I. MOORE TYPE GRAPHS. II. PROPERTIES OF (0,1)-MATRICES WITH FORBIDDEN CONFIGURATIONS. III. PROPERTIES OF (0,1)-MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS .

Thesis by

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Abstract

The thesis consists of three chapters. The first chapter starts by discussing the class $C(S)$ of all (0,1)-matrices A which have the off diagonal entries of AA^T the same as S. This leads to a consideration of the properties of remainder matrices which correspond to graphs without triangles. Our main result concerns special remainder matrices S of order n which satisfy the matrix equation $S + S^2 = k\,I + J - L$, where I is the identity matrix, J is the matrix of all 1's, and L is the direct sum of n/ℓ matrices of all 1's of order ℓ. We call such graphs Moore type graphs because for $\ell = 1$ we obtain Moore graphs and for $\ell = k$ we obtain Moore graphs stripped of some vertices. We obtain a new graph with $k = 6$, $\ell = 4$, $n = 40$ as well as an additional possible parameter set $k = 20$, $\ell = 10$, $n = 410$.

The second chapter considers configurations defined as equivalence classes of matrices, where two matrices represent the same configuration if one is a row and column permutation of the other. A matrix A is said to contain a configuration if a submatrix of A represents that configuration. For certain lists, a (0,1)-matrix A, with no configurations in some list and $AA^T > 0$, is forced to have a column of 1's. We then derive results about row intersections of a matrix. The row intersection of row i and j $(i \neq j)$ is the vector with a 1 in a column if both row i and j do and a 0 otherwise. For matrices with column sums at least 2 satisfying some condition on forbidden configurations, we find that the number of linearly independent row intersections is equal to the number of distinct columns. One possible condition is

"contains no triangles", where a triangle is a configuration of order 3 with row and column sums 2. We study the extremal matrices of size m by $\binom{m}{2}$ with distinct columns and the above properties. In the case of no triangles, we find that there are $m - \ell + 1$ columns of column sum ℓ and they form an $(\ell-1)$-tree. Such matrices have a fascinating inductive buildup. Other conditions are also considered.

The third chapter explores a number of properties of the class $\mathfrak{A}(R,S)$. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$. We define $\mathfrak{A}(R,S)$ to be all $(0,1)$-matrices of size m by n with i^{th} row sum r_i and j^{th} column sum s_j. We obtain a generalization of the Gale-Ryser theorem by finding simple conditions which determine when there exists a matrix $A \in \mathfrak{A}(R,S)$ with $A \geq P$ for some $(0,1)$-matrix P with column sums at most 1. Our second main result extends a theorem of Brualdi and Ross. Consider vectors R', S' with $r_i' = r_i - k$ or $r_i - k - 1$ and $S \geq S'$. If $\mathfrak{A}(R,S)$ and $\mathfrak{A}(R',S')$ are nonempty, then there exists an $A \in \mathfrak{A}(R,S)$ and a $B \in \mathfrak{A}(R',S')$ with $A \geq B$. Our third main result gives that under simple conditions, ($\mathfrak{A}(R,S)$ nonempty, R,S ordered, $r_i \leq n - i + 1$, $s_i \leq n - i + 1$ for all i) there is a matrix $A \in \mathfrak{A}(R,S)$ with 0's in the (i,j) entry for $j > n - i + 1$ i.e. A is in triangular form.

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Section 1. Introduction.

This chapter discusses the class $C(S)$ and uses it to motivate the study of Moore graphs and Moore type graphs. The class $C(S)$ was introduced by Ryser [36]. Let S be a symmetric matrix of order n with positive integral entries and 0's on the main diagonal. Consider a (0,1)-matrix A which has no column of column sum 0 or 1. Then $A \in C(S)$ if there exists a diagonal matrix D such that

$$AA^T = S + D, \tag{1.1}$$

where A^T is the transpose of A.

Section 2 is concerned with the class $C(S)$ and is mainly expository in nature. We discuss equivalent definitions and provide some examples. The class $C(S)$ is used in Chapter 2. Chapter 3 is concerned with a similar object, the class $\mathcal{O}\!\mathcal{C}(R,S)$. The concept of remainder matrices, which are special matrices S, is introduced in Section 3. Remainder matrices correspond to graphs without triangles. A number of results are proven here, including Turan's theorem.

The main new results of the chapter are found in Sections 4 and 5. These concern the existence of certain extremal graphs which can be motivated as certain remainder matrices. We study graphs, whose adjacency matrix S satisfies

$$S + S^2 = kI + J - L, \tag{1.2}$$

where I is the identity matrix of order n, J is the matrix of order n of all 1's, and L is the direct sum of n/ℓ matrices of order ℓ of all 1's. For $\ell = 1$, we obtain Moore graphs. For $\ell = k$, we obtain Moore graphs stripped of some vertices. Thus we call solutions to (1.2), Moore type graphs. Two additional possible parameter sets are determined: $n = 40$, $\ell = 4$, $k = 6$; $n = 420$, $\ell = 10$, $k = 20$. A Moore type graph on the first parameter set is presented in Section 5 and some of its properties are discussed. Murty has reported that a regular graph of girth 5, with the fewest number of vertices, has 40 vertices [30]. Our graph would be an example of such a minimal graph. Most of the results of Section 4 and 5 have been presented in [1].

Section 2. The class C(S).

The class $C(S)$ was introduced by Ryser [36]. We will follow his definitions. The class $C(S)$ will consist of all (0,1)-matrices satisfying a certain matrix equation involving S under a small restriction. Let $S = (s_{ij})$ be a symmetric matrix of order m with positive integral entries and 0's on the main diagonal. We wish to consider (0,1)-matrices A with m rows which satisfy the equation

$$AA^T = S + D, \tag{2.1}$$

where D is a diagonal matrix of order m depending only on the specific matrix A. In other words, the off diagonal entries of AA^T are the same as the off diagonal entries of S. We note that a column of one or no 1's can be inserted or deleted from a matrix A without affecting the off diagonal entries of AA^T. Thus we insist that A have no column of column sum 0 or 1. We then define $C(S)$ to be the set of such matrices. You may think of this as partitioning all (0,1)-matrices, with m rows and with column sums greater than 1, into various classes.

We start by considering the problem of isomorphisms. If $A \in C(S)$, then any column permutation of A is also in $C(S)$. Thus if A is of size m by n and P is a permutation matrix of order n, then $A \in C(S)$ implies $AP \in C(S)$. In addition, if Q is a permutation matrix of order m satisfying

$$S = Q^T S Q, \tag{2.2}$$

then $QAP \in C(S)$. We say that two matrices in $C(S)$ are <u>isomorphic</u> if one is a row and column permutation of the other. The row permutation will necessarily be a matrix Q of order m which satisfies (2.2).

The equation (2.1) can be interpreted in terms of set intersections. Let $X_1, X_2, \ldots, X_m$ be m subsets of an n-set $\{x_1, x_2, \ldots x_n\}$ where each x_i is in at least two of the sets. Let $A = (a_{ij})$ be the (0,1)-matrix of size m by n with $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ otherwise. Then we see that $A \in C(S)$ if and only if

$$|X_i \cap X_j| = s_{ij} \qquad (i \neq j). \tag{2.3}$$

Thus a matrix in $C(S)$ corresponds to a set system with the sizes of the intersections specified by S.

The class $C(S)$ was defined to include a number of combinatorial objects in a more general framework. In particular, a matrix $A \in C(\lambda(J_m - I_m))$ with constant column sums is a block design. In fact, any matrix $A \in C(\lambda(J_m - I_m))$ is a pairwise balanced design. It was hoped that some results in the class $C(S)$ would shed some light on the more familiar combinatorial problems.

There is a procedure to generate all matrices in the class $C(S)$. If $S = 0$, the matrix of order m of all 0's, then $C(S) = \emptyset$. For every $S \neq 0$, we can define a canonical matrix $A_0 \in C(S)$ in the following way. We require A_0 to have $s_{ij}(i < j)$ columns with 1's in row i and row j and 0's elsewhere. To be specific, we choose the lexicographic ordering of the columns. A column with 1's in row i and row j will be to the left of a column with 1's in row p and row q if $i < p$ or $i = p$ and $j < q$. If $S = J_m - I_m$, we denote A_0 by K_m. It is a (0,1)-matrix of size m by $\binom{m}{2}$ and has every possible column of column sum 2. We call it K_m because it is the vertex-edge adjacency matrix of the complete graph on m vertices which is usually denoted K_m. Now the canonical matrix A_0 has the greatest number of columns of any matrix in $C(S)$. To obtain the remaining matrices in $C(S)$, we perform all possible "collapses" of the columns.

Consider the following two matrices where B is of size m by $\binom{p}{2}$ $(p > 2)$ and C is of size m by 1 and has p 1's:

$$B = \begin{bmatrix} 0 \\ \\ K_p \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{2.4}$$

Assume a matrix A has a submatrix which, after a row permutation P and a column permutation Q, is of the form of B. Replace the submatrix by the column $P^{-1}C$ to obtain the matrix A'. We see that

$A \in C(S)$ implies that $A' \in C(S)$. This process of forming A' we call a <u>collapse of a</u> K_p <u>to a column of</u> p <u>1's</u>. The reverse procedure we will call the <u>expansion of a column of</u> p <u>1's to a</u> K_p. It is now possible to see that from any matrix $A \in C(S)$, by expanding all columns of column sum greater than 2 to the K_p's of the correct sizes, one obtains a column permutation of the canonical matrix A_0. Thus any matrix $A \in C(S)$ can be obtained from A_0 by a series of collapses of K_p's. For example, consider

$$S = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{2.5}$$

We collapse the K_3 in the first, second, and fourth columns of A_0 as follows

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{goes to} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{2.6}$$

We obtain the matrix

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{2.7}$$

where $A_1 \in C(S)$. In this particular case, A_0 and A_1 are the only nonisomorphic matrices in $C(S)$.

In the case $S = J_m - I_m$, we have a special interpretation of $C(S)$. Here, arbitrary permutations satisfy (2.2) and so for a matrix

$A_1 \in C(S)$, any row and column permutation of it, say A_1, is also in $C(S)$. We will associate with the rows, <u>points</u>, and with the columns, <u>lines</u>. If $A = (a_{ij})$ and $a_{ij} = 1$ then the i^{th} point is on the j^{th} line. We can draw any matrix $A \in C(J_m - I_m)$ as a "geometry" in the following way. There will be m points and every line on more than two points will be drawn in (as a smooth curve). No two point lines will be drawn. For such a geometry to correspond to a matrix $A \in C(S)$, we must have no pair of points together on more than one line. Enough two point lines are assumed to be present to ensure that every point is on some line. This gives us a pictorial representation of isomorphism classes in $C(J_m - I_m)$. We choose the orientation of the points and lines to give us the most pleasing picture as shown in Figure 1.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

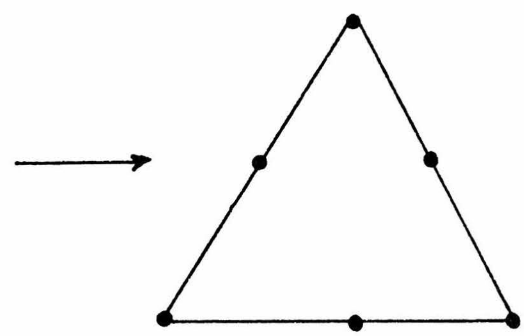


Figure 1.

The matrix on the left corresponds to the diagram on the right. This is a many to one correspondence. We use this method to display $C(J_7 - I_7)$ in Figure 2. There are 24 nonisomorphic members of $C(J_7 - I_7)$.

The collapsing process can be looked at in a slightly different way. We can mimic the collapse of a K_p in rows $i_1, i_2, \ldots, i_p$ as follows.

Let α be the column of m entries with 1's in rows $i_1, i_2, \ldots, i_p$ and zeros elsewhere. Define the matrix <u>offdiag</u> (A) to be the matrix obtained from a matrix A by setting the entries on the main diagonal to 0's. Then let

$$S_1 = S - \text{offdiag}\,(\alpha\alpha^T)\ . \tag{2.8}$$

Any matrix in $C(S)$ which is obtained by a collapse of a K_p in rows $i_1, i_2, \ldots, i_p$ can be formed by taking a matrix in $C(S_1)$ and inserting the column α. Similarly, a further collapse of a K_q in rows $j_i, j_2, \ldots, j_q$ yields

$$S_2 = S_1 - \text{offdiag}\,(\beta\beta^T), \tag{2.9}$$

where β is a column vector having 1's in rows $j_1, j_2, \ldots, j_q$ and 0's elsewhere. Thus a collapse of a K_p roughly corresponds to subtracting a principal submatrix $J_p - I_p$. This way of looking at the collapse process was used in a computer program which generated $C(J_m - I_m)$ for $m = 7, 8, 9$. The results are reported in Table 1 and Figure 2 and will appear after defining the concept of extremal matrices.

The collapsing process continues, forming a sequence of matrices $S = S_0, S_1, S_2, \ldots, S_t$. No further collapses would be possible in the canonical matrix associated with S_t when S_t has no principal submatrices of the form

$$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix} , \tag{2.10}$$

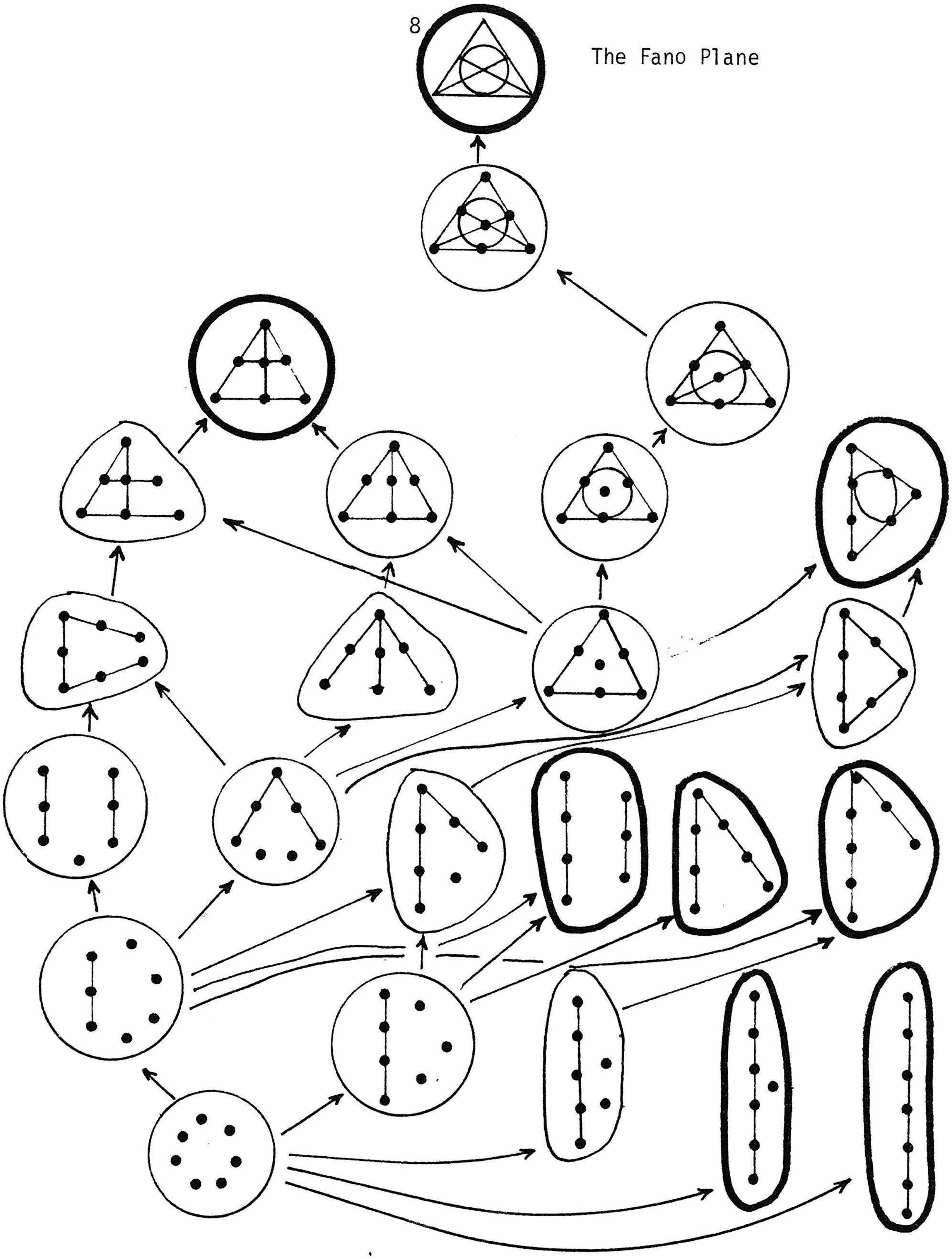


Figure 2. The isomorphism classes of $C(J_7-I_7)$ arranged in a hierarchy according to the lines added. Extremal classes are darkly outlined.

where the entries marked * are arbitrary nonzero positive integers. Thus no collapse would be possible in the matrix $A \in C(S)$ associated with the sequence $S_0, S_1, S_2, \ldots, S_t$. In this case, we say that the matrix $A \in C(S)$ is <u>extremal</u>. The zero-nonzero pattern of the final matrix in the sequence S_t will be defined in Section 3 as a remainder matrix.

Extremal matrices would seem to be the interesting objects in the class $C(S)$. They are specially outlined in Figure 2. The definition of extremal matrices includes the familiar combinatorial objects like block designs. However, for block designs, we have $S_t = 0$ and so this is a rather special case. When $S_t = 0$, then all the columns of the associated matrix $A \in C(S)$ have column sums greater than 2. It is interesting to ask when such matrices exist in $C(J_m - I_m)$. Hanani has answered this question [22]. He shows that $C(J_m - I_m)$ has a matrix with column sums greater than 2, other than the trivial example $J_{m,1}$, for $m = 7$ and $m \geq 9$.

Originally, it was hoped that extremal matrices would be easier to look at than the whole class $C(S)$ since they would be sparse in the class. Such does not appear to be the case since they are quite numerous. The following table gives the results of a computer program which generated all the isomorphism classes of $C(J_m - I_m)$ for $m = 7, 8, 9$.

Table 1. $C(J_m - I_m)$

$m = 7$:	24 nonisomorphic matrices with 8 being extremal.
$m = 8$:	69 nonisomorphic matrices with 15 being extremal.
$m = 9$:	384 nonisomorphic matrices with 40 being extremal.

All the extremal matrices, except those corresponding to the projective plane of order 2 and the Steiner triple system on 9 points, yield a final S_t in the sequence with $S_t \neq 0$.

Section 3. Remainder Matrices and Turan's theorem.

We recall that in a decomposition sequence for A, where A is extremal, the final matrix S_t in the sequence has no principal submatrices of the form given in (2.10). From S_t, we abstract its zero-nonzero pattern. Construct a (0,1)-matrix S' of order m from S_t by replacing the nonzero positions of S_t by 1's. This motivates the following definition which S' will satisfy. Let a symmetric (0,1)-matrix S of order m with zero trace be a <u>remainder</u> matrix if S has no principal submatrix $J_3 - I_3$, i.e. the zero-nonzero pattern of (2.10) :

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} . \tag{3.1}$$

It would be nice to prove results about remainder matrices since they might give us results about the extremal matrices from which they were obtained. Also we note that an equivalent definition is that $S \neq 0$ is a remainder matrix if and only if $C(S)$ has only one element up to isomorphism (the canonical matrix A_0). Thus studying remainder matrices is equivalent to studying classes with this special property. We cannot hope for too much. The extremal matrices that we are most interested in, namely block designs, have the zero matrix as their associated remainder matrix. We recall that we can obtain a symmetric $(0,1)$-matrix S of order m with zeros on the main diagonal as the adjacency matrix of a graph G on m vertices. We associate the i^{th} row and column with the i^{th} vertex, v_i. Vertex v_i and vertex v_j are joined by an edge if and only if the (i,j) entry of S is 1. Thus remainder matrices correspond to graphs without triangles (cycles of length 3). For example, all bipartite graphs can be thought of as remainder matrices. Thus we have a little too much generality which we will remove by considering only the "largest" remainder matrices.

A simple matrix interpretation of remainder matrices is given in the following remark. We first define two matrices to be complementary if when one has a nonzero entry in the (i,j) position, then the other matrix has a zero.

Remark 3.1. *Let S be a symmetric $(0,1)$-matrix. The following are equivalent:*

i) *S is a remainder matrix*

ii) *$tr(S) = 0$ and $tr(S^3) = 0$*

iii) *S and S^2 are complementary.*

Proof. i) $\Rightarrow$ ii) If S is a remainder matrix then $tr(S) = 0$ by definition. We recall that the (i,j) entry of S^k counts the number or paths of length k from vertex i to vertex j. Since S as a graph has no loops or triangles then the (i,i) entry of S^3 is zero and so $tr(S^3) = 0$

ii $\Rightarrow$ iii) If ii) holds, assume iii) does not hold and so in some position (i,j), both S and S^2 have nonzero entries. Then there is an edge joining vertex v_i and vertex v_j and a path of length 2 joining vertex v_i and vertex v_j. But then the graph has a triangle and so $tr(S^3) \neq 0$. This is a contradiction

iii) $\Rightarrow$ i) Assume iii) holds and that S has a triangle in rows i, j, k. Then the (i,j) entry of S and of S^2 are nonzero and so we have a contradiction.

One of the first questions that comes to mind is: what is the maximum number of 1's possible in a remainder matrix of order n? This problem was solved by Turan [38] and Hall [21] and is usually known as Turan's theorem. Turan's original result is actually a generalization of this, namely what graphs have the most number of edges subject to the condition that they do not have a subgraph isomorphic to K_p, the complete graph on p vertices, for some p. I will give my own proof and state the result in terms of remainder matrices.

Theorem 3.2 (Turan [38], Hall [21]). <u>The maximum number of 1's possible in a remainder matrix of order n is achieved by the following examples and they are the unique remainder matrices, up to isomorphism, with this property.</u>

$$n = 2m \quad \begin{bmatrix} 0_m & J_m \\ J_m & 0_m \end{bmatrix} \; ; \; n = 2m+1 \quad \begin{bmatrix} 0_m & J_{m,m+1} \\ J_{m+1,m} & 0_{m+1} \end{bmatrix} \tag{3.2}$$

<u>Proof</u>. Certainly, both of our examples have no triangles. In both our examples, the average column sum is equal to or greater than $n/2$.

Assume that we have a remainder matrix S of order n with at least as many 1's as the appropriate matrix in (3.2). Then S must have a column with k 1's, where $k \geq n/2$. By simultaneous row and column permutations, we can make this the first column and have the 1's occur as follows:

$$S = \begin{array}{r} n-k\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \\ k\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right.\end{array} \left[\begin{array}{cccc|cccc} 0 & & & & & & & \\ 0 & 0 & & & & & & \\ \vdots & P & \ddots & & & & & \\ 0 & & & 0 & & & & \\ \hline 1 & & & & 0 & & & \\ 1 & & & & & 0 & & \\ \vdots & & Q & & & R & \ddots & \\ 1 & & & & & & & 0 \end{array}\right] \qquad (3.3)$$

Recall that S is symmetric. The symbols P and R refer to the triangular blocks of positions below the main diagonal. The 1's in the first column of Q ensure that all the positions in the triangular block R are 0's, otherwise a triangle would be formed. If we could show that P was all 0's, we would be done since then the number of 1's in the matrix would be at most $2k(n-k)$ and this would be maximized precisely as in (3.2).

Now 1's in P force certain positions in Q to be zero. For example, a 1 in position (i,j) in P forces one of (ℓ,i) or (ℓ,j) (or both) to be zero for $n - k + 1 \leq \ell \leq n$ in order that no triangles are formed. Assume that the minimum number of zeros in any row of Q is p. Then we find that at most $p(n - k - 2)$ 1's are allowed in P without triangles being formed. One can see this most clearly for $p = 1$ and Q having a column of zeros.

But there are at least pk 0's in Q and $pk > p(n - k - 2)$.

Thus if S has any 0's in Q, it does not have the maximal number of 1's possible. If it does have all 1's in Q, then P is all 0's and we are done.

This result is usually stated in the alternate form that a graph with greater than $n^2/2$ edges if n is even or $(n^2/2) - 1$ if n is odd will contain a triangle. We are able to use Turan's theorem to prove a result about extremal matrices.

Corollary 3.3. <u>In the class</u> $C(J_n - I_n)$, <u>where</u> $n = 2m$ <u>and</u> $m \equiv 1, 3$ (<u>mod</u> 6), <u>the extremal matrices with the most number of columns have</u> $(4m^2 - m)/3$ <u>columns</u>.

<u>Proof</u>. For $m \equiv 1, 3 \pmod 6$, there exists a Steiner triple system. See [34]. If T is the matrix of a Steiner triple system on m points, then T is a matrix of size $(m^2 - m)/6$ and $T \in C(J_m - I_m)$. Form a matrix $A \in C(J_n - I_n)$ as follows:

$$A = \begin{bmatrix} T & 0 & \\ & & K_{m,m} \\ 0 & T & \end{bmatrix}, \tag{3.4}$$

where $K_{m,m}$ is the vertex-edge adjacency matrix of the complete bipartite graph where each part is of size m or, if you wish, it is the canonical matrix associated with the first matrix given in (3.2). The first part corresponds to the first m rows and the second part corresponds to the last m rows. We have that A is extremal since the only columns of column sum 2 are those in $K_{m,m}$ and we know that this is extremal.

To show that this matrix A has the most number of columns, consider the effect of collapses. A collapse of a K_p reduces $\binom{p}{2}$ columns to 1 column. Considering just the columns that are collapsed, we maximize the number of remaining columns by collapsing K_3's. In our matrix of (3.4) we have done this and also left the maximum number of columns of column sum 2 by Theorem 3.2. This proves our result.

As we have noted, classifying all remainder matrices would be hopeless. We restrict the problem in the following natural way. Consider the partial order on matrices of order n given by $A \leq B$ when $A = (a_{ij})$ and $B = (b_{ij})$ if $a_{ij} \leq b_{ij}$ for $i,j = 1, 2, \ldots, n$. We define a <u>filled</u> remainder matrix as a maximal remainder matrix. Note that if B is a remainder matrix of order n and A is a symmetric (0,1)-matrix of order n with $A \leq B$ then A is also a remainder matrix. Thus filled remainder matrices determine all remainder matrices.

In the graphical terminology, filled remainder matrices are graphs without triangles such that if you join two unjoined vertices then you create a triangle. We may construct new filled remainder matrices from old ones in the following way. Let $R = (r_1, r_2, \ldots, r_n)$ be a vector of positive nonzero integers. Run through the following process for $i = 1, 2, \ldots, n$ in turn. Form r_i copies of v_i where each copy is joined to the same vertices as v_i was joined to. A quick check ensures that the new graph yields a filled remainder matrix. In matrix terms let $S = (s_{ij})$ be a filled remainder matrix of order n. Form a new matrix S' by expanding the (i,j) entry of S into the block $s_{ij}J$, where J is of size r_i by r_j. Then S' is a filled remainder matrix

Remark 3.4. <u>A remainder matrix</u> S <u>is filled if and only if</u> $S + S^2 > 0$.

Proof. Let S be the adjacency matrix of a graph G on n $(n > 1)$ vertices without triangles. Assume S is filled. Consider a pair of vertices v_i and v_j that are not joined. Thus the (i,j) entry of S is zero. If there is no path of length 2 joining v_i and v_j, i.e. the (i,j) entry of S^2 is zero, then joining v_i and v_j will not create a multiple edge or a triangle. But then S would not be filled so we conclude $S + S^2 > 0$.

Similarly if $S + S^2 > 0$ then we can deduce S is filled since any two vertices v_i and v_j, that are not joined, are joined by a path of length 2 and hence joining them would create a triangle.

Thus we have a simple matrix equation to determine when a remainder matrix is filled. We may deduce other properties. For example, a filled remainder matrix S never splits as a direct sum, $A \oplus B$, since the associated graph would be disconnected and an edge joining the two components would not create a triangle.

We recall that the distance between two vertices is the length of the shortest path joining them. We let the distance between a vertex v_i and itself be 0. We will use this in the following results. Also recall that the diameter of a graph is the maximum distance between any two vertices. Thus a filled remainder matrix is a graph without triangles of diameter 2 by Remark 3.4.

Remark 3.5. *A filled remainder matrix S satisfies $S^3 + I > 0$ if and only if S has no repeated columns.*

Proof. Assume S is a filled remainder matrix and that $S^3 + I > 0$. Assume that column i and column j are identical. Since

S has zero trace and is symmetric, we deduce that v_i and v_j are not joined. Thus v_i and v_j are joined by a path of length 3, say (v_i, v_k, v_ℓ, v_j). Now v_i is joined to v_k and so v_j is joined to v_k but this means we have the cycle $(v_j\, v_k\, v_\ell\, v_j)$ which is a triangle. Thus S has no repeated columns.

Assume S has no repeated columns but that $S^3 + I \geq 0$. Thus, for some i and j, $(i \neq j)$, the (i,j) entry of S^3 is zero. Thus the vertices v_i and v_j are not joined by a path of length 3. This implies that v_i and v_j are not joined. Consider v_i joined to v_k. If v_j is not joined to v_k then v_j is joined to v_k by a path of length 2 since S is filled. But then there is a path of length 3 from v_i to v_j which is a contradiction. So v_j is also joined to v_k. Thus column i and column j are identical which is a contradiction. Thus $S^3 + I > 0$.

Remark 3.6. <u>For a filled remainder matrix</u> S, <u>either</u> $S^4 > 0$ <u>or the graph associated with</u> S <u>is complete bipartite but not both</u>.

<u>Proof</u>. Let S be a filled remainder matrix. Assume $S^4 > 0$. Any bipartite graph does not have a path of length 4 between two vertices v_i and v_j where v_i is in one part and v_j is in the other. In this case the (i,j) position of S^4 would be zero so we deduce that the graph associated with S is not bipartite.

Let G be the graph associated with S and assume that the (i,j) entry of S^4 is zero. Assume v_i and v_j are joined by a path of length 2: (v_i, v_k, v_j). But then v_i and v_j are joined by a path of length 4 : $(v_i, v_k, v_i, v_k, f_j)$. Thus we may assume that v_i and v_j are joined. Consider a vertex v_k in G. It is either at distance 2 (or 0) from v_i and at distance 1 from v_j or it is at distance 1 from v_i and at distance 2 (or 0) from v_j. If v_k was at distance 1 from v_i and from v_j then v_i, v_j, v_k would form a triangle. If v_k was at distance 2 from v_i and v_j then v_i and v_j would be joined by a path of length 4 which is a contradiction.

Consider any cycle in G where v_k and v_{k+1} are two adjacent vertices in the cycle. We claim that either v_i is at distance 2 (or 0) from v_k and at distance 1 from v_{k+1} or v_i is at distance 1 from v_k and at distance 2 (or 0) from v_{k+1}. If v_i was at distance 1 from both v_k and v_{k+1} then a triangle would be formed. If v_i was at distance 2 from both v_k and v_{k+1} then by our above remarks v_j would be at distance 1 from both v_k and v_{k+1} and this would yield a triangle. This proves our claim. Thus the adjacent vertices of a cycle alternate at being an even or odd distance from v_i. Thus every cycle is of even length and so G is bipartite. We conclude the proof by noting that a filled bipartite graph is a complete bipartite graph.

Further results along these lines would be nice. The question that was asked at this point is: do some "nice" filled remainder matrices exist? In the next section we consider some remainder matrices S for which $S + S^2$ takes on a particularly nice form.

Section 4. Moore graphs and Moore type graphs.

We recall from Remark 3.1 that a symmetric (0,1)-matrix S is a remainder matrix if and only if the following are true

$$\mathrm{tr}(S) = 0\ , \quad \mathrm{tr}(S^3) = 0. \tag{4.1}$$

In the rest of this chapter, we will be searching for matrices which satisfy (4.1) under an additional constraint. From Remark 3.4, we know that if, in addition to (4.1), $S + S^2 > 0$ then $\mathbf{S}$ is a filled remainder matrix. We will consider a matrix equation where $S + S^2$ is equal to a particularly nice matrix. Examine the following equation

$$S + S^2 = D + \lambda J\ , \tag{4.2}$$

where D is a diagonal matrix. Results of Bridges ensure that $D = (k-1)I$ for some k and that $\lambda = 1$ [9]. Thus studying (4.2) is equivalent to studying the following equation

$$S + S^2 = (k-1)I + J. \tag{4.3}$$

This is the defining equation for Moore graphs which has been studied by Hoffman and Singleton [25] and others. We are going to study a variation of equation (4.3). We are looking for remainder matrices S that satisfy the equation

$$S + S^2 = kI + J - L_{n,\ell} \tag{4.4}$$

where $L_{n,\ell}$ is the direct sum of n/ℓ copies of J_ℓ. Our results have been reported previously in [1].

Equation (4.4) has a simple graph theoretic interpretation. Let G be a graph on n vertices without triangles. The vertices of G are divided into n/ℓ groups of ℓ vertices each. Any pair of vertices in the same group are not joined by a path of length 1 or 2. Any other pair of vertices is joined by either a unique edge or a unique path of length 2 and not both. In addition G is regular, i.e. each vertex has degree k $(k > 0)$. We call G a <u>Moore type</u> graph.

We can give the structure of G as it relates to the diameter of the graph. Choose any vertex v_i of G and place it on level 1 as shown in Figure 3. Then level 2 consists of the k vertices at distance 1 from (joined to) vertex v_i. Level 3 consists of the vertices at distance 2 from v_i. We recall that G has no triangles. Also G has no 4 cycles since paths of length 2 joining a pair of vertices are unique. This gives us the structure shown in Figure 3 where level 3 has $k(k-1)$ vertices. The only vertices at distance more than 2 from v_i are the remaining $\ell-1$ vertices of the same group as v_i. We deduce that the diameter of G is 3, for $\ell > 1$, since vertices in the same group are not joined.

For $\ell = 1$, (4.4) reduces to (4.3) and we have a Moore graph on $k^2 + 1$ vertices. If such a graph exists, it would be the graph with the greatest number of vertices which is regular of degree k and has diameter 2. This is clear from Figure 3. The values of k for which Moore graphs may exist were determined by Hoffman and Singleton [25]. Moore graphs exist for $k = 2, 3, 7$ and possibly for $k = 57$.

For $\ell = k$, we can obtain a Moore graph from a Moore type graph in the following way. For each group of $\ell = k$ vertices, create a new vertex and join it to each vertex in its associated group. We add one more vertex and join it to these $k + 1$ new vertices. The new graph is regular of degree $k + 1$. No triangles or 4 cycles have been formed since two vertices in the same group have no paths of length 1 or 2 between them. We have $k^2 + k + (k+1) + 1 = (k+1)^2 + 1$ vertices in the new graph and so we have formed a Moore graph regular of degree $k + 1$. This is equivalent to bordering the adjacency matrix of the graph to get the adjacency matrix of a Moore graph. The process is reversible. Consider a Moore graph regular of degree k. Delete the first two levels, following Figure 3, to obtain a Moore type graph regular of degree $k - 1$ with $\ell = k - 1$. This was used by Hoffman and Singleton [25]. This process is analogous to the construction of (v, k, λ)-designs from certain group divisible designs and vice versa. See, for example, [8].

The possible parameter sets for Moore type graphs appear to be extremely rare. We determine these by eigenvalue arguments. Such ideas are discussed by Biggs [7]. Let S have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. From (4.1) we have that $\mathrm{tr}(S) = 0$ and so by elementary matrix theory

$$\sum_{i=1}^{n} \lambda_i = 0 . \tag{4.5}$$

We recall that the eigenvalues of a polynomial in S are the polynomials in the eigenvalues of S with multiplicities preserved. Thus $\mathrm{tr}(S^3) = 0$, from (4.1), yields

$$\sum_{i=1}^{n} \lambda_i^3 = 0 . \tag{4.6}$$

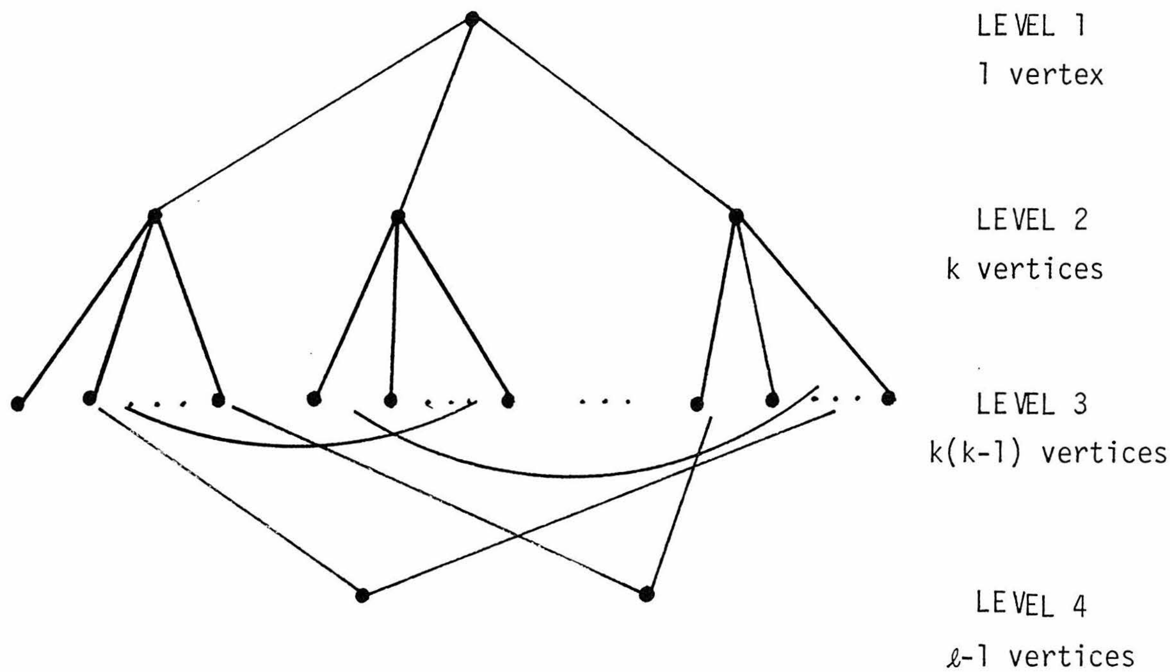


Figure 3. The structure of Moore type graphs.

Also the eigenvalues of $S + S^2$ are $\lambda_i + \lambda_i^2$ for $i = 1, 2, \ldots, n$. These arguments and a certain amount of additional information are enough to determine the possible parameters of Moore type graphs. Hoffman and Singleton [25] only used (4.6) to study Moore graphs but, in our more general problem, (4.7) is needed.

Manipulations of the symmetric polynomials in the eigenvalues yield other equations including

$$\sum_{1 \le i \le j \le k \le n} \lambda_i \lambda_j \lambda_k = 0 \quad ;$$

$$a = \sum_{1 \le i \le j \le k \le \ell \le m \le n} \lambda_i \lambda_j \lambda_k \lambda_\ell \lambda_m = \sum_{i=1}^{n} \lambda_i^5 \; , \tag{4.7}$$

where a is ten times the number of 5 cycles. These equations were first derived using concepts such as the compound of a matrix. Since the equations in (4.7) can be derived directly from (4.5) and (4.6), they yield no new information.

We now determine conditions on the parameters n, k, ℓ in order that solutions to (4.4) may exist. From the definition of $L_{n,\ell}$, we have that

$$\ell | n \; . \tag{4.8}$$

We deduce that S has row and column sums k from (4.4). Thus multiplying both sides of (4.4) by J yields $k + k^2 = k + n - \ell$, which we rewrite as

$$n = k^2 + \ell \; . \tag{4.9}$$

This equation could be deduced from Figure 3. Combining (4.8) and (4.9) yields

$$\ell \mid k^2 . \tag{4.10}$$

Let $A = kI + J - L_{n,\ell}$. We are able to determine the eigenvalues of A by determining n linearly independent eigenvectors. By inspection, we have that the vector of all 1's is an eigenvector of A of eigenvalue $k + k^2$. Let $\alpha = (a_1, a_2, \ldots, a_\ell)$ be any row vector of length ℓ with $a_1 + a_2 + \cdots + a_\ell = 0$. Let $\beta = (0,0, \ldots,0)$ be the zero vector of length ℓ Then the n/ℓ vectors

$$(\alpha,\beta,\beta, \ldots, \beta),\ (\beta,\alpha,\beta, \ldots,\beta),\ \ldots,\ (\beta,\beta, \ldots,\beta,\alpha) \tag{4.11}$$

are all eigenvectors of A of eigenvalue k. Since one can choose $\ell - 1$ linearly independent vectors for α, then there are $n(\ell-1)/\ell$ linearly independent eigenvectors of A of eigenvalue k.

Let $\gamma = (1,1, \ldots,1)$ be the row vector of all 1's of length ℓ. Then the $n|\ell-1$ vectors

$$(\gamma,-\gamma,\beta,\beta,\ldots,\beta),\ (\gamma,\beta,-\gamma,\beta, \ldots,\beta),\ldots,\ (\gamma,\beta,\beta,\ldots,\beta,-\gamma), \tag{4.12}$$

are $(n/\ell)-1$ linearly independent eigenvectors of eigenvalue $k - \ell$. Thus we have determined n linearly independent eigenvectors for A and this determines the eigenvalues of A.

Using equation (4.4) and the fact that the eigenvalues of $S + S^2$ are $\lambda_i + \lambda_i^2$ for $i = 1, 2, \ldots, n$ we obtain the following table

Table 2.

Eigenvalues of A	Multiplicity	Eigenvalues of S	Cubed Eigenvalues of S
$k + k^2$	1	k	k^3
k	$n(\ell-1)/\ell$	$(-1\pm(4k+1)^{1/2})/2$	$(-1-3k\pm(k+1)(4k+1)^{1/2})/2$
$k - \ell$	$(n/\ell) - 1$	$(-1\pm(4k-4\ell+1)^{1/2})/2$	$(3\ell-1-3k\pm(k-\ell)(4k-4\ell+1)^{1/2})/2$

Let p be the multiplicity of $(-1+(4k+1)^{1/2})/2$ as an eigenvalue of S minus the multiplicity of $(-1-(4k+1)^{1/2})/2$ as an eigenvalue of S. Similarly let q be the multiplicity of $(-1+(4k-4\ell+1)^{1/2})/2$ minus the multiplicity of $(-1-(4k-4\ell+1)^{1/2})/2$ as eigenvalues of S. From (4.5) we obtain

$$-2k - k^2 - \ell + 1 + p(4k+1)^{1/2} + q(4k-4\ell)^{1/2} = 0 \quad . \tag{4.13}$$

From (4.6) we obtain

$$-k^3 - (1+3k)(k^2+\ell-1) + 3k^2 + p(k+1)(4k+1)^{1/2} + q(k-\ell+1)(4k-4\ell+1)^{1/2} = 0. \tag{4.14}$$

We may obtain solutions for p and q in terms of k and ℓ from (4.13) and (4.14):

$$p = \frac{(k^2+\ell)(\ell-1)}{\ell(4k+1)^{1/2}} \quad , \qquad q = \frac{k(k-2\ell)}{\ell(4k-4\ell+1)^{1/2}} \quad . \tag{4.15}$$

The definitions of p and q ensure that they are integers and so in (4.15) we can restrict the choices for k and ℓ.

<u>CASE 1</u>. $(4k+1)^{\frac{1}{2}}$ is irrational and hence $p = 0$.

Thus $(k^2+\ell)(\ell-1) = 0$ which implies $\ell = 1$. This gives us Moore graphs. Now $(4k-4\ell+1)^{\frac{1}{2}} = (4k-3)^{\frac{1}{2}}$ being irrational would give $q = 0$. Thus $k(k-2\ell) = 0$ and so $k = 2$. The 5 cycle is the unique Moore graph on the parameters $k = 2$, $\ell = 1$. Otherwise, let $(4k-3)^{\frac{1}{2}} = 2m + 1$ and hence $k = m^2 + m + 1$. Then

$$q = \frac{(m^2+m-1)(m^2+m+1)}{2m+1} = \frac{m^4+2m^3+m^2-1}{2m+1} . \tag{4.16}$$

We compute that

$$(2m+1)(8m^3+12m^2+12m+1) - 16(m^4+2m^3+m^2-1) = 15, \tag{4.17}$$

and hence $(2m+1)\mid 15$. Thus $m = 1, 2,$ or 7 and hence $k = 3, 7,$ or 57. These are the only additional parameters possible for Moore graphs and unique Moore graphs exist for $k = 3, 7$[25]. It is not known whether the Moore graph with $k = 57$ on 3250 vertices exists.

<u>CASE 2</u>. $(4k+1)^{\frac{1}{2}}$ is rational, $(4k-4\ell+1)^{\frac{1}{2}}$ is irrational and hence $q = 0$.

Thus $k(k-2\ell) = 0$ and hence $k = 2\ell$. Let $(4k+1)^{\frac{1}{2}} = 2m + 1 = (8\ell+1)^{\frac{1}{2}}$. Thus $(2m+1)\mid(1+8\ell)$. We have

$$p = \frac{(4\ell+1)(\ell-1)}{2m+1} = \frac{4\ell^2 - 3\ell - 1}{2m+1}, \tag{4.18}$$

and hence $(2m+1)\mid(8\ell^2-6\ell-2)$. Since $(2m+1)\mid(\ell(8\ell+1))$, we obtain that $(2m+1)\mid(7\ell+2)$ and thus $(1+8\ell)\mid(49\ell^2+28\ell+4)$. We compute that

$$64(49\ell^2+28\ell+4) - (1+8\ell)(392\ell+175) = 81 , \tag{4.19}$$

and hence $(1+8\ell) | 81$, Thus $\ell = 1$ or $\ell = 10$. We have already disposed of the first case. The second case allows a Moore type graph with $k = 20$, $\ell = 10$ on $n = 410$ vertices to exist. Such a graph has not been found but a set of eigenvalues satisfying the equations does exist. A possible construction technique is discussed at the end of Section 5.

<u>CASE 3</u>. $(4k+1)^{\frac{1}{2}} = 2m + 1$ and $(4k-4\ell+1) = 2t + 1$ are both integral

Thus $k = m^2 + m$ and $k - \ell = t^2 + t$. Let $d = m - t$. Then $\ell = d(2m+1-d)$. From (4.10) we have $\ell | k^2$ and so $(d(2m+1-d)) | (m^4+2m^3+m^2)$. We compute that

$$\begin{aligned} &d(2m+1-d)(8m^3+(12+4d)m^2 + (2+4d+2d^2)m +(d^3+d^2-d-1)) \\ &\qquad - 16d(m^4+2m^3+m^2) = d(d^2-1)^2 \end{aligned} \tag{4.20}$$

and hence $(2m+1-d) | (d^2-1)^2$. Similarly, using the fact that p and q are integral, we find that

$$\begin{aligned} (2m+1) &| ((16d^2-1)(d^2+1)), \\ (2m+1-2d) &| ((4d^2-1)(4d^2+1)). \end{aligned} \tag{4.21}$$

Using these three divisibility conditions, one can search for possible parameters. A computer search was used for $d \leq 100$, $k \leq 10^{21}$ and only the following four parameter sets were found. We can have $k = \ell = 2$ on 6 vertices which gives the 6 cycle. As described at the beginning of this section, it can be derived from the Moore graph with

$k = 3$ (Petersen graph). We can have $k = \ell = 6$ on 42 vertices which is the graph given in [25] in the construction of the Moore graph with $k = 7$. We can have $k = \ell = 56$ on 3192 vertices. If the graph exists, it would yield the Moore graph with $k = 57$ after a suitable bordering of its adjacency matrix. Finally we can have $k = 6$, $\ell = 4$ on 40 vertices. A Moore type graph on these parameters is given in Section 5.

It is surprising that the possible parameters are so rare, nearly as rare as those of Moore graphs. One would now conjecture that no other parameters are possible. We were unable to combine the three divisibility conditions to prove this conjecture. It must be fairly difficult to prove since we have found pairs k, ℓ with $\ell \mid k^2$, where p is integral and q is not and vice versa.

Section 5. A new Moore type graph with $k = 6$ and $\ell = 4$.

In this section we present a Moore type graph with $k = 6$ and $\ell = 4$ on 40 vertices. We will also comment on the existence of Moore type graphs on the remaining possible parameters. Let

$$S = \left[\begin{array}{ccccc|ccccc}
0 & 0 & I & I & 0 & 0 & C & I & C^2 & C^3 \\
0 & 0 & 0 & I & I & C^3 & 0 & C & I & C^2 \\
I & 0 & 0 & 0 & I & C^2 & C^3 & 0 & C & I \\
I & I & 0 & 0 & 0 & I & C^2 & C^3 & 0 & C \\
0 & I & I & 0 & 0 & C & I & C^2 & C^3 & 0 \\
\hline
0 & C & C^2 & I & C^3 & 0 & I & 0 & 0 & I \\
C^3 & 0 & C & C^2 & I & I & 0 & I & 0 & 0 \\
I & C^3 & 0 & C & C^2 & 0 & I & 0 & I & 0 \\
C^2 & I & C^3 & 0 & C & 0 & 0 & I & 0 & I \\
C & C^2 & I & C^3 & 0 & I & 0 & 0 & I & 0
\end{array}\right], \qquad (5.1)$$

where C is the circulant matrix of order 4. We have

$$S + S^2 = 6I_{40} + J_{40} - K_{40,4} \rightarrow L_{40,4} \tag{5.2}$$

and so S is a Moore type graph with parameters $k = 6$ and $\ell = 4$ on 40 vertices. We will use the fact that S splits into block circulants to shorten our exposition. We do not attempt to prove the uniqueness of S.

Remark 5.1. <u>If we assume that a solution</u> S <u>of</u> (5.2) <u>splits into blocks of order</u> 4 <u>with constant row and column sums</u> i.e. <u>permutation matrices and zero matrices, then the pattern of zero blocks is unique</u>.

<u>Proof</u>. From equation (5.2) we know that there are zero blocks down the diagonal. Form a matrix $P = (p_{ij})$ of order 10 from S, where $p_{ij} = 1$ if the (i,j) block of S is a permutation matrix and $p_{ij} = 0$ if the block is a zero matrix. Then P has line sums (row and column sums) 6. Recalling that the product of permutation matrices is again a permutation matrix and four are required to sum to J_4, then

$$P + P^2 = 2I + 4J \; . \tag{5.3}$$

Looking at the zeros off the main diagonal of P, we set $A = J - P - I$. We compute

$$A + A^2 = (J-P-I)(J-P) = 2I + J. \tag{5.4}$$

The unique solution for A is the Petersen graph, a Moore graph with $k = 3$. Thus the zero blocks of S off the main diagonal form the adjacency pattern of the Petersen graph. We have ordered the blocks in (5.1) to correspond to the labelling of the vertices of the Petersen graph given in Figure 4.

Another block substitution yields an interesting matrix. Form the complex hermitian matrix H of order 10 from S as given in (5.1) by the block substitutions $C^k \to i^k$ and $0 \to 0$. Then H satisfies

$$H + H^2 = 6I \;, \quad HJ = 2J \;. \tag{5.5}$$

The automorphism groups of the Moore graphs has been studied. The automorphism group of the Peterson graph is isomorphic to S_5. The automorphism group of the Moore graph with $k = 7$ is large and very interesting. Aschbacher has shown that the automorphism group of the Moore graph with $k = 57$ is not rank 3 [4]. With this background, we determine the automorphism group of S.

Remark 5.2. The automorphism group of S is isomorphic to $C_4 \times S_5$.

Proof. Any automorphism of the graph splits into a permutation of blocks with the blocks fixed. This follows from the definition of Moore type graphs given at the beginning of Section 4. Two vertices are in the same block if and only if they are at distance 3 from each other. Thus an automorphism takes a block of 4 vertices and maps them into a block of 4 vertices.

The possible permutations of the blocks correspond to the automorphisms of the Petersen graph whose automorphism groups is isomorphic to S_5. We try permutations of order 2 and order 5. In applying the

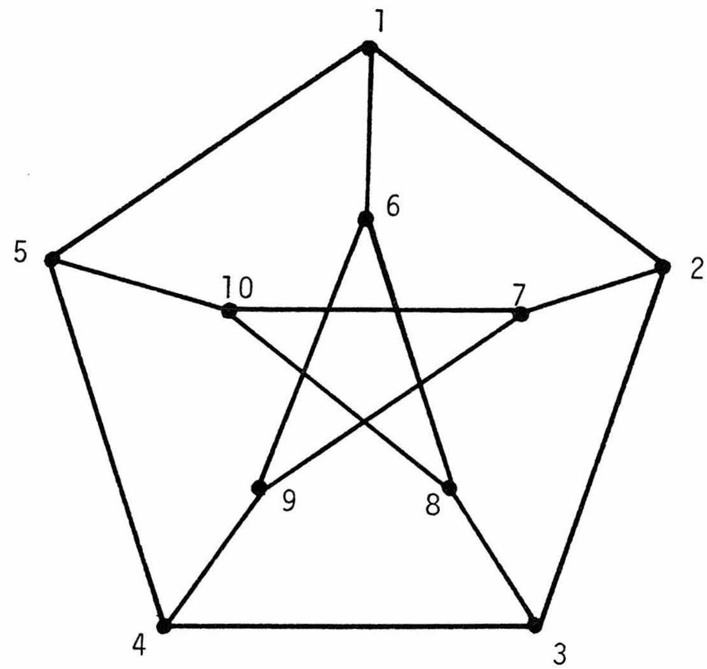


Figure 4.

block permutation (2 5) (3 4) (7 10) (8 9) to S, corresponding to a reflection of the Petersen graph of Figure 4, we obtain (in block circulant form)

$$T = \begin{bmatrix} (0 & 0 & I & I & 0) & (0 & C^3 & C^2 & I & C) \\ (0 & C^3 & I & C^2 & C) & (0 & I & 0 & 0 & I) \end{bmatrix} . \tag{5.6}$$

If we let

$$P = I_{20} \oplus C^2 \oplus C^2 \oplus C^2 \oplus C^2 \oplus C^2 \quad , \tag{5.7}$$

then we find $P^T TP = S$ and so the block permutation does induce an automorphism.

Similarily we try a permutation of order 5: (1 2 3 4 5)(6 7 8 9 10) which corresponds to a rotation of the Petersen graph of Figure 4. This permutation of the blocks of S gives us S back again without the need of any permutations within the blocks. With elements of order 2 and order 5, we conclude that every automorphism of the Petersen graph induces an automorphism of S.

If we fix the blocks, we find that all the remaining automorphisms form a group isomorphic to C_4. Let

$$P = D_1 \oplus D_2 \oplus \cdots \oplus D_{10} \quad , \tag{5.8}$$

be the direct sum of 10 permutation matrices of order 4. If P is an automorphism, then $P^T SP = S$. From the first row of S we have that

$$D_1{}^T D_3 = D_1{}^T D_4 = D_1{}^T D_8 = I \quad . \tag{5.9}$$

From the other rows we obtain

$$D_2^TD_4 = D_2^TD_5 = D_2^TD_9 = D_4^TD_6 = D_6^TD_7 = D_6^TD_{10} = I \ , \tag{5.10}$$

and so we conclude $D_1 = D_2 = \cdots D_{10} = D$. From the first row we have that

$$D^TCD = C \ , \tag{5.11}$$

and thus we conclude $D = I, C, C^2, \text{ or } C^3$. Hence the automorphism group of S is isomorphic to $C_4 \times X_5$.

We note that this group is transitive since it is transitive on the blocks and it is transitive within the first block when the remaining blocks are fixed.

A similar construction to that used for S might yield a Moore type graph T with $k = 20$, $\ell = 10$ on 410 vertices. If we assume that the solution splits into blocks of order 10, either permutation matrices or zero matrices, then we can reduce T to a matrix $P = (p_{ij})$ of order 41 as follows. We set $p_{ij} = 1$ if the (i,j) block of T is a permutation matrix and we set $p_{ij} = 0$ if the block is a zero matrix. Then P satisfies

$$P + P^2 = 10I + 10J. \tag{5.12}$$

We can obtain such a matrix by the Paley construction as follows

$$p_{ii} = 0, \quad p_{ij} = \frac{\left(\frac{i-j}{41}\right) + 1}{2} \quad (i \neq j) \ , \tag{5.13}$$

where $\left(\frac{k}{41}\right)$ is the Legendre symbol. Unfortunately, we have been unable to expand P into a solution T. The block circulant form of S in (5.1) and the above construction were suggested by the referees of [1].

We have discovered a property of the Moore type graph with $k = \ell = 6$ on 42 vertices that might prove helpful in constructing a Moore graph with $k = 57$. We display in Figure 5 the Moore type graph with $k = \ell = 6$ in blocks of order 6 in the manner of Hoffman and Singleton [25]. The blocks are either zero matrices or permutation matrices. We write the non identity permutation matrices as permutations consisting of a product of disjoint cycles (in this case transpositions). Let A be the matrix of order 36 consisting of the last 36 rows and columns. Hoffman and Singleton have pointed out that the blocks of A have some remarkable properties. Let P_{ij} be the (i,j) block of A. Then $P_{ij} = P_{ji}^{T} = P_{ji}^{-1}$ and

$$P_{ij}\, P_{jk}\, P_{ki} = P_{jk} \qquad (k \neq i,\ i \neq j). \tag{5.14}$$

We recall the definition of the direct product of matrices. Let $A = (a_{ij})$ and B be matrices of order n and m respectively. Then $A \otimes B$ is the matrix of order nm divided into n^2 blocks of order m with the (i,j) block being $a_{ij}B$. Let E_{ij} be the matrix of order 6 with a 1 in position (i,j) and zeros elsewhere. Then the block decomposition of A can be written

$$A = \sum_{i,j} E_{ij} \otimes P_{ij} \,. \tag{5.15}$$

$$
S = \left[\begin{array}{c|c|c|c|c|c|c}
0 & I & I & I & I & I & I \\
\hline
I & 0 & \begin{matrix}(12)\\(36)\\(45)\end{matrix} & \begin{matrix}(13)\\(25)\\(46)\end{matrix} & \begin{matrix}(14)\\(23)\\(56)\end{matrix} & \begin{matrix}(15)\\(26)\\(34)\end{matrix} & \begin{matrix}(16)\\(24)\\(35)\end{matrix} \\
\hline
I & \begin{matrix}(12)\\(36)\\(45)\end{matrix} & 0 & \begin{matrix}(14)\\(26)\\(35)\end{matrix} & \begin{matrix}(16)\\(25)\\(34)\end{matrix} & \begin{matrix}(13)\\(24)\\(56)\end{matrix} & \begin{matrix}(15)\\(23)\\(46)\end{matrix} \\
\hline
I & \begin{matrix}(13)\\(25)\\(46)\end{matrix} & \begin{matrix}(14)\\(26)\\(35)\end{matrix} & 0 & \begin{matrix}(15)\\(24)\\(36)\end{matrix} & \begin{matrix}(16)\\(23)\\(45)\end{matrix} & \begin{matrix}(12)\\(34)\\(56)\end{matrix} \\
\hline
I & \begin{matrix}(14)\\(23)\\(56)\end{matrix} & \begin{matrix}(16)\\(25)\\(34)\end{matrix} & \begin{matrix}(15)\\(24)\\(36)\end{matrix} & 0 & \begin{matrix}(12)\\(35)\\(46)\end{matrix} & \begin{matrix}(13)\\(26)\\(45)\end{matrix} \\
\hline
I & \begin{matrix}(15)\\(26)\\(34)\end{matrix} & \begin{matrix}(13)\\(24)\\(56)\end{matrix} & \begin{matrix}(16)\\(23)\\(45)\end{matrix} & \begin{matrix}(12)\\(35)\\(46)\end{matrix} & 0 & \begin{matrix}(14)\\(25)\\(36)\end{matrix} \\
\hline
I & \begin{matrix}(16)\\(24)\\(35)\end{matrix} & \begin{matrix}(15)\\(23)\\(46)\end{matrix} & \begin{matrix}(12)\\(34)\\(56)\end{matrix} & \begin{matrix}(13)\\(26)\\(45)\end{matrix} & \begin{matrix}(14)\\(25)\\(36)\end{matrix} & 0
\end{array}\right]
$$

Figure 5. Moore type graph with $k = \ell = 6$.

It may be verified that A has the property

$$A = \sum_{i,j} E_{ij} \otimes P_{ij} = \sum_{i,j} P_{ij} \otimes E_{ij} . \tag{5.16}$$

This remarkable equation also works for the Petersen graph after a similar reduction to a matrix of order 4. It is possible that this property would hold for the Moore graph with $k = 57$ if it existed. One could use (5.16) to reduce the number of choices required, by a factor of two, in constructing the Moore graph.

Chapter 2. Properties of (0,1)-matrices with forbidden configurations.

Section 1. Introduction.

This chapter considers the properties of (0,1)-matrices which do not contain certain configurations. We make the term configuration more precise using the definition of Ryser [35]. A configuration is an equivalence class of matrices where two matrices represent the same configuration if one of the matrices is a row and column permutation of the other matrix. A matrix A is said to contain a configuration if there is a submatrix of A which represents that configuration. We see that configurations extend the idea of submatrices.

Helly's theorem, in one dimension, says that if you have a collection of intervals such that every pair of intervals has a nonempty intersection, then there is a point in common to all the intervals. We will deal with the case where the collection of intervals is finite and is given by a (0,1)-matrix A. Each row of A corresponds to an interval and has 0's followed by 1's followed by 0's. The position of the string of 1's corresponds to the position of the interval on the real line. Requiring that each pair of intervals has a nonempty intersection is equivalent to requiring that $AA^T > 0$. The conclusion that there is a point in common to all the intervals is equivalent to saying that A has a column of all 1's.

Section 2 is concerned with improvements on Helly's theorem in the (0,1)-matrix setting. We generalize a result of Ryser [35]. Some of our results were reported briefly in [2]. We also provide interpretations of these results in a graph theoretic setting.

In Section 3, we are able to use these configuration theorems to obtain results about $C(S)$. We show that the columns, of certain column sums, of a matrix in $C(S)$ with no configurations in some list, are unique apart from order. This leads to the idea of a "condition" imposed on a matrix which forbids certain configurations. Let the row intersection of row i and row j, $(i \neq j)$, when considered as a vector, have a 1 in a column if both row i and row j do and a 0 otherwise. Using the methods of Ryser [37], we obtain the following striking result. Given a (0,1)-matrix A satisfying some "condition" with column sums at least 2, then the number of linearly independent row intersections (over the rationals) is equal to the number of distinct columns. Thus, if A is of size m by n and has distinct columns, then $n \leq \binom{m}{2}$. When $n = \binom{m}{2}$, we obtain a structure result. Let K_m be a matrix of size m by $\binom{m}{2}$ with distinct columns, all of column sum 2. Then there is a unique permutation matrix P of order $\binom{m}{2}$ such that $A \geq K_m P$.

Section 4 is concerned with matrices without triangles. A triangle is the configuration of order 3 represented by the following matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} . \tag{1.1}$$

Requiring a matrix to have no triangles is a possible "condition" as described above. We define a solution to be a $(0,1)$-matrix of size m by $\binom{m}{2}$ with distinct columns, column sums at least 2. We discover that in a solution there are $m - \ell + 1$ columns of column sum ℓ. Solutions have a nice inductive buildup and an algorithm exists for generating solutions. These results were reported in [3].

Section 5 studies the following configurations

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{1.2}$$

We modify a configuration theorem of Ryser [35]. Similar results to those obtained in Sections 3 and 4 are proven.

Section 2. Configuration theorems with the Helly conclusion.

As explained in the introduction, we are looking for conditions on a (0,1)-matrix A with $AA^T > 0$ such that A is forced to have a column of 1's. The conclusion that A has a column of 1's is the analogue of Helly's theorem. The starting point for all this work is the following theorem of Ryser.

Theorem 2.1 (Ryser [35]). *Let A be a (0,1)-matrix of size m by n with $AA^T > 0$ and A contains no triangles. Then A has a column of m 1's.*

Proof. There are undoubtedly many ways to prove this result but the following is surely the slickest. The proof is by induction on m. The theorem is true for $m = 1,2$. Assume it is true for $m = t \geq 2$. Let A be a (0,1)-matrix, of size $t + 1$ by n, satisfying the hypothesis. Assume A does not have a column of $t + 1$ 1's. Delete the first row of A to obtain a submatrix A_1 which has $A_1A_1^T > 0$, no triangles, and of size t by n. By induction, A_1 has a column of t 1's and so in A there is a column of t 1's with a zero in the first row. Repeat for the second row and the third row. Thus, after a row and column permutation, A has the form

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & \\ 1 & 0 & 1 & * \\ 1 & 1 & 0 & \\ \hline & 1 & & * \\ & & & \end{array}\right] . \qquad (2.1)$$

Then A contains a triangle, which is a contradiction, and this proves the theorem.

We note that the condition A "contains no triangles" is still true in arbitrary submatrices of A and so is ideally suited to inductive arguments.

We can extend this theorem by considering alternate lists of forbidden configurations. Consider the set of (0,1)-matrices A with k rows, with $AA^T > 0$, such that any submatrix B of A, obtained by deleting a column, does not have $BB^T > 0$. Let L_k be the set of configurations represented by such matrices. For example, if A is a (0,1)-matrix of size m by n $(m \geq k)$ with $AA^T > 0$, then any k rows of A contain a configuration in L_k. Simply let B be the k by n submatrix formed by the chosen rows. Then we have $BB^T > 0$ and we delete as many columns of B as possible to obtain a matrix C of size k by ℓ with $CC^T > 0$. By our definition, C represents some configuration in L_k. The matrix C need not be unique.

Viewing the rows of the matrix as indexing vertices and the columns as complete graphs, the elements of L_k correspond to minimal edge coverings of the complete graph K_k with complete subgraphs.

Theorem 2.2. _Let_ A _be a_ $(0,1)$-_matrix of size_ m _by_ n _with_ $AA^T > 0$ _and_ A _contains no configurations in_ $L_k \setminus J_{k,1}$ _for given_ k _with_ $3 \leq k \leq m$. _Then_ A _has a column of_ m _1's_.

Proof. The proof is virtually identical to the proof of Theorem 2.1. The proof is by induction on m. The theorem is true for $m = k$ by the definition of L_k. Let A be a $(0,1)$-matrix of size m by n $(m > k)$ satisfying the hypothesis. Assume A has no column of m 1's. Using the same arguments, we obtain that A has the following configuration of size m by 3

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 1 & 1 \end{bmatrix} . \qquad (2.2)$$

The first k rows of these 3 columns yield a configuration in $L_k \setminus J_{k,1}$ and hence a contradiction is reached. This proves the theorem.

In reference to the preceding theorem, note that A contains no configuration in $L_k \setminus J_{k,1}$ implies that A contains no configurations in $L_{k+1} \setminus J_{k+1,1}$. The reverse implication is not always true.

Examples.

$$L_3 \setminus \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right\}$$

$$L_4 \setminus \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} \tag{2.3}$$

We see that Theorem 2.1 is the case $k = 3$ in Theorem 2.2 and so Theorem 2.2 generalizes Theorem 2.1. The condition "contains no configuration in $L_4 \setminus J_{4,1}$" is weaker than the condition "contains no triangles". For example, consider the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{2.4}$$

Theorem 2.2 for $k = 4$ yields that (2.4) has a column of 1's but Theorem 2.1 says nothing since (2.4) has a triangle. The more general nature of Theorem 2.2 will be demonstrated more concretely later on.

Consider the matrix K_m as defined in Chapter 1. It is the canonical matrix of the class $C(J_m-I_m)$ and is of size m by $\binom{m}{2}$. It has all possible columns of column sum 2. We note that $K_k \in L_k$ and find that the analogue to Theorem 2.2 is true.

Theorem 2.3. Let A be a (0,1)-matrix of size m by n with $AA^T > 0$ and A contains no configuration in $L_k \setminus K_k$ for a given k, $3 \leq k \leq m$. Then A contains a configuration represented by K_m.

Proof. We will show that A has all of its column sums at most 2. Assume A has some column with 3 or more 1's. Select k rows of A, containing at least 3 of these 1's, to form a k by n submatrix B. We have $BB^T > 0$. Delete as many columns as possible to get a submatrix B_1 of size k by ℓ, with the special column not deleted, such that $B_1B_1^T > 0$. Thus B_1 or B_1 without the special column will represent a configuration in L_k. In either case we find that the configuration has a column of at least 3 1's and so is not K_k. This is a contradiction and thus A has column sums at most 2. With this property and the fact that $AA^T > 0$, we deduce that A contains the configurations represented by K_m.

We can prove configuration theorems similar to Theorem 2.2 with a weaker conclusion.

Theorem 2.4. Let A be a (0,1)-matrix of size m by n with $AA^T > 0$ and containing no configurations of the form

$$\begin{bmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 0&1&1 \\ 1&0&1 \\ 1&1&0 \\ 1&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1 \\ 0&1&1 \\ 1&0&1 \\ 1&0&1 \\ 1&1&0 \\ 1&1&0 \end{bmatrix} \qquad (2.5)$$

Then A has a column of at least $m - 1$ 1's.

Proof. We will use induction on m. The theorem is true for $m = 1, 2, 3$ by inspection. Consider the case $m = 4$. If A has column sums 0, 1, or 2 then, since $AA^T > 0$, A contains K_4, the first forbidden configuration in (2.5). Thus the theorem holds for $m = 4$.

Assume $m > 4$ and A has no column of $m - 1$ (or m) 1's. Delete the first row. By induction, A has a column with $m - 2$ 1's. We permute the rows of A so that the two zeros in the column appear in the first two rows. Deleting the third row, we obtain the following two possibilities for two columns in A

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \end{bmatrix} \quad . \tag{2.6}$$

In the former case, by deleting the fourth row, we obtain that A contains the second, third, or fourth configurations in (2.5). In the latter case, by deleting the fifth row, we obtain that A contains the fourth or fifth configurations in (2.5). In either case we have a contradiction and so the theorem is proven.

The following lists of forbidden configurations could replace (2.5) and the theorem would still hold.

$$1.\ \left\{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\} \qquad 2.\ \left\{\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\0&0&1&0&1&1\end{bmatrix},\ \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\right\}$$

$$3.\ \left\{\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\\1&0\\1&1\\1&1\end{bmatrix},\ \begin{bmatrix}0&1\\0&1\\1&0\\1&0\\1&1\end{bmatrix}\right\} \tag{2.7}$$

The given list of 5 forbidden configurations is fourth in this series. This process of constructing forbidden lists could continue but the fifth list would already be unmanageable.

The main interest in the above theorems is in their applications. Theorem 2.2 will be used to prove a result in the class $C(S)$ in Section 3. We will apply Theorem 2.1 to graph theory in three ways. Let A be a $(0,1)$-matrix of size m by n. Then A can be thought of as the vertex-vertex incidence matrix of a bipartite graph B on $m + n$ vertices. One part is of size m and the other is of size n. Let $A = (a_{ij})$. Then vertex i of the part with m vertices is joined to vertex j of the part with n vertices if and only if $a_{ij} = 1$. This correspondence of $(0,1)$-matrices to bipartite graphs is one to one and onto.

The inequality $AA^T > 0$ asserts that every pair of vertices in the part with m vertices are joined to some common vertex in the part with n vertices. A triangle corresponds to a subgraph which is a 6-cycle. Note that the difinition of a subgraph corresponds directly to a configuration. A column of 1's in A corresponds to a vertex in the part with n vertices that is joined to every vertex in the other part. Theorem 2.1 translates to the following result.

Theorem 2.5. *Let* B *be a bipartite graph on two parts* B_1, B_2 *with* $|B_1| = m$, $|B_2| = n$, *where every pair of vertices in* B_1 *is joined to a common vertex in* B_2. *Assume* B *does not have a 6-cycle as a subgraph. Then there exists a vertex in* B_2 *joined to every vertex in* B_1.

Theorem 2.2 could be used to extend this result by altering the list of forbidden subgraphs. For example, $L_4 \setminus J_{4,1}$ would translate as the 4 forbidden configurations given in Figure 1.

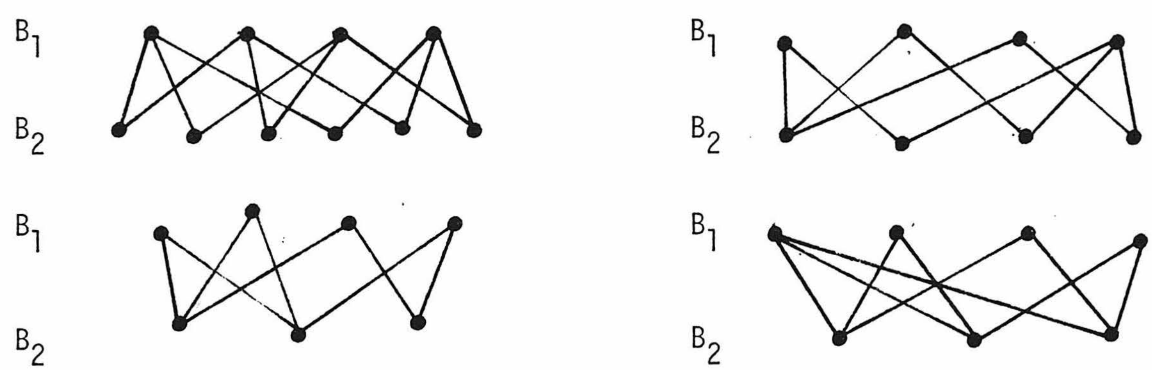


Figure 1.

The full power of Theorem 2.1 is used in Theorem 2.5. Such is not the case when we consider graphs in general. Consider a (0,1)-matrix A of order m to be the usual adjacency matrix of a graph 6 on m vertices. Thus $A = A^T$ and $AA^T = A^2$. Then $A^2 > 0$ asserts that there is a path of length 2 between any pair of vertices in G. We will consider the case where loops are allowed in G and obtain a rather complicated theorem by translating Theorem 2.1. Consider the triangle and the possible subgraphs it could correspond to. Index the rows and columns of the configuration as follows

$$\begin{array}{cc} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} . \tag{2.8}$$

Since we are dealing with a configuration, neither a_1, a_2, a_3 or b_1, b_2, b_3 need be in order. We display the possible identifications of the a_i's and the b_i's in Figure 2 as well as the subgraphs they give rise to. We now try all possibilities for the optional edges. In certain cases, you will obtain a forbidden subgraph that has already been forbidden because it contains another forbidden subgraph. We will ignore such subgraphs. We obtain the following result by translating Theorem 2.1.

CASE 1. $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.

CASE 2. $a_1 = b_2$, $a_2 = b_1$, $a_3 = b_3$.

CASE 3. $a_1 = b_1$, $a_2 = b_2$.

CASE 4. $a_1 = b_1$, $a_2 = b_3$.

CASE 5. $a_1 = b_2$, $a_2 = b_1$.

CASE 6. $a_1 = b_1$.

CASE 7. $a_1 = b_2$.

CASE 8. No identifications.

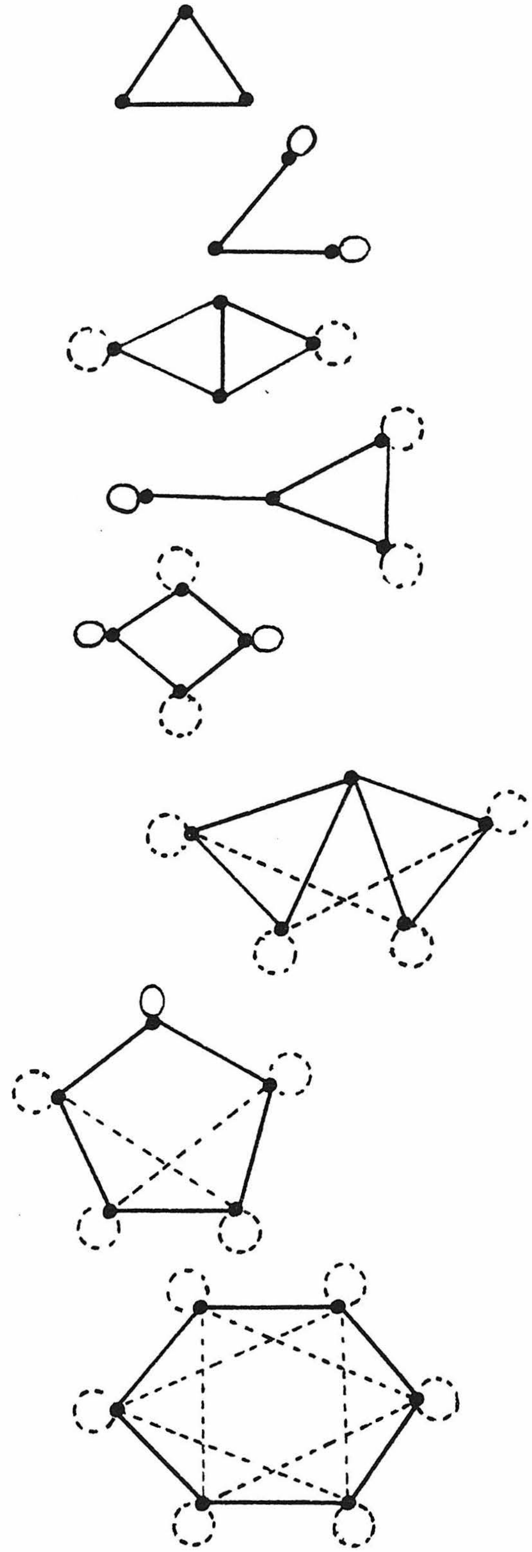


Figure 2. Possible identifications and resulting subgraphs. Dotted edges are optional.

Theorem 2.6. *Let G be a graph for which every pair of vertices is joined by a path of length 2. Loops are allowed in G but G has no subgraphs in Figure 3. Then G has a vertex which is joined to every other vertex (as well as itself by a loop).*

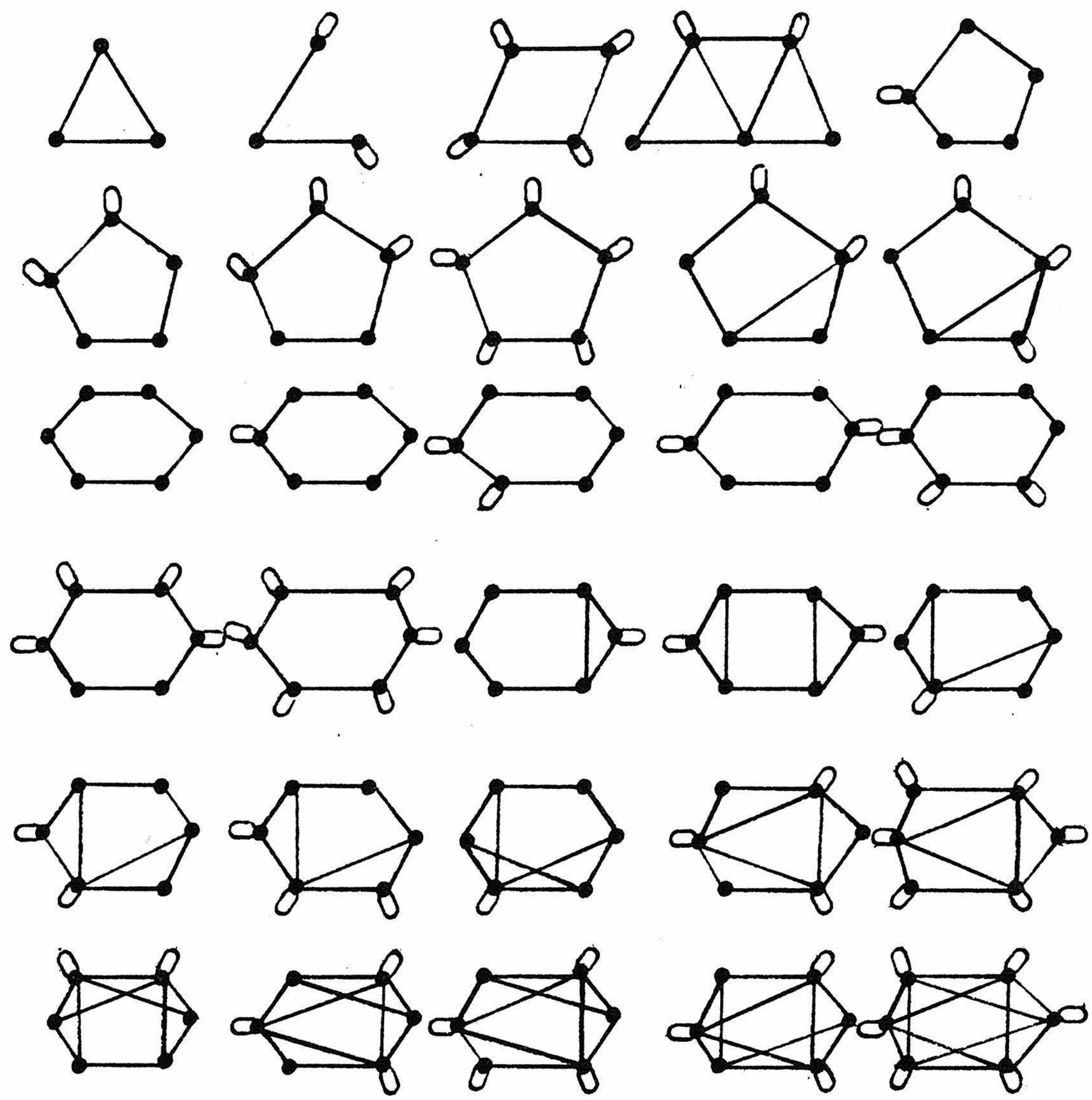


Figure 3. Forbidden configurations in Theorem 2.6.

Loops have made the list of forbidden subgraphs rather unmanageable. We can avoid loops in the following way. Let G be a graph without loops and let B be its associated adjacency matrix. Then B has zeros on the main diagonal. Let $A = B + I$. The condition $A^2 > 0$ is the condition that any pair of vertices in G are joined by a path of length 1 or a path of length 2. Recalling that the diameter of a graph, $d(G)$, is the longest distance between any two vertices in G, we have that $d(G) \leq 2$.

Consider a triangle in A and the possible subgraphs it could correspond to in G. Index the rows and columns of the triangle as in (2.8). We note that certain identifications are impossible here since A has 1's on the main diagonal. Cases 1, 2, 3, 4, and 6 are all impossible. There are three remaining possible identifications.

<u>CASE 1</u>. $a_1 = b_2$, $a_2 = b_1$ yields first graph in Figure 4.

<u>CASE 2</u>. $a_1 = b_2$ yields second graph in Figure 4.

<u>CASE 3</u>. no identifications yields third and fourth graphs in Figure 4.

A column of 1's in A corresponds to a vertex in G joined to every other vertex. We obtain the following result translating Theorem 2.1.

Theorem 2.7. <u>Let</u> G <u>be a graph without loops such that</u> $d(G) \leq 2$ <u>and</u> G <u>contains no subgraphs in Figure</u> 4. <u>Then</u> G <u>has a vertex which is joined to every other vertex in</u> G.

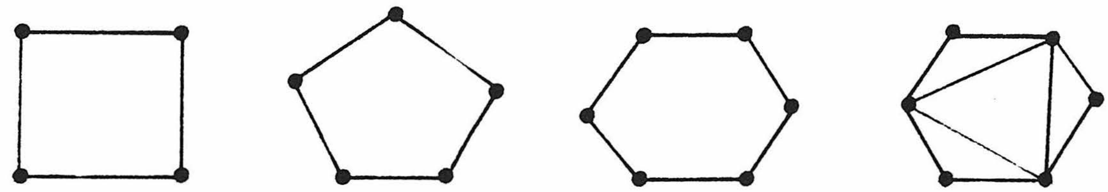


Figure 4. Forbidden subgraphs for Theorem 2.7.

A result similar to Theorem 2.7 can be proven using graph theory directly.

Theorem 2.8. _Let_ G _be a graph without loops such that_ $d(G) \leq 2$ _and_ G _contains no subgraphs in Figure_ 5. _Then_ G _has a vertex which is joined to every other vertex in_ G.

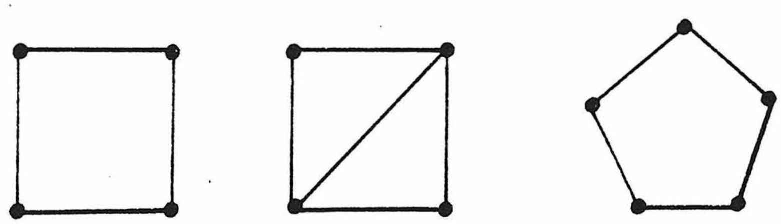


Figure 5. Forbidden subgraphs for Theorem 2.8.

Proof. Let v_1 be a vertex adjacent to the maximum number of vertices. We may assume there is a vertex v_2 not adjacent to v_1 otherwise the theorem is true. Using the fact that $d(G) \leq 2$, let v_3 be a vertex joined to v_1 and v_2. Since v_3 is adjacent to 2 vertices, v_1 must be adjacent to some other vertex v_4. If v_2 is adjacent to v_4, then we have a 4-cycle and so by the forbidden subgraphs v_2 is adjacent to v_1, a contradiction. Thus v_2 is at distance 2 from v_4

and both are joined to some vertex v_5. This forms a pentagon as in Figure 6 unless $v_3 = v_5$. Hence, if

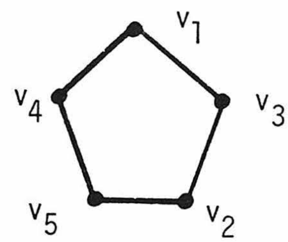


Figure 6.

$v_3 \neq v_5$, by the forbidden subgraphs we have two diagonals in the pentagon. If v_1 is adjacent to v_5 then v_3 is adjacent to v_5 and v_1 is adjacent to v_2, a contradiction. If v_3 is adjacent to v_5 then v_1 is adjacent to v_5 and this leads to the same contradiction. If v_3 is adjacent to v_4 then v_3 is adjacent to v_5 and again this leads to a contradiction.

We conclude that $v_3 = v_5$. As before, v_1 is joined to the maximum number of vertices, and thus v_1 is joined to some new vertex v_6. Repeating the argument, we obtain v_6 is adjacent to v_3. Since v_3 is now joined to 4 vertices, v_1 must be joined to another vertex v_7. This process cannot terminate yet G is finite, thus we have reached a contradiction and proven the theorem.

The above result is different from Theorem 2.7 but it is not clear whether it is a more precise result. One expects that a better result than Theorem 2.7 can be found since we have merely translated Theorem 2.1 and have not used the information that $A = A^T$.

Section 3. The row intersection theorem.

We now apply Theorem 2.2 to the class $C(S)$, which was defined in Chapter 1. Ryser proved the following result using Theorem 2.1 [36].

Theorem 3.1 (Ryser). Every matrix in the class $C(S)$ contains a triangle or else the class contains exactly one matrix, apart from column permutations, without triangles.

Using Theorem 2.2, our generalization of Theorem 2.1, we prove a generalization of Theorem 3.1. The proof is similar to the proof of Ryser [36] with some changes that reveal the structure of S. Let S be a symmetric matrix of order n with positive integral entries and zeros on the main diagonal. Define the possible J-I parts of S to be the matrices B of order n with $B = \text{offdiag}(\beta\beta^T)$, where β is a (0,1)-column vector of length n, such that $B \leq S$. Thus if $A = \text{offdiag}(\alpha\alpha^T)$ is a possible J-I part of S then x would be a column of some matrix in $C(S)$. Simply take a matrix $A' \in C(S-A)$ and append x to it to get a matrix A'' with $A'' \in C(S)$. We use the term J-I since if α has t 1's then the submatrix of A in those rows and colums with nonzero entries is precisely $J_t - I_t$. Define the size of a possible J-I part $B = \text{offdiag}(\beta\beta^T)$ to be the number of 1's in β.

Theorem 3.2. For every matrix $A \in C(S)$ containing no configurations in $L_k \setminus J_{k,1}$, the columns with column sum at least k are unique apart from order.

Proof. Take an arbitrary matrix $A \in C(S)$ with no configurations in $L_k \setminus J_{k,1}$. Let $B_1 = (\beta\beta^T)$ be a maximal possible J-I part of S (using our partial order on matrices) of size at least k. Let β have 1's in rows $i_1, i_2, \ldots, i_r$. Then the rows $i_1, i_2, \ldots, i_r$ of A form a submatrix B with $BB^T > 0$. Certainly B has no configurations in $L_k \setminus J_{k,1}$, since A does not, and thus by Theorem 2.2 B has a column of all 1's. Thus A has a column with 1's in rows $i_1, i_2, \ldots, i_r$. The column has no more 1's since B_1 was maximal. Thus β is a column of A. Delete β from A to form A_1 and let $S_1 = S - B_1$ as in Chapter 1. Then $A_1 \in C(S_1)$. Now A_1 has no configuration in $L_k \setminus J_{k,1}$ and thus we repeat the process on A_1. We continue this until at the t^{th} stage there is no maximal possible J-I part of $S_t = S - B_1 - B_2 - \cdots - B_t$ of size at least k. Thus any matrix $A_t \in S_t$ has column sums all less than k.

Thus the columns of A with column sum at least k are determined precisely by a selection of matrices $B_1, B_2, \ldots, B_t$ where B_i is a maximal possible J-I block of $S - B_1 - B_2 - \cdots - B_{i-1}$ of size at least k for $i = i, 2, \ldots, t$. This selection of B_i's depends only on S. Thus for any matrix in $C(S)$, with no configuration in $L_k \setminus J_{k,1}$, the columns of column sum at least k must be unique apart from order.

We note that $L_3 \setminus J_{3,1}$ is just the triangle from (2.3) and thus Theorem 3.1 follows from Theorem 3.2 by substituting $k = 3$ and by noting that, once the columns of column sum at least 3 are specified, the columns of column sum 2 are also specified. The proof was done this way to show that the selection of B_i's is unique. Let $B_1, B_2, \ldots, T_t$ be a <u>sequence of maximal parts</u> of S when B_i is a maximal possible J-I part of $S - B_1 - B_2 - \cdots - B_{i-1}$ of size at least k for $i = 1, 2, \ldots, t$. In addition, assume that

$S - B_1 - B_2 - \cdots - B_t$ has no possible J-I parts of size k. Let $B_i = \text{offdiag}(\beta_i\beta_i^T)$.

Theorem 3.3. <u>There exists a matrix</u> $A \in C(S)$ <u>where</u> A <u>contains no configurations in</u> $L_k \setminus J_{k,1}$ <u>if and only if the sequence of maximal parts</u> $B_1, B_2, \ldots, B_t$ <u>is unique apart from order.</u>

<u>Proof.</u> Let $A \in C(S)$ where A contains no configurations in $L_k \setminus J_{k,1}$. Following the proof of Theorem 3.2, we know that the columns of column sum at least k are unique apart from order. But a sequence of maximal parts $B_1, B_2, \ldots, B_t$ tells us, using the proof of Theorem 3.2, that $\beta_1, \beta_2, \ldots, \beta_t$ are the columns of A of column sum at least k. Thus $B_1, B_2, \ldots, B_t$ are unique apart from order.

Assume that the sequence of maximal parts is unique apart from order. Let $S' = S - B_1 - B_2 - \cdots - B_t$ and let $A' \in C(S')$. Form the matrix A by appending to A' the columns $\beta_1, \beta_2, \ldots, \beta_t$. Assume A has a submatrix B which represents a configuration in $L_k \setminus J_{k,1}$. Let B occur in rows $i_1, i_2, \ldots, i_k$. Let those columns of A which have 1's in all the rows $i_1, i_2, \ldots, i_k$ be $\alpha_1, \alpha_2, \ldots, \alpha_r$ and let $C_i = \text{offdiag}(\alpha_i\alpha_i^T)$. Then consider the matrix A_1 obtained from A by deleting the columns $\alpha_1, \alpha_2, \ldots, \alpha_r$. Since B represents a configuration in $L_k \setminus J_{k,1}$, B has no column of all 1's and thus B is a submatrix of A_1. We have that $BB^T > 0$. Let $S_1 = S - C_1 - C_2 - \cdots - C_r$ and then $A_1 \in C(S_1)$. We obtain that S_1 has a possible J-I part $C = \text{offdiag}(\alpha\alpha^T)$ where α has 1's in rows $i_1, i_2, \ldots, i_k$. Thus there is a maximal possible J-I part $C_{r+1} = \text{offdiag}(\alpha_{r+1}\,\alpha_{r+1}{}^T)$ where α_{r+1} has 1's in rows $i_1, i_2, \ldots, i_k$. We complete the sequence of possible

J-I parts $C_1, C_2, \ldots, C_r, C_{r+1}$ to a sequence of maximal parts. However, this sequence has one more possible J-I part $C_{r+1} = \text{offdiag}(\alpha_{r+1}\,\alpha_{r+1}^T)$, where α_{r+1} has 1's in all the rows $i_1, i_2, \ldots, i_k$, than the sequence $B_1, B_2, \ldots, B_t$. This is a contradiction to the uniqueness and thus A contains no configurations in $L_k \setminus J_{k,1}$ as desired.

We will now use Theorem 2.2 to establish a lemma which will in turn establish the Row Intersection Theorem. We define the following condition for (0,1)-matrices to satisfy. Let CONDITION(k), for $k \geq 3$, be the condition: "contains no configuration in

$$\{[J_{k,1}C] \mid C \in L_k \setminus J_{k,1}\}\,, \tag{3.1}$$

and for $i = 3,4, \ldots, k-1$, contains no configuration in

$$\{[J_{i,1}C] \mid C \in L_k \setminus J_{k,1}\}\,, \tag{3.2}$$

in those columns with column sum at most i".

Examples. CONDITION(3): "contains no configurations represented by

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{3.3}$$

CONDITION(4): "contains no configuration represented by

$$\begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&1&1&0\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&1&0\\ 1&1&0&0&1\\ 1&0&1&0&1\\ 1&0&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 1&1&1&1 \end{bmatrix}, \qquad (3.4)$$

and no configuration represented in (3.3) in columns of column sum at most 3."

Note that if a matrix satisfies CONDITION(k), then it satisfies CONDITION(ℓ) for $\ell \geq k$. Thus each successive condition is weaker (less restrictive). We are led to define CONDITION(∞) as follows: "for $i \geq 3$, contains no configuration in (3.2) in columns with column sum at most i". We will prove the results in this section using CONDITION(∞) but since CONDITION(∞) is weaker than CONDITION(k) for any k, we can substitute CONDITION(k) for CONDITION(∞) in our results. We note that CONDITION(k) involves only a finite number of forbidden configurations and, for small k, may be more useful than CONDITION(∞). Also, the condition: "contains no triangles" is weaker than CONDITION(3) and can be substituted in the following results.

Lemma 3.4. *Let* A *be a* (0,1-*matrix of size* m *by* n *satisfying* CONDITION(∞). *Let* $A_1, A_2 \in C(S)$ *for some* $S \neq 0$ *where* A_1 *and* A_2 *are obtained from* A *by selecting certain columns of* A *where each column can be repeated a finite number of times. Then* A_1 *and* A_2 *are equal apart from a column permutation.*

Proof. This will be similar to the proof of Theorem 3.2. Let α be a column of both A_1 and A_2. Delete a column α from A_1 and A_2 to obtain the matrices A_1' and A_2' respectively. Then $A_1', A_2' \in C(S)$

for $S' = S\text{-offdiag}(\alpha\alpha^T)$. Hence, without loss of generality, we assume that A_1 and A_2 have no columns in common.

Among all the columns of A_1 and A_2, take α to be the column of largest column sum and that it occurs in A_1. Let α have i 1's. If $i = 2$, then since $A_1, A_2 \in C(S)$ we have that A_1 and A_2 are equal apart from a column permutation to the canonical matrix of S. This is a contradiction, so assume $i \geq 3$. Let α have 1's in rows $j_1, j_2, \ldots, j_i$. Let B be the submatrix of A_2 consisting of rows $j_1, j_2, \ldots, j_i$. We have $A_1, A_2 \in C(S)$ and $A_1 A_1^T \geq \alpha\alpha^T$ and so $BB^T > 0$. If B has a configuration C in $L_i \setminus J_{i,1}$, then appending it to the i 1's of α we have a configuration of (3.2) in columns of column sum at most i. This violates CONDITION(∞). Note that we do not have to worry about repeated columns since no configuration in L_i has repeated columns. Thus B has no configuration in $L_i \setminus J_{i,1}$ and hence, by Theorem 2.2, B has a column of 1's. Thus A_2 has a column with 1's in rows $j_1, j_2, \ldots, j_i$ and this is a contradiction. Thus A_1 and A_2 are the same apart from a column permutation.

Results such as Theorem 3.2 and Lemma 3.4 about the class $C(S)$ are interesting because they operate in such a general framework. One hopes for some applications. Ryser found a nice application which we now generalize [37]. Let A be a (0,1)-matrix of size m by n. We wish to consider the row intersections as vectors. Let the row intersection of row i and row j, regarded as a vector, have a 1 in a given column if both row i and row j do and a 0 otherwise. We can now give the Row Intersection Theorem.

Theorem 3.5. *Let A be a (0,1)-matrix of size m by n with column sums at least 2 and satisfying* CONDITION(∞). *Then the number of linearly independent row intersections (over the rationals) is equal to the number of distinct columns of* A.

Proof. We use some of the ideas of Ryser [37]. We start with what is called the fundamental matrix equation for finite sets. Let A be a (0,1)-matrix of size m by n. Let $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are independent indeterminates. Let

$$AXA^T = Y . \tag{3.5}$$

Recall from Chapter 1 that A can be considered as describing m subsets of a n-set. Let the elements of the n-set be labelled $x_1, x_2, \ldots, x_n$ and the sets be labelled $S_1, S_2, \ldots, S_n$. Then Y contains a great deal of information about the sets. Let $Y = (y_{ij})$. Then y_{ii} is the sum of the elements of S_i and y_{ij} is the sum of the elements of $S_i \cap S_j$. We note that the row intersections as defined above are precisely these set intersections when thought of as vectors in $\mathbb{Q}^n$ with basis $\{x_1, x_2, \ldots, x_n\}$.

Repeated columns can be deleted without affecting the linear independence of the row intersections so we will assume n is the number of distinct columns; i.e., A has no repeated columns. We have immediately that n is greater than or equal to the number of linearly independent row intersections since n is the dimension of the space which contains the row intersections.

Assume n is greater than the number of linearly independent row intersections. We now consider $x_1, x_2, \ldots, x_n$ as variables and solve for $y_{ij} = 0$ $(i \neq j)$. Since the number of variables exceeds the number of linearly independent equations, we can find rational and hence integral values $e_1, e_2, \ldots, e_n$, not all zero, for $x_1, x_2, \ldots, x_n$. Let $E = \text{diag}(e_1, e_2, \ldots, e_n)$. Then

$$AEA^T = D, \tag{3.6}$$

where D is a diagonal matrix. Every variable occurs in some equation (column sums at least 2), thus some e_i's are positive and some are negative. Define A_1 and A_2 as follows. For all i with $e_i > 0$, A_1 is to contain column i of A repeated e_i times. For all j with $e_j < 0$, A_2 is to contain column j of A repeated $-e_j$ times. Then

$$A_1 A_1^T - A_2 A_2^T = D, \tag{3.7}$$

and so $A_1, A_2 \in C(S)$ for some S. By Lemma 3.4, which was created for this purpose, A_1 is the same as A_2 apart from column permutations and so A has a repeated column. This is a contradiction which proves that n, the number of distinct columns, is precisely the number of linearly independent row intersections.

Corollary 3.6. _Let_ A _be a_ (0,1)-_matrix of size_ m _by_ n, _column sums at least_ 2, _distinct columns, satisfying_ CONDITION(∞). _Then_ $n \leq \binom{m}{2}$.

Proof. This follows directly from Theorem 3.5 by noting that there are only $\binom{m}{2}$ row intersections.

This bound was obtained by Ryser when CONDITION(∞) was replaced by "contains no triangle" [37]. Using more direct means, Cunningham obtained this bound for CONDITION (3) [13]. One is immediately interested in the extremal case when $n = \binom{m}{2}$. We will call a (0,1)-matrix A <u>special</u> if A is of size m by $\binom{m}{2}$ with column sums at least 2, distinct columns and satisfies CONDITION(∞). All the row intersections are linearly independent for special matrices.

We are able to obtain a structure result for special matrices. Recall the definition of K_m as a (0,1)-matrix of size m by $\binom{m}{2}$ with all possible columns of column sum 2. We start with a remark which was the first result obtained. We include it, despite the fact that the next theorem is stronger, to show how the ideas of linear independence may be applied.

Remark 3.7. <u>Let</u> A <u>be a</u> (0,1)-<u>matrix of size</u> m <u>by</u> $\binom{m}{2}$ <u>with column sums at least</u> 2, <u>distinct columns, and satisfying</u> CONDITION(∞). <u>Then there exists a permutation matrix</u> P <u>of order</u> $\binom{m}{2}$ <u>such that</u> $A \geq K_m P$.

<u>Proof</u>. This is an application of P. Hall's theorem on a system of distinct representatives (SDR). With each column i in A associate the set S_i where

$$S_i = \{\{j,k\} \mid j \neq k, \text{ column } i \text{ has 1's in row } j \text{ and row } k\} \qquad (3.8)$$

Our remark asserts that these $\binom{m}{2}$ sets have an SDR which corresponds to some reordering, P, of the columns of K_m.

Let $I \subseteq \{1,2, \ldots, \binom{m}{2}\}$ and let $|I| = r$. Let B be the m by

r matrix obtained by selecting from A precisely the columns indicated in I. We have that $U_{i \in I} S_i$ is equal to the number of nonzero row intersections in B which is at least the number of linearly independent row intersections in B which is equal to r by Theorem 3.5. Theorem 3.5 applies here because CONDITION(∞) continues to hold under arbitrary deletion of columns. But then $|U_{i \in I} S_i| > |I| = r$ for all I and thus by P. Hall's SDR theorem our remark follows.

At this point we can construct an infinite family of special matrices. We have that any special matrix can be formed from K_m by adding 1's and then applying a column permutation. Let A be formed from K_m as follows. If column i has 1's in row j and row k with $j < k$, then add 1's in position (ℓ, i) where $j < \ell < k$. Here is the matrix A for $m = 5$:

$$\begin{bmatrix} 1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1&1&1&0 \\ 0&0&1&1&0&1&1&1&1&1 \\ 0&0&0&1&0&0&1&0&1&1 \end{bmatrix} \tag{3.9}$$

A matrix is said to have the <u>consecutive</u> 1's <u>property</u> by columns if some row permutation yields a matrix which has the 1's appearing consecutively (no two 1's separated by a 0) in every column. Our construction yields a matrix A with the consecutive 1's property by colums. This property implies that A has no triangles and hence A satisfies CONDITION(∞). Thus A is special. One may verify that our matrix A is the unique (0,1)-matrix of size m by $\binom{m}{2}$, apart from arbitrary row and column permutations, with column sums at least 2, distinct columns, and having the consecutive 1's property by columns. A generalization of the consecutive 1's property was presented by Ryser and is used in Section 5 [35]. We next consider matrices with the consecutive 1's property by rows to round out our remarks.

Remark 3.8. _Let_ A _be a (0,1)-matrix of size_ m _by_ n, _column sums at least_ 1, _distinct columns, with consecutive_ 1's _by rows. Then_ $n \leq 2m - 1$.

Proof. Consider such a matrix A of size m by n with n as large as possible. Assume that A has its 1's appearing consecutively in each row and consider each row as an interval. Let two rows, row i and row j, have both their intervals start in column i. Then one can form a matrix A' from A, of size m by $n + 1$ in the following way. Place between column $i - 1$ and column i a column identical to column i except for a 0 in row j. One verifies that A' satisfies the hypotheses and this contradicts the maximality of n. We may assume henceforth that no two intervals start or finish in the same column.

For each i, $1 \leq i \leq n$, let S_i denote a subset of $\{s,f\}$ where $s \in S_i$ if an interval starts in column i and $f \in S_i$ if an interval finishes in column i. The following are impossible if we are to have distinct columns

$$\begin{aligned} S_i &= \{s\} \quad , \quad S_{i+1} = \{f\} ; \\ S_i &= \{s\} \quad , \quad S_{i+1} = \emptyset \quad ; \\ S_i &= \emptyset \quad , \quad S_{i+1} = \{f\} ; \\ S_i &= \emptyset \quad , \quad S_{i+1} = \emptyset \quad . \end{aligned} \tag{3.10}$$

Also $S_1 = \{s\}$ and $S_n = \{f\}$ because the column sums are at least 1 and because of the minimality of n. The number of i for which $S_i = \{s,f\}$ is at least one greater than the number of j for which $S_j = \emptyset$. One proves this by looking at the possible sequences of sets using (3.10). For example, a set $\{s\}$ and a set $\{f\}$ must have a set $\{s,f\}$ between them.

Our above inequality forces $n \leq 2m - 1$, proving the remark. The following construction shows that this bound is achieved. Let A be the matrix of size m by $2m - 1$ with the 1's in row i in columns $i, i + 1, \ldots, i + m - 1$. For $m = 4$, we have:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} . \tag{3.11}$$

Quite different matrices also achieve the bound.

We return to our example (3.9). The SDR of A (permutation matrix P) is unique and corresponds exactly to K_m ($P = I$). This is not an isolated occurrence. The following lemma will be used to prove this.

Lemma 3.9. _Let A be a (0,1)-matrix of size m by n with column sums at least 2, distinct columns, and satisfying_ CONDITION(∞). _Then there is an off diagonal entry of AA^T equal to 1._

Proof. Assume there is no off diagonal entry of AA^T equal to 1. Let α be the column in A with the largest column sum, say k, with 1's in rows $i_1, i_2, \ldots, i_k$. If $k = 2$, all the nonzero off diagonal entries of AA^T are zero and we are done. We assume $k \geq 3$. Let A_1 be the matrix consisting of rows $i_1, i_2, \ldots, i_k$ of A. Then $A_1 A_1^T \geq 2J_k$. Let A_2 be the matrix obtained from A_1 by deleting the column of all 1's which comes from α. Then $A_2 A_2^T \geq J_k > 0$ and so A_2 has a configuration in L_k. This configuration cannot be a column of 1's since A has distinct columns. Thus CONDITION(∞) is violated since A has a configuration as in (3.2) for $i = k$ and all the columns of A are of column sum at most k. Thus AA^T has an off diagonal entry equal to 1.

Theorem 3.10. Let A be a (0,1)-matrix of size m by $\binom{m}{2}$ with column sums at least 2, distinct columns, and satisfying CONDITION(∞). Then there exists a unique permutation matrix P of order $\binom{m}{2}$ such that $A \geq K_m P$.

Proof. We define the sets S_i as in Remark 3.7. We create the SDR in the following way. If AA^T has a 1 in position (j,ℓ), $j \neq \ell$, where column i has 1's in row j and row ℓ then we select $\{j,\ell\}$ from S_i. We note that S_i is the only set containing $\{j,\ell\}$ which gives uniqueness. We delete column i from A to obtain a matrix A_1. Assume that $A_1 A_1^T$ has a 1 in the off diagonal position (p,q) and that it arises from column r of A. Thus we select $\{p,q\}$ from S_r and note that $\{p,q\}$ can occur in no other sets apart from S_i. We continue this process until the SDR is complete. Lemma 3.9 insures that we can continue at each stage. Uniqueness follows from the construction.

The uniqueness of the matrix P is a striking condition. Consider the following result of Brualdi.

Lemma 3.11 (Brualdi [10]). _Let_ B _be a_ $(0,1)$-_matrix of order_ n _which has a unique set of_ n _positions with_ 1's _having no two positions in the same row or column. Then, after a row and column permutation,_ B _can be written._

$$B = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ & 1 & * \\ 1 & & \end{bmatrix} . \tag{3.12}$$

We may apply this result to the matrix consisting of the row intersections of a special matrix. The uniqueness of the matrix P given in Theorem 3.10 ensures that such a matrix can be put in the form of (3.12) using row and column permutations and thus the row intersections are all linearly independent.

We also find that the number of 1's in a special matrix A are restricted.

Remark 3.12. _Let_ A _be a_ $(0,1)$-_matrix of size_ m _by_ $\binom{m}{2}$ _such that there is a unique permutation matrix_ P _of order_ $\binom{m}{2}$ _with_ $A \geq K_m P$. _Then_ A _has at most_ $\binom{m}{2} + \binom{m}{3}$ 1'_s_.

Proof. We may assume $P = I$ by replacing A by AP^{-1}. We view A as formed from K_m by adding 1's. We claim that every 3 rows of K_m contain a unique triangle. Let i_1, i_2, i_3 be three rows of K_m. The only columns which have 2 1's in rows i_1, i_2, i_3 are the three columns with 1's in rows i_1 and i_2, rows i_1 and i_3, and rows i_2 and i_3 respectively. We recall that K_m has every possible column of column sum 2 exactly once. We claim that every 0 in K_m is contained in a unique triangle. Let the 0 be in row i_1 and column j where column j has 1's in row i_2 and row i_3. Our above comments ensure that there is a unique triangle in rows i_1, i_2, i_3 and hence involving the given 0.

At most one 1 can be added to each triangle in K_m and still have the matrix P unique. Combining this with our above comments proves the result.

A converse to Theorem 3.10 does not hold. A matrix satisfying the hypotheses of Remark 3.12 can be shown to have distinct columns and column sums at least 2. Yet even having the maximum number of 1's and having a unique P does not yield CONDITION(∞). In the next section we will show (in Corollary 4.4) that special matrices with no triangles have the maximum number of 1's so that this constraint is reasonable. Consider the following matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \tag{3.13}$$

The matrix A satisfies the hypothesis of Remark 3.12, with $P = I$, and has $\binom{4}{2} + \binom{4}{3} = 10$ 1's. However A does not even satisfy CONDITION(∞). The last 4 columns and the last 3 rows violate the second condition (3.2) for $i = 3$.

Section 4. Matrices without triangles.

We define a <u>solution</u> (<u>of</u> <u>size</u> m) to be a (0,1)-matrix of size m by $\binom{m}{2}$ with column sums at least 2, distinct columns, and containing no triangles. Ryser has given an infinite family of solutions [37]. Form a matrix A from K_m as follows. If column i has 1's in row j and row k for $j < k$ then add 1's in positions (ℓ, i) for $1 \le \ell < j$. Here is the matrix A for $m = 5$:

$$A = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1 \\ 1&0&0&0&1&1&1&1&1&1 \\ 0&1&0&0&1&0&0&1&1&1 \\ 0&0&1&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&1&0&1&1 \end{bmatrix} . \tag{4.1}$$

One may verify that A has no triangles in a number of ways. Remark 4.11 provides a simple way of viewing the construction of A.

Theorem 3.10 tells us that there is a unique permutation matrix P of order $\binom{m}{2}$ such that $A \ge K_m P$. In this case, $P = I$. We note that A has the maximum number of 1's possible in a solution, by Remark 3.12. This is the first inkling that matrices without triangles might have considerable structure. Most of the results we present here have been reported in [3].

We are able to show that solutions have other forbidden configurations. Let A be a solution of size m. Theorem 3.5 yields that the $\binom{m}{2}$ row intersections are linearly independent. Let B be a submatrix of A of size r by $\binom{m}{2}$ consisting of r rows of A. Then B has $\binom{r}{2}$ linearly independent row intersections. Deleting columns with one or no 1's or deleting repeated columns does not affect this. We recall that any submatrix of a matrix with no triangles also has no triangles. Hence Theorem 3.5 tells us that we are left with a submatrix B' of size r by $\binom{r}{2}$ which is a solution of size r. The case where the r selected rows are precisely the rows containing 1's for some column is especially interesting and is discussed in Corollary 4.7.

Let C_k denote the configuration represented by the (0,1)-matrix $C = (c_{ij})$ of order k. Let $c_{ij} = 1$ if $i = j$ or $i = j + 1$ or $i = 1$ and $j = k$. Let $c_{ij} = 0$ otherwise. Thus C is the vertex-edge incidence matrix of the cycle of length k.

Remark 4.1. <u>A solution</u> A <u>contains no</u> C_k<u>'s for</u> $3 \leq k \leq m$.

<u>Proof</u>. Certainly A has no C_3's since C_3 is a triangle. Let ℓ be the smallest value of k for which A has a C_k. It follows from the discussion above that the associated ℓ rows, which contain a C_ℓ, contain a solution B of size ℓ by $\binom{\ell}{2}$. The construction of B ensures that B contains a C_ℓ. Any column of column sum t $(2 \leq t < \ell)$ in B creates a smaller C_k in B and hence in A. The column of ℓ 1's is a possible but then B has at most $1 + \ell$ columns and $1 + \ell < \binom{\ell}{2}$ for $\ell > 3$. This is a contradiction which proves the result.

Remark 4.1 sets up the very important lemma on columns of a solution of a given column sum. We will first introduce some hypergraph notation which will use some of the terminology of Berge [6]. Any (0,1)-matrix A can be thought of as a hypergraph H. The rows will correspond to vertices. Each column will be an edge, consisting of a set of

vertices.

Our forbidden configurations, C_k, are akin to cycles. We define a special chain of length n to be a chain $x_1E_1x_2E_2 \ldots x_nE_nx_{n+1}$ of vertices x_i and edges E_j with $x_1, x_2, \ldots, x_n$ distinct, $E_1, E_2, \ldots, E_n$ distinct, and $E_i \cap \{x_1, x_2, \ldots, x_{n+1}\} = \{x_i, x_{i+1}\}$. The definition of chains in hypergraphs only requires $E_i \cap \{x_1, x_2, \ldots, x_{n+1}\} \supseteq \{x_i, x_{i+1}\}$. A special cycle of length n is a special chain as above with $x_1 = x_{n+1}$. Thus A has no configurations C_k for $k > 2$ if and only if H has no special cycles of length greater than 2. Chains and special chains are closely related.

Remark 4.2. *If a pair of vertices are joined by a chain, then they are joined by a special chain. Thus every pair of vertices in a component of a hypergraph is joined by a special chain.*

Proof. Let the shortest chain joining x and y be $xE_1x_1E_2x_2 \ldots x_{n-1}E_ny$. We will show that this is a special chain. If $x \in E_i$ for $i > 1$ then the chain $xE_ix_i \ldots x_{n-1}E_ny$ is a shorter chain joining x and y. This is a contradiction and thus $x \notin E_i$ for $i > 1$. Similarly $y \notin E_i$ for $i < n$. Assume there are three vertices $x_k, x_{i-1}, x_i \in E_i$. If $k < i - 1$, the chain $xE_1x_1 \ldots x_kE_ix_i \ldots x_{n-1}E_ny$ is a shorter chain joining x and y. Similarly for $k > i$. Both cases yield contradictions and thus we conclude that the shortest chain joining x and y is a special chain.

We define a once covered vertex as a vertex belonging to exactly one edge of a hypergraph. Also let $v(H)$ denote the union of the edges in some hypergraph H. We can now tackle our lemma.

Lemma 4.3. *Let* L *be a* (0,1)-*matrix of size* m *by* n, *column sums* ℓ $(\ell \geq 2)$, *distinct columns, and no* C_k*'s for* $3 \leq k \leq m$. *Then* $n \leq m - \ell + 1$ *and, in addition for* $n \geq 2$, *there exist two rows of* L *each of row sum* 1 *with the two* 1*'s in different columns.*

Proof. We regard L as a hypergraph H. The hypergraph H will contain no special cycles of length greater than 2 but it may contain cycles. In the case $\ell = 2$, H is also a graph and the condition "no C_k's" translates exactly as no cycles. Thus, for $\ell = 2$, the lemma is a well known result about forests and trees.

We will prove the lemma by induction on $m + n$ for a given ℓ. If $m = \ell$, since the edges are distinct, $n \leq 1 = m - \ell + 1$. The second conclusion holds vacuously. This establishes the base of the induction for a given ℓ.

Assume the lemma is true for less than m vertices and less than n edges. The hypergraph H, given by L, has m vertices. If H has a once covered vertex, then delete it and the edge which covered it. By induction we have the inequality $n - 1 \leq (m-1) - \ell + 1$ and thus $n \leq m - \ell + 1$. Thus to prove the lemma, it suffices to show that H has two once covered vertices, covered by different edges, for $n \geq 2$. The case $n = 1$ is easy to check.

We may assume that there is an edge E in H all of whose vertices are covered by edges. Otherwise, each edge will contain a once covered vertex and the proof would be complete. Let H' be the hypergraph consisting of the edges of H which intersect E in at most $\ell - 1$

vertices. We define it this way so that the same arguments can be used in Theorem 4.6, where E is an edge of size greater than ℓ. Consider a mapping of the edges of H', $\theta : H' \to H''$, where we define, for an edge $E' \in H'$, $\theta(E') = E' \setminus E$. Then H'' is simply the hypergraph whose edges are the images of edges in H'. Put another way, H'' corresponds to the hypergraph obtained from H by deleting the vertices of E.

We will show that $\theta(E_1) = \theta(E_2)$ implies $E_1 = E_2$. Assume E_1 and E_2 are different. Select a vertex $z \in E_1 \setminus E = E_2 \setminus E$. Since $|E_1| + |E_2| = \ell$, we can find an $x \in E_1 \cap E \setminus E_2$ and a $y \in E_2 \cap E \setminus E_1$. Then H contains the special cycle $x\ E\ y\ E_2\ z\ E_1\ x$ which is a contradiction. Thus θ is one to one on the edges of H'.

We partition the edges of H'' into components, using the usual definition. This partitions the edges of H' into what we denote as E-components. As a consequence of Remark 4.2, a pair of vertices in the same E-component, but not in E, are joined by a special chain none of whose vertices are in E. We will prove the lemma by showing that there are at least two E-components, each with a once covered vertex outside of E or that there is one E-component with two once covered vertices, from different edges, outside of E.

To prove this, take any pair of vertices x,y where $x \in E' \cap E \setminus E''$ and $y \in E'' \cap E \setminus E'$ with E' and E'' in the same E-component. We will show that neither x nor y are once covered vertices in H'. It is true that $(E' \cap E'') \setminus E = \emptyset$, otherwise, by the same argument that θ is one to one, H will contain a special cycle of length 3. Thus we may take two different vertices x',y' with $x' \in E' \setminus E$ and $y' \in E'' \setminus E$. Since these vertices are in the same E-component, they are joined by a special chain $x'E_1x_1E_2x_2 \dots x_{n-1}E_ny'$ with $x_i \notin E$ for $1 \leq i \leq n-1$. We may assume $E',E'' \notin \{E_1,E_2 \dots, E_n\}$. Otherwise we would have either $E' = E_1$ and we could replace x' by x_1 or $E'' = E_n$ and we could replace y' by x_{n-1}. The new shorter special chain would be of the desired form. Other equalities are not allowed since we have a special chain.

By appending to the above special chain $y'E''yExE'x'$, we obtain a cycle $x'E_1x_1E_2x_2 \dots x_{n-1}E_ny'E''yExE'x'$. We would have a contradiction if we obtained a special cycle from this cycle. If both x and y are once covered vertices in H', then one can verify that the cycle is also a special cycle. Consider the case $x \in E_j$ and $y \notin E_j$. We may replace E' by E_j, x' by x_j, and use the shorter special chain $x_jE_{j+1}x_{j+1} \dots x_{n-1}E_ny'$. Similarly if $y \in E_j$ and $x \notin E_j$ then replacing E'' by E_j, y' by x_{j-1}, and use the shorter special chain $x'E_1x_1E_2x_2 \dots x_{j-2}E_{j-1}x_{j-1}$. Assuming there is no edge E_j with $x,y \in E_j$, then the above changes, denoted by *'s would eventually yield the special cycle $x'^*E_1^*x_1^*E_2^* \dots x_{k-1}^*E_k^*y'^*E''^*yExE'^*x'^*$. This is a contradiction, since $x \in E'^*$, $y \notin E'^*$ and $x \notin E''^*$, $y \in E''^*$ ensures that the special cycle is of length at least 4. The only case that remains, and hence it must occur, is that for some j, we have $x,y \in E_j$. Thus neither x nor y are once covered vertices in H'.

By induction, we have that an E-component, as a hypergraph itself, has two once covered vertices, belonging to different edges, or the E-component consists of one edge. Our discussion above proves that an E-component cannot have two of its once covered vertices, covered by different edges, in E. We will now show that if the E-component has one of its once covered vertices in E (this is true for an E-component consisting of a single edge), then there are two E-components. This would show that there are two once covered vertices of E-components outside E and belonging to different edges. These vertices would be once covered vertices of H, which proves the lemma.

Consider an E-component with one once covered vertex in E, say $x \in E_1 \cap E$. If there is another edge E_2 in the E-component with $y \in (E_2 \cap E) \setminus E_1$, then by our previous arguments x is not once covered. Thus the intersection of the union of the vertices of the E-component with E is $E_1 \cap E$. Our construction of H' ensures that $E_1 \cap E$ consists of at most $\ell - 1$ vertices. By our choice of E, the vertices of E are not once covered vertices in H. Thus there must be additional E-components to cover the remaining vertices of E which are $E \setminus E_1$. This completes the proof.

Corollary 4.4. <u>In a solution of size</u> m, <u>the number of columns of of column sum</u> ℓ <u>is precisely</u> $m - \ell + 1$ <u>for</u> $2 \leq \ell \leq m$.

<u>Proof</u>. Let a_ℓ be the number of columns of column sum ℓ. By lemma 4.3,

$$\binom{m}{2} = \sum_{\ell=2}^{m} a_\ell \leq \sum_{\ell=2}^{m} (m-\ell+1) = \binom{m}{2}, \tag{4.2}$$

and so $a_\ell = m - \ell + 1$ for every ℓ.

Before going on to other structure results, some comments on the lemma are in order. A similar lemma, which considers hypergraphs with no cycles (not special cycles) of length 3 or more, was given by Hansen and Las Vergnas [23]. In some respects, our result is stronger than their result. Lemma 4.3 has been proven for $\ell = 3$ by Harary and Palmer by looking at a cyclic simply connected 2-plexes [24]. These correspond to 2-trees which we will describe shortly.

The examples which motivate the following conjecture show that Remark 4.1 is vital to the proof of Lemma 4.3.

Conjecture 4.5. <u>Let</u> L <u>be a</u> (0,1)-<u>matrix of size</u> m <u>by</u> n <u>with column sums</u> ℓ ($\ell \geq 2$), <u>distinct columns, and no triangles. Then</u>

$$n \leq \left\lfloor \frac{m - \ell + 2}{2} \right\rfloor \cdot \left\lfloor \frac{m - \ell + 3}{2} \right\rfloor . \tag{4.3}$$

We use the notation $\lfloor x \rfloor$ to be the largest integer no larger than x. This conjecture was motivated by the following example. Let $a = \lfloor (m-\ell+2)/2 \rfloor$ and $b = \lfloor (m-\ell+3)/2 \rfloor$. Let

$$A = \begin{bmatrix} J_{\ell-2,ab} \\ K_{a,b} \end{bmatrix} , \tag{4.4}$$

where $K_{a,b}$ is the incidence matrix of the complete bipartite graph where one part is of size a and the other is of size b. The example gives equality in (4.3) and one would also conjecture that A is the unique matrix, apart from row and column permutations, to do so.

Note that Conjecture 4.5, for $\ell = 2$, is just Turan's theorem. (Theorem 3.2 of Chapter 1). Corollary 3.6 ensures that $n \leq \binom{m}{2}$ and, since solutions always have a column of 1's, we may improve this to $n < \binom{m}{2}$. The conjecture is also true for $\ell = m - 1$.

We will show later that the columns of column sum ℓ form an $(\ell-1)$-tree in Remark 4.12. We now prove the main structure Theorem for solutions which utilizes the ideas of Lemma 4.3. Numerous structure results will be corollaries to it. Define a column to <u>cover</u> another column if the latter has nonzero entries only when the former does.

Theorem 4.6. <u>Let</u> A <u>be a solution of size</u> m. <u>Then a column of</u> ℓ 1's <u>in</u> A <u>covers precisely</u> $\ell - t + 1$ <u>columns of</u> t 1's <u>of</u> A <u>for</u> $2 \leq t \leq \ell$.

<u>Proof</u>. Since A is a solution, A has the maximum number, $m - t + 1$, of columns of column sum t. Let H be the hypergraph associated with these $m - t + 1$ columns. Let E be an edge consisting of the ℓ vertices specified by the column of ℓ 1's. Delete a once covered vertex in H that lies outside E and delete the edge which covers it. The resulting hypergraph has the maximum number of edges of size t on the remaining vertices. Repeat this process, deleting as many vertices as possible, to obtain a hypergraph H', having the maximum number of edges of size t on the vertices of $v(H')$.

We can show that $v(H') = E$ using the arguments of Lemma 4.3. We consider the E-components of H' which consist of the edges of H' that intersect E in at most $t - 1$ vertices. If one examines the arguments of the lemma, one finds that the fact that $|E| = \ell > t$ causes no difficulties. From the lemma, we have that an E-component, if there is one, will have a once covered vertex outside of E. This is true even when an E-component consists of a single edge. Our construction of H' would have deleted this vertex and hence there are no E-components. Thus

$v(H') = E$. We conclude by noting that in H', and hence in H, there are the maximum number, $\ell - t + 1$, of edges of size t contained in E, which proves the theorem.

Corollary 4.7. _Given any column of column sum_ ℓ _in a solution_ A, _there are_ $\binom{\ell}{2}$ _columns of the solution covered by the given column, which form a solution of size_ ℓ.

Proof. Apply Theorem 4.6 for all possible values of t yielding $\binom{\ell}{2}$ distinct columns of column sum greater than 1 and covered by the given column of ℓ 1's.

Let A be a solution. We may apply Corollary 4.7 as follows for a given column of column sum ℓ. A suitable row permutation will place the ℓ 1's in the first ℓ rows. A suitable column permutation will place A in the form

$$A = \begin{bmatrix} A' & * \\ 0 & * \end{bmatrix} , \qquad (4.5)$$

where A' is a solution of size ℓ. This is the improvement on the result that any ℓ rows of A contain a submatrix B', which is a solution of size ℓ. Corollary 4.7 is a substantial improvement in the case that the ℓ rows are precisely the rows with 1's of some column.

Corollary 4.8. _In a solution_ A _of size_ m _there are two rows of row sum_ $m-1$. _After suitable row and column permutations, we have_

$$A = \left[\begin{array}{c|cccc} & 0 & & & 1 \\ A' & & & \cdot & \\ & & \cdot & & \\ & 1 & & & 1 \\ & 1 & & & \\ \hline 0\ 0\ \cdots\ 0 & 1 & 1 & \cdots & 1 \end{array}\right], \qquad (4.6)$$

where A' *is a solution of size* $m - 1$.

Proof. Consider the two columns of columm sum $m - 1$ in A. Using Corollary 4.7, each will yield a solution A' of size $m - 1$ as above. We note that there is one column of column sum ℓ left over from A' in A, for each ℓ, $2 \leq \ell \leq m$. Each of these $m - 1$ columns must have a 1 in the bottom row because otherwise there would be no permutation matrix P of order $\binom{m}{2}$ with $A \geq K_m P$. This would contradict Theorem 3.10. The uniqueness of the matrix P, from Theorem 3.10, and Lemma 3.11, due to Brualdi, ensure that there is a row and column permutation which leaves the 1's arranged in the triangular pattern given in (4.6).

This gives us a nice inductive buildup of solutions. A different outlook is taken at the end of this section to create an algorithm to generate solutions.

Corollary 4.9. *In a solution* A *of size* m, *a column of column sum* ℓ *is covered by a column of column sum* $\ell + 1$ *for* $\ell < m$.

Proof. If $\ell = m - 1$, the column of column sum ℓ is covered by the column of m 1's. If $\ell < m - 1$, then we use Corollary 4.7 repeatedly. The column of column sum ℓ is covered by one of the two columns of column sum $m - 1$ because otherwise a triangle would be formed. Thus the column of column sum ℓ is contained in a solution A' of size $m - 1$ as written in (4.5). Repeat this argument on the smaller solution until the given column is contained in a solution of size $\ell + 1$. The column of column sum $\ell + 1$, in the solution of size $\ell + 1$, is the desired column.

Corollary 4.10. *In a solution* A, *let a column of column sum* ℓ *and a column of column sum* k *be given. Then* A *has a column having* 1's *precisely in the* t *rows where both columns have* 1's, *for* $t \geq 2$. *In addition, a column with* 1's *in the* $\ell + k - t$ *rows where either given column has a* 1 *would cover precisely* $(\ell+k-t) - s + 1$ *columns of column sum* s *for* $2 \leq s \leq t$.

Proof. Let α be the column with t 1's $(t \geq 2)$ precisely in the rows where the two given columns have 1's. Appending α to A will create no new triangles and, since a solution has as many distinct columns as possible, we deduce that α is a column of A.

By Theorem 4.6, the column of column sum ℓ covers $\ell - s + 1$ columns of column sum s for $2 \leq s \leq t$. The same holds with ℓ replaced by k or t. Thus a column with 1's in the $\ell + k - t$ rows where either given column has a 1 would cover at least

$$(\ell-s+1) + (k-s+1) - (t-s+1) = (\ell+k-t) - s + 1, \tag{4.7}$$

columns of column sum s. By Lemma 4.3, the number in (4.7) is as large as possible and hence the result holds.

Remark 4.11. *Given a solution* A *of size* m, *there exists a solution of size* $m + 1$ *having* A *as a submatrix*.

Proof. Use the following bordering technique:

$$A' = \begin{bmatrix} 1\ 1 \cdot \cdot \cdot 1 \\ A \quad I_m \end{bmatrix} . \tag{4.8}$$

The matrix A' does not have any triangles since a triangle could not involve the top row and hence none of the last m columns. Hence, a triangle in A' is contained in A, but A is a solution and has no triangles. We deduce that A' is a solution of size $m + 1$ with the desired properties.

The following is an example of a solution of size 6. Our results on the structure of solutions may be checked for this matrix. An appropriate submatrix is outlined so that Corollary 4.8 may be verified more easily.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix} . \tag{4.9}$$

We will finish this section by giving an algorithm for constructing an arbitrary solution. This will involve the notion of k-trees as well as the results from Theorem 4.6 that a column of column sum ℓ covers two columns of column sum $\ell - 1$.

We define a k-tree as a graph following Beineke and Pippert [5]. The definition is inductive on the number of vertices n in the graph. A set of k mutually adjacent vertices is a k-tree and is analogous to the initial root vertex in a usual tree (a 1-tree). A k-tree on $n + 1$ vertices is obtained from one on n vertices by joining the $(n+1)$st vertex to a set of k mutually adjacent vertices which creates a set of $k + 1$ mutually adjacent vertices. The induction guarantees that these k mutually adjacent vertices are either the initial k mutually adjacent vertices or k vertices chosen from $k + 1$ mutually adjacent vertices as formed above. We may think of a k-tree as a collection of superimposed complete graphs on $k + 1$ vertices.

Remark 4.12. _In a solution of size_ m, _the columns of column sum_ $k + 1$ $(m > k)$ _can be thought of as a_ k-_tree_.

Proof. Let H be the hypergraph associated with the columns of column sum $k + 1$. Let G be the graph obtained as the set union of the edges of size 2 contained in the edges of H, on the m vertices of H. Using the inductive definition, we verify that G is a k-tree by induction on m.

Since H comes from a solution, it has $m - k$ edges, the maximum number possible. If $m = k + 1$, then H has just one edge and G corresponds to the second stage in the inductive definition of a k-tree, i.e. one has added the $(k+1)$st vertex. Assume $m > k + 1$. Consider a once covered vertex x in some edge E. By Lemma 4.3, we may repeatedly delete other once covered vertices, not in E, as well as the edges that cover them until only $k + 2$ vertices remain. The hypergraph, H', that remains has only two edges, E and E'. Thus we can think of x as the $(n+1)$st vertex in the definition of k-trees being joined, in G, to k other vertices already mutually adjacent, because of E'. Let

H'' be the hypergraph obtained from H by deleting x and E. By induction, the graph G'' associated with H'' is a k-tree. Thus G is a k-tree and the proof is complete.

The algorithm to generate an arbitrary solution A follows below. To generate all solutions, simply try all possibilities at each choice in the algorithm and then check for isomorphisms.

<u>Algorithm to generate solutions</u>.

<u>Step 1</u>. Choose a spanning tree on m vertices. Its vertex-edge incidence matrix yields $m - 1$ columns of column sum 2 for A. Note that a tree is a 1-tree.

<u>Step 2</u>. Repeat for $k = 3,4, \ldots, m$ in turn. We use the fact that the columns of column sum $k - 1$ form a $(k-2)$-tree. For each k, we will add $m - k + 1$ columns of column sum k. Select two columns of column sum $k - 1$ that have exactly $k - 2$ rows where both have 1's. This occurs because the columns of column sum $k - 1$ form a $(k-2)$-tree. Add a column of k 1's covering both columns. Repeat what follows until the remaining $m - k$ columns of column sum k have been added. Select a column of column sum $k - 1$, not already covered by a column of column sum k, such that there is a covered column of column sum $k - 1$ with $k - 2$ rows where both have 1's. There is such a column because the columns of column sum $k - 1$ form a $(k-2)$-tree. Add a column of k 1's covering both of the above columns.

In the above algorithm, a column of column sum $k - 1$ only gets covered when explicitly selected. Again, this follows because the columns of column sum $k - 1$ form a $(k-2)$-tree. Thus Step 2 will be completed for a given k when a $(k-1)$-tree is formed. The columns of column sum k form an arbitrary $(k-1)$-tree as described in Remark 4.12 subject only to the condition that a column of column sum k covers two columns of column sum $k-1$, a result of Theorem 4.6. Thus any A can be generated in the manner described in the algorithm. We need only verify that A has no triangles.

Let H be the hypergraph corresponding to the matrix A where the rows correspond to vertices. A short inductive argument verifies that an edge of size ℓ covers the maximum number, $\ell - 1$, edges of size 2. For $\ell = 2$, this is obvious. For an edge of size ℓ $(\ell > 2)$, we recall that it covers two edges of size $\ell - 1$. Thus, for $\ell = 3$, the result is proved. In the remaining cases, the two edges of size $\ell - 1$ cover $\ell - 2$ edges of size 2 by induction. By the nature of the algorithm, the two edges have $\ell - 2$ vertices in common and these $\ell - 2$ vertices form an edge of H. This edge of size $\ell - 2$ covers $\ell - 3$ edges of size 2. We compute that the edge of size ℓ covers at least $2(\ell-2) - (\ell-3) = \ell - 1$ edges of size 2. It cannot cover more than $\ell - 1$ edges of size 2 otherwise they would form a cycle, contradicting Step 1. Thus every edge covers a spanning tree on its vertices.

Assume H has some special cycle of length 3, say $x_1E_1x_2E_2x_3E_3x_1$. We have that x_1 and x_2 are joined by a path in the spanning tree on the vertices of E_1. This path cannot involve x_3 since $x_3 \notin E_1$. Similarly, we can find paths joining the pair x_2 and x_3 and the pair x_3 and x_1. This yields a cycle in the edges of size 2 of H which is a contradiction to Step 1 of the algorithm. Thus A has no triangles.

This algorithm provides a different viewpoint of the inductive buildup of solutions from that given in Corollary 4.8. It is important to note that an arbitrary k-tree may contain a triangle and hence need not be a part of a solution.

Some of the arguments given in this secion, in particular Remark 4.1, might apply to special matrices with no configuration (3.3). Any general results will be more complicated. For example, K_m and the matrix obtained from K_m by adding a 1, both have no configuration (3.3). Such examples suggest that classifying special matrices in general would be quite difficult.

Section 5. An application of another configuration theorem of Ryser.

Recall the proof of Theorem 3.5 and the fundamental matrix equation $AXA^T = Y$. One might consider what would happen if, rather than setting just the off diagonal elements of Y to zero, one sets $Y = 0$. Continuing along the lines of that proof, one would get two (0,1)-matrices A_1 and A_2, from A, with

$$A_1A_1^T - A_2A_2^T = 0. \tag{5.1}$$

Lemma 3.4 would still apply here but (5.1) is a stronger condition than the previous equation (3.7). Thus one would hope to find weaker conditions on A such that, when (5.1) is satisfied, A_1 and A_2 are forced to be equal apart from a column permutation. The following configuration theorem of Ryser suggests such conditions [35].

Theorem 5.1 (Ryser). <u>Let</u> A <u>and</u> B <u>be</u> (0,1)-<u>matrices of size</u>

m *by* n *such that* $AA^T = BB^T$ *and* A *contains no configurations*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{5.2}$$

Then A *and* B *are the same apart from a column permutation.*

We will be using the following variant of Theorem 5.1.

Theorem 5.2. *Let* A *and* B *be* (0,1)-*matrices of size* m *by* n *such that* $AA^T = BB^T$ *and* A *and* B *contain no configuration*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{5.3}$$

Then A *and* B *are the same apart from a column permutation.*

Proof. We follow Ryser's proof of Theorem 5.1 except that we are dealing with the transposes of the matrices. Use induction on m. The theorem is true for m = 1,2. Let α be the first row of A and let A_1 consist of the remaining rows of A. Let β be the first row of B and let B_1 consist of the remaining rows of B. Then

$$A = \begin{bmatrix} \alpha \\ A_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta \\ B_1 \end{bmatrix}, \tag{5.4}$$

where $A_1 A_1^T = B_1 B_1^T$. By induction, since A_1 and B_1 have m - 1 rows, a column permutation of B yields

$$\tilde{B} = \begin{bmatrix} \tilde{\beta} \\ A_1 \end{bmatrix} . \tag{5.5}$$

We have that $\alpha\alpha^T = \tilde{\beta}\tilde{\beta}^T$ and so the number of 1's in α is equal to the number of 1's in β. Whenever α and $\tilde{\beta}$ have common 1's or common 0's, then the corresponding columns of A and $\tilde{B}$ are identical. Delete all the columns matched in this way. We obtain

$$a^* = \begin{bmatrix} \alpha^* \\ A_1^* \end{bmatrix} , \qquad B^* = \begin{bmatrix} \beta^* \\ A_1^* \end{bmatrix} , \tag{5.6}$$

where α^* and β^* have no common 0's or 1's. We still have $\alpha^*(\alpha^*)^T = \beta^*(\beta^*)^T$. Hence the number of 1's in α^* is equal to the number of 1's in β^* is equal to the number of 0's in α^* is equal to the number of 0's in β^* and we obtain:

$$\overline{A} = \begin{bmatrix} 11 \cdots 100 \cdots 0 \\ E \qquad\quad F \\ \gamma \qquad\quad \delta \end{bmatrix} , \quad \overline{B} = \begin{bmatrix} 11 \cdots 100 \cdots 0 \\ E \qquad\quad F \\ \gamma \qquad\quad \delta \end{bmatrix} , \tag{5.7}$$

where the number of columns in E is equal to the number of columns in F. We still have $\overline{A}\overline{A}^T = \overline{B}\overline{B}^T$.

If we ignore the last row, then by induction, E and F are the same apart from a column permutation. We may assume that $E = F$ in (5.7). As in the above arguments, we delete the columns where γ and δ match since these yield matching columns in $\overline{A}$ and $\overline{B}$. We apply a column permutation to what is left to obtain

$$\hat{A} = \left[\begin{array}{c|c|c|c} 11 \cdots 1 & 11 \cdots 1 & 00 \cdots 0 & 00 \cdots 0 \\ E_1^* & E_2^* & E_1^* & E_2^* \\ 11 \cdots 1 & 00 \cdots 0 & 00 \cdots 0 & 11 \cdots 1 \end{array}\right],$$
$$\hat{B} = \left[\begin{array}{c|c|c|c} 00 \cdots 0 & 00 \cdots 0 & 11 \cdots 1 & 11 \cdots 1 \\ E_1^* & E_2^* & E_1^* & E_2^* \\ 11 \cdots 1 & 00 \cdots 0 & 00 \cdots 0 & 11 \cdots 1 \end{array}\right], \tag{5.8}$$

where the number of columns in E_1^* is the same as the number of columns in E_2^*. If there are no unmatched columns, we are done.

Take the first column from each of the four parts of $\hat{A}$ and $\hat{B}$. We obtain 6 possible rows formed by the four columns in each case.

$$\hat{A}: \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{B}: \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} . \tag{5.9}$$

The first and last rows occur precisely once by our construction. If the third row occurs, then $\hat{A}$ contains the forbidden configuration (5.3). If the fourth row occurs, then $\hat{B}$ contains the forbidden configuration (5.3). Thus neither the third nor fourth rows are possibilities. Then the first column of $\hat{A}$ and the fourth column in (5.9) of $\hat{B}$ match. The remaining 6 columns in (5.9) can be paired off similarly. One can repeat this process and so we get that $\hat{A}$ and $\hat{B}$ are the same apart from a column permutation. This proves the theorem.

A similar argument will prove the following companion result.

Theorem 5.3. *Let* A *and* B *be* (0,1)*-matrices of size* m *by* n *such that* $AA^T = BB^T$ *and* A *and* B *have no configurations.*

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \qquad (5.10)$$

Then A *and* B *are the same apart from a column permutation.*

We may obtain a useful version of Theorem 5.2 which does not require A and B to be of the same size.

Lemma 5.4. *Let* A *and* B *be* (0,1)*-matrices such that* $AA^T = BB^T$ *and* A *and* B *have no configuration* (5.3). *Assume all column sums are greater than* 1. *Then* A *and* B *are the same apart from a column permutation.*

Proof. Assume A is of size m by n and B is of size m by $n + \ell$, with $\ell \geq 0$. Form the matrix A_1 from A by adding ℓ columns of all 0's. Then $A_1A_1^T = AA^T = BB^T$. Also, A_1 has no configuration (5.3) since A does not. Using Theorem 5.2 on A_1 and B, we obtain that A_1 and B are the same apart from a column permutation. Thus B has ℓ columns of all 0's and hence $\ell = 0$. Thus A and B are the same apart from a column permutation.

One wonders why the proof of Theorem 5.2, or indeed Lemma 5.4, is so different from that of Lemma 3.4. We have been unable to find the connection between the two results. A version of Lemma 5.4 does not work for the configuration (5.10). This is because adding a column of 0's may indeed create a configuration (5.10). Lemma 5.4 is used in the proof of the next theorem which mimics the proof of Theorem 3.5 in the manner described at the beginning of this section.

Theorem 5.5. _Let_ A _be a_ $(0,1)$_-matrix of size_ m _by_ n _with column sums at least_ 1 _and_ A _has no configuration_ (5.3). _Then the number of linearly independent row intersections and rows (over the rationals) is equal to the number of distinct columns of_ A.

Proof. We use the fundamental matrix equation

$$AXA^T = Y, \tag{5.11}$$

where $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ we may consider A as indexing m subsets, labelled $S_1, S_2, \ldots, S_m$, of an n-set $\{x_1, x_2, \ldots, x_n\}$. Then if $Y = (y_{ij})$, we have that y_{ij} is the sum of the elements in $S_i \cap S_j$ for all pairs i,j such that $1 \leq i \leq n$, $1 \leq j \leq n$. Thus the entries y_{ij} correspond to the row intersections $(i \neq j)$ and rows $(i=j)$ when considered as vectors in $\mathbb{Q}^n$ with basis $\{x_1, x_2, \ldots, x_n\}$.

Repeated columns may be deleted without affecting the linear independence of the row intersections and rows. Thus we may assume that A has distinct columns. We obtain immediately that the number of distinct columns, n, is at least the number of linearly independent row intersections and rows since n is the dimension of the space containing them.

Assume n is greater than the number of linearly independent row intersections and rows. We now consider $x_1, x_2, \ldots, x_n$ as variables and set $y_{ij} = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n$. Since the number of variables exceeds the number of linearly independent equations, we can find rational, and hence integral values, $e_1, e_2, \ldots, e_n$, not all zero, for $x_1, x_2, \ldots, x_n$. Let $E = \mathrm{diag}(e_1, e_2, \ldots, e_n)$. Then

$$AEA^T = 0, \tag{5.12}$$

where 0 is the zero matrix of order m. Every variable occurs in some equation (column sums at least 1), thus some e_i's are positive and some e_j's are negative. Define A_1 and A_2 as follows. For all i with $e_i > 0$, A_1 contains column i of A repeated e_i times. For all j with $e_j < 0$, A_2 contains column j of A repeated $-e_j$ times. Then

$$A_1A_1^T - A_2A_2^T = 0. \tag{5.13}$$

Now A_1 and A_2 do not contain the forbidden configuration (5.3) since A does not. Thus by Lemma 5.4, A_1 and A_2 are the same apart from a column permutation and so A has repeated columns. This is a contradiction, which proves the theorem.

Corollary 5.6. _Let_ A _be a_ (0,1)-_matrix of size_ m _by_ n _with column sums at least_ 1, _distinct columns, and no configurations_ (5.3) _Then_ $n < \binom{m+1}{2}$.

Proof. This follows from Theorem 5.5 by noting that the total number of row intersections and rows is $\binom{m}{2} + m = \binom{m+1}{2}$.

We will concern ourselves with the case of equality in Corollary 5.6. Define a _restricted_ matrix to be a (0,1)-matrix of size m by $\binom{m+1}{2}$ with column sums at least 1, distinct columns, and with no configuration (5.3). As before, for solutions, all the row intersections and rows are linearly independent for a restricted matrix. Also, any r rows of a restricted matrix contain a submatrix B of size r by $\binom{r+1}{2}$ which is also a restricted matrix. In analogy to Remark 3.7, we may prove the following result.

Theorem 5.7. <u>Let</u> A <u>be a restricted matrix. Then there exists a permutation matrix</u> P <u>of order</u> $\binom{m+1}{2}$ <u>such that</u> $A \geq [K_m I_m]P$.

<u>Proof</u>. Associate with each column of A, the set S_i as follows

$$S_i = \{\{j,k\} | \text{ column } i \text{ has 1's in row } j \text{ and row } k\}, \tag{5.14}$$

where we allow $j = k$. Let $I \subseteq \{1,2, \ldots, \binom{m+1}{2}\}$ and $|I| = r$. Let B be the m by r matrix obtained by selecting from A precisely the columns indicated by I. Then, in $\cup_{i \in I} S_i$, the number of elements is equal to the number of nonzero row intersections and rows in B which is at least the number of linearly independent row intersections and rows which is equal to r by Theorem 5.5 This follows by noting that B has no configuration (5.3) since A does not. But then $|\cup_{i \in I} S_i| >$ $|I| = r$ for all I. Hence, by P. Halls' SDR theorem, our result holds.

We can find restricted matrices easily. Define taking the (0,1)-<u>complement</u> of a matrix as the process of replacing 0's by 1's and 1's by 0's. The following matrix is restricted:

$$A = \left[\begin{array}{c|c|c} 1 & & \\ 1 & & \\ \cdot & J_m - I_m & A' \\ \cdot & & \\ 1 & & \end{array}\right], \tag{5.15}$$

where A' is the (0,1)-complement of a solution of size m with the resulting column of 0's deleted. Another possibility for a restricted matrix is $[K_m I_m]$. It turns out that these are the only possibilities.

We note that if a matrix A is restricted and has column sums at most 2, then it is a column permutation of $[K_m I_m]$. If a restricted matrix A has a column of 1's then consider A_1, the (0,1)-complement of A. Because A has a column of 1's, then A_1 has no triangles. Since A has distinct columns, A_1 has at most $m+1$ columns of column sum 0 or 1. We have that A_1 has no column of 1's since A does not have a column of 0's. Thus if we let A_2 be the submatrix of A_1 of those columns with column sum at least 2, then A_2 is a solution of size m apart from a missing column of all 1's. Also, there are $m+1$ columns of column sum 0 or 1 in A_1 and hence all possible columns of column sum 0 or 1. Thus A has the form given in (5.15) where A' is the (0,1)-complement of A_2.

Theorem 5.8. <u>Let</u> A <u>be a restricted matrix. Then, after a suitable row and column permutation,</u> A <u>is in the form given in</u> (5.15), <u>where</u> A' <u>is the</u> (0,1)-<u>complement of some solution of size</u> m <u>with the resulting column of</u> 0's <u>deleted, or</u> $A = [K_m I_m]$.

<u>Proof</u>. We prove this by induction on m. For $m = 2,3$ the theorem is true by looking at the possible column sums and using the above remarks. For $m = 4$, we are done if either all the columns have column sum at most 2 or there is a column of column sum 4. Consider the remaining case of the largest column sum being 3. One finds that the only possible matrix of size 4 by 10 that has this property and no

obvious configuration (5.3) is the following

$$\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&1&0&1&0&1&0&0 \\ 1&1&1&0&0&1&1&0&0&0 \end{bmatrix} . \tag{5.16}$$

However, this matrix has the configuration (5.3) in rows 1, 2, 4 and columns 3, 4, 6, and 10. Thus the theorem is true for $m = 4$.

We now assume the theorem is true for $m = k$ $(k \geq 4)$. Let A be a restricted matrix of size $k + 1$ by $\binom{k+2}{2}$. Consider a selection of k of the rows. The k rows will contain a submatrix B of size k by $\binom{k+1}{2}$ which is a restricted matrix. By induction, B is of one of the two forms given.

If, after a row and column permutation, $B = [K_k I_k]$, then A has column sums at most 3. Thus any k columns of A contain a restricted matrix of the form $[K_k I_k]$ using the fact that $k \geq 4$ and the other possibility, (5.15), has a column of k 1's. Hence A has column sums at most 2 and so, after a row and column permutation, $A = [K_{k+1} I_{k+1}]$.

If B is of the form given in (5.15), then A has a column of at least k 1's. Thus any selection of k columns yields a submatrix of the form given in (5.15) since the other possibility would require A to have column sums at most 3. Thus any k rows have a column with 1's in those rows. Thus any 3 rows has a column with 1's in those 3 rows. Hence A has no configuration of the form

$$\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} . \tag{5.17}$$

We deduce that A has a column of all 1's since adding a column of 1's to A cannot create a configuration (5.3) and A has the greatest number of distinct columns under the restriction no configurations (5.3). By our remarks before the theorem, we deduce that a row and column permutation will leave A in the form given in (5.15). This proves the theorem.

We note that (5.3) and (5.10) are (0,1)-complements of each other. Thus a (0,1)-complement of a restricted matrix has no configurations (5.10). Similarly, the (0,1)-complement of a matrix with no configuration (5.10), has no configuration (5.3). We note that any matrix with no configurations (5.10) can have a column of 1's added without creating any configuration (5.10). Combining these remarks, we obtain the following result.

Theorem 5.9. _Let_ A _be a_ (0,1)-_matrix of size_ m _by_ n _with column sums at least_ 1, _distinct columns, and no configurations_ (5.10). _Assume that no column may be added to_ A _and yet preserve these properties. Then, after a suitable row and column permutation, we have_ $n = \binom{m+1}{2} + 1$ _and_ $A = [J_{m,1}K']$ _where_ K' _is the_ (0,1)-_complement of_ $[K_m I_m]$ _or_ $n = \binom{m+1}{2}$ _and_ $A = [BI_m]$, _where_ B _is a solution of size_ m.

We could make this result more symmetrical with Theorem 5.8 by allowing columns of column sum 0.

Further results along this line may be obtained by other specializations of the fundamental matrix equation $AXA^T = Y$. For example, if we set all the off diagonal elements of Y to be equal, we would obtain matrices A_1, A_2 satisfying

$$A_1A_1^T - A_2A_2^T = \lambda J + D, \tag{5.18}$$

where λ is a nonzero integer and D is a diagonal matrix. For an arbitrary (0,1)-matrix satisfying CONDITION(∞), this is an interesting result which deserves further study.

Chapter 3. The class $\mathfrak{A}(R,S)$

Section 1. Introduction.

The class $\mathfrak{A}(R,S)$ is the set of (0,1)-matrices with prescribed row and column sums. As with $C(S)$, we are interested in various existence and structure questions involving the matrices of the class.

We define $\mathfrak{A}(R,S)$ as follows. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ with $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ and all the entries r_i and s_j being positive integers. Then the class $\mathfrak{A}(R,S)$ consists of all $m \times n$ (0,1)-matrices with i^{th} row sum r_i and j^{th} column sum s_j. Thus if $A \in \mathfrak{A}(R,S)$ then

$$AJ_{n,m} = RJ_{1,m} \quad ; \quad J_{n,m}A = J_{n,1}S \quad . \tag{1.1}$$

As with most (0,1)-matrices we may interpret the class in graph theory terms. For example, $\mathfrak{A}(R,S)$ corresponds to all bipartite graphs with R as the degree sequence of one part and S the degree sequence of the other part. With no entry of R or S being zero, we may interpret $\mathfrak{A}(R,S)$ as consisting of all hypergraphs on m vertices. The entries of S give the sizes of the n edges and r_i gives the number of edges which contain vertex i. Comments on some graph theory results will be given later.

Section 2 will contain a number of small results as an introduction to $\mathfrak{A}(R,S)$. Invariant 1's, the structure matrix, and possible submatrices and columns will be considered.

Gale and Ryser have found a simple condition to determine whether

$\mathfrak{A}(R,S)$ is nonempty or empty [18,31]. The condition involves the concept of majorization. Let $X = (x_1,x_2, \ldots, x_n)$ and $Y = (y_1,y_2, \ldots, y_n)$ with $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$. We define X to <u>majorize</u> Y (written $X \succ Y$) if after reordering X and Y such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ then

$$\sum_{i=1}^{t} x_i \geq \sum_{i=1}^{t} y_i \qquad (1 \leq t \leq n) . \tag{1.2}$$

We define a vector X to be <u>monotone</u> if $x_1 \geq x_2 \geq \cdots \geq x_n$.

Theorem 1.1. (Gale,Ryser). <u>Let $\bar{A}$ be the $m \times n$ (0,1)-matrix with i^{th} row sum r_i and all 1's in the matrix as far to the left as possible. Let A have j^{th} column sum $\bar{s}_i$ and let $\bar{S} = (\bar{s}_1,\bar{s}_2, \ldots, \bar{s}_n)$. Then $\mathfrak{A}(R,S)$ is nonempty if and only if</u>

$$\bar{S} \succ S . \tag{1.3}$$

We define $\bar{A}$ to be the <u>Ferrer's</u> matrix associated with R in analogy to a Ferrer's graph. Typically we require that R and S be monotone. With S monotone, the condition (1.3) becomes the following inequalities

$$\sum_{i=1}^{t} \bar{s}_i \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n) . \tag{1.4}$$

Section 3 is concerned with a generalization of this result. We find similar necessary and sufficient conditions for there to exist a matrix $A \in \mathfrak{A}(R,S)$ with $A \geq P$ for some matrix P. Our result only works when P is "acceptable". A number of extensions are given. This leads naturally to the subclass $\mathfrak{A}_P(R,S)$ which consists of all matrices $A \in \mathfrak{A}(R,S)$ with $A \geq P$. This is the subject of a brief discussion in Section 4.

Another important result about $\mathfrak{A}(R,S)$ concerns the following matrices:

$$\text{i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{ii)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.5}$$

We note that replacing a submatrix of a matrix A of type i) by a submatrix of type ii) does not alter the row and column sums. Similarly when the roles of i) and ii) are reversed. We define an <u>interchange</u> to be the operation of replacing a submatrix of one of the types in (1.5) by the other. Thus if one can get from a matrix A_1 to a matrix A_2 by a series of interchanges then A_1 and A_2 are in the same class $\mathfrak{A}(R,S)$. More is true:

Theorem 1.2 (Ryser [34]) <u>Interchange theorem</u>. <u>Let</u> $A_1, A_2 \in \mathfrak{A}(R,S)$. <u>Then one can obtain</u> A_2 <u>from</u> A_1 <u>by a series of interchanges</u>.

Interchanges are used in a result of Brualdi and Ross and from their result one may obtain the Interchange theorem. In Section 5 we discuss this while extending the result of Brualdi and Ross [12].

Section 6 is concerned with finding matrices $A \in \mathfrak{A}(R,S)$ in triangular form. A nice application of our result yields a characterization of those classes $\mathfrak{A}(R,S)$ containing matrices with permanent value 1.

Before going on we will give some notation. We will assume that R,S are monotone and that $\mathfrak{A}(R,S)$ is nonempty when its nonemptiness is not the subject of the discussion. With a (0,1)-matrix A we can associate a class $\mathfrak{A}(R,S)$ where the entries of R are the row sums of A and the entries of S are the column sums of A. One could use (1.1) to determine R and S. Let the <u>size</u> of the class $\mathfrak{A}(R,S)$ be the size of the matrices in it. Thus we may say that $\mathfrak{A}(R,S)$ is of size $m \times n$ or, in the case $m = n$, $\mathfrak{A}(R,S)$ is of order n.

An extensive survey article on the class $\mathfrak{A}(R,S)$ is available in Brualdi [11].

Section 2. Invariant 1's, the structure matrix, and submatrices.

Invariant 1's.

Define a 1 in position (e,f) to be an <u>invariant</u> 1 (<u>of</u> $\mathfrak{A}(R,S)$) if every matrix $A \in \mathfrak{A}(R,S)$ has a 1 in position (e,f). We define an <u>invariant</u> 0 similarly. It is easy to construct classes which have invariant 1's and 0's. We would like a way to classify the invariant 1's (and 0's) of a class $\mathfrak{A}(R,S)$.

Define an invariant 1 in position (e,f) to be extremal if there is no invariant 1 in position (e′,f′) not equal (e,f) with $e' \geq e$ and $f' \geq f$. This defines a partial order on the positions in the matrix. We follow the proof of Ryser (Theorem 3.2 [34]) to obtain:

Theorem 2.1. <u>The class</u> $\mathfrak{A}(R,S)$ <u>has an extremal invariant</u> 1 <u>in position</u> (e,f) <u>if and only if every matrix</u> $A \in \mathfrak{A}(R,S)$ <u>decomposes into blocks as follows</u>

$$A = \begin{bmatrix} J & X \\ Y & 0 \end{bmatrix} , \tag{2.1}$$

where J is of size e by f and is all 1's and 0 is of size $n-e$ by $n-f$ and is all 0's. Degenerate blocks with 0 rows or columns are allowed here. X has no column of 1's and Y has no row of 1's.

Proof. Assume $\mathfrak{A}(R,S)$ has an invariant 1 in position (e,f) and let an $A \in \mathfrak{A}(R,S)$ be written as

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} , \tag{2.2}$$

where W is of size e by f and Z is of size $(n-e)$ by $(n-f)$. If a 0 occurs in W then at most two interchanges are required to replace the invariant 1 by a 0. We do this using monotonicity of R and S. For example a submatrix of the form $[0\ 1]$ guarantees the existence of a submatrix $[1\ 0]$ in the same columns. Replacing an invariant 1 by a 0 is a contradiction so $W = J_{ef}$.

If we assume A decomposes as in (2.1) then certainly all the 1's in J are invariant 1's and all the 0's of 0 are invariant 0's. Using our previous discussion and the fact that X has no column of 1's and Y has no row of 1's, we deduce that the 1 in position (e,f) is an extremal invariant 1.

Assume that $\mathfrak{A}(R,S)$ has an extremal invariant 1 in position (e,f). Decompose matrices in the class as in (2.2) with $W = J_{ef}$ and use the discussion at the beginning of the proof. Since (e,f) is extremal, we may select an $A \in \mathfrak{A}(R,S)$ with a 0 in position $(e,f+1)$.

Now consider any row t of Z with a 1. Using monotonicity, an interchange will place a 1 in column 1 of row t, if one is not already there, and will leave a 0 in position $(e,f+1)$. Then, to avoid an interchange that will place a 0 in W, we deduce that row t of Y is all 1's. Thus for any row t, $1 \leq t \leq m-e$, either Y has all 1's or Z has all 0's. But then the 1's of Y are invariant 1's. This contradicts the fact that the invariant 1 in position (e,f) is extremal unless $Z = 0_{m-e,n-f}$. Then all matrices $A \in \mathfrak{A}(R,S)$ decompose as in (2.1). The extremality of (e,f) ensures that X has no column of 1's and Y has no row of 1's.

The following result gives us a quick way of finding all the invariant 1's in a matrix. A similar result is due to Haber [19].

Theorem 2.2. <u>Every matrix</u> $A \in \mathfrak{A}(R,S)$ <u>can be written</u>

$$A = \begin{bmatrix} J & X \\ Y & 0 \end{bmatrix}, \tag{2.3}$$

<u>where</u> J <u>is of size</u> $\bar{s}_f$ <u>by</u> f <u>if and only if</u>

$$\sum_{i=1}^{f} \bar{s}_i = \sum_{i=1}^{f} s_i . \tag{2.4}$$

<u>Proof</u>. If A can be written as in (2.3) where J is of size $\bar{s}_f$ by f, then by counting the 1's in the first f columns of A and $\bar{A}$ (the Ferrer's matrix introduced in Theorem 1.1) we see that (2.4) holds.

Assume (2.4) holds. Consider the following classes (R',S') and

$\mathfrak{A}(R'',S'')$ given by

$$
\begin{aligned}
R' &= (r_1', r_2', \ldots, r_m') & r_i' &= \min(r_i, f) \\
S' &= (s_1', s_2', \ldots, s_n') & s_i' &= s_i \ (i \le f) \\
& & s_{f+1}' = s_{f+2}' &= \cdots = s_n' = 0 \\
R'' &= (r_1'', r_2'', \ldots, r_m'') & r_i'' &= \max\,(r_i - f, 0) \\
& & s_1'' = s_2'' &= \cdots = s_f'' = 0 \\
& & s_i'' &= s_i \quad (i > f)
\end{aligned}
\tag{2.5}
$$

Using (2.4) and the nonemptiness of $\mathfrak{A}(R,S)$, we obtain that both $\mathfrak{A}(R',S')$ and $\mathfrak{A}(R'',S'')$ are nonempty. A matrix $A' \in \mathfrak{A}(R',S')$ is of the form

$$A' = \begin{bmatrix} J & 0 \\ Y & 0 \end{bmatrix} . \tag{2.6}$$

A matrix $A'' \in \mathfrak{A}(R'',S'')$ is of the form

$$A'' = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} . \tag{2.7}$$

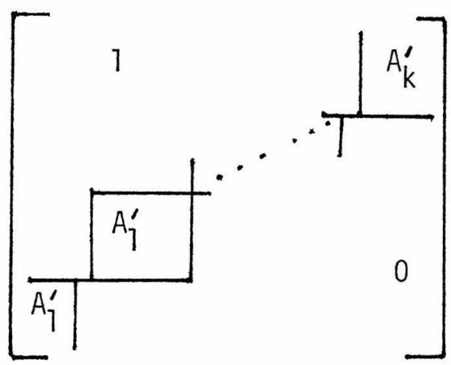
In both cases the sizes of the blocks correspond to those in (2.3). Taking $A = A' + A''$ we obtain a matrix in $\mathfrak{A}(R,S)$ of the form given in (2.3). But then every matrix can be written in this form using the arguments of Theorem 2.1.

Corollary 2.3. <u>If there is an extremal invariant</u> 1 <u>in position</u>

(e,f) <u>then</u> $e = \overline{s}_f$ <u>and</u> (2.4) <u>holds</u>.

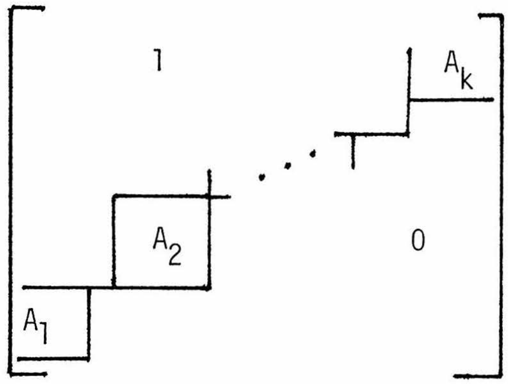
<u>Proof</u>. Use Theorem 2.1 to verify that every matrix $A \in \mathfrak{A}(R,S)$ can be written as in (2.3).

Note that the converse need not hold. The above theorems give us a quick way of determining all invariant 1's and invariant 0's for a given class $\mathfrak{A}(R,S)$. We can determine all the invariant 1's by using Theorem 2.2 and finding for which f (2.4) holds. Then any matrix $A \in \mathfrak{A}(R,S)$ can be written as follows

$A =$ 

. (2.8)

This determines most of the invariant 0's as shown. The remaining invariant 0's consist of the rows of the various blocks A_i $(i=1,2,\ldots,K)$ with row sum 0. This follows since the class associated with each A_i has no invariant 1's and so the only possibility, for invariant 0's is rows of 0's. A column of 0's would already have been taken care of as for example in the columns to the right of A'_1. Thus any matrix $A \in \mathfrak{A}(R,S)$ may be written as follows:

$A =$ 

(2.9)

Of course the invariant 0's could have been determined in the same way as the invariant 1's by looking at the (0,1)-complement of the class. We see that for any class $\mathcal{A}(R,S)$ we may easily determine its invariant 1's and 0's.

In order to examine properties of $\mathcal{A}(R,S)$ we should focus our attention on the classes $\mathcal{A}(R_1,S_1)$, $\mathcal{A}(R_2,S_2)$, ..., $\mathcal{A}(R_k,S_k)$ associated with the matrices $A_1, A_2, \ldots, A_k$ in turn. Each of these classes has no invariant 1's or 0's. Thus if we are able to determine results about matrices without invariant 1's or 0's then we will be able to obtain results about any class $\mathcal{A}(R,S)$.

The Structure Matrix

Ryser has defined an $m+1$ by $n+1$ (0,1)-matrix as follows

$$t_{ef} = ef + (r_{e+1}+r_{e+2}+ \cdots + r_m) - (s_1+s_2 + \cdots + s_f) \tag{2.10}$$

$$(0 \le e \le m\ ,\ 0 \le f \le n).$$

Let $T = (t_{ij})$ be the <u>structure matrix</u> where the index 0 is used. We use the term structure since it gives certain invariants of the class. If we split up a matrix of $\mathfrak{A}(R,S)$ as in (2.2) then

$$t_{ef} = N_0(W) + N_1(Z) \ , \tag{2.11}$$

where $N_i(A)$ is the number of i's in A. We wish to consider the special case where $e = \bar{s}_f$. Examining $\bar{A}$, the Ferrer's matrix, we note that

$$\bar{s}_f \cdot f + r_{\bar{s}_f+1} + r_{\bar{s}_f+2} + \cdots + r_m = \bar{s}_1 + \bar{s}_2 + \cdots + \bar{s}_f . \tag{2.12}$$

Thus we have the following.

Remark 2.4. <u>For</u> $e = \bar{s}_f$, <u>we have</u>

$$t_{ef} = \sum_{i=1}^{f} \bar{s}_i - \sum_{i=1}^{f} s_i \ . \tag{2.13}$$

This observation yields the following result of Ford and Fulkerson very directly. Let T' be the matrix obtained from T by ignoring the 0^{th} row and column.

Theorem 2.5. (Ford and Fulkerson [14]). <u>The class</u> $\mathfrak{A}(R,S)$ <u>is nonempty if and only if</u> $T' \geq 0$.

<u>Proof</u>. Surely if $\mathfrak{A}(R,S)$ is nonempty then $T' \geq 0$ using (2.11). If $T \geq 0$ using (2.11). If $T \geq 0$ then the conditions (1.4) of the Gale-Ryser Theorem are satisfied by Remark 2.4. Thus $\mathfrak{A}(R,S)$ is nonempty.

Note how the structure matrix ties in nicely with the concept of invariant 1's. The positions (e,f) where T' is zero correspond to the positions (e,f) in the decomposition of (2.1) without imposing any restrictions on X and Y as described in Theorem 2.1. Thus determining the structure matrix allows one to determine the pattern of invariant 1's and 0's as in (2.9).

Submatrices.

A reasonable question to ask is whether there exists a matrix $A \in \mathfrak{A}(R,S)$ with a given matrix B as a submatrix of A in certain rows and columns. For example, if $\mathfrak{A}(R,S)$ is of size m by n and B is of size m by 1 then we are asking whether a given column is possible. We start by considering this case.

Let α be a column of 0's and 1's with $\alpha = (a_1,a_2, \ldots, a_m)^T$. We define a column α to be a <u>possible</u> k^{th} column if there exists a matrix $A \in \mathfrak{A}(R,S)$ with α as its k^{th} column. For this to make sense, we require that the number of 1's in α be s_k. Let $R' = (r_1',r_2', \ldots, r_m')$ be defined by $r_i' = r_i - a_i$. Let $S' = (s_1',s_2', \ldots, s_{n-1}')$ be defined by $s_i' = s_i$ for $i < k$ and $s_i' = s_{i+1}$ for $i \geq k$.

Remark 2.6. <u>Column</u> α <u>is a possible</u> k^{th} <u>column if and only if</u> $\mathfrak{A}(R',S')$ <u>is nonempty</u>.

<u>Proof</u>. Note that the existence of a matrix $A \in \mathfrak{A}(R,S)$ with α as its k^{th} column is equivalent to the existence of a matrix $A' \in \mathfrak{A}(R',S)$ which can be obtained from A by deleting the k^{th} column.

Let β be another column of 0's and 1's with

$\beta = (b_1, b_2, \ldots, b_m)^T$ and β has s_k 1's. We now need to assume that R and S are monotone for the following definition. We define a partial order on columns of column sum s_k. Let $\alpha \leq \beta$ if the i^{th} 1 of α (from the top) is in a higher row, say j, than the i^{th} 1 of β which is in row ℓ. Then $r_j \geq r_\ell$. We obtain:

Remark 2.7. <u>If β is a possible k^{th} column and $\alpha \leq \beta$ then α is also a possible k^{th} column.</u>

<u>Proof</u>. Let $R' = (r_1', r_2', \ldots, r_m')$ and $R'' = (r_1'', r_2'', \ldots, r_m'')$ with $r_i' = r_i - a_i$ and $r_i'' = r_i - b_i$. Let S' be as defined above. Consider the conjugate sequences $(\bar{s}_i')$ and $(\bar{s}_i'')$ obtained from the sequences (r_i') and (r_i'') respectively. If we let d_i (respectively e_i) be the number of 1's of α (respectively β) in rows j with $r_j = i$ then we discover that

$$\bar{s}_i' = \bar{s}_i - d_i \quad ; \quad \bar{s}_i'' = \bar{s}_i - e_i, \qquad (1 \leq i \leq n-1) \quad (2.14)$$

where the sequence $(\bar{s}_i)$ is the conjugate of the sequence (r_i). Now if $\alpha \leq \beta$ then we have that

$$\sum_{i=1}^{t} d_i \leq \sum_{i=1}^{t} e_i \qquad (1 \leq t \leq n-1). \quad (2.15)$$

and so for $1 \leq t \leq n-1$ we have

$$\sum_{i=1}^{t} \bar{s}_i' = \sum_{i=1}^{t} (\bar{s}_i - d_i) \geq \sum_{i=1}^{t} (\bar{s}_i - e_i) = \sum_{i=1}^{t} \bar{s}_i'' \geq \sum_{i=1}^{t} s_i' \qquad . \quad (2.16)$$

The last inequality follows from the fact that β is a possible k^{th} column using Remark 2.6 and the Gale-Ryser theorem. Similarily (2.19) yields that α is a possible k^{th} column, completing the proof.

It would be nice to know the maximal possible k^{th} columns using Remark 2.7. We might be able to use this to generate all matrices in $\mathfrak{A}(R,S)$ in an inductive way. Remarkably we can show that there is a maximum possible k^{th} column. Let α and β be possible k^{th} columns. Consider the column $\alpha \vee \beta$ obtained in the partial ordering. If α has its i^{th} 1 (from the top) in row p and β has its i^{th} 1 in row q then $\alpha \vee \beta$ has its i^{th} 1 in row $u = \max(p,q)$.

Remark 2.8. _If_ α _and_ β _are possible_ k^{th} _columns then so is_ $\alpha \vee \beta$.

Proof. We will use the notation of the previous remark. Let $R - (\alpha \vee \beta)^T = \tilde{R} = (\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_m)$. Let the sequence $(\tilde{s}_i)$ be the conjugate of the sequence $(\tilde{r}_i)$. Let f_i be the number of 1's of $\alpha \vee \beta$ in rows j with $r_j = i$.

Assume that the row of largest index j, with $r_j = t$ and with $\alpha \vee \beta$ having a 1 in that row, is row p. Then either $a_p = 1$ or $b_p = 1$. Say $a_p = 1$. Then

$$\sum_{i=1}^{t} f_i = \sum_{i=1}^{t} d_i \quad . \tag{2.17}$$

But then

$$\sum_{i=1}^{t} \tilde{s}_i = \sum_{i=1}^{t} (\bar{s}_i - f_i) = \sum_{i=1}^{t} (\bar{s}_i - d_i) = \sum_{i=1}^{t} \bar{s}_i' \geq \sum_{i=1}^{t} s_i \quad . \tag{2.18}$$

A similar argument works when $b_p = 1$. Thus $\alpha \vee \beta$ is a possible k^{th} column using Remark 2.6.

We can give an algorithm for finding this maximum column. For $t = 1,2, \ldots, n-1$, let

$$\sum_{i=1}^{t} \overline{s}_i - \sum_{i=1}^{t} s_i' = d_t' \quad , \tag{2.19}$$

and then define

$$d_t = \min d_i' \qquad (i+t, t+1, \ldots, n-1). \tag{2.20}$$

Form a column γ by placing 1's in those rows with the largest indices (and smallest row sums r_i) subject to the conditions that there are no more than s_k 1's in γ and that there are no more than d_t 1's in rows j with $r_j \leq t$. Then Remark 2.6 and our other comments ensure that γ is the desired maximum column.

The maximum column need not have its 1's in those rows with smallest row sums;i.e.,not all columns of column sum s_k need be possible k^{th} columns. This is true, even in a class without invariant 1's and for the last column with smallest column sum. For example let $R = (3,3,3,1,1,1)$ and $S = (5,3,2,2)$ then the column $(0,0,0,1,0,1)^T$ is the maximum fourth column.

The minimal column has its 1's in the s_k highest rows, i.e. those with the largest row sums. As long as $\mathfrak{A}(R,S)$ is nonempty, we deduce that the minimal column is a possible k^{th} column using Remark 2.7. Ford and Fulkerson used this idea to give an algorithm which will generate a matrix in $\mathfrak{A}(R,S)$, if one exists [14]. It is called the

"(0,1)-matrix rule".

<u>Algorithm</u>. Select any column, say k, and place s_k 1's in those rows with the largest row sums. Repeat in the reduced problem.

Ford and Fulkerson's algorithm need not generate an arbitrary matrix in $\mathfrak{A}(R,S)$, as the following example demonstrates. Let $R = S = (4,3,3,3,2,2,2)$ and consider the following matrix in $\mathfrak{A}(R,S)$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{2.21}$$

One may verify that no column has its 1's in rows with the largest row sums. Since A is symmetric, the same is true for the rows.

Ford and Fulkerson's (0,1)-matrix rule yields an alternate means of testing whether $\mathfrak{A}(R,S)$ is nonempty or not apart from the Gale-Ryser theorem. It yields a matrix in $\mathfrak{A}(R,S)$ if one exists. A slight specialization of this algorithm yields the matrix $\tilde{A}$ as described by Fulkerson and Ryser [17] as the matrix with minimal α-widths for any α.

The (0,1)-matrix rule can be used to answer the question of whether there exists a matrix $A \in \mathfrak{A}(R,S)$ with B as a submatrix when B has fewer then m rows and n columns. Let $B \in \mathfrak{A}(R',S')$ where B is of size p by q. We reorder R and S so that B is to appear in the last p rows and q columns. Decompose any matrix $A \in \mathfrak{A}(R,S)$ into blocks as follows

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} , \qquad (2.22)$$

where Z is of size p by q. We wish to know whether there is an $A \in \mathfrak{A}(R,S)$ with $Z = B$. Define B to be a <u>possible</u> submatrix of $\mathfrak{A}(R,S)$ if such a matrix A exists. We note that if B is possible and $B' \in \mathfrak{A}(R',S')$, then B' is possible.

Assume such an A exists. Consider the class associated with $[W\ X]$. Using the (0,1)-matrix rule one can fill in the columns of X using the row sums of $[W\ X]$ and the column sums of X, and obtain some matrix X'. The (0,1)-matrix rule guarantees that this partial solution can be extended to a full matrix $[W'X']$ in the same class as $[W\ X]$. But the algorithm only uses the row sums of $[W\ X]$ and the column sums of Y and so is independent of which matrix $A \in \mathfrak{A}(R,S)$ with $Z = B$ that has been chosen. We obtain the following.

Remark 2.9. <u>The matrix B is a possible submatrix of $\mathfrak{A}(R,S)$ if and only if there is matrix $A \in \mathfrak{A}(R,S)$ with $X = X'$ and $Z = B$ in the decomposition</u> (2.22).

To determine whether such an A exists simply determine the row and column sums of the remaining m by $n - q$ block and use the Gale-Ryser theorem. This is our procedure to check whether B is a possible submatrix of $\mathfrak{A}(R,S)$. This result might be useful in the case $B = 0$. Looking for possible large zero blocks is equivalent to computing $\tilde{p}$, the minimum term rank of all matrices in $\mathfrak{A}(R,S)$. We will discuss this in more detail in Section 5.

The theorems of the following sections concern themselves with the existence of a matrix $A \in \mathfrak{A}(R,S)$ with $A \geq B$ for some matrix B. This problem is similar to the above problem and in certain cases we get strong results. The existence conditions that we obtain are much simpler than the algorithms we have just presented.

Section 3. Generalization of the Gale-Ryser theorem.

Our next theorem generalizes the Gale-Ryser theorem as well as a theorem of Fulkerson [16]. It gives necessary and sufficient conditions for there to exist an $A \in \mathfrak{A}(R,S)$ with $A \geq P$ for some specified matrix P. We do not assume that $\mathfrak{A}(R,S)$ is nonempty.

Define P to be an <u>acceptable</u> matrix if it is an m by n (0,1)-matrix with column sums at most 1. In addition, assume that no row (column) sum of P is greater than the corresponding row (column) sum of the class $\mathfrak{A}(R,S)$. Let A^* be the m by n (0,1)-matrix with row sum vector R under the following constraints. A^* has 1's wherever P has 1's and the remaining 1's are as far to the left as possible. We will call A^* the <u>altered Ferrer's</u> matrix. We have $A^* \geq P$. Define s_i^* to be the i^{th} column sum of A^*. Call the sequence (s_i^*) the P-<u>required conjugate</u> of the sequence (r_i).

Theorem 3.1. <u>Let</u> P <u>be an acceptable matrix. There exists an</u> $A \in \mathfrak{A}(R,S)$ <u>with</u> $A \geq P$ <u>if and only if</u>

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n), \tag{3.1}$$

<u>where the sequence</u> (s_i^*) <u>is the</u> P-<u>required conjugate of the sequence</u> (r_i).

<u>Proof</u>. Assume there exists an $A \in \mathfrak{A}(R,S)$ with $A \geq P$. Then (3.1) holds because A can be obtained from the altered Ferrer's matrix A^* by shifting 1's to the right.

Assume (3.1) holds and that P is an acceptable matrix. We will construct an $A \in \mathfrak{A}(R,S)$ with $A \geq P$ following the spirit of Fulkerson-Ryser proof of the Gale-Ryser theorem. Let a 1 in a matrix be <u>free</u> if it is not in the same position as a 1 of P. We will start with A^* and shift free 1's to the right until we obtain a matrix $A \in \mathfrak{A}(R,S)$ using the following algorithm.

<u>Algorithm</u>. Start with the matrix A^*. Fill in columns $n, n-1, \ldots, 2$ in turn so that each has the required column sum given by S. If the r^{th} column does not have s_r 1's then do the following. Among those rows which have a zero in the r^{th} column, choose one which has a free 1 in column p, where p is as large as possible and $1 \leq p < r$. Shift the 1 from column p to column r. Repeat this process until column r has s_r 1's.

Assuming that the algorithm never results in too many 1's in the r^{th} column or has no available free 1's to shift, the process will terminate in a matrix $A \in \mathfrak{A}(R,S)$ with $A \geq P$. This is because the matrix we started with has the proper row sums and we only shift free 1's to the right.

We will establish that the algorithm does not fail by proving the following claim in an inductive way. Let C be a matrix at some stage in the process and let it have column sums c_i. We claim that

$$\sum_{i=1}^{t} c_i \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n). \tag{3.2}$$

Certainly, if C is the first matrix A^* in the algorithm then the inequalities of (3.2) hold by assumption. Otherwise, let B be the matrix just before C in the algorithm, where C is obtained from B by shifting a 1 from column p to column r. Let B have column sums b_i. We assume by induction that

$$\sum_{i=1}^{t} b_i \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n). \tag{3.3}$$

Since column i of C is the same as column i of B for $1 \leq i < p$ and $r < i \leq n$, we have that

$$\sum_{i=1}^{t} c_i = \sum_{i=1}^{t} b_i \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t < p;\ r \leq t \leq n). \tag{3.4}$$

Now we claim that

$$\sum_{i=1}^{t} b_i > \sum_{i=1}^{t} s_i \qquad (p \leq t < r). \tag{3.5}$$

We have that $b_r < s_r$ since we are moving a 1 to column r. We claim that $b_q \leq s_q$ for $p < q < r$. This follows for two reasons. First, S is monotone so $s_q \geq s_r$ for $p < q < r$. Second, $b_q \leq b_r + 1$ for $p < q < r$ since the first free 1 in the same row as a 0 in column r is in column p and not in column q. Also there is at most one 1 of P that is in column q, because P is acceptable, and it might occur in the same row as a 0 in column r. Combining, we obtain $b_q \leq b_r + 1 \leq s_r \leq s_q$ which yields our claim. Using our claim and considering the effect of shifting a 1 from column p to column r, we obtain that

$$\sum_{i=1}^{t} c_i \geq \sum_{i=1}^{t} s_i \qquad (p \leq t < r). \tag{3.6}$$

Combining with (3.4) we prove our claim (3.2).

We may now verify that the algorithm works. Say we have obtained a matrix B as above with $b_n = s_n$, $b_{n-1} = s_{n-1}$, $\ldots$, $b_{r+1} = s_{r+1}$. Letting $t = r$ in (3.3), we obtain $b_r \leq s_r$. Thus there are never too many 1's in a column as we prepare to fill it. If $b_r < s_r$, there must be a free 1 in one of the first $r - 1$ columns in the given rows. Otherwise, the arguments above tell us that $b_q \leq s_q$ for $1 \leq q < r$ and this contradicts the fact that $s_1 + s_2 + \cdots + s_n = b_1 + b_2 + \cdots + b_n$. This completes the proof.

Corollary 3.2 (Gale, Ryser). Theorem 1.1.

<u>Proof</u>. We verify that in (1.3) $\bar{s}_i = s_i^*$ when $P = 0_{mn}$.

Before going on, we will look at some examples of our theorem. Let $R = (3,2,2,1)$ and $S = (2,2,2,2)$. If we define $P = I$, we verify that P is acceptable and determine A^* to be the following

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

The column sums s_i^* of A^* satisfy (3.1) and so there exists an $A \in \mathfrak{A}(R,S)$ with $A \geq P$. The algorithm tells us to move the 1 in position (1,3) of A^* to position (1,4) to give column 4 two 1's. Also one moves a 1 from (1,2) to (1,3) and a 1 from (3,1) to (3,2) to obtain the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{3.8}$$

The matrix $A \in \mathfrak{A}(R,S)$ and $A \geq P$ as desired. The algorithm can be modified to generate an arbitrary matrix $A \in \mathfrak{A}(R,S)$ with $A \geq P$.

Another example shows that the restriction that P have column sums at most 1 cannot be removed without other alterations. Let $R = (2,1,1)$ and $S = (2,2)$ and let P be as follows

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} . \tag{3.9}$$

Now P is acceptable apart from a column sum of 2 and (3.1) is satisfied. Yet there is no matrix $A \in \mathfrak{A}(R,S)$ with $A \geq P$.

We can easily obtain an equivalent theorem which gives necessary conditions for there to exist an $A \in \mathfrak{A}(R,S)$ with A avoiding P, i.e. $A + P \leq J$, for some specified P. Define P to be an <u>avoidable</u> matrix if it is an m by n (0,1)-matrix with column sums at most 1. In addition if row i (column j) of P has row sum (column sum) a, then $r_i \leq n - a$ ($s_j \leq m - a$). Let A^{**} be the m by n (0,1)-matrix with row sum vector R under the following constraints. A^{**} has 0's wherever P has 1's and subject to this condition, the 1's of A^{**} are as far to the left as possible. Let s_i^{**} denote the i^{th} column sum of A^{**}. Call the sequence (s_i^{**}) the P-<u>restricted</u> <u>conjugate</u> of the sequence (r_i).

Theorem 3.3. *Let P be an avoidable matrix. Then there exists an $A \in \mathfrak{A}(R,S)$ with $A + P \leq J$ if and only if*

$$\sum_{i=1}^{t} s_i^{**} \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n) \tag{3.10}$$

where the sequence (s_i^{**}) is the P-restricted conjugate of the sequence (r_i).

Proof. Let P have i^{th} row sum r_i'' and j^{th} column sum s_j''. Define $R' = (r_1', r_2', \ldots, r_m')$ and $S = (s_1', s_2', \ldots, s_n')$ as follows

$$r_i' = r_i + r_i'' \quad ; \quad s_i' = s_i + s_i'' \tag{3.11}$$

Now there exists an $A \in \mathfrak{A}(R,S)$ with $A + P \leq J$ if and only if there exists a $B \in \mathfrak{A}(R',S')$ with $B \geq P$. If such an A exists, take $B = A + P$. If such a B exists, take $A = B - P$. We can now use Theorem 3.1. Such a B exists if and only if

$$\sum_{i=1}^{t} s_i^{*} \geq \sum_{i=1}^{t} s_i' \qquad (1 \leq t \leq n) \tag{3.12}$$

where the sequence (s_i^{*}) is the P-required conjugate of the sequence

(r_i'). But we find that $s_i^{*} = s_i^{**} + s_i''$ and $s_i' = s_i + s_i''$. Thus (3.9) holds if and only if (3.7) holds and this proves the theorem.

Corollary 3.4. (Fulkerson [16]). *There exists an* $A \in \mathfrak{A}(R,S)$, *where* $\mathfrak{A}(R,S)$ *is of order* n, *with* $\mathrm{tr}(A) = 0$ *if and only if*

$$\sum_{i=1}^{t} s_i^{**} \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n), \tag{3.13}$$

where the sequence (s_i^{**}) *is the* I-*restricted conjugate of the sequence* (r_i).

The above result has a nice graph theoretic interpretation. Square (0,1)-matrices with 0's down the diagonal are in a 1-1 correspondence with directed graphs, where no loops or multiple directed edges are allowed. Thus Corollary 3.4 gives simple conditions for the existence of such digraphs with specified indegrees and outdegrees.

Another easy application of our theorems is the determination of the minimal and maximal trace for $\mathfrak{A}(R,S)$ where $\mathfrak{A}(R,S)$ is of order n. We define

$$\tilde{\sigma} = \min_{A \in \mathfrak{A}(R,S)} \mathrm{tr}(A) \quad ; \quad \tilde{\sigma} = \min_{A \in \mathfrak{A}(R,S)} \mathrm{tr}(A) . \tag{3.14}$$

Ryser has determined simple formulae for $\tilde{\sigma}$ and $\tilde{\sigma}$ using the structure matrix. We determine alternate simple formulae using the following theorem of Ryser [33].

Theorem 3.5 (Ryser). *If* $A \in \mathfrak{A}(R,S)$ *has trace* σ *then there exists a matrix* $A' \in \mathfrak{A}(R,S)$ *with trace* σ *and the* σ 1's *occuring* in positions $(1,1)$, $(2,2)$, ..., (σ,σ).

Let L_i denote the (0,1)-matrix of order n with 1's in positions $(1,1)$, $(2,2)$, ..., (i,i) and 0's elsewhere. Then by Theorem 3.5,

$$\begin{aligned} \tilde{\sigma} &= \min i \mid \exists\, A \in \mathfrak{A}(R,S) \text{ with } A + (I-L_i) < J, \\ \bar{\sigma} &= \max i \mid \exists\, A \in \mathfrak{A}(R,S) \text{ with } A \geq L_i . \end{aligned} \tag{3.15}$$

The computations in (3.15) are easily carried out using Theorem 3.3 and Theorem 3.1.

A similar result applies to the computation of the maximal term rank using a result proven by Ryser which is analogous to Theorem 3.5 [34]. We follow his proof to prove the following lemma.

Lemma 3.6. <u>Let there exist an</u> $A \in \mathfrak{A}(R,S)$ <u>with</u> $A \geq P$, <u>where the submatrix of</u> P <u>given by rows</u> $i_1, i_2, \ldots, i_r$ <u>and columns</u> $j_1, j_2, \ldots, j_r$ <u>is a permutation matrix and the rest of</u> P <u>is</u> 0's. <u>Assume</u> $i_1 \geq i_2 \geq \cdots \geq i_r$ <u>and</u> $j_1 \geq j_2 \geq \cdots \geq j_r$. <u>Then there exists a matrix</u> $B \in \mathfrak{A}(R,S)$ <u>with</u> 1's <u>in positions</u> $(i_1,j_r),(i_2,j_{r-1}), \ldots, (i_r,j_1)$.

<u>Proof</u>. Let the 1's of A which are the 1's of the permutation submatrix be called essential. We wish to move by interchanges these essential 1's until they lie in the required positions. Assume that the essential 1's are already in positions (i_1,j_r), (i_2,j_{r-1}), ..., (i_d,j_{r-d+1}) but there is no essential 1 in position (i_{d+1}, j_{r-d}). Then the essential 1's in row i_{d+1} and column j_{r-d} determine a

2 by 2 submatrix of one of the following forms

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{3.16}$$

The 1's on the main diagonals are the essential 1's. Using the monotonocity of R and S, we see that at most one interchange is required to place 1's on the off diagonal positions. These we can now define to be essential 1's and so we have placed an essential 1 in position (i_{d+1}, j_{r-d}). Continuing in this way, we obtain the desired matrix B.

This gives us a theorem concerning the existence of a matrix covering a permutation matrix where part of the permutation matrix is specified. Let Q be a (0,1)-matrix of order n with at most one 1 per row and per column. Let rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_r$ have row sum, respectively, column sum zero. Assume $i_1 \geq i_2 \geq \cdots \geq i_r$ and $j_1 > j_2 \geq \cdots \geq j_r$. Define Q' to be the permutation matrix with 1's in those positions where Q has 1's and 1's in positions $(i_1, j_r), (i_2, j_{r-1}), \ldots, (i_r, j_1)$. Using Lemma 3.6 we obtain the following.

Theorem 3.7. _There exists a matrix_ $A \in \mathfrak{A}(R,S)$ _with_ $A \geq P$ _for some permutation matrix_ P _with_ $P \geq Q$ _if and only if there exists a matrix_ $A' \in \mathfrak{A}(R,S)$ _with_ $A' \geq Q'$.

Proof. Simply note that the interchanges that place the 1's in positions $(i_1, j_r), (i_2, j_{r-1}), \ldots, (i_r, j_1)$ in Lemma 3.6 do not affect the 1's of Q.

We note that Theorem 3.1 provides an easy test for determining the existence of an $A' \in \mathfrak{A}(R,S)$ with $A' \geq Q'$. In the class $\mathfrak{A}(R,S)$, this solves the existence question of an SDR where part of the SDR is already specified.

Let M be the (0,1)-matrix of order n with 1's in positions $(1,n)$, $(2,n-1)$, ..., $(n,1)$. In the notation of Theorem 3.7, $M = 0'$. Lemma 3.6 can be specialized to the following result.

Theorem 3.8. _If there is an_ $A \in \mathfrak{A}(R,S)$ _and a permutation matrix_ P _with_ $A \geq P$ _then there exists a matrix_ $A' \in \mathfrak{A}(R,S)$ _with_ $A' \geq M$.

There is a simple analogue for the identity matrix.

Theorem 3.9. _If there is an_ $A \in \mathfrak{A}(R,S)$ _with_ $A \geq I$ _then, for any permutation matrix_ P, _there exists a matrix_ $A' \in \mathfrak{A}(R,S)$ _with_ $A' \geq P$.

Proof. Applying Theorem 3.1 for $P = I$ we have

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n) \tag{3.17}$$

Consider an arbitrary permutation matrix P. Let the column sums of the associated altered ferrers matrix A^* be p_i^* to avoid confusion. We will show that

$$\sum_{i=1}^{t} p_i^* \geq \sum_{i=1}^{t} s_i^* \qquad (1 \leq t \leq n) \tag{3.18}$$

We will do this by considering how the altered Ferrer's matrices for

P and I can be obtained from the Ferrer's matrix $\bar{A}$. Consider the sum of the first t column sums in each case. The sums for the altered Ferrer's matrices have been reduced by shifting the rightmost 1's from rows with row sum at most t to one of the last $n - t$ columns. At most $n - t$ such rows can be affected in each case, since P and I are permutation matrices. Now

$$\sum_{i=1}^{t} s_i^* - \sum_{i=1}^{t} \bar{s}_i \tag{3.19}$$

will be the minimum of $n - t$ and the number of rows with row sum at most t. Thus (3.19) is as large as possible and so (3.18) holds. Combining with (3.17) and Theorem 3.1 we deduce that there exists an $A' \in \mathfrak{A}(R,S)$ with $A' \geq P$.

Thus M and I can be regarded as the easiest and the most difficult, respectively, of permutation matrices to be covered by a matrix in $\mathfrak{A}(R,S)$.

Another problem concerns the existence of a matrix $A \in \mathfrak{A}(R,S)$ whose entries in the positions given by some matrix P are completely specified by some matrix Q. For example, if P is a block of 1's then this would be the submatrix problem of Section 2. We consider a different case. Let $P = (p_{ij})$ and $Q = (q_{ij})$ by m by n (0,1)-matrices with $P \geq Q$. Let $Q \in \mathfrak{A}(R',S')$ be an acceptable matrix in $\mathfrak{A}(R,S)$ and let P be an avoidable matrix in $\mathfrak{A}(R-R',S-S')$. Then we have the following

Theorem 3.10. <u>There exists a matrix</u> $A \in \mathfrak{A}(R,S)$ <u>with</u> $A = (a_{ij})$ <u>and</u> $a_{ij} = q_{ij}$ <u>for those pairs for which</u> $P_{ij} = 1$ <u>if and only if there</u>

exists a matrix $A' \in \mathfrak{A}(R-R',S-S')$ with $A' + P \leq J$.

Proof. If such an A' exists then take $A = A' + Q$ and A has the desired properties. Similarly if A has the desired properties then the matrix $A' = A - Q$ has $A' \in \mathfrak{A}(R-R',S-S')$ and $A' + P \leq J$.

We can use Theorem 3.3 to test the second condition. Note that we cannot test directly in the manner of Theorem 3.1 or Theorem 3.3. Consider the following example. Let $R = (1,1) = S$ and let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{3.20}$$

Let A^{***} be the 2 by 2 (0,1)-matrix with row sums given by R as follows. The 1's are as far to the left as possible subject to the condition that in those positions where P has 1's, A^{***} has the same entries as those of Q. Then let A^{***} have column sums s_i^{***} and these will satisfy the analogue of (3.1) or (3.10). Yet there is no matrix $A \in \mathfrak{A}(R,S)$ of the form desired in Theorem 3.10.

Section 4. The class $\mathfrak{A}_p(R,S)$

Let P be an acceptable matrix for $\mathfrak{A}(R,S)$. Define the class $\mathfrak{A}_p(R,S)$ to be the set of matrices $A \in \mathfrak{A}(R,S)$ with $A \geq P$. We introduce this class in the hope that some of the properties of $\mathfrak{A}(R,S)$ will carry over to $\mathfrak{A}_p(R,S)$. Note that Theorem 3.1 which determines whether $\mathfrak{A}_p(R,S)$ is nonempty is virtually identical to the Gale-Ryser theorem which determines whether $\mathfrak{A}(R,S)$ is nonempty. Our first result gives an extension of the algorithm in Theorem 3.1 to generate an arbitrary matrix in $\mathfrak{A}_p(R,S)$. Our second result extends Ford and

Fulkerson's (0,1)-matrix rule to $\mathcal{O}\!\mathcal{L}_p(R,S)$. Our third result gives the new version of the Interchange theorem in $\mathcal{O}\!\mathcal{L}_p(R,S)$. We consider $\mathcal{O}\!\mathcal{L}_p(R,S)$ a promising area for further research. In particular, can the definition of acceptable be extended?

It is possible to generate all matrices in $\mathcal{O}\!\mathcal{L}_p(R,S)$. Consider the following algorithm.

Algorithm. Start with the altered ferrer's matrix A^*.

Fill in columns $n, n-1, \ldots, 2$ in turn so that each has the required column sum given by S. If the r^{th} column does not have s_r 1's, then do the following. Among those rows which have a zero in the r^{th} column, shift a free 1 from one of the first $r-1$ columns to column r. Do this only if in the resulting matrix C, with i^{th} column sum c_i, we have

$$\sum_{i=1}^{t} c_i \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n) . \tag{4.1}$$

Repeat this until column r has s_r 1's.

We have proven that, as long as the inequalities of the form (4.1) are satisfied, there will be a free 1 available to shift and in addition, no column ends up with too many 1's. Thus the algorithm will terminate with a matrix $A \in \mathcal{O}\!\mathcal{L}_p(R,S)$. In fact the matrix A will be an arbitrary element of $\mathcal{O}\!\mathcal{L}_p(R,S)$. Consider any matrix $A \in \mathcal{O}\!\mathcal{L}_p(R,S)$. It can be obtained from A^* by shifting free 1's to the right as described in the algorithm. Certainly the inequalities (4.1) will be satisfied at any intermediate stage since they are satisfied at the final stage $A \in \mathcal{O}\!\mathcal{L}_p(R,S)$ when they become equalities. By trying every possibility when a choice is given to you in the algorithm, one could generate all the matrices in $\mathfrak{A}_p(R,S)$. The only reason to preserve the algorithm in Theorem 3.1 as given is that the algorithm is more direct and, suitably phrased, the matrix A that is produced might have interesting properties such as the minimal α-widths given by the $\tilde{A}$ of Fulkerson and Ryser [17].

Let the <u>last free</u> 1 of row i of a matrix A, with $A \geq P$, be the free 1 which occurs as far to the right as possible i.e. in column j where j is as large as possible. The following lemma considers possible columns as was done in Section 2 for $\mathfrak{A}(R,S)$. Let α be a column of 0's and 1's with $\alpha = (a_1, a_2, \ldots, a_m)^T$ such that α has s_k 1's and 1's wherever the k^{th} column of P has 1's. Then we define α to be a <u>possible</u> k^{th} column of $\mathfrak{A}_p(R,S)$ if there is a matrix $A \in \mathfrak{A}_p(R,S)$ with α being the k^{th} column. Here is an analogue of Remark 2.7.

Lemma 4.1. <u>Let</u> β <u>be a possible</u> k^{th} <u>column of</u> $\mathfrak{A}_p(R,S)$. <u>Let</u> α <u>be a column obtained from</u> β <u>by replacing a</u> 0 <u>in row</u> i <u>by a</u> 1 <u>and replacing a</u> 1 (<u>not a</u> 1 <u>of the</u> k^{th} <u>column of</u> P) <u>in row</u> j <u>by a</u> 0 <u>where the last free</u> 1 <u>in row</u> i <u>of</u> A^* <u>is at least as far to the left as the last free</u> 1 <u>in row</u> j <u>of</u> A^*. <u>Then</u> α <u>is a possible</u> k^{th} <u>column of</u> $\mathfrak{A}_p(R,S)$.

<u>Proof</u>. Let $S' = (s'_1, s'_2, \ldots, s'_{n-1})$ with $s'_i = s_i$ for $i < k$ and $s'_i = s_{i+1}$ for $i \geq k$. Let $R'' = (r''_1, r''_2, \ldots, r''_m) = R - \beta^T$. Let P' be the matrix of size m by $n-1$ obtained from P by deleting the k^{th} column. We note that P' is acceptable in $\mathfrak{A}(R'',S')$. Let the

sequence $(\overline{s}_i'')$ be the P'-required conjugate of the sequence (r_i''). In analogy to Remark 2.6, β is a possible k^{th} column if and only if there exists an $A' \in \mathfrak{A}(R'',S')$ with $A' \geq P'$. By Theorem 3.1, this is true if and only if

$$\sum_{i=1}^{t} \overline{s}_i'' \geq \sum_{i=1}^{t} s_i' \qquad (1 \leq t \leq n-1). \qquad (4.2)$$

Let $R' = (r_1', r_2', \ldots, r_m') = R - \alpha^T$, Let the sequence $(\overline{s}_i')$ be the P'-required conjugate of the sequence (r_i'). From the derivation of α, we obtain that

$$\sum_{i=1}^{t} \overline{s}_i' \geq \sum_{i=1}^{t} \overline{s}_i'' \qquad (1 \leq t \leq n-1). \qquad (4.3)$$

Combining this with (4.2) and using Theorem 3.1, we deduce that there exists an $A' \in \mathfrak{A}(R',S')$ with $A' \geq P'$. Thus α is a possible k^{th} column for $\mathfrak{A}_p(R,S)$.

This leads directly to an analogue of the (0,1)-matrix rule.

Theorem 4.2. _Let_ α _be a column of_ m _entries with_ s_k _1's. Let_ α _have 1's where the_ k^{th} _column of_ P _has 1's and place the remaining 1's in those rows whose last free 1 in_ A^* _is furtherst to the right. Then_ α _is a possible_ k^{th} _column for_ $\mathfrak{A}_p(R,S)$ _if_ $\mathfrak{A}_p(R,S)$ _is nonempty._

Proof. If $\mathfrak{A}_p(R,S)$ is nonempty then take a matrix $A \in \mathfrak{A}_p(R,S)$. Let β be its k^{th} colunn and so β is a possible k^{th} column. Repeated applications of Lemma 4.1 yield that α is a possible k^{th} column.

Further work would enable one to prove analogues of the results in Section 2 on submatrices.

It is possible to get a version of the interchange theorem in the new class $\mathcal{O}\!\mathcal{L}_p(R,S)$. We will use the techniques of Brualdi and Ross [12]. Let $A,B \in \mathcal{O}\!\mathcal{L}_p(R,S)$. Define $d(A,B)$ to be the number of -1's in A-B. This corresponds to a distance. We wish to determine whether interchanges are sufficient to get from A to B and yet remain at each stage in the class $\mathcal{O}\!\mathcal{L}_p(R,S)$. Assume that there is no interchange which will reduce $d(A,B)$ and that $d(A,B) > 0$. Let $P = (p_{ij})$, $A = (a_{ij})$, and $B = (b_{ij})$. Consider the following cases.

<u>CASE 1</u>. Assume A and B have corresponding 2×2 submatrices of the form

$$\text{i)} \begin{bmatrix} 0 & 1 \\ 1 & a_{ij} \end{bmatrix}, \quad \text{ii)} \begin{bmatrix} 1 & 0 \\ 0 & b_{ij} \end{bmatrix} . \tag{4.4}$$

If $a_{ij} = 0$, then an interchange in i) will reduce $d(A,B)$. We note that, since $A,B \in \mathcal{O}\!\mathcal{L}_p(R,S)$, all the positions except the lower right cannot be places where P has 1's. If $b_{ij} = 1$ and $p_{ij} = 0$ then an interchange in ii) will reduce $d(A,B)$. Thus we conclude that $a_{ij} = 1$ and $b_{ij} = 0$ or $a_{ij} = b_{ij} = p_{ij} = 1$.

<u>CASE 2</u>. Assume A,B have corresponding 2×2 submatrices of the form

$$\text{iii)} \begin{bmatrix} 1 & 0 \\ 0 & a_{ij} \end{bmatrix}, \quad \text{iv)} \begin{bmatrix} 0 & 1 \\ 1 & b_{ij} \end{bmatrix}. \tag{4.5}$$

Using arguments similar to Case 1, we conclude that $a_{ij} = 0$ and $b_{ij} = 1$

or $a_{ij} = b_{ij} = p_{ij} = 1$.

Let column k of A-B have t -1's where t is as large as possible. Then column k has t 1's since A-B has row and column sums all zero. Let the -1's be in positions $(j_1,k), (j_2,k), \ldots, (j_t,k)$ and let the +1's be in positions $(\ell_1,k),(\ell_2,k), \ldots, (\ell_t,k)$. Since the j_1 st row of A-B has row sum zero, we deduce that there is a column h with the (j_1,h) position of A-B being 1 and so the (j_1,h) position of A being 1 and the (j_1,h) position of B being 0. Using Case 1, we conclude that either that the (ℓ_i,h) positions of A and B are 1 and 0 respectively or that the (ℓ_i,h) positions of A, B, and P are all 1 (for $1 \leq i \leq t$).

If the former case always occurred, then column h of A-B would have $t+1$ 1's and so $t+1$ -1's which is impossible by assumption. Recalling that P has column sums at most 1, we deduce that in some ordering of the indices $\ell_1, \ell_2, \ldots, \ell_t$ we have that the (ℓ_i,h) positions of A and B are 1 and 0 respectively for $2 \leq i \leq t$ and that the (ℓ_1,h) positions of A,B, and P are all 1. Now consider the t -1's of column h. Let position (p,h) of A-B be -1. Then by Case 2, looking at the 2 by 2 array joined by columns k and h and rows j_1 and p, we deduce that the (p,k) position of A-B is -1 or the (p,k) positions of A,B, and P are all 1. Thus we must have the (j_i,h) positions of A-B or be -1 for $2 \leq i \leq t$ and the (q,k) positions of A - B, and P must all be 1 for some q where the (q,h) position of A-B is -1.

Having determined column h, we note that we may assume $t = 1$. If $t > 1$, then we will be unable to have the row sums of A-B to be zero for rows $j_1, j_2, \ldots, j_t$. In fact, with $t = 1$, and by trying to ensure that the row sum of $A-B$ is zero for row q, we deduce that $A-B$ has a 3 by 3 submatrix (after a row and column permutation) of the form

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} . \tag{4.4}$$

The zero's correspond to positions where A, B, and P all have 1's. We are led to define a variation on the idea of an interchange. Consider the following 3 by 3 submatrices.

$$\text{i)} \begin{bmatrix} 1 & 0 & c \\ 0 & b & 1 \\ a & 1 & 0 \end{bmatrix} \quad \text{ii)} \begin{bmatrix} 0 & 1 & c \\ 1 & b & 0 \\ a & 0 & 1 \end{bmatrix} , \tag{4.5}$$

where a, b, c take on the values 0 or 1. Replacing a 3 by 3 submatrix of $A \in \mathcal{A}(R,S)$ of the form i) by one of the form ii) leaves the row and column sums fixed as well as the entries a, b, c. Similarly, the replacement can go in the reverse direction. Similarly if we replace the matrices in (4.5) by a row and column permutation of them performed simultaneously on both. Define a <u>triangle interchange</u> to be such a replacement operation. We note that a triangle interchange would let us reduce $d(A,B)$ when $A-B$ has a submatrix of the form given in (4.4). Thus we have the following analogue of the Interchange theorem.

Theorem 4.3. <u>Let</u> $A, B \in \mathcal{A}_p(R,S)$ <u>with</u> $A \neq B$. <u>Then one can obtain</u> B <u>from</u> A <u>by a series of interchanges and triangle interchanges where each matrix in the series is still in</u> $\mathcal{A}_p(R,S)$.

The concept of invariant 1's in $\mathcal{A}_p(R,S)$ would be defined similarly to invariant 1's in $\mathcal{A}(R,S)$. The 1's of P are invariant 1's. Decompose an $A \in \mathcal{A}_p(R,S)$ as follows

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \qquad (4.6)$$

where W is of size e by f. If $t_{ef} = N_0(W) + N_1(Z)$ is precisely the number of 1's of P in Z, then all the 1's of W are invariant 1's. If $P \in \mathcal{A}(R',S')$, then all the invariant 1's of $\mathcal{A}(R-R',S-S')$ are invariant 1's of $\mathcal{A}_p(R.S)$. But these three cases do not exhaust all the possibilities as the following example shows. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} . \qquad (4.7)$$

where $P = M$ i.e. the 1's in positions (1,3), (2,2) and (3,1). It turns out that all the 1's are invariant 1's. Further work should help in classifying invariant 1's in $\mathcal{A}_p(R,S)$.

Section 5. Generalization of a theorem of Brualdi and Ross.

The following result has been proven by Brualdi and Ross [12]. Let $R = (r_1,r_2, \ldots, r_m)$ and $S = (s_1,s_2, \ldots, s_n)$. Let $\overline{R} = (\overline{r}_1,\overline{r}_2, \ldots,\overline{r}_m)$ with $\overline{r}_i = r_i$ or $r_i - 1$. Let $\overline{S} = (\overline{s}_1,\overline{s}_2, \ldots, \overline{s}_n)$ with $\overline{s}_i = s_i$ or $s_i - 1$.

Theorem 5.1. (Brualdi and Ross). <u>There exists an</u> $A \in \mathcal{A}(R,S)$

<u>and a</u> $B \in \mathcal{A}(\bar{R},\bar{S})$ <u>with</u> $A > B$ <u>if and only if both</u> $\mathcal{A}(R,S)$ <u>and</u> $\mathcal{A}(\bar{R},\bar{S})$ <u>are nonempty</u>.

This is a variation on the problem discussed in Section 3 where we now allow P to be any matrix in a given class. With this extra freedom, the conditions are considerably simplified. Using similar proof techniques to those used by Brualdi and Ross, that were exploited in our Theorem 4.3, we are able to prove a generalization. Let R and S be as above. Let $R' = (r_1', r_2', \ldots, r_m')$ with $r_i' = r_i - k$ or $r_i - (k+1)$ for some specified k. Let $S' = (s_1', s_2', \ldots, s_n')$ with $s_i' \leq s_i$.

Theorem 5.2. <u>There exists an</u> $A \in \mathcal{A}(R,S)$ <u>and a</u> $B \in \mathcal{A}(R',S')$ <u>with</u> $A > B$ <u>if and only if both</u> $\mathcal{A}(R,S)$ <u>and</u> $\mathcal{A}(R',S')$ <u>are nonempty</u>.

<u>Proof</u>. If we assume A and B exist then the classes are nonempty.

Assume the classes are nonempty. Let $A \in \mathcal{A}(R,S)$ and $B \in \mathcal{A}(R',S')$ with $A = (a_{ij})$ and $B = (b_{ij})$. Let $d(A,B)$ be the number of -1's in A-B. We will show that if $d(A,B) > 0$, then there is an interchange of either A or B to reduce $d(A,B)$. Assume otherwise. Consider the following two cases.

<u>CASE 1</u>. Assume A and B have corresponding 2 by 2 submatrices of the form

$$\text{i)} \quad \begin{bmatrix} 0 & 1 \\ 1 & a_{ij} \end{bmatrix}, \qquad \text{ii)} \quad \begin{bmatrix} 1 & 0 \\ 0 & b_{ij} \end{bmatrix}. \qquad (5.1)$$

Then, following the arguments in the discussion before Theorem 4.3, we conclude that $a_{ij} = 1$ and $b_{ij} = 0$.

CASE 2. Assume A and B have corresponding 2 by 2 submatrices of the form

$$\text{i)} \quad \begin{bmatrix} 1 & 0 \\ 0 & a_{ij} \end{bmatrix}, \qquad \text{ii)} \quad \begin{bmatrix} 0 & 1 \\ 1 & b_{ij} \end{bmatrix}. \tag{5.2}$$

We conclude that $a_{ij} = 0$ and $b_{ij} = 1$.

Let row ℓ of A-B contain the largest number of -1's, say t. Thus by the definition of r'_ℓ, row ℓ of A-B contains r 1's in columns $j_1, j_2, \ldots, j_r$ where $r = t + k$ or $r = t + k + 1$. Let the (ℓ, j) entry of A-B be -1. Since $s_j' \leq s_j$, there must be some row h for which the (h,j) entry of A-B is 1. Using Case 1 we deduce that the (h, j_i) positions of A-B are 1 for $i = 1,2, \ldots, r$. Thus row h has at least $r + 1$ 1's. Assume that row h has a -1 in column p. Then using Case 2 on the submatrix given by rows ℓ and h and columns j and p we deduce that the (ℓ, p) position of A-B is a -1. Thus all the -1's of row h lie in columns where row ℓ has -1's. Thus row h has at most $t - 1$ -1's. But then row h has at least $k + 2$ more +1's than -1's and this contradicts the definition of r_h.

Thus we conclude that there is some interchange which will reduce d(A,B). A finite sequence of these interchanges will lead to two matrices $A'B'$ with $A' \in \mathfrak{A}(R,S)$ and $B' \in \mathfrak{A}(R'S')$ and $d(A',B') = 0$ i.e. $A' \geq B'$ as desired.

Corollary 5.3. <u>Let</u> $A,B \in \mathfrak{A}(R,S)$ <u>with</u> $d(A,B) = d$. <u>Then it is possible to obtain</u> B <u>from</u> A <u>by at most</u> $d - 1$ <u>interchanges</u>.

Proof. This result follows from Theorem 5.1 when $R' = R$ and $S' = S$. The proof of the theorem assures us that, for $d(A,B) > 0$, there exists an interchange in one of them that reduces $d(A,B)$. We also note that $d(A,B) = 1$ is impossible since the row and column sums of $A-B$ are all zero.

Walkup has proven the best result of this type [39]. He has shown that the minimum number of interchanges necessary is $d(A,B)$ minus the maximum number of disjoint cycles in the directed bipartite graph given by $A-B$. In certain cases the bound of Corollary 5.3 is tight. For example let $R = (1,1, \ldots, 1) = S$ and consider the number of interchanges required to obtain the identity matrix I_n from the circulant matrix C of order n. It is precisely $n-1$ as can be seen by considering the group theory or using the result of Walkup. An examination of Corollary 5.3 yields the Interchange theorem.

Corollary 5.4 (Ryser [34]) Interchange theorem. Given $A, B \in \mathfrak{A}(R,S)$, it is possible to obtain B from A by a series of interchanges.

Theorem 5.2 is akin to some results in graph theory concerning the existence of subgraphs with vertex degrees k or $k+1$ [26,27,28]. The proof of Brualdi and Ross of Theorem 5.1 was derived from a paper of Lovasc on 1-factors [29].

Fulkerson has examined the following problem [15]. Given a (0,1)-matrix A of order n, what is the maximum number of disjoint permutation matrices it covers. We say that A covers k disjoint permutation matrices if

$$A = P_1 + P_2 + \cdots + P_k + D \; , \tag{5.3}$$

where each P_i is a permutation matrix and D is a (0,1)-matrix Fulkerson obtained a formula for the maximum possible k. We may pose this problem in terms of the class $\mathfrak{A}(R,S)$. We recall the following version of Birkhoff's theorem stated in (0,1)-matrix terms and not in terms of doubly stochastic matrices [34].

Theorem 5.5 (Birkhoff). _Let_ $R = (k,k, \ldots, k) = S$. _Then any matrix_ $A \in \mathfrak{A}(R,S)$ _is a sum of_ k _disjoint permutation matrices_.

Let $T_k = (k,k, \ldots, k)$ be the row of n k's. We can now pose Fulkerson's problem in the class $\mathfrak{A}(R,S)$.

Theorem 5.6. _Let_ $\mathfrak{A}(R,S)$ _be of order_ n. _The maximum number of disjoint permutation matrices covered by any matrix in_ $\mathfrak{A}(R,S)$ _is given by the maximum_ k _for which_ $\mathfrak{A}(R-T_k,S-T_k)$ _is nonempty_.

Proof. If $\mathfrak{A}(R-T_k,S-T_k)$ is nonempty then by Theorem 5.2 there exists a matrix $A \in \mathfrak{A}(R,S)$ and a matrix $B \in \mathfrak{A}(R-T_k,S-T_k)$ with $A \geq B$. Let $T = A - B$. Then $T \in \mathfrak{A}(T_k,T_k)$ and so is the sum of k disjoint permutation matrices by Theorem 5.5. Using (5.3), with $D = B$, we have our result.

A somewhat different sounding problem is determining the minimum number of disjoint permutation matrices required to cover a matrix A (or an arbitrary matrix in $\mathfrak{A}(R,S)$). In either case, merely solve the original problem in the (0,1)-complement, namely $J - A$ (or the class $\mathfrak{A}(T_n-R,T_n-S)$). The maximum number of disjoint permutation matrices contained in $A-J$ is precisely n minus the minimal number of disjoint permutations required to cover A.

Another important concept in $\mathfrak{A}(R,S)$ is the term rank. The *term rank* of a (0,1)-matrix is the maximum number of 1's, no two in the same row or column. We define the minimal and maximal term ranks of the class $\mathfrak{A}(R,S)$ as follows. We assume $\mathfrak{A}(R,S)$ is of order n. Then

$$\begin{aligned} \tilde{\rho} &= \min \{\text{term rank } (A) \mid A \in \mathfrak{A}(R,S)\} , \\ \bar{\rho} &= \max \{\text{term rank } (A) \mid A \in \mathfrak{A}(R,S)\} . \end{aligned} \tag{5.4}$$

Theorem 5.1 can be used to calculate $\bar{\rho}$. We need the following result of Ryser similar to Lemma 3.6.

Theorem 5.7 (Ryser [34]). *Let $A \in \mathfrak{A}(R,S)$ have maximal term rank $\bar{\rho}$. Then there exists a matrix $A' \in \mathfrak{A}(R,S)$ with 1's in positions* $(\bar{\rho},1)$, $(\bar{\rho}-1,2)$, ..., $(1,\bar{\rho})$.

Brualdi and Ross [12] noted that this gives us an easy way to determine $\bar{\rho}$. Let $R_k = (r_1-1, r_2-1, \ldots, r_k-1, r_{k+1}, \ldots, r_n)$ and let $S_k = (s_k-1, s_2-1, \ldots, s_k-1, s_{k+1}, \ldots, s_n)$. Then $\bar{\rho}$ is the maximum k for which $\mathfrak{A}(R_k,S_k)$ is nonempty, using Theorem 5.1. They used this to derive Ryser's formula for the maximal term rank involving the structure matrix.

It is interesting to compare the two ways of determining whether $\bar{\rho} = n$. In the case we have just mentioned, $\bar{\rho} = n$ if and only if the class $\mathfrak{A}(R-T_1,S-T_1)$ is nonempty. Our second method comes from Section 3, Theorem 3.1 for $P = M$. We have $\bar{\rho} = n$ if there is a matrix

$A \in \mathfrak{A}(R,S)$ with $A \geq M$.

We can express the condition $\mathfrak{A}(R-T_1, S-T_1)$ being nonempty as a set of inequalities. Let the sequence $(\tilde{s}_i)$ be the conjugate of the sequence $(r_i - 1)$. Then by the Gale-Ryser Theorem, $\mathfrak{A}(R-T_1, S-T_1)$ is nonempty if and only if

$$\sum_{i=1}^{t} \tilde{s}_i \geq \sum_{i=1}^{t} (s_i - 1) \qquad (1 \leq t \leq n) . \tag{5.5}$$

We rewrite (5.5) as follows

$$\sum_{i=1}^{t} (\tilde{s}_i - 1) \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n). \tag{5.6}$$

If we let the sequence (s_i^*) to be the M-required conjugate of the sequence (r_i), then there exists a matrix $A \in \mathfrak{A}(R,S)$ with $A \geq M$ if and only if

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i \qquad (1 \leq t \leq n), \tag{5.7}$$

by Theorem 3.1. Since both (5.6) and (5.7) determine when $\bar{\rho} = n$, there should be some connection. Certainly s_i^* need not equal $\tilde{s}_i + 1$. For example, let $R = (4, 3, 2, 2)$ then we have $(s_i^*) = (4, 4, 2, 1)$ and $(\tilde{s}_i + 1) = (5, 3, 2, 1)$. It is possible to show that, for $1 \leq t \leq n$,

$$\sum_{i=1}^{t} (\tilde{s}_i + 1) \geq \sum_{i=1}^{t} s_i^* . \tag{5.8}$$

Consider the first t columns of $\bar{A}$ and recall how $\tilde{s}_i + 1$ and s_i^* are computed. They are the column sums of the matrices $\tilde{A}$ (ferrer's matrix associated with $(r_1-1, r_2-1, \ldots, r_n-1))$ and the altered Ferrer's matrix A^* respectively. Let ℓ be the number of rows i with $r_i > t$. Then

$$\sum_{i=1}^{t} (\tilde{s}_i+1) - (n-\ell-t) = \sum_{i=1}^{t} \bar{s}_i \quad , \tag{5.9}$$

where $\bar{s}_i$ is the i^{th} column sum of $\bar{A}$. For $\ell < n - t$ we find that

$$\sum_{i=1}^{t} s_i^* - (n-\ell-t) = \sum_{i=1}^{t} \bar{s}_i \quad , \tag{5.10}$$

but for $\ell \geq n - t$ we have

$$\sum_{i=1}^{t} s_i^* = \sum_{i=1}^{t} \bar{s}_i \quad . \tag{5.11}$$

Thus (5.8) holds as desired.

We recall that (5.7) also implies that $\mathfrak{A}(R,S)$ is nonempty whereas (5.5) does not. Thus it is less surprising that the inequalities (5.5) and (5.7) differ despite the fact that they both compute when $\bar{\rho} = n$.

The computation of $\tilde{\rho}$ remains more difficult than the computation of $\bar{\rho}$. The following is the analogue of theorem 5.7 for $\tilde{\rho}$.

Theorem 5.8 (Haber [20]). <u>Let</u> $\mathfrak{A}(R,S)$ <u>be of order</u> n. <u>There is a matrix</u> $A \in \mathfrak{A}(R,S)$ <u>with term rank at most</u> ρ <u>where</u> $\rho < n$ <u>if and only if there is a matrix</u> $A' \in \mathfrak{A}(R,S)$ <u>which decomposes into blocks as</u> follows

$$A' = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \quad , \tag{5.12}$$

<u>where</u> Z <u>is of size</u> k <u>by</u> $2n - k - \rho$ <u>consisting entirely of</u> 0's.

We note that the term rank of A' is at most ρ. We will say that Z of size k by ℓ is possible if there is a matrix $A' \in \mathfrak{A}(R,S)$ in the form of (5.12) with Z of size k by ℓ where Z is all 0's. Then the term rank of A' is at most $2n - k - \ell$. Note that for $\rho = n$, such large zero blocks need not exist. The term rank of J_n is n yet it has no 0's.

Haber uses a version of Theorem 5.8 and some additional results to obtain a formula for $\tilde{\rho}$ [20]. Brualdi has since simplified this formula [11]. Determining large zero blocks remains the important first step. The existence of large zero blocks was discussed briefly in Section 2 when considering possible submatrices. The algorithm presented there is probably too inefficient to be of much help. Shortly, we will present conjecture 5.10 which would probably give an efficient algorithm for determining whether Z of size k by ℓ is possible. Then we need only find the pair k,ℓ for which Z is possible and in addition $2n - k - \ell$ is as small as possible. This will give us $\tilde{\rho}$, using Theorem 5.8 with ρ replaced by $\tilde{\rho}$. The following algorithm is an efficient way of doing this testing.

Algorithm to determine $\tilde{\rho}$.

Step 1. Z of size 1 by $n - s_n$ is possible using Fulkerson's (0,1)-matrix rule. Set $k = 1$ and $\ell = n - s_n$.

Step 2. Assume Z of size k by ℓ is possible. If k or ℓ equal n to to Step 5. If Z of size $k + 1$ by ℓ is possible to to Step 3. If such a Z is not possible go to Step 4.

Step 3. Set k to $k + 1$ and go to Step 2.

Step 4. Find first t $(t = 1, 2, \ldots, n-k-1)$ for which Z of size $k + 1 + t$ by $\ell - t$ is possible. If no such t is found go to Step 5. If such a t is found, set k to $k + 1 + t$ and set ℓ to $\ell - t$ and go to Step 2

Step 5. $\tilde{\rho} = 2n - k - \ell$.

We note that this algorithm will terminate after testing at most n times for possible zero blocks since at each test, k is incremented.

We now consider a restatement of Theorem 5.2. Let $R'' = (r_1'', r_2'', \ldots, r_m'')$ and $S'' = (s_1'', s_2'', \ldots, s_n'')$ with $R'' \le R$ and $S'' \le S$. Also $r_i'' = k$ or $r_i'' = k + 1$ for some given k. Let $R' = R = R''$ and $S' = S - S''$. Theorem 5.2 becomes the following.

Theorem 5.9. *There exists an* $A \in \mathfrak{A}(R,S)$ *and a* $B \in \mathfrak{A}(R'',S'')$ *with* $A \ge B$ *if and only if* $\mathfrak{A}(R,S)$ *and* $\mathfrak{A}(R',S')$ *are nonempty*.

The matrix B of Theorem 5.2 becomes A-B here. Note that $\mathfrak{A}(R'',S'')$ is necessarily nonempty when $r_1'' + r_2'' + \cdots + r_n'' = s_1'' + s_2'' + \cdots + s_n''$ since the ferrer's matrix $\bar{A}$ associated with R'' is as "smooth" as possible. Its 1's are as far to the left as possible without regard to row sums. One is led to the following conjecture which has also been proposed by Brualdi [11].

Conjecture 5.10. *There exists an* $A \in \mathfrak{A}(R,S)$, *a* $B \in \mathfrak{A}(R',S')$, *and a* $C \in \mathfrak{A}(R+R',S+S')$ *with* $A + B = C$ *if and only if* $\mathfrak{A}(R,S)$, $\mathfrak{A}(R',S')$, *and* $\mathfrak{A}(R+R',S+S')$ *are all nonempty*.

This conjecture does not extend to the case where one class is the sum of three or more other classes. Here is a counterexample. Let $R = S = R' = S' = (1,0)$ and $R'' = S'' = (0,1)$. Then the classes $\mathfrak{A}(R,S)$, $\mathfrak{A}(R',S')$, $\mathfrak{A}(R'',S'')$ and $\mathfrak{A}(R+R'+R'',S+S'+S'')$ are all nonempty yet there is no quadruple of matrices $A \in \mathfrak{A}(R,S)$, $B \in \mathfrak{A}(R',S')$, $C \in \mathfrak{A}(R'',S'')$, and $D \in \mathfrak{A}(R+R'+R'', S+S'+S'')$ with $A + B + C = D$.

It is interesting to note that the proof techniques of Theorem 5.2 involving interchanges do not appear to work here. Consider the following triple of matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (5.13)$$

Using the notation of Theorem 5.2, there is no interchange that will reduce $d(B,C)$. However a triangle interchange, as described just before Theorem 4.3, will reduce $d(B,C)$. Perhaps combining triangle interchanges and interchanges in the proof techniques of Theorem 5.2 will yield a proof of the conjecture.

If the conjecture were true, it would provide us a simple way of testing whether a given block of 0's is possible. For example, if you wished to test for whether Z of size k by ℓ is possible then let R' be a row with k ℓ's as far to the right as possible and S' to be a row with ℓ k's as far to the right as possible. Since we know $\mathfrak{A}(R,S)$ and $\mathfrak{A}(R',S')$ are nonempty, then Conjecture 5.10 would imply that Z of size k by ℓ is possible if and only if

$\mathfrak{A}(R-R',S-S')$ is nonempty. This, combined with the algorithm for computing $\tilde{\rho}$, would yield an efficient algorithm for computing $\tilde{\rho}$.

Section 6. Matrices in triangular form.

This section is concerned with matrices which have a triangular block of 0's. The following theorem tells us that simple necessary conditions are sufficient to ensure the existence of a matrix, in a given class $\mathfrak{A}(R,S)$, in triangular form. Our result is reminiscent of Theorem 3.3 or Theorem 5.2 specialized to a given case.

Theorem 6.1. *Let* $\mathfrak{A}(R,S)$ *be a nonempty class of order* n, *with* R,S *monotone. Also*

$$r_i \le n - i + 1 \quad ; \quad s_i \le n - i + 1 \qquad (1 \le i \le n). \tag{6.1}$$

Then there exists an $A \in \mathfrak{A}(R,S)$ *of the form*

$$A = \begin{bmatrix} * & & * \\ & \diagup & \\ * & & 0 \end{bmatrix} . \tag{6.2}$$

Proof. We note that if such an A exists, then (6.1) holds. We need R,S monotone for our proof to work. The proof is by induction on n. It will yield an inductive construction for generating an A as desired in (6.2).

The theorem is true for $n = 1$ trivially. Assume the theorem is true for $n - 1$, then we will prove it for n. Assume all the conditions are satisfied. We wish to consider possible first rows of matrices

Let P be the $(0,1)$-matrix of order n with a 1 in column i and row 1 for each i for which $s_i = n - i + 1$ and 0's elsewhere. Let Q be the $(0,1)$-matrix of order n with r_1 1's in the first row as far to the left as possible subject to the condition that $Q \geq P$. The remaining rows of Q consist entirely of 0's. It follows from $\mathcal{A}(R,S)$ being nonempty and Theorem 4.2 (transposed) that there is a matrix $A \in \mathcal{A}(R,S)$ with $A \geq P$ if and only if there is a matrix $\tilde{A} \in \mathcal{A}(R,S)$ with $\tilde{A} \geq Q$.

We define a $(0,1)$-matrix Q' in a similar way to Q. Let Q' have r_1 1's in the first row and 0's elsewhere. We require that $Q' \geq P$ and the remaining 1's are placed in the remaining columns with largest column sum. We also require that in the event of a choice, the 1's are pushed as far to the right as possible. For example, let $S = (4, 4, 4, 3, 1, 1)$ and let $r_1 = 4$. Then the first row of P is $(0, 0, 1, 1, 0, 1)$ and the first row of Q is $(1, 0, 1, 1, 0, 1)$. But the first row of Q' is $(0, 1, 1, 1, 0, 1)$. A row permutation is sufficient to verify that there is an $\tilde{A} \in \mathcal{A}(R,S)$ with $\tilde{A} \geq Q$ if and only if there is an $A' \in \mathcal{A}(R,S)$ with $A' \geq Q'$.

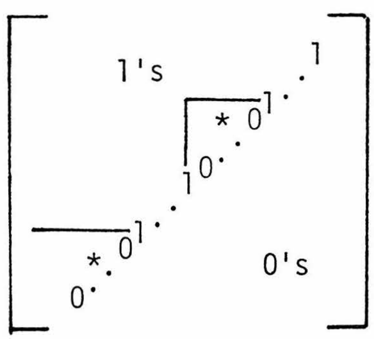
If such an A' exists, consider the submatrix A'' of A' obtained by deleting the first row and last column. We may associate with A'', the class $\mathcal{A}(R',S')$. Now Q' was chosen so that R' and S' are monotone and satisfy the inequalities of (6.1) when n is replaced by $n - 1$. Certainly $\mathcal{A}(R',S')$ is nonempty. Thus, by induction, there is a matrix $B \in \mathcal{A}(R',S')$ in triangular form,

$$B = \begin{bmatrix} * & & * \\ & \diagup & \\ * & & 0 \end{bmatrix} . \qquad (6.3)$$

Let C be the (0,1)-matrix of order n obtained from B by adding a first row of 0's and a last column of 0's. Then $C + Q' \in \mathcal{A}(R,S)$ and has the desired form of (6.2). Thus our induction argument will be completed if we can show that there is a matrix $A \in \mathcal{A}(R,S)$ with $A \geq P$. The remainder of the proof involves showing that P is acceptable and that the inequalities of Theorem 3.1 hold so that there is a matrix $A \in \mathcal{A}(R,S)$ with $A \geq P$.

We must show that P is acceptable. We note that P has column sums at most 1. By definition, P has a 1 in column i only when $s_i = n - i + 1$ but then $s_i \geq 1$. Let P have c 1's. Then we must show that $r_1 \geq c$. We do this by considering the worst case.

Form the (0,1)-matrix with column sum vector S and 1's as far to the top as possible. It will look like



, (6.4)

with c 1's on the reverse diagonal. The entries marked * will be 0's or 1's. The minimum value of r_1, for a given c, is obtained when there are as few 1's as possible for a given c. Thus the entires marked * may be set to zero and we note that monotonicity is kept. Now it can be seen that the c 1's on the reverse diagonal can be shifted, one at a time, as far to the upper right as possible to obtain the matrix

$$\left[\begin{array}{cccc} & & & 1 \\ & 1\text{'s} & & \cdot^{\cdot^{\cdot}} \\ \hline & & 1 & \\ 0 & & 0 & \\ & \cdot^{\cdot^{\cdot}} & & 0\text{'s} \\ 0 & & & \end{array}\right], \qquad (6.5)$$

which has no more 1's than the previous matrix. Having reached this configuration, one can deduce that $r_1 \geq c$ by looking at the transpose of (6.5) and recalling that R is monotone and satisfies (6.1). Thus P is acceptable.

Let the sequence (s_i^*) be the P-required conjugate of the sequence (r_i). Using Theorem 3.1, we must show

$$\sum_{i=1}^{t} s_i^* \geq \sum_{i=1}^{t} s_i \, , \qquad (6.6)$$

for $1 \leq t \leq n$ in order to verify that there exists an $A \in \mathfrak{A}(R,S)$ with $A \geq P$. We will prove that (6.6) holds for a given value of t. Let k be the number of i for which $s_i = n - i + 1$ and $i > t$. We may obtain

$$\sum_{i=1}^{t} \bar{s}_i - \sum_{i=1}^{t} s_i^* = \begin{cases} k & \text{for } r_1 < t \\ \max(0,k-(r_1-t)) & \text{for } r_1 \geq t \end{cases} \, , \qquad (6.7)$$

by noting that the number on the right corresponds to the number of 1's that have to be shifted from the first t columns to the last $n - t$ columns when forming the altered ferrer's matrix A^* from the ferrer's matrix $\bar{A}$.

To obtain (6.6) from (6.7) we consider a decomposition of an $A \in \mathfrak{A}(R,S)$ as follows

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} , \tag{6.8}$$

where W is of size $\bar{s}_t$ by t. It is possible that $\bar{s}_t = 0$. We have, from Remark 2.4, that

$$\sum_{i=1}^{t} \bar{s}_i - \sum_{i=1}^{t} s_i = N_0(W) + N_1(Z) . \tag{6.9}$$

We need only show that this expression is at least as large as that in (6.7) in order to verify (6.6).

<u>CASE 1</u>. $r_1 < t$.

Thus $\bar{s}_t = 0$ and $N_0(W) = 0$. Also $N_1(Z) = s_{t+1} + s_{t+2} + \cdots + s_n$ and so $N_0(W) + N_1(Z) \geq k$.

<u>CASE 2</u>. $r_1 \geq t$.

Let $r_1 = t + u$ with $u \geq 0$. We may assume $u < k$ otherwise the expression in (6.7) is zero and so the expression in (6.9) is as large as the expression in (6.7). Since there are k i's for which $s_i = n - i + 1$ and $i > t$, we deduce that

$$N_1(X) + N_1(Z) \geq (n-k-t)k + k(k+1)/2, \tag{6.10}$$

using the same arguments that showed P was acceptable.

Also, we calculate that

$$N_1(X) - N_0(W) = \sum_{i=t+1}^{n} \bar{s}_i . \tag{6.11}$$

This holds in any class $\mathcal{A}(R,S')$, for arbitrary S' where $\mathcal{A}(R,S')$ is nonempty. We can establish this result by induction. It is true for $\bar{A}$. If a 1 is shifted from W to X then it increases both $N_1(X)$ and $N_0(W)$ by 1. Thus it holds in $\mathcal{A}(R,S')$ by noting that any $A \in \mathcal{A}(R,S')$ can be formed by shifting the 1's of $\bar{A}$ to the right. We may maximize the righthand side by taking $\bar{s}_t = n - t + 1$ and all the r_i's as large as possible, for $1 \leq i \leq n - t + 1$, subject to the monotonicity of R and the inequalities of (6.1). Thus

$$N_1(X) - N_0(W) \leq (n-t+1)u - u(u+1)/2. \tag{6.12}$$

Combining (6.12) and (6.10) we obtain

$$N_0(W) + N_1(Z) \geq (k-u)(n-t+1 - (k+u+1)/2). \tag{6.13}$$

We note that $(k+u+1)/2 < k$ and $n \geq t + k$ and thus

$$N_0(W) + N_1(Z) \geq k - u. \tag{6.14}$$

Since $r_1 = t + u$, we have that $k - u$ is precisely the expression in (6.7). Thus by (6.14), the expression in (6.9) is as large as the expression in (6.7).

Thus we have verified that (6.6) holds and this completes the proof.

A small amount of extra work extends the theorem to a larger class of triangular blocks of zeros.

Corollary 6.2. *Let* $\mathcal{A}(R,S)$ *be a nonempty class of size* m *by* n *with* R *and* S *monotone and*

$$\begin{aligned} r_i &\le q - i + 1 \qquad (1 \le i \le m) \\ s_j &\le q - i + 1 \qquad (1 \le j \le n) \end{aligned} \quad , \tag{6.15}$$

where $q \ge \max(m,n)$. *Then there exists an* $A \in \mathcal{A}(R,S)$ *of the form*

$$A = \begin{bmatrix} * & & \\ & & * \\ & * & 0 \end{bmatrix} , \tag{6.16}$$

where the triangular block of zeros is of size $m + n - q - 1$.

Proof. Define a class $\mathcal{A}(R',S')$ from $\mathcal{A}(R,S)$ as follows. Let $R' = (r_1', r_2', \ldots, r_q')$ with $r_i' = r_i$ for $1 \le i \le m$ and $r_i' = 0$ for $i > m$. Let $S' = (s_1', s_2', \ldots, s_q')$ with $s_j' = s_j$ for $1 \le j \le n$ and $s_j' = 0$ for $j > n$. Then $\mathcal{A}(R',S')$ is of order q and satisfies the conditions of Theorem 6.1. Thus there is a matrix $A' \in \mathcal{A}(R,S)$ in the triangular form given in (6.2). Let A be the submatrix of A' consisting of the first m rows and n columns. But then $A \in \mathcal{A}(R,S)$ and A has the form given in (6.16).

Note that Corollary 6.2 implies Theorem 6.1 and so is equivalent to it. The following lemma bears on the problem of R and S being monotone.

Lemma 6.3. *Let* $R = (r_1, r_2, \ldots, r_n)$ *satisfy the inequality* $r_i \le n - i + 1$ *for* $1 \le i \le n$ *where* R *need not be monotone. Let*

$R' = (r_1', r_2', \ldots r_n')$ *be the reordering of* R *which is monotone. Then* $r_i' \leq n - i + 1$ *for* $i \leq i \leq n$.

Proof. The reordering of R can be achieved by a series of interchanges of adjacent entries. We will verify that the inequalities still hold after such an interchange and hence, after the complete reordering, in R'. Let $r_{k+1} > r_k$ with $r_i \leq n - i + 1$ for $1 \leq i \leq n$. Let $R'' = (r_1'', r_2'', \ldots, r_n'')$ with $r_k'' = r_{k+1}$, $r_{k+1}'' = r_k$ and $r_1'' = r_i$ for $i \neq k, k + 1$. Certainly $r_i'' = r_i \leq n - i + 1$ for $i \neq k, k + 1$. We have $r_k'' = r_{k+1} \leq n - (k+1) + 1 < n - k + 1$ and $r_{k+1}'' = r_k < r_{k+1} \leq n - (k+1) + 1$. This establishes the result.

Corollary 6.4. *Let* $\mathfrak{A}(R,S)$ *be a nonempty class of size* m *by* n *with* R *and* S *not necessarily monotone. In some reordering* R' *of* R *and some reordering* S' *of* S *we have*

$$r_i' \leq n - i + 1 \; ; \quad s_i' \leq n - i + 1 \qquad (1 \leq i \leq n) \,, \tag{6.17}$$

if and only if there exists a matrix $A \in \mathfrak{A}(R,S)$ *which, after a row and column permutation, is in the form*

$$A = \begin{bmatrix} * & & * \\ & \diagup & \\ * & & 0 \end{bmatrix} . \tag{6.18}$$

Proof. Given the inequalities in (6.17), an application Lemma 6.3 followed by Theorem 6.1 yields the matrix A in (6.18). Given the matrix A, it is clear that the row and column permutations that formed it yield the reorderings of R and S that give (6.17).

We state this corollary partly to give the following example. Theorem 6.1 may fail if R and S are not monotone. For example, take $R = (0,1) = S$. In other respects, the conditions of Theorem 6.1 are satisfied. But the class $\mathfrak{A}(R,S)$ has only one member and that matrix has a 1 in the (2,2) position. Thus there is no matrix A in the form of (6.2). We note that this example shows that the reorderings that give (6.17) need not correspond to the row and column permutations that give (6.18).

The following corollary is perhaps the most important result here. We find simple conditions for there to exist a matrix $A \in \mathfrak{A}(R,S)$ with $\mathrm{per}(A) = 1$. Note that if $\mathrm{per}(A) = 1$, then $\det(A) = \pm 1$ and so we have managed to get a result about nonsingularity in a very special case. Further results on nonsingularity, would be very desirable.

Corollary 6.5. *Let* $\mathfrak{A}(R,S)$ *be a nonempty class of order* n *with* R,S *monotone. There exists an* $A \in \mathfrak{A}(R,S)$ *with* $\mathrm{per}(A) = 1$ *if and only if* $\overline{\rho} = n$ *and*

$$r_i \leq n - i + 1 \quad ; \quad s_i \leq n - i + 1 \qquad (1 \leq i \leq n). \qquad (6.19)$$

Proof. Let $A \in \mathfrak{A}(R,S)$ have $\mathrm{per}(A) = 1$. Since $\rho(A) = n$ we have that $\overline{\rho} = n$. Using a theorem of Brualdi, Lemma 3.11 of Chapter 2, we have that after a suitable row and column permutation, A is in the form

$$A = \begin{bmatrix} * & & 1 \\ & \diagup & \\ 1 & & 0 \end{bmatrix} . \qquad (6.20)$$

Let the i^{th} row sum be r'_i and let the j^{th} column sum be s'_j. Then by (6.20) we have that

$$r'_i \leq n - i + 1 \quad ; \quad s'_i \leq n - i + 1 \quad (1 \leq i \leq n). \tag{6.21}$$

Then an application of Lemma 6.3 verifies that (6.19) holds.

Assume $\overline{\rho} = n$ and (6.19) holds. Let $R' = R - T_1$ and $S' = S - T_1$ where $T_1 = (1,1, \ldots, 1)$. Since $\overline{\rho} = n$, we know that $\mathfrak{A}(R',S')$ is nonempty. Also

$$r'_i \leq (n-1) - i + 1 \quad ; \quad s'_i \leq (n-1) - i + 1 \quad (1 \leq i \leq n-1). \tag{6.22}$$

Now R' and S' are monotone. Thus using Theorem 6.1 we obtain that there is a matrix $B \in \mathfrak{A}(R',S')$ of the form

$$B = \begin{bmatrix} * & & 0 \\ & \cdot^{\cdot^{\cdot}} & \\ & 0 & 0 \\ 0 & & \end{bmatrix} . \tag{6.23}$$

Let $A = B + I$. Then $A \in \mathfrak{A}(R,S)$ and $\text{per}(A) = 1$ which establishes our result.

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