_S + v/R_N \tag{A.75}$$

and the time average of this, combined with the time-average of eq.

(A.67) gives

$$I_1 = \bar{I}_S + \bar{v}/R_N \quad (A.76)$$

Note that the left-hand side of eqs. (A.75) and (A.76) differ by a subscript; eq. (A.76) is perhaps obvious. Because the phase-lock has been assumed to be complete ($\bar{\phi} = \omega_\mu/2$), the time-average of eq. (A.3), together with eq. (A.71) and eq. (A.76) can be combined to give

$$\cos \theta_0 = - \frac{\omega_1'}{\omega_2'} \quad (A.77)$$

The steps leading to eq. (A.77) have not been discussed in much detail, since they are completely analogous to the derivation of eq. (A.38). Note that the phase-lock has been produced by the I_C' term in eq. (A.65), not the I_C term.

Now that the current is known as a function of time, as well as θ_0 as a function of the dc bias current, the Fourier components v_μ' and u_μ' of the microwave voltage can be calculated. From eq. (A.75), $v = R_N(I - I_S)$, so that each of the supercurrent terms in eq. (A.70) can contribute to v . Those terms at frequencies other than $2\omega_0'$ do not contribute to v_μ' and u_μ' . Using trigonometric identities such as $\sin(x) \cos(y) = [\sin(x+y) + \sin(x-y)]/2$, it can readily be seen that the only component due to I_C at a frequency of $2\omega_0'$, denoted by $v_{s\mu}$, is from the fourth term in eq. (A.70):

$$v_{s\mu} = -\alpha R_N I_C \cos(2\omega_0' t) \quad (A.78)$$

Here the terms in I_C' have been ignored, since $I_C' \ll I_C$. Using eq. (A.77)

the in and out of phase parts of (A.78) are found to be

$$v'_\mu = \alpha R_N I_C \sqrt{1 - (\omega'_1/\omega'_2)^2} + R_N I_2 \quad (\text{A.79})$$

$$u'_\mu = \alpha R_N I_C \frac{\omega'_1}{\omega'_2} \quad (\text{A.80})$$

Note that the subharmonic step structure given by eqs. (A.79) and (A.80) is the same as that of the step itself, given by eqs. (A.43) and (A.44), except that the roles of the real and imaginary parts have been interchanged and all the symmetries reversed. In addition, the magnitude of v'_μ and u'_μ has been reduced by an extra factor of α , making these results second order (I_C is proportional to α .)

Actually, eq. (A.65) is not as general as possible and could have been replaced by

$$I = I_C \sin(\phi + \rho) + I'_C \sin[2(\phi + \rho) + \delta] + v/R_N \quad (\text{A.81})$$

Note that the results of the calculation presented above corresponded to phase-locking due to I'_C (since the I_C term will not cause any locking at the subharmonic), but that the voltages v'_μ and u'_μ were produced by I_C (since $I_C \gg I'_C$). Therefore, eq. (A.81) will give the same result as before, except that Z will be multiplied by a phase factor which depends on δ . (In other words, the I'_C term fixes the relative phase between the oscillating supercurrent and the microwave current. Since the voltages are produced by the I_C term, the relative phase of the I_C and I'_C terms determines the phase between the microwave current and the microwave voltage. Changing the relative phase of the microwave current

and voltage is equivalent to multiplying Z by a phase factor.) For $\delta = n\pi/2$, with n an integer, this simply interchanges the symmetry of v'_μ and u'_μ . The value of ρ has no effect on the structure of the subharmonic step, just as replacing $\sin(\phi)$ by $\cos(\phi)$ had no effect at the first step in Section A.4.

This concludes the calculation of the structure in Z at the subharmonic step. The results may be summarized by saying that the higher harmonics (I'_c term) in the current-phase relation can cause phase-locking to the external microwaves, but that the fundamental (I_c term) may still provide the largest microwave voltage (excluding of course the voltage $I_2 R_N$, which is of no interest). Under these conditions, the symmetries of the real and imaginary parts of Z depend upon the relative phase between the fundamental and the higher harmonics in the current-phase relation.

APPENDIX B. CALCULATION OF QUASI-PARTICLE CHEMICAL POTENTIAL

B.1 Introduction

The purpose of this appendix is to provide the details of the mathematical calculations outlined in Chapter IV. The calculations begin with a derivation of the modified current-phase relation based on the nonzero gradient in the phase, $\nabla\theta$, of the exponential "tail" of the order parameter. (The nonzero value of $\nabla\theta$ is assumed to be caused by the effects of the microwave electric field, as suggested in Chapter IV.) The nonequilibrium quasi-particle chemical potential is then calculated from this modified current-phase relation as a function of the time-averaged quantum phase difference across the weak link, ϕ_0 , and the amplitude of the microwave voltage, v_0 . Because the weak link is current biased rather than voltage biased, it is then necessary to solve for ϕ_0 and v_0 in terms of the independent (experimentally fixed) variables, I_{dc} and $I_{\mu 0}$. (I_{dc} and $I_{\mu 0}$ are, respectively, the dc bias current and the amplitude of the microwave bias current.) The interdependence of the microwave impedance and ϕ_0 is discussed and is shown to cause ϕ_0 to be a multivalued function of the dc bias current I_{dc} . The range of microwave currents over which this occurs is calculated, and the value of ϕ_0 as a function of I_{dc} is used to evaluate the quasi-particle chemical potential. This gives rise to the discontinuities or "phase-jumps" observed in Chapter IV. The stability of these multivalued solutions for ϕ_0 is considered. The appendix ends with a calculation of the minimum value of the effective critical current (maximum dc supercurrent) as a function of the microwave bias current.

The same notation will be used in this appendix as was used in Chapter IV; the definition of variables given in the text will sometimes not be repeated here. Also, any physical arguments or other discussions which have already appeared in Chapter IV will generally not be repeated.

B.2 Derivation of Modified Current-Phase Relation

From eq. (4.2), generalized to include the contribution from the superconducting region on the right, the order parameter is given by

$$\Psi(z) = \Psi_0 e^{-Kz} e^{i\phi_L} e^{i\nabla\theta z} + \Psi_0 e^{-K(L-z)} e^{i\phi_R} e^{i\nabla\theta(z-L)} \quad (\text{B.1})$$

where Ψ_0 is a real constant, as are K , L , $\nabla\theta$, ϕ_L , and ϕ_R . $\nabla\theta$ is proportional to the applied voltage. Here L is the length of the weak link ($\sim 0.5\mu$), and ϕ_L and ϕ_R are the phases in the bulk superconducting regions on the left and right sides of the weak link. The origin of coordinates has been chosen in such a way that $z = 0$ at the left edge of the normal region of the weak link. The current is evaluated using the operator

$$\vec{j}_s = \frac{e^*\hbar}{2m^*i} (\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*) \quad (\text{B.2})$$

(m^* is twice the mass of an electron; see Chapter I.) The gradient of Ψ is

$$\begin{aligned} \nabla\Psi = & (-K + i\nabla\theta) \Psi_0 e^{-Kz} e^{i\phi_L} e^{i\nabla\theta z} \\ & + (K + i\nabla\theta) \Psi_0 e^{-K(L-z)} e^{i\phi_R} e^{i\nabla\theta(z-L)} \end{aligned} \quad (\text{B.3})$$

Substituting into the current operator gives

$$\begin{aligned}
 \left(\frac{e\hbar}{2mi} \psi_0^2\right)^{-1} j_s = & [e^{-Kz} e^{-i\phi_L} e^{-i\nabla\theta z} + e^{-K(L-z)} e^{-i\phi_R} e^{-i\nabla\theta(z-L)}] \\
 & \cdot [(-K+i\nabla\theta)e^{-Kz} e^{i\phi_L} e^{i\nabla\theta z} + (K+i\nabla\theta)e^{-K(L-z)} e^{i\phi_R} e^{i\nabla\theta(z-L)}] \\
 & - [e^{-Kz} e^{i\phi_L} e^{i\nabla\theta z} + e^{-K(L-z)} e^{i\phi_R} e^{i\nabla\theta(z-L)}] \\
 & \cdot [(-K-i\nabla\theta)e^{-Kz} e^{-i\phi_L} e^{-i\nabla\theta z} + (K-i\nabla\theta)e^{-K(L-z)} e^{-i\phi_R} e^{-i\nabla\theta(z-L)}]
 \end{aligned} \tag{B.4}$$

Multiplying and combining terms yields

$$\begin{aligned}
 \left(\frac{e\hbar}{2mi} \psi_0^2\right)^{-1} j_s = & (e^{-Kz})^2 (2i\nabla\theta) + e^{-KL} (e^{-i(\phi_L-\phi_R+\nabla\theta L)} + e^{i(\phi_L-\phi_R+\nabla\theta L)}) i\nabla\theta \\
 & + e^{-KL} (e^{i(\phi_L-\phi_R+\nabla\theta L)} + e^{-i(\phi_L-\phi_R+\nabla\theta L)}) i\nabla\theta \\
 & + (e^{-K(L-z)})^2 (2i\nabla\theta) + e^{-KL} (e^{-i(\phi_L-\phi_R+\nabla\theta L)} - e^{i(\phi_L-\phi_R+\nabla\theta L)}) K \\
 & - e^{-KL} (e^{i(\phi_L-\phi_R+\nabla\theta L)} - e^{-i(\phi_L-\phi_R+\nabla\theta L)}) K
 \end{aligned} \tag{B.5}$$

This can be reduced to

$$\begin{aligned}
 \left(\frac{e\hbar}{m} \psi_0^2\right)^{-1} j_s = & 2e^{-KL} \nabla\theta \cos(\phi_L - \phi_R + \nabla\theta L) \\
 & - 2K e^{-KL} \sin(\phi_L - \phi_R + \nabla\theta L) \\
 & + \nabla\theta [(e^{-Kz})^2 + (e^{-K(L-z)})^2]
 \end{aligned} \tag{B.6}$$

The last term in eq. (B.6) is independent of the phase difference across the weak link and is of no interest in this experiment. Since it is proportional to the voltage across the weak link and independent of phase, it can be included in the usual quasi-particle current and will be ignored throughout the rest of this appendix. (This term does have a nonzero divergence, but since it is independent of $(\phi_L - \phi_R)$ its time average is zero.) The remaining terms can be rewritten as

$$I_S = I_C \sin(\phi_R - \phi_L - \nabla\theta L) + \left(\frac{\nabla\theta}{K}\right) I_C \cos(\phi_R - \phi_L - \nabla\theta L) \quad (B.7)$$

where I_C is a constant proportional to e^{-KL} . I_S is the supercurrent and is proportional to j_S . The first of these two terms can be shown to have the largest effect by assuming that $\nabla\theta L \ll 1$ and expanding eq. (B.7):

$$I_S = I_C \sin(\phi_R - \phi_L) - I_C (\nabla\theta L) \cos(\phi_R - \phi_L) + I_C \left(\frac{\nabla\theta}{K}\right) \cos(\phi_R - \phi_L) \quad (B.8)$$

The ratio of the last two terms is LK , which is large compared to one in the weak coupling limit being considered here. Thus the second term in eq. (B.7) can be dropped in favor of the first. (Alternatively, the second term could be incorporated in the first, since they both have the same form, as shown by eq. (B.8).) Finally,

$$I_S = I_C \sin(\phi_R - \phi_L - \nabla\theta L) \quad (B.9)$$

Physically, if $KL \gg 1$, the largest effect of $\nabla\theta$ is to modulate the current produced by the gradient in Ψ which is proportional to K . (The exponential factor e^{-Kz} causes a large gradient in Ψ .) The additional gradients in Ψ due to $\nabla\theta \neq 0$ give rise to currents which are negligible.

B.3 Calculation of Nonequilibrium Quasi-Particle Chemical Potential

In this section the current-phase relation of eq. (B.9) will be used to calculate $\bar{V}_T$, the space and time average of μ_{qp} , as a function of ϕ_0 and v_0 . From eq. (4.1), $\mu_{qp} = a' \vec{\nabla} \cdot \vec{j}_s$, where a' is a constant proportional to the appropriate relaxation time. Consequently, this section begins by using eq. (B.9) to calculate the divergence of the supercurrent.

Note that I_s given by eq. (B.9) is independent of position if $\nabla\theta$ is really a constant (independent of z). The dependence on z of eq. (B.1) has not been dropped but simply cancels out. Thus $\vec{\nabla} \cdot \vec{j}_s = 0$ if the parameter $\nabla\theta$ is independent of position, as was assumed in the derivation of eq. (B.9). Recalling that $\vec{j}_s$ is proportional to I_s , eq. (B.8) can be written in vector form as

$$\vec{j}_s = \vec{j}_c \sin(\phi_R - \phi_L) - \vec{j}_c (\nabla\theta L) \cos(\phi_R - \phi_L) \quad (B.10)$$

Here $\vec{j}_c$ is a constant related to I_c , and the last term in eq. (B.8) has been dropped, as already discussed. (Although this result is similar in form to the phase-dependent quasi-particle current discussed in Chapter II, its physical origin is quite different.) Taking the divergence of eq. (B.10) gives

$$\vec{\nabla} \cdot \vec{j}_s = -\vec{j}_c \cos(\phi_R - \phi_L) L \frac{d}{dz} (\nabla\theta) \quad (B.11)$$

Before using this expression for $\vec{\nabla} \cdot \vec{j}_s$ to evaluate μ_{qp} , a comment must be made about the spatial dependence of $\nabla\theta$. As can be seen from eq. (B.11), $\vec{\nabla} \cdot \vec{j}_s$ and thus μ_{qp} are identically zero unless $\nabla\theta$ is at least a weak function of position. Since the spatial dependence of $\nabla\theta$ could not be determined from the discussion given in the text, it will be necessary to assume that $\nabla\theta$ is a slowly varying function of position. (This assumption was also discussed in the text.) As will be seen shortly, it is still possible to calculate a great deal of information about $\bar{V}_T$ even if the exact spatial dependence of $\nabla\theta$ is unknown. However, the calculations leading to eq. (B.9) treated $\nabla\theta$ as a constant; if $\nabla\theta$ is not a constant, then there will be additional terms in $\nabla\Psi$ which will create additional terms in the final result for the current. If the spatial dependence of $\nabla\theta$ is explicitly included in the calculation of the current-phase relation, eq. (B.9), it can readily be shown that the final result is the same as that obtained for a constant value of $\nabla\theta$. The primary reason for this (there are additional reasons as well) is the fact that, as has already been mentioned, the main effect of $\nabla\theta$ is to modulate the current which would exist for $\nabla\theta = 0$, rather than to create additional currents by creating additional gradients. (Recall that $\vec{j}_s$ is proportional to $\Psi^* \vec{\nabla} \Psi$; spatial variations in $\nabla\theta$ will produce extra gradients in Ψ and thus extra currents.) In other words, the effects of these additional gradients are negligible and so it is valid to treat $\nabla\theta$ as a constant in the derivation of eq. (B.9). In order to simplify the algebra, $\nabla\theta$ was considered to be a constant in

the derivation of eq. (B.9), even though $\nabla\theta$ will now be considered to be a slowly varying function of position.

It will now be convenient to define δj_s as the value of j_s given in eq. (B.10), minus its value if $\nabla\theta$ is set to zero. In other words,

$$\begin{aligned}\delta j_s &= j_s(\nabla\theta) - j_s(0) \\ &= -j_c \cos(\phi_R - \phi_L)(\nabla\theta L)\end{aligned}\tag{B.12}$$

Dividing eq. (B.11) by eq. (B.12) gives

$$\vec{\nabla} \cdot \vec{j}_s = \frac{d}{dz}(\nabla\theta) \frac{1}{\nabla\theta} \delta j_s\tag{B.13}$$

Thus the divergence of the supercurrent at any given value of z is proportional to the total supercurrent minus the supercurrent for $\nabla\theta = 0$. This seems reasonable, since only that part of the supercurrent which is due to setting $\nabla\theta \neq 0$ can have a nonzero divergence.

For small applied voltages, v , it seems reasonable to assume that the ratio $[d(\nabla\theta)/dz]/\nabla\theta$ is (at least to a first approximation) independent of v . This is equivalent to assuming that the shape of $\nabla\theta(z)$ is independent of v , while the magnitude of $\nabla\theta$ increases in proportion to v . Actually, this assumption is already contained in the assumption that $\nabla\theta$ be linear in v , which requires that $\nabla\theta = f(z)v$. Here $f(z)$ is some arbitrary function of z . (This assertion can be proven by expanding $\nabla\theta$ in a Taylor series.) From this form of $\nabla\theta$, it follows that $[d(\nabla\theta)/dz]/\nabla\theta$ is indeed independent of v . Equation (B.13) can then be rewritten as

$$\vec{\nabla} \cdot \vec{j}_s = c(z) \delta j_s \quad (B.14)$$

where $c(z)$ is a function of position but independent of voltage.

According to eq. (B.14), the divergence of the supercurrent is proportional to the total supercurrent minus the supercurrent for $\nabla\theta = 0$, where the constant of proportionality is a function of position only.

It might be relevant to ask whether or not it would be meaningful to continue the calculations beyond this point, since, according to eq. (B.14), the results will depend upon the spatial dependence of $\nabla\theta$. As has already been pointed out, nothing is known about this spatial dependence. The text has merely suggested a physical mechanism which might be expected to produce some value of $\nabla\theta$, without giving any details of its spatial variation. As it turns out, a great deal of information can be obtained which is independent of the exact form of $\nabla\theta$. In fact, all the remaining calculations in this appendix are independent of the spatial form of $\nabla\theta$ except for a constant multiplying the results. The value of this constant cannot be determined experimentally because of the unknown degree of asymmetry of the tunnel junction probe. Consequently, if the exact spatial dependence of $\nabla\theta$ were known, the comparison between the analysis and the data would not yield any more information than can be obtained without such knowledge. For these reasons, the calculations of $\bar{V}_T$ will continue, despite the fact that the exact form of $\nabla\theta(z)$ is not known.

The value of $\vec{\nabla} \cdot \vec{j}_s$ given by eq. (B.14) can now be used to find an expression for $\bar{V}_T$. Recalling that I_s is proportional to j_s , and using the definition of δj_s , gives

$$\bar{V}_T = \gamma [\overline{I_S(\nabla\theta)} - \overline{I_S(0)}] \quad (\text{B.15})$$

Here γ is a constant which includes the spatial average of $c(z)$ over the area covered by the tunnel junction probe. In writing down eq. (B.15), use was also made of the fact that μ_{qp} , and thus $\bar{V}_T$, are proportional to $\vec{V} \cdot \vec{j}_S$. The constant γ also includes the value of a' in eq. (4.1), as well as the constant relating I_S to j_S . Unfortunately, the constant γ cannot be determined experimentally or theoretically from the work described in this thesis. (Theoretically, the function $c(z)$ is unknown; experimentally, the asymmetry of the tunnel junction is unknown so that the weighting function used in the spatial average of $c(z)$ cannot be determined.)

Equation (B.15) gives $\bar{V}_T$ as a function of the time-averaged supercurrent. The next step in the analysis is the calculation of the time-average of I_S , $\bar{I}_S$, for an arbitrary value of $\nabla\theta$. These results can then be used in eq. (B.15) to evaluate $\bar{V}_T$.

The time average of eq. (B.9) will now be calculated for a current biased weak link. That is, it will be assumed (as in Appendix A) that

$$I = I_{dc} + I_{\mu 0} \sin(\omega t) \quad (\text{B.16})$$

$$I = I_S + v/R_N \quad (\text{B.17})$$

$$\frac{d\phi}{dt} = 2ev/\hbar \quad (\text{B.18})$$

Here I is the total instantaneous current, I_{dc} is the dc bias current, and $I_{\mu 0}$ is the amplitude of the microwave bias current at frequency ω ;

ϕ is the phase difference across the weak link; $\phi = \phi_R - \phi_L$. I_S is given by eq. (B.9). (Recall that $\nabla\theta$ and ϕ are two unrelated parameters.) As pointed out in Chapter IV, $\nabla\theta$ is, in the model under consideration, proportional to the applied voltage. Consequently, it will be convenient to define ϵ' by

$$\nabla\theta L = \epsilon' v \quad (\text{B.19})$$

Because of the spatial variation of $\nabla\theta$, ϵ' is also a weak function of position. This value of $\nabla\theta L$ is to be inserted into eq. (B.9) for I_S .

As suggested in Appendix A, the easiest way to treat the case of a current-biased weak link for $|I_{dc}| < I_c$ (for the approximate case to be considered here) is by the use of successive iteration, starting with the voltage-biased solution. Thus, to a first approximation,

$$v = v_0 \sin(\omega t) \quad (\text{B.20})$$

where v_0 is the amplitude of the microwave voltage across the weak link. (To be more precise, v_0 is the Fourier component of v at frequency ω .) As will be discussed in detail shortly, the microwave impedance is approximately equal to R_N . Consequently, v_0 is approximately equal to $I_{\mu 0} R_N$, although deviations from this value will at times be extremely important. The calculations of this section will be carried out in terms of v_0 , although the value of v_0 need not (and will not) be determined until one of the later sections. It should also be pointed out that there may be additional (smaller) voltages at angular frequencies other than ω . As will also be shown in a later section, the approximate form for v given by eq. (B.20) (which neglects voltages

at frequencies other than ω) will be sufficiently accurate for the purposes of this appendix. It might also be pointed out that eq. (B.20) would be exact if the weak link were voltage biased rather than current biased at microwave frequencies. Equation (B.20) would also be exact if the microwave impedance were exactly equal to R_N , in which case there is no distinction between current bias and voltage bias.

Having approximated v by eq. (B.20), eq. (B.18) can immediately be integrated to give

$$\phi = \phi_0 - \frac{2ev_0}{\hbar\omega} \cos(\omega t) \quad (B.21)$$

Here ϕ_0 is a constant of integration, which can be seen to be the time-average of ϕ . (The value of ϕ_0 is largely determined by the value of I_{dc} . Also, the symbol ϕ_0 should not be confused with the flux quantum, $h/2e$, which is sometimes denoted ϕ_0 .) If desired, eq. (B.21) could be put back into all the previous equations and the next approximation for v_0 could be obtained by iteration. As has already been mentioned, eq. (B.21) gives all the accuracy required here. Putting eq. (B.21) into eq. (B.9) and using eq. (B.19) gives

$$I_s = I_c \sin[\phi_0 - \frac{2ev_0}{\hbar\omega} \cos(\omega t) - \epsilon' v_0 \sin(\omega t)] \quad (B.22)$$

The sine of the term in brackets in eq. (B.22) can be expanded using the usual trigonometric identities to obtain

$$\begin{aligned}
 I_s/I_c = & \sin \phi_0 \cos[\epsilon' v_0 \sin(\omega t)] \cos\left[\frac{2ev_0}{\hbar\omega} \cos(\omega t)\right] \\
 & - \sin \phi_0 \sin[\epsilon' v_0 \sin(\omega t)] \sin\left[\frac{2ev_0}{\hbar\omega} \cos(\omega t)\right] \\
 & - \cos \phi_0 \sin[\epsilon' v_0 \sin(\omega t)] \cos\left[\frac{2ev_0}{\hbar\omega} \cos(\omega t)\right] \\
 & - \cos \phi_0 \cos[\epsilon' v_0 \sin(\omega t)] \sin\left[\frac{2ev_0}{\hbar\omega} \cos(\omega t)\right] \quad (B.23)
 \end{aligned}$$

The trigonometric functions whose arguments are themselves trigonometric functions can be further expanded using the generating function and associated series for Bessel functions (Ref. 1):

$$\begin{aligned}
 I_s/I_c = & \sin(\phi_0) [J_0(\epsilon' v_0) + 2 \sum_{k=1}^{\infty} J_{2k}(\epsilon' v_0) \cos(2k\omega t)] \\
 & \cdot [J_0(x) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(x) \cos(2k\omega t)] \\
 & - \sin(\phi_0) [2 \sum_{k=0}^{\infty} J_{2k+1}(\epsilon' v_0) \sin((2k+1)\omega t)] \\
 & \cdot [2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x) \cos((2k+1)\omega t)] \\
 & - \cos(\phi_0) [2 \sum_{k=0}^{\infty} J_{2k+1}(\epsilon' v_0) \sin((2k+1)\omega t)] \\
 & \cdot [J_0(x) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(x) \cos(2k\omega t)] \\
 & - \cos(\phi_0) [J_0(\epsilon' v_0) + 2 \sum_{k=1}^{\infty} J_{2k}(\epsilon' v_0) \cos(2k\omega t)] \\
 & \cdot [2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x) \cos((2k+1)\omega t)] \quad (B.24)
 \end{aligned}$$

Here x has been defined by

$$x = \frac{2ev_0}{\hbar\omega} \quad (\text{B.25})$$

The orthonormality of the sine and cosine can be used to determine (by inspection) the time-average of eq. (B.24). The last three terms can be seen to contribute nothing and the first term gives

$$\begin{aligned} \bar{I}_S = & I_C \sin \phi_0 [J_0(\epsilon'v_0) J_0(x) \\ & + 4 \sum_{k=1}^{\infty} (-1)^k J_{2k}(\epsilon'v_0) J_{2k}(x) \left(\frac{1}{2}\right)] \end{aligned} \quad (\text{B.26})$$

For the case of $\epsilon'v_0 \ll 1$, this expression can be expanded in power series in $\epsilon'v_0$, where the lowest (nonzero) order turns out to be $(\epsilon'v_0)^2$. This is also a convenient point in the calculation to subtract off $\bar{I}_S(0)$ in eq. (B.15) to evaluate V_T :

$$\begin{aligned} \bar{V}_T = & \gamma I_C \sin \phi_0 [(J_0(\epsilon'v_0) - 1) J_0(x) \\ & - 2 J_2(\epsilon'v_0) J_2(x)] \end{aligned} \quad (\text{B.27})$$

Here use was made of the fact that $J_N(x)$ contains only terms of order x^N or higher, and that $J_0(0) = 1$. For $(\epsilon'v_0) \ll 1$, $J_2(\epsilon'v_0) = (\epsilon'v_0)^2/8$ and $J_0(\epsilon'v_0) = 1 - (\epsilon'v_0)^2/4$ (Ref. 2). Equation (B.27) then reduces to

$$\bar{V}_T = -\frac{\gamma}{4} I_C \sin(\phi_0) (\epsilon'v_0)^2 [J_0(x) + J_2(x)] \quad (\text{B.28})$$

Equation (B.28) gives $\bar{V}_T$ in terms of ϕ_0 and the microwave voltage across the weak link. (From eq. (B.25), x is proportional to v_0 .) It should be noted that the derivation of this equation is valid even for $x > 1$. This concludes one logical step in the analysis. If the weak link were voltage biased, rather than current biased, then x (or v_0) would be an independent experimental parameter and ϕ_0 could be readily calculated, thus determining $\bar{V}_T$. However, the fact that the microwave impedance (as will be discussed shortly) is a function of ϕ_0 , and the fact that the weak link is current biased, cause the evaluation of ϕ_0 and x in terms of $I_{\mu 0}$ and I_{dc} (the independent experimental parameters) to be a difficult (or at least lengthy) task. This process is begun in the next section.

B.4 Dependence of Microwave Impedance on Quantum Phase

In order to determine v_0 from $I_{\mu 0}$ it is, by definition, necessary to know the value of the microwave impedance. In this section some properties of the microwave impedance which were not needed in Chapter II and not dealt with in Appendix A will be calculated. First, the Josephson inductance was calculated in Appendix A only for the limit of very small microwave voltages. This calculation needs to be generalized to the case of reasonably large voltages. As before, suppose that the amplitude (Fourier component at frequency ω) of the microwave voltage is v_0 , and that all other voltages across the weak link are negligible compared to v_0 . (The self-consistency of this assumption will be examined shortly.) Then eq. (B.18) can be integrated to give

$$\phi = \phi_0 - \frac{2ev_0}{\hbar\omega} \cos \omega t \quad (B.29)$$

(This is identical to eq. (B.21) but has been repeated here for the sake of completeness.) When this result is inserted into eq. (B.9), the supercurrent which results is

$$I_s = I_c \sin[\phi_0 - \frac{2ev_0}{\hbar\omega} \cos(\omega t)] \quad (B.30)$$

Here the term in $\nabla\theta$ has been dropped, since its effects can be shown to be higher order in ϵ' than is necessary here.

Equation (B.30) can be expanded using trigonometric and Bessel function identities as in the previous section and yields

$$\begin{aligned} I_s = I_c \sin(\phi_0) [J_0(x) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(x) (\cos 2k\omega t)] \\ - I_c \cos(\phi_0) [2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x) \cos((2k+1)\omega t)] \end{aligned} \quad (B.31)$$

Only the component $I_{s\omega}$ at frequency ω is of interest here and is

$$I_{s\omega} = -2I_c \cos(\phi_0) J_1(x) \cos(\omega t) \quad (B.32)$$

This corresponds to an inductive impedance $\omega L_J = v_0/I_{s\omega}$ given by

$$L_J = \frac{R_N I_{\mu 0}}{2e R_N I_{\mu 0} \cos(\phi_0)} \quad (B.33)$$

In writing down eq. (B.33) the value of v_0 was approximated by $R_N I_{\mu 0}$. As will be seen, this approximation is very good for the case of

$|R_N/\omega L_J| \ll 1$. However, there are times when it will be essential to use a more exact expression for v_0 . Throughout the remainder of this appendix only those terms in $R_N/\omega L_J$ will be kept which are needed to give a final result for $\bar{V}_T$ which is accurate to lowest order in $R_N/\omega L_J$. The most important feature in eq. (B.33) is the dependence of L_J on ϕ_0 . The deviation of v_0 from $R_N I_{\mu 0}$ is not important in eq. (B.33), although it will be important elsewhere, and so v_0 has been approximated by $R_N I_{\mu 0}$ in this equation. The remaining equations in this appendix will also keep only the necessary terms in $R_N/\omega L_J$, generally without emphasizing that this has been done.

The quantity which will actually be needed throughout the remainder of this appendix is the magnitude of the microwave impedance, Z . (Although Z is a complex number, the relative phase of the current and voltage is not important.) Equations (B.16), (B.17), and (B.32), which govern the currents passing through the weak link, are equivalent to that of a circuit consisting of an inductor L_J in parallel with a resistor R_N . Consequently,

$$Z = \frac{1}{\frac{1}{R_N} + \frac{1}{j\omega L_J}} \quad (\text{B.34})$$

and

$$|Z| = \frac{R_N |\omega L_J|}{[R_N^2 + (\omega L_J)^2]^{1/2}} \quad (\text{B.35})$$

Note that $|Z|$ is always less than either R_N or ωL_J , as expected. For $R_N \ll |\omega L_J|$, eq. (B.35) can be expanded to give

$$|Z| \doteq R_N \left[1 - \frac{1}{2} \left(\frac{R_N}{\omega L_J} \right)^2 \right] \quad (\text{B.36})$$

At this point it is essential to note that, since L_J is a function of ϕ_0 (see eq. (B.33)), $|Z|$ is also a function of ϕ_0 . For $R_N \ll |\omega L_J|$, however, $|Z|$ is only a weak function of ϕ_0 . This weak dependence of $|Z|$ on ϕ_0 will be crucial in understanding the origin of the phase-jump structures in $\bar{V}_T$. Equation (B.36) shows that, for $R_N \ll |\omega L_J|$ the other harmonics in eq. (B.31) do indeed give rise to voltages which are negligible compared to v_0 (since $Z \doteq R_N$). Thus the assumption made in eq. (B.20) is seen to be self-consistent.

B.5 Range of Multivalued Quantum Phase

The task of solving for v_0 (or, equivalently, x) and ϕ_0 in terms of I_{dc} and $I_{\mu 0}$ continues in this section. As shown in the previous section, $|Z|$ is a function of ϕ_0 . This will give rise to a set of nonlinear coupled equations which need to be solved in order to obtain x and ϕ_0 , and thereby $\bar{V}_T$. The method used to solve these equations is first to eliminate x as a variable by finding an equation which gives x as a function of ϕ_0 (and $I_{\mu 0}$). This leads to a nonlinear equation in ϕ_0 alone (considering $I_{\mu 0}$ and I_{dc} as known), which can then be solved for ϕ_0 . Although a computer program was written which was capable of solving these nonlinear equations exactly, it is more instructive to solve them analytically. In addition, the model already contains enough approximations that there is little benefit in obtaining an exact numerical solution. Consequently, this appendix (as has already been mentioned) only considers the case where $R_N \ll |\omega L_J|$, which allows an

approximate analytic solution even though the equations remain non-linear even then.

First of all, if the microwave bias current gives a value of x which is not too near one of the zeros of $J_0(x)$, then a reasonable approximation is to take $|Z| = R_N$, so that $v_0 = I_{\mu 0} R_N$. Equation (B.25) then gives x . From Appendix A the time-averaged supercurrent is given by

$$\bar{I}_s = I_c \sin(\phi_0) J_0(x) \quad (B.37)$$

which can be solved for ϕ_0 (since $I_{dc} = \bar{I}_s$ below the critical current). This value of ϕ_0 , along with x , can be inserted into eq. (B.28) to give $\bar{V}_T$. The value of $\bar{V}_T$ obtained will be a good approximation, since the percent change in the values of all the relevant parameters due to a slight error in the value of v_0 will be small. Thus there is no difficulty in calculating $\bar{V}_T$ for $R_N \ll |\omega L_J|$ if x is not too near a zero of $J_0(x)$.

Suppose, however, that x is near z_1 , the first zero of $J_0(x)$. (This appendix will restrict itself to the structures near z_1 , although the results are easily generalized to include all of the zeros.) In that case, ϕ_0 becomes a very strong function of x (for a fixed value of I_{dc}) because of the $J_0(x)$ term in eq. (B.37). In particular, the fractional error $\delta J_0(x)/J_0(x)$ which would result from setting $v_0 = R_N I_{\mu 0}$ becomes very large as J_0 approaches zero. (This was discussed in detail in the text.) In fact, the nonlinearity of the equations involved causes ϕ_0 to become a multivalued and discontinuous

function of $I_{\mu 0}$ and I_{dc} . A systematic approach to the solution of these equations for values of x near z_1 will now be described.

The first step is to find x as a function of ϕ_0 (considering $I_{\mu 0}$ as given.) From the definition given in eq. (B.25)

$$x = \frac{2e|Z|I_{\mu 0}}{\hbar \omega} \quad (B.38)$$

where use has been made of the fact that $v_0 = |Z|I_{\mu 0}$. (Eq. (B.38) is exact.) Using eq. (B.36) for Z , this becomes

$$x = \frac{2eI_{\mu 0}R_N}{\hbar \omega} \left[1 - \frac{1}{2} \left(\frac{R_N}{\omega L_J} \right)^2 \right] \quad (B.39)$$

It will be convenient to define the parameter x_0 (which is very nearly equal to x) by:

$$x_0 = \frac{2e I_{\mu 0} R_N}{\hbar \omega} \quad (B.40)$$

Then eq. (B.35) becomes

$$x = x_0 \left[1 - \frac{1}{2} \left(\frac{R_N}{\omega L_J} \right)^2 \right] \quad (B.41)$$

It will also be convenient to define L_0 and ϵ by

$$L_0 = \frac{R_N I_{\mu 0}}{2\omega J_1(x_0) I_c} \quad (B.42)$$

$$\epsilon = R_N / \omega L_0 \quad (B.43)$$

Note that x_0 , L_0 , and ϵ all depend on only $I_{\mu 0}$, and not ϕ_0 , v_0 , etc., so that they can all be considered to be "known." Also, recall that

$|\epsilon| \ll 1$ has already been assumed. (ϵ and ϵ' are independent parameters, both of which have been assumed to be small.) Substitution into eq. (B.41) gives

$$x = x_0 \left[1 - \frac{1}{2} \epsilon^2 \cos^2(\phi_0) \right] \quad (\text{B.44})$$

This equation, which gives x in terms of ϕ_0 , will be extensively used throughout the rest of the appendix. Since x_0 is proportional to $I_{\mu 0}$, x_0 can be considered to be an independent parameter (experimentally fixed), replacing $I_{\mu 0}$. In other words, x_0 and $I_{\mu 0}$ are, in some sense, equivalent parameters, just as x and v_0 are in some sense equivalent.

Near z_1 , J_0 can be expanded in Taylor series to give

$$J_0(x) \doteq -(x-z_1) J_1(z_1) \quad (\text{B.45})$$

where the identity

$$\frac{dJ_0(x)}{dx} = -J_1(x) \quad (\text{B.46})$$

has been used. Equations (B.44) and (B.45) can now be used to put all of the nonlinear relations which have to be solved in terms of x_0 and ϕ_0 .

The range of values of x_0 for which phase-jumps can occur will now be calculated. (The equations which will be derived will be useful for other purposes as well.) From Fig. 4.7 it can be seen that the necessary requirement for phase-jumps to occur (or for ϕ_0 to be a multivalued function of $\bar{I}_S$) is that $d\bar{I}_S/d\phi_0 = 0$. This condition also gives the location of the relative maxima in $\bar{I}_S$, which will later be needed in order

to calculate the maximum dc supercurrent, $\tilde{I}_c$, as a function of x_0 .

Differentiating eq. (B.37) yields

$$\frac{d\tilde{I}_s}{d\phi_0} = I_c \cos(\phi_0) J_0(x) + I_c \sin(\phi_0) \frac{dJ_0(x)}{dx} \frac{dx}{d\phi_0} = 0 \quad (B.47)$$

From eq. (B.44)

$$\frac{dx}{d\phi_0} = x_0 \epsilon^2 \cos(\phi_0) \sin(\phi_0) \quad (B.48)$$

and using eq. (B.45), (B.46), and (B.47) gives

$$I_c \cos(\phi_0)(x-z_1) J_1(z_1) + I_c \sin(\phi_0) x_0 J_1(z_1) \epsilon^2 \cos(\phi_0) \sin(\phi_0) = 0 \quad (B.49)$$

This can be rewritten as

$$I_c \cos(\phi_0) J_1(z_1) [x-z_1 + x_0 \epsilon^2 \sin^2(\phi_0)] = 0 \quad (B.50)$$

Note that eq. (B.50) is always satisfied for $\phi_0 = 90^\circ$, so that a relative maximum or minimum is always located there.

An additional extremum will occur at $\phi_0 \neq 90^\circ$ if

$$x_0 \epsilon^2 \sin^2 \phi_0 = (z_1 - x) \quad (B.51)$$

(from eq. (B.50)). Using eq. (B.44) to eliminate x in favor of ϕ_0 gives

$$x_0 \epsilon^2 \sin^2 \phi_0 = z_1 - x_0 [1 - \frac{1}{2} \epsilon^2 \cos^2(\phi_0)] \quad (B.52)$$

This equation can be solved to find ϕ_0^* , the value of ϕ_0 at which an extremum occurs. The result is

$$\phi_0^* = \sin^{-1} \left[\pm \left(\frac{1 + \frac{2}{\epsilon^2} \frac{z_1 - x_0}{x_0}}{3} \right)^{1/2} \right] \quad (\text{B.53})$$

Equation (B.53) gives ϕ_0^* as a function of the independent variable x_0 , which is defined in terms of $I_{\mu 0}$ through eq. (B.40).

The range of values of x_0 over which ϕ_0^* has a real solution is given by the requirement that the argument of the square root in eq. (B.53) lie between 0 and 1, which yields

$$z_1(1 - \epsilon^2) \leq x_0 \leq z_1 \left(1 + \frac{\epsilon^2}{2} \right) \quad (\text{B.54})$$

For this range of x_0 , ϕ_0 will be a multivalued function of $\bar{I}_S = I_{dc}$, and phase-jumps will give rise to the observed discontinuities in $\bar{V}_T$.

At this point we digress a bit from the calculation of $\bar{V}_T$, since the extremum value of $\bar{I}_S$, denoted $\tilde{I}_C^*$ (since it may give the maximum effective supercurrent) can be found by substituting eqs. (B.53), (B.44), and (B.45) into eq. (B.37) for $\bar{I}_S$. The result reduces to

$$\tilde{I}_C^* = \epsilon^2 x_0 I_C J_1(z_1) \left(\frac{1 - \frac{2}{\epsilon^2} \frac{x_0 - z_1}{x_0}}{3} \right)^{3/2} \quad (\text{B.55})$$

where a considerable amount of algebraic manipulation has been omitted.

From eq. (B.37), the value of $\bar{I}_S$ at the extremum at $\phi_0 = 90^\circ$ is given by

$$\begin{aligned}\tilde{I}'_c &= I_c |J_0(x_0)| \\ &= I_c J_1(z_1) |x_0 - z_1|\end{aligned}\tag{B.56}$$

where eq. (B.45) has been used. The actual effective critical current $\tilde{I}_c$ is then the larger of $\tilde{I}_c^*$ or $\tilde{I}'_c$. This can be used to plot $\tilde{I}_c$ as a function of x_0 or $I_{\mu 0}$, such as is shown in Fig. 4.8. Since it is not possible for both eqs. (B.55) and (B.56) to be simultaneously zero, the effective critical current is never reduced to zero by any value of $I_{\mu 0}$. Note that this result depends only on the properties of the microwave impedance, and so is not dependent on quasi-particle non-equilibrium or the value of $\nabla\theta$.

Having calculated the effective critical current as a function of $I_{\mu 0}$, we now return to the calculation of $\bar{V}_T$.

B.6 Calculation of Quantum Phase as a Function of Supercurrent

Equation (B.28) gives $\bar{V}_T$ as a function of ϕ_0 and x , and eq. (B.44) gives x in terms of ϕ_0 . Thus $\bar{V}_T$ could be evaluated if ϕ_0 were determined. In this section ϕ_0 will be calculated as a function of I_{dc} for the range of values of x_0 found in the preceding section. (Actually, these calculations are valid sufficiently close to z_1 that $J_0(x)$ can be expanded in Taylor series; thus they are valid outside the region of phase-jumps as well.) The relevant equation is found by simply setting $I_{dc} = \bar{I}_s$, which is a function of ϕ_0 . From eq. (B.37)

$$I_{dc} = \bar{I}_s = -I_c \sin(\phi_0)(x - z_1) J_1(z_1)\tag{B.57}$$

where eq. (B.45) has been used as well. Using eq. (B.44) to eliminate x in favor of ϕ_0 gives

$$I_{dc} = -I_c \sin(\phi_0) [x_0 (1 - \frac{1}{2} \epsilon^2 \cos^2 \phi_0) - z_1] J_1(z_1) \quad (B.58)$$

Replacing $\cos^2(\phi_0)$ by $[1 - \sin^2(\phi_0)]$ and rearranging terms yields

$$\sin^3(\phi_0) - \left[1 - \frac{2}{\epsilon^2} \frac{x_0 - z_1}{x_0} \right] \sin(\phi_0) + \frac{2I_{dc}}{I_c J_1(z_1) x_0 \epsilon^2} = 0 \quad (B.59)$$

The roots of this cubic equation give ϕ_0 as a function of I_{dc} and x_0 (the independent parameters). Standard formulas for the roots of cubic equations exist in math tables (see, for example, Ref. 3). It will be convenient to define

$$a = -\left(1 - \frac{2}{\epsilon^2} \frac{x_0 - z_1}{x_0}\right) \quad (B.60)$$

$$b = \frac{2I_{dc}}{I_c J_1(z_1) x_0 \epsilon^2} \quad (B.61)$$

$$g = \frac{b^2}{4} + \frac{a^3}{27} \quad (B.62)$$

There are two cases, depending on the sign of g . (The degenerate case $g = 0$ can be included with the case for $g < 0$.) Only the real roots are of physical significance, and so the imaginary roots will be ignored.

For $g > 0$, it is convenient to define

$$A = \left(-\frac{b}{2} + g^{1/2}\right)^{1/3} \quad (B.63)$$

$$B = \left(-\frac{b}{2} - g^{1/2}\right)^{1/3} \quad (\text{B.64})$$

Here the $1/3$ power of a negative number is taken to be minus the $1/3$ power of its absolute value; in other words, the complex roots are ignored. In this case there is only one real root of the cubic equation, given by

$$\phi_0 = \sin^{-1}(A+B) \quad (\text{B.65})$$

Both branches of arcsin must be taken, since the two branches are not physically equivalent. Thus there are two solutions for ϕ_0 for $g > 0$ (one of which will later be shown to be physically unstable).

For $g < 0$, it will be convenient to define

$$\rho' = \cos^{-1}\left[-\frac{b}{2} / \left(-\frac{a^3}{27}\right)^{1/2}\right] \quad (\text{B.66})$$

Then there are three roots of the cubic equation given by

$$\phi_{01} = \sin^{-1}\left[2 \sqrt{\frac{-a}{3}} \cos\left(\frac{\rho'}{3}\right)\right] \quad (\text{B.67})$$

$$\phi_{02} = \sin^{-1}\left[2 \sqrt{\frac{-a}{3}} \cos\left(\frac{\rho'}{3} + 120^\circ\right)\right] \quad (\text{B.68})$$

$$\phi_{03} = \sin^{-1}\left[2 \sqrt{\frac{-a}{3}} \cos\left(\frac{\rho'}{3} + 240^\circ\right)\right] \quad (\text{B.69})$$

Once again, both branches of arcsin must be used, giving six possible values of ϕ_0 , half of which will later be shown to be physically unstable. In addition, whenever the argument of the arcsin is greater than 1 or less than -1, those roots must be discarded. Thus the actual number of solutions for ϕ_0 may be 0, 1, 2, or 3.

B.7 Stability of Multivalued Solutions

Having solved for ϕ_0 , $\bar{V}_T$ can now be evaluated using eqs. (B.44) and (B.28). First, however, it is necessary to consider the stability of the solutions for ϕ_0 .

As pointed out in the text, the solutions for ϕ_0 are physically unstable if $d\bar{I}_s/d\phi_0 < 0$. This can be seen to be the case by supposing that ϕ_0 initially has exactly the right value to make $\bar{I}_s = I_{dc}$ in eq. (B.37). Now suppose that the situation is slightly perturbed; for example, I_{dc} might be increased slightly. Since ϕ_0 cannot change instantly (see eq. (B.18)), I_{dc} will exceed $\bar{I}_s$ and produce a positive voltage across the weak link (in accordance with eqs. (B.16) and (B.17)) which then begins to increase the value of ϕ_0 . (The time dependence of ϕ_0 is given by eq. (B.18).) If $d\bar{I}_s/d\phi_0$ is positive, then $\bar{I}_s$ will be increased and eventually will make up for the change in I_{dc} . (The return to equilibrium can be shown to be an exponential decay in time of the quantity $(I_{dc} - \bar{I}_s)$.) If, however, $d\bar{I}_s/d\phi_0$ is negative, then the voltage developed will cause $\bar{I}_s$ to decrease, making the difference between I_{dc} and $\bar{I}_s$ even larger. This further increases the voltage, etc., so that this case corresponds to an instability. Thus solutions for ϕ_0 with $d\bar{I}_s/d\phi_0 < 0$ are physically unstable.

From eq. (B.50)

$$\frac{d\bar{I}_s}{d\phi_0} = -I_c \cos(\phi_0) J_1(z_1) [x - z_1 + x_0 \epsilon^2 \sin^2 \phi_0] \quad (B.70)$$

Equation (B.44) can then be used to eliminate x in favor of x_0 and ϕ_0 . The resulting expression can be used to reject those solutions for

which $d\bar{I}_s/d\phi_0 < 0$. The evaluation of this formula, as well as all the other formulas in the appendix, is sufficiently tedious that all the calculated graphs shown in Chapter IV were plotted by computer.

Equation (B.28) gave $\bar{V}_T$ as a function of ϕ_0 and x , both of which have now been evaluated in terms of the independent variables, $I_{\mu 0}$ and I_{dc} . As pointed out in the text, there is reason to believe that all of the stable solutions for ϕ_0 are equally likely, so that $\bar{V}_T$ was taken to be the average corresponding to all the stable solutions for ϕ_0 . Once again, the number of stable solutions was counted and the average of $\bar{V}_T$ was taken using a computer program, even though all the necessary formulas exist in closed analytic form.

This concludes the calculation of $\bar{V}_T$ as a function of I_{dc} and $I_{\mu 0}$. The results have been plotted in the text. It might be pointed out that the phase-jumps occur when the number of real, stable solutions for ϕ_0 changes from one integer to another.

B.7 Calculation of Minimum Effective Critical Current

The primary purpose of this appendix was to calculate $\bar{V}_T$ as a function of I_{dc} and $I_{\mu 0}$; this was completed in the previous section. The purpose of this last section is to find an analytic expression for the minimum value of $\tilde{I}_c$, which will be denoted $\tilde{I}_{cm}$, and the value of x_0 which produces it, denoted x_{0m} . $\tilde{I}_{cm}$ is the minimum value of the effective dc critical current which results from the application of a microwave current. (For a voltage biased weak link, $\tilde{I}_{cm} = 0$.) Once again, the calculation given here is restricted to the region near z_1 , but the

results are easily generalized to the other zeros of J_0 .

The relation which determines x_{0m} can most easily be understood by simply noticing in Fig. 4.8 that the minimum in $\tilde{I}_c$ occurs when $\tilde{I}_c^* = \tilde{I}_c'$. (See eqs. (B.55) and (B.56).) In other words, x_{0m} corresponds to the point at which the relative maximum in $\tilde{I}_s$ at $\phi_0 \neq 90^\circ$ just equals the relative maximum at $\phi_0 = 90^\circ$. This result can also be shown more directly by considering the derivatives involved. Setting $\tilde{I}_c^* = \tilde{I}_c'$ gives

$$I_c J_1(z_1)(x_0 - z_1) = x_0 \epsilon^2 I_c J_1(z_1) \left[\frac{1 - \frac{2}{\epsilon^2} \frac{x_0 - z_1}{x_0}}{3} \right]^{3/2} \quad (B.71)$$

It will be convenient to define

$$\delta = \frac{1}{\epsilon^2} \frac{x_0 - z_1}{x_0} \quad (B.72)$$

Then eq. (B.71) reduces to

$$\delta^3 + \frac{15}{8} \delta^2 + \frac{3}{4} \delta - \frac{1}{8} = 0 \quad (B.73)$$

This cubic equation in δ is more complicated than that of eq. (B.59) due to the presence of the δ^2 term. Its solution can, nevertheless, be found algebraically (Ref. 3). Only the final result is of interest here, which is

$$\delta = 1/8 \quad (B.74)$$

Substituting back into eq. (B.72) and using $\epsilon \ll 1$ (an assumption already made) gives

$$x_{0m} = z_1 \left(1 + \frac{\epsilon^2}{8} \right) \quad (\text{B.75})$$

The minimum value of $\tilde{I}_c$ is most easily found by substitution of x_{0m} into $\tilde{I}_c'$, with the result that

$$\tilde{I}_{cm} = I_c z_1 J_1(z_1) \epsilon^2 / 8 \quad (\text{B.76})$$

This gives the first minimum in the effective critical current due to microwave irradiation, which, for a current-biased weak link, is nonzero. Note that this result depends only on the properties of the microwave impedance and so is not dependent on quasi-particle nonequilibrium or the value of $\nabla\theta$.

References for Chapter I

1. R. K. Kirschman, Ph.D. thesis, California Institute of Technology, 1972.
2. M. L. Yu, Ph.D. thesis, California Institute of Technology, 1974.
3. R. H. Wang, Ph.D. thesis, California Institute of Technology, 1975.
4. S. K. Decker, Ph.D. thesis, California Institute of Technology, 1975.
5. D. W. Palmer, Ph.D. thesis, California Institute of Technology, 1975.
6. T. Ganz, Ph.D. thesis, California Institute of Technology, 1976.
7. V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. 20, 1064 (1950).
8. F. London, Superfluids (Wiley, New York, 1950), Vol. I.
9. A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957) [Sov. Phys. JETP 5, 1174 (1957)].
10. J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).
11. L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959); [Sov. Phys. JETP 9, 1364 (1959)].
12. L. N. Cooper, Phys. Rev. 104, 1189 (1956).
13. Strictly speaking, only the N-particle projection of the wavefunction can be put in this form; see J. R. Schrieffer, Theory of Superconductivity (W. A. Benjamin, Inc., Reading, Mass., 1964), p. 42.
14. P. G. de Gennes, Superconductivity of Metals and Alloys (W. A. Benjamin, Inc., New York, 1966), p. 105.
15. B. D. Josephson, Phys. Lett. 1, 251 (1962).
16. I. Giaever, Phys. Rev. Lett. 5, 147 (1960).

17. N. F. Pedersen, T. F. Finnegan, and D. N. Langenberg, Phys. Rev. B 6, 4151 (1972).
18. C. M. Falco, W. H. Parker, and S. E. Trullinger, Phys. Rev. Lett. 31, 933 (1973); 31, 1476 (1973).
19. D. A. Vincent and B. S. Deaver, Jr., Phys. Rev. Lett. 32, 212 (1974).
20. R. E. Harris, Phys. Rev. B 10, 84 (1974).
21. Actually, the correct boundary conditions give a discontinuity in the order parameter; see, for example, P. G. de Gennes, Superconductivity of Metals and Alloys (W. A. Benjamin, Inc., New York, 1966), p. 228.
22. H. Meissner, Phys. Rev. 117, 672 (1960).
23. H. A. Notarys and J. E. Mercereau, Intern. Conf. on Science of Superconductivity, Stanford, Palo Alto, California, Aug. 1969.
24. D. W. Palmer and S. K. Decker, Rev. Sci. Instrum. 44, 1621 (1973).
25. M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975).

References for Chapter 2

1. J. D. Franson and J. E. Mercereau, Bull. Am. Phys. Soc. 20, 1493 (1975).
2. J. D. Franson and J. E. Mercereau, J. Appl. Phys. 47, 3261 (1976).
3. B. D. Josephson, Phys. Lett. 1, 251 (1962).
4. N. F. Pedersen, T. F. Finnegan, and D. N. Langenberg, Phys. Rev. B 6, 4151 (1972).
5. C. M. Falco, W. H. Parker, and S. E. Trullinger, Phys. Rev. Lett. 31, 933 (1973); 31, 1476 (1973).
6. D. A. Vincent and B. S. Deaver, Jr., Phys. Rev. Lett. 32, 212 (1974).
7. S. Shapiro, Phys. Rev. Lett. 11, 80 (1963).
8. A. N. Vystavkin, V. N. Gubankov, L. S. Kuz'min, K. K. Likharev, V. V. Miqulin, and A. M. Spitsyn, ZhETF Pis. Red. 17, 284 (1973) [Sov. Phys. JETP Lett. 17, 204 (1973)].
9. C. G. Montgomery, R. H. Dicke, and E. M. Purcell, Principles of Microwave Circuits (Dover, New York, 1965), p. 214.
10. J. A. Lane, Proc. Phys. Soc. 70B, 1173 (1957); J. A. Lane and D. M. Evans, Proc. IEE 108B, 133 (1961).
11. The capacitance was estimated at roughly 0.1 pf using a formula from W. R. Smythe, Static and Dynamic Electricity (McGraw Hill, New York, 1939), 3d Ed., p. 104.
12. D. W. Palmer and S. K. Decker, Rev. Sci. Instrum. 44, 1621 (1973).
13. D. W. Palmer, Ph.D. thesis, California Institute of Technology, 1975.
14. W. C. Stewart, Appl. Phys. Lett. 12, 277 (1968).
15. D. E. McCumber, J. Appl. Phys. 39, 3113 (1968).
16. R. K. Kirschman, Ph.D. thesis, California Institute of Technology, 1972.

17. R. K. Kirschman, H. A. Notarys, and J. E. Mercereau, Phys. Lett. 34A, 209 (1971).
18. T. J. Rieger, D. J. Scalapino, and J. E. Mercereau, Phys. Rev. Lett. 27, 1787 (1971); Phys. Rev. B 6, 1734 (1972).
19. F. Auracher and T. Van Duzer, J. Appl. Phys. 44, 848 (1973).
20. H. Kanter and F. L. Vernon, Jr., J. Appl. Phys. 43, 3174 (1972).
21. E. D. Thompson, J. Appl. Phys. 44, 5587 (1973).
22. H. Ohta, J. Appl. Phys. 43, 5161 (1972).
23. A. F. Volkov and F. Ya Nad', ZhETF Pis'ma Red. 11, 92 (1970).
[Sov. Phys. JETP Lett. 11, 56 (1970)].
24. W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Low Temp. Phys. 16, 145 (1974).
25. N. R. Werthamer, Phys. Rev. 147, 255 (1966).
26. R. E. Harris, Phys. Rev. B 10, 84 (1974).

References for Chapter 3

1. A.F.G. Wyatt, V. M. Dmitriev, W. S. Moore, and F. W. Sheard, Phys. Rev. Lett. 16, 1166 (1966).
2. A. H. Dayem and J. J. Wiegand, Phys. Rev. 155, 419 (1967).
3. T. K. Hunt and J. E. Mercereau, Phys. Rev. Lett. 18, 551 (1967).
4. T. C. Tredwell and E. H. Jacobsen, Phys. Rev. Lett. 35, 244 (1975); Phys. Rev. B 13, 2931 (1976).
5. T. M. Klapwijk and J. E. Mooij, Physica 81B, 132 (1976).
6. G. M. Eliashberg, ZhETF Pis. Red. 11, 186 (1970) [Sov. Phys. JETP Lett. 11, 114 (1971)].
7. D. M. Ginsberg, Phys. Rev. Lett. 8, 204 (1962).
8. B. I. Miller and A. H. Dayem, Phys. Rev. Lett. 18, 1000 (1967).
9. J. L. Levine and S. Y. Hsieh, Phys. Rev. Lett. 20, 994 (1968).
10. S. Y. Hsieh and J. L. Levine, Phys. Rev. Lett. 20, 1502 (1968).
11. K. E. Gray, A. R. Long, and C. J. Adkins, Phil. Mag. 20, 273 (1969).
12. J. L. Levine and S. Y. Hsieh, Physica 55, 471 (1971).
13. K. E. Gray, J. Phys. F 1, 290 (1971).
14. L. R. Testardi, Phys. Rev. B 4, 2189 (1971).
15. J. Clarke, Phys. Rev. Lett. 28, 1363 (1972).
16. W. H. Parker and W. D. Williams, Phys. Rev. Lett. 29, 924 (1972).
17. P. Hu, R. C. Dynes, and V. Narayanamurti, Phys. Rev. B 10, 2786 (1974).
18. G. A. Sai-Halasz, C. C. Chi, A. Denenstein, and D. N. Langenberg, Phys. Rev. Lett. 33, 215 (1974).
19. W. H. Parker, Sol. Stat. Comm. 15, 1003 (1974).

20. I. Schuller and K. E. Gray, Phys. Rev. Lett. 36, 429 (1976).
21. A. Rothwarf and B. N. Taylor, Phys. Rev. Lett. 19, 27 (1967).
22. C. S. Owen and D. J. Scalapino, Phys. Rev. Lett. 28, 1559 (1972).
23. W. H. Parker, Phys. Rev. B 12, 3667 (1975).
24. J. J. Chang and D. J. Scalapino, NATO Advanced Study Institute on Small Scale Superconducting Devices, Lago di Garda, Italy, 1976.
25. S. B. Kaplan, C. C. Chi, D. N. Langenberg, J. J. Chang, S. Jafarey, and D. J. Scalapino, to be published.
26. D. W. Palmer, Ph.D. thesis, California Institute of Technology, 1975.
27. M. Tinkham, M. Octavio, and W. J. Skocpol, to be published.
28. For a recent review, see M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975), Chap. 5.
29. W. E. Eisenmenger and A. H. Dayem, Phys. Rev. Lett. 18, 125 (1967).
30. H. Kinder, K. Lassmann, and W. Eisenmenger, Phys. Lett. A 31, 475 (1970).
31. A. H. Dayem, B. I. Miller, and J. J. Wiegand, Phys. Rev. B 3, 2949 (1971).
32. R. C. Dynes, V. Narayanamurti, and M. Chin, Phys. Rev. Lett. 26, 181 (1971).
33. V. Narayanamurti and R. C. Dynes, Phys. Rev. Lett. 27, 410 (1971).
34. A. H. Dayem and J. J. Wiegand, Phys. Rev. B 5, 4390 (1972).
35. R. C. Dynes and V. Narayanamurti, Phys. Rev. B 6, 143 (1972).
36. M. Welte, K. Lassmann, and W. Eisenmenger, J. de Physique 33, C4-25 (1972).
37. A. R. Long, J. de Physique 33, C4-74 (1972).

38. H. J. Trumpp, K. Lassmann, and W. Eisenmenger, Phys. Lett. A 41, 431 (1972).
39. H. Kinder, Phys. Rev. Lett. 28, 1564 (1972).
40. J.W.M. Bakker, H. Van Kempen, H.W.M. Salemink, and P. Wyder, Phys. Lett. A 42, 193 (1972).
41. H. Kinder, Z. Phys. 262, 295 (1973).
42. H. Kinder and J. P. Dietsche, Phys. Rev. Lett. 33, 578 (1974).
43. A.F.G. Wyatt, N. A. Lockerbie, and R. A. Sherlock, Phys. Rev. Lett. 33, 1425 (1974).
44. M. G. Hauser and D. W. Palmer, Rev. de Phys. Appl. 9, 53 (1974).
45. M. Octavio, W. J. Skocpol, and M. Tinkham, Applied Superconductivity Conference, Stanford, Ca., 1976, and preprint.
46. D. W. Palmer and J. E. Mercereau, Appl. Phys. Lett. 25, 467 (1974).
47. D. W. Palmer and J. E. Mercereau, IEEE Trans. Mag. 11, 667 (1975).
48. D. W. Jillie, J. E. Lukens, Y. H. Kao, and G. J. Dolan, Phys. Lett. 55A, 381 (1976).
49. D. N. Langenberg in Low Temperature Physics-LT14, Vol. V, M. Krusius and M. Vuorio, Eds. (North Holland/American Elsevier Pub. Co., Amsterdam/New York, 1975), p. 223.
50. J.T.C. Yeh and D. N. Langenberg, to be published.
51. G. H. Wood and B. L. White, Appl. Phys. Lett. 15, 237 (1969).
52. J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).
53. For example, see J. R. Schrieffer, Theory of Superconductivity (W. A. Benjamin, Inc., Reading, Mass., 1964), p. 47.
54. L. D. Landau, J. Exp. Theor. Phys. (USSR) 30, 1058 (1956) [Sov. Phys. JETP 3, 920 (1957)].

55. M. Tinkham and J. Clarke, Phys. Rev. Lett. 28, 1366 (1972).
56. A. Schmid and G. Schön, J. Low Temp. Phys. 20, 207 (1974).
57. A. R. Long and C. J. Adkins, Phil. Mag. 27, 865 (1973).
58. R. C. Dynes and V. Narayanamurti, Sol. Stat. Comm. 12, 341 (1973).
59. W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Low Temp. Phys. 16, 145 (1974).
60. E. Abrahams and T. Tsuneto, Phys. Rev. 152, 416 (1966).
61. R. V. Carlson and A. M. Goldman, Phys. Rev. Lett. 31, 880 (1973).
62. R. V. Carlson and A. M. Goldman, Phys. Rev. Lett. 34, 11 (1975).
63. A. M. Goldman and J. C. Solinsky in Low Temperature Physics-LT14, Vol. V, M. Krusius and M. Vuorio, Eds. (North Holland/American Elsevier Pub. Co., Amsterdam/New York, 1975), p. 379.
64. A. M. Goldman and R. V. Carlson, op. cit., p. 383.
65. G. Schön, preprint.
66. M. Tinkham, Introduction to Superconductivity (McGraw Hill, New York, 1975), p. 273.
67. M. Tinkham, Phys. Rev. Lett. 13, 804 (1964).
68. A. Schmid, Phys. kondens. Mater. 5, 302 (1966).
69. L. P. Gor'kov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 54, 612 (1968) [Sov. Phys. JETP 27, 328 (1968)].
70. J. R. Schrieffer, Theory of Superconductivity (W. A. Benjamin, Inc., Reading, Mass., 1964), pp. 42,43.
71. T. Wong, J.T.C. Yeh, and D. N. Langenberg, Phys. Rev. Lett. 37, 150 (1976).
72. R. Janik, L. Morelli, N. C. Cirillo, Jr., J. N. Lechevet, W. D. Gregory, and W. L. Goodman, IEEE Trans. MAG-11, 687 (1975).

73. The apparatus used is known as a "plasmod" and is manufactured by the Tegal Corp., Richmond, Ca.
74. P. K. Hansma, J. Appl. Phys. 45, 1472 (1974).
75. J. Lambe and R. C. Jaklevic in Tunneling Phenomena in Solids, E. Burstein and S. Lundqvist, Eds. (Plenum Press, New York, 1969), p. 233.
76. G. Deutscher and P. G. de Gennes in Superconductivity, R. D. Parks, Ed. (Marcel Dekker, Inc., New York, 1969), Vol. II, p. 1005.
77. M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975), Chap. 7
78. R. K. Kirschman, J. A. Hutchby, J. W. Burgess, R. P. McNamara, and H. A. Notarys, Applied Superconductivity Conference, Stanford, Cal, 1976; and to be published.
79. M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975), p. 112
80. M. Tinkham, op. cit., p. 238.
81. R. A. Buhrman and W. P. Halperin, Phys. Rev. Lett. 30, 692 (1973).
82. H. Nyquist, Phys. Rev. 29, 614 (1927).
83. R. H. Parmenter, Phys. Rev. Lett. 7, 274 (1961).
84. A. G. Aronov and V. L. Gurevich, Zh. Eksp. Teor. Fiz. 63, 1809 (1972) [Sov. Phys. JETP 36, 957 (1973)].

References for Chapter 4

1. M. L. Yu and J. E. Mercereau, Phys. Rev. Lett. 37, 1148 (1976).
2. M. L. Yu and J. E. Mercereau, Phys. Rev. Lett. 28, 1117 (1972).
3. M. L. Yu, Ph.D. thesis, California Institute of Technology, 1974.
4. M. L. Yu and J. E. Mercereau, Phys. Rev. B 12, 4909 (1975).
5. T. J. Rieger, D. J. Scalapino, and J. E. Mercereau, Phys. Rev. Lett. 27, 1787 (1971).
6. T. J. Rieger, D. J. Scalapino, and J. E. Mercereau, Phys. Rev. B 6, 1734 (1972).
7. J. Clarke, Phys. Rev. Lett. 28, 1363 (1972).
8. M. Tinkham and J. Clarke, Phys. Rev. Lett. 28, 1366 (1972).
9. D. W. Palmer, Ph.D. thesis, California Institute of Technology, 1975.
10. H. A. Notarys, M. L. Yu, and J. E. Mercereau, Phys. Rev. Lett. 30, 743 (1973).
11. J. E. Mercereau, Rev. de Phys. Appl. 9, 47 (1974).
12. B. D. Josephson, Phys. Lett. 1, 251 (1962).
13. M. Tinkham, Phys. Rev. Lett. 13, 804 (1964).
14. A. Schmid, Phys. kondens. Mater. 5, 302 (1966).
15. G. M. Eliashberg, ZhETF Pis. Red. 11, 186 (1970) [Sov. Phys. JETP Lett. 11, 114 (1971)].

References for Appendix A

1. J. Mathews and R. L. Walker, Mathematical Methods of Physics (W. A. Benjamin, Inc., New York, 1970), 2d Ed., p. 25.
2. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965), p. 361.
3. Ibid, p. 360.
4. J. Mathews and R. L. Walker, op. cit., p. 3.
5. A. F. Volkov and F. Ya Nad', ZhETF Pis'ma Red 11, 92 (1970) [Sov. Phys. JETP Lett. 11, 56 (1970)].
6. W. C. Stewart, Appl. Phys. Lett. 12, 277 (1968).
7. See, for example, N. F. Pedersen, T. F. Finnegan, and D. N. Langenberg, Phys. Rev. B 6, 4151 (1972).

References for Appendix B

1. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965), p. 361.
2. Ibid, p. 360.
3. S. M. Selby, Ed., Standard Mathematical Tables (Chemical Rubber Co., Cleveland, 1964), 14th Ed., p. 392.