

F-bundles and the mirror symmetry of flag varieties

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ABSTRACT

This thesis consists of three projects related to enumerative geometry and mirror symmetry, with particular emphasis on the mirror symmetry of flag varieties.

The first project introduces and develops the theory of F-bundles, a framework for formulating mirror symmetry type results. We prove a spectral decomposition theorem for maximal F-bundles, and use it to obtain reconstruction and uniqueness results for certain decompositions of quantum cohomology related to birational geometry, complementing existing the existence theorem in the literature. In the third project, we further extend the F-bundle formalism to the equivariant setting and establish an unfolding theorem strengthening mirror symmetry statements from small to big quantum cohomology. As an application, we derive the equivariant mirror symmetry of general flag varieties in the third project, extending previous results which were previously known only at the level of small quantum cohomology.

The second project focuses on the mirror symmetry of flag varieties. For complex partial flag varieties, we provide an explicit Plücker coordinate superpotential formula that is sufficient to recover their small quantum cohomology on A-side, and we prove a folklore conjecture in mirror symmetry. Namely, we show that the eigenvalues of the action of the first Chern class on quantum cohomology are equal to the critical values of the superpotential.

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INTRODUCTION

1.1 Motivation

Enumerative geometry seeks to determine how many geometric objects satisfy specified incidence or tangency conditions within a given ambient space. Classical problems include counting the number of lines intersecting a prescribed collection of subvarieties, or determining the number of rational curves of fixed degree passing through general points in projective space. For centuries, such questions have inspired deep developments in algebraic geometry. For example, one classical question asks how many lines intersect four given lines in \mathbb{P}^3 , and this classical question can be solved by studying the cohomology ring structure of the complex Grassmannian $\text{Gr}(2, 4)$ ([47]). Yet, beyond lines cases, traditional algebraic methods quickly reach their limits, and more sophisticated frameworks are required to handle families of curves in higher dimensions.

At the turn of the twentieth century, the emergence of Gromov–Witten theory provided a unifying language for curve counting in both algebraic and symplectic geometry. Building on ideas from quantum field theory, Gromov–Witten invariants encode intersection numbers on moduli spaces of stable maps and furnish powerful deformation-invariant curve counts. What began as a physically inspired construction has now been transformed into a rigorous mathematical theory, forming one of the cornerstones of modern enumerative geometry. ([90? ?]).

In parallel, mirror symmetry emerged as a remarkable duality that originated in string theory, relating the symplectic geometry of one Calabi-Yau to the complex geometry of another ([19, 80]). Early physical predictions suggested that Gromov–Witten invariants on one side correspond to period integrals of holomorphic forms on the mirror side, giving rise to deep and unexpected identities between enumerative invariants and analytic quantities. These conjectures, once purely physical, have been reformulated and proven in several settings, revealing a profound link between algebraic geometry, symplectic topology, and mathematical physics ([51, 98, 99]) .

Since then, mirror symmetry has developed into a wide-reaching field of study.

A major goal is to understand conceptually the numerical correspondences originally observed in concrete examples. This has led to two primary formulations of mirror symmetry. The first, Homological Mirror Symmetry, was proposed by Kontsevich ([90]) and predicts an equivalence between the derived categories naturally associated to the two sides of the mirror pair. The second, often referred to as Hodge-theoretic or D-module mirror symmetry, assigns to each side a system of differential equations and asserts that these systems correspond ([36], [5]). In this formulation, numerical identities arise by comparing the solutions of the differential systems. Although these perspectives are distinct, they are closely related. In particular, in several cases the validity of homological mirror symmetry implies the Hodge-theoretic form ([82]). These general formulations apply to broader classes of varieties beyond Calabi–Yau varieties. Understanding and proving them in important cases, such as toric varieties ([1, 6, 31, 32]) and flag varieties ([28, 50, 52, 65, 96, 103, 117, 118]), has become crucial.

Flag varieties form a fundamental class of algebraic varieties that lie at the intersection of representation theory, symplectic geometry, and algebraic geometry. Given a complex semisimple Lie group G , a (partial) flag variety G/P parametrizes the filtrations of a vector space compatible with the parabolic subgroup P . These varieties are smooth, projective, and have rich geometric and combinatorial structure encoded by the Weyl group. Their cohomology rings admit a canonical basis of Schubert classes, and the corresponding intersection theory is governed by the combinatorics of Bruhat order and the geometry of Schubert varieties. Flag varieties also appear naturally as parameter spaces in geometric representation theory and play a central role in the study of quantum cohomology, integrable systems, and mirror symmetry. In particular, the quantum cohomology of G/P exhibits deep structural features that reflect the representation theory of the Langlands dual group, and this structure is mirrored by Landau–Ginzburg models constructed on the dual flag variety. Some of the results in this thesis contribute to the Hodge-theoretic mirror symmetry of flag varieties.

Recent advances in Hodge-theoretic mirror symmetry have provided new tools for studying interactions between enumerative and birational geometry ([72, 78]). Part of the results in this thesis contribute to these developments. Arising from Hodge-theoretic mirror symmetry, the Hodge atoms theory ([84]) is developed

and used to prove a long-standing conjecture in birational geometry: a very general 4-dimensional cubic hypersurface is not rational. In particular, the main results of this thesis are foundational for the definition of Hodge atoms.

In this thesis, we develop the theory of F-bundles based on both non-archimedean geometry and formal geometry. F-bundles can be viewed as a generalization of quantum D-module. An automorphism given by a gauge transformation is a functor in the category of F-bundle but is not in the category of quantum D-module. The theory of F-bundles plays a crucial role in Atom theory ([84]), and we used it to prove the big mirror symmetry of flag varieties as another important application. Non-archimedean geometry is a generalization of analytic geometry, where the notion of convergence is made a lot weaker by the use of non-archimedean absolute values. The significance of non-Archimedean geometry in mirror symmetry was emphasized in ([91]). Meanwhile, formal geometry has long played a central role in mirror symmetry, providing a natural framework for encoding and interpreting generating functions of Gromov–Witten invariants.

1.2 Overview of the main results

In this section, we provide an overview of the results of the thesis. The general background will be provided in Chapter 2.

Decomposition and framing of F-bundles and application to quantum cohomology

In Chapter 3, we introduce F-bundle, which is a formal/non-archimedean version of variation of nc-Hodge structures. We establish the spectral decomposition theorem for F-bundles according to the generalized eigenspaces of the Euler vector field action (see Theorem 3.3.32 and Theorem 3.3.42).

We establish the existence and uniqueness of the extension of framing for logarithmic F-bundles. (see Theorem 3.4.2 and Theorem 3.4.26). As an application, we prove the uniqueness of the decomposition map for the A-model F-bundle (hence quantum D-module and quantum cohomology) associated to a projective bundle, as well as to a blowup of an algebraic variety. (see Theorem 3.5.16). This complements the existence results by Iritani-Koto and Iritani ([72, 78]) and also plays a crucial role in my joint paper ([65]), where we proved the big mirror symmetry for general type flag varieties. (see Theorem 5.1.4)

A Plücker coordinate mirror for partial flag varieties and quantum Schubert calculus

In Chapter 4, we construct a Plücker coordinate superpotential \mathcal{F}_- that is mirror to a partial flag variety $\mathbb{F}\ell(n_\bullet)$. (see Theorem 4.3.10). Its Jacobi ring recovers the small quantum cohomology of $\mathbb{F}\ell(n_\bullet)$, (see Theorem 4.4.9), and we prove a folklore conjecture in mirror symmetry. (see Theorem 4.4.11). Namely, we show that the eigenvalues for the action of the first Chern class $c_1(\mathbb{F}\ell(n_\bullet))$ on quantum cohomology are equal to the critical values of \mathcal{F}_- . We achieve this by proving new identities in quantum Schubert calculus that are inspired by our formula for \mathcal{F}_- and the mirror symmetry conjecture.

Unfolding of equivariant F-bundles and application to the mirror symmetry of flag varieties

In Chapter 5, we establish an unfolding theorem for equivariant F-bundles, (see Theorem 5.3.36), generalizing Hertling-Manin's universal unfolding of meromorphic connections. As an application, we obtain the mirror symmetry theorem for the big quantum cohomology of flag varieties, from the recent works on the small quantum cohomology mirror symmetry, via the equivariant unfolding theorem. (see Theorem 5.4.35).

Chapter 2

BACKGROUND

In this chapter, we summarize the background material that will be used throughout the thesis. Gromov–Witten theory appears in several places in our work, and section 2.1 reviews the basic definitions of moduli spaces of stable maps and the associated Gromov–Witten invariants. Section 2.2 discusses Hodge-theoretic mirror symmetry, we review the construction of A-model, B-model, and the expected mirror statement between them.

2.1 Gromov Witten theory

In this section, we provide a brief introduction for the rough idea of Gromov–Witten theory, which is used throughout the thesis. The reader may look at standard references for details. [40, 89?]. Let X be a complex projective variety. We are interested in counting curves in X subject to various incidence conditions. The basic idea is to define a moduli space of the curves, so that the incidence conditions restrain some subspaces of the moduli space, and then counting curves that satisfy these incidence conditions can be translated to study the intersection theory of the moduli space. However, this rough idea has some technique issues, which we will explain starting from the introduction of the moduli space.

The curves we want to count can be presented as maps $f : (C, x_1, \dots, x_n) \rightarrow X$, where C is a smooth proper curve with marked points (x_1, \dots, x_n) and f is a smooth proper map, then $f(C)$ is a curve inside X passing through $f(x_1), \dots, f(x_n)$. However, the moduli space of the above maps is not compact and does not have good algebraic structure. This motivates Kontsevich’s definition of stable maps, which allows nodal singularities in the domain curve to fix the compactness issue and impose a stability condition to kill infinitesimal automorphisms to make the moduli space a Deligne-Mumford stack.

Definition 2.1.1. Let X be a complex projective variety. Then a genus g , n -marked stable curve consists of the data (C, x_1, \dots, x_n) where

- C is a curve of arithmetic genus g with at worst nodal singularities and x_i are smooth points in C .

- C is stable: there are finitely many automorphisms that preserve x_1, \dots, x_n and carry nodes to nodes.

The special points of a curve C are the fixed points x_1, \dots, x_n and the nodes. The stability condition for curves (C, x_1, \dots, x_n) can be deduced from the genus and number of special points:

Proposition 2.1.2. ([58])

- genus 0 curves are stable if the number of special points ≥ 3 .
- genus 1 curves are stable if the number of special points ≥ 1 .
- genus ≥ 2 curves are all stable.

The stable maps to X are (C, x_1, \dots, x_n, f) where (C, x_1, \dots, x_n) are stable curves and $f(x_i)$ are distinct. If $f_*([C]) = \beta \in NE(X, \mathbb{Z})$, we say (C, x_1, \dots, x_n, f) is of class β . Now we are ready to introduce the moduli stack of stable maps.

Definition 2.1.3. For X a smooth complex projective variety, we denote $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to be the moduli stack parametrizing n -pointed, genus g stable maps to X of class β .

While the moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ may be singular and have components of varying dimensions, it is a proper Deligne-Mumford stack, (See [40, 89]), and it has an expected dimension called virtual dimension which is given by ([?])

$$vdim \overline{\mathcal{M}}_{g,n}(X, \beta) = n + (1 - g)(dim X - 3) + \int_{\beta} c_1(T_X)$$

The intersection theory for Deligne-Mumford stacks was developed by Vistoli [128], extending the classical construction of Fulton [42].

Let $V_1, \dots, V_n \subset X$ be subvarieties, and let $\gamma_i \in H^*(X)$ denote their Poincaré dual cohomology classes. For a stable map $f : (C, x_1, \dots, x_n) \rightarrow X$, the evaluation morphisms

$$ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X, \quad (C, x_1, \dots, x_n, f) \longmapsto f(x_i)$$

allow us to consider the cohomology class

$$ev_1^* \gamma_1 \cup \dots \cup ev_n^* \gamma_n \in H^*(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

This class represents the locus of stable maps whose images intersect each V_i at the marked point x_i . If the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ were smooth and of the expected (finite) dimension, then the corresponding Gromov–Witten invariant would be given by the intersection number

$$\int_{\overline{\mathcal{M}}_{g,n}(X, \beta)} \text{ev}_1^* \gamma_1 \cup \cdots \cup \text{ev}_n^* \gamma_n.$$

In general, however, the moduli space is rarely smooth and may not be equidimensional. It is only proper as a Deligne–Mumford stack. To remedy this, Behrend and Fantechi [9] constructed a *virtual fundamental class*

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} \in A_{\dim \overline{\mathcal{M}}_{g,n}(X, \beta)}(\overline{\mathcal{M}}_{g,n}(X, \beta)),$$

defined using a perfect obstruction theory. The Gromov–Witten invariant is then defined by integrating against this virtual class:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} \text{ev}_1^* \gamma_1 \cup \cdots \cup \text{ev}_n^* \gamma_n.$$

2.2 Hodge theoretic Mirror symmetry

This section serves as an index of the preliminary of Hodge theoretic mirror symmetry discussed in this thesis.

The preliminary of the A-model quantum F-bundle is provided in section 3.2, and in 5.4 we focus on the equivariant version for flag varieties cases.

The preliminary on the construction of the B-side mirror space for flag varieties is provided in 4.2, and the small D -module mirror symmetry for G/P is reviewed in section 5.4.

DECOMPOSITION AND FRAMING OF F-BUNDLES AND APPLICATIONS TO QUANTUM COHOMOLOGY

This chapter is based on [66], joint work with Thorgal Hinault, Tony Yue Yu, and Shaowu Zhang.

3.1 Introduction

Motivations

Let X be a smooth projective complex variety. The enumeration of curves in X is a classical subject in algebraic geometry, enjoying a variety of approaches (see [109]). Gromov–Witten theory is one of the most widely known and the most general (e.g. no restriction on the dimension of X), (see [97? ?]). The Gromov–Witten invariants of X are rational numbers depending on the genus g , number of marked points n and cohomology classes ϕ_1, \dots, ϕ_n of X . They satisfy a notable relation called the WDVV equation, which allows them to be packaged into differential geometric data, such as Frobenius manifolds by Dubrovin ([36]) or semi-infinite Hodge structures by Barannikov ([4]). The differential geometric framework facilitates intuitions from geometry and mirror symmetry and contributes tremendously to the development of the subject. The framework was further extended to incorporate the integral/rational structure via the notion of nc-Hodge structure by Katzarkov–Kontsevich–Pantev [82]. They established a profound gluing/decomposition theorem using the Fourier-Laplace transform of the associated D-modules (see §2.4.2 in loc. cit.). This motivated the development of the theory of atoms for applications to birational geometry (see [84]). The idea is to apply the decomposition to the A-model nc-Hodge structure (defined using Gromov–Witten invariants) associated to a smooth projective variety at a generic point of the base, and view the resulting pieces as elementary pieces of the variety called atoms. The collection of atoms (modulo an equivalence relation induced by blowups) is expected to serve as a powerful birational invariant.

While the notions of nc-Hodge structure and atom are natural and beautiful, it is still conjectural that Gromov–Witten invariants actually give rise to an nc-Hodge structure satisfying all the axioms in [82, §2.1.5]. The difficulties

include the convergence of the Gromov–Witten potential ([74]), the Gamma conjecture ([46]) and the opposedness axiom ([116]). This means that the theory of nc-Hodge structure cannot yet be unconditionally employed for the study of Gromov–Witten invariants and their applications in general. In this paper, we consider a formal/non-archimedean distilled version of variation of nc-Hodge structures, which we call F-bundles (see Section 3.2, and see Section 3.1 for related notions). We establish the spectral decomposition theorem for F-bundles, according to the generalized eigenspaces of the Euler vector field action, motivated by the gluing theorem for nc-Hodge structures via Fourier-Laplace transform, see Section 3.3. The comparison of the F-bundle decomposition and the nc-Hodge structure decompositions is studied in [130, §8].

Furthermore, we study the notion of framing of F-bundles, analogous to the decoration on variations of nc-Hodge structures, and prove the existence and uniqueness of the extension of framing, see Section 3.4. This allows us to identify F-bundles via maps on the base (analogous to a mirror map) together with a gauge transformation on the bundle. As an application, we prove the uniqueness of the decomposition map for the A-model F-bundle (hence quantum D-module and quantum cohomology) associated to a projective bundle, as well as to a blowup of an algebraic variety. This complements the existence result by Iritani-Koto [72] and Iritani [78].

Main results

Below we give a more detailed description of our results.

Throughout the paper, we fix a field \mathbb{k} of characteristic 0. In the non-archimedean setting, we assume that \mathbb{k} has a nontrivial valuation whose restriction to \mathbb{Q} is trivial. Let B be a smooth \mathbb{k} -analytic space, and \mathbb{D}_u the germ at 0 in a \mathbb{k} -analytic closed unit disk with coordinate u .

An *F-bundle* (\mathcal{H}, ∇) over B consists of a vector bundle \mathcal{H} over $B \times \mathbb{D}_u$ and a meromorphic flat connection ∇ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B . We refer to Definition 3.2.2 for the definition of logarithmic F-bundle in the formal case.

For any $b \in B$, we have a natural action

$$\mu_b: T_b B \longrightarrow \text{End}(\mathcal{H}_{b,0})$$

$$\xi \longmapsto \nabla_{u\xi}|_{\mathcal{H}_{b,0}}.$$

The F-bundle is called *maximal* at b if the action induces an isomorphism between $T_b B$ and $\mathcal{H}_{b,0}$ via a cyclic vector, see Definition 3.2.6. This gives a commutative product on $T_b B$ by the flatness of ∇ .

Spectral decomposition theorems

Let $K_b := \nabla_{u^2 \partial_u}|_{b,0}$, it is the action of the Euler vector field on the fiber $\mathcal{H}_{b,0}$. We show that the generalized eigenspace decomposition of K_b extends locally to a product decomposition of the F-bundle. Here are the precise statements.

Theorem 3.1.1 (Formal spectral decomposition, Theorem 3.3.32). *Let B be a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety, and (\mathcal{H}, ∇) an F-bundle over B maximal at b . Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$ stable under K_b , and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then $(\mathcal{H}, \nabla)/B$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}|_{b,0}$.*

Theorem 3.1.2 (Non-archimedean spectral decomposition, Theorem 3.3.42). *Let B be an admissible open neighborhood of a rational point b in a smooth \mathbb{k} -analytic space, and (\mathcal{H}, ∇) an F-bundle over B maximal at b . Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$ stable under K_b , and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then there exists an admissible open neighborhood U of b such that $(\mathcal{H}|_U, \nabla|_U)/U$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ extending the decomposition of $\mathcal{H}_{b,0}$.*

For proving the spectral decomposition, first we establish a formal and a non-archimedean version of the Frobenius theorem (see Theorems 3.3.7 and 3.3.10), by solving recursively a system of partial differential equations (see Proposition 3.3.4). By the maximality assumption, we obtain an F-manifold structure on the base B of the F-bundle (see Definition 3.3.11 and Lemmas 3.3.24 and 3.3.35). The eigenspaces of K_b induce a decomposition of the tangent space $T_b B$ as a \mathbb{k} -algebra, and we show that this decomposition extends locally around b (Theorems 3.3.13 and 3.3.20). To do so, we first prove that the algebra structure on the tangent spaces decomposes via deformations of \mathbb{k} -algebras (Lemmas 3.3.15 and 3.3.22). Then, using the F-identity (3.3.12) of the F-manifold, we prove that the induced decomposition of the tangent bundle

is a decomposition into commuting integrable distributions (Proposition 3.3.19). Finally, using the formal and non-archimedean versions of the Frobenius theorem, we integrate those distributions and produce a decomposition of the F-manifold $B \simeq \prod_{i \in I} B_i$.

Having decomposed the base B , we use maximality again to obtain a splitting of $\mathcal{H}|_{u=0}$. The link between the connection ∇ and the F-manifold structure implies that this decomposition is stable under the residue endomorphisms $\nabla_{u^2 \partial_u}|_{u=0}$ and $\nabla_{u\xi}|_{u=0}$ for any $\xi \in TB$. Using the disjoint spectra assumption, we extend this decomposition to a decomposition of \mathcal{H} stable under $\nabla_{u^2 \partial_u}$ by a recursive procedure, and obtain the decomposition in the formal case, see Proposition 3.3.26. In the non-archimedean case, through a careful analysis of the recursion and the norms of the coefficients, we show that the decomposition is convergent over an admissible open neighborhood of b , see Proposition 3.3.36. Finally, using flatness, we prove that the connection also decomposes according to the splitting of \mathcal{H} (Proposition 3.3.29), and that each piece is the pullback of a maximal F-bundle on B_i from the decomposition of the base B .

Extension of framing

A *framing* for an F-bundle $(\mathcal{H}, \nabla)/B$ is roughly a local trivialization of \mathcal{H} in which the connection involves no positive powers of u (Definition 5.2.4). It is analogous to the notion of decoration on variations of nc-Hodge structures in [82, §4.1.3]. Framings do not exist in general. We prove in the following that if a framing exists at a point $b \in B$ and is strong in the logarithmic case, then it extends uniquely and explicitly to a formal or non-archimedean analytic neighborhood.

Theorem 3.1.3 (Theorem 3.4.2). *Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. A framing at b extends to a framing over B if and only if it is strong with respect to D (see Definition 3.4.1). In this case, the extension is unique and explicitly determined.*

Theorem 3.1.4 (Theorem 3.4.26). *Let B be an admissible open neighborhood of a rational point b in a smooth \mathbb{k} -analytic space. Let (\mathcal{H}, ∇) be a non-archimedean F-bundle over B . Then every framing at b extends uniquely and explicitly to a framing over an admissible open neighborhood U of b in B .*

The proofs are carried out by reformulating the problem into a system of partial differential equations ((3.4.4)-(3.4.7)), which is then solved inductively on the number of variables. If there are no logarithmic directions and (t_1, \dots, t_n) are coordinates on B centered at b , we first solve (3.4.6) in the t_1 -direction at $t_2 = \dots = t_n = 0$ order by order in t_1 , by observing that the equation provides a recursive relation. We use this solution as an initial condition, and then solve (3.4.6) in the t_2 -direction at $t_3 = \dots = t_n = 0$. Using flatness of the connection, we prove that the solution obtained solves the equation in the t_1 -direction as well, for all t_2 . In this way, we solve (3.4.6) for all directions, and we show that the solution also solves (3.4.4) using flatness again. In the non-archimedean case, we prove that the solution converges by bounding its coefficients using (3.4.6).

The extension in the formal setting works also for logarithmic F-bundles, under the assumption that the framing at b is strong with respect to D (see Definition 3.4.1). This condition implies that the residues $\mu_b(v)$ at b along $u = 0$ have nilpotent adjoint endomorphism for $v \in T_b D$, a property we call the nilpotency condition (see Definition 3.4.11). This nilpotency condition allows us to extract a recursive relation from (3.4.5), so we can reconstruct a solution to the equation order by order. We proceed as in the non-logarithmic case and solve the system of PDEs inductively on the number of variables, this time starting from the logarithmic directions.

Based on the extension of framing theorem, we give a reconstruction result for isomorphisms of logarithmic F-bundles with framing in Section 3.4. We can always reconstruct the bundle isomorphism from its restriction to a point and the framing. In the maximal case, we can also reconstruct the map on the base from its restriction to a point, up to some multiplicative constants in the logarithmic directions.

Proposition 3.1.5 (Proposition 3.4.31). *For $i = 1, 2$, let $(\mathcal{H}_i, \nabla_i)/(B_i, D_i)$ be a logarithmic F-bundle where B_i is the formal neighborhood of a rational point in a smooth \mathbb{k} -variety. Let $(f, \Phi): (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \rightarrow (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ be an isomorphism between logarithmic F-bundles with $f(b_1) = b_2$. Assume $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ has a framing ∇_1^{fr} .*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $\mathcal{H}_1|_{b_1 \times \text{Spf } \mathbb{k}[u]}$.*

2. If $(\mathcal{H}_1, \nabla_1)$ and $(\mathcal{H}_2, \nabla_2)$ are maximal, then the base map f is also uniquely determined by its restriction to b_1 , up to some multiplicative constants in the logarithmic directions. The reconstruction is explicit after fixing compatible cyclic vectors at b_1 and b_2 .

Motivated by the extension of framing theorem and our applications in Section 3.5, we prove the following classification result for framed F-bundles over a point.

Proposition 3.1.6 (Corollary 3.4.35). *Let $\mathcal{H} \simeq H \times \mathbb{k}[[u]]$ be a trivialized rank m vector bundle over $\mathbb{k}[[u]]$. Let (\mathcal{H}, ∇) and (\mathcal{H}, ∇') be two F-bundles framed in the given trivialization, and write*

$$\begin{aligned}\nabla_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K} + \mathbf{G}, \\ \nabla'_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K}' + \mathbf{G}'.\end{aligned}$$

Assume \mathbf{K} has simple eigenvalues. Then (\mathcal{H}, ∇) is isomorphic to (\mathcal{H}, ∇') if and only if there exists $\phi \in \mathrm{GL}(H)$ such that

1. $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$, and
2. in an eigenbasis of \mathbf{K} , we have $(\mathbf{G})_{ii} = (\phi^{-1} \circ \mathbf{G}' \circ \phi)_{ii}$ for $1 \leq i \leq m$.

Furthermore, the gauge equivalence is uniquely and explicitly determined by the initial condition ϕ at $u = 0$.

Proceeding order by order in u , we reformulate the gauge equivalence of the two connections as a system of equations (3.4.38)-(3.4.39) involving the adjoint map $[\mathbf{K}, \cdot]$. When the eigenvalues are not simple, the equations are hard to solve because the map $[\mathbf{K}, \cdot]$ does not have an easy description. We provide a partial classification in Theorem 3.4.34, under the assumption that all the generalized eigenspaces of \mathbf{K} have the same dimension, and by restricting the type of coefficients we allow in the connections. The assumption on the coefficients allows us to work relative to a universal algebra. Relative to this algebra, the endomorphism \mathbf{K} has simple eigenvalues, and we are able to solve the system.

We illustrate an application of these results in the next paragraph. The reconstruction of isomorphisms also has applications in the reconstruction of

mirror maps in Hodge-theoretic mirror symmetry, which we plan to explore in a subsequent work.

Application to the decomposition of quantum cohomology

Let $V \rightarrow X$ be a rank m vector bundle on a smooth complex projective variety X , $P := \mathbb{P}(V) \xrightarrow{\pi} X$ the associated projective bundle, and $h := c_1(\mathcal{O}_P(1))$. We have a natural splitting

$$iso: \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i] \xrightarrow{\sum h^i \cup \pi^*} H^*(P, \mathbb{Q}). \quad (3.1.7)$$

Fix an ample class $\omega_X \in H^2(X, \mathbb{Z})$, and a homogeneous basis $\{T_j\}_{0 \leq j \leq N}$ of $H^*(X, \mathbb{Q})$ extending $\{\mathbf{1}, \omega_X\}$. We obtain the A-model maximal F-bundle (\mathcal{H}, ∇) for P over a formal base B with closed point b given by $0 \in H^*(X, \mathbb{Q})$ (see Example 3.2.25). Let $X' := \coprod_{0 \leq i \leq m-1} X$, and (\mathcal{H}', ∇') the A-model maximal F-bundle over a formal base B' with closed point b' given by $\Delta(a) \in H^*(X', \mathbb{C})$. We denote by $(a_{i,j})$ the coordinates of $\Delta(a)$ in the basis of $H^*(X', \mathbb{C})$ induced from $\{T_j\}$ using (3.1.7).

Our first result shows the existence and uniqueness of a gauge equivalence over the base points.

Theorem 3.1.8 (Theorem 3.5.16). *There exists an F-bundle isomorphism*

$$\Phi(u): (\mathcal{H}, \nabla)|_b \rightarrow (\mathcal{H}', \nabla')|_{b'},$$

whose components Φ_{ij} (as power series in u) are given by the cup-product with elements in $H^*(X, \mathbb{C})$ if and only if the coordinates of the base point $\Delta(a)$ satisfy

$$\sum_{j: \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j = c_1 V + m \lambda_i, \quad (3.1.9)$$

where λ_i was defined in Lemma 3.5.8.

Furthermore, in this case Φ is uniquely determined by the H^0 -components of $\Phi_{ij}|_{u=0}$, and $\Delta(a)$ is uniquely determined by (3.1.9), up to a shift in $\bigoplus_{i=1}^m H^2(X, \mathbb{C})$.

Next, we extend the uniqueness result over the bases B and B' . The existence is shown by Iritani-Koto [72].

Theorem 3.1.10 (Theorem 3.5.20). *Let $(f, \Phi): (\mathcal{H}, \nabla)/B \rightarrow (\mathcal{H}', \nabla')/B'$ be an isomorphism of F-bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*
2. *The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.*

In Theorems 3.5.22 and 3.5.24, we state the analogous results in the case of blowups of smooth complex projective varieties.

Related works

Various related but slightly different concepts of F-bundles have been studied in the literature. We refer to [36, 102] for Frobenius manifolds, [119, 121] for Saito structures, [35, 63] for (TE)-structures and variations, [4] for semi-infinite variations of Hodge structures, [61, 62] for F-manifolds, [82, 83] for nc-Hodge structures, and [21, 107, 123] for other related works. Logarithmic variants of Frobenius manifolds and (TE)-structures were introduced in [114, 115].

Works related to our decomposition theorems for F-bundles include [36] for the decomposition of semisimple Frobenius manifolds, [119] for the decomposition of meromorphic connections, [62] for the decomposition of F-manifolds, and [130] for the comparison of the spectral decomposition and the vanishing cycle decomposition of nc-Hodge structures. Analogs of our extension of framing theorem were studied in [32, 73] for the q -direction, in [35] for the t -direction, and in [75] under different assumptions. We refer to [72, 78] for the decomposition of quantum D-modules for projective bundles and blowups.

Acknowledgements

Non-archimedean F-bundles were considered in [84], and play a pivotal role in the application from Gromov–Witten theory to birational geometry. Discussions with Ludmil Katzarkov, Maxim Kontsevich and Tony Pantev provided invaluable motivations, perspectives and ideas for this paper. Special thanks to Hiroshi Iritani who generously answered our questions and pointed out relevant literature. We are grateful to Sheel Ganatra, Mark Gross, Yuki Koto, Todor Milanov, Yong-Geun Oh, Constantin Teleman, Yukinobu Toda, Song Yu and Xiaolei Zhao for inspiring discussions around the subject. The authors were partially supported by NSF grants DMS-2302095 and DMS-2245099.

3.2 Basic definitions and examples

In this section, we give the basic definitions regarding F-bundles and give the example of the A-model F-bundle.

Notion of F-bundle

Let \mathbb{D}_u denote the germ at 0 in a \mathbb{k} -analytic closed unit disk with coordinate u .

Definition 3.2.1 (F-bundle). Let B be a smooth \mathbb{k} -analytic space (resp. a smooth formal scheme over \mathbb{k}). An *F-bundle* (\mathcal{H}, ∇) over B consists of a vector bundle \mathcal{H} over $B \times \mathbb{D}_u$ (resp. over $B \times \mathrm{Spf} \mathbb{k}[[u]]$), and a meromorphic flat connection ∇ on \mathcal{H} with poles along $u = 0$, such that $\nabla_{u^2 \partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B .

For applications to Gromov–Witten theory (see Section 3.2), the base B , the vector bundle \mathcal{H} and the connection ∇ should all be understood in the context of supergeometry (see [?, §4]).

Given a map $f: B' \rightarrow B$, the pullback $f^*(\mathcal{H}, \nabla) := ((f \times \mathrm{id}_u)^*\mathcal{H}, (f \times \mathrm{id}_u)^*\nabla)$ is an F-bundle on B' .

In the formal case, we introduce the notion of logarithmic F-bundle.

Definition 3.2.2 (Logarithmic F-bundle). Let B be a smooth formal scheme over \mathbb{k} together with a normal crossing divisor $D \subset B$. A *logarithmic F-bundle* over (B, D) consists of a vector bundle \mathcal{H} over $B \times \mathrm{Spf} \mathbb{k}[[u]]$ and a meromorphic flat connection ∇ on \mathcal{H} with poles along $u = 0$, such that $\nabla_{u^2 \partial_u}$ and $\nabla_{u\xi}$ are regular for any log tangent vector field ξ on B .

Below we formulate several definitions for logarithmic F-bundles, which also apply to non-archimedean F-bundles, up to replacing logarithmic tangent vectors by analytic tangent vectors.

Remark 3.2.3 (Restriction to $u = 0$). Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle and ξ a logarithmic vector field on B . The failure of \mathcal{O}_B -linearity of the operator $\nabla_{u\xi}$ is given by the symbol

$$\begin{aligned} \sigma(\nabla_{u\xi}): T^*B \otimes_{\mathcal{O}_{B \times \mathrm{Spf} \mathbb{k}[[u]]}} \mathcal{H} &\longrightarrow \mathcal{H} \\ df \otimes h &\longmapsto [\nabla_{u\xi}, f]h. \end{aligned}$$

We have $\sigma(\nabla_{u\xi})(df \otimes h) = df(u\xi)h$, which vanishes at $u = 0$. We thus obtain a map

$$\begin{aligned} \mu: TB(-\log D) &\longrightarrow \text{End}_{\mathcal{O}_B}(\mathcal{H}|_{u=0}) \\ \xi &\longmapsto \nabla_{u\xi}|_{u=0}. \end{aligned} \quad (3.2.4)$$

In a similar way, the restriction of $\nabla_{u^2\partial_u}$ to $\mathcal{H}|_{u=0}$ is \mathcal{O}_B -linear.

Let $b = \text{Spec } \mathbb{k} \rightarrow B$ be a closed point. The map (3.2.4) induces a map

$$\mu_b: T_b B(-\log D) \longrightarrow \text{End}(\mathcal{H}_{b,0}). \quad (3.2.5)$$

Let K_b denote the action of $\nabla_{u^2\partial_u}$ on $\mathcal{H}_{b,0}$. The flatness of ∇ implies that the image of μ_b consists of commuting operators, which also commute with K_b .

Definition 3.2.6. A logarithmic F-bundle $(\mathcal{H}, \nabla)/(B, D)$ is called *maximal* (resp. *over-maximal*) at a closed point $b = \text{Spec } \mathbb{k} \rightarrow B$ if there exists a *cyclic vector* for the action μ_b , i.e. a vector $h \in \mathcal{H}_{b,0}$ such that the map

$$T_b B(-\log D) \longrightarrow \mathcal{H}_{b,0}, \quad v \longmapsto \mu_b(v)(h)$$

is an isomorphism (resp. epimorphism). It is called maximal (resp. over-maximal) if it is maximal (resp. over-maximal) everywhere.

In the maximal case, the dimension of $T_b B(-\log D)$ is equal to the rank of \mathcal{H} , and μ_b induces an inclusion from $T_b B(-\log D)$ to $\text{End}(\mathcal{H}_{b,0})$. We obtain a commutative associative product structure on $T_b B$, given by

$$\mu_b(v_1 \star v_2)(h) = \mu_b(v_2) \circ \mu_b(v_1)(h). \quad (3.2.7)$$

Definition 3.2.8. Let $(\mathcal{H}, \nabla)/(B, D)$ be a maximal logarithmic F-bundle. The unique logarithmic vector field Eu on B with $\mu(\text{Eu}) = K := \nabla_{u^2\partial_u}|_{u=0}$ is called the *Euler vector field*.

Definition 3.2.9. For a logarithmic F-bundle (\mathcal{H}, ∇) over (B, D) , a *framing* is another flat connection ∇^{fr} on \mathcal{H} without poles, such that in the local trivializations of \mathcal{H} given by ∇^{fr} , if we denote by H the vector space of local flat sections, the original connection ∇ has the form

$$\nabla_{\partial_u} = \partial_u + \frac{1}{u^2} \mathbf{K} + \frac{1}{u} \mathbf{G}, \quad \nabla_{\xi} = \xi + \frac{1}{u} \mathbf{A}(\xi) \quad (3.2.10)$$

for any logarithmic vector field ξ on B , where \mathbf{K}, \mathbf{G} are $\text{End}(H)$ -valued functions on B , and \mathbf{A} is an $\text{End}(H)$ -valued 1-form on B .

We give the definition of product of logarithmic F-bundles. The definition is analogous in the non-archimedean case.

Definition 3.2.11 (Product of F-bundles). The product of two logarithmic F-bundles $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ and $(\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ is the F-bundle $(\mathcal{H}, \nabla)/(B, D)$ defined over $B = B_1 \times B_2$, with divisor $D = (D_1 \times B_2) \cup (B_1 \times D_2)$, by

$$\begin{aligned}\mathcal{H} &= pr_1^* \mathcal{H}_1 \oplus pr_2^* \mathcal{H}_2, \\ \nabla &= pr_1^* \nabla_1 \oplus pr_2^* \nabla_2,\end{aligned}$$

where $pr_i: B_1 \times B_2 \times \mathrm{Spf} \mathbb{k}[[u]] \rightarrow B_i \times \mathrm{Spf} \mathbb{k}[[u]]$ denotes the projection for $i = 1, 2$.

Example of A-model F-bundle

Let X be a smooth projective variety over \mathbb{C} . The rational Gromov–Witten invariants of X can be encoded in an F-bundle, called the A-model F-bundle associated to X , also known as the quantum D-module (see [49]). Here we will give a logarithmic version and a non-archimedean version of the A-model F-bundle.

Gromov–Witten potential and quantum product

Fix a homogeneous basis $(T_i)_{0 \leq i \leq N}$ of $H^*(X, \mathbb{Q})$, such that $T_0 = \mathbf{1}$ is the unit, and (T_1, \dots, T_k) is a basis of $H^2(X, \mathbb{Q})$. Let $(T^i)_{0 \leq i \leq N}$ denote the dual basis with respect to the cup product pairing.

Let $\mathbb{Q}[[\mathrm{NE}(X, \mathbb{Z})]]$ denote the completion of $\mathbb{Q}[\mathrm{NE}(X, \mathbb{Z})] = \mathbb{Q}[q^\beta \mid \beta \in \mathrm{NE}(X, \mathbb{Z})]$ with respect to the maximal ideal $(q^\beta, \beta \neq 0)$. We write $\mathbb{k}[[\mathrm{NE}(X, \mathbb{Z})]] := \mathbb{Q}[[\mathrm{NE}(X, \mathbb{Z})]] \otimes_{\mathbb{Q}} \mathbb{k}$.

The *genus 0 Gromov–Witten potential* is

$$\Phi = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \dots, i_n} \langle T_{i_1} \cdots T_{i_n} \rangle_{0, n}^\beta t_{i_1} \cdots t_{i_n} \in \mathbb{Q}[[\mathrm{NE}(X, \mathbb{Z})]][[t_0, \dots, t_N]], \quad (3.2.12)$$

where $\langle \cdots \rangle_{0, n}^\beta$ denotes the Gromov–Witten invariants of X of genus 0, class β and observables T_{i_1}, \dots, T_{i_n} .

The *(big) quantum product* is given by

$$\begin{aligned}\star: H^*(X, \mathbb{Q}) \otimes H^*(X, \mathbb{Q}) &\longrightarrow H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[[\mathrm{NE}(X, \mathbb{Z})]][[t_0, \dots, t_N]] \\ T_i \star T_j &\longmapsto \sum_r \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_r} T^r,\end{aligned} \quad (3.2.13)$$

where

$$\frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_r} = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \dots, i_n} \langle T_i T_j T_r T_{i_1} \cdots T_{i_n} \rangle_{0, n+3}^\beta t_{i_1} \cdots t_{i_n}. \quad (3.2.14)$$

In Section 3.5, we will use a quantum product at a shifted origin, which we explain in the following lemma.

Lemma 3.2.15. *Let $\Delta(a) = \sum_{0 \leq i \leq N} a_i T_i \in H^*(X, \mathbb{k})$. Assume $\Delta(a)$ has no terms of degree 1 or 2. Then applying the shift $t = (t_0, \dots, t_N) \mapsto t + a = (t_0 + a_0, \dots, t_N + a_N)$ to Φ produces a well-defined element $\Phi(t + a) \in \mathbb{k}[[\text{NE}(X, \mathbb{Z})]][[t_0, \dots, t_N]]$.*

Proof. Before the shift, for $\alpha = (\alpha_0, \dots, \alpha_N) \in \mathbb{N}^{N+1}$, the coefficient of the monomial $q^\beta t_0^{\alpha_0} \cdots t_N^{\alpha_N}$ in Φ is $\frac{1}{\alpha_0! \cdots \alpha_N!} \langle T_0^{\alpha_0} \cdots T_N^{\alpha_N} \rangle_{0, |\alpha|}^\beta$. The coefficient of the monomial $t_0^{\alpha_0} \cdots t_N^{\alpha_N}$ of $\Phi(t + a)$ is given by evaluating

$$\frac{1}{\alpha_0! \cdots \alpha_N!} \frac{\partial^{|\alpha|} \Phi(t + a)}{\partial^{\alpha_0} t_0 \cdots \partial^{\alpha_N} t_N}$$

at $t = 0$. By the chain rule, this is the same as evaluating the derivative of the unshifted potential Φ at $t = a$. So, it is enough to check that this evaluation makes sense, i.e. that it gives an element in $\mathbb{k}[[\text{NE}(X, \mathbb{Z})]]$. The coefficient of q^β in the derivative of Φ is

$$\sum_{\gamma \in \mathbb{N}^{N+1}} \frac{1}{\gamma_0! \cdots \gamma_N!} \langle T_0^{\alpha_0 + \gamma_0} \cdots T_N^{\alpha_N + \gamma_N} \rangle_{0, |\alpha| + |\gamma|}^\beta t_0^{\gamma_0} \cdots t_N^{\gamma_N}. \quad (3.2.16)$$

We claim that the above sum is finite when evaluated at a . By the unit axiom of Gromov–Witten invariants, the part of the sum with $\gamma_0 > 0$ is finite: if T_0 appears in a nonzero n -pointed Gromov–Witten invariant, then $n = 3$ and $\beta = 0$. We now prove that there are finitely many terms with $\gamma_0 = 0$. If a nonzero Gromov–Witten invariant $\langle T_0^{\alpha_0} T_1^{\alpha_1 + \gamma_1} \cdots T_N^{\alpha_N + \gamma_N} \rangle_{0, |\alpha| + |\gamma|}^\beta$ contributes to the sum, the formula for the virtual dimension gives

$$\sum_{0 \leq i \leq N} \alpha_i \text{codim } T_i + \sum_{1 \leq i \leq N} \gamma_i \text{codim } T_i = 2(\dim X - 3 + |\alpha| + |\gamma| + \beta \cdot c_1 T_X).$$

Since we assume that there is no shift in the H^1 and H^2 -directions, the monomial $t_0^{\gamma_0} \cdots t_N^{\gamma_N}$ evaluated at $t = a$ is 0, unless $\gamma_i = 0$ for $\text{codim } T_i \in \{1, 2\}$. Then, when evaluating the $\gamma_0 = 0$ part of (3.2.16) at $t = a$, nonzero terms satisfy $\text{codim } T_i \geq 3$ for $\gamma_i \neq 0$. We deduce

$$3|\gamma| \leq \sum_{1 \leq i \leq N} \gamma_i \text{codim } T_i = 2|\gamma| + \text{constant}.$$

It follows that the sum (3.2.16) is finite at $t = a$, completing the proof. \square

By Lemma 3.2.15, the quantum product is also well-defined on a formal neighborhood of the shifted point $\Delta(a) \in H^*(X, \mathbb{k})$.

Logarithmic A-model F-bundle

Let U be the formal neighborhood of a cohomology class $\Delta(a) \in H^*(X, \mathbb{k})$ at which the quantum potential is well-defined. Using the basis $(T_i)_{0 \leq i \leq N}$, we write $U = \mathrm{Spf} \mathbb{k}[[t_0, \dots, t_N]]$.

For $\xi \in H^2(X, \mathbb{k})$, we define a derivation $\xi q \partial_q$ of $\mathbb{k}[[\mathrm{NE}(X, \mathbb{Z})]]$ by

$$\xi q \partial_q(q^\beta) := (\beta \cdot \xi) q^\beta.$$

Definition 3.2.17. The *logarithmic A-model F-bundle of X at base point $\Delta(a)$* is the logarithmic F-bundle (\mathcal{H}, ∇) over $\mathrm{Spf} \mathbb{k}[[\mathrm{NE}(X, \mathbb{Z})]] \times U$ defined as follows:

1. The bundle \mathcal{H} is trivial with fiber $H^*(X, \mathbb{k})$.
2. Let

$$\begin{aligned} \mathbf{K} &:= \left[c_1(T_X) + \sum_{i: \deg T_i \neq 2} \frac{\deg T_i - 2}{2} t_i T_i \right] \star, \\ \mathbf{G} &:= \frac{1}{2}(\deg_X - \dim X), \\ \mathbf{A}(\tau) &:= \tau \star, \quad \tau \in H^*(X, \mathbb{k}), \\ \mathbf{A}(\xi) &:= \xi \star, \quad \xi \in H^2(X, \mathbb{k}), \end{aligned}$$

where $\deg_X(\alpha) = i\alpha$ for $\alpha \in H^i(X, \mathbb{k})$, and \star is the quantum product shifted at $\Delta(a)$. The connection ∇ is given by

$$\begin{aligned} \nabla_{\partial_u} &= \partial_u - \frac{1}{u^2} \mathbf{K} + \frac{1}{u} \mathbf{G}, \\ \nabla_{\partial_\tau} &= \partial_\tau + \frac{1}{u} \mathbf{A}(\tau), \\ \nabla_{\xi q \partial_q} &= \xi q \partial_q + \frac{1}{u} \mathbf{A}(\xi). \end{aligned}$$

Non-archimedean A-model F-bundle

In the non-archimedean setting, \mathbb{k} is a complete non-archimedean field of characteristic 0 with a nontrivial valuation whose restriction to \mathbb{Q} is trivial.

Let $N^1(X)/\text{Tor}$ denote the Néron-Severi group of X modulo torsion. The valuation of \mathbb{k} induces a map

$$v: (N^1(X)/\text{Tor}) \otimes_{\mathbb{Z}} \mathbb{G}_{\text{m}/\mathbb{k}} \rightarrow (N^1(X)/\text{Tor}) \otimes_{\mathbb{Z}} \mathbb{R}. \quad (3.2.18)$$

Since the ample cone $\text{Amp}(X)$ is open in $N^1(X)_{\mathbb{R}}$, its preimage $B_q := v^{-1}(\text{Amp}(X))$ is a \mathbb{k} -analytic space. Let B_t^{even} be the product of a \mathbb{k} -analytic affine line and an open polyunit disk, where the affine line has coordinate t_0 and the polyunit disk has coordinates t_i for $\deg T_i \in \{2, 4, 6, \dots\}$. Let B_t^{odd} be the purely odd vector space with coordinates t_i for $\deg T_i \in \{1, 3, 5, \dots\}$. Let $B := B_q \times B_t^{\text{even}} \times B_t^{\text{odd}}$.

Lemma 3.2.19. *The genus 0 Gromov–Witten potential*

$$\Phi = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \dots, i_n} \langle T_{i_1} \cdots T_{i_n} \rangle_{\beta} t_{i_1} \cdots t_{i_n} \in \mathbb{Q}[[\text{NE}(X, \mathbb{Z})]][[t_i]],$$

defines an analytic function over B .

Proof. Let $\sigma \subset N^1(X)_{\mathbb{R}}$ be any simplicial cone generated by ample classes $\omega_1, \dots, \omega_m$. Let B'_q be the preimage of σ under the valuation map (3.2.18), and $B' = B'_q \times B_t^{\text{even}} \times B_t^{\text{odd}}$. Then Φ is analytic over B' , since the restriction of Φ to B' is given by the power series with rational coefficients

$$\Phi = \sum_{n \geq 0, \beta} \frac{1}{n!} q_1^{\beta \cdot \omega_1} \cdots q_m^{\beta \cdot \omega_m} \sum_{i_1, \dots, i_n} \langle T_{i_1} \cdots T_{i_n} \rangle_{\beta} t_{i_1} \cdots t_{i_n} \in \mathbb{Q}[[\{q_j\}, \{t_i\}],$$

which is polynomial in t_0 by the unit axiom. Since the union of all such σ covers the ample cone, the proof is complete. \square

Lemma 3.2.19 implies that the quantum product is convergent over the non-archimedean base space B .

Definition 3.2.20. The *non-archimedean A-model F-bundle of X* is the F-bundle (\mathcal{H}, ∇) over B defined by the same formulas as in Definition 3.2.17.

Maximal logarithmic F-bundle

The F-bundles defined above are not maximal because the base has larger dimension than the fibers. We can cut down the base dimension by choosing one q -variable and eliminating one t -variable as follows.

Fix a nef class $\omega \in N^1(X)$. It induces a projection

$$\mathbb{k}[\mathrm{NE}(X, \mathbb{Z})] \rightarrow \mathbb{k}[q], \quad q^\beta \mapsto q^{\beta \cdot \omega}. \quad (3.2.21)$$

Assumption 3.2.22. Assume that for any i_1, \dots, i_n and d , there are finitely many β such that $\beta \cdot \omega = d$ and $\langle T_{i_1} \cdots T_{i_n} \rangle_{0,n}^\beta \neq 0$.

Lemma 3.2.23. *Assumption 3.2.22 holds if there exists $\epsilon \in \mathbb{Q}$ such that $\omega + \epsilon c_1(T_X)$ is ample. In particular, it holds if ω is ample, or if X is Fano.*

Proof. Recall that the virtual dimension of $\mathcal{M}_{0,n}(X, \beta)$ is equal to $\dim X - 3 + \beta \cdot c_1(T_X) + n$. If $\langle T_{i_1} \cdots T_{i_n} \rangle_{0,n}^\beta \neq 0$, we have $\dim_{\mathrm{vir}} \mathcal{M}_{0,n}(X, \beta) = \sum_{j=1}^n \mathrm{codim} T_{i_j}$. So $\beta \cdot c_1(T_X)$ is fixed given T_{i_1}, \dots, T_{i_n} . If $\beta \cdot \omega$ is also given, then $\beta \cdot (\omega + \epsilon c_1(T_X))$ is fixed too. This is only possible for finitely many β , since $\omega + \epsilon c_1(T_X)$ is assumed ample. \square

Lemma 3.2.24. *Under Assumption 3.2.22, the Gromov–Witten potential $\Phi \in \mathbb{Q}[\mathrm{NE}(X, \mathbb{Z})][[t_0, \dots, t_N]]$ as in (3.2.12) induces an element $\Phi^\omega \in \mathbb{Q}[[q]][[t_0, \dots, t_N]]$, via the projection (3.2.21). Conversely, Φ is uniquely determined by Φ^ω .*

Proof. Assumption 3.2.22 guarantees that Φ^ω is well-defined. Let us prove the other direction. Fix i_1, \dots, i_n and d . Knowing Φ^ω , we can form the following series

$$\Psi = \sum_{r_1, \dots, r_k} \frac{1}{r_1! \cdots r_k!} \sum_{\beta \cdot \omega = d} \langle T_{i_1} \cdots T_{i_n} T_1^{r_1} \cdots T_k^{r_k} \rangle_{0, n+r_1+\dots+r_k}^\beta s_1^{r_1} \cdots s_k^{r_k} \in \mathbb{Q}[[s_1, \dots, s_k]],$$

where T_1, \dots, T_k constitute a basis of $H^2(X, \mathbb{Q})$. By the divisor axiom, we have

$$\Psi = \sum_{r_1, \dots, r_k} \frac{1}{r_1! \cdots r_k!} \sum_{\beta \cdot \omega = d} \langle T_{i_1} \cdots T_{i_n} \rangle_{0,n}^\beta (\beta \cdot T_1)^{r_1} s_1^{r_1} \cdots (\beta \cdot T_k)^{r_k} s_k^{r_k} \in \mathbb{Q}[[s_1, \dots, s_k]].$$

Comparing the coefficients, we conclude that every $\langle T_{i_1} \cdots T_{i_n} \rangle_{0,n}^\beta$ is uniquely determined by Ψ , therefore by Φ^ω . \square

Example 3.2.25 (Maximal A-model F-bundle). Assume $\omega = T_1$ satisfies Assumption 3.2.22. Let $\Delta(a) \in H^*(X, \mathbb{k})$ be a cohomology class at which the quantum potential is well-defined. Let $U = \mathrm{Spf} \mathbb{k}[[t_0, \dots, t_N]]$ be the formal neighborhood of $\Delta(a)$ in $H^*(X, \mathbb{k})$, $U' \subset U$ the closed subspace given by $t_1 = 0$, and $B' = \mathrm{Spf} \mathbb{k}[[q]] \times U'$. Then the potential Φ^ω in Lemma 3.2.24 produces a maximal logarithmic F-bundle over B' by the same formulas as in Definition 3.2.17. Indeed, the multiplicative unit $\mathbf{1}$ is a cyclic vector at 0 by the unit axiom.

3.3 Spectral decomposition of maximal F-bundles

In this section, we establish the spectral decomposition theorem for maximal F-bundles in the formal and non-archimedean settings, see Theorems 3.3.32 and 3.3.42. We first prove in §3.3 formal and non-archimedean analogs of the Frobenius theorem in differential geometry using an argument that we call “generalized flatness”. We study the decomposition of the base as F-manifolds in Section 3.3. The spectral decomposition theorems are presented and proved in Section 3.3.

Recall that \mathbb{k} is a field of characteristic 0. In the non-archimedean setting, we equip \mathbb{k} with a complete nontrivial valuation whose restriction to \mathbb{Q} is trivial.

Frobenius theorem

Generalized flatness for systems of PDEs

We prove a criterion ensuring the existence of a unique formal solution to some systems of quasi-linear PDEs in Proposition 3.3.4. We also prove a non-archimedean version in a special case in Lemma 3.3.6. Throughout, we set $M_0 := \mathbb{k}^m$. We denote by \mathfrak{m} the maximal ideal (t_1, \dots, t_n) in $\mathbb{k}[[t_1, \dots, t_n]]$.

Notation 3.3.1. We use the following notations for tuples of integers:

1. Let \preceq denote the partial order on \mathbb{N}^n defined by

$$(r_i)_{1 \leq i \leq n} \preceq (s_i)_{1 \leq i \leq n} \iff \forall 1 \leq i \leq n, r_i \leq s_i.$$

2. For $r = (r_i) \in \mathbb{N}^n$, let $|r| := \sum_{1 \leq i \leq n} r_i$.
3. For $r = (r_i)_{1 \leq i \leq n} \in \mathbb{N}^n$ and $1 \leq j \leq n$, we set

$$\tau_j(r) := (r_1, \dots, r_{j-1}, r_j + 1, r_{j+1}, \dots, r_n) \in \mathbb{N}^n.$$

Definition 3.3.2. Let $\mathcal{D} = (D_i: M_0 \otimes_{\mathbb{k}} \mathfrak{m} \rightarrow M_0 \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]])_{1 \leq i \leq n}$ be a system of differential operators of the form $D_i = \partial_{t_i} - f_i$, with $f_i: M_0 \otimes_{\mathbb{k}} \mathfrak{m} \rightarrow M_0 \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]]$ an arbitrary map. We say that the system \mathcal{D} is *generalized flat* if the two following conditions are satisfied:

1. For every $d \in \mathbb{N}$ and every $1 \leq i \leq n$, the composition

$$M_0 \otimes_{\mathbb{k}} \mathfrak{m} \xrightarrow{f_i} M_0 \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]] \longrightarrow M_0 \otimes_{\mathbb{k}} (\mathbb{k}[[t_1, \dots, t_n]]/\mathfrak{m}^d)$$

factors through $M_0 \otimes_{\mathbb{k}} (\mathfrak{m}/\mathfrak{m}^d)$.

2. If $\varphi \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$ satisfies $D_i(\varphi) = 0 \pmod{\mathfrak{m}^d}$ for all $1 \leq i \leq n$, then $\partial_{t_i}(f_j(\varphi)) = \partial_{t_j}(f_i(\varphi)) \pmod{\mathfrak{m}^d}$ for all $1 \leq i, j \leq n$.

Remark 3.3.3. Condition (1) means that for $\varphi \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$, the total t -degree d terms of $f_i(\varphi)$ depend on terms in φ of total t -degree at most d . This assumption allows to solve the associated system of PDEs recursively. It is automatically satisfied if the components of $f(\varphi)$ are power series in the components of φ .

Our notion of generalized flat systems of PDEs allows to prove the following existence and uniqueness result.

Proposition 3.3.4. *Let $(D_i: M_0 \otimes_{\mathbb{k}} \mathfrak{m} \rightarrow M_0 \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]])_{1 \leq i \leq n}$ be a generalized flat system of differential operators. Then there exists a unique $\varphi \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$ satisfying $D_i(\varphi) = 0$ for all $1 \leq i \leq n$.*

Proof. In this proof, for $\varphi \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$ and $\ell \in \mathbb{N}^n$, we denote by $[f_i(\varphi)]_{\ell}$ the coefficient of t^{ℓ} in $f_i(\varphi)$.

For the uniqueness, if $\varphi = \sum_{\ell \in \mathbb{N}^n} \varphi_{\ell} t^{\ell}$ is a solution of the differential system, then φ satisfies the recursive relations with respect to t -monomials

$$(\ell_i + 1)\varphi_{\tau_i(\ell)} = [f_i(\varphi)]_{\ell}. \tag{3.3.5}$$

This uniquely determines the coefficients of φ from the initial condition $\varphi_0 = 0$.

For the existence, we construct inductively on $d \in \mathbb{N}$ an element $\varphi^{(d)} \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$ such that

1. $\varphi^{(d)}$ has terms of degree at most $d + 1$,

2. if $d \geq 1$, then $\varphi^{(d)} = \varphi^{(d-1)} \pmod{\mathfrak{m}^{d+1}}$,
3. $D_i(\varphi^{(d)}) = 0 \pmod{\mathfrak{m}^{d+1}}$ for all $1 \leq i \leq n$.

Set $\varphi^{(0)} := \sum_{i=1}^n [f_i(0)]_0 t_i$, it satisfies (1), (2) and (3) for $d = 0$.

For the inductive step, fix $d \in \mathbb{N}$ and assume $\varphi^{(d)}$ is constructed. Given $\ell \in \mathbb{N}^n$ with $|\ell| = d + 2$, there exists a minimal index i_0 and a unique $\ell' \in \mathbb{N}^n$ such that $\ell = \tau_{i_0}(\ell')$. The index i_0 corresponds to the first nonzero component of ℓ . We set $\varphi_\ell := \frac{1}{\ell_{i_0}} [f_{i_0}(\varphi^{(d)})]_{\ell'}$, and define

$$\varphi^{(d+1)} := \varphi^{(d)} + \sum_{\substack{\ell \in \mathbb{N}^n \\ |\ell|=d+2}} \varphi_\ell t^\ell.$$

By construction $\varphi^{(d+1)}$ satisfies (1) and (2), it remains to check (3). By the inductive assumption (2) and Condition (1) of generalized flatness, we have $[f_i(\varphi^{(d+1)})]_\ell = [f_i(\varphi^{(d)})]_\ell$ for all $\ell \in \mathbb{N}^n$ such that $|\ell| \leq d + 1$. Thus we only need to check that the added coefficients φ_ℓ with $|\ell| = d + 2$ satisfy the recursive relations (3.3.5) for all $1 \leq i \leq n$.

Fix $\ell \in \mathbb{N}^n$ with $|\ell| = d + 2$, and an index i . Let i_0 be as in the definition of φ_ℓ , then there exists a unique $\ell_0 \in \mathbb{N}^n$ with $|\ell_0| = d$ such that $\ell = \tau_i \tau_{i_0}(\ell_0) = \tau_{i_0} \tau_i(\ell_0)$. By the construction of φ_ℓ , the recursive relation (3.3.5) in the t_i -direction is equivalent to

$$\ell_i [f_{i_0}(\varphi^{(d+1)})]_{\tau_i(\ell_0)} = \ell_{i_0} [f_i(\varphi^{(d+1)})]_{\tau_{i_0}(\ell_0)}.$$

Since $|\tau_i(\ell_0)| = |\tau_{i_0}(\ell_0)| = d + 1$, the induction hypothesis (2) and Condition (1) of generalized flatness imply $[f_{i_0}(\varphi^{(d+1)})]_{\tau_i(\ell_0)} = [f_{i_0}(\varphi^{(d)})]_{\tau_i(\ell_0)}$, and similarly for the right hand side. Then the recursion relation for ℓ is equivalent to

$$[\partial_{t_{i_0}} f_i(\varphi^{(d)})]_{\ell_0} = [\partial_{t_i} f_{i_0}(\varphi^{(d)})]_{\ell_0},$$

which follows from Condition (2) of generalized flatness. We conclude that $\varphi^{(d+1)}$ satisfies (3), proving the inductive step.

Condition (2) of the construction implies that $\{\varphi^{(d)} \pmod{\mathfrak{m}^d}\}_{d \geq 0}$ is an inductive system producing a well-defined element $\tilde{\varphi} \in M_0 \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]]$ such that $\tilde{\varphi} = \varphi^{(d)} \pmod{\mathfrak{m}^{d+2}}$ for all $d \geq 0$. Condition (3) of the construction implies that $\tilde{\varphi}$ satisfies the recursive relations (3.3.5) for all $\ell \in \mathbb{N}^n$, hence it is a solution of $D_i(\varphi) = 0$. Thus $\tilde{\varphi}$ satisfies $D_i(\tilde{\varphi}) = 0$ for $1 \leq i \leq n$, completing the proof. \square

We denote by T_n the Tate \mathbb{k} -algebra in n variables. For $\rho \in \sqrt{|\mathbb{k}^\times|}$ (the value group of the algebraic closure of \mathbb{k}), we denote by $T_n(\rho)$ the \mathbb{k} -affinoid algebra associated to the closed polydisk of radius ρ and dimension n ([15, §6.1.5]), consider the norm

$$\left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha t^\alpha \right|_\rho := \max_\alpha |a_\alpha| \rho^{|\alpha|}.$$

Lemma 3.3.6. *For $1 \leq i \leq n$ and $1 \leq k \leq m$, let $Y_i^k \in T_m = \mathbb{k}\langle x_1, \dots, x_m \rangle$. Let $|Y| := \max_{1 \leq i, k \leq n} |Y_i^k|$, assume $|Y| > 0$. Let $f = (f_k)_{1 \leq k \leq m} \in M_0 \otimes_{\mathbb{k}} \mathfrak{m}$ satisfying ($1 \leq i \leq n$, $1 \leq k \leq m$)*

$$\partial_{t_i} f_k = Y_i^k(f_1, \dots, f_m).$$

Then the components of f are convergent on the open polydisk of radius $|Y|^{-1}$ and have norms bounded by 1. Equivalently, f induces a map $\mathrm{Sp} T_n(\rho) \rightarrow \mathrm{Sp} T_n$ for all $\rho \in \sqrt{|\mathbb{k}^\times|}$ with $0 < \rho < |Y|^{-1}$.

Proof. Write $f_i = \sum_{\alpha \in \mathbb{N}^n} f_{i,\alpha} t^\alpha$ and $Y_i^k = \sum_{r \in \mathbb{N}^m} y_r^{(i,k)} x^r$. We have $|Y| = \sup |y_r^{(i,k)}|$. By assumption we have $f_{i,0} = 0$, which ensures that the composition $Y_i^k(f_1, \dots, f_m)$ is well-defined.

For $d \in \mathbb{N}$, we set $v_d := \max_{1 \leq i \leq m, |\alpha|=d} |f_{i,\alpha}|$. We will prove $v_d \leq |Y|^d$ by induction on d . We have $v_0 = 0 \leq 1 = |Y|^0$. Next, fix $d > 0$ and assume we have proved $v_e \leq |Y|^e$ for all $e < d$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = d - 1$. Then for $1 \leq k \leq n$, as in (3.3.5), we have the recursion

$$f_{i,\tau_k(\alpha)} = \frac{1}{\alpha_k + 1} [Y_i^k(f_1, \dots, f_m)]_\alpha,$$

where the right hand side is the coefficient of t^α in $Y_i^k(f_1, \dots, f_m)$. We now express this coefficient. For $r \in \mathbb{N}^m$, let $\mathcal{P}(r, \alpha)$ denote the set of partitions of α into $|r|$ -tuples. We write an element of $\mathcal{P}(r, \alpha)$ as $\{\alpha_1^{(1)}, \dots, \alpha_{r_1}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{r_m}^{(m)}\}$, where $\alpha_p^{(q)} \in \mathbb{N}^n$ for each p, q . The coefficient can then be expressed as the finite sum

$$[Y_i^k(f_1, \dots, f_m)]_\alpha = \sum_{r \in \mathbb{N}^m} y_r^{(i,k)} \sum_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r, \alpha)} \prod_{1 \leq q \leq m} \prod_{1 \leq p \leq r_q} f_{q, \alpha_p^{(q)}}.$$

We deduce

$$|f_{i,\tau_k(\alpha)}| \leq |Y| \max_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r, \alpha)} \prod_{1 \leq q \leq m} \prod_{1 \leq p \leq r_q} |f_{q, \alpha_p^{(q)}}|$$

$$\begin{aligned}
&\leq |Y| \max_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r, \alpha)} \prod_{1 \leq q \leq m} \prod_{1 \leq p \leq r_q} |Y|^{\alpha_p^{(q)}} \\
&\leq |Y| \times |Y|^{|\alpha|} = |Y|^d.
\end{aligned}$$

Let $0 < \rho < |Y|^{-1}$ in $\sqrt{|\mathbb{k}^\times|}$, we then have $|f_{i,\alpha}| \rho^{|\alpha|} \leq (\rho|Y|)^{|\alpha|} \leq 1$. This implies that $f_i \in T_n(\rho)$, since $\rho|Y| < 1$, and that $|f_i|_\rho \leq 1$, and the lemma follows. \square

Frobenius theorem

We prove the formal and non-archimedean analogs of the Frobenius theorem in differential geometry, which states that a local basis of commuting vector fields comes from coordinates.

Theorem 3.3.7. *Let $B = \text{Spf } \mathbb{k}[[t_1, \dots, t_n]]$ and let $(Y_i)_{1 \leq i \leq n}$ be a commuting basis of vector fields on B . Then there exists a unique automorphism $\varphi: B \rightarrow B$ such that $d\varphi(\partial_{t_i}) = \varphi^* Y_i$ for all $1 \leq i \leq n$.*

Proof. Let b be the closed point of B , given by $t_1 = \dots = t_n = 0$. Let $\mathfrak{m} = (t_1, \dots, t_n)$ denote the maximal ideal of $\mathbb{k}[[t_1, \dots, t_n]]$. We write $Y_i = \sum_k Y_i^k \partial_{t_k}$, with $Y_i^k \in \mathbb{k}[[t_1, \dots, t_n]]$. Working in coordinates, giving $\varphi: B \rightarrow B$ is equivalent to giving $\varphi_1, \dots, \varphi_n \in \mathbb{k}[[t_1, \dots, t_n]]$ such that $\varphi_i(0) = 0$. Furthermore, φ is invertible if and only if the differential at b is invertible, i.e. if and only if the matrix $\left(\frac{\partial \varphi_i}{\partial t_j}\right)_{1 \leq i, j \leq n}$ is invertible at $t = 0$. The condition $d\varphi(\partial_{t_i}) = \varphi^* Y_i$ is equivalent to

$$\sum_{1 \leq k \leq n} \frac{\partial \varphi_i}{\partial t_k}(t) \partial_{t_k} = \sum_{1 \leq k \leq n} Y_i^k(\varphi_1(t), \dots, \varphi_n(t)) \partial_{t_k}. \quad (3.3.8)$$

Since $\varphi_i(0) = 0$, the composition on the right hand side is well-defined.

For $1 \leq i \leq n$, consider the first-order quasi-linear differential operator

$$\begin{aligned}
D_i: \mathbb{k}^n \otimes_{\mathbb{k}} \mathfrak{m} &\longrightarrow \mathbb{k}^n \otimes_{\mathbb{k}} \mathbb{k}[[t_1, \dots, t_n]] \\
(\varphi_1, \dots, \varphi_n) &\longmapsto \left(\frac{\partial \varphi_k}{\partial t_i} - Y_i^k(\varphi_1, \dots, \varphi_n) \right)_{1 \leq k \leq n}.
\end{aligned}$$

Equation (3.3.8) is equivalent to $D_i(\varphi_1, \dots, \varphi_n) = 0$ for $1 \leq i \leq n$. We will prove that the system $\{D_i = 0\}$ is generalized flat, and apply Proposition 3.3.4. Condition (1) of Definition 3.3.2 is satisfied because the components of Y_i are power series in the argument. We now check Condition (2). Assume

$(\varphi_1, \dots, \varphi_n) \in \mathbb{k}^n \otimes_{\mathbb{k}} \mathfrak{m}$ satisfies $D_i(\varphi_1, \dots, \varphi_n) = 0 \pmod{\mathfrak{m}^d}$ for all $1 \leq i \leq n$. Then, since $[Y_i, Y_j] = 0$, we have ($1 \leq i, k \leq n$)

$$\begin{aligned} \frac{\partial(Y_j^k(\varphi_1, \dots, \varphi_n))}{\partial t_i} &= \sum_s \frac{\partial \varphi_s}{\partial t_i} \frac{\partial Y_j^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \pmod{\mathfrak{m}^d} \\ &= \sum_s Y_i^s(\varphi_1, \dots, \varphi_n) \frac{\partial Y_j^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \pmod{\mathfrak{m}^d} \\ &= \sum_s Y_j^s(\varphi_1, \dots, \varphi_n) \frac{\partial Y_i^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \pmod{\mathfrak{m}^d} \\ &= \frac{\partial(Y_i^k(\varphi_1, \dots, \varphi_n))}{\partial t_j} \pmod{\mathfrak{m}^d}. \end{aligned}$$

We deduce from Proposition 3.3.4 that the components $(\varphi_1, \dots, \varphi_n)$ of φ are uniquely determined and that they can be constructed inductively. The associated morphism $\varphi: B \rightarrow B$ is automatically an automorphism, because its differential at b is given by the matrix $(Y_j^i(0))_{1 \leq i, j \leq n}$, which is invertible by assumption. \square

Lemma 3.3.9. *Let X be a \mathbb{k} -analytic space, and $x \in X$ a smooth \mathbb{k} -rational point. There exists an admissible open neighborhood $U \subset X$ of x and an open immersion $U \hookrightarrow \mathrm{Sp} T_n$, where $n = \dim_x X$.*

Proof. Since x is a smooth rigid point, there exists an admissible affinoid neighborhood $U \subset X$ of x and an étale map $U \rightarrow Y := \mathrm{Sp} T_n$, with $n = \dim_x X$. Up to shrinking U , we may assume that $f^{-1}(f(x)) = \{x\}$. We will show that $f_x^*: \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ is an isomorphism, then f restricts to an open immersion on an affinoid open neighborhood of x by [15, 7.3.3/Corollary 6]. By [15, 7.3.3/Proposition 5], it is enough to check that the induced morphism $\hat{f}_x^*: \hat{\mathcal{O}}_{Y, f(x)} \rightarrow \hat{\mathcal{O}}_{X, x}$ on the completed local rings is an isomorphism.

Since f is étale, we have $f_x^*(\mathfrak{m}_{f(x)}) = \mathfrak{m}_x$, in particular $\hat{\mathcal{O}}_{X, x}$ is a complete $\hat{\mathcal{O}}_{Y, f(x)}$ -module. Since x is a \mathbb{k} -rational, the map \hat{f}_x^* is an isomorphism modulo $\mathfrak{m}_{f(x)}$, hence \hat{f}_x^* is surjective by [125, Tag 0315]. The Krull dimension of noetherian local rings is invariant under completion. Since $\dim_x X = \dim_{f(x)} Y$, necessarily \hat{f}_x^* is injective using [125, Tag 00KW]. This concludes the proof. \square

Theorem 3.3.10. *Let B be a smooth \mathbb{k} -analytic space, and (Y_1, \dots, Y_n) be a commuting basis of local vector fields around a rational point $b \in B$. Then, there exists admissible open neighborhoods $V \subset B$ of b and $U \subset \mathrm{Sp} T_n$ of 0 and an isomorphism $\varphi: U \rightarrow V$ such that $\varphi(b) = 0$ and $d\varphi(\partial_{t_i}|_U) = \varphi^*(Y_i|_V)$.*

Proof. By Lemma 3.3.9, we may assume that $B \simeq \mathrm{Sp} T_n$. We start by applying Theorem 3.3.7 to the restriction of the vector fields $(Y_i)_{1 \leq i \leq n}$ to a formal neighborhood $\widehat{B} = \mathrm{Spf} \mathbb{k}[[t_1, \dots, t_n]]$ of $0 \in \mathrm{Sp} T_n$. This produces a unique formal automorphism $\widehat{\varphi} = (\varphi_1, \dots, \varphi_n): \widehat{B} \rightarrow \widehat{B}$ satisfying the relations (3.3.8). We will prove that $\widehat{\varphi}$ extends to admissible open neighborhoods of 0.

Let $|Y| := \max_i |Y_i|$, and let $\rho \in \sqrt{|\mathbb{k}^\times|}$ such that $\rho < \min(1, |Y|^{-1})$. By Lemma 3.3.6, $\widehat{\varphi}$ extends to a map $\varphi: \mathrm{Sp} T_n(\rho) \rightarrow \mathrm{Sp} T_n$. The truncations of φ coincide with the truncations of $\widehat{\varphi}$. In particular, they induce isomorphisms $T_n/\mathfrak{m}^d \xrightarrow{\sim} T_n(\rho)/\mathfrak{m}^d$ for all $d \geq 0$. We conclude the proof using [14, §3.3 Lemma 18(ii)]. \square

Decomposition theorems for F-manifolds

In this subsection, we prove the decomposition theorems for formal and non-archimedean versions of F-manifolds, see Theorems 3.3.13 and 3.3.20.

Decomposition theorem for formal F-manifolds

The notion of F-manifold was introduced by Hertling and Manin as a weaker version of Frobenius manifolds, see [61] and the monograph [102, I.§5].

Definition 3.3.11 (F-manifold). Let B be a smooth formal scheme or a smooth \mathbb{k} -analytic space. An F-manifold structure on B is a \mathcal{O}_B -bilinear commutative associative product \star on the tangent bundle TB , satisfying the *F-identity*: for any (local) vector fields X, Y, Z, W we have

$$P_{X \star Y}(Z, W) = X \star P_Y(Z, W) + (-1)^{|X||Y|} Y \star P_X(Z, W), \quad (3.3.12)$$

where

$$P_X(Z, W) := [X, Z \star W] - [X, Z] \star W - (-1)^{|X||Z|} Z \star [X, W].$$

We prove the following decomposition result for formal F-manifolds.

Theorem 3.3.13. *Let B be a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. Let \star denote an F-manifold structure with unit on B . Assume that there exists a splitting as \mathbb{k} -algebras*

$$T_b B = \bigoplus_{i \in I} A_i. \quad (3.3.14)$$

Then there exists formal F-manifolds (B_i, \star_i) such that

1. (B, \star) is isomorphic to $\prod_{i \in I} (B_i, \star_i)$ as F -manifolds with unit,
2. and the induced decomposition of (TB, \star) restricts to (3.3.14) at b .

The idea of the proof is the following. We obtain a decomposition of TB into sheaves of subalgebras in Lemma 3.3.15, induced from that of $T_b B$. Proposition 3.3.19 will show that the direct summands of TB define commuting foliations (in the sense of [3, Definition 2.1]). We can then integrate them using the Frobenius theorem (Theorem 3.3.7).

Lemma 3.3.15. *Let A be a unital associative commutative \mathbb{k} -algebra and I a finite set. Assume A admits a splitting $A \simeq \bigoplus_{i \in I} A_i$ as \mathbb{k} -algebras. Then the splitting extends over any deformation of A over $\mathbb{k}[[t_1, \dots, t_n]]$.*

Proof. Let $\tilde{R} := \mathbb{k}[[t_1, \dots, t_n]]$ and let \tilde{A} be an \tilde{R} -algebra which is a deformation of A . Let $\mathfrak{m} = (t_1, \dots, t_n)$, and for $k \in \mathbb{N}$, $A_k := \tilde{A}/\mathfrak{m}^{k+1}\tilde{A}$ and $B_k := (\tilde{R}/\mathfrak{m}^{k+1})^{\oplus I}$.

We will prove by induction on $\ell \geq 0$ that for any $0 \leq k \leq \ell$, there are \tilde{R} -algebra maps $B_k \rightarrow A_k$ that fit into a commutative diagram

$$\begin{array}{ccccccc}
 A_\ell & \longrightarrow & A_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & A_0 \\
 \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
 B_\ell & \longrightarrow & B_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0.
 \end{array} \tag{3.3.16}$$

For $\ell = 0$, the \tilde{R} -algebra structures on $A_0 \simeq A$ and $B_0 \simeq \mathbb{k}^{\oplus I}$ are induced by the compositions of the quotient map $\tilde{R} \rightarrow \mathbb{k}$ with the structural maps $\mathbb{k} \rightarrow A$ and $\mathbb{k} \rightarrow \mathbb{k}^{\oplus I}$. In particular, the map $B_0 \rightarrow A_0$ provided by the splitting $A \simeq \bigoplus_{i \in I} A_i$ is a map of \tilde{R} -algebras.

Now assume that the maps $B_k \rightarrow A_k$ are constructed for $k \leq \ell$. Let us prove that the dashed arrow exists in the commutative diagram of \tilde{R} -algebras

$$\begin{array}{ccc}
 A_{\ell+1} & \longrightarrow & A_\ell \\
 \uparrow \text{---} & & \uparrow \\
 B_{\ell+1} & \longrightarrow & B_\ell.
 \end{array}$$

In other words, we are looking for a lift of the composite map $B_{\ell+1} \rightarrow A_\ell$ to $A_{\ell+1}$. Since $\ker(A_{\ell+1} \rightarrow A_\ell) = \mathfrak{m}^{\ell+1}A_{\ell+1}$, the algebra $A_{\ell+1}$ is a square-zero

extension of A_ℓ . Then, the obstruction to the existence of this lift is a class in $\text{Ext}_{\tilde{R}}^1(\mathbb{L}_{B_{\ell+1}/\tilde{R}} \otimes_{B_{\ell+1}} A_\ell, \mathfrak{m}^{\ell+1} A_{\ell+1})$. Since

$$\mathbb{L}_{B_{\ell+1}/\tilde{R}} \simeq \mathbb{L}_{B_{\ell+1}/(\tilde{R}/\mathfrak{m}^{\ell+1})} = 0,$$

the obstruction vanishes, and the lift always exists, concluding the induction.

By functoriality of limits in the category of \tilde{R} -algebras, we obtain a map of \tilde{R} -algebras

$$\tilde{R}^{\oplus I} \simeq \lim_k B_k \longrightarrow \lim_k A_k \simeq \tilde{A},$$

concluding the proof. \square

We now state two lemmas needed to prove Proposition 3.3.19.

Lemma 3.3.17. *Let R be a local domain. Let $f: M \rightarrow N$ be a surjective morphism of finite free R -modules, and $D \subset M$ a free submodule. Assume (1) $D \cap \ker f = 0$, (2) $\text{rk } D = \text{rk } N$ and (3) M/D is torsion-free. Then f restricts to an isomorphism $D \xrightarrow{\sim} N$.*

Proof. Let $S := \text{Frac}(R)/R$. We have $N/f(D) \simeq M/(\ker f + D)$. We prove that this module is torsion-free. Since $\ker f \cap D = 0$, we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M/D \longrightarrow M/(\ker f + D) \longrightarrow 0.$$

Applying $\otimes_R S$ gives the exact sequence

$$0 = \text{Tor}_1^R(M/D, S) \longrightarrow \text{Tor}_1^R(M/(\ker f + D), S) \longrightarrow \ker f \otimes_R S \xrightarrow{\varphi} M/D \otimes_R S,$$

and we see that $M/(\ker f + D)$ is torsion-free if and only if φ is injective. Since M/D is torsion-free, the module $D \otimes_R S$ is identified with a submodule of $M \otimes_R S$ and we have $M/D \otimes_R S \simeq (M \otimes_R S)/(D \otimes_R S)$. Since $M/\ker f \simeq N$ is torsion-free, the module $\ker f \otimes_R S$ is identified with a submodule of $M \otimes_R S$, and φ corresponds to the composition

$$\ker f \otimes_R S \longrightarrow M \otimes_R S \longrightarrow (M \otimes_R S)/(D \otimes_R S),$$

where the first map is the canonical inclusion and the second map is the canonical projection. Then, since $\ker f + D$ is torsion-free, we have

$$\ker(\varphi) \simeq (\ker f \otimes_R S) \cap (D \otimes_R S) \simeq (\ker f \cap D) \otimes_R S = 0.$$

We deduce that $N/f(D)$ is torsion-free. But since $\text{rk } N = \text{rk } f(D)$, the quotient $N/f(D)$ is a torsion module. We conclude that $N/f(D) = 0$, and the lemma follows. \square

Lemma 3.3.18. *Let $B = \text{Spf } \mathbb{k}[[t_1, \dots, t_n]]$. Let \mathcal{D} be a free \mathcal{O}_B -subsheaf of TB stable under the Lie bracket and such that TB/\mathcal{D} is torsion-free. Then \mathcal{D} admits an \mathcal{O}_B -basis of commuting vector fields.*

Proof. We denote by ∂_i the vector field associated to t_i . The coordinates $(t_i)_{1 \leq i \leq n}$ provide a trivialization $TB = \bigoplus_{1 \leq i \leq n} \mathcal{O}_B \partial_i$.

Let m denote the rank of \mathcal{D} , then up to reordering the coordinates we may assume $\mathcal{D} \cap \bigoplus_{m+1 \leq i \leq n} \mathcal{O}_B \partial_i = 0$. If $m = n$ there is nothing to show. Assume $m < n$, then there exists i_1 such that $\mathcal{O}_B \partial_{i_1} \cap \mathcal{D} = 0$. Then $\mathcal{D}^{(1)} := \mathcal{D} \oplus \mathcal{O}_B \partial_{i_1}$ is a free \mathcal{O}_B -module of rank $m + 1$. We can thus apply the same argument inductively until we obtain a free \mathcal{O}_B -module of rank n , and obtain in this way vector fields $(\partial_{i_1}, \dots, \partial_{i_{n-m+1}})$ such that $\mathcal{D} \cap \bigoplus_{m+1 \leq j \leq n} \mathcal{O}_B \partial_{i_j} = 0$.

Let $B' = \text{Spf } \mathbb{k}[[t_1, \dots, t_m]]$ and $\pi: B \rightarrow B'$ denote the canonical projection. Let $\psi: \mathcal{D} \rightarrow \pi^*TB'$ denote the restriction of $d\pi: TB \rightarrow \pi^*TB'$. The kernel of $d\pi$ is $\bigoplus_{m+1 \leq i \leq n} \mathcal{O}_B \partial_i$, so ψ is injective. By Lemma 3.3.17, ψ is an isomorphism. Let ∂'_i denote the vector field of B' associated to t_i . We define $X_i := \psi^{-1}(\pi^* \partial'_i)$. By construction $(X_i)_{1 \leq i \leq m}$ is an \mathcal{O}_B -basis of \mathcal{D} .

We now check that $[X_i, X_j] = 0$. The \mathcal{O}_B -linearity of $d\pi$ and π^* implies

$$d\pi[X_i, X_j] = \pi^*[\partial'_i, \partial'_j] = 0.$$

Since $[X_i, X_j]$ is a section of \mathcal{D} and $d\pi$ restricted to \mathcal{D} is an isomorphism, we deduce that X_i and X_j commute. \square

Proposition 3.3.19. *Let $B = \text{Spf } \mathbb{k}[[t_1, \dots, t_n]]$ and \star an F -manifold structure with unit on B . Assume that we have a decomposition into sheaves of subalgebras $(TB, \star) = \bigoplus_{i \in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i})$. Then:*

1. For all i we have $[\mathcal{D}_i, \mathcal{D}_i] \subset \mathcal{D}_i$.
2. For $i \neq j$ we have $[\mathcal{D}_i, \mathcal{D}_j] = 0$.
3. There exists an automorphism $\varphi: B \rightarrow B$ and a partition $\{1, \dots, n\} = \bigsqcup_{i \in I} J_i$ such that, for each $i \in I$, the pullback $\varphi^* \mathcal{D}_i$ is generated by $\{d\varphi(\partial_{t_j})\}_{j \in J_i}$.

Proof. For $i \in I$, let $p_i: TB \rightarrow \mathcal{D}_i$ denote the projection, corresponding to the multiplication by the identity section e_i of \mathcal{D}_i . We have $p_i^2 = p_i$, $p_i \circ p_j = \delta_{ij}$ and $\bigoplus_{i \in I} p_i = \text{id}$, thus $\ker p_i = \bigoplus_{j \neq i} \mathcal{D}_j$.

Let $i \in I$, we prove that \mathcal{D}_i is stable under Lie bracket. Let X, Y be sections of \mathcal{D}_i . Since $e_i \star X = X$, the F-identity gives

$$P_X(e_i, Y) = e_i \star P_X(e_i, Y) + X \star P_{e_i}(e_i, Y).$$

The left-hand side equals

$$P_X(e_i, Y) = [X, Y] - [X, e_i] \star Y - e_i \star [X, Y],$$

and the terms on the right-hand side are

$$\begin{aligned} e_i \star P_X(e_i, Y) &= e_i \star ([X, Y] - [X, e_i] \star Y - e_i \star [X, Y]) \\ &= -e_i \star [X, e_i] \star Y \\ &= -Y \star [X, e_i], \end{aligned}$$

and

$$\begin{aligned} X \star P_{e_i}(e_i, Y) &= X \star ([e_i, Y] - e_i \star [e_i, Y]) \\ &= X \star [e_i, Y] - X \star e_i \star [e_i, Y] \\ &= 0, \end{aligned}$$

where we used $e_i \star X = X$, $e_i \star Y = Y$, $e_i \star e_i = e_i$ and the commutativity of the product. Thus, the F-identity above reduces to $[X, Y] = e_i \star [X, Y]$. Equivalently, $[X, Y]$ is a section of \mathcal{D}_i , proving (1).

Fix $i, j \in I$ with $i \neq j$. Let X and Y be sections of \mathcal{D}_i and \mathcal{D}_j respectively, in particular $e_i \star X = X$ and $e_j \star Y = Y$. We need to show $[X, Y] = 0$. We have

$$\begin{aligned} [X, Y] &= [e_i \star X, e_j \star Y] \\ &= P_{e_i \star X}(e_j, Y) + [e_i \star X, Y] \star Y + [e_i \star X, Y] \star e_j \\ &= P_{e_i \star X}(e_j, Y) + \left(P_{e_j}(e_j, X) + [e_j, e_i] \star X + [e_j, X] \star e_i \right) \star Y \\ &\quad + (P_Y(e_i, X) + [Y, e_i] \star X + [Y, X] \star e_i) \star e_j \\ &= e_i \star P_X(e_j, Y) + X \star P_{e_i}(e_j, Y) + Y \star P_{e_j}(e_i, X) + e_j \star P_Y(e_i, X) \\ &= (e_i - e_j) \star [X, Y] + X \star [e_i, Y] + Y \star [e_j, X]. \end{aligned}$$

Multiplication by e_i shows that $X \star [e_i, Y] = 0$. By symmetry, we also have $Y \star [e_j, X] = 0$, so the equation reduces to

$$[X, Y] = (e_i - e_j) \star [X, Y].$$

Multiplication by e_k for k different from i and j gives $e_k \star [X, Y] = 0$, so $[X, Y]$ is a section of $\mathcal{D}_i \oplus \mathcal{D}_j$. We then have

$$(e_i + e_j) \star [X, Y] = [X, Y] = (e_i - e_j) \star [X, Y].$$

We deduce $e_j \star [X, Y] = 0$, and by symmetry $e_i \star [X, Y] = -e_i \star [Y, X] = 0$. Thus $[X, Y] = 0$, and (2) is proved.

By (1) and (2), the decomposition $TB = \bigoplus_{i \in I} \mathcal{D}_i$ is a decomposition into commuting subsheaves of Lie algebras. For each $i \in I$, we have $TB/\mathcal{D}_i \simeq \bigoplus_{j \neq i} \mathcal{D}_j$, which is torsion-free. By Lemma 3.3.18, \mathcal{D}_i admits an \mathcal{O}_B -basis of commuting vector fields. By (2), these bases assemble into a basis of commuting vector fields for sections of TB . Then (3) follows by applying Theorem 3.3.7 to the union of these bases. \square

Proof of Theorem 3.3.13. By [125, Tag 0C0S(2)], we may assume that the base B has the form $\mathrm{Spf} \mathbb{k}[[t_1, \dots, t_n]]$. The sheaf of algebras (TB, \star) corresponds to a formal deformation of $(T_b B, \star|_b)$ over $\mathbb{k}[[t_1, \dots, t_n]]$. By Lemma 3.3.15, we obtain a decomposition into sheaves of subalgebras $(TB, \star) = \bigoplus_{i \in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i})$ extending the decomposition of the fiber at b . Let $\varphi: B \rightarrow B$ be the change of coordinates provided by Proposition 3.3.19(3) and let $\{1, \dots, n\} = \coprod_{i \in I} J_i$ be the associated partition. Let $\mathcal{E}_i := \bigoplus_{j \in J_i} \mathcal{O}_B \partial_{t_j} \subset TB$, its image under $d\varphi$ generates $\varphi^* \mathcal{D}_i$.

Since φ is an automorphism of the formal neighborhood of a point, the differential $d\varphi: TB \rightarrow \varphi^* TB$ is an isomorphism. Then, we can produce another F-manifold structure on B , which we denote by $\varphi^*(\star)$, such that $\varphi: (B, \varphi^*(\star)) \rightarrow (B, \star)$ is an isomorphism of F-manifolds. Let $B_i := \mathrm{Spf} \mathbb{k}[[t_j, j \in J_i]]$, let $\iota_i: B_i \rightarrow B$ be the canonical closed immersion. By construction the subsheaves \mathcal{E}_i are stable under $\varphi^*(\star)$. Thus the restriction $\varphi^*(\star)|_{\mathcal{E}_i}$ is well-defined, and induces an F-manifold structure \star_i on B_i , such that $\iota_i: (B_i, \star_i) \rightarrow (B, \varphi^*(\star))$ is a closed immersion of F-manifolds. Since $(B, \varphi^*(\star)) \simeq \prod_{i \in I} (B_i, \star_i)$, we obtain (1), and (2) holds by construction. \square

Decomposition theorem for non-archimedean F-manifolds

Theorem 3.3.20. *Let B be a smooth \mathbb{k} -analytic space endowed with an F-manifold structure \star with unit, and $b \in B$ a \mathbb{k} -rational point. Assume there exists a splitting as \mathbb{k} -algebras*

$$T_b B = \bigoplus_{i \in I} A_i. \quad (3.3.21)$$

Then there exist an admissible open neighborhood U of b and non-archimedean F-manifolds with unit (U_i, \star_i) such that

1. $(U, \star|_U)$ is isomorphic to $\prod_{i \in I} (U_i, \star_i)$ as F-manifolds with unit,
2. and the induced decomposition of $(TU, \star|_U)$ restricts to (3.3.21) at b .

Lemma 3.3.22. *Let (B, \star) and b be as in Theorem 3.3.20. Assume there exists a splitting as \mathbb{k} -algebras*

$$T_b B = \bigoplus_{i \in I} A_i.$$

Then there exists an admissible open neighborhood U of b , and a decomposition into sheaves of subalgebras $(TU, \star|_U) = \bigoplus_{i \in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i})$ extending the decomposition of $T_b B$.

Proof. In this proof, we view the rigid \mathbb{k} -analytic spaces as Berkovich spaces. Then the base B is Hausdorff. Let $X := \text{Spec}^{\text{an}} TB$ be the relative analytic spectrum. Since TB is a finite free \mathcal{O}_B -module, the structural map $f: \text{Spec}^{\text{an}} TB \rightarrow B$ is proper as Berkovich spaces, in particular proper as topological spaces.

The splitting of $T_b B$ produces a surjection $X_b = \text{Spec}^{\text{an}} T_b B \rightarrow \prod_{i \in I} \text{Sp } \mathbb{k}$. This implies that $X_b = \prod_{i \in I} X_{b,i}$, where $X_{b,i}$ is the preimage of the i -th copy of $\text{Sp } \mathbb{k}$. Let $U \subset B$ be the open neighborhood of b given by Lemma 3.3.23, with $f^{-1}(U) = \prod_{i \in I} W_i$. We obtain a map $X \times_B U \rightarrow \prod_{i \in I} U$ extending $X_b \rightarrow \prod_{i \in I} \text{Sp } \mathbb{k}$ by mapping W_i to the i -th copy of U under f . This is equivalent to a map of sheaves of \mathcal{O}_U -algebras $\mathcal{O}_U^{\oplus I} \rightarrow TU$, producing the desired splitting. \square

Lemma 3.3.23. *Let $f: X \rightarrow B$ be a proper map between Hausdorff topological spaces. Let $b \in B$, assume that $f^{-1}(b) = \prod_{i \in I} X_{b,i}$ for a finite set I . Then, there exists an open neighborhood $U \subset B$ of b such that $f^{-1}(U)$ is a disjoint union $\prod_{i \in I} W_i$, and $W_i \cap f^{-1}(b) = X_{b,i}$.*

Proof. Since f is proper, the fiber $f^{-1}(b)$ is compact. Hence, each $X_{b,i}$ is compact. Since X is Hausdorff, there exists open subsets $V_i \subset X$ containing $X_{b,i}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$. Since f is proper, it is closed, so $U := f\left(\left(\bigcup_i V_i\right)^{\mathbb{C}}\right)^{\mathbb{C}}$ is open in B . Let $W_i := V_i \cap f^{-1}(U)$. Since $f^{-1}(U) \cap \left(\bigcup_{i \in I} V_i\right)^{\mathbb{C}} = \emptyset$, we have $f^{-1}(U) = \coprod_{i \in I} W_i$. By construction of V_i , we have $W_i \cap f^{-1}(b) = X_{b,i}$, completing the proof. \square

Proof of Theorem 3.3.20. By Lemma 3.3.22, there exists an admissible open neighborhood U_1 of b and a decomposition into sheaves of subalgebras

$$(TU_1, \star|_{U_1}) = \bigoplus_{i \in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i}),$$

extending the decomposition of $T_b B$. As in the proof of Proposition 3.3.19, the F-identity implies that $\{\mathcal{D}_i\}_{i \in I}$ define commuting integrable distributions on TU_1 .

Up to shrinking U_1 , we can choose a local basis of commuting vector fields $(Y_j)_{j \in J_i}$ of \mathcal{D}_i at b , and assemble them into a local commuting basis of TU_1 at b . By Theorem 3.3.10, there exists admissible opens $U_2 \subset U_1$ and $V \subset \mathrm{Sp} T_n$ and an isomorphism $\varphi: V \rightarrow U_2$ such that $d\varphi(\partial_{t_j}) = \varphi^*(Y_j)$, where $\{t_j\}$ are the analytic coordinates on V centered at 0. We conclude as in the formal case (see Theorem 3.3.13).

\square

Decomposition theorems for maximal F-bundles

In this subsection, we establish the spectral decomposition theorem for maximal F-bundles (see Theorems 3.3.32 and 3.3.42).

We consider a maximal F-bundle (\mathcal{H}, ∇) over a formal (resp. admissible open) neighborhood of a rational point b in a smooth \mathbb{k} -variety (resp. \mathbb{k} -analytic space). Let $K_b := \nabla_{u^2 \partial_u}|_{b,0}$. Consider a decomposition of the fiber $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$ stable under K_b , such that the induced endomorphisms $K_b|_{H_i}$ and $K_b|_{H_j}$ have disjoint spectra for each $i \neq j$. Our spectral theorem asserts that this produces a decomposition of (\mathcal{H}, ∇) into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}_{b,0}$. We refer to Section 3.1 for an outline of the proof.

The formal case

Lemma 3.3.24. *Let B be a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. Let $(\mathcal{H}, \nabla)/B$ be an F -bundle maximal at b , and let $h: B \rightarrow \mathcal{H}|_{u=0}$ be a section of cyclic vectors (see Definition 3.2.6). The data $\{(\mathcal{H}, \nabla), h\}$ induce a formal F -manifold structure on B with identity.*

Proof. Evaluation on the section of cyclic vectors h provides an isomorphism $\eta := \mu(\cdot)(h): TB \rightarrow \mathcal{H}|_{u=0}$, and a commutative and associative product on TB as in (3.2.7). Furthermore $e := \eta^{-1}(h)$ is an identity for this product since for a vector field X we have

$$\eta(X \star e) = \mu(X) \circ \eta(\eta^{-1}(h)) = \mu(X)(h) = \eta(X).$$

We refer to [34, Lemma 10] for the proof of the F -identity, which is given there for (TE)-structures. \square

Lemma 3.3.25. *Let H be a \mathbb{k} -vector space of finite dimension, and $U \in \text{End}_{\mathbb{k}}(H)$. Assume we have a decomposition $H = \bigoplus_{i \in I} H_i$ stable under U , such that the induced endomorphisms $U|_{H_i}$ and $U|_{H_j}$ have disjoint spectra for $i \neq j$. Then*

1. $\ker[\cdot, U] \subset \bigoplus_{i \in I} \text{End}_{\mathbb{k}}(H_i)$, and
2. $[\cdot, U]$ restricts to an isomorphism of $\bigoplus_{i \neq j} \text{Hom}_{\mathbb{k}}(H_j, H_i)$ onto itself.

Proof. Let \mathbb{k}^a denote an algebraic closure of \mathbb{k} . The disjoint spectra assumption implies that $H_i \otimes_{\mathbb{k}} \mathbb{k}^a$ is a direct sum of generalized eigenspaces for U . In particular, any endomorphism that commutes with U preserves this decomposition, proving (1). It follows that the restriction $[\cdot, U]: \bigoplus_{i \neq j} \text{Hom}_{\mathbb{k}}(H_j, H_i) \rightarrow \bigoplus_{i \neq j} \text{Hom}_{\mathbb{k}}(H_j, H_i)$ is injective, hence an isomorphism by comparing dimensions, proving (2). \square

Proposition 3.3.26. *Let (\mathcal{H}, ∇) be an F -bundle over a formal neighborhood $B = \text{Spf } \mathbb{k}[[t_1, \dots, t_n]]$ of $b = 0$ in an affine space. Let $K = \nabla_{u^2 \partial_u}|_{u=0}$ and $\mathcal{H}_{b,0} = \bigoplus_{i \in I} H_i$ a decomposition stable under K_b such that the induced endomorphisms on H_i have disjoint spectra.*

Let $\mathcal{H}|_{u=0} = \bigoplus_{i \in I} \mathcal{H}_{i,0}$ be a decomposition extending the decomposition of $\mathcal{H}_{b,0}$, and stable under K . Then it extends to a decomposition $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ such that $u^2 \nabla_{\partial_u}(\mathcal{H}_i) \subset \mathcal{H}_i$.

Proof. Write $t = (t_1, \dots, t_n)$ and $H = \mathcal{H}_{b,0}$. Choose a trivialization $\Phi: \mathcal{H} \simeq H \times \text{Spf } \mathbb{k}[[t, u]]$ such that $\mathcal{H}_{i,0} \simeq H_i \times \text{Spf } \mathbb{k}[[t]]$. Write the connection in the u direction as

$$\nabla_{\partial_u} = \frac{\partial}{\partial u} + \frac{U(t)}{u^2},$$

where $U(t, u) = \sum_{k \geq 0} U_k(t)u^k$ for $U_k(t) \in \text{End}(H)[[t]]$. By assumption, $U_0(t) \in \bigoplus_{i \in I} \text{End}(H_i)[[t]]$.

We will construct an automorphism $P(t, u) \in \text{Aut}(H \times \text{Spf } \mathbb{k}[[t, u]])$ with $P(t, 0) = \text{id}$, such that $P(t, 0) = \text{id}$ and $P^{-1}UP + P^{-1}\frac{\partial P}{\partial u} \in \bigoplus_{i \in I} \text{End}(H_i)[[t, u]]$. Given such a $P(t, u)$, defining \mathcal{H}_i to be the constant extension of H_i in the trivialization $P^{-1} \circ \Phi$ provides the desired splitting of \mathcal{H} .

For $m \geq 1$, $T_m(t) \in \text{End}(H)[[t]]$ and $P(t, u) = \text{id} + u^m T_m(t) \in \text{GL}(H)[[t, u]]$, write $(P^* \nabla)_{\partial_u} = \frac{\partial}{\partial u} + u^{-2} \tilde{U}(t, u)$. We have

$$\tilde{U}(t, u) - U(t, u) = \sum_{k \geq 0} (-1)^{k+1} u^{m(k+1)} T_m(t)^k [T_m(t), U] + \sum_{k \geq 0} (-1)^k m u^{m(k+1)+1} T_m(t)^{k+1}. \quad (3.3.27)$$

Note that the right-hand side of (3.3.27) has degree at least degree m in u , and the coefficient of u^m is $-[T_m(t), U_0(t)]$.

Let $<$ denote the degree lexicographic order on \mathbb{N}^n . For $v = (v_1, \dots, v_n)$, we write $t^v = t_1^{v_1} \cdots t_n^{v_n}$. Now for $T_m(t) = t^v T_{m,v}$ with $T_{m,v} \in \text{End}(H)$, we have $-[T_m(t), U_0(t)] = -[T_{m,v}, U_0(0)]t^v + T' t^{v'}$ where $T' \in \text{End}(H)[[t]]$ and $v < v'$. Write $U_k(t) = \sum_{w \in \mathbb{N}^n} U_{k,w} t^w$. By Lemma 3.3.25, we can choose $T_{m,v}$ such that $U_{m,v} - [T_{m,v}, U_0(0)] \in \bigoplus_{i \in I} \text{End}(H_i)$. By induction on $v \in \mathbb{N}^n$ using the lexicographic order on \mathbb{N}^n , we can assume $U_m(t) \in \bigoplus_{i \in I} \text{End}(H_i)[[t]]$. By induction on $m \geq 1$, we can further make $\tilde{U}(t, u) \in \bigoplus_{i \in I} \text{End}(H_i)[[t, u]]$, completing the proof. \square

Lemma 3.3.28. *Write $t = (t_1, \dots, t_n)$. Let \tilde{H} be a finite free $\mathbb{k}[[t]]$ -module, and $U(t) \in \text{End}(\tilde{H})$. Let $\tilde{H} = \bigoplus_{i \in I} \tilde{H}_i$ be a decomposition of \tilde{H} stable under $U(t)$. Assume that for $i \neq j$, the induced endomorphisms $U(t)|_{\tilde{H}_i}$ and $U(t)|_{\tilde{H}_j}$ have disjoint spectra. Let $X(t) \in \text{End}(\tilde{H})$ such that $[X(t), U(t)] \in \bigoplus_{i \in I} \text{End}(\tilde{H}_i)$, then $X(t) \in \bigoplus_{i \in I} \text{End}(\tilde{H}_i)$.*

Proof. Let $R := \mathbb{k}[[t_1, \dots, t_n]]$ and $K := \text{Frac}(R)$ its fraction field. Working over K , Lemma 3.3.25 implies that $\ker[\cdot, U] \subset \bigoplus_{i \in I} \text{End}_R(\tilde{H}_i)$.

We have a decomposition $\text{End}_R(\widetilde{H}) = \bigoplus_{i,j \in I} \text{Hom}_R(\widetilde{H}_j, \widetilde{H}_i)$. Let $X_{i,j}$ denote the components of X with respect to this splitting. Let $Y := \sum_{i \neq j} X_{i,j}$ denote the off-diagonal part of X . We will prove that $Y = 0$. Since $U \in \bigoplus_{i \in I} \text{End}(\widetilde{H}_i)$, the commutator $[Y, U]$ has vanishing diagonal, i.e. it lies in $\bigoplus_{i \neq j} \text{Hom}_R(\widetilde{H}_j, \widetilde{H}_i)$. Furthermore, using the assumption, we see that $[Y, U] = [X, U] - \sum_{i \in I} [X_{i,i}, U]$ is block diagonal. It follows that $[Y, U] = 0$, hence $Y \in \bigoplus_{i \in I} \text{End}_R(\widetilde{H}_i)$. By definition, Y is off-diagonal, so $Y = 0$, proving the lemma. \square

The following proposition implies that the decomposition in Proposition 3.3.26 induces a decomposition of F-bundle $(\mathcal{H}, \nabla) \simeq \bigoplus_{i \in I} (\mathcal{H}_i, \nabla_i)$ over B , where ∇_i is the restriction of ∇ to \mathcal{H}_i .

Proposition 3.3.29. *In the setting of Proposition 3.3.26, we have $u\nabla_\xi(\mathcal{H}_i) \subset \mathcal{H}_i$ for any vector field ξ on B .*

Proof. Write $t = (t_1, \dots, t_n)$. Let $H := \mathcal{H}|_{b,0}$, and $H = \bigoplus_{i \in I} H_i$ the splitting induced by the decomposition of \mathcal{H} . Fix a trivialization $\mathcal{H} \simeq H \times \text{Spf } \mathbb{k} - \llbracket t, u \rrbracket$ such that $\mathcal{H}_i \simeq H_i \times \text{Spf } \mathbb{k} \llbracket t, u \rrbracket$, and write

$$\nabla = d + u^{-1} \sum_{1 \leq i \leq n} T_i(t, u) dt_i + u^{-2} U(t, u) du,$$

with $U(t, u) = \sum_{k \geq 0} U_k(t) u^k$ and $T_i(t, u) = \sum_{k \geq 0} T_{i,k}(t) u^k$. By assumption, we have $U(t, u) \in \bigoplus_{i \in I} \text{End}(H_i) \llbracket t, u \rrbracket$. In particular, $U_0(t)$ induces endomorphisms in $\text{End}(H_i) \llbracket t \rrbracket$ for all $i \in I$, and the assumption on the decomposition at $t = u = 0$ implies that those have disjoint spectra.

Fix $i \in \{1, \dots, n\}$. The flatness equation $[\nabla_{\partial_u}, \nabla_{\partial_{t_i}}] = 0$ reads

$$\frac{\partial(u^{-1}T_i)}{\partial u} - \frac{\partial(u^{-2}U)}{\partial t_i} = u^{-3}[T_i, U].$$

Splitting this equation according to powers of u gives $[T_{i,0}, U_0] = 0$, and for $k \geq 1$:

$$[T_{i,k}, U_0] = (k-2)T_{i,k-1} - \frac{\partial U_{k-1}}{\partial t_i} - \sum_{\substack{k_1+k_2=k \\ k_1 < k}} [T_{k_1}, U_{k_2}]. \quad (3.3.30)$$

We prove by induction on $k \geq 0$ that $T_{i,k}$ is block diagonal, i.e. $T_{i,k} \in \bigoplus_{i \in I} \text{End}(H_i) \llbracket t \rrbracket$. The base case $k = 0$ follows from Lemma 3.3.28, because $T_{i,0}$ commutes with $U_0(t)$. Now, let $k \geq 1$ and assume $T_{i,\ell}(t)$ is block diagonal

for $\ell < k$. Since each $U_\ell(t)$ is assumed block diagonal, the right-hand side of (3.3.30) is block diagonal. Applying Lemma 3.3.28, we obtain that $T_{i,k}(t)$ is also block diagonal, completing the proof. \square

It remains to show that the above decomposition $(\mathcal{H}, \nabla) \simeq \bigoplus_{i \in I} (\mathcal{H}_i, \nabla_i)$ is compatible with the decomposition of the base.

Lemma 3.3.31. *Let $B \simeq B_1 \times B_2$ be a formal neighborhood of $b = 0$ in a product of affine spaces, and $(\mathcal{H}, \nabla)/B$ be an F -bundle over B . Assume that $\nabla_{u\xi}|_{u=0} = 0$ for all vector fields ξ in the directions of B_2 . Then there exists an F -bundle $(\mathcal{H}_1, \nabla_1)/B_1$ such that $pr_1^*(\mathcal{H}_1, \nabla_1) \simeq (\mathcal{H}, \nabla)$, where pr_1 is the projection $B \simeq B_1 \times B_2 \rightarrow B_1$.*

Proof. For $i = 1, 2$, let $t_i = (t_{i,j}, 1 \leq j \leq n_i)$ denote coordinates on B_i . Let $\mathcal{H}_1 := \mathcal{H}|_{B_1 \times \{0\} \times \mathrm{Spf} \mathbb{k}[[u]]}$. By assumption, ∇ has no pole at $u = 0$ in the directions of B_2 . Since ∇ is flat, given any trivialization of \mathcal{H}_1 we can extend it uniquely by ∇ to a trivialization of \mathcal{H} over $B_1 \times B_2 \times \mathrm{Spf} \mathbb{k}[[u]]$. This defines an isomorphism $pr_1^* \mathcal{H}_1 \simeq \mathcal{H}$, and in this trivialization we have

$$\nabla = d + u^{-1} \sum_{1 \leq j \leq n_1} T_{1,j}(t_1, t_2, u) dt_{1,j} + u^{-2} U(t_1, t_2, u) du.$$

Since ∇ is flat, we have for all $1 \leq j \leq n_1$ and $1 \leq k \leq n_2$

$$\frac{\partial(u^{-1}T_{1,j})}{\partial t_{2,k}} = 0, \quad \frac{\partial(u^{-2}U)}{\partial t_{2,k}} = 0.$$

Hence, the connection matrices in the directions of B_1 and the u -direction are independent of t_2 . This means that the connection is equal to the pullback of a connection on $B_1 \times \mathrm{Spf} \mathbb{k}[[u]]$, completing the proof. \square

Theorem 3.3.32 (Spectral decomposition theorem). *Let B be a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety, and (\mathcal{H}, ∇) an F -bundle over B maximal at b . Write $K_b = \nabla_{u^2 \partial_u}|_{b,0}$. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$ stable under K_b , and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then $(\mathcal{H}, \nabla)/B$ decomposes into a product of maximal F -bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}|_{b,0}$.*

Proof. As in the proof of Theorem 3.3.13, we may assume the base B has the form $\mathrm{Spf} \mathbb{k}[[t_1, \dots, t_n]]$. Let $h : B \rightarrow \mathcal{H}|_{u=0}$ be a section of cyclic vectors,

providing an isomorphism

$$\eta := u\nabla|_{u=0}(h) : TB \xrightarrow{\sim} \mathcal{H}|_{u=0}.$$

This induces an F-manifold structure (B, \star) on B by Lemma 3.3.24. In particular, we have a decomposition $T_b B = \bigoplus_{i \in I} E_i$ with $E_i = \eta_b^{-1}(H_i)$. Since the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint, up to extending the base field, each H_i is a direct sum of generalized eigenspaces for K_b . Since ∇ is flat, it follows that $T_b B = \bigoplus_{i \in I} E_i$ is a splitting of \mathbb{k} -algebra. By Theorem 3.3.13, we obtain a decomposition of F-manifold $B \simeq \prod_{i \in I} (B_i, \star_i)$, extending the decomposition at $T_b B$. This induces a decomposition of $TB = \bigoplus_{i \in I} \mathcal{E}_i$ as \mathcal{O}_B -algebras. We refer to sections of \mathcal{E}_i as being in the directions of B_i .

Under η , we obtain a decomposition $\mathcal{H}|_{u=0} \simeq \bigoplus_{i \in I} \mathcal{H}_{i,0}$. Since the action of K corresponds to multiplication by the Euler vector field, this decomposition is stable under K , and extends the decomposition of $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$. By Propositions 3.3.26 and 3.3.29, this further extends to a decomposition $(\mathcal{H}, \nabla) \simeq \bigoplus_{i \in I} (\mathcal{H}_i, \nabla_i)$.

For each $i \in I$ and ξ not in the directions of B_i , the action of $(\nabla_i)_{u\xi}|_{u=0}$ on $\mathcal{H}_{i,0}$ under η is the restriction of $\xi\star$ to the subalgebra \mathcal{E}_i , hence it vanishes. Then by Lemma 3.3.31, $(\mathcal{H}_i, \nabla_i)/B$ isomorphic to a pullback of F-bundle from B_i , which we also denote as $(\mathcal{H}_i, \nabla_i)/B_i$. We thus have a decomposition of F-bundle

$$(\mathcal{H}, \nabla) \simeq \bigoplus_{i \in I} pr_i^*(\mathcal{H}_i, \nabla_i),$$

where $pr_i : B \simeq \prod_{j \in I} B_j \rightarrow B_i$ is the projection to the i -th component.

It remains to check that each F-bundle in the decomposition is maximal. Let $j_i : B_i \hookrightarrow B$ be the canonical closed immersion, and $h_i := j_i^* h$. We claim that h_i is a section of cyclic vectors for $(\mathcal{H}_i, \nabla_i)/B_i$, i.e. the map $\eta_i : \xi \mapsto (\nabla_i)_{u\xi}|_{u=0}(h_i)$ is an isomorphism $TB_i \xrightarrow{\sim} \mathcal{H}_i|_{u=0}$. Since B_i is the formal neighborhood of a point in an affine space, it is enough to check that the stalk of η_i at the closed point b_i of B_i is an isomorphism. This stalk is the composition of the isomorphisms

$$T_{b_i} B_i \longrightarrow E_i \xrightarrow{\eta_b|_{E_i}} H_i,$$

hence it is an isomorphism, completing the proof. \square

Example 3.3.33 (rank 1 maximal F-bundle). Let $B = \mathrm{Spf} \mathbb{k}[[t]]$ and $b = 0 \in B$. Let $(\mathcal{H}, \nabla)/B$ be an F-bundle, maximal at b . Fixing a trivialization of \mathcal{H} , we

write the connection as $\nabla = d + u^{-2}p(t, u)du + u^{-1}q(t, u)dt$. Flatness of ∇ reduces to the equation $\frac{\partial(u^{-2}p)}{\partial t} = \frac{\partial(u^{-1}q)}{\partial u}$. Solutions are parameterized by pairs $(\psi(t, u), c) \in \mathbb{k}[[t, u]] \times \mathbb{k}$ by the rule

$$p = u \frac{\partial \psi}{\partial u} - \psi + uc, \quad q = \frac{\partial \psi}{\partial t}.$$

The F-bundle is maximal at $t = 0$ if and only if $q(0, 0) \neq 0$ or, in terms of ψ , $\frac{\partial \psi}{\partial t}(0, 0) \neq 0$.

Example 3.3.34 (simple eigenvalues). Let B be the formal neighborhood of $b = 0$ in an n -dimensional affine space. Let $(\mathcal{H}, \nabla)/B$ be an F-bundle, maximal at b . Assume that $K_b = u^2 \nabla_{\partial_u}|_{b, u=0}$ has simple eigenvalues. Then $(\mathcal{H}, \nabla)/B$ is isomorphic to a product of rank 1 maximal F-bundles.

Concretely, there exists a change of coordinates $f: \prod_{1 \leq i \leq n} \text{Spf } \mathbb{k}[[t_i]] \xrightarrow{\sim} B$, and a trivialization of $f^*(\mathcal{H}, \nabla)$ in which the connection takes the form

$$f^* \nabla = d + u^{-1} \begin{pmatrix} \frac{\partial \psi_1}{\partial t_1} dt_1 & & 0 \\ & \ddots & \\ 0 & & \frac{\partial \psi_n}{\partial t_n} dt_n \end{pmatrix} + u^{-2} \begin{pmatrix} u \frac{\partial \psi_1}{\partial u} - \psi_1 + uc_1 & & 0 \\ & \ddots & \\ 0 & & u \frac{\partial \psi_n}{\partial u} - \psi_n + uc_n \end{pmatrix} du,$$

with $(\psi_i, c_i) \in \mathbb{k}[[t_i, u]] \times \mathbb{k}$ such that $-\psi_i(0, 0)$ is an eigenvalue of K_b , and $\frac{\partial \psi_i}{\partial t_i}(0, 0) \neq 0$ (see Example 3.3.33).

When K_b has simple eigenvalues, the change of coordinates is obtained by integrating a basis of sections of eigenvectors for the connection in the u -direction.

The non-archimedean case

Next, we prove the spectral decomposition theorem in the non-archimedean case. The proof builds on the formal case, but an additional challenge lies in bounding the norms of the coefficients of the gauge transform and establishing non-archimedean convergence. We achieve these bounds through a detailed analysis of the recursive relations of the coefficients, see Proposition 3.3.36.

Lemma 3.3.35. *Let B be an admissible open neighborhood of a rational point b in a smooth \mathbb{k} -analytic space. Let $(\mathcal{H}, \nabla)/B$ be a non-archimedean F-bundle maximal at b . Then there exists an admissible open neighborhood $U \subset B$ of b such that (\mathcal{H}, ∇) admits a section of cyclic vectors, and the data $\{(\mathcal{H}, \nabla), h\}$ induces a non-archimedean F-manifold structure on U with identity.*

Proof. Being maximal is an open condition, so there exists an admissible open neighborhood $U \subset B$ of b over which a section of cyclic vector h exists. The proof is then identical to the formal case, and relies on explicit computations in local analytic coordinates centered at b . \square

Proposition 3.3.36. *Let (\mathcal{H}, ∇) be an F -bundle over $B = \mathrm{Sp} \mathbb{k}\langle t_1, \dots, t_n \rangle$, and let $b = 0 \in B$. Let $K = \nabla_{u^2 \partial_u}|_{u=0}$ and $\mathcal{H}_{b,0} = \bigoplus_{i \in I} H_i$ a decomposition stable under K_b such that the induced endomorphisms on H_i have disjoint spectra.*

Let $\mathcal{H}|_{u=0} = \bigoplus_{i \in I} \mathcal{H}_{i,0}$ be a decomposition extending the decomposition of $\mathcal{H}_{b,0}$, and stable under K . Then, there exists an admissible open neighborhood $U \subset B$ of b and a decomposition $\mathcal{H}|_U = \bigoplus_{i \in I} \mathcal{H}_i$ such that $\mathcal{H}_i|_{u=0} = \mathcal{H}_{i,0}|_U$ and $u^2 \nabla_{\partial_u}(\mathcal{H}_i) \subset \mathcal{H}_i$.

Proof. We keep the setting and notations of Proposition 3.3.26, in particular $H := \mathcal{H}_{b,0}$. Let \leq denote the degree lexicographic order on \mathbb{N}^n . We denote by $\tau(v)$ the direct successor of $v \in \mathbb{N}^n$ for this order. The gauge transformation P constructed in the formal case is an ordered product

$$P = \prod_{m \geq 1} P_m, \quad P_m = \prod_{v \in \mathbb{N}^n} P_{m,v},$$

where $P_{m,v} = \mathrm{id} + u^m t^v T_{m,v}$ and $T_{m,v} \in \mathrm{End}(H)$. Let ϕ denote the inverse of the restriction of $[\cdot, U_0(0)]$ to $\bigoplus_{i \neq j} \mathrm{Hom}(H_j, H_i)$. The gauge transformations $P_{m,v}$ are constructed inductively, and characterized by the following relations:

$$T_{m,v} = \phi(\text{off-diagonal part of the term } u^m t^v \text{ in } \tilde{U}_{m,v}), \quad (3.3.37)$$

$$\tilde{U}_{m,\tau(v)} = P_{m,v}^{-1} \tilde{U}_{m,v} P_{m,v} + u^2 P_{m,v}^{-1} \frac{\partial P_{m,v}}{\partial u}, \quad (3.3.38)$$

$$\tilde{U}_{m+1,0} = P_m^{-1} \tilde{U}_{m,0} P_m + u^2 P_m^{-1} \frac{\partial P_m}{\partial u}, \quad (3.3.39)$$

and $\tilde{U}_{1,0} = U(t, u)$ is the initial connection matrix. For an element $M(t, u) = \sum_{m,v} M_{m,v} u^m t^v \in \mathrm{End}(H)[[t, u]]$ and $\delta, \varepsilon > 0$, we let

$$|M(t, u)|_{\delta, \varepsilon} := \sup_{m \in \mathbb{N}, v \in \mathbb{N}^n} |M_{m,v}| \delta^m \varepsilon^{|v|}.$$

We denote by $\mathbb{D}(\delta, \varepsilon)$ the polydisk $\{|u| \leq \delta, |t| \leq \varepsilon\}$.

Since the gauge transformations restrict to id at $u = 0$, all the matrices $\tilde{U}_{m,v}(t, u)$ have the same constant term. We denote this common value by U_0 ,

and set $\tilde{V}_{m,v}(t, u) := \tilde{U}_{m,v}(t, u) - U_0$. Fix $\delta \leq 1$ and $\varepsilon \leq 1$ such that $\delta|\phi| \leq 1$ and $|\phi|\tilde{V}_{1,0}|_{\delta,\varepsilon} < 1$. This is possible, since $\tilde{V}_{1,0}(0, 0) = 0$.

We prove by a double induction on m and v the inequalities

$$|u^m t^v T_{m,v}|_{\delta,\varepsilon} \leq |\phi| |\tilde{V}_{m,v}|_{\delta,\varepsilon} \leq |\phi| |\tilde{V}_{m,0}|_{\delta,\varepsilon} \leq |\phi| |\tilde{V}_{1,0}|_{\delta,\varepsilon} < 1. \quad (3.3.40)$$

We use the lexicographic order on the product $\mathbb{N}_{>0} \times \mathbb{N}^n$, i.e. $(m, v) < (m', v')$ if and only if $m < m'$ or $m = m'$ and $v < v'$. For $m = 1, v = 0$, the inequalities follow from (3.3.37) and the choice of (δ, ε) . Now fix $(m, v) \in \mathbb{N}_{>0} \times \mathbb{N}^n$ with $(m, v) > (1, 0)$, and assume all the inequalities proved for $(m', v') < (m, v)$. Equation (3.3.37) gives

$$|u^m t^v T_{m,v}|_{\delta,\varepsilon} \leq |\phi| |\tilde{V}_{m,v}|_{\delta,\varepsilon}.$$

We now bound $|\tilde{V}_{m,v}|_{\delta,\varepsilon}$. If $v > 0$, then we can write $v = \tau(w)$ for some $w \geq 0$. The difference between $\tilde{V}_{m,\tau(w)}$ and $\tilde{V}_{m,w}$ is given by (3.3.27):

$$\begin{aligned} \tilde{V}_{m,\tau(w)} - \tilde{V}_{m,w} &= \tilde{U}_{m,\tau(w)} - \tilde{U}_{m,w} \\ &= \sum_{k \geq 0} (-1)^{k+1} (u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, \tilde{V}_{m,w}] \\ &\quad + \sum_{k \geq 0} (-1)^{k+1} (u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, U_0] \\ &\quad + \sum_{k \geq 0} (-1)^k u (u^m t^w)^{k+1} T_{m,w}^{k+1}. \end{aligned}$$

Let us bound each term on the right hand side. Since $|u^m t^w T_{m,w}|_{\delta,\varepsilon} < 1$, we have for all $k \geq 0$

$$|(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, \tilde{V}_{m,w}]|_{\delta,\varepsilon} \leq |u^m t^w T_{m,w}|_{\delta,\varepsilon}^{k+1} |\tilde{V}_{m,w}|_{\delta,\varepsilon} < |\tilde{V}_{m,w}|_{\delta,\varepsilon}.$$

By the definition of ϕ and (3.3.37), we have

$$|u^m t^w [T_{m,w}, U_0]|_{\delta,\varepsilon} = |u^m t^w \phi^{-1}(T_{m,w})|_{\delta,\varepsilon} \leq |\tilde{V}_{m,w}|_{\delta,\varepsilon}. \quad (3.3.41)$$

We can then bound the second term for all $k \geq 0$

$$|(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, U_0]|_{\delta,\varepsilon} \leq |u^m t^w T_{m,w}|_{\delta,\varepsilon}^k |[T_{m,w}, U_0]|_{\delta,\varepsilon} < |\tilde{V}_{m,w}|_{\delta,\varepsilon}.$$

For the third term, using the induction hypothesis and $\delta|\phi| \leq 1$, we obtain for all $k \geq 0$

$$|u^{m(k+1)+1} t^{w(k+1)} T_{m,w}^{k+1}|_{\delta,\varepsilon} \leq \delta (|\phi| |\tilde{V}_{m,w}|_{\delta,\varepsilon})^{k+1} \leq \delta |\phi| |\tilde{V}_{m,w}|_{\delta,\varepsilon} \leq |\tilde{V}_{m,w}|_{\delta,\varepsilon},$$

where we used $|\phi||\tilde{V}_{m,w}|_{\delta,\varepsilon} \leq 1$ in the second inequality. Using those bounds, we obtain the inequalities

$$|\tilde{V}_{m,\tau(w)}|_{\delta,\varepsilon} \leq \max(|\tilde{V}_{m,w}|_{\delta,\varepsilon}, |\tilde{V}_{m,\tau(w)} - \tilde{V}_{m,w}|_{\delta,\varepsilon}) \leq |\tilde{V}_{m,w}|_{\delta,\varepsilon} \leq |\tilde{V}_{m,0}|_{\delta,\varepsilon},$$

proving the inductive step when $v > 0$. If $v = 0$, then necessarily $m > 1$ and we can write $m = m' + 1$. We compare $\tilde{V}_{m'+1,0}$ to $\tilde{V}_{m',0}$. To do so, write $P_{m'} = \text{id} + u^{m'} R_{m'}(t, u)$. Similarly to the previous case, using (3.3.27) we obtain

$$\begin{aligned} \tilde{V}_{m'+1,0} - \tilde{V}_{m',0} &= \tilde{U}_{m'+1,0} - \tilde{U}_{m',0} \\ &= \sum_{k \geq 0} (-1)^{k+1} u^{m'(k+1)} R_{m'}^k [R_{m'}, \tilde{V}_{m,w}] \\ &\quad + \sum_{k \geq 0} (-1)^{k+1} u^{m'(k+1)} R_{m'}^k [R_{m'}, U_0] \\ &\quad + \sum_{k \geq 0} (-1)^k u^{m'(k+1)+1} R_{m'}^{k+1}, \end{aligned}$$

and we will use the induction hypothesis to bound each term. Since $|u^{m'} t^v T_{m',v}|_{\delta,\varepsilon} < 1$ for all $v \in \mathbb{N}^n$, we have $|u^{m'} R_{m'}|_{\delta,\varepsilon} < 1$. In particular, similarly to the case $v > 0$, the first term is bounded by $|\tilde{V}_{m',0}|_{\delta,\varepsilon}$. To handle the other terms, we use the explicit formula

$$u^{m'} R_{m'} = \sum_{\substack{k \geq 1 \\ w \in \mathbb{N}^n}} u^{km'} t^w \sum_{\substack{w_1 + \dots + w_k = w \\ w_1 > \dots > w_k}} T_{m',w_1} \cdots T_{m',w_k}.$$

Using this formula, we obtain

$$\begin{aligned} |[u^{m'} R_{m'}, U_0]|_{\delta,\varepsilon} &\leq \max_{\substack{k \geq 1, w \in \mathbb{N}^n \\ w_1 + \dots + w_k = w}} |u^{km'} t^w [T_{m',w_1} \cdots T_{m',w_k}, U_0]|_{\delta,\varepsilon} \\ &\leq \max_{\substack{k \geq 1, w \in \mathbb{N}^n \\ w_1 + \dots + w_k = w}} \max_{1 \leq i \leq k} \left(\prod_{\substack{1 \leq j \leq k \\ j \neq i}} |u^{m'} t^{w_j} T_{m',w_j}|_{\delta,\varepsilon} \times |u^{m'} t^{w_i} [T_{m',w_i}, U_0]|_{\delta,\varepsilon} \right) \\ &\leq \max_{\substack{k \geq 1, w \in \mathbb{N}^n \\ w_1 + \dots + w_k = w}} \max_{1 \leq i \leq k} \left(\prod_{\substack{1 \leq j \leq k \\ j \neq i}} |\phi||\tilde{V}_{m',w_j}|_{\delta,\varepsilon} \times |\tilde{V}_{m',w_i}|_{\delta,\varepsilon} \right) \\ &\leq \max_{\substack{k \geq 1, w \in \mathbb{N}^n \\ w_1 + \dots + w_k = w}} |\tilde{V}_{m',w_i}|_{\delta,\varepsilon} \leq |\tilde{V}_{m',0}|_{\delta,\varepsilon}. \end{aligned}$$

For the second inequality, we used the formula for the commutator of a product. The third inequality follows from the induction hypothesis at step (m', w_j) , and the inequality (3.3.41) applied to T_{m',w_i} . The fourth and fifth inequalities follow from the induction hypothesis. Then, similarly to the case $v > 0$, we

obtain that the second term is bounded by $|\tilde{V}_{m',0}|_{\delta,\varepsilon}$. We now consider the third term. For $k \geq 1$ and $w_1, \dots, w_k \in \mathbb{N}^n$, since $|\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon} \leq 1$ by the induction hypothesis, we have

$$|u^{km'} t^{w_1 + \dots + w_k} T_{m',w_1} \cdots T_{m',w_k}|_{\delta,\varepsilon} \leq (|\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon})^k \leq |\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon}.$$

In particular, we have the better bound $|u^{m'} R_{m'}|_{\delta,\varepsilon} \leq |\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon}$. Since $|\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon} \leq 1$, we obtain the bound on the third term for all $k \geq 0$

$$|u^{m'(k+1)+1} R_{m'}^{k+1}|_{\delta,\varepsilon} \leq \delta (|\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon})^{k+1} \leq \delta |\phi| |\tilde{V}_{m',0}|_{\delta,\varepsilon} \leq |\tilde{V}_{m',0}|_{\delta,\varepsilon}.$$

Similarly to the case $v > 0$, we deduce

$$|\tilde{V}_{m'+1,0}|_{\delta,\varepsilon} \leq \max(|\tilde{V}_{m'+1,0}|_{\delta,\varepsilon}, |\tilde{V}_{m'+1,0} - \tilde{V}_{m',0}|_{\delta,\varepsilon}) \leq |\tilde{V}_{m',0}|_{\delta,\varepsilon} \leq |\tilde{V}_{1,0}|_{\delta,\varepsilon},$$

concluding the induction.

Now, (3.3.40) implies that the product defining P is convergent on the polydisk $\mathbb{D}(\delta, \varepsilon)$, that P^{-1} is also convergent on $\mathbb{D}(\delta, \varepsilon)$, and that $|P|_{\delta,\varepsilon} = |P^{-1}|_{\delta,\varepsilon} = 1$. In particular, the decomposition constructed in the formal case extends to an admissible open neighborhood of $(b, 0)$, completing the proof. \square

Theorem 3.3.42 (Non-archimedean spectral decomposition theorem). *Let B be a \mathbb{k} -analytic space, $b \in B$ a smooth \mathbb{k} -rational point, and (\mathcal{H}, ∇) an F -bundle over B maximal at b . Write $K_b = \nabla_{u^2 \partial_u}|_{b,0}$. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i \in I} H_i$ stable under K_b , and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then there exists an admissible open neighborhood U of b such that the restriction $(\mathcal{H}|_U, \nabla|_U)/U$ decomposes into a product of maximal F -bundles $(\mathcal{H}_i, \nabla_i)/U_i$ extending the decomposition of $\mathcal{H}_{b,0}$.*

Proof. By Lemma 3.3.9, we can find an admissible neighborhood U of b isomorphic to an admissible open neighborhood of 0 in a \mathbb{k} -analytic affine space. Hence, we may assume that $B = \mathrm{Sp} T_n$ and $b = 0$. By Lemma 3.3.35, up to shrinking B we can find a section of cyclic vectors $h: B \rightarrow \mathcal{H}|_{u=0}$, providing an isomorphism

$$\eta := (u\nabla)|_{u=0}(h): TB \longrightarrow \mathcal{H}|_{u=0},$$

and an F -manifold structure \star on B . The splitting of $\mathcal{H}_{b,0}$ induces a splitting of $T_b B$ as a \mathbb{k} -algebra. By Theorem 3.3.20, there exists an admissible neighborhood U of b such that $(U, \star|_U)$ is isomorphic to a product of F -manifolds $\prod_{i \in I} (U_i, \star_i)$, and the induced decomposition of TU extends the decomposition of $T_b B$.

We keep denoting by (\mathcal{H}, ∇) the restriction of the F-bundle to U . The decomposition of TU induces a decomposition $\mathcal{H}|_{u=0} \simeq \bigoplus_{i \in I} \mathcal{H}_{i,0}$ satisfying the assumptions of Proposition 3.3.36. As in the formal case, this implies that there exists F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ such that

$$(\mathcal{H}, \nabla) \simeq \bigoplus_{i \in I} pr_i^*(\mathcal{H}_i, \nabla_i),$$

where $pr_i: U \simeq \prod_{j \in I} U_j \rightarrow U_i$ is the projection.

Let b_i denote the image of b under the projection $U \rightarrow U_i$, let $j_i: U_i \hookrightarrow U$ denote the canonical closed immersion and $h_i := j_i^* h$. As in the formal case, the stalk at b_i of the map $\eta_i := (u\nabla_i)|_{u=0}(h_i): TU_i \rightarrow \mathcal{H}_i|_{u=0}$ is an isomorphism. Hence $(\mathcal{H}_i, \nabla_i)/U_i$ is maximal at b_i . Up to shrinking U_i , this implies that $(\mathcal{H}_i, \nabla_i)/U_i$ is maximal, completing the proof. \square

3.4 Framing of F-bundles

In this section, we prove the extension of framing theorems (Theorems 3.4.2 and 3.4.26). In Section 3.4, we apply the extension of framing to obtain a uniqueness result for isomorphisms between maximal F-bundles admitting a framing (Proposition 3.4.31). In Section 3.4, we provide a partial classification of framed F-bundles over a point, up to gauge equivalence, under some assumptions on the coefficients of the connection (Theorem 3.4.34). When the K -operator of the F-bundle has simple eigenvalues, we obtain a full classification in Corollary 3.4.35. We will apply those results to the A-model F-bundles in Section 3.5.

Extension of framing for logarithmic formal F-bundles

Main result

Here we state the theorem of extension of framing, and fix the notations for the proof.

Definition 3.4.1. Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle and $b \in B$ a rational point. We say that a framing ∇_b^{fr} for the restricted F-bundle $(\mathcal{H}, \nabla)|_b$ is *strong* with respect to D if for any function q vanishing on D , the endomorphism $\nabla_{uq\partial_q}|_{b \times \text{Spf } \mathbb{k}[[u]]}$ is independent of u in a ∇_b^{fr} -flat trivialization of $\mathcal{H}|_{b \times \text{Spf } \mathbb{k}[[u]]}$.

Theorem 3.4.2 (Extension of framing). *Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth*

\mathbb{k} -variety. A framing ∇_b^{fr} for the restricted F -bundle $(\mathcal{H}, \nabla)|_b$ extends to a framing for (\mathcal{H}, ∇) if and only if ∇_b^{fr} is strong with respect to D . In this case, the extension is uniquely and explicitly determined from ∇_b^{fr} and (\mathcal{H}, ∇) .

We refer to Example 3.4.25 for a counter-example to the existence part of Theorem 3.4.2 without assuming the framing is strong with respect to D .

Write $B = \text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$, where $\prod_{1 \leq i \leq s} q_i = 0$ is a local equation for D at b . Let m be the rank of \mathcal{H} and $H := \mathcal{H}_{b,0}$ the fiber of \mathcal{H} . We start with any trivialization $iso: \mathcal{H} \simeq H \times B \times \text{Spf } \mathbb{k}[[u]]$ extending a ∇_b^{fr} -flat trivialization of $\mathcal{H}|_{b \times \text{Spf } \mathbb{k}[[u]]}$. Let Ω denote the connection form of ∇ in the trivialization iso . Fix a basis of H , and write

$$\Omega = \sum_{1 \leq i \leq s} u^{-1} q_i^{-1} Q^i(q, t, u) dq_i + \sum_{1 \leq j \leq n} u^{-1} T^j(q, t, u) dt_j + u^{-2} U(q, t, u) du, \quad (3.4.3)$$

where $U, Q^i, T^j \in \text{Mat}(m \times m, \mathbb{k}[[q_i, t_j, u]])$. The framing assumption at b allows us to assume that $U(0, 0, u)$ is linear in u . The assumption that the endomorphism $\nabla_{uq_i \partial_{q_i}}|_{q=t=0}$ is ∇_b^{fr} -flat means that $Q^i(0, 0, u)$ is independent of u .

We want to modify the trivialization iso by an automorphism of $H \times B \times \text{Spf } \mathbb{k}[[u]]$, to produce a new trivialization extending $iso|_{b \times \text{Spf } \mathbb{k}[[u]]}$ and in which ∇ is framed. Equivalently, we seek a gauge transformation $P(q, t, u) \in \text{GL}(m, \mathbb{k}[[q_i, t_j, u]])$ and matrices $K(q, t), G(q, t), \tilde{Q}^i(q, t), \tilde{T}^j(q, t)$ in $\text{Mat}(m \times m, \mathbb{k}[[q_i, t_j]])$ such that

$$P^{-1} \partial_u P + u^{-2} P^{-1} U P = u^{-2} K + u^{-1} G, \quad (3.4.4)$$

$$P^{-1} \partial_{q_i} P + u^{-1} q_i^{-1} P^{-1} Q^i P = u^{-1} q_i^{-1} \tilde{Q}^i, \quad (3.4.5)$$

$$P^{-1} \partial_{t_j} P + u^{-1} P^{-1} T^j P = u^{-1} \tilde{T}^j, \quad (3.4.6)$$

and satisfying $P(0, 0, u) = \text{id}$. By identifying the polar part at $u = 0$, we get an expression for the matrices $K, G, \tilde{Q}^i, \tilde{T}^j$. In particular, setting $P_0 := P(q, t, 0)$, we have the following expressions

$$\tilde{Q}^i = P_0^{-1} Q_{-1}^i P_0 \quad \text{and} \quad \tilde{T}^j = P_0^{-1} T_{-1}^j P_0, \quad (3.4.7)$$

with $Q_{-1}^i = \nabla_{uq_i \partial_{q_i}}|_{u=0}$ and $T_{-1}^j = \nabla_{u \partial_{t_j}}|_{u=0}$. We will construct P in Section 3.4 order by order in each variable, starting with the logarithmic directions.

Two matrix lemmas

We now state two matrix lemmas that we will use for the proof of Theorem 3.4.2.

Lemma 3.4.8. *Let R be a ring.*

1. *Let $T \in \text{Mat}(m \times m, R[[t]])$. Let $(X_k(t))_{k \in \mathbb{N}}$ be a sequence of matrices in $\text{Mat}(m \times m, R[[t]])$ satisfying*

$$\partial_t X_k = -[T, X_{k+1}].$$

Then $(X_k(t))_{k \in \mathbb{N}}$ is uniquely determined by $(X_k(0))_{k \in \mathbb{N}}$. In particular, if $X_k(0) = 0$ for all $k \geq 0$, then $X_k(t) = 0$ for all $k \geq 0$.

2. *Let $n \in \mathbb{N}$, and $T_1, \dots, T_n \in \text{Mat}(m \times m, R[[t_1, \dots, t_n]])$. Let $(X_k(t))_{k \in \mathbb{N}}$ be a sequence of matrices in $\text{Mat}(m \times m, R[[t_1, \dots, t_n]])$ satisfying for all $1 \leq i \leq n$*

$$\partial_{t_i} X_k = -[T_i, X_{k+1}].$$

Then $(X_k(t))_{k \in \mathbb{N}}$ is uniquely determined by $(X_k(0))_{k \in \mathbb{N}}$. In particular, if $X_k(0) = 0$ for all $k \geq 0$, then $X_k(t) = 0$ for all $k \geq 0$.

Proof. For (1), we write $X_k(t) = \sum_{\ell \in \mathbb{N}} X_{\ell,k} t^\ell$. For $d \geq 0$, we have

$$(d+1)! X_{d+1,k} = \left. \frac{\partial^{d+1} X_k}{\partial^{d+1} t} \right|_{t=0} = - \left. \frac{\partial^d}{\partial t^d} [T, X_{k+1}] \right|_{t=0} = - \sum_{s=0}^d \binom{d}{s} \left[\frac{\partial^{d-s} T}{\partial^{d-s} t}, \frac{\partial^s X_{k+1}}{\partial^s t} \right] \Big|_{t=0}.$$

This provides a recursive relation for $\{X_{d+1,k}\}_{k \in \mathbb{N}}$ in terms of $\{X_{r,k}, r \leq d\}_{k \in \mathbb{N}}$. Thus, $(X_k)_{k \geq 0}$ is uniquely determined by $(X_k(0))_{k \geq 0}$.

For (2), we apply inductively on $1 \leq i \leq n$ the single variable case with the ring $R[[t_1, \dots, t_{i-1}]]$. In this way, we prove that for $1 \leq i \leq n$, the sequence $(X_k|_{t_{i+1}=\dots=t_n=0})_{k \in \mathbb{N}}$ is uniquely determined by the sequence $(X_k|_{t_i=\dots=t_n=0})_{k \in \mathbb{N}}$. Thus $(X_k)_{k \in \mathbb{N}}$ is uniquely determined by the initial condition $(X_k|_{t_1=\dots=t_n=0})_{k \in \mathbb{N}}$.

For both (1) and (2), choosing $X_k(t) = 0$ for all $k \geq 0$ provides a sequence that satisfies the assumptions of the lemma, with the initial condition $X_k(0) = 0$. It follows from the uniqueness that this is the only solution to the equations such that $X_k(0) = 0$ for all $k \geq 0$. \square

Lemma 3.4.9. *Let R be a ring. For $1 \leq i \leq s$, let $Q_i \in \text{Mat}(m \times m, R[[q_1, \dots, q_s]])$ such that $\phi_i := \text{ad}(Q_i)|_{q=0}$ is nilpotent. Let $(X_k(q))_{k \in \mathbb{N}}$ be a sequence of matrices in $\text{Mat}(m \times m, R[[q_1, \dots, q_s]])$ satisfying for all $1 \leq i \leq s$*

$$q_i \partial_{q_i} X_k = [Q_i, X_{k+1}].$$

Then, for any initial condition $(X_k(0))_{k \in \mathbb{N}}$, there exists at most one solution $(X_k(q))_{k \in \mathbb{N}}$. In particular, if $X_k(0) = 0$ for all $k \geq 0$, then $X_k(q) = 0$ for all $k \geq 0$.

Proof. We use Notation 3.3.1. In particular, given tuples of integers $\ell = (\ell_i)_{1 \leq i \leq n}$ and $r = (r_i)_{1 \leq i \leq n}$, the length of ℓ is $|\ell| = \ell_1 + \cdots + \ell_n$, and we write $r \preceq \ell$ if $r_i \leq \ell_i$ for all $1 \leq i \leq n$. We denote the linear differential operator $q_i \partial_{q_i}$ by D_i , so the equations are $D_i X_k = [Q_i, X_{k+1}]$.

First, a direct induction shows that for all $n \in \mathbb{N}$ we can express $D_i^{n+1} X_k$ as a linear combination of terms of the form

$$[D_i^{a_1} Q_i, [\cdots, [D_i^{a_u} Q_i, X_{k+u}] \cdots]], \quad (3.4.10)$$

with $1 \leq u \leq n+1$ and $(a_v)_{1 \leq v \leq u} \in \mathbb{N}^u$ satisfying $a_1 + \cdots + a_u + u = n+1$. If we denote the coefficient of such a term by $\alpha_n(a_1, \dots, a_u)$, it is elementary to see that the sequence $(\alpha_n)_{n \in \mathbb{N}}$ is fully determined by the initial condition $\alpha_0(0) = 1$ and the recursion relation

$$\alpha_{n+1}(a_1, \dots, a_u) = \sum_{a_v \neq 0} \alpha_n(a_1, \dots, a_v - 1, \dots, a_u) + \delta_{a_u, 0} \alpha_n(a_1, \dots, a_{u-1}).$$

Write $X_k(q) = \sum_{r \in \mathbb{N}^s} X_{r,k} q_1^{r_1} \cdots q_s^{r_s}$. We will show that for $d \geq 1$, the terms $\{X_{\ell,k}, |\ell| = d\}_{k \in \mathbb{N}}$ are determined by $\{X_{r,k}, |r| < d\}_{k \in \mathbb{N}}$. It will follow directly that $(X_k(q))_{k \in \mathbb{N}}$ is uniquely determined by the initial term $(X_k(0))_{k \in \mathbb{N}}$. Fix $\ell \in \mathbb{N}^s$ with $|\ell| = d$ and $k \in \mathbb{N}$. We express $X_{\ell,k}$ in terms of $\{X_{r,k+s}, |r| < d, s \geq 1\}$. Fix i such that $\ell_i \neq 0$, and let $n \in \mathbb{N}$. We note that the coefficient of q^ℓ in $D_i^{n+1} X_k$ is $\ell_i^{s+1} X_{\ell,k}$. On the other hand, by the previous paragraph $D_i^{n+1} X_k$ is a linear combination of terms of the form (3.4.10). The coefficient of q^ℓ in (3.4.10) is expressed in terms of derivatives of Q_i and coefficients $X_{r,k+u}$ with $r \preceq \ell$ and $u \geq 1$. If $X_{\ell,k+u}$ appears in a term, then only the constant term of the terms involving Q_i contribute. If $a > 0$, then $D_i^a Q_i$ has no constant term, so $X_{\ell,k+u}$ appears in the relation if and only if $a_1 = \cdots = a_u = 0$. Given the condition $a_1 + \cdots + a_u + u = n+1$, this implies $u = n+1$ and we conclude that

$$\ell_i^{n+1} X_{\ell,k} = \phi_i^{n+1}(X_{\ell,k+n+1}) + \{\text{terms involving derivatives of } Q_i \text{ and } X_{r,k+u} \text{ with } |r| < d\}.$$

Since ϕ_i is nilpotent, for n large enough the right hand side does not depend on $\{X_{\ell,k}\}_{k \in \mathbb{N}}$, and we obtain a recursive relation determining uniquely $X_{\ell,k}$ as a function of terms already known. This completes the proof. \square

Proof of Theorem 3.4.2

We formulate a condition under which we are able to solve the system of PDEs (3.4.4)-(3.4.6) recursively.

Definition 3.4.11 (Nilpotency condition). Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. We say that $(\mathcal{H}, \nabla)/(B, D)$ satisfies *the nilpotency condition at b* if for all vector $v \in T_b D$, the adjoint $\text{ad } \mu_b(v)$ is nilpotent (see (3.2.5) for μ_b).

Lemma 3.4.12. *Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. If there exists a framing for (\mathcal{H}, ∇) at b that is strong with respect to D , then $\mu_b(v)$ is nilpotent for every $v \in T_b B$. In particular, (\mathcal{H}, ∇) satisfies the nilpotency condition at b .*

Proof. Write $B = \text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$, with q_i the logarithmic directions. Let ∇_b^{fr} be a framing at b that is strong with respect to D , fix a trivialization of \mathcal{H} extending a ∇_b^{fr} -flat trivialization. Fix $1 \leq i \leq s$ and write

$$\nabla_{q_i \partial_{q_i}} = q_i \partial_{q_i} + u^{-1} Q(q, t, u).$$

By the assumption, $Q_0 := Q(0, 0, u)$ is independent of u . Since ∇_b^{fr} is a framing, we have $\nabla_{\partial_u}|_{b \times \text{Spf } \mathbb{k}[[u]]} = \partial_u + u^{-2} K + u^{-1} G$, with K and G constant endomorphisms of $\mathcal{H}_{b,0}$. In this trivialization, the flatness equation $[\nabla_{\partial_u}, \nabla_{q_i \partial_{q_i}}] = 0$ restricted to $b \times \text{Spf } \mathbb{k}[[u]]$ reads

$$-Q_0 = u^{-1} [Q_0, K] + [Q_0, G].$$

In particular $[Q_0, G] = -Q_0$. It follows that $[Q_0, [Q_0, -G]] = [Q_0, Q_0] = 0$. Jacobson's lemma ([79, Lemma 4, p. 44]) implies that $[Q_0, -G] = Q_0$ is nilpotent, proving the first part of the lemma. Since the adjoint of a nilpotent endomorphism is nilpotent, the second part follows. \square

The next series of lemmas will enable us to prove Theorem 3.4.2 by framing the connection inductively in each direction. Given a logarithmic F-bundle $(\mathcal{H}, \nabla)/(B, D)$ over $B = \text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$, a closed subscheme $B' \subset B$ and a subsheaf $\mathcal{F} \subset T_B(-\log D)$, we will say that (\mathcal{H}, ∇) is *framed in the directions of \mathcal{F} at B'* if there exists a trivialization of \mathcal{H} such that $\nabla_\xi|_{B'}$ takes the form (3.2.10) for any section ξ of \mathcal{F} , i.e. the restriction of the connection matrix in the direction ξ to B' has no positive powers of u . If we formulate

multiple conditions involving several subsheaves and closed subschemes, we mean that there exists a trivialization in which the connection form satisfies all the formulated conditions.

Lemma 3.4.13. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\mathrm{Spf} \mathbb{k}[[q_1, \dots, q_s]]$ (without t -variables) satisfying the nilpotency condition (Definition 3.4.11), fix $1 \leq i \leq s$. Assume it is framed in all q -directions at $\{q_j = 0, i \leq j \leq s\}$. Then there exists a gauge transformation $P(q_1, \dots, q_s, u)$ such that $P|_{q_i=\dots=q_s=0} = \mathrm{id}$ and $P^*\nabla$ is framed in the q_i -direction at $\{q_j = 0, i+1 \leq j \leq s\}$. In particular, $P^*\nabla$ is still framed in all q -directions at $\{q_j = 0, i \leq j \leq s\}$.*

Proof. We let $q := \{1, \dots, s\}$, $q^{\leq i} := \{q_1, \dots, q_i\}$, $q^{\geq i} := \{q_i, \dots, q_s\}$ and $q^{> i} := \{q_{i+1}, \dots, q_s\}$. Let $u^{-1}q_i^{-1}Q(q, u)$ denote the connection matrix in the q_i -direction in a trivialization of \mathcal{H} provided by the partial framing assumption. Write $Q(q, u) = \sum_{\ell, k \geq 0} Q_{\ell, k-1} q_i^\ell u^k$, by the framing assumption we have $Q|_{q^{\geq i}=0} = Q_{0, -1}|_{q^{\geq i}=0}$.

We seek a gauge transformation $P(q, u)$ such that

$$\begin{aligned} \partial_{q_i} P|_{q^{> i}=0} &= u^{-1} q_i^{-1} \left(-QP + PP_0^{-1} Q_{-1} P_0 \right) |_{q^{> i}=0}, \\ P|_{q^{\geq i}=0} &= \mathrm{id}, \end{aligned}$$

where $P_0 := P(q, 0)$ and $Q_{-1} := Q(q, 0)$. We look for P of the form $P(q, u) = \sum_{\ell, k \geq 0} P_{\ell, k} q_i^\ell u^k$, where $P_{\ell, k}$ depends on $\{q_1, \dots, q_{i-1}\}$, We construct the solution P order by order in powers of q_i , by expressing $\{P_{\ell+1, k}\}_{k \in \mathbb{N}}$ in terms of $\{P_{\ell', k}, \ell' \leq \ell\}_{k \in \mathbb{N}}$ for $\ell \in \mathbb{N}$.

The initial condition gives $P_{0,0} = \mathrm{id}$ and $P_{0,k} = 0$ for $k > 0$. Let $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$. We isolate a monomial $q_i^\ell u^k$ in the differential equation and obtain

$$(\ell+1)P_{\ell+1, k} = - \sum_{\substack{\ell_1 + \ell_2 = \ell + 1 \\ k_1 + k_2 = k + 1}} Q_{\ell_1, k_1 - 1} |_{q^{> i}=0} P_{\ell_2, k_2} + \sum_{\ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell + 1} P_{\ell_1, k+1} (P_0^{-1})_{\ell_2} Q_{\ell_3, -1} |_{q^{> i}=0} P_{\ell_4, 0},$$

where $(P_0^{-1})_{\ell_2}$ is the coefficient of $q_i^{\ell_2}$ in P_0^{-1} . Using the framing assumption at $q^{\geq i} = 0$ and the initial condition for P , we isolate terms involving $\{P_{\ell+1, k'}\}_{k' \in \mathbb{N}}$ and obtain the relation for all $k \geq 0$

$$P_{\ell+1, k} = \psi_{\ell, k}(P) - \frac{1}{\ell+1} [Q|_{q^{\geq i}=0}, P_{\ell+1, k+1}], \quad (3.4.14)$$

where

$$\psi_{\ell,k}(P) := \frac{1}{\ell+1} \left(- \sum_{\substack{k_1+k_2=k+1 \\ \ell_1+\ell_2=\ell+1 \\ \ell_2 < \ell+1}} Q_{\ell_1, k_1-1} |_{q^{>i}=0} P_{\ell_2, k_2} \right. \\ \left. + \sum_{\substack{\ell_1+\ell_2+\ell_3+\ell_4=\ell+1 \\ 0 < \ell_1 < \ell+1}} P_{\ell_1, k+1} (P_0^{-1})_{\ell_2} Q_{\ell_3, -1} |_{q^{>i}=0} P_{\ell_4, 0} \right).$$

Note that $\psi_{\ell,k}(P)$ only depends on $\{P_{\ell', k'}, \ell' < \ell+1, k' \leq k+1\}$.

Let $E := \text{Mat}(m \times m, \mathbb{k}[[q_1, \dots, q_{i-1}]])^{\mathbb{N}}$. Consider the linear maps $\tau: E \rightarrow E$ given by the shift $\{M_k\}_{k \in \mathbb{N}} \mapsto \{M_{k+1}\}_{k \in \mathbb{N}}$ and $\Phi: E \rightarrow E$ given by $\{M_k\}_{k \in \mathbb{N}} \mapsto \{[Q|_{q^{\geq i}=0}, M_k]\}_{k \in \mathbb{N}}$. The relations (3.4.14) give

$$\left(\text{id}_E + \frac{1}{\ell+1} \Phi \circ \tau \right) \{P_{\ell+1, k}\}_{k \in \mathbb{N}} = \{\psi_{\ell, k}(P)\}_{k \in \mathbb{N}}. \quad (3.4.15)$$

We prove that $\text{id}_E + \frac{1}{\ell+1} \Phi \circ \tau$ is invertible. To do so, it is enough to prove that it is invertible at $q_1 = \dots = q_{i-1} = 0$. The map $\Phi|_{q_1 = \dots = q_{i-1} = 0}$ is nilpotent, since $\text{ad}(Q|_{q=u=0})$ is. The maps τ and Φ commute, so the composition $\Phi \circ \tau: E \rightarrow E$ is also nilpotent at $q_1 = \dots = q_{i-1} = 0$. Hence $\text{id}_E + \frac{1}{\ell+1} \Phi \circ \tau$ is invertible at $q_1 = \dots = q_{i-1} = 0$. It follows that $\text{id}_E + \frac{1}{\ell+1} \Phi \circ \tau$ is invertible, and composing (3.4.15) with its inverse provides a recursive relation determining the coefficient of $q_i^{\ell+1}$ from lower order terms. Hence the differential equation admits a solution $P(q, u)$ such that $P|_{q^{\geq i}=0} = \text{id}$. The initial condition implies that the connection $P^* \nabla$ is still framed in all q -directions at $q^{\geq i} = 0$. This completes the proof. \square

Lemma 3.4.16. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\text{Spf } \mathbb{k}[[q_1, \dots, q_s]]$ (with no t -variables) satisfying the nilpotency condition (Definition 3.4.11), fix $1 \leq i \leq s$. Assume it is framed in all q -directions at $\{q_j = 0, i \leq j \leq s\}$, and framed in the q_i -direction at $\{q_j = 0, i+1 \leq j \leq s\}$. Then (\mathcal{H}, ∇) is framed in all the q -directions at $\{q_j = 0, i+1 \leq j \leq s\}$.*

Proof. Let $q^{\leq i} := \{q_1, \dots, q_i\}$. The partial framing assumption provides a trivialization of \mathcal{H} . For $1 \leq i' \leq s$, let $u^{-1} q_i'^{-1} Q^{i'}(q^{\leq i}, u) = q_i'^{-1} \sum_{k \geq 0} Q_{k-1}^{i'}(q^{\leq i}) u^{k-1}$ denote the restriction of the connection matrix in the $q_{i'}$ -direction to $q_{i+1} = \dots = q_s = 0$. The framing assumption means that $Q_k^{i'}|_{q_i=0} = 0$ and $Q_k^i = 0$ for all $k \geq 0$ and $1 \leq i' \leq s$.

Fix $1 \leq i' \leq s$, with $i' \neq i$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{q_{i'} \partial_{q_{i'}}}, \nabla_{q_i \partial_{q_i}}] = 0$ provides the equation

$$q_i \partial_{q_i} Q_k^{i'} = -[Q_{-1}^i, Q_{k+1}^{i'}].$$

Since $\text{ad}(Q_{-1}^s(0))$ is nilpotent, we can apply Lemma 3.4.9 with $R = \mathbb{k}[[q_1, \dots, q_{i-1}]]$ and $X_k = Q_k^{i'}$. We deduce that $Q_k^{i'} = 0$ for all $k \geq 0$, proving that the connection is also framed in the $q_{i'}$ -direction at $q_{i+1} = \dots = q_s = 0$. \square

Lemma 3.4.17. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$ framed in the q -directions at $t = 0$. Then there exists a gauge transformation P such that $P|_{t=0} = \text{id}$ and $P^*\nabla$ is framed in all the q -directions and t -directions at $t = 0$.*

Proof. We work in a trivialization of \mathcal{H} provided by the partial framing assumption. For $1 \leq i \leq s$, let $u^{-1}q_i^{-1}Q^i(q, t, u)$ denote the connection matrix in the q_i -direction in this trivialization. For $1 \leq j \leq n$, let $u^{-1}T^j(q, t, u)$ denote the connection matrix in the t_j -direction in this trivialization. Let

$$P(q, t, u) := \prod_{j=1}^n \left(\text{id} - t_j \frac{T^j(q, 0, u) - T^j(q, 0, 0)}{u} \right).$$

Note that $P(q, t, u)$ only has non-negative powers of u , because $T^j(q, 0, u) - T^j(q, 0, 0)$ has no constant term in u . We have $P|_{t=0} = P^{-1}|_{t=0} = \text{id}$, and we compute $\frac{\partial P}{\partial q_i}|_{t=0} = 0$ and $\frac{\partial P}{\partial t_j}|_{t=0} = -u^{-1}(T^j(q, 0, u) - T^j(q, 0, 0))$. The connection matrix of $P^*\nabla$ in the t_j -direction at $t = 0$ is

$$\left[P^{-1} \frac{\partial P}{\partial t_j} + u^{-1} P^{-1} T^j P \right] \Big|_{t=0} = u^{-1}(-T^j(q, 0, u) + T^j(q, 0, 0) + T^j(q, 0, u)) = u^{-1}T^j(q, 0, 0),$$

which is framed. The connection matrix of $P^*\nabla$ in the q_i -direction at $t = 0$ is

$$\left[P^{-1} \frac{\partial P}{\partial q_i} + u^{-1} q_i^{-1} P^{-1} Q^i P \right] \Big|_{t=0} = u^{-1} q_i^{-1} Q^i(q, 0, u),$$

which is also framed. The lemma is proved. \square

Lemma 3.4.18. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$ framed in the q -directions at $t = 0$, fix $1 \leq j \leq n$. Assume it is framed in all t -directions at $\{t_i = 0, j \leq i \leq n\}$. Then there exists a gauge transformation $P(q, t, u)$ such that $P|_{t_j=\dots=t_n=0} = \text{id}$ and $P^*\nabla$ is framed in the t_j -direction at $\{t_i = 0, j+1 \leq i \leq n\}$, framed in all the q -directions at $t = 0$, and in all the t -directions at $\{t_i = 0, j \leq i \leq n\}$.*

Proof. Let $t^{\leq j} := \{t_1, \dots, t_j\}$, $t^{\geq j} := \{t_j, \dots, t_n\}$ and $t^{> j} := \{t_{j+1}, \dots, t_n\}$. Let $u^{-1}T(q, t, u)$ denote the connection matrix in the t_j -direction in a trivialization of \mathcal{H} provided by the partial framing assumption. Write $T(q, t, u) = \sum_{\ell, k \in \mathbb{N}} T_{\ell, k-1} t_j^\ell u^k$, by the framing assumption we have $T|_{t^{\geq j}=0} = T_{0, -1}|_{t^{\geq j}=0}$.

We seek a gauge transformation $P(q, t, u)$ such that

$$\begin{aligned}\partial_{t_j} P|_{t>j=0} &= u^{-1} \left(-TP + PP_0^{-1}T_{-1}P_0 \right) |_{t>j=0}, \\ P|_{t\geq j=0} &= \text{id},\end{aligned}$$

where $P_0 := P(q, t, 0)$ and $T_{-1} := T(q, t, 0)$. We look for P of the form $P(q, t, u) = \sum_{\ell, k \geq 0} P_{\ell, k} t_j^\ell u^k$, where $P_{\ell, k}$ depends on the variables $\{q_1, \dots, q_s, t_1, \dots, t_{j-1}\}$. The differential equation provides a recursive relation for $\{P_{\ell, k}\}_{k \in \mathbb{N}}$. By isolating the coefficient of $t_j^\ell u^k$ we obtain

$$(\ell+1)P_{\ell+1, k} = - \sum_{\substack{\ell_1 + \ell_2 = \ell \\ k_1 + k_2 = k+1}} T_{\ell_1, k_1-1} |_{t>j=0} P_{\ell_2, k_2} + \sum_{\ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell} P_{\ell_1, k+1} (P_0^{-1})_{\ell_2} T_{\ell_3, -1} |_{t>j=0} P_{\ell_4, 0}, \quad (3.4.19)$$

where $(P_0^{-1})_{\ell_2}$ denotes the coefficient of $t_j^{\ell_2}$ in P_0^{-1} . This determines P from the initial data $\{P_{0, k}\}_{k \in \mathbb{N}}$, i.e. from $P|_{t \geq j=0} = \text{id}$. Hence the differential equation admits a solution $P(q, t, u)$ such that $P|_{t \geq j} = \text{id}$. By construction, $P^* \nabla$ is framed in the t_j -direction at $t^{>j} = 0$.

We now check that the other t -directions are still framed at $t^{\geq j} = 0$, and that the q -directions are still framed at $t = 0$. Since $P|_{t \geq j=0} = \text{id}$, the connection matrices at $t^{\geq j} = 0$ are modified by the first derivatives of $\sum_{k \geq 0} P_{1, k} u^k$. From the recursion (3.4.19), the initial condition for P and the framing assumption for T we obtain that $P_{1, k} = -T_{0, k} |_{t>j=0} = 0$ for all $k \geq 0$. We conclude that $P^* \nabla$ remains framed in all the t -directions at $t^{\geq j} = 0$ and in all the q -directions at $t = 0$, concluding the proof. \square

Lemma 3.4.20. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\text{Spf } \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$, fix $1 \leq j \leq n$. Assume it is framed in all the t -directions at $\{t_i = 0, j \leq i \leq n\}$, and framed in the t_j -direction at $t_{j+1} = \dots = t_n = 0$. Then (\mathcal{H}, ∇) is framed in all the t -directions at $\{t_i = 0, j+1 \leq i \leq n\}$.*

Proof. Let $t^{\leq j} := \{t_1, \dots, t_j\}$. The partial framing assumption provides a trivialization of \mathcal{H} . For $1 \leq j' \leq n$, let $u^{-1} T^{j'}(q, t^{\leq j}, u) = \sum_{k \geq 0} T_{k-1}^{j'}(q, t^{\leq j}) u^{k-1}$ denote the restriction of the connection matrix in the $t_{j'}$ -direction to $t_{j+1} = \dots = t_n = 0$. The framing assumption means that $T_k^{j'} |_{t_j=0} = 0$ and $T_k^j = 0$ for all $k \geq 0$ and $1 \leq j' \leq n$.

Fix $1 \leq j' \leq n$, with $j' \neq j$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_{t_{j'}}}, \nabla_{\partial_{t_j}}] = 0$ provides the equation

$$\partial_{t_j} T_k^{j'} = -[T_{-1}^j, T_{k+1}^{j'}].$$

We apply Lemma 3.4.8(1) with $R = \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_{j-1}]]$, $X_k = T_k^{j'}$ and the initial condition $T_k^{j'}|_{t_j=0} = 0$, and deduce that $T_k^{j'}(q, t^{\leq j}) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the $t_{j'}$ -direction at $t_{j+1} = \dots = t_n = 0$. \square

Lemma 3.4.21. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\mathrm{Spf} \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$. Assume it is framed in the t -directions and framed in the q -directions at $t = 0$. Then (\mathcal{H}, ∇) is also framed in the q -directions.*

Proof. In a trivialization provided by the framing assumption, denote by $u^{-1}T_{-1}^j(q, t)$ the connection matrix in the t_j -direction ($1 \leq j \leq n$) and by $u^{-1}q_i^{-1}Q^i(q, t, u)$ the connection matrix in the q_i -direction ($1 \leq i \leq s$). Write $Q^i = \sum_{k \geq 0} Q_{k-1}^i(q, t)u^k$. The framing assumption means that $Q_k^i|_{t=0} = 0$ for $1 \leq i \leq s$ and $k \geq 0$.

Fix $1 \leq i \leq s$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_{t_j}}, \nabla_{q_i \partial_{q_i}}] = 0$ is

$$\partial_{t_j} Q_k^i = -[T_{-1}^j, Q_{k+1}^i].$$

We apply Lemma 3.4.8(2) with $R = \mathbb{k}[[q_1, \dots, q_s]]$, $X_k = Q_k^i$ and the initial condition $Q_k^i|_{t=0} = 0$, and deduce that $Q_k^i(q, t) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the q_i -direction. \square

Lemma 3.4.22. *Let (\mathcal{H}, ∇) be a logarithmic F -bundle over $\mathrm{Spf} \mathbb{k}[[q_1, \dots, q_s, t_1, \dots, t_n]]$ satisfying the nilpotency condition (Definition 3.4.11). Assume it is framed in the q -directions and t -directions, and framed in the u -direction at $q = t = 0$. Then (\mathcal{H}, ∇) is also framed in the u -direction.*

Proof. In a trivialization provided by the framing assumption, let $u^{-1}q_i^{-1}Q^i(q, t)$ (resp. $u^{-1}T^j(q, t)$) denote the connection matrix in the q_i -direction (resp. t_j -direction). Let $u^{-2}U(q, t, u)$ denote the connection matrix in the u -direction. Write $U(q, t, u) = \sum_{k \geq 0} U_{k-2}(q, t)u^k$. The framing assumption means that for $k \geq 0$, we have $U_k(0, 0) = 0$.

For $k \geq 0$, and $1 \leq i \leq s$, the u^k term of the flatness equation $[\nabla_{\partial_u}, \nabla_{q_i \partial_{q_i}}] = 0$ provides the equation

$$q_i \partial_{q_i}(U_k) = -[Q^i, U_{k+1}].$$

We restrict this equation to $t = 0$. Since $\mathrm{ad}(Q^i(0, 0))$ is nilpotent, we can apply Lemma 3.4.9 with $R = \mathbb{k}$ and $X_k = U_k(q, 0)$ to deduce that $U_k(q, 0) = 0$ for all $k \geq 0$.

Next, for $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_u}, \nabla_{\partial_{t_j}}] = 0$ provides the equation

$$\partial_{t_j}(U_k) = -[T^j, U_{k+1}].$$

We apply Lemma 3.4.8(2) with $R = \mathbb{k}[[q_1, \dots, q_s]]$, $X_k = U_k(q, t)$ and the initial condition $U_k(q, 0) = 0$, and deduce that $U_k(q, t) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the u -direction. \square

We can now finish the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. Fix a trivialization $\mathcal{H} \simeq H \times (B \times \mathrm{Spf} \mathbb{k}[[u]])$ extending the trivialization of $\mathcal{H}|_{B \times \mathrm{Spf} \mathbb{k}[[u]]}$ induced by ∇_b^{fr} . As explained after Theorem 3.4.2, the content of the theorem reduces to proving existence and uniqueness of a solution $P(q, t, u)$ to the overdetermined nonlinear system of PDEs (3.4.4)-(3.4.6) with initial condition $P(0, 0, u) = \mathrm{id}$.

We prove the existence part of the statement. If there exists a framing ∇^{fr} extending ∇_b^{fr} , then we see that ∇_b^{fr} is strong with respect to D by working in a ∇^{fr} -flat trivialization. Conversely assume that ∇_b^{fr} is strong with respect to D , in particular the nilpotency condition is satisfied by Lemma 3.4.12. We first frame the restricted F-bundle $(\mathcal{H}', \nabla') := (\mathcal{H}, \nabla)|_{t=0}$, defined over the base $B' := \mathrm{Spf} \mathbb{k}[[q_1, \dots, q_s]]$. Applying inductively Lemmas 3.4.13 and 3.4.16 on $i \in \{1, \dots, s\}$, we obtain a gauge transformation $P(q, u)$ such that $P(0, u) = \mathrm{id}$ and $P^*\nabla'$ is framed in all the q -directions. Note that to apply the lemmas for the base case $i = 1$, we use that ∇_b^{fr} is strong with respect to D . Extending this gauge transformation constantly in the t -directions, we obtain a gauge transformation $P_1(q, u) \in \mathrm{Aut}(\mathcal{H})$ with $P_1(0, u) = \mathrm{id}$ such that $\nabla_1 := P_1^*\nabla$ is framed in all the q -directions at $t = 0$. By Lemma 3.4.17, we obtain a gauge transformation $P_2(q, t, u) \in \mathrm{Aut}(\mathcal{H})$ with $P_2(q, 0, u) = \mathrm{id}$ such that $\nabla_2 := P_2^*\nabla_1$ is framed in all the q -directions and t -directions at $t = 0$. Applying inductively Lemmas 3.4.18 and 3.4.20 on $j \in \{1, \dots, n\}$, we obtain a gauge transformation $P_3(q, t, u) \in \mathrm{Aut}(\mathcal{H})$ with $P_3(q, 0, u) = \mathrm{id}$ such that $\nabla_3 := P_3^*\nabla_2$ is framed in all the q -directions at $t = 0$, and in all the t -directions along B . By Lemma 3.4.21, the connection ∇_3 is also framed in all the q -directions along B . Since $\nabla_{3, \partial_u}|_{q=t=0} = \nabla_{\partial_u}|_{q=t=0}$, the connection ∇_3 is framed in the u -directions at $q = t = 0$. We conclude by Lemma 3.4.22 that ∇_3 is framed in the u -direction as well. Thus the gauge transformation $\tilde{P} := P_3 P_2 P_1$ solves the

system (3.4.4)-(3.4.6) with the initial condition $\tilde{P}(0, 0, u) = \text{id}$, concluding the proof of existence.

We now prove uniqueness. Assume the system of PDEs is written in a trivialization in which the connection is framed. In particular, the nilpotency condition is satisfied by Lemma 3.4.12. From the equations in the directions of B we obtain recursive relations as in (3.4.15) and (3.4.19). Hence, any solution is uniquely determined by the condition $P(0, 0, u) = \text{id}$. \square

Framings on rank 1 F-bundles

F-bundles do not admit framings in general (see [119, §IV.5.b] for a sufficient condition), even though we established the extension of framing in Theorem 3.4.2. Here we discuss the existence of framing on rank 1 F-bundles.

Proposition 3.4.23. *Let B be a formal neighborhood of a rational point b in a smooth \mathbb{k} -variety. Let $(\mathcal{H}, \nabla)/B$ be a (non-logarithmic) formal F-bundle of rank 1. Then it admits a framing.*

Proof. We keep the notations of the proof of Theorem 3.4.2. In the non-logarithmic case there are no q -variables, and in the rank 1 case the matrices are elements of $\mathbb{k}[[t, u]]$, so they commute. Then $K = U_{-2}$, $G = U_{-1}$ and $\tilde{T}^i = T_{-1}^i$ for $1 \leq i \leq n$. The system of PDEs (3.4.4)-(3.4.6) is then

$$\begin{aligned} \partial_u P(t, u) + P(t, u)U_{\geq 0}(t, u) &= 0, \\ \partial_{t_i} P(t, u) + P(t, u)T_{\geq 0}^i(t, u) &= 0, \end{aligned}$$

where $U_{\geq 0} = \sum_{k \geq 0} U_k u^k$ and $T_{\geq 0}^i = \sum_{k \geq 0} T_k^i u^k$. We furthermore need $P(0, 0) \neq 0$ in order for $P(t, u)$ to be invertible.

It is readily checked, using flatness, that the ansatz

$$\begin{aligned} P(t, u) = & \\ \exp \left(- \sum_{i=1}^n \int_0^{t_i} \left(T_{\geq 0}^i(t_1, \dots, t_{i-1}, s_i, 0, \dots, 0, u) + T_{\geq 0}^i(0, u) \right) ds_i - \int_0^u U_{\geq 0}(0, v) dv \right) & \end{aligned} \tag{3.4.24}$$

solves the system of PDEs, and is invertible since $P(0, 0) = 1$. \square

In the following example, we discuss the case of rank 1 logarithmic F-bundle, and provide a counter-example to the existence part of Theorem 3.4.2 without assuming the framing is strong with respect to D .

Example 3.4.25. Let \mathcal{H} be the trivial rank 1 bundle over $\mathrm{Spf} \mathbb{k}[[q, u]]$. Let $\nabla = d + \Omega$ be the connection on \mathcal{H} with $\Omega = \alpha \frac{dq}{q}$, where $\alpha \in \mathbb{k}$. Then (\mathcal{H}, ∇) is a F-bundle and $\nabla_0^{\mathrm{fr}} = d$ is a framing for $(\mathcal{H}, \nabla)|_{q=0}$. It is strong with respect to D if and only if $\alpha = 0$. The differential system to solve in order to extend the framing is

$$\begin{aligned} \frac{\partial P}{\partial u} &= 0, \\ q \frac{\partial P}{\partial q} + \alpha P &= 0. \end{aligned}$$

If $\alpha \neq 0$, all solutions to this system are scalar multiples of αq^{-1} . In particular, they are not well-defined at $q = 0$.

Extension of framing for non-archimedean F-bundles

In this subsection, we establish the theorem of extension of framing for non-archimedean F-bundles, building on the results of the previous subsection.

Theorem 3.4.26. *Let B be a smooth \mathbb{k} -analytic space, and $b \in B$ a \mathbb{k} -rational point. Let (\mathcal{H}, ∇) be a non-archimedean F-bundle over B . Then every framing of (\mathcal{H}, ∇) at b extends uniquely and explicitly to a framing over an admissible open neighborhood U of b in B .*

We need to show that the gauge transformation $P(t, u)$ constructed in the formal case is convergent on an admissible open neighborhood of $t = 0$, $u = 0$. This gauge transformation is characterized by $P(0, u) = 0$ and the equations (3.4.6) for $1 \leq j \leq n$. We use these equations to obtain estimates on the coefficients of $P(t, u)$.

Lemma 3.4.27. *Let $(R, |\cdot|)$ be a Banach \mathbb{k} -algebra. Let $Q = \mathrm{id} + \sum_{r \geq 1} Q_r t^r \in \mathrm{Mat}(m \times m, R)[[t]]$, and write $Q^{-1} = \mathrm{id} + \sum_{r \geq 1} (Q^{-1})_r t^r$. For $\ell \geq 1$ we have*

$$|(Q^{-1})_\ell| \leq \max_{\substack{\ell \geq k \geq 1, r_i \geq 1 \\ r_1 + \dots + r_k = \ell}} \prod_{i=1}^k |Q_{r_i}|.$$

Proof. We have $Q^{-1} = \sum_{k \geq 0} (-1)^k \left(\sum_{r \geq 1} Q_r t^r \right)^k$. Isolating the coefficient of t^ℓ ($\ell \geq 1$) we obtain

$$(Q^{-1})_\ell = \sum_{k \geq 0} \sum_{\substack{r_1 + \dots + r_k = \ell \\ r_i \geq 1}} \prod_{1 \leq i \leq k} Q_{r_i},$$

and we see that only the range $1 \leq k \leq \ell$ contributes. This completes the proof. \square

Proposition 3.4.28. *Let $(R, |\cdot|)$ be a Banach \mathbb{k} -algebra and let $T \in \text{Mat}(m \times m, R)\langle t, u \rangle$. Let $P(t, u) \in \text{Mat}(m \times m, R)\llbracket t, u \rrbracket$ be the unique solution of the system*

$$\begin{aligned}\partial_t P &= u^{-1} \left(-TP + PP_0^{-1}T_{-1}P_0 \right), \\ P(0, u) &= \text{id},\end{aligned}$$

where $P_0 := P(t, 0)$. Then P is convergent on the open disk of radius $\min\left(1, \frac{1}{|T|}\right)$, meaning that for all $0 < \rho < \min\left(1, \frac{1}{|T|}\right)$ in $\sqrt{|\mathbb{k}^\times|}$ we have $P \in \text{Mat}(m \times m, R)\langle \rho^{-1}t, \rho^{-1}u \rangle$.

Proof. We write $T = \sum_{\substack{\ell \geq 0 \\ k \geq -1}} T_{\ell, k} t^\ell u^{k+1}$. Since we assume T is convergent on the closed unit disk, we have for all $\ell \geq 0, k \geq -1$

$$|T_{\ell, k}| \leq |T|. \quad (3.4.29)$$

Let $P := \text{id} + \sum_{\substack{\ell \geq 1 \\ k \geq 0}} P_{\ell, k} t^\ell u^k$ and $v_{\ell, k} := |P_{\ell, k}|$. If we show $v_{\ell, k} \leq \alpha^{\ell+k}$ for $\alpha > 0$, then $P(t, u)$ converges on the open polydisk of radius $\frac{1}{\alpha}$.

We have seen in Lemma 3.4.18 that P is uniquely determined by the recursion

$$(\ell+1)P_{\ell+1, k} = - \sum_{\substack{\ell_1 + \ell_2 = \ell \\ k_1 + k_2 = k+1}} T_{\ell_1, k_1-1} P_{\ell_2, k_2} + \sum_{\ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell} P_{\ell_1, k_1+1} (P_0^{-1})_{\ell_2} T_{\ell_3, -1} P_{\ell_4, 0}.$$

Applying the norm, we obtain

$$\begin{aligned}(\ell+1)v_{\ell+1, k} &\leq \max \left(\max_{\substack{\ell_1 + \ell_2 = \ell \\ k_1 + k_2 = k+1}} |T_{\ell_1, k_1-1}| |P_{\ell_2, k_2}|, \max_{\substack{\ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell \\ \ell_1 \neq 0}} |P_{\ell_1, k_1+1}| |(P_0^{-1})_{\ell_2}| |(T_{-1})_{\ell_3}| |(P_0)_{\ell_4}| \right) \\ &\leq |T| \cdot \max \left(\max_{\substack{\ell_1 + \ell_2 = \ell \\ k_1 + k_2 = k+1}} v_{\ell_2, k_2}, \max_{\substack{\ell_1 + \ell_2 + \ell_3 + \ell_4 = \ell \\ \ell_1 \neq 0}} v_{\ell_1, k_1+1} v_{\ell_4, 0} |(P_0^{-1})_{\ell_2}| \right),\end{aligned}$$

where on the second inequality we use (3.4.29).

Let $\alpha := \max(1, |T|)$. We use the above inequality to prove by induction on $\ell \geq 0$ that

$$\forall k \geq 0, \quad v_{\ell, k} \leq \alpha^\ell.$$

For $\ell = 0$, we have $v_{0, k} = \delta_{0, k}$ so the inequality is obvious. Now assume $v_{r, k} \leq \alpha^r$ for all $r \leq \ell$. By Lemma 3.4.27, we then have $|(P_0^{-1})_r| \leq \max_{1 \leq i \leq r} \alpha^i = \alpha^r$ for

all $r \leq \ell$. Since $\alpha \geq |T|$, we deduce that

$$\begin{aligned} (\ell + 1)v_{\ell+1,k} &\leq |T| \max \left(\max_{\substack{s \leq \ell \\ k_2 \leq k+1}} \alpha^s, \max_{\substack{\ell_1 + \ell_2 + \ell_4 \leq \ell \\ \ell_1 \neq 0}} \alpha^{\ell_1 + \ell_2 + \ell_4} \right) \\ &= |T| \alpha^\ell \leq \alpha^{\ell+1} \leq (\ell + 1) \alpha^{\ell+1}. \end{aligned}$$

This concludes the inductive step.

Since $\alpha \geq 1$, we have $v_{\ell,k} \leq \alpha^\ell \leq \alpha^{\ell+k}$ for all $\ell, k \geq 0$. We deduce that P converges on the open disk of radius $\frac{1}{\alpha}$, completing the proof. \square

We can now finish the proof of Theorem 3.4.26.

Proof of Theorem 3.4.26. Up to restricting to an open neighborhood of b , we may assume that $B = \mathrm{Sp} T_n$ by Lemma 3.3.9. Let (t_1, \dots, t_n) be local analytic coordinates centered at b . After rescaling we can assume that the connection matrices converge on $\mathrm{Sp} \mathbb{k}\langle t_1, \dots, t_n, u \rangle$.

As in the formal case, we can reformulate the extension of framing problem into a system of PDEs (3.4.4) and (3.4.6). We can solve the equations (3.4.6) inductively on the number of t -variables, and by Lemma 3.4.22 the equation (3.4.4) will be automatically satisfied. Using Proposition 3.4.28 inductively, we obtain that at each step the solution, i.e the gauge transformation, converges on an admissible open neighborhood of b . \square

Reconstruction of isomorphism of framed maximal F-bundles

In this subsection, we explain how to use the extension of framing for logarithmic F-bundles (Theorem 3.4.2) to reconstruct an isomorphism of framed maximal F-bundles compatible with the framings. This is useful for establishing the uniqueness of mirror maps in applications to enumerative geometry.

Definition 3.4.30 (Compatibility of framings). For $i = 1, 2$ let $(\mathcal{H}_i, \nabla_i, \nabla_i^{\mathrm{fr}})/(B_i, D_i)$ be two framed logarithmic F-bundles. A morphism $(f, \Phi): (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \rightarrow (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ of logarithmic F-bundles is said to be *compatible with the framings* if $\Phi \circ \nabla_1^{\mathrm{fr}} = (f \times \mathrm{id}_u)^* \nabla_2^{\mathrm{fr}} \circ \Phi$.

Proposition 3.4.31. For $i = 1, 2$, let $(\mathcal{H}_i, \nabla_i)/(B_i, D_i)$ be a logarithmic F-bundle where B_i is the formal neighborhood of a rational point in a smooth \mathbb{k} -variety. Let $(f, \Phi): (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \rightarrow (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ be an isomorphism

of logarithmic F -bundles with $f(b_1) = b_2$. Assume $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ has a framing ∇_1^{fr} .

1. The bundle map Φ is uniquely determined by its restriction to $\mathcal{H}_1|_{b_1 \times \text{Spf } \mathbb{k}[[u]]}$.
2. If $(\mathcal{H}_1, \nabla_1)$ and $(\mathcal{H}_2, \nabla_2)$ are maximal, then the map on the bases f is also uniquely determined by its restriction to b_1 , up to some multiplicative constants in the logarithmic directions. The reconstruction is explicit after fixing compatible cyclic vectors at b_1 and b_2 .

Proof. For (1), let H_i denote the fiber of \mathcal{H}_i over b_i , and $\phi \in \text{Hom}(H_1, H_2)$ the restriction of Φ at b_1 . Fix a ∇_1^{fr} -flat trivialization Ψ_1 of \mathcal{H}_1 and an arbitrary trivialization Ψ_2 of $(f \times \text{id}_u)^*\mathcal{H}_2$, producing the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\Psi_1} & H_1 \times B_1 \times \text{Spf } \mathbb{k}[[u]] \\ \downarrow \Phi & & \downarrow \tilde{\Phi} \\ (f \times \text{id}_u)^*\mathcal{H}_2 & \xrightarrow{\Psi_2} & H_2 \times B_1 \times \text{Spf } \mathbb{k}[[u]]. \end{array}$$

Denote by $\varphi: \mathcal{H}_1 \rightarrow (f \times \text{id}_u)^*\mathcal{H}_2$ the map obtained from ϕ by taking its constant extension with respect to the trivializations Ψ_1 and Ψ_2 . If $\tilde{\phi} = \tilde{\Phi}|_{(b_1, 0)}$, then $\varphi = \Psi_2^{-1} \circ (\tilde{\phi} \times \text{id}_{B_1 \times \text{Spf } \mathbb{k}[[u]])} \circ \Psi_1$. Define two connections on $(f \times \text{id}_u)^*\mathcal{H}_2$

$$\begin{aligned} \nabla'_1 &:= (f \times \text{id}_u)^*\nabla_2 = \Phi \circ \nabla_1 \circ \Phi^{-1}, \\ \nabla'_2 &:= \varphi \circ \nabla_1 \circ \varphi^{-1}. \end{aligned}$$

In the trivialization Ψ_2 we see that ∇'_1 is framed over all B_1 , and ∇'_2 is framed only at b_1 . Furthermore ∇'_1 and ∇'_2 are gauge equivalent under $\Phi \circ \varphi^{-1}$, and $\Phi \circ \varphi^{-1}|_{b_1} = \text{id}$. We conclude from Theorem 3.4.2 that $\Phi \circ \varphi^{-1}$ is unique, then so is Φ provided that we know $\Phi|_{b_1 \times \text{Spf } \mathbb{k}[[u]]}$. This proves (1).

Next we prove (2), and assume that the F -bundles are maximal. The framing ∇_1^{fr} induces unique framings ∇_2^{fr} (resp. $\nabla_2^{\text{fr}'}$) on $(\mathcal{H}_2, \nabla_2)$ (resp. $f^*(\mathcal{H}_2, \nabla_2)$) such that in the diagram

$$(\mathcal{H}_1, \nabla_1) \xrightarrow{(\text{id}_{B_1}, \Phi)} f^*(\mathcal{H}_2, \nabla_2) \xrightarrow{(f, \text{id})} (\mathcal{H}_2, \nabla_2),$$

all the morphisms are compatible with the framings. Furthermore, the framing $\nabla_2^{\text{fr}'}$ is determined by ∇_1^{fr} and Φ , hence is already known by (1).

Let h_1 be a ∇_1^{fr} -flat section of cyclic vectors for $(\mathcal{H}_1, \nabla_1)$. Because of the compatibility of the framings, h_1 induces a $\nabla_2^{\text{fr}'}$ -flat section of cyclic vectors $h'_2 := \Phi(h_1)$

for $f^*(\mathcal{H}_2, \nabla_2)$, and a ∇_2^{fr} -flat section of cyclic vectors $h_2 := (f^{-1} \times \text{id}_u)^*(h'_2)$ for $(\mathcal{H}_2, \nabla_2)$. We obtain isomorphisms $\eta_i := \mu_{\mathcal{H}_i}(\cdot)(h_i): TB_i(-\log D_i) \rightarrow \mathcal{H}_i|_{u=0}$ that fit into a commutative diagram

$$\begin{array}{ccc} TB_1(-\log D_1) & \xrightarrow{df} & f^*TB_2(-\log D_2) \\ \downarrow \eta_1 & & \downarrow f^*(\eta_2) \\ \mathcal{H}_1|_{u=0} & \xrightarrow{\Phi|_{u=0}} & (f \times \text{id}_u)^*\mathcal{H}_2|_{u=0}, \end{array}$$

where all arrows are isomorphisms. The maps η_1 and $\Phi|_{u=0}$ are already known. We have $f^*(\eta_2) = \mu_{(f \times \text{id}_u)^*\mathcal{H}_2}(\cdot)(h'_2)$ by construction and compatibility of (f, id) with the framings. So $f^*(\eta_2)$ is determined by h_1 , ∇_2^{fr} and Φ , hence is known. We deduce that df is determined by h_1 , ∇_1^{fr} and Φ .

Since B_i are formal neighborhoods of points, the differential df determines f uniquely, up to some multiplicative constants in the logarithmic directions. To see this, choose coordinates $(q, t) = (q_1, \dots, q_r, t_1, \dots, t_n)$ for (B_1, D_1) , centered at b_1 , where $\prod_{1 \leq i \leq r} q_i = 0$ is a local equation for D_1 . Similarly, choose coordinates $(p, s) = (p_1, \dots, p_r, s_1, \dots, s_n)$ for (B_2, D_2) centered at $f(b_1)$. In coordinates, the restriction of f to B_1 is given by $f = (f_1, \dots, f_{r+n})$ where $q_i = f_i(p, s)$ and $t_j = f_{r+j}(p, s)$. The differential df corresponds to a map of $\mathbb{k}[[q, t]]$ -modules

$$\Psi: \Gamma(B_1, f^*\Omega_{B_2}^1(\log D_2)) \rightarrow \Gamma(B_1, \Omega_{B_1}^1(\log D_1)),$$

given by the pullback of differential forms, i.e.

$$\Psi(d \log p_i) = d \log f_i = \frac{df_i}{f_i}, \quad \Psi(ds_j) = df_{r+j}.$$

We conclude the proof by integrating the differential forms. □

Equivalence of F-bundles over a point

For applications in Section 3.5, we present some results here for the classification of framed F-bundles over a point up to gauge equivalence, see Theorem 3.4.34 and Corollary 3.4.35.

Let $(\mathcal{H}, \nabla, \nabla^{\text{fr}})$ be a framed F-bundle over a point. Fix a ∇^{fr} -flat trivialization $\mathcal{H} \simeq H \otimes_{\mathbb{k}} \mathbb{k}[[u]]$ and write

$$\nabla_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K} + \mathbf{G},$$

with $\mathbf{K}, \mathbf{G} \in \text{End}_{\mathbb{k}}(H)$.

We assume that the endomorphism \mathbf{K} induces a \mathbb{k} -vector space decomposition $H = \bigoplus_{1 \leq k \leq m} H_k$ into generalized eigenspaces, and all H_k have same dimensions. Then we have a \mathbb{k} -vector space H_0 and a splitting of the fiber

$$iso: H_0^{\oplus m} \xrightarrow{\sim} H. \quad (3.4.32)$$

So we can represent endomorphisms on H as $m \times m$ matrices with coefficients in $\text{End}_{\mathbb{k}}(H_0)$. In particular we write $\mathbf{K} = (\mathbf{K}_{ij})_{1 \leq i, j \leq m}$ and $\mathbf{G} = (\mathbf{G}_{ij})_{1 \leq i, j \leq m}$. By construction $\mathbf{K}_{ij} = 0$ if $i \neq j$ and $\mathbf{K}_{ii} = \xi_i \text{id}_{H_0} + \mathbf{N}_i$ with $\xi_i \in \mathbb{k}$ and \mathbf{N}_i a nilpotent endomorphism.

Fix $\mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{d} \in \text{End}_{\mathbb{k}}(H_0)$ such that \mathbf{c}_i are nilpotent endomorphisms, $[\mathbf{c}_i, \mathbf{c}_j] = 0$ and $[\mathbf{d}, \mathbf{c}_i] = d_i \mathbf{c}_i$ for $d_i \in \mathbb{N}_{>0}$.

Definition 3.4.33. We denote by $\mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r})$ the space of connections ∇' on \mathcal{H} which, in the fixed ∇^{fr} -flat trivialization, are of the form

$$\nabla'_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K}' + (\mu'\mathbf{D} + \mathbf{H}'),$$

where

1. $\mu' \notin \mathbb{Q}_{<0} \subset \mathbb{k}$,
2. $\mathbf{K}', \mathbf{D}, \mathbf{H}' \in \text{End}_{\mathbb{k}}(H)$,
3. $\mathbf{K}'_{ij}, \mathbf{H}'_{ij} \in \mathbb{k}[\mathbf{c}_1, \dots, \mathbf{c}_r]$, and
4. $\mathbf{D}_{ii} = \mathbf{d}$ and $\mathbf{D}_{ij} = 0$ for $i \neq j$.

Theorem 3.4.34. Let $(\mathcal{H}, \nabla, \nabla^{\text{fr}})$ be as above. Assume $\nabla \in \mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r})$ and let $\nabla' \in \mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r})$. Write

$$\begin{aligned} \nabla_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K} + (\mu\mathbf{D} + \mathbf{H}), \\ \nabla'_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K}' + (\mu'\mathbf{D} + \mathbf{H}'). \end{aligned}$$

Then ∇ is gauge-equivalent to ∇' under $\Phi(u) \in \text{GL}(H[[u]])$ with $\Phi_{ij}(u) \in \mathbb{k}[\mathbf{c}_1, \dots, \mathbf{c}_r][[u]]$ if and only if the following three conditions are satisfied:

1. there exists $\phi \in \text{GL}(H)$ with $\phi_{ij} \in \mathbb{k}[\mathbf{c}_1, \dots, \mathbf{c}_r]$ such that $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$,
2. $\mu = \mu'$, and
3. for all $1 \leq i \leq m$, $\mathbf{H}_{ii} = (\phi^{-1} \circ \mathbf{H}' \circ \phi)_{ii} \pmod{(\mathbf{c}_1, \dots, \mathbf{c}_r)}$.

Furthermore, Φ is then uniquely determined by the initial condition $\Phi|_{u=0} = \phi \bmod (\mathbf{c}_1, \dots, \mathbf{c}_r)$.

The assumptions on the form of the operators allow us to work in the non-commutative subalgebra $\mathbb{k}[\mathbf{d}, \mathbf{c}_1, \dots, \mathbf{c}_r] \subset \text{End}_{\mathbb{k}}(H_0)$. We then reduce to the case of simple eigenvalues by treating the operators $\mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r}$ as formal variables. In the simple eigenvalues case, the gauge equivalence can be constructed inductively.

As a corollary, in the simple eigenvalue case we obtain a classification of F-bundles over a point with a fixed framing.

Corollary 3.4.35. *Let $\mathcal{H} \simeq H \times \mathbb{k}[[u]]$ be a trivialized rank m vector bundle over $\mathbb{k}[[u]]$. Let (\mathcal{H}, ∇) and (\mathcal{H}, ∇') be two F-bundle structures framed in the given trivialization, and write*

$$\begin{aligned}\nabla_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K} + \mathbf{G}, \\ \nabla'_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K}' + \mathbf{G}'.\end{aligned}$$

Assume \mathbf{K} has simple eigenvalues. Then (\mathcal{H}, ∇) is isomorphic to (\mathcal{H}, ∇') if and only if there exists $\phi \in \text{GL}(H)$ such that

1. $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$, and
2. in an eigenbasis of \mathbf{K} , we have $(\mathbf{G})_{ii} = (\phi^{-1} \circ \mathbf{G}' \circ \phi)_{ii}$ for $1 \leq i \leq m$.

Furthermore, the gauge equivalence is uniquely and explicitly determined by the initial condition ϕ at $u = 0$.

Proof. The choice of an eigenbasis for \mathbf{K} produces a splitting *iso*: $\mathbb{k}^{\oplus m} \xrightarrow{\sim} H$ as in (3.4.32). Since there are no nilpotent operators in $\text{End}_{\mathbb{k}}(\mathbb{k}) \simeq \mathbb{k}$, and this algebra is commutative, the content of Definition 3.4.33 becomes empty, and the corollary is just a reformulation of Theorem 3.4.34 in this special case. \square

Proof of Theorem 3.4.34. Let $R_0 = \mathbb{k}[[c_1, \dots, c_r]]$ and $R = \mathbb{k}[\text{deg}][[c_1, \dots, c_r]]$ where $\{(c_i)_{1 \leq i \leq r}, \text{deg}\}$ are formal variables satisfying the commutation relations $[c_i, c_j] = 0$ and $[\text{deg}, c_i] = d_i c_i$. There is a specialization map $R \rightarrow \mathbb{k}[\mathbf{d}, \mathbf{c}_1, \dots, \mathbf{c}_r]$. Using *iso* we also have a specialization map

$$\text{Mat}(m \times m, R) \rightarrow \text{End}_{\mathbb{k}}(H).$$

By the definition of $\mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r})$, the connections $\nabla_{u\partial_u}$ and $\nabla'_{u\partial_u}$ lift to differential operators of the form

$$\begin{aligned} u\partial_u + u^{-1}K + \mu D + H, \\ u\partial_u + u^{-1}K' + \mu' D + H', \end{aligned}$$

with $K, K', H, H' \in \text{Mat}(m \times m, R_0)$ and $D = \text{deg} \cdot \text{Id}_m$. A gauge equivalence Φ as in the theorem also lifts along the specialization map, so we have reduced the problem to finding $\Phi(u) \in \text{GL}(m, R_0[[u]])$ such that

$$\Phi^{-1}(u\partial_u + u^{-1}K + \mu D + H)\Phi = u\partial_u + u^{-1}K' + \mu' D + H'. \quad (3.4.36)$$

The conditions (1)-(3) also lift under the specialization map, so we are left to prove the following lemma. \square

Lemma 3.4.37. *There exists a gauge equivalence $\Phi(u) \in \text{GL}(m, R_0[[u]])$ solving (3.4.36) if and only if there exists $Q \in \text{GL}(m, R_0)$ such that*

- (a) $K = Q^{-1}K'Q$,
- (b) $\mu = \mu'$, and
- (c) $H_{ii} = (Q^{-1}H'Q)_{ii} \pmod{(c_1, \dots, c_r)}$.

In this case, $\Phi(u)$ is uniquely determined by the initial condition $\Phi|_{u=0} = Q \pmod{(c_1, \dots, c_r)}$.

Proof. By construction of the splitting (3.4.32) and Definition 3.4.33(2), the matrix K is diagonal, and $K_{ii} = \xi_i \pmod{(c_1, \dots, c_r)}$, where $\{\xi_1, \dots, \xi_m\}$ are the distinct eigenvalues of \mathbf{K} . In particular K has simple eigenvalues, so $\text{ad}_K = [K, \cdot]$ has kernel given by diagonal matrices, and image given by matrices with vanishing diagonal.

Let us first prove that the conditions (a)-(c) are sufficient. Fix $Q \in \text{GL}(m, R_0)$ satisfying (a) and (c). We are looking for $\Phi(u)$ such that $\Phi|_{u=0} = Q \pmod{(c_1, \dots, c_r)}$ solving (3.4.36). Write $\Phi(u) = QP(u)$ with $P(u) = \sum_{k \geq 0} P_k u^k$ satisfying $P_0 = \text{Id}_m \pmod{(c_1, \dots, c_r)}$. Equation (3.4.36) then reduces to the system

$$[K, P_0] = 0, \quad (3.4.38)$$

$$[K, P_{k+1}] = \varphi(P_k) - kP_k, \quad (3.4.39)$$

where

$$\varphi: M \mapsto M(\mu'Q^{-1}DQ + Q^{-1}H'Q) - (\mu D + H)M.$$

Before analyzing the existence of solutions, let us rewrite (3.4.39) in order to isolate the terms involving the non-commutative variable deg . Define the \mathbb{k} -linear operator $\text{Eu}(\cdot) := [\text{deg}, \cdot]$ on R . The commutations relations in R give $\text{Eu}(\cdot) = \sum_{1 \leq i \leq r} d_i c_i \partial_{c_i}$. For $M \in \text{Mat}(m \times m, R)$, we write $\text{Eu}(M) := (\text{Eu}(M_{ij}))_{1 \leq i, j \leq m}$. We have $\text{Eu}(M) = [D, M]$, so

$$\begin{aligned} \varphi(M) &= M(\mu'D + Q^{-1}\text{Eu}(Q) + Q^{-1}H'Q) - (\mu D + H)M \\ &= \mu'MD - \mu DM + M(Q^{-1}\text{Eu}(Q) + Q^{-1}H'Q) - HM \\ &= (\mu' - \mu)MD - \mu\text{Eu}(M) + M(Q^{-1}\text{Eu}(Q) + Q^{-1}H'Q) - HM. \end{aligned}$$

Since $\mu = \mu'$, the term involving D vanishes.

We now prove by induction on k the following: there exists a unique sequence of matrices (P_0, \dots, P_k) such that (i) $P_0 = \text{Id}_m \pmod{(c_1, \dots, c_r)}$ and $[K, P_0] = 0$, (ii) $(P_\ell, P_{\ell+1})$ solves (3.4.39) for $0 \leq \ell \leq k-1$, and (iii) $\varphi(P_k) - kP_k \in \text{im ad}_K$.

We construct P_0 satisfying (i), (ii) and (iii). The condition $[K, P_0] = 0$ implies that P_0 is a diagonal matrix, $P_0 = \text{Diag}(\delta_1, \dots, \delta_n)$. The initial condition $P_0 = \text{Id}_m \pmod{(c_1, \dots, c_r)}$ gives $\delta_i = 1 \pmod{(c_1, \dots, c_r)}$. To ensure that we can solve the recursion for P_1 , we need $\varphi(P_0)_{ii} = 0$ for all i . This provides the relation

$$\mu \text{Eu}(\delta_i) = \alpha_i \delta_i, \tag{3.4.40}$$

for all i , where $\alpha_i = (Q^{-1}\text{Eu}(Q) + Q^{-1}H'Q - H)_{ii}$. For any $x \in R_0$, we have $\text{Eu}(x) \in (c_1, \dots, c_r)R_0$. Together with Condition (c), this implies that $\alpha_i = 0 \pmod{(c_1, \dots, c_r)}$. We can then solve for δ_i order by order in (c_1, \dots, c_r) and determine P_0 uniquely from the initial condition $P_0 = \text{Id}_m \pmod{(c_1, \dots, c_r)}$. Note that the condition on μ in Definition 3.4.33 ensures that we obtain a recursion that we can solve.

Let $k \geq 1$, and assume (P_0, \dots, P_{k-1}) are constructed. The existence of a matrix P such that $[K, P] = \varphi(P_{k-1}) - (k-1)P_{k-1}$ is guaranteed by Condition (iii) of the induction hypothesis. The matrix P is determined up to a diagonal matrix. We first prove that for any choice of P , there exists a unique diagonal matrix Δ such that $\varphi(P + \Delta) - k(P + \Delta) \in \text{im ad}_K$, i.e. has vanishing diagonal. Let $\Delta = \text{Diag}(\delta_1, \dots, \delta_n)$ be a diagonal matrix, the vanishing of the i -th diagonal

term of $\varphi(P + \Delta) - k(P + \Delta)$ is equivalent to an equation of the form

$$\mu \text{Eu}(\delta_i) + k\delta_i = \alpha\delta_i + \beta, \quad (3.4.41)$$

with

$$\begin{aligned} \alpha &= (Q^{-1} \text{Eu}(Q))_{ii} + (Q^{-1} H' Q)_{ii} - H_{ii}, \\ \beta &= (\varphi(P) - kP)_{ii}. \end{aligned}$$

As in the initial step ($k = 0$), we have $\alpha = 0 \pmod{(c_1, \dots, c_r)}$. Since δ_i is a power series in (c_1, \dots, c_r) , (3.4.41) provides a recursion relation on the coefficients of δ_i . Since $k \geq 1$ the constant term of δ_i is uniquely determined by looking at the equation modulo (c_1, \dots, c_r) , where it gives $k\delta_i = \beta \pmod{(c_1, \dots, c_r)}$. The other coefficients are then uniquely determined inductively. The condition on μ in Definition 3.4.33 ensures that we obtain a recursion that we can solve, thus δ_i is uniquely determined from P . We have proved the existence of a matrix P_k satisfying Conditions (ii) and (iii) of the induction. Now we prove uniqueness. Let P_k and \widetilde{P}_k be two matrices satisfying (ii) and (iii). In particular, they are solutions of the equation $[K, P] = \varphi(P_{k-1}) - (k-1)P_{k-1}$, so there exists a diagonal matrix Δ such that $P_k = \widetilde{P}_k + \Delta$. Condition (iii) gives $\varphi(\widetilde{P}_k + \Delta) - k(\widetilde{P}_k + \Delta) \in \text{im ad}_K$. Since \widetilde{P}_k already satisfies (iii) and Δ is diagonal, we deduce from the uniqueness in the previous paragraph that $\Delta = 0$. Hence $P_k = \widetilde{P}_k$, concluding the induction.

Now we prove Conditions (a)-(c) assuming that there exists $\Phi(u) = \sum_{k \geq 0} P_k u^k \in \text{GL}(m, R_0[[u]])$ solving (3.4.36). In particular $P_0 \in \text{GL}(m, R_0)$. Multiplying (3.4.36) on the left by Φ and isolating the u^k term, we obtain for $k = -1$ and $k \geq 0$ respectively:

$$\begin{aligned} KP_0 &= P_0 K', \\ kP_k + KP_{k+1} + (\mu D)P_k + HP_k &= P_{k+1}K' + P_k(\mu' D) + P_k H'. \end{aligned}$$

Let $Q = P_0^{-1}$, it satisfies Condition (a). For any $1 \leq i \leq r$ we have $\text{deg} \cdot c_i = c_i \cdot \text{deg} + d_i c_i$. By comparing the coefficient of the formal variable deg we obtain $\mu = \mu'$, verifying Condition (b). Looking at the u^0 term, using $K = P_0 K' P_0^{-1}$ and modding out $(c_i)_{1 \leq i \leq r}$, we obtain

$$K(P_1 P_0^{-1}) - P_1 P_0^{-1} K + H = P_0 H' P_0^{-1} \pmod{(c_1, \dots, c_r)}.$$

Since $[K, P_1(P_0)^{-1}]$ has vanishing diagonal, Condition (c) follows. \square

3.5 Application: quantum cohomology of projective bundle

In this section, we study the decomposition of the maximal A-model F-bundle associated to a projective bundle. We prove the existence of the decomposition when restricting the F-bundle to a point, as well as the uniqueness of the decomposition (Theorems 3.5.16 and 3.5.20). In Section 3.5, we state the analogous results in the case of a blowup of algebraic varieties (Theorems 3.5.22 and 3.5.24).

Let X be a smooth complex projective variety of dimension d , $V \rightarrow X$ a vector bundle of rank m on X , $P := \mathbb{P}(V)$ the associated projective bundle of lines in V , and write $\pi: P \rightarrow X$. We fix an ample divisor class $\omega_X \in H^2(X, \mathbb{Z})$, and a homogeneous basis $\{T_i\}_{0 \leq i \leq N}$ of $H^*(X, \mathbb{Q})$ extending $\{\mathbf{1}, \omega_X\}$.

A-model F-bundle of P at the limiting point

We have the following classical decomposition of the cohomology of P , as a special case of Leray-Hirsch theorem (see [59, Theorem 4D-1]).

Proposition 3.5.1. *Let $h := c_1(\mathcal{O}_P(1))$. We have the splitting isomorphism of cohomology groups*

$$iso: H_{\text{split}} := \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i] \xrightarrow{\sum h^i \cup \pi^*} H^*(P, \mathbb{Q}). \quad (3.5.2)$$

Lemma 3.5.3. *We have*

$$K_P = \pi^* K_X - mh - \pi^* c_1 V.$$

Proof. It follows from the relative Euler sequence

$$0 \rightarrow \Omega_{P/X} \rightarrow \mathcal{O}_P(-1) \otimes \pi^* V^\vee \rightarrow \mathcal{O}_P \rightarrow 0$$

that

$$K_{P/X} = -mh - \pi^* c_1 V.$$

Hence

$$K_P = \pi^* K_X + K_{P/X} = \pi^* K_X - mh - \pi^* c_1 V. \quad \square$$

Recall that we fixed an ample class ω_X on X . Let $\omega_P := \pi^* \omega_X$.

Lemma 3.5.4. *The class ω_P is nef and satisfies Assumption 3.2.22.*

Proof. Since ω_X is ample, its pullback ω_P is nef. Furthermore, there exists $\varepsilon > 0$ such that $\omega_X + \varepsilon(c_1T_X + c_1V)$ is ample. Then, by Lemma 3.5.3, we have $\omega_P + \varepsilon c_1P = \pi^*(\omega_X + \varepsilon(c_1T_X + c_1V)) + \varepsilon mh$. It is ample, since it is the sum of a nef class and an ample class ([94, Corollary 1.4.10]). We conclude by Lemma 3.2.23. \square

Using the homogeneous basis $\{T_i\}_{0 \leq i \leq N}$ of $H^*(X, \mathbb{Q})$, we produce a homogeneous basis

$$\{\pi^*(T_i)h^j, 0 \leq i \leq N, 0 \leq j \leq m-1\}$$

of $H^*(P, \mathbb{C})$ extending ω_P . We denote by $\{t_{i,j}\}$ the induced linear coordinates on $H^*(P, \mathbb{C})$.

Let $(\mathcal{H}, \nabla)/B$ denote the maximal A-model F-bundle of P constructed from ω_P , with base point $0 \in H^*(P, \mathbb{C})$ (see Example 3.2.25). Write $(q, t = \{t_{i,j}, (i, j) \neq (1, 0)\})$ for the coordinates on B . Let b denote the closed point of B , given by $q = 0, t = 0$, which we refer to as the limiting point in this section. Let \mathbf{K}_{lim} and \mathbf{G}_{lim} denote the restrictions of the operators \mathbf{K} and \mathbf{G} at the limiting point (see Definition 3.2.17).

Let us compute the matrices of \mathbf{K}_{lim} and \mathbf{G}_{lim} under the splitting *iso* in (3.5.2).

We have

$$\mathbf{G}_{\text{lim}} = \begin{pmatrix} \mathbf{G}_X - \frac{m-1}{2} & & & & \\ & \mathbf{G}_X - \frac{m-3}{2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \mathbf{G}_X + \frac{m-1}{2} \end{pmatrix},$$

and \mathbf{K}_{lim} is computed in the following proposition.

Proposition 3.5.5. *The operator \mathbf{K}_{lim} on $H^*(P, \mathbb{C})$ has the following matrix with respect to the splitting in (3.5.2):*

$$\mathbf{K}_{\text{lim}} = \begin{pmatrix} c_1T_X + c_1V & & & & m(1 - c_mV) \\ m & c_1T_X + c_1V & & & -mc_{m-1}V \\ & m & \ddots & & \vdots \\ & & \ddots & c_1T_X + c_1V & -mc_2V \\ & & & m & c_1T_X + c_1V - mc_1V \end{pmatrix}.$$

Proof. Consider four operators K_1, \dots, K_4 on H_{split} such that for $\gamma \in H^*(P, \mathbb{C}) \simeq H_{\text{split}}$, we have

1. $K_1(\gamma) = \pi^*(c_1 T_X) \cup \gamma$,
2. $K_2(\gamma) = h \cup \gamma$,
3. $K_3(\gamma) = \pi^* c_1 V \cup \gamma$, and
4. $K_4(\gamma) = p_* q^* \gamma$, where $p, q: P \times_X P \rightarrow P$ are the projections.

By Lemma 3.5.3, the classical multiplication by $c_1 T_P$ has matrix $K_1 + mK_2 + K_3$.

The non-classical part of \mathbf{K}_{lim} is expressed in terms of 3-pointed Gromov–Witten invariants of the form $\langle c_1 P, \gamma_1, \gamma_2 \rangle_{0,3}^\beta$ for an effective curve class $\beta \neq 0$ such that $\beta \cdot \omega_P = 0$ and cohomology classes $\gamma_1, \gamma_2 \in H^*(P, \mathbb{C})$. Fix such a β , by the projection formula, we have $\beta \cdot \omega_P = (\pi_* \beta) \cdot \omega_X$. Since ω_X is ample, this implies that $\pi_* \beta = 0$, i.e. $\beta = \delta[L]$ for $[L]$ the class of a line in a fiber of π and $\delta \in \mathbb{N}_{>0}$ ($\delta = 0$ gives the classical contribution). By the divisor axiom and Lemma 3.5.3, we have

$$\langle c_1 P, \gamma_1, \gamma_2 \rangle_{0,3}^\beta = (\beta \cdot c_1 P) \langle \gamma_1, \gamma_2 \rangle_{0,2}^\beta = \delta m \langle \gamma_1, \gamma_2 \rangle_{0,2}^\beta.$$

Let $M := \overline{\mathcal{M}}_{0,2}(P, \delta[L])$ denote the moduli stack of 2-pointed rational stable maps of class β . By the Riemann-Roch formula, the virtual dimension $\dim_{\text{vir}} M$ of M is equal to $\dim P - 3 + \int_\beta c_1(P) + 2 = d - 2 + m(\delta + 1)$. Since β is a fiber class, the evaluation map

$$\text{ev}_1 \times \text{ev}_2: M \rightarrow P \times P$$

factors through

$$P \times_X P \subset P \times P.$$

In order to have nonzero counts, we need $\dim_{\text{vir}} M \leq \dim P \times_X P$ which implies that $\delta = 1$, i.e. the curve class can only be $[L]$. We then have an isomorphism

$$\text{ev}_1 \times \text{ev}_2: M \xrightarrow{\sim} P \times_X P \subset P \times P.$$

In particular, M is smooth, so $[M]^{\text{vir}} = [M]$. Under this isomorphism, the operator

$$\gamma \mapsto \text{ev}_{1,*} \left(\text{ev}_2^* \gamma \cup [M]^{\text{vir}} \right) = \text{ev}_{1,*} \text{ev}_2^* \gamma$$

is equal to mK_4 . Therefore, the non-classical contribution to \mathbf{K}_{lim} is mK_4 .

We obtain

$$\mathbf{K}_{\text{lim}} = \text{iso} \circ (K_1 + mK_2 + K_3 + mK_4) \circ \text{iso}^{-1}. \quad (3.5.6)$$

Now let us calculate the four matrices K_1, \dots, K_4 . For any $\alpha_i \in H^*(X, \mathbb{C})[-2i]$, we have

$$\pi^*(c_1 T_X) \cup (h^i \cup \pi^* \alpha_i) = h^i \cup \pi^*(c_1 T_X \cup \alpha_i),$$

hence $K_1 = (c_1 T_X \cup) \cdot \text{id}_{H_{\text{split}}}$. Similarly, we have that $K_3 = (c_1 V \cup) \cdot \text{id}_{H_{\text{split}}}$. For $i = 0, \dots, m-1$, we have

$$h \cup (h^i \cup \pi^* \alpha_i) = h^{i+1} \cup \pi^* \alpha_i.$$

When $i = m-1$, by [16, Eq. (20.6)] we have

$$h \cup (h^{m-1} \cup \pi^* \alpha_i) = h^m \cup \pi^* \alpha_{m-1} = - \sum_{j=0}^{m-1} h^j \cup \pi^*(c_{m-j} V \cup \alpha_{m-1}).$$

So

$$K_2 = \begin{pmatrix} & & -c_m V \\ 1 & & -c_{m-1} V \\ & \ddots & \vdots \\ & & 1 & -c_2 V \\ & & & 1 & -c_1 V \end{pmatrix}.$$

For any $\alpha_i \in H^*(X, \mathbb{C})[-2i]$, $i = 0, \dots, m-1$, since $\pi \circ p = \pi \circ q$, by the projection formula we have

$$p_* q^*(h^i \cup \pi^* \alpha_i) = p_*(q^*(h^i) \cup q^* \pi^* \alpha_i) = p_*(q^*(h^i) \cup p^* \pi^* \alpha_i) = p_* q^*(h^i) \cup \pi^* \alpha_i.$$

Since $p_* q^*(h^i) \in H^{2(i-(m-1))}(P, \mathbb{C})$, it vanishes unless $i = m-1$, in which case it is equal to the identity. We deduce that the matrix of K_4 has only one nonzero block: the top-right corner, which is $\text{id}_{H^*(X, \mathbb{C})}$.

Substituting the above computations into (3.5.6), we conclude the proof. \square

Decomposition of \mathbf{K}_{lim}

In this subsection, we study the generalized eigenspaces of \mathbf{K}_{lim} . We will consider the commutative subalgebra $\mathbb{C}[\mathbf{t}, \mathbf{c}_1, \dots, \mathbf{c}_m]$ of $\text{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$ generated by the commuting nilpotent operators

$$\mathbf{t} := c_1 T_X \cup \quad \text{and} \quad \mathbf{c}_i := c_i V \cup \quad (1 \leq i \leq m).$$

Let $\mathbf{d} := \mathbf{G}_X = \frac{1}{2}(\deg_X - \dim X)$, where $\deg_X(\alpha) = i\alpha$ for $\alpha \in H^i(X, \mathbb{C})$. We have the commutation relations

$$[\mathbf{d}, \mathbf{t}] = \mathbf{t}, \quad [\mathbf{d}, \mathbf{c}_i] = i\mathbf{c}_i. \quad (3.5.7)$$

Lemma 3.5.8. *1. There exists $\phi = (\phi_{ij}) \in \mathrm{GL}(H_{\mathrm{split}})$ with entries $\phi_{ij} \in \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m]$, and $\boldsymbol{\lambda}_i = \lambda_i \cup \in \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m] \subset \mathrm{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$ such that*

$$\mathbf{K}_{\mathrm{split}} := \phi^{-1} \mathbf{K}_{\mathrm{lim}} \phi = \begin{pmatrix} \mathbf{t} + \mathbf{c}_1 & & & & \\ & \ddots & & & \\ & & \mathbf{t} + \mathbf{c}_1 & & \\ & & & \ddots & \\ & & & & \mathbf{t} + \mathbf{c}_1 \end{pmatrix} + m \begin{pmatrix} \boldsymbol{\lambda}_1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \boldsymbol{\lambda}_m \end{pmatrix}.$$

2. Up to reordering the blocks, for $1 \leq i \leq m$ we have

$$\boldsymbol{\lambda}_i = \xi^{i-1} - \frac{\mathbf{c}_1}{m} \pmod{(\mathbf{c}_1^2, \mathbf{c}_2, \dots, \mathbf{c}_m)}, \quad \xi = e^{\frac{2\pi}{m}}.$$

In particular the i -th diagonal block of $\mathbf{K}_{\mathrm{split}}$ is the cup-product with an element in $H^(X, \mathbb{C})$ whose H^2 -component is $c_1 T_X$.*

Proof. As an element of $\mathrm{Mat}(m \times m, \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m])$, we have $\mathbf{K}_{\mathrm{lim}} = (\mathbf{t} + \mathbf{c}_1)\mathrm{Id}_m + mM$, where M is the companion matrix

$$M = \begin{pmatrix} 0 & & & & 1 - \mathbf{c}_m \\ 1 & & & & -\mathbf{c}_{m-1} \\ & 1 & & & -\mathbf{c}_{m-2} \\ & & \ddots & & \vdots \\ & & & 1 & -\mathbf{c}_1 \end{pmatrix}. \quad (3.5.9)$$

The characteristic polynomial of M is $\lambda^m + \sum_{i=1}^{m-1} \mathbf{c}_{m-i} \lambda^i + (\mathbf{c}_m - 1)$. Modulo $(\mathbf{c}_1, \dots, \mathbf{c}_m)$ this polynomial has simple roots given by m -th roots of unity. Since it is monic, we can lift these roots to $\mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m]$ by solving the equation order by order. (1) follows.

For (2), the characteristic polynomial of M modulo $(\mathbf{c}_1^2, \mathbf{c}_2, \dots, \mathbf{c}_m)$ is

$$\lambda^m + \mathbf{c}_1 \lambda^{m-1} - 1 = \left(\lambda + \frac{\mathbf{c}_1}{m} \right)^m - 1.$$

We deduce that $m\boldsymbol{\lambda}_i = m e^{\frac{2\pi(i-1)}{m}} - \mathbf{c}_1$ modulo those classes, proving (2). \square

Lemma 3.5.8 implies that \mathbf{K}_{lim} has m generalized eigenspaces, all isomorphic to $H^*(X, \mathbb{C})$, matching the setup of Section 3.4. The splitting considered in (3.4.32) is given by the modified isomorphism

$$\text{iso} \circ \phi^{-1}: H_{\text{split}} \xrightarrow{\sim} H^*(P, \mathbb{C}). \quad (3.5.10)$$

We will use the following lemma to check Condition (c) of Theorem 3.4.34.

Lemma 3.5.11. *Let $H = \text{Diag}(\mu_1, \dots, \mu_m) \in \text{GL}(H_{\text{split}})$ be a block diagonal matrix with scalar entries. Let $\phi = (\phi_{ij}) \in \text{GL}(H_{\text{split}})$ be as in Lemma 3.5.8. Assume that $\sum_{1 \leq j \leq m} \mu_j = 0$. Then $(\phi^{-1} \circ H \circ \phi)_{ii} = 0$ for all $1 \leq i \leq m$.*

Proof. As in the previous lemma, we view ϕ and H as elements in $\text{Mat}(m \times m, \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m])$. By construction, ϕ diagonalizes the companion matrix $M \in \text{Mat}(m \times m, \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m])$ from (3.5.9). Let $\Lambda = \text{Diag}(\lambda_1, \dots, \lambda_m)$. By construction we have $M\phi = \phi\Lambda$. For every $1 \leq i \leq m$, we deduce

$$\phi_{mi} = \lambda_i \phi_{1i}, \quad \phi_{1i} = \lambda_i \phi_{2i}, \quad \phi_{2i} = \lambda_i \phi_{3i}, \quad \dots, \quad \phi_{m-1,i} = \lambda_i \phi_{mi}.$$

Similarly, for $\psi := \phi^{-1}$ we have that $\Lambda\psi = \psi M$, and we obtain for all $1 \leq i \leq m$

$$\lambda_i \psi_{i1} = \psi_{i2}, \quad \lambda_i \psi_{i2} = \psi_{i3}, \quad \dots, \quad \lambda_i \psi_{i,m-1} = \psi_{im}, \quad \lambda_i \psi_{im} = \psi_{i1}.$$

In particular for $1 \leq i \leq m$, we have

$$\psi_{i1} \phi_{1i} = \psi_{i2} \phi_{2i} = \dots = \psi_{im} \phi_{mi}.$$

We deduce

$$(\phi^{-1} \circ H \circ \phi)_{ii} = \sum_{1 \leq j \leq m} \psi_{ij}(H)_{jj} \phi_{ji} = \psi_{i1} \phi_{1i} \sum_{1 \leq j \leq m} \mu_j = 0. \quad \square$$

Remark 3.5.12. The automorphism $\phi \bmod (\mathbf{c}_1, \dots, \mathbf{c}_m)$ gives the initial condition for the gauge equivalence in Theorem 3.5.16. Since it diagonalizes the (block) circulant matrix $M \bmod (\mathbf{c}_1, \dots, \mathbf{c}_m)$ it can be chosen to be the matrix

$$Q = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & \xi^{-1} & \dots & \xi^{-(m-1)} \\ 1 & \xi^{-2} & \dots & (\xi^{-2})^{m-1} \\ \vdots & & & \vdots \\ 1 & \xi^{-(m-1)} & \dots & (\xi^{-(m-1)})^{m-1} \end{pmatrix}.$$

Example 3.5.13 (Trivial bundle case). If $V = \mathcal{O}_X^{\oplus m}$ is a trivial vector bundle, then $\mathbf{c}_i = 0$ for $1 \leq i \leq m$. In particular, we have $\lambda_i = \xi^{i-1}$, where $\xi = e^{\frac{2\pi i}{m}}$.

Example 3.5.14 (\mathbb{P}^1 -bundle case). Let V be a rank 2 bundle over X of dimension d . Then the classes (λ_1, λ_2) are obtained by solving the quadratic equation

$$\lambda^2 + \mathbf{c}_1 \lambda + \mathbf{c}_2 - 1 = 0,$$

where \mathbf{c}_i is the cup product with $c_i V$. Since $(\mathbf{c}_1^2)^{\frac{d}{2}} = (\mathbf{c}_2)^{\frac{d}{2}} = 0$, the discriminant $\Delta = \mathbf{c}_1^2 - 4\mathbf{c}_2 + 4$ admits a square-root in $\mathbb{C}[\mathbf{c}_1, \mathbf{c}_2]$ given by

$$\sqrt{\Delta} = 2\sqrt{1 + \frac{\mathbf{c}_1^2}{4} - \mathbf{c}_2} = 2 \left(1 + \sum_{1 \leq n \leq \frac{d}{2}} \binom{1/2}{n} \left(\frac{\mathbf{c}_1^2}{4} - \mathbf{c}_2 \right)^n \right).$$

Using the quadratic formula, we obtain the roots ($i = 1, 2$)

$$\lambda_i = (-1)^{i-1} - \frac{c_1 V}{2} + (-1)^{i-1} \sum_{1 \leq n \leq \frac{d}{2}} \binom{1/2}{n} \left(\frac{(c_1 V)^2}{4} - c_2 V \right)^n.$$

Uniqueness of the decomposition

In this subsection, we prove the uniqueness of the decomposition of the maximal A-model associated to a projective bundle, as well as its existence at the limiting point (Theorems 3.5.16 and 3.5.20). We will consider a maximal A-model F-bundle (\mathcal{H}', ∇') of $X' := \coprod_{i=1}^m X$ with a shifted base point, and use Theorem 3.4.34 to construct a gauge equivalence between the F-bundle (\mathcal{H}, ∇) of P and (\mathcal{H}', ∇') over the base points. The uniqueness results will follow from Theorem 3.4.34 and the extension of framing theorem.

We have

$$H^*(X', \mathbb{Q}) \xrightarrow{\sim} \bigoplus_{i=1}^m H^*(X, \mathbb{Q}). \quad (3.5.15)$$

Let $\omega' \in H^2(X', \mathbb{Q})$ denote the class corresponding to $(\omega_X, \dots, \omega_X)$ under (3.5.15), it is ample so Assumption 3.2.22 is satisfied.

Fix a homogeneous basis of $H^2(X', \mathbb{Q})$ extending ω' . Complete it to a homogeneous basis of $H^*(X', \mathbb{Q})$ by adding the elements $\{T_i, \deg T_i \neq 2\}$ in each copy of $H^*(X, \mathbb{Q})$.

Let $\Delta(a) \in H^*(X', \mathbb{C})$ be a cohomology class at which the quantum product is well-defined. We produce $(\mathcal{H}', \nabla')/B'$, the maximal A-model F-bundle of X' associated to ω' with base point $\Delta(a)$ as in Example 3.2.25. Let (q, t)

denote the coordinates on B' , and let b' denote the closed point of B' , given by $t = 0, q = 0$, which we refer to as the limiting point for X' .

Using the last observation of Lemma 3.5.8, we will interpret $\mathbf{K}_{\text{split}}$ as the \mathbf{K} -operator of (\mathcal{H}', ∇') for certain values of $\Delta(a)$.

For $i \in \{1, \dots, m\}$ and j such that $\deg T_j \neq 2$, we denote by $a_{i,j}$ the coordinate of $\Delta(a)$ along the basis element T_j in the i -th copy of $H^*(X, \mathbb{C})$ in $H^*(X', \mathbb{C})$.

Theorem 3.5.16. *There exists an F -bundle isomorphism*

$$\Phi(u): (\mathcal{H}, \nabla)|_b \rightarrow (\mathcal{H}', \nabla')|_{b'},$$

whose components Φ_{ij} (as power series in u) are given by the cup-product with elements in $H^*(X, \mathbb{C})$ if and only if the coordinates of the base point $\Delta(a)$ satisfy

$$\sum_{j: \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j = c_1 V + m \lambda_i, \quad (3.5.17)$$

where λ_i was defined in Lemma 3.5.8.

Furthermore, in this case Φ is uniquely and explicitly determined by the H^0 -components of $\Phi_{ij}|_{u=0}$, and $\Delta(a)$ is uniquely determined by (3.5.17), up to a shift in $\bigoplus_{i=1}^m H^2(X, \mathbb{C})$.

Proof. The bundles $\mathcal{H}|_b$ and $\mathcal{H}'|_{b'}$ are trivial by definition, their fibers are identified with H_{split} through (3.5.2) and (3.5.15), and the connections ∇ and ∇' are framed. We use Theorem 3.4.34 to prove the proposition.

The matrices of \mathbf{K}_{lim} , \mathbf{G}_{lim} were computed in Section 3.5. Write $\nabla_{u\partial_u}|_b = u\partial_u - u^{-1}\mathbf{K}_{\text{split}} + \mathbf{G}_{\text{split}}$, we have

$$\mathbf{G}_{\text{split}} = \begin{pmatrix} \mathbf{G}_X & & \\ & \ddots & \\ & & \mathbf{G}_X \end{pmatrix}.$$

To compute $\mathbf{K}_{\text{split}}$, note that the class ω' is ample. In particular, the restriction to $q = t = 0$ of the quantum product associated to $\Phi^{\omega'}$ is the classical cup-product. Then, $\mathbf{K}_{\text{split}}$ is block diagonal, and its i -th block is given by

$$(\mathbf{K}_{\text{split}})_{ii} = \left(c_1 T_X + \sum_{j: \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j \right) \cup. \quad (3.5.18)$$

Thus, after identifying the fibers with H_{split} , the connections $\nabla|_b$ and $\nabla'|_{b'}$ lie in $\mathcal{F}(H_{\text{split}}, \text{id}, \mathbf{G}_X, (T_j \cup)_{0 \leq j \leq N})$, see Definition 3.4.33. We apply Theorem 3.4.34 with $\mathbf{K} = -\mathbf{K}_{\text{split}}$, $\mathbf{D} = \mathbf{G}_{\text{split}}$, $\mathbf{H} = 0$, $\mathbf{K}' = -\mathbf{K}_{\text{lim}}$ and

$$\mathbf{H}' = \mathbf{G}_{\text{lim}} - \mathbf{G}_{\text{split}} = \begin{pmatrix} -\frac{m-1}{2} & & & & \\ & -\frac{m-3}{2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{m-1}{2} \end{pmatrix}.$$

Assume first that the coordinates of $\Delta(a)$ satisfy (3.5.17). Let $\phi = (\phi_{ij}) \in \text{GL}(H_{\text{split}})$ denote the automorphism from Lemma 3.5.8. Equations (3.5.17) and (3.5.18) imply that $\phi^{-1}\mathbf{K}_{\text{lim}}\phi = \mathbf{K}_{\text{split}}$, which is Condition (1) of the theorem. Condition (2) is satisfied with $\mu = \mu' = 1$. Condition (3) follows from Lemma 3.5.11 and our choice of \mathbf{H} . We conclude that the connections $\nabla|_b$ and $\nabla'|_{b'}$ are gauge equivalent through a bundle isomorphism $\Phi(u)$ satisfying the conditions of the theorem.

Now, assume that there exists a bundle isomorphism $\Phi(u)$ as in the theorem, in particular each component ϕ_{ij} of $\Phi|_{u=0}$ is given by the cup-product with a cohomology class. Let $\phi := (\phi_{ij}) \in \text{GL}(H_{\text{split}})$. Since $\Phi(u)$ is a gauge equivalence, we have in particular $\phi^{-1}\mathbf{K}_{\text{lim}}\phi = \mathbf{K}_{\text{split}}$. Recall from (3.5.18) that $\mathbf{K}_{\text{split}}$ is block diagonal, and that its coefficients are given by the cup-product with cohomology classes in $H^*(X, \mathbb{C})$. The assumption on the components of $\Phi|_{u=0}$ implies that ϕ diagonalizes \mathbf{K}_{lim} viewed as an element of $\text{Mat}(m \times m, R)$, where $R = \{\alpha \mapsto x \cup \alpha \mid x \in H^*(X, \mathbb{C})\}$. The eigenvalues of \mathbf{K}_{lim} as an R -linear map were computed in Lemma 3.5.8, they are $(c_1 T_X + c_1 V + m\lambda_i) \cup$ with $1 \leq i \leq m$. In particular, $\Delta(a)$ satisfies (3.5.17).

The uniqueness part of the theorem follows from the uniqueness of Theorem 3.4.34, and the non-degeneracy of the Poincaré pairing. \square

Remark 3.5.19. If the H^2 -component of the base point $\Delta(a)$ is 0, then the quantum product converges at $\Delta(a)$ by Lemma 3.2.15.

Theorem 3.5.20. *Let $(f, \Phi): (\mathcal{H}, \nabla)/B \rightarrow (\mathcal{H}', \nabla')/B'$ be an isomorphism of F -bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*

2. The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.

Proof. The F-bundle $(\mathcal{H}', \nabla')/B'$ is framed by definition. Since ω' is ample, at the point b' the quantum product reduces to the classical cup-product. In particular $(\mathbf{1}, \dots, \mathbf{1}) \in H^*(X', \mathbb{C})$ is a cyclic vector. The theorem thus follows from a direct application of Proposition 3.4.31. \square

We refer to [72] regarding the existence of the isomorphism.

Case of blowups of algebraic varieties

In this subsection, we state the analogs of the results in Section 3.5 in the case of blowups of algebraic varieties.

Let X be a smooth projective complex algebraic variety, and $\sigma: Z \hookrightarrow X$ a smooth closed subvariety of codimension $m \geq 2$. Let $\pi: \widetilde{X} \rightarrow X$ be the blowup of X along Z . Similar to the projective bundle case, we have a classical decomposition

$$iso: H^*(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{m-1} H^*(Z, \mathbb{Q})[-2i] \xrightarrow{\sim} H^*(\widetilde{X}, \mathbb{Q}). \quad (3.5.21)$$

Let $X' := X \sqcup \coprod_{i=1}^{m-1} Z$. Fix an ample class $\omega_X \in H^2(X, \mathbb{Q})$.

Let $(\mathcal{H}, \nabla)/B$ denote the maximal A-model F-bundle of X associated to the nef class $\pi^*\omega_X$, with base point $b = 0 \in H^*(\widetilde{X}, \mathbb{Q})$ and coordinates (q, t) . Fix a class $\Delta(a) \in H^*(X', \mathbb{C}) \simeq H^*(X, \mathbb{C}) \oplus \bigoplus_{1 \leq i \leq m-1} H^*(Z, \mathbb{C})$ at which the quantum product is well-defined. Let $(\mathcal{H}', \nabla')/B'$ denote the maximal A-model F-bundle associated to the class $(\omega_X, \sigma^*\omega_X, \dots, \sigma^*\omega_X)$, with base point $b' = \Delta(a)$ and coordinates (q, t) such that $q = t = 0$ at b' . Since X' is a disjoint union, (\mathcal{H}', ∇') is the product of a maximal A-model F-bundle associated to X and ω_X , and $m - 1$ copies of maximal F-bundles associated to Z and $\sigma^*\omega_X$.

We can prove a result analogous to Theorem 3.5.16. For $1 \leq i \leq m$, let \mathbf{c}_i denote the cup-product with $c_i(N_{Z/X})$. The polynomial $\lambda^m + \sum_{i=0}^{m-1} \mathbf{c}_{m-i} \lambda^i + \lambda$ has m distinct roots $\boldsymbol{\lambda}_i = \lambda_i \cup \in \mathbb{C}[\mathbf{c}_1, \dots, \mathbf{c}_m]$, with

$$\lambda_1 = 0, \quad \lambda_i = \xi^{2(i-1)-1} - \frac{c_1 N_{Z/X}}{m-1} \quad \text{mod } H^{\geq 3}(X, \mathbb{C}),$$

where $\xi = e^{\frac{\pi i}{m-1}}$ and $2 \leq i \leq m$, up to a permutation of the indices $\{2, \dots, m\}$. Those are the analogs of the eigenvalues computed for \mathbf{K}_{lim} in the projective bundle case.

Let $\{S_j\}_{1 \leq j \leq \dim H^*(Z, \mathbb{C})}$ be a basis of $H^*(Z, \mathbb{C})$ extending $\sigma^*\omega_X$. For $1 \leq i \leq m$, let $\Delta_i(a)$ denote the component of $\Delta(a)$ in the i -th summand of (3.5.21), and for $2 \leq i \leq m$ decompose it as

$$\Delta_i(a) = \sum_j a_{i,j} S_j.$$

Using the splitting (3.5.21), we can view an element $\Phi \in \text{End}_{\mathbb{C}}(H^*(\widetilde{X}, \mathbb{C}))$ as a matrix $(\Phi_{i,j})_{1 \leq i,j \leq m}$, with $\Phi_{1,1} \in \text{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$, and $\Phi_{i,i} \in \text{End}_{\mathbb{C}}(H^*(Z, \mathbb{C}))$ for $2 \leq i \leq m$. The following result is analogous to Theorem 3.5.16.

Theorem 3.5.22. *Let $\Delta(a) \in H^*(X', \mathbb{C})$ be a cohomology class at which the quantum product is well-defined, such that $\Delta_1(a) \in H^2(X, \mathbb{C})$, and for $2 \leq i \leq m$, we have*

$$\sum_{j: \deg S_j \neq 2} \frac{\deg_Z S_j - 2}{2} a_{i,j} S_j = c_1 N_{Z/X} + (m-1)\lambda_i. \quad (3.5.23)$$

Then, there exists an F -bundle isomorphism $\Phi: (\mathcal{H}, \nabla)|_b \rightarrow (\mathcal{H}', \nabla')|_{b'}$.

Furthermore, if we restrict the coefficients of Φ to lie in a universal algebra as in the projective bundle case, then Φ is uniquely determined by its restriction to $u = 0$, and the base point $\Delta(a)$ is uniquely determined up to a shift in $H^2(X, \mathbb{C}) \oplus \bigoplus_{i=1}^{m-1} H^2(Z, \mathbb{C})$.

A direct consequence of Proposition 3.4.31 is the following, which is analogous to Theorem 3.5.20.

Theorem 3.5.24. *Let $(f, \Phi): (\mathcal{H}, \nabla)/B \rightarrow (\mathcal{H}', \nabla')/B'$ be an isomorphism of F -bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*
2. *The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.*

We refer to [72] regarding the existence of the isomorphism.

A PLÜCKER COORDINATE MIRROR FOR PARTIAL FLAG VARIETIES AND QUANTUM SCHUBERT CALCULUS

This chapter is based on [96], joint work with Changzheng Li, Konstanze Rietsch, and Mingzhi Yang.

4.1 Introduction

Mirror symmetry is a fascinating phenomenon arising in string theory: two apparently completely different objects on A-model and B-model give rise to equivalent physics. Mathematical descriptions of mirror symmetry, in terms of equivalence of mathematical structures, were first made for pairs of Calabi-Yau manifolds in early 1990s (see e.g. [68]). The (closed string) mirror symmetry was extended to Fano manifolds X on the topological A-model soon after by Givental [49, 51] and Eguchi-Hori-Xiong [37]. In this case, the topological B-model is given by a Landau-Ginzburg model (\check{X}, W) , consisting of a non-compact Kähler manifold \check{X} and a holomorphic function $W : \check{X} \rightarrow \mathbb{C}$ called the superpotential. Mirror symmetry predicts equivalences between both sides on various levels. For instance on one level, the (small) quantum cohomology ring $QH^*(X)$ should be isomorphic to the Jacobi ring $Jac(W)$ of W .

Studying mirror symmetry for X a priori requires a good construction of the mirror superpotential W . However, this is only known for certain Fano manifolds, with toric Fano manifolds and complete intersections inside toric manifolds being typical examples, following work of Givental [49, 51] and Hori-Vafa [69]. In this article, we will focus on the case when $X = \mathbb{F}\ell(n_\bullet)$ is a partial flag variety parameterizing flags of quotient vector subspaces of \mathbb{C}^n . Special cases include complex Grassmannians $Gr(k, n)$ and complete flag variety $\mathbb{F}\ell_n$. Candidate Landau-Ginzburg models for $Gr(k, n)$ and $\mathbb{F}\ell_n$ were constructed by Eguchi-Hori-Xiong [37] and Givental [50] respectively. They were later generalized to $\mathbb{F}\ell(n_\bullet)$ by Batyrev-Ciocan-Fontanine-Kim-van Straten [7]. See also [108] for a construction using holomorphic disk counts. Here different approaches turned out to result in identical versions of the superpotential, namely arriving at a particular Laurent polynomial W_{tor} defined on a complex torus of dimension $\dim \mathbb{F}\ell(n_\bullet)$. It turned out that there is a

toric degeneration of $\mathbb{F}\ell(n_\bullet)$ with the central fiber a singular toric variety X_0 , and the superpotential W_{tor} coincides with the superpotential mirror to X_0 as constructed by Givental and Hori-Vafa. This superpotential, however, does not contain enough information to be an honest mirror superpotential for $\mathbb{F}\ell(n_\bullet)$ sometimes. For instance for $Gr(2, 4)$ the correct mirror superpotential would be expected to have $6 = \dim H^*(Gr(2, 4))$ critical points, but W_{tor} only has 4.

In [118], the second-named author wrote down a Lie theoretical superpotential, namely a function $\mathcal{F}_{Lie} : Z_P \rightarrow \mathbb{C}$ defined on a subvariety Z_P of B_- . Here G is a connected complex reductive Lie group, and P is a parabolic subgroup of G containing a Borel subgroup B_- . This function had appeared separately earlier in a different context, as part of a theory of geometric crystals [10]. It is shown in [118] that the fiberwise critical locus of \mathcal{F}_{Lie} is isomorphic to (an open dense part of) the so-called Peterson variety stratum Y_P in the flag variety G/B_- . This relates the superpotential \mathcal{F}_{Lie} to quantum cohomology via the remarkable isomorphism of Dale Peterson's, described in his unpublished lecture notes [113], between $\mathbb{C}[Y_P]$ and the small quantum cohomology ring $QH^*(G^\vee/P^\vee)$ of the Langlands dual flag variety. A proof of Peterson's isomorphism for the type A case, that is for $G^\vee/P^\vee = \mathbb{F}\ell(n_\bullet)$, was given in [117]. Some other cases were covered in [92?], and the general case was proved in a recent preprint [26]. The combination of both isomorphisms leads to mirror symmetry for flag varieties on the level of small quantum cohomology. Namely, the ring $QH^*(G^\vee/P^\vee)$, with inverse quantum parameters adjoined, is isomorphic to the (fiberwise) Jacobi ring of \mathcal{F}_{Lie} .

This is not the end of the story, but only the end of the beginning. The function \mathcal{F}_{Lie} is defined quite indirectly, and it is desirable to find a compact expression in terms of coordinates on the mirror space Z_P . In the special case of $\mathbb{F}\ell(n_\bullet) = Gr(n - k, n)$, a natural isomorphic interpretation of the mirror space was given in [103]. There, Z_P was identified with a trivial family over \mathbb{C}^* with fiber a particular open log Calabi-Yau subvariety in the Langlands dual Grassmannian $Gr(k, n)$. Moreover, [103] gave a very compact and clean expression for \mathcal{F}_{Lie} using the Plücker coordinates of $Gr(k, n)$. This also led to an improved mirror symmetry result on the higher level of D -modules.

One generalization of this $Gr(n - k, n)$ construction is to cominuscule Grassmannians of other types. The fiber of the mirror space is then inside the Langlands dual minuscule Grassmannian, which has (generalized) Plücker coordinates,

due to its embedding into the projective space of a minuscule representation. Corresponding coordinate presentation of \mathcal{F}_{Lie} have been individually obtained for quadrics [111], Lagrangian Grassmannians [110], the Cayley plane and the Freudenthal variety [124].

The generalization of $Gr(n - k, n)$ of interest to us here is the partial flag variety $X = \mathbb{F}l(n_\bullet)$. As the first main result of this paper, we provide a Plücker coordinate formula version \mathcal{F}_- of the superpotential \mathcal{F}_{Lie} for this case. To construct the domain we consider the Langlands dual partial flag variety $Fl_{n_\bullet} = Fl_{n_1, \dots, n_r; n} = P \backslash G$ that parameterizes flags of vector subspaces V_{n_j} in the dual vector space of \mathbb{C}^n . Let $(P \backslash G)^\circ$ denote the complement of the Knutson-Lam-Speyer anti-canonical divisor $-K_{Fl_{n_\bullet}}$ [88], which consists of $(n - 1 + r)$ irreducible components (see Proposition 4.3.5).

Theorem 4.1.1. There is an isomorphism

$$\psi_- : Z_P \longrightarrow (P \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^*, \quad \text{where } I^P := \{n_1, \dots, n_r\}.$$

The superpotential $\mathcal{F}_- := \mathcal{F}_{Lie} \circ \psi_-^{-1} : (P \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^* \rightarrow \mathbb{C}$ is of the form

$$\mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i \in I^P} q_i v_{i, i+1} + \sum_{i=1}^{n-1} u_{i, i+1},$$

consisting of $(n - 1 + r)$ summands, and these satisfy

1. the $v_{i, i+1}$ are all of the form $\frac{p_{j'}}{p_j}$ for some Plücker coordinates;
2. the $u_{i, i+1}$ are of the form $\frac{p_{j'}}{p_j}$ if $i \in I^P$, or if $1 \leq i \leq n_1$ or $n_r \leq i \leq n - 1$. Otherwise, if $n_j < i < n_{j+1}$ for some $j \in \{1, \dots, r - 1\}$, then $u_{i, i+1}$ is of the form $\frac{f_1}{f_2}$ with each f_i a quadratic polynomial in the Plücker coordinates;
3. all $v_{i, i+1}$ and $u_{i, i+1}$ have pole of order 1 along a (unique) irreducible component of $-K_{Fl_{n_\bullet}}$.

In fact, the denominators in the summands of \mathcal{F}_- are precisely the defining equations of the irreducible components of $-K_{Fl_{n_\bullet}}$ [95].

The isomorphism Ψ_- will be constructed explicitly in Definition 4.3.3. Explicit expressions for $v_{i, i+1}$ and $u_{i, i+1}$ will be given in **Theorem 4.3.10**. Here we provide an example to give a first impression.

Example 4.1.1. For $F\ell_{n_\bullet} = F\ell_{2,4;7} \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^7) \times \mathbb{P}(\wedge^4 \mathbb{C}^7)$, we have

$$\mathcal{F}_- = q_2 \frac{p_{46}}{p_{67}} + q_4 \frac{p_{1467}}{p_{4567}} + \frac{p_{27}}{p_{17}} + \frac{p_{24}p_{1567} - p_{14}p_{2567} + p_{12}p_{4567}}{p_{23}p_{1567} - p_{13}p_{2567} + p_{12}p_{3567}} + \frac{p_{2346}}{p_{2345}} + \frac{p_{3457}}{p_{3456}} + \frac{p_{13}}{p_{12}} + \frac{p_{1235}}{p_{1234}}.$$

Remark 4.1.2. A straightforward generalization of the superpotential in [103] leads to another superpotential \mathcal{F}_+ defined on $(P_+ \setminus G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^*$. The Plücker coordinate expression of \mathcal{F}_+ , however, appears to be complicated and does not have the similar good properties (especially property (3) above). We refer to Examples 4.3.2 and 4.3.24 for a comparison of \mathcal{F}_+ and \mathcal{F}_- in the case of the complete flag variety $F\ell_3$.

Remark 4.1.3. In [57], Gu and Sharpe proposed a construction of ‘non-abelian’ mirrors, examples of which included $\mathbb{F}\ell(n_\bullet)$. Their approach is closely related to [69] but involves more variables than the dimension of $\mathbb{F}\ell(n_\bullet)$ (see also [56]). Their mirror diverges already in the Grassmannian case from the mirror constructions [37, 103, 118] that are related to ours here, see [57, Section 4.9].

Another mirror construction inspired by viewing flag varieties as non-abelian GIT quotients was given by Kalashnikov [81]. Namely, Kalashnikov proposed a generalization of the superpotential from [103] for $Gr(n-k, n)$ to partial flag varieties $\mathbb{F}\ell(n_\bullet)$ in the form of a rational function on a product of Grassmannians, expressed explicitly in terms of Plücker coordinates, which recovers the aforementioned Laurent polynomial W_{tor} in a cluster chart. Kalashnikov also described a relation (on the level of critical points) between her superpotential and Gu-Sharpe’s superpotential in a special case.

To compare the Kalashnikov formula with our \mathcal{F}_- , consider the partial flag variety $\mathbb{F}\ell(n_\bullet) = \mathbb{F}\ell(4; 2, 1)$. Kalashnikov’s superpotential is a rational function on $Gr(2, 4) \times Gr(1, 2) \times (\mathbb{C}^*)^2$ described in terms of Plücker coordinates $[p_{ij}; \hat{p}_k]$ by

$$W_{\text{Kal}} = \frac{p_{13}}{p_{12}} + \frac{p_{24} + q_2}{p_{23}} + \frac{p_{24}}{p_{14}} + \frac{q_2 p_{13} \hat{p}_2}{p_{34}} + \frac{\hat{p}_2}{\hat{p}_1} + \frac{q_1}{\hat{p}_2}.$$

Our superpotential \mathcal{F}_- is a rational function on $F\ell_{1,2;4} \times (\mathbb{C}^*)^2$, given in terms of $[p_k; p_{ij}]$ by

$$\mathcal{F}_- = q_1 \frac{p_3}{p_4} + q_2 \frac{p_{14}}{p_{34}} + \frac{p_2}{p_1} + \frac{p_{13}}{p_{12}} + \frac{p_{24}}{p_{23}}.$$

Apart from the formula looking more complicated, the superpotential W_{Kal} turns out not to have the full set of critical points in this example. Namely, while \mathcal{F}_- has 12 critical points along its $q_1 = q_2 = 1$ fiber, in agreement with

$\dim H^*(\mathbb{F}\ell(4; 2, 1)) = 12$, one of these critical points, the one with critical value -3 , is missing for $W_{\text{Kal}}|_{\mathbf{q}=(1,1)}$.

Let us now recall that, on the A-side, the (small) quantum cohomology ring $QH^*(X) = (H^*(X, \mathbb{C}) \otimes \mathbb{C}[\mathbf{q}], \cdot)$ of the Fano manifold X is a deformation of the classical cohomology ring $H^*(X, \mathbb{C})$ by incorporating genus zero, 3-point Gromov–Witten invariants. The quantum multiplication by the first Chern class of X induces a linear operator

$$\hat{c}_1(\mathbf{q}) : QH^*(X) \longrightarrow QH^*(X); \beta \mapsto c_1(X) \cdot \beta$$

depending on the values of the deformation parameters $\mathbf{q} = (q_i)_i$, also called quantum parameters. Here we treat the q_i as nonzero complex numbers, so that $QH^*(X) = H^*(X)$ as vector spaces. On the B-side, we consider the superpotential $W = W_{\mathbf{q}}$ with the quantum parameters fixed correspondingly. Now let us state a celebrated folklore conjecture in mirror symmetry.

Conjecture 4.1.4. *The eigenvalues of the first Chern class operator $\hat{c}_1(\mathbf{q})$ coincide with the critical values of the mirror superpotential $W_{\mathbf{q}}$.*

There has been very little progress on this conjecture in the past two decades. The case of toric Fano manifolds was first proved by Auroux [?], which was also known to Kontsevich and Seidel. Recently, Yuan [131] proved that the critical values of the family Floer mirror Landau-Ginzburg superpotential are the eigenvalues of the first Chern class, under certain assumptions. The cases of complex Grassmannians and quadrics were proved implicitly in [103] and [70] respectively.

As a central result of this paper, we prove a theorem that implies this conjecture for any partial flag variety $X = \mathbb{F}\ell(n_{\bullet})$.

Let us write $\mathbf{q} = (q_{n_1}, \dots, q_{n_r})$ for the quantum parameters associated to $\mathbb{F}\ell(n_{\bullet})$, and view them as coordinates on an algebraic torus that we denote by $\prod_{i \in I^P} \mathbb{C}_q^*$. Let us consider the (fiberwise) Jacobi ring,

$$\text{Jac}(\mathcal{F}_-) := \mathcal{O} \left((P \setminus G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^* \right) / (\partial_{(P \setminus G)^\circ} \mathcal{F}_-), \quad (4.1.5)$$

where we are taking partial derivatives of \mathcal{F}_- in the $(P \setminus G)^\circ$ directions only. Using Theorem 4.1.1, and the isomorphism between fiberwise Jacobi ring of

\mathcal{F}_{Lie} and quantum cohomology resulting from [113, 117, 118], we obtain an isomorphism of rings

$$\Theta : Jac(\mathcal{F}_-) \xrightarrow{\sim} QH^*(X)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}]. \quad (4.1.6)$$

See Section 4.4 for a more detailed description. We can now state our second main theorem.

Theorem 4.1.2. For the class $[\mathcal{F}_-]$ of \mathcal{F}_- in the Jacobi ring $Jac(\mathcal{F}_-)$, we have

$$\Theta([\mathcal{F}_-]) = c_1(X),$$

where $c_1(X)$ is the first Chern class of $X = \mathbb{F}\ell(n_\bullet)$, as element of the small quantum cohomology ring.

The above theorem is stated again in an isomorphic form in **Theorem 4.4.11**, using a version \mathcal{F}_R of the superpotential whose domain relates more directly to the Peterson variety. By interpreting the critical values of \mathcal{F}_- as eigenvalues for the operator of multiplication by $[\mathcal{F}_-]$ on $Jac(\mathcal{F}_-)$, we obtain the following corollary.

Corollary 4.1.3. Conjecture 4.1.4 holds for $X = \mathbb{F}\ell(n_\bullet)$ and the mirror superpotential \mathcal{F}_- .

We note that isomorphically changing the domain of the superpotential does not affect the critical values. Therefore the same corollary holds for \mathcal{F}_{Lie} , and \mathcal{F}_R . We also note that \mathcal{F}_R and \mathcal{F}_- look to be related by the chiral map in [43].

An exciting aspect of this part of our paper is the interaction between the Conjecture 4.1.4 in mirror symmetry and identities in quantum Schubert calculus. The quantum cohomology ring $QH^*(\mathbb{F}\ell(n_\bullet))$ has a $\mathbb{C}[\mathbf{q}]$ -basis of Schubert classes σ_w , that is indexed by permutations in S_n with descents at most in n_j , for $j \in \{1, \dots, r\}$. The study of the ring structure of $QH^*(\mathbb{F}\ell(n_\bullet))$ in terms of this basis, referred to as (type A) quantum Schubert calculus, is an area of great independent interest from the viewpoint of enumerative geometry. One of the most central problems is to find a manifestly positive formula for the Schubert structure constants in the quantum product of two Schubert classes. Another topic of interest is the study of identities among quantum products of Schubert classes. For example, the quantum Schubert polynomials [29, 38] are expressions for general Schubert classes as polynomials

in special Schubert classes. The quantum Giambelli formula [11] for complex Grassmannians is another example. It turns out, that mirror symmetry also helps us find identities of this kind. Let us illustrate this from the perspective of the following natural question. Consider the isomorphism Θ from (4.1.6)

Question 4.1.4. What are the preimages of the Schubert classes in $QH^*(\mathbb{F}\ell(n_\bullet))$ under Θ ?

Assuming the answer, one may expect to find relations in $QH^*(\mathbb{F}\ell(n_\bullet))$ simply by studying the mirror superpotential. Indeed, in the special case of $\mathbb{F}\ell(n_\bullet) = Gr(n-k, n)$, our \mathcal{F}_- turns out to coincide with the superpotential \mathcal{F}_+ of [103] (see Example 4.3.25). Therefore, we have $\Theta^{-1}(\sigma_w) = [p_w]$, where p_w denotes the (suitably normalised) Plücker coordinate corresponding to the Grassmannian permutation w , as described in [103]. Each term in \mathcal{F}_- turns out to reveal a quantum cohomology relation, see [103, Remark 6.2], recovering an instance of the known quantum Pieri-Chevalley formula in this case. For more general partial flag varieties the question above can be answered for certain Schubert classes using work of Peterson, see Section 4.4. This means that quantum Schubert calculus relations involving these classes can be viewed as relations in the Jacobi ring.

The approach of using the superpotential for understanding quantum cohomology was used also in [23]. Namely, one can consider partial derivatives of the superpotential which naturally represent the zero class in the Jacobi ring, and translate these into quantum cohomology relations via the mirror isomorphism. Following this approach, [23] obtained certain relations involving ‘quantum hooks’ via W_{Kal} .

Our final result is a set of quantum Schubert calculus identities related to our formula for \mathcal{F}_- . The proof of Theorem 4.1.2, turns out to involve showing each term in \mathcal{F}_- corresponds to specific class in quantum cohomology. This requires certain relations to be proved in $QH^*(\mathbb{F}\ell(n_\bullet))$. For instance, the term $\frac{p_{24}p_{1567} - p_{14}p_{2567} + p_{12}p_{4567}}{p_{23}p_{1567} - p_{13}p_{2567} + p_{12}p_{3567}}$ in Example 4.1.1 relates to an identity

$$\sigma_{24} \cdot \sigma_{1567} - \sigma_{14} \cdot \sigma_{2567} + \sigma_{12} \cdot \sigma_{4567} = (\sigma_{23} \cdot \sigma_{1567} - \sigma_{13} \cdot \sigma_{2567} + \sigma_{12} \sigma_{3567}) \cdot (\sigma_{13} + \sigma_{1235})$$

in quantum Schubert calculus. Note that we have simplified the notations, for instance by σ_{24} above we mean the Schubert class $\sigma_{2413567}$ indexed by the Grassmannian permutation 2413567 in one-line notation. The above identity

is equivalent to the following simpler one by using quantum Chevalley-Monk formula [18, 29, 104],

$$\sigma_{1526347} \cdot \sigma_{2314567} - \sigma_{2516347} \cdot \sigma_{1324567} + \sigma_{3516247} \cdot \sigma_{1234567} = 0. \quad (4.1.7)$$

Our final theorem, that we prove concurrently with Theorem 4.1.2, gives new relations in quantum Schubert calculus of $QH^*(\mathbb{F}\ell(n_\bullet))$, including identity (4.1.7) as one example.

Theorem 4.1.5. In $QH^*(\mathbb{F}\ell(n_\bullet))$, there are quantum relations of the form

$$\sum_J (-1)^{|J|} \sigma_{w_J} \sigma_{[1, n_j + d] \setminus J} = 0.$$

We will postpone the explanations of the relevant notations to Section 5, and will restate the identity fully in **Theorem 4.5.2**. The proof of the above theorem goes via the complete flag variety $\mathbb{F}\ell_n$ using Peterson's remarkable extension property (see Proposition 4.5.11). The proof of Theorem 4.1.2 is closely linked to the above result.

Remarks for further directions

Closed string mirror symmetry in full generality at genus zero predicts an isomorphism on the level of Frobenius manifolds. The notion of a *Frobenius manifold* was first introduced by B. Dubrovin in 1990s [36], while the first construction of a Frobenius manifold could date back to K. Saito [120] in early 1980s in the name of *flat structures* using his primitive form theory. Mirror symmetry predicts that the Frobenius manifold associated to the Gromov–Witten theory of a Fano manifold X (the *big* quantum cohomology ring of X) should be isomorphic to the Frobenius manifold of the mirror Landau-Ginzburg model (\check{X}, W) associated to an appropriate Saito's primitive form. This was indirectly proved for complex Grassmannians in [30, 86] by a reduction to the case of projective spaces [4]. The case of quadrics was recently proved in [70], where the verification of Conjecture 4.1.4 is an important step. We expect that our Theorem 4.1.2 will play an important role in studying mirror symmetry $\mathbb{F}\ell(n_\bullet)$ on such level as well.

For the mirror symmetry on the intermediate level of D -modules, the Plücker coordinate versions of the superpotential of \mathcal{F}_{Lie} play a very important role in proving an explicit injective morphism of D -modules for complex Grassmannians and quadrics [103, 111]. An implicit isomorphism of D -modules

for minuscule Grassmannians was proved in [93]. A proof for an equivariant D -module isomorphism for general G^\vee/P^\vee was recently given in [28]. However, the isomorphism therein seems not explicit enough either. Moreover, verification of the Gauss-Manin connection along z -direction was missing, which is an indispensable piece in the mirror symmetry on the level of Frobenius manifolds. We believe that our Theorem 4.1.2 will be helpful towards getting a better understanding of the D -module mirror symmetry for $\mathbb{F}\ell(n_\bullet)$.

Conjecture 4.1.4 also appeared in the context of Kontsevich's *homological mirror symmetry* [90], which is one main approach to (open string) mirror symmetry (in addition to another main approach by Strominger-Yau-Zaslow [126]). For G^\vee/P^\vee , homological mirror symmetry was so far only proved for complex Grassmannians $G(n-k, n)$ with n prime [20], beyond the projective space case covered earlier [1]. Here it is important to understand the superpotential \mathcal{F}_- in a Floer theoretical way, which has only been achieved for very few cases including $Gr(2, n)$ [67]. It will be desirable to understand the superpotential more deeply.

Another closely related direction is about the Gamma conjecture I and its underlying conjecture \mathcal{O} proposed by Galkin-Golyshev-Iritani [44]. Here conjecture \mathcal{O} concerns the eigenvalues of the aforementioned linear operator $\hat{c}_1|_{\mathbf{q}=1}$. For flag varieties G^\vee/P^\vee , conjecture \mathcal{O} has already been proved in [24], while the Gamma conjecture I was only known for very few cases including complex Grassmannians and quadrics. One main approach to Gamma conjecture I in [45] relies on a B-side analogy of conjecture \mathcal{O} and a conifold condition. Our Theorem 4.4.11, together with [24], ensures the B-side analogy of conjecture \mathcal{O} . Therefore it will play an important role in the study of the Gamma conjecture I for $\mathbb{F}\ell(n_\bullet)$ via this approach.

Finally, we would propose a deeper interaction between mirror symmetry and quantum Schubert calculus for G^\vee/P^\vee . Indeed, for the type C case, some conjectural quantum relations in the quantum cohomology of a Lagrangian Grassmannian were given in [110, Conjecture 4.1], inspired by Conjecture 4.1.4. Even in type A , we would ask which quantum relations arise from the natural partial derivatives of the mirror superpotential \mathcal{F}_- via the mirror isomorphism. We also note that some new quantum relations in $QH^*(\mathbb{F}\ell(n_\bullet))$ related with a cluster algebra structure were discovered in [60]. It will be interesting to investigate whether these relations could also be revealed using cluster charts

in the domain of \mathcal{F}_- .

The paper is organized as follows. In Section 2, we introduce the basic notions. In Section 3, we construct two superpotentials \mathcal{F}_\pm , and provide the Plücker coordinate expression of \mathcal{F}_- . In Section 4, we prove Theorem 4.1.2 by assuming Lemma 4.4.12 first. Section 5 is devoted to a proof of Lemma 4.4.12 in terms of equivalent identities on quantum product of Schubert classes. Finally, in the Appendix we provide a description of \mathcal{F}_- using Young diagrams.

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4.2 Preliminaries

We review some background in Lie theory (see e. g. [13] for details).

Notation

Let $G = GL_n(\mathbb{C})$. Let B_+ and B_- denote the upper-triangular and lower-triangular Borel subgroups of G with unipotent radicals U_+ and U_- , respectively. Then $T = B_- \cap B_+$ is the maximal torus of diagonal matrices in G .

Let $\mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_-, \mathfrak{u}_+, \mathfrak{h}$ be the Lie algebras of B_-, B_+, U_-, U_+ and T respectively. Let $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ be the standard base of simple roots, and R (resp. R_+) be the set of (positive) roots. That is, we have the Cartan decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad \text{with} \quad \mathfrak{g}_{\alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}} = \mathbb{C}E_{ij} \quad \forall 1 \leq i < j \leq n,$$

where $E_{i,j}$ is the matrix with entry 1 in row i and column j and zeros elsewhere. We will view elements of Δ as lying in the character group of T , so that

$$\alpha_i : T \rightarrow \mathbb{C}^* := \mathbb{C} \setminus \{0\}; \quad t = \text{diag}(t_1, \dots, t_n) \mapsto \alpha_i(t) = \frac{t_i}{t_{i+1}}.$$

For any positive integers k, m with $k < m$, we denote the integral interval $[k, m] := \{k, k+1, \dots, m\}$, and simply denote $[m] := [1, m]$. Set $I = [n-1]$. The Weyl group W of $\mathfrak{gl}(n, \mathbb{C})$ is generated by simple reflections $s_i = s_{\alpha_i}$, $i \in I$. We will freely identify W with the Weyl group $N_G(T)/T$ of G as well as the symmetric group S_n , by using the isomorphisms

$$S_n \xrightarrow{\cong} W \xrightarrow{\cong} N_G(T)/T,$$

where $(i, i+1) \mapsto s_i \mapsto \dot{s}_i T$ for $\dot{s}_i = \exp(E_{i,i+1}) \exp(-E_{i+1,i}) \exp(E_{i,i+1})$.

Moreover, we let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the standard length function. For $w = s_{i_1} \cdots s_{i_m}$ with $m = \ell(w)$, the element $\dot{w} := \dot{s}_{i_1} \cdots \dot{s}_{i_m} \in N_G(T)$ is independent of the reduced expression chosen.

We let $P \supseteq B_-$ be a parabolic subgroup of G . Set $I_P = \{i \in I \mid \dot{s}_i \in P\}$ and $I^P = I \setminus I_P$. We write

$$I^P = \{n_1, \dots, n_r\},$$

where $1 \leq n_1 < n_2 < \cdots < n_r \leq n-1$.

Let W_P be the Weyl subgroup of W associated to P , and W^P be the set of minimal length coset representatives in W/W_P . Precisely,

$$W_P = \langle s_i \mid i \in I_P \rangle, \quad W^P = \{u \in W \mid \ell(us_i) > \ell(u), \forall i \in I_P\}.$$

Denote by w_P (resp. w^P, w_0) the longest element in W_P (resp. W^P, W).

The Langlands dual group G^\vee to G is again $GL(n, \mathbb{C})$, but plays a different role. Let $B_\pm^\vee, P_\pm^\vee, T^\vee, \Delta^\vee$ be the Langlands dual versions of B_\pm, P_\pm, T, Δ , respectively. The base Δ^\vee for G^\vee are canonically identified with the set $\{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\}$ of simple coroots for G . In particular, we have $s_{\alpha_i} = s_{\alpha_i^\vee}$. The Weyl group for G^\vee is again W , and $I_{P^\vee} = I_P$ for any parabolic subgroup P of G containing B_+ or B_- . The deeper relationship between the original group $GL_n(\mathbb{C})$ and its Langlands dual group is described by the geometric Satake correspondence [48, 100, 106].

Langlands dual flag varieties

A partial flag variety is a quotient of $GL(n, \mathbb{C})$ by a parabolic subgroup on the left or right. We can think of it as parameterizing flags of subspaces (of row vectors) in $\text{Hom}(\mathbb{C}^n, \mathbb{C})$ or flags of quotients of the space \mathbb{C}^n (in column vectors), to be precisely described below. Since we will focus more on the B-side of mirror symmetry, we will use G there, and leave G^\vee for the A-side of mirror symmetry.

On the B-side, recall that $P \supseteq B_-$ is the parabolic subgroup of G with $I^P = \{n_1, \dots, n_r\}$. Denote $n_0 := 0$ and $n_{r+1} := n$, and set

$$a_j := n_j - n_{j-1}, \quad \forall j \in [r+1].$$

Then P consists of block-lower-triangular matrices in G with block-diagonal matrices of the form $\text{diag}\{M_1, M_2, \dots, M_{r+1}\}$, where each M_j is an $a_j \times a_j$ invertible matrix.

Consider the partial flag variety $Fl_{n_\bullet} = Fl_{n_1, \dots, n_r; n}$ that parameterizes flag of vector subspaces V_{n_j} in $Hom(\mathbb{C}^n, \mathbb{C})$, namely

$$Fl_{n_\bullet} = \{V_{n_1} \subset \dots \subset V_{n_r} \subset Hom(\mathbb{C}^n, \mathbb{C}) \mid \dim V_{n_j} = n_j, 1 \leq j \leq r\}.$$

The Lie group G transitively acts on Fl_{n_\bullet} on the right, inducing an isomorphism

$$\tau_P : P \backslash G \xrightarrow{\cong} Fl_{n_\bullet}.$$

The isomorphism τ_P sends Pb to the partial flag V_\bullet such that V_{n_j} is spanned by the first n_j row vectors of the matrix b for all $j \in [r]$.

On the A-side, we consider the partial flag variety $\mathbb{F}l(n_\bullet) = \mathbb{F}l(n; n_r, \dots, n_1)$ that parameterizes flag of quotients, namely

$$\mathbb{F}l(n_\bullet) = \{\mathbb{C}^n \twoheadrightarrow \Lambda_{n_r} \twoheadrightarrow \dots \twoheadrightarrow \Lambda_{n_1} \rightarrow 0 \mid \dim \Lambda_{n_j} = n_j, \forall j \in [r]\} / \sim.$$

Here $\Lambda_\bullet \sim \Lambda'_\bullet$ if and only if there are isomorphisms $\Lambda_{n_j} \rightarrow \Lambda'_{n_j}$ making the diagram of the two flags of quotients commutative. There is a canonical isomorphism

$$G^\vee / P^\vee \xrightarrow{\cong} \mathbb{F}l(n_\bullet),$$

which sends gP^\vee to the class of a flag Λ_\bullet of quotients such that the kernel of $\mathbb{C}^n \twoheadrightarrow \Lambda_{n_j}$ is the vector subspace spanned by the last $n - n_j$ column vectors of the matrix g for all j .

We remark that there are many canonical isomorphisms between group quotients and different parameterizations of flag varieties floating around. Here we are taking the above isomorphisms, to fit the philosophy of Langlands dual as well as the word "mirror". It does not matter much if different isomorphisms are taken.

Open Richardson varieties

For $v, w \in W$, with $v \leq w$ with respect to the Bruhat order, we have the *open Richardson subvarieties*,

$$\mathcal{R}_{v,w}^- := (B_- \backslash B_- \dot{v}^{-1} B_+) \cap (B_- \backslash B_- \dot{w}^{-1} B_-) \subset B_- \backslash G,$$

$$\mathcal{R}_{v,w}^+ := (B_+ \setminus B_+ \dot{v}^{-1} B_-) \cap (B_+ \setminus B_+ \dot{w}^{-1} B_+) \subset B_+ \setminus G.$$

These are smooth and irreducible of dimension $\ell(w) - \ell(v)$ [85]. Note that the intersections above are empty if $v \not\leq w$. The Zariski closures are called a (closed) Richardson varieties and denoted by $\overline{\mathcal{R}}_{v,w}^-$ and $\overline{\mathcal{R}}_{v,w}^+$, respectively. We will also consider the following (open) Richardson varieties in G/B_- .

$$\begin{aligned} \mathcal{R}_{v,w}^{R^-} &:= (B_+ \dot{v} B_- / B_-) \cap (B_- \dot{w} B_- / B_-) \subset G/B_-, \\ \mathcal{R}_{v,w}^{R^+} &:= (B_- \dot{v} B_+ / B_+) \cap (B_+ \dot{w} B_+ / B_+) \subset G/B_+. \end{aligned}$$

Open Richardson varieties and their projections will be our main target spaces on the B-side.

We use the notation $(P \setminus G)^\circ$ for the top-dimensional projected open Richardson variety inside $P \setminus G$, namely $(P \setminus G)^\circ = \text{pr}_P(\mathcal{R}_{\text{id}, w_0 w_P}^-)$ under the projection $\text{pr}_P : B_- \setminus G \rightarrow P \setminus G$.

Schubert varieties

On the A-side, we consider the Bruhat decompositions

$$X = G^\vee / P^\vee = \bigsqcup_{v \in W^P} B_+^\vee \dot{v} P^\vee / P^\vee = \bigsqcup_{w \in W^P} B_-^\vee \dot{w} P^\vee / P^\vee.$$

The Zariski closures of the (opposite) Schubert cells $B_+^\vee \dot{v} P^\vee / P^\vee$ and $B_-^\vee \dot{w} P^\vee / P^\vee$,

$$X^v = \overline{B_+^\vee \dot{v} P^\vee / P^\vee}, \quad X_w := \overline{B_-^\vee \dot{w} P^\vee / P^\vee}$$

are (opposite) Schubert varieties in X . They are of codimension $\ell(v)$ and dimension $\ell(w)$ respectively. It is well-known that the classical cohomology ring $H^*(X, \mathbb{Z})$ has a \mathbb{Z} -basis of Schubert classes σ_v :

$$H^*(X, \mathbb{Z}) = \bigoplus_{v \in W^P} \mathbb{Z} \sigma_v, \quad \text{where } \sigma_v := \text{P.D.}[X^v] \in H^{2\ell(v)}(X, \mathbb{Z}),$$

and $\text{P.D.}[X^v]$ stands for the Poincaré dual of the fundamental homology class of X^v .

4.3 The Plücker coordinate superpotentials

In this section, we will construct two versions of a superpotential for $X = G^\vee / P^\vee$ defined on projected open Richardson varieties for G . The first superpotential \mathcal{F}_+ is a straightforward extension of a construction for complex

Grassmannians given in [103]. It has a natural formula in terms of Plücker coordinates in the Grassmannian case, as was shown in [103], but this formula does not generalise well to more general partial flag varieties. The construction of the second superpotential \mathcal{F}_- , which has a natural Plücker coordinate presentation in general, is the main outcome of this section.

The superpotential \mathcal{F}_+ generalising [103]

Recall we have the following (open) Richardson variety in $B_+ \backslash G$,

$$\mathcal{R}_{\text{id}, w_0 w_P}^+ := (B_+ \backslash B_+ B_-) \cap (B_+ \backslash B_+ \dot{w}_P^{-1} w_0^{-1} B_+) \subset B_+ \backslash G$$

and the projection map $\text{pr}_{P_+} : B_+ \backslash G \rightarrow P_+ \backslash G$, where P_+ is upper-triangular parabolic subgroup with $I^{P_+} = \{n_1, \dots, n_r\}$. It is shown in [88, Lemma 3.1] that $\text{pr}_{P_+} : \mathcal{R}_{\text{id}, w_0 w_P}^+ \rightarrow (P_+ \backslash G)^\circ = \text{pr}_{P_+}(\mathcal{R}_{\text{id}, w_0 w_P}^+)$ is an isomorphism. Moreover, implicitly from [88], the projected open Richardson variety $\text{pr}_{P_+}(\mathcal{R}_{\text{id}, w_0 w_P}^+)$ is the complement of an anti-canonical divisor in $P_+ \backslash G$.

As in [103], we use $GL(n, \mathbb{C})$ as the starting point instead of $PSL(n, \mathbb{C})$ used in [118], and thus need to cut down to a codimension one subtorus in T . The torus T has an adjoint action by W . Consider the invariant subtorus

$$\mathcal{T}^{W_P} = \{t \in T \mid t_n = 1, \dot{w} t \dot{w}^{-1} = t, \forall w \in W_P\}.$$

We define a map

$$\begin{aligned} \psi_+ : B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ &\longrightarrow (P_+ \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^* \\ b_- = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2 &\longmapsto (P_+ b_-, \mathbf{q}(t)) \end{aligned}$$

where

$$\begin{aligned} \mathcal{T}^{W_P} &\xrightarrow{\sim} \prod_{i \in I^P} \mathbb{C}_q^* \\ t &\longmapsto \mathbf{q}(t) := (\alpha_{n_1}(t), \dots, \alpha_{n_r}(t)). \end{aligned} \tag{4.3.1}$$

It follows from [118, Section 4] and [88] that ψ_+ is an isomorphism.

Definition 4.3.1. We define the superpotential \mathcal{F}_+ by

$$\begin{aligned} \mathcal{F}_+ : (P_+ \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^* &\xrightarrow{\psi_+^{-1}} B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ \longrightarrow \mathbb{C} \\ (P_+ b_-, \mathbf{q}(t)) &\longmapsto b_- = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2 \longmapsto \sum_{i=1}^{n-1} e_i^*(u_1) + e_i^*(u_2). \end{aligned}$$

Where $e_k^* : U_+ \rightarrow \mathbb{C}$ is the map that sends $u = (u_{ij})$ in U_+ to its $(k, k+1)$ -entry, namely $e_k^*(u) = u_{k, k+1}$. This is well-defined by [118, Lemma 5.2].

This definition is a direct translation of the Lie-theoretic superpotential defined in [118] via the isomorphism ψ_+ . If P_+ is a maximal parabolic, then this definition agrees with the one used to give a Plücker coordinate superpotential for Grassmannians in [103]. In general, viewing \mathcal{F}_+ , as a rational function on the partial flag variety $P_+ \backslash G$ (depending additionally on parameters q_i), there will be a Plücker coordinate formula also in the partial flag setting. However, it turns out that this construction gives a superpotential that is not as well-suited for being expressed in terms of Plücker coordinates as we would like.

Example 4.3.2. Consider the complete flag variety G^\vee/B_-^\vee for $GL_3(\mathbb{C})$ and the associated superpotential \mathcal{F}_+ . Fix a representative b_- of P_+b_- . For a subset I of $\{1, 2, 3\}$ let p_I denote the minor of b_- with column set I and row set the last $|I|$ many rows. Then

$$\mathcal{F}_+(P_+b_-, (q_1, q_2)) = \frac{p_2}{p_1} + \frac{p_{13}}{p_{12}} + q_2 \frac{p_1 p_{13}}{p_3 p_{12}} + q_1 \frac{p_2 p_{12}}{p_1 p_{23}}.$$

This example will be useful for comparison between \mathcal{F}_+ and our alternative version of the superpotential that we construct in Section 4.3 (see Example 4.3.24).

Superpotential \mathcal{F}_{Lie}

We now give the construction of the original Lie theoretic superpotential \mathcal{F}_{Lie} in a form that is suitable for our applications. The following definition is a slight change of conventions on [103, Definition 6.3] and [118].

We recall the definition of the torus \mathcal{T}^{W_P} and the isomorphism from (4.3.1), as well as the maps $e_k^* : U_+ \rightarrow \mathbb{C}$. We also recall that $P \supseteq B_-$ is the parabolic subgroup of G with $I^P = \{n_1, \dots, n_r\}$.

Definition 4.3.2 (The Lie-theoretic superpotential). Let $Z_P := B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+$.

Define the map $\mathcal{F}_{\text{Lie}} : Z_P \rightarrow \mathbb{C}$ by

$$b_- = u_1 t \dot{w}_P^{-1} \dot{w}_0 u_2 \mapsto - \left(\sum_{i=1}^{n-1} e_i^*(u_1) + \sum_{i=1}^{n-1} e_i^*(u_2) \right).$$

It is an important fact that the map \mathcal{F}_{Lie} is well-defined even though u_1 and u_2 are not uniquely determined by b_- , see [118, Lemma 5.2].

The superpotential \mathcal{F}_-

In this section we give a non-standard isomorphism from Z_P to the product of the projected open Richardson variety $(P \backslash G)^\circ = \text{pr}_P(\mathcal{R}_{id, w_0 w_P}^-)$ with $\prod_{i \in I^P} \mathbb{C}_q^*$. The superpotential \mathcal{F}_- will then be defined as a function on $(P \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^*$.

We start by considering the isomorphism

$$\begin{aligned} \gamma : Z_P = B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+ &\longrightarrow (B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+) \times \prod_{i \in I^P} \mathbb{C}_q^* \\ b_- = u_1 t \dot{w}_P^{-1} \dot{w}_0 u_2 &\mapsto (\hat{b} = t^{-1} b_-, \mathbf{q}(t)), \end{aligned}$$

where $\mathbf{q}(t)$ is as in (4.3.1). Here the first factor of the right-hand side may be considered as fiber of Z_P where t equals to the identity element. The $q_{n_i} = \alpha_{n_i}(t)$ can also be obtained directly using minors of b_- .

We translate the superpotential \mathcal{F}_{Lie} to a function on the right-hand side above, and write down a formula for it for future reference. Namely, we have $\hat{b} = \hat{v} \dot{w}_P^{-1} \dot{w}_0 \hat{u}$ for $\hat{b} \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$. Then

$$\mathcal{F}_{\text{Lie}} \circ \gamma^{-1}(\hat{b}, (q_i)_{i \in I^P}) = - \left(\sum_{i \in I^P} \hat{v}_{i,i+1} + \sum_{i \in I^P} q_i \hat{v}_{i,i+1} + \sum_{i=1}^{n-1} \hat{u}_{i,i+1} \right). \quad (4.3.3)$$

The main step in the construction now is to make a choice for \hat{v} for which only the quantum terms in the formula above will appear, and the other $\hat{v}_{i,i+1}$ vanish. Recall that $\mathcal{R}_{\text{id}, w_P w_0}^{R+} = B_- B_+ \cap B_+ w_P w_0 B_+ / B_+$. We define the variety

$$\mathcal{Z} := U_+ \dot{w}_P^{-1} \dot{w}_0 \cap \dot{w}_P^{-1} \dot{w}_0 U_- \cap B_- B_+, \quad (4.3.4)$$

which will play a central role in our constructions.

Lemma 4.3.5. *Consider the intersection $B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$ and the variety \mathcal{Z} as defined above. We have the following isomorphisms*

$$\begin{aligned} \zeta_1 : B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+ &\longrightarrow \mathcal{R}_{\text{id}, w_P w_0}^{R+} \\ \hat{b} &\mapsto \hat{b} B_+, \\ \zeta_2 : \mathcal{Z} &\longrightarrow \mathcal{R}_{\text{id}, w_P w_0}^{R+} \\ z &\mapsto z B_+. \end{aligned}$$

We consider the composition $\zeta = \zeta_2^{-1} \circ \zeta_1$,

$$\begin{aligned} \zeta : B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+ &\longrightarrow \mathcal{Z} \\ \hat{b} &\mapsto z, \end{aligned}$$

where $z \in \mathcal{Z}$ is the unique representative with $z B_+ = \hat{b} B_+$.

Proof. The map ζ_1 is just the restriction to the fiber over $e \in T^{W_P}$ of the isomorphism $B_- \cap U_+ T^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+ \cong \mathcal{R}_{\text{id}, w_P w_0}^{R+} \times T^{W_P}$ from [118, Section 4.1]. We now show that ζ_2 is an isomorphism.

Note that $\mathcal{R}_{id, w_P w_0}^{R^+}$ is the open dense subset of the Bruhat cell $B_+ w_P w_0 B_+ / B_+$ obtained by intersecting with opposite big cell $B_- B_+ / B_+$. We have the factorisation $U_+ = U_+^P U_{+,P}$, where

$$\begin{aligned} U_+^P &:= U_+ \cap \dot{w}_P^{-1} U_+ \dot{w}_P, \\ U_{+,P} &:= U_+ \cap \dot{w}_P^{-1} U_- \dot{w}_P, \end{aligned} \quad (4.3.6)$$

and the map $u \mapsto u \dot{w}_P^{-1} \dot{w}_0 B_+$ from U_+ restricts to an isomorphism $U_+^P \rightarrow B_+ w_P w_0 B_+ / B_+$. Equivalently, the projection map $g \mapsto g B_+$ restricts to an isomorphism

$$U_+^P \dot{w}_P^{-1} \dot{w}_0 \xrightarrow{\sim} B_+ w_P w_0 B_+ / B_+. \quad (4.3.7)$$

We now rewrite the definition of \mathcal{Z} as follows,

$$\begin{aligned} \mathcal{Z} &= U_+ \dot{w}_P^{-1} \dot{w}_0 \cap \dot{w}_P^{-1} \dot{w}_0 U_- \cap B_- B_+ \\ &= (U_+ \cap \dot{w}_P^{-1} U_+ \dot{w}_P) \dot{w}_P^{-1} \dot{w}_0 \cap B_- B_+ \\ &= U_+^P \dot{w}_P^{-1} \dot{w}_0 \cap B_- B_+. \end{aligned}$$

It follows that (4.3.7) restricts to an isomorphism $\mathcal{Z} \rightarrow \mathcal{R}_{id, w_P w_0}^{R^+}$, and this is precisely the map ζ_2 . \square

Lemma 4.3.8. *Recall that $(P \backslash G)^\circ = \text{pr}_P(\mathcal{R}_{id, w_0 w_P}^-)$ and let \mathcal{Z} be as defined in (4.3.4). We have an isomorphism*

$$\begin{aligned} \pi : \mathcal{Z} &\longrightarrow (P \backslash G)^\circ, \\ z &\longmapsto Pz. \end{aligned}$$

Proof. Note that

$$\mathcal{R}_{id, w_0 w_P}^- = \mathcal{R}_{w_P, w_0}^- \dot{w}_0 = B_- \backslash (B_- \dot{w}_P^{-1} B_+ \cap B_- \dot{w}_0^{-1} B_-) \dot{w}_0.$$

Consider U_+^P and $U_{+,P}$ as defined in (4.3.6) and the factorisation $U_+ = U_{+,P} U_+^P$. The Bruhat cell $B_- \backslash B_- \dot{w}_P^{-1} B_+$ is isomorphic to U_+^P via the map $u \mapsto B_- \dot{w}_P^{-1} u$. It follows that

$$\begin{aligned} \dot{w}_P^{-1} U_+^P \cap B_- \dot{w}_0^{-1} B_- &\rightarrow B_- \backslash (B_- \dot{w}_P^{-1} B_+ \cap B_- \dot{w}_0^{-1} B_-) \\ \dot{w}_P^{-1} u &\mapsto B_- \dot{w}_P^{-1} u. \end{aligned} \quad (4.3.9)$$

is an isomorphism. We can rewrite the definition of \mathcal{Z} as follows,

$$\mathcal{Z} = U_+ \dot{w}_P^{-1} \dot{w}_0 \cap \dot{w}_P^{-1} \dot{w}_0 U_- \cap B_- B_+$$

$$\begin{aligned}
&= \dot{w}_P^{-1} \left(\dot{w}_P U_+ \dot{w}_P^{-1} \cap \dot{w}_0 U_- \dot{w}_0^{-1} \right) \dot{w}_0 \cap (B_- \dot{w}_0^{-1} B_-) \dot{w}_0. \\
&= \left(\dot{w}_P^{-1} U_+^P \cap B_- \dot{w}_0^{-1} B_- \right) \dot{w}_0.
\end{aligned}$$

We now translate both sides of the isomorphism from (4.3.9) by \dot{w}_0 and obtain an isomorphism

$$\begin{aligned}
\pi' : \mathcal{Z} &\rightarrow \mathcal{R}_{id, w_0 w_P}^-, \\
z &\mapsto B_- z.
\end{aligned}$$

The composition of π' above with the isomorphism $\mathcal{R}_{id, w_0 w_P}^- \xrightarrow{\sim} (P \backslash G)^\circ$ from [88] is the map $\pi : \mathcal{Z} \rightarrow (P \backslash G)^\circ$, which proves that π is an isomorphism. \square

Definition 4.3.3 (The superpotential \mathcal{F}_-). We denote by ψ_- the composition of isomorphisms, where $\mathcal{B} = B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$,

$$\psi_- : Z_P \xrightarrow{\gamma} \mathcal{B} \times \prod_{i \in I^P} \mathbb{C}_q^* \xrightarrow{\zeta \times \text{id}} \mathcal{Z} \times \prod_{i \in I^P} \mathbb{C}_q^* \xrightarrow{\pi \times \text{id}} (P \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^*.$$

We now define the superpotential \mathcal{F}_- by

$$\mathcal{F}_- := \mathcal{F}_{\text{Lie}} \circ \psi_-^{-1} : (P \backslash G)^\circ \times \prod_{i \in I^P} \mathbb{C}_q^* \rightarrow \mathbb{C}.$$

Notations for \mathcal{F}_-

In summary, we have shown above that we may write any element of $(P \backslash G)^\circ$ uniquely as Pz for some $z \in \mathcal{Z}$. We can then write

$$z = v^{-1} \dot{w}_P^{-1} \dot{w}_0 \tag{4.3.10}$$

for a unique $v \in U_+$. Next let $\hat{b} := \zeta^{-1}(z) \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$. We can write

$$\hat{b} = zu^{-1} = v^{-1} \dot{w}_P^{-1} \dot{w}_0 u^{-1}, \tag{4.3.11}$$

for a unique $u \in U_+$. Finally, the superpotential \mathcal{F}_- is computed by

$$\mathcal{F}_-(Pz, q) = \mathcal{F}_{\text{Lie}} \circ \gamma^{-1}(\hat{b}, (q_i)_{i \in I^P}) = \sum_{i \in I^P} v_{i, i+1} + \sum_{i \in I^P} q_i v_{i, i+1} + \sum_{i=1}^{n-1} u_{i, i+1}, \tag{4.3.12}$$

using the description of \mathcal{F}_{Lie} in (4.3.3).

Definition 4.3.4. Given $z \in \mathcal{Z}$ we will always use the notations above for the related elements \hat{b}, v, u with $\hat{b} \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$ and $u, v \in U^+$, satisfying

$$\begin{aligned}
\hat{b} B_+ &= z B_+, \\
z &= v^{-1} \dot{w}_P^{-1} \dot{w}_0, \\
\hat{b} &= zu^{-1} = v^{-1} \dot{w}_P^{-1} \dot{w}_0 u^{-1}.
\end{aligned}$$

We will also let $b := \hat{b}^{-1} = u \dot{w}_0^{-1} \dot{w}_P v$.

We can immediately simplify the formula (4.3.12) using the following lemma.

Lemma 4.3.13. *Let $z \in \mathcal{Z}$. For $v \in U_+$ as in Definition 4.3.4, we have that $v \in \dot{w}_P^{-1}U_+\dot{w}_P$, and therefore the entry $v_{i,i+1} = 0$ for all $i \in I_P$.*

Remark 4.3.14. This lemma says that $v \in U_+^P$, in the notation (4.3.6) from Lemma 4.3.5.

Proof of Lemma 4.3.13. By (4.3.4) we have that $z \in \dot{w}_P^{-1}\dot{w}_0U_-$. The lemma follows from this and the fact that $v = \dot{w}_P^{-1}\dot{w}_0z^{-1}$. \square

As a consequence of Lemma 4.3.13 we have the formula,

$$\mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i \in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1}, \quad (4.3.15)$$

for \mathcal{F}_- in terms of u, v .

We can now use the formula (4.3.15) as our starting point for studying \mathcal{F}_- . The rest of this section will be devoted to giving a compact description of \mathcal{F}_- in terms of Plücker coordinates.

The description of \mathcal{Z}

We first give a concrete description of our variety

$$\mathcal{Z} = U_+\dot{w}_P^{-1}\dot{w}_0 \cap \dot{w}_P^{-1}\dot{w}_0U_- \cap B_-B_+. \quad (4.3.16)$$

Recall that $a_j = n_j - n_{j-1}$ where $I^P = \{n_1, \dots, n_r\}$.

Lemma 4.3.17. *Let I_{a_j} denote the $a_j \times a_j$ identity matrix. If $z \in \mathcal{Z}$ then z is of the following form,*

$$\begin{pmatrix} * & * & \dots & * & * & I_{a_1} \\ * & * & \dots & * & (-1)^{n_1}I_{a_2} & 0 \\ * & * & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ * & (-1)^{n_{r-1}}I_{a_r} & 0 & \dots & 0 & 0 \\ (-1)^{n_r}I_{a_{r+1}} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{n \times n}. \quad (4.3.18)$$

We write $z = v^{-1}\dot{w}_P^{-1}\dot{w}_0$ as in Definition 4.3.4. Then the matrix $v \in U_+$ has its non-zero above-diagonal entries given by

$$v_{n_j, n_{j+1}} = (-1)^{n_j+1} z_{n_j, n-n_{j+1}+1}.$$

A Plücker coordinate formula for \mathcal{F}_-

For positive integers $j \leq m$, we denote by $\binom{[m]}{j}$ the set of subsets J of $[m]$ of cardinality j . We denote the complement $J^c_{(m)} = [m] \setminus J$ simply as J^c whenever $J \subset [m]$ is well understood. We always write J, J^c as increasing sequences, and define $|J| := \sum_{i \in J} i$.

We consider the Plücker embedding

$$\begin{aligned} \text{Pl} : P \setminus G &\longrightarrow \mathbb{P}^{\binom{n}{n_1}-1} \times \dots \times \mathbb{P}^{\binom{n}{n_r}-1} \\ P g &\mapsto \left([p_{K_1}(g)]_{K_1 \in \binom{[n]}{n_1}}, \dots, [p_{K_r}(g)]_{K_r \in \binom{[n]}{n_r}} \right) \end{aligned}$$

where the Plücker coordinate $p_{K_j}(g)$ of Pg is the determinant of the $n_j \times n_j$ sub-matrix of g with first n_j rows and the columns from K_j .

The next proposition is a combination of Propositions 2.2 and 3.9 in [95] with respect to the quotient $P \setminus G$, which was also implicitly contained in [88].

Proposition 4.3.5. The projected open Richardson variety $(P \setminus G)^\circ = \text{pr}_P(\overline{\mathcal{R}}_{id, w_0 w_P}^-)$ is the complement of the anti-canonical divisor $-K_{P \setminus G}$ in $P \setminus G$, where

$$-K_{P \setminus G} = \sum_{i \in I} \text{pr}_P(\overline{\mathcal{R}}_{s_i, w_0 w_P}^-) + \sum_{i \in I^P} \text{pr}_P(\overline{\mathcal{R}}_{id, w_0 s_i w_P}^-).$$

Definition 4.3.6. For any homogeneous polynomial p in Plücker coordinates, we denote $\mathcal{V}(p) := \{Pg \in P \setminus G \mid p([p_{K_1}]_{K_1}, \dots, [p_{K_r}]_{K_r})(Pg) = 0\}$ and define

$$D_k := \begin{cases} \mathcal{V}(p_{[k]}), & \text{if } k \in \{n_1, \dots, n_r\}, \\ \mathcal{V}(p_{[n] \setminus [k+1, n-n_1+k]}), & \text{if } 1 \leq k < n_1, \\ \mathcal{V}(p_{[k-n_r+1, k]}), & \text{if } n_r < k \leq n-1, \\ \mathcal{V}\left(\sum_{J \in \binom{[\min\{k, \hat{k}\}]}{k-n_j}} (-1)^{|J|} p_{[k] \setminus J} \cdot p_{J \cup [\hat{k}+1, n]}\right), & \text{if } n_j < k < n_{j+1} \text{ with } j \in [r-1] \\ & \text{where } \hat{k} := n - n_{j+1} + k - n_j \\ \mathcal{V}(p_{[n-n_k-n+1, n]}), & \text{if } k \in \{n, \dots, n-1+r\}. \end{cases}$$

The following proposition from [95], provides explicit equations for the irreducible components of the the anti-canonical divisor $-K_{P \setminus G}$ in terms of Plücker coordinates.

Proposition 4.3.7 ([95, Theorem 4.1]). We have

$$D_k = \begin{cases} \text{pr}_P(\overline{\mathcal{R}}_{s_k, w_0 w_P}^-), & \text{if } 1 \leq k \leq n-1, \\ \text{pr}_P(\overline{\mathcal{R}}_{id, w_0 s_{n_k-n+1} w_P}^-) & \text{if } n \leq k \leq n-1+r, \end{cases}$$

and $\sum_{k=1}^{n-1+r} D_k$ is an anti-canonical divisor in $P \setminus G$, denoted as $-K_{P \setminus G}$.

We will give a Plücker coordinate expansion of \mathcal{F}_- where each summand has a pole of order 1 along a (unique) irreducible component D_k of $-K_{P \setminus G}$, giving rise to a bijection between divisors D_k and summands of \mathcal{F}_- .

For any $n \times m$ matrix $A \in M_{n \times m}(\mathbb{C})$, we let $\Delta_K^J(A)$ denote the minor with row set J and column set K , whenever the sub-matrix is a square matrix. We will need the following generalization in [53] of the famous Cramer's rule in linear algebra.

Lemma 4.3.23 (Generalized Cramer's rule). *Let $A \in GL_n(\mathbb{C})$ and $X, Y \in M_{n \times m}(\mathbb{C})$ such that $AX = Y$. For any $J \in \binom{[n]}{l}$ and $K \in \binom{[m]}{l}$ where $l \leq \min\{n, m\}$, we have*

$$\Delta_K^J(X) = \frac{\det(A_Y(J, K))}{\det A}$$

where $A_Y(J, K)$ denotes the matrix constructed from A by order-preserving replacing the column set J of A by the column set K of Y .

The special case of $X = A^{-1}$ in the generalized Cramer's rule yields Jacobi Theorem for the minors of an inverse matrix immediately as follows.

Corollary 4.3.8 (Jacobi Theorem). *Let $A \in GL_n(\mathbb{C})$. For any $J, K \in \binom{[n]}{l}$ where $l \in [n]$, we write $J^c = [n] \setminus J$ and $K^c = [n] \setminus K$ in increasing sequences. We have*

$$\Delta_K^J(A^{-1}) = \frac{(-1)^{|J|+|K|}}{\det A} \Delta_{J^c}^{K^c}(A).$$

Using Lemma 4.3.17, we have the next key proposition.

Proposition 4.3.9. *Let $z \in \mathcal{Z}$. We define b, u and v as in Definition 4.3.4, so that $b = u\dot{w}_0^{-1}\dot{w}_P v = uz^{-1}$. Then*

$$u_{i,i+1} = \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)}, \quad \text{for any } i \in [n-1];$$

$$v_{i,i+1} = \begin{cases} \frac{\Delta_{\{n-n_j+1\} \cup [n-n_j+1, n] \setminus \{n-n_j-1\}}^{[n_j]}(z)}{\Delta_{[n-n_j+1, n]}^{[n_j]}(z)}, & \text{if } i = n_j \text{ with } j \in [r], \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $u \in U_+$ and $b \in B_-$, we have $\hat{u} := u^{-1} \in U_+$ and $\hat{b} := b^{-1} \in B_-$. For $m = n - i$, we let $\hat{u}^{(m)}$ (resp. $\hat{b}^{(m)}$) be the $n \times m$ matrix obtained by taking

the last m columns of \hat{u} (resp. \hat{b}). Since $b = uz^{-1}$, we have $z\hat{u}^{(m)} = \hat{b}^{(m)}$. By using Lemma 4.3.23 and noting $\hat{b} \in B_-$ and $\det z = 1$, we have

$$\begin{aligned}\hat{u}_{n-m, n-m+1} &= \Delta_{[m]}^{\{n-m\} \cup [n-m+2, n]}(\hat{u}^{(m)}) \\ &= \det(z_{\hat{b}^{(m)}}(\{n-m\} \cup [n-m+2, n], [m])) \\ &= -\left(\prod_{j=n-m+1}^n \hat{b}_{jj}\right) \Delta_{[i-1] \cup \{i+1\}}^{[i]}(z),\end{aligned}$$

$$1 = \Delta_{[m]}^{[n-m+1, n]}(\hat{u}^{(m)}) = \left(\prod_{j=n-m+1}^n \hat{b}_{jj}\right) \Delta_{[i]}^{[i]}(z).$$

Thus for $i \in [n-1]$, we have

$$u_{i, i+1} = -\hat{u}_{i, i+1} = -\frac{\hat{u}_{i, i+1}}{1} = \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)}.$$

By Lemma 4.3.13, we have $v_{i, i+1} = 0$ if $i \neq \{n_1, \dots, n_r\}$; We recall that by Lemma 4.3.17, z is of the form (4.3.18) and therefore for $j \in [r]$ we have

$$\begin{aligned}v_{n_j, n_j+1} &= (-1)^{n_j+1} z_{n_j, n-n_j+1+1} \\ &= (-1)^{n_j+1} \Delta_{\{n-n_j+1+1\}}^{\{n_j\}}(z) \\ &= (-1)^{n_j+1} (-1)^{a_j-1} (-1)^{n_j-1} \frac{\Delta_{\{n-n_j+1+1\} \cup [n-n_j+1, n] \setminus \{n-n_j-1\}}^{\{n_j\}}(z)}{\Delta_{[n-n_j+1, n]}^{\{n_j\}}(z)} \\ &= \frac{\Delta_{\{n-n_j+1+1\} \cup [n-n_j+1, n] \setminus \{n-n_j-1\}}^{\{n_j\}}(z)}{\Delta_{[n-n_j+1, n]}^{\{n_j\}}(z)}.\end{aligned}$$

□

Theorem 4.3.10. Let $(q_{n_1}, \dots, q_{n_r})$ denote the coordinates of $(\mathbb{C}^*)^r = \prod_{i \in I^P} \mathbb{C}_q^*$. The superpotential $\mathcal{F}_- : (P \setminus G)^\circ \times (\mathbb{C}^*)^r \rightarrow \mathbb{C}$ is given by the explicit formula

$$\begin{aligned}\mathcal{F}_- &= \sum_{i=1}^{n_1-1} \frac{\mathcal{P}_{[i-1] \cup \{i+1\} \cup [n-n_1+i+1, n]}}{\mathcal{P}_{[i] \cup [n-n_1+i+1, n]}} + \sum_{j=1}^{r-1} \sum_{i=n_j+1}^{n_{j+1}-1} S_i^{(j)} + \sum_{i=n_r+1}^{n-1} \frac{\mathcal{P}_{[i-n_r+1, i+1] \setminus \{i\}}}{\mathcal{P}_{[i-n_r+1, i]}} \\ &+ \sum_{j=1}^r \frac{\mathcal{P}_{[n_j-1] \cup \{n_j+1\}}}{\mathcal{P}_{[n_j]}} + \sum_{j=1}^r q_{n_j} \frac{\mathcal{P}_{\{n-n_j+1+1\} \cup [n-n_j+1, n] \setminus \{n-n_j-1\}}}{\mathcal{P}_{[n-n_j+1, n]}},\end{aligned}$$

where

$$S_i^{(j)} = \frac{\sum_{J \in \binom{[\min\{i+1, i\}] \setminus \{i\}}{i-n_j}} \epsilon(J) (-1)^{|J|} \mathcal{P}_{[i-1] \cup \{i+1\} \setminus J} \cdot \mathcal{P}_{J \cup [i+1, n]}}{\sum_{J \in \binom{[\min\{i, i\}]}{i-n_j}} (-1)^{|J|} \mathcal{P}_{[i] \setminus J} \cdot \mathcal{P}_{J \cup [i+1, n]}}$$

with $\hat{i} = n - n_{j+1} + i - n_j$ and $\epsilon(J) = \begin{cases} 1, & \text{if } i+1 \notin J, \\ -1, & \text{if } i+1 \in J. \end{cases}$

Proof. We let \hat{b}, b, u, v be defined as in Definition 4.3.4 for given $z \in \mathcal{Z}$. By Lemma 4.3.13, we have

$$\mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i \in I^P} q_i v_{i, i+1} + \sum_{i=1}^{n-1} u_{i, i+1},$$

Since by Proposition 4.3.9 $u_{i, i+1}$ and $v_{i, i+1}$ are quotients of minors of z , it suffices to interpret those minors by the Plücker coordinates. Recall that for subsets $K \in \binom{[n]}{n_j}$, $p_K(z) = \Delta_K^{[n_j]}(z)$ is the determinant of the submatrix of z with first n_j rows and columns from K . Therefore if $i \in \{n_1, \dots, n_r\}$, this is already done for both $u_{i, i+1}$ and $v_{i, i+1}$, as given in the last two sums of the expression of \mathcal{F}_- . Recall that z is of the form (4.3.18). For $i < n_1$ then both $\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z) = \Delta_{[i-1] \cup \{i+1\} \cup [n-n_1+i+1, n]}^{[n_1]}(z)$ and $\Delta_{[i]}^{[i]}(z) = \Delta_{[i] \cup [n-n_1+i+1, n]}^{[n_1]}(z)$ hold. Then we have

$$u_{i, i+1} = \frac{\mathcal{P}_{[i-1] \cup \{i+1\} \cup [n-n_1+i+1, n]}}{\mathcal{P}_{[i] \cup [n-n_1+i+1, n]}}.$$

Next we consider the case $i > n_r$. Let $0_{\mu, \nu}$ denote the zero matrix with μ rows and ν columns. then the last $(i - n_r)$ rows for minors $\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)$ and $\Delta_{[i]}^{[i]}(z)$ are both given by $\left((-1)^{n_r} I_{i-n_r}, 0_{i-n_r, n-i+n_r} \right)_{(i-n_r) \times i}$. The Laplace expansion on the last $(i - n_r)$ rows leads to the third sum in the expression of \mathcal{F}_- immediately.

It remains to discuss $u_{i, i+1}$ for the case $n_j < i < n_{j+1}$ for some $j \in [r-1]$. Denote $k = i - n_j$. The first i rows of z is given by

$$\begin{pmatrix} * & * & * & \dots & * & * & I_{a_1} \\ * & * & * & \dots & * & (-1)^{n_1} I_{a_2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ * & * & * & (-1)^{n_{j-1}} I_{a_j} & 0 & 0 & 0 \\ * & (-1)^{n_j} I_k & 0_{k, a_{j+1}-k} & 0 & \dots & 0 & 0 \end{pmatrix}_{i \times n}.$$

Here the first column block has size $n - n_{j+1}$. Using Laplace expansion on the last k rows, we obtain

$$\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z) = \sum_{J \in \binom{[i-1] \cup \{i+1\}}{k}} (-1)^{|[i-n_j+1, i]| + |J|} \epsilon(J) \Delta_{[i-1] \cup \{i+1\} \setminus J}^{[n_j]}(z) \cdot z_J,$$

where z_J is the determinant of the submatrix with columns from J and last k rows. Since the last k row is $\left(* \quad (-1)^{n_{k-1}} I_k \quad 0_{k, a_{j+1-k}} \quad 0 \quad \cdots \quad 0 \right)$. Its last nonzero column is in position $\hat{i} := n - n_{j+1} + k$, so we may assume that $J \subset [l'] \setminus \{i\}$ where $l' = \min\{i+1, \hat{i}\}$, as otherwise $z_J = 0$ for J occurring in the above sum. Similarly and more easily, we set $l = \min\{i, \hat{i}\}$ and have

$$\Delta_{[i]}^{[i]}(z) = \sum_{\tilde{J} \in \binom{[l]}{k}} (-1)^{|[i-n_{j+1}, i]| + |\tilde{J}|} \Delta_{[i] \setminus \tilde{J}}^{[n_j]}(z) \cdot z_{\tilde{J}}.$$

Since z is of the specified form as above, we have $z_J = \varepsilon p_{J \cup [n-n_{j+1}+k+1, n]}$ and $z_{\tilde{J}} = \varepsilon p_{\tilde{J} \cup [n-n_{j+1}+k+1, n]}$, in which $\varepsilon = \pm 1$ depends only on $\{n_1, \dots, n_j\}$. Hence, we have $u_{i, i+1} = \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)} = S_i^{(j)}$ and the proof is complete. \square

Example 4.3.24. When $I^P = I$, we have that $P \setminus G$ is a complete flag variety. In this case, there is no $S_i^{(j)}$ -term, and the superpotential \mathcal{F}_- is simply given by

$$\mathcal{F}_- = \sum_{i=1}^{n-1} p_{\frac{[i-1] \cup \{i+1\}}{p[i]}} + \sum_{i=1}^{n-1} q_i \frac{p_{[n-i, n] \setminus \{n-i+1\}}}{p_{[n-i+1, n]}}.$$

Example 4.3.25. When $I^P = \{k\}$, we have that $P \setminus G$ is the complex Grassmannian $Gr(k, n)$. In this case, there are no $S_i^{(j)}$ -terms, and the superpotential \mathcal{F}_- is simply given by

$$\mathcal{F}_- = \sum_{i=1}^{k-1} \frac{p_{[i-1] \cup \{i+1\} \cup [n-k+i+1, n]}}{p_{[i] \cup [n-k+i+1, n]}} + \sum_{i=k+1}^{n-1} \frac{p_{[i-k+1, i+1] \setminus \{i\}}}{p_{[i-k+1, i]}} + \frac{p_{[k-1] \cup \{k+1\}}}{p_{[k]}} + q_k \frac{p_{[n-k, n] \setminus \{n-k+1\}}}{p_{[n-k+1, n]}}.$$

4.4 Quantum cohomology of partial flag varieties

The main result of this section will be to show that the image of the superpotential in the Jacobi ring agrees with the the first Chern class under the known isomorphism with quantum cohomology (Section 4.4). This result is Theorem 4.4.11.

On the A-side, we consider the small quantum cohomology ring of $X = G^\vee / P^\vee$, denoted by $QH^*(X)$. It is an associative and commutative algebra over $\mathbb{C}[q_{n_1}, \dots, q_{n_r}]$ with a basis given by the Schubert classes σ_v , where q_{n_j} are formal variables.

$$QH^*(X) = \mathbb{C}[q_{n_1}, \dots, q_{n_r}] \otimes H^*(X, \mathbb{Z})$$

The structure constants are defined through the 3-point, genus-zero Gromov–Witten invariants of X . There is also an enumerative meaning of these

constants (see e.g. [41]). Recall that the number r occurs in the isomorphism $G^\vee/P^\vee \xrightarrow{\cong} \mathbb{F}\ell(n_\bullet) = \mathbb{F}\ell(n; n_r, \dots, n_1)$.

A permutation $w \in S_n$ is an element in W^P if and only if $w(n_j + 1) < \dots < w(n_{j+1})$ for all $0 \leq j \leq r$. In particular, if $w = s_{n_j-i+1} \cdots s_{n_j}$, $1 \leq j \leq r$, $1 \leq i \leq n_j$, then σ_w is called a special Schubert class. The following is a special case of the quantum Pieri rule in [29].

Proposition 4.4.1 (Ciocan-Fontanine). Let w be a Grassmannian permutation with $w(1) < \dots < w(m)$ and $w(m + 1) < \dots < w(n)$ for some $m = n_j$, $1 \leq j \leq r$. Then in $QH^*(X)$, we have

$$\sigma_w \cdot \sigma_{s_{n_j}} = \sum_{\substack{a \leq n_j < b, \\ \ell(wt_{ab}) = \ell(w) + 1}} \sigma_{wt_{ab}} + \sum_{\ell(w\tau) = \ell(w) - \ell(\tau)} q_{n_j} \sigma_{w\tau},$$

where t_{ab} is the transposition interchanging a and b , $\tau := s_{n_j} \cdot s_{n_j+1} \times \dots \times s_{n_{j+1}-1} \cdot s_{n_j-1} \cdot s_{n_j-2} \times \dots \times s_{n_{j-1}+1}$.

Remark 4.4.1. Since w is a Grassmannian permutation, there is at most one quantum term in the expansion of the product $\sigma_w \cdot \sigma_{s_{n_j}}$. Note that the condition $\ell(w\tau) = \ell(w) - \ell(\tau)$ is equivalent to $w(n_j) > w(n_{j+1}), w(n_j + 1) < w(n_{j-1} + 1)$.

Peterson isomorphism

In this section we state three theorems of D. Peterson, of which the proofs may be found in [117, Section 4].

Definition 4.4.2. For $1 \leq i \leq m < n$, we define a rational function G_i^m on G/B_- as follows:

$$G_i^m(gB_-) := \frac{\Delta_{[m+1,n]}^{\{m-i+1\} \cup [m+2,n]}(g)}{\Delta_{[m+1,n]}^{[m+1,n]}(g)}.$$

Definition 4.4.3 (Peterson variety). Let \mathcal{Y} denote the (type A) Peterson variety, which is the projective subvariety of G/B_- cut out by the relation

$$g^{-1}fg \in \mathfrak{b}_- \oplus \left(\bigoplus_{i \in I} \mathfrak{g}_{\alpha_i} \right) = \left\{ \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ * & \dots & \dots & * & * \end{pmatrix} \right\}, \quad (4.4.2)$$

where g represents gB_- and f is the principal nilpotent

$$f = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{pmatrix}.$$

We set

$$\mathcal{Y}_P := \mathcal{Y} \cap B_+ \dot{w}_P B_- / B_-,$$

and refer to this intersection as the Peterson variety associated to the parabolic subgroup P .

Theorem 4.4.4 (D. Peterson). Let \mathcal{Y}_P be the Peterson variety associated to the parabolic subgroup P . There is a unique isomorphism

$$\begin{aligned} \mathcal{O}(\mathcal{Y}_P) &\xrightarrow{\sim} QH^*(G^\vee/P^\vee), \\ G_i^{n_j} &\mapsto (-1)^i \sigma_{s_{n_j-i+1} \cdots s_{n_j}} \end{aligned}$$

where $1 \leq j \leq r$, $1 \leq i \leq n_j$ and $G_i^{n_j}$ is as constructed in Definition 4.4.2.

Remark 4.4.3. The isomorphism we are using differs from the one used in [117] by signs.

Remark 4.4.4. The rational function G_i^m is a regular function on the Schubert cell $B_+ \dot{w}_P B_- / B_-$ if $m \in I^P$.

More generally, for a Grassmannian permutation w , with $w(1) < \cdots < w(m)$ and $w(m+1) < \cdots < w(n)$ for some $1 < m < n$, we can define a rational function G_w on G/B_- as follows:

$$G_w(gB_-) := \frac{\Delta_{[m+1, n]}^{\{w(m+1), \dots, w(n)\}}(g)}{\Delta_{[m+1, n]}^{[m+1, n]}(g)}.$$

We use $G^{\{w(m+1), \dots, w(n)\}}(gB_-) := G_w(gB_-)$ for short. This will not lead to any misunderstanding since our n is fixed throughout the paper. Then we have the following result, by [117, Prop 11.3].

Proposition 4.4.5. If w is a Grassmannian permutation with descent m and $m \in I^P$, then the isomorphism in Theorem 4.4.4 sends G_w to $(-1)^{\ell(w)} \sigma_w$.

Theorem 4.4.6 (D. Peterson). Let $\mathcal{X}_P := \mathcal{Y}_P \cap (B_- \dot{w}_0 B_- / B_-)$. Then the map in Theorem 4.4.4 induces an isomorphism between $\mathcal{O}(\mathcal{X}_P)$ and $QH^*(G^\vee / P^\vee)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}]$.

We may also refer to \mathcal{X}_P as Peterson variety. We recall that \mathcal{X}_P can be described using Toeplitz matrices using an idea going back to B. Kostant.

Theorem 4.4.7 (D. Peterson). Consider the following variety of lower-triangular Toeplitz matrices given by

$$X_P := \left\{ b_- = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_n & \cdots & x_2 & x_1 \end{pmatrix} \mid b_- \in B_+ \dot{w}_P \dot{w}_0 B_+ \right\}.$$

The map $X_P \rightarrow \mathcal{X}_P$ sending b_- to $b_- \dot{w}_0 B_-$ is an isomorphism.

Critical points of the superpotential

We have the following isomorphism ψ_R which is a version of an isomorphism from [118, Section 4.1],

$$\begin{aligned} \psi_R : B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+ &\xrightarrow{\cong} \mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^*; \\ b_- = u_1 t \dot{w}_P^{-1} \dot{w}_0 u_2 &\mapsto \psi_R(b_-) = (b_- \dot{w}_0 B_-, (\alpha_{n_i}(t))_{i=1}^r). \end{aligned}$$

Define \mathcal{F}_R by

$$\mathcal{F}_R := \mathcal{F}_{\text{Lie}} \circ \psi_R^{-1} : \mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^* \longrightarrow \mathbb{C}.$$

We now consider the ring

$$\text{Jac}(\mathcal{F}_R) := \mathcal{O} \left(\mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^* \right) / (\partial_{\mathcal{R}_{w_P, w_0}^{R^-}} \mathcal{F}_R), \quad (4.4.5)$$

where we are taking partial derivatives of \mathcal{F}_R in the $\mathcal{R}_{w_P, w_0}^{R^-}$ directions, and which we refer to as the (fiberwise) Jacobi ring of \mathcal{F}_R . This ring describes the critical points of \mathcal{F}_R along the fibres of the projection $\text{pr}_2 : \mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^* \rightarrow \prod_{i \in I^P} \mathbb{C}_q^*$.

The next theorem shows that the Jacobi ring of \mathcal{F}_R is isomorphic to the coordinate ring of the Peterson variety \mathcal{X}_P . Namely, Consider the subvariety

of $\mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^*$ cut out by the ideal $(\partial_{\mathcal{R}_{w_P, w_0}^{R^-}} \mathcal{F}_R)$ of partial derivatives of \mathcal{F}_R along $\mathcal{R}_{w_P, w_0}^{R^-}$. We denote the corresponding subvariety in $Z_P = B_- \cap U_+ \mathcal{T}^{W_P} \dot{w}_P^{-1} \dot{w}_0 U_+$ by Z_P^{crit} . The theorem stated below is a direct translation of [118, Theorem 4.1] with our notation.

Theorem 4.4.8. We have that $Z_P^{crit} = X_P$. Moreover, the subvariety

$$\psi_R(Z_P^{crit}) \subset \mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^*,$$

which is defined by the ideal $(\partial_{\mathcal{R}_{w_P, w_0}^{R^-}} \mathcal{F}_R)$ is isomorphic to \mathcal{X}_P via the restriction of the first projection $\text{pr}_1 : \mathcal{R}_{w_P, w_0}^{R^-} \times \prod_{i \in I^P} \mathbb{C}_q^* \rightarrow \mathcal{R}_{w_P, w_0}^{R^-}$.

Proof. We have the following result proved in [118] that we state for the parabolic subgroup $Q := \dot{w}_0 P \dot{w}_0^{-1}$. Let us set $\tilde{\mathcal{T}}^{W_Q} := \dot{w}_0^{-1} \mathcal{T}^{W_P} \dot{w}_0$. We define

$$\begin{aligned} \tilde{\mathcal{F}}_Q : \tilde{Z}_Q = B_- \cap U_+ \tilde{\mathcal{T}}^{W_Q} \dot{w}_Q \dot{w}_0^{-1} U_+ &\rightarrow \mathbb{C}, \\ \tilde{b}_- = u_1 \tilde{t} \dot{w}_Q \dot{w}_0^{-1} u_2 &\mapsto \sum_i e_i^*(u_1) + \sum_i e_i^*(u_2), \end{aligned}$$

and its restrictions

$$\tilde{\mathcal{F}}_{Q, \tilde{t}} : \tilde{Z}_{Q, \tilde{t}} = B_- \cap U_+ \tilde{t} \dot{w}_Q \dot{w}_0^{-1} U_+ \rightarrow \mathbb{C}.$$

Then it is shown in [118] that the critical points of $\tilde{\mathcal{F}}_{Q, \tilde{t}}$ lie in X_Q . Namely,

$$\tilde{Z}_{Q, \tilde{t}}^{crit} := \{\tilde{b}_- \in B_- \cap U_+ \tilde{t} \dot{w}_Q \dot{w}_0^{-1} U_+ \mid \partial \tilde{\mathcal{F}}_{Q, \tilde{t}}(\tilde{b}) = 0\} = X_Q \cap \tilde{Z}_{Q, \tilde{t}},$$

where X_Q is as in Theorem 4.4.7. Moreover the fiberwise critical point variety $\tilde{Z}_{Q, \tilde{t}}^{crit}$ (union over all fibers $\tilde{Z}_{Q, \tilde{t}}^{crit}$) agrees with X_Q .

We can now compare $\tilde{\mathcal{F}}_Q$ with our superpotential \mathcal{F}_{Lie} . We have that

$$\mathcal{F}_{\text{Lie}}(b_- = u_1 t \dot{w}_P^{-1} \dot{w}_0 u_2) = \tilde{\mathcal{F}}_Q(b_-^{-1} = u_2^{-1} \tilde{t} \dot{w}_Q \dot{w}_0^{-1} u_1^{-1})$$

where $\tilde{t} = \dot{w}_0^{-1} \dot{w}_P t^{-1} \dot{w}_P^{-1} \dot{w}_0 = \dot{w}_0^{-1} t^{-1} \dot{w}_0$. Therefore, b_- is a critical point of the analogous restriction $\mathcal{F}_{\text{Lie}, t}$ if and only if b_-^{-1} is a critical point of $\tilde{\mathcal{F}}_{Q, \tilde{t}}$ and we have that the critical point variety Z_P^{crit} of \mathcal{F}_{Lie} is equal to the inverse of X_Q . Finally, the inverse of X_Q is precisely X_P (using that the inverse of a Toeplitz matrix is again Toeplitz). It follows that $\psi_R(Z_{\text{Lie}}^{crit})$ projects to \mathcal{X}_P , thanks to Theorem 4.4.7 \square

Corollary 4.4.9. The fiberwise Jacobi ring $\text{Jac}(\mathcal{F}_R)$ of the superpotential \mathcal{F}_R is isomorphic to the quantum cohomology ring $QH^*(X)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}]$.

Proof. The Jacobi ring is related to $QH^*(X)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}]$ by

$$\begin{aligned} \text{Jac}(\mathcal{F}_R) &\xrightarrow{\sim} \mathcal{O}(\mathcal{X}_P) && \text{(Theorem 4.4.8)} \\ &\xrightarrow{\sim} QH^*(G^\vee/P^\vee)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}] && \text{(Theorem 4.4.6)}. \end{aligned}$$

□

We have the following corollary of Theorem 4.4.8 that we record for use later on. Suppose $\mathbf{q} \in \prod_{i \in I^P} \mathbb{C}_q^*$. Define

$$\mathcal{F}_{\mathbf{q}} : B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+ \rightarrow \mathbb{C} \quad (4.4.6)$$

by $\mathcal{F}_{\mathbf{q}}(\hat{b}) = \mathcal{F}_{\text{Lie}} \circ \gamma^{-1}(\hat{b}, \mathbf{q})$. This is precisely the function from (4.3.3), but now with quantum parameters fixed.

Corollary 4.4.10. If \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $t\hat{b} = \gamma^{-1}(\hat{b}, \mathbf{q})$ is a Toeplitz matrix.

Proof. In the notation from the proof of Theorem 4.4.8, we have \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$ if and only if $t\hat{b}$ is a critical point of $\mathcal{F}_{\text{Lie}, t}$. But the critical points of $\mathcal{F}_{\text{Lie}, t}$ lie in Z_P^{crit} , which equals to X_P by Theorem 4.4.8. Therefore $t\hat{b}$ is a Toeplitz matrix. □

Image of first Chern class

The following theorem is the main result in this section, the proof of which is in the end of this subsection.

Theorem 4.4.11. Let θ be the isomorphism $\text{Jac}(\mathcal{F}_R) \xrightarrow{\sim} QH^*(X)[q_{n_1}^{-1}, \dots, q_{n_r}^{-1}]$ in Corollary 4.4.9. Let $[\mathcal{F}_R]$ be the class of superpotential \mathcal{F}_R in the Jacobi ring. Then we have

$$\theta([\mathcal{F}_R]) = c_1(X).$$

The proof of Theorem 4.4.11 will occupy the rest of the paper.

Fix $\mathbf{q} \in \prod_{i \in I^P} \mathbb{C}_q^*$ and recall the map $\mathcal{F}_{\mathbf{q}} : B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+ \rightarrow \mathbb{C}$ defined in (4.4.6). Let us consider $\hat{b} \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$ and let z, u, v and b be as in Definition 4.3.4. We then have

$$b = u \dot{w}_0^{-1} \dot{w}_P v = uz^{-1}. \quad (4.4.7)$$

Also recall that $b_- = t\hat{b} = tb^{-1}$, with $t \in \mathcal{T}^{W_P}$ corresponding to \mathbf{q} via (4.3.1). The following lemmas are key ingredients in the proof of Theorem 4.4.11.

Lemma 4.4.8. *Let $n_j \in I^P$. Then $v_{n_j, n_j+1} = -\frac{t_{n_j+1}}{t_{n_j}} G_1^{n_j}(b_- \dot{w}_0 B_-)$.*

Proof. Since $u, v \in U_+$ and $\dot{w}_0^{-1} \dot{w}_P = \text{antidiag}\{(-1)^{n_r} I_{a_{r+1}}, \dots, (-1)^{n_1} I_{a_2}, I_{a_1}\}$, we have

$$\begin{aligned} v_{n_j, n_j+1} &= \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n_j]}(v)}{\Delta_{[n_j]}^{[n_j]}(v)} \\ &= \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n-n_j+1, n]}(\dot{w}_0^{-1} \dot{w}_P v)}{\Delta_{[n_j]}^{[n-n_j+1, n]}(\dot{w}_0^{-1} \dot{w}_P v)} \\ &= \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n-n_j+1, n]}(b)}{\Delta_{[n_j]}^{[n-n_j+1, n]}(b)} \end{aligned}$$

Note that for the diagonal entries of t we have that $t_{n_j+1} = t_{n_j+2} = \dots = t_{n_{j+1}}$, where $0 \leq j \leq r$. Therefore, we have

$$\begin{aligned} v_{n_j, n_j+1} &= \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n-n_j+1, n]}(b)}{\Delta_{[n_j]}^{[n-n_j+1, n]}(b)} \\ &= \frac{t_{n_j+1}}{t_{n_j}} \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n-n_j+1, n]}(bt^{-1})}{\Delta_{[n_j]}^{[n-n_j+1, n]}(bt^{-1})} \\ &= -\frac{t_{n_j+1}}{t_{n_j}} \frac{\Delta_{[n_j] \cup [n_j+2, n]}^{[n_j] \cup [n_j+2, n]}(tb^{-1})}{\Delta_{[n-n_j]}^{[n_j+1, n]}(tb^{-1})} \\ &= -\frac{t_{n_j+1}}{t_{n_j}} G_1^{n_j}(b_- \dot{w}_0 B_-). \end{aligned}$$

where in the second to last equality one needs to apply Corollary 4.3.8. \square

The situation for $u_{i, i+1}$ is slightly more complicated.

Lemma 4.4.9. *Let $1 \leq i < n - n_r$. Suppose \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $u_{i, i+1} = u_{n-n_r, n-n_r+1}$.*

Proof. Since $u \in U_+$, we have $u_{i, i+1} = \Delta_{[n] \setminus \{i\}}^{[n] \setminus \{i+1\}}(u)$. By Corollary 4.3.8, we have that $\Delta_{[n] \setminus \{i\}}^{[n] \setminus \{i+1\}}(u) = -\Delta_{\{i+1\}}^{\{i\}}(u^{-1})$. Applying generalized Cramer's rule, see Lemma 4.3.23, to the equation $b^{-1} = zu^{-1}$, we have $\Delta_{\{i+1\}}^{\{i\}}(u^{-1}) = \det z_{b^{-1}}(\{i\}, \{i+1\})$. Now since \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, we have that $t\hat{b} = tb^{-1}$ is a Toeplitz matrix by Corollary 4.4.10. Note that $t_{n_r+1} = t_{n_r+2} = \dots = t_n$. Therefore for $1 \leq i < n - n_r$ we have

$$\det z_{b^{-1}}(\{i\}, \{i+1\}) = \det z_{t^{-1}(tb^{-1})}(\{i\}, \{i+1\})$$

$$\begin{aligned}
&= t_n^{-1} \det z_{(tb^{-1})}(\{i\}, \{i+1\}) \\
&= t_n^{-1} \det z_{(tb^{-1})}(\{n-n_r\}, \{n-n_r+1\}) \\
&= \det z_{b^{-1}}(\{n-n_r\}, \{n-n_r+1\}).
\end{aligned}$$

Therefore, we have shown that $u_{i,i+1} = u_{n-n_r, n-n_r+1}$ whenever b is a critical point of $\mathcal{F}_{\mathbf{q}}$ for $1 \leq i < n - n_r$.

□

Lemma 4.4.10. *Let $i = n - n_j$ for some $1 \leq j \leq r$. Suppose \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $u_{i,i+1} = -G_1^{n_j}(b_- \dot{w}_0 B_-)$.*

Proof. Since $u, v \in U_+$ and $\dot{w}_0^{-1} \dot{w}_P = \text{antidiag}\{(-1)^{n_r} I_{a_{r+1}}, \dots, (-1)^{n_1} I_{a_2}, I_{a_1}\}$, we have

$$\begin{aligned}
u_{n-n_j, n-n_j+1} &= \frac{\Delta_{[n-n_j+1, n]}^{\{n-n_j\} \cup [n-n_j+2, n]}(u)}{\Delta_{[n-n_j+1, n]}^{[n-n_j+1, n]}(u)} \\
&= \frac{\Delta_{[n_j]}^{\{n-n_j\} \cup [n-n_j+2, n]}(u \dot{w}_0^{-1} \dot{w}_P)}{\Delta_{[n_j]}^{[n-n_j+1, n]}(u \dot{w}_0^{-1} \dot{w}_P)} \\
&= \frac{\Delta_{[n_j]}^{\{n-n_j\} \cup [n-n_j+2, n]}(b)}{\Delta_{[n_j]}^{[n-n_j+1, n]}(b)} \\
&= \frac{\Delta_{[n_j]}^{\{n-n_j\} \cup [n-n_j+2, n]}(bt^{-1})}{\Delta_{[n_j]}^{[n-n_j+1, n]}(bt^{-1})}.
\end{aligned}$$

Now since \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, we have that $t\hat{b} = tb^{-1}$ is a Toeplitz matrix by Corollary 4.4.10. So that its inverse bt^{-1} is also a Toeplitz matrix. Therefore, we have

$$\frac{\Delta_{[n_j]}^{\{n-n_j\} \cup [n-n_j+2, n]}(bt^{-1})}{\Delta_{[n_j]}^{[n-n_j+1, n]}(bt^{-1})} = \frac{\Delta_{[n_j-1] \cup \{n_j+1\}}^{[n-n_j+1, n]}(bt^{-1})}{\Delta_{[n_j]}^{[n-n_j+1, n]}(bt^{-1})}.$$

Then it follows as in the proof of Lemma 4.4.8 that $u_{n-n_j, n-n_j+1} = -G_1^{n_j}(b_- \dot{w}_0 B_-)$.

□

Lemma 4.4.11. *Let $n - n_1 < i \leq n - 1$. Suppose \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then $u_{i,i+1} = -G_1^{n_1}(b_- \dot{w}_0 B_-)$.*

Proof. By Proposition 4.3.9, we have $u_{i,i+1} = \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)}$. Since z is of the form (4.3.18), if we set $d = i - (n - n_1)$, then $d < \min\{i, n_1\}$ and we have

$$\begin{aligned} \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)} &= -\frac{\Delta_{[i-d]}^{\{d\} \cup \{d+2, i\}}(z)}{\Delta_{[i-d]}^{[d+1, i]}(z)} \\ &= -\frac{\Delta_{[i-d]}^{\{d\} \cup \{d+2, i\}}(b^{-1})}{\Delta_{[i-d]}^{[d+1, i]}(b^{-1})}. \end{aligned}$$

Recall that $b_- = t\hat{b} = tb^{-1}$. Note that $t_1 = t_2 = \cdots = t_{n_1}$ and $1 \leq d < d+1 \leq n_1$. So we have

$$\begin{aligned} u_{i,i+1} &= -\frac{\Delta_{[i-d]}^{\{d\} \cup \{d+2, i\}}(b^{-1})}{\Delta_{[i-d]}^{[d+1, i]}(b^{-1})} \\ &= -\frac{\Delta_{[n-n_1]}^{\{d\} \cup \{d+2, i\}}(tb^{-1})}{\Delta_{[n-n_1]}^{[d+1, i]}(tb^{-1})} \\ &= -\frac{G^{\{d\} \cup \{d+2, i\}}(b_- \dot{w}_0 B_-)}{G^{[d+1, i]}(b_- \dot{w}_0 B_-)} \\ &= -G_1^{n_1}(b_- \dot{w}_0 B_-). \end{aligned}$$

The last equality follows from Proposition 4.4.1 and the isomorphism in Corollary 4.4.9. \square

Lemma 4.4.12. *Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$. Suppose \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then*

$$u_{i,i+1} = -(G_1^{n_j}(b_- \dot{w}_0 B_-) + G_1^{n_{j+1}}(b_- \dot{w}_0 B_-)).$$

We leave the proof of this lemma to the next section. Now we are ready to prove Theorem 4.4.11 assuming the above lemmas.

Proof of Theorem 4.4.11. Let $\hat{b} \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$ and let z, u, v and b be as in Definition 4.3.4. Also recall that $b_- = t\hat{b}$ and $q_{n_j} = \frac{t_{n_j}}{t_{n_{j+1}}}$. By Equation 4.3.15, we have $\mathcal{F}_R(b_- \dot{w}_0 B_-, \mathbf{q}) = \mathcal{F}_{\text{Lie}}(b_-) = \mathcal{F}_-(Pz, \mathbf{q}) = \sum_{i \in I^P} q_i v_{i,i+1} + \sum_{i=1}^{n-1} u_{i,i+1}$. Suppose that \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$. Then by Corollary 4.4.9 and the above lemmas, we have

$$\theta([\mathcal{F}_R]) = \sum_{j=1}^r (n_{j+1} - n_{j-1}) \sigma_{s_{n_j}} = c_1(X) \in QH^*(X).$$

\square

4.5 Proof of Lemma 4.4.12 and Quantum Schubert calculus

The goal of this section is to prove the most difficult of the lemmas, which is Lemma 4.4.12.

Equivalence of Lemma 4.4.12 and an identity in quantum Schubert calculus

We first prove that Lemma 4.4.12 is equivalent to the identity in quantum Schubert calculus stated in the following theorem.

Definition 4.5.1. Recall that we are considering $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$, and let $d := i - (n - n_{j+1})$. We set

$$\Xi := \left\{ J \in \binom{[i]}{d} \mid J \cap [n_j + d + 1, n] = \emptyset \right\},$$

and define Weyl group elements $w_J \in W^P$ for certain $J \in \Xi$ as follows. For $J = \{j_1 < j_2 < \cdots < j_d\} \in \Xi$, let $\{x_1 < x_2 < \cdots < x_{i-d}\} := [i] \setminus J$.

1. If $n_j \geq d$, then w_J is the following permutation

$$\begin{aligned} \{w(1) < \cdots < w(n_j)\} &= \{j_1 < j_2 < \cdots < j_d < i + 1 < i + 2 < \cdots < i + n_j - d\} \\ \{w(n_j + 1) < \cdots < w(n_{j+1})\} &= \{x_1 < i + n_j - d + 1 < i + n_j - d + 2 < \cdots < n - 1\} \\ \{w(n_{j+1} + 1) < \cdots < w(n_{j+2})\} &= \{x_2 < \cdots < x_{n_{j+2} - n_{j+1}} < n\} \\ \{w(n_{j+2} + 1) < \cdots < w(n)\} &= \{x_{n_{j+2} - n_{j+1} + 1} < \cdots < x_{i-d}\} \end{aligned}$$

2. If $n_j < d$, and $x_1 < j_{n_j+1}$ then w_J is defined by

$$\begin{aligned} \{w(1) < \cdots < w(n_j)\} &= \{j_1 < \cdots < j_{n_j}\} \\ \{w(n_j + 1) < \cdots < w(n_{j+1})\} &= \{x_1 < j_{n_j+1} < \cdots < j_d < i + 1 < \cdots < n - 1\} \\ \{w(n_{j+1} + 1) < \cdots < w(n_{j+2})\} &= \{x_2 < \cdots < x_{n_{j+2} - n_{j+1}} < n\} \\ \{w(n_{j+2} + 1) < \cdots < w(n)\} &= \{x_{n_{j+2} - n_{j+1} + 1} < \cdots < x_{i-d}\}. \end{aligned}$$

Example 4.5.1. Suppose $n_1 = 2, n_2 = 4, n = 7$. Let $j = 1$ and $i = 4$ (which indeed satisfies $n - n_{j+1} < i < n - n_j$). Then $d = 1$ and

$$\Xi = \left\{ J \in \binom{[4]}{1} \mid J \cap [4, 7] = \emptyset \right\} = \{\{1\}, \{2\}, \{3\}\}.$$

Since $n_j = 2 \geq d = 1$ we have a Weyl group element w_J for each $J \in \Xi$. Suppose $J = \{j_1\}$ and $[4] \setminus J = \{x_1, x_2, x_3\}$. Then the definition of w_J is

$$w_J(1) = j_1, \quad w_J(2) = 5, \quad w_J(3) = x_1, \quad w_J(4) = 6, \quad w_J(5) = x_2, \quad w_J(6) = x_3, \quad w_J(7) = 7.$$

For our three choices of J this gives the following three Weyl group elements,

$$w_{\{1\}} = 1526347, \quad w_{\{2\}} = 2516347, \quad w_{\{3\}} = 3516247,$$

with descents at 2 and 4, so in W^P .

We can now state our theorem. Note that since $w_J \in W^P$, we have an associated Schubert class σ_{w_J} . If a permutation $w \in W^P$ is Grassmannian, so that it is determined by the values $w(1) < \dots < w(m)$ up to $m = n_k$ for some k , then we may write $\sigma_{\{w(1), \dots, w(m)\}}$ for σ_w . Recall that we set $|J| := \sum_{i \in J} i$.

Theorem 4.5.2. Consider $X = G^\vee/P^\vee$ and fix i such that $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$. Let $d := i - (n - n_{j+1})$. For each w_J defined above we consider $\sigma_{w_J} \in QH^*(X)$, and we set $\sigma_{w_J} := 0$ for $J \in \Xi$ where w_J is not defined. Then the identity

$$\sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J} \sigma_{[1, n_j + d] \setminus J} = 0 \quad (4.5.2)$$

holds in $QH^*(X)$.

Remark 4.5.3. The quantum product $\sigma_{J \cup [i+1, n]} \cdot \sigma_{s_{n_{j+1}}}$ consists of at most one quantum part, say $q_{n_{j+1}} \sigma_{w_J}$. The above definition of w_J is an explicit description of such a class.

Lemma 4.5.4. *The formula for $u_{i, i+1}$ in Lemma 4.4.12 is equivalent to the corresponding identity in Theorem 4.5.2.*

Proof. We use determinantal identities to rewrite the $u_{i, i+1}$ in Lemma 4.4.12.

By Proposition 4.3.9, we have $u_{i, i+1} = \frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)}$. Using Laplace expansion on the first $n - n_{j+1}$ columns, we have

$$\frac{\Delta_{[i-1] \cup \{i+1\}}^{[i]}(z)}{\Delta_{[i]}^{[i]}(z)} = \frac{\sum_{J \in \binom{[i]}{d}} (-1)^{|J|} \Delta_{[n-n_{j+1}] \setminus J}^{[i]}(z) \cdot \Delta_{[n-n_{j+1}+1, i-1] \cup \{i+1\}}^J(z)}{\sum_{J \in \binom{[i]}{d}} (-1)^{|J|} \Delta_{[n-n_{j+1}] \setminus J}^{[i]}(z) \cdot \Delta_{[n-n_{j+1}+1, i]}^J(z)}$$

where $d := i - (n - n_{j+1})$. Since z is of the form (4.3.18), we have that the determinant $\Delta_{[n-n_{j+1}+1, i-1] \cup \{i+1\}}^J(z)$ vanishes if $J \cap (\{n_j + d\} \cup [n_j + d + 2, n]) \neq \emptyset$, and $\Delta_{[n-n_{j+1}+1, i]}^J(z)$ vanishes if $J \cap [n_j + d + 1, n] \neq \emptyset$. If we set

$$A := \left\{ J \in \binom{[i]}{d} \mid J \cap (\{n_j + d\} \cup [n_j + d + 2, n]) = \emptyset \right\}$$

and $\Xi = \{J \in \binom{[i]}{d} \mid J \cap [n_j + d + 1, n] = \emptyset\}$ as already defined, then we have

$$\begin{aligned} & \frac{\sum_{J \in \binom{[i]}{d}} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_{j+1}+1, i-1] \cup \{i+1\}}^J(z)}{\sum_{J \in \binom{[i]}{d}} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_{j+1}+1, i]}^J(z)} \\ &= - \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_j]}^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(z)}{\sum_{J \in \Xi} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_j]}^{J \cup [n_j+d+1, n]}(z)} \end{aligned}$$

in which $\eta(J)$ is the function

$$\eta(J) = \begin{cases} 1, & \text{if } n_j + d + 1 \notin J, \\ -1, & \text{if } n_j + d + 1 \in J. \end{cases}$$

Since $b^{-1} = zu^{-1}$ and $u \in U_+$, we have

$$\begin{aligned} & \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_j]}^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(z)}{\sum_{J \in \Xi} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(z) \cdot \Delta_{[n-n_j]}^{J \cup [n_j+d+1, n]}(z)} \\ &= \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(b^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(b^{-1})}{\sum_{J \in \Xi} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(b^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup [n_j+d+1, n]}(b^{-1})} \end{aligned}$$

We recall that $b_- = \hat{t}b = tb^{-1}$. Note that $t_{n_j+1} = t_{n_j+2} = \dots = t_{n_{j+1}}$ for all $0 \leq j \leq r$. Since $n_j < n_j + d < n_j + d + 1 \leq n_{j+1}$, we have

$$\begin{aligned} u_{i, i+1} &= - \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(b^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(b^{-1})}{\sum_{J \in \Xi} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(b^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup [n_j+d+1, n]}(b^{-1})} \\ &= - \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(tb^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(tb^{-1})}{\sum_{J \in \Xi} (-1)^{|J|} \Delta_{[n-n_{j+1}]}^{[i] \setminus J}(tb^{-1}) \cdot \Delta_{[n-n_j]}^{J \cup [n_j+d+1, n]}(tb^{-1})} \\ &= - \frac{\sum_{J \in A} \eta(J) (-1)^{|J|} G^{[i] \setminus J}(b_- \dot{w}_0 B_-) \cdot G^{J \cup \{n_j+d\} \cup [n_j+d+2, n]}(b_- \dot{w}_0 B_-)}{\sum_{J \in \Xi} (-1)^{|J|} G^{[i] \setminus J}(b_- \dot{w}_0 B_-) \cdot G^{J \cup [n_j+d+1, n]}(b_- \dot{w}_0 B_-)} \end{aligned}$$

Applying the isomorphism in Corollary 4.4.9, we see that the formula for $u_{i, i+1}$ in Lemma 4.4.12 is equivalent to the following identity in $QH^*(X)$.

$$\sum_{J \in \Xi} (-1)^{|J|} (\sigma_{J \cup [i+1, n]} \cdot \sigma_{s_{n_{j+1}}}) \cdot \sigma_{[1, n_j+d] \setminus J} + \sum_{J \in \Xi} (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot (\sigma_{[1, n_j+d] \setminus J} \cdot \sigma_{s_{n_j}})$$

$$= \sum_{J \in A} \eta(J) (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot \sigma_{([1, n_j + d - 1] \cup \{n_j + d + 1\}) \setminus J}. \quad (4.5.5)$$

It remains to show that this is exactly the identity in Theorem 4.5.2.

Assume that $J = \{j_1 < j_2 < \cdots < j_d\}$, then by Proposition 4.4.1,

$$\sigma_{J \cup [i+1, n]} \cdot \sigma_{s_{n_{j+1}}} = \sum_{1 \leq s \leq d} \sigma_{\{j_1, \dots, j_{s-1}, j_s+1, j_{s+1}, \dots, j_d, i+1, \dots, n\}} + q_{n_{j+1}} \sigma_{w_J}.$$

Here, we set $\sigma_{\{j_1, \dots, j_{s-1}, j_s+1, j_{s+1}, \dots, j_d, i, \dots, n\}} := 0$ if either $s = d$ and $j_d = i$ or $j_s + 1 = j_{s+1}$ holds. We divide the above sum into two parts as follows

$$C_1(J) := \sum_{1 \leq s \leq d-1} \sigma_{\{j_1, \dots, j_{s-1}, j_s+1, j_{s+1}, \dots, j_d, i+1, \dots, n\}}$$

$$C_2(J) := \sigma_{\{j_1, \dots, j_{d-1}, j_d+1, i+1, \dots, n\}}$$

Similarly, we have

$$\begin{aligned} \sigma_{[1, n_j + d] \setminus J} \cdot \sigma_{s_{n_j}} &= \sum_{1 \leq j \leq d} \sigma_{[1, n_j + d] \setminus \{j_1, \dots, j_{s-1}, j_s-1, j_{s+1}, \dots, j_d\}} + D_2(J) \\ &= D_1(J) + D_2(J) \end{aligned}$$

Here, $D_1(J)$ is defined as

$$D_1(J) := \sum_{1 \leq j \leq d} \sigma_{[1, n_j + d] \setminus \{j_1, \dots, j_{s-1}, j_s-1, j_{s+1}, \dots, j_d\}}$$

where we set $\sigma_{[1, n_j + d] \setminus \{j_1, \dots, j_{s-1}, j_s-1, j_{s+1}, \dots, j_d\}} := 0$ if either $s = 1$ and $j_1 = 1$ or $j_s - 1 = j_{s-1}$ holds. And $D_2(J)$ is defined as follows

$$D_2(J) := \begin{cases} \sigma_{[1, n_j + d - 1] \cup \{n_j + d + 1\} \setminus J}, & \text{if } n_j + d \notin J, \\ 0, & \text{if } n_j + d \in J. \end{cases}$$

Note that since $n_{j+1} > n_j + d$, we have $w(n_{j+1}) > w(n_j)$ and therefore there are no quantum terms in the product $\sigma_{[1, n_j + d] \setminus J} \cdot \sigma_{s_{n_j}}$ by the remark after Proposition 4.4.1.

If $n_j + d \in J$, namely, $j_d = n_j + d$, then directly from the definition of A and Ξ we have

$$\sum_{\substack{J \in \Xi \\ n_j + d \in J}} (-1)^{|J|} C_2(J) \cdot \sigma_{[1, n_j + d] \setminus J} = \sum_{\substack{J \in A \\ n_j + d + 1 \in J}} \eta(J) (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot \sigma_{([1, n_j + d - 1] \cup \{n_j + d + 1\}) \setminus J}.$$

If $n_j + d \notin J$, then we have

$$\sum_{\substack{J \in \Xi \\ n_j + d \notin J}} (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot D_2(J) = \sum_{\substack{J \in A \\ n_j + d + 1 \notin J}} \eta(J) (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot \sigma_{([1, n_j + d - 1] \cup \{n_j + d + 1\}) \setminus J}.$$

Moreover, for $J = \{j_1 < j_2 < \cdots < j_d\}$, we denote $J_s^+ := \{j_1, \cdots, j_{s-1}, j_s + 1, j_{s+1}, \cdots, j_d\}$ and $J_s^- := \{j_1, \cdots, j_{s-1}, j_s - 1, j_{s+1}, \cdots, j_d\}$. Then

$$\sum_{J \in \Xi} (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot D_1(J) = \sum_{J \in \Xi} \sum_{s=1}^d (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot \sigma_{[1, n_j + d] \setminus J_s^-}.$$

Since $\sigma_{[1, n_j + d] \setminus J_s^-} \neq 0$ only if $J_s^- \in J$, in which case $(J_s^-)^+ = J \in \Xi$, we have

$$\begin{aligned} \sum_{J \in \Xi} \sum_{s=1}^d (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot \sigma_{[1, n_j + d] \setminus J_s^-} &= \sum_{J \in \Xi} \sum_{s=1}^{d-1} (-1)^{|J|+1} \sigma_{J_s^+ \cup [i+1, n]} \cdot \sigma_{[1, n_j + d] \setminus J^+} \\ &\quad + \sum_{\substack{J \in \Xi \\ n_j + d \notin J}} (-1)^{|J|+1} \sigma_{J_d^+ \cup [i+1, n]} \cdot \sigma_{[1, n_j + d] \setminus J}. \end{aligned}$$

Therefore, we have

$$\sum_{J \in \Xi} (-1)^{|J|} C_1(J) \cdot \sigma_{[1, n_j + d] \setminus J^+} + \sum_{\substack{J \in \Xi \\ n_j + d \notin J}} (-1)^{|J|} C_2(J) \cdot \sigma_{[1, n_j + d] \setminus J^+} + \sum_{J \in \Xi} (-1)^{|J|} \sigma_{J \cup [i+1, n]} \cdot D_1(J) = 0.$$

We therefore see that the identity (4.5.5) is equivalent to the identity

$$\sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J} \sigma_{[1, n_j + d] \setminus J} = 0$$

in Theorem 4.5.2. Hence, the statement follows. \square

The structure of the proof of Lemma 4.4.12 and Theorem 4.5.2 is now the following. We will first prove Theorem 4.5.2 in the special case where $n_j + n_{j+1} \leq n$. It then follows that Lemma 4.4.12 holds whenever $n_j + n_{j+1} \leq n$, because of Lemma 4.5.4. Next, we introduce a symmetry on the domain of the superpotential \mathcal{F}_R that allows us to deduce the statement of Lemma 4.4.12 for $n_j + n_{j+1} \geq n$ from the one for $n_j + n_{j+1} \leq n$. Finally, we obtain Theorem 4.5.2 for $n_j + n_{j+1} \leq n$, since this proposition and Lemma 4.4.12 are equivalent in every case. This strategy also shows an interaction between mirror symmetry and quantum Schubert calculus.

A special case of Theorem 4.5.2

In this section we prove Theorem 4.5.2 in the case where $n_j + n_{j+1} \leq n$. To do this we first prove a version of the identity (4.5.2) in the quantum cohomology of the complete flag variety $\mathbb{F}l_n$ in the following key lemma.

Lemma 4.5.6. *Assume that $n_j + n_{j+1} \leq n$. With notations as in Theorem 4.5.2, the following identity, obtained simply by replacing the Schubert classes in (4.5.2) with corresponding ones for $\mathbb{F}l_n$, holds in the quantum cohomology ring $QH^*(\mathbb{F}l_n)$.*

$$\sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J}^B \sigma_{[1, n_j + d] \setminus J}^B = 0.$$

To prove the above lemma, we need some preparation. Recall that a permutation w is called *321-avoiding* if there does not exist $i < j < k$ such that $w(i) > w(j) > w(k)$. The key observation in the proof of Lemma 4.5.6 is the following lemma, that the permutations w_J arising above are all 321-avoiding.

Lemma 4.5.7. *The permutations w_J constructed in Definition 4.5.1 are 321-avoiding.*

Proof. We will argue by contradiction. Consider the case $n_j \geq d$ first. Suppose that there exists $i_0 < j_0 < k_0$ satisfying $w_J(i_0) > w_J(j_0) > w_J(k_0)$. Since $i_0 < j_0$ and $w_J(i_0) > w_J(j_0)$, we must have $n_j + 1 \leq j_0$. If we assume that $j_0 \leq n_{j+1}$, then we must have $j_0 = n_j + 1$ since $w_J(i_0) > w_J(j_0)$. So we have $w_J(j_0) = x_1$. However, this is in contradiction with $j_0 < k_0$ and $w_J(j_0) > w_J(k_0)$. Therefore we must have $j_0 > n_{j+1}$. Since $j_0 < k_0$ and $w_J(j_0) > w_J(k_0)$, we have $j_0 = n_{j+2}$ and $w_J(j_0) = n$. But this is in contradiction with $w_J(i_0) > w_J(j_0)$. In conclusion, for the case $n_j \geq d$, w_J is a 321-avoiding permutation. The case $n_j < d$ can be proved similarly. \square

Remark 4.5.8. For example, the identity in Lemma 4.5.6 coming from $\mathbb{F}l(7; 4, 2)$, where $n_j = 2, n_{j+1} = 4$ and $i = 4, d = 1$, is

$$\sigma_{1526347}^B \cdot \sigma_{2314567}^B - \sigma_{2516347}^B \cdot \sigma_{1324567}^B + \sigma_{3516247}^B \cdot \sigma_{1234567}^B = 0,$$

and it involves only 321-avoiding permutations.

Definition 4.5.3. Let $w \in S_n$ be a permutation, then the code of w is defined as

$$c(w) = (c_1, c_2, \dots, c_n)$$

where $c_i := \#\{j \mid i < j, w(j) < w(i)\}$.

Definition 4.5.4. Let w be a 321-avoiding permutation with code $c(w) = (c_1, \dots, c_n)$. The flag of the partition is defined as $\phi(w) = \{j_1 < j_2 < \dots < j_l\} := \{j \mid c_j > 0\}$. We define a skew partition λ/μ by embedding it into $\mathbb{Z} \times \mathbb{Z}$ as follows:

$$\begin{aligned} \lambda_k - \mu_k &= c_{j_k} \\ \lambda/\mu &= \{(k, h) : 1 \leq k \leq l, k - j_k - c_{j_k} + 1 \leq h \leq k - j_k\} \end{aligned}$$

Example 4.5.9. We continue with the example from Remark 4.5.8.

1. If $w = 1526347$, then the code of w is $c(w) = (0, 3, 0, 2, 0, 0, 0)$. And we have $\{j_1 < j_2\} = \{2 < 4\}$ with $l = 2$. Then we have

$$\begin{aligned} k = 1, 1 - 2 - 3 + 1 &\leq h \leq 1 - 2 \\ k = 2, 2 - 4 - 2 + 1 &\leq h \leq 2 - 4 \end{aligned}$$

Therefore, the skew partition is $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ with $\lambda = (3, 2)$ and $\mu = (0, 0)$.

2. If $w = 2516347$, then the code of w is $c(w) = (1, 3, 0, 2, 0, 0, 0)$ with flag $\phi(w) = \{1, 2, 4\}$. Then

$$\begin{aligned} k = 1, 1 - 1 - 1 + 1 &\leq h \leq 1 - 1 \\ k = 2, 2 - 2 - 3 + 1 &\leq h \leq 2 - 2 \\ k = 3, 3 - 4 - 2 + 1 &\leq h \leq 3 - 4 \end{aligned}$$

Therefore, the skew partition is $\lambda/\mu = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$ with $\lambda = (3, 3, 2)$ and $\mu = (2, 0, 0)$.

3. If $w = 3516247$, then the code of w is $c(w) = (2, 3, 0, 2, 0, 0, 0)$ with flag $\phi(w) = \{1, 2, 4\}$. Then

$$\begin{aligned} k = 1, 1 - 1 - 2 + 1 &\leq h \leq 1 - 1 \\ k = 2, 2 - 2 - 3 + 1 &\leq h \leq 2 - 2 \\ k = 3, 3 - 4 - 2 + 2 + 1 &\leq h \leq 3 - 4 \end{aligned}$$

Therefore, the skew partition is $\lambda/\mu = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$ with $\lambda = (3, 3, 2)$ and $\mu = (1, 0, 0)$.

In [12], Schubert polynomial \mathfrak{S}_w is explicitly written down in a determinantal formula for a 321-avoiding permutation as follows.

Theorem 4.5.5 (Corollary 2.3 of [12]). Let w be a 321-avoiding permutation with flag $\phi(w) = (\phi_1 < \cdots < \phi_k)$ and skew partition λ/μ . Let $X_i = (x_1, x_2, \dots, x_i)$. Then we have

$$\mathfrak{S}_w = \det(h_{\lambda_i - \mu_j - i + j}(X_{\phi_i}))_{1 \leq i, j \leq k}$$

where $h_r(X_i)$ is the complete homogeneous symmetric polynomial of degree r in variables X_i .

In [87], A.N. Kirillov defines quantum Schubert polynomials and conjectures that the quantum version of the above determinantal formula holds as well, see [87, Conjecture 1]. We will verify this conjecture in the quantum cohomology ring of the complete flag variety $\mathbb{F}\ell_n$ using the work of [38]. This should have been known to the experts (see e.g. [23, formula (6)]). Since we are not aware of this formula appearing in form of a theorem, we state it here as Theorem 4.5.10 and provide a detailed argument for completeness.

Definition 4.5.6. Let G_k be the matrix

$$\begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_k \end{pmatrix}$$

The quantum elementary polynomial E_i^k is defined by the following formula

$$\det(1 + \lambda G_k) = \sum_{i=0}^k E_i^k \lambda^i$$

And we set $E_i^k = 0$ if $i < 0$ or $i > k$.

By setting $q_1 = q_2 = \cdots = q_{k-1} = 0$, E_i^k recovers the ordinary elementary symmetric polynomial $e_i^k = e_i^k(x_1, \dots, x_k)$. Let $e_{i_1 \dots i_m} := e_{i_1}^1 \cdots e_{i_m}^m$ be standard elementary monomial. The following lemma is a classical result, and can be found in [101].

Lemma 4.5.10. *Let I_n be the ideal in $\mathbb{Z}[x_1, \dots, x_n]$ generated by e_1^n, \dots, e_n^n , then each of the following forms a \mathbb{Z} -basis in $\mathbb{Z}[x_1, \dots, x_n]/I_n$:*

1. *the standard elementary monomials $e_{i_1 \dots i_{n-1}}$, with $0 \leq i_k \leq k$;*
2. *the Schubert polynomials \mathfrak{S}_w for $w \in S_n$.*

Moreover, each of these families spans the same vector space $L_n \subset \mathbb{Z}[x_1, \dots, x_n]$ which is complementary to I_n .

Therefore, any Schubert polynomial \mathfrak{S}_w is uniquely a linear combination of standard elementary monomials with integer coefficients. In [38], the quantum Schubert polynomial is defined as the linear combination of the quantum elementary monomials $E_{i_1 \dots i_m} := E_{i_1}^1 \cdots E_{i_m}^m$ with the same coefficients. Namely, we have

Definition 4.5.7. The quantum Schubert polynomial \mathfrak{S}_w^q for a permutation $w \in S_n$ is defined as

$$\mathfrak{S}_w^q = \sum \alpha_{i_1 \dots i_{n-1}} E_{i_1 \dots i_{n-1}}$$

where the coefficients $\alpha_{i_1 \dots i_{n-1}}$ are the same as the coefficients found in the classical expansion $\mathfrak{S}_w = \sum \alpha_{i_1 \dots i_{n-1}} e_{i_1 \dots i_{n-1}}$.

We recall the quantum analogue of Lemma 4.5.10 proved in [38].

Lemma 4.5.11. *Let I_n^q be the ideal in $\mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]$ generated by E_1^n, \dots, E_n^n , then each of the following determines a $\mathbb{Z}[q]$ -basis in $\mathbb{Z}[q, x]/I_n^q$:*

1. *the quantum standard elementary monomials $E_{i_1 \dots i_{n-1}}$, with $0 \leq i_k \leq k$;*
2. *the quantum Schubert polynomials \mathfrak{S}_w^q for $w \in S_n$.*

Moreover, each of these families spans the same vector space $L_n^q \subset \mathbb{Z}[q, x]$ which is complementary to I_n^q .

One of the main results in [38] is the following

Theorem 4.5.8 (Theorem 1.2 of [38]). The map

$$\pi : \mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n] \longrightarrow QH^*(\mathbb{F}\ell_n)$$

sending $x_1 + \cdots + x_i$ to $\sigma_{s_i}^B \in QH^*(\mathbb{F}\ell_n)$ is a surjective ring homomorphism with kernel I_n^q generated by E_1^n, \dots, E_n^n . Under the induced isomorphism $\mathbb{Z}[q, x]/I_n^q \cong QH^*(\mathbb{F}\ell_n)$, the coset of the quantum Schubert polynomial \mathfrak{S}_w^q is sent to the corresponding quantum Schubert class σ_w^B .

Now we are ready to prove the quantum version of the determinantal formula for a 321-avoiding permutation.

Definition 4.5.9. We call $H_l^k := \det(E_{j-i+1}^{k+l-1})_{1 \leq i, j \leq l}$ the quantum complete homogeneous polynomial in k variables of degree l . Set $H_{i_1, \dots, i_{n-1}} := H_{i_1}^1 \cdots H_{i_{n-1}}^{n-1}$.

Remark 4.5.12. $H_{i_k}^k \in I_n^q$ if $i_k > n - k$.

Theorem 4.5.10. Let w be a 321-avoiding permutation with flag $\phi(w) = (\phi_1 < \cdots < \phi_k)$ and skew partition λ/μ . Let $X_i = (x_1, x_2, \dots, x_i)$. Then in $\mathbb{Z}[q, x]/I_n^q$ we have

$$\mathfrak{S}_w^q = \det(H_{\lambda_i - \mu_j - i + j}(X_{\phi_i}))_{1 \leq i, j \leq k}.$$

Proof. We consider the involution ω of $\mathbb{Z}[q_1, \dots, q_{n-1}][x_1, \dots, x_n]$ defined by $\omega(x_k) = -x_{n+1-k}$ and $\omega(q_k) = q_{n-k}$, for $1 \leq k \leq n$. According to [38], I_n^q is an invariant subspace for the involution ω . Therefore ω induces an automorphism on $\mathbb{Z}[q, x]/I_n^q$. Moreover, we have

$$\omega(E_{i_1 \dots i_{n-1}}) = H_{i_{n-1} \dots i_1}; \quad \omega(H_{i_1 \dots i_{n-1}}) = E_{i_{n-1} \dots i_1}; \quad \omega(\mathfrak{S}_w^q) = \mathfrak{S}_{w_0 w w_0}^q.$$

Therefore it suffices to show

$$\mathfrak{S}_{w_0 w w_0}^q = \det(E_{\lambda_i - \mu_j - i + j}(X_{n - \phi_i}))_{1 \leq i, j \leq k}$$

Note that the right hand side of the equality is a linear combination of quantum standard elementary monomials by the definition of determinants. Then by Lemma 4.5.11 it suffices to show that the coefficient of any standard elementary monomial $E_{i_1, \dots, i_{n-1}}$ with $0 \leq i_k \leq k$ on the right hand side is the same as in the definition of the quantum Schubert polynomial. But by Definition 4.5.7, it suffices to show this in the classical case. However, by applying involution to Theorem 4.5.5, we have the following equality in $\mathbb{Z}[x_1, \dots, x_n]/I_n$

$$\mathfrak{S}_{w_0 w w_0} = \det(e_{\lambda_i - \mu_j - i + j}(X_{n - \phi_i}))_{1 \leq i, j \leq k}$$

Since the right hand side is a linear combination of $e_{i_1, \dots, i_{n-1}}$ and the standard elementary monomials $e_{i_1, \dots, i_{n-1}}$ with $0 \leq i_k \leq k$ span a vector space complementary to I_n , by discarding the other monomials in the expansion of the determinant, we get the formula $\mathfrak{S}_{w_0 w w_0} = \sum_{0 \leq i_k \leq k} \alpha_{i_1 \dots i_{n-1}} e_{i_1 \dots i_{n-1}}$ as wanted. \square

Remark 4.5.13. In the proof we used the involution ω , therefore we are only able to prove the identity $\mathfrak{S}_w^q = \det(H_{\lambda_i - \mu_j - i + j}(X_{\phi_i}))_{1 \leq i, j \leq k}$ in the quotient ring $\mathbb{Z}[q, x]/I_n^q$. However, the original conjectural identity in [87] is stated in the ring $\mathbb{Z}[q, x]$.

We now use this theorem to prove Lemma 4.5.6.

Proof of Lemma 4.5.6. Using the isomorphism $\mathbb{Z}[q, x]/I_n^q \cong QH^*(\mathbb{F}\ell_n)$, we may identify \mathfrak{S}_w^q with σ_w^B , and treat $H_r(X_i)$ as an element in $QH^*(\mathbb{F}\ell_n)$. Also we use \times for the multiplication. Since w_J is a 321-avoiding permutation by Lemma 4.5.7, we are able to apply Theorem 4.5.10. The proof is divided into two cases: $n_j \geq d$ and $n_j < d$.

1. Consider the case $n_j \geq d$ first. Then for $J = \{j_1 < \dots < j_d\} \in \Xi = \{J \in \binom{[n]}{d} \mid J \cap [n_j + d + 1, n] = \emptyset\}$, let $\{x_1 < x_2 < \dots < x_{i-d}\} := [i] \setminus J$, w_J is the following permutation

$$\begin{aligned} \{w(1) < \dots < w(n_j)\} &= \{j_1 < j_2 < \dots < j_d < i + 1 < i + 2 < \dots < i + n_j - d\} \\ \{w(n_j + 1) < \dots < w(n_{j+1})\} &= \{x_1 < i + n_j - d + 1 < i + n_j - d + 2 < \dots < n - 1\} \\ \{w(n_{j+1} + 1) < \dots < w(n_{j+2})\} &= \{x_2 < \dots < x_{n_{j+2} - n_{j+1}} < n\} \\ \{w(n_{j+2} + 1) < \dots < w(n)\} &= \{x_{n_{j+2} - n_{j+1} + 1} < \dots < x_{i-d}\}. \end{aligned}$$

The code of w_J is $c(w_J) = (j_1 - 1, j_2 - 2, \dots, j_d - d, i - d, i - d, \dots, i - d, 0, i - d - 1, \dots, i - d - 1, 0, \dots, 0, n - n_{j+2}, 0, \dots, 0)$ with flag $\phi(w_J) = (1, 2, \dots, n_j, n_j + 2, n_j + 3, \dots, n_{j+1}, n_{j+2})$. Then it determines a skew partition λ/μ , where

$$\begin{aligned} \lambda &= (i - d, \dots, i - d, i - d - 1, \dots, i - d - 1, n - n_{j+2}) \text{ with } n_j \text{ many } i - d \\ &\quad \text{and } n_{j+1} - n_j - 1 \text{ many } i - d - 1; \\ \mu &= (i - d - (j_1 - 1), i - d - (j_2 - 2), \dots, i - d - (j_d - d), 0, \dots, 0). \end{aligned}$$

Then by Theorem 4.5.10, we have $\sigma_{w_J}^B = \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq r, s \leq n_{j+1}}$. Here we do assume $n_{j+2} < n$, the case $n_{j+2} = n$ can be dealt with

similarly. We are going to use Laplace expansion on the first d columns of this determinant. Let $R = (r_1 < \cdots < r_d) \in \binom{[n_{j+1}]}{d}$ be row index for the expansion, and denote M_R for the cofactor (with sign) obtained by removing the first d columns and rows indexed by R . Then Laplace expansion says that

$$\det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq r, s \leq n_{j+1}} = \sum_R M_R \times \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R}.$$

We observe that M_R is independent of J since it involves only the last $n_{j+1} - d$ columns of $\det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq r, s \leq n_{j+1}}$ and only the first d items of μ depend on J . Therefore, we have

$$\begin{aligned} & \sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J}^B \sigma_{[1, n_j + d] \setminus J}^B \\ &= \sum_{J \in \Xi} \sum_R (-1)^{|J|} M_R \times \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma_{[1, n_j + d] \setminus J}^B \\ &= \sum_R M_R \sum_{J \in \Xi} (-1)^{|J|} \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma_{[1, n_j + d] \setminus J}^B. \end{aligned}$$

The Schubert class $\sigma_{[1, n_j + d] \setminus J}^B$ is indexed by a Grassmannian permutation, which in particular is a 321-avoiding permutation. Let $\alpha = (\alpha_1, \cdots, \alpha_{n_j})$ be the corresponding partition such that $J \cup \{\alpha_1 + n_j, \cdots, \alpha_{n_j} + 1\} = [1, n_j + d]$. Then we have

$$\sigma_{[1, n_j + d] \setminus J}^B = \det(H_{\alpha_a - a + b}(X_{n_j + 1 - b}))_{1 \leq a, b \leq n_j}.$$

We will construct an $(n_j + d) \times (n_j + d)$ matrix A_R . We define the first d row vectors of A_R to be

$$(H_{\lambda_r - r - i + d + 1}(X_{\phi_r}), H_{\lambda_r - r - i + d + 2}(X_{\phi_r}), \cdots, H_{\lambda_r - r - i + d + n_j + d}(X_{\phi_r}))$$

where r runs through $R = (r_1 < \cdots < r_d)$. And we define the last n_j row vectors of A_R to be

$$(H_{1 - n_j - 1 + b}(X_{n_j + 1 - b}), H_{2 - n_j - 1 + b}(X_{n_j + 1 - b}), \cdots, H_{n_j + d - n_j - 1 + b}(X_{n_j + 1 - b}))$$

where b run through $[1, n_j]$.

Next we show that $\det A_R = 0$. We will prove this by showing that either A_R contains two identical row vectors or A_R contains a zero row vector. We observe that $\lambda_r - r - i + d + \phi_r = 0$, therefore, in order to show that A_R contains two identical row vectors it suffices to prove that $\phi_r = n_j + 1 - b$

for some $r \in R$ and $b \in [1, n_j]$, namely, $R \cap [1, n_j] \neq \emptyset$. Now suppose we have the opposite, namely $R \cap [1, n_j] = \emptyset$. Then we have $r_1 \geq n_j + 1$ and $r_d \geq n_j + d$. So we have $\lambda_{r_d} - r_d - i + d = -\phi_{r_d} < -(n_j + d)$. Therefore the d^{th} row of A_R is a zero vector since $H_m(X) := 0$ for $m < 0$. In conclusion, we have $\det A_R = 0$.

Note that λ_r is independent of $J \in \Xi$ and $\mu_s = i - d - (j_s - s)$, so $\lambda_r - \mu_s - r + s = \lambda_r - r - i + d + j_s$. Also note that $\alpha_a - a + b = \alpha_a + (n_j + 1 - a) - n_j - 1 + b$, while $\alpha_a + (n_i + 1 - a)$ lies in the complement of $J \subseteq [1, n_j + d]$. By our assumption that $n_j + n_{j+1} \leq n$, we have $i \geq n_j + d = n_j + i - (n - n_{j+1})$. Therefore we have $\Xi = \binom{[n_j+d]}{d}$. Then by taking the Laplace expansion on the first d rows of A_R , we see that

$$\det A_R = \sum_{J \in \Xi} (-1)^{|J|} \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma_{[1, n_j + d] \setminus J}^B.$$

Therefore, under the assumption $n_j + n_{j+1} \leq n$, we have

$$\begin{aligned} & \sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J}^B \sigma_{[1, n_j + d] \setminus J}^B \\ &= \sum_R M_R \sum_{J \in \Xi} (-1)^{|J|} \det(H_{\lambda_r - \mu_s - r + s}(X_{\phi_r}))_{1 \leq s \leq d, r \in R} \times \sigma_{[1, n_j + d] \setminus J}^B \\ &= \sum_R M_R \det A_R \\ &= 0. \end{aligned}$$

2. For the case $n_j < d$, the proof is similar. Let $J = \{j_1 < \cdots < j_d\} \in \Xi = \{J \in \binom{[i]}{d} \mid J \cap [n_j + d + 1, n] = \emptyset\}$. Let $\{x_1 < \cdots < x_{i-d}\} = [i] \setminus J$. We consider those J with $x_1 < j_{n_j+1}$ only. Then w_J is defined as

$$\begin{aligned} \{w(1) < \cdots < w(n_j)\} &= \{j_1 < \cdots < j_{n_j}\} \\ \{w(n_j + 1) < \cdots < w(n_{j+1})\} &= \{x_1 < j_{n_j+1} < \cdots < j_d < i + 1 < \cdots < n - 1\} \\ \{w(n_{j+1} + 1) < \cdots < w(n_{j+2})\} &= \{x_2 < \cdots < x_{n_{j+2} - n_{j+1}} < n\} \\ \{w(n_{j+2} + 1) < \cdots < w(n)\} &= \{x_{n_{j+2} - n_{j+1} + 1} < \cdots < x_{i-d}\}. \end{aligned}$$

The code of w_J is $c(w_J) = (j_1 - 1, j_2 - 2, \cdots, j_{n_j} - n_j, 0, j_{n_j+1} - n_j - 2, j_{n_j+2} - n_j - 3, \cdots, j_d - d - 1, i - d - 1, i - d - 1, \cdots, i - d - 1, 0, \cdots, 0, n - n_{j+2}, 0, \cdots, 0)$ with flag $\phi(w_J) = (1, 2, \cdots, n_j, n_j + 2, n_j + 3, \cdots, n_{j+1}, n_{j+2})$. Then it determines a skew partition λ/μ , where

$$\lambda = (i - d, \cdots, i - d, i - d - 1, \cdots, i - d - 1, n - n_{j+2}) \text{ with } n_j \text{ many } i - d$$

and $n_{j+1} - n_j - 1$ many $i - d - 1$;

$$\mu = (i - d - (j_1 - 1), i - d - (j_2 - 2), \dots, i - d - (j_d - d), 0, \dots, 0).$$

We notice that the flag $\phi(w_J)$ and the skew partition λ/μ are the same as the case $n_j \geq d$. Therefore, the rest of the proof is similar to the case $n_j \geq d$.

□

Example 4.5.14. We demonstrate the idea of the above proof in the following identity.

$$\sigma_{1526347}^B \cdot \sigma_{2314567}^B - \sigma_{2516347}^B \cdot \sigma_{1324567}^B + \sigma_{3516247}^B \cdot \sigma_{1234567}^B = 0.$$

Applying the determinantal formula, we see that

$$\begin{aligned} \sigma_{1526347}^B &= \det \begin{pmatrix} H_3(X_2) & H_4(X_2) \\ H_1(X_4) & H_2(X_4) \end{pmatrix}, & \sigma_{2314567}^B &= \det \begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_1(X_1) \end{pmatrix}, \\ \sigma_{2516347}^B &= \det \begin{pmatrix} H_1(X_1) & H_4(X_1) & H_5(X_1) \\ H_0(X_2) & H_3(X_2) & H_4(X_2) \\ H_{-2}(X_4) & H_1(X_4) & H_2(X_4) \end{pmatrix}, & \sigma_{1324567}^B &= \det \begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix}, \\ \sigma_{3516247}^B &= \det \begin{pmatrix} H_2(X_1) & H_4(X_1) & H_5(X_1) \\ H_1(X_2) & H_3(X_2) & H_4(X_2) \\ H_{-1}(X_4) & H_1(X_4) & H_2(X_4) \end{pmatrix}, & \sigma_{1234567}^B &= \det \begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix}. \end{aligned}$$

We write

$$\sigma_{1526347}^B = \det \begin{pmatrix} H_3(X_2) & H_4(X_2) \\ H_1(X_4) & H_2(X_4) \end{pmatrix} = \det \begin{pmatrix} 1 & H_4(X_1) & H_5(X_1) \\ 0 & H_3(X_2) & H_4(X_2) \\ 0 & H_1(X_4) & H_2(X_4) \end{pmatrix}.$$

Notice that the last two columns of these 3×3 matrix are the same, so it suffices to prove that

$$\begin{aligned} 1 \times \det \begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_1(X_1) \end{pmatrix} - H_1(X_1) \det \begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix} + H_2(X_1) \det \begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix} &= 0, \\ -H_0(X_2) \det \begin{pmatrix} H_1(X_2) & H_2(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix} + H_1(X_2) \det \begin{pmatrix} H_0(X_2) & H_1(X_1) \\ H_{-1}(X_2) & H_0(X_1) \end{pmatrix} &= 0. \end{aligned}$$

These follow from the Laplace expansion of the following identities respectively.

$$\det \begin{pmatrix} H_2(X_1) & H_1(X_2) & H_2(X_1) \\ H_1(X_1) & H_0(X_2) & H_1(X_1) \\ 1 = H_0(X_1) & H_{-1}(X_2) & H_0(X_1) \end{pmatrix} = 0,$$

$$\det \begin{pmatrix} H_1(X_2) & H_1(X_2) & H_2(X_1) \\ H_0(X_2) & H_0(X_2) & H_1(X_1) \\ 0 = H_{-1}(X_2) & H_{-1}(X_2) & H_0(X_1) \end{pmatrix} = 0.$$

It remains in this section to deduce the identity (4.5.2) also in the partial flag variety setting. We use the following result due to Dale Peterson.

Proposition 4.5.11 (Proposition 11.1 in [117]). Let $w \in W$ and let σ_w^B be the corresponding quantum Schubert class regarded as a function on the Peterson variety \mathcal{Y}_{B_-} for the complete flag variety. Let $\tilde{\sigma}_w^B$ be the rational function on the closure $\mathcal{Y} = \overline{\mathcal{Y}}_{B_-}$ that agrees with σ_w^B on \mathcal{Y}_B . If $w \in W^P$, then $\tilde{\sigma}_w$ restricts to a regular function on $\overline{\mathcal{Y}}_P \subset \mathcal{Y}$, and this restriction represents the quantum Schubert class $\sigma_w^P \in QH^*(G^\vee/P^\vee)$ associated to w .

This proposition implies that any identity in quantum Schubert calculus for the complete flag variety $G^\vee/B^\vee = \mathbb{F}\ell_n$ involving only Schubert classes of the form σ_w^B for $w \in W^P$ and without quantum parameters, holds also in $QH^*(G^\vee/P^\vee)$ with σ_w^B replaced by σ_w^P . As a consequence we have the following corollary; namely we obtain Theorem 4.5.2 in the case $n_j + n_{j+1} \leq n$.

Corollary 4.5.12. Let $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$ and $d := i - (n - n_{j+1})$. Let Ξ and w_J be as defined in Definition 4.5.1. Assume that $n_j + n_{j+1} \leq n$. Set $\sigma_{w_J} := 0$ if w_J is not defined. Then the following identity holds in $QH^*(X)$,

$$\sum_{J \in \Xi} (-1)^{|J|} \sigma_{w_J} \sigma_{[1, n_j + d] \setminus J} = 0.$$

Proof of Lemma 4.4.12

Lemma 4.5.15. Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$. Assume additionally that $n_j + n_{j+1} \leq n$. If \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then

$$u_{i, i+1} = -(G_1^{n_j}(b_- \dot{w}_0 B_-) + G_1^{n_j+1}(b_- \dot{w}_0 B_-)).$$

Proof. The statement is a direct consequence of Corollary 4.5.12 combined with Lemma 4.5.4. \square

Definition 4.5.13. We define a group involution on $G = GL_n(\mathbb{C})$ using a combination of inverse, transpose and conjugation by \dot{w}_0 ,

$$g \mapsto \tau(g) := \dot{w}_0(g^{-1})^T \dot{w}_0^{-1}.$$

Let $Q \supseteq B_-$ be the parabolic subgroup with $I^Q = n - I^P = \{n - n_r, \dots, n - n_1\}$. It is straightforward to check that our involution has the following properties.

1. $\tau(P) = Q$ and $\tau(U_+) = U_+$.
2. $\tau(\dot{w}_P) = \dot{w}_Q$.
3. for $x \in U_+$ we have the relationship $\tau(x)_{i,i+1} = x_{n-i,n-i+1}$, for the entries just above the diagonal.

Lemma 4.5.16. *Suppose $n - n_{j+1} < i < n - n_j$ for some $1 \leq j \leq r - 1$. Assume additionally that $n_j + n_{j+1} \geq n$. If \hat{b} is a critical point of $\mathcal{F}_{\mathbf{q}}$, then*

$$u_{i,i+1} = -(G_1^{n_j}(b_- \dot{w}_0 B_-) + G_1^{n_{j+1}}(b_- \dot{w}_0 B_-)). \quad (4.5.17)$$

Proof. Since $\hat{b} \in B_- \cap U_+ \dot{w}_P^{-1} \dot{w}_0 U_+$, we have that $\tau(\hat{b}) \in B_- \cap U_+ \dot{w}_Q^{-1} \dot{w}_0 U_+$. We can now apply Lemma 4.5.15 to $\tau(\hat{b})$, where we must replace P by Q . Namely for $\tau(\hat{b})$, Lemma 4.5.15 says that, if $n_j = n - (n - n_j) < n - i < n - (n - n_{j+1}) = n_{j+1}$, and $(n - n_j) + (n - n_{j+1}) \leq n$ (which is equivalent to our assumptions on i), then

$$\tau(u)_{n-i,n-i+1} = -(G_1^{n-n_j}(\tau(b_-) \dot{w}_0 B_-) + G_1^{n-n_{j+1}}(\tau(b_-) \dot{w}_0 B_-)).$$

Recall that

$$G_1^m(g B_-) := \frac{\Delta_{[m+1,n]}^{\{m\} \cup [m+2,n]}(g)}{\Delta_{[m+1,n]}^{[m+1,n]}(g)}.$$

Now we deduce that

$$G_1^{n-m}(\tau(b_-) \dot{w}_0 B_-) = G_1^m(b_- \dot{w}_0 B_-),$$

using Jacobi's theorem. Moreover by property (3) above, we have $\tau(u)_{n-i,n-i+1} = u_{i,i+1}$. Therefore the identity (4.5.17) holds. \square

Proof of Lemma 4.4.12 and Theorem 4.5.2. Lemma 4.4.12 follows from the combination of Lemmas 4.5.15 and 4.5.16. We showed in Lemma 4.5.4, that Theorem 4.5.2 is true if and only if Lemma 4.4.12 holds. Since Lemma 4.4.12 has now been proved, we are done. \square

4.6 Appendix

In this Appendix we give a translation of the Plücker coordinate formula for the superpotential \mathcal{F}_- using Young diagrams.

For $1 \leq k < n$, we consider the set of partitions inside $k \times (n - k)$ rectangle,

$$\mathcal{P}_{k,n} := \{(\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k \mid n - k \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0\}.$$

There is a bijection

$$\binom{[n]}{k} \rightarrow \mathcal{P}_{k,n}; \quad J = (j_1, \dots, j_k) \mapsto \lambda(J) = (j_k - k, \dots, j_2 - 2, j_1 - 1).$$

Geometrically, we consider the $k \times (n - k)$ rectangle of $k(n - k)$ unit boxes. A positive path of such rectangle is a path starting from the lower left hand corner and moving either upward or to the right along edges, towards the upper right hand corner. In particular, a Plücker coordinate $p_{j_1 \dots j_k}$ is naturally viewed as the positive path that moves upwards precisely at the j_1, j_2, \dots, j_k -th steps. Moreover, the boxes above the positive path p_J form the partition $\lambda(J)$. We therefore use the following notation convention

$$p_J = p_\lambda = p_{\text{YD}(\lambda)}^{(k)},$$

where the superscript (k) is used to indicate that the Young diagram $\text{YD}(\lambda)$ of the partition λ is inside $k \times (n - k)$ rectangle. In particular,

$$p_{[k]} = p_{(0, \dots, 0)} = p_\emptyset^{(k)}.$$

Example 4.6.1. The Young diagrams of the partitions $(4, 4, 4)$ and $(3, 2, 0)$ in $\mathcal{P}_{3,7}$ are given as follows.

$$\begin{array}{cc} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \\ (4, 4, 4) & (3, 2, 0) \end{array}$$

The Plücker coordinate p_{146} for $Gr(3, 7)$ corresponds to the partition $(3, 2, 0)$.

By $(m^l, 0^{k-l})$ we mean the partition $(m, \dots, m, 0, \dots, 0) \in \mathcal{P}_{k,n}$ with l copies of m . The Young diagram $\text{YD}(m^l, 0^{k-l})$ is an $l \times m$ rectangle $\square_{l \times m}$, and $\text{YD}(1, 0^{k-1}) = \square$. We call $(m^l, 0^{k-l})$ a *maximal partition* in $\mathcal{P}_{k,n}$ if $l = k$ or $m = n - k$ holds.

Definition 4.6.1. Let $\lambda \in \mathcal{P}_{k,n}$ and $\nu \in \mathcal{P}_{k-a,n-a}$. We define

$$p_{\square_{k \times m}, YD(\lambda)}^{(k)} := \begin{cases} p_{YD(m^k + \lambda)}^{(k)}, & \text{if } m^k + \lambda \in \mathcal{P}_{k,n}, \\ 0, & \text{otherwise;} \end{cases}$$

$$p_{\square_{a \times (n-k)}, YD(\nu)}^{(k)} := \begin{cases} p_{YD((n-k)^a, \nu)}^{(k)}, & \text{if } ((n-k)^a, \nu) \in \mathcal{P}_{k,n}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.6.2. Let $k < l < n$ and $1 \leq m < l - k$. We define

$$L(p_{\square_{k \times m}}^{(k)} \cdot p_{\square_{(l-m) \times (n-l)}}^{(l)}) := \sum_{\mu \leq m^k} (-1)^{|\mu| + km} p_{YD(\mu)}^{(k)} \cdot p_{\square_{(l-m) \times (n-l)}, YD((m^k/\mu)^c)}^{(l)},$$

where $(m^k/\mu)^c \in \mathcal{P}_{m, m+k}$ denotes the conjugate of $(m - \mu_k, \dots, m - \mu_1)$. We define

$$L(p_{\square_{k \times m}, \square}^{(k)} \cdot p_{\square_{(l-m) \times (n-l)}}^{(l)}) := \sum_{\mu \leq \lambda'; \mu_1 \neq m} (-1)^{|\mu| + km} p_{YD(\mu)}^{(k)} \cdot p_{\square_{(l-m) \times (n-l)}, YD((\lambda'/\mu)^c)}^{(l)},$$

where $\lambda' = (m + 1, m^{k-1})$; $(\lambda'/\mu)^c \in \mathcal{P}_{m, m+k+1}$ denotes the conjugate of $(m - \mu_k, \dots, m - \mu_2, 0)$ if $\mu_1 = m + 1$, or of $(m - \mu_k, \dots, m - \mu_1, 1)$ if $\mu_1 < m$.

Theorem 4.6.3. In terms of the Plücker coordinates indexed by Young diagrams,

$$\begin{aligned} \mathcal{F}_- = & \sum_{i=1}^{n_1-1} \frac{p_{\square_{i \times (n-n_1)}, \square}^{(n_1)}}{p_{\square_{i \times (n-n_1)}}^{(n_1)}} + \sum_{j=1}^{r-1} \sum_{m=1}^{n_{j+1}-n_j-1} \frac{L(p_{\square_{n_j \times m}, \square}^{(n_j)} \cdot p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}}^{(n_{j+1})})}{L(p_{\square_{n_j \times m}}^{(n_j)} \cdot p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}}^{(n_{j+1})})} \\ & + \sum_{i=1}^{n-n_r-1} \frac{p_{\square_{n_r \times i}, \square}^{(n_r)}}{p_{\square_{n_r \times i}}^{(n_r)}} + \sum_{j=1}^r \frac{p_{\square}^{(n_j)}}{p_{\emptyset}^{(n_j)}} + \sum_{j=1}^r q_{n_j} \frac{p_{\square_{n_j \times (n-n_j)} \setminus q_{n_j}}^{(n_j)}}{p_{\square_{n_j \times (n-n_j)}}^{(n_j)}} \end{aligned}$$

where $\square_{n_j \times (n-n_j)} \setminus q_{n_j}$ denotes the Young diagram obtained by removing $n_j - n_{j-1}$ boxes from the last column of $\square_{n_j \times (n-n_j)}$ and removing $n_{j+1} - n_j$ boxes from the last row, with the removal of the box at the bottom-right corner double counted.

Proof. It suffices to discuss the $S_i^{(j)}$ -terms in Theorem 4.3.10. (Other terms therein are direct translations to Young diagrams.)

For the denominator $L(p_{\square_{n_j \times m}}^{(n_j)} \cdot p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}}^{(n_{j+1})})$ as above, where $m = i - n_j$, we define a map $\alpha : \{J | J \in \binom{[i]}{m}\} \rightarrow \{\mu | \mu \leq m^{n_j}\}$ as follows: it sends

$J = \{a_1, \dots, a_m\}$ to the Young diagram $\alpha(J)$ with a_1, \dots, a_m steps horizontal. It follows directly that α is a bijection. It remains to check the following facts:

1. $J \in \binom{[\min\{i, \hat{i}\}]}{m}$ if and only if the join $\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((m^k/\alpha(J))^c)$ is inside the $n_{j+1} \times (n - n_{j+1})$ rectangle.
2. For $J \in \binom{[\min\{i, \hat{i}\}]}{m}$, we have $p_{J \cup [\hat{i}+1, n]} = p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((m^k/\alpha(J))^c)}^{(n_{j+1})}$ and $p_{[i] \setminus J} = p_{YD(\alpha(J))}^{(n_j)}$. In particular for $J = [m]$, the corresponding product is the leading term $p_{\square_{n_j \times m}}^{(n_j)} \cdot p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}}^{(n_{j+1})}$.

By definition, $J \in \binom{[\min\{i, \hat{i}\}]}{m}$ if and only if the numbering of the first m vertical steps of the Young diagram $J \cup [\hat{i}+1, n]$ are a_1, \dots, a_m and $J \cup [\hat{i}+1, n]$ is inside the $n_{j+1} \times (n - n_{j+1})$ rectangle. Notice that $YD((m^{n_j}/\alpha(J))^c)$ is the Young diagram $(a_m - m, \dots, a_1 - 1)$. Thus when the join $\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((m^{n_j}/\alpha(J))^c)$ is inside the $n_{j+1} \times (n - n_{j+1})$ rectangle, the numbering of its first m vertical steps are exactly a_1, \dots, a_m , and hence coincides with the Young diagram of $\alpha(J \cup [\hat{i} + 1, n])$. Therefore in this case, the Plücker coordinates are also identified.

The arguments for the numerators are similar. Let $\lambda' = (m + 1, m^{n_j-1})$. Here we define $\alpha' : \{J | J \in \binom{[i+1] \setminus i}{m}\} \rightarrow \{\mu | \mu \leq \lambda', \mu_1 \neq m\}$ as follows: α' sends $\{a_1, \dots, a_m\}$ to the (unique) Young diagram $\alpha'(J)$ inside λ' with $[i + 1] \setminus \{i, a_1, \dots, a_m\}$ steps vertical and $\mu_1 \neq m$. Such map is a bijection. Again we can similarly check the following facts:

1. $J \in \binom{[\min\{i+1, \hat{i}\}] \setminus i}{m}$ if and only if the join $\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((\lambda'/\alpha'(J))^c)$ is inside the $n_{j+1} \times (n - n_{j+1})$ rectangle.
2. For $J \in \binom{[\min\{i+1, \hat{i}\}] \setminus i}{m}$, we have $p_{[i-1] \cup \{i+1\} \setminus J} = p_{YD(\alpha'(J))}^{(n_j)}$ and $p_{J \cup [\hat{i}+1, n]} = p_{\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((\lambda'/\alpha'(J))^c)}^{(n_{j+1})}$.

When $\alpha'(J)_1 = m + 1$. Let $J = \{a_1, \dots, a_m\}$. $(\lambda'/\alpha'(J))^c$ is a partition given by the conjugate of $(m - \mu_k, \dots, m - \mu_2, 0)$, and we have the fact that $a_m \neq i + 1$ and $YD((\lambda'/\alpha'(J))^c)$ is the Young diagram $(a_m - m, \dots, a_1 - 1)$. Thus when the join $\square_{(n_{j+1}-m) \times (n-n_{j+1})}, YD((\lambda'/\alpha'(J))^c)$ is inside the $n_{j+1} \times (n - n_{j+1})$

rectangle, the numbering of its first m vertical steps are exactly a_1, \dots, a_m and hence it coincides with the Young diagram of $J \cup [\hat{i} + 1, n]$. Therefore the Plücker coordinates are also identified. The argument about other parts is similar. \square

Example 4.6.2. For $F\ell_{2,4;7}$, we have

$$\begin{aligned} \mathcal{F}_- &= \frac{p_{27}}{p_{17}} + \frac{p_{24}p_{1567} - p_{14}p_{2567} + p_{12}p_{4567}}{p_{23}p_{1567} - p_{13}p_{2567} + p_{12}p_{3567}} + \frac{p_{2346}}{p_{2345}} + \frac{p_{3457}}{p_{3456}} + \frac{p_{13}}{p_{12}} + \frac{p_{1235}}{p_{1234}} + q_2 \frac{p_{46}}{p_{67}} + q_4 \frac{p_{1467}}{p_{4567}} \\ &= \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)}} + \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)} - p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)} + p_{\emptyset}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)} - p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)} + p_{\emptyset}^{(2)} p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}} + \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}} + \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}} + \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)}}{p_{\emptyset}^{(2)}} + \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}{p_{\emptyset}^{(4)}} + q_2 \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(2)}} + q_4 \frac{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}{p_{\begin{array}{|c|} \hline \square \\ \hline \square \square \square \square \\ \hline \end{array}}^{(4)}}. \end{aligned}$$

UNFOLDING OF EQUIVARIANT F-BUNDLES AND APPLICATION TO THE MIRROR SYMMETRY OF FLAG VARIETIES

This chapter is based on [65], joint work with Thorgal Hinault, Changzheng Li, Tony Yue YU, and Shaowu Zhang.

5.1 Introduction

Motivations

For a smooth complex projective variety X , the Gromov–Witten invariants of X are roughly counts of algebraic curves in X with given genus, class, and constraints (see [9, 54, 129]). We can organize the rational (i.e. genus zero) Gromov–Witten invariants into a generating series as follows.

Fix a homogeneous basis $(T_i)_{0 \leq i \leq N}$ of $H^*(X, \mathbb{Q})$, and let $(T_i^*)_{0 \leq i \leq N}$ denote the dual basis with respect to the Poincaré pairing. Let $\mathbb{Q}[[\text{NE}(X, \mathbb{Z})]]$ denote the completion of $\mathbb{Q}[\text{NE}(X, \mathbb{Z})] = \mathbb{Q}[q^\beta \mid \beta \in \text{NE}(X, \mathbb{Z})]$ with respect to the maximal ideal $(q^\beta, \beta \neq 0)$.

The *genus 0 Gromov–Witten potential* is a formal power series

$$\Phi = \sum_{n \geq 0, \beta} \frac{q^\beta}{n!} \sum_{i_1, \dots, i_n} \langle T_{i_1} \cdots T_{i_n} \rangle_{0, n}^\beta t_{i_1} \cdots t_{i_n} \in \mathbb{Q}[[\text{NE}(X, \mathbb{Z})]][[t_0, \dots, t_N]],$$

where $\langle \cdots \rangle_{0, n}^\beta$ denotes the Gromov–Witten invariants of X of genus 0, class β and cohomological constraints T_{i_1}, \dots, T_{i_n} . It gives rise to the *big quantum cohomology* of X , i.e. a deformation of the classical cup product on $H^*(X, \mathbb{Q})$:

$$\begin{aligned} \star: H^*(X, \mathbb{Q}) \otimes H^*(X, \mathbb{Q}) &\longrightarrow H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[[\text{NE}(X, \mathbb{Z})]][[t_0, \dots, t_N]] \\ T_i \star T_j &\longmapsto \sum_r \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_r} T_r^*. \end{aligned}$$

A simpler version called *small quantum cohomology* is the restriction of the big quantum cohomology to $t_i = 0$, for all $i = 0, \dots, N$ (or equivalently, by the divisor axiom of Gromov–Witten invariants, for all i with $\deg T_i \neq 2$). The idea of small quantum cohomology appeared before the big version, first in

[19], where the small quantum cohomology of a quintic Calabi-Yau threefold was computed using the mirror manifold's periods. This computation led to the curve counting invariants of the quintic that were previously unknown, and sparked decades of research of enumerative geometry and mirror symmetry from the mathematical viewpoint.

The small quantum cohomology mirror symmetry was proved in various cases, such as complete intersections in projective spaces in [98?] and toric complete intersections in [51, 99, 116]. Given that the small quantum cohomology is the restriction of the big quantum cohomology, a natural question is whether mirror symmetry still holds for the big quantum cohomology. The big quantum cohomology mirror symmetry was proved for projective spaces in [5], for quadric hypersurfaces in [71], for \mathbb{P}^2 via tropical geometry in [55], for toric varieties in [76, 77, 115], and for toric Deligne-Mumford stacks in [32].

One tool for such an extension is the reconstruction theorem for Gromov–Witten invariants by Kontsevich-Manin [?], which is the prototype of the universal unfolding of Frobenius manifolds by Hertling-Manin [64] and that of logarithmic Frobenius manifolds by Reichelt [114]. This is the essential ingredient in the proof of big quantum cohomology mirror symmetry for projective spaces in [4, 5]. It is shown that the big quantum cohomology can be reconstructed from the small under the condition that the small quantum cohomology (or the classical cohomology) is H^2 -generated. The Hertling-Manin unfolding theorem applies more generally to so called (TE)-structures, or F-bundles $(\mathcal{H}, \nabla)/B$, where \mathcal{H} is a vector bundle over $B \times \mathrm{Spf} \mathbb{k}[[u]]$ and ∇ is a flat connection on \mathcal{H} with poles at $u = 0$, such that $\nabla_{u^2 \partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B . The H^2 -generation condition is then replaced by two conditions called (IC) and (GC). For $b \in B$, the residues $\nabla_{u\xi}|_{(b,0)}$ and $\nabla_{u^2 \partial_u}|_{(b,0)}$ are endomorphisms of the fiber $\mathcal{H}_{b,0}$. An element $v \in \mathcal{H}_{b,0}$ satisfies the (GC) condition if the iterated action of these endomorphisms on v generate $\mathcal{H}_{b,0}$. It satisfies the (IC) condition if the map $\xi \in T_b B \mapsto \nabla_{u\xi}|_{(b,0)}(v)$ is injective. Under those two conditions, the F-bundle admits a universal unfolding into a maximal F-bundle.

Another tool for such an extension from small to big quantum cohomology is the reconstruction from a semisimple point. In the context of Frobenius manifolds, the structure around a semisimple point was studied in [33, 36], and a reconstruction result was proved in [8, 105]. Teleman also studied

semisimplicity in the context of topological field theories in [127].

In this paper, we aim to establish the big quantum cohomology mirror symmetry for flag varieties, in the sense of isomorphism of big quantum D -modules. The small quantum cohomology mirror symmetry for general flag varieties was recently established in [28], as an isomorphism of small quantum D -modules. In general the small quantum cohomology of flag varieties is neither H^2 -generated, nor semisimple, so neither of the above reconstruction methods can be applied here.

The *main discovery* of this paper is that an analogous H^2 -generation condition can be recovered if we work equivariantly with respect to a torus action.

We first extend the definition of F-bundle (from [66, 84]) to equivariant F-bundle (see Definition 5.2.10). Since the connection ∇_{∂_u} is not linear with respect to the equivariant variables, we need to work with infinite rank F-bundles over an infinite dimensional base. Nevertheless, most of the data can still be reduced to a finite rank (T)-structure relative to $H_T^*(\text{pt}, \mathbb{Q})$.

Next we extend the (IC) and (GC) conditions to the equivariant setting, and establish an unfolding theorem for equivariant F-bundles under these conditions (see Theorem 5.3.36).

For application to the mirror symmetry of flag varieties, we produce an unfolding of the B -model by constructing an appropriate unfolding of the Landau-Ginzburg superpotential. We further check the various conditions on the big quantum D -module of flag varieties, and apply our equivariant unfolding theorem to obtain the mirror symmetry theorem for the big quantum cohomology of flag varieties.

Main results

An F -bundle (\mathcal{H}, ∇) over some base B consists of a vector bundle \mathcal{H} over $B \times \text{Spf } \mathbb{k}[[u]]$ and a meromorphic flat connection ∇ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B . If the connection ∇ is only defined in the directions of B , we call (\mathcal{H}, ∇) a \mathbb{k} -linear (T) -structure. In order not to create confusion in the infinite rank/dimension setting, we formulate F-bundles and (T)-structures in purely algebraic terms in Section 5.2, replacing the vector bundle by a free module, and the connection by derivations.

Let us explain the various conditions involved in our equivariant unfolding theorem. Let \mathbb{k} be a field of characteristic zero, R a \mathbb{k} -algebra and (\mathcal{H}, ∇) an F-bundle (resp. a (T)-structure) over $R[[t_i, i \in I]]$ for a countable set I , with fiber H at $t = 0, u = 0$. Residues of ∇ induce $K := \nabla_{u^2 \partial_u}|_{u=t=0} \in \text{End}_R(H)$,

$$\mu: \bigoplus_{i \in I} R \partial_{t_i} \longrightarrow \text{End}_R(H), \quad \partial_{t_i} \longmapsto \nabla_{u \partial_{t_i}}|_{u=t=0},$$

and for any $v \in H$,

$$\mu_v: \bigoplus_{i \in I} R \partial_{t_i} \longrightarrow H, \quad \partial_{t_i} \longmapsto \nabla_{u \partial_{t_i}}|_{u=t=0}(v).$$

The F-bundle (\mathcal{H}, ∇) is called *maximal* if there exists $v \in H$ such that μ_v is an isomorphism, and v is called a cyclic vector. We further define the following conditions on v (see Definition 5.3.15)

- (IC) The map μ_v is injective.
- (GC) The orbit of v under the action of the subalgebra $R[\text{im } \mu, K] \subset \text{End}_R(H)$ (resp. $R[\text{im } \mu] \subset \text{End}_R(H)$ in the case of a (T)-structure) is H .
- (GC') The condition (GC) is satisfied after base change to $\text{Frac}(R)$.

The conditions (IC) and (GC) were originally formulated in [64] as necessary conditions to obtain the existence and uniqueness of a maximal unfolding. We find that when working relative to a ring, condition (GC') is enough for uniqueness, while conditions (IC) and (GC) need to be complemented by the assumption that $\text{coker } \mu_v$ is free in order to construct a maximal unfolding (see Theorem 5.1.3 for a precise statement).

Equivariant unfolding theorem

For our application to the mirror symmetry of a flag variety $X = G/P$, the F-bundle associated to the quantum cohomology of X does not satisfy conditions (GC) or (GC'). Our new idea is to consider the equivariant quantum cohomology of X induced by the natural torus action. Note that while the associated (T)-structure is linear over $R := H_T^*(\text{pt}, \mathbb{k})$ and of finite rank, the connection ∇_{∂_u} in the u -direction connection is *not* R -linear due to the nontrivial grading on R . Therefore, the associated F-bundle can only be defined over the base field \mathbb{k} ,

and hence has infinite rank and depends on infinitely many variables, indexed by a \mathbb{k} -basis of $H_T^*(X, \mathbb{k})$.

We introduce the notion of *equivariant F-bundle* in Definition 5.2.10. Let I be a finite set and choose a \mathbb{k} -linear basis of a \mathbb{k} -algebra R indexed by a countable set K . Let $\mathbf{t}_I = \{t_{i,k} \mid (i, k) \in I \times K\}$ denote formal parameters over \mathbb{k} , and $t_I = \{t_i \mid i \in I\}$ formal parameters over R . An equivariant F-bundle consists of the data $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$, where (\mathcal{H}, ∇) is a \mathbb{k} -linear F-bundle over $\mathbb{k}[[\mathbf{t}_I]]$ and $\{(\mathcal{H}_R, \nabla_R), \alpha\}$ is an R -linear lift over $R[[t_I]]$ of the \mathbb{k} -linear (T)-structure underlying (\mathcal{H}, ∇) . An unfolding of an equivariant F-bundle is an extension over a bigger formal base (see Definition 5.3.33). We also generalize the notion of framing (from [66, Definition 2.9]) to equivariant F-bundles (Definition 5.2.13), which consists of framings for the \mathbb{k} -linear F-bundle and the R -linear (T)-structure that are compatible under the lift.

We extend the (IC), (GC), (GC') and maximality conditions to equivariant F-bundles by requiring that the R -linear (T)-structure satisfy those conditions. Our main theorem is the following unfolding theorem for equivariant F-bundles.

Theorem 5.1.1 (Unfolding of equivariant F-bundles, Theorem 5.3.36). *Let $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ be an equivariant F-bundle over $\mathbb{k}[[\mathbf{t}_I]]$, and fix $v \in \mathcal{H}_R|_{u=t_I=0}$.*

1. *If v satisfies (IC), (GC) and $\text{coker } \mu_v$ is free, then \mathcal{F} admits a maximal unfolding with a cyclic vector induced from v .*
2. *If v satisfies (GC'), then any two maximal unfoldings of \mathcal{F} with cyclic vectors induced from v are isomorphic under a unique isomorphism.*

Furthermore, any framing for \mathcal{F} induces a unique framing on a maximal unfolding.

The first step in our proof is to establish a formal version of the Hertling-Manin unfolding theorem in the finite rank case (see Theorem 5.1.3). Then we use it to unfold the R -linear (T)-structure. Finally we conclude by unfolding the \mathbb{k} -linear F-bundle in the u -direction. The key observation for the last step is the very useful Lemma 5.3.1. It states that an equivariant F-bundle is uniquely determined by the underlying (T)-structure and the value of the u -direction connection at one point, under the assumption that the (T)-structure admits a

framing. This assumption always holds for the \mathbb{k} -linear (T)-structure associated to an equivariant F-bundle by Proposition 5.3.4 and Lemma 5.2.6.

Proposition 5.1.2 (Lemma 5.3.1). *For $k = 1, 2$, let I_k be a countable set and $(\mathcal{H}_k, \nabla_k)/R[[t_j, j \in I_k]]$ an F-bundle. Let $(f, \Phi): (\mathcal{H}_1, \nabla_1)_0 \rightarrow (\mathcal{H}_2, \nabla_2)_0$ be a morphism of (T)-structures. Assume the (T)-structure $(\mathcal{H}_1, \nabla_1)_0$ admits a framing. Then*

1. ∇_1 is uniquely determined by the underlying (T)-structure and $\nabla_{1, \partial_u}|_{t_{I_1}=0}$, and any such data determine a unique F-bundle connection extending the (T)-structure.
2. (f, Φ) is an isomorphism of F-bundles if and only if $(f, \Phi)|_{t_{I_1}=0}$ is an isomorphism of F-bundles.

Here is our formal version of the Hertling-Manin unfolding theorem we mentioned above. We also deduce a version for (T)-structures in Corollary 5.3.30.

Theorem 5.1.3 (Formal Hertling-Manin unfolding, Theorem 5.3.28). *Let R be an integral domain containing \mathbb{Q} . Let $(\mathcal{H}, \nabla)/R[[t_1, \dots, t_d]]$ be a finite rank F-bundle. Let $v \in \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$.*

1. *If v satisfies (IC), (GC) and $\text{coker } \mu_v$ is free, then there exists a maximal unfolding with a cyclic vector induced from v .*
2. *If v satisfies (GC'), then any two maximal unfoldings of (\mathcal{H}, ∇) with cyclic vectors induced from v are isomorphic under a unique isomorphism.*

Furthermore, any framing for (\mathcal{H}, ∇) induces a unique framing on a maximal unfolding.

Our proof follows mostly the original proof of Hertling and Manin, which was carried out in the complex analytic setting. In particular, we produce unfoldings using the (GC) condition in Lemma 5.3.16, which is the formal analogue of [64, Lemma 2.9]. While the original proof uses analytic methods to construct a framing of the (T)-structure in which the u -direction has a logarithmic pole at $u = 1$, we show that the proof actually works in any framing trivialization.

Since we are working over an integral domain R , the (IC) and (GC) conditions are not sufficient to prove existence, and we have to require that $\text{coker } \mu_v$ is free

in order to construct a maximal unfolding. This ensures that we can extend a basis of $\text{im } \mu_v$ to a basis of $\mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$. We prove the uniqueness by observing that under (GC'), an unfolding (\mathcal{H}', ∇') is characterized by a choice of framing before unfolding and the action of ∇' on a section that extends v . This allows us to compare unfoldings through their action on a section extending v , and to establish the isomorphism. A priori, the isomorphism we produce is only defined over $\text{Frac}(R)$, but we note that it is in fact defined over R if the unfoldings are. A key result is the canonical extension of framing for (T)-structures (Proposition 5.3.4), which was essentially proved in [66].

Application to mirror symmetry of flag varieties

We apply Theorem 5.1.1 to the mirror symmetry of flag varieties G/P , where G is a simply-connected complex simple Lie group and P is a parabolic subgroup of G . We begin by reviewing some relevant progress on the mirror symmetry of flag varieties.

On the A -side, there was a remarkable presentation of the small quantum cohomology ring $\text{QH}^*(G/P)$ in terms of Peterson variety given in the unpublished lecture notes [113] by Peterson. This was partially verified in [25, 92, 117], and was recently proved in [26] in full generality. On the B -side, Rietsch constructed a mirror Landau-Ginzburg model (X_P^\vee, \mathcal{W}) for G/P in [118], and showed the coincidence between the critical loci of W and the Peterson variety strata. As a consequence, we obtain a first level of small quantum cohomology mirror symmetry in the sense of a ring isomorphism $\text{QH}^*(G/P) \cong \text{Jac}(\mathcal{W})$. We refer to [7?] and the references therein for more relevant studies in the special case $G = SL(n+1, \mathbb{C})$.

Furthermore, on the A -side, we can consider the Dubrovin connection on the trivial $\text{QH}^*(G/P)$ -bundle over \mathbb{C}^* , which endows the vector bundle with a quantum D -module structure. The flag variety G/P admits a natural torus action by the maximal torus T of G , so that we can consider the equivariant quantum D -module structure as well. On the B -side, we consider the Brieskorn lattice $G_0(X_P^\vee, \mathcal{W}, p)$ associated to Rietsch's equivariant superpotential mirror to G/P (see Section 5.4 for more details). The small quantum cohomology mirror symmetry in the sense of isomorphism of small quantum D -modules has been studied for certain Grassmannians in [93, 103, 111, 112], and was recently established in [28]. In the present paper, we first reformulate this in terms of

an isomorphism $\mathcal{F}^A \cong \mathcal{F}^B$ of equivariant F-bundles. Then, as an application of Theorem 5.1.1, we obtain the following.

Theorem 5.1.4 (Big quantum D -module mirror symmetry, Theorem 5.4.35). *The A -model big equivariant F -bundle $\mathcal{F}^{A,\text{big}}$ is isomorphic to the B -model big equivariant F -bundle $\mathcal{F}^{B,\text{big}}$. The isomorphism is uniquely determined by the small equivariant quantum D -module mirror symmetry.*

By taking the non-equivariant limit of the isomorphism in Theorem 5.1.4, we obtain a non-equivariant version of the big quantum cohomology mirror symmetry for flag varieties, see Theorem 5.4.38.

Note that the small quantum cohomology $\text{QH}^*(G/P)$ can be neither semisimple nor H^2 -generated, such as is the case when $G/P = SG(2, 2n)$ is the Grassmannian of isotropic planes in Lie type C_n (see [22]). Therefore, the application of our unfolding theorem is essential in such cases, for which neither the unfolding in [64] nor the semisimple reconstruction in [127] can be applied.

In addition to the mirror statement above, we further anticipate the complex analytic convergence of the mirror map, as well as the compatibility with the pairings on the F-bundles. These aspects present promising directions for future research.

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5.2 (T)-structures and equivariant F-bundles

We fix a field \mathbb{k} of characteristic zero, a \mathbb{k} -algebra R and a \mathbb{k} -linear basis $(\lambda_k)_{k \in K}$ of R , with K a countable set.

Completions

We set the conventions for completions of rings of polynomials in infinitely many variables, following [76, §2.1]. Our reference for topological algebra is [39, 0§7].

Let I be a countable set indexing indeterminates $t_I = \{t_i, i \in I\}$. We denote by $\mathbb{N}^{(I)}$ the set of almost zero integer sequences indexed by I . Let M be a module or ring, we denote by $M[[t_I]]$ the module consisting of formal power series $\sum_{\alpha \in \mathbb{N}^{(I)}} a_\alpha t_I^\alpha$, where $t_I^\alpha := \prod_{i \in I} t_i^{\alpha_i}$ and $a_\alpha \in M$. It is the projective limit of the modules $M[[t_i, i \in I']]$, where $I' \subset I$ runs through finite subsets. For two countable sets I and I'' , we have $M[[t_i, i \in I]][[t_i, i \in I'']] \simeq M[[t_i, i \in I \cup I'']]$.

If M is linearly topologized by the descending chain of submodules $\{M_\lambda\}_{\lambda \in \Lambda}$, we equip $M[[t_I]]$ with the linear topology induced by the submodules

$$M[[t_I]]_{\lambda, \mathcal{I}} := \left\{ \sum_{\alpha \in \mathbb{N}^{(I)}} a_\alpha t_I^\alpha, a_\alpha \in M_\lambda \text{ for all } \alpha \in \mathcal{I} \right\}, \quad (5.2.1)$$

where $\lambda \in \Lambda$ and $\mathcal{I} \subset \mathbb{N}^{(I)}$ is a finite set of exponents. The convergence of a sequence for this topology means that the sequence of coefficients of each monomial converges in M . Hence, if M is complete, so is $M[[t_I]]$. If R is a topological ring and M is a topological R -module, then $R[[t_I]]$ is a topological ring and $M[[t_I]]$ is a topological $R[[t_I]]$ -module. If R is a topological \mathbb{k} -algebra, then $R[[t_I]]$ is a topological $\mathbb{k}[[t_I]]$ -algebra.

Let R be a discrete ring, let M be an R -module. The closure of the monomial ideal $(t_i, i \in I) \subset R[[t_I]]$ is the ideal $\mathcal{J} := \{f \in R[[t_I]], f|_{t_i=0} = 0\}$. If I is finite, those two ideals coincide and the topology on $M[[t_I]]$ is equivalent to the usual $(t_i, i \in I)$ -adic topology. When I is infinite, the \mathcal{J} -adic topology is finer, which means that for any finite subset $\mathcal{I} \subset \mathbb{N}^{(I)}$ there exists $n \in \mathbb{N}$ such that $\mathcal{J}^n M[[t_I]] \subset M[[t_I]]_{\mathcal{I}}$.

Remark 5.2.2. Let I be a countable set, $t_I = \{t_i, i \in I\}$ a set of indeterminates. Let R be a topological ring. Here are a few facts we will use about modules over $R[[t_I]]$.

1. If M is a free R -module, then $M[[t_I]]$ is free, and we have a canonical isomorphism $M \otimes_R R[[t_I]] \simeq M[[t_I]]$ given by $m \otimes 1 \mapsto m$.
2. If M and M' are free R -modules, there is a canonical isomorphism of $R[[t_I]]$ -modules

$$\mathrm{Hom}_{R[[t_I]]}(M[[t_I]], M'[[t_I]]) \simeq \mathrm{Hom}_R(M, M')[[t_I]].$$

3. If R is discrete and M is a free R -module, an element $\Phi \in \mathrm{End}_{R[[t_I]]}(M[[t_I]])$ is an isomorphism if and only if $\Phi|_{t_i=0} \in \mathrm{End}_R(M)$ is an isomorphism.

For (3), we may reduce to the case $\Phi = 1 + f$ with $f \in \mathcal{J} \operatorname{End}_R(M)[[t_I]]$. Then it suffices to prove that the sequence $\Psi_n := \sum_{k=0}^n (-1)^k f^k$ converges in $\operatorname{End}_R(M)[[t_I]]$. For $m \geq n$ we have $\Psi_m - \Psi_n = \sum_{k=n+1}^m (-1)^k f^k \in \mathcal{J}^{n+1} \operatorname{End}_R(M)[[t_I]]$. Since the \mathcal{J} -adic topology is finer than the topology (5.2.1), the sequence $(\Psi_n)_n$ is a Cauchy sequence. Since $\operatorname{End}_R(M)$ is a discrete space, it is complete. We conclude that $(\Psi_n)_n$ converges to an element Ψ such that $\Phi \circ \Psi = \Psi \circ \Phi = 1$.

F-bundles and (T)-structures

We equip R with the discrete topology. Given a countable set I , the derivations $\partial_{t_j}: R[[t_i, i \in I]] \rightarrow R[[t_i, i \in I]]$ are continuous and linearly independent. Hence, it makes sense to define a (partial) connection in the t -directions on a $R[[t_i, i \in I]]$ -module \mathcal{H} by specifying its action on ∂_{t_j} for all $j \in I$.

Definition 5.2.3 (F-bundle, (T)-structure). Let I be a countable set and $t_I := \{t_i, i \in I\}$.

1. An (R -linear) *F-bundle* (\mathcal{H}, ∇) over $R[[t_I]]$ is a free $R[[t_I, u]]$ -module \mathcal{H} together with a (R -linear) connection

$$\begin{aligned} \nabla_{\partial_{t_i}}: \mathcal{H} &\rightarrow u^{-1}\mathcal{H}, \\ \nabla_{\partial_u}: \mathcal{H} &\rightarrow u^{-2}\mathcal{H} \end{aligned}$$

satisfying the flatness condition.

2. An (R -linear) *(T)-structure* (\mathcal{H}, ∇) over $R[[t_I]]$ is a free $R[[t_I, u]]$ -module \mathcal{H} together with a (R -linear) connection in the t -directions

$$\nabla_{\partial_{t_i}}: \mathcal{H} \rightarrow u^{-1}\mathcal{H}$$

satisfying the flatness condition.

A *morphism of F-bundles (resp. (T)-structures)* $(f, \phi): (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}', \nabla')/R[[t_I]]$ consists of a continuous map of R -algebras $f: R[[t_I]] \rightarrow R[[t_I]]$, and a continuous map of $R[[t_I, u]]$ -modules $\phi: \mathcal{H} \rightarrow f^*\mathcal{H}' := \mathcal{H}' \otimes_{R[[t_I, u]]} R[[t_I, u]]$ such that $\phi \circ \nabla = f^*\nabla' \circ \phi$.

Underlying an F-bundle (\mathcal{H}, ∇) over $R[[t_I]]$ is an R -linear (T)-structure $(\mathcal{H}, \nabla)_0$ over $R[[t_I]]$ obtained by forgetting ∇_{∂_u} . This defines a functor $(\cdot)_0$ from R -linear F-bundles to R -linear (T)-structures.

Let (\mathcal{H}, ∇) be an R -linear (\mathbb{T}) -structure over $R[[t_I]]$. A *trivialization* of the (\mathbb{T}) -structure is a choice of isomorphism $\mathcal{H} \simeq H \otimes_R R[[t_I, u]]$, where H is a free R -module (necessarily isomorphic to $\mathcal{H}/J\mathcal{H}$, where J is the closure of the ideal (t_I, u)). Under such an isomorphism, the connection ∇ decomposes as $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}\mathbf{A}_i(t_I, u)$, with $\mathbf{A}_i \in \text{End}_R(H)[[t_I, u]]$. We refer to \mathbf{A}_i as the *connection matrix in the direction t_i* . Different choices of trivialization produce connection matrices related by the usual gauge-transformation formula.

We introduce special trivializations called framings.

Definition 5.2.4 (Framing). 1. A *framing* for an R -linear F-bundle (resp. an R -linear (\mathbb{T}) -structure) $(\mathcal{H}, \nabla)/R[[t_I]]$ is a trivialization in which the connection matrices only have negative powers of u .

2. A morphism of framed F-bundles (resp. (\mathbb{T}) -structures)

$$(f, \phi): (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}', \nabla')/R[[t_J]]$$

is *compatible with the framings* if it is constant when read in framing trivializations. More precisely, the framings $\mathcal{H} \simeq H \otimes_R R[[t_I, u]]$ and $\mathcal{H}' \simeq H' \otimes_R R[[t_J, u]]$ induce an isomorphism

$$\text{Hom}_{R[[t_I, u]]}(\mathcal{H}, f^*\mathcal{H}') \simeq \text{Hom}_R(H, H')[[t_I, u]].$$

The condition is that the image of ϕ is independent of t_I and u .

Lift of (\mathbb{T}) -structures

Recall that we have fixed R a \mathbb{k} -algebra and a \mathbb{k} -basis $\boldsymbol{\lambda} = (\lambda_k, k \in K)$ of R . Let I be a countable set, we introduce two sets of formal variables

$$t_I := \{t_i, i \in I\}, \quad \mathbf{t}_I := \{t_{i,k}, (i, k) \in I \times K\}.$$

There is a continuous morphism of R -algebras

$$\psi_{\boldsymbol{\lambda}}: R[[t_I]] \longrightarrow R[[\mathbf{t}_I]], \quad t_i \mapsto \sum_{k \in K} \lambda_k t_{i,k}. \quad (5.2.5)$$

This induces a functor $(\mathcal{H}, \nabla)/R[[t_I]] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\mathbb{k}[[\mathbf{t}_I]]$ from R -linear (\mathbb{T}) -structures to \mathbb{k} -linear (\mathbb{T}) -structures.

Lemma 5.2.6. *Let R be a \mathbb{k} -algebra with a fixed \mathbb{k} -basis $\boldsymbol{\lambda} = (\lambda_k, k \in K)$.*

1. There exists a functor $(\mathcal{H}, \nabla)/R[[t_I]] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\mathbb{k}[[\mathbf{t}_I]]$ from R -linear (T) -structures to \mathbb{k} -linear (T) -structures. It is obtained by applying the change of variable (5.2.5) and forgetting the R -linear structure.

2. Any framing for $(\mathcal{H}, \nabla)/R[[t_I]]$ induces a framing for $(\widetilde{\mathcal{H}}, \widetilde{\nabla})/\mathbb{k}[[\mathbf{t}_I]]$.

Proof. Let (\mathcal{H}, ∇) be an R -linear (T) -structure over $R[[t_I]]$. We define $\widetilde{\mathcal{H}}$ to be the $\mathbb{k}[[\mathbf{t}_I]]$ -module obtained by forgetting the R -linear structure on $\mathcal{H} \otimes_{R[[t_I]]} R[[t_I]]$.

To define the (T) -structure connection $\widetilde{\nabla}$ we fix a trivialization $\mathcal{H} \simeq H \otimes_R R[[t_I]]$. This induces an isomorphism $\widetilde{\mathcal{H}} \simeq \widetilde{H} \otimes_{\mathbb{k}} \mathbb{k}[[\mathbf{t}_I]]$, where \widetilde{H} denotes the \mathbb{k} -module obtained from H by forgetting the R -linear structure. We have a map of $\mathbb{k}[[t_I, u]]$ -algebras

$$\Psi_\lambda: \text{End}_R(H)[[t_I, u]] \longrightarrow \text{End}_{\mathbb{k}}(\widetilde{H})[[\mathbf{t}_I, u]], \quad (5.2.7)$$

given by applying the change of variable ψ_λ and forgetting the R -linear structure. Fix $(i, k) \in I \times K$, and write $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1} \mathbf{A}_i(t_I, u)$, with $\mathbf{A}_i(t_I, u) \in \text{End}_R(H)[[t_I, u]]$. We then set

$$\widetilde{\nabla}_{\partial_{t_{i,k}}} := \partial_{t_{i,k}} + u^{-1} \lambda_k \widetilde{\mathbf{A}}_i(\mathbf{t}_I, u),$$

where $\widetilde{\mathbf{A}}_i := \Psi_\lambda(\mathbf{A}_i)$. The chain rule and the flatness of ∇ imply that $\widetilde{\nabla}$ is flat, producing a \mathbb{k} -linear (T) -structure $(\widetilde{\mathcal{H}}, \widetilde{\nabla})$ over $\mathbb{k}[[\mathbf{t}_I]]$. It is easily checked that this (T) -structure is independent of the choice of trivialization for (\mathcal{H}, ∇) .

We now check functoriality. Let $(f, \phi): (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}', \nabla')/R[[s_J]]$ be a morphism of (T) -structures. Let $(\widetilde{\mathcal{H}}, \widetilde{\nabla})/\mathbb{k}[[\mathbf{t}_I]]$ and $(\widetilde{\mathcal{H}'}, \widetilde{\nabla}')/\mathbb{k}[[\mathbf{s}_J]]$ denote the induced \mathbb{k} -linear (T) -structures. There exists a unique morphism of \mathbb{k} -algebras $\widetilde{f}: \mathbb{k}[[\mathbf{s}_J]] \rightarrow \mathbb{k}[[\mathbf{t}_I]]$ making the following diagram of R -algebras commutative

$$\begin{array}{ccc} R[[s_J]] & \xrightarrow{f} & R[[t_I]] \\ \downarrow \psi'_\lambda & & \downarrow \psi_\lambda \\ R[[s_J]] & \xrightarrow{\widetilde{f} \otimes_{\mathbb{k}} 1} & R[[t_I]]. \end{array}$$

It is characterized by the relations $\psi_\lambda \circ f(s_j) = \sum_{k \in K} \lambda_k \widetilde{f}(s_{j,k})$ for all $j \in J$, and is automatically continuous. The morphism of $R[[t_I, u]]$ -modules $\phi: \mathcal{H} \rightarrow \mathcal{H}' \otimes_{R[[s_J, u]]} R[[t_I, u]]$ induces a morphism of $\mathbb{k}[[\mathbf{t}_I, u]]$ -modules $\widetilde{\phi}: \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}'} \otimes_{\mathbb{k}[[\mathbf{s}_J, u]]} \mathbb{k}[[\mathbf{t}_I, u]]$ obtained by forgetting the R -linear structure of the map of $R[[t_I, u]]$ -modules

$$\mathcal{H} \otimes_{R[[t_I, u]]} R[[\mathbf{t}_I, u]] \xrightarrow{\phi \otimes 1} \mathcal{H}' \otimes_{R[[s_J, u]]} R[[t_I, u]] \otimes_{R[[t_I, u]]} R[[\mathbf{t}_I, u]]$$

$$\simeq \left(\mathcal{H}' \otimes_{R[[s_J, u]]} R[[s_J, u]] \right) \otimes_{R[[s_J, u]]} R[[t_I, u]].$$

Forgetting the R -linear structure, the right-hand side is naturally isomorphic to $\widetilde{\mathcal{H}}' \otimes_{\mathbb{k}[[s_J, u]]} \mathbb{k}[[t_I, u]]$. Fixing trivializations of the (T)-structures, we directly check that the pair $(\widetilde{f}, \widetilde{\phi})$ is compatible with the connections. We omit the check that this is compatible with composition of morphisms. By construction, a framing trivialization for (\mathcal{H}, ∇) induces a framing trivialization for $(\widetilde{\mathcal{H}}, \widetilde{\nabla})$, concluding the proof. \square

Remark 5.2.8. The functor $(\mathcal{H}, \nabla)/R[[t_I]] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\mathbb{k}[[t_I]]$ defined above for (T)-structures is analogous to the composition of inverse image functor ψ_λ^* and the restriction of scalars from R to \mathbb{k} in the theory of D -modules.

Definition 5.2.9. An R -linear lift of a \mathbb{k} -linear (T)-structure $(\mathcal{H}, \nabla)/\mathbb{k}[[t_I]]$ is the data of an R -linear (T)-structure $(\mathcal{H}_R, \nabla_R)/R[[t_I]]$ and an isomorphism of \mathbb{k} -linear (T)-structures $\alpha: (\mathcal{H}, \nabla)_0 \xrightarrow{\sim} (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R)$.

Equivariant F-bundles

Definition 5.2.10 (Equivariant F-bundle). Let I and J be countable sets. An R -equivariant F-bundle over $\mathbb{k}[[t_I]]$ consists of the following data $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$.

1. (\mathcal{H}, ∇) is a \mathbb{k} -linear F-bundle over $\mathbb{k}[[t_I]]$, and
2. $\alpha: (\mathcal{H}, \nabla)_0 \xrightarrow{\sim} (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R)$ is an R -linear lift of the underlying (T)-structure $(\mathcal{H}, \nabla)_0$, where \mathcal{H}_R has finite rank as a $R[[t_I, u]]$ -module.

A *morphism* of equivariant F-bundles

$$\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\mathbb{k}[[t_I]] \xrightarrow{(\beta, \beta_R)} \{(\mathcal{H}', \nabla'), (\mathcal{H}'_R, \nabla'_R), \alpha'\}/\mathbb{k}[[t_J]]$$

consists of

1. a morphism of \mathbb{k} -linear F-bundles $\beta: (\mathcal{H}, \nabla) \rightarrow (\mathcal{H}', \nabla')$, and
2. a morphism of R -linear (T)-structures $\beta_R: (\mathcal{H}_R, \nabla_R) \rightarrow (\mathcal{H}'_R, \nabla'_R)$,

such that the following diagram of \mathbb{k} -linear (T)-structures commutes

$$\begin{array}{ccc} (\mathcal{H}, \nabla)_0 & \xrightarrow{\beta_0} & (\mathcal{H}', \nabla')_0 \\ \downarrow \alpha & & \downarrow \alpha' \\ (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R) & \xrightarrow{\widetilde{\beta}_R} & (\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R). \end{array} \quad (5.2.11)$$

Remark 5.2.12. 1. We identify a \mathbb{k} -linear F-bundle (\mathcal{H}, ∇) with the \mathbb{k} -equivariant F-bundle $\{(\mathcal{H}, \nabla), (\mathcal{H}, \nabla)_0, \text{id}\}$, where we choose $1 \in \mathbb{k}$ as a \mathbb{k} -basis of \mathbb{k} . This defines a fully faithful functor.

2. When $\dim_{\mathbb{k}} R = 1$, equivariant F-bundles correspond to \mathbb{k} -linear F-bundles of finite dimension and parametrized by finitely many variables, up to isomorphism. Indeed, after choosing the basis given by $1 \in R$ the change of coordinate (5.2.5) is the identity and the formal variables t_I and \mathbf{t}_I agree. Given an equivariant F-bundle $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\mathbb{k}[[t_I]]$, using α we see that \mathcal{H} has finite rank over $\mathbb{k}[[t_I]]$ because \mathcal{H}_R does, and we can define a u -direction connection on \mathcal{H}_R compatible with the (T)-structure, making α an isomorphism of F-bundles.

Definition 5.2.13. 1. A *framing* for an equivariant F-bundle $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ is the data of framings for (\mathcal{H}, ∇) and $(\mathcal{H}_R, \nabla_R)$, such that $\alpha: (\mathcal{H}, \nabla)_0 \rightarrow (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R)$ is compatible with the induced framings.

2. A morphism (β, β_R) of framed equivariant F-bundles is *compatible with the framings* if both β and β_R are compatible with the framings.

Remark 5.2.14. A morphism of equivariant F-bundles (β, β_R) is uniquely determined by β_R and the R -linear lifts through (5.2.11). Similarly, a framing of equivariant F-bundle is uniquely determined by the framing on the R -linear lift.

5.3 Unfolding of equivariant F-bundles

Recall the setting of Section 5.2, \mathbb{k} is a field of characteristic 0 and R is a \mathbb{k} -algebra of countable dimension.

Framing of (T)-structures

In this subsection, we prove that an F-bundle (\mathcal{H}, ∇) is characterized by the underlying (T)-structure and the restriction of the F-bundle to a point using framing of (T)-structures (see Lemma 5.3.1). We deduce a criterion for lifting a morphism of (T)-structures to a morphism of F-bundles. We also prove the existence of framing and extension of framing results for (T)-structures over a noetherian base.

Lemma 5.3.1. *For $k = 1, 2$, let I_k be a countable set and $(\mathcal{H}_k, \nabla_k)/R[[t_j, j \in I_k]]$ be an F-bundle. Let $(f, \Phi): (\mathcal{H}_1, \nabla_1)_0 \rightarrow (\mathcal{H}_2, \nabla_2)_0$ be a morphism of (T)-structures. Assume that the (T)-structure $(\mathcal{H}_1, \nabla_1)_0$ admits a framing.*

1. ∇_1 is uniquely determined by the underlying (T)-structure and $\nabla_{1,\partial_u}|_{t_{I_1}=0}$, and any such data determine a unique F-bundle connection extending the (T)-structure.

2. (f, Φ) is an isomorphism of F-bundles if and only if $(f, \Phi)|_{t_{I_1}=0}$ is an isomorphism of F-bundles.

Proof. For (1), fix a framing trivialization $\mathcal{H} \simeq H \otimes R[[t_i, i \in I_1, u]]$ of the underlying (T)-structure. In this trivialization, write $\nabla_{1,\partial_{t_i}} = \partial_{t_i} + u^{-1}T^i$ and $\nabla_{1,\partial_u} = \partial_u + u^{-2}U$. By assumption, the endomorphism T^i is independent of u . The flatness equations for the u -direction and t_i -direction give for all $i \in I_1$

$$\frac{\partial U}{\partial t_i} = -T^i + u \frac{\partial T^i}{\partial u} + u^{-1}[U, T^i] = -T^i + u^{-1}[U, T^i]. \quad (5.3.2)$$

Any U solving this system of equations gives rise to an F-bundle structure extending the (T)-structure. Then (1) reduces to proving that for any initial condition $U_0(u) \in \text{End}_{R[[u]]}(H[[u]])$, there exists a unique $U(t, u)$ solving (5.3.2) with $U(0, u) = U_0(u)$. Introduce the differential operators $D_i: X \mapsto \frac{\partial X}{\partial t_i} + u^{-1} \text{ad}_{T^i}(X)$, where $\text{ad}_{T^i} = [T^i, \cdot]$. Then (5.3.2) can be written as $D_i(U) = -T^i$, and we need to prove that the system is compatible for any initial condition.

Since ∇_1 is flat, by comparing degrees in u , we have for all $i, j \in I_1$

$$[T^j, T^i] = u \left(\frac{\partial T^j}{\partial t_i} - \frac{\partial T^i}{\partial t_j} \right) = 0. \quad (5.3.3)$$

It follows that

$$\begin{aligned} [D_i, D_j] &= [\partial_{t_i}, \partial_{t_j}] + u^{-1} \left(\left[\frac{\partial}{\partial t_i}, \text{ad}_{T^j} \right] + \left[\text{ad}_{T^i}, \frac{\partial}{\partial t_j} \right] \right) + u^{-2} [\text{ad}_{T^i}, \text{ad}_{T^j}] \\ &= u^{-1} \left(\text{ad}_{\partial_{t_i} T^j} - \text{ad}_{\partial_{t_j} T^i} \right) + u^{-2} \text{ad}_{[T^i, T^j]} = 0. \end{aligned}$$

Hence, by the usual theory of linear system of ODEs, the system is compatible if and only if for all $i, j \in I_1$, we have $D_i(T^j) = D_j(T^i)$. This follows from the flatness equations (5.3.3). We can thus construct a unique solution inductively on the number of variables from any initial condition. If I_1 is finite, we obtain a solution in finitely many steps. If $I_1 \simeq \mathbb{N}$ is infinite, we construct a solution in the projective limit $\varprojlim \text{End}_R(H)[[t_1, \dots, t_n, u]] = \text{End}_R(H)[[t_i, i \in I_1, u]] \simeq \text{End}_{R[[t_i, i \in I_1, u]]}(H \otimes R[[t_i, i \in I_1, u]])$. (1) is proved.

For (2), the first direction is obvious. For the converse, if $\Phi|_{t_{I_1}=0}$ is an isomorphism, then the $R[[t_{I_1}]]$ -module map Φ is an isomorphism (see Remark 5.2.2).

The connection $\nabla'_2 := \Phi^{-1} \circ f^* \nabla_2 \circ \Phi$ defines an F-bundle structure on \mathcal{H} . By assumption, the underlying (T)-structure agrees with $(\mathcal{H}, \nabla)_0$ and $\nabla_{1, \partial_u}|_{t_I=0} = \nabla'_{2, \partial_u}|_{t_I=0}$. It follows from the uniqueness in (1) that $\nabla'_2 = \nabla_1$, hence (f, Φ) is a morphism of F-bundles. \square

For (T)-structures defined over a Noetherian base $R[[t_1, \dots, t_n]]$, results from [66, §4.1] imply the existence of framing trivializations.

Proposition 5.3.4. *Let $(\mathcal{H}, \nabla)/R[[t_1, \dots, t_n]]$ be an R -linear (T)-structure. Any trivialization of $\mathcal{H}|_{t=0}$ extends uniquely to a framing of (\mathcal{H}, ∇) .*

Proof. Fix a trivialization $\mathcal{H} \simeq H \otimes R[[t_1, \dots, t_n, u]]$ lifting the trivialization of $\mathcal{H}/(t_1, \dots, t_n)\mathcal{H}$. Write the connection as $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}T^i(t, u)$. We want to show that there exists a unique gauge transformation $P(t, u) \in \text{GL}(H[[t_1, \dots, t_n, u]])$ with $P(0, u) = \text{id}$ such that $uP^{-1}\frac{\partial P}{\partial t_i} + P^{-1}T^iP$ is independent of u for all $1 \leq i \leq n$. This amounts to solving the system of PDEs ($1 \leq i \leq n$)

$$\frac{\partial P}{\partial t_i} = u^{-1}(-T^iP + PP_0^{-1}T_{-1}^iP_0),$$

where $P_0 = P(t, 0)$ and $T_{-1}^i = T^i(t, 0)$, with the initial condition $P(0, u) = \text{id}$. Uniqueness is clear, as the system provides recursive relations for the coefficients of P , and existence follows from [66, Lemmas 4.17, 4.18, 4.20]. The arguments there still apply, because we assume that R contains \mathbb{Q} . \square

Fix I a finite set, let $(\mathcal{H}, \nabla)/R[[t_I]]$ be a (T)-structure of finite rank $n \in \mathbb{N}$. Let $v_1 \in \mathcal{H}/(t_I, u)\mathcal{H}$. Any choice (h_1, \dots, h_n) of $R[[t_I, u]]$ -basis for \mathcal{H} provides a trivialization through the isomorphisms

$$\mathcal{H} \simeq \bigoplus_{1 \leq i \leq n} R[[t_I, u]]h_i \simeq R^{\oplus n} \otimes_R R[[t_I, u]].$$

We call a basis (h_1, \dots, h_n) *good for (\mathcal{H}, ∇)* if it induces a framing trivialization. We say that it *extends v_1* if h_1 is a lift of v_1 . Proposition 5.3.4 implies that any basis of $\mathcal{H}/(t_I, u)\mathcal{H}$ lifts uniquely to a good basis of (\mathcal{H}, ∇) . More generally, we have the following.

Lemma 5.3.5. *Let I and J be finite sets. Let $(f, \Phi): (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}', \nabla')/R[[t_J]]$ be a morphism of finite rank (T)-structures. Assume that $\Phi|_{t_I=0}$ is an isomorphism.*

1. Any good basis (h_1, \dots, h_n) of (\mathcal{H}, ∇) induces a unique good basis (h'_1, \dots, h'_n) of (\mathcal{H}', ∇') such that $\Phi(h_k) = f^*(h'_k)$ for all $1 \leq k \leq n$.

2. Φ is uniquely determined by its restriction to $\mathcal{H}|_{t_I=0}$.

Proof. The assumptions imply that Φ is an isomorphism of $R[[t_I, u]]$ -modules. In particular, we have isomorphisms of $R[[u]]$ -modules

$$\mathcal{H}/(t_I)\mathcal{H} \simeq f^*\mathcal{H}'/(t_I)f^*\mathcal{H}' \simeq \mathcal{H}'/(t_J)\mathcal{H}'. \quad (5.3.6)$$

A good basis (h'_1, \dots, h'_n) for (\mathcal{H}', ∇') is uniquely characterized by its projection to $\mathcal{H}'/(t_J)\mathcal{H}'$. This value is uniquely specified by the condition $\Phi(h_k) = f^*(h'_k)$ using the isomorphism (5.3.6), which proves (1).

For (2), we note that Φ is uniquely determined by the image of a good basis (h_1, \dots, h_n) of (\mathcal{H}, ∇) . By (1), the image $(\Phi(h_1), \dots, \Phi(h_n))$ is a good basis for $f^*(\mathcal{H}', \nabla')$. In particular, it is uniquely determined by its restriction to $t_I = 0$, which only depends on $\Phi|_{t_I=0}$. The proof is complete. \square

Formal Hertling-Manin unfolding theorem

In this subsection, we prove an analogue of the Hertling-Manin unfolding theorem for (TE)-structures (see [64, Theorem 2.5]) for formal R -linear F-bundles and (T)-structures.

Definition 5.3.7 (Unfolding of (T)-structure, F-bundle). Let R be a \mathbb{k} -algebra, I and J countable sets. Let $(\mathcal{H}, \nabla)/R[[t_I]]$ be an R -linear (T)-structure (resp. F-bundle). An *unfolding* of (\mathcal{H}, ∇) is a morphism of (T)-structures (resp. F-bundles) $(i, \phi): (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}', \nabla')/R[[t_J]]$, where

1. $I \subset J$ and $i: R[[t_J]] \rightarrow R[[t_I]]$ is the quotient by the closure of the ideal $(t_j, j \in J \setminus I)$, and
2. $\phi: \mathcal{H} \rightarrow i^*\mathcal{H}'$ is an isomorphism of $R[[t_I, u]]$ -modules.

A *morphism between two unfoldings* $\iota_k: (\mathcal{H}, \nabla) \rightarrow (\mathcal{H}_k, \nabla_k)$ for $k = 1, 2$, is a morphism of (T)-structures (resp. F-bundles) $(f, \psi): (\mathcal{H}_2, \nabla_2) \rightarrow (\mathcal{H}_1, \nabla_1)$ such that ψ is an isomorphism and the following diagram commutes

$$\begin{array}{ccc} & (\mathcal{H}, \nabla) & \\ \iota_2 \swarrow & & \searrow \iota_1 \\ (\mathcal{H}_2, \nabla_2) & \xrightarrow{(f, \psi)} & (\mathcal{H}_1, \nabla_1). \end{array}$$

Remark 5.3.8. In the above commutative diagram, assume that $(\mathcal{H}_k, \nabla_k)$ depends on finitely many variables indexed by a finite set J_k for $k = 1, 2$, and write

$$\iota_k = (i_k, \phi_k) : (\mathcal{H}, \nabla)/R[[t_I]] \rightarrow (\mathcal{H}_2, \nabla_2)/R[[t_{J_k}]].$$

Then for any two morphisms (f, ψ_k) , $k = 1, 2$, between the unfoldings ι_2 and ι_1 , we have $\psi_1 = \psi_2$. In other words, the morphism on the base f determines the bundle map. Indeed, the commutativity of the diagram implies that $i_2^* \psi_k \circ \phi_2 = \phi_1$. This determines $\psi_k|_{t_{J_2}=0} = \phi_1 \circ \phi_2^{-1}|_{t_{J_2}=0}$. By Lemma 5.3.5, ψ_k is uniquely determined by $\psi_k|_{t_{J_2}=0}$, thus $\psi_1 = \psi_2$.

Remark 5.3.9. When I and J are finite, given an unfolding of R -linear (T) -structures

$$(i, \phi) : (\mathcal{H}, \nabla)/R[[t_I]] \longrightarrow (\mathcal{H}', \nabla')/R[[t_J]],$$

any framing for $(\mathcal{H}, \nabla)/R[[t_I]]$ induces a unique framing for $(\mathcal{H}', \nabla')/R[[t_J]]$, and vice versa. Indeed, ϕ takes the framing trivialization for (\mathcal{H}, ∇) to a framing trivialization for $i^*(\mathcal{H}', \nabla')$, which is uniquely determined by its restriction to the fiber $i^*\mathcal{H}'|_{t_I=0} = \mathcal{H}'|_{t_J=0}$. We can extend this to a framing trivialization for (\mathcal{H}', ∇') by Proposition 5.3.4.

Lemma 5.3.10. *For $k = 1, 2$, let I_k be countable sets, and let*

$$(f, \Phi) : (\mathcal{H}_1, \nabla_1)/R[[t_1]] \rightarrow (\mathcal{H}_2, \nabla_2)/R[[t_2]]$$

be an unfolding of R -linear (T) -structures. Assume the (T) -structure $(\mathcal{H}_2, \nabla_2)$ admits a framing. Given an F -bundle structure $(\mathcal{H}_1, \nabla_1^F)$ on $(\mathcal{H}_1, \nabla_1)$, there exists a unique F -bundle structure $(\mathcal{H}_2, \nabla_2^F)$ on $(\mathcal{H}_2, \nabla_2)$ such that (f, Φ) is an unfolding of F -bundles.

Proof. Since (f, Φ) is an unfolding of (T) -structures, we have isomorphisms of $R[[u]]$ -modules

$$\mathcal{H}_1|_{t_1=0} \simeq f^*\mathcal{H}_2|_{t_1=0} \simeq \mathcal{H}_2|_{t_2=0}. \quad (5.3.11)$$

Under this isomorphism, the restriction $\nabla_1^F|_{t_1=0}$ produces a F -bundle connection on $\mathcal{H}_2|_{t_2=0}$. Since the latter admits a framing, applying Lemma 5.3.1(1) we obtain a unique F -bundle $(\mathcal{H}_2, \nabla_2^F)$ extending the (T) -structure $(\mathcal{H}_2, \nabla_2)$. We now check that (f, Φ) is a morphism of F -bundles. By construction, the connections $f^*\nabla_2^F$ and $\Phi \circ \nabla_1^F \circ \Phi^{-1}$ are F -bundle connections on $f^*\nabla_2$ which coincide at $t_1 = 0$, and with the same underlying (T) -structures. The framing

for $(\mathcal{H}_2, \nabla_2)$ induces a framing on $f^*(\mathcal{H}_2, \nabla_2)$, as can be seen by fixing a framing trivialization of $(\mathcal{H}_2, \nabla_2)$ and pulling it back under f . Then, it follows from Lemma 5.3.1(1) that those two F-bundle structures agree. Hence, (f, Φ) is a morphism of F-bundles.

For uniqueness, note that the F-bundle connection ∇_2^F is uniquely determined by its restriction to $t_2 = 0$ since $(\mathcal{H}_2, \nabla_2)$ admits a framing, and that $\nabla_2^F|_{t_2=0}$ is uniquely specified by $\nabla_1^F|_{t_1=0}$ through the isomorphisms (5.3.11). \square

For an R -linear (T)-structure $(\mathcal{H}, \nabla)/R[[t_I]]$, there is a morphism of R -modules [66, Remark 2.3]

$$\begin{aligned} \mu: \bigoplus_{i \in I} R\partial_{t_i} &\longrightarrow \text{End}_R(H), \\ \partial_{t_i} &\longmapsto \nabla_{u\partial_{t_i}}|_{u=0, t_I=0}, \end{aligned} \quad (5.3.12)$$

where $H := \mathcal{H}/J\mathcal{H}$ with J the closure of the ideal (t_I, u) . For each $v \in H$ we obtain an evaluation map of R -modules:

$$\begin{aligned} \mu_v: \bigoplus_{i \in I} R\partial_{t_i} &\longrightarrow H, \\ \xi &\longmapsto \mu(\xi)(v). \end{aligned} \quad (5.3.13)$$

Furthermore, if (\mathcal{H}, ∇) is an F-bundle, we also have a residue endomorphism in the u -direction $K := [u^2\nabla_{\partial_u}]|_{u=t=0} \in \text{End}_R(H)$. We introduce the notion of maximal (T)-structure and maximal F-bundle, analogous to [66, Definition 2.6].

Definition 5.3.14 (Maximal (T)-structure, maximal F-bundle). Let R be a \mathbb{k} -algebra, I a countable set, and $J \subset R[[t_I, u]]$ the closure of the ideal (t_I, u) . An R -linear (T)-structure, or F-bundle, $(\mathcal{H}, \nabla)/R[[t_I]]$ is *maximal* if there exists $v \in \mathcal{H}/J\mathcal{H}$ such that the map μ_v is an isomorphism. We call such a v a *cyclic vector*.

The Hertling-Manin unfolding theorem guarantees the existence and uniqueness of a maximal unfolding under certain conditions, which we introduce in the next definition.

Definition 5.3.15. Let I be a countable set, $(\mathcal{H}, \nabla)/R[[t_I]]$ an R -linear (T)-structure (resp. F-bundle), and $J \subset R[[t_I, u]]$ the closure of the ideal (t_I, u) . We define the following conditions on an element $v \in H := \mathcal{H}/J\mathcal{H}$:

- (IC) The map μ_v in (5.3.13) is injective.
- (GC) The orbit of v under the action of the subalgebra $R[\text{im } \mu] \subset \text{End}_R(H)$ (resp. $R[\text{im } \mu, K] \subset \text{End}_R(H)$) defined by evaluation on v is H .
- (GC') The condition (GC) is satisfied after base change to $\text{Frac}(R)$.

If v satisfies (GC), we say that v is a *generating vector* for (\mathcal{H}, ∇) .

The following lemma provides a construction of unfoldings under the (GC) condition. It is analogous to [64, Lemma 2.9], except that we use framings of (T)-structures to avoid the analytic argument used there.

Lemma 5.3.16. *Let $(\mathcal{H}^{(0)}, \nabla^{(0)})/R[[t_1, \dots, t_d]]$ be an F -bundle of rank n satisfying the (GC) condition, let $v_1 \in \mathcal{H}^{(0)}/(t_I, u)\mathcal{H}^{(0)}$ be a generating vector. Let $(h_1^{(0)}, \dots, h_n^{(0)})$ be a good basis of $(\mathcal{H}^{(0)}, \nabla^{(0)})$ extending v_1 . Fix $\ell \geq 1$ and let $f_1, \dots, f_n \in R[[t_1, \dots, t_d, s_1, \dots, s_\ell]]$ whose restrictions to $s = 0$ are 0.*

Then there exists an unfolding $\iota: (\mathcal{H}^{(0)}, \nabla^{(0)})/R[[t_1, \dots, t_d]] \rightarrow (\mathcal{H}, \nabla)/R[[t_1, \dots, t_d, s_1, \dots, s_\ell]]$ such that, if (h_1, \dots, h_n) denotes the good basis of (\mathcal{H}, ∇) induced from $(h_1^{(0)}, \dots, h_n^{(0)})$ (see Lemma 5.3.5), we have for $1 \leq j \leq \ell$

$$[u\nabla_{\partial_{s_j}}]|_{u=0}(h_1|_{u=0}) = \sum_{i=1}^n \frac{\partial f_i}{\partial s_j} h_i|_{u=0}. \quad (5.3.17)$$

Any two unfoldings satisfying (5.3.17) are isomorphic under a morphism (id, ψ) , where ψ identifies the canonical extensions of the good basis $(h_i^{(0)})_{1 \leq i \leq n}$.

Proof. Set $t := \{t_1, \dots, t_d\}$. We consider the case $\ell = 1$, as we can always decompose an unfolding as a sequence of 1-dimensional unfoldings.

Let $H := R^{\oplus n}$. The good basis $(h_i^{(0)})_{1 \leq i \leq n}$ provides an isomorphism $\phi: \mathcal{H}^{(0)} \xrightarrow{\sim} H \otimes_R R[[t, u]]$. Let $\mathcal{H} := H \otimes_R R[[t, s, u]]$. We first prove that there exists a unique connection ∇ on \mathcal{H} such that $\iota = (i, \phi): (\mathcal{H}^{(0)}, \nabla^{(0)}) \rightarrow (\mathcal{H}, \nabla)$ is an unfolding satisfying (5.3.17). This is equivalent to constructing unique matrices $T^i(t, s), S(t, s), U_k(t, s) \in \text{Mat}(n \times n, R[[t, s]])$ such that the connection form

$$\Omega := \frac{1}{u} \sum_{i=1}^n T^i(t, s) dt_i + \frac{1}{u} S(t, s) ds + \frac{1}{u^2} \sum_{k \geq 0} U_{k-2}(t, s) u^k du$$

satisfies:

- (a) the flatness equation $d\Omega + \Omega \wedge \Omega = 0$,
- (b) $T^i(t, 0)$ and $U_k(t, 0)$ coincide with the connection matrix of $\nabla^{(0)}$ in $(h_i^{(0)})_{1 \leq i \leq n}$,
and
- (c) $S(t, s)e_1 = \sum_{i=1}^n \frac{\partial f_i}{\partial s} e_i$, where $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathcal{H}/(u)\mathcal{H} = R^{\oplus n} \otimes_R R[[t, s]]$.

We further decompose the matrices into powers of s , and write $T_w^i(t)$ (resp. $S_w(t)$, $U_{k,w}(t)$) for the coefficient of s^w in $T^i(t, s)$ (resp. $S(t, s)$, $U_k(t, s)$). We will construct the matrices order by order in s .

Condition (a) is equivalent to the following system of equations:

$$[S, T^i] = 0 \quad (5.3.18)$$

$$[S, U_{-2}] = 0 \quad (5.3.19)$$

$$\partial_s T^i = \partial_{t_i} S \quad (5.3.20)$$

$$\partial_s U_{-2} = [U_{-1}, S] - S \quad (5.3.21)$$

$$\partial_s U_k = [U_{k+1}, S] \quad (k \geq -1) \quad (5.3.22)$$

$$[T^i, T^j] = 0 \quad (5.3.23)$$

$$[U_{-2}, T^i] = 0 \quad (5.3.24)$$

$$\partial_{t_i} T^j = \partial_{t_j} T^i \quad (5.3.25)$$

$$\partial_{t_i} U_{-2} = [U_{-1}, T^i] - T^i \quad (5.3.26)$$

$$\partial_{t_i} U_k = [U_{k+1}, T^i] \quad (k \geq -1) \quad (5.3.27)$$

We prove by induction on $m \in \mathbb{N}$ that there exists unique matrices $T_w^i(t)$ and $U_{k,w}(t)$ for $0 \leq w \leq m$ and $S_w(t)$ for $0 \leq w \leq m - 1$ such that the equations (5.3.18) through (5.3.22) are satisfied modulo s^m , the equations (5.3.23) through (5.3.27) are satisfied modulo s^{m+1} , condition (b) is satisfied and condition (c) is satisfied modulo s^m .

For $m = 0$, condition (b) provides the matrices $T_0^i(t)$ and $U_{k,0}(t)$, and the equations (5.3.23)-(5.3.27) are satisfied modulo s by flatness of $\nabla^{(0)}$.

Now assume the induction carried out until step m , we prove step $m + 1$. We only need to construct the matrices T_{m+1}^i , $U_{k,m+1}$ and S_m so that the various conditions of the induction are satisfied. The construction of a unique matrix S_m such that (5.3.18), (5.3.19) and condition (c) are satisfied modulo s^{m+1} is

as in (i) of the proof of [64, Lemma 2.9]. The matrices T_{m+1}^i and $U_{k,m+1}$ are uniquely determined by imposing equations (5.3.20)-(5.3.22) modulo s^{m+1} .

It remains to check that equations (5.3.23)-(5.3.27) hold modulo s^{m+2} , assuming that equations (5.3.18)-(5.3.27) hold modulo s^{m+1} . Since they hold at $s = 0$, we simply check that the s -derivative of these equations is zero modulo s^{m+1} .

For (5.3.23) we have modulo s^{m+1}

$$\begin{aligned}\partial_s[T^i, T^j] &= [\partial_s T^i, T^j] + [T^i, \partial_s T^j] \\ &= [\partial_{t_i} S, T^j] + [T^i, \partial_{t_j} S] \\ &= -[S, \partial_{t_i} T^j] - [\partial_{t_j} T^i, S] \\ &= 0.\end{aligned}$$

For (5.3.24) we have modulo s^{m+1}

$$\begin{aligned}\partial_s[U_{-2}, T^i] &= [\partial_s U_{-2}, T^i] + [U_{-2}, \partial_s T^i] \\ &= [[U_{-1}, S], T^i] + [U_{-2}, \partial_{t_i} S] \\ &= [[U_{-1}, S], T^i] - [\partial_{t_i} U_{-2}, S] \\ &= [[U_{-1}, S], T^i] - [[U_{-1}, T^i], S] \\ &= 0.\end{aligned}$$

For (5.3.25) we have modulo s^{m+1}

$$\partial_s(\partial_{t_i} T^j - \partial_{t_j} T^i) = \partial_{t_i} \partial_s T^j - \partial_{t_j} \partial_s T^i = \partial_{t_i} \partial_{t_j} S - \partial_{t_j} \partial_{t_i} S = 0.$$

For (5.3.26) we have modulo s^{m+1}

$$\begin{aligned}\partial_s(\partial_{t_i} U_{-2} + T^i + [T^i, U_{-1}]) &= \partial_{t_i} [U_{-1}, S] - \partial_{t_i} S + \partial_{t_i} S + [\partial_s T^i, U_{-1}] + [T^i \partial_s U_{-1}] \\ &= [\partial_{t_i} U_{-1}, S] + [U_{-1}, \partial_{t_i} S] + [\partial_{t_i} S, U_{-1}] + [T^i, [U_0, S]] \\ &= [[U_0, T^i], S] + [T^i, [U_0, S]] \\ &= 0,\end{aligned}$$

where on the first line we used (5.3.20) and (5.3.21), on the second line we used (5.3.20) and (5.3.22), on the third line we used (5.3.27), and on the last line we used the Jacobi identity and (5.3.18). For (5.3.27) we have modulo s^{m+1}

$$\begin{aligned}\partial_s(\partial_{t_i} U_k + [T^i, U_{k+1}]) &= \partial_{t_i} [U_{k+1}, S] + [\partial_s T^i, U_{k+1}] + [T^i, \partial_s U_{k+1}] \\ &= [\partial_{t_i} U_{k+1}, S] + [T^i, \partial_s U_{k+1}] \\ &= [[U_{k+2}, T^i], S] + [T^i, [U_{k+2}, S]]\end{aligned}$$

$$= 0.$$

This finishes the induction step, and proves the existence.

For uniqueness up to isomorphism, assume that $\iota': (\mathcal{H}^{(0)}, \nabla^{(0)}) \rightarrow (\mathcal{H}', \nabla')$ is another unfolding satisfying (5.3.17). We prove that it is isomorphic to the unfolding (\mathcal{H}, ∇) constructed above. Let $\psi: \mathcal{H} \rightarrow \mathcal{H}'$ denote the $R[[t, s, u]]$ -module isomorphism obtained by identifying the good bases obtained from $(h_i^{(0)})_{1 \leq i \leq n}$. Then the connection form of $\psi^{-1} \circ \nabla' \circ \psi$ in the trivialization of \mathcal{H} given by (e_1, \dots, e_n) satisfies conditions (a), (b) and (c) above. Thus $\psi^{-1} \circ \nabla' \circ \psi = \nabla$, and we conclude that $(\text{id } \psi): (\mathcal{H}, \nabla) \rightarrow (\mathcal{H}', \nabla')$ is an isomorphism of unfoldings. \square

Lemma 5.3.16 says that under the (GC) assumption, an unfolding $\iota: (\mathcal{H}^{(0)}, \nabla^{(0)}) \rightarrow (\mathcal{H}, \nabla)$ is uniquely determined up to isomorphism by the choice of a good basis (h_1, \dots, h_n) extending a cyclic vector, and the action of the connection on h_1 .

Theorem 5.3.28 (Hertling-Manin for F-bundles). *Let R be an integral domain containing \mathbb{Q} . Let $(\mathcal{H}, \nabla)/R[[t_1, \dots, t_d]]$ be a finite rank F-bundle. Let $v \in \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$.*

1. *If v satisfies (IC), (GC) and $\text{coker } \mu_v$ is free, then there exists a maximal unfolding with cyclic vector induced from v .*
2. *If v satisfies (GC'), then any two maximal unfoldings of (\mathcal{H}, ∇) with cyclic vector induced from v are isomorphic under a unique isomorphism.*

Furthermore, any framing for (\mathcal{H}, ∇) induces a unique framing on a maximal unfolding.

Proof. Let n denote the rank of \mathcal{H} , and $\ell := n - d$. We assume $\ell \geq 0$, as otherwise the evaluation map μ_v cannot be injective and a maximal unfolding of (\mathcal{H}, ∇) does not exist. Write $t = \{t_1, \dots, t_d\}$ and $s = \{s_1, \dots, s_\ell\}$. Fix a good basis (h_1, \dots, h_n) for (\mathcal{H}, ∇) extending v , i.e. with $h_1|_{t=u=0} = v$.

For (1), let $N \in \text{Mat}(n \times d, R)$ denote the matrix of the evaluation map μ_v . Let $f_1, \dots, f_n \in R[[t, s]]$ with $f_i(t, 0) = 0$. Applying Lemma 5.3.16 we obtain an unfolding $\iota: (\mathcal{H}, \nabla)/R[[t]] \rightarrow (\mathcal{H}', \nabla')/R[[t, s]]$. Let $v' \in \mathcal{H}'/(t, s, u)\mathcal{H}'$

corresponding to v , the matrix of the evaluation map μ_v in the good basis obtained from $(h_i)_{1 \leq i \leq n}$ is

$$\left(N \left(\frac{\partial f_i}{\partial s_j} \Big|_{t=s=0} \right)_{1 \leq i \leq n, 1 \leq j \leq \ell} \right) \in \text{Mat}(n \times n, R). \quad (5.3.29)$$

Since v satisfies (IC), the columns of N form a basis of $\text{im } \mu_v \subset \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$. Since $\text{coker } \mu_v$ is free, by the basis extension theorem, we can extend this basis to a basis of $\mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$ by adding elements $\{v_1, \dots, v_\ell\}$. Any choice (f_1, \dots, f_n) such that the vector $(\frac{\partial f_i}{\partial s_k} \Big|_{t=s=0})_{1 \leq i \leq n}$ corresponds to v_k for all $1 \leq k \leq \ell$ gives rise to a maximal unfolding, since the columns of (5.3.29) then form a basis of H . This proves (1).

We now prove (2). For $k = 1, 2$, let $\iota_k = (i_k, \phi_k): (\mathcal{H}, \nabla) \rightarrow (\mathcal{H}'_k, \nabla'_k)$ be a maximal unfolding. In the good bases induced from $(h_i)_{1 \leq i \leq n}$ the 1-forms defining the (T)-structures are closed by (5.3.25), hence can be written as $u^{-1}dA_k$ for a unique $A_k \in \text{Mat}(n \times n, R)[[t, s]]$ satisfying $A_k(0, 0) = 0$. The first column of A_k provides n elements of $R[[t, s]]$ that define a map of R -algebras $\psi_k: R[[t, s]] \rightarrow R[[t, s]]$. Since the unfoldings are assumed to be maximal, $d\psi_k|_{t=s=0}$ is an isomorphism. This follows from the fact that, by construction, its matrix in the basis $(dt_1, \dots, dt_d, ds_1, \dots, ds_\ell)$ coincides with the matrix of the evaluation map for (\mathcal{H}', ∇') . We deduce that $\psi_k \in \text{Aut}_R(R[[t, s]])$. If $(f, j): (\mathcal{H}_1, \nabla_1) \rightarrow (\mathcal{H}_2, \nabla_2)$ is an isomorphism of unfoldings, then $f^*dA_2 = dA_1$ which implies $A_2 \circ f = A_1$. In particular $\psi_2 \circ f = \psi_1$, and this determines f uniquely, since ψ_2 is an isomorphism. In turn, this determines j uniquely by Remark 5.3.8. Conversely, let $f = \psi_2^{-1} \circ \psi_1$ and define $j: \mathcal{H}'_1 \rightarrow f^*\mathcal{H}'_2$ by identifying the good bases induced from $(h_i)_{1 \leq i \leq n}$. In particular, we have $d\psi_1 = d\psi_2 \circ df$, therefore $f^*(\mathcal{H}'_2, \nabla'_2)$ is a maximal unfolding whose action on the cyclic section that extends h_1 agrees with that of $(\mathcal{H}'_1, \nabla'_1)$. After base changing to $\text{Frac}(R)$, the (GC) condition is satisfied. It follows from Lemma 5.3.16 that (f, j) is compatible with the connections and is an isomorphism of unfoldings after base changing to $\text{Frac}(R)$. But f (resp. j) is invertible over R (resp. $R[[t, s, u]]$) by construction, so the unfoldings are isomorphic over R .

The last claim follows from the extension of framing result in [66, Theorem 1.3]. The proof is complete. \square

Corollary 5.3.30 (Hertling-Manin for (T)-structures). *Let R be an integral domain containing \mathbb{Q} . Let $(\mathcal{H}, \nabla)/R[[t_1, \dots, t_d]]$ be a finite rank (T)-structure. Let $v \in \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$.*

1. If v satisfies (IC), (GC) and $\text{coker } \mu_v$ is free, then there exists a maximal unfolding with cyclic vector induced from v .

2. If v satisfies (GC'), then any two maximal unfoldings of (\mathcal{H}, ∇) with cyclic vector induced from v are isomorphic under a unique isomorphism.

Proof. Let n denote the rank of \mathcal{H} . Write $t = \{t_1, \dots, t_d\}$ and $s = \{s_1, \dots, s_{n-d}\}$. We choose an F-bundle structure $(\mathcal{H}, \nabla^F)/R[[t]]$ lifting the (T)-structure (\mathcal{H}, ∇) . Then (\mathcal{H}, ∇^F) satisfies the conditions of Theorem 5.3.28(1), producing a maximal unfolding of F-bundle. Since being maximal is a property of the (T)-structure, the unfolding of the underlying (T)-structure is maximal, proving (1).

For (2), let $\iota_1: (\mathcal{H}, \nabla)/R[[t]] \rightarrow (\mathcal{H}_1, \nabla_1)/R[[t, s]]$ and $\iota_2: (\mathcal{H}, \nabla)/R[[t]] \rightarrow (\mathcal{H}_2, \nabla_2)/R[[t, s]]$ be two maximal unfoldings of (T)-structures, with cyclic vector induced from v . Since the base of (\mathcal{H}, ∇) has finitely many variables, it follows from Proposition 5.3.4 that it admits a framing. This induces a framing on any unfolding by Remark 5.3.9. Thus, we can apply Lemma 5.3.10 and extend the two unfoldings ι_1 and ι_2 uniquely to maximal unfoldings of the F-bundle (\mathcal{H}, ∇^F) . We conclude from Theorem 5.3.28 that they are isomorphic under a unique isomorphism, hence the same holds for the underlying unfoldings of (T)-structures. This concludes the proof. \square

Remark 5.3.31 (Existence when R is not a field). Let R be an integral domain, $(\mathcal{H}, \nabla)/R[[t_1, \dots, t_d]]$ be a finite rank F-bundle, and $v \in H := \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$.

1. If v only satisfies (IC) and (GC'), we know that a maximal unfolding exists after base change to $\text{Frac}(R)$. In fact, the maximal unfolding is defined over any localization R' of R such that $\text{coker } \mu_v \otimes_R R'$ is a free module, by Theorem 5.3.28(1).

2. Let $(\mathcal{H}, \nabla) \rightarrow (\mathcal{H}', \nabla')$ be an unfolding. We obtain maps μ and μ' as in (5.3.12). Let $\mathcal{A} := R[\text{im } \mu]$ and $\mathcal{A}' := R[\text{im } \mu']$ denote the associated commuting subalgebras of $\text{End}_R(H)$. We have $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{C}(\mathcal{A}') \subset \mathcal{C}(\mathcal{A})$, where $\mathcal{C}(\cdot)$ denotes the commutant algebra. Let $\tilde{\mu}_v: \mathcal{A} \rightarrow H$ and $\tilde{\mu}'_v: \mathcal{A}' \rightarrow H$ denote the

evaluation on v . From the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{A}'/\mathcal{A} \longrightarrow 0 \\ & & \downarrow \tilde{\mu}_v & & \downarrow \tilde{\mu}'_v & & \downarrow \\ 0 & \longrightarrow & H & \xrightarrow{\text{id}} & H & \longrightarrow & 0, \end{array}$$

we obtain the long exact sequence

$$0 \longrightarrow \ker \tilde{\mu}_v \longrightarrow \ker \tilde{\mu}'_v \longrightarrow \mathcal{A}'/\mathcal{A} \longrightarrow \text{coker } \tilde{\mu}_v \longrightarrow \text{coker } \tilde{\mu}'_v \longrightarrow 0.$$

If the unfolding is maximal, we have $\mathcal{A}' = \text{im } \mu'$ and $\tilde{\mu}'_v$ is an isomorphism. We deduce that $\ker \tilde{\mu}_v = 0$ and $\text{coker } \tilde{\mu}_v \simeq \mathcal{A}'/\mathcal{A}$. Then, v satisfies the (IC) condition but not necessarily the (GC) condition. In the special case when $\mathcal{A} = \mathcal{C}(\mathcal{A})$, a maximal unfolding exists if and only if v satisfies (IC) and (GC).

This is illustrated in Example 5.3.32.

Example 5.3.32. Let $R = \mathbb{k}[[\lambda_1, \lambda_2]]$, $H = R^{\oplus 3}$ and $\mathcal{H} = H \otimes_R R[[t_1, t_2]]$. Let (e_1, e_2, e_3) denote the canonical basis of H . We consider the matrices

$$A = \text{Id}_3, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}, \quad C = B^2 = \begin{pmatrix} 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \\ \lambda_1 \lambda_2 & 0 & 0 \end{pmatrix}.$$

Assume ∇ is an F-bundle connection on \mathcal{H} such that $\mu(\partial_{t_1}) = A$ and $\mu(\partial_{t_2}) = B$. We have $R[\text{im } \mu] = RA \oplus RB \oplus RC$ and $R[\text{im } \mu] = \mathcal{C}(R[\text{im } \mu])$. It follows from Remark 5.3.31(2) that there exists a maximal unfolding with cyclic vector $v = \alpha e_1 + \beta e_2 + \gamma e_3$ if and only if v satisfies (IC) and (GC). The matrix of the evaluation map $\tilde{\mu}_v: R[\text{im } \mu] \rightarrow H$ with respect to the bases (A, B, C) and (e_1, e_2, e_3) is

$$\tilde{\mu}_v = \begin{pmatrix} \alpha & \gamma & \lambda_2 \beta \\ \beta & \lambda_1 \alpha & \lambda_1 \gamma \\ \gamma & \lambda_2 \beta & \lambda_1 \lambda_2 \alpha \end{pmatrix},$$

whose determinant is $\lambda_1^2 \lambda_2 \alpha^3 + \lambda_2^2 \beta^3 + \lambda_1 \gamma^3 - 3 \lambda_1 \lambda_2 \alpha \beta \gamma$. The vector v satisfies (IC) and (GC) if and only if this determinant is invertible. For $v = e_3$, this determinant is λ_1 and we conclude that the associated maximal unfolding is defined over $\mathbb{k}[[\lambda_1, \lambda_2]][[\lambda_1^{-1}]]$. For $v = e_2$, this determinant is λ_2^2 and the associated maximal unfolding is defined over $\mathbb{k}[[\lambda_1, \lambda_2]][[(\lambda_2^2)^{-1}]]$.

Unfolding theorem for equivariant F-bundles

In this subsection, we prove the unfolding theorem for equivariant F-bundles. The strategy is to unfold the R -linear (T)-structure using Corollary 5.3.30, and then extend it in the u -direction using Lemma 5.3.1.

Definition 5.3.33. Let R be a \mathbb{k} -algebra, and let I be a countable set.

1. An *unfolding of \mathbb{k} -linear equivariant F-bundle* $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\mathbb{k}[[t_I]]$ is a morphism of equivariant F-bundles (ι, ι_R) such that ι is an unfolding of \mathbb{k} -linear F-bundles and ι_R is an unfolding of R -linear (T)-structure. In particular, ι and ι_R are compatible with the R -linear lifts as in (5.2.11).
2. A *morphism of unfoldings* is a morphism (β, β_R) of equivariant F-bundles such that both β and β_R are morphisms of unfoldings. In particular, (β, β_R) commutes with the unfolding maps.
3. An equivariant F-bundle is *maximal* if the underlying R -linear (T)-structure is maximal.

Lemma 5.3.34. *Let I be a countable set. Let $(\mathcal{H}, \nabla)/R[[t_I]]$ be an F-bundle. A framing for the (T)-structure $(\mathcal{H}, \nabla)_0$ is a framing for the F-bundle if and only if it restricts to a framing of F-bundles at $t_I = 0$.*

Proof. The framing provides a trivialization $\mathcal{H} \simeq H \otimes_R R[[t_I, u]]$. Write $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}T_i(t)$ and $\nabla_{\partial_u} = \partial_u + u^{-2}U(t, u)$. By Lemma 5.3.1(1), $U(t, u)$ is uniquely determined by the system of differential equations (5.3.2) and the initial condition $U(0, u)$. Write $U(t, u) = \sum_{k \geq 0} U_{k-2}(t)u^k$. The differential equation implies for all $k \geq 0$

$$\frac{\partial U_k}{\partial t_i} = -[T_i, U_{k+1}].$$

Since we have the initial condition $U_k(0) = 0$, we deduce that $U_k(t) = 0$ for all $k \geq 0$ by applying [66, Lemma 4.8(1)] inductively on the number of variables. The reverse direction is obvious. \square

Proposition 5.3.35. *Let I and J be finite sets, and R be a \mathbb{k} -algebra without zero divisors equipped with a fixed basis. Let $\mathcal{F} \rightarrow \mathcal{F}'$ be an unfolding of \mathbb{k} -linear equivariant F-bundles. Then any framing on \mathcal{F} extends uniquely to a framing on \mathcal{F}' .*

Proof. Uniqueness follows from the uniqueness of extension of framing for $(\mathcal{H}_R, \nabla_R)$, together with Remark 5.2.14.

We now prove the existence part. Assume \mathcal{F} admits a framing and

$$(\beta, \beta_R): \mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\mathbb{k}[[t_I]] \longrightarrow \mathcal{F}' = \{(\mathcal{H}', \nabla'), (\mathcal{H}'_R, \nabla'_R), \alpha'\}/\mathbb{k}[[t_J]]$$

is an unfolding. By Remark 5.3.9, the framing for $(\mathcal{H}_R, \nabla_R)$ produces a unique framing on $(\mathcal{H}'_R, \nabla'_R)$. By Lemma 5.2.6(2), this framing induces a framing on $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)$, thus a framing on the (T)-structure $(\mathcal{H}', \nabla')_0$ under α' . By construction, under $\beta|_{t_I=0}$, the framing constructed on (\mathcal{H}', ∇') coincides with the initial framing of (\mathcal{H}, ∇) . We conclude from Lemma 5.3.34 that it is a framing of F-bundle. This concludes the proof. \square

Theorem 5.3.36 (Unfolding of equivariant F-bundles). *Let $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ be an equivariant F-bundle over $\mathbb{k}[[t_I]]$, and fix $v \in \mathcal{H}_R/(t_I, u)\mathcal{H}_R$.*

1. *If v satisfies (IC), (GC) and $\text{coker } \mu_v$ is free, then \mathcal{F} admits a maximal unfolding with cyclic vector induced from v .*
2. *If v satisfies (GC'), then any two maximal unfoldings of \mathcal{F} with cyclic vector induced from v are isomorphic under a unique isomorphism.*

Furthermore, any framing for \mathcal{F} induces a unique framing on a maximal unfolding.

Proof. We prove (1). For the R -linear (T)-structures, there exists a maximal unfolding by Corollary 5.3.30

$$\beta_R: (\mathcal{H}_R, \nabla_R)/R[[t_I]] \longrightarrow (\mathcal{H}'_R, \nabla'_R)/R[[t_J]].$$

By functoriality, we obtain an unfolding of \mathbb{k} -linear (T)-structures

$$\widetilde{\beta}_R \circ \alpha: (\mathcal{H}, \nabla)_0/\mathbb{k}[[t_I]] \longrightarrow (\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)/\mathbb{k}[[t_J]].$$

By Proposition 5.3.4, the R -linear (T)-structures admit framings. Those framings induce framings on the \mathbb{k} -linear (T)-structures by Lemma 5.2.6(2). Hence we can apply Lemma 5.3.10 to define an F-bundle structure $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)$ extending the \mathbb{k} -linear (T)-structure $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)$, such that $\widetilde{\beta}_R \circ \alpha$ becomes an unfolding of F-bundles. Then $\{(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R), (\mathcal{H}'_R, \nabla'_R), \text{id}\}$ is an equivariant F-bundle and $(\widetilde{\beta}_R \circ \alpha, \beta_R)$ is a maximal unfolding of \mathcal{F} with cyclic vector v .

We now prove (2). For $k = 1, 2$, let

$$(\beta_k, \beta_{R,k}): \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\} \rightarrow \{(\mathcal{H}_k, \nabla_k), (\mathcal{H}_{R,k}, \nabla_{R,k}), \alpha_k\}$$

be two maximal unfoldings of equivariant F-bundles, with cyclic vectors $v_k \in \mathcal{H}_{R,k}/(t_J, u)\mathcal{H}_{R,k}$ induced from v . By Corollary 5.3.30, there exists a unique isomorphism of R -linear (T)-structures

$$iso_R: (\mathcal{H}_{R,1}, \nabla_{R,1}) \rightarrow (\mathcal{H}_{R,2}, \nabla_{R,2})$$

such that $\beta_{R,2} = iso_R \circ \beta_{R,1}$. This induces an isomorphism for the underlying \mathbb{k} -linear (T)-structures

$$iso := \alpha_2^{-1} \circ \widetilde{iso}_R \circ \alpha_1: (\mathcal{H}_1, \nabla_1) \rightarrow (\mathcal{H}_2, \nabla_2),$$

and it satisfies $\beta_2 = \beta_1 \circ iso$. It suffices to show that iso is compatible with the u -direction. Since β_k are unfoldings of F-bundles, they restrict to isomorphisms of F-bundles at $\mathfrak{t}_J = 0$. Hence iso is compatible with the u -direction at $\mathfrak{t}_J = 0$. Since the \mathbb{k} -linear (T)-structures come from finite R -linear (T)-structures, they admit framings. Then Lemma 5.3.1(2) implies that iso is an isomorphism of F-bundles. We conclude that (iso, iso_R) is an isomorphism of equivariant F-bundles compatible with the unfoldings. This isomorphism is unique, since (iso, iso_R) is uniquely determined by iso_R . (2) is proved.

The last statement is a special case of Proposition 5.3.35. The theorem is proved. \square

5.4 Application to mirror symmetry of flag varieties

In this section, we apply our equivariant unfolding theorem to obtain the big D -module mirror symmetry for flag varieties G/P of general Lie type (Theorem 5.4.35).

We start with the \mathbb{k} -linear F-bundles given by the equivariant small quantum D -module for G/P on the A-side, and another one by the equivariant Gauss-Manin system with respect to Rietsch's superpotential on the B-side (see [118]). Note that both F-bundles are of infinite rank, as the equivariant parameters are not yet included in the base ring. Moreover, their R -linear (T)-structure lifts coincide with the D -module structures defined in [28], and are thus isomorphic to each other as shown therein. We will construct a suitable unfolding on the B-side, and apply our equivariant unfolding theorem to deduce the isomorphism

between the unfoldings on both sides. We remark that in general, the classical cohomology of G/P is not generated by the divisor classes and the small quantum cohomology is not semisimple, so that neither the unfolding in [64] nor the semisimple reconstruction in [127] is directly applicable.

Equivariant F-bundles for G/P

Equivariant quantum cohomology ring of G/P

Let G be a simply-connected complex simple Lie group, and P be a parabolic subgroup of G containing a Borel subgroup $B \subset G$. Let B_- denote the opposite Borel subgroup, and then $T := B \cap B_-$ is a maximal torus of G . Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a basis of simple roots, and $\{\omega_1, \dots, \omega_n\}$ be the fundamental weights. The Weyl group $W := N_G(T)/T$ is generated by simple reflections $s_i := s_{\alpha_i}$. The Weyl subgroup W_P of P is generated by the simple reflections s_α with $\alpha \in \Delta_P := \{\alpha_i \in \Delta \mid s_i P \subset P\}$. Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ denote the standard length function, and w_0 (resp. w_P) denote the longest element in W (resp. W_P). Denote by $W^P \subset W$ the subset of minimal length representative of the cosets W/W_P .

The flag variety $X := G/P$ is a Fano manifold. It parametrizes partial flags (resp. isotropic partial flags) in a complex vector space when G is of type A (resp. B, C, D). For each $w \in W^P$, there are Schubert varieties $X^w := \overline{BwP/P}$ (resp. $X_w := \overline{B_-wP/P}$) of (co)dimension $\ell(w)$ inside X . We have

$$H^*(X, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z} \text{PD}([X_w]),$$

where $\text{PD}(\cdot)$ denotes the Poincaré dual, and

$$H_2(X, \mathbb{Z}) = \bigoplus_{\alpha \in \Delta \setminus \Delta_P} \mathbb{Z}[X^{s_\alpha}].$$

For each $w \in W^P$, the Schubert variety X_w (resp. X^w) is invariant under the natural T -action on X , so that it defines a fundamental class in the T -equivariant Borel-Moore homology. This class is identified with a T -equivariant cohomology class in $H_T^{2\ell(w)}(X, \mathbb{C})$ (resp. $H_T^{2(\dim X - \ell(w))}(X, \mathbb{C})$) denoted as σ_w (resp. σ^w). The fundamental weights produce equivariant parameters for the T -action which we denote by $\lambda = (\lambda_1, \dots, \lambda_n)$. We have identifications

$$H_T^*(pt, \mathbb{C}) = \mathbb{C}[\lambda_1, \dots, \lambda_n] =: \mathbb{C}[\lambda], \quad (5.4.1)$$

$$H_T^*(X, \mathbb{C}) = \bigoplus_{w \in W^P} \mathbb{C}[\lambda] \sigma_w. \quad (5.4.2)$$

To be more precise, we view ω_i as a character in $\text{Hom}(T, \mathbb{C}^*)$, and denote by $\mathbb{C}_{-\omega_i}$ the one-dimensional representation of T viewed a vector bundle over a point. Then we take $\lambda_i := c_1^T(\mathbb{C}_{-\omega_i})$ and consequently we have $\lambda_i = -\omega_i$. We denote by (\cdot, \cdot) the equivariant Poincaré pairing on $H_T^*(X, \mathbb{C})$. The $\mathbb{C}[\lambda]$ -bases $\{\sigma^w\}_w$ and $\{\sigma_w\}_w$ are dual with respect to the Poincaré pairing, i.e. $(\sigma_u, \sigma^v) = \delta_{u,v}$. In the following, we denote by $\mathbb{C}(\lambda)$ the fraction field of $\mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \dots, \lambda_n]$.

Lemma 5.4.3 ([17, Lemma 5.11]). *The localized equivariant cohomology of X , $H_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ is generated by the element $\sum_{\alpha \in \Delta \setminus \Delta_P} \sigma_{s_\alpha}$ as a $\mathbb{C}(\lambda)$ -algebra.*

Remark 5.4.4. The above lemma shows that $H_T^*(X, \mathbb{C})$ is generated by $H_T^2(X, \mathbb{C})$ after localization. This also follows from [30, Lemma 4.1.3], and can be generalized to any smooth projective variety admitting a torus action with finitely many attractive torus-fixed points by [2, Lemma 1].

Let $\overline{\mathcal{M}}_{0,m}(X, d)$ denote the moduli space of m -pointed stable maps to X of genus zero and degree $d \in H_2(X, \mathbb{Z})$, and $ev_i: \overline{\mathcal{M}}_{0,m}(X, d) \rightarrow X$ denote the i -th T -equivariant evaluation map. The moduli space $\overline{\mathcal{M}}_{0,m}(X, d)$ carries a T -action, and has a T -equivariant virtual fundamental class $[\overline{\mathcal{M}}_{0,m}(X, d)]^{\text{vir}}$. For $\gamma_1, \dots, \gamma_m \in H_T^*(X, \mathbb{C})$, we have the genus-zero, m -point equivariant Gromov–Witten invariant

$$\langle \gamma_1, \dots, \gamma_m \rangle_d := \int_{[\overline{\mathcal{M}}_{0,m}(X, d)]^{\text{vir}}} ev_1^*(\gamma_1) \cup \dots \cup ev_m^*(\gamma_m) \in \mathbb{C}[\lambda]. \tag{5.4.5}$$

We introduce the necessary choices of bases, and associated coordinates, in order to define the equivariant big quantum cohomology ring of X . Write $\Delta \setminus \Delta_P = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ and $W^P = \{v_1, \dots, v_N\}$ with $v_j = s_{i_j}$ for $1 \leq j \leq r$. We introduce Novikov variables $q = (q_1, \dots, q_r)$ corresponding to the basis $\{[X^{s_\alpha}] \mid \alpha \in \Delta \setminus \Delta_P\}$ of $H_2(X, \mathbb{Z})$. For $d \in H_2(X, \mathbb{Z})$, we have $d = \sum_j d_j [X^{s_{i_j}}]$ and denote $q^d := \prod_{j=1}^r q_j^{d_j}$. We use $\{\tau_i\}$ for the $\mathbb{C}[\lambda]$ -linear coordinates of $H_T^*(X)$, whose elements are of the form $\alpha = \sum_{i=1}^N \tau_i \sigma_{v_i}$.

As a module, the equivariant big quantum cohomology ring is

$$\text{QH}_T^{*, \text{big}}(X) := H_T^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[q][[\tau]].$$

It encodes all genus zero Gromov–Witten invariants in the quantum product \star_τ^{big} , defined by

$$\sigma_v \star_\tau^{\text{big}} \sigma_w = \sum_{\eta \in W^P} \sum_{m \geq 0} \sum_{i_1, \dots, i_m} \sum_{d \in H_2(X, \mathbb{Z})} \frac{\tau_{i_1} \cdots \tau_{i_m}}{m!} \langle \sigma_v, \sigma_w, \sigma^\eta, \sigma_{v_{i_1}}, \dots, \sigma_{v_{i_m}} \rangle_d q^d \sigma_\eta.$$

Here the coefficient of $\tau_{i_1} \cdots \tau_{i_m}$ is indeed a polynomial in q since X is Fano.

Denote $\tilde{q}_j := q_j e^{\tau_j}$ and $\tilde{q}^d := \prod_j \tilde{q}_j^{d_j}$. Letting $\tau_i = 0$ for all $i > r$ and using the divisor axiom for Gromov–Witten invariants, we obtain the equivariant small quantum cohomology ring

$$\mathrm{QH}_T^*(X) = H_T^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{q}] \quad \text{with} \quad \sigma_v \star \sigma_w = \sum_{\eta, d} \langle \sigma_v, \sigma_w, \sigma^\eta \rangle_d \tilde{q}^d \sigma_\eta.$$

The next lemma follows directly from Lemma 5.4.3 and [122, Lemma 2.1].

Lemma 5.4.6. *The localized equivariant small quantum cohomology of X , $\mathrm{QH}_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ is generated by $\{\sigma_{s_\alpha} \mid \alpha \in \Delta \setminus \Delta_P\}$ as a $\mathbb{C}(\lambda)[\tilde{q}]$ -algebra.*

Remark 5.4.7. By further taking the nonequivariant limit $\lambda = 0$, we obtain the small quantum cohomology $\mathrm{QH}^*(X)$, which could be non-semisimple. For instance for G of type C_n and $\Delta_P = \Delta \setminus \{\alpha_2\}$, we obtain the isotropic Grassmannian $SG(2, 2n) = \{V \leq \mathbb{C}^{2n} \mid \dim V = 2, \Omega(V, V) = 0\}$, where Ω is a symplectic form on \mathbb{C}^{2n} . It is shown in [22] that $\mathrm{QH}^*(SG(2, 2n))$ is not semisimple. It is easy to see that $\mathrm{QH}^*(SG(2, 2n))$ is not generated by $H^2(SG(2, 2n), \mathbb{C})$ either.

Equivariant F-bundle structures for \mathbf{G}/\mathbf{P}

We recall that $\tau = (\tau_1, \dots, \tau_N)$ are the $\mathbb{C}[\lambda]$ coordinates of $H_T^*(X)$ dual to the standard basis we chose, $q = (q_1, \dots, q_r)$ are the Novikov variables and $\lambda = (\lambda_1, \dots, \lambda_n)$ are the equivariant variables. For $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, we set $\lambda^k := \prod_{i=1}^n \lambda_i^{k_i}$ and $|k| := \sum_{i=1}^n k_i$. It is expected but remains unsolved in general that the big quantum cohomology is convergent around $\tau = 0$. Therefore we work on the formal neighborhood of $\tau = 0$.

Let $\mathbb{k} := \mathbb{C}(q)$ be the fraction field of $\mathbb{C}[q]$, and let $R := \mathbb{k}[\lambda]$. We fix the \mathbb{k} -basis $\boldsymbol{\lambda} = (\lambda^k, k \in \mathbb{N}^n)$ of R . We obtain \mathbb{k} -linear coordinates $\boldsymbol{\tau} = \{\tau_{i,k}, 1 \leq i \leq N, k \in \mathbb{N}^n\}$ on $H_T^*(X, \mathbb{k})$ associated to the \mathbb{k} -basis $(\sigma_{v_i} \lambda^k, 1 \leq i \leq N, k \in \mathbb{N}^n)$. There is a continuous morphism of R -algebras

$$\psi_{\boldsymbol{\lambda}} : R[[\boldsymbol{\tau}]] \rightarrow R[[\boldsymbol{\tau}]], \tau_i \mapsto \sum_{k \in \mathbb{N}^n} \lambda^k \tau_{i,k}.$$

We define a \mathbb{k} -linear equivariant F-bundle equivariant F-bundle

$$\mathcal{F}^{A, \text{big}} := \{(\mathcal{H}^{A, \text{big}}, \nabla^{A, \text{big}}), (\mathcal{H}_R^{A, \text{big}}, \nabla_R^{A, \text{big}}), \alpha\} / \mathbb{k}[[\boldsymbol{\tau}]].$$

associated to the equivariant big quantum cohomology as follows. The R -linear (T)-structure $(\mathcal{H}_R^{A,\text{big}}, \nabla_R^{A,\text{big}})$ is given by the $R[[\tau, u]]$ -module

$$\begin{aligned}\mathcal{H}_R^{A,\text{big}} &= H_T^*(X, \mathbb{k}) \otimes_R R[[\tau, u]], \\ \nabla_{R, \partial \tau_j}^{A,\text{big}} &= \partial_{\tau_j} + u^{-1} \left((\sigma_{v_j} + \lambda_{i_j}) \star_{\tau}^{\text{big}} \right),\end{aligned}$$

where $1 \leq j \leq N$ and we set $\lambda_{i_j} = 0$ for $j > r$. Here i_j are the index of $\Delta \setminus \Delta_P = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$. The \mathbb{k} -linear F-bundle $(\mathcal{H}^{A,\text{big}}, \nabla^{A,\text{big}})$ has underlying $\mathbb{k}[[\tau, u]]$ -module

$$\mathcal{H}^{A,\text{big}} = H_T^*(X, \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{k}[[\tau, u]],$$

and the connection $\nabla^{A,\text{big}}$ is specified by:

$$\begin{aligned}\nabla_{\partial \tau_{j,k}}^{A,\text{big}} &= \partial_{\tau_{j,k}} + u^{-1} \left(\lambda^k (\sigma_{v_j} + \lambda_{i_j}) \star_{\tau}^{\text{big}} \right), \\ \nabla_{u \partial_u}^{A,\text{big}} &= \text{Gr}^{A,\text{big}} - \nabla_{E_A^{\text{big}}}^{A,\text{big}}.\end{aligned}$$

Here,

$$\text{Gr}^{A,\text{big}} = u \partial_u + E_A^{\text{big}} + \mu_A,$$

where μ_A is the $\mathbb{k}[[\tau, y]]$ -linear grading operator on the fiber $H_T^*(X, \mathbb{k})$ linear defined by

$$\mu_A(\lambda^k \sigma_{v_i}) = (\ell(v_i) + |k|) \lambda^k \sigma_{v_i},$$

and E_A^{big} is the Euler vector field measuring degree on the base $\mathbb{k}[[\tau]]$, given by

$$E_A^{\text{big}} = \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{\tau_{j,0}} + \sum_{\substack{1 \leq j \leq N \\ k \in \mathbb{N}^n}} (1 - \ell(v_j) - |k|) \tau_{j,k} \partial_{\tau_{j,k}},$$

where $\ell(v_j) = 1$ for $1 \leq j \leq r$ and the degree $\deg(q_j)$ is defined as

$$\deg(q_j) := 2 \int_{[X^{s_{i_j}}]} c_1(T_{G/P}).$$

Under the change of variables $R[[\tau]] \rightarrow R[[\boldsymbol{\tau}]]$, $\tau_i \mapsto \sum_{k \in \mathbb{N}^n} \lambda^k \tau_{i,k}$, the data $\{(\mathcal{H}_R^{A,\text{big}}, \nabla_R^{A,\text{big}}), \text{id}\}$ provides an R -linear lift of the underlying (T)-structure $(\mathcal{H}^{A,\text{big}}, \nabla^{A,\text{big}})_0$. We obtain the A -model big equivariant F-bundle

$$\mathcal{F}^{A,\text{big}} = \{(\mathcal{H}^{A,\text{big}}, \nabla^{A,\text{big}}), (\mathcal{H}_R^{A,\text{big}}, \nabla_R^{A,\text{big}}), \text{id}\} / \mathbb{k}[[\boldsymbol{\tau}]].$$

Remark 5.4.8. For $1 \leq j \leq r$, consider the line bundle $L_j = G \times_P \mathbb{C}_{-\omega_{i_j}}$ over G/P . Since $c_1^T(L_j) = \sigma_{s_{i_j}} - \omega_{i_j} = \sigma_{v_j} + \lambda_{i_j}$, we can write $\nabla_{R, \partial \tau_j}^{A,\text{big}} = \partial_{\tau_j} + u^{-1} c_1^T(L_j) \star_{\tau}^{\text{big}}$. For $j > r$, $\nabla_{R, \partial \tau_j}^{A,\text{big}}$ are not weighted.

Remark 5.4.9. The flatness of $\nabla^{A,big}$ in the u -direction follows from a similar argument to that in [32, Section 3.2] .

By restricting to the small locus $\tau_j = 0$ for $(\mathcal{H}^{A,big}, \nabla^{A,big})$, and $\tau_{j,k} = 0$ for $(\mathcal{H}_R^{A,big}, \nabla_R^{A,big})$ when $j > r$, we obtain the A -model small equivariant F-bundle

$$\mathcal{F}^A := \{(\mathcal{H}^A, \nabla^A), (\mathcal{H}_R^A, \nabla_R^A), \text{id}\} / \mathbb{k}[[\tau_{\leq r}]],$$

where $\tau_{\leq r} = \{\tau_{i,k}, 1 \leq i \leq r, k \in \mathbb{N}^n\}$ parametrizes the \mathbb{k} -linear F-bundle, and $\tau_{\leq r} = \{\tau_i, 1 \leq i \leq r\}$ parametrizes the R -linear (T)-structure.

The quotient maps $\mathbb{k}[[\tau]] \rightarrow \mathbb{k}[[\tau_{\leq r}]]$ and $R[[\tau]] \rightarrow R[[\tau_{\leq r}]]$ together with the natural identification of the fibers produce an unfolding of the equivariant F-bundle $\iota: \mathcal{F}^A \rightarrow \mathcal{F}^{A,big}$.

Proposition 5.4.10. *The morphism $\iota: \mathcal{F}^A \rightarrow \mathcal{F}^{A,big}$ is a maximal unfolding of \mathbb{k} -linear equivariant F-bundles, with cyclic vector given by $1 \in H_T^*(X, \mathbb{k})$.*

Proof. We have already proven the morphism $\iota: \mathcal{H}^A \rightarrow \mathcal{F}^{A,big}$ is an unfolding, and it remains to check that $(\mathcal{H}_R^A, \nabla_R^A)$ is maximal. We take the cyclic vector $v = 1 \in \mathcal{H}_R^{A,big}|_{\tau=u=0} = H_T^*(X, \mathbb{k})$. The R -linear evaluation map is the

$$\begin{aligned} \mu_{v=1}: \bigoplus_{j=1}^N R\partial_{\tau_j} &\longrightarrow H_T^*(X, \mathbb{k}) \\ \partial_{\tau_j} &\longmapsto \nabla_{u\partial_{\tau_j}}|_{\tau=u=0}(1) = \sigma_{v_j} + \lambda_{i_j}. \end{aligned}$$

Since $\{\sigma_{v_i}\}_i$ is an $R = \mathbb{k}[\lambda]$ basis of $H_T^*(X, \mathbb{k})$ by equation (5.4.2), $\mu_{v=1}$ is an R -isomorphism and we conclude that ι is maximal unfolding. \square

The small D -module mirror symmetry for G/P

In this section, we review the B -side of mirror symmetry for G/P as in [118], construct a R -linear (T)-structure $(\mathcal{H}_R^B, \nabla_R^B)$ from the Gauss-Manin connection, and state the small mirror symmetry as in [28].

Small D -module mirror symmetry

Let G^\vee be the Langlands dual group of G , and T^\vee, B^\vee, P^\vee be the Langlands dual of T, B, P respectively. Rietsch’s equivariant mirror superpotential is a triple $(X_P^\vee, \mathcal{W}, p)$. Here X_P^\vee is a subvariety of $G^\vee \times Z$ isomorphic to

$$\left((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee} \right) \times \text{Spec } \mathbb{C}[\tilde{q}_i^{\pm 1} | \alpha_i \in \Delta \setminus \Delta_P],$$

where Z is the center of the Levi subgroup of P^\vee , $-K_{G^\vee/P^\vee}$ is the anti-canonical divisor of the dual partial flag variety G^\vee/P^\vee given in [88]. The holomorphic function $\mathcal{W} : X_P^\vee \rightarrow \mathbb{C}$ is the non-equivariant mirror superpotential of G/P , and $p : X_P^\vee \rightarrow T^\vee$ is a morphism which gives information on the equivariant part of Rietsch's original mirror superpotential $\mathcal{W} + \ln \phi(\cdot; h)$ (see [118]).

Denote by $\Omega^i(X_P^\vee/Z)$ the space of holomorphic i -forms over X_P^\vee with respect to

$$Z \cong \text{Spec } \mathbb{C}[\tilde{q}_i^{\pm 1} \mid \alpha_i \in \Delta \setminus \Delta_P]$$

via the aforementioned isomorphism. Identify the Lie algebra $\mathfrak{t}^\vee = \text{Lie}(T^\vee)$ with \mathfrak{t}^* . Let $\{(\lambda_i)^*\} \subset (\mathfrak{t}^\vee)^*$ be the dual base of $\{\lambda_i\} \subset \mathfrak{t}^\vee$, and $\text{mc}_{T^\vee} \in \Omega^1(T^\vee; \mathfrak{t}^\vee)$ denote the Maurer-Cartan form of T^\vee .

In [28], the B -model D -module $(G_0(X_P^\vee, \mathcal{W}, p), \nabla)$ consists of a $\mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}]$ -module defined by

$$G_0(X_P^\vee, \mathcal{W}, p) = \text{coker} \left(\mathbb{C}[\lambda, u] \otimes_{\mathbb{C}} \Omega^{\text{top}-1}(X_P^\vee/Z) \xrightarrow{\partial} \mathbb{C}[\lambda, u] \otimes_{\mathbb{C}} \Omega^{\text{top}}(X_P^\vee/Z) \right),$$

$$\partial = 1 \otimes \left(ud + d\mathcal{W} \wedge - \sum_{j=1}^n \lambda_j (p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle) \wedge \right).$$

It is equipped with a meromorphic connection having a logarithmic pole in the \tilde{q}_i -direction

$$\nabla_{\partial_{\tilde{q}_i}}([\omega]) = \left[\mathcal{L}_{\partial_{\tilde{q}_i}}(\omega) + u^{-1} \frac{\partial \mathcal{W}}{\partial \tilde{q}_i} \omega - u^{-1} \sum_{j=1}^n \lambda_j \left(\iota_{\partial_{\tilde{q}_i}} p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle \right) \omega \right].$$

Here $p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle \in \Omega^1(X_P^\vee/Z)$, and ∂, ∇ are linear on $\mathbb{C}[\lambda, u]$.

Remark 5.4.11. For any $\omega \in \mathbb{C}[\lambda, u] \otimes \Omega^{\text{top}}(X_P^\vee/Z)$, $\omega = g\omega_0$ for some $g \in \mathcal{O}(X_P^\vee)$ and $\omega_0 \in \Omega^{\text{top}}((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee})$. We have the Lie derivative $\mathcal{L}_{\partial_{\tilde{q}_i}}(\omega) = \frac{\partial g}{\partial \tilde{q}_i} \omega_0$.

The small quantum D -module mirror symmetry holds for G/P in the following sense.

Proposition 5.4.12 ([28, Theorem 1.2]). *There exists a unique $\mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}]$ -linear map*

$$\Phi_{\text{mir}} : G_0(X_P^\vee, \mathcal{W}, p) \longrightarrow \text{QH}_T^*(G/P)[u, \tilde{q}_1^{-1}, \dots, \tilde{q}_r^{-1}] \quad (5.4.13)$$

satisfying the following:

1. Φ_{mir} is bijective, and preserves the connection,
2. $\Phi_{\text{mir}}([\Omega]) = 1$, where Ω is the unique (up to sign) volume form in $\Omega^{\text{top}}(X_P^\vee/Z)$, whose restrictions to every torus chart of $(G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee}$ is equal to the standard volume form $\pm dz_1 \wedge \cdots \wedge dz_K / z_1 \cdots z_K$,
3. at the semi-classical limit, we have a ring isomorphism

$$\Phi_{\text{mir}}^{u=0} : \text{Jac}(X_P^\vee, \mathcal{W}, p) \xrightarrow{\cong} \text{QH}_T^*(G/P)[\tilde{q}_1^{-1}, \dots, \tilde{q}_r^{-1}], \quad (5.4.14)$$
4. Φ_{mir} intertwines the shift operators (see [28, Sections 3.3 and 4.5]), and
5. Φ_{mir} preserves the \mathbb{Z} -grading.

In (5.4.14), $\text{Jac}(X_P^\vee, \mathcal{W}, p)$ denotes the Jacobi ring, which is the coordinate ring of the scheme-theoretic zero locus of certain relative 1-forms in $\Omega^1(X_P^\vee \times \mathfrak{t}/Z \times \mathfrak{t})$ (see [28, Definition 4.9] for more details). It corresponds to setting $u = 0$ in the B -model D -module.

We remark that the above isomorphism is a bit implicit. Below we provide an example with explicit isomorphism of small quantum D -modules from [103].

Example 5.4.15. For $G/P = \text{Gr}(3, 5) = \{V \leq \mathbb{C}^5 \mid \dim V = 3\}$, the Langlands dual flag variety G^\vee/P^\vee is the Grassmannian $\text{Gr}(2, 5) \hookrightarrow \mathbb{P}^9$, whose image is defined by the Plücker relations $p_{a_1 a_2} p_{a_3 a_4} - p_{a_1 a_3} p_{a_2 a_4} + p_{a_1 a_4} p_{a_2 a_3} = 0$ for $1 \leq a_1 < a_2 < a_3 < a_4 \leq 5$. In this case, $-K_{G^\vee/P^\vee} = \{p_{12} p_{23} p_{34} p_{45} p_{15} = 0\}$. The \mathcal{W} -part of Rietsch's equivariant superpotential is given by

$$\mathcal{W} = \frac{p_{13}}{p_{12}} + \frac{p_{24}}{p_{23}} + \frac{p_{35}}{p_{34}} + \tilde{q} \frac{p_{14}}{p_{45}} + \frac{p_{25}}{p_{15}}.$$

The degree of the inhomogeneous coordinate $\theta_{ij} = \frac{p_{ij}}{p_{12}}$ is equal to $2(i + j - 3)$. The volume form $\Omega = \frac{d\theta_{13} d\theta_{14} d\theta_{15} d\theta_{23} d\theta_{24} d\theta_{25}}{\theta_{23} \theta_{34} \theta_{45} \theta_{15}}$ is of degree 0. For $1 \leq i < j \leq 5$, $\Phi_{\text{mir}}([\theta_{ij} \Omega]) = \sigma_w$ with $w \in S_5$ the unique permutation satisfying $w(4) = 6 - j$, $w(5) = 6 - i$ and $w(1) < w(2) < w(3)$.

Equivariant F-bundles formulation

In our setting of (T)-structures, we need to replace the logarithmic \tilde{q}_i -directions with a regular meromorphic connection in y_i -directions. This is achieved by the D -module inverse image under

$$\psi_1 : \mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}] \longrightarrow \mathbb{C}[\lambda, u][q_i^{\pm 1}][[y_{\leq r}]]$$

$$\tilde{q}_i \longmapsto q_i e^{y_i},$$

where $y_{\leq r} = \{y_i, 1 \leq i \leq r\}$. For the purpose of applying our reconstruction theorem to obtain big mirror symmetry, we need to further take the fraction field of q and formalize u . So we compose ψ_1 with the following base change

$$\psi_2: \mathbb{C}[\lambda, u][q_i^{\pm 1}][[y_{\leq r}]] \longrightarrow \mathbb{C}[\lambda, u][q_i^{\pm 1}][[y_{\leq r}]] \otimes_{\mathbb{C}[q_i^{\pm 1}, u]} \mathbb{C}(q)[[u]].$$

Namely we have the following, where we recall $\mathbb{k} = \mathbb{C}(q)$ and $R = \mathbb{k}[\lambda]$.

$$\psi := \psi_2 \circ \psi_1: \mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}] \longrightarrow R[[y_{\leq r}, u]]. \quad (5.4.16)$$

The B -model R -linear (T)-structure $(\mathcal{H}_R^B, \nabla_R^B)$ is the D -module inverse image $\psi^*\left(\left(G_0(X_P^\vee, \mathcal{W}, p), \nabla\right)\right)$, whose underlying $R[[y_{\leq r}, u]]$ -module is

$$\mathcal{H}_R^B = G_0(X_P^\vee, \mathcal{W}, p) \otimes R[[y_{\leq r}, u]],$$

and the connection is given by

$$\nabla_{R, \partial_{y_i}}^B([\omega]) = \left[\mathcal{L}_{\partial_{y_i}}(\omega) + u^{-1} \frac{\partial \mathcal{W}}{\partial y_i} \omega - u^{-1} \sum_{j=1}^n \lambda_j \left(\iota_{\partial_{y_i}} p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle \right) \omega \right]. \quad (5.4.17)$$

Next we define the B -model \mathbb{k} -linear F-bundle. Fix the \mathbb{k} -basis of R given by $\boldsymbol{\lambda} = (\lambda^k, k \in \mathbb{N}^n)$ and let $\mathbf{y}_{\leq r} := \{y_{i,k}, 1 \leq i \leq r, k \in \mathbb{N}^n\}$. Consider the following change of variables

$$\begin{aligned} \psi_{\boldsymbol{\lambda}}: R[[y_{\leq r}, u]] &\longrightarrow R[[\mathbf{y}_{\leq r}, u]] \\ y_i &\longmapsto \sum_{k \in \mathbb{N}^n} \lambda^k y_{i,k}. \end{aligned} \quad (5.4.18)$$

The underlying (T)-structure $(\mathcal{H}^B, \nabla^B)$ of the B -model \mathbb{k} -linear F-bundle is defined as the D -module inverse image $\psi_{\boldsymbol{\lambda}}^*(\mathcal{H}_R^B, \nabla_R^B)$ and restrict scalars from R to \mathbb{k} , as in Lemma 5.2.6. Explicitly, the underlying $\mathbb{k}[[\mathbf{y}_{\leq r}, u]]$ -module is

$$\mathcal{H}^B = G_0(X_P^\vee, \mathcal{W}, p) \otimes R[[\mathbf{y}_{\leq r}, u]],$$

equipped with a regular meromorphic connection

$$\nabla_{\partial_{y_{i,k}}}^B([\omega]) = \left[\mathcal{L}_{\partial_{y_{i,k}}}(\omega) + u^{-1} \frac{\partial \mathcal{W}}{\partial y_{i,k}} \omega - u^{-1} \sum_{j=1}^n \lambda_j \left(\iota_{\partial_{y_{i,k}}} p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle \right) \omega \right],$$

where \mathcal{W} is in variables $y_{i,k}$. The u -direction is defined in Eq. (5.4.21), its definition uses the grading operator on the B -model, which we now define. We first construct a grading on $\Omega^{\text{top}}(X_P^\vee/Z) \otimes \mathbb{k}[[\mathbf{y}_{\leq r}, u]]$.

Construction 5.4.19 (Grading on differential forms). We construct a $\mathbb{k}[\mathbf{y}_{\leq r}, u]$ -linear operator μ_B on $\Omega^{\text{top}}(X_P^\vee/Z) \otimes \mathbb{k}[\mathbf{y}_{\leq r}, u]$, which defines a grading on differential forms.

Recall that $\Omega^{\text{top}}(X_P^\vee/Z) \otimes \mathbb{k}[\mathbf{y}_{\leq r}, u]$ is a rank 1 free module over $\mathcal{O}(X_P^\vee)[\mathbf{y}_{\leq r}, u]$. The choice of Ω in Proposition 5.4.12(2) produces a basis of this module. For any differential form $\omega = h\Omega$ with $h \in \mathcal{O}(X_P^\vee)[\mathbf{y}_{\leq r}, u]$, we define

$$\mu_B(h\Omega) := \frac{\deg_B(h)}{2}\Omega,$$

where the degree operator \deg_B is defined on functions in $\mathcal{O}(X_P^\vee)$ using the \mathbb{G}_m -action on X_P^\vee constructed in [28, Lemma 4.6], and extended by $\mathbb{k}[\mathbf{y}_{\leq r}, u]$ -linearity. Using the Jacobian isomorphism (5.4.14) and the grading operator μ_A , we can describe \deg_B as follows. For any local chart, we take a coordinate system $(z_i)_i$ so that z_i are homogeneous. For a monomial function $h \in \mathcal{O}((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee})$, we have $\deg_B(h) = 2dh$ where $d \in \mathbb{Z}$ is given by:

$$\left(\sum_{i=1}^r \frac{\deg(q_i)}{2} q_i \frac{\partial}{\partial q_i} + \mu_A \right) \Phi_{\text{mir}}^{u=0, y=0}(\bar{h}) = d\Phi_{\text{mir}}^{u=0, y=0}(\bar{h}),$$

where \bar{h} denotes the image of h in $\text{Jac}(X_P^\vee, \mathcal{W}, p)$ and $\Phi_{\text{mir}}^{u=0, y=0}$ is the Jacobi isomorphism (5.4.14). It is then extended to a $\mathbb{k}[\mathbf{y}_{\leq r}, u]$ -linear operator on $\mathcal{O}(X_P^\vee)[\mathbf{y}_{\leq r}, u]$.

We use the simple example of $X = \mathbb{C}\mathbb{P}^1$ case to illustrate the definition of \deg_B .

Example 5.4.20 (μ_B for $\mathbb{C}\mathbb{P}^1$). For $X = \mathbb{C}\mathbb{P}^1$, we have $\mathbb{k} = \mathbb{C}(q)$. The equivariant (small) quantum cohomology $\text{QH}_T^*(X)$ is isomorphic to $\mathbb{k}[H, \lambda]/(H^2 - H\lambda - q)$, where q has degree 4. The mirror X_P^\vee is the family $\mathbb{G}_m \times \text{Spec } \mathbb{C}[q^{\pm 1}] \rightarrow \text{Spec } \mathbb{C}[q^{\pm 1}]$, the superpotential is $\mathcal{W} = z + \frac{q}{z}$ and the Jacobi ring $\text{Jac}(X_P^\vee, \mathcal{W}, p)$ is isomorphic to $\mathbb{k}[z, z^{-1}, \lambda]/(1 - \frac{\lambda}{z} - \frac{q}{z^2})$. The mirror isomorphism $\Phi_{\text{mir}}^{u=0, y=0}$ at $u = 0, y = 0$ is given the morphism of $\mathbb{k}[\lambda]$ -modules defined by $z \mapsto H$.

We have $\deg_B(qz^3) = q \deg_B(z^3) = 6qz^3$, where the last equality follows from the computation:

$$\begin{aligned} \left(\frac{\deg(q)}{2} q \frac{\partial}{\partial q} + \mu_A \right) \Phi_{\text{mir}}^{u=0, y=0}(z^3) &= \left(\frac{\deg(q)}{2} q \frac{\partial}{\partial q} + \mu_A \right) (\lambda^2 H + qH + q\lambda) \\ &= 3(\lambda^2 H + qH + q\lambda). \end{aligned}$$

The u -direction connection of the B -model \mathbb{k} -linear F-bundle $(\mathcal{H}^B, \nabla^B)$ is defined as

$$\begin{aligned}\nabla_{u\partial_u}^B &:= \text{Gr}^B - \nabla_{E_B}, \\ \text{Gr}^B &:= u\partial_u + E_B + \mu_B,\end{aligned}\tag{5.4.21}$$

where $u\partial_u$ measures the degree in u , μ_B is the grading operator on differential forms (see Construction 5.4.19), and E_B is the Euler vector field measuring the degree of the \mathbf{y} -variables and accounting for the degree of q :

$$E_B := \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{y_{j,0}} - \sum_{1 \leq j \leq r, k \in \mathbb{N}^n} |k| y_{j,k} \partial_{y_{j,k}}.$$

In the following proposition, we note that even though μ_B is only defined on differential forms, the total grading operator Gr^B is well-defined on equivalence classes.

Proposition 5.4.22. *The grading operator Gr^B produces a well-defined operator on \mathcal{H}^B .*

Proof. Let K denote the rank of $\Omega^1(X_P^\vee/Z)$. We only need to prove that for any $(K-1)$ -form η , there exists a $(K-1)$ -form η' such that

$$\text{Gr}^B((ud + d\mathcal{W} \wedge) \eta) = (ud + d\mathcal{W} \wedge) \eta' \tag{5.4.23}$$

Fix a torus-invariant local chart with coordinates $(z_i)_i$ and write

$$\eta = \sum_i g_i(z, q, \mathbf{y}_{\leq r}, u) \iota_{\partial_{z_i}} \left(\bigwedge_{j=1}^K dz_j \right).$$

Let

$$\eta' := \sum_i \left((u\partial_u + E_B + \deg_B)(g_i) + \left(1 - \frac{\deg_B(z_i)}{2z_i} \right) g_i \iota_{\partial_{z_i}} \left(\bigwedge_{j=1}^K dz_j \right) \right).$$

A direct computation shows that this choice of η' satisfies (5.4.23). We conclude that Gr^B descends to an operator on \mathcal{H}^B , completing the proof. \square

We note the following property of μ_B , which we will use in Proposition 5.4.31.

Lemma 5.4.24 (Leibniz rule for μ_B). *The grading operator μ_B on differential forms μ_B (see Construction 5.4.19) satisfies the Leibniz rule:*

$$\mu_B(g_1 g_2 \Omega) = g_1 \mu_B(g_2 \Omega) + g_2 \mu_B(g_1 \Omega),$$

for any $g_1, g_2 \in \mathcal{O}(X_P^\vee) \otimes \mathbb{k}[\mathbf{y}_{\leq r}, u]$.

Proof. The statement follows from the facts that the Jacobian isomorphism $\Phi_{\text{mir}}^{u=0, y=0}$ is a ring isomorphism, and that $(\sum_{j=1}^r \frac{\deg(q_j)}{2} q_j \frac{\partial}{\partial q_j} + \mu_A)$ satisfies the Leibniz rule. \square

By construction, the data $\{(\mathcal{H}_R^B, \nabla_R^B), \text{id}\}$ is an R -linear lift of the \mathbb{k} -linear (T)-structure $(\mathcal{H}^B, \nabla^B)_0$, and we obtain the B -model small equivariant F-bundle

$$\mathcal{F}^B = \{(\mathcal{H}^B, \nabla^B), (\mathcal{H}_R^B, \nabla_R^B), \text{id}\} / \mathbb{k}[\mathbf{y}_{\leq r}].$$

Lemma 5.4.25 (Proposition 5.4.12(5)). *For $h \in \mathcal{O}((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee})$, we have*

$$\left(u\partial_u + \sum_{i=1}^r \frac{\deg(q_j)}{2} q_j \frac{\partial}{\partial q_j} + \mu_A \right) \Phi_{\text{mir}}^{y=0}(h\Omega) = \Phi_{\text{mir}}^{y=0}((u\partial_u + \mu_B)(h\Omega)).$$

The following proposition essentially follows from Proposition 5.4.12. Here we add the explanation in detail for completeness.

Proposition 5.4.26. *The A -model small equivariant F-bundle \mathcal{F}^A over $\mathbb{k}[\tau_{\leq r}]$ is isomorphic to the B -model small equivariant F-bundle \mathcal{F}^B over $\mathbb{k}[\mathbf{y}_{\leq r}]$ under an isomorphism $((\text{mir}_{\mathbb{k}}, \Phi_{\text{mir}, \mathbb{k}}), (\text{mir}, \Phi_{\text{mir}}))$ where Φ_{mir} is as in Proposition 5.4.12, $\Phi_{\text{mir}, \mathbb{k}}$ is the induced \mathbb{k} -linear map, and $\text{mir}_{\mathbb{k}}: y_{i,k} \mapsto \tau_{i,k}$, $\text{mir}: y_i \mapsto \tau_i$ identify the variables of the equivariant F-bundles.*

Proof. Our construction of B -model small equivariant F-bundle

$$\mathcal{F}^B = \{(\mathcal{H}^B, \nabla^B), (\mathcal{H}_R^B, \nabla_R^B), \text{id}\} / \mathbb{k}[\mathbf{y}_{\leq r}]$$

consists of the (T)-structure obtained as D -module inverse image:

$$\begin{aligned} (\mathcal{H}_R^B, \nabla_R^B) &= \psi^*(G_0(X_P^\vee, \mathcal{W}, p), \nabla), \\ (\mathcal{H}^B, \nabla^B) &= (\psi_\lambda \circ \psi)^*(G_0(X_P^\vee, \mathcal{W}, p), \nabla), \end{aligned}$$

where ψ and ψ_λ are base change maps defined in equations (5.4.16) and (5.4.18), together with a u -direction on $(\mathcal{H}^B, \nabla^B)$.

Our construction of A -model small equivariant F-bundle $\mathcal{F}^A = \{(\mathcal{H}^A, \nabla^A), (\mathcal{H}_R^A, \nabla_R^A), \text{id}\} / \mathbb{k}[\tau_{\leq r}]$ in Section 5.4, can also be written as (T)-structures $(\mathcal{H}^A, \nabla^A)$ and $(\mathcal{H}_R^A, \nabla_R^A)$ obtained as D -module inverse image under the same maps, together with a u -direction on $(\mathcal{H}^A, \nabla^A)$.

By the functoriality of pullback, the isomorphism in Proposition 5.4.12 induces an isomorphism

$$((\mathrm{mir}_{\mathbb{k}}, \Phi_{\mathrm{mir}, \mathbb{k}}), (\mathrm{mir}, \Phi_{\mathrm{mir}})): \mathcal{F}^B \longrightarrow \mathcal{F}^A$$

of the underlying (T)-structures. It suffices to prove that in our setting Φ_{mir} is also graded, i.e. that $(\mathrm{mir}_{\mathbb{k}})_* E_B = E_A$ and

$$\Phi_{\mathrm{mir}, \mathbb{k}} \circ \mathrm{Gr}^B = \mathrm{mir}_{\mathbb{k}}^* \mathrm{Gr}^A \circ \Phi_{\mathrm{mir}, \mathbb{k}}, \quad (5.4.27)$$

where $\mathrm{mir}_{\mathbb{k}}^* \mathrm{Gr}^A$ is the grading operator on $\mathrm{mir}_{\mathbb{k}}^* \mathcal{H}_A$ induced from Gr^A . The compatibility of the Euler vector fields is clear from their definition and the fact that $\mathrm{mir}_{\mathbb{k}}$ identifies the variables $\mathbf{y}_{\leq r}$ and $\boldsymbol{\tau}_{\leq r}$. For (5.4.27), we need to check that for all $f \in O(X_P^\vee)[[\mathbf{y}_{\leq r}, u]]$, we have

$$\Phi_{\mathrm{mir}, \mathbb{k}}([(u\partial_u + E_B + \mu_B)(f\Omega)]) = (u\partial_u + E_B + \mu_A)(\Phi_{\mathrm{mir}, \mathbb{k}}([f\Omega])).$$

Since both sides are linear in q and satisfy the Leibniz rule for y and u , it suffices to check the equation for $f \in \mathcal{O}((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee})$.

In this case, the left-hand side reduces to $\Phi_{\mathrm{mir}, \mathbb{k}}([\mu_B(f\Omega)])$. For the right-hand side, observe that $\Phi_{\mathrm{mir}, \mathbb{k}}([f\Omega])$ is obtained by first applying the mirror map (5.4.13) from [28] to $[f\Omega]$, and then pulling back under the change of variables $\tilde{q}_i \mapsto q_i e^{\sum_k \lambda^k y_{i,k}}$. Hence $\Phi_{\mathrm{mir}, \mathbb{k}}([f\Omega])$ is a power series in the variables $\{q_i e^{\sum_k \lambda^k y_{i,k}}\}_{1 \leq i \leq r}$ with coefficients in $H_T^*(X, \mathbb{C})$. For an equivariant cohomology class $\alpha \in H_T^*(X, \mathbb{C})$, we have

$$q_i \frac{\partial}{\partial q_i} \left(q_i e^{\sum_k \lambda^k y_{i,k}} \alpha \right) = \frac{\partial}{\partial y_{i,0}} \left(q_i e^{\sum_k \lambda^k y_{i,k}} \alpha \right).$$

It follows that at $y = 0$, the right hand side is

$$\begin{aligned} (u\partial_u + E_B + \mu_A)(\Phi_{\mathrm{mir}, \mathbb{k}}([f\Omega])) &= \left(u\partial_u + \sum_{i=1}^r \frac{\deg(q_i)}{2} q_i \frac{\partial}{\partial q_i} + \mu_A \right) (\Phi_{\mathrm{mir}, \mathbb{k}}([f\Omega])) \\ &= \Phi_{\mathrm{mir}, \mathbb{k}}([\mu_B(f\Omega)]), \end{aligned}$$

where the last equality follows from Lemma 5.4.25. So this implies that the bundle map is compatible with u -direction at $y = 0$. This implies by Lemma 5.3.5 that Eq. (5.4.27) holds for any y . We deduce that $(\mathrm{mir}_{\mathbb{k}}, \Phi_{\mathrm{mir}, \mathbb{k}})$ is an isomorphism of F-bundles, concluding the proof. \square

The big D -module mirror symmetry for G/P

In this subsection, we prove the big D -module mirror symmetry for G/P in the framework of equivariant F-bundles. We first construct a maximal unfolding of the small B-model equivariant F-bundle by unfolding the superpotential \mathcal{W} (Construction 5.4.28). In particular, we discuss freeness in Lemma 5.4.30 and Proposition 5.4.29, and flatness in Propositions 5.4.29 and 5.4.31. We obtain the equivariant big mirror symmetry for flag varieties in Theorem 5.4.35, and deduce a non-equivariant version in Theorem 5.4.38.

Unfolding of the B-model

We fix formal variables $y = \{y_1, \dots, y_N\}$ and $\mathbf{y} = \{y_{i,k}, 1 \leq i \leq N, k \in \mathbb{N}^n\}$.

Construction 5.4.28 (Unfolded superpotential). For any $r < j \leq N$, let $\bar{f}_j := (\Phi_{\text{mir}}^{u=0, y=0})^{-1}(\sigma_{v_j}) \in \text{Jac}(X_P^\vee, \mathcal{W}, p)$. Let $f_j \in \mathcal{O}(X_P^\vee)$ be a lift of \bar{f}_j such that the function f_j is independent of y_1, \dots, y_r . The unfolded superpotential $\widetilde{\mathcal{W}}$ is

$$\widetilde{\mathcal{W}} := \mathcal{W} + \sum_{j=r+1}^N y_j f_j.$$

Similar to the previous section on the small B -model equivariant F-bundle, we now associate to $\widetilde{\mathcal{W}}$ the big B -model equivariant F-bundle

$$\mathcal{F}^{B, \text{big}} := \{(\mathcal{H}^{B, \text{big}}, \nabla^{B, \text{big}}), (\mathcal{H}_R^{B, \text{big}}, \nabla_R^{B, \text{big}}), \text{id}\} / \mathbb{k}[\mathbf{y}].$$

We first construct an R -linear (T)-structure $(\mathcal{H}_R^{B, \text{big}}, \nabla_R^{B, \text{big}})$, consisting of a $R[[y, u]]$ -module defined by

$$\begin{aligned} \mathcal{H}_R^{B, \text{big}} &:= \text{coker} \left(R[[y, u]] \otimes_{\mathbb{C}[\bar{q}_i^{\pm 1}]} \Omega^{\text{top}-1}(X_P^\vee/Z) \xrightarrow{\partial} R[[y, u]] \otimes_{\mathbb{C}[\bar{q}_i^{\pm 1}]} \Omega^{\text{top}}(X_P^\vee/Z) \right), \\ \partial &:= 1 \otimes \left(ud + d\widetilde{\mathcal{W}} \wedge - \sum_{j=1}^n \lambda_j (p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle) \wedge \right), \end{aligned}$$

equipped with a connection defined by

$$\nabla_{R, \partial y_i}^{B, \text{big}}([\omega]) := \left[\mathcal{L}_{\partial y_i}(\omega) + u^{-1} \frac{\partial \widetilde{\mathcal{W}}}{\partial y_i} \omega - u^{-1} \sum_{j=1}^n \lambda_j \left(\iota_{\partial y_i} p^* \langle (\lambda_j)^*, \text{mc}_{T^\vee} \rangle \right) \omega \right].$$

Proposition 5.4.29. *The data $(\mathcal{H}_R^{B, \text{big}}, \nabla_R^{B, \text{big}}) / R[[y]]$ defines an R -linear (T)-structure.*

Proof. We first check that $\mathcal{H}_R^{B,\text{big}}$ is a free $R[[y, u]]$ -module. By construction we have $\mathcal{H}_R^{B,\text{big}}|_{y_{\geq r+1}=0} = \mathcal{H}_R^B$, which is a finite free $R[[y_{\leq r}, u]]$ -module (see [28, p. 52]). Since $y_j\omega \in \text{im}(\partial)$ if and only if $\omega \in \text{im}(\partial)$, the element y_j is torsion-free. Hence, the freeness of $\mathcal{H}_R^{B,\text{big}}$ follows from Lemma 5.4.30 below. The flatness of $\nabla_R^{B,\text{big}}$ follows from the facts that $\frac{\partial \widetilde{\mathcal{W}}}{\partial y_i} = f_i$ is independent of y_1, \dots, y_N , that $[\mathcal{L}_{\partial y_s}, \mathcal{L}_{\partial y_t}] = 0$, and that $[\mathcal{L}_{\partial y_s}, \iota_{y_t}] = 0$. \square

Lemma 5.4.30. *Let R_0 be a commutative unital ring and M be an $R_0[[z_1, \dots, z_m]]$ -module such that z_i is torsion-free for all $1 \leq i \leq m$. Let $\Omega_1, \dots, \Omega_N \in M$. If $\{\bar{\Omega}_1, \dots, \bar{\Omega}_N\}$ is an R_0 -basis of $M/(z_1, \dots, z_m)M$, then $\{\Omega_1, \dots, \Omega_N\} \subset M$ is an $R_0[[z_1, \dots, z_m]]$ -basis of $M[[z_1, \dots, z_m]]$.*

Proof. By Nakayama lemma [132, §VIII.3, Corollary 2], $\{\Omega_1, \dots, \Omega_N\}$ generates M as an $R_0[[z_1, \dots, z_m]]$ -module. It remains to prove the freeness.

We first treat the case $m = 1$. Let $g_i(z_1) \in R_0[[z_1]]$ be coefficients such that $\sum_i g_i(z_1)\Omega_i = 0$. Since $\{\Omega_i\}$ induces a basis of M/z_1M , we have $g_i(0) = 0$. Assume g_i has no terms of degree less than or equal to $b - 1$ in z_1 , i.e. we can write $g_i = z_1^b h_i$. Then we have

$$\sum_i z_1^b h_i \Omega_i = 0.$$

Since z_1^b is torsion-free in M , we deduce that $\sum_i h_i \Omega_i = 0$, which implies $h_i(0) = 0$. Hence, g_i has no terms of degree less than or equal to b . By induction on b , this implies $g_i = 0$.

The case $m \geq 2$ follows from the case $m = 1$ by a direct induction on the number of variables. The proof is complete. \square

The \mathbb{k} -linear F-bundle $(\mathcal{H}^{B,\text{big}}, \nabla^{B,\text{big}})/\mathbb{k}[[\mathbf{y}]]$ is given by the D -module inverse image of $(\mathcal{H}_R^{B,\text{big}}, \nabla_R^{B,\text{big}})$ under the map of $R[[u]]$ -algebras $R[[y, u]] \rightarrow R[[\mathbf{y}, u]]$ given by $y_i \mapsto \sum_{k \in \mathbb{N}^n} \lambda^k y_{i,k}$. More explicitly:

$$\begin{aligned} \mathcal{H}^{B,\text{big}} &:= \mathcal{H}_R^{B,\text{big}} \otimes_{\mathbb{k}[[y, u]]} \mathbb{k}[[\mathbf{y}, u]], \\ \nabla_{\partial y_{i,k}}^{B,\text{big}}([\omega]) &:= \left[\mathcal{L}_{\partial y_{i,k}}(\omega) + u^{-1} \frac{\partial \widetilde{\mathcal{W}}}{\partial y_{i,k}} \omega - u^{-1} \sum_{j=1}^n \lambda_j \left(\iota_{\partial y_{i,k}} p^* \langle (\lambda_j)^*, \text{mc}_{T^V} \rangle \right) \omega \right], \end{aligned}$$

equipped with the u -direction connection

$$\nabla_{u\partial u}^{B,\text{big}} := \text{Gr}^{B,\text{big}} - \nabla_{E_B^{\text{big}}},$$

$$\mathrm{Gr}^{B,\mathrm{big}} := u\partial_u + E_B^{\mathrm{big}} + \mu_B,$$

where the Euler vector field is given by

$$E_B^{\mathrm{big}} := \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{y_{j,0}} + \sum_{1 \leq j \leq N, k \in \mathbb{N}^n} (1 - \ell(v_j) - |k|) y_{j,k} \partial_{y_{j,k}},$$

and where the grading on differential forms μ_B (see Construction 5.4.19) is extended to a $\mathbb{k}[\mathbf{y}, u]$ -linear operator. Similar to Proposition 5.4.22, the total grading operator $\mathrm{Gr}^{B,\mathrm{big}}$ lifts to $\mathcal{H}^{B,\mathrm{big}}$.

Proposition 5.4.31. *The big Gauss-Manin connection $\nabla^{B,\mathrm{big}}$ is flat.*

Proof. The underlying (T)-structure of $(\mathcal{H}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}})$ is flat, as it is obtained from the R -linear (T)-structure $(\mathcal{H}_R^{B,\mathrm{big}}, \nabla_R^{B,\mathrm{big}})$, whose flatness was established in Proposition 5.4.29. We check the flatness in the u -direction as follows. We have:

$$\left[\nabla_{u\partial_u}^{B,\mathrm{big}}, \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}} \right] = \left[\mathrm{Gr}^{B,\mathrm{big}}, \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}} \right] - \left[\nabla_{E_B^{\mathrm{big}}}^{B,\mathrm{big}}, \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}} \right]$$

Since the underlying (T)-structure of $(\mathcal{H}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}})$ is flat, we have

$$\left[\nabla_{E_B^{\mathrm{big}}}^{B,\mathrm{big}}, \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}} \right] = \nabla_{[E_B^{\mathrm{big}}, \partial_{y_{j,k}}]}^{B,\mathrm{big}} = -(1 - \ell(v_j) - |k|) \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}}. \quad (5.4.32)$$

We claim that the grading structure is compatible with the connection, in the sense that

$$\left[\mathrm{Gr}^{B,\mathrm{big}}, \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}} \right] = -(1 - \ell(v_j) - |k|) \nabla_{\partial_{y_{j,k}}}^{B,\mathrm{big}}. \quad (5.4.33)$$

When $1 \leq j \leq r$, the equality holds because ∇^B is flat. When $r + 1 \leq j \leq N$, the equality holds because for any $[\omega] \in \mathcal{H}^{B,\mathrm{big}}$ we have

$$\begin{aligned} \mathrm{Gr}^{B,\mathrm{big}}([f_j \omega]) - f_j \mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= [\ell(v_j) f_j \omega], \\ \mathrm{Gr}^{B,\mathrm{big}}([u^{-1} \omega]) - u^{-1} \mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= -[u^{-1} \omega], \\ \mathrm{Gr}^{B,\mathrm{big}}([\lambda^k \omega]) - \lambda^k \mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= [|k| \lambda^k \omega], \end{aligned}$$

where we use Lemma 5.4.24 and the homogeneity of the elements f_j , u^{-1} and λ^k . Note that f_j is homogeneous of degree $\ell(v_j)$ by our choice of lift in Construction 5.4.28. Now, flatness follows from equations (5.4.32) and (5.4.33), concluding the proof. \square

Similar to the A -model, we note that the quotient maps $\mathbb{k}[\mathbf{y}] \rightarrow \mathbb{k}[\mathbf{y}_{\leq r}]$ and $R[y] \rightarrow R[y_{\leq r}]$ induce an unfolding of equivariant F-bundles $\iota_B: \mathcal{F}^B \rightarrow \mathcal{F}^{B,\mathrm{big}}$.

Proposition 5.4.34. *The morphism $\iota_B: \mathcal{F}^B \rightarrow \mathcal{F}^{B,\text{big}}$ is a maximal unfolding of \mathbb{k} -linear equivariant F -bundles, with cyclic vector given by $[\Omega] := \Phi_{\text{mir}}^{-1}(1)$, where $1 \in H_T^*(X, \mathbb{k})$ is the cyclic vector on the A -side.*

Proof. We only need to check that $\mathcal{F}^{B,\text{big}}$ is a maximal unfolding of \mathcal{F}^B with cyclic vector induced from $[\Omega]$. By definition of $\mathcal{F}^{B,\text{big}}$, for $r < j \leq N$ we have

$$\Phi_{\text{mir}} \left(\left[u \nabla_{R, \partial_{y_j}}^{B,\text{big}} \right] \Big|_{y=u=0}([\Omega]) \right) = \Phi_{\text{mir}}^{u=0, y=0}(\bar{f}_j) = \sigma_{v_j}.$$

For $1 \leq j \leq r$, by Proposition 5.4.26 we have

$$\Phi_{\text{mir}} \left(\left[u \nabla_{R, \partial_{y_j}}^{B,\text{big}} \right] \Big|_{y=u=0}([\Omega]) \right) = \left[u \nabla_{\partial t_j}^A \right] \Big|_{t=u=0}(1) = \sigma_{v_j}.$$

Since Φ_{mir} is an isomorphism and $\{\sigma_{v_j}, 1 \leq j \leq N\}$ is a basis of $H_T^*(X, \mathbb{k})$, we conclude that the unfolding is maximal. \square

Mirror symmetry

Now we can show the big D -module mirror symmetry for $X = G/P$ in the following sense.

Theorem 5.4.35 (Equivariant big mirror symmetry). *There exists a unique isomorphism of equivariant F -bundles*

$$\left(\left(\text{mir}_{\mathbb{k}}^{\text{big}}, \Phi_{\text{mir}, \mathbb{k}}^{\text{big}} \right), \left(\text{mir}^{\text{big}}, \Phi_{\text{mir}}^{\text{big}} \right) \right): \mathcal{F}^{B,\text{big}} \longrightarrow \mathcal{F}^{A,\text{big}}$$

extending the small mirror map $\left((\text{mir}_{\mathbb{k}}, \Phi_{\text{mir}, \mathbb{k}}), (\text{mir}, \Phi_{\text{mir}}) \right)$ in Proposition 5.4.26.

Proof. By Proposition 5.4.10, we have a maximal unfolding of the small A -model equivariant F -bundle, given by $\mathcal{F}^A \rightarrow \mathcal{F}^{A,\text{big}}$. Composing the small mirror isomorphism $\left((\text{mir}_{\mathbb{k}}, \Phi_{\text{mir}, \mathbb{k}}), (\text{mir}, \Phi_{\text{mir}}) \right): \mathcal{F}^A \rightarrow \mathcal{F}^B$ of Proposition 5.4.26, with the B -model maximal unfolding $\iota: \mathcal{F}^B \rightarrow \mathcal{F}^{B,\text{big}}$, we obtain another maximal unfolding $\mathcal{F}^A \rightarrow \mathcal{F}^{B,\text{big}}$.

To apply Theorem 5.3.36 and obtain a unique isomorphism $\mathcal{F}^{A,\text{big}} \rightarrow \mathcal{F}^{B,\text{big}}$, we need to check the small A -model equivariant F -bundle satisfies the (GC') condition as in Definition 5.3.15. We take $v = 1 \in H_T^*(X, \mathbb{k}) = \mathcal{H}^A|_{\tau=u=0}$. After base change to $\text{Frac}(R) = \mathbb{k}(\lambda)$, we have the evaluation map of $\mathbb{k}(\lambda)$ -modules:

$$\mu_{v=1}: \bigoplus_{1 \leq j \leq r} \mathbb{k}(\lambda) \partial_{\tau_j} \longrightarrow H_T^*(X, \mathbb{k}) \otimes \mathbb{k}(\lambda), \quad (5.4.36)$$

$$\partial_{\tau_j} \mapsto \mu(\partial_{\tau_j})(1) = \sigma_{v_j} + \lambda_{i_j}.$$

By Lemma 5.4.6, $H_T^*(X, \mathbb{k}) \otimes \mathbb{k}(\lambda)$ is generated as a $\mathbb{k}(\lambda)$ -algebra by σ_{v_j} for $1 \leq j \leq r$, so the orbit of $v = 1$ under the action of $R[\text{im } \mu]$ is the fiber $\mathcal{H}_R^A/(\tau_1, \dots, \tau_N, u)\mathcal{H}_R^A \otimes_R \text{Frac}(R)$, and (GC') is verified. The proof is complete. \square

We deduce a non-equivariant limit of the theorem by applying the base-change associated to the quotient map $R \rightarrow R/(\lambda)$. This corresponds to setting $\lambda = 0$ in all the previous formulas. We use the superscript λ_0 to indicate that the non-equivariant limit is taken. We note that since $R/(\lambda) \simeq \mathbb{k}$, in the non-equivariant limit the equivariant F-bundles can be reduced to \mathbb{k} -linear F-bundles (see Remark 5.2.12).

On the A-side, we have $\tau_i = \sum \lambda^k \tau_{i,k} = \tau_{i,0}$. The big quantum D -module $(\mathcal{H}^{A,\text{big},\lambda_0}, \nabla^{A,\text{big},\lambda_0})$ is an F-bundle over $\mathbb{k}[[\tau]] = \mathbb{k}[[\tau_1, \dots, \tau_N]]$ defined by

$$\begin{aligned} \mathcal{H}^{A,\text{big},\lambda_0} &= H^*(X, \mathbb{C}) \otimes \mathbb{k}[[\tau, u]], \\ \nabla_{\partial_{\tau_j}}^{A,\text{big},\lambda_0} &= \partial_{\tau_j} + u^{-1} \sigma_{v_j} \star^{\text{big},\lambda_0}, \\ \nabla_{u\partial_u}^{A,\text{big},\lambda_0} &= \text{Gr}^{A,\text{big},\lambda_0} - \nabla_{E_A^{\text{big},\lambda_0}}^{A,\text{big},\lambda_0}, \end{aligned}$$

where $\text{Gr}^{A,\text{big},\lambda_0} = u\partial_u + E_A^{\text{big},\lambda_0} + \mu_A$, with the Euler vector field

$$E_A^{\text{big},\lambda_0} := \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{\tau_j} + \sum_{1 \leq j \leq N} (1 - \ell(v_j)) \tau_j \partial_{\tau_j}.$$

On the B-side, we have $y_i = \sum \lambda^k y_{i,k} = y_{i,0}$. The non-equivariant big Gauss-Manin system is an F-bundle $(\mathcal{H}^{B,\text{big},\lambda_0}, \nabla^{B,\text{big},\lambda_0})$ over $\mathbb{k}[[y]] = \mathbb{k}[[y_1, \dots, y_N]]$ defined by

$$\begin{aligned} \mathcal{H}^{B,\text{big},\lambda_0} &= \text{coker} \left(\mathbb{k}[[y, u]] \otimes \Omega^{\text{top}-1}(X_P^\vee/Z) \xrightarrow{1 \otimes (ud + d\widetilde{\mathcal{W}} \wedge)} \mathbb{k}[[y, u]] \otimes \Omega^{\text{top}}(X_P^\vee/Z) \right), \\ \nabla_{\partial_{y_j}}^{B,\text{big},\lambda_0} &= \mathcal{L}_{\partial_{y_j}} + \frac{\partial \widetilde{\mathcal{W}}}{\partial y_j}, \\ \nabla_{u\partial_u}^{B,\text{big},\lambda_0} &= u\partial_u - u^{-1} \widetilde{\mathcal{W}} \quad (\text{by Proposition 5.4.37 below}). \end{aligned}$$

Proposition 5.4.37. *At the non-equivariant limit, we have $\nabla_{u\partial_u}^{B,\text{big},\lambda_0} = u\partial_u - u^{-1} \widetilde{\mathcal{W}}$.*

Proof. In this proof, to simplify notations we drop the superscript λ_0 . For any $[\omega] \in \mathcal{H}^{B,\text{big}}$, we can write $\omega = g\Omega$ where $g \in \mathcal{O}(X_P^\vee)[[y, u]]$ and Ω is given in Proposition 5.4.12(2). We have

$$\begin{aligned} \nabla_{u\partial_u}^{B,\text{big}}([g\Omega]) &= \left[u\partial_u(g\Omega) + E_B^{\text{big}}(g)\Omega + \mu_B(g\Omega) \right] - \nabla_{E_B^{\text{big}}}^{B,\text{big}}([g\Omega]) \\ &= \left[u\partial_u(g\Omega) + \mu_B(g\Omega) - u^{-1}E_B^{\text{big}}(\widetilde{\mathcal{W}})g\Omega \right] \\ &= \left[u\partial_u(g\Omega) + \mu_B(g\Omega) - u^{-1}g\text{Gr}^{B,\text{big}}(\widetilde{\mathcal{W}}\Omega) + g\partial_u(\widetilde{\mathcal{W}})\Omega + g\mu_B(\widetilde{\mathcal{W}}\Omega) \right]. \end{aligned}$$

We claim that $\text{Gr}^{B,\text{big}}(\widetilde{\mathcal{W}}\Omega) = \widetilde{\mathcal{W}}\Omega$, or equivalently that $\widetilde{\mathcal{W}}$ is homogeneous of degree 1 as an element of $\mathcal{O}(X_P^\vee)[[y, u]]$. By [27, Lemma 4.3], we have $\Phi_{\text{mir}}([\mathcal{W}\Omega]) = c_1(G/P)$, hence \mathcal{W} is homogeneous of degree 1 since Φ_{mir} is graded (Proposition 5.4.12(5)). For $r < j \leq N$, our choice of lift in Construction 5.4.28 shows that $\text{Gr}^{B,\text{big}}(y_j f_j \Omega) = E_B^{\text{big}}(y_j) f_j \Omega + y_j \mu_B(f_j \Omega) = y_j f_j \Omega$. Since $\widetilde{\mathcal{W}}$ is independent of u , we have

$$\nabla_{u\partial_u}^{B,\text{big}}([g\Omega]) = \left[u\partial_u(g\Omega) - u^{-1}\widetilde{\mathcal{W}}g\Omega \right] + \left[u\mu_B(g\Omega) + g\mu_B(\widetilde{\mathcal{W}}\Omega) \right].$$

By [28, Remark 4.13] and the references therein, after fixing local coordinates $(z_i)_i$ we can write $\Omega = \frac{\bigwedge_{i=1}^K dz_i}{z_1 \dots z_K}$. By Lemma 5.4.24, we have

$$\mu_B(g\Omega) = \sum_{i=1}^K \frac{\partial g}{\partial z_i} \mu_B(z_i \Omega).$$

Consider the $K-1$ form $\eta := \sum_{i=1}^K g \iota_{\partial_{z_i}} \mu_B(z_i \Omega)$. We have:

$$\begin{aligned} u d\eta &= u \sum_{i,j=1}^K \frac{\partial g}{\partial z_j} dz_j \wedge \iota_{\partial_{z_i}} \mu_B(z_i \Omega) + u g \sum_{i=1}^K d(\iota_{\partial_{z_i}} \mu_B(z_i \Omega)) \\ &= u \sum_{i=1}^K \frac{\partial g}{\partial z_i} \mu_B(z_i \Omega) = u \mu_B(g\Omega), \end{aligned}$$

and

$$\begin{aligned} d\widetilde{\mathcal{W}} \wedge \eta &= \sum_{i=1}^K \frac{\partial \widetilde{\mathcal{W}}}{\partial z_i} dz_i \wedge (g \iota_{\partial_{z_i}} \mu_B(z_i \Omega)) \\ &= g \mu_B(\widetilde{\mathcal{W}}\Omega). \end{aligned}$$

We deduce that

$$\left[u\mu_B(g\Omega) + g\mu_B(\widetilde{\mathcal{W}}\Omega) \right] = \left[(ud + d\widetilde{\mathcal{W}} \wedge) \eta \right] = 0,$$

concluding the proof. \square

As a direct consequence of Theorem 5.4.35, we obtain the following non-equivariant big D -module mirror symmetry.

Theorem 5.4.38 (Non-equivariant big mirror symmetry). *The non-equivariant limit of $(\text{mir}_{\mathbb{k}}^{\text{big}}, \Phi_{\text{mir}, \mathbb{k}}^{\text{big}})$ in Theorem 5.4.35 gives an isomorphism of \mathbb{k} -linear F -bundles*

$$\left(\mathcal{H}^{A, \text{big}, \lambda_0}, \nabla^{B, \text{big}, \lambda_0} \right) \xrightarrow{\sim} \left(\mathcal{H}^{B, \text{big}, \lambda_0}, \nabla^{B, \text{big}, \lambda_0} \right).$$

This isomorphism is uniquely determined by the non-equivariant small mirror isomorphism.

Proof. The existence is clear. The uniqueness follows from [66, Proposition 4.27], which applies because the F -bundles are maximal and admit framings. \square

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