# PUSHING UP IN FINITE GROUPS

Thesis by

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To the Memory of Russ and Emily Forester

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## Abstract

The main result is a pushing up theorem for $L_3(2^n)$. Let G be a finite group with $F^*(G) = O_2(G)$. Suppose there exists a normal subgroup H of G with $O_2(G) \leq H$ such that $H/O_2(G)$ has no partial complement in $G/O_2(G)$ and G/H is isomorphic to $L_3(2^n)$. Let T be a Sylow 2-subgroup of G and let X be a maximal 2-local subgroup of G containing H with XH/H a maximal parabolic subgroup of G/H. Let $P = O_2(X)$ and $E = \Omega_1(Z(Q))$.

<u>Theorem.</u> If $G \neq \langle C_G(\Omega_1(Z(T))), N_G(J(P))\rangle$, then

either (i) $[G,E^\alpha] \leq E$ for each automorphism $\alpha$ of P

or (ii) G has exactly one noncentral chief factor within $O_2(G)$.

We also derive some new pushing up theorems for $L_2(2^n)$ which are variations on the results of Baumann, Glauberman and Niles.

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# I. Introduction

## 1. Pushing Up

The classification problem for simple groups[1] is an important part of finite group theory and is a large subject in itself. Gorenstein [15] gives a survey of the field in which he describes a multitude of special methods for investigating simple groups. This thesis is concerned with one such method, the theory of pushing up.

The known nonabelian simple groups are the Chevalley groups[2], the alternating groups and 24 sporadic groups.[3] The Chevalley groups are perhaps the most representative of these examples.

If X is a group and p is a prime, then the largest normal p-subgroup of X is denoted by $O_p(X)$ and the smallest normal subgroup of X with index in X a power of p is denoted by $O^p(X)$. We follow the usual convention that the normalizer in X of a nonidentity p-subgroup is referred to as a p-local subgroup of X.

The structure of a Chevalley group is determined by the collection of maximal parabolic subgroups. If the group is defined over a finite field of characteristic p, then the collection of maximal parabolic subgroups is precisely the collection of maximal p-local subgroups. Furthermore

$$C_H(O_p(H)) \leq O_p(H)$$

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[1] All groups will be understood to be finite in this thesis.

[2] We include the twisted groups of Lie type in this designation.

[3] Janko's fourth group and the Fischer monster are sometimes counted as sporadic groups, but it is not yet known whether they actually exist. The inclusion of these groups would make a total of 26 sporadic groups as of May 1979.

for each p-local subgroup H. There is an equivalent formulation of this property in terms of the generalized Fitting subgroup $F^*(H)$, namely

$$F^*(H) = O_p(H).$$

This alternate notation is used in later chapters.

Thus a finite group G is said to be of characteristic p type if

$$C_M(O_p(M)) \leq O_p(M)$$

for each p-local subgroup M of G. Characteristic 2 type groups play an important role in the program to classify the finite simple groups.

There is a well established strategy for studying an arbitrary finite simple group of characteristic 2 type. One attempts to force a resemblance between the collection of maximal 2-local subgroups of G and the collection of maximal parabolic subgroups in some Chevalley group. The ultimate goal is to show under suitable circumstances that G is a Chevalley group.

The necessary local analysis is quite technical and requires various special techniques. Pushing up is one of these techniques. Suppose X is a 2-local subgroup of G. Let $U \in Syl_2(X)$, where $Syl_2(X)$ is the set of Sylow 2-subgroups of X. Also assume that $N_G(U) \not\leq X$. When can X be shown to be properly contained in some larger 2-local subgroup of G? The theory of pushing up is concerned with finding suitable conditions under which the question can be answered in the affirmative; that is, X can be "pushed up" to a larger 2-local subgroup. In particular, one asks when there exists a 2-local subgroup containing $\langle X, N_G(U)\rangle$, the group generated by X and $N_G(U)$. There is always a 2-local subgroup if G is in fact a Chevalley group.

Pushing up is particularly applicable to the characterization of Chevalley groups of low Lie rank in which a maximal parabolic subgroup has one nonabelian composition factor which is itself a Chevalley group of low rank. In this thesis we shall be concerned with problems associated with characteristic 2 type groups containing a maximal 2-local subgroup M such that M has a composition factor isomorphic to $L_2(2^n)$ or $L_3(2^n)$, the projective special linear groups of dimensions two and three respectively over a field of order $2^n$. The theorems we obtain are described in sections four and five of this first chapter. The main result concerns $L_3(2^n)$ and is applicable to the theory of quasithin groups. These are the groups for which no 2-local subgroup contains an elementary abelian p-group of order $p^3$ for any prime p.

Let M be a maximal 2-local subgroup of G and let $T \in Syl_2(M)$. One hopes to find conditions which assure that $N_G(T) \leq M$. This is the case in Chevalley groups of characteristic 2 type. The maximal 2-local subgroups can be examined by considering the size of their Sylow 2-subgroups. In several situations M and T can be chosen so that T is a Sylow 2-subgroup of G and M is a unique maximal 2-local subgroup containing T. As M is unique, any 2-local subgroup of G which contains T is contained in M. Choose L to be a maximal 2-local subgroup other than M for which $T \cap L$ has maximum possible order. It follows that $T \cap L$ is a Sylow subgroup of L. Let $S = T \cap L$. Now the normalizer of S is not in L. This is a situation in which L cannot be pushed up to a larger 2-local subgroup, and a situation in which the normalizer of a Sylow 2-subgroup of a maximal 2-local subgroup is not contained in that local subgroup.

This situation was first encountered by Sims [18] in his work on primitive permutation groups in which the one point stabilizer has an orbit of length 3. Sims associated trivalent graphs with the groups in question and then studied the groups via their graphs. Sims' arguments are related to earlier work by Tutte [21,22] on vertex transitive groups of automorphisms of trivalent graphs. Recent work of Goldschmidt [12] yields the classification of the vertex stabilizers in an edge transitive group of automorphisms of a finite, connected, trivalent graph. Examples of such graphs are afforded by buildings associated to Chevalley groups defined over the field of two elements [20].

Given a subgroup X of G and $T \in Syl_2(X)$, there is a 2-local subgroup containing $\langle X, N_G(T)\rangle$ if there exists a nonidentity characteristic subgroup R of T with $R \triangleleft X$. Namely $\langle X, N_G(T)\rangle \leq N_G(R)$. Baumann [6] and Niles [17] showed the existence of such a characteristic subgroup in a case of special interest:

<u>Thoerem</u> (Baumann and Niles) Let X be a finite group with $C_X(O_2(X)) \leq O_2(X)$, $X/O_2(X) \cong L_2(2^N)$ and $T \in Syl_2(X)$. Then

there exists a nonidentity characteristic subgroup R of T such that $R \triangleleft X$

or (i) If $E = \Omega_1(Z(O_2(X)))$, then $V/C_V(X)$ is a natural module for $X/O_2(X)$ where $V = [X,E]$.

(ii) G has exactly one noncentral chief factor within $O_2(X)$.

(iii) T has class 2.

Here $Z(O_2(X))$ is the center of $O_2(X)$ and $\Omega_1(Z(O_2(X)))$ is the subgroup of the center generated by the elements of order 2. The term "natural module" used in part (i) of the above theorem also requires some explanation. If A is any group of matrices over the field $GF(2^n)$, then A can be made to act on the vector spaces of row and column vectors. Each of these spaces can be considered as vector spaces over a field K of order 2. The resulting KA-modules are referred to as the "natural modules" for A. We now consider the group X of the Baumann-Niles theorem. Let W denote the quotient group $V/C_V(X)$. The groups V and $C_V(X)$ are normal in X so X acts on W by conjugation. The group $O_2(X)$ centralizes V and hence centralizes W. Thus $X/O_2(X)$ acts on W. Now W is an elementary abelian 2-group and can be considered as a vector space over K. The quotient group $X/O_2(X)$ is isomorphic to $L_2(2^n)$, so W can be viewed as a module for $L_2(2^n)$. Now W is said to be a natural module for $X/O_2(X)$ if it is a natural module for $L_2(2^n)$.

Niles also proved a similar result for groups involving $L_2(p^n)$ for an odd prime p, and Goldschmidt [13] has extended the theorem given above to the case that $X/O_2(X)$ is an alternating or symmetric group of odd degree. Glauberman and Niles have proved additional pushing up theorems for $L_2(2^n)$ similar to this theorem. These are described in §3 of this chapter. Aschbacher [3,4] used the results of Baumann, Glauberman, Niles and Goldschmidt in developing his theory of blocks[4] and groups with proper characteristic generated cores. Aschbacher [2] proved a different sort of pushing up theorem for $L_2(2^n)$:

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[4] Aschbacher's blocks have no connection with Brauer's character theoretic blocks.

<u>Theorem</u> (Aschbacher) Let G be a finite group of characteristic 2 type, $H \leq G$, $M = N_G(O^2(H))$ and $T \in Syl_2(H)$. Assume $H^* = O^2(H)/O_2(H) \cong Z_3$ or $L_2(2^n)$, $O_2(H) \in Syl_2(C_M(H^*))$, and M is the unique maximal 2-local subgroup of G containing H. Then

either (i) $N_G(T) \leq M$

or (ii) G has sectional 2-rank at most 4.

The statement of Aschbacher's theorem involves the concept of sectional 2-rank, which we now define. The 2-rank of an elementary abelian 2-group is defined to be the dimension of the group considered as a vector space over the field GF(2). The 2-rank of a group G is the maximum 2-rank of an elementary abelian 2-subgroup. A group K is called a section of G if K is the homomorphic image of some subgroup of G. The sectional 2-rank of G is just the maximum 2-rank of a section of G. The simple groups of sectional 2-rank at most 4 have been classified by Gorenstein and Harada [16], so Aschbacher's theorem provides a definitive answer to the problem it treats.

We now develop some necessary notation for the statement of the next theorem. For a given group X, the set of elementary abelian 2-subgroups of X of maximal order is denoted by $\mathcal{A}(X)$. The Thompson subgroup J(X) is defined by $J(X) = \langle \mathcal{A}(X) \rangle$ and the subgroup O(X) is the largest normal subgroup of X of odd order. The notation $\Sigma_n$ is used for the symmetric group of degree n. A group Y is said to be involved in X if Y is isomorphic to some section of X.

<u>Theorem</u> (Thompson [19]) Let X be a solvable group with $O(X) = 1$ and

let $T \in \mathrm{Syl}_2(X)$. If $\Sigma_3$ is not involved in X, then

$$X = C_X(Z(T))N_X(J(T)).$$

The theorem above is only a special case of the theorem Thompson proved. He also considered the situation where T is a Sylow subgroup for an odd prime. We will not require these further considerations.

A given group X is said to satisfy the Thompson factorization if the conclusion of the above theorem holds, that is

$$X = C_X(Z(T))N_X(J(T))$$

for a Sylow 2-subgroup T of X. The Thompson factorization is a starting point for a theory of factorizations of groups. One of the triumphs of this theory is the classification [10] of simple groups not involving $\Sigma_4$. This classification includes as a special case the determination of the simple groups with orders not divisible by 3.

The previously mentioned pushing up problems for $L_2(2^n)$ are closely related to failure of the Thompson factorization. Let X and T be as in the theorem of Baumann and Niles. If the Thompson factorization holds, then either $Z(T) \leq Z(X)$ or $J(T) \lhd X$. In either case we may choose $R \in \{Z(T), J(T)\}$ such that $R \lhd G$. The subgroup R is automatically characteristic in T. Thus failure of the Thompson factorization may be assumed in these pushing up problems. It is a fortunate circumstance that failure of the Thompson factorization gives technical information about the group X which can be exploited.

The second chapter of this thesis is primarily devoted to proving Theorem A, a pushing up theorem for $L_2(2^n)$ which yields a factorization related to the Thompson factorization that fails only in a degenerate case. Theorem A is similar in spirit to the theorem of Baumann and Niles, but we consider characteristic subgroups of Thompson subgroups as well as characteristic subgroups of Sylow subgroups. A short argument gives Corollary 1, which interprets Theorem A in terms of Sylow subgroups. Corollary 2 is related to a theorem of Glauberman and Niles, which is described in §3. Theorem A and its corollaries are discussed in §4.

Niles [17] has obtained results related to Aschbacher's theorem. A monograph of Glauberman [10] includes an account of some of Niles' work.

One of the main results of this thesis is Theorem C, a pushing up theorem for $L_3(2^n)$. The statement and proof are discussed in §5 of this first chapter and the proof constitutes most of the third chapter.

Chapters two and three are logically independent except that some well known lemmas are spelled out in §1 of chapter two which are also used in chapter three. However, an appreciation for the proof of Theorem A would facilitate an understanding of the proof of Theorem C.

## 2. Notation

Most of the notation is standard, as in Gorenstein [14]. We mention some exceptions and also repeat some previously defined notation for the sake of completeness.

For a given group X, we define $\mathcal{A}(X)$ to be the set of elementary abelian 2-subgroups of X of maximal order. $\mathcal{A}_a(X)$ is the set of abelian 2-subgroups of maximal order. $J(X) = \langle \mathcal{A}(X) \rangle$ and $J_a(X) = \langle \mathcal{A}_a(X) \rangle$. We use $\mathcal{U}(X)$ to denote $\{W \leq X \mid \exists\, A \in \mathcal{A}(X) \text{ with } W \leq A\}$. If S is a 2-group, then $\tilde{J}_a(S) = C_S(Z(J_a(S)))$ and $\tilde{J}(S) = C_S(\Omega_1(Z(J(S))))$.

If G is any group and E is a subgroup of G, then a partial complement of E in G is a proper subgroup F of G satisfying EF = G.

We take the liberty of supressing an occasional parenthesis. For example ZJ(X) means Z(J(X)).

Each chapter has its own internal numbering system for lemmas. Thus any mention in chapter two of lemma 4.1 refers to lemma 4.1 of chapter two unless stated otherwise.

## 3. Theorems of Glauberman and Niles

In this section we quote two pushing up theorems of Glauberman and Niles for $L_2(2^n)$ [11]. They also considered $L_2(p^n)$ for odd primes p, but we will restrict our attention to the characteristic 2 type problem. Their results apply to the following hypothesis:

(E) G is a finite group .

$C_G(O_2(G)) \leq O_2(G)$.

S is a Sylow 2-subgroup of G.

There exists $K \lhd G$ with $O_2(G) \leq K$ such that $K/O_2(G)$ has no partial complement in $G/O_2(G)$ and $G/K \cong L_2(2^n)$ for some n.

Glauberman and Niles consider a fixed 2-group S and the set $\mathcal{S}$ of groups G that contain S and satisfy (E). Under certain conditions they prove that for each $G \in \mathcal{S}$ there exists a characteristic subgroup R of S with $R \lhd G$. As 11.4 of [9] demonstrates, there exist choices of S and G for which there is no such subgroup R. Glauberman and Niles find suitable choices of R for a large class of 2-groups and do it somewhat independently of the choice of G.

Theorem (Glauberman and Niles) If either $\tilde{J}_a(S) \neq S$ or $\Phi(S) \not\leq \Omega_1 Z(S)$, then there exist nonidentity characteristic subgroups $S_1$ and $S_2$ of S having the following properties:

(1) $S_1 \leq Z(S)$.

(2) $S_2$ is a characteristic subgroup of $\tilde{J}_a(S)$.

(3) Given $G \in \mathscr{G}$, either $S_1 \trianglelefteq G$ or $S_2 \triangleleft G$.

Now let $\mathscr{G}_1 = \{G \in \mathscr{G} \mid Z(S) \neq Z(G)$ and $\tilde{J}_a(S) \ntriangleleft G\}$. The second main theorem of Glauberman and Niles deals with this situation.

Theorem (Glauberman and Niles) Let G and $G^*$ be elements of $\mathscr{G}_1$. Assume that

(i) $\tilde{J}_a(S) = S$.

(ii) $Z(S)$ is elementary abelian.

(iii) $\Omega_1(Z_2(S)) \leq O_2(G) \cap O_2(G^*)$.

Then $O_2(G) \cap O_2(G^*)$ is normal in both G and $G^*$.

## 4. The $L_2(2^n)$ Problem.

Theorem A deals with Hypothesis (E) of §3, as do the theorems of Glauberman and Niles. This result, however, focuses on J(S) instead of S.

Theorem A. Let G and S be groups satisfying hypothesis (E). Define $Q = O_2(G)$. Let $F = \Omega_1 Z(J(Q))$ and $V = [G, \Omega_1(Z(Q))]$. Also let $\Lambda$ be a subgroup of $\mathrm{Aut}(J(S))$ such that $\Lambda$ contains the action of $S \cap J(G)$ on $J(S)$.

Then

either (i) $\Omega_1(Z(S)) \leq Z(G)$

or (ii) $F \leq J(S)$ and $[J(G),F^{\alpha}] \leq F$ for each $\alpha \in \Lambda$.

or (iii) $V \leq J(S)$ and there exists $\alpha \in \Lambda$ such that $S = V^{\alpha}Q$.

We now consider a special case of Theorem A. Let the groups G, S, Q, F and V be defined as in Theorem A. Choose $\Lambda$ to be the restriction of Aut(S) to Aut(J(S)) and note that $\Lambda$ contains the action of $S \cap J(G)$ on J(S). Now Theorem A is applicable to this situation. Suppose conclusion (ii) of Theorem A holds, namely $F \leq T$ and $[J(G),F^{\alpha}] \leq F$ for each $\alpha \in$ Aut(S). It follows that J(G) is contained in the normalizer of $FF^{\alpha}$. The group S normalizes F and hence normalizes $F^{\alpha}$. Then J(G)S normalizes $FF^{\alpha}$. If $FF^{\alpha}$ is not normal in G, then $J(G)S \neq G$. If $J(G)S \neq G$, then J(G) is contained in Q since J(G) is normal in G and generated by 2-groups. It follows that $J(S) = J(Q)$ and $F = \Omega_1(Z(J(S)))$. Hence F is characteristic in S and $F^{\alpha} = F$ for each $\alpha \in \Lambda$. Therefore G normalizes $FF^{\alpha}$ if part (ii) of Theorem A holds for our choice of $\Lambda$. We have proved the following corollary to Theorem A:

Corollary 1. If G, S, Q, F and V are groups defined as in the hypothesis of Theorem A, then

either $\Omega_1(Z(S)) \leq Z(G)$

or $FF^{\alpha} < G$ for each $\alpha \in$ Aut(S)

or There exists $\alpha \in$ Aut(S) such that $S = V^{\alpha}Q$.

If G and S are groups satisfying Hypothesis (E) such that

$$G \neq C_G(\Omega_1(Z(S)))N_G(J(S))$$

and Corollary 1 (iii) holds, then it can be shown that G has exactly one noncentral chief factor within $O_2(G)$. A simple argument of Baumann [6] yields the Baumann-Niles theorem (of Section 1) as a consequence of Corollary 1.

Define $R = \langle F^{\Lambda} \rangle$. If part (ii) of Theorem A holds, then the group $\langle F^{\Lambda} \rangle$ is normal in G. We can interpret Corollary 1 as saying that if G has more than one noncentral chief factor within $O_2(G)$, then $\Omega_1(Z(S)) \leq Z(G)$ or $R \lhd G$.

Suppose $G_1$ is another group containing S such that $G_1$ and S satisfy Hypothesis (E) with G replaced by $G_1$. Assume that $G_1$ has more than one noncentral chief factor within $O_2(G_1)$. We apply Corollary 1 to construct a characteristic subgroup $R_1$ of S such that $\Omega_1(Z(S)) \leq Z(G_1)$ or $R_1 \lhd G$.

The construction of the group R depends on the embedding of S in G and the construction of the group $R_1$ depends on the embedding of the group S in $G_1$. It is desirable to produce another factorization theorem in which the choice of the characteristic subgroups of S is not dependent on the embedding of S in G. We are led to the following definition:

<u>Definition 1</u>. Let G and S be finite groups. G is an S-embeddable group if G and S satisfy Hypothesis (E) of §3.

Our next result provides appropriate characteristic subgroups of S which do not depend on the particular S-embeddable group under consideration.

<u>Corollary 2</u>. Let S be a 2-group and

let $\mathcal{S}$ = $\{G | G$ is a S-embeddable group and G has more than one noncentral chief factor within $O_2(G)\}$.

Then there exist nonidentity characteristic subgroups $S_1$ and $S_2$ of $S$ such that

(1) $S_1$ is characteristic in $\tilde{J}(S)$.

(2) $S_2 \leq \Omega_1 Z(S)$.

(3) If $G \in \mathcal{S}$, then either $S_1 \lhd G$ or $S_2 \lhd G$.

Corollary 2 is closely related to the first theorem of Glauberman and Niles in §3. Let S be any 2-group. Corollary 2 provides two non-identity characteristic subgroups $S_1$ and $S_2$ of S such that for any S-embeddable group G with more than one noncentral chief factor within $O_2(G)$, either $S_1$ is normal in G or $S_2$ is normal in G. The theorem of Glauberman and Niles gives restrictions on S under which characteristic subgroups $T_1$ and $T_2$ of S are constructed such that for any S-embeddable group G either $T_1$ is normal in G or $T_2$ is normal in G. Thus Corollary 2 focuses on the groups G containing S while the theorem of Glauberman and Niles focuses on the 2-group S.

Corollary 2 concerns groups with a common Sylow 2-subgroup. Theorem A suggests that we should consider groups sharing the Thompson subgroup of a Sylow subgroup.

Definition 2. Let T be a 2-group and let F, G, S and V be finite groups. (G, S, F, V) is a T-embeddable quadruple if

(i) G and S satisfy hypothesis (E) of §3.

(ii) $T = J(S)$.

(iii) $F = \Omega_1 Z(J(O_2(G)))$.

(iv) $V = [G, \Omega_1 Z(O_2(G))]$.

If T is a 2-group, and $J(T) \neq T$, then there can be no T-embeddable quadruples. If S is a 2-group and G is an S-embeddable group, then $(G, S, F, V)$ is a $J(S)$-embeddable quadruple for appropriate subgroups F and V of S.

We now give a result which provides a link between T-embeddable quadruples.

<u>Theorem B.</u> Let $(G, S, F, V)$ and $(G_\alpha, S_\alpha, F_\alpha, V_\alpha)$ be T-embeddable quadruples such that

$$G \neq C_G(\Omega_1 Z(S))N_G(T)$$

$$\text{and} \quad G_\alpha \neq C_{G_\alpha}(\Omega_1 Z(S_\alpha))N_{G_\alpha}(T).$$

If $V_\alpha \leq O_2(G)$ and $F_\alpha \not\leq F$, then $[S_\alpha, V_\alpha] \leq ZJ(G)$.

Theorem B is proved before Theorem A and is used in conjunction with Theorem A in the proof of Corollary 2. Theorem B is also used in the proof of Theorem A. This may be surprising since Theorem A does not seem to involve two distinct quadruples. Let T be a 2-group, let $(G, S, F, V)$ be a T-embeddable quadruple and let $\alpha$ be an automorphism of T. If

$$G \neq \Omega_1(Z(S))N_G(T)$$

it can be shown that T contains F and V, so the automorphism $\alpha$ can be applied to F and V. There are groups $G_\alpha$ and $S_\alpha$ such that

$(G_\alpha, S_\alpha, F^\alpha, V^\alpha)$ is a T-embeddable quadruple with $G_\alpha$ isomorphic to G and $S_\alpha$ isomorphic to S. We may apply Theorem B to yield the following result.

Proposition. Let T be a 2-group and let (G, S, F, V) be a T-embeddable quadruple such that

$$G \neq C_G(\Omega_1(Z(S)))N_G(T).$$

If $V^\alpha \leq O_2(G)$ and $F^\alpha \neq F$, then $[S,V]^\alpha \leq Z(G)$.

This is the form in which Theorem B is applied to the proof of Theorem A. Theorem A could be proved with no mention of Theorem B or T-embeddable quadruples. Fortunately the proof of Theorem B is no more difficult, except for unpleasant notation, than a direct proof of the proposition. Thus Theorem B is proved in §5 of the second chapter although no essential use of embeddable quadruples is made until the proof of Corollary 2 in §8.

We now describe some of the ideas involved in the proof of Theorem A. Let T be a 2-group with $J(T) = T$ and let (G, S, F, V) be a T-embeddable quadruple. Also let $\Lambda \leq \mathrm{Aut}(T)$ such that $\Lambda$ contains the action of $S \cap J(G)$ on T. Define $Q = O_2(G)$. Assume that

$$G \neq C_G(S)N_G(T).$$

We prove in this situation that $[J(G),F^\alpha] \leq F$ for each $\alpha \in \Lambda$ or there exists $\alpha \in \Lambda$ such that $S = V^\alpha Q$.

The failure of the Thompson factorization implies that $G/C_G(V)$ is isomorphic to $L_2(2^n)$ and $V/C_V(G)$ is a natural module for $G/C_G(V)$. This gives considerable information about $\mathcal{A}(S)$. If A is any element of $\mathcal{A}(S)$, then $(A \cap Q)F = (A \cap Q)V$ and $(A \cap Q)V$ is an element of $\mathcal{A}(S)$. The group $T$ contains $F$ and $V$. The collection $\mathcal{A}(S)$ contains $\mathcal{A}(Q)$. It is also shown that

$$S = AQ = TQ.$$

We mention the simplest possible candidate for the group $G$. Let $Q$ be an elementary abelian 2-group order $2^{2n}$ and let $G$ be a semidirect product of $Q$ by $L_2(2^n)$ such that $Q$ is a natural module for $L_2(2^n)$. Now let $S$ be a Sylow 2-subgroup of $G$ and define $T = J(S)$. It follows that $(G, S, Q, Q)$ is a T-embeddable quadruple. This example occurs as a parabolic subgroup of $L_3(2^n)$. Hence $T = S$ and $\mathcal{A}(S)$ has exactly two elements, one of which is $Q$. The other element of $\mathcal{A}(S)$ is the group generated by matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in GF(2^n)$$

and column vectors of the form

$$\begin{pmatrix} b \\ 0 \end{pmatrix}, \quad b \in GF(2^n).$$

There is an automorphism of $S$ which interchanges the two elements of $\mathcal{A}(S)$. Thus whether parts (i) or (ii) of Theorem A apply to this example depends on whether the group $\Lambda$ contains such an automorphism.

We attempt to mirror this in the general case. The important point is that the existence of a unique noncentral chief factor within $V$ as a natural module for $L_2(2^n)$ forces strong conditions on commutators of 2-subgroups, and this is what we utilize. In particular, if $Z = C_V(S)$, then it is shown that

$$[S,V] = [T,V] = Z.$$

If A is any element of $\mathcal{A}(S)$ not contained in $\mathcal{A}(Q)$, then

$$[A,F] = [A,V] = Z.$$

Also $F = V(A \cap F)$. These facts are developed in §4 of the second chapter.

The next step is the proof of Theorem B in §5 of the second chapter. In the context of our present discussion this asserts that if $\alpha$ is any automorphism of T such that $V^{\alpha} \leq Q$ and $F^{\alpha} \neq F$, then $Z^{\alpha} \leq Z(J(G))$. We fix an automorphism $\alpha$ and assume that $Z^{\alpha} \not\leq Z(J(G))$. We show that there is an element g of J(G) such that J(G) is generated by the subgroup T and the element g. A study of the subgroups F, $F^{\alpha}$ and $F^{\alpha g}$ and their commutators eventually forces a contradiction.

We now make some general remarks concerning Theorem A. If $V^{\alpha}$ is not contained in Q for some element $\alpha$ of $\Lambda$, it is shown that $S = V^{\alpha}Q$ which is one of the conclusions of Theorem A. We assume therefore that $V^{\alpha} \leq Q$ for each automorphism $\alpha$ of T contained in $\Lambda$. We now let $\alpha$ be an element of $\Lambda$. We show that $F^{\alpha} \leq Q$, and assume that $[J(G), F^{\alpha}] \not\leq F$.

We now let g and x be any elements of J(G) such that x is a 2-element. The main purpose of §6 of chapter two is to prove that $F^{\alpha g}$ and $F^{\alpha gx}$ commute. The groups $F^{\alpha g}$ and $F^{\alpha gx}$ are normal in J(Q), and the action of $F^{\alpha g}$ on $F^{\alpha gx}$ is highly restricted by the embedding of V in T and vice versa. A technical argument exhibits the incompatibility of these group actions with the embeddings of the subgroups $F^{\alpha g}$ and $F^{\alpha gx}$ in the group T. We conclude that $F^{\alpha g}$ and $F^{\alpha gx}$ commute. This information is applied in lemma 6.4 to yield some other commuting subgroups.

The proof of Theorem A given in §8 of chapter two is based on the methods of Baumann. Assume $[J(G), F^{\alpha}] \not\leq F$. Let A be an element of $\mathcal{A}(S)$ not contained in Q and let g be an involution of J(G) such that J(G) is generated by T and g.

The commutators computed in lemma 6.4 are used to construct an element B of $\mathcal{A}(Q)$ with special properties. In particular $[B, F^{\alpha g \alpha^{-1} g}]$ is a nonidentity A-invariant subgroup contained in $\mathcal{A}(Q)^{\alpha^{-1} g}$. It is shown that $[B, F^{\alpha g \alpha^{-1} g}] = Z^g$. A contradiction is obtained by observing that A does not act on $Z^g$ since $V/C_V(G)$ is a natural module for $G/C_G(V)$.

## 5. The $L_3(2^n)$ problem

Let G be a finite group with $C_G(O_2(G)) \leq O_2(G)$ and define $Q = O_2(G)$. Let $X \lhd G$ such that $Q \leq X$ and X/Q has no partial complement in G/Q. Define $\overline{G} = G/X$ and suppose $\overline{G} \cong L_3(q)$ where $q = 2^n$. Let $T \in Syl_2(G)$.

One might conjecture that either there exists a characteristic subgroup of T that is normal in G or else there is exactly one noncentral chief factor of G within Q. Unfortunately the situation with $L_3(q)$ is more complicated than with $L_2(q)$. We construct a counterexample involving $SL_5(q)$.

Let Q be the group of matrices in $SL_5(q)$

of the form $$\begin{pmatrix} 1 & a & b & c & d \\ 0 & 1 & e & f & g \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let S be the group of upper triangular matrices in $SL_5(q)$. Let H be the group of matrices in $SL_5(q)$

of the form $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & g & h & i \end{pmatrix}.$$

Let $G = SH$. Now $Q = O_2(G) = F^*(G)$ and there are two noncentral chief factors of $G$ within $Q$.

$$\text{Let } g = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and let $\beta$ be the automorphism of $SL_5(q)$ induced by conjugation by $g$. Now let $\varphi$ be the contragredient automorphism of $SL_5(q)$, and let $\alpha = \beta\varphi$. $S^\alpha = S$ so $\alpha$ induces an automorphism of $S$. Suppose there exists a characteristic subgroup $R$ of $S$ such that $R \lhd G$. Then $R^\alpha \lhd G^\alpha$ and $R^\alpha \lhd S$ so $R^\alpha \lhd SL_5(q)$ since $\langle S, G^\alpha \rangle = SL_5(q)$. But $O_2(SL_5(q)) = 1$, a contradiction.

Of course this example can also be explained by considering graph automorphisms of Dynkin diagrams. This topic is treated in section 12.2 of Carter's book on Chevalley groups [7].

We now give the main result of the third chapter.

Theorem C. Let $G$ be a finite group with $C_G(O_2(G)) \leq O_2(G)$ and define $Q = O_2(G)$. Let $X \lhd G$ such that $Q \leq X$ and $X/Q$ has no partial complement in $G/Q$. Assume $G/X \cong L_3(2^n)$ and let $T \in Syl_2(G)$. Let $M_1$ and $M_2$ be the maximal proper 2-local subgroups of $G$ containing $T$ such that $M_1X/X$ and

We assume that $E^{\alpha}$ is contained in Q for each automorphism $\alpha$ of P contained in the group $\Lambda$, and let F be the centralizer in E of $O_2(K)$. Our goal is to prove that $[G,E^{\alpha}]$ is contained in E for each $\alpha$ in $\Lambda$. The normal closure of F in $N_G(P)$ is the group E, so it suffices to prove that $[K,F^{\alpha}]$ is contained in F.

The conjugates of $F^{\alpha}$ in the group K are studied via the embedding of these conjugates in various elements of the set $\mathcal{O}(S)$. The structure of $\mathcal{O}(S)$ is highly restricted by the action of the group G on the normal subgroup V. Our analysis requires the methods used in the proof of Theorem A.

We mention a key intermediate result.

<u>Theorem D.</u> Assume the hypothesis and notation of Theorem C. If $\alpha$ is an automorphism of P such that $E^{\alpha} \neq E$ and $E^{\alpha} \leq Q$, then $Z^{\alpha} \leq Z(G)$, where $Z = C_V(S)$.

Theorem D plays the same role in the proof of Theorem C as Theorem B plays in the proof of Theorem A.

Almost every argument used in the proof of Theorem A reappears in the proof of Theorem C. Those arguments are insufficient to prove Theorem C because the set $\mathcal{O}(P)$ is properly contained in the set $\mathcal{O}(T)$. This leads to many difficulties not present in the proof of Theorem A. In particular, it is necessary to study the embedding of $N_G(P)/C_G(P)$ in $\Lambda$.

# II. Pushing Up $L_2(2^n)$

## 1. $L_2(q)$ Modules

In this section $G$ is a group isomorphic to $L_2(q)$ where $q = 2^n$. Let $S \in Syl_2(G)$ and let $F$ be the field of 2 elements. $V$ is an indecomposable $FG$ module and $D = C_V(G)$. Assume $V/D$ is a natural $L_2(q)$ module for $G$ and $Z = C_V(S)$.

Lemma 1.1. $D = 1$ if $q = 2$.

Proof. $V = D \oplus [O(G), V]$ if $q = 2$. But $G$ acts on $[O(G),V]$ so $[G,V] = [O(G),V]$. From $V = [G,V]$ we get $V = [O(G),V]$ and $D = 1$.

Lemma 1.2. $|Z : D| = q$.

Proof. Let $g \in S^\#$. Then $g^2 = 1$ so $[V,g,g] = 1$. But $[V,g]D/D = C_{V/D}(g)$. Then $C_V(g)/_D = C_{V/D}(g)$. Hence $|Z : D| = q$.

Lemma 1.3. If $g \in S^\#$, then $Z = [g,V] \oplus D$.

Proof. $|V : C_V(g)| = |[g,V]|$. But $|V : C_V(g)| = q$ by 1.2 so $|[g,V]| = q$ and $[g,V] \cap D = 1$. But $|Z : D| = q$ so $Z = [g,V] \oplus D$.

Lemma 1.4. $[S,V] = Z$.

Proof. Let $x \in G \setminus N_G(S)$ and let $M = [S,V] + [S^x,V]$. Now $V = [G,V] = [\langle S,S^x\rangle, V] \geq M$, so $V = M$. But $[S,V] \cap [S^x,V] = [S,V] \cap [S,V]^x = [S,V] \cap D$,

$$\text{so } q^2|D| = |V| = \frac{|[S,V]|^2}{|[S,V] \cap D|} = q^2 \cdot |[S,V] \cap D|$$

Then $|D| = |[S,V] \cap D|$ and $D \leq [S,V]$. Then $Z = [S,V]$.

## 2. The Main Hypothesis

We recall the definition of a T-embeddable quadruple from §4 of the first chapter.

Definition 2. Let F, G, S, T and V be finite groups. (G, S, F, V) is a T-embeddable quadruple if

$G$ is a finite group with $F^*(G) = O_2(G)$.

$S \in Syl_2(G)$.

There exists $K \lhd G$ such that K contains $O_2(G)$ where K has no partial complement in G and $G/K \cong L_2(2^n)$ for some n.

$T = J(S)$.

$F = \Omega_1(Z(J(O_2(G))))$.

$V = [G,\Omega_1(Z(O_2(G)))]$ .

In the rest of this chapter let T be a 2-group with (G, S, F, V) a T-embeddable quadruple. Let K and n be as in Definition 2, define $H = J(G)$ and let $\Lambda \leq Aut(T)$ such that $\Lambda$ contains the action of $S \cap H$ on T. Define $Q = O_2(G)$ and $R = J(Q)$. Represent $C_V(S)$ by Z and $C_V(G)$ by D. Also let $q = 2^n$. Let $\overline{G} = G/K$, and $E = \Omega_1(Z(Q))$.

## 3. The Subgroup K.

Lemma 3.1. The following statements are true in the group G.

(i) $KN_G(S)$ is the unique maximal subgroup of G containing S.

(ii) $(H \cap K)N_H(S)$ is the unique maximal subgroup of H containing S.

(iii) $K/Q$ has odd order.

Proof. Suppose W is a proper subgroup of G that contains S. K has no partial complement in X so $KW \neq G$. But $\overline{KS} \leq \overline{KW} < \overline{G}$. Then $\overline{KW} \leq N_{\overline{G}}(\overline{KS})$ since $\overline{KS} \in Syl_2(\overline{G})$ and $\overline{G} \cong L_2(q)$. But $N_{\overline{G}}(\overline{KS}) = \overline{KN_G(S)}$ by the Frattini argument. Then $\overline{KW} \leq \overline{KN_G(S)}$ and $W \leq KN_G(S)$. Thus part (i) holds. Part (ii) follows from part (i). $G = KN_G(S \cap K)$ by the Frattini argument. Then $N_G(S \cap K) = G$ since K has no partial complements in G. But $O_2(G) = Q$ so $S \cap K = Q$. Then $K/Q$ has odd order.

Lemma 3.2. Suppose $G \neq C_G(\Omega_1(Z(S)))N_G(T)$. Let $A \in \mathcal{A}(S) \setminus \mathcal{A}(Q)$ and $W = \langle \Omega_1(Z(S))^G \rangle$. Then

(i) $[K,W] = 1$.

(ii) $W/C_W(G)$ is a natural module for $\overline{G}$.

(iii) $S = AQ$.

Proof. Lemma 3.2 follows directly from a theorem of Aschbacher [5].

## 4. The Thompson Subgroup.

Assume $G \neq C_G(\Omega_1(Z(S)))N_G(T)$.

Lemma 4.1. Let $A \in \mathcal{A}(S) \setminus \mathcal{A}(Q)$. Then

(i) $\mathcal{A}(S) = \mathcal{A}(T)$. In particular $A \leq T$.

(ii) $[K,E] = 1$.

(iii) $V/D$ is a natural $L_2(q)$ module for $\bar{G}$.

(iv) $(A \cap Q)E \in \mathcal{A}(Q)$.

(v) $\mathcal{A}(Q) \subset \mathcal{A}(S)$.

(vi) $E = V \cdot \Omega_1(Z(G))$.

Proof. $T = J(S)$ so $\mathcal{A}(S) = \mathcal{A}(T)$. Let $W = \langle \Omega_1(Z(S))^G \rangle$ and let $\tilde{W} = W/C_W(G)$. Lemma 3.2 implies that $[K,W] = 1$ and $\tilde{W}$ is a natural $L_2(q)$ module for $\bar{G}$. Also $S = AQ$. Now $|E : C_E(A)| \geq |W : C_W(A)| \geq |\tilde{W} : C_{\tilde{W}}(A)| = q$. But $(A \cap Q)E$ is elementary abelian and $A \in \mathcal{A}(S)$ so $|A| \geq |(A \cap Q)E| = |A \cap Q| \cdot |E : A \cap E|$. But $|E : A \cap E| \geq q$ since $|\tilde{W} : C_{\tilde{W}}(A)| = q$. Then $|A| = |(A \cap Q)E|$ and $|E : A \cap E| = q$. Now $(A \cap Q)E \in \mathcal{A}(S)$ so $(A \cap Q)E \in \mathcal{A}(Q)$ and $\mathcal{A}(Q) \subset \mathcal{A}(S)$. Let $g \in G \setminus KN_G(S)$. Then $\langle S, S^g \rangle = G$ so $\langle Q, A, A^g \rangle = G$ and $C_E(\langle A, A^g \rangle) = C_E(G)$. Now $|E : A \cap A^g \cap E| \leq q^2$ since $|E : A \cap E| = q$. But $|\tilde{W} : C_{\tilde{W}}(G)| = q^2$ so $|E : A \cap A^g \cap E| = q^2$ and $E = WC_E(G)$. Now $[K,E] = 1$ and $E = VC_E(G)$ where $C_E(G) = \Omega_1(Z(G))$. Also $V/D$ is a natural $L_2(q)$ module for $\bar{G}$.

Lemma 4.2. $G = HQ$ and $O^2(G) = O^2(H)$.

Proof. $T \neq R$ so $H \not\leq Q$. But $|K : Q|$ is odd by 3.1 so $HK = G$. Then $HQ = G$. $O^2(G) = O^2(H)$ since $H \trianglelefteq G$.

Lemma 4.3. (i) $D = 1$ if $q = 2$.

(ii) $|Z : D| = q$.

(iii) If $g \in S \setminus Q$, then $Z = [g,V] \times D$.

(iv) $[T,V] = [S,V] = Z$.

Proof. 4.3 follows directly from 4.1 and the lemmas in §1 of this chapter.

Lemma 4.4. Let $A \in \mathcal{A}(S)$.

(i) $F \cdot C_A(F) = (A \cap Q)F \in \mathcal{A}(Q)$.

(ii) $F = V \cdot (A \cap F)$.

Proof. We may assume $A \not\leq Q$.

(i) $(A \cap Q)E \in \mathcal{A}(Q)$ by 4.1 so $F \leq (A \cap Q)E$. Now $E \leq F$ so $(A \cap Q)F = (A \cap Q)E$ and $(A \cap Q)F \in \mathcal{A}(Q)$. Notice that $A \cap Q \leq C_A(F)$. But $A \cap Q = C_A(E) \geq C_A(F)$ so $A \cap Q = C_A(F)$.

(ii) $(A \cap Q)F = (A \cap Q)E = (A \cap Q)V$ so $|F : A \cap F| = |V : A \cap F|$. Then $F = (A \cap Q)V$.

Lemma 4.5. Let $A \in \mathcal{A}(S) \setminus \mathcal{A}(Q)$. Then

(i) $[T,F] = [A,F] = [A,V] = Z$.

(ii) $[R,F] = V$.

(iii) If $Y \in \mathcal{U}(T)$, then $Y \cap Q \in \mathcal{U}(Q)$ and $[Y \cap Q, F] = 1$. Also $F = VC_F(Y)$.

Proof. (i) $[A,F] = [A, V \cdot (A \cap F)] = [A,V]$ by 4.4. But $AQ = S$ so $[A,V] = [S,V] = Z$ by 4.3. Then $[T,F] = Z$.

(ii) $[T,F] = Z$ so $[R,F] = V$.

(iii) This follows from 4.4 (i).

## 5. The Proof of Theorem B.

In this section $G \neq C_G(\Omega_1(Z(S)))N_G(T)$. Let $(G_\alpha, S_\alpha, F_2, V_\alpha)$ be a T-embeddable quadruple and $Z_\alpha = C_{V_\alpha}(S_\alpha)$. Let $Q_\alpha = O_2(G_\alpha)$ with $E_\alpha = \Omega_1(Z(Q_\alpha))$ and $R_\alpha = J(Q_\alpha)$.

Theorem B. If $V_\alpha \leq Q$ and $F_\alpha \not\leq F$, then

either $\quad G_\alpha = C_{G_\alpha}(\Omega_1(Z(S_\alpha))N_{G_\alpha}(T)$

or $\quad Z_\alpha \leq Z(H)$.

We prove Theorem B in a sequence of lemmas. Suppose $V_\alpha \leq Q$ and $F_\alpha \not\leq F$. Also assume $G_\alpha \neq C_{G_\alpha}(\Omega_1(Z(S_\alpha))N_{G_\alpha}(T)$ and $Z_\alpha \not\leq Z(H)$.

Notice that $F \in \mathcal{U}(T)$ so $F_\alpha = V_\alpha C_{F_\alpha}(F)$ by lemma 4.5(iii). Then $F_\alpha = V_\alpha(F_\alpha \cap Q)$, and $V_\alpha \leq Q$, so $F_\alpha \leq Q$.

Let $A \in \mathcal{A}(T) \setminus \mathcal{A}(Q)$ with $B_0 = (A \cap Q)F$ and $B = F^\alpha C_{B_0}(F_\alpha)$. Let $g \in H \setminus N_H(KS)$.

Lemma 5.1. $N_G(Z_\alpha) \leq N_G(SK)$.

Proof. Let $Y = N_G(Z_\alpha)$ and suppose $Y \not\leq N_G(KS)$. $T \leq N_G(Z_\alpha)$ so $\overline{Y} = \overline{G}$. Then $YQ = G$. $Q \leq N_G(T)$ so $H = \langle T^Y \rangle$. Then $Z_\alpha \trianglelefteq H$. But $[T, Z_\alpha] = 1$, so $[H, Z_\alpha] = 1$, a contradiction.

Lemma 5.2. (i) $B_0 \in \mathcal{A}(T)$ and $B \in \mathcal{A}(T)$.

(ii) $[A,B] \leq ZZ_\alpha \leq C_B(F_\alpha^g)$

(iii) $\langle F_\alpha^G \rangle \leq R$.

Proof. (i) follows from 4.4(i).

(ii) $[A,B] \leq [A,FF_\alpha] \leq [T,FF_\alpha] \leq ZZ_\alpha$. Now $ZZ^\alpha \leq \Omega_1 Z(T) \leq F \leq B$. But $[F_\alpha,F] = 1$ by 4.5(iii), so $[F_\alpha^g,F] = 1$. Then $ZZ_\alpha \leq C_B(F_\alpha^g)$.

(iii) $F_\alpha \trianglelefteq R \triangleleft G$ so $\langle F_\alpha{}^G\rangle \leq R$.

Lemma 5.3. $\langle F_\alpha{}^{gA}\rangle$ is elementary abelian.

Proof. $F_\alpha \triangleleft T$ so $F_\alpha \triangleleft R \triangleleft G$ and $F_\alpha^g \triangleleft R$. Then $A \cap Q$ acts on $F_\alpha^g$. Suppose 5.3 is false and let $x \in A \setminus Q$ sit. $[F_\alpha^g, F_\alpha^{gx}] \neq 1$. But $F_\alpha^g \cdot C_B(F_\alpha^g) \in \mathcal{A}(Q)$ and A acts on $C_B(F_\alpha^g)$ by 5.2(ii), so $F_\alpha^{gx} \cdot C_B(F_\alpha^g) = F_\alpha^{gx} \cdot C_B(F_\alpha^{gx}) \in \mathcal{A}(Q)$. Now $Z_\alpha^g = [F_\alpha^g, F_\alpha^{gx} \cdot C_B(F_\alpha^g)] = [F_\alpha^g, F_\alpha^{gx}] = [F_\alpha^g \cdot C_B(F_\alpha^{gx}), F_\alpha^{gx}) = Z_\alpha^{gx}$ by 4.5(i). Then $gxg^{-1} \in N_G(Z_\alpha) \leq N_G(SK)$ by 5.1. But $x \in S \setminus Q$ and $g \in H \setminus N_H(SK)$, which gives a contradiction.

Lemma 5.4. $F_\alpha^g \leq B$.

Proof. A acts on $C_B(F_\alpha^g)$ by 5.2 so $C_B(F_\alpha^g) = C_B(\langle F^{\alpha gA}\rangle)$. But $\langle F^{\alpha gA}\rangle$ is abelian by 5.3 so $\langle F_\alpha^{gA}\rangle \cdot C_B(F_\alpha^g) = F_\alpha^g \cdot C_B(F_\alpha^g) \in \mathcal{A}(Q)$. But A acts on $\langle F_\alpha^{gA}\rangle$ and $C_B(F_\alpha^g)$, so A acts on $F_\alpha^g \cdot C_B(F_\alpha^g)$. Then A acts on $[B, F_\alpha^g \cdot C_B(F_\alpha^g)] = [B, F_\alpha^g]$. But $B \in \mathcal{A}(Q)$. If $[B,F_\alpha^g] \neq 1$, then $[B,F_\alpha^g] = Z_\alpha^g$ and A acts on $Z_\alpha^g$. Then $A \leq N_G(KS^g)$ by 5.1, a contradiction.

Lemma 5.5. $FF_\alpha \triangleleft H$.

Proof. $[A^{g^{-1}}, F_\alpha] = [A, F_\alpha^g]^{g^{-1}} \leq [A,B]^{g^{-1}} \leq (ZZ_\alpha)^{g^{-1}} \leq (\Omega_1 Z(T))^{g^{-1}} \leq F^{g^{-1}} = F$. But $[T,F_\alpha] = Z_\alpha \leq F$ so $H = \langle T, A^{g^{-1}}\rangle \leq N_G(FF_\alpha)$.

We now prove Theorem B. Suppose $F_\alpha \not\leq F$. Then there exists $x \in \mathcal{A}(Q)$ such that $[X,F_\alpha] \neq 1$. Then $[X,F_\alpha] = Z_\alpha = [T,F_\alpha]$ by 4.5. But $X \leq R \leq T$ so $[R,F_\alpha] = Z_\alpha$, and H acts on $FF_\alpha$ by 5.5, so H acts on $[R,FF_\alpha] = [R,F_\alpha] = Z_\alpha$, contradicting 5.1.

We now prove an important corollary to Theorem B.

Corollary 5.6. If $\langle F^\Lambda \rangle \leq Q$ and $\alpha \in \Lambda$, then $F^\alpha \lhd S \cap H$.

Proof. Suppose $x \in S \cap H$ such that $F^{\alpha x} \neq F^\alpha$. Then $F^{\alpha x \alpha^{-1}} \neq F$. But $\Lambda$ contains the action of $S \cap H$ on T. $F^{\alpha x \alpha^{-1}} \leq Q$ by hypothesis so $Z^{\alpha x \alpha^{-1}} \leq Z(H)$ by Theorem S. But $Z^\alpha \leq Z(H)$ since $F^\alpha \neq F$, so $Z^{\alpha x} = Z^\alpha$. Then $Z^{\alpha x \alpha^{-1}} = Z$ and $Z \leq Z(H)$, a contradiction.

## 6. $\langle F^{\alpha H} \rangle$ .

In this section suppose $G \neq C_G(\Omega_1(Z(S)))N_G(T)$ and $\langle V^\Lambda \rangle \leq Q$. Let $\alpha \in \Lambda$ and let $W = \langle F^{\alpha H} \rangle$. If $F^\alpha \neq F$, then $Z^\alpha \leq Z(H)$ by theorem B.

Lemma 6.1. (i) $W \leq R$.

(ii) $\Phi(W) \leq Z^\alpha$.

Proof. (i) $F^\alpha \leq R \lhd H$ so $W \leq R$.

(ii) $[F^\alpha,W] \leq [F^\alpha,R] \leq [F^\alpha,T] \leq Z^\alpha \leq Z(H)$ if $F^\alpha \neq F$. Then $\Phi(W) \leq Z^\alpha$ if $F^\alpha \neq F$. If $F^\alpha = F$, then $\Phi(W) = 1 < Z^\alpha$.

Lemma 6.2. If $u \in H$ and $g \in V^{\alpha u \alpha^{-1}}$, then $[h,h^g] = 1$ if h is an involution of W.

Proof. We may assume $F^\alpha \neq F$. Let $X = \langle g,h \rangle$ and suppose $[h,h^g] \neq 1$.

$X$ is dihedral since $g^2 = h^2 = 1$. But $\Phi(X \cap W) \leq \Phi(W) \leq Z^{\alpha} \leq Z(H)$ and $\Phi(X \cap W^{\alpha^{-1}}) \leq \Phi(W)^{\alpha^{-1}} \leq Z$. Now $g \in X \cap W^{\alpha^{-1}} \trianglelefteq X$ and $h \in X \cap W \trianglelefteq X$. But $[h, h^g] \neq 1$, so $g \notin X \cap W$ and $h \notin X \cap W^{\alpha^{-1}}$. Then $X/X \cap W \cong X/X \cap W^{\alpha^{-1}} \cong \mathbb{Z}_2$. But $h \notin Z(X \cap W)$ since $[h, h^g] \neq 1$. Also $\Phi(\Phi(X \cap W)) = 1$. Then $X \cap W \cong D_8$ and $X \cong D_{16}$. Also $\Phi(X \cap W) = Z(X)$. But $|X : X \cap W^{\alpha^{-1}}| = 2$ with $g^2 = 1$ and $g \notin Z(X)$ so $X \cap W^{\alpha^{-1}} \cong D_8$. Then $Z(X) = \Phi(X \cap W^{\alpha^{-1}}) \leq Z$ and $Z(X) \leq Z \cap Z^{\alpha} \leq Z \cap Z(H) = D$. Let $y$ be an element of order 4 in $X \cap W^{\alpha^{-1}}$. Now $g^{\alpha}$, $y^{\alpha} \in W$ and $G^{\alpha u}$, $y^{\alpha u} \in W$ and $W \leq T$. Now $1 \neq [y^{\alpha u 2^{-1}}, g^{\alpha u \alpha^{-1}}] \in [y^{\alpha u \alpha^{-1}}, V] \cap D^{\alpha u \alpha^{-1}}$, which equals $[y^{\alpha u \alpha^{-1}}, V] \cap D$, contradicting 4.3.

<u>Lemma 6.3</u>. If $x \in S \cap H$ and $g \in H$, then $[F^{\alpha g}, F^{\alpha g x}] = 1$.

<u>Proof</u>. We may assume $F^{\alpha} \neq F$. Suppose false. Then there exists $g \in H$ and $t \in S \cap H$ such that $[F^{\alpha g}, F^{\alpha g t}] \neq 1$. Let $u = gth^{-1}$. Then $[F^{\alpha u}, F^{\alpha}] \neq 1$ and $[F^{\alpha u \alpha^{-1}}, F] \neq 1$. But $F^{\alpha} \in \mathcal{U}(T)$ and $F^{\alpha} \leq Q$ so $F^{\alpha} \in \mathcal{U}(Q)$. Then $F^{\alpha u} \in \mathcal{U}(Q) \subseteq \mathcal{U}(T)$ and $F^{\alpha u \alpha^{-1}} \in \mathcal{U}(T)$. $F^{\alpha u \alpha^{-1}} \notin \mathcal{U}(Q)$ since $[F^{\alpha u \alpha^{-1}}, F] \neq 1$. Then $F^{\alpha u \alpha^{-1}} \not\leq Q$ with $[F^{\alpha u \alpha^{-1}}, V] \neq 1$ and $[F, V^{\alpha u \alpha^{-1}}] \neq 1$. But $V^{\alpha u \alpha^{-1}} \in \mathcal{U}(T)$. Now $V^{\alpha u \alpha^{-1}} \notin \mathcal{U}(Q)$ since $[F, V^{\alpha u \alpha^{-1}}] \neq 1$, so $V^{\alpha u \alpha^{-1}} \not\leq Q$. Then $[V^{\alpha u \alpha^{-1}}, V] \neq 1$ and $C_V(V^{\alpha u \alpha^{-1}}) = Z$. Now $V^{\alpha u \alpha^{-1}} \cap C(V) = Z^{\alpha u \alpha^{-1}}$.

$Z^{\alpha u\alpha^{-1}} = Z$ so $V^{\alpha u\alpha^{-1}} \cap C(V) = Z$. Then $V^{\alpha u\alpha^{-1}} \cap Q = Z$ and $S \cap H = O_2(H)\, V^{\alpha u\alpha^{-1}}$. Notice $F^{\alpha} \lhd S \cap H$ by 5.6 so $F^{\alpha} \lhd O_2(H)$ and $F^{\alpha g} \lhd O_2(H)$. Then there exists $x \in V^{\alpha u\alpha^{-1}}$ such that $F^{\alpha gx} = F^{\alpha gt}$.

Let $y \in V^{\alpha g} \setminus Z^{\alpha}$. Then $[y,y^x] = 1$ by 6.2. But $y^{g^{-1}\alpha^{-1}} \in V \setminus Z$, so $C_T(y^{g^{-1}\alpha^{-1}}) = C_T(V)$. Then $C_T(y^{g^{-1}}) = C_T(V^{\alpha})$. So $C_R(y^{g^{-1}}) = C_R(V^{\alpha})$. But $y \in R$ and $V^{\alpha} \leq R$ with $R \lhd H$, so $C_R(y) = C_R(V^{\alpha g})$. Then $y^x \in C_R(V^{\alpha g})$, so $y \in C_R(V^{\alpha gx})$ and $V^{\alpha gx} \leq C_R(y)$. But $C_R(y) = C_R(V^{\alpha g})$. Then $[V^{\alpha g}, V^{\alpha gx}] = 1$. But $V^{\alpha g} \in \mathcal{U}(Q) \leq \mathcal{U}(T)$ so $[V^{\alpha g}, F^{\alpha gx}] = 1$. Use $F^{\alpha gx} \in \mathcal{U}(Q)$ to get $[F^{\alpha g}, F^{\alpha gx}] = 1$.

Theorem 6. If $x \in H$ is a 2-element and $h \in H$, then $[F^{\alpha h}, F^{\alpha hx}] = 1$.

Proof. Theorem 6 follows directly from 6.3.

Lemma 6.4. If $x \in T \setminus Q$ and $g,h$ are involutions of $H \setminus N_H(SK)$, then

(a) $F^{\alpha g} \in \mathcal{U}(Q)$.

(b) $F^{\alpha g\alpha^{-1}} \in \mathcal{U}(Q)$ and $F^{\alpha g\alpha^{-1}h} \in \mathcal{U}(Q)$.

(c) $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(Q)$.

(d) $[F^{\alpha g\alpha^{-1}h}, F^{\alpha g\alpha^{-1}}] = 1$.

(e) $[F^{\alpha g\alpha^{-1}gx}, F^{\alpha g\alpha^{-1}g}] = 1$.

(f) $[F^{\alpha g\alpha^{-1}g}, F^{\alpha^{-1}gx}] = 1$.

Proof. We may assume $F^{\alpha} \neq F$.

(a) $F \in \mathcal{U}(Q) \subseteq \mathcal{U}(T)$ so $F^{\alpha} \in \mathcal{U}(T)$. But $F^{\alpha} \leq Q$, so $F^{\alpha} \in \mathcal{U}(Q)$ by lemma 4.5. It follows that $F^{\alpha g} \in \mathcal{U}(Q)$.

(b) $[F^{\alpha g\alpha^{-1}}, F] = [F^{\alpha g}, F^{\alpha}]^{\alpha^{-1}} = 1$ by Theorem 6, so $F^{\alpha g\alpha^{-1}} \leq Q$. But $F^{\alpha g} \in \mathcal{U}(Q)$ and $\mathcal{U}(Q) \subseteq \mathcal{U}(T)$ so $F^{\alpha g\alpha^{-1}} \in \mathcal{U}(T)$. Then $F^{\alpha g\alpha^{-1}} \in \mathcal{U}(Q)$ and $F^{\alpha g\alpha^{-1}h} \in \mathcal{U}(Q)$.

(c) $[F^{\alpha g\alpha^{-1}h\alpha}, F] = [F^{\alpha g}, F^{\alpha^{-1}h\alpha}]^{\alpha^{-1}h\alpha} \leq [F^{\alpha g}, J(Q)]^{\alpha^{-1}h\alpha}$ by (b). But $[F^{\alpha g}, J(Q)] = [F^{\alpha}, J(Q)]^{g} \leq [F^{\alpha}, T]^{g} = Z^{\alpha g} = Z^{\alpha}$. Then $[F^{\alpha g\alpha^{-1}h\alpha}, F] \leq (Z^{\alpha})^{\alpha^{-1}h\alpha} = Z^{h\alpha}$. But $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(T)$ by (b), so $[F^{\alpha g\alpha^{-1}h\alpha}, F] \leq Z$. Then $[F^{\alpha g\alpha^{-1}h\alpha}, F] \leq Z \cap Z^{h\alpha} \leq (Z(T) \cap Z^{h})^{\alpha} = D^{\alpha}$ and $[F^{\alpha g}, F^{\alpha^{-1}h\alpha}] \leq (D^{\alpha})^{\alpha^{-1}h\alpha} = D^{h\alpha} = D^{\alpha} = D^{\alpha g}$. Then $[V^{\alpha g}, F^{\alpha^{-1}h\alpha}] = 1$. But $F^{\alpha^{-1}h\alpha} \in \mathcal{U}(Q)$, so $[F^{\alpha g}, F^{\alpha^{-1}h\alpha}] = 1$ and $[F^{\alpha g\alpha^{-1}h\alpha}, F] = 1$. But $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(T)$ by (b), so $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(Q)$.

(d) $[F^{\alpha g\alpha^{-1}h}, F^{\alpha g\alpha^{-1}}] = [F^{\alpha g\alpha^{-1}h\alpha}, F^{\alpha g}]^{\alpha^{-1}} \leq [J(Q), F^{\alpha g}]^{\alpha^{-1}}$ by (c). But $[J(Q), F^{\alpha g}]^{\alpha^{-1}} = [J(Q), F^{\alpha}]^{g\alpha^{-1}} \leq [T, F^{\alpha}]^{g\alpha^{-1}} = (Z^{\alpha})^{g\alpha^{-1}} = Z$. Notice $[F^{\alpha g\alpha^{-1}h}, F^{\alpha g\alpha^{-1}}] = [F^{\alpha g}, F^{\alpha g\alpha^{-1}h\alpha}]^{\alpha^{-1}h} \leq [F^{\alpha g}, J(Q)]^{\alpha^{-1}h} \leq (Z^{\alpha})^{\alpha^{-1}h} = Z^{h}$. Then $[F^{\alpha g\alpha^{-1}h}, F^{\alpha g\alpha^{-1}}] \leq Z \cap Z^{h} = D$ and $[F^{\alpha g\alpha^{-1}h\alpha}, F^{\alpha g}] \leq D^{\alpha} = D^{\alpha g}$. Now $[F^{\alpha g\alpha^{-1}h\alpha}, V^{\alpha g}] = 1$. But $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(Q)$ by (c) so $F^{\alpha g\alpha^{-1}h\alpha} \in \mathcal{U}(Q)^{\alpha g}$ and $[F^{\alpha g\alpha^{-1}h\alpha}, F^{\alpha g}] = 1$. Then $[F^{\alpha g\alpha^{-1}h}, F^{\alpha g\alpha^{-1}}] = 1$.

(e) $[F^{\alpha g\alpha^{-1}gx}, F^{\alpha g\alpha^{-1}g}] = [F^{\alpha g\alpha^{-1}(gxg)}, F^{\alpha g\alpha^{-1}}] = 1$ by (d).

(f) $[F^{\alpha g\alpha^{-1}g}, F^{\alpha^{-1}gx}] = [F^{\alpha g\alpha^{-1}(gxg)\alpha}, F] = 1$ by (c).

## 7. The Proof of Theorem A.

Theorem A is proved in this section. If $G = C_G(\Omega_1(Z(S)))N_G(T)$, then either $\Omega_1(Z(S)) \leq Z(S)$ or $T \lhd G$. If $T \lhd G$, then $F = \Omega_1(Z(T))$ and $F \lhd G$ and Theorem A holds. So we may assume that $G \neq C_G(\Omega_1(Z(S)))N_G(T)$.

The proof of Theorem A is obtained in a sequence of lemmas. We suppose Theorem A is false and proceed by way of contradiction.

Lemma 7.1. $F^{\alpha} \leq Q$ for each $\alpha \in \Lambda$ and $\Phi(\langle F^{\Lambda}\rangle) = 1$.

Proof. Suppose there exists $\alpha \in \Lambda$ such that $F^{\alpha} \not\leq Q$. Of course $F \in \mathcal{U}(T)$ so $F^{\alpha} = V^{\alpha} \cdot C_{F^{\alpha}}(F)$ by lemma 4.4. But $F^{\alpha} \in \mathcal{U}(T)$ so $C_{F^{\alpha}}(F) = F^{\alpha} \cap Q$ and $F^{\alpha} = V^{\alpha} \cdot (F^{\alpha} \cap Q)$. Then $V^{\alpha} \not\leq Q$ and $[V^{\alpha}, V] \neq 1$. Now $|V^{\alpha} : V^{\alpha} \cap Q| = q$ and $S = QV^{\alpha}$ and Theorem A holds, a contradiction. Therefore $F^{\alpha} \leq Q$ for each $\alpha \in \Lambda$. Then $[F^{\alpha}, F] = 1$ since $F^{\alpha} \in \mathcal{U}(T)$. Then $F \leq Z(\langle F^{\Lambda}\rangle)$ and $\langle F^{\Lambda}\rangle$ is elementary abelian.

We may assume there exists $\alpha \in \Lambda$ such that $[H, F^{\alpha}] \not\leq F$. Let $A \in \mathcal{A}(T) \setminus \mathcal{A}(Q)$ and $B_0 = FF^{\alpha} \cdot C_A(FF^{\alpha})$. Let $g \in H \setminus N_H(SK)$ such that $g^2 = 1$ and $B = F^{\alpha^{-1}g} \cdot C_{B_0}(F^{\alpha^{-1}g})$. Notice that $B_0$, $B \in \mathcal{A}(Q)$ and $[A, B_0] \leq [A, FF^{\alpha}] \leq ZZ^{\alpha}$.

Lemma 7.2.

(i) $C_{B_0}(F^{\alpha^{-1}g}) = C_{B_0}(\langle F^{\alpha^{-1}gA}\rangle)$.

(ii) $[A, B] \leq C_B(F^{\alpha g\alpha^{-1}g}) \leq B$.

(iii) $[R,F^{\alpha g}] = [B_0,F^{\alpha g}] = Z^{\alpha}$.

Proof. (i) $[A,B_0] \leq ZZ^{\alpha} \leq C_{B_0}(F^{\alpha^{-1}g})$ so $C_{B_0}(F^{\alpha^{-1}g})$ and $C_{B_0}(F^{\alpha^{-1}g}) = C_{B_0}(\langle F^{\alpha^{-1}gA}\rangle)$.

(ii) $\Phi(\langle F^{\alpha^{-1}gA}\rangle) = 1$ by Theorem 6, so $\Phi(B\langle F^{\alpha^{-1}gA}\rangle) = 1$ by (i). It follows from $B \in \mathcal{A}(T)$ that $\langle F^{\alpha^{-1}gA}\rangle \leq B$ and $[A,B] \leq [A,F^{\alpha^{-1}g}B_0] \leq B \cdot ZZ^{\alpha} \leq B$.

Let $x \in A \setminus Q$. Now $[x,B] \leq [h,F^{\alpha^{-1}g}B_0] \leq F^{\alpha^{-1}g}F^{\alpha^{-1}gx}ZZ^{\alpha}$. Therefore $[F^{\alpha^{-1}g},F^{\alpha g\alpha^{-1}g}] = [F,F^{\alpha g}]^{\alpha^{-1}g} = 1$ and $[F^{\alpha^{-1}gx},F^{\alpha g\alpha^{-1}g}] = 1$ by lemma 6.4(f). But $F^{\alpha g\alpha^{-1}g} \leq T$ by 6.4(b), so $ZZ^{\alpha} \leq C(F^{\alpha g\alpha^{-1}g})$. Therefore $[x,B] \leq C_B(F^{\alpha g\alpha^{-1}g})$ for each $x \in A \setminus Q$. Then $[A,B] \leq C_B(F^{\alpha g\alpha^{-1}g})$.

(iii) If $F^{\alpha g} \leq B_0$, then $[A^g,F^{\alpha}] = [A,F^{\alpha g}]^g \leq [A,B_0]^g \leq (ZZ^{\alpha})^g = Z^gZ^{\alpha} \leq F$ and $[H,F^{\alpha}] = [\langle T,A^g\rangle,F^{\alpha}] \leq F$, a contradiction. Then $F^{\alpha g} \not\leq B_0$. Now $B_0 \in \mathcal{A}(Q)$ so $[B_0,F^{\alpha g}] = Z^{\alpha g} = Z^{\alpha}$. Now we have $[R,F^{\alpha g}] = [R,F^{\alpha}]^g \leq [T,F^{\alpha}]^g = Z^{\alpha g} = Z^{\alpha}$ with $B_0 \leq R$ and $[R,F^{\alpha g}] = Z^{\alpha}$.

Lemma 7.3. If $F^{\alpha g\alpha^{-1}g} \leq B$ then $B_0^{g\alpha} \in \mathcal{A}(Q)$ and $F^{\alpha^{-1}g} \leq B_0$. Also $B = B_0$.

Proof. Suppose $F^{\alpha g\alpha^{-1}g} \leq B$. Then $[B_0,F^{\alpha g\alpha^{-1}g}] \leq [B_0,B] = [B_0,F^{\alpha^{-1}g}] \leq [R,F^{\alpha^{-1}g}] = [R,F^{\alpha^{-1}}]^g \leq Z^{\alpha^{-1}g}$. Then $[B_0^{g\alpha g},F^{\alpha}] \leq (Z^{\alpha^{-1}g})^{g\alpha g} = Z^g \leq F$. Then $[T,F^{\alpha}] \leq F$ and $[H,F^{\alpha}] \not\leq F$ yields $B_0^{g\alpha g} \leq S$. From $B_0^{g\alpha} \leq S$, we get

$B_0^{g\alpha g} \le S \cap S^g = Q$. Then $B_0^{g\alpha} \le Q$ and $B_0^{g\alpha} \le R$. Hence $B_0 \le R^{\alpha^{-1}g}$, so $[B_0, F^{\alpha^{-1}g}] = 1$ and $B_0 = B$.

Lemma 7.4. If $F^{\alpha g\alpha^{-1}g} \le B$, then $[H, F^{\alpha^{-1}}] \le F$.

Proof. Suppose $F^{\alpha g\alpha^{-1}g} \le B$. Then $F^{\alpha^{-1}g} \le B_0$ by 7.4. Now $[A^g, F^{\alpha^{-1}}] = [A, F^{\alpha^{-1}g}]^g \le [A, B_0]^g \le (ZZ^{\alpha})^g = Z^g Z^{\alpha} \le F$. But $[T, F^{\alpha^{-1}}] = Z^{\alpha^{-1}} \le F$ and $\langle A^g, T\rangle = H$. Then $[H, F^{\alpha^{-1}}] \le F$.

Lemma 7.5. $F^{\alpha g\alpha^{-1}g} \not\le B$.

Proof. Suppose $F^{\alpha g\alpha^{-1}g} \le B$. $B_0 = B$ by 7.3 so $[A, F^{\alpha g\alpha^{-1}g}] \le [A, B_0] \le ZZ^{\alpha} \le F$ and $[A^g, F^{\alpha g\alpha^{-1}}] \le F$. Then $A^g$ acts on $FF^{\alpha g\alpha^{-1}}$.

$B = B_0 \le R \cap R^{\alpha}$. Let $X = \langle \mathcal{A}(Q) \cap \mathcal{A}(Q)^{\alpha}\rangle$. We have $B^{\alpha^{-1}} \le X \le R \cap R^{\alpha^{-1}}$ and $X = J(C_Q(FF^{\alpha^{-1}}))$. Now $A^g$ acts on $FF^{\alpha^{-1}}$ by 7.4 so $A^g$ acts on $X$. However, $A^g$ acts on $FF^{\alpha g\alpha^{-1}}$ so $A^g$ acts on $[X, FF^{\alpha g\alpha^{-1}}] = [X^{\alpha}, F^{\alpha g}]^{\alpha^{-1}}$. From $B_0 \le X^{\alpha} \le R$, it follows that $[X^{\alpha}, F^{\alpha g}] = Z^{\alpha}$ by 7.2. Then $A^g$ acts on $Z$, a contradiction.

Lemma 7.6. $[B, F^{\alpha g\alpha^{-1}g}]$ is A-invariant.

Proof. Let $x \in A \setminus Q$, and $Y = F^{\alpha g\alpha^{-1}g}$. But $[Y, Y^x] = 1$ by 6.4 (e). Now $F^{\alpha^{-1}g} \le B$ so $B \in \mathcal{A}(Q) \cap \mathcal{A}(Q)^{\alpha^{-1}g}$. Then $Y \cdot C_B(Y) \in \mathcal{A}(Q)$. Lemma 7.2 (ii) yields $C_B(Y) = C_B(Y^x)$ and then $YY^xC_B(Y)$ is elementary abelian.

But $Y \cdot C_B(Y) \in \mathcal{A}(Q)$ so $Y^x \leq Y \cdot C_B(Y)$. Then x acts on $Y \cdot C_B(Y)$ so x acts on $[B, Y \cdot C_B(Y)] = [B,Y]$ for all $x \in A \setminus Q$. Now A acts on $[B,Y]$.

We now establish the final contradiction that proves Theorem A. We have $F^{\alpha^{-1}g} \leq B$ so $B \in \mathcal{A}(Q) \cap \mathcal{A}(Q)^{\alpha^{-1}g}$. But $[B, F^{\alpha g \alpha^{-1}g}] \neq 1$ by 7.5 so $[B, F^{\alpha g \alpha^{-1}g}] = Z^{\alpha g \alpha^{-1}g} = Z^g$. Then A acts on $Z^g$ by 7.6, a contradiction.

## 8. Corollaries to Theorem A

We give two corollaries to Theorem A. The first corollary is closely related to a theorem of Baumann [6].

Corollary 1. Either (i) $\Omega_1(Z(S)) \leq Z(G)$,

or (ii) $FF^\alpha \lhd G$ for each $\alpha \in \mathrm{Aut}(S)$

or (iii) there exists $\alpha \in \mathrm{Aut}(S)$ such that $S = V^\alpha Q$.

Corollary 1 was proved in §4 of the first chapter. Notice that if neither (i) or (ii) holds, then $G \neq C_G(\Omega_1(Z(S)))N_G(T)$. Now if $\alpha \in \mathrm{Aut}(S)$ such that $V^\alpha \not\leq Q$, then $[O^2(G),Q] = V$ since $[V^\alpha, Q] \leq Z^\alpha \leq P$.

The second corollary is similar to a result of Glauberman and Niles described in §3 of the first chapter. The proof involves an application of both Theorem A and Theorem B. We give some lemmas and additional notation before proceeding with Corollary 2.

Lemma 8.1. Assume that $G \neq C_G(\Omega_1(Z(S)))N_G(T)$ and let $Z = \bigcap_{a \in G} \Omega_1(Z(T))^a$.

Then (i) $VZ = V \cdot \Omega_1(Z(T))$

and (ii) $\widetilde{J}(S) \in \mathrm{Syl}_2(\langle \widetilde{J}(S)^G \rangle)$.

Proof. Let Y be a Sylow 2-subgroup of G with $\langle S,Y\rangle = G$. Let $B \in \mathcal{A}(Y) \setminus \mathcal{A}(Q)$, choose $g \in B \setminus Q$ and define $\tilde{Z} = \Omega_1(Z(T))$. Clearly g normalizes $V\tilde{Z}$ since $[g,F] \leq C_V(Y)$ and $V\tilde{Z} \leq F$. Then $V\tilde{Z}$ is normal in G since G is generated by S and g. We know $Z = V \cap \tilde{Z}$ and $ZZ^g = V$. Then $\tilde{Z}\tilde{Z}^g = V\tilde{Z}$ and $|\tilde{Z}^g : \tilde{Z}^g \cap \tilde{Z}| = |V : V \cap \tilde{Z}| = q$. It follows from $|V : Z \cap Z^g| = q^2$ that $V\tilde{Z} = V(\tilde{Z} \cap \tilde{Z}^g)$.

Of course $\tilde{Z} \cap \tilde{Z}^g$ is normalized by Q and g. Then $\tilde{Z} \cap \tilde{Z}^g$ is normal in G since $\tilde{Z}$ is centralized by T and $G = \langle T,Q,g\rangle$. Hence $\tilde{Z} \cap \tilde{Z}^g = Z$ and $VZ = V\tilde{Z}$.

Let $X = \langle \tilde{J}(S)^G\rangle$. It follows from (i) that $C_Q(Z)$ is contained in $\tilde{J}(S)$. Then $X \cap Q = \tilde{J}(S) \cap Q$ and hence $\tilde{J}(S)$ is a Sylow subgroup of X.

Lemma 8.2. If $T \triangleleft G$, then either $\tilde{J}(S) \triangleleft G$ or $\Omega_1(Z(S)) \leq Z(G)$.

Proof. Suppose $T \triangleleft G$ and $\tilde{J}(S) \ntriangleleft G$. Then $C_G(\Omega_1(Z(T)) \triangleleft G$ and

$$O^2(G) \leq \langle \tilde{J}(S)^G\rangle \leq C_G(\Omega_1(Z(T))) \leq C_G(\Omega_1(Z(S))).$$

Hence $\Omega_1(Z(S)) \leq Z(G)$.

Definition 8. A finite group X is a $(\tilde{J},S)$-embeddable group if $(X,S_X,F_X,V_X)$ is a T-embeddable quadruple for some subgroups $S_X$, $F_X$ and $V_X$ of X such that $\tilde{J}(S) = \tilde{J}(S_X)$ and $X \neq C_G(\Omega_1(Z(S_X)))N_G(T)$. We then say $(X,S_X,F_X,V_X)$ is a $(\tilde{J},S)$-embeddable quadruple. Let $\mathcal{B}$ be the collection $(\tilde{J},S)$-embeddable groups with more than one noncentral chief factor in the Frattini subgroup. Let $\mathcal{T} = \{\Omega_1 Z(J(O_2(X)))| X \in \mathcal{B}\}$.

Corollary 2. Let $\mathcal{S} = \{X|X$ is an S-embeddable group and X has more than one noncentral chief factor within $O_2(X)\}$.

Then there exist nonidentity characteristic subgroups $S_1$ and $S_2$ of S such that

(1) $S_1$ is characteristic in $\tilde{J}(S)$.

(2) $S_2 \leq \Omega_1(Z(S))$.

(3) If $X \in \mathcal{S}$, then either $S_1 \triangleleft G$ or $S_2 \trianglelefteq G$.

Proof. Let $A = \mathrm{Aut}(S)$ and $\Lambda = \mathrm{Aut}(T)$. If $\mathcal{S}$ is empty then Corollary 2 holds with $S_1 = \tilde{J}(S)$ and $S_2 = \Omega_1(Z(S))$. Suppose $\mathcal{S}$ is nonempty and let $(X_1, S_1, F_1, F_1)$ be a $(\tilde{J}, S)$-embeddable quadruple with $F_1$ of maximal possible order such that $X_1 \in \mathcal{S}$. Now let $S_1 = \langle F_1{}^{\Lambda}\rangle$ and $S_2 = \langle [T, V_1]^A\rangle \cap \Omega_1(Z(S))$. Clearly $S_1 \neq 1$. Also $\langle [T, V_1]^A\rangle$ is a nonidentity normal subgroup of S, so $S_2 \neq 1$.

Notice that $\Lambda$ permutes the elements of $\mathcal{F}$. Let $\mathcal{F}_1$ be the orbit of $F_1$ and let $X_2 \in \mathcal{S}$. If $X_2 \in \mathcal{F}_1$, then $S_1 \triangleleft X_2$ by Theorem A. Suppose $X_2 \in \mathcal{F} \setminus \mathcal{F}_1$. It follows from lemma 8.1 that $O_2(X_2) \cap O^2(X_2) \leq O_2(X_2) \cap \langle \tilde{J}(S)^{X_2}\rangle = \tilde{J}(S) \cap O_2(X_2)$. If $V_1 \nleq X_2$, then $[O^2(X_2), O_2(X_2)] \leq F$ since $[V_1, O_2(X_2) \cap O^2(X^2)] \leq [V_1, \tilde{J}(S)] \leq \Omega_1(Z(T)) \leq F$. Then $[O^2(X_2), O_2(X_2)] \leq [O^2(X_2), F] \leq [J(X_2), F] = V$ and $X_2$ has exactly one noncentral chief factor within $O_2(X)$, which gives a contradiction. Hence $V_1 \leq X_2$. Now $[T, V_1] \leq Z(J(X_2))$ by Theorem B. Hence $S_2 \leq Z(J(X_2))$. Then $S_2 \triangleleft X_2$ since $X_2 = J(X_2)N_{X_2}(\tilde{J}(S))$.

If $X_2 \notin \mathcal{S}$, then $X_2$ clearly normalizes either $X_1$ or $X_2$.

# Chapter III. Pushing Up $L_3(2^n)$

## 1. The Basic Configuration.

Let G be a finite group with $F^*(G) = O_2(G) = Q$, suppose $\mathcal{L} \triangleleft G$ such that $Q \leq \mathcal{L}$ and $\mathcal{L}/Q$ has no partial complement in $G/Q$. Let $G/\mathcal{L} \cong SL_3(q)$ with $q = 2^n$. Define $\hat{G} = G/Q$ and $T \in Syl_2(G)$. Of course $|\mathcal{L} : Q|$ is odd by the Frattini argument. $M_1$ and $M_2$ are maximal 2-locals containing T and $\hat{M}_1, \hat{M}_2$ are the parabolic subgroups of $\hat{G}$ containing $\hat{T}$ and let $P \leq T$ such that $P = O_2(M_1)$. Let $E = \Omega_1 Z(Q)$, $V = [G,E]$ and $\Lambda \leq \mathrm{Aut}\ P$ such that $\Lambda$ contains the action of $N_G(P)$ on P.

The main purpose of this chapter is to prove the following result:

<u>Theorem 1</u>. If $C_G(\Omega_1 Z(T)) \leq M_1$ and $J(P) \neq J(Q)$, then

<u>either</u> (i) $[G,E^{\alpha}] \leq E$ for each $\alpha \in \Lambda$

<u>or</u> (ii) $V = [O^2(G),Q]$ and there exists $\alpha \in \Lambda$ such that $V^{\alpha} \not\leq Q$.

In the rest of this chapter assume $C_G(\Omega_1(Z(T))) \leq M_1$ and $J(P) \neq J(Q)$. We make use of the following lemma.

<u>Lemma 1.1.</u>

$G/C_G(E) \cong SL_3(q)$ and $E/C_E(G)$ is a natural $SL_3(q)$ module for $G/C_G(E)$. There exists $A \in \mathcal{A}(P)$ such that $\bar{A}$ is a root group of $\bar{G}$.

<u>Proof.</u> We use the notation and results of Aschbacher [5]. There exists $A \in \mathcal{A}(P) \setminus \mathcal{A}(Q)$ such that $AC(E)/C(E) \in \mathcal{P}^*(\bar{G},E)$. But $\langle A^{M_1}\rangle Q = P$, so if $B = \langle A^g | g \in G \text{ and } A^g \leq P\rangle$, then QB contains P and $N_G(B) \leq M_1$. Then $\langle N_G(B), C_G(E \cap Z(T))\rangle \neq G$. Then lemma 1.1 follows from the factorization

theorem of Aschbacher.

Let $H = C_G(E)$ and define $\overline{G} = G/H$. Choose $L \leq M_1$ and $K \leq M_2$ such that $\overline{L} = O^{2'}(\overline{M}_1)$ and $\overline{K} = O^{2'}(\overline{M}_2)$ with $T \leq L \cap K$ such that $(L \cap H)/Q$ has no partial complement in $L/Q$ and $(K \cap H)/Q$ has no partial complement in $K/Q$. Let $Z = C_V(T)$ with $D = C_V(G)$ and $M = O_2(K)$, $F = \Omega_1 Z(M)$ and $U = [K,V]$. Of course $P = O_2(L)$ with $\overline{M} = O_2(\overline{K})$ and $\overline{L}/\overline{P} \cong \overline{K}/\overline{M} \cong L_2(q)$. Also $H$ has no partial complement in $G$.

## 2. Properties of E and $\mathcal{A}(P)$.

Lemma 2.1. $E = V \cdot \Omega_1(Z(G))$ and $V/D$ is a natural module for $\overline{G}$ and $[G,V] = V$.

Proof. Lemma 2.1 follows directly from lemma 1.1.

Lemma 2.2. If $A \in \mathcal{A}(P)$, then

(i) If $A \not\leq Q$, then $\overline{A} = \overline{P}$ or $\overline{A}$ is a root group of $\overline{G}$.

(ii) $(A \cap Q)V \in \mathcal{A}(P)$.

(iii) $\mathcal{A}(Q) \subset \mathcal{A}(P)$.

Proof. If $\overline{A}$ is not contained in a root group, then $|V : A \cap V| \geq |V/D : C_{V/D}(A)| = q^2$ and $|A| = q^2 \cdot |A \cap Q| \leq |(A \cap Q|V|$. But $A \in \mathcal{A}(P)$ and $(A \cap Q)V$ is elementary abelian so $|A| = |(A \cap Q)V|$ and $(A \cap Q)V \in \mathcal{A}(P)$ with $|A| = q^2 \cdot |A \cap Q|$ and $\overline{A} = \overline{P}$.

If $\overline{A}$ is contained in a root group, then $|V : A \cap V| \geq |V/D : C_{V/D}(A)| = q$ and $|A : A \cap Q| \leq q$. Now $A \in \mathcal{A}(P)$ and $(A \cap Q)V$ is elementary abelian so $|A| = |(A \cap Q)V|$ and $(A \cap Q)V \in \mathcal{A}(P)$, with $|A : A \cap Q| = q$. Then $\overline{A}$ is a root group.

Lemma 2.3. If $R \leq \overline{P}$ and $R$ is a root group of $\overline{G}$, then there exists $A \in \mathcal{Q}(P)$ such that $\overline{A} = R$.

Proof. By lemma 1.1, there exists $B \in \mathcal{Q}(P)$ such that $\overline{B}$ is a root group of $\overline{G}$. $\overline{L}$ acts transitively on the root subgroups of $\overline{P}$, so lemma 2.3 follows.

Lemma 2.4. $C_V(g)/D = C_{V/D}(g)$ for each $g \in G$.

Proof. If $g \in P$, then there exists some root subgroup $\overline{R}$, of $\overline{P}$ such that $\overline{g} \in \overline{R}$. By lemma 2.3, there exists $A \in \mathcal{Q}(P)$ such that $\overline{A} = R$. Now $|V : C_V(A)| = |A : A \cap Q| = q$ by lemma 2.2. Then $|V : C_V(g)| \leq q$ since $g \in \overline{A}$. If $\overline{g} \neq 1$, then $|V/D : C_{V/D}(g)| = q$ by lemma 2.1. Then $C_V(g)/D \leq C_{V/D}(g)$ and $|V : C_V(g)| = q$. It follows that $C_V(g)/D = C_{V/D}(g)$ for each $g \in P$.

$\overline{P} \cap \overline{M} \neq 1$ and $\overline{K}$ acts transitively on $\overline{M}^{\#}$. Hence $C_V(g)/D = C_{V/D}(g)$ for each $g \in M$. We have $T = \langle P, M\rangle$, so

$$Z/D = (C_V(P)/D) \cap (C_V(M)/D) = C_{V/D}(P) \cap C_{V/D}(M) = C_{V/D}(T).$$

Then $|Z : D| = q$.

If $g \in T$ such that $g \notin P \cup M$, then $|C_{V/D}(g)| = q$. Now $|Z : D| = q$ and $Z \leq C_V(g)$, so $C_{V/D}(g) = C_V(g)/D$. Therefore lemma 2.4 holds for each $g \in G$ such that $\overline{g} \in \overline{T}$. By conjugation, 2.4 holds for each $g \in G$ such that $\overline{g}$ is a 2-element.

If $g \in G$ such that $\overline{g}$ is of odd order, then $C_V(g)/D = C_{V/D}(g)$ holds trivially. If $g \in G$ such that $\overline{g}$ is of even order and $\overline{g}$ is not a

2-element, then $C_{V/D}(g) = 1$. Hence $C_V(g) = D$ and $C_V(g)/D = C_{V/D}(g)$.

<u>Lemma 2.5.</u> $[K,U] = U$

Proof. If $q \neq 2$, then $O^2(\bar{K}) = \bar{K}$. Then

$$[U,K] = [V,K,K] = [V,O^2(K),O^2(K)] = [V,O^2(K)] = [V,K] = U.$$

Suppose $q = 2$ and let $g \in P \setminus M$. Then $V = [V,O^2(K)] \cdot C_V(g)$ by lemma 2.4. Now $[V,O^2(K)] \trianglelefteq K$ and $[g,V] \leq [V,O^2(K)]$. Then

$$U = [V,K] = [V,\langle g,O^2(K)\rangle] = [V,O^2(K)].$$

Now $[U,O^2(K)] = [[V,O^2(K)],O^2(K)] = [V,O^2(K)] = U$ so $[U,K] = U$.

<u>Lemma 2.6.</u> $U = C_V(M)$ and $F = U \cdot \Omega_1 Z(G)$.

<u>Proof.</u> Let $g \in G$ such that $G = \langle K,K^g\rangle$ and $P \leq K^g$. Now

$$V = [G,V] = [\langle K,K^g\rangle,V] = [K,V][K^g,V] = UU^g.$$

Then $|V:U| = |V:U^g| = |U:U \cap U^g|$.

$U^g \cap D = (U \cap D)^g = U \cap D$. We know that $[P,V] \leq U \cap U^g$ since $P \leq K \cap K^g$. Then $|U \cap U^g : U \cap D| \geq |[P,V]:[P,V] \cap D| = q$ by lemma 2.1. Of course $|(UD/_D) \cap (U^gD/_D)| = q$, so $|U \cap U^g| = q \cdot |U \cap D|$. Then $|V:U| = |U:U \cap U^g| = \frac{|U|}{q \cdot |U \cap D|} = q$ since $|U:U \cap D| = q^2$.

$UD/_D = C_{V/D}(M)$ by lemma 2.1 and $C_{V/D}(M) = C_V(M)/_D$ by lemma 2.4. Then $UD = C_V(M)$. But $|V/_D : UD/_D| = q$ and $|V:U| = q$ so $D \leq U$. Then $U = C_V(M)$. $F = U \cdot \Omega_1 Z(G)$ follows from lemma 2.1.

Lemma 2.7. Let $g \in P \setminus M$ and $A \in \mathcal{A}(P)$. The following are true.

(i) $V = U \cdot C_V(g)$.

(ii) $D = 1$ if $q = 2$.

(iii) $Z = [g,U] \times D$.

(iv) $[P,V] = [P,U] = Z$.

(v) $[A,U] \in \{1,Z\}$.

(vi) $[A,V] \in \{1,Z\}$.

Proof. Part (i) follows from lemmas 2.1 and 2.4. Of course $[K,U] = U$ and $[MH,U] = 1$ with $U/D$ a natural $L_2(q)$ module for $\bar{K}/\bar{M}$. Now $\bar{P} \in Syl_2(\bar{K})$ and $Z = C_U(P)$, so we may apply lemmas 1.1, 1.3 and 1.4 of the first chapter, to conclude that $Z = [g,U] \times D$ and $[P,U] = Z$ with $D = 1$ if $q = 2$. Now $[P,V] = Z$ follows from (i). Therefore parts (ii), (iii) and (iv) hold.

Let $A \in \mathcal{A}(P)$. If $\bar{A} = \bar{P}$, then $[A,U] = [A,V] = Z$ by part (iv). If $\bar{A} \cap \bar{M} = 1$, then $P = AM$ by lemma 2.2. Then $[A,U] = [P,U] = Z$ and $[A,V] = Z$ follows from part (i). If $A \leq Q$, then $[A,V] = [A,U] = 1$. By 2.2, we may assume that $\bar{A} = \bar{M} \cap \bar{P}$. Then $[A,U] = 1$. There exists $g \in L$ such that $A^g \not\leq M$. Then $[A^g,V] \in \{1,Z\}$ by the above. $Z^g = Z$, hence $[A,V] \in \{1,Z\}$.

Lemma 2.8. Let $A \in \mathcal{A}(P)$, $B \in \mathcal{A}(Q)$, $g \in G$ and $\alpha \in \text{Aut } P$ such that $E^\alpha = Q$. Then

(a) $(A \cap Q^\alpha)E^\alpha \in \mathcal{A}(Q^\alpha)$.

(b) $(A \cap Q \cap Q^\alpha)\, EE^\alpha \in \mathcal{A}(Q) \cap \mathcal{A}(Q^\alpha)$.

(c) $E^{\alpha g} \cdot C_B(E^{\alpha g}) \in \mathcal{A}(Q) \cap \mathcal{A}(Q^{\alpha g})$.

(d) If $E^{\alpha^{-1}g} \leq B$ and $E^{\alpha g \alpha^{-1}} \leq Q$, then $E^{\alpha g \alpha^{-1}g} \cdot C_B(E^{\alpha g \alpha^{-1}g}) \in \mathcal{A}(Q)$.

Proof.

(a) $A^{\alpha^{-1}} \in \mathcal{A}(P)$, so $(A^{\alpha^{-1}} \cap Q)E \in \mathcal{A}(Q)$ by lemma 2.2. Then $(A \cap Q^{\alpha})E^{\alpha} \in \mathcal{A}(Q^{\alpha})$.

(b) $(A \cap Q^{\alpha})\, E^{\alpha} \in \mathcal{A}(Q^{\alpha})$ by part (a). Then $((A \cap Q^{\alpha})E^{\alpha} \cap Q)E \in \mathcal{A}(Q)$ by lemma 2.2. Notice that $(A \cap Q^{\alpha})E^{\alpha} \cap Q = (A \cap Q \cap Q^{\alpha})E^{\alpha}$, so $(A \cap Q \cap Q^{\alpha})EE^{\alpha} \in \mathcal{A}(Q)$ and $\mathcal{A}(Q) \subset \mathcal{A}(P)$. From $[E,E^{\alpha}] = 1$ we conclude $E \leq Q^{\alpha}$ and $(A \cap Q \cap Q^{\alpha})EE^{\alpha} \in \mathcal{A}(Q) \cap \mathcal{A}(Q^{\alpha})$.

(c) Now $B^{g^{-1}} \in \mathcal{A}(Q)$, so $E^{\alpha} \cdot C_{B^{g^{-1}}}(E^{\alpha}) \in \mathcal{A}(Q) \cap \mathcal{A}(Q^{\alpha})$ by part (b). Then $E^{\alpha g} \cdot C_B(E^{\alpha g}) \in \mathcal{A}(Q) \cap \mathcal{A}(Q^{\alpha g})$.

(d) Let $X = E^{\alpha g \alpha^{-1} g}$. If $E^{\alpha^{-1}g} \leq B$, then $[B^{g^{-1}\alpha},E] = [B,E^{\alpha^{-1}g}]^{g^{-1}\alpha} = 1$, $B^{g^{-1}\alpha} \leq Q$ and $B^{g^{-1}\alpha} \in \mathcal{A}(Q)$. Then $E^{\alpha g} \cdot C_{B^{g^{-1}\alpha}}(E^{\alpha g}) \in \mathcal{A}(Q)$ by part (c) and $X \cdot C_B(X) \in \mathcal{A}(Q^{\alpha^{-1}g})$. From $E^{\alpha g \alpha^{-1}} \leq Q$ it follows that $X \leq Q$. Now $B \leq Q$ and $X \cdot C_B(X) \leq Q$. Then $X \cdot C_B(X) \in \mathcal{A}(Q)$.

Lemma 2.9. Let $A \in \mathcal{A}(P)$, $g \in G$ and $\alpha \in \operatorname{Aut} P$.

(a) $[A,E^{\alpha}] \in \{1,Z^{\alpha}\}$.

(b) If $A \leq Q$, then $[A,E^{\alpha g}] \in \{1,Z^{\alpha g}\}$.

(c) If $A \leq Q^{\alpha^{-1}g}$ and $E^{\alpha} \leq Q$, then $[A,E^{\alpha g \alpha^{-1} g}] \in \{1,Z^{\alpha g \alpha^{-1} g}\}$.

Proof.

(a) With $A^{\alpha^{-1}} \in \mathcal{A}(P)$, we see that $[A^{\alpha^{-1}}, E] \in \{1, Z\}$ follows from lemma 2.7 (vi). Then $[A, E^{\alpha}] \in \{1, Z^{\alpha}\}$.

(b) If $A \in \mathcal{A}(Q)$, then $[A^{g^{-1}}, E^{\alpha}] \in \{1, Z^{\alpha}\}$ by part (a). Then $[A, E^{\alpha g}] \in \{1, Z^{\alpha g}\}$.

(c) If $A \in Q^{\alpha^{-1} g}$, then $A^{g^{-1}} \leq P$ and $A^{g^{-1}\alpha} \leq Q$. Hence $[A^{g^{-1}\alpha}, E^{\alpha g}] \in \{1, Z^{\alpha g}\}$ by part (b). Now $E^{\alpha g} \leq Q \leq P$ and $[A, E^{\alpha g \alpha^{-1} g}] \in \{1, Z^{\alpha g \alpha^{-1} g}\}$.

Lemma 2.10. If $g, h \in G$ and $\alpha \in \operatorname{Aut} P$ such that $F^{\alpha} \leq Q$ with $x \in F^{\alpha g} \setminus \Omega_1(Z(P))^g$ and $y \in F^{\alpha h} \setminus \Omega_1(Z(P))^h$, then $[x, y] = 1$ if and only if $[F^{\alpha g}, F^{\alpha h}] = 1$.

Proof. From $x^{g^{-1}\alpha^{-1}} \in F \setminus \Omega_1(Z(P))$ and $y^{h^{-1}\alpha^{-1}} \in F \setminus \Omega_1(Z(P))$ it follows that $C_P(x^{g^{-1}\alpha^{-1}}) = C_P(F) = C_P(y^{h^{-1}\alpha^{-1}})$ and $C_P(x^{g^{-1}}) = C_P(F^{\alpha}) = C_P(y^{h-1})$. In particular, $C_Q(x^{g^{-1}}) = C_Q(F^{\alpha}) = C_Q(y^{h^{-1}})$, $C_Q(x) = C_Q(F^{\alpha g})$ and $C_Q(y) = C_Q(F^{\alpha h})$. Lemma 2.10 follows.

Lemma 2.11. If $g, h \in G$ and $\alpha \in \operatorname{Aut} P$ such that $F^{\alpha} \leq Q$ and $[F^{\alpha g}, F^{\alpha h}] \neq 1$, then $|F^{\alpha g} : C_{F^{\alpha g}}(F^{\alpha h})| = q$.

Proof. 2.11 follows directly from 2.10.

Lemma 2.12. If $X \leq P$ and $[X, E] \leq D$, then $[X, E] = 1$.

Proof. Lemma 2.12 is a direct consequence of lemmas 2.1 and 2.4.

Lemma 2.13. If $A \in \mathcal{A}(P) \setminus \mathcal{A}(Q)$, then $\{[g,E] \mid g \in P\} = \{[g,E] \mid g \in A\}$.

Proof. Let $g \in P$. By lemma 2.2, there exists $h \in L$ such that $\overline{g^h} \in \bar{A}$. With $[g,V] \leq Z \leq Z(L)$ we find that $[g^h,V] = [g,V]^h = [g,V]$. Lemma 2.13 follows.

## 3. The Proof of Theorem D.

Theorem D. If $\alpha \in \mathrm{Aut}(P)$ such that $E^\alpha \leq Q$ and $E \neq E^\alpha$, then $Z^\alpha \leq Z(G)$.

The purpose of this section is to prove Theorem D. Suppose $\alpha \in \mathrm{Aut}\ P$ such that $E^\alpha \leq Q$ and $E^\alpha \neq E$ such that $Z^\alpha \not\leq Z(G)$. Let $A \in \mathcal{A}(P) \setminus \mathcal{A}(Q)$ with $A \cap M = A \cap Q$. Let $g \in K \setminus N_G(HT)$ and define $B_0 = (A \cap Q)E$ and $B = (B_0 \cap Q^\alpha)E^\alpha$.

Lemma 3.1. (i) $B_0 \in \mathcal{A}(Q)$.

(ii) $B \in \mathcal{A}(Q)$.

(iii) $E^{\alpha g} \cdot C_B(E^{\alpha g}) \in \mathcal{A}(Q)$.

(iv) $[A,B] \leq ZZ^\alpha \leq C_B(E^{\alpha g})$.

Proof. Parts (i) and (ii) follow from lemma 2.8. Part (iii) follows from (ii) and lemma 2.8. Notice that

$$[A,B] \leq [A,EE^\alpha] \leq ZZ^\alpha \leq \Omega_1 Z(P) \leq C_B(E^{\alpha g}),$$

so that (iv) holds.

Lemma 3.2. $\langle E^{\alpha g A} \rangle$ is elementary abelian.

Proof. Suppose lemma 3.2 is false. $E^{\alpha g} \triangleleft Q$, so there exists $x \in A \setminus Q$

such that $[E^{\alpha g}, E^{\alpha gx}] \neq 1$. Now $E^{\alpha g} \cdot C_B(E^{\alpha g}) \in \mathcal{A}(Q)$ by lemma 3.1. But $[A,B] \leq C_B(E^{\alpha g})$ by 3.1, so A acts on B and $C_B(E^{\alpha g})$. Now $E^{\alpha gx} \cdot C_B(E^{\alpha gx}) \in \mathcal{A}(Q)$ by lemma 2.8 and $C_B(E^{\alpha g}) = C_B(E^{\alpha gx})$, so $Z^{\alpha g} = [E^{\alpha g}, E^{\alpha gx} \cdot C_B(E^{\alpha g})] = [E^{\alpha g}, E^{\alpha gx}] = [E^{\alpha g} \cdot C_B(E^{\alpha g}), E^{\alpha gx}] = Z^{\alpha gx}$ by lemma 2.9. From $Z^{\alpha g} \leq \Omega_1 Z(P)^g$ and $Z^{\alpha gx} = Z^{\alpha g}$ it follows that $Z^{\alpha g} \leq Z(G)$ and $Z^{\alpha} \leq Z(G)$, a contradiction.

Lemma 3.3. $E^{\alpha g} \leq B$.

Proof. Suppose $E^{\alpha g} \nleq B$. Since $B \in \mathcal{A}(Q)$, we have that $[B, E^{\alpha g}] = Z^{\alpha g}$ by lemma 2.9. Now A acts on B and $C_B(E^{\alpha g})$ by lemma 3.1, so $C_B(E^{\alpha g}) = C_B(\langle E^{\alpha gA} \rangle)$. Notice that $\langle E^{\alpha gA} \rangle$ is elementary abelian by lemma 3.2. Then $\langle E^{\alpha gA} \rangle \cdot C_B(E^{\alpha g})$ is elementary abelian. $E^{\alpha g} \cdot C_B(E^{\alpha g}) \in \mathcal{A}(Q)$ by lemma 3.1 and $E^{\alpha g} \cdot C_B(E^{\alpha g}) = \langle E^{\alpha gA} \rangle \cdot C_B(E^{\alpha g})$. Now A acts on $E^{\alpha g} \cdot C_B(E^{\alpha g})$. Then A acts on $Z^{\alpha g}$ since $[B, E^{\alpha g} \cdot C_B(E^{\alpha g})] = [B, E^{\alpha g}] = Z^{\alpha g}$. From $Z^{\alpha g} \leq \Omega_1 Z(P)^g$, it follows that $Z^{\alpha g} \leq Z(G)$ and $Z^{\alpha} \leq Z(G)$, which gives a contradiction.

Lemma 3.4. $EE^{\alpha} \lhd K$.

Proof. $[A^{g^{-1}}, E^{\alpha}] = [A, E^{\alpha g}]^{g^{-1}} \leq [A,B]^{g^{-1}} \leq (ZZ^{\alpha})^{g^{-1}} \leq \Omega_1 Z(P)^{g^{-1}} \leq E$.

Hence $A^{g^{-1}} \leq N_K(EE^{\alpha})$. With $EE^{\alpha} \lhd P$ and $\langle P, A^{g^{-1}} \rangle = K$, we get that $EE^{\alpha} \lhd K$.

Lemma 3.5. $J(Q)^{\alpha} = J(Q)$ and $E^{\alpha} \leq Z(J(Q))$.

Proof. $EE^{\alpha} \lhd K$ by lemma 3.4. If $E^{\alpha} \nleq Z(J(Q))$, then $1 < [J(Q), E^{\alpha}] = [J(Q), EE^{\alpha}] \lhd K$. If $E^{\alpha} \nleq Z(J(Q))$, then $[J(Q), E^{\alpha}] = Z^{\alpha}$ by lemma 2.9. Then $Z^{\alpha} \lhd K$. Notice that $Z^{\alpha} \leq \Omega_1(Z(P) = \Omega_1 Z(L)$. Then $Z^{\alpha} \leq Z(G)$, which gives a contradiction. Therefore $E^\alpha \leq Z(J(Q))$. Then $[J(Q),E^\alpha] = 1$ and $J(Q) \leq Q^\alpha$. Recall that $\mathcal{A}(Q) \subset \mathcal{A}(P)$. This yields $J(Q) \leq J(Q)^\alpha$. Now $J(Q) = J(Q)^\alpha$ since $|J(Q)| = |J(Q)^\alpha|$.

<u>Lemma 3.6.</u> (i) $\{[x,E^\alpha] | x \in P\} = \{[x,E] | x \in P\}$.

(ii) $Z^\alpha = Z$.

<u>Proof.</u> $B_0 \in \mathcal{A}(Q)$ by lemma 2.1, but $E^\alpha \geq Z(J(Q))$ by lemma 3.5. This yields that $E^\alpha \leq B_0$. Let $x \in A$. Then $[x,E^\alpha] \leq [x,B_0] = [x,E]$. We have that $E \leq Z(J(Q)) = Z(J(Q))^\alpha$ by lemma 3.5 and $(A \cap Q^\alpha)E^\alpha \in \mathcal{A}(Q^\alpha)$ by lemma 2.8, so $E \leq (A \cap Q^\alpha)E^\alpha$ and $[x,E] \leq [x,(A \cap Q^\alpha)E^\alpha] = [x,E^\alpha]$. Therefore $[x,E^\alpha] = [x,E]$ for each $x \in A$. But $J(Q)^\alpha = J(Q)$, so $A \not\leq Q^\alpha$. Then $\{[x,E^\alpha] | x \in P\} = \{[x,E^\alpha] | x \in A\} = \{[x,E] | x \in A\} = \{[x,E] | x \in P\}$ by 2.11. In particular, $Z = [A,E] = [A,E^\alpha] = Z^\alpha$ by lemma 2.9.

<u>Lemma 3.7.</u> $1 < [Q,E^\alpha] \leq D$.

<u>Proof.</u> $E \neq E^\alpha$ so $[Q,E^\alpha] \neq 1$. Then $[Q,E^\alpha] \leq [P,E^\alpha] = Z^\alpha = Z$ by lemma 3.6. But $EE^\alpha \lhd K$ by lemma 3.4 and $[Q,E^\alpha] = [Q,EE^\alpha] \lhd K$. Then $[Q,E^\alpha] \leq D$.

We now establish a contradiction which proves Theorem D. We know that $1 < [Q,E^\alpha] \leq D$ from lemma 3.7, so there exists $y \in Q$ such that $1 < [y,E^\alpha] \leq D$. It follows from lemma 3.6 that there exists $x \in P$ with $1 < [x,E] \leq D$. This contradicts lemma 2.7 (iii) and theorem 3 is proved.

## 4. The Normalizer of $E^{\alpha}$.

Lemma 4.1. If $\langle E^{\Lambda}\rangle \leq Q$ and $\alpha \in \Lambda$, then $E^{\alpha} \lhd N_G(P)$ and $Q^{\alpha} \lhd N_G(P)$.

Proof. We may assume that $E^{\alpha} \neq E$. If $g \in N_G(P)$ such that $E^{\alpha g} \neq E^{\alpha}$, then $E^{\alpha g \alpha^{-1}} \neq E$. Now $E^{\alpha g \alpha^{-1}} \leq Q$ since $\langle E^{\Lambda}\rangle \leq Q$ and $\Lambda$ contains the action of $N_G(P)$ on P. Then $Z^{\alpha g \alpha^{-1}} \leq Z(G)$ by theorem D. Also $Z^{\alpha} \leq Z(G)$ and $Z^{\alpha g \alpha^{-1}} = (Z^{\alpha g})^{\alpha^{-1}} = Z^{\alpha\alpha^{-1}} = Z \nleq Z(G)$, which yields a contradiction. Therefore $E^{\alpha} \lhd N_G(P)$. Then $Q^{\alpha} = C_P(E)^{\alpha} = C_P(E^{\alpha}) \lhd N_G(P)$.

## 5. The action of $N_G(P)$ on $E^{\beta}$.

Let $\theta : N_G(P) \to \Lambda$ be the natural homomorphism. Suppose $\langle E^{\Lambda}\rangle \leq Q$ and let $\beta \in \Lambda$. Now $E^{\beta} < N_G(P)$ and $Q^{\beta} \lhd N_G(P)$ by lemma 4.1. Then $N_G(P)$ permutes the elements of S, where

$$S = \left\{C_{E^{\beta}}(x) \mid x \in P \setminus Q^{\beta}\right\}$$

$$= \left\{(C_E(x))^{\beta} \mid x \in P \setminus Q\right\}$$

$$= \left\{F^{g\beta} \mid g \in L\right\} .$$

Define $W = F^{\beta}/\Omega_1(Z(P))$ and let $\psi : E^{\beta} \to W$ be the natural homomorphism. $E/\Omega_1(Z(P))$ is a natural $L_2(q)$ module for $L/P(L \cap H)$, so a $GF(q)$ vector space structure is induced on W such that $\psi(S)$ is the collection of one dimensional subspaces of W. Now $N_G(P)$ acts on $E^{\beta}$ and $\Omega_1(Z(P))$, so $N_G(P)$ acts on W. Of course $[P,E^{\beta}] = Z^{\beta} \leq \Omega_1(Z(P))$, so $[P,W] = 1$. We use $\theta(L)\cdot\beta$ to mean the coset of $\theta(L)$ in $\Lambda$ that contains $\beta$.

We recall some basic facts about semilinear transformations. Let X be a finite dimensional vector space over GF(q) with dimension greater than one. An additive function $\lambda : X \to X$ is called semilinear if there exists an automorphism $\varphi$ of GF(q) such that $\lambda(av) = a^{\varphi}\lambda(v)$ for each scalar a and each vector v. An additive function $\mu : X \to X$ is semilinear if it permutes the subspaces of X. Define $\Gamma L(X)$ to be the group of semilinear transformations on X. The group $\Gamma L(X)$ contains the projective linear group PGL(X) as a normal subgroup. $\Gamma L(X)$ is the semidirect product of PGL(X) and a subgroup A, where A is isomorphic to the automorphism group of GF(q). A treatment of semilinear transformations can be found in Artin [1].

Lemma 5.1. The action of $N_G(P)$ on W is semilinear.

Proof. $N_G(P)$ permutes the elements of S, so $N_G(P)$ permutes the subspaces of W. Then $N_G(P)$ acts semilinearly on W.

Lemma 5.2. If $[L,W] \neq 1$, then

either (i) $[P(L \cap H),W] = 1$ and W is a natural $L_2(q)$ module for $L/P(L \cap H)$.

or (ii) $q = 2$ and $|L : C_L(W)| = 2$.

Proof. Let $[L,W] \neq 1$. Suppose $\overline{C_L(W)} = \overline{L}$. Now $(L \cap H)/Q$ has no partial complement in $L/Q$, so $[L,W] = 1$, which yields a contradiction.

Suppose that $C_L(W) \leq P\cdot(L \cap H)$. Since $L/P(L \cap H) \cong L_2(q)$, L is semilinear on W and $(L \cap H)/Q$ has no partial complement in $L/Q$, it follows that $[P(L \cap H),W] = 1$ and W is a natural $L_2(q)$ module for

$L/P(L \cap H)$.

$C_L(W) \lhd L$ with $[P,W] = 1$ and $L/P(L \cap H) \cong L_2(q)$. If $q \neq 2$, then either $\overline{C_L(W)} = \overline{L}$ or $\overline{C_L(W)} = \overline{P}$, and lemma 5.2 holds by the above arguments.

We may assume that $q = 2$ and $\overline{C_L(W)} = O^2(\overline{L})$. The group of semilinear transformations on W is isomorphic to $S_3$. Since $(L \cap H)/Q$ has no partial complement in $L/Q$, lemma 5.2 follows.

Lemma 5.3. There exists $\alpha \in \theta(L) \cdot \beta$ such that $[T,F^{\alpha}] \leq F \leq \Omega_1(Z(P))$ and $[T,E^{\alpha}] \leq F^{\alpha}$.

Proof. L is transitive on the elements of $S^{\beta^{-1}}$ so lemma 5.3 follows from lemma 5.2.

Lemma 5.4. If $\alpha \in \Lambda$ such that $F^{\alpha} \leq E^{\beta}$, then there exists $g \in L$ such that $F^{\alpha} = F^{g\beta}$.

Proof. Let $\alpha \in \Lambda$ such that $F^{\alpha} \leq E^{\beta}$ and let $\gamma = \alpha\beta^{-1}$. Notice that $F^{\gamma} \leq E$ and $|P : C_P(F^{\gamma})| = |P : C_P(F)| = q$. There exists $h \in P \setminus Q$ such that $F^{\gamma} \leq C_E(h)$. Now $F^{\gamma} = C_E(h)$ since $|F^{\gamma} : \Omega_1(Z(P))| = q$. There exists $g \in L$ such that $F^{\gamma} = F^{g}$.

Lemma 5.4 is applied several times in the next section.

## 6. Commuting Conjugates of $E^{\beta}$.

Theorem 6. If $\langle E^{\Lambda} \rangle \leq Q$, $\beta \in \Lambda$, and x is an involution of K, then $[E^{\beta}, E^{\beta x}] = 1$.

The purpose of this section is to prove theorem 6. Suppose that

$\langle E^{\Lambda}\rangle \leq Q$ and $\beta \in \Lambda$. Let x be an involution of K such that $[E^{\beta}, E^{\beta x}] \neq 1$. Lemma 4.1 implies that $x \notin M$. Let $\alpha$ be as in lemma 5.3 and let $\theta : N_G(P) \to \Lambda$ be the natural map. Let $W = \langle E^{\alpha K}\rangle$ and $R = \langle F^{\alpha K}\rangle$. Of course $E^{\alpha} = E^{\beta} \neq E$, so $E^{\alpha} \lhd N_G(P)$ by 4.1. Let $s \in K \setminus N_K(HT)$ such that $s^2 = 1$. Now $F = \Omega_1(Z(P))\Omega_1(Z(P))s \leq F^{\alpha}F^{\alpha s}$ and $F \leq R \leq W$. Define

$$I = \{\lambda \in \Lambda | E^{\delta\lambda} = E^{\delta} \text{ and } [Z^{\delta}, \lambda] = 1 \text{ for each } \delta \in \Lambda\}.$$

Notice that $\theta(L) \leq I \lhd \Lambda$ and $Q^{\delta\lambda} = C_P(E^{\delta\lambda}) = C_P(E^{\delta}) = Q^{\delta}$ for each $\delta \in \Lambda$ and $\lambda \in I$.

<u>Lemma 6.1</u>. $[M,W] \leq R$ and $[M,R] \leq F$.

<u>Proof</u>. $[M,E^{\alpha}] \leq F^{\alpha}$ by the choice of $\alpha$, so $[M,W] \leq R$. Also $[M,R] \leq F$ since $[M,F^{\alpha}] \leq F$.

<u>Lemma 6.2</u>. If $\delta, \varepsilon \in \Lambda$ such that $[F^{\delta x}, F^{\delta}] \neq 1$, then $T = F^{\delta x\varepsilon^{-1}} \cdot M$.

<u>Proof</u>. $|F^{\delta x} : C_{F^{\delta x}}(F^{\varepsilon})| = q$ by lemma 2.11, so $|F^{\delta x\varepsilon^{-1}} : F^{\delta x\varepsilon^{-1}} \cap M| = |F^{\delta x\varepsilon^{-1}} : F^{\delta x\varepsilon^{-1}} \cap C_P(F)| = q$. Then $T = F^{\delta x\varepsilon^{-1}} \cdot M$.

<u>Lemma 6.3</u>. $\Phi(W) \leq Z^{\alpha}$ and $[Q,W] \leq Z^{\alpha}$.

<u>Proof</u>. $[P,E^{\alpha}] = Z^{\alpha}$, so $[Q,E^{\alpha}] \leq Z^{\alpha}$. It follows that $[Q,W] \leq Z^{\alpha}$ and $[W,W] \leq Z^{\alpha}$. We conclude that $\Phi(W) \leq Z^{\alpha}$ since $\Phi(E) = 1$.

<u>Lemma 6.4</u>. Let $\delta, \varepsilon \in I \cdot \alpha$ and $g \in F^{\delta x\varepsilon^{-1}}$. If h is an involution of W, then $[h^g, h] = 1$.

Proof. Let $Y = \langle g,h\rangle$ and suppose $[h^g, h] \neq 1$. Then $[g,h] \neq 1$, so Y is dihedral since $g^2 = h^2 = 1$. Now $\Phi(Y \cap W) \leq \Phi(W) \leq Z^{\alpha}$ by lemma 6.3 and $\Phi(Y \cap W^{\varepsilon^{-1}}) \leq \Phi(W)^{\varepsilon^{-1}} \leq (Z^{\alpha})^{\varepsilon^{-1}} = Z$. Notice that $g \in Y \cap W^{\varepsilon^{-1}} \triangleleft Y$ and $h \in Y \cap W \trianglelefteq Y$. We are given that $[h,h^g] \neq 1$, so $g \notin Y \cap W$ and $h \notin Y \cap W^{\varepsilon^{-1}}$. Then $Y/Y \cap W \cong Y/Y \cap W^{\varepsilon^{-1}} \cong \mathbb{Z}_2$. Now $h \notin Z(Y \cap W)$ since $[h,h^g] \neq 1$. Notice that $\Phi(\Phi(Y \cap W)) = 1$. It follows that $Y \cap W \cong D_8$ and $Y \cong D_{16}$. Also $Z(Y) = \Phi(Y \cap W) \leq Z^{\alpha}$. We have $|Y : Y \cap W^{\varepsilon^{-1}}| = 2$ with $g^2 = 1$ and $g \notin Z(Y)$, so $Y \cap W^{\varepsilon^{-1}} \cong D_8$. Therefore $Z(Y) = \Phi(Y \cap W^{\varepsilon^{-1}}) \leq Z$ and $Z(Y) \leq Z^{\alpha} \cap Z \leq Z(G) \cap Z = D$ by theorem D. Let $y$ be an element of order 4 in $Y \cap W^{\varepsilon^{-1}}$. Of course $g^{\varepsilon}$ and $y^{\varepsilon}$ are contained in $W$. Also $g^{\varepsilon x^{-1}}$ and $y^{\varepsilon x^{-1}}$ are in $W$. Now $F$ contains $g^{\varepsilon x^{-1}\delta^{-1}}$ and $P$ contains $y^{\varepsilon x^{-1}\delta^{-1}}$. It follows that $1 = [g^{\varepsilon x^{-1}\delta^{-1}}, y^{\varepsilon x\delta^{-1}}] \in [y^{\varepsilon x^{-1}\delta^{-1}},F] \cap D^{\varepsilon x^{-1}\delta^{-1}} = [y^{\varepsilon x^{-1}\delta^{-1}},U] \cap D$, contradicting lemma 2.7 (iii).

Lemma 6.5. $x$ may be chosen such that $[F^{\delta},F^{\delta x}] = 1$ for each $\delta \in I \cdot \alpha$.

Proof. $[E^{\alpha x},E^{\alpha}] \neq 1$, so there exists $\psi, \varphi \in I \cdot \alpha$ such that $[F^{\psi x},F^{\varphi}] \neq 1$. Let $Y = F^{\psi x \varphi^{-1}}$. Now $T = YM$ by lemma 6.2. There exists $a \in K$ such that $x \in T^a$. Now $x \in MY^a$ and $E^{\alpha} \triangleleft M$, so there exists $y \in Y$ such that $E^{\alpha x} = E^{\alpha(y^a)}$. Recall from lemma 6.4 that $[h^y,h] = 1$ for each involution $h$ of $W$. Then $[h^{(y^a)},h] = 1$ for each involution $h$ of $W$. Let $\delta \in I \cdot \alpha$ and choose $h \in F^{\delta} \setminus \Omega_1(Z(P))$. $[h^{(y^a)},h] = 1$ so $[F^{\delta(y^a)},F^{\delta}] = 1$ by lemma 2.10. Now lemma 6.5 holds with $x$ replaced by $y^a$. In the remainder of this section, choose $x$ as in lemma 6.5.

Lemma 6.6. If $\delta \in I\cdot\alpha$, then $C_{E^{\alpha x}}(F^{\delta}) = F^{\delta x}$ and $C_{E^{\alpha}}(F^{\delta x}) = F^{\delta}$.

Proof. Let $\delta \in I\cdot\alpha$. We have that $[F^{\delta x},F^{\delta}] = 1$ by lemma 6.5. Suppose $F^{\delta x} \neq C_{E^{\delta x}}(F^{\delta})$. There exists $g \in E^{\delta x} \setminus F^{\delta x}$ such that $[g,F^{\delta}] = 1$. There exists $\varepsilon \in I\cdot\alpha$ such that $g \in F^{\varepsilon x}$. Now $g \notin F^{\delta x}$, so $g \notin \Omega_1(Z(P))^x$. Hence $[F^{\delta},F^{\varepsilon x}] = 1$ by lemma 2.10. Notice that $E^{\alpha x} = F^{\delta x}F^{\varepsilon x}$ and $[F^{\delta},E^{\alpha x}] = 1$. Also $[F^{\delta x},F^{\varepsilon}] = [F^{\delta},F^{\varepsilon x}]^x = 1$. Now $[F^{\varepsilon x},F^{\varepsilon}] = 1$ by lemma 6.5, so $[E^{\alpha x},F^{\varepsilon}] = [F^{\delta x}F^{\varepsilon x},F^{\varepsilon}) = 1$. Now $[E^{\alpha x},E^{\alpha}] = [E^{\alpha x},F^{\delta},F^{\varepsilon}] = 1$, a contradiction. Therefore $F^{\delta x} = C_{E^{\alpha x}}(F^{\delta})$. We conclude the $F^{\delta} = C_{E^{\alpha}}(F^{\delta x})$ since $x^2 = 1$.

Lemma 6.7. $C_{E^{\alpha x}}(E^{\alpha}) = \Omega_1(Z(P))^x$.

Proof. Let $\delta \in I\cdot\alpha$ such that $E^{\alpha} = F^{\alpha}F^{\delta}$. Observe that $C_{E^{\alpha x}}(E^{\alpha}) = C_{E^{\alpha x}}(F^{\alpha}) \cap C_{E^{\alpha x}}(F^{\delta}) = F^{\alpha x} \cap F^{\delta x}$ by lemma 6.6. Now $F^{\alpha x} \cap F^{\delta x} = (F^{\alpha} \cap F^{\delta})^x = \Omega_1(Z(P))^x$ and 6.7 follows.

Lemma 6.8. $FF^{\alpha} \ntriangleleft K$.

Proof. If $FF^{\alpha} \triangleleft K$, then $C_W(FF^{\alpha}) \triangleleft K$. We conclude that $[FF^{\alpha},W] = 1$ since $E^{\alpha} \leq C_W(FF^{\alpha})$. This contradicts the choice of $\alpha$.

Lemma 6.9. $W \nleq Q^{\alpha^{-1}}$.

Proof. Let $\delta \in I\cdot\alpha$ such that $F^{\delta} \neq F^{\alpha}$. Then $[F^{\alpha x},F^{\delta}] \neq 1$ by lemma 6.6 and $T = F^{\alpha x\delta^{-1}}\cdot M$ by lemma 6.2. Now $Q^{\alpha^{-1}} = Q^{\delta^{-1}}$ by lemma 4.1. If $W \leq Q^{\alpha^{-1}}$, then $W \leq Q^{\delta^{-1}}$ and $[W,F^{\alpha x\delta^{-1}}] \leq [Q^{\delta^{-1}},F^{\alpha x\delta^{-1}}] \leq [Q,F^{\alpha x}]^{\delta^{-1}} \leq$

$(Z^{\alpha})^{\delta^{-1}} = Z \leq F$. But $[M,R] \leq F$ by lemma 6.1, so $[T,R] = [F^{\alpha x \delta^{-1}} \cdot M, R] \leq F$ and $FF^{\alpha} = R < K$, contradicting lemma 6.8.

Lemma 6.10. $E^{\alpha^2} = E$.

Proof. $[W, E^{\alpha^{-1}}] \leq [P, E^{\alpha^{-1}}] \cap [W,Q] \leq Z^{\alpha^{-1}} \cap Z^{\alpha} = (Z \cap Z^{\alpha^2})^{\alpha^{-1}}$. If $E^{\alpha^2} \neq E$, then $Z^{\alpha^2} \leq Z(G)$ by theorem D. Now $Z \cap Z(G) = D$, so $[W, E^{\alpha^{-1}}] \leq D^{\alpha^{-1}}$. Then $[W^{\alpha}, E] \leq D$ and it follows from lemma 2.12 that $W^{\alpha} \leq Q$ and $W \leq Q^{\alpha^{-1}}$, contradicting lemma 6.9.

Lemma 6.11. If $\delta \in I \cdot \alpha$, then there exists $\varepsilon \in I \cdot \alpha$, such that $F^{\varepsilon} = F^{\delta^{-1}}$.

Proof. $E^{\alpha^{-1}} = E^{\alpha}$ by lemma 6.10. Now $F^{\delta^{-1}} \leq E^{\delta^{-1}} = E^{\alpha^{-1}}$, so $F^{\delta^{-1}} \leq E^{\alpha}$. By lemma 5.4, there exists $\varepsilon \in I \cdot \alpha$ such that $F^{\varepsilon} = F^{\delta^{-1}}$.

Lemma 6.12. If $\delta, \varepsilon \in I \cdot \alpha$ such that $E^{\alpha} = F^{\delta} F^{\varepsilon}$, then $W = C_W(F^{\delta}) \cdot (C_W(F^{\varepsilon})$.

Proof. $|W : C_W(F^{\delta})| \leq q$, but $|E^{\alpha x} : E^{\alpha x} \cap C_W(F^{\delta})| = q$ by lemma 6.6. Then $W = E^{\alpha x} \cdot C_W(F^{\delta})$. We likewise conclude that $W = E^{\delta x} \cdot C_W(F^{\varepsilon})$ and observe that $E^{\alpha x} = F^{\delta x} \cdot F^{\varepsilon x} \leq C_W(F^{\delta}) \cdot C_W(F^{\varepsilon})$. It is now obvious that $W = C_W(F^{\delta}) \cdot C_W(F^{\varepsilon})$.

Lemma 6.13. There exist elements $\lambda$ and $\varphi$ of $I$ such that $F \notin \{F^{\lambda}, F^{\varphi}\}$ and $F^{\alpha^{-1}\lambda^{-1}} \neq F^{\alpha^{-1}\varphi^{-1}}$.

Proof. Suppose lemma 6.13 is false and define $Y = \{\delta \in \Lambda | F^{\delta} = F\}$. Let

$\lambda \in I \setminus Y$. If $\varphi \in I \setminus Y$, then $F^{\alpha^{-1}\lambda^{-1}} = F^{\alpha^{-1}\varphi^{-1}}$. Now $\alpha^{-1}\lambda^{-1}\varphi\alpha$ is in $Y$ and $\varphi$ is in $\lambda \cdot Y^{\alpha^{-1}}$. This shows that $I \subseteq Y \cup (\lambda \cdot Y^{\alpha^{-1}})$. Let $A = I \cap Y$. Now $|I : A| \leq 2$ since $I \trianglelefteq \Lambda$. We conclude that $|\theta(L) : \theta(L) \cap A| \geq 2$ and $|L : N_L(F)| \leq 2$, which yields a contradiction.

<u>Lemma 6.14.</u> There exist $\varepsilon, \delta \in I \cdot \alpha$ such that $F^{\alpha} \notin \{F^{\varepsilon}, F^{\delta}\}$ and $E^{\alpha} = F^{\varepsilon^{-1}} F^{\delta^{-1}}$.

<u>Proof.</u> Let $\varphi$ and $\lambda$ be as in lemma 6.13 and let $\varepsilon = \lambda\alpha$ and $\delta = \varphi\alpha$. Now $F \notin \{F^{\lambda}, F^{\varphi}\}$, so $F^{\alpha} \notin \{F^{\lambda\alpha}, F^{\varphi\alpha}\} = \{F^{\varepsilon}, F^{\delta}\}$. We have that $F^{\varepsilon^{-1}} = F^{\alpha^{-1}\lambda^{-1}} \neq F^{\alpha^{-1}\varphi^{-1}} = F^{\delta^{-1}}$. Now $F^{\varepsilon^{-1}} \leq E^{\alpha^{-1}}$ and $F^{\delta^{-1}} \leq E^{\delta^{-1}}$. Since $E^{\alpha^{-1}} = E^{\alpha}$ by lemma 6.10, it follows that $F^{\varepsilon^{-1}} \leq E^{\alpha}$ and $F^{\delta^{-1}} \leq E^{\alpha}$. Now $E^{\alpha} = F^{\varepsilon^{-1}} F^{\delta^{-1}}$ follows from lemma 5.4.

<u>Lemma 6.15.</u> If $\delta$ and $\varepsilon$ are as in 6.14, then $[T, C_W(F^{\delta^{-1}})] \leq F^{\delta^{-1}} \cdot [M,W]$ and $[T, C_W(F^{\varepsilon^{-1}})] \leq F^{\varepsilon^{-1}} \cdot [M,W]$.

<u>Proof.</u> $F^{\delta} \neq F^{\alpha}$, so $[F^{\delta x}, F^{\alpha}] \neq 1$ by lemma 6.6. Then $T = F^{\alpha x \delta^{-1}} \cdot M$ by lemma 6.2. Now $[F^{\alpha x \delta^{-1}}, C_W(F^{\delta^{-1}})] \leq [F^{\alpha x}, C_P(F)]^{\delta^{-1}} \leq [F^{\alpha x}, P \cap M]^{\delta^{-1}} \leq [R,M]^{\delta^{-1}} \leq F^{\delta^{-1}}$ by lemma 6.1 and $[M, C_W(F^{\delta^{-1}})] \leq [M,W]$, so $[T, C_W(F^{\delta^{-1}})] \leq F^{\delta^{-1}} \cdot [M,W]$. We similarly conclude that $[T, C_W(F^{\varepsilon^{-1}})] \leq F^{\varepsilon^{-1}} \cdot [M,W]$.

<u>Lemma 6.16.</u> $[T,W] \leq E^{\alpha} \cdot [M,W]$.

<u>Proof.</u> Let $\delta$ and $\varepsilon$ be as in lemma 6.14. Now $W = C_W(F^{\delta^{-1}}) \cdot C_W(F^{\varepsilon^{-1}})$ by

lemmas 6.12 and 5.4. It follows that $[T,W] \leq F^{\delta^{-1}} F^{\varepsilon^{-1}} \cdot [M,W] = E^{\alpha}[M,W]$ by lemma 6.15.

Lemma 6.17. $[T,W] \leq E^{\alpha}R$ and $W = E^{\alpha}E^{\alpha s}R$.

Proof. $[M,W] \leq R$ by lemma 6.1, so it follows from lemma 6.16 that $[M,W] \leq E^{\alpha}R$ and $[T^{s},W] \leq E^{\alpha s}R$. We conclude that $[K, E^{\alpha}E^{\alpha s}R] = [\langle T,T^{s}\rangle, E^{\alpha}E^{\alpha s}R] \leq E^{\alpha}E^{\alpha s}R$ and $W = E^{\alpha}E^{\alpha s}R$.

Lemma 6.18. $[T,W] \leq E^{\alpha}F^{\alpha s}$.

Proof. $[M,W] = [M, E^{\alpha}E^{\alpha s}R] \leq F^{\alpha}F^{\alpha s}F = F^{\alpha}F^{\alpha s}$ by lemma 6.17. Now lemma 6.18 follows from lemma 6.16.

Lemma 6.19. $W = E^{\alpha}E^{\alpha s}$.

Proof. It follows from lemma 6.18 that $[T,W] \leq E^{\alpha}F^{\alpha s} \leq E^{\alpha}E^{\alpha s}$ and $[T^{s},W] \leq (E^{\alpha}F^{\alpha s})^{s} = F^{\alpha s}E^{\alpha} \leq E^{\alpha}E^{\alpha s}$. Therefore $E^{\alpha}E^{\alpha s} \lhd \langle T,T^{s}\rangle = K$ and $W = E^{\alpha}E^{\alpha s}$.

Lemma 6.20. $W = E^{\alpha}E^{\alpha x}F$.

Proof. $W = E^{\alpha}E^{\alpha s}$ and $|W : C_W(E^{\alpha})| \leq |E^{\alpha s} : E^{\alpha s} \cap C(E^{\alpha})| \leq |E^{\alpha s} : E^{\alpha s} \cap F| \leq |E^{\alpha s} : \Omega_1(Z(P))^{s}| = q^2$. We know that $|E^{\alpha x} : C_{E^{\alpha x}}(E^{\alpha})| = q^2$ from lemma 6.7. It follows that $C_W(E^{\alpha}) = FE^{\alpha}$ and $W = FE^{\alpha}E^{\alpha x}$.

Observe that lemma 6.20 cannot be obtained by merely specializing s to x, as x might be contained in SH.

The two previous lemmas show that $[E^{\alpha}, E^{\alpha x}] = 1$ if and only if $[E^{\alpha}, E^{\alpha s}] = 1$. By the same argument as in lemma 6.5, we may choose $x \in M \cdot s$ such that x satisfies the conclusion of 6.5. Then $W = E^{\alpha}E^{\alpha x}$ and

$\langle T,T^x\rangle = K$.

Lemma 6.21. $R = F^{\alpha}F^{\alpha x}$ and $\Phi(R) = 1$.

Proof. Let $g \in E^{\alpha}$ and $h \in F^{\alpha x} \setminus \Omega_1(Z(P))^x$ such that $(gh)^2 = 1$ and hence $[g,h] = 1$. It follows from lemma 2.10 that $[g,F^{\alpha x}] = 1$. Lemma 6.6 yields that g is contained in $F^{\alpha}$. Notice that $\Omega_1(Z(P))^x$ is contained in $F^{\alpha x}$. If y is any involution of $E^{\alpha}F^{\alpha x}$, then $y \in E^{\alpha} \cup (F^{\alpha}F^{\alpha x})$.

$C_{E^{\alpha x}}(F^{\alpha}) = F^{\alpha x}$ by lemma 6.6 and we have that $[E^{\alpha},F^{\alpha}] = 1$, so $C_W(F^{\alpha}) = E^{\alpha}F^{\alpha x}$. We have that $F^{\alpha} \triangleleft T$, and it follows that $C_W(F^{\alpha}) \triangleleft T$ and $E^{\alpha}F^{\alpha x} \triangleleft T$. We conclude that $F^{\alpha}F^{\alpha x} \triangleleft T$ since $[T,E^{\alpha}] \leq F^{\alpha}$ and each involution of $E^{\alpha}F^{\alpha x}$ is contained in $E^{\alpha} \cup (F^{\alpha}F^{\alpha x})$. Now $(F^{\alpha}F^{\alpha x})^x = F^{\alpha x}F^{\alpha}$ and $K = \langle T,x\rangle$, so $F^{\alpha}F^{\alpha x} \triangleleft K$ and $R = F^{\alpha}F^{\alpha x}$. Then we also have that $\Phi(R) = 1$.

Lemma 6.22. $P \cap M = Q \cdot F^{\alpha x\alpha^{-1}}$.

Proof. $|F^{\alpha x} : C_{F^{\alpha x}}(E^{\alpha})| = q$, so $|F^{\alpha x\alpha^{-1}} : F^{\alpha x\alpha^{-1}} \cap Q| = q$ and $P \cap M = Q \cdot F^{\alpha x\alpha^{-1}}$.

Lemma 6.23. $F^{\alpha^{-1}} = F^{\alpha}$.

Proof. Suppose that $F^{\alpha^{-1}} \neq F^{\alpha}$ and notice that $[F^{\alpha x\alpha^{-1}},C_W(F^{\alpha^{-1}})] \leq [F^{\alpha x\alpha^{-1}},C_P(F^{\alpha^{-1}})] \cap [M,W] \leq [F^{\alpha x},P \cap M]^{\alpha^{-1}} \cap R \leq [R,P]^{\alpha^{-1}} \cap R \leq F^{\alpha^{-1}} \cap R$.

Lemmas 6.6 and 6.10 imply that $C_{\sim}(F^{\alpha x}) = F^{\alpha}$ and $F^{\alpha^{-1}} < E^{\alpha}$. We now conclude from lemma 5.4 that $F^{\alpha^{-1}} \cap R \leq F^{\alpha^{-1}} \cap C_{E^\alpha}(F^{\alpha x}) = F^{\alpha^{-1}} \cap F^\alpha = \Omega_1(Z(P))$ and $F^{\alpha^{-1}} \cap R = \Omega_1(Z(P))$.

It follows that $[F^{\alpha x\alpha^{-1}}, C_W(F^{\alpha^{-1}})] \leq \Omega_1(Z(P))$. Recall that $[Q,W] \leq Z^\alpha \leq \Omega_1(Z(P))$. Lemma 6.22 yields that $[P \cap M],C_W(F^{\alpha^{-1}})] \leq \Omega_1(Z(P)) \leq F$.

It follows from lemma 6.7 that $C_{F^{\alpha x}}(F^{\alpha^{-1}}) = C_{F^{\alpha x}}(E^\alpha) = \Omega_1(Z(P))^x$ and $|F^{\alpha x} : C_{F^{\alpha x}}(F^{\alpha^{-1}})| = q$. Now $W = F^{\alpha x}C_W(F^{\alpha^{-1}})$ since $|P : C_P(F^{\alpha^{-1}})| = q$. We conclude that $W = RC_W(F^{\alpha^{-1}})$ with $[P \cap M,W] \leq [P \cap M,R]F \leq F$ and $[M,W] \leq F$.

Since $E^\alpha = E^{\alpha^{-1}}$, it follows from lemma 5.4 that $F^{\alpha^{-1}} = F^{i\alpha}$ for some $i \in I$. Lemma 6.5 shows that $F^{\alpha^{-1}x} = F^{i\alpha x} \leq C_W(F^{i\alpha}) = C_W(F^{\alpha^{-1}})$.

Observe that $[F^{\alpha^{-1}x\alpha^{-1}}, C_W(F^{\alpha^{-1}})] \leq [C_W(F^{\alpha^{-1}}),P \cap M]^{\alpha^{-1}} \leq \Omega_1(Z(P))^{\alpha^{-1}} = \Omega_1(Z)P))$. Now $[F^{\alpha^{-1}x\alpha^{-1}},W] \leq R$ since $W = RC_W(F^{\alpha^{-1}})$. Lemma 6.6 shows that $[F^{\alpha^{-1}x},F^\alpha] \neq 1$ and lemma 6.2 yields that $T = F^{\alpha^{-1}x\alpha^{-1}}M$. Now $[T,W] \leq R$ and $[K,W] \leq R$ since $[M,W] \leq R$ A contradiction to lemma 6.6 is obtained by noticing that $W = E^\alpha R$ and hence $F^\alpha \leq Z(W)$.

<u>Lemma 6.24</u>. $[P \cap M,R] \leq \Omega_1(Z(P))$.

<u>Proof</u>. $F^\alpha = F^{\alpha^{-1}}$ by 6.23. So $R \leq C_W(F^{\alpha^{-1}}) \leq (P \cap M)^{\alpha^{-1}}$. Then $[F^{\alpha x\alpha^{-1}},R] \leq [P \cap M,R] \cap [F^{\alpha x},P \cap M]^{\alpha^{-1}} \leq F \cap F^{\alpha^{-1}}$. Recall from lemma 6.23 that $F^{\alpha^{-1}} = F^\alpha$. Lemma 6.6 implies that

$F \cap F^{\alpha} \leq C_{F^{\alpha}}(E^{\alpha x}) = \Omega_1(Z(P))$ and $F \cap F^{\alpha^{-1}} \leq \Omega_1(Z(P))$. The lemma follows from lemma 6.24, since $[Q,R] \leq \Omega_1(Z(P))$.

We now prove theorem 6. It follows from lemma 6.14 that there exists $\lambda \in I \cdot \alpha$ such that $F^{\lambda^{-1}} \neq F^{\alpha}$ and $F^{\lambda} \neq F^{\alpha}$. Now $F^{\alpha x} \cap C(F^{\lambda}) = \Omega_1(Z(P))$, so $T = F^{\alpha x \lambda^{-1}} \cdot M$ by lemma 6.2. Notice that $[F^{\alpha x \lambda^{-1}}, C_W(F^{\lambda^{-1}})] \leq [F^{\alpha x}, C_P(F)]^{\lambda^{-1}} \leq [F^{\alpha x}, P \cap M]^{\lambda^{-1}} \leq \Omega_1(Z(P))^{\lambda^{-1}} = \Omega_1(Z(P))$ by lemma 6.24. Now $[T,W] \leq R$ and $[K,W] \leq R$ since $W = RC_W(F^{\lambda^{-1}})$ and $[M,W] \leq R$. A contradiction to lemma 6.6 is reached by observing that $W = RE^{\alpha}$ and hence $F^{\alpha} \leq Z(W)$. Theorem 6 has now been proved.

## 7. Some Important Commutators.

Lemma 7. If $\langle E^{\Lambda} \rangle \leq Q$, $\alpha \in \Lambda$, $x$ is an involution of $T \setminus M$, and $g,h$ are involutions of $K \setminus N_K(TH)$, then

(a) $E^{\alpha g \alpha^{-1}} \leq Q$.

(b) $E^{\alpha g \alpha^{-1} h \alpha} \leq Q$.

(c) $[E^{\alpha g^{-1} \alpha h}, E^{\alpha g \alpha^{-1}}] = 1$.

(d) $[E^{\alpha g \alpha^{-1} h x}, E^{\alpha g \alpha^{-1} g}] = 1$.

(e) $[E^{\alpha g \alpha^{-1} g}, E^{\alpha^{-1} g x}] = 1$.

Proof. We may assume that $E^{\alpha} \neq E$.

(a) $[E^{\alpha g}, E^{\alpha}] = 1$ by theorem 6. Then we conclude that $[E^{\alpha g\alpha^{-1}}, E] = 1$ and $E^{\alpha g \alpha^{-1}} \leq Q$.

(b) Notice that $E^{\alpha g\alpha^{-1}} \leq Q$ and $E^{\alpha^{-1}h\alpha} \leq Q$ by (a), and that $[E^{\alpha g\alpha^{-1}h\alpha}, E] = [E^{\alpha g}, E^{\alpha^{-1}h\alpha}]^{\alpha^{-1}h\alpha} \leq [E^{\alpha g}, Q]^{\alpha^{-1}h\alpha} \leq (Z^{\alpha g})^{\alpha^{-1}h\alpha}$ $= (Z^{\alpha})^{\alpha^{-1}h\alpha} = Z^{h\alpha}$. Now $E^{\alpha g\alpha^{-1}h\alpha} \leq Q^{\alpha} \leq P$ and $[P,E] = Z$, so $[E^{\alpha g\alpha^{-1}h\alpha}, E] \leq Z^{h\alpha} \cap Z \leq Z^{h\alpha} \cap Z(P) = (Z^{h} \cap Z(P))^{\alpha} = D^{\alpha}$. Observe that $E^{\alpha^{-1}h\alpha g} \leq P$ and $[E, E^{\alpha^{-1}h\alpha g\alpha^{-1}}] =$ $[E^{\alpha g\alpha^{-1}h\alpha}, E]^{\alpha^{-1}h\alpha g\alpha^{-1}} \leq (D^{\alpha})^{\alpha^{-1}h\alpha g\alpha^{-1}} = D^{h\alpha g\alpha^{-1}} = D^{\alpha g\alpha^{-1}} =$ $D^{\alpha\alpha^{-1}} = D$. It follows from lemma 2.12 that $[E, E^{\alpha^{-1}h\alpha g\alpha^{-1}}] = 1$ and $[E^{\alpha g\alpha^{-1}h\alpha}, E] = 1$. Now (b) is satisfied.

(c) Part (b) implies that $[E^{\alpha g\alpha^{-1}h}, E^{\alpha g\alpha^{-1}}] =$ $[E^{\alpha g\alpha^{-1}h\alpha}, E^{\alpha g}]^{\alpha^{-1}} \leq [Q, E^{\alpha g}]^{\alpha^{-1}}$. Notice that $[Q, E^{\alpha g}] \leq Z^{\alpha g} = Z^{\alpha}$ and $[E^{\alpha g\alpha^{-1}h}, E^{\alpha g\alpha^{-1}}] \leq (Z^{\alpha})^{\alpha^{-1}} = Z$.

We also have that $[E^{\alpha g\alpha^{-1}h}, E^{\alpha g\alpha^{-1}}] = [E^{\alpha g}, E^{\alpha g\alpha^{-1}h\alpha}]^{\alpha^{-1}h}$ $\leq [E^{\alpha g}, Q]^{\alpha^{-1}h} \leq (Z^{\alpha g})^{\alpha^{-1}h} = (Z^{\alpha})^{\alpha^{-1}h} = Z^{h}$. Now we see that $[E^{\alpha g\alpha^{-1}h}, E^{\alpha g\alpha^{-1}}] \leq Z \cap Z^{h} = D$. It follows from (b) that $E^{\alpha g\alpha^{-1}h\alpha g} < Q^{g} = Q < P$. We also have that

$$[E^{\alpha g\alpha^{-1}h\alpha g\alpha^{-1}},E] = [E^{\alpha g\alpha^{-1}h},E^{\alpha g\alpha^{-1}}]^{\alpha g\alpha^{-1}} \leq D^{\alpha g\alpha^{-1}} = (D^{\alpha})^{\alpha^{-1}} = D.$$

Then $[E^{\alpha g\alpha^{-1}h\alpha g\alpha^{-1}},E] = 1$ by lemma 2.12, and $[E^{\alpha g\alpha^{-1}h},E^{\alpha g\alpha^{-1}}] = 1$.

(d) $[E^{\alpha g\alpha^{-1}gx},E^{\alpha g\alpha^{-1}g}] = [E^{\alpha g\alpha^{-1}(gxg)},E^{\alpha g\alpha^{-1}}]^{g} = 1$ by (c).

(e) $[E^{\alpha g\alpha^{-1}g},E^{\alpha^{-1}gx}] = [E^{\alpha g\alpha^{-1}(gxg)\alpha},E]^{\alpha^{-1}gx} = 1$ by (b).

## 8. The Proof of Theorem C.

The purpose of this section is to prove Theorem C. Suppose there exists an automorphism $\alpha$ in $\Lambda$ such that $E^{\alpha}$ is not contained in Q. Now $E^{\alpha}$ centralizes $P/Z^{\alpha}$ and have $E^{\alpha}$ centralizes Q/E. The centralizer in G of Q/E is normal in G, the group G/H is simple and H/Q has no partial complement in G/Q. Therefore $O^2(G)$ centralizes Q/V since $O^2(G)$ centralizes Q/E and E/V. It follows that $[O^2(G),Q] = V$.

Suppose Theorem C is false. We may assume that the group $\langle E^{\Lambda}\rangle$ is contained in Q, and that $[G,E^{\beta}]$ is not contained in E for some automorphism $\beta$ in $\Lambda$.

Lemma 4.1 implies that $E^{\beta} \lhd N_G(P)$. It follows that $[K,E^{\beta}] \not\leq E$, since G is generated by K and $N_G(P)$.

Let $I = \{\delta \in \Lambda | E^{\alpha\delta} = E \text{ for each } \alpha \in \Lambda\}$ and $A \in \mathcal{A}(P)$ s.t. $|A : A \cap M| = |A : A \cap Q| = q$. Define $B_0 = (A \cap Q \cap Q^{\beta})EE^{\beta}$, let $g \in K \setminus N_K(TH)$ such that $g^2 = 1$ and define $B = E^{\beta^{-1}g} \cdot C_{B_0}(E^{\beta^{-1}g})$. Lemma 2.8 yields that $B_0 \in \mathcal{A}(Q)$ and $B \in \mathcal{A}(Q)$.

If $\alpha \in I \cdot \beta$, it follows that $E^{\alpha^{-1}} = E^{\beta^{-1}}$ and $B = E^{\alpha^{-1}g} \cdot C_{B_0}(E^{\alpha^{-1}g})$.

We also obtain that $Z^{\alpha} = [P,E^{\alpha}] = [P,E^{\beta}] = Z^{\beta}$ and $[A,B_0] \leq [A,EE^{\alpha}] \leq ZZ^{\alpha}$.

Lemma 8.1. If $\alpha \in I \cdot \beta$, then

(a) $\langle E^{\alpha^{-1}gA} \rangle$ is elementary abelian.

(b) $C_{B_0}(E^{\alpha^{-1}g}) = C_{B_0}(\langle E^{\alpha^{-1}gA} \rangle)$.

(c) $[A,B] \leq C_B(E^{\alpha g \alpha^{-1}g}) \leq B$.

(d) $[Q^{\alpha} \cap Q^{\alpha}, E^{\alpha g}] = [B_0, E^{\alpha g}] = Z^{\alpha}$.

(e) If $E^{\alpha g \alpha^{-1}g} \leq B$, then $B_0^{g\alpha} \leq M$ and $F^{\alpha^{-1}g} \leq B_0$.

(f) If $E^{\alpha g \alpha^{-1}g} \leq B$, then $[K, F^{\alpha^{-1}}] \leq E$.

Proof.

(a) If $x,y \in A$, then $[E^{\alpha^{-1}gx}, E^{\alpha^{-1}gy}] = [E^{\alpha^{-1}gxyg}, E^{\alpha^{-1}}]^{gy} = 1$ by theorem 6 since $(gxyg)^2 = 1$. Therefore $\langle E^{\alpha^{-1}gA} \rangle$ is elementary abelian.

(b) $[A,B_0] \leq ZZ^{\alpha} \leq C_{B_0}(E^{\alpha^{-1}g})$, so $C_{B_0}(E^{\alpha^{-1}g}) = C_{B_0}(\langle E^{\alpha^{-1}gA} \rangle)$.

(c) Part (a) says that $\Phi(\langle E^{\alpha^{-1}gA} \rangle) = 1$ by (a). Then $\Phi(B\langle E^{\alpha^{-1}gA} \rangle) = 1$ follows from (b) and the definition of B. Now $B \in \mathcal{A}(Q)$ and it follows that $\langle E^{\alpha^{-1}gA} \rangle \leq B$ and $[A, E^{\alpha^{-1}g}] \leq B$. Notice that $[A,B_0] \leq ZZ^{\alpha} \leq B$ and $[A,B] \leq B$.

Let $x \in A \setminus M$. Now $[x,B] \leq [x, E^{\alpha^{-1}g}B_0] \leq E^{\alpha^{-1}g} \cdot E^{\alpha^{-1}gx} \cdot ZZ^{\alpha}$ and $[E^{\alpha^{-1}g}, E^{\alpha g \alpha^{-1}g}] = [E, E^{\alpha g}]^{\alpha^{-1}g} = 1$. Notice that $[E^{\alpha^{-1}gx}, E^{\alpha g \alpha^{-1}g}] = 1$ by lemma 7 (e). Now $[ZZ^{\alpha}, E^{\alpha g\alpha^{-1}g}] \leq [ZZ^{\alpha}, Q] = 1$, so $[x,B] \leq C_B(E^{\alpha g\alpha^{-1}g})$. We conclude that $[A,B] \leq C_B(E^{\alpha g\alpha^{-1}g})$ since $A = \langle x \in A | x \notin M \rangle$.

(d) If $E^{\alpha g} \leq B_0$, then $[A^g, E^{\alpha}] = [A, E^{\alpha g}]^g \leq [A, B_0]^g \leq (ZZ^{\alpha})^g \leq E^g = E$ and $[K, E^{\alpha}] = [\langle P, A^g \rangle, E^{\alpha}] \leq Z^{\alpha}E = E$, which yields a contradiction. Therefore $E^{\alpha g} \not\leq B_0$. By lemma 2.9 we have that $[B_0, E^{\alpha g}] = Z^{\alpha}$. Since $B_0 \leq Q \cap Q^{\alpha}$ and $[Q, E^{\alpha g}] \leq Z^{\alpha}$, we conclude that $[Q \cap Q^{\alpha}, E^{\alpha g}] = Z_{\alpha}$.

(e) If $E^{\alpha g\alpha^{-1}g} \leq B$, then $[B_0, E^{\alpha g\alpha^{-1}g}] \leq [B_0, B] \leq [B_0, E^{\alpha^{-1}g}] \leq [Q, E^{\alpha^{-1}g}] \leq Z^{\alpha^{-1}}$. It follows that $[B_0^{g\alpha}, E^{\alpha g}] \leq (Z^{\alpha^{-1}})^{g\alpha} = Z$. If $B_0^{g\alpha} \not\leq M$, then $[K, E^{\alpha}] = [\langle B_0^{g\alpha}, P^g \rangle, E^{\alpha g}]^g \leq (ZZ^{\alpha g})^g = Z^g Z^{\alpha} \leq E$, which gives a contradiction. Therefore $B_0^{g\alpha} \leq P \cap M = C_P(F)$ and $B_0^g \leq Q \cap C_P(F^{\alpha^{-1}})$. Notice that $B_0 \leq C_Q(F^{\alpha^{-1}g})$ and $B_0 \in \mathcal{A}(Q)$. We conclude that $F^{\alpha^{-1}g} \leq B_0$.

(f) Suppose that $E^{\alpha g\alpha^{-1}g} \leq B$. It follows from (e) that $F^{\alpha^{-1}g} \leq B_0$ and $[A^g, F^{\alpha^{-1}}] = [A, F^{\alpha^{-1}g}]^g \leq [A, B_0]^g \leq (ZZ^{\alpha})^g \leq E^g = E$. Now $[P, F^{\alpha^{-1}}] = Z^{\alpha^{-1}} \leq E$ and $K = \langle A^g, P \rangle$, so $[K, F^{\alpha^{-1}}] \leq E$.

Lemma 8.2. There exist elements $\alpha$ and $\delta$ of the coset $I\beta$ such that

(i) $E^{\alpha} = F^{\alpha}F^{\delta}$.

(ii) $E^{\alpha^{-1}} = F^{\alpha^{-1}}F^{\delta^{-1}}$.

(iii) $[K, F^{\alpha}] \not\leq E$.

(iv) $[K, F^{\delta}] \not\leq E$.

Proof. Suppose lemma 8.2 is false and let $J = \{\psi \in I | F^{\psi} = F\}$. Also let $\theta$ be the natural homomorphism from L to $\Lambda$. Lemma 4.1 implies that the group $\theta(L)$ is contained in the group I. Since $[K,E^{\beta}] \not\leq E$. There exists an element $\delta$ of $I\beta$ and an element x of $\theta(L) \setminus J$ such that $[K,F^{\delta}] \not\leq E$ and $[K,F^{x\delta}] \not\leq E$. The hypothesis yields that $F^{\alpha^{-1}} = F^{\delta^{-1}}$ for each $\alpha \in (JS) \cup (Jx\delta)$. Now $F^{\delta^{-1}} = F^{(j\delta)^{-1}} = F^{\delta^{-1}j^{-1}}$ for each $j \in J$, and it follows that $F^{\delta^{-1}j\delta} = F$. Notice that $J^{\delta} \leq I$ since $J \leq I \lhd \Lambda$. We obtain $J^{\delta} = J$ and observe that $F^{\delta^{-1}} = F^{(x\delta)^{-1}} = F^{\delta^{-1}x^{-1}}$ and $F^{\delta^{-1}x\delta} = F$. Now J contains $x^{\delta}$ since I is normal in $\Lambda$ and I contains x. A contradiction is obtained by noticing that $J^{\delta} = J$ and J contains x.

Lemma 8.3. Let $\alpha$ and $\delta$ be as in lemma 8.2. If $E^{\alpha g \alpha^{-1} g} \leq B$ and $E^{\delta g \delta^{-1} g} \leq B$, then $B = B_0$ and $[K,E^{\beta^{-1}}] \leq E$.

Proof. Lemma 8.1 (e) implies that $F^{\alpha^{-1} g} \leq B_0$ and $F^{\delta^{-1} g} \leq B_0$. Notice that $E^{\beta^{-1}} = E^{\alpha^{-1}} = F^{\alpha^{-1}} F^{\delta^{-1}}$ with $E^{\beta^{-1} g} \leq B_0$ and $B = B_0$. Now $[K,F^{\alpha^{-1}}] \leq E$ and $[K,F^{\delta^{-1}}] \leq E$ by lemma 8.1(f), so $[K,E^{\beta^{-1}}] = [K,F^{\alpha^{-1}} F^{\alpha^{-1}}] \leq E$.

Lemma 8.4. There exists $\lambda \in I \cdot \beta$ such that $E^{\lambda g \lambda^{-1} g} \not\leq B$.

Proof. Suppose lemma 8.4 is false. It follows from lemma 8.3 that $[K,E^{\beta^{-1}}] \leq E$ with $C_Q(E^{\beta^{-1}}) = C_Q(EE^{\beta^{-1}}) \lhd K$ and $Q \cap Q^{\beta^{-1}} \lhd K$. We conclude from lemma 8.3 that $B = B_0$ and $[A,E^{\beta g \beta^{-1} g}] \leq [A,B] = [A,B_0] \leq ZZ^{\beta}$.

Therefore $[A^g, E^{\beta g\beta^{-1}}] \leq Z^g Z^\beta \leq E$ and $A^g$ acts on $EE^{\beta g\beta^{-1}}$. Now $A^g$ acts on $[Q \cap Q^{\beta^{-1}}, E^{\beta g\beta^{-1}}] = [Q^\beta \cap Q, E^{\beta g}]^{\beta^{-1}}$. It follows from lemma 8.1 (d) that $[Q^\beta \cap Q, E^{\beta g}] = Z^\beta$ and $A^g$ acts on $Z$, which gives the desired contradiction.

Lemma 8.5. If $\alpha \in I\beta$ such that $E^{\alpha g\alpha^{-1}g} \nleq B$, then $[B, E^{\alpha g\alpha^{-1}g}]$ is $A$-invariant.

Proof. Let $R = E^{\alpha g\alpha^{-1}g}$. If $x \in A \setminus M$, then $[R^x, R] = 1$ by lemma 7 (d). $E^{\alpha^{-1}g} \leq B \leq Q$, so $B \leq Q \cap Q^{\alpha^{-1}g}$ and $B \in \mathcal{A}(Q) \cap \mathcal{A}(Q)^{\alpha^{-1}g}$. Now lemma 2.8 implies that $RC_B(R)$ is contained in $\mathcal{A}(Q)$. Now $C_B(R^x) = C_B(R)$ by lemma 8.1 (c), and $RR^xC_B(R)$ is elementary abelian. Notice that $RC_B(R) \in \mathcal{A}(Q)$ with $R^x \leq RC_B(R)$ and $x \in N_K(R \cdot C_B(R))$. Therefore $x$ acts on $[B, R \cdot C_B(R)]$ and $x$ acts on $[B,R]$. Now $x$ is contained in $N_A([B,R])$ for each element $x$ of $A$ which is not contained in $M$, so $A$ acts on $[B,R]$.

We now prove Theorem C. By lemma 8.4, there exists $\alpha \in I\beta$ such that $E^{\alpha g\alpha^{-1}g} \nleq B$. Now $E^{\alpha^{-1}g} \leq B$ and $B \in \mathcal{A}(Q) \cap \mathcal{A}(Q^{\alpha^{-1}g})$ with $[B, E^{\alpha g\alpha^{-1}g}] \neq 1$. It follows that $[B, E^{\alpha g\alpha^{-1}g}] = Z^{\alpha g\alpha^{-1}g} = Z^g$ by lemma 2.9. Then $Z^g$ is $A$-invariant by lemma 8.5, which yields a contradiction. Theorem C is proved.

## Chapter IV. Open Problems

### 1. The Pushing Up Problem for $Sp_4(2^n)$.

We have described some pushing up problems for $L_2(2^n)$ and $L_3(2^n)$. It is also of interest to consider the four dimensional sympletic groups $Sp_4(2^n)$.

Let G be a finite group with $F^*(G) = O_2(G)$ and suppose that $G/O_2(G)$ is isomorphic to $Sp_4(q)$, where $q = 2^n$. Define $Q = O_2(G)$ with $E = \Omega_1(Z(Q))$ and $\overline{G} = G/Q$. Let T be a Sylow 2-subgroup of G and let $M_1$ and $M_2$ be the maximal 2-local subgroups of G which contain T. Define $P = O_2(M_1)$ and $M = O_2(M_2)$, and assume that $G \neq C_G(\Omega_1(Z(T)))N_G(J(P))$. A theorem of Cooperstein [8] asserts that $E/C_E(G)$ is a natural module for $\overline{G}$.

If G has more than one noncentral chief factor within Q, is $[G,E^{\alpha}]$ contained in E for each automorphism $\alpha$ of P? We propose that the methods of Chapter III might be applicable to this problem.

We mention a reduction in the problem that illustrates the usage of Theorem A. The methods of Chapter III require that the Thompson subgroup J(P) is not contained in M. We suppose that J(P) is contained in M and observe that there exists a subgroup H of G, with H containing $\widetilde{J}(P)Q$ such that $\overline{H}$ is isomorphic to $L_2(2^n)$ and G is generated by H and $M_1$. The subgroup H contains $Q\widetilde{J}(P)$ and $\overline{H}$ is isomorphic to $L_2(2^n)$. The Thompson subgroup J(P) of P is also the Thompson subgroup $J(Q\widetilde{J}(P))$ of $Q\widetilde{J}(P)$. Suppose that G has more than one noncentral chief factor within Q. Theorem A yields that $[H,E^{\alpha}]$ is contained in E for each automorphism $\alpha$ of P. We conclude that $[G,E^{\alpha}]$ is also contained in E since $M_1$ normalizes P.

The above argument allows us to assume that J(P) is not contained in

M. The methods of the third chapter are now relevant.

We note that lemma 2.8.1 (ii) and Baumann's theorem [6] are sufficient to show the existence of a characteristic subgroup K of $\tilde{J}(P)$ with K normal in G if J(P) is contained in M but not in Q.

## 2. A Triple Factorization Problem.

Let S be a 2-group and let $\mathcal{S}$ be as in the statement of Corollary 2. We would like to find some suitably large subcollection $\mathcal{B}$ of $\mathcal{S}$ and some nonidentity characteristic subgroups $S_1$, $S_2$ and $S_3$ of S such that at least two of the $S_i$'s are normal in each group G of $\mathcal{B}$. This can also be written as

$$G = N_G(S_1)N_G(S_2)$$

$$= N_G(S_1)N_G(S_3)$$

$$= N_G(S_\alpha(N_G(S_3))).$$

Any results of this kind would be of considerable interest in the study of characteristic 2 type groups.

This question is partially motivated by Glauberman's use of a triple factorization to classify the simple groups not involving the symmetric group $\Sigma_4$ of degree four.

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