# Conformal Laminations on the Circle

Thesis by

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# Abstract

A lamination $L$ on $\mathbf{T}$ is an equivalence relation on $\mathbf{T}$. In this paper, we consider laminations $L(\varphi)$ induced by some continuous mapping $\varphi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ which is one-to-one in $\mathbf{D}$, i.e., $\xi \overset{L(\varphi)}{\sim} \eta$ if and only if $\varphi(\xi) = \varphi(\eta)$ for any $\xi$, $\eta \in \mathbf{T}$. The laminations arise as in above can be characterized topologically as continuous and flat laminations. Our major question is what conditions on $L$ can ensure that $L{=}L(\varphi)$ for some continuous mapping $\varphi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ which is conformal in $\mathbf{D}$. In this paper, we study various aspects of conformal laminations and get conditions for conformality in several configurations. We relate the conformal welding problem with the classical conformal sewing problem. By the extremal length method, we obtain a generalization of Oikawa's condition for conformal sewings to a sufficient condition for conformal laminations and also obtain necessary conditions for conformal laminations. We prove that a continuous, flat lamination $L$ on $\mathbf{T}$ is conformal if cap$\bar{M_L}{=}0$ and the quotient space $\bar{M_L}/L$ is a totally disconnected set where $M_L$ is the set of multiple points of $L$. Let $E$ be a compact subset of $\mathbf{T}$. Suppose that $I_n = (a_n, b_n)$ are the components of the set $\mathbf{T} \setminus E$. We define $L$ to be the lamination that identifies $a_n$ and $b_n$ for each $n$. We prove that if cap$E > 0$ and $E$ is Dirichlet regular, then the lamination $L$ as described above is conformal. Furthermore, the quotient space $E/L$ is homeomorphic to the unit circle. We also conjecture that: If cap$M_L = 0$, then $L$ is conformal.

# Contents

# Chapter 1 Introduction

A lamination $L$ on $\mathbf{T}$ is an equivalence relation on $\mathbf{T}$. In other words, a lamination $L$ on $\mathbf{T}$ is a way to identify points on the unit circle. For any lamination $L$ on $\mathbf{T}$, $L$ can be extended to an equivalence relation on $\bar{\mathbf{D}}$ where $\mathbf{D}=\{|z| < 1 : z \in \mathbf{C}\}$ by including $(z, z)$ in $L$ for all $z \in \mathbf{D}$. In the remaining parts of this paper, a lamination on $\mathbf{T}$ is also considered as an equivalence relation on $\bar{\mathbf{D}}$ in the above sense. In this paper, we consider laminations induced by some continuous mapping $\varphi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ which is one-to-one in $\mathbf{D}$.

**Definition 1** *Given a mapping $\varphi$ as in above, the lamination $L(\varphi)$ induced by $\varphi$ on $\mathbf{T}$ is defined by*

$$\xi \overset{L(\varphi)}{\sim} \eta \text{ if and only if } \varphi(\xi) = \varphi(\eta)$$

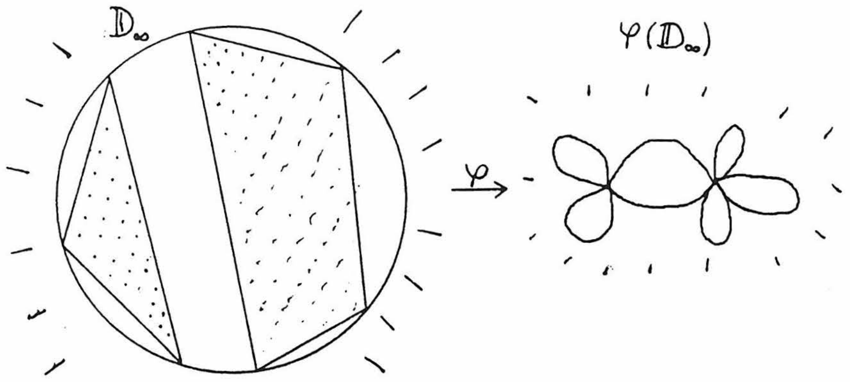
*for any $\xi$, $\eta \in \mathbf{T}$.*

The laminations arise as in the above definition can be characterized topologically as continuous and flat laminations. The condition of continuity comes from the continuity of the mapping $\varphi$ and the flatness condition comes from the separation properties of the Riemann sphere.

**Definition 2** *A lamination $L$ on $\mathbf{T}$ is continuous (or closed) if $L$ is closed in $\mathbf{T}\times\mathbf{T}$, i.e., $\xi_n, \eta_n \in \mathbf{T}$, $\xi_n \sim \eta_n$ and $\xi_n, \eta_n \to \xi, \eta$ respectively as $n \to \infty$ implies $\xi = \eta$.*

**Definition 3** *A lamination $L$ on $\mathbf{T}$ is flat if for any $\xi_1, \xi_2, \eta_1, \eta_2$ on $\mathbf{T}$ such that $\xi_1 \sim \xi_2$ and $\eta_1 \sim \eta_2$, $\eta_1$ and $\eta_2$ lie on different components of $\mathbf{T}\backslash\{\xi_1, \xi_2\}$ implies $\xi_1 \sim \eta_1$.*

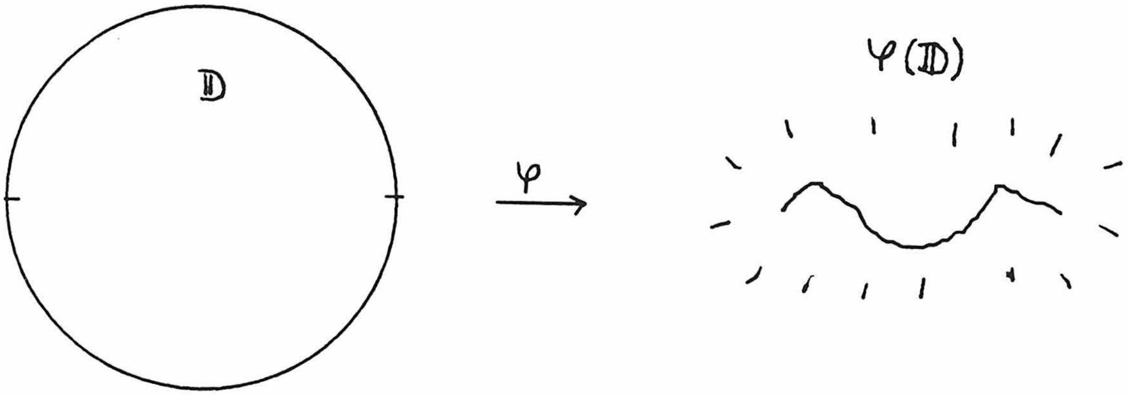
In [23], Thurston regarded the flatness condition to be the disjointness of convex hulls of equivalence classes in $\bar{\mathbf{D}}$. Intuitively, we collapse each convex hull to a point to get the quotient space $\bar{\mathbf{D}}_\infty/L$ which can be embedded in the sphere topologically where $\mathbf{D}_\infty = \{z \in \mathbf{C} : |z| > 1\}$. We are going to give more details in this in the next chapter.



**Definition 4** *A lamination* $L$ *on* $\mathbf{T}$ *is* ***conformal*** *if* $L=L(\varphi)$ *for some continuous mapping* $\varphi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ *which is conformal in* $\mathbf{D}$.

### Examples of Laminations

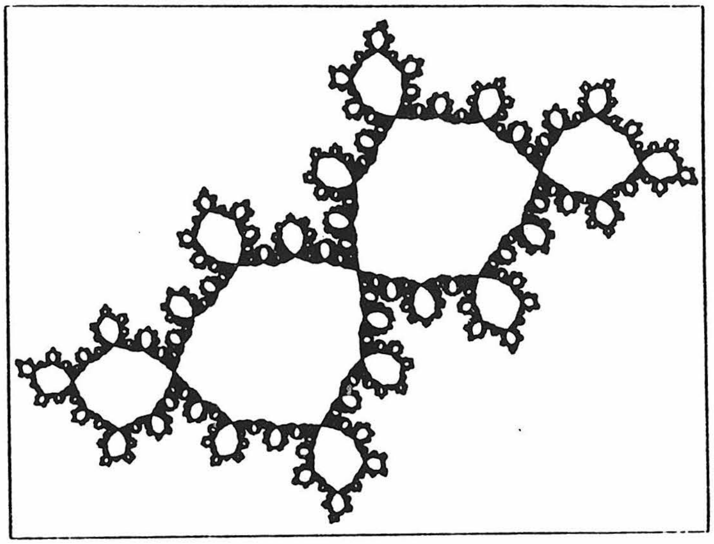
(1) (Welding) If the upper and lower semi-circles are welded by mean of a homeomorphism, we have a lamination on $\mathbf{T}$. The conformality of this type of laminations is closely related to Riemann-Hilbert boundary value problem and the type problem of open Riemann surfaces. This was studied by many authors before and the problem is only partially solved. There is not much known about conformality beyond the quasisymmetric case (cf **[4, 11, 19, 20, 21, 24]**).



(2) (Julia Set) There is a very important type of laminations introduced by Thurston in [**23**] which is obtained from Julia sets of quadratic polynomials. This is closely related to the local connectivity of Julia sets which has been one of the central questions in complex dynamics.

Let $J(c)$ be the Julia set for the polynomial $z^2 + c$ $(c \in \mathbf{C})$, $\Omega = \hat{\mathbf{C}} \setminus \overline{J}(c)$ ($\overline{J}(c)$ is the filled-in Julia set). If $\partial\Omega$ is locally connected, then $c \in J(c)$. Then let $\theta$ be the dyadic expression of an external angle of $c$. We can obtain a lamination on $\bigcup T^{-n}\theta$ $(T : z \mapsto z^2)$ from the conjugate dynamic of iterating $z^2 + c$ and the flatness condition. Thus we have a lamination on $\mathbf{T}$ by continuous extension.

This lamination is determined by combinatorics. The problem is whether there is a locally connected set corresponding to this lamination. If not, $J(c)$ is not locally connected.



Julia set for $z \mapsto z^2 + e^{2\pi i t}z$ with $t = (\sqrt{5} - 1)/2$.

Let $M_L = \{x \in \mathbf{T} : x \overset{L}{\sim} y \text{ for some } y \neq x\}$ be the set of multiple points of $L$. We formulate our main conjecture as follows:

**Main Conjecture** *Let $L$ be a continuous, flat lamination on $\mathbf{T}$. If* $\operatorname{cap} M_L = 0$, *then $L$ is conformal.*

The zero capacity condition is essential since we cannot collapse a set with positive capacity to a point via conformal mappings.

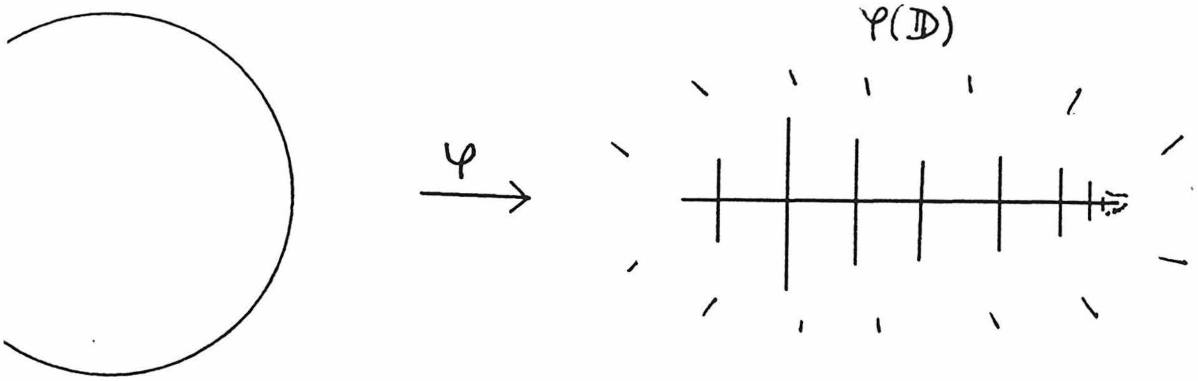
In this paper, we study various aspects of conformal laminations and get conditions for conformality in several configurations including some special cases of the conjecture above. Our first result is on a special configuration of quadrilaterals. Let $0 < \tilde{x}_{n+1} < x_n < \tilde{x}_n < \ldots < \tilde{x}_1 < \pi$, $-\pi < y_1 < \ldots < y_n < \tilde{y}_n < y_{n+1} < 0$, and $\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n = \lim_{n\longrightarrow\infty} \tilde{x}_n = \lim_{n\longrightarrow\infty} \tilde{y}_n = 0$.

**Theorem 1** *There is a continuous mapping $\varphi$ from $\bar{\mathbf{D}}$ onto $\hat{\mathbf{C}}$ which is conformal in $\mathbf{D}$ such that*

$$\varphi^{-1}(z_n) = \{e^{ix_n}, e^{i\tilde{x}_n}, e^{iy_n}, e^{i\tilde{y}_n}\}$$

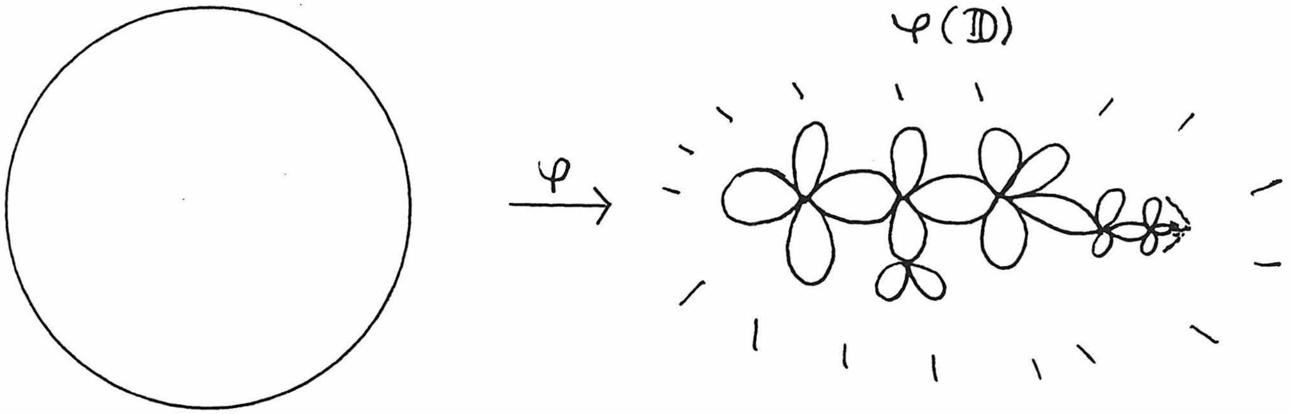
*where $z_n = \varphi(x_n)$ for each $n > 0$.*

The proof of this employs the extremal length and type problem method which is extremely useful in similar kinds of problems. We are going to show the application of this technique to the conformal lamination problem in chapter 3. The proof of the above theorem is also presented to illustrate how the technique is applied.

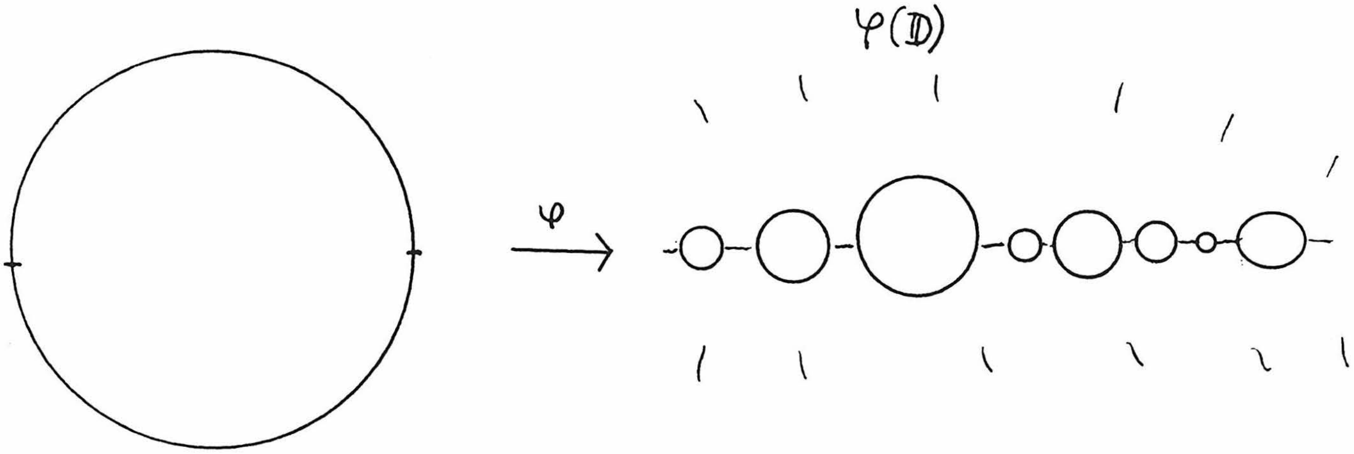


**Theorem 2** *Let $L$ be a continuous, flat lamination on $\mathbf{T}$. If $M_L$ has finitely many limit points, then $L$ is conformal.*

This theorem is a non-trivial fact. Indeed, there can be countably many multiple points which seem to be an obstruction at the first glance. The above theorem can be derived as a corollary of the following which we are going to prove in chapter 4 with a potential theory approach and perturbation methods.



**Theorem 3** *Let $L$ be a continuous, flat lamination on* $\mathbf{T}$. *If* $\mathrm{cap}\bar{M}_L=0$ *and the quotient space* $\bar{M}_L/L$ *is a totally disconnected set, then $L$ is conformal.*



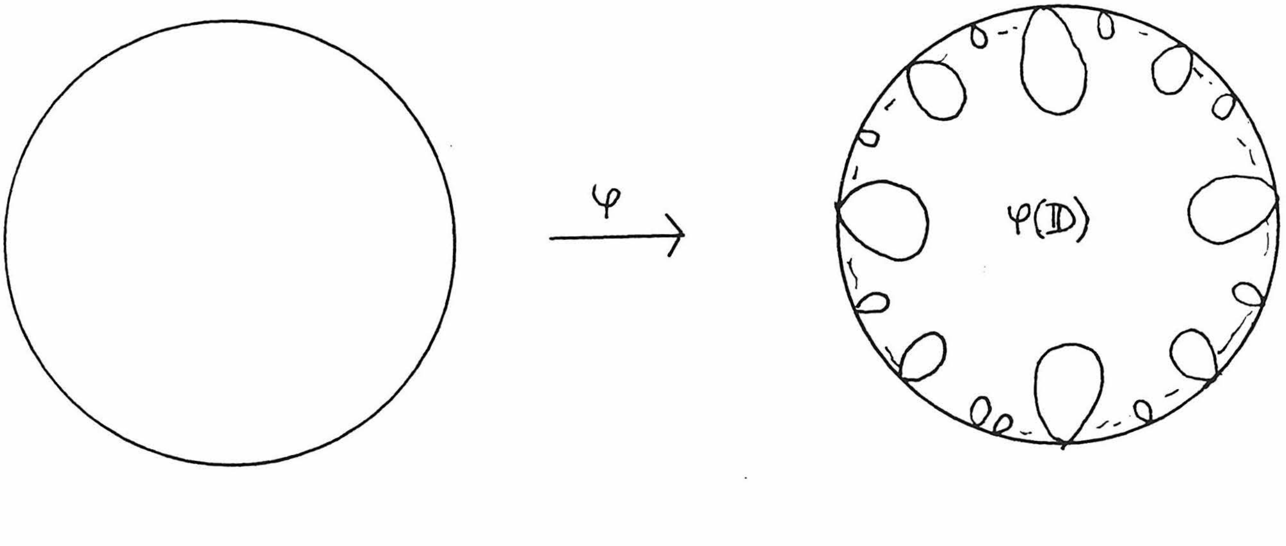
We shall also give an example which demonstrates the importance of the zero capacity condition for laminations with possibly only double points. Let $h : I_- \longrightarrow I_+$ be a homeomorphism fixing endpoints, $E_- \subset I_-$ be a Cantor set, and $E_+ = h(E_-)$.

Let $L$ be defined by $x \sim y$ if $x \in E_-$ and $y = h(x)$, or $y \in E_-$ and $x = h(y)$, or $x = y$. If $\text{cap}(E_- \cup E_+)=0$, then $L$ is conformal. If $\Lambda_1(E_- \cup E_+) > 0$, then we have examples of $h$ and $E_-$ such that we do not have a conformal lamination $L$ (the idea is given in chapter 3). From chapter 2, any two quasisymmetrically equivalent laminations are conformal or not simultaneously. Therefore we have a bad lamination with possibly only double points on a set of arbitrarily small Hausdorff dimension.

On the other hand, there is a completely different type of laminations with countably many multiple points. Let $E$ be a compact subset of $\mathbf{T}$. Suppose that $I_n = (a_n, b_n)$ are the components of the set $\mathbf{T} \setminus E$. We define $L$ to be the lamination that identifies $a_n$ and $b_n$ for each $n$.

**Theorem 4** *If* $\text{cap}E > 0$ *and* $E$ *is Dirichlet regular, then the lamination* $L$ *as described above is conformal. Furthermore, the quotient space* $E/L$ *is homeomorphic to the unit circle.*

This type of laminations are topologically on the other extreme end as those in Theorem 3. The proof is also from potential theory and perturbation methods and so we shall do it in chapter 4. Finally, it should be noted that our main conjecture has not been completely proven in this paper yet. Further work can be done on improving our results or giving counterexamples that turn down our conjecture.



# Chapter 2 Background

We are going to present a brief history of the conformal lamination problem. In addition, we prove a couple of simple properties of conformal laminations which extend the corresponding properties in conformal sewings. Then we try to reduce the conformal welding problem to the classical conformal sewing problem. Finally, we characterize John disk condition in terms of laminations as an application of this approach.

## 2.1 Quasiconformal Mappings

Let **C** be the complex plane and $\Omega \subseteq \mathbf{C}$ be a plane domain. Then a sense-preserving homeomorphism $h : \Omega \longrightarrow \mathbf{C}$ is quasiconformal if

(1) $h(x+iy)$ is absolutely continuous in $x$ for almost all $y$ and absolutely continuous in $y$ for almost all $x$,

(2) the partial derivatives are locally integrable and satisfy the Beltrami equation

$$\frac{\partial h}{\partial \bar{z}} = \mu(z)\frac{\partial h}{\partial z} \text{ for almost all } z \in \Omega$$

where $\mu$ is a complex measurable function with $|\mu(z)| \leq \kappa < 1$ for $z \in \Omega$. The function $\mu$ is often called the complex dilation of $h$.

There is a completely geometrical definition of quasiconformal maps in terms of (conformal) modules. A quadrilateral in $\hat{\mathbf{C}}$ consists of a Jordan domain $Q$ and a sequence $z_1, z_2, z_3, z_4$ of boundary points of $Q$ with positive orientation with respect to $Q$, and we shall denote it by $Q(z_1, z_2, z_3, z_4)$. It follows from the Riemann mapping theorem and classical theory of elliptic integrals that any quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be mapped conformally onto a rectangle. All such rectangles are corresponded under similarity transformations and therefore have the same ratio $\frac{a}{b}=\mathrm{M}(Q)$ where $a$ is the length of sides corresponding to the subarcs of $\partial Q$ that joins $z_1$ to $z_2$ and $z_3$ to $z_4$ respectively with positive orientations with respect to $Q$, and $b$ is the length of sides corresponding to the subarcs of $\partial Q$ that joins $z_2$ to $z_3$ and $z_4$ to $z_1$ respectively with positive orientations with respect to $Q$. $\mathrm{M}(Q)$ is called the module of $Q$. A sense-preserving homeomorphism $h : \Omega \longrightarrow \mathbf{C}$ is quasiconformal if $K(\Omega) = \sup_{\bar{Q}\subseteq\Omega} \frac{\mathrm{M}(h(Q))}{\mathrm{M}(Q)}$ is finite (cf **[2]**, **[12]**).

For any complex measurable function $\mu(z)$ in a plane domain $\Omega$, we have the following fundamental theorem in the theory of quasiconformal mappings which is known as the "Existence Theorem" in **[12]**.

**Existence Theorem** *If $\mu$ is a complex measurable function in a plane domain $\Omega$ with $\sup_{z\in\Omega} |\mu(z)| < 1$, then there exists a quasiconformal map $h$ of $\Omega$ which satisfies the Beltrami equation*

$$\frac{\partial h}{\partial \bar{z}} = \mu(z)\frac{\partial h}{\partial z} \text{ for almost all } z \in \Omega.$$

Concerning the boundary behavior of quasiconformal mappings in the unit disk, Beurling-Ahlfors **[4]** found a condition for boundary maps for quasiconformal mappings in the unit disk which is called quasisymmetry nowadays. Let $\mathbf{T} = \{|z| = 1 : z \in \mathbf{C}\}$ and $f : \mathbf{T} \longrightarrow \mathbf{C}$ be a sense-preserving embedding. $f$ is quasisymmetric if there is an $M > 0$ such that

$$\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|} \leq M\frac{|f(z_2)-f(z_3)|}{|z_2-z_3|} \text{ for any } z_1, z_2, z_3 \in \mathbf{T}$$

It was first proved by Beurling-Ahlfors **[4]** that any quasisymmetric map from $\mathbf{T}$ to $\mathbf{T}$ can be extended to a homeomorphism of the closed unit disk which is a quasiconformal mapping in $\mathbf{D}$. There is a better extension by Douady-Earle **[7]**.

**Theorem (Douady-Earle)** *The sense-preserving homeomorphism $\varphi$ of $\mathbf{T}$ onto $\mathbf{T}$ can be extended to a homeomorphism $\tilde{\varphi}$ of $\bar{\mathbf{D}}$ onto $\bar{\mathbf{D}}$ that is real analytic in $\mathbf{D}$ and has the following properties:*

*(1) If $\sigma$, $\tau \in M\ddot{o}b(\mathbf{D})$, then the extension of $\sigma \circ \varphi \circ \tau$ is given by $\sigma \circ \tilde{\varphi} \circ \tau$.*

*(2) If $\varphi$ is quasisymmetric, then $\tilde{\varphi}$ is quasiconformal.*

It follows that (for a proof, see [**12**])

**Theorem** *Let $\varphi : \mathbf{T} \longrightarrow \mathbf{C}$ be a sense-preserving embedding. Then $\varphi$ can be extended to a quasiconformal map from $\mathbf{C}$ onto $\mathbf{C}$ if and only if $\varphi$ is quasisymmetry.*

The image of $\mathbf{T}$ under a quasiconformal map from $\mathbf{C}$ onto $\mathbf{C}$ is called a quasicircle (cf [**9**]).

**Quasicircle Theorem** *Let $J$ be a Jordan curve in $\mathbf{C}$ and let $f$ be a conformal map from $\mathbf{D}$ onto the inner domain of $J$. Then the following statements are equivalent:*

*(1) $J$ is a quasicircle;*

*(2) $\mathrm{diam} J(a,b) \leq M|a-b|$ for some $M > 0$ and $a, b \in J$ where $J(a,b)$ is the smaller arc of $J$ between $a$ and $b$;*

*(3) $f$ has a quasiconformal extension to $\mathbf{C}$;*

*(4) $f$ is quasisymmetric in $\mathbf{T}$.*

## 2.2 The Conformal Sewing Problem

Two plane domains or bounded Riemann surfaces can be conformally sewn together into a single Riemann surface by an identification of two boundary arcs (cf [**1, p.4,118**]). If the sewing takes place on the unit circle (or the real axis), the condition can be replaced by much weaker conditions than analyticity. The problem is equivalent to find conditions on the identification such that there are conformal mappings $\varphi_1$, $\varphi_2$ which map the interior and the exterior of the unit disk onto the interior and exterior of some Jordan domain and $\varphi_2^{-1} \circ \varphi_1$ equals the given identification. Based on the existence and uniqueness for quasiconformal mappings with prescribed dilation, Pfluger and Oikawa proved that a quasisymmetric map from $\mathbf{T}$ onto $\mathbf{T}$ always admits sewing (cf [**12**], [**19**], [**21**]).

**Theorem** *Let $f : \mathbf{T} \longrightarrow \mathbf{T}$ be a sense-preserving homeomorphism. Then there exist* *a quasicircle $J$ and conformal mappings $\varphi_1 : \mathbf{D} \longrightarrow \Omega$, $\varphi_2 : \mathbf{D}_\infty \longrightarrow \Omega_\infty$ such that $\varphi_2^{-1} \circ \varphi_1|_{\mathbf{T}} = f$ if and only if $f$ is quasisymmetric where $\Omega$ and $\Omega_\infty$ are the interior and exterior domain of $J$.*

Lehto [**11**], Oikawa [**19**], Pfluger [**21**] and David [**6**] proved conformal sewing for other classes of homeomorphisms.

It should be noted that conformal sewing is not always possible for a homeomorphism from $\mathbf{T}$ onto $\mathbf{T}$.

**Examples:**

(1) For any $0 < a \leq 1$ and $b > a$, let

$$f(e^{i\theta}) = \begin{cases} e^{i\pi(\theta/\pi)^a} & \text{if } 0 \leq \theta \leq \pi \\ e^{-i\pi(-\theta/\pi)^b} & \text{if } -\pi \leq \theta \leq 0. \end{cases}$$

There is no conformal sewing with the identification under $f$. It was shown in Oikawa [**19**] by using extremal length method on the type of Riemann surface formed by sewing the upper half-plane to the lower half-plane with 0 as a singularity.

(2) There is another example function with no sewing. Let

$$\begin{aligned} g_1 &: \{x+iy : x > 0, y < \sin\frac{1}{x}\} \longrightarrow \{x+iy : x > 0, y < 0\} \\ g_2 &: \{x+iy : x > 0, y > \sin\frac{1}{x}\} \longrightarrow \{x+iy : x > 0, y > 0\} \end{aligned}$$

be conformal. In [**24**], Vainio showed that

$$\varphi(x) = \begin{cases} g_2 \circ g_1^{-1}(x) & \text{if } x > 0 \\ 0 & x = 0 \\ -g_2 \circ g_1^{-1}(-x) & \text{if } x < 0 \end{cases}$$

is a homeomorphism. There is no sewing by considering the module of the ring domain formed by gluing the upper and lower half-plane along the real line with singularity at point 0.

## 2.3 Laminations

When $\varphi$ is a continuous mapping from $\bar{\mathbf{D}}$ into a compact, locally path-connected metric space $(X, d)$ and $\varphi|_{\mathbf{D}}$ is one-to-one, we can see that $\varphi$ defines a lamination on $\mathbf{T}$ by identifying points which are mapped to same point in $X$.

**Definition 2.3.1** *Let $(X, d)$ be a compact, locally path-connected, metric space. For any continuous mapping $\varphi$ from $\bar{\mathbf{D}}$ into $X$ which is one-to-one in $\mathbf{D}$, the lamination $L(X, \varphi)$ (when $X$ is the complex plane or Riemann Sphere, we do not specify it in the notation and just write $L(\varphi)$) on $\mathbf{T}$ induced by $\varphi$ is defined by*

$$\xi \sim \eta \text{ if } \varphi(\xi) = \varphi(\eta) \text{ for } \xi, \eta \in \mathbf{D}.$$

There is a complete characterization for laminations induced by the $\varphi$'s in the above definition. If $L$ is a lamination induced by some continuous mapping from $\mathbf{D}$ into a metric space, then it is not hard to show that $L$ is continuous. On the other hand, for a continuous lamination $L$ on $\mathbf{T}$, the quotient space $\bar{\mathbf{D}}/L$ is metrizable (c.f. [5]). Thus we have the following:

**Theorem 2.3.2** *Let $L$ be a lamination on $\mathbf{T}$. Then $L = L(X, \varphi)$ for some compact, locally path-connected, metric space $X$ and some $\varphi \in C(\bar{\mathbf{D}}, X)$ which is one-to-one in $\mathbf{D}$ if and only if $L$ is continuous.*

When $\varphi$: $\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ is continuous in $\bar{\mathbf{D}}$ and one-to-one in $\mathbf{D}$, there is a big restriction on $L(\varphi)$ due to the separation properties of the sphere.

**Theorem 2.3.3 (Thurston)** *Let $L$ be a continuous lamination on $\mathbf{T}$. Then $L$ is flat if and only if the convex hulls of the points which are equivalent under $L$ are disjoint.*

Proof: Suppose that $L$ is flat. Let $\xi_1, ..., \xi_n \sim \xi$ and $\eta_1, ..., \eta_m \sim \eta$ such that $\xi$ is not equivalent to $\eta$ under $L$. Then $\eta_1, ..., \eta_m$ lie on the same component $I$ of $\mathbf{T}\backslash\{\xi_1, ..., \xi_n\}$ and so the convex hull of $\{\eta_1, ..., \eta_m\}$ is a subset of the convex hull of $I$ which is disjoint from $\{\sum_{i=1}^{n} t_i\xi_i : 0 \leq t_i \leq 1, \sum_{i=1}^{n} t_i = 1\}$. Hence the convex hull of the points equivalent to $\xi$ is disjoint from the convex hull of the points equivalent to $\eta$.

Conversely, if there are $\xi_1, \xi_2, \eta_1, \eta_2$ on $\mathbf{T}$ such that $\xi_1 \sim \xi_2$, $\eta_1 \sim \eta_2$ and $\eta_1$, $\eta_2$ lie on different components of $\mathbf{T}\backslash\{\xi_1, \xi_2\}$, then the convex hull of the points equivalent to $\xi$ is not disjoint from the convex hull of the points equivalent to $\eta$. Thus $\xi \sim \eta$.

*Q.E.D.*

It is not hard to show that continuity and flatness are necessary for a lamination $L$ on $\mathbf{T}$ to be coincide with $L(\varphi)$ where $\varphi$: $\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ continuous in $\bar{\mathbf{D}}$ and one-to-one in $\mathbf{D}$. The converse is also true by a theorem of Moore (cf **[14, 15]**) which became extremely important in the theory of foliation and geodesic laminations later.

**Theorem 2.3.4** *Let $L$ be a lamination on $\mathbf{T}$. There is some $\varphi$: $\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ continuous in $\bar{\mathbf{D}}$ and one-to-one in $\mathbf{D}$ such that $L = L(\varphi)$ if and only if $L$ is continuous and flat.*

Proof: If there is some $\varphi$: $\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ continuous in $\bar{\mathbf{D}}$ and one-to-one in $\mathbf{D}$ such that $L = L(\varphi)$, then $L$ is continuous by **2.3.2**. If $L$ is not flat, there would be smooth arcs $\alpha_i$ beginning and ending in $\mathbf{T}$ which intersect transversely in $\mathbf{D}$. In the image, they would become closed curves which cross exactly once. Contradiction occurs.

Conversely, the collection of all the convex hulls of equivalent points under $L$ is a collection of continua which satisfies the condition in **[14, 15]** which is called "Upper Semi-Continuity". In our case, the condition is just the quotient topology of $\hat{\mathbf{C}}/L$ is equivalent to the topology induced by the Hausdorff metric. In turn, the quotient space $\hat{\mathbf{C}}/L$ is homeomorphic to the Riemann sphere.

*Q.E.D.*

There is another theorem of Moore (cf **[16, 22]**) which gives a strong condition on the multiple points of $L$, namely, there are at most countably many of them.

**Theorem 2.3.5** *Let $f$ be a homeomorphism of $\mathbf{D}$ into $\mathbf{C}$. Then there are at most countably many points $a \in \mathbf{C}$ such that*

$$f(r\zeta_j) \rightarrow a \text{ as } r \rightarrow 1_- \ (j=1,2,3)$$

*for three distinct points* $\zeta_1, \zeta_2, \zeta_3$ *on* $\mathbf{T}$.

If we consider $\varphi$'s in the class of mappings $D=\{\varphi : \varphi(\bar{\mathbf{D}}) = \hat{\mathbf{C}}$, $\varphi$ is continuous in $\bar{\mathbf{D}}$ and one-to-one in $\mathbf{D}\}$, then $\partial\varphi(\mathbf{D})$ is a dendrite, i.e., a locally connected continuum which does not contain simple closed curves.

**Theorem 2.3.6** *Let* $L$ *be a lamination on* $\mathbf{T}$. *There is some* $\varphi \in D$ *such that* $L = L(\varphi)$ *if and only if* $\bar{\mathbf{D}}/L$ *is homeomorphic with the sphere if and only if* $L$ *is continuous, flat and* $\mathbf{T}/L$ *does not contain simple closed curves.*

We want to look at continuous, flat laminations on $\mathbf{T}$ which are induced by univalent maps that can be extended continuously to $\bar{\mathbf{D}}$.

**Definition 2.3.7** *Let* $C_A = \{\varphi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ *continuous in* $\bar{\mathbf{D}}$ *and univalent in* $\mathbf{D}\}$, $D_A = \{\varphi \in C_A\text{:}\ \varphi(\bar{\mathbf{D}}) = \hat{\mathbf{C}}\}$. $L$ *is conformal if* $L = L(\varphi)$ *for some* $\varphi \in C_A$.

There are some simple analogies of conformal laminations to conformal sewings. The most natural one is the invariance of conformality under a quasisymmetric map from $\mathbf{T}$ onto $\mathbf{T}$.

**Theorem 2.3.8** *Let* $L$, $\tilde{L}$ *be laminations on* $\mathbf{T}$ *such that there is a quasisymmetric map* $f : \mathbf{T} \longrightarrow \mathbf{T}$ *with* $x \sim y$ *under* $L$ *if and only if* $f(x) \sim f(y)$ *under* $\tilde{L}$. *Then* $L$ *is conformal if and only if* $\tilde{L}$ *is conformal.*

Proof: It suffices to show that $L$ is conformal implies that $\tilde{L}$ is conformal. By the Douady-Earle extension, $f$ can be extended to a homeomorphism of $\bar{\mathbf{D}}$ onto $\bar{\mathbf{D}}$ which is quasiconformal in $\mathbf{D}$.

Suppose that $L$ is conformal. Then there is some continuous $\varphi : \hat{\mathbf{D}} \longrightarrow \mathbf{C}$ which is univalent in $\mathbf{D}$ such that $L = L(\varphi)$. Let $\mu$ be defined on $\mathbf{C}$ by

$$\mu(z) = \begin{cases} \mu_f(\varphi^{-1}(z)) & z \in \varphi(\mathbf{D}) \\ 0 & \text{otherwise.} \end{cases}$$

$\mu$ is complex measure and is bounded by $\sup|\mu_f| < 1$. Therefore there exists a quasiconformal homeomorphism $W$ from $\mathbf{C}$ onto $\mathbf{C}$ with complex dilation $\mu$ by the "Existence Theorem".

Let $\tilde{\varphi} = W \circ \varphi \circ f^{-1}$ which is quasiconformal in $\mathbf{D}$ and continuous in $\bar{\mathbf{D}}$. Since $f$ is a homeomorphism,

$$\tilde{\varphi} \circ f = W \circ \varphi \qquad \text{in } \bar{\mathbf{D}}.$$

This implies that $L(\tilde{\varphi}) = L$. Also,

$$\begin{aligned} \mu_{\tilde{\varphi} \circ f}(z) &= \mu_W(\varphi(z)) \qquad \text{almost everywhere in } \mathbf{D} \text{ (as } \varphi \text{ is conformal)} \\ &= \mu_f(z). \end{aligned}$$

Hence $\mu_{\tilde{\varphi}} = 0$ almost everywhere. Thus $\tilde{\varphi}$ is conformal.

*Q.E.D.*

The restriction of any conformal lamination to a closed subarc of $\mathbf{T}$ is still a conformal lamination.

**Theorem 2.3.9** *Let $L$ be a conformal lamination on $\mathbf{T}$. Then the restriction $L|_{\bar{I}}$ to any closed subarc $\bar{I}$ is conformal where $L|_{\bar{I}}$ is defined by $\xi \sim \eta$ if and only if $\xi, \eta \in \bar{I}$ and $\xi$ is equivalent to $\eta$ under $L$.*

Proof: By applying a conformal mapping, we can assume that the arc $I = I_+$ the upper semi-circle. Let $f : \mathbf{T} \longrightarrow \{z : -1 \leq z \leq 1\} \cup I_+$ be defined by

$$f(e^{i\theta}) = \begin{cases} e^{i\theta} & \text{if } 0 \leq \theta \leq \pi \\ \cos\theta & \text{if } -\pi \leq \theta \leq 0. \end{cases}$$

Then as $f$ is quasisymmetric, it can be extended into a quasiconformal mapping in $\mathbf{D}$. Let $\varphi \in C_A$ with $L = L(\varphi)$, we can get a conformal map $\tilde{\varphi}$ such that $L(\tilde{\varphi}) = L(\varphi \circ f)$ which is exactly $L|_{\bar{I}_+}$.

*Q.E.D.*

## 2.4 A Simple Configuration: Weldings

A welding function is a homeomorphism from $I_-$ to $I_+$ fixing the endpoints 1, $-1$. A welding function $h$ determines a lamination by identifying $x$ and $h(x)$. If the lamination determined by a welding function $h$ is conformal, we call $h$ a conformal welding. In other words, $h$ is conformal if there is a $\varphi \in D_A$ such that $\varphi \circ h = h$ on $I_-$. This is closely related to the classical problem on conformal sewings of the interior of the unit disk to the exterior of the unit disk.

The converse of **2.3.9** is not true in general. An example is given by the welding $\exp(i\pi h)$:$I_- \longrightarrow I_+$ where

$$h(\theta) = \begin{cases} \theta & \text{if } -1/2 \geq \theta \geq -1 \\ 2(1/2+\theta)^2 - 1/2 & \text{if } 0 \geq \theta \geq -1/2. \end{cases}$$

It is a classical example (c.f. [19]) that there is a singularity at the point $-1/2$. However, the weldings defined by $h_1(\theta) = \theta$ and

$$h_2(\theta) = \begin{cases} -2(1/2+\theta)^2 - 1/2 & \text{if } -1/2 \geq \theta \geq -1 \\ 2(1/2+\theta)^2 - 1/2 & \text{if } 0 \geq \theta \geq -1/2 \end{cases}$$

are conformal. Actually, $h_1$ corresponds to $\varphi(z) = z - 1/z$ and $h_2$ is analytic except at points $0, -1/2, -1$ and it can be shown by extremal length method that all of them are weak components.

**Theorem 2.4.1** *Let $h : I_- \longrightarrow I_+$ be a welding function. Then the following statements are equivalent:*

(1) *$h$ is conformal.*

(2) *For some simple curve $J$ in $\mathbf{D}$ with $\sigma$-finite linear measure which joins $-1$ to $1$, the interior and exterior of the unit disk are sewn conformally via the homeomorphism $g$ on $\mathbf{T}$ defined by*

$$g(e^{i\theta}) = \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \textit{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1} \circ h \circ f(e^{i\theta}) & \textit{if } 2\pi \geq \theta \geq \pi \end{cases}$$

*where* $D_1$ *and* $D_2$ *are the components of* $\mathbf{D} \backslash J$ *with* $I_-$ *and* $I_+$ *contained in the boundary respectively,* $f = F|_{\mathbf{T}}$ *for* $F : \mathbf{D} \longrightarrow D_1$ *the Riemann map from the unit disk to* $D_1$ *and* $\tilde{f} = \tilde{F}|_{\mathbf{T}}$ *for* $\tilde{F} : \mathbf{D}_\infty \longrightarrow D_2$ *the Riemann map from the exterior of the unit disk to* $D_2$ *such that* $f(1) = \tilde{f}(1) = 1$, $f(-1) = \tilde{f}(-1) = 1$ *and* $f(I_-) = \tilde{f}(I_-) = J$.

(3) *For any simple curve* $J$ *in* $\mathbf{D}$ *which joins* $-1$ *to* $1$, *the interior and exterior of the unit disk are sewn conformally via the homeomorphism* $g$ *on* $\mathbf{T}$ *defined by*

$$g(e^{i\theta}) = \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \textit{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1} \circ h \circ f(e^{i\theta}) & \textit{if } 2\pi \geq \theta \geq \pi \end{cases}$$

*where* $D_1$ *and* $D_2$ *are the components of* $\mathbf{D} \backslash J$ *with* $I_-$ *and* $I_+$ *contained in the boundary respectively,* $f = F|_{\mathbf{T}}$ *for* $F : \mathbf{D} \longrightarrow D_1$ *the Riemann map from the unit disk to* $D_1$ *and* $\tilde{f} = \tilde{F}|_{\mathbf{T}}$ *for* $\tilde{F} : \mathbf{D}_\infty \longrightarrow D_2$ *the Riemann map from the exterior of the unit disk to* $D_2$ *such that* $f(1) = \tilde{f}(1) = 1$, $f(-1) = \tilde{f}(-1) = 1$ *and* $f(I_-) = \tilde{f}(I_-) = J$.

Proof: Suppose $h$ is conformal. Then there is a conformal map $\varphi : \mathbf{D} \longrightarrow \hat{\mathbf{C}}$ which can be extended continuously up to $\mathbf{T}$ such that $\varphi(h(z)) = \varphi(z)$ for $z \in I_-$. Let

$$\varphi_1 = \varphi \circ F, \ \varphi_2 = \varphi \circ \tilde{F}.$$

Therefore $\varphi_1$, $\varphi_2$ are conformal maps onto the interior and exterior of the Jordan domain $\Omega = \varphi(\mathbf{D}_1)$. Then

$$\begin{aligned} \varphi_2^{-1} \circ \varphi_1(e^{i\theta}) &= \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \varphi_2^{-1} \circ \varphi(f(e^{i\theta})) & \text{if } 2\pi \geq \theta \geq \pi \end{cases} \\ &= \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \varphi_2^{-1} \circ \varphi(h(f(e^{i\theta}))) & \text{if } 2\pi \geq \theta \geq \pi \end{cases} \end{aligned}$$

$$= \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1} \circ h \circ f(e^{i\theta}) & \text{if } 2\pi \geq \theta \geq \pi. \end{cases}$$

Therefore (1) implies (3). It is clear that (3) implies (2). So it remains to show (2) implies (1).

Suppose that there is some simple curve $J$ in $\mathbf{D}$ with $\sigma$-finite linear measure which joins $-1$ to 1 and that $D_1$ and $D_2$ are the components of $\mathbf{D}\backslash J$. Let $\varphi_1$, $\varphi_2$ be the conformal maps from the interior and exterior of the unit disk onto the interior and exterior of the Jordan domain $\Omega=\varphi(\mathbf{D})$ such that

$$\varphi_2^{-1} \circ \varphi_1(e^{i\theta}) = \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1} \circ h \circ f(e^{i\theta}) & \text{if } 2\pi \geq \theta \geq \pi \end{cases}$$

where $f = F|_{\mathbf{T}}$ for $F : \mathbf{D} \longrightarrow D_1$ the Riemann map from the unit disk to $D_1$ and $\tilde{f} = \tilde{F}|_{\mathbf{T}}$ for $\tilde{F} : \mathbf{D}_\infty \longrightarrow D_2$ the Riemann map from the exterior of the unit disk to $D_2$ such that $f(1) = \tilde{f}(1) = 1$, $f(-1) = \tilde{f}(-1) = 1$ and $f(I_-) = \tilde{f}(I_-) = J$. Now let

$$\varphi(z) = \begin{cases} \varphi_1 \circ F^{-1}(z) & z \in D_1 \\ \varphi_2 \circ \tilde{F}^{-1}(z) & z \in D_2. \end{cases}$$

As $\varphi_1 \circ f^{-1}(z)=\varphi_2 \circ \tilde{f}^{-1}(z)$ for $z \in J$, $\varphi$ can be extended to a homeomorphism on $\mathbf{D}$ by setting $\varphi(z)=\varphi_1 \circ f^{-1}(z)=\varphi_2 \circ \tilde{f}^{-1}(z)$. Therefore $\varphi$ is a homeomorphism on $\mathbf{D}$ with maximum dilation 0 except for a set of $\sigma$-finite linear measure. Hence $\varphi$ is conformal in $\mathbf{D}$. Now for $z \in I_-$,

$$\begin{aligned} \varphi(z) &= \varphi_1 \circ f^{-1}(z) \\ &= \varphi_2 \circ \tilde{f}^{-1} \circ h(z) \\ &= \varphi \circ h(z). \end{aligned}$$

Hence $h$ is conformal.

*Q.E.D.*

**Corollary 2.4.2** *Let $h : I_- \longrightarrow I_+$ be a welding function. Then the following statements are equivalent:*

*(1) $h$ is conformal.*

*(2) The interior and exterior of the unit disk are sewn conformally via the homeomorphism $g$ on $\mathbf{T}$ defined by*

$$g(e^{i\theta}) = \begin{cases} e^{i\theta} & \textit{if } \pi \geq \theta \geq 0 \\ f^{-1}(\overline{h \circ f(e^{i\theta})}) & \textit{if } 2\pi \geq \theta \geq \pi \end{cases}$$

*where $f = F|_{\mathbf{T}}$ and $F : \mathbf{D} \longrightarrow \mathbf{D}_-$ is the Riemann map from the unit disk to the lower unit disk $\mathbf{D}_-=\{z \in \mathbf{C} : |z| < 1, \Im(z) < 0\}$ such that $F(-1) = -1$, $F(1) = 1$ and $F(i) = 0$.*

*(3) The interior and exterior of the unit disk are sewn conformally via the homeomorphism $\tilde{g}$ on $\mathbf{T}$ defined by*

$$\tilde{g}(e^{i\theta}) = \begin{cases} e^{i\theta} & \textit{if } \pi \geq \theta \geq 0 \\ \overline{h(e^{i\theta})} & \textit{if } 2\pi \geq \theta \geq \pi. \end{cases}$$

Proof: By the reflection principle, $F$ can be extended to a conformal map from $\hat{\mathbf{C}} \setminus I_+$ onto $\mathbf{D}$ by letting $F(z)=\overline{F(1/\bar{z})}$ on $\mathbf{D}_\infty \cup I_-$ where $D_\infty = \{z \in \hat{\mathbf{C}} : |z| > 1\}$. Let $\tilde{F}$ be the restriction of the extension of $F$ to $\mathbf{D}_\infty$. Then $\tilde{F}$ is analytic and so there is a continuous extension to $\mathbf{T}$. Let $\tilde{f}=\tilde{F}|_{\mathbf{T}}=\bar{f}$. We have $\tilde{f}^{-1}(z)=f^{-1}(\bar{z})$ for any $z \in \mathbf{T}$. Let g be the welding as defined in (2) of **2.4.1**. Therefore,

$$\begin{aligned} g(e^{i\theta}) &= \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1}(h \circ f(e^{i\theta})) & \text{if } 2\pi \geq \theta \geq \pi \end{cases} \\ &= \begin{cases} e^{i\theta} & \text{if } \pi \geq \theta \geq 0 \\ f^{-1}(\overline{h \circ f(e^{i\theta})}) & \text{if } 2\pi \geq \theta \geq \pi. \end{cases} \end{aligned}$$

It follows from **2.4.1** that (1) is equivalent to (2). We are going to show that (2) is equivalent to (3). Let

$$k(e^{i\theta}) = \begin{cases} e^{i\pi(1-f(e^{i\theta}))/2} & \text{if } \pi \geq \theta \geq 0 \\ f(e^{i\theta}) & \text{if } 2\pi \geq \theta \geq \pi. \end{cases}$$

Therefore $k$ is a homeomorphism of $\mathbf{T}$ and $g = k^{-1} \circ \tilde{g} \circ k$. Therefore if $k$ is quasisymmetric, then the interior and exterior of the unit disk are sewn conformally via $g$ if and only if the interior and exterior of the unit disk are sewn conformally via $\tilde{g}$. So it suffices to show that $k$ is quasisymmetric.

Let us consider the function $l(t) = e^{i\pi(1-t)/2}$ for $1 \geq t \geq -1$. Because

$$|l(t_1) - l(t_2)| = 2\sin \pi|t_1 - t_2|/4,$$

we have

$$\pi|t_1 - t_2|/2 \geq |l(t_1) - l(t_2)| \geq |t_1 - t_2|.$$

Next for any $z \in I_-$ and $t_0 \in [-1, 1]$, the ratio $|l(t_0) - z|/|t_0 - z|$ attains maximum when the points $z$, $t_0$ and $l(t_0)$ are colinear, and it attains minimum when $z$ is either $-1$ or $1$. Therefore,

$$2|t_0 - z| \geq |l(t_0) - z| \geq |t_0 - z|.$$

Hence we can deduce that

$$2|f(z_1) - f(z_2)|/|f(z_2) - f(z_3)| \geq |k(z_1) - k(z_2)|/|k(z_2) - k(z_3)|$$

for any $z_1$, $z_2$, $z_3 \in \mathbf{T}$. As $f$ is quasisymmetric (since $\mathbf{D}_-$ is a quasidisk), $k$ is also quasisymmetric.

*Q.E.D.*

A John disk is a conformal disk $D'$ in $\hat{\mathbf{C}}$ such that for any crosscut $\alpha$ of $D'$ dividing $D$ into subdomains $D'_1$ and $D'_2$, we have $\min\{d(D'_1), d(D'_2)\} \leq c\,d(\alpha)$ where $c$ is some positive constant and $d(E)$ is the diameter of the set $E$. In [17], *R.Nakki* and *J.Väisälä* proved the following:

**Theorem 2.4.3** (*R.Nakki, J.Väisälä*) *A closed Jordan arc $A \subset \hat{\mathbf{C}}$ with one endpoint at infinity is a quasiconformal arc if and only if $\hat{\mathbf{C}} \setminus A$ is a John disk.*

For any conformal welding function $h$, the image of $\mathbf{D}$ under $\varphi$ is a conformal disk for any $\varphi$ associated with $h$. If there is some $\varphi$ such that the image of $\mathbf{D}$ under $\varphi$ is a John disk, then there is a bigger constraint on $h$.

**Theorem 2.4.4** *Let $h : I_- \longrightarrow I_+$ be a welding function which is conformal. Then there exists some $\varphi \in D_A$ such that $\varphi(\mathbf{D})$ is a John disk and $\varphi(x) = \varphi \circ h(x)$ for any $x \in I_-$ if and only if for some open quasiconformal arc $J$ in $\mathbf{D}$ which joins $-1$ to $1$, the interior and exterior of the unit disk are sewn conformally via the quasisymmetric homeomorphism $g$ on $\mathbf{T}$ defined by*

$$g(e^{i\theta}) = \begin{cases} \tilde{f}^{-1} \circ f(e^{i\theta}) & \text{if } \pi \geq \theta \geq 0 \\ \tilde{f}^{-1} \circ h \circ f(e^{i\theta}) & \text{if } 2\pi \geq \theta \geq \pi \end{cases}$$

*where $D_1$ and $D_2$ are the components of $\mathbf{D} \setminus J$ with $I_-$ and $I_+$ contained in the boundary respectively, $f = F|_{\mathbf{T}}$ for $F : \mathbf{D} \longrightarrow D_1$ the Riemann map from the unit disk to $D_1$ and $\tilde{f} = \tilde{F}|_{\mathbf{T}}$ for $\tilde{F} : \mathbf{D}_\infty \longrightarrow D_2$ the Riemann map from the exterior of the unit disk to $D_2$ such that $f(1) = \tilde{f}(1) = 1$, $f(-1) = \tilde{f}(-1) = 1$ and $f(I_-) = \tilde{f}(I_-) = J$.*

Proof: If $\varphi(\mathbf{D})$ is a John disk, without loss of generality, we can assume that one of $\varphi(1)$, $\varphi(-1)$ is $\infty$. From **2.4.3**, we can have a quasidisk $D' \subseteq \varphi(\mathbf{D})$ such that $\varphi(\mathbf{T}) \subset \partial D'$. Then let $J = \varphi^{-1}(\partial D' \setminus \varphi(\mathbf{T}))$. From our construction as in **2.4.1**, the images of the interior and exterior of the unit disk under the conformal maps $\varphi_1$ and $\varphi_2$ are the two components of $\hat{\mathbf{C}} \setminus \partial D'$. Therefore $g = \varphi_2^{-1} \circ \varphi_1$ is quasisymmetric. The converse is just similar.

*Q.E.D.*

**Corollary 2.4.5** *Let $h : I_- \longrightarrow I_+$ be a welding function which is conformal. If the homeomorphism $\tilde{g}$ on $\mathbf{T}$ defined by*

$$\tilde{g}(e^{i\theta}) = \begin{cases} e^{i\theta} & \text{if } \pi \geq \theta \geq 0 \\ \overline{h(e^{i\theta})} & \text{if } 2\pi \geq \theta \geq \pi, \end{cases}$$

*where* $f = F|_{\mathbf{T}}$ *and* $F : \mathbf{D} \longrightarrow \mathbf{D}_-$ *is the Riemann map from the unit disk to the lower unit disk* $\mathbf{D}_-=\{z \in \mathbf{C} : |z| < 1, \Im(z) < 0\}$ *such that* $F(-1) = -1$, $F(1) = 1$ *and* $F(i) = 0$, *is quasisymmetric, then there exists some* $\varphi \in D_A$ *such that* $\varphi(\mathbf{D})$ *is a John disk and* $\varphi(x) = \varphi \circ h(x)$ *for any* $x \in I_-$.

Proof: From **2.4.2** and **2.4.4**, if $g$ in **2.4.2** is quasisymmetric, then there exists some $\varphi \in D_A$ such that $\varphi(\mathbf{D})$ is a John disk and $\varphi(x) = \varphi \circ h(x)$ for any $x \in I_-$. As $\tilde{g}=k^{-1} \circ g \circ k$ where $k$ is quasisymmetric, $\tilde{g}$ is quasisymmetric implies that $g$ is quasisymmetric.

*Q.E.D.*

# Chapter 3 Extremal Length Conditions for Conformality

In this chapter, we relate conformality of a lamination with the extremal length of certain curve families which are defined in terms of the lamination. By considering the type of the ideal component corresponding to a singular point, we get a generalization of the Oikawa condition (cf **[19]**). It is not hard to get the local version of this criterion for an isolated multiple point. Then we prove Theorem 1 in chapter 1.

## 3.1 The Extremal Length of a Curve Family

Let $B$ be a Borel set in $\hat{\mathbf{C}}$. A curve family $\Gamma$ in $B$ consists of open, half open or closed arcs or curves in $B$. Any non-negative, Borel measurable function $\rho$ defines a metric on $B$. Let

$$l(\rho)=\inf_{\gamma\in\Gamma}\int_\gamma \rho(z)|dz|,$$
$$A(\rho)=\int\int_B \rho(z)^2 dxdy.$$

Thus $l(\rho)$ is the infimum of the arc lengths over all curves in $\Gamma$ and $A(\rho)$ is the area of $B$ under the metric induced by $\rho$.

We define the extremal length of the curve family $\Gamma$ by

$$\lambda(\Gamma) = \sup_\rho \frac{l(\rho)^2}{A(\rho)}$$

where the supremum is taken over all non-negative, Borel measurable functions $\rho$ with $A(\rho) < \infty$. For the proofs and more properties of the extremal length of a curve family, we refer to **[3, 18, 22]**.

**Elementary Properties of Extremal Lengths 3.1.1** *Let $\Gamma$, $\Gamma'$, $\Gamma_k$ be families of curves in Borel sets $B$, $B'$, $B_k$ respectively.*

(1) *(Conformal Invariance) If* $F$ *is a conformal map from a domain* $\Omega\subseteq B$ *into* $\hat{\mathbf{C}}$, *then* $\lambda(F(\Gamma)) = \lambda(\Gamma)$.

(2) *If for every* $\gamma \in \Gamma$, *there exists* $\gamma' \in \Gamma'$ *with* $\gamma' \subseteq \gamma$, *then* $\lambda(\Gamma) \geq \lambda(\Gamma')$. *In particular, if* $\Gamma \subseteq \Gamma'$, *then* $\lambda(\Gamma) \geq \lambda(\Gamma')$.

(3) $\frac{1}{\lambda(\bigcup_k \Gamma_k)} \leq \sum_k \frac{1}{\lambda(\Gamma_k)}$.

(4) *If the Borel sets* $B_k$ *are disjoint and* $\bigcup_k \Gamma_k \subseteq \Gamma$, *then* $\sum_k \frac{1}{\lambda(\Gamma_k)} \leq \frac{1}{\lambda(\Gamma)}$. *In particular,* $\sum_k \frac{1}{\lambda(\Gamma_k)} = \frac{1}{\lambda(\bigcup_k \Gamma_k)}$ *for* $\Gamma_k$ *in disjoint Borel sets* $B_k$.

(5) *If for each* $\gamma \in \Gamma$, *there is a* $\gamma_k \in \Gamma_k$ *such that* $\gamma_k \subseteq \gamma$ *for every* $k$, *then*

$$\lambda(\Gamma) \geq \textstyle\sum_k \lambda(\Gamma_k).$$

## 3.2 Modules of Continuity

**Definition 3.2.1** *Let* $\varphi \in C_A$, $K$ *be a continuum in* $\mathbf{D}$, $E \subseteq \mathbf{T}$ *closed and* $e = \varphi(E) \subseteq J = \varphi(\mathbf{T})$. *We define*

$\Gamma(e) = \{\gamma : \gamma \text{ joins } \varphi(K),\ e \text{ in } \hat{\mathbf{C}}\}$,

$\Gamma_n(e) = \{\gamma : \gamma \text{ joins } \varphi(K),\ e \text{ in } \varphi(\bar{\mathbf{D}}) \text{ and } \gamma \cap J \text{ has at most } n \text{ points }\}$,

$\Gamma_f(e) = \{\gamma : \gamma \text{ joins } \varphi(K),\ e \text{ in } \varphi(\bar{\mathbf{D}}) \text{ and } \gamma \cap J \text{ has finitely many points }\}$.

$\Gamma_\infty(e) = \{\gamma : \gamma \text{ joins } \varphi(K),\ e \text{ in } \varphi(\bar{\mathbf{D}}) \text{ and } \gamma \cap J \text{ has countably many points }\}$.

$\Gamma_\omega(e) = \{\gamma : \gamma \text{ joins } \varphi(K),\ e \text{ in } \varphi(\bar{\mathbf{D}}) \text{ and } \omega(\gamma \cap J) = 0\ \}$.

Since $\Gamma_n(e) \subseteq \Gamma_f(e) \subseteq \Gamma_\infty(e) \subseteq \Gamma_\omega(e) \subseteq \Gamma(e)$, it follows from **3.1.1** that

$$\lambda(\Gamma(e)) \leq \lambda(\Gamma_\omega(e)) \leq \lambda(\Gamma_\infty(e)) \leq \lambda(\Gamma_f(e)) \leq \lambda(\Gamma_n(e)).$$

By Beurling's estimate, $\lambda(\Gamma(e)) = \frac{1}{2\pi}\log\frac{1}{cap e} + O(1)$. Because $\varphi \in C_A$, diam$e \longrightarrow 0$ as diam$E \longrightarrow 0$ and so $\lambda(\Gamma(e)) \longrightarrow \infty$ as diam$E \longrightarrow 0$.

**Corollary 3.2.2** *If* $\varphi \in C_A$, *then*

$\lambda(\Gamma(e)),\ \lambda(\Gamma_\omega(e)),\ \lambda(\Gamma_\infty(e)),\ \lambda(\Gamma_f(e)),\ \lambda(\Gamma_n(e)) \longrightarrow \infty$ *as* diam$E$ *tends to 0.*

We can define $\lambda_n$, $\lambda_f$ and $\lambda_\infty$ in terms of the lamination $L$. Suppose $\pi : \bar{\mathbf{D}} \longrightarrow \bar{\mathbf{D}}/L$ is the natural projection and $\tau : [0,1] \longrightarrow \bar{\mathbf{D}}/L$ joins $\pi(K)$ and $\pi(E)$, let $\tau^{\mathbf{D}}$ denote $\pi^{-1} \circ \tau|_{\tau^{-1}\circ\pi(\mathbf{D})}$. $\tau^{\mathbf{D}}$ consists of a countable family of open arcs. Let

$\Gamma_n(E, L) = \{\tau^{\mathbf{D}} : \tau$ joins $\pi(K)$ and $\pi(E)$ in $\bar{\mathbf{D}}/L$ and $\tau \cap \mathbf{T}/L$ has at most $n$ points$\}$.

We can define $\Gamma_f(E, L)$ and $\Gamma_\infty(E, L)$ in a similar way.

**Proposition 3.2.3** *$\lambda_n(E, L) = \lambda(\Gamma_n(e))$ and similar equalities hold for $\lambda_f$ and $\lambda_\infty$ where $\lambda_n(E, L) = \lambda(\Gamma_n(E, L))$.*

Proof: We are going to prove the equality for $\lambda_\infty$. The proof for other cases are just similar.

$\varphi^{-1}(\{\gamma|_{\varphi\mathbf{D}} : \gamma \in \Gamma_\infty(e)\}) = \Gamma_\infty(E, L)$, so by conformal invariance of extremal length, $\lambda_\infty(E, L) = \lambda(\{\gamma|_{\varphi\mathbf{D}} : \gamma \in \Gamma_\infty(e)\})$. Therefore it suffices to show that $\lambda(\Gamma_\infty(e)) = \lambda(\{\gamma|_{\varphi\mathbf{D}} : \gamma \in \Gamma_\infty(e)\})$. Since $\Lambda_1(\gamma \cap J) = 0$, for any Borel measurable function $\rho$,

$$\int_\gamma \rho(z)|dz| = \int_{\gamma\cap\varphi\mathbf{D}} \rho(z)|dz|.$$

Therefore $l(\rho)$ and $A(\rho)$ for the two families are the same. Hence we have the same extremal length for both of them.

*Q.E.D.*

**Corollary 3.2.4** *Let $L$ be a closed and flat lamination on $\mathbf{T}$. If $L \subseteq L(\varphi)$ for some $\varphi \in C_A$, then $\lambda_\infty(E, L)$, $\lambda_f(E, L)$ and $\lambda_n(E, L)$ tend to $\infty$ as diamE tends to zero.*

Proof: If $L \subseteq L(\varphi)$, then $\Gamma_n(E, L) \subseteq \Gamma_n(E, L(\varphi))$. Hence

$$\lambda_n(E, L) \geq \lambda_n(E, L(\varphi))$$

by **3.1.1**. The other cases are similar.

*Q.E.D.*

**An Example** We are going to construct a continuous, flat lamination $L = L(\varphi)$ for some univalent $\varphi$ which is continuous up to the boundary except at a countable set. It is equivalent to give a lamination of the real line since the upper half-plane is conformally equivalent to the unit disk. Let $\chi_0(t) = t$ and

$$\chi_n(t) = \begin{cases} \frac{1-2^{-n^2}}{1-2^{-(n+1)^2}}(t - 2^{-n^2}) + 2^{-(n+1)^2} & 2^{-n^2} \leq t \leq 1 \\ t^{\frac{(n+1)^2}{n^2}} & 4^{-n^2} \leq t \leq 2^{-n^2} \\ 2^{n^2-(n+1)^2}t & 0 \leq t \leq 2^{n^2}. \end{cases}$$

Therefore $\chi_n$ is an increasing homeomorphism from [0,1] onto [0,1]. Then let

$$\phi_n(x) = 2n + 2 - \chi_n(x - 2n)$$

for $2n \leq x \leq 2n+1$ $(n \geq 0)$ which is a decreasing homeomorphism from $[2n, 2n+1]$ onto $[2n+1, 2n+2]$. We can extend the definition of $\phi_n$ to the interval $[-2n-1, -2n]$ by taking the reflection at the origin of $\phi_n$. The lamination $L$ defined by

$$x \sim y \text{ if } \phi_n(x) = y \text{ or } \phi_n(y) = x \text{ for some integer } n \text{ or } x \text{ and } y \text{ are even integers}$$

is continuous and closed. Under the lamination, each interval $[2n, 2n+1]$ is glued to the interval $[2n+1, 2n+2]$ with reverse orientation. Since the welding function on $(2n, 2n+1]$ is bi-Lipschitz, a simply connected, open Riemann surface is obtained by welding each pair of intervals $(2n, 2n+1]$ and $[2n+1, 2n+2)$ by means of $\phi_n$. By the uniformization theorem, there is a univalent $\varphi$ which is continuous up to the boundary except at the even integers such that $L{=}L(\varphi)$.

Consider the family of curves $\{\gamma_t\}_{\frac{1}{4} \leq t \leq \frac{1}{2}}$ where

$$\gamma_t = \{2 + t + is : 0 \leq s \leq 1\} \cup (\cup_{n=1}^{N} \{2n + 2 - \chi_n \circ \chi_{n-1} \circ \ldots \circ \chi_1(t) e^{i\pi s} : 0 \leq s \leq 1\})$$

for each $N > 0$. As $\{\gamma_t\}_{\frac{1}{2} \leq t \leq \frac{1}{2}} \subset \Gamma_f([2N + 2 + 4^{-(N+1)^2}, 2N + 2 + 2^{-(N+1)^2}], L)$, by **3.1.1**, $\lambda(\{\gamma_t\}_{\frac{1}{2} \leq t \leq \frac{1}{2}}) \geq \lambda_N([2N + 2 + 4^{-(N+1)^2}, 2N + 2 + 2^{-(N+1)^2}], L)$.

Let $f_n(s,t) = 2n + 2 - \chi_n \circ \chi_{n-1} \circ \ldots \circ \chi_1(t) e^{i\pi s}$ and $J_n(s,t) = |\mathrm{D} f_n|(s,t)$. Since $f_n$ is a homeomorphism in $[0,1] \times [\frac{1}{4}, \frac{1}{2}]$ which is absolutely continuous in lines,

$$\int \int_{f_n([0,1]\times[\frac{1}{4},\frac{1}{2}])} \rho(z)^2 dxdy = \int \int_{[0,1]\times[\frac{1}{4},\frac{1}{2}]} \rho^2(f_n(s,t)) J_n(s,t) dsdt$$

for every $n$. Then let

$$f(s,t) = \begin{cases} 2 + t + is & \text{if } 0 \leq s \leq 1 \\ f_n(s - n, t) & \text{if } n \leq s \leq n+1 \end{cases}$$

and $J(s,t){=}|\mathrm{D} f(s,t)|$. Now because

$$\begin{aligned} l(\rho) &\leq \int_0^1 \rho(s,t) dsdt + \sum_{n=1}^{N} \int_0^1 \rho(f_n(s,t)) |\frac{\partial}{\partial s} f_n(s,t)| ds \\ &= \int_0^{N+1} \rho(f(s,t)) |\frac{\partial}{\partial s} f(s,t)| ds \end{aligned}$$

we have

$$l(\rho)^2 \le \textstyle\int_0^{N+1} \rho^2(f(s,t))J(s,t)ds \times \int_0^{N+1} |\frac{\partial}{\partial s}f(s,t)|^2/J(s,t)ds.$$

Thus

$$\begin{aligned}
A(\rho) &\ge \int\int_{[0,1]\times[\frac{1}{4},\frac{1}{2}]} \rho(z)^2dxdy + \sum_{n=1}^{N}\int\int_{f_n([0,1]\times[\frac{1}{4},\frac{1}{2}])} \rho(z)^2dxdy \\
&\ge \int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^1 [\rho^2(s,t) + \sum_{n=1}^{N}\rho^2(f_n(s,t))J_n(s,t)]dsdt \\
&\ge \int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^{N+1} \rho^2(f(s,t))J(s,t)dsdt \\
&\ge l(\rho)^2\int_{\frac{1}{4}}^{\frac{1}{2}}(\int_0^{N+1}|\frac{\partial}{\partial s}f(s,t)|^2/J(s,t)ds)^{-1}dt \\
&= l(\rho)^2\int_{\frac{1}{4}}^{\frac{1}{2}}(1+\sum_{n=1}^{N}\int_0^1|\frac{\partial}{\partial s}f_n(s,t)|^2/J_n(s,t)ds)^{-1}dt
\end{aligned}$$

Therefore, we have

$$\lambda(\{\gamma_t\}_{e^{-1}\le t\le\frac{1}{2}}) \le \{\textstyle\int_{\frac{1}{4}}^{\frac{1}{2}}(1+\sum_{n=1}^{N}\int_0^1|\frac{\partial}{\partial s}f_n(s,t)|^2/J_n(s,t)ds)^{-1}dt\}^{-1}.$$

For any $n > 0$ and $4^{-n^2} \le t \le 2^{-n^2}$,

$$\begin{aligned}
\int_0^1|\frac{\partial}{\partial s}f_n(s,t)|^2/J_n(s,t)ds &= \int_0^1|\frac{\partial}{\partial s}f_n(s,t)|/|\frac{\partial}{\partial t}f_n(s,t)|ds \\
&= \int_0^1 \pi g_n(t)/g_n'(t)ds \\
&\quad \text{where} \quad g_n = \chi_n\circ\chi_{n-1}\circ\ldots\circ\chi_1 \\
&= \pi g_n(t)/g_n'(t) \\
&= \pi t/(n+1)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda(\{\gamma_t\}_{\frac{1}{4}\le t\le\frac{1}{2}}) &\le \{\int_{\frac{1}{4}}^{\frac{1}{2}}(1+\sum_{n=2}^{N+1}\frac{\pi t}{n^2})^{-1}dt\}^{-1} \\
&\le \{\int_{\frac{1}{4}}^{\frac{1}{2}}(4t+\sum_{n=2}^{N+1}\frac{\pi t}{n^2})^{-1}dt\}^{-1} \\
&\le (4+\pi\sum_{n=2}^{\infty}\frac{1}{n^2})/\log 2.
\end{aligned}$$

for any $N > 0$ and so

$$\lim_{N \longrightarrow \infty} \lambda_N([2N+2+4^{-(N+1)^2}, 2N+2+2^{-(N+1)^2}], L) \leq (4 + \pi \textstyle\sum_{n=2}^{\infty} \frac{1}{n^2}) / \log 2.$$

It shows that there does not exist any $\varphi \in C_A$ such that $L$=$L(\varphi)$ by **3.2.4.**

**Remark** We can define a lamination on the set

$$E' = E \cup (\cup_{n=0}^{\infty}(2n+2+\chi_n \circ \ldots \circ \chi_0 E)) \cup (\cup_{n=0}^{\infty}(2n+2-\chi_n \circ \ldots \circ \chi_0 E))$$

by restricting the lamination in our example above to $E'$ where $E$ is a subset of $[\frac{1}{4}, \frac{1}{2}]$ with positive linear measure. By considering the family of curves $\{\gamma_t\}$ with $t \in E$, we can get the same estimation of extremal lengths except a factor of $(\int_E \frac{1}{t} dt)^{-1}$. Hence the lamination is not conformal. This supplies us examples of laminations on a compact subset $E$ of the unit circle with $\Lambda_1(E) > 0$. We can define a lamination which welds a compact subset $E_-$ of $I_-$ with a compact subset $E_+$ of $I_+$ similarly. By translating this to a strip $[0, \infty) \times [0, 1]$, we can define the welding on the set $(\cup_{n=0}^{\infty} \chi_n \circ \ldots \circ \chi_0 E) \cup (\cup_{n=0}^{\infty}(-\chi_n \circ \ldots \circ \chi_0)E)$ by setting $h(x) = x + i$ when $x \in \cup_{n=0}^{\infty}(-\chi_n \circ \ldots \circ \chi_0)E$, and $h(x) = \chi_n(x) + i$ when $x \in \cup_{n=0}^{\infty} \chi_n \circ \ldots \circ \chi_0 E$. By a similar method, we can see that this lamination is not conformal. Thus we have the examples as described in the first chapter.

## 3.3 Laminations with Singularity

Let $L$ be a closed and flat lamination on $\mathbf{T}$ such that $\bar{\mathbf{D}}/L$ is homeomorphic to the sphere. Suppose $E = \{\xi_n$: $2\pi >\arg\xi_{n+1 \,(\mathrm{mod}\, N)} \geq \arg\xi_n \geq 0$, $n = 1, \ldots, N\}$ $(N \geq 2)$ is an equivalence class under $L$ and there is no equivalence class under $L$ with more than two points except possibly $E$. The lamination is very simple combinatorially, namely, there are points $\eta_n \in I_n$ $(n = 1, \ldots, N)$ where $I_n$ is the component of $\mathbf{T} \setminus E$ with endpoints $\xi_n$, $\xi_{n+1 \,(\mathrm{mod}\, N)}$ such that the lamination is just obtained by identifying points in $E$ and welding the intervals $[\xi_n, \eta_n]$, $[\eta_n, \xi_{n+1 \,(\mathrm{mod}\, N)}]$ with reverse orientations.

Suppose that there is an univalent function $\varphi$ which is continuous in $\bar{\mathbf{D}} \setminus E$ such that for any two points $\zeta_1, \zeta_2 \in \mathbf{T} \setminus E$, $\zeta_1 \sim \zeta_2$ under $L$ if and only if $\varphi(\zeta_1) = \varphi(\zeta_2)$.

Then $\varphi(\bar{\mathbf{D}} \setminus E)$ is a simply connected, open set in the complex plane. It is conformally equivalent to the unit disk or the complex plane by the uniformization theorem. If it is the first case, the lamination $L$ cannot be conformal. Otherwise $\varphi$ can be extended to a continuous function from $\bar{\mathbf{D}}$ onto $\hat{\mathbf{C}}$ by letting $\varphi(\xi_n) = \infty$ $(n = 1, ..., N)$ and $L = L(\varphi)$. This is a type problem of the simply connected, open Riemann surface obtained by welding the intervals $(\xi_n, \eta_n]$, $[\eta_n, \xi_{n+1\ (\mathrm{mod}\ N)})$ for $n = 1, ..., N$ with the identifications defined by $L|_{\mathbf{T}\setminus E}$. The ideal boundary component at $E$ of the Riemann surface thus obtained corresponds to $\partial\Omega$ where $\Omega = \varphi(\bar{\mathbf{D}} \setminus E)$. In classical terms the ideal boundary component at $E$ is parabolic if $\hat{\mathbf{C}} \setminus \varphi(\bar{\mathbf{D}} \setminus E)$ is a point. Otherwise the ideal boundary component is hyperbolic.

We can have a branch of logarithm defined on the sector $\arg\xi_n \leq \theta \leq \arg\xi_{n+1\ (\mathrm{mod}\ N)}$. Then the arc $I_n$ becomes the line segment $[i\arg\xi_n, i\arg\xi_{n+1\ (\mathrm{mod}\ N)}]$ and we can write the welding to be $\phi_n(ix) = i(\arg\xi_{n+1\ (\mathrm{mod}\ N)} - \chi_n(x - \arg\xi_n))$ where $\chi_n$ is an increasing homeomorphism from $[0, \arg\eta_n - \arg\xi_n]$ onto $[0, \arg\xi_{n+1\ (\mathrm{mod}\ N)} - \arg\eta_n]$. Then we have a generalization of theorem 2 in **[19]** and corollary 2.2 in **[24]** which gives a criterion for parabolicity. If we take $N = 2$ in our case, the following condition is exactly Vaino's re-statement of Oikawa's condition where the function $\varphi$ in section 2.1 of **[24]** is expressed by

$$\varphi(x) = \begin{cases} \chi_1^{-1}(x) & \text{if } x > 0 \\ -\chi_2(|x|) & \text{if } x \leq 0 \end{cases}$$

in our case for $x$ close to 0. Then

$$\begin{aligned} &\int_0 (t + \frac{\chi_1(t)}{\chi_1'(t)} + \frac{\chi_2 \circ \chi_1(t)}{(\chi_2 \circ \chi_1)'(t)})^{-1}(1 + \log^2 \frac{\chi_2 \circ \chi_1(t)}{t})^{-1} dt \\ &= \int_0 (x + \frac{\varphi(x)}{\varphi'(x)} + \frac{|\varphi(-x)|}{\varphi'(-x)})^{-1}(1 + \log^2 \frac{|\varphi(-x)|}{\varphi'(-x)})^{-1} dx \end{aligned}$$

by substituting $x = \chi_1(t)$ in the integral.

**Theorem 3.3.1** *Let the lamination $L$ satisfies the assumptions at the beginning of this section and the functions $\phi_n, \chi_n$ as described earlier in this section. If the condition*

$$\int_0[\sum_{n=0}^{N-1} \frac{\psi_n(t)}{\psi'_n(t)} + (t + \frac{\psi_N(t)}{\psi'_N(t)})(1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})]^{-1}dt = \infty$$

*is satisfied for*

$$\psi_n(t) = \begin{cases} \chi_{\sigma(n)} \circ \chi_{\sigma(n-1)} \circ \ldots \circ \chi_{\sigma(1)}(t) & \text{if } n > 0 \\ t & \text{if } n = 0 \end{cases}$$

*for any cyclic shift $\sigma$ of the cycle $(N, N-1, ..., 1, 0)$, then $L$ is conformal.*

Proof: Consider the family of curves $\{e^{i\gamma_t(s)} : 0 \le s \le 1\}_{t_0 \le t \le t_1}$ where

$$\gamma_t(s) = (\cup_{n=1}^{N-1}\{\arg\xi_{n+1} + \psi_n(t)e^{i\pi s}\}) \cup \{\arg\xi_1 + te^{s\log\frac{\psi_N(t)}{t} + i\pi s}\}.$$

If we take $t_1$ small enough, then each curve in the family becomes a simple closed curve and each of the $N$ segments of a curve in the family lies in disjoint open sets. Since the exponential map is conformal, we have $\lambda(\{e^{i\gamma_t(s)}\}) = \lambda(\{\gamma_t\})$.

Let

$$f_n(s,t) = \begin{cases} \arg\xi_{n+1} - \psi_n(t)e^{i\pi s} & \text{if } 1 \le n \le N-1 \\ \arg\xi_1 + te^{s\log\frac{\psi_N(t)}{t} + i\pi s} & \text{if } n = N \end{cases}$$

and $J_n(s,t) = |\mathrm{D}f_n|(s,t)$. Since $f_n$ is a homeomorphism in $[0,1]\times[t_0, t_1]$ and $J_n(s,t)$ exists almost everywhere,

$$\int\int_{f_n([0,1]\times[t_0,t_1])} \rho(z)^2 dxdy \ge \int\int_{[0,1]\times[t_0,t_1]} \rho^2(f_n(s,t))J_n(s,t)dsdt$$

for every $n$. Since

$$l(\rho) \le \sum_{n=1}^{N} \int_0^1 \rho(f_n(s,t))|\frac{\partial}{\partial s}f_n(s,t)|ds,$$

we have

$$l(\rho)^2 \le \int_0^1[\sum_{n=1}^{N} \rho^2(f_n(s,t))J_n(s,t)]ds \times \int_0^1[\sum_{n=1}^{N} |\frac{\partial}{\partial s}f_n(s,t)|^2/J_n(s,t)]ds.$$

It is not hard to check that

$$|\frac{\partial}{\partial s}f_n(s,t)|^2/J_n(s,t) = \begin{cases} \frac{\pi\psi_n(t)}{\psi'_n(t)} & \text{if } 1 \le n \le N-1 \\ (\pi + \frac{1}{\pi}\log^2\frac{\psi_N(t)}{t})[\frac{s}{t} + (1-s)\frac{\psi'_N(t)}{\psi_N(t)}]^{-1} & \text{if } n = N. \end{cases}$$

Therefore

$$\begin{aligned}\int_0^1 [\sum_{n=1}^{N} |\frac{\partial}{\partial s} f_n(s,t)|^2 / J_n(s,t)]ds &= \int_0^1 (\pi + \frac{1}{\pi}\log^2 \frac{\psi_N(t)}{t})[\frac{s}{t} + (1-s)\frac{\psi'_N(t)}{\psi_N(t)}]^{-1} ds \\ &\quad + \sum_{n=1}^{N-1} \frac{\pi\psi_n(t)}{\psi'_n(t)} \\ &\leq \sum_{n=1}^{N-1} \frac{\pi\psi_n(t)}{\psi'_n(t)} + (\pi + \frac{1}{\pi}\log^2 \frac{\psi_N(t)}{t})(t + \frac{\psi_N(t)}{\psi'_N(t)}),\end{aligned}$$

and so

$$l(\rho)^2 [\sum_{n=1}^{N-1} \frac{\psi_n(t)}{\psi'_n(t)} + (1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})(t + \frac{\psi_N(t)}{\psi'_N(t)})]^{-1} \leq \pi \int_0^1 [\sum_{n=1}^{N} \rho^2(f_n(s,t)) J_n(s,t)]ds$$

Integrating both sides of the inequality above from $t_0$ to $t_1$, we get

$$\begin{aligned}\pi A(\rho) &= \pi \sum_{n=1}^{N} \int\int_{f_n([0,1]\times[t_0,t_1])} \rho(z)^2 dxdy \\ &\geq \pi \sum_{n=1}^{N} \int\int_{[0,1]\times[t_0,t_1]} \rho^2(f_n(s,t)) J_n(s,t) dsdt \\ &\geq l(\rho)^2 \int_{t_0}^{t_1} [\sum_{n=1}^{N-1} \frac{\psi_n(t)}{\psi'_n(t)} + (1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})(t + \frac{\psi_N(t)}{\psi'_N(t)})]^{-1} dt\end{aligned}$$

If $\int_0 [\sum_{n=1}^{N-1} \frac{\psi_n(t)}{\psi'_n(t)} + (1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})(t + \frac{\psi_N(t)}{\psi'_N(t)})]^{-1} dt = \infty$, then the extremal length $\lambda(\{\gamma_t\}_{0\leq t\leq t_1}) = 0$. Hence $L$ is conformal.

*Q.E.D.*

**Corollary 3.3.2** *Let the lamination $L$ satisfies the assumptions at the beginning of this section and the functions $\phi_n, \chi_n$ as described earlier in this section. If the condition*

$$\int_0 (\sum_{n=0}^{N} \frac{\psi_n(t)}{\psi'_n(t)})^{-1} (1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})^{-1} dt = \infty$$

*is satisfied for*

$$\psi_n(t) = \begin{cases} \chi_{\sigma(n)} \circ \chi_{\sigma(n-1)} \circ \ldots \circ \chi_{\sigma(1)}(t) & if\, n > 0 \\ t & if\, n = 0 \end{cases}$$

*for any cyclic shift* $\sigma$ *of the cycle* $(N, N-1, ..., 1, 0)$*, then* $L$ *is conformal.*

Proof: Because

$$(\sum_{n=0}^{N} \frac{\psi_n(t)}{\psi_n'(t)})^{-1}(1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})^{-1} \le [\sum_{n=0}^{N-1} \frac{\psi_n(t)}{\psi_n'(t)} + (t + \frac{\psi_N(t)}{\psi_N'(t)})(1 + \frac{1}{\pi^2}\log^2 \frac{\psi_N(t)}{t})]^{-1},$$

it follows from **3.3.1**.

*Q.E.D.*

**Remark** The integral criteria in this section are applicable to any multiple point without any triple or higher order multiple points nearby. The configuration is just similar since we only consider the local behavior of the lamination.

## 3.4 The Quadratuple Problem

In this section, we are going to prove Theorem 1 in the first chapter. For our convenience, we translate the problem to a strip setting. Let $I = [0, \infty) \times [0, 1]$ be the standard strip in $\mathbf{C}$. Let $a_n$, $\tilde{a}_n$, $b_n$, $\tilde{b}_n$ be non-negative real numbers ($n \geq 0$) such that

(1)$a_0 = b_0 = 0$

(2)$a_n < \tilde{a}_n < a_{n+1}, b_n < \tilde{b}_n < b_{n+1}$ for any $n \geq 0$

(3)$a_n, b_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

Our goal is to show that there is a continuous mapping $\varphi$ from $I \cup \{\infty\}$ onto $\hat{\mathbf{C}}$ which is conformal in Int$I$ such that $\varphi^{-1}(\varphi(a_n)) = \{a_n, \tilde{a}_n, b_n + i, \tilde{b}_n + i\}$ for any $n$.

**Lemma 3.4.1** *There are vertical strips* $I_k = [c_k, \tilde{c}_k] \times [0, 1]$ *with* $\tilde{c}_k < c_{k+1}$ *for all* $k \geq 0$ *and* $\sum_{k=0}^{\infty}(\tilde{c}_k - c_k) = \infty$ *such that either one of the following holds for all* $k$*:*

(1) $[c_k, \tilde{c}_k] \subset (a_{n_k}, \tilde{a}_{n_k}) \cap (b_{m_k}, \tilde{b}_{m_k})$ *for some* $n_k, m_k \geq 0$*;*

(2) $[c_k, \tilde{c}_k] \subset (\tilde{a}_{n_k}, a_{n_k+1}) \cap (\tilde{b}_{m_k}, b_{m_k+1})$ *for some* $n_k, m_k \geq 0$*;*

(3) $[c_k, \tilde{c}_k] \subset (\tilde{a}_{n_k}, a_{n_k+1}) \cap (b_{m_k}, \tilde{b}_{m_k})$ *for some* $n_k, m_k \geq 0$*;*

(4) $[c_k, \tilde{c}_k] \subset (a_{n_k}, \tilde{a}_{n_k}) \cap (\tilde{b}_{m_k}, b_{m_k+1})$ *for some* $n_k, m_k \geq 0$.

Proof: Since

$$\begin{aligned}
[0,\infty)\setminus\{a_n,\tilde{a}_n,b_m,\tilde{b}_m\}_{m,n=0}^{\infty} &= [\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}(a_n,\tilde{a}_n)\cap(b_m,\tilde{b}_m)]\cup\\
&\quad [\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}(\tilde{a}_n,a_{n+1})\cap(\tilde{b}_m,b_{m+1})]\cup\\
&\quad [\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}(\tilde{a}_n,a_{n+1})\cap(b_m,\tilde{b}_m)]\cup\\
&\quad [\bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty}(a_n,\tilde{a}_n)\cap(\tilde{b}_m,b_{m+1})],
\end{aligned}$$

we have either one of the following:

$$(1)\qquad \Lambda_1(\cup_{n=0}^{\infty}\cup_{m=0}^{\infty}(a_n,\tilde{a}_n)\cap(b_m,\tilde{b}_m))=\infty$$

$$(2)\qquad \Lambda_1(\cup_{n=0}^{\infty}\cup_{m=0}^{\infty}(\tilde{a}_n,a_{n+1})\cap(\tilde{b}_m,b_{m+1}))=\infty$$

$$(3)\qquad \Lambda_1(\cup_{n=0}^{\infty}\cup_{m=0}^{\infty}(\tilde{a}_n,a_{n+1})\cap(b_m,\tilde{b}_m))=\infty$$

$$(4)\qquad \Lambda_1(\cup_{n=0}^{\infty}\cup_{m=0}^{\infty}(a_n,\tilde{a}_n)\cap(\tilde{b}_m,b_{m+1}))=\infty.$$

Hence we can make the appropriate choice of strips in each case.

*Q.E.D.*

We are assuming that (1) in **3.4.1** is the case and the proof for other cases are just similar. Moreover, we can assume that $n_k \geq m_k$ for any $k$ without loss of generality.

**Definition 3.4.2** *Let* $k_n = \max\{k \geq 0 : \tilde{c}_k < \tilde{a}_n\}$ *and* $\tilde{k}_m = \max\{k \geq 0 : \tilde{c}_k < \tilde{b}_m\}$ *for any* $m$, $n$.

**Lemma 3.4.3** *For any integers* $m$, $n$ *and* $k$, $k_n$ *and* $\tilde{k}_m$ *as defined are monotonically increasing sequences of integers with*

(1) $k_n \leq \tilde{k}_n$ *for each* $n$,

(2) $n_k = n$ *if and only if* $k_{n-1}+1 \leq k \leq k_n$,

(3) $m_k = m$ *if and only if* $\tilde{k}_{m-1}+1 \leq k \leq \tilde{k}_m$, *and*

(4) $m_k \leq n \leq n_k$ *if and only if* $k_{n-1}+1 \leq k \leq \tilde{k}_n$.

Let $u_n$ be chosen such that $a_n < u_n < \tilde{a}_n$ and $[a_n, u_n] \cap [c_k, \tilde{c}_k] = \phi$ for any $k$. Let $v_n$ be chosen such that $b_n < v_n < \tilde{b}_n$ and $[b_n, v_n] \cap [c_k, \tilde{c}_k] = \phi$ for any $k$. Let $S_k = 3(n_k - m_k) + 2$, $l_k = \tilde{c}_k - c_k$, and $r_{nj}$ $(j = 1, \ldots, \tilde{k}_n - k_{n-1})$ be taken such that

$$r_{n1} = 1 \text{ and } r_{n(j+1)} < r_{nj} e^{-\pi s_k l_k}.$$

For any $n \geq 1$, the intervals

$$[a_n + r_{n(k-k_{n-1})} e^{-\pi s_k l_k} \min(\tfrac{u_n - a_n}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4}), a_n + r_{n(k-k_{n-1})} \min(\tfrac{u_n - a_n}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4})]$$

for $k_{n-1} + 1 \leq k \leq \tilde{k}_n$ are contained in $(a_n, u_n)$. Also, the intervals $J_k =$

$$[\tilde{a}_n - r_{n(k-k_{n-1})} e^{-\pi s_k l_k} \min(\tfrac{\tilde{a}_n - \tilde{c}_{k_n}}{2}, \tfrac{a_{n+1} - \tilde{a}_n}{4}), \tilde{a}_n - r_{n(k-k_{n-1})} \min(\tfrac{\tilde{a}_n - \tilde{c}_{k_n}}{2}, \tfrac{a_{n+1} - \tilde{a}_n}{4})]$$

for $k_n + 1 \leq k \leq \tilde{k}_n$ and the intervals $I_k$ for $k_{n-1} + 1 \leq k \leq k_n$ are contained in $(u_n, \tilde{a}_n)$. Therefore, we can have a piecewise analytic, bi-Lipschitz homeomorphism $f_n$ from $[u_n, \tilde{a}_n]$ onto $[a_n, u_n]$ such that $f_n(u_n) = u_n$, $f_n(\tilde{a}_n) = a_n$ and

$$f_n(t) = \begin{cases} a_n + r_{n(k-k_{n-1})} \min(\frac{u_n - a_n}{2}, \frac{a_n - \tilde{a}_{n-1}}{4}) e^{-\pi s_k (t - c_k)} & t \in I_k \text{ for } k_{n-1} + 1 \leq k \leq k_n \\ a_n + \frac{\min(\frac{u_n - a_n}{2}, \frac{a_n - \tilde{a}_{n-1}}{4})}{\min(\frac{\tilde{a}_n - \tilde{c}_{k_n}}{2}, \frac{a_{n+1} - \tilde{a}_n}{4})} (\tilde{a}_n - t) & t \in J_k \text{ for } k_n + 1 \leq k \leq \tilde{k}_n. \end{cases}$$

Similarly, we can define a piecewise analytic, bi-Lipschitz homeomorphism $g_n$ from $[v_n, \tilde{b}_n]$ onto $[b_n, v_n]$ such that $g_n(v_n) = v_n$, $g_n(\tilde{b}_n) = b_n$ and

$$g_n(t) = b_n + r_{n(k-k_{n-1})} \min(\tfrac{v_n - b_n}{2}, \tfrac{b_n - \tilde{b}_{n-1}}{4}) e^{-\pi s_k (t - c_k)} \text{ for } t \in I_k \text{ and } \tilde{k}_{n-1} + 1 \leq k \leq \tilde{k}_n.$$

The intervals $I'_k =$

$$[\tilde{a}_{n-1} + r_{n(k-k_{n-1})} \min(\tfrac{\tilde{a}_{n-1} - \tilde{c}_{k_{n-1}}}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4}) e^{-\pi s_k l_k}, \tilde{a}_{n-1} +$$
$$r_{n(k-k_{n-1})} \min(\tfrac{\tilde{a}_{n-1} - \tilde{c}_{k_{n-1}}}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4})],$$

for $k_{n-1} + 1 \leq k \leq \tilde{k}_{n-1}$ and $I''_k =$

$$[a_n - r_{n(k-k_{n-1})} \min(\tfrac{u_n - a_n}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4}), a_n - r_{n(k-k_{n-1})} \min(\tfrac{u_n - a_n}{2}, \tfrac{a_n - \tilde{a}_{n-1}}{4}) e^{-\pi s_k l_k}]$$

for $k_{n-1} + 1 \leq k \leq \tilde{k}_n$ are contained in $(\tilde{a}_{n-1}, a_n)$. Also the intervals

$$[\frac{b_n - \tilde{b}_{n-1}}{2} - \frac{r_{n(\tilde{k}_n - k_{n-1})}(b_n - \tilde{b}_{n-1})}{4 r_{n(k-k_{n-1})}}, \frac{b_n - \tilde{b}_{n-1}}{2} - \frac{r_{n(\tilde{k}_n - k_{n-1})}(b_n - \tilde{b}_{n-1})}{4 r_{n(k-k_{n-1})}} e^{-\pi s_k l_k}],$$

$$[\tfrac{b_n-\tilde{b}_{n-1}}{2} + \tfrac{r_{n(\tilde{k}_n-k_{n-1})}(b_n-\tilde{b}_{n-1})}{4r_{n(k-k_{n-1})}}e^{-\pi s_k l_k}, \tfrac{b_n-\tilde{b}_{n-1}}{2} + \tfrac{r_{n(\tilde{k}_n-k_{n-1})}(b_n-\tilde{b}_{n-1})}{4r_{n(k-k_{n-1})}}]$$

for $k_{n-1}+1 \leq k \leq \tilde{k}_{n-1}$ and

$$[b_n - r_{n(k-k_{n-1})}\min(\tfrac{v_n-b_n}{2}, \tfrac{b_n-\tilde{b}_{n-1}}{4}), b_n - r_{n(k-k_{n-1})}\min(\tfrac{v_n-b_n}{2}, \tfrac{b_n-\tilde{b}_{n-1}}{4})e^{-\pi s_k l_k}]$$

for $\tilde{k}_{n-1}+1 \leq k \leq \tilde{k}_n$ are contained in $(\tilde{b}_{n-1}, b_n)$. Therefore, we can have a piecewise analytic, bi-Lipschitz homeomorphism $h_n$ from $[\tilde{a}_{n-1}, a_n]$ onto $[\tilde{b}_{n-1}, b_n]$ such that $h_n(\tilde{a}_{n-1}) = \tilde{b}_{n-1}$, $h_n(a_n) = b_n$ and

$$h_n(t) = \begin{cases} y_n - \dfrac{r_{n(\tilde{k}_n-k_{n-1})}(b_n-\tilde{b}_{n-1})e^{-\pi s_k l_k}}{4r^2_{n(k-k_{n-1})}\min(\frac{\tilde{a}_{n-1}-\tilde{c}_{k_{n-1}}}{2}, \frac{a_n-\tilde{a}_{n-1}}{4})}(t-\tilde{a}_{n-1})^{-1} & t \in I'_k, k_{n-1}+1 \leq k \leq \tilde{k}_{n-1} \\ y_n + \dfrac{r_{n(\tilde{k}_n-k_{n-1})}(b_n-\tilde{b}_{n-1})e^{-\pi s_k l_k}}{4r^2_{n(k-k_{n-1})}\min(\frac{u_n-a_n}{2}, \frac{a_n-\tilde{a}_{n-1}}{4})}(a_n-t)^{-1} & t \in I''_k, k_{n-1}+1 \leq k \leq \tilde{k}_{n-1} \\ b_n - \dfrac{\min(\frac{v_n-b_n}{2}, \frac{b_n-\tilde{b}_{n-1}}{4})}{\min(\frac{u_n-a_n}{2}, \frac{a_n-\tilde{a}_{n-1}}{4})}(a_n-t)^{-1} & t \in I''_k, \tilde{k}_{n-1}+1 \leq k \leq \tilde{k}_n \end{cases}$$

where $y_n = \frac{b_n-\tilde{b}_{n-1}}{2}$.

Any finite part of the strip $I$ with the boundary identified under $f_n$, $g_n$ and $h_n$ is an open Riemann surface with finitely many singularities. Since the mappings $f_n$, $g_n$ and $h_n$ are bi-Lipschitz, all of them are removable by a similar argument as in the last section. Hence the strip $I$ with the boundary identified under $f_n$, $g_n$ and $h_n$ is a doubly connected Riemann surface with one ideal component corresponding to $\infty$. If this component is parabolic, then our proposition is true by the uniformization theorem. It suffices to show that $\lambda(\Gamma) = 0$ for some curve family $\Gamma$ consists of closed curves after the identification which separate the ideal component at $\infty$ from a compact subset of $I$.

Let $\Gamma_k = \{\gamma_t^{(k)}\}_{0\leq t\leq 1}$ where

$$\begin{aligned} \gamma_t^{(k)} = {} & (\bigcup_{n=m_k,\ldots,n_k}\{a_n + r_{n(k-k_{n-1})}\min(\frac{u_n-a_n}{2}, \frac{a_n-\tilde{a}_{n-1}}{4})e^{-\pi(s_k l_k t+is)} : 0 \leq s \leq 1\}) \cup \\ & (\bigcup_{n=m_k+1,\ldots,n_k}\{\tilde{a}_{n-1} + r_{n(k-k_{n-1})}\min(\frac{\tilde{a}_{n-1}-\tilde{c}_{k_{n-1}}}{2}, \frac{a_n-\tilde{a}_{n-1}}{4})e^{-\pi(s_k l_k t+is)} : \\ & 0 \leq s \leq 1\}) \cup \\ & (\bigcup_{n=m_k+1,\ldots,n_k}\{\frac{b_n-\tilde{b}_{n-1}}{2} + \frac{r_{n(\tilde{k}_n-k_{n-1})}(b_n-\tilde{b}_{n-1})}{4r_{n(k-k_{n-1})}}e^{-\pi(s_k l_k(1-t)+is)} : 0 \leq s \leq 1\}) \cup \end{aligned}$$

$$\{b_{m_k} + r_{m_k(k-k_{m_k-1})} \min(\frac{v_{m_k} - b_{m_k}}{2}, \frac{b_{m_k} - \tilde{b}_{m_k-1}}{4}) e^{-\pi(s_k l_k t + is)} : 0 \leq s \leq 1\} \cup$$
$$\{c_k + (1-t)l_k + is : 0 \leq s \leq 1\}$$

By similar methods as in the last section, it is not difficult to get $\lambda(\Gamma_k) \leq \frac{2}{l_k}$. Therefore, $\lambda(\Gamma) \leq (\sum \frac{l_k}{2})^{-1} = 0$ where $\Gamma = \bigcup \Gamma_k$ by **3.1.1 (4)**. Hence the component at $\infty$ is parabolic.

# Chapter 4 A Potential Theory Approach

We are going to prove Theorem 3 and Theorem 4 in the first chapter. When the set $\bar{M}_L$ is small (of capacity zero), the topological condition we impose on the lamination in this chapter guarantees conformality. When the set is bigger, we can only show conformality with a stronger regularity on the set (namely, Dirichlet regularity). We refer to **[3, 10, 22]** for our reference in potential theory and related topics in complex analysis.

## 4.1 The Logarithmic Capacity of a Compact Set

Let $E$ be a compact set in $\mathbf{C}$. For any positive integer $n$, we define

$$d_n = d_n(E) = \max_{z_1,\ldots,z_n \in E} \prod_{j<k} |z_k - z_j|^{\frac{2}{n(n-1)}}.$$

It is not hard to show that $d_n$ is monotonically decreasing and we call

$$\mathrm{cap}E = \lim_{n\to\infty} d_n$$

the logarithmic capacity of $E$. It is also called the "transfinite diameter". The points $z_{nk} \in E$ $(k=1,\ldots,n)$ where $d_n$ assumes the maximum are called the Fekete points and the the polynomial $q_n(z) = \prod_{k=1}^{n}(z - z_{nk})$ is called the $n$th Fekete polynomial. We have the property (cf **[3, ch.2]** and **[22, ch.9]**) that

$$\mathrm{cap}E = \lim_{n\to\infty} q_n(z).$$

Next we consider the trivial lamination on a compact set $E$ of capacity zero which identifies all points in $E$.

**Theorem 4.1.1** *Let $E$ be any compact subset of $\mathbf{T}$ with zero capacity. The lamination $L_E$ on $\mathbf{T}$ defined by identifying all points in $E$ is conformal.*

We are going to prove the theorem above in the next section. Before doing that, we give an important lemma which is given in **[22, p.211]** for more general $E$'s. We have a better function in the lemma below than in **[22]** since we are considering only compact sets.

**Lemma 4.1.2** *Let $E \subset \mathbf{T}$ be a compact set with zero capacity. Then there is a starlike function $h(z) = z + \ldots$ $(z \in \mathbf{D})$ such that*

(1) *$h(z) \to \infty$ as $z \to \xi$ if and only if $\xi \in E$ and*

(2) *$h$ can be extended to a continuous function from $\bar{\mathbf{D}}$ to $\hat{\mathbf{C}}$.*

Proof: For each $l > 0$, we can find $n_l$ such that

$$|q_{n_l}(z)|^{\frac{1}{n_l}} < e^{-4^l} \text{ for } z \in H_l$$

where $H_l$ is some open set containing $E$. Let $z_{n_l k}$ $(k = 1, \ldots, n_l)$ be the $n_l$-th Fekete points of $E$. Then let

$$h(z) = z \prod_{l=0}^{\infty} \prod_{k=1}^{n_l} (1 - \bar{z}_{n_l k} z)^{-\frac{1}{n_l 2^l}} \ (z \in \bar{\mathbf{D}} \setminus E).$$

As $z \in \bar{\mathbf{D}} \setminus E$, $Re(1 - \bar{z}_{n_l k} z) > 0$. And so we have $\sum_{l=0}^{\infty} \sum_{k=1}^{n_l} \frac{1}{n_l 2^l} \log(1 - \bar{z}_{n_l k} z)$ converges absolutely and uniformly on compact subsets of $\bar{\mathbf{D}} \setminus E$ if and only if $\prod_{l=0}^{\infty} \prod_{k=1}^{n_l} (1 - \bar{z}_{n_l k} z)^{-\frac{1}{n_l 2^l}}$ converges absolutely and uniformly on compact subsets of $\bar{\mathbf{D}} \setminus E$. For $z \in \bar{\mathbf{D}} \setminus E$, we have

$$|\log(1 - \bar{z}_{n_l k} z)| \leq |\log|1 - \bar{z}_{n_l k} z|| + 2\pi \text{ and}$$
$$d(z, E) \leq |z_{n_l k} - z| = |1 - \bar{z}_{n_l k} z| \leq 2.$$

Therefore

$$|\log|1 - \bar{z}_{n_l k} z|| \leq \max(\log 2, |\log d(z, E)|).$$

From the inequality above, we have

$$\sum_{l=N}^{\infty} \sum_{k=1}^{n_l} \frac{1}{n_l 2^l} |\log(1 - \bar{z}_{n_l k} z)| \leq 2^{1-N}(\max\{\log 2, |\log d(z, E)|\} + 2\pi).$$

Hence $\prod_{l=0}^{\infty} \prod_{k=1}^{n_l} (1 - \bar{z}_{n_l k} z)^{-\frac{1}{n_l 2^l}}$ converges absolutely and uniformly on compact subsets of $\bar{\mathbf{D}} \setminus E$ and thus $h(z)$ is continuous on $\bar{\mathbf{D}} \setminus E$.

Let $\xi \in E$. Since $|1 - \bar{z}_{n_l k} z| \leq 2$ for $z \in \bar{\mathbf{D}} \setminus E$ and $|z_{n_l k}| = 1$, for $z \in (\bar{\mathbf{D}} \setminus E) \cap H_l$,

$$|h(z)| \geq \frac{|z|}{4}|q_{n_l}(z)|^{-\frac{1}{n_l 2^l}} \geq \frac{|z|}{4}e^{2^l}$$

which implies that $h(z) \to \infty$ as $z \to \xi$ if $\xi \in E$. This proves that $h$ is analytic in $\mathbf{D}$ and continuous in $\bar{\mathbf{D}}$.

Finally, by **[22, theorem 3.18]**, $h$ is starlike because the sum of all the exponents is $-2$.

*Q.E.D.*

## 4.2 A Family of Mappings on $\hat{\mathbf{C}}\setminus$ [-1,1]

For any subset $A$ of the complex plane, we denote the convex hull of $A$ by $coA$, i.e., $coA=\{\sum_{j=1}^{n} t_j a_j : \sum_{j=1}^{n} t_j = 1 \text{ and } a_j \in A,\ t_j \geq 0 \text{ for all } j\}$.

Let $K_n=co\{-1,1,i/n,-i/n\}$, $K_n^+=co\{-1,1,i/n\}$. Then let $g_n^+$: $\mathbf{H}_+ \longrightarrow \mathbf{H}_+\setminus K_n^+$ be the conformal mapping with $g_n^+(-1) = -1$, $g_n^+(1) = 1$ and $g_n^+(\infty) = \infty$. Since $\mathbf{H}_+ \setminus K_n^+$ is a sub-domain of the upper half-plane, we can extend $g_n^+$ by reflection to a conformal map $g_n$: $\hat{\mathbf{C}} \setminus [-1,1] \longrightarrow \hat{\mathbf{C}} \setminus K_n$ such that $g_n(-1) = -1$, $g_n(1) = 1$ and $g_n(\infty) = \infty$.

**Lemma 4.2.1** *Let $\rho_S$ be the spherical metric on the Riemann sphere. There exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that for any $\varepsilon > 0$, there is $N > 0$ such that for all $n_k \geq N$, $\rho_S(g_{n_k}(z), z) < \varepsilon$ for any $z \in \hat{\mathbf{C}} \setminus [-1,1]$.*

Proof: Let $k(z) = z + \frac{1}{z}$ be the *Koebe* function from $\mathbf{D}$ onto $\hat{\mathbf{C}} \setminus [-1,1]$. Let $f_n = k^{-1} \circ g_n \circ k$. Then $f_n$ is an univalent function from $\mathbf{D}$ into $\mathbf{D}$ fixing 0, $-1$ and 1. Also, $G_n = f_n(\mathbf{D}) = k^{-1}(\hat{\mathbf{C}} \setminus K_n)$.

As $k$ is continuous on $\bar{\mathbf{D}}$ and $K_n$ is uniformly locally connected, $\bar{\mathbf{D}}\setminus G_n$ is uniformly locally connected. Consequently, $\hat{\mathbf{C}}\setminus G_n = \mathbf{D}_\infty \cup (\bar{\mathbf{D}}\setminus G_n)$ is uniformly locally connected. By **Proposition 2.3** in **[22, p.23]**, The family of functions $\{f_n\}$ is equicontinuous in $\bar{\mathbf{D}}$ and thus it has a uniformly convergent subsequence $\{f_{n_k}\}$.

Let us denote the limit function by $f$. Then $f(\mathbf{D}) = \lim_{n\to\infty} G_n = \mathbf{D}$ and thus $f$ is a *Möbius* transform. Because $f$ fixes 0, $-1$ and 1, $f$ must be the identity map.

For any $z \in \hat{\mathbf{C}} \setminus [-1,1]$,

$$\begin{aligned} \rho_S(g_n(z), z) &= \rho_S(k \circ f_n \circ k^{-1}(z), z) \\ &= \rho_S(k \circ f_n(\omega), k(\omega)) \qquad \text{where } \omega = k^{-1}(z) \in \mathbf{D} \\ &= \rho_S(k \circ f_n(\omega), k \circ f(\omega)). \end{aligned}$$

Since $k$ is a continuous mapping from $\bar{\mathbf{D}}$ to $(\hat{\mathbf{C}}, \rho_S)$, it is uniformly continuous. Therefore, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\rho_S(k(\omega_1), k(\omega_2)) < \varepsilon \text{ whenever } |\omega_1 - \omega_2| < \delta.$$

As $\{f_{n_k}\}$ is uniformly convergent in $\bar{\mathbf{D}}$, there is an $N > 0$ such that

$$|f_{n_k}(\omega) - f(\omega)| < \delta \text{ for any } n_k \geq N.$$

Hence, for any $z \in \hat{\mathbf{C}} \setminus [-1,1]$, we have

$$\rho_S(g_{n_k}(z), z) < \varepsilon \text{ for any } n_k \geq N.$$

*Q.E.D.*

**Lemma 4.2.2** *Let $\gamma$ be a compact continuous curve in $\hat{\mathbf{C}}$ containing the origin and $\Omega$ be the unbounded component of $\hat{\mathbf{C}} \setminus \gamma$. Suppose $0$ is accessible from $\Omega$ and is not a cut point, then for any $\epsilon > 0$, there is a conformal mapping $g : \Omega \longrightarrow \hat{\mathbf{C}}$ such that*

(1) *$g$ fixes $0, \infty$ and the image of $\Omega$ under $g$ is a Jordan domain, and*

(2) *$\rho_S(g(z), z) < \epsilon$ for any $z \in \Omega$.*

Proof: Let $\Phi : \Omega \longrightarrow \hat{\mathbf{C}} \setminus [-1,1]$ be the conformal map such that $\Phi(\infty) = \infty$ and $\Phi(0) = -1$ and $k$ be the *Keobe* function. As $\Phi^{-1} \circ k : \mathbf{D} \longrightarrow (\Omega, \rho_S)$ is a conformal mapping from the unit disk to a simply connected domain with locally connected boundary, it can be extended to a continuous function up to $\bar{\mathbf{D}}$. And by compactness of $\bar{\mathbf{D}}$, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\rho_S(\Phi^{-1} \circ k(\omega_1), \Phi^{-1} \circ k(\omega_2)) < \epsilon \text{ whenever } |\omega_1 - \omega_2| < \delta.$$

Consider $\tilde{g}_k = \Phi^{-1} \circ g_{n_k} \circ \Phi$ where $g_{n_k}$ is defined as in the beginning of this section. We can see that each $g_k$ satisfies (1) in **4.2.1**. For any $z \in \Omega$,

$$\begin{aligned}
\rho_S(\tilde{g}_k(z), z) &= \rho_S(\Phi^{-1} \circ g_{n_k} \circ \Phi(z), z) \\
&= \rho_S(\Phi^{-1} \circ g_{n_k}(\omega), \Phi^{-1}(\omega)) \text{ where } \omega = \Phi(z) \in \hat{\mathbf{C}} \setminus [-1, 1] \\
&= \rho_s(\Phi^{-1} \circ g_{n_k} \circ k(\xi), \Phi^{-1} \circ k(\xi)) \text{ where } \xi = k^{-1}(\omega) \\
&= \rho_S(\Phi^{-1} \circ k \circ f_{n_k}(\xi), \Phi^{-1} \circ k(\xi)) \ (f_n \text{ as defined in } \mathbf{4.2.1}) \\
&< \epsilon
\end{aligned}$$

if $|f_{n_k}(\xi) - \xi| < \delta$. However, this is true for large $n_k$ by **4.2.1**. Hence we can just take $g$ to be $\tilde{g}_k$ for a large $k$.

*Q.E.D.*

Now we are ready to prove **4.1.1**.

Proof of **4.1.1**: Let $h : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ be the starlike function in **4.1.2** which is continuous up to $\mathbf{T}$ and let $\tilde{h} = \frac{1}{h}$.

We are going to define $\varphi_n{:}\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ inductively. Let $\{I_n\}_{n=0}^{\infty}$ be the collection of components of $\mathbf{T} \setminus E$. Take $\varphi_0 = \tilde{h}$. Suppose $\varphi_{n-1}{:}\bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ is defined. From **4.2.2**, there is a conformal mapping $g_n : \hat{\mathbf{C}} \setminus \varphi_{n-1}(\bar{I}_{n-1}) \longrightarrow \hat{\mathbf{C}}$ fixing 0, $\infty$ such that $g_n \circ \varphi_{n-1}|_{\bar{I}_n}$ is the boundary of a Jordan domain and $\rho_S(g_n(\omega), \omega) < \frac{1}{2^n}$. Take $\varphi_n = g_n \circ \varphi_{n-1}$. It is clear that each $\varphi_n$ is continuous up to the boundary and $\varphi_n^{-1}(0) = E$.

For any $\epsilon > 0$, take $N > 0$ such that $\frac{1}{2^N} < \epsilon$. For any $z \in \mathbf{D}$, $m > n \geq N$,

$$\begin{aligned}
\rho_S(\varphi_m(z), \varphi_n(z)) &= \rho_S(g_m \circ \ldots \circ g_{n+1}(\varphi_n(z)), \varphi_n(z)) \\
&\leq \rho_S(g_m(g_{m-1} \circ \ldots \circ g_{n+1} \circ \varphi_n(z)), g_{m-1} \circ \ldots \circ g_{n+1} \circ \varphi_n(z)) \\
&\quad + \ldots + \rho_S(g_{n+1}(\varphi_n(z)), \varphi_n(z)) \\
&= \sum_{k=n+1}^{m} \rho_S(g_k(\varphi_{k-1}(z)), \varphi_{k-1}(z)) \\
&< \sum_{k=n+1}^{m} \frac{1}{2^k}
\end{aligned}$$

$$\leq \frac{1}{2^N}.$$

Since the function $\chi_{mn}(z) = \rho_S(\varphi_m(z), \varphi_n(z))$ is continuous on $\bar{\mathbf{D}}$, for any $z \in \bar{\mathbf{D}}$,

$$\rho_S(\varphi_m(z), \varphi_n(z)) \leq \tfrac{1}{2^N} < \epsilon.$$

Therefore, $\varphi_n$ is uniformly convergent in $\bar{\mathbf{D}}$. Let $\varphi = \lim_{n\to\infty} \varphi_n$. $\varphi \in C_A$ and $\varphi(E) = \{0\}$. It remains to show that $\varphi|_{I_n}$ is one-to-one and $\varphi(\xi) \neq 0$ if $\xi \in I_n$.

For any positive integer $n$, let $G_n = \lim_{m\to\infty} g_m \circ \ldots \circ g_n$ which is conformal in the unbounded component of $\hat{\mathbf{C}} \setminus \varphi_{n-1}(\mathbf{T} \setminus (\cup_{k=0}^{n-1} I_k))$. Observe that $\varphi = g_n \circ \varphi_{n-1}$. Because $\varphi_{n-1}(I_{n-1})$ is a simple curve with endpoints 0 which lies in the unbounded component $\Omega_{n-1}$ of $\hat{\mathbf{C}} \setminus \varphi_{n-1}(\mathbf{T} \setminus (\cup_{k=0}^{n-1} I_k))$, $G_n \circ \varphi_{n-1}(I_{n-1})$ is a simple curve with endpoints 0 which lies in $G_n(\Omega_{n-1})$.

*Q.E.D.*

## 4.3 Totally Disconnected Laminations

For any closed and flat laminations on $\mathbf{T}$, we call any point in the closure of the set of multiple points a branch point. In the case $L = L(\varphi)$ for $\varphi \in D_A$, all points in $\mathbf{T}$ are branch points. And conversely, if $L = L(\varphi)$ for $\varphi \in C_A$ and all points in $\mathbf{T}$ are branch points, then $L = L(\varphi)$ for some $\varphi \in D_A$ by **2.3.6**.

**Definition 4.3.1** *Let $L$ be a continuous and flat lamination on $\mathbf{T}$. $L$ is totally disconnected if $\bar{M}_L/L$ is totally disconnected, i.e., the image of the set of branch points under the projection $\pi : \bar{\mathbf{D}} \longrightarrow \hat{\mathbf{C}}$ is a totally disconnected set.*

**Lemma 4.3.2** *Let $L$ be a closed and flat lamination on $\mathbf{T}$. If there exists an open subarc $J$ of $\mathbf{T}$ which satisfies the following:*

(1) *The endpoints of $J$ lie in $\bar{M}_L$ and are not equivalent under $L$.*

(2) *For any component $I$ of $\bar{J} \setminus \bar{M}_L$, there is an open arc $I \subseteq \tilde{I} \subseteq J$ such that the endpoints of $\tilde{I}$ are equivalent under $L$.*

*Then $L$ is not totally disconnected.*

Proof: Let $I_n$ be the components of $\bar{J} \setminus \bar{M}_L$. Let $\tilde{I}_n = \bigcup \tilde{I}$ where the union is over all $I_n \subseteq \tilde{I} \subseteq J$ such that the endpoints of $\tilde{I}$ are equivalent under $L$. From the flatness condition, $\tilde{I}_n$ and $\tilde{I}_m$ are either disjoint or equal. We pick out all the disjoint ones and still denote them by $\tilde{I}_n$. It is easy to show that $\gamma_N = (\bar{J} \setminus (\bigcup_{n=1}^{N} \tilde{I}_n))/L$ is connected. Since

$$\gamma = (\bar{J} \setminus (\bigcup_{n=1}^{\infty} \tilde{I}_n))/L = \bigcap_{N=1}^{\infty} \gamma_N \subseteq (\bar{J} \setminus (\bigcup_{n=1}^{\infty} I_n))/L = (\bar{J} \cap \bar{M}_L/L$$

is an intersection of countably many compact, connected sets $\gamma_{N+1} \subset \gamma_N$, $\gamma$ is connected. Thus there is a connected subset of $\bar{M}_L/L$ which contains at least two points. Hence, $L$ is not totally disconnected.

*Q.E.D.*

**Lemma 4.3.3** *For any closed and flat lamination $L$ on $\mathbf{T}$, $L$ is totally disconnected if and only if for any two non-equivalent points $\xi_1, \xi_2 \in \bar{M}_L$, there exist components $I_1$, $I_2$ of $\mathbf{T} \setminus \bar{M}_L$ such that each of the two components of $\mathbf{T} \setminus (I_1 \cup I_2)$ contains all the equivalent points under $L$ and one of the $\xi_i$'s.*

Proof: Suppose $L$ is totally disconnected. Let $\xi_1, \xi_2 \in \bar{M}_L$ be two non-equivalent points on $\mathbf{T}$ such that for any components $I_1$, $I_2$ of $\mathbf{T} \setminus \bar{M}_L$, the two components of $\mathbf{T} \setminus (I_1 \cup I_2)$ contain points equivalent to some points in the other one, i.e., $(\mathbf{T} \setminus (I_1 \cup I_2))/L$ is connected.

Let $K_i = \{\xi \in \mathbf{T} : \xi \sim \xi_i\}$ and $J_1$, $J_2$ be the components of $\mathbf{T} \setminus (K_1 \cup K_2)$ with non-equivalent endpoints. Then $(\bar{J}_1 \cup \bar{J}_2)/L|_{K_1 \cup K_2}$ is homeomorphic to the circle and $L|_{\bar{J}_1 \cup \bar{J}_2}$ induces a totally disconnected lamination on $(\bar{J}_1 \cup \bar{J}_2)/L|_{K_1 \cup K_2}$ with exactly the same property as described in the previous paragraph. It follows also that the two points $\eta_1$, $\eta_2$ in the circle corresponding to $K_1$, $K_2$ respectively must be branch points, otherwise one of the sets $K_1$, $K_2$ is bounded away from $(J_1 \cup J_2) \cap \bar{M}_L$ which gives a contradiction.

Therefore, without loss of generality, we can assume that $\xi_1 = -1$, $\xi_2 = 1$ and the $\xi_i$'s do not have any other equivalent points. Let $I_+$ be the upper semi-circle and

$I_-$ be the lower semi-circle. From **4.3.2**, there is a component $I$ of $I_+$ such that any open arc $I \subseteq J \subseteq I_+$ has non-equivalent endpoints. Take

$$J_+ = \mathrm{Int}(\bigcap\{\bar{J} : I \subseteq J \subseteq I_+, \text{ endpoints of } J \text{ are equivalent to some points in } I_-\}).$$

Since endpoints of $J_+$ cannot be equivalent, there is an open arc $J_- \subseteq I_-$ with non-equivalent endpoints which are equivalent to endpoints of $J_+$ under $L$.

For any component $I'$ of $J_- \setminus \bar{M}_L$, let $\gamma_1$, $\gamma_2$ be the components of $\mathbf{T} \setminus (I \cup I')$. From our assumption, there are points $\xi_i \in \gamma_i$ $(i = 1, 2)$ such that $\xi_1 \sim \xi_2$ under $L$. We have one of the following: $\xi_1 \in \gamma_1 \cap J_+$, $\xi_1 \in \gamma_1 \cap J_-$ or $\xi_1 \in \gamma_1 \setminus (J_+ \cup J_-)$. In the first case, the definition of $J_+$ gives $\xi_2 \in \gamma_2 \cap I_+$ which violates our choice of $I$. In the last case, either $\xi_2 \in \gamma \cap I_+$ or $\xi_2 \in \gamma_2 \cap I_-$ contradicts our choice of $I$ and $J_+$. Therefore $\xi_1 \in \gamma_1 \cap J_-$. It follows from the definitions of $J_+$, $J_-$ and the flatness condition on $L$ that $\xi_2 \in \gamma_2 \cap J_-$. Since this is true for any component $I'$ of $J_- \setminus \bar{M}_L$, $L$ is not totally disconnected by **4.3.2**. We have a contradiction.

Conversely, if for any two non-equivalent points $\xi_1, \xi_2 \in \bar{M}_L$, there exists components $I_1$, $I_2$ of $\mathbf{T} \setminus \bar{M}_L$ such that each of the two components of $\mathbf{T} \setminus (I_1 \cup I_2)$ contains all the equivalent points under $L$ and one of the $\xi_i$'s, let $\eta_1$ and $\eta_2$ be the midpoints of $I_1$ and $I_2$ respectively. Then the image of the two components of $\mathbf{T} \setminus \{\eta_1, \eta_2\}$ under the projection onto the quotient space would be two disjoint open sets containing the image of $\xi_1$ and $\xi_2$ respectively. Thus any two distinct points in $\bar{M}_L/L$ are separated. Hence $L$ is totally disconnected.

*Q.E.D.*

We are going to prove Theorem 3 in the first chapter now.

**Theorem 4.3.4** *Let $L$ be a totally disconnected lamination on $\mathbf{T}$. If $\bar{M}_L$ is of capacity zero, then $L$ is conformal.*

Proof: From **2.1.1**, there is a $\varphi' \in D_A$ such that $L(\varphi') = L_{\bar{M}_L}$. From the proof of **2.1.1**, $\varphi'(\bar{I})$ is a Jordan curve for each component $I$ of $\mathbf{T} \setminus \bar{M}_L$. Also $\bar{\Omega}_n \cap \bar{\Omega}_m = \{0\}$ where $\Omega_n$ is the interior of the Jordan curve $\varphi'(\bar{I}_n)$, $\Omega_m$ is the interior of the Jordan curve $\varphi'(\bar{I}_m)$ and $m \neq n$. Take $\gamma_n$ to be a Jordan arc in $\Omega_n$ joining an interior point of $\Omega_n$ to 0.

Let us consider all pairs of components $(I_m, I_n)$ such that each component of $\mathbf{T} \setminus (I_m \cup I_n)$ contains all its equivalent points under $L$. Let the pair $P_{mn} = (\gamma_m, \gamma_n)$ be the corresponding pair of Jordan arcs which lie in $\Omega_m$ and $\Omega_n$ respectively. Since there are countably many of them, we just write them as $P_k$.

We are going to define $\varphi_n : \hat{\mathbf{C}} \setminus (\bigcup \bar{\gamma}_n) \longrightarrow \hat{\mathbf{C}}$ and $\varphi_n P_k$ $(k > n)$ inductively (note that $\bigcup \bar{\gamma}_n$ is a closed set). Take the pair $P_0 = (\gamma_{m_0}, \gamma_{n_0})$. Then $\bar{\gamma}_{n_0} \cup \bar{\gamma}_{m_0}$ is a compact continuous curve. By **4.2.2**, there is a conformal $g_0 : \hat{\mathbf{C}} \setminus (\bar{\gamma}_{n_0} \cup \bar{\gamma}_{m_0}) \longrightarrow \hat{\mathbf{C}}$ fixing $\infty$ such that the image of $\hat{\mathbf{C}} \setminus (\bar{\gamma}_{n_0} \cup \bar{\gamma}_{m_0})$ is $\hat{\mathbf{C}} \setminus \bar{D}$ where $D$ is a bounded Jordan domain and

$$\rho_S(g_0(z), z) < 1 = d_0 \text{ for any } z \in \hat{\mathbf{C}} \setminus (\gamma_{n_0} \cup \gamma_{m_0}).$$

Take $\varphi_0 = g_0$. Let $\gamma_1'$, $\gamma_2'$ be two disjoint Jordan arcs in $D$ with one endpoint in the boundary that corresponds to an impression of a prime end at $\bar{\gamma}_{m_0} \cap \bar{\gamma}_{n_0}$. For $P_k = (\gamma_m, \gamma_n)$, if neither $\gamma_{m_0}$ nor $\gamma_{n_0}$ are in the pair, we will define

$$\varphi_0 P_k = (\varphi_0(\gamma_m), \varphi_0(\gamma_n));$$

otherwise one of $\gamma_{m_0}$ and $\gamma_{n_0}$ is in the pair $P_k$, say, $\gamma_m = \gamma_{m_0}$, and we will define

$$\varphi_0 P_k = (\varphi_0(\gamma_{m_0}), \gamma_i')$$

where $i$ is either 1 or 2 with the choice made such that $\varphi_0(\gamma_{m_0})$ and $\gamma_i'$ have one endpoint in common.

Suppose $\varphi_{n-1} : \hat{\mathbf{C}} \setminus (\bigcup_{\gamma \in P_k, 0 \leq k \leq n-1} \bar{\gamma}) \longrightarrow \hat{\mathbf{C}}$ is defined so that the image of $\hat{\mathbf{C}} \setminus (\bigcup_{\gamma \in P_k, 0 \leq k \leq n-1} \bar{\gamma})$ is the complement of a bounded connected set which is the union of finitely many closures of Jordan domains with gluing at a finite set $K_{n-1} = \{z \in \hat{\mathbf{C}} : z \text{ is the impression of a prime end at } 0 \text{ for } \varphi_{n-1}\}$. Let

$$d_n = \min(\min_{k=0,\ldots,n-1} d_k, \min_{z \neq z' \in K_{n-1}} \rho_S(z, z')) > 0.$$

Suppose also that $\varphi_{n-1} P_k$ is defined for $k > n-1$ so that the Jordan arcs in each pair $P_k$ have one endpoint in common. By **4.2.2**, there is a $g_n$ conformal on the complement of the closure of the union of these two arcs, fixing $\infty$, with image the exterior of a Jordan domain and

$$\rho_S(g_n(z), z) < d_n/4.$$

Take $\varphi_n = g_n \circ \varphi_{n-1}|_{\hat{\mathbf{C}}\setminus(\bigcup_{\gamma\in P_k, 0\le k\le n}\bar{\gamma})}$. We can define $\varphi_n P_k$ for $k > n$ in a similar way as in the case $n = 0$. Then let $\tilde{\varphi}_n = \varphi_n \circ \varphi'$ which is conformal in $\mathbf{D}$ and continuous up to the boundary.

For any $\epsilon > 0$, take $N > 0$ such that $\frac{1}{4^N} < \epsilon$. For any $z \in \mathbf{D}$, $m > n \geq N$,

$$\begin{aligned}
\rho_S(\tilde{\varphi}_m(z), \tilde{\varphi}_n(z)) &= \rho_S(g_m \circ \ldots \circ g_{n+1}(\tilde{\varphi}_n(z)), \tilde{\varphi}_n(z)) \\
&\leq \rho_S(g_m(g_{m-1} \circ \ldots \circ g_{n+1} \circ \tilde{\varphi}_n(z)), g_{m-1} \circ \ldots \circ g_{n+1} \circ \tilde{\varphi}_n(z)) \\
&\quad + \ldots + \rho_S(g_{n+1}(\tilde{\varphi}_n(z)), \tilde{\varphi}_n(z)) \\
&= \sum_{k=n+1}^{m} \rho_S(g_k(\tilde{\varphi}_{k-1}(z)), \tilde{\varphi}_{k-1}(z)) \\
&< \sum_{k=n+1}^{m} d_k/4 \\
&\leq \sum_{k=n+1}^{m} \frac{d_0}{4^{k+1}} \\
&< \frac{1}{4^{N+1}}.
\end{aligned}$$

Since $\chi_{mn}(z) = \rho_S(\tilde{\varphi}_m(z), \tilde{\varphi}_n(z))$ is continuous on $\bar{\mathbf{D}}$, for any $z \in \bar{\mathbf{D}}$,

$$\rho_S(\tilde{\varphi}_m(z), \tilde{\varphi}_n(z)) \leq \tfrac{1}{4^N} < \epsilon.$$

Therefore, $\tilde{\varphi}_n$ is uniformly convergent in $\bar{\mathbf{D}}$. Let $\varphi = \lim_{n\longrightarrow\infty} \tilde{\varphi}_n$ which is conformal in $\mathbf{D}$ and continuous in $\bar{\mathbf{D}}$. It remains to check that $L(\varphi) = L$.

Because $\varphi'(\mathbf{T} \setminus \bar{M}_L)$ is in the domain of $\varphi_n$ for each $n$, $\varphi$ is one-to-one in $\mathbf{T} \setminus \bar{M}_L$, i.e., $L(\varphi) = L$ on $\mathbf{T} \setminus \bar{M}_L$. Now for $\xi_1, \xi_2 \in \bar{M}_L$, if $\xi_1 \sim \xi_2$, then any two $I_m, I_n$ in the above construction lie in the same component of $\mathbf{T} \setminus \{\xi_1, \xi_2\}$ and in turn, $\tilde{\varphi}_n(\xi_1) = \tilde{\varphi}_n(\xi_2)$ for each $n$. Thus $L \subseteq L(\varphi)$. If $\xi_1$ and $\xi_2$ are not equivalent, take

$$n_0 = \min\{n : \tilde{\varphi}_n(\xi_1) \neq \tilde{\varphi}_n(\xi_2)\}.$$

Since for each $m > n_0$,

$$\begin{aligned} d_{n_0} &\leq \rho_S(\tilde{\varphi}_{n_0}(\xi_1), \tilde{\varphi}_{n_0}(\xi_1)) \\ &\leq \rho_S(\tilde{\varphi}_{n_0}(\xi_1), \tilde{\varphi}_m(\xi_1)) + \rho_S(\tilde{\varphi}_m(\xi_1), \tilde{\varphi}_m(\xi_2)) + \rho_S(\tilde{\varphi}_m(\xi_2), \tilde{\varphi}_{n_0}(\xi_2)), \end{aligned}$$

letting $m$ tend to infinity,

$$\begin{aligned} \rho_S(\varphi(\xi_1), \varphi(\xi_2)) &\geq d_{n_0} - \rho_S(\tilde{\varphi}_{n_0}(\xi_1), \varphi(\xi_1)) - \rho_S(\tilde{\varphi}_{n_0}(\xi_2), \varphi(\xi_2)) \\ &\geq d_{n_0} - 2 \sup_{z \in \mathbf{D}} \sum_{k=n+1}^{\infty} \rho_S(g_k(\tilde{\varphi}_{k-1}(z)), \tilde{\varphi}_{k-1}(z)) \\ &\geq d_{n_0} - 2 \sum_{k=1}^{\infty} \frac{d_{n_0}}{4^k} \\ &= \frac{d_{n_0}}{3} \\ &> 0. \end{aligned}$$

Therefore $L(\varphi) = L$.

*Q.E.D.*

As an application of **4.3.4**, we have

**Corollary 4.3.5** *Let $h : I_+ \longrightarrow I_-$ be a homeomorphism fixing $-1, 1$ and $E$ a compact subset of $I_+$ with zero capacity. If $h(E)$ is of capacity zero, then the lamination $L$ is conformal where $L$ is defined by $x \sim y$ if $x = y$ ,or $x \in E$ and $y = h(x)$, or $y \in E$ and $x = h(y)$.*

Proof: Since $\bar{M}_L = E \cup h(E)$ and $\bar{M}_L / L$ is homeomorphic to $E$ which is totally disconnected, by **4.3.4**, $L$ is conformal.

*Q.E.D.*

**Corollary 4.3.6** *Let $\{\xi_i^{(k)}\}_{i=1,\dots,n_k}$ $(k = 1, \dots)$ be a sequence of distinct $n_k$-tuples on the unit circle such that $\{\xi_i^{(k')}\}_{i=1,\dots,n_{k'}}$ lie in one component of $\mathbf{T} \setminus \{\xi_i^{(k)}\}_{i=1,\dots,n_k}$ for each $k \neq k'$. Let $L$ be the equivalence relation defined by*

$\xi_1 \sim \xi_2$ *if* $\xi_1 = \xi_2$, *or* $\xi_1 = \lim_{l \longrightarrow \infty} \xi_{i_{1l}}^{(k_l)}$ *and* $\xi_2 = \lim_{l \longrightarrow \infty} \xi_{i_{2l}}^{(k_l)}$ *for some sequence of positive integers* $k_l$.

*If the set of limit points of* $\bigcup_{k=1,\ldots,\infty}\{\xi_i^{(k)}\}_{i=1,\ldots,n_k}$ *is a finite set, then* $L$ *is conformal.*

Proof: The quotient space $\bar{M}_L/L$ is homeomorphic to a sequence of points with finitely many limit points in $\mathbf{C}$ which is totally disconnected, so by **4.3.4**, $L$ is conformal.

*Q.E.D.*

## 4.4 A Lamination on Regular Compact Sets

We are going to prove Theorem 4 in the first chapter. Suppose that $E$ is a regular compact subset of $\mathbf{T}$ with positive capacity and $\mu$ is the equilibrium measure determined by $E$. Then the Green's function

$$g(z) = \int_E \log|z-\zeta| d\mu(\zeta) - \log \operatorname{cap} E$$

with pole at infinity corresponding to $E$ is equal to 0 everywhere on $E$ (cf **[10, ch.4]**). We need the following lemma from **[22, p.66]**:

**Lemma 4.4.1** *If* $g$ *is starlike, then the limits*

$$\beta(t) = \lim_{r \longrightarrow 1-} \arg g(re^{it}),\ g(e^{it}) = \lim_{r \longrightarrow 1-} g(re^{it}) \in \hat{\mathbf{C}}$$

*exist for all* $t$ *and*

$$g(z) = zg'(0)\exp(-\tfrac{1}{\pi}\textstyle\int_0^{2\pi} \log(1-e^{-it}z)d\beta(t))\ (z \in \mathbf{D}).$$

*Conversely, if* $\beta(t)$ *is increasing and* $\beta(2\pi)-\beta(0) = 2\pi$, *then the above expression represents a starlike function.*

**Theorem 4.4.2** *Suppose that* $E \subset \mathbf{T}$ *is compact, regular and* $\operatorname{cap}E > 0$. *Let* $\mu$ *be the equilibrium measure determined by* $E$. *Then the function*

$$h(z) = z(\operatorname{cap}E)^2 \exp[-2\textstyle\int_E \log(1-\bar{\zeta}z)d\mu(\zeta)]\ (z \in \mathbf{D})$$

*maps* $\mathbf{D}$ *conformally onto a starlike domain*

$$h(\mathbf{D}) = \{se^{i\theta} : 0 \le s \le R(\theta), 0 \le \theta \le 2\pi\}$$

*where* $h$ *can be extended continuously to* $\bar{\mathbf{D}}$ *and* $|h(\zeta)| = 1$ *if and only if* $\zeta \in E$.

Proof: The function $h(z)$ is starlike from **4.4.1** with $\beta(t) = \mu(E \cap I_t)$ where $I_t$ is the image of the interval $[0, t]$ under the mapping $e^{is}$. Hence the angular limit $h(\zeta)$ exists for all $\zeta \in \mathbf{T}$. Since $|h(z)| = |z|e^{-2g(z)} < 1$ for $z \in \mathbf{D}$ and $\lim_{r \longrightarrow 1-} |h(re^{it})| = \lim_{r \longrightarrow 1-} e^{-2g(r\zeta)} = e^{-2g(\zeta)} = 1$ if and only if $\zeta = e^{it} \in E$, $|h(\zeta)| = 1$ if and only if $\zeta \in E$. In order to show that $h$ can be extended continuously to $\bar{\mathbf{D}}$, it suffices to show that both $\arg h(\zeta)$ and $|h(\zeta)|$ are continuous for $\zeta \in \mathbf{T}$. As $\mu$ is the equilibrium measure, there is no atom in its support. Thus $\beta(t)$ is an increasing function without jumps. Therefore, $\arg h(e^{it}) = \beta(t)$ is continuous.

Moreover, $g(\zeta) = 0$ on $E$ which is the support of $\mu$. Therefore $g$ is continuous in $\mathbf{C}$ (cf [**10, p.69**]). Hence $|h(\zeta)|$ is also continuous in $\mathbf{T}$.

*Q.E.D.*

The lamination $L(h)$, where $h$ is the starlike function in the theorem above, is very close to the lamination we stated in Theorem4. It is not hard to see that $h(\zeta) = h(\eta)$ if and only if $\zeta$, $\eta \in \bar{I}$ for some component $I$ of $\mathbf{T} \setminus E$ whenever $\zeta$, $\eta \in \mathbf{T}$. However, each $\bar{I}$ is mapped to a line segment joining an interior point of the unit disc to the boundary. We can do perturbations similar to those we did in the previous section to get a $\varphi$ which is one-to-one on each $I$. Thus $L(\varphi)$ is exactly the lamination we want. This proves Theorem 4.

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