THREE THEOREMS OF PALEY AND WIENER
FOR LOCALLY COMPACT ABELIAN GROUPS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology
Pasadena, California

1979

(Submitted April 25, 1979)

Acknowledgments

My thanks go first to Professor W. A. J. Luxemburg, whose direction, encouragement, and patience regarding the preparation of this thesis has been indispensible. Professors F. Brock Fuller and Herbert Ryser have also imparted valuable insights during the course of several conversations. Of course, the California Institute of Technology mathematics community as a whole deserves thanks for its role in providing both intellectual stimulation and a sense of common purpose.

I have also benefited greatly from the emotional support and understanding expressed by my wife and family, as well as from the fellowships and teaching assistantships provided by the California Institute of Technology. Finally, Mrs. Frances Williams is to be commended for her rapid and accurate preparation of the manuscript.

Abstract

The representation theorems of Paley and Wiener concerning analytic functions on a vertical strip, analytic functions on the right half-plane, and entire functions of exponential type are generalized within the context of abstract harmonic analysis. The holomorphic Fourier transform is generalized as in Liepins (7), but a different, vector-valued abstraction of analyticity is employed. The proofs of these results use standard facts from harmonic analysis, integration theory, and the theory of vector-valued analytic functions, but are otherwise elementary; in particular, no use is made of structure theorems for locally compact abelian groups.

Finally, the appendix contains several analogous results on maximal ideal spaces of related convolution algebras.

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INTRODUCTION

Abstract harmonic analysis has developed historically as the natural extension of the classical theory of Fourier series and transforms, and the older subject has for the most part been readily absorbed within more recent and general formulations. However, there remain a number of aspects of the original viewpoint which have proven less agreeable to incorporation within modern results; perhaps the most immediate and important of these concerns the analyticity of the classical Fourier kernal. e^{ixt} extends naturally to an analytic function of two variables, and this simple fact means that it is often possible to extend the Fourier transform of a function to an analytic function defined on some domain in the complex plane. The best known results along these lines are the Paley-Wiener theorems, which give necessary and sufficient conditions for certain types of analytic functions to be representable as holomorphic Fourier transforms. These theorems (three of which are listed below for future reference) can be viewed as analytic analogues of Plancherel's theorem, and have a number of important applications in classical analysis. In spite of this, the question of their possible generalization has yet to receive more than occasional attention.

Our aim in the following is to deduce suitable forms for three of the theorems when the real line is replaced by an arbitrary locally compact abelian group Γ. The case $\Gamma = \mathbb{R}^n$ has received some attention, notably from Plancherel & Polya (9), and Stein & Wiess (12). The general case was apparently first considered by Liepins (7), and there are certain simularities between his approach and ours. We believe, however, that the present treatment contains an essential improvement in technique resulting from the use of the concept of vector-valued holomorphic functions as presented in Hille (5). This advance allows for the more natural statement of assumptions and results, and more importantly leads towards considerable simplifications in many of our proofs. In particular, we are able to dispense entirely with the use of structure theorems for locally compact abelian groups in the proofs of our three main theorems, and are able to give shorter, "group independant" proofs for these results.

Before proceeding, we stop to recall the three theorems of Paley and Wiener under discussion.

Theorem A. Let λ and μ be real numbers. The following two classes of analytic functions on the strip $\{z: -\lambda \leq \mathrm{Re}(z) \leq \mu\}$ are identical:

(1) The class of analytic functions $F(z)$ which can be written in the form

$$F(\sigma+it) = \underset{A\to\infty}{\mathrm{l.\ i.\ m.}}\ (2\pi)^{-\frac{1}{2}} \int_{-A}^{A} f(x)e^{x(\sigma+it)}dx \ [-\lambda \leq \sigma \leq \mu],$$

where $f(x)$ is a measurable function on $\mathbb{R}$ with both $f(x)e^{\mu x}$ and $f(x)e^{-\lambda x}$ square integrable.

(2) The class of functions $F(z)$ for which

$$\sup_{\sigma\in[-\lambda,\mu]} \left\{\int_{-\infty}^{\infty} |F(\sigma+it)|^{2}dt\right\}$$

is finite.

Theorem B. The following two classes of functions analytic on the half plane $\{z : \mathrm{Re}(z) > 0\}$ are identical:

(1) The class of functions $F(z)$ for which

$$\sup_{\sigma>0} \left\{\int_{-\infty}^{\infty} |F(\sigma+it)|^{2} dt\right\} < \infty.$$

(2) The class of functions $F(z)$ which may be written in the form

$$F(\sigma+it) = \underset{A\to\infty}{\text{l. i. m.}}\ (2\pi)^{-\frac{1}{2}} \int_{-A}^{0} f(x)e^{x(\sigma+it)}dx \ [\sigma > 0],$$

where $f(x)$ is square integrable on $(-\infty,0)$.

Theorem C. The following two classes of entire functions are identical:

(1) The class of entire functions which can be written in the form

$$F(z) = \int_{-A}^{A} f(u)e^{iuz}du,$$

where $f(u)$ is square integrable on the interval $[-A,A]$.

(2) The class of entire functions $F(z)$ whose restriction to the real axis is square integrable and which satisfy $F(z) = o(e^{A|z|})$.

Finally, the organization of our work is as follows: Chapter 1 contains a short review of some basic notions from the theory of vector-valued holomorphic functions, and also introduces the additional concepts necessary for the generalization of theorems A and B. Chapter 2 contains the proofs of these generalizations, and also illustrates their relationship with the classical results. Lastly, Chapter 3 presents our generalization of theorem C, and considers the special case of this generalization obtained when $\Gamma = \mathbf{Z}$, the integers.

CHAPTER 1 Holomorphic Vector-Valued Functions on a Strip.

Theorem A concerns the Fourier transform of an analytic function defined on a vertical strip in the complex plane; of these several ideas, only the first can be transferred directly to the more general setting of an arbitrary locally compact abelian group. Since a function's domain must obviously first be defined before the function itself can be considered, our immediate task is to find a suitable analogue for the complex plane.

Notation. From now on, Γ and G will denote a locally compact abelian group and its dual group. The normalized Haar measures on Γ and G will be written $d\gamma$ and dg, respectively. X will be the vector space of all continuous homomorphisms from Γ into the additive group of complex numbers, and $X_{\mathbb{R}}$ will be the real subspace of X consisting of real-valued homomorphisms. Elements of $X_{\mathbb{R}}$ will be called real characters. We topologize X and $X_{\mathbb{R}}$ with the usual compact-open (henceforth CO) topologies. If C is a subset of X we will abbreviate $e^C = \{e^\psi : \psi \in C\}$. p will be a real number in the interval $[1,2]$, and p' the conjugate exponent. Finally, let $\mathcal{F}_p : L^p(G) \to L^{p'}(\Gamma)$ be the Fourier transform. When no confusion can arise we will write $\mathcal{F}$ for $\mathcal{F}_p$.

Our interest in X and $X_{\mathbb{R}}$ is motivated by the role played by $\mathbb{C}$ in the classical case $\Gamma = \mathbb{R}$. In this situation, each $z \in \mathbb{C}$ is associated with the continuous multiplicative function $f_z(t) = e^{izt}$, and all continuous, multiplicative, nowhere zero functions on $\mathbb{R}$ are obtained this way. This suggests that a possible replacement for $\mathbb{C}$ in the general case is the set of all continuous multiplicative nonzero functions on Γ, and it is easy to see that this is just $\{ge^{\psi} : g \in G, \psi \in X_{\mathbb{R}}\}$. With the above as motivation, the following definition should appear reasonably natural.

Definition 1. A Γ-strip S is a set of continuous complex-valued functions on Γ of the form $\{ge^{\psi} : g \in G, \psi \in C\}$, where C is a convex subset of X. We write $S = G \times e^{C}$.

It should be clear how this definition generalizes the vertical strip $S_0 = \{z : -\lambda \leq \mathrm{Re}(z) \leq \mu\}$ appearing in theorem A. In this simple case we have $\Gamma = \mathbb{R}$, and each point $x + iy \in S_0$ implicitly corresponds to the function $f(t) = e^{(x+iy)t} = e^{xt}e^{iyt}$. Here $g(t) = e^{iyt}$ is of course a character on $\mathbb{R}$, and $\psi(t) = xt$ is a real character on $\mathbb{R}$ contained in the convex set of real characters $C = \{h(t) : h(t) = st, -\lambda \leq s \leq \mu\}$.

Armed with this definition of a Γ-strip, we are now ready to consider the more difficult and ambiguous problem of finding a suitable analogue for analytic functions. It is of course desirable that this definition be as closely related as possible to regular analytic functions: One possible approach to this requirement is to actually provide for the embedding of complex domains within Γ-strips.

Definition 2. Let S be a Γ-strip. A Γ-domain in S is a function $d : D \to S$ of the form $d(z) = ge^{\psi + z\theta}$, where $D \subseteq \mathbb{C}$ is a domain, $g \in G$, $\psi \in X_{\mathbb{R}}$, and $\theta \in X$. If $d(z) \in S$ for all $z \in \overline{D}$, we say that d is a

Γ-domain with boundary contained in S.

This definition is not as contrived as it may seem at first glance; indeed, it is a straightforward generalization of the implicit identification between the complex number x + iy and the function $f(t) = e^{(x+iy)t}$ already found in theorem A.

At this point it is tempting to define a Γ-analytic function on the Γ-strip S to be a complex-valued function f on S such that $F(z) = f(d(z))$ is analytic in the usual sense for every Γ-domain d contained in S. However, theorems A, B, and C appear to be essentially L^2-type theorems, and a slightly more involved vector-valued definition will prove to be more tractible in what follows. Towards this definition, we first recall some elementary notions from Hille (5):

Definition 3. Let D be a domain in C, let B be a Banach space, and let b be a function from D into B. b(z) is said to be holomorphic on D if $b^*[b(z)]$ is a holomorphic function on D in the usual sense for every $b^* \in B^*$, the dual space of B.

N. Dunford has shown that the above condition already implies the seemingly stronger properties of norm continuity and norm differentiability. Given this, it should not be surprising that B-valued holomorphic functions retain many of the elementary properties of the simpler scalar-valued variety. We summarize the facts we will need in a lemma.

Lemma A. Let b be a B-valued holomorphic vector function on the complex domain D.

(a) b is uniformly norm continuous and uniformly norm differentiable on every compact set K contained in D.

(b) b is infinitely differentiable on D, and its derivatives $b^{(n)}(z)$ can be determined by the usual Cauchy integral formula.

(c) If $\{z : |z - z_o| < R\} = D_o \subseteq D$ for some $R > 0$ and $\sup_{z \in D_o} \|b(z)\| \leq M$, we have $\|b^{(n)}(z_o)\| \leq MR^{-n}n!$ for each $n > 0$.

(d) With D_o and z_o as in (c), we also have

$$b(z) = \sum_{n=o}^{\infty} b^{(n)}(z_o)\frac{(z-z_o)^n}{n!},$$

for all $z \in D_o$, where the infinite sum is norm convergent.

Proof. See Hille (5), pages 52-58.

We actually need only facts (c) and (d), but (a) and (b) have been included to give an idea of a possible approach to a proof. Next, one more bit of notation.

Notation. Let f be a complex-valued function defined on the Γ-strip $S = G \times e^C$, and let $p \in [1,2]$. Assume that for each $\psi \in C$, the function f_ψ defined by $f_\psi(g) = f(ge^\psi)$ is an element of $L^p(G)$. Then the function $L_p(f) : S \to L^p(G)$ is defined by $[L_p(f)(ge^\psi)]\ (g_1) = f(gg_1e^\psi)$ for each $\psi \in C$ and $g, g_1 \in G$. When no confusion is possible, we abbreviate $L_p(f)$ as $L(f)$.

We are finally ready to define our analogue of an analytic function defined on a strip:

Definition 4. A function f defined on a Γ-strip $S = G \times e^C$ is said to be L^p, Γ-analytic on S if

(a) $f_\psi(g) = f(ge^\psi)$ is an element of $L^p(G)$ for every $\psi \in C$, and

(b) $F(z) = L(f)(d(z))$ is a $L^p(G)$-valued holomorphic function on the domain of d for every Γ-domain d contained in S.

When p = 2, f will be called simply <u>Γ-analytic</u>.

Note that no reference has been made to the continuity of L(f). In fact, continuity at interior points of S will follow automatically from the above definition for the type of f in which we are most interested.

<u>Lemma 1</u>. Let f be a L^p, Γ-analytic function on the Γ-strip $S = G \times e^C$, and assume that $\|L_p(f)(ge^\psi)\|_p \leq M < \infty$ for all $ge^\psi \in S$. Topologize (for this lemma only) e^C with the topology given by the subbasis of sets of the form

$$V(e^\psi, R) = \{e^{\psi+\theta} : e^{\psi+t\theta} \in e^C \text{ for all } t \in (-R,R)\},$$

were $R \geq 1$ and $\psi \in C$. Topologize S with the product topology obtained from the CO topology on G and the above topology on e^C. Then L(f) is continuous on S with respect to this topology.

<u>Proof</u>. Whenever g, h $\in$ G and $\psi, \psi + \theta \in C$, we have

$$\|L(f)(ge^\psi) - L(f)(he^{\psi+\theta})\|_p = \|L(f)(gh^{-1}e^\psi) - L(f)(e^{\psi+\theta})\|_p$$

$$\leq \|L(f)(gh^{-1}e^\psi) - L(f)(e^\psi)\|_p + \|L(f)(e^\psi) - L(f)(e^{\psi+\theta})\|_p.$$

The first term clearly approaches 0 as h approaches g by the continuity of translation in $L^p(G)$. To bound the second term, we will use the

Cauchy estimates.

We know that $\|L(f)(ge^{\psi})\|_p \leq M$ for all $ge^{\psi} \in S$. Suppose $e^{\psi+\theta} \in V(e^{\psi},R)$ for some $R > 1$. Then $d(z) = e^{\psi+z\theta}$ defined for $|z| < R$ is a Γ-domain in S, and $L(f)(d(z))$ is $L^p(G)$-valued analytic on $\{z : |z| < R\}$ by hypothesis. Hence parts (c) and (d) of lemma A may be used to estimate

$$\|L(f)(e^{\psi}) - L(f)(e^{\psi+\theta})\|_p \leq \sum_{m=o}^{\infty} \|L(f)(d(z))^{(n)}(0)\|_p/n!$$

$$\leq \sum_{m=o}^{\infty} M/R^n$$

$$= M/(R-1).$$

Therefore

$$\lim_{h\to g}\lim_{R\to\infty} \sup_{e^{\psi+\theta}\in V(e^{\psi},R)} \|L(f)(ge^{\psi}) - L(f)(he^{\psi+\theta})\|_p = 0,$$

and the lemma is proved.

Corollary 1. Let f, S, and p be as in lemma 1. Consider S as a subset of $G \times e^{X}\mathbb{R}$ equipped with the CO topology. Then L(f) is continuous with respect to this topology at all interior points of S.

Proof. It is enough to show that the CO topology on S is stronger at interior point of S than the topology used on S in lemma 1. Since the CO topology on $S = G \times e^{C}$ is obviously the product of the CO topologies on its two factors (because all functions in G have modulus identically 1, and all functions in e^{C} are positive-valued), we see that it is actually enough to show that the CO topology on e^C is stronger at interior points than the topology used on e^C in lemma 1. Let e^ψ be such an interior point. Then there is an open set $U \subseteq \mathbb{R}$ and a compact $K \subseteq \Gamma$ so that $V = \{e^\varphi : e^{\varphi(\gamma)} \in U \text{ for all } \gamma \in K\}$ is an open set in e^C containing e^ψ. We may easily find $\varepsilon > 0$ so that $V_1 =$ $\{e^{\psi+\theta} : |\theta(\gamma)| < \varepsilon \text{ for all } \gamma \in K\} \subseteq V \subseteq e^C$. Now note that if $e^{\psi+\theta} \in V_1 \subseteq e^C$, then so is $e^{\psi+r\theta}$ whenever $|r| \leq 1$. Hence $e^{\psi+\theta} \in V_1$ implies $e^{\psi+\theta} \in V(e^\psi,1)$. Similarly $V_R = \{e^{\psi+\theta} : |\theta(\gamma)| < R^{-1}\varepsilon$ for all $\gamma \in K\}$ satisfies the implications $e^{\psi+\theta} \in V_R \Rightarrow e^{\psi+R\theta} \in V_1 \Rightarrow e^{\psi+R\theta} \in V(e^\psi,1) \Rightarrow$ $e^{\psi+\theta} \in V(e^\psi,R)$, so that $V_R \subseteq V(e^\psi,R)$. It is straightforward to check that V_1 and each V_R is open in the CO topology, so the assertion is proved.

In effect, the above result corresponds to Liepins' inclusion of a continuity condition in the definition of his "analytic II" functions (recall that he works only with open subsets of $X_{\mathbb{R}}$).

We need one more set of preliminary results before we can proceed to our analogue of theorem A. This has to do with the derivatives of our Γ-analytic functions.

Lemma 2. Let $F(z)$ be a $L^p(G)$-valued analytic function on the complex domain D. Assume that there exists a $\theta \in X_{\mathbb{R}}$ so that $F(z+it)(g) =$ $F(z)(e^{it\theta}g)$ almost everywhere on G whenever z and z + it are points in D. Then for each $n \geq 0$ and all $z \in D$, we have

$$\mathcal{F}\left(\frac{d^nF}{dz^n}(z)\right) = \theta^n \mathcal{F}(F(z)).$$

Proof. We prove the result first for n = 1 and proceed by induction. We have

$$\mathcal{F}\left(\frac{dF}{dz}(z)\right) = \mathcal{F}\left(\lim_{t\to 0} (it)^{-1}(F(z+it) - F(z))\right)$$

$$= \lim_{t\to 0} (it)^{-1}(\mathcal{F}(F(z+it)) - \mathcal{F}(F(z)))$$

$$= \lim_{t\to 0} (it)^{-1}(e^{it\theta}-1)\ \mathcal{F}(F(z)),$$

where we have used the fact that the above limits are strong limits, the continuity of $\mathcal{F}$, and the elementary formula for the Fourier transform of the translate of a function. Now

$\lim_{t\to 0} (it)^{-1}(e^{it\theta(\gamma)}-1)\ \mathcal{F}(F(z))(\gamma) = \theta(\gamma)\mathcal{F}(F(z))(\gamma)$ pointwise almost everywhere, and the pointwise and $L^{p'}(\Gamma)$ limits must agree as elements of $L^{p'}(\Gamma)$. Hence we have $\mathcal{F}\left(\frac{dF}{dz}(z)\right) = \theta\mathcal{F}(F(z))$.

Now assume that the result is true for n = k and prove it for n = k + 1. $\frac{d^kF}{dz^k}(z)$ is an $L^p(G)$-valued analytic function on D satisfying the same hypothesis as F does. Hence combining the result for n = 1 and n = k we see

$$\mathcal{F}\left(\frac{d^{k+1}F}{dz^{k+1}}(z)\right) = \theta\mathcal{F}\left(\frac{d^kF}{dz^k}(z)\right)$$

$$= \theta^{k+1}\ \mathcal{F}(F(z)),$$

as desired.

Corollary 2. Let f be a L^p, Γ-analytic function on the Γ-strip S and let $d(z) = ge^{\psi+z\theta}$ be a Γ-domain in S with $\theta \in X_{\mathbb{R}}$. Then

$$\mathcal{F}\left(\frac{d^n}{dz^n}(L(f)(d(z)))\right) = \theta^n \mathcal{F}(L(f)(d(z))).$$

Proof. $F(z) = L(f)(d(z))$ is a $L^p(G)$-valued analytic function satisfying the hypothesis of lemma 2.

CHAPTER 2 Generalization of Theorems A and B.

Proof of the Generalizations

We are nearly ready to generalize theorem A to the context of locally compact abelian groups. First, however, we pause to prove an elementary lemma.

Lemma 3. Let z be complex, and let $r > 0$. Then for any complex numbers t and w with $|z-w| \leq r$, we have

$$\left|\frac{e^{zt}-e^{wt}}{z-w}\right| \leq \frac{1}{r}2\left(\sum_{i=1}^{4} e^{\mathrm{Re}(z_i t)}\right),$$

where $z_1 = z + 2r$, $z_2 = z - 2r$, $z_3 = z + 2ir$, and $z_4 = z - 2ir$.

Proof. $f(w) = (e^{zt}-e^{wt})/(z-w)$ is analytic on $\mathbb{C}$ (save for a removable singularity at $w = z$), so the maximum modulus principle insures that it is sufficient to check the desired inequality on the set $\{w : |z-w| = r\}$. For such a w we have

$$\left|\frac{e^{zt}-e^{wt}}{z-w}\right| \leq \frac{2}{r}e^{\tau},$$

where $\tau = \sup_{|w-z|\leq r} \mathrm{Re}(tw)$. Therefore it is enough to show

$$\tau \leq \sup_{1\leq i\leq 4} \mathrm{Re}(tz_i).$$

But this is clear, since $\sup_{1 \leq i \leq 4} \mathrm{Re}(tz_i) = \sup_{z \in \overline{\mathrm{conv}}(\{z_1,\ldots,z_4\})} \mathrm{Re}(tz)$, and $\{w : |w-z| \leq r\}$ is contained in this second set.

Finally, we arrive at our theorem.

Theorem 1. Let $S = G \times e^C$ be a Γ-strip. Then following two classes of Γ-analytic functions on S are identical:

(1) The class of functions which can be written in the form

(*) $\quad L(f)(ge^{\psi}) = \mathcal{F}^{-1}(Fge^{\psi})$,

where F is a measurable function on Γ satisfying

$\sup_{ge^{\psi} \in S} \|Fge^{\psi}\|_2 = M < \infty$.

(2) The class of Γ-analytic functions f such that

(a) $\sup_{ge^{\psi} \in S} \|L(f)(ge^{\psi})\|_2 = M < \infty$, and

(b) $L(f)(d(z))$ is continuous on each Γ-domain $d(z)$ with boundary contained in S of the form $d(z) = ge^{\psi+z\theta}$, with $\theta \in X_{\mathbb{R}}$.

Recall that corollary 1 makes condition (b) of case (2) an automatic consequence of condition (a) when S is open. We are in general neither able to weaken the conditions on class (2) by removing (b) altogether, or to strengthen them by replacing $\theta \in X_{\mathbb{R}}$ with the more general $\theta \in X$.

The proof of theorem 1 is by a series of lemmas. We first show that the first class of functions is contained within the second. Let f be a function defined as in (*).

Lemma 4. Let $d : D \to S$ be a Γ-domain in S, and suppose that $d(z) = ge^{\psi+z\theta}$ with $\theta \in X_{\mathbb{R}}$. Let $\mu = \inf\{Re(z) : z \in D\}$, $\lambda = \sup\{Re(z) : z \in D\}$. Then if $\mu > -\infty$, $Fe^{\psi+\mu\theta} \in L^2(\Gamma)$. Similarly for λ.

Proof. It is enough to exhibit the method by showing $Fe^{\psi+\lambda\theta} \in L^2(\Gamma)$ when $\lambda < \infty$. By a translation of D if necessary, we may assume that $\lambda > 0$. For each positive integer i, pick $z_i \in D$ so that $Re(z_i) > 0$, $Re(z_i)$ increases monotonically, and $\lim_{i\to\infty} Re(z_i) = \lambda$. Write $\Gamma = \Gamma_+ \cup \Gamma_-$, where $\Gamma_+ = \{\gamma : \theta(\gamma) \geq 0\}$ and $\Gamma_- = \{\gamma : \theta(\gamma) < 0\}$. On Γ_+ we have $\lim_{i\to\infty} |F(\gamma)e^{\psi(\gamma)+z_i\theta(\gamma)}|^2 = |F(\gamma)e^{\psi(\gamma)+\lambda\theta(\gamma)}|^2$ monotonically, so the monotone convergence theorem insures

$$\int_{\Gamma_+} |Fe^{\psi+\lambda\theta}|^2 d\gamma = \lim_{i\to\infty} \int_{\Gamma_+} |Fe^{\psi+z_i\theta}|^2 d\gamma \leq M^2 < \infty.$$

On Γ_-, we have $|F(\gamma)e^{\psi(\gamma)+\lambda\theta(\gamma)}|^2 \leq |F(\gamma)e^{\psi(\gamma)+z_1\theta(\gamma)}|^2$. Hence $\int_{\Gamma_-} |Fe^{\psi+\lambda\theta}|^2 d\gamma < M^2$ as well, and we are done.

Lemma 4 shows that $L(f)(d(z)) = \mathcal{F}^{-1}(Fd(z))$ can be extended to $\bar{D}$ even when $d(\partial D)$ is not contained in S. The following lemma shows that this extension is continuous.

Lemma 5. $L(f)(d(z))$ is continuous on $\bar{D}$, even when $d(\partial D) \not\subset S$.

Proof. We show the continuity of $L(f)(d(z))$ at an arbitrary point $z_o \in \bar{D}$. Let λ and μ be defined as in lemma 4; since z_o is fixed, we may assume that λ and μ are finite. If $w \in \bar{D}$, Plancherel's theorem gives

(*) $$\|L(f)(d(z_o)) - L(f)(d(w))\|_2 = \|Fe^{\psi}(e^{z_o\theta} - e^{w\theta})\|_2.$$

Now if $t \in \bar{D}$, then $|e^{t\theta(\gamma)}| \leq e^{\lambda\theta(\gamma)} + e^{\mu\theta(\gamma)}$ for all $\gamma \in \Gamma$. Hence

$$|F|^2 e^{2\psi}|e^{z_o\theta} - e^{w\theta}|^2 \leq |F|^2 e^{2\psi} 4(e^{\lambda\theta} + e^{\mu\theta})^2 \in L^1(\Gamma),$$

by lemma 4. Therefore we may apply the dominated convergence theorem to (*) as w approaches z_o through $\bar{D}$, and we are done.

Lemma 5 shows that f satisfies condition (b) of case (2). It actually shows a tiny bit more; this additional information will be used in the next section .

Lemma 6. f is Γ-analytic on S, even if $\|Fge^{\psi}\|_2$ is not bounded for $ge^{\psi} \in S$.

Proof. Let $d(z) = ge^{\psi+z\theta}$ be a Γ-domain in S with domain D. We must show that L(f)(d(z)) is a $L^2(G)$-valued analytic function. To do this, it is enough to show that $\int_G L(f)(d(z))\bar{h}dg$ is an ordinary analytic function on D for each choice of $h \in L^2(G)$. Again by Plancherel's theorem, it is enough to show that $H(z) = \int_\Gamma Fge^{\psi+z\theta}hd\gamma$ is an ordinary analytic function on D for each choice of $h \in L^2(\Gamma)$. To show H(z) differentiable at $z \in D$, we write

(*) $$\frac{H(w)-H(z)}{w-z} = \int_\Gamma Fge^{\psi} \frac{e^{w\theta}-e^{z\theta}}{w-z} hd\gamma.$$

Let $r = 1/4 \inf\{|z - t| : t \in \partial D\}$. If $|w - z| \leq r$, lemma 3 gives

$$\left|Fge^{\psi}\ \frac{e^{w\theta}-e^{z\theta}}{w-z}h\right| \leq \left|Fge^{\psi}h\right| \sum_{i=1}^{4} \left|e^{z_i\theta}\right| \frac{2}{r}$$

$$= \frac{2}{r} \sum_{i=1}^{4} |Fge^{\psi}e^{z_i\theta}h|,$$

where each z_i is derived from z as in lemma 3. Now $z_i \in D$ for each i, so $Fge^{\psi+z_i\theta} \in L^2(\Gamma)$. Hence the last sum of functions appearing above is integrable, so that we may apply the dominated convergence theorem to (*) as w approaches z and conclude

$$H'(z) = \int_{\Gamma} Fge^{\psi}\theta\ e^{z\theta}hd\gamma,$$

where $Fge^{\psi}\theta e^{z\theta}h \in L^1(\Gamma)$. H(z) is therefore an analytic function on D as desired, and the lemma is proved.

Finally, it is clear that f satisfies condition (a) of class (2) whenever $\sup_{ge^{\psi}\in S} \|Fge^{\psi}\|_2 \leq M$, so we have shown that class (1) of theorem 1 is contained within class (2).

We now prove the reverse inclusion. Let f be a Γ-analytic function on S satisfying the conditions of class (2). As in the classical theorem, we wish to show that we can define F on Γ by $F = (ge^{\psi})^{-1}\ \mathcal{J}(L(f)(fe^{\psi}))$, where $ge^{\psi} \in S$ is arbitrary; the difficulty is of course to show that such an F is well defined. We again will proceed by a series of simple lemmas. For reasons which will become apparent in Chapter 3, we formulate these lemmas for L^p,Γ-analytic functions.

<u>Lemma 7</u>. Let f be a L^p,Γ-analytic function on S satisfying

(a′) $\sup_{ge^{\psi} \in S} \|L(f)(ge^{\psi})\|_p = M < \infty$, and

(b′) $L(f)(d(z))$ is continuous on each Γ-domain $d(z)$ with boundary contained in S of the form $d(z) = ge^{\psi + z\theta}$, where $\theta \in X_{\mathbb{R}}$.

Then if θ and ψ are real characters with $\psi + \theta$ and $\psi - \theta$ elements of C, we have $e^{-\theta}\, \mathcal{J}(L(f)(e^{\psi+\theta})) = e^{\theta}\, \mathcal{J}(L(f)(e^{\psi-\theta}))$.

Proof. Let $d: \{z : |z| \leq 1\} \to S$ be defined by $d(z) = e^{\psi + z\theta}$. Since $e^{\psi+\theta}$ and $e^{\psi-\theta}$ are in S, the convexity of C assures that $d(z)$ is a Γ-domain with boundary contained in S. By corollary 2, we have

$$\mathcal{J}\left(\frac{d^n}{dz^n}(L(f)(d(z)))(0)\right) = \theta^n\, \mathcal{J}(L(f)(d(0)))$$

for each $n \geq 0$. Hence if $0 < r < 1$, the fact that the Taylor series expansion for $L(f)(d(r))$ is norm convergent gives

$$\mathcal{J}(L(f)(d(r))) = \mathcal{J}\left(\sum_{n=o}^{\infty} \frac{r^n}{n!} \frac{d^n}{dz^n} L(f)(d(z))(0)\right)$$

$$= \sum_{n=o}^{\infty} \frac{r^n}{n!}\, \theta^n\, \mathcal{J}(L(f)(d(0))),$$

where the last sum converges in $L^{p'}(\Gamma)$. But obviously the last sum is pointwise convergent to the function $e^{r\theta}\, \mathcal{J}(L(f)(d(0)))$ as well, and the two limits must be equal. Hence

$$\mathcal{J}(L(f)(d(r))) = e^{r\theta}\, \mathcal{J}(L(f)(d(0))).$$

By hypothesis, $L(f)(d(z))$ is norm continuous on $\{z : |z| \leq 1\}$, so

$\mathcal{F}(L(f)(d(1))) = \lim_{r\to 1} e^{r\theta}\, \mathcal{F}(L(f)(d(0)))$, where the limit is again taken in $L^{p'}(\Gamma)$. But $\lim_{r\to 1} e^{r\theta}\, \mathcal{F}(L(f)(d(0))) = e^{\theta}\, \mathcal{F}(L(f)(d(0)))$ pointwise, and again the limits must be equal. Hence

$$\mathcal{F}(L(f)(d(1))) = e^{\theta}\, \mathcal{F}(L(f)(d(0))), \text{ and likewise}$$

$$\mathcal{F}(L(f)(d(-1))) = e^{-\theta}\, \mathcal{F}(L(f)(d(0))).$$

This yields the desired result.

<u>Lemma 8</u>. With f as in lemma 7, the function $F = (ge^{\psi})^{-1}\, \mathcal{F}(L(f)(ge^{\psi}))$ is independent of the choice of $ge^{\psi} \in S$.

<u>Proof.</u> Let ge^{ψ} and he^{φ} be two arbitrary points in S. Obviously $(e^{\psi}g)^{-1}\, \mathcal{F}(L(f)(ge^{\psi})) = e^{-\psi}\, \mathcal{F}(L(f)(e^{\psi}))$ and similarly with he^{φ} in place ge^{ψ} by an elementary fact about Fourier transforms, so it is enough to show $e^{-\psi}\, \mathcal{F}(L(f)(e^{\psi})) = e^{-\varphi}\, \mathcal{F}(L(f)(e^{\varphi}))$. For this, simply apply lemma 8 to the real characters $(\psi+\varphi)/2$ and $(\psi-\varphi)/2$.

Finally, we note that F defined above satisfies $\sup_{ge^{\psi}\in S} \|Fge^{\psi}\|_{p'} \leq M$ by the Hausdorff-Young inequality. Specializing to $p = 2$ and invoking the inversion formula once more proves the second inclusion and the theorem.

We now discuss several simple applications. It is well known that for f as in theorem A, $\left(\int_{-\infty}^{\infty} |f(s+it)|^2 dt\right)^{\frac{1}{2}}$ is a logrithmically convex function of s. This result is true in the context of theorem 1 as well.

Corollary 3. Let $S = G \times e^C$ be a Γ-strip, and let f be Γ-analytic on S and satisfy the conditions of theorem 1. Then $N(\psi) = \|L(f)(e^\psi)\|_2$ is a logrithmically convex function on C.

Proof. By theorem 1, we can write $L(f)(ge^\psi) = \mathcal{F}^{-1}(Fge^\psi)$, where $Fge^\psi \in L^2(\Gamma)$, and so $|F|^{2t}e^{2t\psi} \in L^{t^{-1}}(\Gamma)$. Similarly $F^{2(1-t)}e^{2(1-t)\varphi} \in L^{(1-t)^{-1}}(\Gamma)$. Hence we have

$$\int_\Gamma |F|^2 e^{2(1-t)\varphi+2t\psi} d\gamma = \int_\Gamma |F|^{2(1-t)} e^{2(1-t)\varphi} |F|^{2t} e^{2t\psi} d\gamma$$

$$\leq \left(\int_\Gamma |F|^2 e^{2\varphi} d\gamma\right)^{1-t} \left(\int_\Gamma |F|^2 e^{2\psi} d\gamma\right)^t,$$

and taking square roots and then logrithms in this inequality shows

$$\log\|Fe^{t\psi+(1-t)\varphi}\|_2 \leq (1-t)\log\|Fe^\varphi\|_2 + t\log\|Fe^\psi\|_2.$$

Finally, yet another use of Plancherel's theorem leads to the desired result.

The second application is our generalization of theorem B. Its precise connection with theorem B will be discussed in the next section.

Theorem 2. Let $C \subseteq X_{\mathbb{R}}$ be an open cone. Then the following two classes of Γ-analytic functions on $S = G \times e^C$ are identical:

(1) The class of functions f such that

$$\sup_{ge^\psi \in S} \|L(f)(ge^\psi)\|_2 < \infty.$$

(2) The class of functions f which can be written in the form

$$(*) \quad L(f)(ge^{\psi}) = \mathcal{F}^{-1}(Fge^{\psi}),$$

where $F \in L^2(\Gamma)$ is 0 almost everywhere off the closed semigroup $\Gamma_- = \{\gamma \in \Gamma : \psi(\gamma) \leq 0 \text{ for all } \psi \in C\}$.

Proof. Denote Haar measure on Γ by σ for the proof of this theorem. C is open by hypothesis, so S is open. Therefore theorem 1 asserts that a Γ-analytic function f on S is of class (1) above if and only if we can write

$$L(f)(ge^{\psi}) = \mathcal{F}^{-1}(Fge^{\psi}),$$

where F is a measurable function on Γ satisfying $\sup_{ge^{\psi} \in S} \|Fge^{\psi}\|_2 < \infty$; hence we will be done if it is shown that such F are precisely those in $L^2(\Gamma_-)$.

First suppose $F \in L^2(\Gamma_-)$. Then if $ge^{\psi} \in S$, the fact that $\psi(\gamma) \leq 0$ for $\gamma \in \Gamma_-$ insures $\|Fge^{\psi}\|_2 < \|F\|_2$, so that one of the necessary inclusions is established.

Conversely, suppose F is measurable on Γ and $\sup_{ge^{\psi} \in S} \|Fge^{\psi}\|^2 < \infty$. If F is not zero almost everywhere off Γ_-, the regularity of σ implies that there is a compact set $K \subseteq \Gamma - \Gamma_-$ of positive Haar measure and an $\varepsilon > 0$ such that $|F(\gamma)| > \varepsilon$ for all $\gamma \in K$. Now $\{\{\gamma : \psi(\gamma) > \delta\} : \psi \in C, \delta > 0\}$ is an open cover for $\Gamma - \Gamma_-$. Since $K \subseteq \Gamma - \Gamma_-$ is compact, we may find positive $\{\delta_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n \subseteq C$ so that $K \subseteq \sum_{i=1}^n \{\gamma : \psi_i(\gamma) > \delta_i\}$. Since $\sigma(K) > 0$, we see that $\sigma(K \cap \{\gamma : \psi_j(\gamma) > \delta_j\}) > 0$ for some j. Define

$K' = K \cap \{\gamma : \psi_j(\gamma) > 0\}$. Then for each positive integer m, we have

$$\|Fe^{m\psi_j}\|_2^2 \geq \int_{K'} |Fe^{m\psi_j}|^2 \, d\gamma$$

$$\geq \sigma(K') \, \varepsilon^2 e^{2m\delta_j},$$

and this is unbounded as $m \to \infty$. Since $m\psi_j \in C$ for all positive m, this contradiction with $\sup_{ge^{\psi} \in S} \|Fge^{\psi}\|_2 < M$ shows that $F = 0$ almost everywhere off Γ_-.

It remains to show that $F \in L^2(\Gamma_-)$. To see this, let $\psi \in C$ be arbitrary. We know that $\|Fe^{\varepsilon\psi}\|_2 \leq M$ for all positive ε and that $\lim_{\varepsilon \to 0} |Fe^{\varepsilon\psi}|^2 = |F|^2$ monotonically. Therefore the monotone convergence theorem insures $F \in L^2(\Gamma_-)$, and we are done.

Connection with Theorems A and B.

Strictly speaking, we have yet to show that theorems 1 and 2 actually generalize theorems A and B. More precisely, we have yet to exhibit any connection between regular analytic functions and Γ-analytic functions when $\Gamma = \mathbb{R}$. The next lemma shows the relationship between the two classes in the slightly more general case $\Gamma = \mathbb{R}^n$.

Notation. Let n be a fixed positive integer. Points in $\mathbb{R}^n$ will be denoted x or t; for $z \in \mathbb{C}^n$, Re(z) and Im(z) are defined by coordinates. (z_1, z_2) will be the usual bilinear pairing on $\mathbb{C}^n \times \mathbb{C}^n$. $C \subseteq \mathbb{R}^n$ will be a closed, convex set, and $C_0 = \text{int } C$. Define $S = \{z \in \mathbb{C}^n : \text{Re}(z) \in C\}$, and define S_0 analogously. Finally, each $z \in S$ will be implicitly identified with the multiplicative function $f_z(x) = e^{(z,x)}$ on $\mathbb{R}^n$.

Lemma 9. Let $f(z)$ be a regular analytic function on S_o, and suppose

$$\sup_{x \in C_o} \left(\int_{\mathbb{R}^n} |f(x+iy)|^2 \, dy \right)^{\frac{1}{2}} = M < \infty.$$

Then f satisfies the conditions of class (2)(and therefore of class (1)) in theorem 1.

Proof. Since C_o is open, it is enough to show that f is $\mathbb{R}^n$-analytic on S_o. To do so, let $d : D \to S_o$ by a $\mathbb{R}^n$-domain. We must show that $L(f)(d(z))$ is a $L^2(\mathbb{R}^n)$-valued holomorphic function; to do this, it in turn suffices to show that

$$G(z) = \int_{\mathbb{R}^n} f(d(z)+iy)g(y)dy$$

is an ordinary analytic function on D for each $g \in L^2(\mathbb{R}^n)$. We do this by first proving that $G(z)$ is continuous on D, and then invoking Morerra's theorem.

Since $f(z)$ is uniformly continuous on each compact subset of S_o, $G(z)$ will certainly be continuous on D when g is the characteristic function of a bounded set, or a linear combination of such functions. For $g \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$ arbitrary, choose $g_\varepsilon \in L^2(\mathbb{R}^n)$ to be a linear combination of such characteristic functions satisfying $\|g - g_\varepsilon\|_2 < \varepsilon$. Now if z, $w \in D$ we have

$$|G(z)-G(w)| \leq \|L(f)(d(z))\|_2 \|g-g_\varepsilon\|_2 + |G_\varepsilon(z)-G_\varepsilon(w)| + \|L(f(d(w))\|_2 \|g-g_\varepsilon\|_2,$$

where

$$G_\varepsilon(z) = \int_{\mathbb{R}^n} f(d(z)+iy)g_\varepsilon(y)dy.$$

Since $G_\varepsilon(z)$ is a continuous function on D, we have

$$\lim_{z\to w} \sup |G(z) - G(w)| < 2M\varepsilon,$$

and letting ε approach 0 shows that G(z) is continuous on D.

We may now apply Morrera's theorem: G(z) will be analytic on D if $\int_{\partial T} G(z)dz = 0$ for all triangles $T \subseteq D$. We have

$$\int_{\partial T}\int_{\mathbb{R}^n} |f(d(z)+it)g(t)|dt\, d|z| \leq KM\|g\|_2,$$

where $K < \infty$ is just the length of ∂T. Hence we may also use Fubini's theorem, and compute

$$\int_{\partial T}\int_{\mathbb{R}^n} f(d(z)+it)g(t)dtdz = \int_{\mathbb{R}^n}\int_{\partial T} f(d(z)+it)dzdt$$
$$= 0,$$

since g(t)f(d(z)+it) is an analytic function of z for each t. Hence the lemma is proved.

We now sketch briefly how theorem 1 can be used to show that the second class of analytic functions in theorem A is contained within the first class. Let f(z) be such a function. Lemma 9 and theorem 1 show that there is a function F defined on R with $\sup_{-\lambda<s<\mu} \|F(x)e^{sx}\|_2 \leq M$ and

$$L(f)(z) = \mathcal{F}^{-1}(F(x)e^{zx})$$

whenever $-\lambda < \mathrm{Re}(z) < \mu$. Lemma 5 now shows that $F_1(z) = \mathcal{F}^{-1}(F(x)e^{zx})$ is actually a continuous $L^2(\mathbb{R})$-valued function on all of S. Finally (as in the original proof of theorem A), the fact that

$$\lim_{s \to -\lambda+} f(s+it) = f(-\lambda+it) \text{ almost everywhere,}$$

and similarly for μ in place of $-\lambda$, shows that $F_1(z) = L(f)(z)$ for all $z \in S$. Therefore the desired integral representation is established.

The connection between theorems 2 and B is also straightforward. Here we have $C = \{\psi_x(t) = xt: x > 0\}$. If f is an analytic function in class (1) of theorem B, lemma 9 shows that f actually belongs to class (1) of theorem 2. Now theorem 2 asserts that there exists a function $F \in L^2(\mathbb{R}_-)$ such that $L(f)(z) = \mathcal{F}^{-1}(F(x)e^{zx})$ for all z with $\mathrm{Re}(z) > 0$. Since $\mathbb{R}_- = \{t \in \mathbb{R}: tx \leq 0 \text{ for all } x > 0\} = \{t \in \mathbb{R}: t < 0\}$, the significant inclusion of theorem B is established.

CHAPTER 3 Generalization of Theorem C.

At first glance, the fact that theorem C deals with entire functions which are square integrable over only a single line may appear to make it unsuitable for generalization within the present context. However, the following lemma (which is of course obvious after theorem C but which may be proven independently from it) indicates the proper approach to take.

Lemma B. Let $f(z)$ be an entire function with $\int_{-\infty}^{\infty} |f(it)|^2\, dt < \infty$ and $f(z) = o(e^{A|z|})$ for some $A > 0$. Then we have for any $\sigma \in \mathbb{R}$

$$\left(\int_{-\infty}^{\infty}|f(\sigma+it)|^2dt\right)^{\frac{1}{2}} \le e^{|\sigma|A}\left(\int_{-\infty}^{\infty}|f(it)|^2dt\right)^{\frac{1}{2}}.$$

Proof. See Stein (12), page 109, for a proof which is independent of theorem C.

The above lemma suggests that theorem C is indeed in some sense a $L^2(\mathbb{R})$-type theorem, and that it perhaps admits some sort of generalization within the present context. At this point, the last remaining obstacle to such a generalization is the lack of a suitable analogue for the interval $[-A,A]$. The following definition contains our solution.

Definition 5. Let Γ and $X_{\mathbb{R}}$ be as before, and let $X_{\mathbb{R}}^*$ denote the dual space of $X_{\mathbb{R}}$ equipped with the $\sigma(X_{\mathbb{R}}, X_{\mathbb{R}}^*)$ topology. Let $i : \Gamma \to X_{\mathbb{R}}^*$ be the canonical map from Γ into $X_{\mathbb{R}}^*$. We call a set $I \subseteq \Gamma$ a Γ-interval if $I = i^{-1}(K)$, where K is a compact, convex set in $X_{\mathbb{R}}^*$.

Inspection of the original proof of theorem C suggests that the compactness of $[-A,A]$ in $\mathbb{R}$ is important only because every real character on $\mathbb{R}$ is bounded on $[-A,A]$. This motivates our passage to the condition of compactness in $X_{\mathbb{R}}^*$ in the general case.

Example. Let $\Gamma = \mathbb{Q}$ with the discrete topology, and let $I = \mathbb{Q} \cap [0,1]$. Then since $X_{\mathbb{R}} = X_{\mathbb{R}}^* = \mathbb{R}$, we see $I = i^{-1}([0,1])$ is a noncompact $\mathbb{Q}$-interval. $I - \{1\}$ is not a $\mathbb{Q}$-interval.

If $E \subseteq \mathbb{R}$ satisfies $m(E) > 0$ and $E \cap [-A,A] = \emptyset$, then for some $\delta > 0$ we have either $m(E \cap (A+\delta,\infty)) > 0$ or $m(E \cap (-\infty,-A-\delta)) > 0$. This simple fact generalizes as follows:

Lemma 10. Let $K \subseteq X_{\mathbb{R}}^{*}$ be a convex, compact set, and let $I = i^{-1}(K)$ be a Γ-interval. Suppose $E \subseteq \Gamma$ has nonzero Haar measure and that $E \cap I = \emptyset$. Then we can find $E' \subseteq E$ of nonzero measure, $\psi \in X_{\mathbb{R}}$ and $\delta > 0$ so that

$$\inf_{\gamma \in E'} \psi(\gamma) \geq (\sup_{x \in K} \psi(x)) + \delta.$$

Proof. Denote Haar measure on Γ by σ for the rest of this lemma. Since σ is regular, we may assume that E is compact. One form of the Hahn-Banach theorem for locally convex topological vector spaces implies that $\{V_{\delta,\psi} : \delta > 0, \psi \in X_{\mathbb{R}}\}$ is an open cover for $X_{\mathbb{R}}^{*}$-K, where $V_{\delta,\psi} = \{x \in X_{\mathbb{R}}^{*} : \psi(x) > (\sup_{x' \in K} \psi(x')) + \delta\}$. Hence $\{i^{-1}(V_{\delta,\psi}): \delta > 0, \psi \in X_{\mathbb{R}}\}$ is an open cover for $i^{-1}(X_{\mathbb{R}}^{*}-K) = \Gamma - I$. Since E is compact, we know that $E \subseteq \bigcup_{i=1}^{n} i^{-1}(V_{\delta_i,\psi_i})$ for some finite set of $\delta_i > 0$ and $\psi_i \in X_{\mathbb{R}}$. Since $\sigma(E) > 0$, we must have $\sigma(E \cap i^{-1}(V_{\delta_i,\psi_i})) > 0$ for some i as well. Since $i^{-1}(V_{\delta_i,\psi_i}) = \{\gamma \in \Gamma : \psi_i(\gamma) > (\sup_{x \in K} \psi_i(x)) + \delta_i\}$, we can take $E' = E \cap i^{-1}(V_{\delta_i,\psi_i})$, $\psi = \psi_i$, $\delta = \delta_i$, and the lemma is proved.

Finally, one last definition.

Definition 6. Let $K \subseteq X_{\mathbb{R}}^{*}$ be convex and compact. A function f defined on $G \times e^{X_{\mathbb{R}}}$ is said to be L^p,Γ-entire of exponential type of type K if f is L^p,Γ-analytic on $G \times e^{X_{\mathbb{R}}}$ and satisfies the inequality

$$\|L(f)(ge^{\psi})\|_p = \|L(f)(e^{\psi})\|_p \leq L \exp(\sup_{x \in K} \psi(x))$$

for all $ge^{\psi} \in G \times e^{X_{\mathbb{R}}}$ and some constant L. When p = 2, the prefix L^2

will be dropped.

Note that the compactness of K in $X^*_{\mathbb{R}}$ assures that the supremum appearing above is always finite. Lemma B illustrates that this is a legitimate generalization of entire functions of exponential type which are square integrable on a single line. After this definition, the generalization of theorem C is straightforward:

Lemma 11. Let $K \subseteq X^*_{\mathbb{R}}$ be convex and compact, and let $I = i^{-1}(K)$. Suppose that f is a function on $G \times e^{X_{\mathbb{R}}}$ which is L^p, Γ-entire of exponential type of type K. Then the function $(ge^{\psi})^{-1}\,\mathcal{F}(L(f)(ge^{\psi})) \in L^{p'}(\Gamma)$ is independent of the choice of $ge^{\psi} \in G \times e^{X_{\mathbb{R}}}$, and is zero almost everywhere off I.

Proof. If $d : D \to G \times e^{X_{\mathbb{R}}}$ is a Γ-domain, then $L(f)(d(z))$ is a $L^p(G)$-valued analytic function on D by definition. Since each $d(z)$ may be trivially extended to all of $\mathbb{C}$, it is obvious that $L(f)(d(z))$ is continuous for each Γ-domain d with boundary contained in $G \times e^{X_{\mathbb{R}}}$. Hence we may apply lemmas 7 and 8 to the restriction of f to the Γ-strip $G \times e^{C_n}$, where $C_n \subseteq X_R$ is the convex subset $\{\psi \in X_{\mathbb{R}} : \sup_{x \in K} \psi(x) \leq n\}$.

Doing so, we obtain functions F_n on Γ so that $\mathcal{F}(L(f)(ge^{\psi})) = F_n ge^{\psi}$ whenever $\psi \in C_n$. Clearly $F_n = F_m = F$ almost everywhere for any n and m, so we have

$$\mathcal{F}(L(f)(ge^{\psi})) = Fge^{\psi}$$

for all $ge^{\psi} \in G \times e^{X_{\mathbb{R}}}$.

$F \in L^{p'}(\Gamma)$ by the Hausdorff-Young inequality; it remains to show that F is 0 almost everywhere off I. Suppose not; then there is an $\varepsilon > 0$

and a set E disjoint from I of positive measure such that $|F(\gamma)| \geq \varepsilon$ for all $\gamma \in E$. Applying lemma 10 to E, we find that there is a set $E' \subseteq E$ of measure $\alpha > 0$, $\delta > 0$, and $\psi \in X_{\mathbb{R}}$ so that $\inf_{\gamma \in E'} \psi(\gamma) \geq (\sup_{x \in K} \psi(x)) + \delta$. For each positive n, we then have

$$\|L(f)(e^{n\psi})\|_p \geq \|Fe^{n\psi}\|_{p'},$$

$$\geq \left(\int_{E'} |F|^{p'} e^{p'n\psi}\, d\gamma\right)^{1/p'}$$

$$\geq \alpha^{1/p'}\, \varepsilon \exp(n(\sup_{x \in K} \psi(x) + \delta)).$$

In particular, we have $\lim_{n\to\infty} \|L(f)(e^{n\psi})\|_p / \exp(\sup_{x\in K} n\psi(x)) = \infty$, so that f is not L^p, Γ-entire of exponential type of type K. This contradiction with our assumption shows that F is 0 almost everywhere off I, and we are done.

Specializing to p = 2 and invoking Plancherel's theorem one last time gives an immediate proof of the following theorem.

Theorem 3. Let $K \subseteq X^*_{\mathbb{R}}$ be convex and compact, and let $I = i^{-1}(K)$. Then the following two classes of Γ-analytic functions on $G \times e^{X_{\mathbb{R}}}$ are equivalent:

(1) Those which are Γ-entire of exponential type of type K.

(2) Those which can be written in the form $L(f)(ge^{\psi}) = \mathcal{F}^{-1}(Fge^{\psi})$, where $F \in L^2(\Gamma)$ is zero almost everywhere off I.

One of the earliest corollaries to theorem C was the result that if

$f(z)$ is an entire function of exponential type which satisfies

$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty,$$

for some $p \in [1,2]$, then actually

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

(see, for example, (8)). This fact and its proof transfer almost directly to a significant special case of the present context.

Corollary 4. Let $K \subseteq X_{\mathbb{R}}^*$ be convex and compact, and suppose that $I = i^{-1}(K)$ has finite measure. Then any L^p, Γ-entire function f of exponential type of type K is actually Γ-entire of exponential type of type K.

Proof. By lemma 11, there is a function $F \in L^{p'}(\Gamma)$ so that $F = (ge^{\psi})^{-1} \mathcal{F}(L(f)(ge^{\psi}))$ is well defined and has support contained in I. Since I has finite measure, the Minkowski inequality shows that $Fge^{\psi} \in L^2(\Gamma)$ for each $ge^{\psi} \in G \times e^{X}\mathbb{R}$. Hence we have $L(f)(ge^{\psi}) = \mathcal{F}^{-1}(Fge^{\psi})$ for all $ge^{\psi} \in G \times e^{X}\mathbb{R}$, and theorem 3 now shows that f is Γ-entire of exponential type of type K.

Corollary 5. If Γ is compactly generated, then every L^p, Γ-entire function of exponential type is Γ-entire of exponential type.

Proof. Recall that if Γ is compactly generated, then $\Gamma \cong C \oplus \mathbb{Z}^a \oplus \mathbb{R}^b$, where C is a compact group and a and b are nonnegative integers. Hence $X_{\mathbb{R}} \cong \mathbb{R}^a \oplus \mathbb{R}^b$, and $X_{\mathbb{R}}^* \cong \mathbb{R}^a \oplus \mathbb{R}^b$. Moreover, if $(c,n_1,\ldots,n_a,r_1,\ldots,r_b) =$

$= \gamma \in \Gamma$, then $i(\gamma) = (n_1, \ldots, n_a, r_1, \ldots, r_b)$. Therefore if $K \subseteq X^*_{\mathbb{R}}$ is compact and convex, we have $i^{-1}(K) = C \oplus (K \cap (Z^a \oplus \mathbb{R}^b))$. Clearly this set is of finite measure, so that any L^p, Γ-entire function of exponential type of type K is Γ-entire of exponential type of type K by the previous corollary.

It is interesting to consider the form that theorem 3 takes when $\Gamma = \mathbb{Z}$. We begin with a lemma relating $\mathbb{Z}$-analytic functions and analytic functions on $\mathbb{C} - \{0\}$.

Lemma 12. Let $\Gamma = \mathbb{Z}$, and let $S = T \times e^{X_{\mathbb{R}}}$ (where T is the circle group). Then S can be identified with $\mathbb{C} - \{0\}$, and $\mathbb{Z}$-analytic functions on S correspond to regular analytic functions on $C - \{0\}$ under this identification.

Proof. Let $g(n) = e^{in\theta} \in T$, and let $\psi(n) = rn \in X_{\mathbb{R}}$. Then we identify $ge^{\psi} \in S$ with $e^{i\theta+r} \in \mathbb{C} - \{0\}$, since $g(n)e^{\psi(n)} = (e^{i\theta+r})^n$ for all $n \in \mathbb{Z}$. Clearly this identification is 1 - 1 and onto, and implicit use will be made of it for the rest of the lemma.

Let f be a $\mathbb{Z}$-analytic function on S. Consider the $\mathbb{Z}$-domain d: $\mathbb{C} \to S$ defined by $d(z)(n) = (e^{iz})^n$. d maps each strip $\{z: |\mathrm{Im}(z)| \leq n\} = S_n$ onto the compact annulus $\{z: |\log|z|| \leq n\} = A_n$. Since A_n is the image of a compact subset of S_n, lemma A shows that L(f) is bounded on A_n; since int A_n is open, theorem 1 now shows that there is a function $F_n \in L^2(\mathbb{Z})$ so that

$$L(f)(e^{i\theta+r}) = \mathcal{F}^{-1}(F_n e^{i\theta+r})$$

whenever $|r| < n$. Clearly $F_n = F$ must be independent of n, and hence

$Fe^{i\theta+r} \in L^2(\mathbb{Z})$ for all $e^{i\theta+r} \in S$. But this actually shows that $Fe^{i\theta+r} \in L^1(\mathbb{Z})$ for all $e^{i\theta+r}$, since

$$\sum_{n=0}^{\infty} |F(n)e^{in\theta+rn}| \leq \Big(\sum_{n=0}^{\infty} |F(n)e^{in\theta+2rn}|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} |e^{-rn}|^2\Big)^{\frac{1}{2}},$$

and similarly for $\sum_{n=-\infty}^{0} |F(n)e^{in\theta+rn}|$. Therefore $f(z) = \sum_{n=-\infty}^{\infty} F(n)z^n$ is absolutely convergent for each $z \in \mathbb{C} - \{0\}$, and is easily seen to be analytic on this domain.

Conversely, let f be analytic on $\mathbb{C} - \{0\}$. Then if $g(\theta) \in L^2(T)$ is arbitrary and $d(z)$ is an arbitrary $\mathbb{Z}$-domain, we may show that

$$G(z) = \int_0^{2\pi} g(\theta)L(f)(d(z))(\theta)d\theta$$

is a regular analytic function on the domain of d as in lemma 9. Therefore f is actually Γ-analytic on S, so that the equivalence is established.

Theorem 3 can now be interpreted for $\Gamma = \mathbb{Z}$. It turns out to be an obvious fact, but it is perhaps interesting to see its connection with theorem C.

Corollary 6. Let $f(z)$ be analytic on $\mathbb{C} - \{0\}$. Suppose there is a $\delta > 0$ and n,m two integers with $n \leq m$ so that

$$(*) \qquad \Big(\int_0^{2\pi} |f(e^{r+i\theta})|^2 d\theta\Big)^{\frac{1}{2}} = \begin{cases} O(e^{(n-1+\delta)r}) & \text{as } r \to 0 \\ O(e^{(m+1-\delta)r}) & \text{as } r \to \infty. \end{cases}$$

Then $f(z)$ is a rational function of the form $f(z) = \sum_{i=n}^{m} a_i z^i$.

Proof. We have $X_{\mathbb{R}}^{*} = \mathbb{R}$. Set $K = [n-1+\delta, m+1-\delta]$, so that $I = i^{-1}(K) = \{n, n+1, \ldots, m-1, m\}$. Then (*) just says that f is $\mathbb{Z}$-entire of exponential type of type K. Applying theorem 3 shows that we have

$$L(f)(e^{r+i\theta}) = \mathcal{F}^{-1}(Fe^{i\theta+r}),$$

where $F = 0$ almost everywhere and hence everywhere off I. This is the desired result.

Appendix: The Maximal Ideal Spaces of Some Related Convolution Algebras

Many important theorems of harmonic analysis can be organized into pairs of corresponding "L^1" and "L^2" results. Plancherel's theorem, for example, can be viewed as the "L^2" analog to the result that the maximal ideal space of the algebra $L^1(\Gamma)$ may be identified with G, the dual group of Γ. Since the classical Paley–Wiener theorems are analytic forms of Plancherel's theorem, it is natural to ask if these results posess a corresponding "L^1" formulation. In particular, do the strips and halfplanes appearing in theorems A and B (or, more generally, the Γ-strips appearing in theorems 1 and 2) arise as the maximal ideal spaces of related convolution algebras?

We have not been able to obtain an entirely precise answer to this question, but we feel that the following partial results suggest a positive answer and are therefore worthy of attention. Notation and definitions are as in the main text unless specified otherwise.

Notation. For $g \in L^1(\Gamma)$ and $\gamma \in \Gamma$, let g_γ be the translation of g by γ: $g_\gamma(\gamma') = g(\gamma' - \gamma)$ for all $\gamma' \in \Gamma$. Let $\Lambda \subseteq \Gamma$ be a closed semigroup of positive Haar measure which contains the identity of Γ. Of course, Λ may be a subgroup of Γ or Γ itself. Finally, denote convolution in $L^1(\Gamma)$ by $*$.

Definition 7. Let $\| \ \|_A$ be a $[0,\infty]$-valued function on $L^1(\Lambda)$. Define $A = \{f \in L^1(\Lambda): \|f\|_A < \infty\}$. A is called a quasi-$L^1$ algebra provided that A and $\| \ \|_A$ satisfy the following properties:

(1) If $f, g \in L^1(\Lambda)$ and $|f| \leq |g|$ almost everywhere, then $\|f\|_A \leq \|g\|_A$.

(2) A is a Banach algebra with norm $\| \ \|_A$ and multiplication given by convolution.

(3) $\|g\|_1 \leq \|g\|_A$ for all $g \in A$.

(4) If $g \in A$, then $\lambda \to g_\lambda$ is a continuous A-valued function on Λ.

(5) If $g \in A$ and $g \geq 0$, then g is the limit in A of an increasing sequence of positive simple functions with compact support.

(6) If $E \subseteq \Lambda$ and $\Lambda - E$ has nonzero measure, then there is a $g \in A$ which is not supported on E.

We denote the maximal ideal space of A by $\Sigma(A)$.

Our primary goal now is to prove that if A is a quasi-L^1 algebra, then $\Sigma(A)$ may be identified with the set of functions $A_m = \{\varphi\colon \varphi$ is continuous and multiplicative on Λ, and $g\varphi \in L^1(\Lambda)$ for all $g \in \Lambda\}$. It is simplest to proceed by a series of straightforward lemmas. In what follows, A is always assumed to be quasi-L^1.

<u>Lemma 13</u> Let $g \in A$. Then there is a sequence $\{g_n\}_{n=1}^{\infty} \subseteq A$ of simple functions with compact support so that $\lim_{n\to\infty} g_n = g$ in A and also pointwise almost everywhere. Additionally, we have:

(a) If $g \geq 0$, then $\{g_n\}$ increases to g monotonically.

(b) If $g \in A$ is arbitrary, then $|g_n| \leq |g|$ almost everywhere for each n.

Proof. If $g \geq 0$, pick a sequence $\{g_n\}_{n=1}^{\infty} \subseteq A$ as guaranteed by condition 5) in the definition of A. Then $\{g_n\}$ increases monotonically and $\lim_{n\to\infty} \|g_n - g\|_A = 0$. Since $\|g-g_n\|_1 \leq \|g-g_n\|_A$ for all n and $\{g_n\}$ increases monotonically, we must have $g_n \to g$ pointwise almost everywhere as well.

If $g \in A$ is arbitrary, apply the above result to the positive and negative parts of the real and imaginary parts of g. This yields four sequences of simple functions with compact support which approximate these four functions in A, monotonically, and pointwise almost everywhere. The obvious linear combination of these four sequences then approximates g in the desired fashion.

Lemma 14. Let $g \in A$, and let $E \subseteq \Lambda$ be measurable and have compact closure. Suppose that χ_E, the characteristic function of E, is a function in A. Then

$$\chi_E * g = \int_E g_\gamma d\gamma \text{ as elements of } A,$$

where the integral on the right is the integral of a continuous A-valued function.

Proof. Since E has compact closure and $\gamma \to g_\gamma$ is a continuous A-valued function on E, the integral

$$\int_E g_\gamma d\gamma$$

is defined as an element of A (and hence of $L^1(\Lambda)$). Let $\langle\ ,\ \rangle$ denote the bilinear pairing on $L^\infty(\Lambda) \times L^1(\Lambda)$. If $h \cdot \in L^\infty(\Lambda)$ is arbitrary, we see

$$\langle h, \int_E g_\gamma d\gamma\rangle = \int_E \langle h, g_\gamma\rangle \, d\gamma$$

$$= \int_E \int_\Lambda h(\gamma')g(\gamma'-\gamma)d\gamma' d\gamma$$

$$= \int_\Lambda h(\gamma') \int_E g(\gamma'-\gamma)d\gamma d\gamma'$$

$$= \langle h, \chi_E * g\rangle .$$

Since $h \in L^\infty(\Lambda)$ was arbitrary, we must have

$$\chi_E * g = \int_E g_\gamma d\gamma$$

as elements of $L^1(\Lambda)$, and hence as elements of A.

Lemma 15. Let $\varphi \in \Sigma(A)$, and let $g \in A$ satisfy $\varphi(g) = 1$. If $h \in A$ has compact support, then

$$\varphi(h) = \int_\Lambda h(\gamma)\varphi(g_\gamma)d\gamma.$$

Proof. Suppose first that $h = \chi_E$, where E has compact closure. Then by lemma 14, we have

$$\varphi(h) = \varphi(h)\varphi(g) = \varphi(h * g)$$

$$= \varphi(\int_E g_\gamma d\gamma)$$

$$= \int_E \varphi(g_\gamma)d\gamma$$

$$= \int_\Lambda h(\gamma)\varphi(g_\gamma)d\gamma.$$

The desired result now follows for any simple function $h \in A$ with compact support by linearity. If $h \in A$ has compact support $E \subseteq \Lambda$ but is otherwise arbitrary, let $\{h_n\}_{n=1}^{\infty}$ be a sequence of simple functions in A with $|h_n| \leq |h|$ almost everywhere and $\lim_{n\to\infty} h_n = h$ in A and also pointwise almost everywhere. Since E has compact closure, the function $\gamma \to \varphi(g_\gamma)$ is bounded on E, so that $|h(\gamma)\varphi(g_\gamma)|$ is an integrable function. Therefore the dominated convergence theorem yields

$$\varphi(h) = \lim_{n\to\infty} \varphi(h_n) = \lim_{n\to\infty} \int_\Lambda h_n(\gamma)\varphi(g_\gamma)d\gamma$$

$$= \int_\Lambda h(\gamma)\varphi(g_\gamma)d\gamma.$$

This is the desired result.

<u>Notation</u>. Let $\varphi \in \Sigma(A)$ and let ψ be a bounded, measurable, multiplicative function on Λ. Define the functional φ_ψ on A by $\varphi_\psi(g) = \varphi(\psi g)$. ψ is bounded, so $\psi g \in A$ whenever $g \in A$, and so φ_ψ is defined on all of A. Since ψ is multiplicative, it is easy to check that $(\psi g) * (\psi h) = \psi(g * h)$ for all g, $h \in L^1(\Lambda)$; therefore $\varphi_\psi \in \Sigma(A)$.

<u>Lemma 16</u>: Let $\varphi \in \Sigma(A)$ and suppose $g \in A$ satisfies $\varphi(g) = 1$. Then the function ψ defined on Λ by

$$\psi(\gamma) = \begin{cases} \overline{\varphi(g_\gamma)}/|\varphi(g_\gamma)| & \text{when } \varphi(g_\gamma) \neq 0. \\ 0 & \text{when } \varphi(g_\gamma) = 0. \end{cases}$$

is measurable, bounded, and multiplicative on Λ.

Proof: $\gamma \to \varphi(g_\gamma)$ is continuous, so ψ is measurable. It is also clearly bounded. To see that $\psi(\gamma_1 + \gamma_2) = \psi(\gamma_1)\psi(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Lambda$, it is enough to note that

$$\varphi(g_{\gamma_1+\gamma_2}) = \varphi(g_{\gamma_1})\varphi(g_{\gamma_2})$$

(because $g_{\gamma_1+\gamma_2} * g = g_{\gamma_1} * g_{\gamma_2}$, and $\varphi(g) = 1$), and then verify the desired equality for each possible pair of definitions for $\psi(\gamma_1)$ and $\psi(\gamma_2)$.

Lemma 17. Let $\varphi \in \Sigma(A)$, and suppose $g \in A$ satisfies $\varphi(g) = 1$. Then for any $f \in A$, we have

(a) the function $F(\gamma) = f(\gamma)\varphi(g_\gamma)$ is integrable, and

(b) $\varphi(f) = \int_\Lambda f(\gamma)\varphi(g_\gamma)d\gamma$.

Proof: Let ψ be defined as in lemma 16. We first prove the result for φ_ψ in place of φ. Let $f \in A$ with $f \geq 0$, and pick $\{f_n\}_{n=1}^{\infty}$ to be simple functions with compact support so that $\lim_{n\to\infty} f_n = f$ in A, monotonically, and pointwise almost everywhere. By lemma 15, we have

$$\varphi_\psi(f_n) = \varphi(\psi f_n) = \int_\Lambda f_n(\gamma)\psi(\gamma)\varphi(g_\gamma)d\gamma$$

$$= \int_\Lambda f_n(\gamma)|\varphi(g_\gamma)|d\gamma.$$

Applying the monotone convergence theorem, we find that $f(\gamma)|\varphi(g_\gamma)|$ is integrable, and that

$$\varphi_\psi(f) = \lim_{n\to\infty} \varphi_\psi(f_n) = \lim_{n\to\infty} \varphi(\psi f_n)$$

$$= \lim_{n\to\infty} \int_\Lambda f_n(\gamma)|\varphi(g_\gamma)|d\gamma$$

$$= \int_\Lambda f(\gamma)|\varphi(g_\gamma)|d\gamma.$$

This formula has been proved for positive $f \in A$, but this automatically implies its validity for all $f \in A$ by linearity.

We can now show that $\varphi(f) = \int_\Lambda f(\gamma)\varphi(g_\gamma)d\gamma$ for all $f \in A$. First note that if χ is the characteristic function of the set $\{\gamma\colon \varphi(g_\gamma) \neq 0\}$, then $\chi = \psi\bar{\psi}$ is multiplicative and that

$$\varphi_\chi(f) = \varphi_{\psi\bar{\psi}}(f) = \varphi_\psi(\bar{\psi}f)$$

$$= \int_\Lambda \bar{\psi}(\gamma)f(\gamma)|\varphi(g_\gamma)|d\gamma$$

$$= \int_\Lambda f(\gamma)\varphi(g_\gamma)d\gamma$$

for all $f \in A$. Hence we will be done if

$$\varphi(\chi f) = \varphi(f)$$

for all $f \in A$.

To prove this, it is enough to show that if $f \in A$ is supported by $\Lambda_0 = \{\gamma \in \Lambda\colon \varphi(g_\gamma) = 0\}$, then $\varphi(f) = 0$. If f is a simple function with compact support, this is obvious by lemma 15. If $f \in A$ is supported on Λ_0 but otherwise arbitrary, the result now follows by lemma 13. Hence the lemma is proved.

Theorem 4. If A is a quasi-L^1 algebra, then there is a 1-1 correspondence between $\Sigma(A)$ and A_m: Each $\psi \in A_m$ corresponds to the functional $\Phi(\psi) \in \Sigma(A)$ defined by

$$\Phi(\psi)(f) = \int_\Lambda f(\gamma)\psi(\gamma)d\gamma.$$

Proof: Clearly each $\psi \in A_m$ defines a functional $\Phi(\psi) \in \Sigma(A)$ by the above formula. To show that Φ is 1-1, suppose that $\psi_1, \psi_2 \in A_m$ with $\Phi(\psi_1) = \Phi(\psi_2)$. Then for $f \in A$ and $g \in G$, we have

$$\Phi(\psi_1)(gf) = \Phi(\psi_2)(gf)$$

or

$$\int_\Lambda f(\gamma)(\psi_1(\gamma) - \psi_2(\gamma)) \quad g(\gamma)\, d\gamma = 0.$$

Hence the function $f(\psi_1 - \psi_2)$ has null Fourier transform, so that $f(\psi_1 - \psi_2) \equiv 0$ whenever $f \in A$. Since Λ is the smallest set which supports A, we see $\psi_1 = \psi_2$ almost everywhere on Λ. Since ψ_1 and ψ_2 are continuous, we have $\psi_1 = \psi_2$.

Conversely, let $\varphi \in \Sigma(A)$. Choose $g \in A$ so that $\varphi(g) = 1$. Then lemma 17 shows that $F(\gamma) = f(\gamma)\varphi(g_\gamma) \in L^1(\Lambda)$ for all $f \in A$ and that

$$\varphi(f) = \int_\Lambda f(\gamma)\varphi(g_\gamma)d\gamma.$$

It has already been shown that $\gamma \to \varphi(g_\gamma)$ is multiplicative and continuous, so that the function $\psi(\gamma) = \varphi(g_\gamma)$ is an element of A_m. Hence Φ is onto, and the theorem is proved.

The following two corollaries illustrate the relationship between the above result and theorems 1 and 2 of the main text.

Corollary 7. Let $\{\theta_1, \ldots, \theta_n\}$ be a finite subset of $X_{\mathbb{R}}$ containing the real character 0. Let $\Lambda = \Gamma$, and define $\| \ \|_A$ by $\|f\|_A = \sup_{1 \leq j \leq n} \|e^{\theta_j} f\|_1$ for $f \in L^1(\Gamma)$. Then A is a quasi-L^1 algebra, and $\Sigma(A)$ may be identified with the set $\{ge^{\theta} : g \in G, \theta$ a convex combination of $\{\theta_1, \ldots, \theta_n\}\}$.

Proof: It is not difficult to see that A is a quasi-L^1 algebra. Conditions 1) and 3) are obvious, and 2) follows from the observation that $e^{\theta_i}(f * g) = (e^{\theta_i} f) * (e^{\theta_i} g)$ for all $f, g \in A$ and $1 \leq i \leq n$. Condition 4) follows from the calculations

$$\|e^{\theta_i} g_{\gamma+\lambda} - e^{\theta_i} g_{\gamma}\|_1 = |e^{\theta_i(\gamma)}| \, \|e^{\theta_i} g_{\lambda} - e^{\theta_i} g\|_1$$

and

$$\begin{aligned} \|e^{\theta_i} g_{\lambda} - e^{\theta_i} g\|_1 &\leq \|(e^{\theta_i} g)_{\lambda} - e^{\theta_i} g\|_1 + \|(e^{\theta_i} g)_{\lambda} - e^{\theta_i} g_{\lambda}\|_1 \\ &= \|(e^{\theta_i} g)_{\lambda} - e^{\theta_i} g\|_1 + |e^{-\theta_i(\lambda)} - 1| \, \|e^{\theta_i} g_{\lambda}\|_1 \\ &= \|(e^{\theta_i} g)_{\lambda} - e^{\theta_i} g\|_1 + |e^{-\theta_i(\lambda)} - 1| \, |e^{\theta_i(\lambda)}| \, \|e^{\theta_i} g\|_1, \end{aligned}$$

which approaches 0 as λ approaches the identity element of Γ. Condition 5) follows quickly from the monotone convergence theorem, and 6) from the fact that the characteristic function of an arbitrary compact subset of Γ is an element of A. Hence A is a quasi-L^1 algebra.

It follows that $\Sigma(A)$ may be identified with $A_m = \{\psi : \psi$ is multiplicative and continuous on Γ, $\psi f \in L^1(\Gamma)$ whenever $e^{\theta_i} f \in L^1(\Gamma)$ for $1 \leq i \leq n\}$. We will show $A_m = \{ge^{\theta}$: $g \in G$, θ a convex combination of $\{\theta_1, \ldots, \theta_n\}\}$. One inclusion is easy; if $\theta = \sum_{i=1}^{n} t_i \theta_i$ with $\sum_{i=1}^{n} t_i = 1$ and $t_i \geq 0$ for $1 \leq i \leq n$, then for any $f \in A$ we have

$$\int_{\Gamma} |f| e^{\theta} d\gamma = \int_{\Gamma} \prod_{i=1}^{n} |f|^{t_i} e^{t_i \theta_i} d\gamma$$

$$\leq \prod_{i=1}^{n} \left(\int_{\Gamma} |f| e^{\theta_i} d\gamma \right)^{t_i}$$

$$< \infty.$$

Hence $f \in A \Rightarrow fe^{\theta} \in L^1(\Gamma)$, and so $g\, e^{\theta} \in A_m$ whenever θ is a convex combination of $\{\theta_1, \ldots, \theta_n\}$ and $g \in G$.

The reverse inclusion is somewhat lengthier. Suppose $\psi \in A_m$. Then since ψ is multiplicative on Γ, ψ is nowhere 0; hence we may write $\psi = ge^{\theta}$, where $g \in G$ and $\theta \in X_{\mathbb{R}}$. It remains to show $\theta \in \operatorname{conv}\{\theta_1, \ldots, \theta_n\}$.

To show this, we begin by noticing that $\inf_{1 \leq j \leq n} \theta_j(\gamma) \leq \theta(\gamma) \leq \sup_{1 \leq j \leq n} \theta_j(\gamma)$ for all $\gamma \in \Gamma$. To see this, suppose first that $\theta(\gamma) > \sup_{1 \leq j \leq n} \theta_j(\gamma)$ for all $\gamma \in E$, where $E \subseteq \Gamma$ has positive measure. By regularity, we may assume E is compact. Let χ_E be the characteristic function of E. Then

$$\Phi(\psi)(\bar{g}\chi_E) = \int_E e^{\theta}\, d\theta > \|\bar{g}\chi_E\|_A,$$

which contradicts the fact that $\|\Phi(\psi)\| \leq 1$. Hence $\theta(\gamma) \leq \sup_{1 \leq j \leq n} \theta_j(\gamma)$

almost everywhere; since θ and $\sup\limits_{1 \leq j \leq n} \theta_j$ are both continuous, the inequality is true for all $\gamma \in \Gamma$. Applying this result to $-\gamma$ yields $\inf\limits_{1 \leq j \leq n} \theta_j(\gamma) \leq \theta(\gamma)$ for all $\gamma \in \Gamma$, which is the second half of the desired inequality.

The above result may be written $\theta(\gamma) \in \text{conv } \{\theta_j(\gamma)\}_{j=1}^{n}$ for all $\gamma \in \Gamma$. If this can be strengthened to read $\alpha(\theta) \in \text{conv } \{\alpha(\theta_j)\}_{j=1}^{n}$ for all $\alpha \in X_{\mathbb{R}}^*$, the Hahn-Banach theorem will allow us to conclude that $\theta \in \text{conv } \{\theta_j\}_{j=1}^{n}$. This will complete the proof of the corollary.

First let $\alpha \in X_{\mathbb{R}}^*$ be of the form $\sum\limits_{j=1}^{m} r_j\gamma_j$, where $\gamma_j \in \Gamma$ and $r_j \in \mathbb{Q}$ for $1 \leq j \leq m$. Write $r_j = \frac{s_j}{k}$, where k and $s_1, \ldots, s_m$ are integers with $k \geq 0$. Then we have

$$\frac{1}{k} \inf_{1 \leq i \leq n} \theta_i\Big(\sum_{j=1}^{m} s_j\gamma_j\Big) \leq \frac{1}{k}\, \theta\Big(\sum_{j=1}^{m} s_j\gamma_j\Big) \leq \frac{1}{k} \sup_{1 \leq i \leq n} \theta_i\Big(\sum_{j=1}^{m} s_j\gamma_j\Big)$$

by the previous paragraphs, so that

$$\inf_{1 \leq i \leq n} \Big(\sum_{j=1}^{m} r_j\gamma_j\Big)(\theta_i) \leq \Big(\sum_{j=1}^{m} r_j\gamma_j\Big)(\theta) \leq \sup_{1 \leq i \leq n} \Big(\sum_{j=1}^{m} r_j\gamma_j\Big)(\theta_i).$$

This is the desired result when $\alpha \in X_{\mathbb{R}}^*$ is a rational linear combination of elements of Γ. The result easily extends to all $\alpha \in X_{\mathbb{R}}^*$ which are linear combinations of elements of Γ by continuity. Finally, the result extends to all $\alpha \in X_{\mathbb{R}}^*$ by noting that the linear span of Γ in $X_{\mathbb{R}}^*$ is weak* dense in $X_{\mathbb{R}}^*$. This concludes the corollary.

It is unknown what, if any, conditions on Γ are necessary to extend this result to infinite set of real characters. The above, however, shows that at least some of the Γ-strips appearing in theorem 1 of the main text arise as maximal ideal spaces of convolution algebras.

The next corollary exhibits a somewhat weaker connection between theorems 4 and 2.

Corollary 8. Let $\Lambda \subseteq \Gamma$ be a closed semigroup which contains the identity, and let $\| \|_A = \| \|_1$ on $L^1(\Lambda)$. Then $L^1(\Lambda) = A$ is a quasi-L^1 algebra and the maximal ideal space of $L^1(\Lambda)$ can be identified with the set of functions $\{\psi : \psi$ is continuous and multiplicative on Λ, and $\|\psi\|_\infty \leq 1\}$.

Proof: It is elementary that $L^1(\Lambda) = A$ is a quasi-L^1 algebra. Hence $\Sigma(A)$ may be identified with $A_m = \{\psi : \psi$ is multiplicative and continuous on Λ, and $f\psi \in L^1(\Lambda)$ whenever $f \in L^1(\Lambda)\}$. Now if $\psi \in A_m$, we must have $1 \geq \|\Phi(\psi)\| = \|\psi\|_\infty$. Conversely, if $\|\psi\|_\infty \leq 1$ and ψ is continuous and multiplicative, then clearly $\psi \in A_m$. This proves the corollary.

To see the connection between the above corollary and theorem 2, suppose $\Lambda \subseteq \Gamma$ is a closed semigroup with the additional property that if $\gamma \in \Gamma - \Lambda$, then there is a $\theta \in X_{\mathbb{R}}$ so that $\theta(\gamma) > 0$ and $\theta(\lambda) \leq 0$ for all $\lambda \in \Lambda$. Let $C \subseteq X_{\mathbb{R}}$ be the cone $\{\theta \in X_{\mathbb{R}} : \theta(\lambda) \leq 0$ for all $\lambda \in \Lambda\}$. Then (in the notation of theorem 2) we have $\Lambda_- = \Lambda$, and the Γ strip appearing in theorem 2 corresponds exactly to the nowhere zero functions in A_m.

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