REPLICATIVE FUNCTIONS

Thesis by
Michael Franz Yoder

In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1978
(Submitted July 15, 1977)

ACKNOWLEDGEMENTS

I would like to thank my thesis adviser, Professor John Todd, for his helpful advice and suggestions.

I would also like to thank Dr. W. A. J. Luxembourg and Dr. Charles DePrima for their patience and attention.

Finally, I would like to thank the California Institute of Technology, the Jorgenson Foundation, and, of course, my father, Harry W. Yoder, for their financial support.

ABSTRACT

Functions are considered which satisfy the set of equations

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)+b_n$$

for given sequences $\{a_n\}$, $\{b_n\}$. In the first part of the dissertation f is assumed to be a continuous function from the real line to the complex plane; the cases where f is periodic and aperiodic are considered separately. All possible aperiodic functions are determined; for f periodic the Fourier series is determined. Some miscellaneous results are also given.

In the second part f is considered to be a function from the complex plane to itself which is holomorphic except at isolated points. Again the cases where f is periodic and aperiodic are considered separately, and all possible f are determined for both cases.

TABLE OF CONTENTS

1. Introduction

Knuth ([1], Vol. 1, p. 42) proposes the problem of examining the class of functions which satisfy

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)+b_n \tag{1}$$

for all positive integers n and all real x, a_n and b_n being possibly complex and independent of x.

A great many well-known functions satisfy an equation (1).

The psi function, defined by

$$\psi(x)=\frac{d}{dx}\log\Gamma(x)$$

satisfies

$$\frac{1}{n}\sum_{k=0}^{n-1} \psi\left(x+\frac{k}{n}\right)=\psi(nx)-\ln n,$$

so (1) holds with $a_n=1$, $b_n=\ln n$.

The Bernoulli polynomials $B_m(x)$, defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!},$$

satisfy

$$\frac{1}{n}\sum_{k=0}^{n-1} B_m\left(x+\frac{k}{n}\right)=n^{-m}B_m(nx),$$

which is (1) with $a_n=n^{-m}$, $b_n=0$.

The generalized zeta function, defined for Re $s>1$ by

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s};\, x>0,$$

satisfies

$$\frac{1}{n}\sum_{k=0}^{n-1}\zeta\left(s,x+\frac{k}{n}\right)=n^{s-1}\zeta(s,nx),\, x>0,$$

this being valid for every complex s by analytic continuation; this is (1) with $a_n=n^{s-1}$, $b_n=0$.

The function $F(x)=\ln|2\sin\pi x|$

satisfies

$$\frac{1}{n}\sum_{k=0}^{n-1}\ln\left|2\sin\pi\left(x+\frac{k}{n}\right)\right|=\frac{1}{n}\ln\left|2\sin\pi nx\right|,$$

which is (1) with $a_n=\frac{1}{n}$, $b_n=0$.

(This follows from the identity

$$\prod_{k=0}^{n-1}2\sin\pi\left(x+\frac{k}{n}\right)=2\sin\pi nx).$$

Trigonometric examples of functions satisfying (1) include

$$\frac{1}{n}\sum_{k=0}^{n-1}\cot\pi\left(x+\frac{k}{n}\right)=\cot\pi nx$$

$$\frac{1}{n}\sum_{k=0}^{n-1}\csc^2\pi\left(x+\frac{k}{n}\right)=n\ \csc^2\pi nx$$

$$\frac{1}{n}\sum_{k=0}^{n-1}\csc\ 2\pi\left(x+\frac{k}{n}\right)=\csc\ 2\pi nx\quad(n\text{ odd})$$

$$= 0\quad(n\text{ even}).$$

In each of these cases $b_n=0$; for the first we have $a_n=1$, for the second $a_n=n$, and for the third $a_n=1$ for n odd and 0 for n even. In addition we have

$$\frac{1}{n}\sum_{k=0}^{n-1}\sin\ 2\pi\left(x+\frac{k}{n}\right)=\sin\ 2\pi x\quad(n=1)$$

$$= 0\qquad(n\neq 1),$$

and $\cos 2\pi x$ satisfies a similar identity. Here $a_n=0$ if $n\neq 1$.

Two problems are considered here. The first is that of finding all continuous functions f which satisfy (1) for some $\{a_n\}$, $\{b_n\}$; f is regarded as a complex-valued function on the real line, and $\{a_n\}$, $\{b_n\}$ are complex sequences. The other is that of finding all functions on the complex plane which are holomorphic except at isolated points and which satisfy an equation (1) for all $x\in\mathbb{C}$. The latter problem admits of a complete solution.

PART 1. FUNCTIONS ON THE REAL LINE

2. Preliminary Definitions and Results

DEFINITION.[1] A function $f\colon \mathbb{R}\to\mathbb{C}$ is $\{a_n, b_n\}$-replicative if for all integers $n\geq 1$ and $x\in\mathbb{R}$, the equation (1) holds. If all $b_n=0$, we may abbreviate this to say f is a_n-replicative.

Remarks. (1) Either $f=c$ is constant, whence f is $\{a_n, c-ca_n\}$-replicative for arbitrary $\{a_n\}$, or $\{a_n\}$, $\{b_n\}$ are uniquely determined by f.

(2) If f is $\{a_n, b_n\}$-replicative, f+c is $\{a_n, b_n+c-ca_n\}$-replicative.

(3) Let f be $\{a_n, b_n\}$-replicative and g be $\{a_n, b'_n\}$-replicative. Than c_1f+c_2g is $\{a_n, c_1b_n+c_2b'_n\}$-replicative.

(4) If f is $\{a_n, b_n\}$-replicative, df/dx is na_n-replicative; and if f is a_n-replicative, $\int_a^x f(u)\,du$ is $\{a_n/n, b_n\}$-replicative for some $\{b_n\}$.

(5) If f is $\{a_n, b_n\}$-replicative, then so is $g(x)=f(x-[x])$ where $[x]$ is the greatest integer less than or equal to x. To prove this note that both $a_n g(nx)+b_n$ and $(1/n)\sum_{k=0}^{n-1} g(x+k/n)$ are periodic with period $1/n$; hence, it suffices to prove (1) with f replaced by g for $0\leq x<1/n$. In such a case $0\leq nx<1$ and $0\leq x+k/n<1$ for all $0\leq k<n$, so the equation is equivalent to

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)+b_n .$$

[1]) The normal definition of 'replicative function' is equivalent in my terminology to being '$1/n$-replicative'.

which is true by assumption. Note that $g(x)$ has period 1.

In view of the first remark, f may be assumed to be nonconstant without loss of generality. In addition we may assume $a_1=1$, $b_1=0$.

THEOREM 1. If f is $\{a_n, b_n\}$-replicative (and nonconstant), then a_n is totally multiplicative; i.e., for all m and n $a_{mn}=a_m a_n$. Furthermore, at least one of the following two cases always holds:

1. $a_n=1$ for all n and the b_n satisfy $b_{mn}=b_m+b_n$.

2. There exists $c\in\mathbb{C}$ such that $b_n=c\left(a_n-1\right)$; in this case $f+c$ is a_n-replicative.

Proof. We have

$$a_{mn}f(mnx)+b_{mn}=\frac{1}{mn}\sum_{k=o}^{mn-1} f\left(x+\frac{k}{mn}\right)$$

$$=\frac{1}{m}\sum_{k=o}^{m-1}\frac{1}{n}\sum_{j=0}^{n-1} f\left(x+\frac{j}{n}+\frac{k}{mn}\right)$$

$$=\frac{1}{m}\sum_{k=o}^{m-1}\left(a_n f\left(n\left(x+\frac{k}{mn}\right)\right)+b_n\right)$$

$$=b_n+\frac{a_n}{m}\sum_{k=o}^{m-1} f\left(nx+\frac{k}{m}\right)$$

$$=b_n+a_n\left(a_m f(mnx)+b_m\right)$$

$$=a_m a_n f(mnx)+\left(a_n b_m+b_n\right).$$

Since f is not a constant, $a_{mn}=a_m a_n$ and $b_{mn}=a_n b_m+b_n$. By symmetry, $b_{mn}=a_m b_n+b_m$; hence

$$\left(a_n-1\right)b_m=\left(a_m-1\right)b_n.$$

If some $a_m\neq 1$, then

$$b_n=\frac{b_m}{a_m-1}\left(a_n-1\right)=c\left(a_n-1\right)$$

for all n, as desired, and by remark (a) above f+c is a_n-replicative. If all $a_n=1$, then

$$b_{mn}=a_n b_m+b_n=b_m+b_n.$$

3. <u>The Periodic Solutions</u>

In view of the fifth remark, any solution to (1) can be made into a periodic solution; if the original f was continuous, the new g will also be continuous except perhaps for a jump at every integer. Hence, g has a Fourier expansion which converges (C, 1) at every point to $1/2\left(g\left(x^+\right)+g\left(x^-\right)\right)$; this is always equal to g(x) except possibly when x is an integer.

With this in mind, some definitions are in order. Let

$$\varphi_m(x)=e^{2\pi imx}$$

$$\langle f, g\rangle = \int_0^1 f^* g\,dx$$

so that

$$\langle \varphi_m, \varphi_n\rangle = \delta_{mn}.$$

Let

$$P\{a_n; x\} = \sum_{n=1}^{\infty} a_n \varphi_n(x) \qquad (C, 1)$$

$$Q\{a_n; x\} = \sum_{n=1}^{\infty} a_n \varphi_{-n}(x) \qquad (C, 1)$$

provided only that the series P, Q are Cesàro-summable. Also, define $\mathcal{U}\{a_n\}$ as the vector space (over $\mathbb{C}$) of all continuous a_n-replicative functions, and $\mathcal{P}\{a_n\}$ as the subspace of $\mathcal{U}\{a_n\}$ containing all functions of period 1.

We are now almost ready to investigate the consequences of restricting f to be continuous; before proceeding, however, one minor point must be cleared up.

THEOREM 2. Let f be continuous and $\{1, b_n\}$-replicative. Then $b_n=0$ for all n.

Proof. We have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = f(0) + b_n. \tag{2}$$

Now f is Riemann integrable on [0, 1]; hence, the left side of (2) approaches $\int_0^1 f(u)du$ as $n\to\infty$; therefore, b_n must approach a limit also. But if some $b_m\neq 0$, then from Theorem 1

$$b_{m^2}=2b_m,\ b_{m^3}=3b_m,\ldots,b_{m^k}=kb_m$$

which diverges as $k\to\infty$. Hence, all $b_n=0$.

Theorems 1 and 2 imply that without loss of generality we may assume all continuous f to be a_n-replicative if they are (a_n, b_n)-replicative.

THEOREM 3. Let $\{a_n\}$ be a totally multiplicative sequence. If $f\in\mathfrak{D}(a_n)$, then f is a linear combination of $P(a_n;x)$ and $Q(a_n;x)$ except possibly when all $a_n=1$; in this case $\mathfrak{D}(1)$ is just the space of constant functions. Thus, $\mathfrak{D}(a_n)$ has dimension ≤ 2. Conversely, any linear combination of $P(a_n;x)$ and $Q(a_n;x)$ which is continuous is a_n-replicative.

Proof. We have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx),$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)\varphi^*_{mn}(x)=a_n f(nx)\varphi^*_{mn}(x).$$

So, by periodicity,

$$\frac{1}{n}\sum_{k=o}^{n-1} f\left(x+\frac{k}{n}\right)\varphi^*_{mn}\left(x+\frac{k}{n}\right)=a_n f(nx)\varphi^*_m(nx)$$

$$\frac{1}{n}\int_0^1\left(\sum_{k=o}^{n-1} f\left(x+\frac{k}{n}\right)\varphi^*_{mn}\left(x+\frac{k}{n}\right)\right)dx=\int_0^1 a_n f(nx)\varphi^*_m(nx)\,dx$$

For each k we have

$$\int_o^1 f\left(x+\frac{k}{n}\right)\varphi^*_{mn}\left(x+\frac{k}{n}\right)dx=\int_0^1 f(u)\varphi^*_{mn}(u)\,du,$$

so

$$\int_o^1 f(u)\varphi^*_{mn}(u)\,du=\frac{a_n}{n}\int_o^n f(u)\varphi^*_m(u)\,du$$

$$\langle\varphi_{mn},f\rangle=a_n\langle\varphi_m,f\rangle.$$

In particular

$$\langle\varphi_n,f\rangle=a_n\langle\varphi_1,f\rangle$$

$$\langle\varphi_{-n},f\rangle=a_n\langle\varphi_{-1},f\rangle$$

$$\langle\varphi_o,f\rangle=a_n\langle\varphi_o,f\rangle$$

so if any $a_n\neq 1$, $\langle\varphi_o,f\rangle=0$. If all $a_n=1$, we know $\langle\varphi_o,f\rangle$ is arbitrary since 1 is 1-replicative. So if $a_n\neq 1$,

$$f(x)=\langle\varphi_1,f\rangle P(a_n;x)+\langle\varphi_{-1},f\rangle Q(a_n;x).$$

For the converse, note

$$\frac{1}{m}\sum_{k=o}^{m-1} P\left(a_n; x+\frac{k}{m}\right) = \frac{1}{m}\sum_{k=o}^{m-1}\sum_{n=1}^{\infty} a_n e^{2\pi i n(x+k/m)}$$

$$= \sum_{n=1}^{\infty} a_n e^{2\pi i n x}\left(\frac{1}{m}\sum_{k=o}^{m-1} e^{2\pi i k(n/m)}\right)$$

$$= \sum_{n=1}^{\infty} a_{mn} e^{2\pi i m n x}$$

$$= \sum_{n=1}^{\infty} a_m a_n e^{2\pi i n(mx)}$$

$$= a_m P\left(a_n; mx\right)$$

where all sums are (C,1). Thus, $P\left(a_n;x\right)$ is a_n-replicative, and similarly $Q\left(a_n;x\right)$ is also, whence the converse follows immediately.

Unfortunately, given a particular totally multiplicative sequence $\{a_n\}$, there seems to be no effective way of determining whether $\mathcal{D}\left(a_n\right)$ is the null space, or of dimension one, or of dimension two (obviously the latter is true if and only if $P\left(a_n;x\right)$ is continuous, since $Q\left(a_n;x\right) = P\left(a_n;-x\right)\big)$. However, a few specialized results can be proven.

COROLLARY 3.1. If $f \in \mathcal{D}\left(a_n\right)$ is not a constant, then all of the following hold:

(3) $$\sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

$$a_n \to 0 \text{ as } n \to \infty,$$

$$|a_n| < 1 \text{ for all } n \neq 1.$$

Note that the second follows from the first, and the third from the second upon considering Theorem 1. To prove (3), take $f \in \mathcal{P}(a_n)$ and assume $f = c_o + c_1 P(a_n; x) + c_2 Q(a_n; x)$ where c_1, c_2 are not both zero. (In general $c_o = 0$, but if all $a_n = 1$ it might be nonzero). Then

$$\langle f, \bar{f} \rangle = |c_o|^2 + \left(|c_1|^2 + |c_2|^2 \right) \sum_{n=1}^{\infty} |a_n|^2 < \infty$$

since f is bounded on [0,1]; as $|c_1|^2 + |c_2|^2 > 0$, the inequality (3) follows. This clearly shows that $\mathcal{P}(1)$ consists of just constant functions.

COROLLARY 3.2. If

$$\sum_{n=1}^{\infty} |a_n| < \infty, \text{ then } \dim \mathcal{P}(a_n) = 2.$$

This is clear, since the sums $\sum_{k=1}^{m} a_k \varphi_{\pm k}(x)$ will converge uniformly to $P(a_n; x)$ and $Q(a_n; x)$.

COROLLARY 3.3. Assume $\dim \mathcal{P}(a_n) = 1$ where $a_n \neq 1$. Then

$\mathcal{P}\{a_n\}$ consists entirely of either even or odd functions; i.e., all elements are multiples of $P\{a_n;x\} \pm Q\{a_n;x\}$.

Proof. If dim $\mathcal{P}\{a_n\}=1$, all functions in $\mathcal{P}\{a_n\}$ are multiples of some $f(x)=c_1 P\{a_n;x\}+c_2 Q\{a_n;x\}$. But $f(-x)= c_1 Q\{a_n;x\}+c_2 P\{a_n;x\}$ is also a_n-replicative and is continuous if f is, so we must have $f(-x)=cf(x)$; then obviously $c^2=1$, so $c=\pm 1$ and f is either even or odd, as desired.

COROLLARY 3.4. Let $s=\sigma+it$. Then

(a) if $\sigma<-1$, dim $\mathcal{P}(n^s)=2$

(b) if $\sigma\geq -1$ and $s\neq 0$, dim $\mathcal{P}(n^s)=0$

(c) if $s=0$, $\mathcal{P}(n^s)$ is the space of constant functions.

Ilan Amit has shown that functions in L_2 which satisfy (1) a.e. exist iff $\sum|a_n|^2<\infty$ and in this case form a subspace of dimension 2. See [7].

4. Aperiodic Solutions

Let $\Delta f(x)=f(x+1)-f(x)$. Then subtracting

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)$$

from

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{1}{n}+\frac{k}{n}\right)=a_n f(nx+1)$$

we find

(4) $$\Delta f(x)=na_n\Delta f(nx)$$

Thus, if any $a_n=0$, then $\Delta f(x)=0$ and f is periodic. We assume here that f is aperiodic, so that Δf is not identically zero. Then $a_m\neq 0$, so that by replacing x in (4) by mx/n and then using (4) with n replaced by m we obtain

(5) $$\Delta f\left(\frac{m}{n}x\right)=\frac{na_n}{ma_m}\Delta f(x).$$

We now derive

THEOREM 4. Assume $\mathcal{P}\{a_n\}\neq\mathcal{U}\{a_n\}$. Then there is an $s=\sigma+it$ such that $a_n=n^s$ for all n; and also either $\sigma<-1$ or $s=-1$. Conversely, if $\sigma<-1$, then $\dim \mathcal{U}(n^s)=4$, and $\dim\mathcal{U}(n^{-1})=1$ so that for these cases $\mathcal{P}\{a_n\}\neq\mathcal{U}\{a_n\}$; i.e., there exist aperiodic solutions.

Proof. By (4), if $f\in\mathcal{U}\{a_n\}-\mathcal{P}\{a_n\}$ then $a_n\neq 0$ for all n; if we could also have $\Delta f(1)=\Delta f(-1)=0$ then (5) and continuity would imply $\Delta f(x)=0$ for all x. We may, therefore, take $\Delta f(1)\neq 0$ without loss of generality, so $\Delta f(x)\neq 0$ for $x>0$; hence, there exists a continuous function h on $(0,\infty)$ such that $\exp(h(x))=\Delta f(x)$.

Equation (4) then implies

$$\exp(h(x)-h(nx))=na_n$$

$$h(x)-h(nx)=\ln na_n+2\pi m_x i, m_x\in\mathbb{Z}.$$

But h is continuous, so m_x must be a constant (depending perhaps on n), and we may therefore define

$$h(x)-h(nx)=\lambda_n \ln n.$$

Then in particular, $h(x)-h(mnx)=(h(x)-h(mx))+(h(mx)-h(mnx))$

$$\lambda_{mn}\ln mn=\lambda_m \ln m+\lambda_n \ln n$$

and by induction

$$\lambda_{m^k} \ln m^k=k\lambda_m \ln m$$

$$\lambda_{m^k}=\lambda_m.$$

Let $p>2$; if p is a power of 2, then by the above argument $\lambda_p=\lambda_2$. If not, then $Q=\ln p/\ln 2$ is irrational; hence, there are infinitely many continued fraction convergents to Q; denote them by h_i/k_i. Then it is well known that

$$\left|\frac{h_i}{k_i}-\frac{\ln p}{\ln 2}\right|<\frac{1}{k_i^2};$$

hence

$$|h_i \ln 2-k_i \ln p|<\frac{\ln 2}{k_i}\to 0,\quad i\to\infty,$$

so that

$$\exp\left(h_i \ln 2 - k_i \ln p\right) = 2^{h_i} p^{-k_i} \to 1, \quad i \to \infty.$$

So

$$h(1) - h\left(2^{h_i} p^{-k_i}\right) \to 0$$

or

$$h_i \lambda_2 \ln 2 - k_i \lambda_p \quad p \to 0.$$

If $\lambda_2 = 0$, this implies $\lambda_p = 0$; otherwise

$$\frac{h_i}{k_i} - \frac{\lambda p}{\lambda_2} \frac{\ln p}{\ln 2} \to 0.$$

But $h_i/k_i \to \ln p/\ln 2$, whence $\lambda_p/\lambda_2 = 1$ and $\lambda_p = \lambda_2$.

So in either case $\lambda_p = \lambda_2$; since all λ_i's are the same we may let $\lambda_2 = 1+s$ to derive

$$na_n = \exp(h(x) - h(nx)) = n^{1+s}$$

$$h(x) - h(nx) = (1+s) \ln n$$

i.e.,

$$a_n = n^s.$$

Now, by (5)

$$\Delta f\left(\frac{m}{n}\right) = \left(\frac{m}{n}\right)^{-s-1} \Delta f(1)$$

and by continuity

$$\Delta f(x) = x^{-s-1} \Delta f(1) \text{ for } x > 0;$$

similarly

$$\Delta f(x)=x^{-s-1}\Delta f(-1) \text{ for } x<0.$$

But Δf is continuous, so we must have either $\sigma<-1$ or $s=-1$ and $\Delta f(1)=\Delta f(-1)$; in all other cases f would be periodic.

For the converse, note that for $s=-1$, $f=x-\frac{1}{2}\in\mathcal{U}(n^s)-\mathcal{P}(n^s)$; for $\sigma<-1$, the same is true of the functions e_1, e_2 defined by

$$e_1(x)=\sum_{0<n\leq x} n^{-s}$$

$$e_2(x)=\sum_{0<n\leq 1-x} n^{-s}.$$

It is easy to see that all aperiodic solutions are linear combinations of these with periodic solutions and the statement about the dimensionality of $\mathcal{U}(n^s)$ is verified. In particular, the conjecture by Knuth that multiples of $x-\frac{1}{2}$ are the only continuous solutions to (1) with $a_n=\frac{1}{n}$, $b_n=0$ is true.

5. Miscellaneous Results

If the condition that f is continuous is replaced with the milder one that f have only isolated discontinuities, significant deductions can still be made about the character of f. Let $\mathcal{V}(a_n)$ be the space of a_n-replicative functions with this property; then we have

THEOREM 5. If $f \in \mathcal{M}\{a_n\}$, and all $a_n \neq 0$, then f has discontinuities only at integers. (This holds even when f is considered as defined over $\mathbb{C}$).

Proof. By (1), if f has a discontinuity at x, it has one at one of $x/n, x/n+1/n, \ldots, x/n+(n-1)/n$. If x is not a real rational, this immediately implies that there are an infinity of discontinuities within a closed disc of radius $1+|x|$ about the origin upon considering $n=p_1, p_2 \ldots$ where the p_i are distinct primes. If $x=p/q$ in lowest terms, $q \neq 1$, then one of

$$\frac{p}{q^2}, \frac{p+q}{q^2}, \ldots, \frac{p+(q-1)q}{q^2}$$

is a discontinuity with a larger denominator, so the same applies.

THEOREM 6. Let $\{a_n\}$ satisfy $a_{mn}=a_m a_n$. If $\sum_{n=1}^{\infty} a_n e^{\pm 2\pi i n x}$ is summable (A) to a finite limit except at isolated points, then it defines an a_n-replicative function f, provided that we allow $f(x)=\infty$ at these points.

Proof. The proof is trivial and is left as an exercise to the reader. Note that if F is any method of summation which satisfies

$$\sum (a_n+b_n) = \sum a_n + \sum b_n \ (F)$$
$$\sum c a_n = c \sum a_n \ (F) \quad .$$

if $b_{mn}=a_n$ (m a fixed positive integer), $b_{mn+k}=0$ for $0<k<m$, then

$$\sum b_n = \sum a_n \ (F)$$

then Theorem 6 is true with A replaced by F.

COROLLARY 6.1. The functions

$$P(n^s;x) = \sum_{n=1}^{\infty} n^s e^{2\pi inx} \ (A)$$

$$Q(n^s;x) = \sum_{n=1}^{\infty} n^s e^{-2\pi inx} \ (A)$$

which are defined (when x is not an integer) for all $s \in \mathbb{C}$, are n^s-replicative.

See [4], Sections 5.12, 6.10, and 6.11 for verification.

An interesting unresolved problem is to determine for which sequences $\{b_n\}$ there exists an f, continuous except at the integers, such that f is $(1, b_n)$-replicative. The obvious possibility $b_n = \ln n$ actually occurs; the psi function defined by

$$\psi(x) = -\lim_{n\to\infty}\left(\sum_{m=1}^{n} \frac{1}{x+m} - \ln n\right)$$

is $(1, \ln n)$-replicative (see [5], p. 330).

PART 2. REPLICATIVE FUNCTIONS IN THE COMPLEX PLANE

The problem solved here is that of finding all functions $f:\mathbb{C}\to\mathbb{C}$ which are holomorphic, except at isolated points in $\mathbb{C}$, and satisfy the equations (1) for all $z\in\mathbb{C}$.

Frequent use will be made of the results from Part One and the Introduction; it is easy to see that they are as valid for the complex plane as for the real line.

We consider separately the cases where f is periodic (of period 1) and aperiodic.

6. The Periodic Case

THEOREM 7. If f is analytic and periodic, $f\not\equiv 0$, and $(\{a_n\},\{b_n\})$-replicative, then $a_n=\delta_{n1}$ and there are constants c_o, c_1, c_2 such that

$$f(z)=c_o+c_1e^{2\pi iz}+c_2e^{-2\pi iz}$$

Proof. Assume some $a_m\neq 0$, $m>1$. Choose $k\geqslant 1$ such that $|m^ka^k|>1$; we have that D^kf is $\{n^ka_n\}$-replicative and is also analytic and periodic. By considering the restriction of f to the real line and applying Corollary 3.1 of [1] we see that $D^kf\equiv 0$. Then f is both periodic and a polynomial, so f is constant, a contradiction.

Thus, $a_n=\delta_{n1}$; considering the restriction of f to $\mathbb{R}$ and using Theorem 3 we obtain, for $x\in\mathbb{R}$,

$$f(x)=c_o+c_1\sum_{n=1}^{\infty}\delta_{n1}e^{2\pi inx}+c_2\sum_{n=1}^{\infty}\delta_{n1}e^{-2\pi inx}\ (C,1)$$

$$=c_o+c_1e^{2\pi ix}+c_2e^{-2\pi ix}$$

which gives us the desired result by analytic continuation.

THEOREM 8. Let f be $(\{a_n\}, \{b_n\})$-replicative with only isolated singularities. Then either

(a) f is analytic,

or

(b) there is an integer b such that all singularities of f are multiples of b^{-1} <u>and</u> such that for any p dividing b, $a_p=0$.

Proof. There are many cases to consider.

Case 1. For infinitely many primes p_i, $a_{p_i}=0$. Then by (1) if z is a singularity so is one of

$$\left\{z+\frac{1}{p_i}, z+\frac{2}{p_i}, \ldots, z+\frac{p_i-1}{p_i}\right\}.$$

As each of these sets is disjoint we would have somewhere on the line from z to z+1 an accumulation of singularities; hence, f must be analytic.

Case 2. f has a singularity z with Im $z\neq 0$. By Case 1 we can find infinitely many primes p with $a_p\neq 0$. Using (1) with x replaced by $\frac{z}{p}$ we see that one of

$$\left\{\frac{z}{p}, \frac{z+1}{p}, \ldots, \frac{z+p-1}{p}\right\}$$

is also a singularity. This would give us an infinity of singularities in a compact region (a circle of radius $1+|z|$, say), which is a contradiction, since these sets are disjoint.

Case 3. f has a real irrational singularity α. Take primes p with $a_p \neq 0$ and again look at the sets

$$\left\{\frac{\alpha}{p}, \frac{\alpha+1}{p}, \ldots, \frac{\alpha+p-1}{p}\right\}.$$

If $p_1 = p_2$, then

$$\frac{\alpha+j_1}{p_1} = \frac{\alpha+j_2}{p_2} \Rightarrow \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\alpha = \frac{j_2}{p_2} - \frac{j_1}{p_1}$$

which would imply α is rational; hence, the sets are disjoint and we get the same contradiction.

Case 4. f has a singularity at $\frac{n}{k}$ in lowest terms, and p is a prime dividing k. I assert $a_p = 0$; otherwise one of

$$\left\{\frac{h}{pk}, \frac{h+k}{pk}, \ldots, \frac{h+(p-1)k}{pk}\right\}$$

is a singularity with a higher power of p in the denominator since $p \nmid h+jk$ for all j. Repeating the process gives an infinite set of singularities all with successively higher powers of p in the denominator, and we get a contradiction.

This gives us the theorem. If f is not periodic, then from formula (5) all $a_p \neq 0$ and all singularities of f must be integers; so if f is not analytic, we may take b=1. If f is periodic, then all singularities of f are congruent modulo 1 to a singularity of f in [0,1]; there can only be a finite number of these, and we let b be the lcm of their denominators. Then $p \nmid b$ means p divides some denominator of a singularity, hence $a_p = 0$.

In preparation for the next theorem I introduce a new set of functions.

Definition. The function $P_m(z)$, $m \geq 1$, is the symmetric sum

$$P_m(z) = \sum_{n=-\infty}^{\infty} (z+n)^{-m}$$

$$= \frac{(-1)^{m-1} D^{m-1}}{(m-1)!} \pi \cot \pi z .$$

It is easily verified that $P_m(z)$ is $\{n^{m-1}\}$-replicative.

THEOREM 9. Let f be $(\{a_n\}, \{b_n\})$-replicative and not analytic. Let b be as in Theorem 2. Then there is an integer $m \geq 0$ and coefficients c_i $(0 \leq i < b)$ not all zero such that

$$f(z) = g(z) + \sum_{k=0}^{b-1} c_k P_m\left(z - \frac{k}{b}\right)$$

where $g(z)$ is analytic. Thus all singularities of f are poles of the same order m.

Proof. Let $R_k(f;a)$ mean the coefficient of $(z-a)^{-k}$ in the Laurent expansion of f at a (this must exist at every point). It is easy to see that for general $f, h \neq 0, a$ we have

$$R_m(f(z+h);a) = R_m(f;a+h)$$

and

$$R_m(f(hz);a) = h^{-m} R_m(f;ha).$$

Take some p with $a_p \neq 0$; from

$$\frac{1}{p}\sum_{k=0}^{p-1} f\left(z+\frac{k}{p}\right) = a_p f(pz) + b_p$$

we get (for $m \geq 1$)

$$\frac{1}{p}\sum_{k=0}^{p-1} R_m\left(f\left(z+\frac{k}{p}\right); z_0\right) = a_p R_m\left(f(pz); z_0\right)$$

$$\frac{1}{p}\sum_{k=0}^{p-1} R_m\left(f; z_0+\frac{k}{p}\right) = a_p p^{-m} R_m\left(f; pz_0\right).$$

Choosing $z_0 = \frac{h}{b}$ a singular point one notes that as $(p,b)=1$ all the points $z_0 + \frac{k}{p}, k \neq 0$ will be points where f is analytic, hence

$$\frac{1}{p} R_m\left(f;\frac{h}{b}\right)=a_p p^{-m} R_m\left(f;\frac{hp}{b}\right)$$

(7) $$R_m\left(f;\frac{h}{b}\right)=\left(a_p p^{1-m}\right) R_m\left(f;\frac{hp}{b}\right).$$

and by an easy induction

$$R_m\left(f;\frac{h}{b}\right)=\left(a_p p^{1-m}\right)^{\alpha} R_m\left(f;\frac{hp^{\alpha}}{b}\right).$$

for $\alpha \geq 1$. Take α so that $p^{\alpha} \equiv 1 \pmod{b}$; then

$$R_m\left(f;\frac{h}{b}\right)=\left(a_p p^{1-m}\right)^{\alpha} R_m\left(f;\frac{h}{b}\right),$$

so that either $R_m\left(f;\frac{h}{b}\right)=0$ or $|a_p|=p^{m-1}$.
The latter can be true for at most one m, but some $R_m\left(f;\frac{h}{b}\right)\neq 0$, else f would be analytic. Therefore, at every point where f is not holomorphic it has a pole of order m, and if

(8) $$c_k=R_m\left(f;\frac{k}{b}\right)$$

it is easy to see that

$$g(z)=f(z)-\sum_{k=0}^{b-1} c_k P_m\left(z-\frac{k}{b}\right)$$

is analytic in $\mathbb{C}$.

THEOREM 10. Let f be $(\{a_n\}, \{b_n\})$-replicative, periodic, and not analytic. Let m and b be as in Theorem 3. Define $\theta_n=a_n n^{1-m}$ and let $\gamma=e^{2\pi i z}$. Then $\{\theta_n\}$ is totally multiplicative, and

(a) $\{\theta_n\}$ is periodic modulo b;

(b) for all n either $\theta_n=0$ or $|\theta_n|=1$;

(c) there are constants k_o, k_1, such that

$$f(z)=F(\tau)=k_o+k_1\sum_{n=1}^{\infty}a_n\tau^n, k_1\neq 0. \tag{9}$$

Proof. $\{\theta_n\}$ is completely determined by the values θ_p for p prime. If $p|b$ we have $a_p=0$, so $\theta_p=0$; if $p\nmid b$ we have from the proof of Theorem 3 that $|a_p|=p^{m-1}$, hence $|\theta_p|=1$. This proves (b). Let c_i be as in Theorem 3; from (7) and (8) we have for $(p,b)=1$ (even if p is not prime!)

$$c_h=\theta_p c_{hp}.$$

Since for some h, $c_h\neq 0$, this implies that $\theta_{n_1}=\theta_{n_2}$ whenever $n_1\equiv n_2 \pmod b$ and $(n_1,b)=1$. If $(n,b)\neq 1$ then $\theta_n=0$ from the total multiplicativity of $\{\theta_n\}$, whence (a) is true.

Finally, since f is periodic, the F defined in (9) exists and is holomorphic in $0<|\tau|<1$ since f is holomorphic in the upper half-plane. Then F has a Laurent expansion convergent in $0<|\tau|<1$ (f is also holomorphic in the lower half-plane; we arbitrarily choose to use the upper one).

$$F(\tau)=\sum_{j=-\infty}^{\infty}s_j\tau^j. \tag{10}$$

Rewriting the equation (1) in terms of F gives

$$\frac{1}{n}\sum_{k=0}^{n-1} F\left(\omega_n{}^k \tau\right) = a_n F\left(\tau^n\right) + b_n$$

where $\omega_n = e^{2\pi i/n}$. Substituting in (10) yields

$$a_n F\left(\tau^n\right) + b_n = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{j=-\infty}^{\infty} s_j \omega_n{}^{jk} \tau^j$$

(11)
$$= \sum_{j=-\infty}^{\infty} s_j \tau^j \left(\frac{1}{n}\sum_{k=0}^{n-1} \omega_n{}^{jk}\right)$$

$$= \sum_{j=-\infty}^{\infty} s_{nj} \tau^{nj}$$

since the parenthesized term in (11) is 0 when $n \nmid j$ and 1 when $n \mid j$. Equating coefficients gives, if $j \neq 0$,

$$a_n s_j = s_{nj}$$

whence

$$s_n = a_n s_1$$

$$s_{-n} = a_n s_{-1}$$

and to have convergence for $|\tau| < 1$ we must have $s_{-1} = 0$ (for infinitely many n, $\left|a_n\right| = n^{1-m}$, so $\lim\sup \left|a_n\right|^{1/n} = 1$). Then let $k_o = s_o, k_1 = s_1$ to get (c).

We now have enough to completely specify all periodic solutions. Summarizing previous results we get

THEOREM 11. Let f be a periodic (period 1) nonconstant solution to (1). Then either

(a) $a_n = \delta_{n1}$ and there are k_o, k_1, k_2 such that

$$F(z)=k_o+k_1e^{2\pi iz}+k_2e^{-2\pi iz},$$

k_1 and k_2 not both zero,

or (b) there are positive integers m, b and a character χ modulo b such that $a_n=n^{m-1}\chi(n)$ and for some k_o and $k_1 \neq 0$

$$(12) \qquad F(z)=k_o+k_1D^{m-1}\left(\frac{\sum_{j=1}^{b} \chi(j)e^{2\pi ijz}}{1-e^{2\pi ibz}}\right).$$

A character χ modulo b is a totally multiplicative function which is periodic mod b and such that $\chi(m)=0$ whenever $(m,b)\neq 1$.

Proof. If case (a) does not hold, by the preceding theorems, if we define $\chi(n)=n^{1-m}a_n$, χ is a character. Also, if

$$g(z)=G(\tau)$$

we get

$$(13) \qquad g'(z)=2\pi i\tau G'(\tau);$$

applying (13) m-1 times to (a) and "absorbing" the factors of $2\pi i$ into k_1 (it is an arbitrary nonzero constant) we have

$$f(z)=k_o+k_1D_z^{m-1}\left(\sum_{j=1}^{\infty}\chi(j)\tau^j\right).$$

But χ is periodic modulo b, whence

$$f(z)=k_o+k_1D_z^{m-1}\left(\frac{\sum_{j=1}^{b}\chi(j)\tau^j}{1-\tau^b}\right).$$

Replacing τ with $e^{2\pi iz}$ we have (12).

The converse of this theorem also holds by a simple scrutiny of the proof of the theorem.

7. <u>The Aperiodic Case</u>

Assume $\Delta f(z)=f(z+1)-f(z)\not\equiv 0$. By replacing z with $z+\frac{1}{n}$ in (1) and subtracting (1) one gets

(14) $$\Delta f(z)=na_n\Delta f(nz);$$

it follows that all $a_n\neq 0$. Furthermore

(15) $$\Delta f\left(\frac{m}{n}z\right)=\frac{na_n}{ma_m}\Delta f(z).$$

This implies that Δf has no singularities or zeros except perhaps at z=0, since in the latter case continuity and analytic continuation would imply $\Delta f\equiv 0$, and in the former Δf would be singular at every point on the ray from 0 through the given singular point.

We now show

THEOREM 12. Let f satisfy (1) with $\Delta f \equiv 0$. Then there is an integer m and a $c \neq 0$ such that

(16) $$a_n = n^{-m-1} \text{ and}$$

(17) $$\Delta f(z) = cz^m.$$

Proof. Select an $n > 1$; then in the annulus between circles of radii $\frac{1}{n}$ and 1 around 0, Δf has neither zeros nor singularities, so there are real numbers M_1, M_2 such that

$$0 < M_1 \leq |\Delta f(z)| \leq M_2 < \infty.$$

Then by (14) plus a simple induction, if b is an integer $\geq$ 0, for z in the annulus between circles of radii n^{-b-1} and n^{-b} we have

$$|na_n|^b M_1 \leq |\Delta f(z)| \leq |na_n|^b M_2.$$

This shows that Δf cannot have an essential singularity at 0, so Δf is meromorphic. Let the principal part of Δf at 0 be cz^m; then

$$\lim_{z \to 0} \frac{\Delta f(nz)}{\Delta f(z)} = \frac{cn^m z^m}{cz^m} = n^m$$

but by (15) $\Delta f(nz) = \frac{1}{na_n} \Delta f(z)$, so (16) follows. Equation (15) shows that whenever z is a positive rational,

(18) $$\Delta f(z)=z^m \Delta f(1)$$

and analytic continuation shows that (18) holds everywhere, whence $c=\Delta f(1)$ and (17) is true.

There are in fact examples of functions satisfying (1) with $\Delta f(z)=z^m$ for all integral values of m. Define

$$E_m(z)=\begin{cases} -\sum_{n=0}^{\infty}(x+n)^m & \text{if } m<-1 \\ \lim_{M\to\infty}\left(\ln M-\sum_{n=0}^{M}(x+n)^{-1}\right) & \text{if } m=-1 \\ B_{m+1}(z) & \text{if } m\geq 0 \end{cases}$$

Here $B_k(z)$ means the kth Bernoulli polynomial, and $E_{-1}(z)$ is recognizable as the psi function. These are all $\left(\{n^{-m-1}\}, \{0\}\right)$-replicative except for E_{-1}, which is $\left(\{1\},\{\ln n\}\right)$-replicative. For $m<-1$ this is left as an exercise to the reader; for $m\geq -1$ see [1], vol. 1, p. 42, problems 39,40.

THEOREM 13. If f is aperiodic and $\left(\{a_n\},\{b_n\}\right)$-replicative, then for some $m\in\mathbb{Z}$, $a_n=n^{-m-1}$, and f is of the form

$$f(z)=g(z)+cE_m(z)$$

where $c\neq 0$ and g is $\left(\{a_n\},\{b'_n\}\right)$-replicative for some b'_n. Also g has period 1.

Proof. This follows trivially from Theorem 6 together with the properties of the E_m.

This completes the solution to the original problem.

8. Summary

Let $\{a_n\}$ be an arbitrary given sequence, and let $W(\{a_n\})$ be the linear space of all functions which satisfy (1) for some sequence $\{b_n\}$. All constant functions will be in this space; combining the previous results we see there are five cases.

THEOREM 14.

(a) If $a_n=\delta_{n1}$, then

$$W(\{a_n\})=\text{span}\left(1,e^{2\pi iz},e^{-2\pi iz}\right).$$

(b) If $a_n=n^{m-1}$, $m\in\mathbb{Z}$, $m\geq 1$, then

$$W(\{a_n\})=\text{span}\left(1,P_m(z),E_{-m}(z)\right).$$

(c) If $a_n=n^{m-1}$, $m\in\mathbb{Z}$, $m\leq 0$, then

$$W(\{a_n\})=\text{span}\left(1,E_{-m}(z)\right).$$

(d) If $a_n=n^{m-1}\chi(n)$, χ a character mod $b\neq 1$, then

$$W(\{a_n\})=\text{span}\left(1,D^{m-1}\left(\frac{\sum_{j=1}^{b}\chi(j)\,e^{2\pi ijz}}{1-e^{2\pi ibz}}\right)\right).$$

(e) In all other cases

$$W\left(\{a_n\}\right)=\text{span }(1).$$

It is easily seen that the given spanning elements are also linearly independent in each case. As an example, let us take m=1, b=2 and χ such that $\chi(2n)=0$, $\chi(2n+1)=1$. Then $a_n=0$ if n is even, $a_n=1$ if n is odd, and

$$D^{m-1}\left(\frac{\sum_{j=1}^{b}\chi(j)e^{2\pi ijz}}{1-e^{2\pi ibz}}\right)=\frac{e^{2\pi iz}}{1-e^{4\pi iz}}$$

$$=\frac{1}{e^{-2\pi iz}-e^{2\pi iz}}=-\frac{1}{2i}\csc 2\pi z$$

so $W\left(\{a_n\}\right)=\text{span}(1,\csc 2\pi z)$ verifying the identity given for $\csc 2\pi z$ in the introduction.

REFERENCES

[1] KNUTH, D. E., The Art of Computer Programming, Vol. I. Addison-Wesley, Reading, Mass., 1969.

[2] MORDELL, L. J., Integral Formulae of Arithmetical Character, J. London Math. Soc. 33, 371-375 (1958).

[3] MIKOLAS, M., Integral Formulae of Arithmetical Characteristics Relating to the Zeta-Function of Hurwitz, Publ. Math. Debrecen 5, 44-53 (1957).

[4] HARDY, G. H., Divergent Series. Oxford University Press, London, 1949.

[5] JORDAN, C., Calculus of Finite Differences, Chelsea Publ. Co., New York, N. Y., 1947.

[6] YODER, M. F., Continuous Replicative Functions, Aeq. Math. 13 (1975), 251-261.

[7] AMIT, ILAN, On Replicative Functions, Tel-Aviv University Dept. of Math Sciences, Feb., 1974.