

The ring of modular forms on a unitary Shimura surface

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ABSTRACT

We study the ring of the algebraic automorphic forms of parallel weights on the Deligne–Mostow Shimura variety associated to a family of cyclic covers of the projective line. This Shimura surface defined over the cyclotomic field $\mathbb{Q}(\zeta_5)$ with signature $(3, 0)$ and $(2, 1)$. Moreover, it is isomorphic to the closure of the Torelli image of the Hurwitz space of ramified $\mathbb{Z}/5\mathbb{Z}$ covers of $\mathbb{P}^1$. We compute the pullback of the Hodge line bundle in terms of the boundary divisors of the Hurwitz space, and identify a specific level supported at ramified primes, for which a finite cover of the Hurwitz space embeds into. We use these results to obtain the graded ring structure of algebraic automorphic forms at specific levels. As an application, we express the well-known Klein j -function on the sub-Shimura curve in terms of the generators of this graded ring. We also explore the general framework to compute such algebraic automorphic forms on other Deligne–Mostow Shimura varieties.

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Chapter 1

INTRODUCTION

This paper is motivated by the problem of computing the ring of algebraic automorphic forms on Shimura varieties. Algebraic automorphic forms have long been studied in number theory, since they provide abundant representations of reductive groups over $\mathbb{Q}$. However, the lack of explicit arithmetic models for Shimura varieties makes it difficult to describe the ring structure.

In this paper, we focus on a particular Shimura surface of PEL type, which we denote by $\text{Sh}[16]$. This is the Shimura surface associate to the unitary similitude group over the fifth cyclotomic field $\mathbb{Q}(\zeta_5)$ of signature $(3, 0)$ and $(2, 1)$. It arises as the moduli space of abelian 6-folds with multiplication by $\mathbb{Z}[\zeta_5]$ and signature type $(3, 0)$ and $(2, 1)$.

The primary aim of this paper is to compute the ring structure of algebraic automorphic forms with parallel weights on this Shimura surface. That is, we want to compute $M(\text{Sh}[16])$,

$$M(\text{Sh}[16]) := \bigoplus_{k \geq 0} H^0(\text{Sh}[16], \lambda^k)$$

Where λ is the Hodge line bundle on $\text{Sh}[16]$.

In particular, we prove the following statement:

Theorem 4.3.2. *The ring of automorphic forms with parallel weights on $\text{Sh}[16]$ is*

$$M(\text{Sh}[16]) = \mathbb{Q}(\zeta_5)[f_2, f_4, f_6, \varepsilon^2] / (\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4))$$

Where f_{2i} have of weight $5i$ and ε^2 has weight 2.

Note that the defining relation $\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4)$ implies that the ring is freely generated by generators f_2, f_6, ε^2 .

Beyond the classical modular forms, ring of algebraic automorphic forms of parallel weights have been explicitly computed for certain Hilbert modular surface in (Geer and Zagier, 1977) and for Picard modular surface in (Dimitrov and Ramakrishnan, 2025). In these computations, a concrete arithmetic model of the corresponding

Shimura surface is crucial. By realizing the Shimura surface as a projective variety, the authors can explicitly write the desired automorphic forms as certain homogeneous polynomials in the projective coordinate, and describe the ring as a quotient of a polynomial ring. In our case, we have access to a concrete arithmetic model of the Shimura surface $\text{Sh}[16]$ as it is identified with a Hurwitz space of ramified cyclic covers of $\mathbb{P}^1$ of degree 5.

Let us describe this association in a greater generality. Let μ_m be the cyclic group of size m , let $\zeta_m = e^{\frac{2\pi i}{m}}$ be the primitive root of unity. The Hurwitz space of μ_m -covers of $\mathbb{P}^1$ with inertia type $\underline{a}$ as defined in Chapter 3 is a smooth, proper and irreducible Deligne-Mumford stack defined over $\text{Spec}(\mathbb{Z}[\frac{1}{m}, \zeta_m])$. We denote such Hurwitz spaces by $\mathcal{M}(m, r, \underline{a})$. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g . Over $\mathbb{C}$, the image of $\mathcal{M}(m, r, \underline{a})_{\mathbb{C}}$ under the Torelli map is contained in a special subvariety $S(m, r, \underline{a})$ of $\mathcal{A}_{g, \mathbb{C}}$, of PEL-type. By (Deligne and Mostow, 1986), $S(m, r, \underline{a})$ is an irreducible component of a PEL-type moduli space $\text{Sh}(\mathcal{D})_{\mathbb{C}}$, where $\text{Sh}(\mathcal{D})$ is a smooth proper Deligne-Mumford stack defined over $\text{Spec}(\mathbb{Z}[\frac{1}{m}, \zeta_m])$; furthermore, the Torelli morphism $\mathcal{M}(m, r, \underline{a})_{\mathbb{C}} \rightarrow \text{Sh}(\mathcal{D})_{\mathbb{C}}$ extends to a morphism $\mathcal{M}(m, r, \underline{a}) \rightarrow \text{Sh}(\mathcal{D})$ defined over $\text{Spec}(\mathbb{Z}[\frac{1}{m}, \zeta_m])$. In this paper, we refer to the Shimura varieties arising in this way as the Deligne-Mostow Shimura varieties and denote them as $\text{Sh}(\mathcal{D})$.

The Torelli morphism from Hurwitz space of curves to the ambient Shimura variety is an open immersion. In (Moonen, 2010, Table 1), Moonen has identified 20 special families of Hurwitz spaces where the Torelli locus is dense in the associated Deligne-Mostow Shimura varieties. For these Shimura varieties, if in addition we know that they are compact, then the Torelli morphism in fact extends to an isomorphism between the compactified Hurwitz space and a connected component of the ambient Shimura variety. Moreover, the Hodge line bundle on the ambient Shimura variety pulls back to the Hodge line bundle of the Hurwitz space. Hence, the modular forms may be identified with global sections of the Hodge line bundle on the compactified Hurwitz space.

In particular, the Shimura surface $\text{Sh}[16]$ fits into this setting. It is the Deligne-Mostow Shimura variety associated to the Hurwitz space $\mathcal{M}(5, 5, (1, 1, 1, 1, 1))$. This family is the 16th in Moonen's 20 special families in (Moonen, 2010, Table 1), and we will refer to this family as $\mathcal{M}[16]$ in the following.

To get a better understanding of the geometry of Hurwitz space, we consider its finite cover where the automorphism of the symmetric group is rigidified. This

finite cover $\widetilde{\mathcal{M}}(m, r, \underline{a})$ parameterize $\mathbb{Z}/m\mathbb{Z}$ cover of $\mathbb{P}^1$ together with an labeling of the i th ramification point for each $1 \leq i \leq r$. It has a natural forgetful map to $\overline{\mathcal{M}}_{0,r}$, where $\overline{\mathcal{M}}_{0,r}$ is the moduli spaces of r -marked point on genus 0 curve. The moduli spaces $\overline{\mathcal{M}}_{0,r}$ have explicit arithmetic models, and their boundary divisors and projective embeddings are well known. Hence, we are led to the following problem:

Problem 1.0.1. Compute the Hodge line bundle λ on the finite cover $\widetilde{\mathcal{M}}(m, r, \underline{a})$ in terms of its boundary divisors and compute the global sections as homogeneous polynomials in some projective coordinate.

This can be solved in general. In Chapter 5, we give an algorithm to compute the Hodge line bundle λ on a general Hurwitz space $\widetilde{\mathcal{M}}(m, r, \underline{a})$ and express it as a linear combination of the boundary divisors of $\widetilde{\mathcal{M}}(m, r, \underline{a})$. When $\widetilde{\mathcal{M}}(m, r, \underline{a}) = \widetilde{\mathcal{M}}[16]$, we compute the ring of global sections of λ as a polynomial ring.

While the finite cover $\widetilde{\mathcal{M}}(m, r, \underline{a})$ is not isomorphic to the associated Deligne-Mostow Shimura variety $\text{Sh}(\mathcal{D})$ at level 1, it is expected to map to a finite Shimura cover $\text{Sh}_{\Gamma}(\mathcal{D}) \rightarrow \text{Sh}(\mathcal{D})$ parameterizing a suitable level structure α on $(\mathcal{A}, \lambda, i)$. Indeed, $\widetilde{\mathcal{M}}(m, r, \underline{a})$ parameterize labeling on the ramification points, and these labeled points on the curve should generate a torsion subgroup of the corresponding Jacobian, which is a level structure. Hence, we are led to the following problem:

Problem 1.0.2. Find the smallest finite cover $\text{Sh}_{\Gamma_m}(\mathcal{D}) \rightarrow \text{Sh}(\mathcal{D})$ for which the following factorization exists

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{\tilde{T}} & \text{Sh}_{\Gamma_m}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \text{Sh}(\mathcal{D}) \end{array}$$

and determine if the square is exact.

What is this level Γ_m ? For the Deligne-Mostow Shimura varieties associated to $\mathcal{M}(m, r, \underline{a})$, its integral datum can be computed from the cohomology and Theta divisor of the curves in $\mathcal{M}(m, r, \underline{a})$. In particular, the integral datum has good reduction everywhere. This allows us to define a certain level $(1 - \zeta_m)$ -level structure supported at the ramified primes, which corresponds to a Shimura cover $\text{Sh}_{\Gamma_m}(\mathcal{D}) \rightarrow$

$\text{Sh}(\mathcal{D})$ as defined in Definition 4.2.2. In Chapter 6, we show that when m is an odd prime, the factorization in Problem 1.0.2 through this specific level $\text{Sh}_{\Gamma_m}(\mathcal{D})$ exists.

In the special case of $\mathcal{M}[16]$, we verify that the square is exact. As an application, we compute the ring of automorphic forms of parallel weight on the Shimura surface $\text{Sh}_{\Gamma_5}[16]$. Here we introduce a notation: If R is a graded polynomial ring, we denote by R_{ev} the even graded sub-ring of R .

Theorem 4.3.1. *The ring of automorphic forms with parallel weights on $\text{Sh}_{\Gamma_5}[16]$ is*

$$M(\text{Sh}_{\Gamma_5}[16]) = (\mathbb{Q}(\zeta_5)[s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}, \varepsilon]/\mathcal{I})_{\text{ev}}$$

where

$$\mathcal{I} = (\varepsilon^5 - Q_{12}, Q_{12} - Q_{21}, Q_{12} - Q_{23}, Q_{12} - Q_{32}, Q_{12} - Q_{13}, Q_{12} - Q_{13});$$

s_{ij} has degree 1 and weight $\frac{5}{2}$, ε has degree $\frac{2}{5}$ and weight 1, and Q_{ij} is the following quadratic polynomial:

$$Q_{ij} = \text{sign} \begin{bmatrix} 1 & 2 & 3 \\ i & j & k \end{bmatrix} s_{ij}(s_{ik} - s_{ki} - s_{jk})$$

We obtain the ring on $\text{Sh}[16]$ as a subring of $M(\text{Sh}_{\Gamma_5}[16])$. The generators f_{2i} are explicitly defined in Notation 7.2.1. They are certain polynomials of degree $2i$ in s_{ij} invariant under the natural $\mathfrak{S}_5$ action. The choice of ε is unique up to scalar in K . In particular, the fraction $\frac{1}{52}$ appearing in the defining relation is due to our choice of generator that $\varepsilon^5 = Q_{12}$.

Inside $\text{Sh}[16]$, there is a compact unitary Shimura curve that we denote by $\text{Sh}[11]$. This is the Shimura curve associated to the unitary similitude group over the fifth cyclotomic field of signature $(2, 0)$ and $(1, 1)$, and it is the Deligne-Mostow Shimura variety associated to the Hurwitz space $\mathcal{M}(5, 4, (1, 1, 1, 2))$. This family is also special, and it's the 11th in Moonen's 20 special families in (Moonen, 2010, Table 1). We will refer to this family as $\mathcal{M}[11]$ in the following and refer to its associated Deligne-Mostow Shimura variety as $\text{Sh}[11]$. As an application, we obtain the ring of modular forms at level $1 - \zeta_5$ and at level 1 for the sub Shimura curve $\text{Sh}[11]$ as a quotient of the rings above. We also obtain the Klein j -function on $\mathcal{M}[11]$ defined in (Li, Mantovan, Pries, and Tang, 2025, Lemma 2.2) as an automorphic form expressed in terms of generators f_2 and f_6 :

Proposition 7.3.7. *Let j be the Klein j -function on $\text{Sh}[11]$. Then*

$$j = \frac{3}{2} \cdot \frac{f_2^3}{16f_6 - f_2^3}$$

1.1 Outline

The paper is organized as follows. In Chapter 2, we recall the relevant definitions of PEL-type Shimura varieties and define the algebraic automorphic forms. We also give the integral datum for the Shimura surface $\text{Sh}[16]$. In Chapter 3, we recall the relevant definition of Hurwitz space of cyclic covers of $\mathbb{P}^1$ and prove some properties of these Hurwitz spaces that will be useful later. In Chapter 4, we define the Torelli morphism from Hurwitz spaces to Deligne-Mostow Shimura varieties as well as the integral $(1 - \zeta_m)$ structure supported at ramified primes. Then, we give a high-level summary of our approach. In Chapter 5, we focus on Problem 1.0.1. We compute the Hodge line bundle on these Hurwitz spaces and express it in terms of the boundary divisors. We also explicitly compute the ring of global sections of the Hodge line bundle of the Hurwitz space associated to the Shimura surface $\text{Sh}[16]$. In Chapter 6, we focus on Problem 1.0.2. We consider the subgroup of the Jacobian generated by the marked points. We compute its group structure and identify it with a certain invariant subgroup of m -torsion of the Jacobian. In Chapter 7, we apply the results of Problem 1.0.1 and Problem 1.0.2 in the case of $\text{Sh}[16]$ and obtain the ring at level $(1 - \zeta_5)$ and level 1. We also obtain the two rings for the Shimura subvariety $\text{Sh}[11]$ as quotients of the ring of $\text{Sh}[16]$, and express the canonical j -function in terms of the generators.

Chapter 2

THE MODULI SPACE OF ABELIAN VARIETIES

In this section, we recall the relevant definitions for a general PEL-type Shimura variety. These include the rational and integral PEL datum, and the moduli problem that they represent. Then, we give the integral datum of specific Shimura surface Sh[16] that will be the subject of study of this paper. Finally, we give the definition of the ring of modular forms that will be used in this paper.

2.1 Hermitian forms and its associated unitary groups

These part recall relevant definitions on Hermitian forms on CM vector space and its associated unitary group. This part rephrases materials from (Milne, 2005) and (Eischen, 2022).

To give a complex vector space amounts to give a real vector space V together with a complex structure J . Then given (V, J) , a Hermitian form on V is a bilinear pairing

$$H : V \times V \rightarrow \mathbb{C}$$

such that $H(Ju, v) = iH(u, v)$ and $H(u, v) = \overline{H(v, u)}$. It can be represented by a Hermitian matrix H , and the pairing is given by $H(u, v) = \bar{u}^T H v$.

We can write $H(u, v) = \varphi(u, v) + i\psi(u, v)$ where $\varphi, \psi : V \times V \rightarrow \mathbb{R}$, φ is symmetric and ψ is alternating (symplectic).

$$H(u, v) = \varphi(u, v) + i\varphi(u, Jv)$$

$$H(u, v) = \psi(u, Jv) - i\psi(u, v)$$

Now let K be a CM field in the sense of (Milne, 2006, Proposition 1.4), and let K_0 be the totally real subfield. Let V be finite dimensional K -vector space. Then an **K -valued Hermitian form** $H : V \times V \rightarrow K$ is a pairing on V linear in the first variable and conjugate linear in the second variable. Then we can choose $\beta \in K$ such that $-i\beta$ is totally real and totally positive. For such Hermitian form H , we can define the associated symplectic form as $\psi(x, y) = \text{Tr}(\beta H(x, y))$.

Let (V, H) be a K vector space together with a K valued Hermitian form H . Then the **unitary group** associated with (V, H) , denoted as $U(V, H)$, is the group variety

defined over K_0 such that for any K_0 algebra R ,

$$U(R) = \{g \in \mathrm{GL}_{K \otimes_{K_0} S}(V \otimes_{K_0} S) : H(gu, gv) = H(u, v)\}$$

Now, let $2r = [K : \mathbb{Q}]$. Fix a CM type $\Sigma = (\tau_1, \dots, \tau_r) : K \rightarrow \mathbb{C}$ extending the totally real embeddings $(\rho_1, \dots, \rho_r)$. Let $n = \dim_K(V)$. Then Σ gives the isomorphism

$$V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{\tau_i \in \Sigma} V \otimes_{K_0, \tau_i} \mathbb{R} \cong \bigoplus_{\tau_i \in \Sigma} V_{\tau_i, \mathbb{R}} \cong \bigoplus_{\tau_i \in \Sigma} \mathbb{C}_{\tau_i}^n$$

Notice that $V_{\tau_i, \mathbb{R}} \cong \mathbb{C}_{\tau_i}^n$, with the isomorphism given by $v \otimes r \rightarrow \tau_i(v)r$. The action of $\mathrm{Gal}(K_0/\mathbb{Q})$ is such that $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q})$ acts by swapping factors, sending $\mathbb{C}_{\tau_i} \rightarrow \mathbb{C}_{\sigma\tau_i}$ and the conjugate of CM types acts by conjugation on the $\mathbb{C}^n$ factor.

Under this isomorphism $V_{\mathbb{R}} \cong \bigoplus_{\tau_i \in \Sigma} \mathbb{C}_{\tau_i}^n$, the Hermitian form also decompose as

$$H_{\mathbb{R}} = \prod_{\tau_i \in \Sigma} H_{\tau_i, \mathbb{R}}$$

Where $H_{\tau_i, \mathbb{R}}$ has the entries given by $\tau_i(H)$ and only acts on $V_{\tau_i, \mathbb{R}}$. Since $V_{\tau_i, \mathbb{R}}$ are n -dimensional $\mathbb{C}$ -vector space, $H_{\tau_i, \mathbb{R}}$ is conjugate to a matrix of the form $\begin{bmatrix} 1_{a_{\tau_i}} & 0 \\ 0 & -1_{b_{\tau_i}} \end{bmatrix}$ where $a_{\tau_i} + b_{\tau_i} = n, a_{\tau_i^*} = b_{\tau_i}$. In this case we say that $U(V, H)$ has **signature** $\prod_{\tau_i \in \Sigma} (a_{\tau_i}, b_{\tau_i})$. Hence we get that

$$\mathrm{Res}_{\mathbb{Q}}^{K_0} U(\mathbb{R}) \cong \prod_{\tau_i \in \Sigma} U(a_{\tau_i}, b_{\tau_i})$$

2.2 PEL datum

The unitary group and Hermitian pairing naturally arises from PEL datum, which gives rise to Shimura variety of PEL type. In this section, we recall the general definition of PEL datum and its integral model. In the later section, we will explain what exactly is the PEL datum in our setting. The definitions are mostly rephrasing (Li, Mantovan, and Pries, 2021) and (Kottwitz, 1992, section 4, 5).

An **PEL datum** is a tuple $(B, *, V, \Psi, h_0)$ such that:

- B is a semi-simple algebra over $\mathbb{Q}$;
- $*$ is a positive involution on B ;
- V is a finite dimensional vector space over $\mathbb{Q}$ that is a B -module.

- $\Psi : V \times V \rightarrow \mathbb{Q}$ is an alternating form, such that $\Psi(bx, y) = \Psi(x, b^*y)$ for all $x, y \in V, b \in B$.
- h_0 is an $\mathbb{R}$ algebra homomorphism $h_0 : \mathbb{C} \rightarrow \text{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$. It satisfies the following properties:
 - $h_0(z)^* = h_0(\bar{z})$;
 - $\Psi_{\mathbb{R}}(x, h_0(i)x) > 0$ for $x \neq 0$.

To an PEL datum $(B, *, V, \Psi, h_0)$, one can attach the **group of similitude** $G/\mathbb{Q}$:

$$G(R) = \{g \in \text{End}_{O \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) : \Psi(gv, gw) = r\Psi(v, w) \text{ for } r \in R^\times\}$$

It has a subgroup $G_1/\mathbb{Q}$:

$$G_1(R) = \{g \in \text{End}_{O \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) : \Psi(gv, gw) = \Psi(v, w)\}$$

There is the following exact sequence of algebraic group over $\mathbb{Q}$:

$$1 \rightarrow G_1 \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 1$$

A PEL datum defines a reflex field E , given as follows:

Definition 2.2.1 (Reflex field). *Given a PEL datum $(B, *, V, \Psi, h_0)$, $V_{\mathbb{C}} = V_0 \oplus \overline{V_0}$, where $V_0 = \{v \in V_{\mathbb{C}} \mid h_0(z)v = (1 \otimes z)v\}$.*

*The **reflex field** attached to this PEL datum is $E = \mathbb{Q}(\text{Tr}(b|_{V_0}), b \in B)$.*

Remark 2.2.2. The reflex field E is the field of definition of V_0 as a $O \otimes_{\mathbb{Z}} \mathbb{C}$ module. The proof is given in (Lan, 2008, corollary 1.1.2.12).

To a PEL datum $(B, *, V, \Psi, h_0)$, there is an associated PEL type Shimura variety $\text{Sh}(B, *, V, \Psi, h_0)$ over the reflex field E . E is exactly the field of definition of the associated moduli problem.

For a PEL datum, one can find an integral model for it. Given $(B, *, V, \Psi, h_0)$, an **integral PEL datum** is the tuple $(O_B, *, L, \Psi, h_0)$ such that:

- O_B is an order in B , stable under $*$.
- L is an $\mathbb{Z}$ -lattice of V stable under the O_B action.
- Ψ restricted to L takes value in $\mathbb{Z}$.

Then to the integral Shimura datum $(O_B, *, L, \Psi, h_0)$, we attach the following group scheme over $\mathbb{Z}$:

$$\mathcal{G}(R) = \{g \in \text{End}_{O_B \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) : \Psi(gv, gw) = r\Psi(v, w) \text{ for } r \in R^\times\}$$

Which has subgroup

$$\mathcal{G}_1(R) = \{g \in \text{End}_{O_B \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) : \Psi(gv, gw) = \Psi(v, w)\}$$

Notice that $G = \mathcal{G}_{\mathbb{Q}}$ and $G_1 = \mathcal{G}_{1\mathbb{Q}}$

Remark 2.2.3. How is the PEL datum we introduced in this section relates to unitary groups in the pervious section? In Section 4.1, we will see that the Shimura varieties studied in this paper all have $B = \mathbb{Q}[\mu_m]$ and $O_B = \mathbb{Z}[\mu_m]$ for some positive integer m . Let $K = \mathbb{Q}(\zeta_m)$ and $K_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. Then, $\Psi(x, y) = \text{Tr}_K^K(H(x, y))$ for some K -valued Hermitian form H . For these Shimura varieties, we have $G_1 = \text{Res}_{\mathbb{Q}}^{K_0} \text{U}(V, H)$ and $\mathcal{G}_1 = \text{Res}_{\mathbb{Z}}^{O_{K_0}} \text{U}(L, H)$, and $\mathcal{G}$ looks like the product of $\mathcal{G}_1$ with the torus $\mathbb{G}_m$. These Shimura varieties are hence called **unitary type**, because $\mathcal{G}_1(\mathbb{R})$ are products of unitary group, and the tuples $\prod_{\tau_i \in \Sigma} (a_{\tau_i}, b_{\tau_i})$ such that $\mathcal{G}_1(\mathbb{R}) = \prod_{\tau_i \in \Sigma} \text{U}(a_{\tau_i}, b_{\tau_i})$ are the **signature type** of the Shimura variety.

For an integral PEL datum, we have the notion of when is its associated canonical model is well behaved when reduced mod p . This gives us the notion of a "good" prime.

Definition 2.2.4. Given an integral PEL datum $(O_B, *, L, \Psi, h_0)$, a rational prime p is said to be **good**, if $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is maximal in $B \otimes \mathbb{Q}_p$ and is self-dual under Ψ .

Given $(O_B, *, L, \Psi, h_0)$, let S be the finite set of primes that is bad. Then there is a canonical model $\text{Sh}(O_B, *, L, \Psi, h_0)$ defined over $O_E[\frac{1}{S}]$ and is smooth over $\text{Spec}(O_E[\frac{1}{S}])$. Hence, it has good reduction at all the prime $p \notin S$.

If we take the generic fibre of $\text{Sh}(O_B, *, L, \Psi, h_0)/O_E[\frac{1}{S}]$, we will get $\text{Sh}(B, *, V, \Psi, h_0)/E$. Conversely, choose an order and a lattice (O_B, L) inside (B, V) gives an extension of $\text{Sh}(B, *, V, \Psi, h_0)/E$ over $O_E[\frac{1}{S}]$. Different choices of lattices gives different extension, but they all have the same generic fibre.

2.3 Level structure away from ramified primes

Fix an integral PEL datum $(O_B, *, L, \psi, h_0)$ and its set of bad primes S . In this section, we define the notion of level structure $\mathcal{K}$ away from S .

Recall in the case of modular curves, we have the notion of principal level n structure, which is to choose a trivialization $\eta_n : L/nL \rightarrow A[n]$ of the parameterized abelian scheme. But there are cases where we hope to only trivialize a subgroup inside $A[n]$. Hence, we adopt the following more general notion of level $\mathcal{K}$ structure.

Let E be the reflex field. Let $R = \mathcal{O}_E[\frac{1}{S}]$ the localization at S , so the primes in S do not exist in R . Let T be a locally Noetherian scheme over R .

Definition 2.3.1 (Integral level $\mathcal{K}$ structure). *Let the integral PEL datum, the triple $(\mathcal{A}, \lambda, i)$ and S be the finite set of bad primes. Let $\mathcal{K}$ be a compact open in $\mathcal{G}(\hat{\mathbb{Z}}^S)$.*

*For such $(\mathcal{A}, \lambda, i)$, we choose a geometric point $\bar{t} \in T$. Then, an **integral level $\mathcal{K}$ structure** is a $\mathcal{K}$ -orbit of the pair (η, ν) where η, ν are isomorphisms*

$$\eta : L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^S \rightarrow T^S(A_{\bar{t}})$$

$$\nu : \hat{\mathbb{Z}}^S(1) \rightarrow T^S \mathbb{G}_{m\bar{t}}$$

and the following diagram commute:

$$\begin{array}{ccc} L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^S \times L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^S & \xrightarrow{\psi} & \hat{\mathbb{Z}}^S(1) \\ \downarrow \eta \times \eta & & \downarrow \nu \\ T^S(A_{\bar{t}}) \times T^S(A_{\bar{t}}) & \xrightarrow{e_\lambda} & T^S \mathbb{G}_{m\bar{t}} \end{array}$$

and that (η, ν) is fixed under the action of $\pi_1(T, \bar{t})$. Here, the $\mathcal{K}$ orbit is the action of $\mathcal{G}(\hat{\mathbb{Z}})$ acting from the right, and we are taking elements of the form $(\eta_0, \nu_0) \cdot k$ for $k \in \mathcal{K}$. The action of $\pi_1(T, \bar{t})$ is from the left.

Remark 2.3.2. The level n structure is a special type of level $\mathcal{K}$ structure. How to understand this? Suppose $n = \prod p_i^{e_i}$. Then, we can take $\mathcal{K} = \mathcal{K}_0 \mathcal{K}(n)$, where $\mathcal{K}_0 = \prod_{p \nmid n} \mathcal{G}(\mathbb{Z}_p)$, $\mathcal{K}(n) = \prod_{p_i | n} \Gamma(p_i^{e_i}) \subseteq \prod_{p_i | n} \mathcal{G}(\mathbb{Z}_{p_i})$. Here, $\Gamma(l) = \{g \in \mathcal{G}(\mathbb{Z}_l) \mid g \equiv I \pmod{l}\}$. That is, $\Gamma(l)$ is the subgroup that fixes the basis of $\frac{1}{l}\Lambda_l/\Lambda_l$. Take a η in the orbit and reduced it modulo $p^k \Lambda_p$, we get an isomorphism $\eta_p : L/p^k L \rightarrow A[p^k]$. Moreover, any other isomorphism $\eta' : L_p \rightarrow T_p(A)$ reduces to η_p if and only if $\eta' = \eta \cdot k$ for some $k \in \Gamma(p^k)$. Similarly for other primes $p' \mid n$. Hence, the choice of an $\mathcal{K}$ -orbit of η exactly gives us integral level $\mathcal{K}$ -structure as defined in Definition 2.3.1.

Now, we can define the smooth moduli problem. Given in integral PEL datum $(\mathcal{O}, *, L, \Psi, h_0)$, let S be the set of bad primes. Recall that $R = \mathcal{O}_E[\frac{1}{S}]$, where E is the reflex field of $(\mathcal{O}, *, L, \Psi, h_0)$. Let $\mathcal{K}^S \subseteq \mathcal{G}(\hat{\mathbb{Z}}^S)$ be an open compact.

Definition 2.3.3 (moduli problem of level $\mathcal{K}^S$ structure $M_{\mathcal{K}^S}$). *The moduli problem $M_{\mathcal{K}^S}$ is a category fibred in the groupoid*

$$(\text{Sch}/R) \rightarrow \text{Grp}$$

$$M_{\mathcal{K}^S}(T) = (\mathcal{A}, \lambda, i, \eta)$$

such that:

- $\mathcal{A}$ is an abelian scheme over T ;
- $\lambda : \mathcal{A} \rightarrow \mathcal{A}^\vee$ is a polarization with degree divisible only by $p \in S$.
- $i : \mathcal{O} \rightarrow \text{End}_T(\mathcal{A})$ compatible with Rosati involution.
- $\text{Lie}(\mathcal{A}/T)$ as a $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}$ module satisfies the **Kottwitz determinantal condition** for $(L \otimes_{\mathbb{Z}} \mathbb{R}, h_0)$.
- η is an integral level $\mathcal{K}^S$ structure. In other words, it is a $\mathcal{K}^S$ orbit of $\mathcal{O}$ -equivariant symplectic isomorphism

$$\eta : L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^S \rightarrow \mathcal{T}^S(A)$$

Here $\mathcal{T}(A)$ means the Tate module of A at all primes, whereas $\mathcal{T}^S(A)$ means the Tate module of A at primes away from S .

Two quadruples (A, λ, i, η) and $(A', \lambda', i', \eta')$ are said to be isomorphic if there is an isogeny $f : A \rightarrow A'$, commuting with the action of $\mathcal{O}_B$ carrying $\eta \rightarrow \eta'$, and carrying λ into a scalar multiple of λ' . The degree of f can only have prime factors subset of the prime factors of m , and the scalar multiple also has prime factor subset of the prime factors of m .

Remark 2.3.4. Let us explain the Kottwitz determinantal condition. Let $\alpha_1, \dots, \alpha_n$ be a $\mathbb{Z}$ -basis of $\mathcal{O}$. Then, $f(X_1, \dots, X_n) = \det(X_1\alpha_1 + \dots X_n\alpha_n \mid V_0)$ is a homogeneous polynomial of degree g , where $2g = [V : \mathbb{Q}]$. In fact, the coefficient is in $\mathcal{O}_{E,S}$.

On the other hand, $\text{Lie}(A/S)$ is a locally free $\mathcal{O}_S$ module of dimension g with a action of $\mathcal{O}$. We also consider $g(X_1, \dots, X_n) := \det(X_1\alpha_1 + \dots X_n\alpha_n \mid \text{Lie}_{A/S})$. Then $\text{Lie}(A/S)$ is said to satisfy Kottwitz determinant condition if $g(X_1, \dots, X_n) = f(X_1, \dots, X_n)$. That is, V_0 is isomorphic to $\text{Lie}(A/S)$ as a $\mathcal{O} \otimes \mathbb{C}$ representation.

Now notice that when we say no level, we mean $\mathcal{K} = \mathcal{G}(\hat{\mathbb{Z}}^S)$. Then all symplectic isomorphism are in one $\mathcal{K}$ orbit, so there is no need to specify the data.

$\text{Sh}(\mathcal{O}_B, *, L, \Psi, h_0)$ is the moduli space that represent this functor, for $\mathcal{K}^S = \mathcal{G}(\hat{\mathbb{Z}}^S)$.

2.4 Integral datum for the Shimura surface

In this section, we define the PEL-type Shimura surface $\text{Sh}[16]$ which is the interest of study of this paper. To define an integral model of $\text{Sh}[16]$, we just need to explicitly write down its integral PEL datum $(\mathcal{O}, *, L, \Psi, h_0)$. In the later Section 4.1, we will see how this Shimura surface naturally arises from a family of curves and how these integral datum are obtained.

Definition 2.4.1 (Integral Datum for $\text{Sh}[16]$). *We consider the following integral datum $(\mathcal{O}, *, L, \psi, h_0)$. Let $K = \mathbb{Q}[\zeta_5]$, $K_0 = \mathbb{Q}[\sqrt{5}]$, consider the tuple $(\mathcal{O}, *, L, \Psi, h_0)$ such that:*

1. $\mathcal{O} = \mathcal{O}_K$
2. $*$: $\zeta_5 \rightarrow \zeta_5^{-1}$
3. $L = \mathcal{O}_K^3$
4. $\Psi : L \times L \rightarrow \mathbb{Z}$ is given by

$$\Psi(x, y) = \text{Tr}_{K/\mathbb{Q}}(\bar{x}^T B y)$$

$$\text{where } B = \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})} \text{ and } \alpha = \zeta_5 + \zeta_5^{-1}.$$

5. The $\mathbb{R}$ algebra homomorphism

$$h_0 : \mathbb{C} \rightarrow \text{End}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R})$$

is defined on $\mathbb{C}^*$ as:

$$h_0 : \mathbb{C}^* \rightarrow U(3, 0) \times U(2, 1)$$

$$h_0(x + yi) = \begin{bmatrix} x - yi & 0 & 0 \\ 0 & x - yi & 0 \\ 0 & 0 & x - yi \end{bmatrix}, \begin{bmatrix} x - yi & 0 & 0 \\ 0 & x - yi & 0 \\ 0 & 0 & x + yi \end{bmatrix}$$

The reflex field E of $(O, *, L, \Psi, h_0)$ is $E = \mathbb{Q}(\zeta_5)$.

Then, $\text{Sh}[16]/O_E[\frac{1}{5}]$ is the PEL type moduli space associated to $(O, *, L, \Psi, h_0)$ as above.

For $K = \mathbb{Q}(\zeta_5)$, $\text{Hom}(K, \mathbb{C}^*) = \{\tau_i\}_{i=1}^4$, such that $\tau_i(\zeta_5) = \zeta_5^i$. We can fix a CM type $\Sigma = \{\tau_1, \tau_2\}$. Then, the associated group $\mathcal{G}_1 = \text{Res}_{\mathbb{Z}}^{O_{K_0}} \text{U}(L, H)$ has the property that $\mathcal{G}_1(\mathbb{R}) = \text{U}(3, 0) \times \text{U}(2, 1)$. That is, $\text{Sh}[16]$ is unitary of signature type $(3, 0) \times (2, 1)$.

From the signature type, we see that $\text{Sh}[16]$ is of dimension 2 and compact. The main goal of this paper is to compute the ring of scalar valued automorphic forms of parallel weight for $\text{Sh}[16]$, and the precise definition of these automorphic forms will be given in the following section.

Remark 2.4.2. Just for the sake of completion, let's check that Ψ has the required signature and is alternating.

Notice that β is totally imaginary with imaginary part negative, and α is totally real. The two real embeddings $\tau_i : F_0 \rightarrow \mathbb{R}$ has the property that $\tau_1(\alpha) > 0, \tau_2(\alpha) < 0$. Therefore, the signature of the Hermitian pairing defined by B is $(3, 0, 2, 1)$, which is the desired signature.

We also check that Ψ is alternating. It suffices to check that $\Psi(x, x) = 0$. Suppose $x = ae_1$ or ae_2 for $a \in O_K$. Then $\text{Tr}_{K/\mathbb{Q}}(\beta a^* a) = \tau_1(\beta + \beta^*)(a^* a) + \tau_2(\beta + \beta^*)(a^* a)$. Since $\beta + \beta^* = 0$ in both extension of the τ_1, τ_2 , we have that $\Psi(x, x) = 0$. If $x = ae_2$, then since α is totally real, we compute $\text{Tr}_{K/\mathbb{Q}}(\beta \alpha a^* a)$, and $\alpha\beta$ is still totally imaginary. So the same reasoning applies to show that $\text{Tr}_{K/\mathbb{Q}}(x^* B y) = 0$. Hence, Ψ is alternating. The perfectness of the symplectic forms follow from the perfectness of trace pairing. $\Psi(bx, y) = \Psi(x, b^* y)$ since the B is K -linear.

We also check that h_0 satisfies

$$\Psi_{\mathbb{R}}(-, h_0(i)-) > 0$$

To check this, fix a CM type $\Phi = \{\tau_1, \tau_2\}$, where $\tau_1 : \zeta \rightarrow \zeta$ and $\tau_2 : \zeta \rightarrow \zeta^2$. This CM type gives us the isomorphism

$$V_{\mathbb{R}} = V \otimes_{K_0, \tau_1} \mathbb{R} \oplus V \otimes_{K_0, \tau_2} \mathbb{R}$$

Notice that $V \otimes_{K_0, \tau_i} \mathbb{R} \cong \mathbb{C}^3$ with the isomorphism $v \otimes a \rightarrow a\tau_i(v)$.

To evaluate the trace, notice that $x^*By \in K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \times \mathbb{C}$. $\text{Tr}_{K/\mathbb{Q}}$ extended to $\mathbb{R}$ sends (z_1, z_2) to $\text{Tr}_{\mathbb{C}/\mathbb{R}}(z_1) + \text{Tr}_{\mathbb{C}/\mathbb{R}}(z_2) = 2\Re(z_1) + 2\Re(z_2)$. That is

$$\text{Tr}_{K/\mathbb{Q}}(x^*By) = \text{Tr}_{\mathbb{C}/\mathbb{R}}(\tau_1(x^*By), \tau_2(x^*By)) = 2\Re(\tau_1(x^*By)) + 2\Re(\tau_2(x^*By))$$

The reason is, the Galois action on $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ is as follows: The complex conjugation conjugate each factor, while the extension $\text{Gal}(K_0/\mathbb{Q})$ acts by swapping the factor. Hence, the actions are:

- $\tau_1 : \zeta \rightarrow \zeta : (z_1, z_2) \rightarrow (z_1, z_2)$.
- $\tau_4 : \zeta \rightarrow \zeta^{-1} : (z_1, z_2) \rightarrow (\overline{z_1}, \overline{z_2})$.
- $\tau_2 : \zeta \rightarrow \zeta^2 : (z_1, z_2) \rightarrow (z_2, z_1)$
- $\tau_3 : \zeta \rightarrow \zeta^3 : (z_1, z_2) \rightarrow (\overline{z_2}, \overline{z_1})$

Notice the above action depends on the choice of the CM type. If we choose τ_3 instead of τ_2 for the CM type, then we end up with $\tau_3(z_1, z_2) = (z_2, z_1)$.

Also, under the isomorphism $V \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^3 \oplus \mathbb{C}^3$. This is an $\mathbb{R}$ linear isomorphism. The complex conjugation just conjugate each entry, it does not swop the factor.

Now, we verify the positive definiteness on the bases. Let e_1, e_2, e_3 be the basis of $V \otimes_{K_0, \tau_1} \mathbb{R}$, f_1, f_2, f_3 be the basis of $V \otimes_{K_0, \tau_2} \mathbb{R}$. That is, $e_1 = (e_1, 0) \in V \otimes_{K_0, \tau_1} \mathbb{R} \oplus V \otimes_{K_0, \tau_2} \mathbb{R}$. We have:

$$\begin{aligned} \Psi(e_1, h_0(i)e_1) &= \text{Tr}_{K/\mathbb{Q}}(e_1^* B h_0(i)e_1) \\ &\cong \text{Tr}_{\mathbb{C}/\mathbb{R}}(-i\tau_1(\beta)) + \text{Tr}_{\mathbb{C}/\mathbb{R}}(0) \\ &= 2\Im(\tau_1(\beta)) > 0 \end{aligned}$$

$$\begin{aligned} \Psi(e_3, h_0(i)e_3) &= \text{Tr}_{K/\mathbb{Q}}(e_3^* B h_0(i)e_3) \\ &\cong \text{Tr}_{\mathbb{C}/\mathbb{R}}(-i\tau_1(\alpha\beta)) + \text{Tr}_{\mathbb{C}/\mathbb{R}}(0) \\ &= 2\Im(\tau_1(\alpha\beta)) > 0 \end{aligned}$$

$$\begin{aligned} \Psi(f_1, h_0(i)f_1) &= \text{Tr}_{K/\mathbb{Q}}(f_1^* B h_0(i)f_1) \\ &\cong \text{Tr}_{\mathbb{C}/\mathbb{R}}(0) + \text{Tr}_{\mathbb{C}/\mathbb{R}}(-i\tau_1(\beta)) \\ &= 2\Im(\tau_2(\beta)) > 0 \end{aligned}$$

$$\begin{aligned} \Psi(f_3, h_0(i)f_3) &= \text{Tr}_{K/\mathbb{Q}}(f_3^* B h_0(i)f_3) \\ &\cong \text{Tr}_{\mathbb{C}/\mathbb{R}}(0) + \text{Tr}_{\mathbb{C}/\mathbb{R}}(i\tau_1(\alpha\beta)) \\ &= -2\Im(\tau_2(\alpha\beta)) > 0 \end{aligned}$$

Hence $\Psi(x, h_0(i)x)$ is positive definite.

2.5 Modular forms on PEL-type moduli spaces

In this section, we define the Hodge line bundle λ on PEL-type moduli spaces $\mathrm{Sh}(\mathcal{D})$ and more generally on its Shimura covers $\mathrm{Sh}_\Gamma(\mathcal{D})$. Using this, we define its ring of modular forms, which is the main focus of study of this paper.

We first define the Hodge line bundle λ . This definition follows from (Lan, 2008, Definition 7.1.1.1).

Definition 2.5.1. *Let $\mathrm{Sh}(\mathcal{D})$ be the PEL-type moduli space as defined in Section 2.4. Let $\Gamma \subseteq \mathcal{G}(\hat{\mathbb{Z}})$ be an open compact, and $\mathrm{Sh}_\Gamma(\mathcal{D}) \rightarrow \mathrm{Sh}(\mathcal{D})$ be the corresponding Shimura cover at level Γ . Let $(\mathcal{A}, \lambda, i, \alpha) \rightarrow \mathrm{Sh}_\Gamma(\mathcal{D})$ be the corresponding family of principally polarized abelian variety with level structure α . Suppose $\pi : (\mathcal{A}, \lambda, i, \alpha) \rightarrow \mathrm{Sh}_\Gamma(\mathcal{D})$ is a family of dimension g .*

- The Hodge vector bundle $\mathbb{E}_{\mathrm{Sh}}$ is given as

$$\mathbb{E}_{\mathrm{Sh}} = \pi_* \Omega_{\mathcal{A}/\mathrm{Sh}}$$

Here, $\Omega_{\mathcal{A}/\mathrm{Sh}}$ is the relative sheaf of differential of the family π .

If $[A] \in \mathrm{Sh}_\Gamma(\mathcal{D})$ is a point representing an abelian variety A , then $\mathbb{E}_{\mathrm{Sh}, [A]} = H^0(A, \Omega_A)$ is the vector space of global differentials on A . The stalk of the Hodge vector bundle is $\mathbb{E}_{\mathrm{Sh}, [A]} = H^0(A, \Omega_A)$.

- The Hodge line bundle is given as λ_{Sh} :

$$\lambda_{\mathrm{Sh}} = \bigwedge^g \pi_* \Omega_{\mathcal{A}/\mathrm{Sh}}$$

The global sections of $\lambda_{\mathrm{Sh}_\Gamma(\mathcal{D})}$ are the (scalar-valued) weight modular forms on $\mathrm{Sh}_\Gamma(\mathcal{D})$.

Definition 2.5.2. *Let $\mathrm{Sh}_\Gamma(\mathcal{D})$ and λ be as defined in Definition 2.5.1.*

- The (parallel) weight- k modular forms $M_k(\mathrm{Sh}_\Gamma(\mathcal{D}))$ are the global sections of k -fold product of λ_{Sh} :

$$M_k(\mathrm{Sh}_\Gamma(\mathcal{D})) := H^0(\mathrm{Sh}_\Gamma(\mathcal{D}), \lambda^k)$$

- The ring of (scalar-valued weight) modular forms on $\mathrm{Sh}_\Gamma(\mathcal{D})$ is:

$$M(\mathrm{Sh}_\Gamma(\mathcal{D})) := \bigoplus_k H^0(\mathrm{Sh}_\Gamma(\mathcal{D}), \lambda^k)$$

Remark 2.5.3. The above definition works for all the PEL type moduli space Sh . In most of the literature, $M(\text{Sh}_\Gamma(\mathcal{D}))$ is referred to as the ring of scalar valued automorphic forms of parallel weight. We mainly focus on the computation of the ring for the Shimura surface $\text{Sh}[16]$ and other Deligne-Mostow Shimura varieties.

From now on, when we say "the ring of modular forms", we refer to Definition 2.5.2.

Remark 2.5.4. In this paper, we will only consider the ring at two levels for Deligne-Mostow Shimura varieties $\text{Sh}(\mathcal{D})$

- when $\Gamma = \mathcal{G}(\hat{\mathbb{Z}})$, then $\text{Sh}_\Gamma(\mathcal{D}) = \text{Sh}(\mathcal{D})$;
- when $\Gamma = \mathcal{G}(\hat{\mathbb{Z}}^m)\Gamma_m$ as in the later Definition 4.2.2, then $\text{Sh}_\Gamma(\mathcal{D}) = \text{Sh}_{\Gamma_m}(\mathcal{D})$.

The main goal of this paper is to compute the ring $M(\text{Sh}[16])$ as well as $M(\text{Sh}_{\Gamma_5}[16])$.

Chapter 3

THE MODULI SPACES OF CURVES

In this section, we define various moduli spaces of curves. We then review the geometry of $\overline{M}_{0,5}$ and the geometry on cyclic covers of $\mathbb{P}^1$ in detail.

3.1 Definition as moduli functors

In this sub-section, we give the moduli problem that define the moduli space $\mathcal{M}(m, r, \underline{a})$, $\overline{\mathcal{M}}(m, r, \underline{a})$ and $\widetilde{\mathcal{M}}(m, r, \underline{a})$ and $\overline{M}_{0,r}$. The following is a summary of relevant definition from (Achter and Pries, 2007)[Section 2.1-2.2].

Let $\mathbb{Z}/m\mathbb{Z}$ be a cyclic group of size m . Let S be irreducible over $\text{Spec}(\mathbb{Z}[\zeta_m, \frac{1}{m}])$. Fix an embedding $\alpha : \mathbb{Z}[\zeta_m, \frac{1}{m}] \rightarrow k$ where k is algebraically closed of characteristic prime to m .

Let $\pi : C \rightarrow S$ be a family of **semi-stable curves** with $\mathbb{Z}/m\mathbb{Z}$ action. That is, π is flat, proper, for every closed point $s \in S$, C_s is connected, reduced, with singularity being ordinary double point. Moreover, there is an embedding $\iota : \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Aut}_S(C)$.

A **mark** Δ on C/S is a closed subscheme $\Delta \subseteq (C - \text{Sing}_S(C))$ such that it is finite and etale over S . The **degree** of the marking Δ is $\text{Deg}(\Delta) = \#(\Delta \cap C_s)$, for any geometric point $s \in S$.

Let R be the ramification locus of the quotient $q : C \rightarrow C/\iota(\mathbb{Z}/m\mathbb{Z})$, so R is a divisor on C . Let $R_{sm} = R - R \cap \text{Sing}_S(C)$.

We first define the notion of stable family of curves.

Definition 3.1.1. *The family $\pi : C \rightarrow S$ with $\mathbb{Z}/m\mathbb{Z}$ action is **stable** if $(C/S, R_{sm})$ is stably marked. That is, at every geometric fibre C of C , the marking $\Delta = R_{sm} \cup \text{Sing}(C)$ is such that for every component C_0 of genus 0, $\deg(\Delta \cap C_0) \geq 3$.*

We consider the stable family where the $\mathbb{Z}/m\mathbb{Z}$ action is admissible.

Definition 3.1.2. *Consider a stable family with $\mathbb{Z}/m\mathbb{Z}$ action $(\pi : C \rightarrow S, \iota)$. For a point $z \in \text{Sing}_S(C)$, Let $C_z = C_{\pi(z)}$ be the fibre above $\pi(z)$. Since C_z only has ordinary double point, we have that $C_z = C_{z,1} \sqcup C_{z,2}$, where $C_{z,1}, C_{z,2}$ intersect at z transversally. Let $\iota : \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Aut}_{k(z)}(C_z)$.*

Then, the family is said to be **admissible**, if ι satisfies the following:

- $\iota(\mathbb{Z}/m\mathbb{Z})$ stabilizes $C_{z,1}$ and $C_{z,2}$
- Let x be the local uniformizer of $C_{z,1}$ around point z and y be the local uniformizer of $C_{z,2}$ around point z . The representation of $\mathbb{Z}/m\mathbb{Z}$ induced via ι at the tangent space of C at z , $\iota : \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Aut}_{k(z)}(T_z(C_z))$ is such that

$$\iota(1)(xy) = xy$$

in the tangent space. In other words, $\chi_{C_{1,z}} = -\chi_{C_{2,z}}$, where $\chi_{C_{i,z}}$ is the character of $\mathbb{Z}/m\mathbb{Z}$ at $T_z(C_{i,z})$.

Notice that this notion of admissible is necessary for the $\mathbb{Z}/m\mathbb{Z}$ action on $C_{z,1}$ and $C_{z,2}$ to extends to C_z at the node.

Now, given a stable family $\pi : C \rightarrow S$, $\iota : \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Aut}_S(C)$, we further assume that $C/\iota(C)$ has arithmetic genus 0. In other words, $q : C \rightarrow C/\iota(\mathbb{Z}/m\mathbb{Z})$ is a $\mathbb{Z}/m\mathbb{Z}$ cover of $\mathbb{P}^1$.

For each ramification point $a \in C$, the notion of local monodromy captures the action of $\mathbb{Z}/m\mathbb{Z}$ at tangent space $T_a(C)$.

Definition 3.1.3 (local monodromy at P). Let $s : \text{Spec}(k) \rightarrow S$ be a field valued point. Let $C = C_s$ be the fibre at s , and $P \in R_{sm,s}$ be a ramification point on C . Then the induced representation of $\mathbb{Z}/m\mathbb{Z}$ at the tangent space at a :

$$\iota : \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Aut}_k(T_P(C))$$

is via a character $\chi_P : \mathbb{Z}/m\mathbb{Z} \rightarrow k^*$. Then there exists $a_P \in (\mathbb{Z}/m\mathbb{Z})^*$ such that

$$\chi_P(1) = \zeta_m^{a(P)}$$

$a(P)$ is called the **local monodromy** at P .

Remark 3.1.4. Notice that if P_1, P_2 are above the same $Q \in C/\iota(\mathbb{Z}/m\mathbb{Z})$, then $a(P_1) = a(P_2)$. So the local monodromy only depends on the projection of P on $\mathbb{P}^1$ on the base. By Riemann existing Theorem, $\sum_P a(P) = 0 \pmod{m}$, where the sum is over the branching locus and we choose exactly one preimage per branching locus.

To fix the irreducible component of the moduli space, we further specify a data called inertia type, that specifies the action of $\mathbb{Z}/m\mathbb{Z}$ at the tangent space of C .

Now, let $R \subseteq C$ be the ramification divisor and let $B \subseteq C/\iota(\mathbb{Z}/m\mathbb{Z})$ be the branching locus. Let r be the degree of B . That is, for $s \in S$, the covering $q : C_s \rightarrow \mathbb{P}^1$ has r branching points.

Let's first define labeling on the smooth part of the ramification locus.

Definition 3.1.5 (Labeling on R). *Let $(\pi : C \rightarrow S, \iota)$ be a family $\mathbb{Z}/m\mathbb{Z}$ cover of $\mathbb{P}^1$. Let R be the ramification divisor and B be the branching locus of degree r . Then, an labeling of R_{sm} is a map*

$$\eta : \{1, 2, \dots, r\} \rightarrow R_{sm}$$

η induces a labeling of B_{sm}

$$\eta_0 : \{1, 2, \dots, r\} \rightarrow B_{sm}$$

Such that $\eta_0(i) = \pi(\eta(i))$. Notice that the choice of η_0 is the same as an ordering on the branching locus, and the choice of η is the same as choosing one prime per branching point.

Now we can define the inertia type of the family, which is an r -tuple $\underline{a}$.

Definition 3.1.6 (Inertia type). *Let $(C/S, \iota, \eta)$ be a family of stable $\mathbb{Z}/m\mathbb{Z}$ family with an labeling η on the R_{sm} . Then the ***inertia type*** of C is the r -tuple $\underline{a}$:*

$$\underline{a} : \{1, 2, \dots, r\} \rightarrow (\mathbb{Z}/m\mathbb{Z})^*$$

$$a(i) := a_{\eta(i)}$$

That is, $a(i)$ is the character of $\mathbb{Z}/m\mathbb{Z}$ acting on the i^{th} ramification point in the ordering.

Remark 3.1.7. (Mantovan, 2020, section 2.2) The inertia type $\underline{a}$ satisfies the following properties:

- $a(i) \not\equiv 0 \pmod{m}$, for all $1 \leq i \leq r$
- $\gcd(m, a(1), \dots, a(r)) = 1$
- $\sum_{i=1}^r a(i) \equiv 0 \pmod{m}$

Remark 3.1.8. Notice that the data of inertia type $\underline{a}$ gives a **class vector**

$$\begin{aligned}\underline{\gamma}_a &: \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z} \\ \underline{\gamma}_a(h) &= \#(a^{-1}(h))\end{aligned}$$

Fixing the data of $\underline{\gamma}_a$ is equivalent to having an equivalence class of inertia type $\underline{a}$, where $\underline{a}_1 \sim \underline{a}_2$ if there exists $\sigma \in \mathfrak{S}_r$ such that $a_1(i) = a_2(\sigma(i))$, for all $1 \leq i \leq r$.

Definition 3.1.9 ($\widetilde{\mathcal{M}}(m, r, \underline{a})$ and $\overline{\mathcal{M}}(m, r, \underline{a})$). We define the following moduli functors:

- $\widetilde{\mathcal{M}}(m, r, \underline{a})/\mathbb{Z}[\frac{1}{m}, \zeta_m]$ is the moduli space of admissible curves with $\mathbb{Z}/m\mathbb{Z}$ action, with inertia type given by $\underline{a}$. That is,

$$\widetilde{\mathcal{M}}(m, r, \underline{a})(S) = \{(C/S, \iota, \eta) : \deg(B_{sm}) = r, C \text{ has inertia type } \underline{a}\}$$

- $\overline{\mathcal{M}}(m, r, \underline{a})/\mathbb{Z}[\frac{1}{m}, \zeta_m]$ is the moduli space of admissible $\mathbb{Z}/m\mathbb{Z}$ curves with $\mathbb{Z}/m\mathbb{Z}$ action, with class vector given by $\underline{\gamma}_a$. That is,

$$\overline{\mathcal{M}}(m, r, \underline{a})(S) = \{(C/S, \iota, \eta) : \deg(B_{sm}) = r, C \text{ has inertia type } \sim \underline{a}\}$$

- $\mathcal{M}(m, r, \underline{a})/\mathbb{Z}[\frac{1}{m}, \zeta_m]$ is the locus of smooth curves in $\overline{\mathcal{M}}(m, r, \underline{a})$. That is,

$$\mathcal{M}(m, r, \underline{a})(S) = \{(C/S, \iota, \eta) : \deg(B_{sm}) = r, C \text{ has inertia type } \sim \underline{a}, C_s \text{ is smooth}\}$$

- $\overline{\mathcal{M}}_{0,r}/\mathbb{Z}$ is the moduli space of genus 0 curve with r marked point. That is,

$$\overline{\mathcal{M}}_{0,r}(S) = \{(\mathcal{P}/S, \{\sigma_i\}_{i=1}^r) : \mathcal{P}_s \text{ has genus 0, } \sigma_i : S \rightarrow \mathcal{P} \text{ are disjoint sections on } \mathcal{P}\}$$

We summarize some properties of $\mathcal{M}(m, r, \underline{a})$.

Proposition 3.1.10. (Mantovan, 2020, section 2.1) We have the following properties of the family $\mathcal{M}(m, r, \underline{a})$:

1. The coarse moduli space of $\mathcal{M}(m, r, \underline{a})$ is $\mathcal{U} = \mathbb{A}^r - D$, where D is the union of weak diagonals $Z(x_i - x_j)$.
2. $\mathcal{M}(m, r, \underline{a})$ has the family of universal curve:

$$C \rightarrow \mathcal{M}(m, r, \underline{a})$$

Where the fibre at $(x_1, \dots, x_r)$ is given by:

$$C_{\underline{x}} : y^m = \prod_{j=1}^r (x - x_j)^{a(j)}$$

3. The genus of the curves in $\mathcal{M}(m, r, \underline{a})$ is given by:

$$g(m, r, \underline{a}) = 1 + \frac{(r-2)m - \sum_{j=1}^r \gcd(a(j), m)}{2}$$

4. For a curve C in the family, let $V_0 = H^0(C, \Omega_C)$. It is a $\mathbb{Q}[\mu_m] \otimes \mathbb{C}$ module. Let $\tau_i \in \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*)$ be such that

$$\tau_i(\zeta_m) = \zeta_m^i$$

Then $\mathbb{Q}[\mu_m] \otimes \mathbb{C}$ module structure of V_0 is given by the **signature type**

$$\underline{f} : \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*) \rightarrow \mathbb{Z}$$

$$f(\tau_i) = \dim(V_{0, \tau_i})$$

And $\underline{f}$ is computed by the Chevalley-Weil formula:

$$f(\tau_i) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod{m} \\ -1 + \sum_{j=1}^r \left\langle \frac{-ia(j)}{m} \right\rangle & \text{otherwise} \end{cases}$$

3.2 Hodge line bundle on Hurwitz spaces

Similar to PEL-type moduli spaces Sh , the Hurwitz spaces $\mathcal{M}$ has a family of curves $C \rightarrow \mathcal{M}$ associated to it. On any such moduli space of curves $\mathcal{M}$ (which includes $\widetilde{\mathcal{M}}(m, r, \underline{a})$, $\overline{\mathcal{M}}(m, r, \underline{a})$ and $\overline{\mathcal{M}}_{0,r}$), we can similarly define the Hodge line bundle.

Definition 3.2.1. Let $\mathcal{M}$ be a moduli space of curves and let $C \rightarrow \mathcal{M}$ be the corresponding family. In the context of this paper, $\mathcal{M}$ can be taken as either $\mathcal{M}(m, r, \underline{a})$, $\overline{\mathcal{M}}(m, r, \underline{a})$ or $\widetilde{\mathcal{M}}(m, r, \underline{a})$ as in Definition 3.1.9. Suppose $\pi : C \rightarrow \mathcal{M}$ is a family of curves with genus g .

- The Hodge vector bundle is given as

$$\mathbb{E}_{\mathcal{M}} = \pi_* \Omega_{C/\mathcal{M}}$$

Here, $\Omega_{C/\mathcal{M}}$ is the relative dualizing sheaf of the family π .

If $[C] \in \mathcal{M}$ is a point representing a smooth curve C , then $\mathbb{E}_{\mathcal{M}, [C]} = H^0(C, \Omega_C)$ is the vector space of global differentials on C . If $[C] \in \mathcal{M}$ is a point representing a nodal curve, then $\mathbb{E}_{C/\mathcal{M}, [C]} = H^0(C, \omega_C)$ is the global sections of the dualizing sheaf of C .

- The Hodge line bundle is given as:

$$\lambda_{\mathcal{M}} = \bigwedge^g \pi_* \Omega_{C/\mathcal{M}}$$

Remark 3.2.2. Notice that in our context, $\lambda_{\mathcal{M}}$ might not have non-zero global sections. Indeed, for a generic curve $C_{\eta} \in \widetilde{\mathcal{M}}(m, r, \underline{a})$, C_{η} has non-trivial automorphism $\mathbb{Z}/m\mathbb{Z}$. For any global section $f \in H^0(\widetilde{\mathcal{M}}, \lambda)$, f should be fixed under the action of $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$. On the other hand, the $\mathbb{Z}/m\mathbb{Z}$ action on $\lambda_{[C]} = H^0(C, \Omega_C)$ is encoded in the signature type as defined in Proposition 3.1.10 Equation (4). So, the character of $\mathbb{Z}/m\mathbb{Z}$ acting on λ is given by $\sum_{i \in \mathbb{Z}/m\mathbb{Z}} i \cdot f(\tau_i)$. So, the smallest k for which $\lambda^{\otimes k}$ has global section should be the smallest k such that

$$k \cdot \sum_{i=1}^m i f(\tau_i) \equiv 0 \pmod{m}$$

If we consider $C_{\eta} \in \overline{\mathcal{M}}(m, r, \underline{a})$, then C_{η} has automorphism $\mathbb{Z}/m\mathbb{Z} \times G$, where $G = \{\sigma \in \mathfrak{S}_r \mid \sigma \cdot \underline{a} \sim \underline{a}\}$, and the equivalence of inertia type is as defined in Remark 3.1.8. Hence, we can identify the global sections of $\lambda_{\overline{\mathcal{M}}}$ as the G -invariant subspace of $\lambda_{\widetilde{\mathcal{M}}}$:

$$H^0(\overline{\mathcal{M}}, \lambda^k) = (H^0(\widetilde{\mathcal{M}}, \lambda^k))^G$$

3.3 Divisors of $\overline{M}_{0,5}$

In this section, we review the geometry on $\overline{M}_{0,5}$. The materials from these sections are from (Kock and Vainsencher, 2007, section 1) and (Hartshorne, 2013, section V.3).

$\overline{M}_{0,5}$ is the compactified moduli space of genus 0 with five marked points. It is defined over $\mathbb{Q}$ and can be realized explicitly as an algebraic variety. By (Kock and Vainsencher, 2007, p. I.2.6), $\overline{M}_{0,5}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blow up at three points, which is also isomorphic to $\mathbb{P}^2$ blow up at four points. We use $\text{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1)$ to denote $\mathbb{P}^1 \times \mathbb{P}^1$ blow up at three points $Q_1 = [0, 0]$, $Q_2 = [1, 1]$, $Q_3 = [\infty, \infty]$ and use $\text{Bl}_4(\mathbb{P}^2)$ to denote the blow up of $\mathbb{P}^2$ at four points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$, $P_4 = [1 : 1 : 1]$. Since $\text{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1)$ has a clearer moduli interpretation and the intersection theory on $\text{Bl}_4(\mathbb{P}^2)$ is described in (Hartshorne, 2013, Section V.3) in detail, we list their divisors as below and give the explicit correspondence among them.

On $\overline{M}_{0,5} \cong \text{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1)$, we list the 10 divisors as below:

Proposition 3.3.1. (Kock and Vainsencher, 2007, Example I.2.6) Let $(0, 1, \infty, \alpha, \beta)$ be the five marked points on $\mathbb{P}^1$. The 10 boundary divisors in $\overline{M}_{0,5}$ are:

1. $E_1 = (0, \infty; 1, \alpha, \beta)$
2. $E_0 = (1, \infty; 0, \alpha, \beta)$
3. $E_\infty = (0, 1; \infty, \alpha, \beta)$
4. $\Delta = (0, 1, \infty; \alpha, \beta)$
5. $U_0 = (0, \alpha; 1, \beta, \infty)$
6. $T_0 = (0, \beta; 1, \alpha, \infty)$
7. $U_1 = (1, \alpha; 0, \beta, \infty)$
8. $T_1 = (1, \beta; 0, \alpha, \infty)$
9. $U_\infty = (\infty, \alpha; 1, \beta, 0)$
10. $T_\infty = (\infty, \beta; 1, \alpha, 0)$

Here, the boundary divisor $(0, \infty; 1, \alpha, \beta)$ means that it is the clutching of $(\mathbb{P}^1, (0, \infty, t))$ and $(\mathbb{P}^1, (t, 1, \alpha, \beta))$ glued along the section t .

Since $\text{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1)$ is a monoidal transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\text{Bl}_4(\mathbb{P}^2)$ is a monoidal transformation of $\mathbb{P}^2$, we have the following proposition on their picard group:

Proposition 3.3.2. (Kock and Vainsencher, 2007, Example I.2.6) $\text{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1)$ has 10 special divisor that generates the Picard group. They are:

- The exceptional divisors $E_{Q_1}, E_{Q_2}, E_{Q_3}$;
- The strict transformation of the diagonal $L_{Q_1 Q_2 Q_3}$,
- The strict transformations of the natural copies of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$: $\{0\} \times \mathbb{P}^1, \mathbb{P}^1 \times \{0\}, \{1\} \times \mathbb{P}^1, \mathbb{P}^1 \times \{1\}, \{\infty\} \times \mathbb{P}^1, \mathbb{P}^1 \times \{\infty\}$.

Proposition 3.3.3. (Hartshorne, 2013, Proposition 3.2)

1. On $\text{Bl}_4(\mathbb{P}^2)$ there are 10 special divisors that generates the Picard group. They are:

- The exceptional divisors $E_{P_1}, E_{P_2}, E_{P_3}, E_{P_4}$
- The strict transformation of the lines connecting the each pairs of the point: $L_{P_1, P_4}, L_{P_1, P_3}, L_{P_1, P_2}, L_{P_2, P_4}, L_{P_3, P_4}, L_{P_2, P_3}$.

2. The intersection theory on $Bl_4(\mathbb{P}^2)$ are:

- $E_{P_i}^2 = -1$
- $E_{P_i} \cdot E_{P_j} = 0$
- $E_{P_i} \cdot L_{P_i P_j} = 1$

Under the isomorphism $\overline{M}_{0,5} \cong Bl_3(\mathbb{P}^1 \times \mathbb{P}^1) \cong Bl_4(\mathbb{P}^2)$, we identify the 10 divisors.

Proposition 3.3.4. *The 10 divisors on $\overline{M}_{0,5}$, $Bl_3(\mathbb{P}^1 \times \mathbb{P}^1)$, $Bl_4(\mathbb{P}^2)$ have the following correspondence:*

1. $E_0 = (1, \infty; 0, \alpha, \beta) \iff E_{Q_1} \iff L_{P_2, P_3}$
2. $E_1 = (0, \infty; 1, \alpha, \beta) \iff E_{Q_2} \iff E_{P_4}$
3. $E_\infty = (0, 1; \infty, \alpha, \beta) \iff E_{Q_3} \iff E_{P_1}$
4. $\Delta = (\alpha, \beta; 0, 1, \infty) \iff L_{Q_1 Q_2 Q_3} \iff L_{P_1, P_4}$
5. $U_0 = (0, \alpha; 1, \beta, \infty) \iff 0 \times \mathbb{P}^1 \iff E_{P_3}$
6. $T_0 = (0, \beta; 1, \alpha, \infty) \iff \mathbb{P}^1 \times 0 \iff E_{P_2}$
7. $U_1 = (1, \alpha; 0, \beta, \infty) \iff 1 \times \mathbb{P}^1 \iff L_{P_2, P_4}$
8. $T_1 = (1, \beta; 0, \alpha, \infty) \iff \mathbb{P}^1 \times 1 \iff L_{P_3, P_4}$
9. $U_\infty = (\infty, \alpha; 1, \beta, 0) \iff \infty \times \mathbb{P}^1 \iff L_{P_1, P_2}$
10. $T_\infty = (\infty, \beta; 1, \alpha, 0) \iff \mathbb{P}^1 \times \infty \iff L_{P_1, P_3}$

Now, given a divisor D on $\overline{M}_{0,5}$, D descent to $\overline{M}_{0,5}/\Sigma_5$ iff the intersection number of D with the 10 boundary divisor are the same. Notice that $\text{Pic}(Bl_4(\mathbb{P}^2))$ can be generated by H, E_1, E_2, E_3, E_4 , where $H = E_i + L_{ij} + E_j$ for $i, j \in \{1, 2, 3, 4\}$. Suppose D has intersection number b with all the boundary divisor. Then $D = 3bH - b \sum E_i$. For D to be ample, we have $b > 0$. Notice that $D \cdot E_i = b$

The canonical divisor K on $\text{Bl}_4(\mathbb{P}^2)$ is:

$$K = -3H + \sum_{i=1}^4 E_{P_i}$$

so the anti-canonical divisor is $3H - \sum_{i=1}^4 E_{P_i}$. We denote the sum of boundary divisor as Δ . Notice that $\Delta \cdot E_1 = -1 + 1 + 1 + 1 = 2$. Hence $\Delta \equiv -2K$. The anti-canonical embedding $|-K_{\text{Bl}_4(\mathbb{P}^2)}| : \overline{M}_{0,5} \rightarrow \mathbb{P}^5$ realizes $\overline{M}_{0,5}$ as the degree 5 Del-Pezzo surface in $\mathbb{P}^5$. Its defining equations in $\mathbb{P}^5$ will be given explicitly in Proposition 5.5.3.

3.4 Boundary of Hurwitz spaces

In this section, we discuss the clutching morphism on $\widetilde{\mathcal{M}}(m, r, \underline{a})$ and the configuration of curve at the boundary.

We use $[C \rightarrow P, \{P_i\}]$ to denote an object of $\widetilde{\mathcal{M}}(m, r, \underline{a})$. Here C is the admissible $\mathbb{Z}/m\mathbb{Z}$ cover, and $P = C/\iota(\mathbb{Z}/m\mathbb{Z})$ is a curve of genus 0. When C is smooth, then $P \cong \mathbb{P}^1$.

Definition 3.4.1. *We have the following clutching morphism:*

$$\kappa : \widetilde{\mathcal{M}}(m, r_1, \underline{a}_1) \times \widetilde{\mathcal{M}}(m, r_2, \underline{a}_2) \rightarrow \widetilde{\mathcal{M}}(m, r, \underline{a})$$

source curve: $(C_1, \{P_i\}_{i=1}^{r_1}) \times (C_2, \{Q_i\}_{i=1}^{r_2}) \mapsto (C_1, \{P_i\}_{i=1}^{r_1-1}) \cup_{P_{r_1}=Q_1} (C_2, \{Q_i\}_{i=1}^{r_2-1})$

target curve: $(P_1, \{p_i\}_{i=1}^{r_1}) \times (P_2, \{q_i\}_{i=1}^{r_2}) \mapsto (P_1, \{p_i\}_{i=1}^{r_1-1}) \cup_{p_{r_1}=q_1} (P_2, \{q_i\}_{i=1}^{r_2-1})$

The monodromy data need to satisfy the following condition:

1. $r_1 + r_2 = r + 2$
2. $3 \leq r_1 \leq r - 1$
3. $\underline{a}_1(r_1) \equiv -\underline{a}_2(1) \pmod{m}$
4. $\underline{a} = (\underline{a}_1(1), \dots, \underline{a}_1(r_1 - 1), \underline{a}_2(2), \dots, \underline{a}_2(r_2))$.

Here $\pi_1 : C_1 \rightarrow P_1$ is a $\mathbb{Z}/m\mathbb{Z}$ cover with markings $\{P_i\}$ parametrized by $\widetilde{\mathcal{M}}(m, r_1, \underline{a}_1)$ and $\pi_2 : C_2 \rightarrow P_2$ is a $\mathbb{Z}/m\mathbb{Z}$ cover with markings $\{Q_i\}$ parametrized by $\widetilde{\mathcal{M}}(m, r_2, \underline{a}_2)$.

Remark 3.4.2. The conditions for monodromy data are exactly the admissible condition in (Li, Mantovan, Pries, and Tang, 2018, Definition 2.2, 3.3, 3.5). When

the condition is satisfied, by (Li, Mantovan, Pries, and Tang, 2018, Proposition 3.9), the clutching morphism is well defined, and the image sits inside the boundary of $\widetilde{\mathcal{M}}(m, r, \underline{a})$.

In the rest of this section, we use $\partial\widetilde{\mathcal{M}}$ to denote the boundary of the moduli space $\widetilde{\mathcal{M}}$.

Proposition 3.4.3. *Let $S \subseteq \{1, 2, \dots, r\}$ be such that $2 \leq \#S \leq \lfloor \frac{r}{2} \rfloor$. Let S^c be the complement of S in $\{1, 2, \dots, r\}$. Then we have that*

$$\partial\widetilde{\mathcal{M}}(m, r, \underline{a}) = \bigcup_S \delta_S$$

where $\delta_S = \kappa(\widetilde{\mathcal{M}}(m, r_1, \underline{a}_1) \times \widetilde{\mathcal{M}}(m, r_2, \underline{a}_2))$, and the monodromy datum $(m, r_1, \underline{a}_1)$ is constructed from S as follows:

- $r_1 = \#S + 1$;
- for $1 \leq i \leq \#S$, $a_1(i) = a(s(i))$. Here $s(i)$ is the i^{th} element of S .
- $a_1(r_1) \equiv -\sum_{i=1}^{r_1-1} a_1(i) \pmod{m}$ is the residue mod m .

, and $r_2, \underline{a}_2$ are similarly defined from S^c . Notice that the monodromy data defined as such satisfies the condition in Definition 3.4.1.

In other words, the boundary consists of the image of clutching morphism.

Notice that in the above notation, if r is even, then $\delta_S = \delta_{S^c}$.

Proof. We have the two forgetful map:

$$q : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \overline{\mathcal{M}}_{0,r}$$

$$q : [C \rightarrow P, \{P_i\}] \mapsto [P, \{p_i\}]$$

Now, suppose $[\pi : C \rightarrow P, \{P_i\}]$ is in $\partial\widetilde{\mathcal{M}}(m, r, \underline{a})$. Then $[P, \{p_i\}] \in \partial\overline{\mathcal{M}}_{0,r}$. By (Kock and Vainsencher, 2007, p. 1.5.9), $\partial\overline{\mathcal{M}}_{0,r} = \bigcup_S \Delta_S$, where $2 \leq \#S \leq r - 2$ and $\Delta_S = \kappa(\overline{\mathcal{M}}_{0,S \cup \{x\}} \times \overline{\mathcal{M}}_{0,S^c \cup \{x\}})$. Now, suppose $[P, \{p_i\}] \in \Delta_S$. Then, $P = P_1 \sqcup_x P_2$, where P_1 is marked by $p_{s(1)}, p_{s(2)}, \dots, p_{s(r_1+1)}$ and x being the node, and P_2 is marked by $\{p_{s^c(i)}\} \cup \{x\}$.

Now consider $\pi : C \rightarrow P$ restricted to P_1 and denote the preimage as $\pi_1 : C_1 \rightarrow P_1$. Then π_1 is a $\mathbb{Z}/m\mathbb{Z}$ cover and the local monodromy on C_1 on the smooth branching points are given by $\underline{a} \mid_{C_1}$. By Definition 3.1.3, the local monodromy at the node X above x is given by $a_1(r_1) + \sum_{i=1}^{r_1} a_1(i) \equiv 0 \pmod{m}$ and $0 \leq a_1(r_1) \leq m-1$. Similarly for C_2 . Hence, $[C \rightarrow P, \{P_i\}] \in \kappa(\widetilde{\mathcal{M}}(m, r_1, \underline{a}_1) \times \widetilde{\mathcal{M}}(m, r_2, \underline{a}_2)) = \delta_S$.

□

3.5 Local structure at a smooth point

In this section, we demonstrate the local equation of C in C at a smooth point. This gives us the ramification index at the ramification point of local monodromy given by $a(i)$, and also gives us the action of $\mathbb{Z}/m\mathbb{Z}$ at the tangent space.

Similar to last section, we fix $\pi : C \rightarrow P$ a closed point in $\widetilde{\mathcal{M}}(m, r, \underline{a})$. We further fix a point $q \in B_{sm}$ and a point $Q \in R_{sm}$ above q . Let a be the local monodromy at Q as in definition 3.1.3.

We in fact have $\pi : C \rightarrow C' \rightarrow P$. C' is the scheme cut out by $y^m = \prod_{i=1}^r (x - x_i)^{a(i)}$ and C is its normalization. Let $Q' \in C'$ be such that $\pi : Q \rightarrow Q' \rightarrow q$. This induced morphism on the local rings

$$\pi^\# : \mathcal{O}_{P,q} \rightarrow \mathcal{O}_{C',Q'} \rightarrow \mathcal{O}_{C,Q}$$

Let t be the local parameter around q on P . One can take $t = x - x(q)$. Let y be a local parameter around Q' . Then in a neighborhood of Q' , the local equation of C' is $y^m - ut^a$ and the coordinate ring of C' around Q' is $R' = \mathcal{O}_{P,q}[y]/y^m - ut^a$.

Now consider the normalization of R' , denoted as R . In (ZHANG, 2012) there is an algorithm to compute R :

$$R = \bigoplus_{j=0}^{m-1} y^j t^{-\lfloor \frac{aj}{m} \rfloor} R'$$

We demonstrate the ring morphism $R' \rightarrow R$ in the following two cases:

1. If $\gcd(m, a) = 1$. Then, there exists s, l such that $sa = 1 + lm$. Consider $g = y^s t^{-\lfloor \frac{as}{m} \rfloor} \in R$. From the relation $y^m = ut^a$, we see that $v_P(y) = a, v_P(t) = m$, and $v_P(g) = 1$. So g is the local uniformizer at P . We also have $g^m = u^s t$ and $y = u^{-l} g^a$. This gives the morphism

$$\begin{aligned} \pi^\# : \mathcal{O}_{P,q} &\rightarrow \mathcal{O}_{C,Q} \\ t &\mapsto \frac{g^m}{u^s} \end{aligned}$$

Hence, $\pi^\#$ is totally ramified of degree m , and the maximal ideal $\mathfrak{m}_q$ has image $\pi^\#(\mathfrak{m}_q) = (\mathfrak{m}_Q)^m$. So $\pi^{-1}(q)$ consists only of one point Q .

2. In general, if $\gcd(m, a) = d$. Then write $m = dm_0$ and $a = db$. We have

$$y^m - t^a = \prod_{i \in \mathbb{Z}/d\mathbb{Z}} (y^{m_0} - \zeta_d^i t^b)$$

In particular, $R' = \bigoplus_{i=0}^{d-1} R'_i$, where $R'_i = \mathcal{O}_{P,q}[y]/(y^{m_0} - \zeta_d^i t^b)$. Let R_i be the normalization of R'_i in its function field.

Since $\gcd(m_0, b) = 1$, there exists s, l be such that $sb = 1 + lm_0$. We denote ζ_d^i by u_i . By the same reasoning as above, for each $i \in \mathbb{Z}/d\mathbb{Z}$, we can take $g_i = y^s t^{-\lfloor \frac{sb}{m_0} \rfloor} \in R_i$. That is, $g_i = (1, 1, \dots, y^s t^{-\lfloor \frac{sb}{m_0} \rfloor}, 1, \dots, 1) \in \bigoplus_{i=0}^{d-1} R_i$. Then, we have that $g_i^{m_0} = u_i^s t$ and $y = u_i^{-l} g_i^b$.

$$\begin{aligned} \pi^\# : \mathcal{O}_{P,q} &\rightarrow \mathcal{O}_{C,Q} = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} R_i \\ t &\mapsto \bigoplus \left(\frac{g_i^{m_0}}{u_i^s} \right) \end{aligned}$$

In particular, the maximal ideal $\mathfrak{m}_q$ has image $\pi^\#(\mathfrak{m}_q) = \prod_{i \in \mathbb{Z}/d\mathbb{Z}} (\mathfrak{m}_{Q_i})^{m_0}$, where $\mathfrak{m}_{Q_i} \subseteq R$ is the ideal generated by $(1, \dots, 1, g_i, \dots, 1)$. So, there are d pre-images of q in C , each with ramification index $m_0 = \frac{m}{\gcd(m,a)}$.

As a sanity check, we can compute the genus by Riemann-Hurwitz formula.

$$\begin{aligned} 2g(C) - 2 &= m(-2) + \sum_Q (e_Q - 1) \\ &= -2m + \sum_{i=1}^r (m - \gcd(m, a(i))) \\ g(C) &= 1 + \frac{(r-2)m - \sum_{i=1}^r \gcd(a(i), m)}{2} \end{aligned}$$

We can also verify that the inertia type is indeed $a(i)$. Without loss of generality, say $\gcd(a(j), m) = 1$, then the local uniformizer is given by g and for $\tau_1 \in \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*)$ such that $\tau_1(1) = 1$, we have that $\tau_1(g) = \zeta_m^s g$, where $s \equiv a(j)^{-1} \pmod{m}$. So, the character of $\mathbb{Z}/m\mathbb{Z}$ on $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ is ζ_m^s . But $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ is the cotangent space, so the representation of $\mathbb{Z}/m\mathbb{Z}$ on g is dual of the representation of $\mathbb{Z}/m\mathbb{Z}$ on the $T_Q(C)$. Let $g^\vee$ be the dual basis on $T_Q(C)$, then $\tau_1(g^\vee) = \zeta_m^{a(i)} g^\vee$.

3.6 Local structure at a node

In this section, we briefly describe the morphism on the local ring near a node of a singular curve $C \in \partial \widetilde{\mathcal{M}}(m, r, \underline{a})$.

we fix $\pi : C \rightarrow P$ a closed point in $\delta_S \subseteq \partial \widetilde{\mathcal{M}}(m, r, \underline{a})$. We suppose $[C \rightarrow P]$ is a general point of δ_S , so C only has one node. Let Q be the node on C and q be the projection of Q on P .

Suppose $\delta_S = \kappa(\widetilde{\mathcal{M}}(m, r_1, \underline{a}_1) \times \widetilde{\mathcal{M}}(m, r_2, \underline{a}_2))$. Then by Section 3.5, the inertia type a_Q is given by $a_Q \equiv -(\sum_{i=1}^{r_1-1} a_1(i)) \pmod{m}$. Then, $C = C_1 \sqcup_Q C_2$ and $P = P_1 \sqcup_q P_2$. Let q_1, q_2 be the preimages of the node q under normalization.

The covering map $\pi : C_1 \rightarrow P_1$ is $\mathbb{Z}/m\mathbb{Z}$ cover. There are $d_S = \gcd(a_Q, m)$ preimages of q_1 , each with ramification index $e_S = \frac{m}{d_S}$. Fix one preimage and denote it as Q_1 . Let u be the local parameter of C_1 near Q_1 and x be the local parameter of P_1 near p_1 . Then $\pi_1 : (C_1, Q_1) \rightarrow (P_1, q_1)$ induces the map in the local neighborhood:

$$\begin{aligned} \pi_1^\# : k[x] &\rightarrow k[u] \\ x &\rightarrow u^{e_S} \end{aligned}$$

Similarly, Let Q_2 be the preimage on C_2 that maps to the same point as Q_1 under the normalization. Let v be the local parameter of C_2 near Q_2 and y be the local parameter of P_2 near q_2 . Then

$$\begin{aligned} \pi_2^\# : k[y] &\rightarrow k[v] \\ y &\rightarrow v^{e_S} \end{aligned}$$

Now let $C \rightarrow \widetilde{\mathcal{M}}$ denote the total space of the family, and take an affine chart in C near the point Q . The coordinate ring $\mathcal{P} = C/\iota(\mathbb{Z}/m\mathbb{Z})$ near q is $\mathcal{O}_S[x, y, t]/(xy - t)$, and the locus of singular curve Δ_S in $\overline{\mathcal{M}}_{0,r+1}$ in this affine chart is given by $(t = 0)$. The coordinate ring of C near Q is $\mathcal{O}_{\mathcal{P}}[\tau, u, v]/(uv - \tau, u^{m_0} - x, v^{m_0} - y, \tau^{m_0} - t)$, with δ_S cut out by $(\tau = 0)$ and the morphism of between the total families $C \rightarrow \overline{\mathcal{M}}_{0,r+1}$ is locally given by:

$$\begin{aligned} \pi^\# : \mathcal{O}_{\mathcal{P}} &\rightarrow \mathcal{O}_C \\ k[x, y, t]/xy - t &\rightarrow \mathcal{O}_{\mathcal{P}}[u, v, \tau]/uv - \tau \\ x &= u^{e_S} \\ y &= v^{e_S} \\ t &= \tau^{e_S} \end{aligned}$$

The morphism $k[t] \rightarrow k[t, \tau]/t - \tau^{e_s}$ gives is the morphism on coordinate ring induced by the projection map $\widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \overline{M}_{0, r+1}$, near the point $[C \rightarrow P]$.

Chapter 4

OUTLINE OF THE PROOF

In this section, we introduce the Torelli morphism that gives rise to the PEL type Shimura variety that we study in this paper. We also define the level structure at the ramified prime, which exists because of our integral model comes from the Jacobian of the universal family of curves. These parts follow the preliminary of (Mantovan, 2020) and (Li, Mantovan, and Pries, 2021). Using the Torelli morphism, we describe our strategy and the main results we aim to prove in this paper.

In this section, we will use $\mathrm{Sh}(\mathcal{D})$ to denote **all** the Deligne-Mostow Shimura varieties that arise from Hurwitz spaces $\mathcal{M}(m, r, \underline{a})$ via the Torelli morphism.

4.1 Torelli morphism and the associated PEL type moduli space

In general, the modular forms on an abstract Shimura variety is hard to access. Since the PEL datum does not give us an arithmetic model, it's hard to describe what the global section of Hodge line bundle is. In our case, the computation is made possible because the Shimura variety at interest is associated to a Hurwitz space as defined in Section 3.1 via the Torelli morphism, and the Hurwitz spaces have more explicit arithmetic models. In this section, we will introduce the Torelli morphism in our setting and how it gives rise to Shimura varieties.

Notation 4.1.1. We first fix the notation for the following moduli spaces:

- $\mathcal{M}(m, r, \underline{a})/\mathbb{Z}[\frac{1}{m}]$: This is the moduli functor as in Definition 3.1.9. It is the moduli space of μ_m Galois cover with $\mathbb{P}^1$ branched at r points with inertia type $\underline{a} = (a(1), \dots, a(r))$. When describing them in general, we sometimes drop $(m, r, \underline{a})$ and only denote it as $\mathcal{M}$.
- $\overline{\mathcal{M}}(m, r, \underline{a})$: This is the moduli functor as in Definition 3.1.9. It differs from $\widetilde{\mathcal{M}}(m, r, \underline{a})$ in the sense that we do not put a label on the ramification locus, so the inertia type $\underline{a}$ is defined up to equivalence.
- $\mathcal{C} \rightarrow \mathcal{M}$ is the universal curve. For $x \in \mathcal{M}$ such that $x = (x_1, \dots, x_r)$, the affine equation of $C_x : y^m = \prod_{i=1}^r (x - x_i)^{a(i)}$.

- $\mathcal{J}(C) \rightarrow \mathcal{M}$: This is the relative Jacobian of the universal family of curve C . For $x \in \mathcal{M}$, the fiber is the Jacobian $\mathcal{J}(C_x)$.
- $\mathcal{M}_g/\mathbb{Z}$ is the moduli space of smooth projective genus g curves.
- $\mathcal{A}_g/\mathbb{Z}$ is the moduli space of principally polarized abelian varieties of dimension g .

Recall the Torelli morphism that maps an isomorphism class of curve $[C]$ to its corresponding Jacobian $J(C)$, along with the principal polarization given by the theta divisor:

$$\begin{aligned} T : \mathcal{M}_g &\rightarrow \mathcal{A}_g \\ [C] &\mapsto [J(C), \Theta] \end{aligned}$$

We restrict the Torelli morphism to the image of $\mathcal{M}(m, r, \underline{a})$ in $\mathcal{M}_g$ and consider the PEL type subvariety of $\mathcal{A}_g$ containing this image.

Definition 4.1.2 (Deligne-Mostow Shimura variety). *Given the Hurwitz space $\mathcal{M}(m, r, \underline{a})$, $\text{Sh}(\mathcal{D})$ is the smallest PEL type subvariety of $\mathcal{A}_g$ that contains $T(\mathcal{M}(m, r, \underline{a}))$. It is also the largest closed, reduced and irreducible substack of $\mathcal{A}_g$ where the $\mathbb{Z}[\mu_m]$ action on $\mathcal{J}(C)$ extends to the universal abelian scheme over $\text{Sh}(\mathcal{D})$.*

Remark 4.1.3. Notice that $\mathcal{M}(m, r, \underline{a})$ is connected. Hence, $\text{Sh}(\mathcal{D})$ defined as such is by definition connected. On the other hand, if we define $\text{Sh}(\mathcal{D})$ by specifying its PEL datum, the resulting moduli space might have multiple connected components. If that happens, we take a connected component of $\text{Sh}(\mathcal{D})$. The ring $M(\text{Sh}(\mathcal{D}))$ and $M(\text{Sh}_\Gamma(\mathcal{D}))$ are the modular forms on a connected component of $\text{Sh}(\mathcal{D})$.

Notice that $\text{Sh}(\mathcal{D})$ is always attached to an $\mathcal{M}(m, r, \underline{a})$. It is defined over its reflex field $E = \mathbb{Q}(\zeta_m)$, and it has integral model over $\mathcal{O}_E[\frac{1}{m}]$. The Torelli morphism $T : \mathcal{M}(m, r, \underline{a}) \rightarrow \text{Sh}(\mathcal{D})$ gives an **open immersion**:

$$T : \mathcal{M}(m, r, \underline{a}) \otimes_{\mathbb{Z}[\frac{1}{m}]} \mathcal{O}_E[\frac{1}{m}] \rightarrow \text{Sh}(\mathcal{D})/\mathcal{O}_E[\frac{1}{m}]$$

Normally, only a subset of the abelian varieties in $\text{Sh}(\mathcal{D})$ (namely the image of T) are of the form $\mathcal{J}(C_x)$ for some $x \in \mathcal{M}$. When it is the case that all the abelian varieties in the family are of the form $\mathcal{J}(C_x)$, we say that $\mathcal{M}(m, r, \underline{a})$ is **special**. This is precisely when $\mathcal{J}(C)$ along with its $\mathbb{Z}[\mu_m]$ action becomes the

universal abelian scheme of $\text{Sh}(\mathcal{D})$. Equivalently, this can be classified when $\dim(\mathcal{M}(m, r, \underline{a})_{\mathbb{C}}) = \dim(\text{Sh}(\mathcal{D})_{\mathbb{C}})$.

In (Moonen, 2010), Moonen classifies 20 families of cyclic covers of $\mathbb{P}^1$ that are special. In this paper, we will primarily focus on $\text{Sh}(\mathcal{D})$ attached to special families $\mathcal{M}(m, r, \underline{a})$, though some of the conclusions also apply for $\text{Sh}(\mathcal{D})$ attached to non-special families. When $\mathcal{M}(m, r, \underline{a})$ is special, it provides us explicit geometry of $\text{Sh}(\mathcal{D})$, which makes the computation of Hodge line bundle and automorphic forms possible.

Corollary 4.1.4 (Torelli isomorphism). *Given $\mathcal{M}(m, r, \underline{a})$ and its associated $\text{Sh}(\mathcal{D})$. Assuming that:*

- $\mathcal{M}(m, r, \underline{a})$ is special;
- $\text{Sh}(\mathcal{D})$ is compact

Then $T : \overline{\mathcal{M}}(m, r, \underline{a})/O_E[\frac{1}{m}] \rightarrow \text{Sh}(\mathcal{D})/O_E[\frac{1}{m}]$ is an isomorphism.

The PEL datum for $\text{Sh}(\mathcal{D})$ can be determined from $\mathcal{M}(m, r, \underline{a})$. Choose any $[C] \in \mathcal{M}(m, r, \underline{a})$, the rational PEL datum $(B, *, V, \Psi, h_0)$ is as follows:

- $B = \mathbb{Q}[\mu_m]$ with $*$: $\zeta_m \rightarrow \zeta_m^{-1}$.
- $V = H^1(C(\mathbb{C}), \mathbb{Q})$. It is of dimension $2g$ over $\mathbb{Q}$, where g is the genus of C and is computed via Proposition 3.1.10
- Ψ is the pairing on V induced by the principal polarization Θ .
- h_0 is the isomorphism class $V_{\mathbb{C}}$ as a $B_{\mathbb{C}}$ module determined by Chevalley-Weil formula.

Let's elaborate on the last condition a bit more. h_0 determines the complex structure on $V_{\mathbb{R}}$. So to give the data of h_0 is equivalent to give the data of $V_{\mathbb{C}}$ as a representation of $\mathbb{Q}[\mu_m] \otimes \mathbb{C}$. Recall that in Definition 2.2.1 of reflex field, we define $V_0 = \{v \in V_{\mathbb{C}} \mid h_0(z)v = (1 \otimes z)v\}$, and we have $V_{\mathbb{C}} = V_0 \oplus \overline{V_0}$. So h_0 is completely determined by the $B_{\mathbb{C}}$ structure of V_0 .

On the other hand, notice that $\mathbb{Q}[\mu_m] \otimes \mathbb{C} \cong \bigoplus_{\tau_i \in \text{Hom}(\mu_m, \mathbb{C}^*)} \mathbb{C}_{\tau_i}$. So, $V_0 \cong \bigoplus V_{0, \tau_i}$, where $V_{0, \tau_i} = \{v \in V_0 : \zeta_m \cdot v = \tau_i(\zeta_m)v\}$. Then, to give $B_{\mathbb{C}}$ representation is

the same as giving $\dim(V_{0,\tau_i})$ for each $\tau_i \in \text{Hom}(\mu_m, \mathbb{C}^*)$. These dimensions are computed from the monodromy data $(m, r, \underline{a})$ via the **Chevalley-Weil formula**:

$$f(\tau_i) = \begin{cases} 0 & \text{if } i = 0 \\ -1 + \sum_{j=1}^r \left\langle \frac{-ia(j)}{m} \right\rangle & \text{otherwise} \end{cases} \quad (4.1)$$

And $\dim(V_{0,\tau_i}) = f(\tau_i)$. The tuple $\underline{f} = (f(\tau_1), \dots, f(\tau_{m-1}))$ is called the **signature** of the monodromy datum $(m, r, \underline{a})$, and in this case $\underline{f}$ completely determine h_0 . Notice that the above formula is exactly the same as Proposition 3.1.10 Equation (4). This is due to the isomorphism $H^0(C, \Omega_C) \cong H^0(J(C), \Omega_{J(C)})$.

Heuristically, we can think of $\text{Sh}(\mathcal{D})$ as classifying triples (A, λ, i) where λ is isomorphic to Ψ and $i : \mathbb{Q}[\mu_m] \rightarrow \text{End}^0(A)$, such that the τ_i -eigenspace of $H^0(A, \Omega_A)$ has dimension given by $f(\tau_i)$.

Normally, given a rational PEL datum $(B, *, V, \Psi, h_0)$, different choice of integral model gives different extension of $\text{Sh}(\mathcal{D})/E$ over to $\mathcal{O}_E[\frac{1}{m}]$. However, since $\text{Sh}(\mathcal{D})$ we study comes from Jacobian of curves, there is a canonical integral PEL datum $(\mathcal{O}, *, L, \Psi, h_0)$:

- $\mathcal{O} = \mathbb{Z}[\mu_m]$ with $*$,
- $L = H_1(C(\mathbb{C}), \mathbb{Z})$ with Ψ given by the theta divisor, and it takes integral value on Λ .

For this particular choice of integral PEL datum, notice that (Λ, Ψ) is maximal at all the prime p . This allows us to define the level structure at ramified prime and the associated moduli problem, which will be described in detail of the following section.

Now let's recall the compact Shimura surface $\text{Sh}[16]$ defined in Definition 4.1.2. It is the Deligne-Mostow Shimura variety associated to the special family $\mathcal{M}[16]$, whose monodromy datum is given by $(5, 5, (1, 1, 1, 1, 1))$ and the signature type is $(3, 2, 1, 0)$. Its integral datum comes from $\mathcal{M}[16]$. In (Li, Mantovan, and Pries, 2021) it is described in detail how to compute the lattice L and Hermitian pairing Ψ for the Deligne-Mostow Shimura varieties from the monodromy datum of the associated Hurwitz spaces.

In (Li, Mantovan, and Pries, 2021, Theorem 5.5), when m is an odd prime and $\mathbb{Q}(\zeta_m)$ has class number 1, then the lattice L can be explicitly determined. Indeed,

if m is a prime, then $O = \mathbb{Z}[\mu_m] \cong \mathbb{Z} \oplus \mathbb{Z}[\zeta_m]$ is maximal in B , and since $f(\tau_1) = 0$, L is in fact a $\mathbb{Z}[\zeta_m]$ module. If $\mathbb{Q}(\zeta_m)$ has class number 1, then $\mathbb{Z}[\zeta_m]$ is a principal ideal domain, and so L is a free $\mathbb{Z}[\zeta_m]$ module, and so $L \cong \mathbb{Z}[\zeta_m]^{\frac{2g}{m-1}}$. Then, the Hermitian form Ψ is computed by identifying the distinguish point $C_P \in \overline{\mathcal{M}}(m, r, \underline{a})$ whose $J(C_P)$ is direct sum of CM abelian varieties, and for a CM abelian variety the Hermitian pairing is uniquely determined and explicitly computed.

Example 4.1.5. Following (Li, Mantovan, and Pries, 2021), let's explain how to obtain the integral datum $(O, *, \Psi, L, h_0)$ of $\text{Sh}[16]$ in Definition 4.1.2 from $\mathcal{M}[16]$, whose monodromy datum is $(5, 5, (1, 1, 1, 1, 1))$.

Let $K = \mathbb{Q}(\zeta_5)$ and $K_0 = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$.

Since m is a prime and K has class number 1, we see that $L = O_K^3$ with involution $*$: $\zeta_5 \rightarrow \zeta_5^{-1}$. The h_0 is determined by the signature type. So, we just have to write down the Hermitian pairing $H_B : V \times V \rightarrow K$.

We know the following facts specifically about K :

1. Let $u = \zeta_5 + \zeta_5^{-1} = \frac{\sqrt{5}-1}{2}$ be the fundamental unit. By (Li, Mantovan, and Pries, 2021, Lemma 3.6), we know that K_0 has unit of independent sign, and K_0 has narrow class number 1, and u is such a unit. Let $\tau_1, \tau_2 : F_0 \rightarrow \mathbb{R}$ be the two real embeddings, then $\tau_1(u) > 0, \tau_2(u) < 0$.
2. Let $\beta_0 = \frac{5}{\zeta_5^3 - \zeta_5^2}$. By (Li, Mantovan, and Pries, 2021, Lemma 3.7), we know that β_0 generates the different ideal $\mathcal{D}_{F/\mathbb{Q}}$ and $\overline{\beta_0} = -\beta_0$. Moreover, $\mathfrak{I}(\beta_0) > 0$.
3. Since $m = 5$, by (Li, Mantovan, and Pries, 2021, Lemma 3.8), any CM type Φ on K is simple.

Now, consider a distinguished point $C_P \in \partial \overline{\mathcal{M}}[16]$. C_P is the image of clutching morphism of C_1, C_2, C_3 , where C_1, C_3 has monodromy datum $\mathcal{M}(5, 3, (1, 1, 3))$ and C_2 has monodromy datum $\mathcal{M}(5, 3, (2, 1, 2))$. In particular, $A_i = J(C_i)$ are CM abelian varieties. By computing the signature, we have:

$$f_1 = f_3 = (1, 1, 0, 0), f_2 = (1, 0, 1, 0)$$

$$\Phi_1 = \Phi_3 = \{\tau_1, \tau_2\}, \Phi_2 = \{\tau_1, \tau_3\}$$

Now, notice that A_i are of type (O_K, Φ_j) . Indeed, $\mathbb{Z}[\zeta_5] = O_K \subseteq \text{End}(A_j)$, and the representation on $\text{Lie}(A_j)$ is exactly given by signature f_j .

1. Since Φ_j are simple type and A_j of type $(\mathcal{O}_K, \Phi_j)$, and K has class number 1, by (Li, Mantovan, and Pries, 2021, Theorem 4.1), $A_j \cong \mathbb{C}^n / \Phi_j(\mathcal{O}_K)$.
2. By (Li, Mantovan, and Pries, 2021, Corollary 4.3), every principal polarization on $\mathbb{C}^n / \Phi_j(\mathcal{O}_K)$ is given by $\text{Tr}_{K/\mathbb{Q}}(x\xi_j\bar{y})$, where $\beta_j = \xi_j^{-1}$ satisfies:
 - β_j generates the different $\mathcal{D}_{K/\mathbb{Q}}$;
 - $\overline{\beta_j} = -\beta_j$
 - $\Im(\tau(\beta)) < 0, \forall \tau \in \Phi_j$

and ξ_j, ξ'_j corresponds to isomorphic principal polarization iff $\xi'_j = u\bar{u}\xi_j$ for some $u \in \mathcal{O}_K$. In particular, the number of isomorphism classes of principal polarization is given by $|\mathcal{U}_{K_0}^+ / \text{Nm}_{K/K_0}(\mathcal{U}_K)|$, which is 1 since K has unit of independent sign.

Hence, the principal polarization on A_j must be of the one given by ξ_j satisfying the requirement. By (Li, Mantovan, and Pries, 2021, theorem 5.5), $\xi_j = \beta_j^{-1}$ where $\beta_j = u_j\beta_0$ for u_j have the required signs when embedded by $\tau \in \Phi_j$.

Now we can compute the Hermitian form in our case. Recall that $u = \zeta_5 + \zeta_5^{-1} = \frac{\sqrt{5}-1}{2}$ and $\beta_0 = \frac{5}{\zeta_5^3 - \zeta_5^2}$. Notice that $\Im(\tau_1\beta_0) > 0$, $\Im(\tau_2\beta_0) < 0$, and $\Im(\tau_3\beta_0) > 0$.

Consider the case when $j = 1, 3$. $\Phi_j = \{\tau_1, \tau_2\}$. To make $\Im(\tau(u_j\beta_0)) < 0$ for $\tau \in \Phi_j$, we want u_1 such that $\tau_1(u_1) < 0$ and $\tau_2(u_1) > 0$. we can take $u_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_1 = u_1\beta_0 = \beta_3$.

Consider the case when $j = 2$. $\Phi_2 = \{\tau_1, \tau_3\}$. To make $\Im(\tau(u_2\beta_0)) < 0$ for $\tau \in \Phi_2$, we want u_2 such that $\tau_1(u_2) < 0$ and $\tau_2(u_2) < 0$. Hence, we can have $u_2 = -1$, $\beta_2 = u_2\beta_0 = -\beta_0$.

Hence, the Hermitian form H_B is given by the diagonal matrix

$$\text{diag}[u_1^{-1}\beta_0^{-1}, u_2^{-1}\beta_0^{-1}, u_1^{-1}\beta_0^{-1}] = \frac{1}{u_1\beta_0} \text{diag}[1, \frac{\sqrt{5}-1}{2}, 1]$$

Here $\frac{1}{u_1\beta_0} = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})}$. Hence, we obtain the integral datum as in Definition 4.1.2

Recall that when $T : \mathcal{M}(m, r, \underline{a}) \rightarrow \text{Sh}(\mathcal{D})$ is open and dense, we say that the family $\mathcal{M}(m, r, \underline{a})$ is **special**. Corollary 4.1.4 says that if $\mathcal{M}(m, r, \underline{a})$ is special and $\text{Sh}(\mathcal{D})$ is compact, then the Torelli morphism T gives an isomorphism from $\overline{\mathcal{M}}(m, r, \underline{a})$ to $\text{Sh}(\mathcal{D})$. The following proposition shows that, when T is an isomorphism, we can identify $M(\text{Sh}(\mathcal{D}))$ with $\bigoplus_k H^0(\overline{\mathcal{M}}, \lambda^k)$.

Proposition 4.1.6. *Suppose $T : \mathcal{M}(m, r, \underline{a}) \rightarrow \text{Sh}(\mathcal{D})$ is such that $\mathcal{M}(m, r, \underline{a})$ is special and $\text{Sh}(\mathcal{D})$ is compact.*

Let λ_{Sh} be the Hodge line bundle on $\text{Sh}(\mathcal{D})$, and let $\lambda_{\overline{\mathcal{M}}}$ be the Hodge line bundle on $\overline{\mathcal{M}}$. Then there is a isomorphism of line bundles:

$$i^* : T^* \lambda_{\text{Sh}} \cong \lambda_{\overline{\mathcal{M}}}$$

Consequently, $M(\text{Sh}(\mathcal{D})) \cong \bigoplus_k H^0(\overline{\mathcal{M}}, \lambda^k)$.

Proof. Let $\mathbb{E}_{\text{Sh}} = e^* \Omega_{\mathcal{A}/\text{Sh}}$ be the Hodge bundle on Sh , let $\mathbb{E}_{\overline{\mathcal{M}}} = \pi_* \Omega_{C/\overline{\mathcal{M}}}$ be the Hodge bundle on $\overline{\mathcal{M}}$. Then we have that $\lambda_{\text{Sh}} = \wedge^g \mathbb{E}_{\text{Sh}}$ and $\lambda_{\overline{\mathcal{M}}} = \wedge^g \mathbb{E}_{\overline{\mathcal{M}}}$.

Let $\mathcal{T}_{\mathcal{J}/\overline{\mathcal{M}}}$ be the relative tangent sheaf of the family $\mathcal{J} \rightarrow \overline{\mathcal{M}}$. We first show that $e^* \mathcal{T}_{\mathcal{J}/\overline{\mathcal{M}}} \cong \pi_* \Omega_{C/\overline{\mathcal{M}}}^\vee$. Indeed, if X is a proper variety, then $\mathcal{T}_{\text{Pic}(X), e} \cong H^1(X, \mathcal{O}_X)$. By Serre duality, $\pi_* \Omega_{C/\overline{\mathcal{M}}}^\vee = \mathcal{R}^1 \pi_* \mathcal{O}_{C/\overline{\mathcal{M}}}$, so its fibre over $[C]$ is $H^1(C, \mathcal{O}_C)$. On the other hand, $e^* \mathcal{T}_{\mathcal{J}/\overline{\mathcal{M}}}$ has fibre at $[C]$ being $T_{J[C], e}$, and so these two are isomorphic. The morphism on the family is the pull back of the Abel-Jacobi map.

Now, we take wedge product on both sides of the equation. Since wedge product commutes with the pull back,

$$\begin{aligned} \wedge^g e^* \mathcal{T}_{\mathcal{J}/\overline{\mathcal{M}}} &= T^* \wedge^g e^* \mathcal{T}_{\mathcal{A}/\text{Sh}} \\ &= T^* \wedge^g e^* (\Omega_{\mathcal{A}/\text{Sh}})^\vee \\ &= T^* (\wedge^g e^* (\Omega_{\mathcal{A}/\text{Sh}}))^\vee \\ &= T^* \omega_{\mathcal{A}/\text{Sh}}^\vee \\ &= e^* \mathcal{R}^g \pi_* \mathcal{O}_{\mathcal{J}/\text{Sh}} \end{aligned}$$

On the left hand side we have:

$$\wedge^g (\mathcal{R}^1 \pi_* \mathcal{O}_{C/\overline{\mathcal{M}}})$$

Now we take the dual. By Serre-duality, the left hand side becomes $T^* e^* \omega_{\mathcal{A}/\text{Sh}} = T^* \lambda_{\text{Sh}}$. The right hand side becomes $\wedge^g \pi_* \Omega_{C/\overline{\mathcal{M}}} = \lambda_{\overline{\mathcal{M}}}$. Hence, $T^* \lambda_{\text{Sh}} \cong \lambda_{\overline{\mathcal{M}}}$ with the isomorphism induced by Abel-Jacobi map. $\square$

4.2 Level structure at the ramified prime

Now we consider $\text{Sh}(\mathcal{D})$ associated with $\mathcal{M}(m, r, \underline{a})$ where m is an odd prime. In this case, $(1 - \zeta_m)$ is a prime in $\mathbb{Z}[\zeta_m]$, and $(m) = (1 - \zeta_m)^{m-1}$. We define level $(1 - \zeta_m)$ structure for principally polarized abelian scheme with $\mathbb{Z}[\zeta_m]$ action.

Fix the canonical integral PEL datum $(\mathbb{Z}[\mu_m], *, L, \Psi, h_0)$. Notice that $\mathbb{Z}[\mu_m]$ is maximal in $\mathbb{Q}(\mu_m)$ and L is self dual under Ψ over $\mathbb{Z}$. In other words, the associated integral model is truly a sublocus of $\mathcal{A}_g \otimes \mathcal{O}_E[\frac{1}{m}]$.

Definition 4.2.1 (principal level $(1 - \zeta_m)$ structure). *Given a triple (A, λ, i) such that*

- *A is an abelian scheme over T ;*
- *$\lambda : A \rightarrow A^\vee$ is a principal polarization.*
- *$i : \mathcal{O} \rightarrow \text{End}_T(A)$ compatible with Rosati involution.*

An integral principal level $(1 - \zeta_m)$ structure is an isomorphism

$$\eta : (L/(1 - \zeta_m)L)_T \rightarrow A[1 - \zeta_m]$$

that's compatible with the Weil pairing and is symplectic liftable.

Notice that we no longer include the isomorphism $\nu : \mathbb{Z}/m\mathbb{Z} \cong \mu_m$, and we don't write down the compatibility condition on the Weil pairing. By choosing a roots of unity, we have already fixed a choice of the isomorphism $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$. As we will show in Chapter 6, for (A, λ, i) that arises from $\mathcal{J}(C_x)$, the Weil pairing restricted to $A[1 - \zeta_m]$ is trivial, so the compatibility on the Weil pairing is automatically satisfied.

Now we want to define a cover of $\text{Sh}(\mathcal{D})$ corresponding to the level $1 - \zeta_m$ structure. We can not directly use the $\mathcal{M}_{\mathcal{K}^S}$ in Definition 2.3.3. Indeed, in this case $S = \{m\}$, so $\mathcal{M}_{\mathcal{K}^S}$ does not parameterize level structure supported at m . However, in our case, since the integral datum $(\mathbb{Z}[\mu_m], *, L, \Psi, h_0)$ comes from $\mathcal{J}(C_x)$, the moduli space $S_{\mathcal{K}^S}$ representing $\mathcal{M}_{\mathcal{K}^S}$ actually comes with a principally polarized universal abelian scheme $(\mathcal{A}, \lambda, i)$ that extends $\mathcal{J}(C)$. In the case where $\mathcal{M}(m, r, \underline{a})$ is special and $\text{Sh}(\mathcal{D})$ is compact, $\mathcal{A} = \mathcal{J}(C)$. This allows us to define the following relative moduli problem:

Definition 4.2.2 (Relative moduli problem $\mathcal{M}_{\Gamma_m}/S_{\mathcal{K}^m}$). *Let $(\mathbb{Z}[\mu_m], *, L, \Psi, h_0)$ be the integral datum obtained from $\mathcal{J}(C_x)$ as above. Let $\mathcal{K}^m \subseteq \mathcal{G}(\hat{\mathbb{Z}}^m)$ be an open compact. Let $S_{\mathcal{K}^m}$ be the moduli space representing the functor $\mathcal{M}_{\mathcal{K}^m}$ in Definition 2.3.3. Let $\mathcal{A} \rightarrow S_{\mathcal{K}^m}$ be the principally polarized abelian scheme with principal polarization λ and endomorphism i .*

The relative moduli problem $\mathcal{M}_{\Gamma_m}$ is a category fibred in the groupoid

$$(\text{Sch}/S_{\mathcal{K}^m}) \rightarrow \text{Grp}$$

$$\mathcal{M}_{\Gamma_m}(T) = \{\eta : \eta \text{ is a level } 1 - \zeta_m \text{ structure on } (\mathcal{A}_T, \lambda_T, i_T)\}$$

Consider the case where $\mathcal{K}^m = \mathcal{G}(\hat{\mathbb{Z}}^m)$. Then $S_{\mathcal{K}^m} = \text{Sh}(\mathcal{D})/O_E[\frac{1}{m}]$. Let $\text{Sh}(\Gamma_m) \rightarrow \text{Sh}(\mathcal{D})$ be the moduli space representing $\mathcal{M}_{\Gamma_m}$. From the definition, we see that $\text{Sh}_{\Gamma_m}(\mathcal{D}) \rightarrow \text{Sh}(\mathcal{D})$ is a finite covering.

What is the open compact $\mathcal{K} \subseteq \mathcal{G}(\hat{\mathbb{Z}})$ corresponding to $\text{Sh}_{\Gamma_m}(\mathcal{D})$? If we write $\mathcal{G}(\mathbb{Z}_m) = \{g \in \text{Aut}(L_m) \mid \bar{g}^T H g = c(g)H\}$, then $\mathcal{G}(\mathbb{Z}_m)$ acts on the set of all $1 - \zeta_m$ level structure, and the stabilizer of a particular level $1 - \zeta_m$ structure is $\Gamma_m = \{g \in \mathcal{G}(\mathbb{Z}_m) \mid g \equiv I \pmod{(1 - \zeta_m)L_m}\}$. Hence, the $\mathbb{Z}_m$ component of the level subgroup of $\text{Sh}(\Gamma_m)$ is $\Gamma_m = \{g \in \mathcal{G}(\mathbb{Z}_m) \mid g \equiv I \pmod{(1 - \zeta_m)L_m}\}$

If it is the case that $L \cong \mathbb{Z}[\zeta_m]^r$ for some r , then let $K = \mathbb{Q}(\zeta_m)$, $O_K = \mathbb{Z}[\zeta_m]$, and $\omega = (1 - \zeta_m)$ be the maximal ideal in O_K . We can further write $\mathcal{G}(\mathbb{Z}_m)$ as $r \times r$ matrices with entries in $O_{K,\omega}$. Then, Γ_m consists of those matrices congruent to I_r modulo ω .

We will use $\text{Sh}_{\Gamma_m}(\mathcal{D})$ and $\text{Sh}(\mathcal{D})$ again in Chapter 6, 7. We summarize the notation here:

Notation 4.2.3.

Name	$\mathbb{Z}_m$ -component of the level $\mathcal{K}$	Level $\mathcal{K} \subseteq \mathcal{G}(\hat{\mathbb{Z}})$
$\text{Sh}(\mathcal{D})$	$\mathcal{G}(\mathbb{Z}_m) = \{g \in \text{Aut}(L_m) \mid \bar{g}^T H g = c(g)H\}$	$\mathcal{G}(\mathbb{Z}_m)\mathcal{G}(\hat{\mathbb{Z}}^m)$
$\text{Sh}_{\Gamma_m}(\mathcal{D})$	$\Gamma_m = \{g \in \mathcal{G}(\mathbb{Z}_m) \mid g \equiv I \pmod{(1 - \zeta_m)L_m}\}$	$\Gamma_m\mathcal{G}(\hat{\mathbb{Z}}^m)$

4.3 Main results and the approach

The main results of this paper are the following two theorem:

Theorem 4.3.1. *The ring of modular forms on $\text{Sh}(\Gamma_m)$ is given as R_{ev} , where*

$$R = \mathbb{Q}(\zeta_5)[s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}, \varepsilon]/\mathcal{I}$$

$$\mathcal{I} = (\varepsilon^5 - Q_{12}, Q_{12} - Q_{21}, Q_{12} - Q_{23}, Q_{12} - Q_{32}, Q_{12} - Q_{13}, Q_{12} - Q_{13})$$

Where s_{ij} are of degree 1 and weight $\frac{5}{2}$, ε is of degree $\frac{2}{5}$ and weight 1,

$$Q_{ij} = \text{sign} \begin{bmatrix} 1 & 2 & 3 \\ i & j & k \end{bmatrix} s_{ij}(s_{ik} - s_{ki} - s_{jk})$$

Theorem 4.3.2. *The ring of modular form on $\text{Sh}(\mathcal{D})$ is*

$$M(\text{Sh}(\mathcal{D})) = \mathbb{Q}(\zeta_5)[f_2, f_4, f_6, \varepsilon^2] / (\varepsilon^{10} + \frac{1}{52}(f_2^2 - f_4))$$

Where f_{2i} are of weight $5i$ and ε^2 is of weight 2.

Our approach can be summarized into the following steps:

1. Compute the Hodge line bundle λ on $\widetilde{\mathcal{M}}[16]$ (Chapter 5).
2. Compute the ring $\Gamma_0(\lambda) := \bigoplus_k H^0(\widetilde{\mathcal{M}}[16], \lambda^k)$ (Section 5.5).
3. Show that $\widetilde{\mathcal{M}}[16] \rightarrow \text{Sh}(\mathcal{D})$ has a factorization

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[16] & \xrightarrow{\quad \tilde{T} \quad} & \text{Sh}(\Gamma_m) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}[16] & \xrightarrow{\quad T \quad} & \text{Sh}(\mathcal{D}) \end{array}$$

(Chapter 6)

4. Show that the diagram is exact (Section 7.1). This gives us Theorem 4.3.1.
5. Compute the $\mathfrak{S}_5$ -invariant subring of $\Gamma_0(\lambda)$ (Section 7.1). This gives us Theorem 4.3.2.

In the above steps, relevant statements in (1), (3) are proven in greater generality than $\mathcal{M}[16]$. The steps (2), (4), (5) are proven only for $\mathcal{M}[16]$ but is expected to be generalized to other special families.

Chapter 5

COMPUTING THE HODGE LINE BUNDLE

In this section, we focus on the computation on the Hodge line bundle on the compactified Hurwitz space $\widetilde{\mathcal{M}}(m, r, \underline{a})$. These computations hold for any families of cyclic covers of $\mathbb{P}^1$. Since we will be extensively using the computation on the family $\mathcal{M}[16] = \mathcal{M}(5, 5, (1, 1, 1, 1, 1))$, I will give the explicit on $\mathcal{M}[16]$ explicitly, and omit the computation for other families of cyclic covers of $\mathbb{P}^1$. I will also give the computation results for Moonen's special families, but the detail would be omitted.

Notation 5.0.1. For $\widetilde{\mathcal{M}}(m, r, \underline{a})$, we fix the following notations.

On the boundary of $\widetilde{\mathcal{M}}(m, r, \underline{a})$: Let $J \subseteq \{1, 2, \dots, r\}$ and $|J| \leq \frac{r}{2}$.

- δ_J is the boundary component that arises from the clutching of curves whose inertia type corresponds to index J and J^c .

For example, for $\widetilde{\mathcal{M}}(4, 4, (1, 2, 2, 3))$, we have the following boundaries

1. $J = \{1, 2\}, J^c = \{3, 4\}$: $\delta_{(12)} = \kappa(\widetilde{\mathcal{M}}(4, 3, (1, 2, 1)) \times \widetilde{\mathcal{M}}(4, 3, (3, 2, 3)))$.
2. $J = \{1, 3\}, J^c = \{2, 4\}$: $\delta_{(13)} = \kappa(\widetilde{\mathcal{M}}(4, 3, (1, 2, 1)) \times \widetilde{\mathcal{M}}(4, 3, (3, 2, 3)))$.
3. $J = \{1, 4\}, J^c = \{2, 3\}$: $\delta_{(14)} = \kappa(\widetilde{\mathcal{M}}(4, 3, (1, 3, 0)) \times \widetilde{\mathcal{M}}(4, 3, (0, 2, 2)))$.

- d_J : let $a(r_1) = -\sum_{j \in J} a(j)$. Then $d_J = \gcd(m, a(r_1))$. It is the number of preimages of the gluing point on $[C] \in \delta_J$.
- e_J : $e_J = \frac{m}{d_J}$. It is the ramification index on each preimage of the gluing point on $[C] \in \delta_J$.
- Δ_J : $\Delta_J \subseteq \overline{\mathcal{M}}_{0,r}$ is the image of δ_J under the forgetful map $q : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \overline{\mathcal{M}}_{0,r}$. It is boundary component whose on which the curves are the clutching of $\mathbb{P}^1$ marked at J and $\mathbb{P}^1$ marked at J^c .

For a curve $[C \rightarrow P] \in \widetilde{\mathcal{M}}(m, r, \underline{a})$, for $i \in \{1, 2, \dots, r\}$

- B_i : The i^{th} branching point on P . If $[C]$ has equation $C : y^m = \prod_{i=1}^r (x - x(i))$, then $B_i = [x(i) : 1]$.

- B_∞ : Sometimes we move the r^{th} branching point to ∞ . B_∞ is the point at ∞ .
- $d(i) = \gcd(m, a(i))$. It is the number of preimages above the i^{th} branching point B_i
- $e(i) = \frac{m}{d(i)}$. It is the ramification index at each of the preimage above the i^{th} branching point B_i
- P_{ij} : It is the j^{th} preimage of the i^{th} branching point. Here $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, d(i)\}$.

When $d(i) = 1$: We sometimes write P_i for $P_{i,1}$. P_i is the unique preimage of B_i on C .

- P_∞ : When $\gcd(m, a(r)) = 1$, we move $B_r = B_\infty$ and P_∞ is the unique preimage on C of B_∞ . By default, this is the base point for the Abel-Jacobi map from $C \rightarrow J(C)$.

On the Jacobian $J(C)$:

- $Q_{ij} = P_{ij} - P_\infty$. This is the image P_{ij} in $J(C)$
- $Q_i = \sum_{j=1}^{d(i)} Q_{ij}$

5.1 Hodge class up to 12-torsion

In this part, we apply the method in (Geer and Kouvidakis, 2010). Throughout this section, we extensively use the following two diagrams:

$$\begin{array}{ccccc}
 & & C & \xrightarrow{f} & \overline{M}_{0,r+1} \\
 & & \downarrow t & & \downarrow \pi_{r+1} \\
 \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xleftarrow{h} & S & \xrightarrow{q} & \overline{M}_{0,r}
 \end{array}$$

$$\begin{array}{ccccccc}
 C & \xrightarrow{g} & \mathcal{P} & \xrightarrow{q'} & \overline{M}_{0,r+1} \\
 & \searrow t & \downarrow \pi'_{r+1} & & \downarrow \pi_r \\
 & & S & \xrightarrow{q} & \overline{M}_{0,r}
 \end{array}$$

with the right square being Cartesian. Here, $C \rightarrow S$ is a family of cyclic covers given by $\widetilde{\mathcal{M}}(m, r, \underline{a})$, and $C \rightarrow \mathcal{P}$ is the family of covering map, such that on each fiber of $s \in S$, it is the μ_m cover $C \rightarrow P$. We have that $f = q' \circ g$ and $t = \pi'_{r+1} \circ g$.

We make use of the Mumford's relation on families of curves:

$$12\lambda = t_*[\omega_t^2] + [\text{Sing}(C)]$$

Here, ω_t is the relative dualizing sheaf of the family $t : C \rightarrow S$, and $[\text{Sing}(C)]$ is the locus in S where the fibre is singular. Our goal is then to express $t_*[\omega_t^2]$ and $[\text{Sing}(C)]$ in terms of the boundary divisors δ of $\widetilde{\mathcal{M}}(m, r, \underline{a})$.

Now, let R be the closure of the Ramification locus of t restricted to the open part of S where C is smooth. (i.e., restriction of t to the preimage of $\widetilde{\mathcal{M}}(m, r, \underline{a})^o$). For $C \in \mathcal{C}$, we have the μ_m -cover $f : C \rightarrow P$, where P is a genus zero curve. One has the exact sequence:

$$0 \rightarrow f^*\Omega_P \rightarrow \Omega_C \rightarrow \Omega_{C/P} \rightarrow 0$$

$\Omega_{C/P}$ is supported on the ramification points $P_{i,j}$ with length equal to $e_i - 1$. Hence, one has:

$$\omega_t = f^*\omega_{\pi_{r+1}} + \sum_{i=1}^r (e_i - 1) \left(\sum_{j=1}^{d(i)} R_{ij} \right) \quad (5.1)$$

Here $R_{i,j}$ is the component of R mapping to the i^{th} section $S_{r+1}^{i,r+1}$ of the universal family $\pi_{r+1} : \overline{\mathcal{M}}_{0,r+1} \rightarrow \overline{\mathcal{M}}_{0,r}$. Restricted to $C \in \mathcal{C}$, $R_{i,j}$ becomes $P_{i,j}$. Let $\tau_{i,j} : S \rightarrow C$ be the sections that gives the j^{th} ramification point above the i^{th} branching point. In particular, $t \circ \tau_{ij} = \text{id}$ on S , and $f \circ \tau_{ij} = s_i \circ q$ as morphism from $S \rightarrow \overline{\mathcal{M}}_{0,r+1}$.

Now we express $t_*[\omega_t^2]$ as boundary divisor using Equation 5.1.

Lemma 5.1.1. *Let ψ_i be the line bundle on $\overline{\mathcal{M}}_{0,r}$ such that, its stalk on the point represented by $[P]$ is the tangent space on P at the i^{th} special point. That is, $\psi_{i,[P]} = T_{P,B_i}$.*

Let $\psi = \sum_{i=1}^r \psi_i$. Then we have:

$$\begin{aligned} t_*(f^*\omega_{\pi_{r+1}} \cdot R_{ij}) &= q^*\psi_i \\ t_*(R^2) &= - \sum_{i=1}^r \frac{(e(i) - 1)^2 d(i)}{e(i)} q^*\psi_i \end{aligned} \quad (5.2)$$

Proof. $R_{ij} = \tau_{ij}(S)$ in C . R_{ij} is reduced and irreducible, hence by definition of intersection product:

$$(D \cdot C) = i_*[i^*\mathcal{O}_X(D)]$$

Hence

$$\begin{aligned} f^* \omega_{\pi_{r+1}} \cdot R_{ij} &= \tau_{ij*} \tau_{ij}^* (f^* \omega_{\pi_{r+1}}) \\ t_*(f^* \omega_{\pi_{r+1}} \cdot R_{ij}) &= t_* \tau_{ij*} \tau_{ij}^* f^* \omega_{\pi_{r+1}} = q^* s_i^* \omega_{\pi_{r+1}} = q^* \psi_i \end{aligned}$$

We get the first part of the equation 5.2. In particular, since $R = \sum_{i=1}^r (e(i) - 1)(\sum_{j=1}^{d(i)} R_{i,j})$, we get that

$$t_*(f^* \omega_{\pi_{r+1}} \cdot R) = \sum_{i=1}^r (e(i) - 1) d(i) \psi_i$$

Next, we compute $t_*(R^2) = \sum_{i=1}^r (e(i) - 1)^2 \sum_{j=1}^{d(i)} t_*(R_{ij}^2)$. Since t ramifies at $P_{i,j}$ with index $e(i)$, we have

$$\begin{aligned} f^* S_{r+1}^{i,r+1} &= e(i) \sum_{j=1}^{d(i)} R_{ij} \\ \left(\sum_{j=1}^{d(i)} R_{ij} \right) \cdot R_{ij} &= \frac{1}{e(i)} f^* S_{r+1}^{i,r+1} \cdot R_{ij} = \frac{1}{e(i)} \tau_{ij*} \tau_{ij}^* f^* S_{r+1}^{i,r+1} = \frac{1}{e(i)} \tau_{ij*} q^* s_i^* S_{r+1}^{i,r+1} \\ \sum_{j=1}^{d(i)} t_*(R_{ij}^2) &= t_* \left(\sum_{j=1}^{d(i)} R_{ij} \right) \cdot \left(\sum_{j=1}^{d(i)} R_{ij} \right) = \frac{d(i)}{e(i)} q^* s_i^* S_{r+1}^{i,r+1} = -\frac{d(i)}{e(i)} q^* \psi_i \end{aligned}$$

The last equation is the adjunction formula. Hence $t_*(R^2) = -\sum_{i=1}^r \frac{(e(i)-1)^2 d(i)}{e(i)} q^* \psi_i$ $\square$

Lemma 5.1.2. *Let $\Delta = \sum_J \Delta_J$ be the sum of the boundary divisors in $\overline{M}_{0,r}$. Then*

$$t_*(f^* \omega_{\pi_{r+1}}^2) = -m q^* \Delta$$

Proof. Let $S = \sum_i S_{r+1}^{i,r+1}$. Then one has the identity

$$\pi_{r+1,*}([\omega_{\pi_{r+1}} + S]^2) = \psi - \Delta$$

Combining with adjunction formula

$$\pi_{r+1,*}([S]^2) = -\psi$$

and

$$\pi_{r+1,*}(\omega_{\pi_{r+1}} \cdot S) = \psi$$

We get

$$\pi_{r+1,*}(\omega_{\pi_{r+1}}^2) = -\Delta$$

We have the commutative diagram

$$\begin{array}{ccccc}
 C & \xrightarrow{g} & \mathcal{P} & \xrightarrow{q'} & \overline{M}_{0,r+1} \\
 & \searrow t & \downarrow \pi'_{r+1} & & \downarrow \pi_{r+1} \\
 & & S & \xrightarrow{q} & \overline{M}_{0,r}
 \end{array}$$

with the right square being Cartesian. Hence

$$q^* \pi_{r+1,*} \omega_{\pi_{r+1}}^2 = \pi'_{r+1,*} q'^* \omega_{\pi_{r+1}}^2$$

Hence

$$\begin{aligned}
 t_*(f^* \omega_{\pi_{r+1}}^2) &= \pi'_{r+1,*} g^* g'^* q'^* \omega_{\pi_{r+1}}^2 \\
 &= m \pi'_{r+1,*} q'^* \omega_{\pi_{r+1}}^2 = -mq^* \Delta
 \end{aligned}$$

□

Lemma 5.1.3. *The ramification degree of the map $q : \delta_J \rightarrow \Delta_J$ is $\frac{1}{e_J}$. Hence, one has the equation:*

$$q^*(\Delta_J) = e_J h^*(\delta_J)$$

Proof. Let γ be a generic point of the a irreducible component of $\delta_J \subseteq \widetilde{\mathcal{M}}(m, r, \underline{a})$. Suppose $e_J = e$ and $d_J = d$. γ correspondes to an admissible μ_m cover $C \rightarrow S, \iota : \mu_m \rightarrow \text{Aut}(C)$ such that $C/\iota(\mu_m)$ is stable and genus 0, $P = C/\iota(\mu_m)$ has two components P_1, P_2 intersection at Q . Let $Q_1, \dots, Q_d$ denote the primages of Q in C , and we fix one preimage, say Q_1 . The ramification index of Q_1 is $e_J = e$.

Then, locally near Q_1 , the equation of C is given by $\tilde{x}\tilde{y} = \tau$, and near Q the equation of P is given by $xy = t$. Here, $x = \zeta_m^{i_1} \tilde{x}^e, y = \zeta_m^{-i_1} \tilde{y}^e, t = \tau^e$. The formal neighbourhood of γ in $\widetilde{\mathcal{M}}(m, r, \underline{a})$ that prorepresents the infinitesimal deformation of C_γ is $R = k[t_1, t_2, \dots, t_{r-3}, \tau]/(\tau^e - t_1)$. The divisor δ_J is then cut out by (τ) . Here $t_1, \dots, t_{r-3}$ are parameters of P . The formal in $\overline{M}_{0,r}$ that prorepresents infinitesimal deformation of P_γ is $S = k[t_1, \dots, t_{r-3}]$, and the divisor Δ_J is cut out by (t_1) . Hence, the extension of (t_1) to R vanishes at τ with order e . Hence, $q^*(\Delta_J) = eh^*(\delta_J)$ □

Lemma 5.1.4. *The class of $[\text{Sing}(C)]$ in $\widetilde{\mathcal{M}}(m, r, \underline{a})$ is $\sum_J d_J \delta_J$*

Proof. We focus on the contribution of $[\text{Sing}(C)]$ to the coefficient of δ_J . Suppose the boundary divisor δ_J is of type d, e .

Assume that image of S in $\widetilde{\mathcal{M}}(m, r, \underline{a})$ intersect δ_J at sufficiently general point (so no intersection in codimension 2). Let $s \in S$ whose image in $\widetilde{\mathcal{M}}(m, r, \underline{a})$ is a general point of δ_J . Then the corresponding μ_m cover is $\pi : C_s \rightarrow P_s$, where P_s has two components P_1, P_2 meeting at Q , with local equation $xy - t_1$, x is local parameter of P_1 , y is local equation of P_2 . The preimage Q_1 has local equation $uv = \tau$, with change of variable $u^e = x, v^e = y, \tau^e = t$. The admissibility of μ_m covers implies that if $u = \zeta_e^i x^{\frac{1}{e}}$ then $v = \zeta_e^{-i} y^{\frac{1}{e}}$. Since τ is the parameter at S whose vanishing indicate the singular locus, the contribution of Q_1 to the singular locus is 1. Since there are d branches, the total contribution to the singular locus is d . Hence, the coefficient of $[\text{Sing}(C)]$ at δ_J is d_J . $\square$

Recall that we have the Mumford's relation

$$\lambda = \frac{1}{12}(t_*(\omega_t^2) + [\text{Sing}(C)])$$

By Lemma 5.1.2 and Lemma 5.1.1, we get that:

$$\begin{aligned} t_*(\omega_t^2) &= -mq^*\left(\sum_J \Delta_J\right) + \sum_{i=1}^r \left(2(e(i) - 1)d(i) - \frac{(e(i) - 1)^2 d(i)}{e(i)}\right) q^* \psi_i \\ [\text{Sing}(C)] &= \sum_J d_J \delta_J \end{aligned} \quad (5.3)$$

And we have the relation on $\overline{M}_{0,r}$:

$$\psi_i = \sum_{j=1}^{r-3} \frac{(r-1-j)(r-2-j)}{(r-1)(r-2)} \sum_{A \subseteq \{1,2,\dots,r\} - \{i\}, \#A=j} \Delta_{A \cup \{i\}} \quad (5.4)$$

So, these formula allows us to express the λ in terms of the boundary classes.

Proposition 5.1.5. *When $r = 4$, we have:*

$$\lambda = \frac{1}{12} \sum_J \left(-me_J + \frac{1}{3} \sum_{i=1}^4 \left(2(e(i) - 1)d(i) - \frac{(e(i) - 1)^2 d(i)}{e(i)} \right) + d_J \right) \delta_J \quad (5.5)$$

Proof. When $r = 4$, By Equation 5.4 we have:

$$\psi_1 = \frac{1}{3}(\Delta_{12} + \Delta_{13} + \Delta_{14}) = \frac{1}{3}\Delta$$

And so are ψ_i for $i = 1, 2, 3, 4$. Plug in Lemma 5.1.2 and Lemma 5.1.1 and Lemma 5.1.4, we get the desired equality. $\square$

The general formula for $r \geq 5$ will be more complicated. Instead, we give the list of computation results for all Moonen's 20 special families, all listed as below. In the following table, $\delta_{(e,e,e,\dots,e)} = \sum_J \delta_J$ where $e_J = e$ and $d_J = d$, which is the number of times that d repeats. When all the divisors are of the same type, we omit J and write $\delta = \sum_J \delta_J$.

Theorem 5.1.6. *From the computation above, we can compute the Hodge class of all families $\widetilde{\mathcal{M}}(m, r, \underline{a})$. Below, I list the computation result for all of Moonen's 20 special families.*

Family	m	r	$\underline{a}$	Hodge class up to torsion
1 (non-compact)	2	4	(1,1,1,1)	$\lambda = \frac{1}{6}\delta$
2 (non-compact)	2	6	(1,1,1,1,1,1)	$\lambda = \frac{1}{5}\delta$
3 (non-compact)	3	4	(1,1,2,2)	$\lambda = \frac{2}{9}\delta_{(3)} + \frac{8}{27}\delta_{(1,1,1)}$
4	4	4	(1,2,2,3)	$\lambda = \frac{1}{4}\delta_{(4)} + \frac{3}{8}\delta_{(1,1,1,1)}$
5	6	4	(2,3,3,4)	$\lambda = \frac{13}{36}\delta_{(6)} + \frac{59}{108}\delta_{(1,1,1,1,1,1)}$
6(non-compact)	3	5	(1,1,1,1,2)	$\lambda = \frac{1}{3}\delta$
7	4	4	(1,1,1,1)	$\lambda = \frac{1}{3}\delta$
8	4	5	(1,1,2,2,2)	$\lambda = \frac{3}{8}\delta_{(2,2)} + \frac{5}{12}\delta_{(4)} + \frac{19}{48}\delta_{(1,1,1,1)}$
9	6	4	(1,3,4,4)	$\lambda = \frac{5}{12}\delta_{(3,3)} + \frac{7}{12}\delta_{(6)}$
10(non-compact)	3	6	(1,1,1,1,1,1)	$\lambda = \frac{2}{5}\delta$
11	5	4	(1,3,3,3)	$\lambda = \frac{2}{3}\delta$
12	6	4	(1,1,1,3)	$\lambda = \frac{1}{2}\delta$
13	6	4	(1,1,2,2)	$\lambda = \frac{53}{108}\delta_{(2,2,2)} + \frac{19}{36}\delta_{(3,3)}$
14	6	5	(2,2,2,3,3)	$\lambda = \frac{43}{72}\delta_{(1,1,1,1,1,1)} + \frac{29}{36}\delta_{(6)} + \frac{43}{72}\delta_{(3,3)}$
15	8	4	(2,4,5,5)	$\lambda = \frac{5}{4}\delta_{(8)} + \frac{3}{4}\delta_{(4,4)}$
16	5	5	(1,1,1,1,1)	$\lambda = \delta$
17	7	4	(2,4,4,4)	$\lambda = \frac{4}{3}\delta$
18	10	4	(3,5,6,6)	$\lambda = \frac{23}{12}\delta_{(10)} + \frac{13}{12}\delta_{(5,5)}$
19	9	4	(3,5,5,5)	$\lambda = 2\delta$
20	12	4	(4,6,7,7)	$\lambda = \frac{17}{12}\delta_{(6,6)} + \frac{31}{12}\delta_{(12)}$

Example 5.1.7. We carry out the computation for $\widetilde{\mathcal{M}}(m, r, \underline{a}) = \widetilde{\mathcal{M}}[16] = \widetilde{\mathcal{M}}(5, 5, (1, 1, 1, 1, 1))$ explicitly as an example.

There are 10 boundary divisors δ_J , where $J \subseteq \{1, 2, 3, 4, 5\}$, $|J| = 2$. However, notice that all δ_J has $e_J = 5$ and $d_J = 1$. Indeed, $d_J = \gcd(5, 3) = 1$ and

$e_J = \frac{5}{1} = 5$. Also, for $i = 1, 2, 3, 4, 5$, we have $d(i) = \gcd(m, a(i)) = 1$ and $e(i) = \frac{m}{d(i)} = 5$.

By Equation 5.3, we get:

$$t_*(\omega_t^2) = -5q^*\Delta + \sum_{i=1}^5 \frac{24}{5}q^*\psi_i$$

$$[\text{Sing}(C)] = \delta$$

By Equation 5.4, we get that

$$\psi_1 = \frac{1}{2}\Delta_1 + \frac{1}{6}(\Delta - \Delta_1)$$

Here, $\Delta_1 = \Delta_{1,i}$ for $i \in \{2, 3, 4, 5\}$. Hence,

$$\begin{aligned} \sum_{i=1}^5 \psi_i &= \frac{3}{2}\Delta \\ \sum_{i=1}^5 \frac{24}{5}q^*\psi_i &= 36\delta \end{aligned}$$

Therefore,

$$\lambda = \frac{1}{12}(-25\delta + 36\delta + \delta) = \delta$$

Remark 5.1.8. The class of λ is also computed in (Cavalieri, Owens, and Somerstep, 2021, Theorem 1.3) and my computation is compatible with their computation. Indeed, their computation says that:

$$\lambda = \frac{1}{24m} \left(\sum_{J \subset P(\{r\})} \gcd^2\left(\sum_{j \in J} a(j), m\right) \Delta_J \right) \quad (5.6)$$

Where:

- For $2 \leq |J| \leq r - 2$, Δ_J denotes the boundary divisor generically parameterizing one-nodal curves with the branch points labelled by J sitting on one component, and the branch points in J^c ;
- For $J = \{j\}$, $[r] - j$, $\Delta_J := -\psi_j$;
- For $J = \emptyset$, $[n]$, $\Delta_J := \kappa_1$.

For example, in the case of $\widetilde{\mathcal{M}}[16] = \widetilde{\mathcal{M}}(5, 5, (1, 1, 1, 1, 1))$, $\Delta_J = q^* \Delta$. Then when $|J| = 2, 3$, we get a copy of Δ . Hence, coefficient of $\Delta = 2$. When $|J| = 1, 4$, we get a copy of $\psi = \sum_i \psi_i$. Hence, coefficient of ψ is 2. Also, $\psi = \sum_{i=1}^r \psi_i = \frac{3}{2} \Delta$, so $q^* \psi = \frac{3}{2} q^* \Delta$. When $|J| = 0, 5$, we get 25 copies of κ_1 . By (Geer and Kouvidakis, 2010, proof of Lemma 4.3), $\kappa_1 = \psi - \Delta$, so $q^* \kappa_1 = \frac{1}{2} q^* \Delta$. Therefore, $\lambda = \frac{1}{120} (25 q^* \Delta - 3 q^* \Delta + 2 q^* \Delta) = \frac{1}{5} q^* \Delta = \delta$.

5.2 The torsion part of Hodge class

Notice that the Mumford relation gives Hodge line bundle λ up to 12-torsion. Indeed, the multiple of 12 kills of any torsion part whose order divides 12. In this section, we compute the torsion part of λ .

Notation 5.2.1. Throughout the section, we use δ_J to denote the boundary divisor of type J . Recall that if $[\overline{C}] \in \delta$ is such that $\overline{C} = C_1 \sqcup C_2$, then C_1 has inertia type $(a_1, a_2, \dots, a_{r_1})$ and C_2 had inertia type $(-a_{r_1}, a_{r_1+1}, \dots, a_r)$. Then, we take $d = \gcd(m, a_{r_1})$ and $e = \frac{m}{d}$. We say that δ is of type $J = (e, e, \dots, e)$ where e repeats for d times. The number d should be understood as the number of the primages on C_1 of the r_1^{th} branching points, each with ramification index e .

In the rest of this section, we will be using δ_J, d, e without repeating the definition.

Recall that we have the forgetful morphism:

$$q : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \overline{M}_{0,r}$$

Hence, λ can be expressed by:

$$\lambda = q^* \mathcal{L} \otimes \mathcal{O}(\sum_J a_J \delta_J)$$

Where $\mathcal{L} \in \text{Pic}(\overline{M}_{0,r})$ and the sum $\sum_J a_J \delta_J$ is over all the boundary divisors of $\widetilde{\mathcal{M}}(m, r, \underline{a})$. In this section, we give the algorithm to compute a_J for each boundary divisor δ_J . This is what we call as the torsion part of the Hodge line bundle.

We have the adjunction sequence:

$$0 \rightarrow q^* \mathcal{L} |_{\delta_J} \rightarrow \lambda |_{\delta_J} \rightarrow \mathcal{O}(\delta_J)^{\otimes a_J} |_{\delta_J} \rightarrow 0$$

To compute the coefficient a_J , we make use of the action of the extra isotropic group at the boundary.

Notice that on a generic point $[C] \in \widetilde{\mathcal{M}}(m, r, \underline{a})$, there is a generic isotropic group $\mathbb{Z}/m\mathbb{Z}$ acting on $[C]$, given by the known action $\zeta_m(x, y) = (x, \zeta_m y)$. However, for a singular curve $[\overline{C}] \in \delta_J$, suppose that $\overline{C}$ is the clutching of C_1, C_2 . Then both C_1 and C_2 have an action of $\mathbb{Z}/m\mathbb{Z}$, since each of them is a $\mathbb{Z}/m\mathbb{Z}$ cover of $\mathbb{P}^1$. For an element $(a, b) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ acting on $C_1 \times C_2$, its action extends to $\overline{C}$ when the action on the gluing points is "compatible".

Definition 5.2.2. Let $\overline{C} = C_1 \sqcup C_2$ be a generic curve in δ_J . Let $P_1, \dots, P_d$ be the d points on C_1 , $Q_1, \dots, Q_d$ be the d points on C_2 , such that $\overline{C}$ is the gluing of C_1 and C_2 by identifying the points P_i with Q_i . Let $(a, b) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Then, we say that the action of (a, b) on $C_1 \times C_2$ is **compatible**, if

$$a \cdot P_i = P_j \iff b \cdot Q_i = Q_j$$

for all $i, j \in \{1, 2, \dots, d\}$.

Lemma 5.2.3 (Extra isotropic subgroup at the boundary). Let I be the subgroup of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ that is compatible. Then, $I/(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/e\mathbb{Z}$.

Proof. We can choose a generator $1 \in \mathbb{Z}/m\mathbb{Z}$ and index the points $\{P_i\}, \{Q_i\}$ such that $i \cdot P_j = P_{i+j \pmod{d}}$ and $i \cdot Q_j = Q_{i+j \pmod{d}}$. Since the action of $\mathbb{Z}/m\mathbb{Z}$ is transitive on both the sets, we just need to find (a, b) such that for all $i \in \mathbb{Z}/d\mathbb{Z}$, we have

$$a \cdot P_1 = P_i \iff b \cdot Q_1 = Q_i$$

But then:

$$\begin{aligned} a \cdot P_1 = P_i &\iff b \cdot Q_1 = Q_i \\ &\iff 1 + a \equiv 1 + b \pmod{d} \\ &\iff a \equiv b \pmod{d} \end{aligned}$$

Hence, we can define a surjective map:

$$\begin{aligned} I &\rightarrow \mathbb{Z}/e\mathbb{Z} \\ (a, b) &\rightarrow \frac{a - b}{d} \end{aligned}$$

On the other hand, the generic isotropic group $\mathbb{Z}/m\mathbb{Z}$ embeds into I via the diagonal:

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} &\rightarrow I \\ t &\rightarrow (t, t) \end{aligned}$$

And the image is exactly the kernel of $I \rightarrow \mathbb{Z}/e\mathbb{Z}$. This shows that $I/(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/e\mathbb{Z}$. $\square$

From now on, we will refer to $\mathbb{Z}/e\mathbb{Z}$ as the **extra isotropic group** at boundary δ_J .

We want to understand how $1 \in \mathbb{Z}/e\mathbb{Z}$ acts on λ_{δ_J} and $\mathcal{O}(\delta_J)|_{\delta_J}$, and the coefficient a_J can be computed as the ratio of these two weights. But there is one issue: $\mathbb{Z}/e\mathbb{Z}$ does not act on λ or $\mathcal{O}(\delta_J)$ directly, but rather it is the compatible subgroup $I \subseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ that acts on these two line bundles. If we pick different lifts of 1 in I , then they might act differently on λ_{δ_J} . The action of $\mathbb{Z}/e\mathbb{Z}$ only makes sense if $\mathbb{Z}/m\mathbb{Z}$ acts trivially on λ_{δ_J} .

How do we reconcile this? Notice that we want to compute the class of λ because we want to compute the global sections of λ^k . But as we mentioned in Remark 3.2.2, λ^k has global sections only if $k \cdot \sum_{i \in \mathbb{Z}/m\mathbb{Z}} if(i) \equiv 0 \pmod{m}$. If we let k_0 to be the smallest positive integer such that $k_0 \cdot \sum_{i \in \mathbb{Z}/m\mathbb{Z}} if(i) \equiv 0 \pmod{m}$, then we only care about k such that $k_0 \mid k$. For such λ^k , $\mathbb{Z}/e\mathbb{Z}$ has a well-defined action on it. Since $\mathbb{Z}/m\mathbb{Z}$ acts trivially, every lift of $1 \in \mathbb{Z}/e\mathbb{Z}$ has the same action on $\lambda_{\delta_J}^k$ and $\mathcal{O}(\delta_J)|_{\delta_J}$.

To this end, we pick the lift $(d, 0) \in I$ that lifts $1 \in \mathbb{Z}/e\mathbb{Z}$. We denote this lift as $\tilde{1}$ and the group it generates as $\tilde{H}$. Notice that $\tilde{H} \cong \mathbb{Z}/e\mathbb{Z}$. Then, $\tilde{1}$ fixes C_2 and acts as $d\mathbb{Z}/m\mathbb{Z}$ on C_1 . Then, we look at the weight of $\tilde{1}$ acting on λ_{δ_J} divided by the weight of $\tilde{1}$ acting on $\mathcal{O}(\delta_J)|_{\delta_J}$. This ratio multiplied by k_0 will be the same as the desired $k_0 a_J$.

Now, we will compute the character of $\tilde{1}$ on both λ and $\mathcal{O}(\delta_J)$ restricted to δ_J .

Lemma 5.2.4. *Let $\overline{C} = C_1 \sqcup C_2$ and $\{P_i\}, \{Q_i\}$ be the set of gluing points. Let $\omega_{\overline{C}}$ be the dualizing sheaf of $\overline{C}$. Then we have the $\tilde{H}$ representation:*

$$0 \rightarrow H^0(\overline{C}, \omega_{\overline{C}}) \rightarrow H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i)) \oplus H^0(C_2, \Omega_{C_2}(\sum_{i=1}^d Q_i)) \rightarrow (k^*)^{\oplus(d-1)} \rightarrow 0$$

And the weight of $\tilde{1}$ acting on λ is just the same as the weight of $\tilde{1}$ acting on $\bigwedge^{g_1} H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i))$.

Proof. From the definition of dualizing sheaf $\omega_{\overline{C}}$, we have the following exact sequence of sheaves:

$$0 \rightarrow \omega_{\overline{C}} \rightarrow \Omega_{C_1} \oplus \Omega_{C_2} \rightarrow (k^*)^d \rightarrow 0$$

Where the second map sends $(f, g) \rightarrow \oplus_i (\text{res}_{P_i}(f) + \text{res}_{Q_i}(g))$. By taking global sections, we get the following LES:

$$0 \rightarrow H^0(\bar{C}, \omega_{\bar{C}}) \rightarrow H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i)) \oplus H^0(C_2, \Omega_{C_2}(\sum_{i=1}^d Q_i)) \rightarrow (k^*)^{\oplus d} \rightarrow H^1(\bar{C}, \omega_{\bar{C}}) \rightarrow 0$$

By Serre duality, $H^1(\bar{C}, \omega_{\bar{C}}) = H^0(\bar{C}, \mathcal{O}_{\bar{C}}) \cong k^*$. These are the constant functions, where the action of $\tilde{I}$ is trivial. Hence, we get the designed SES on global sections. Taking the top wedge, we see that the weight on λ is just the same as the weight on the top wedge of $H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i))$ $\square$

Proposition 5.2.5 (weight of $\tilde{H}$ on λ). *Let f_{C_1} be the signature of $\mathbb{Z}/m\mathbb{Z}$ acting on C_1 . Then, the weight of $\tilde{I}$ acting on $\wedge^{g_1} H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i))$ is*

$$w(\tilde{I})_{\lambda|_{\delta_J}} = \sum_{i=1}^{e-1} \left(\sum_{k=0}^{d-1} f_{C_1}(ke + i) \right) \cdot id \pmod{m}$$

Proof. Notice $\tilde{I}$ acts as d on $H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i))$. We have that $H^0(C_1, \Omega_{C_1}(\sum_{i=1}^d P_i)) \cong H^1(C_1, \mathcal{O}_{C_1}(-\sum_{i=1}^d P_i))^\vee$. We have the basis of the i^{th} eigenspace of $H^1(C_1, \mathcal{O}_{C_1})$. We are going to compute the weight of $\mathbb{Z}/e\mathbb{Z}$ acting on $H^1(C_1, \mathcal{O}_{C_1}(-\sum P_i))$ and take the dual.

Following the notations in (Lin, Mantovan, and Singhal, 2024, Notation 5.2), for $1 \leq i \leq m-1$, set

$$v_i = \frac{y^i}{(x - x_1)^{\lfloor \frac{ia(1)}{m} \rfloor} \dots (x - x_{r_1})^{\lfloor \frac{ia(r_1)}{m} \rfloor}}.$$

We use the affine charts $U = \pi^{-1}(\mathbb{P}^1 - \{\infty\})$ and $V = \pi^{-1}(\mathbb{P}^1 - \{0\})$. Then, we identify

$$H^1(C, \mathcal{O}_C) = \frac{\mathcal{O}_C(U \cap V)}{\mathcal{O}_C(U) \oplus \mathcal{O}_C(V)}.$$

Then, the rings of regular functions on U , V and $U \cap V$ are

$$\mathcal{O}_C(U) \cong \bigoplus_{i=1}^{m-1} k[x]v_i, \quad \mathcal{O}_C(V) \cong \bigoplus_{i=1}^{m-1} k\left[\frac{1}{x}\right]x^{-f(i^*)-1}v_i, \quad \mathcal{O}_C(U \cap V) \cong \bigoplus_{i=1}^{m-1} k\left[x, \frac{1}{x}\right]v_i$$

WOLG, we assume that $P_1, \dots, P_d \in C_1$ are above the point $(1, 0)$. Then, we want the regular functions on $U, V, U \cap V$ that vanishes to at least order 1 at $(1, 0)$. We look at the basis v_i of the i^{th} eigenspace. If $\text{ord}_{(1,0)} v_i = 0$, then for these vector spaces we need to multiply the basis by the function $(x - 1)$.

By Proposition 6.1.1 proven in later section, we have that:

$$\begin{aligned}\text{ord}_{(1,0)}(y) &= a(r_1) \\ \text{ord}_{(1,0)}(x-1) &= m\end{aligned}$$

Where $a(r_1)$ is the inertia type of the point $P_1, \dots, P_d$. Hence

$$\begin{aligned}\text{ord}_{(1,0)}(v_i) &= 0 \\ \iff ia(r_1) - m \lfloor \frac{ia(r_1)}{m} \rfloor &= 0 \\ \iff m \mid ia(r_1) \\ \iff \frac{m}{\gcd(m, a(r_1))} \mid \frac{ia(r_1)}{\gcd(m, a(r_1))} \\ \iff e \mid i\end{aligned}$$

Here $e = \frac{m}{\gcd(m, a(r_1))}$. Hence,

$$\begin{aligned}O_C(U)(-\sum P_i) &\cong \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \nmid i}}^{m-1} k[x]v_i \oplus \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \mid i}}^{m-1} k[x](x-1)v_i \\ O_C(V)(-\sum P_i) &\cong \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \nmid i}}^{m-1} k[\frac{1}{x}]x^{-f(i^*)-1}v_i \oplus \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \mid i}}^{m-1} k[\frac{1}{x}](\frac{1}{x}-1)x^{-f(i^*)-1}v_i \\ O_C(U \cap V)(-\sum P_i) &\cong \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \nmid i}}^{m-1} k[x, \frac{1}{x}]v_i \oplus \bigoplus_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \mid i}}^{m-1} k[x, \frac{1}{x}](\frac{1}{x}-1)v_i\end{aligned}$$

Now, consider $H^1(C_1, O_{C_1}(-\sum P_i))$. At i such that $e \nmid i$, the eigenspace is $\bigoplus_{j=1}^{f(i^*)} k\{\frac{v_i}{x^j}\}$. However, at i such that $e \mid i$, We get $\frac{k[x, \frac{1}{x}](\frac{1}{x}-1)}{k[x](x-1) \oplus k[\frac{1}{x}](\frac{1}{x}-1)x^{-1-f(i^*)}}v_i = \bigoplus_{j=1}^{f(i^*)+1} k\{\frac{v_i}{x^j}\}$ In summary, we get dimension of the eigenspaces of H^1 is:

$$\dim H^1(C_1, O_{C_1}(\sum P_i))_{\tau_i} = \begin{cases} 0 & \text{if } i = 0 \\ f(i^*) & \text{if } e \nmid i \\ f(i^*) + 1 & \text{if } e \mid i \end{cases}$$

So the dimension of the eigenspaces of H^0 is:

$$\dim H^0(C_1, \Omega_{C_1}(\sum P_i))_{\tau_i} = \begin{cases} 0 & \text{if } i = 0 \\ f(i) & \text{if } e \nmid i \\ f(i) + 1 & \text{if } e \mid i \end{cases}$$

Hence, the $\mathbb{Z}/m\mathbb{Z}$ weight of $\wedge^{g_1} H^0(C_1, \Omega_{C_1}(\sum P_i))$ is

$$\sum_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e|i}} (f(i) + 1)i + \sum_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \nmid i}} (f(i))i \pmod{m}$$

Hence, the $d\mathbb{Z}/m\mathbb{Z}$ weight is:

$$\begin{aligned} w(\tilde{I})_{\lambda|_{\delta_J}} &= \sum_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e|i}} (f(i) + 1)id + \sum_{\substack{i \in \mathbb{Z}/m\mathbb{Z}, \\ e \nmid i}} (f(i))id \pmod{e} \\ &= \sum_{i=1}^{e-1} \left(\sum_{k=1}^{d-1} f(ke + i) \right) \cdot id \pmod{m} \end{aligned}$$

This finishes the proof of Proposition 5.2.5. $\square$

Now, we want to compute the weight of $\tilde{I}$ on $\mathcal{O}(\delta_J)|_{\delta_J}$.

Proposition 5.2.6 (weight of $\mathbb{Z}/e\mathbb{Z}$ on $\mathcal{O}(\delta_J)$). *The weight of $\tilde{I}$ acting on $\mathcal{O}(\delta_J)$ is:*

$$w(\tilde{I})_{\mathcal{O}(\delta_J)|_{\delta_J}} = -sd \pmod{m}$$

Where $s = b^{-1} \pmod{e}$ and $b = \frac{a(r_1)}{\gcd(m, a(r_1))}$.

Proof. $\mathcal{O}(\delta_J)$ is a line bundle on $\widetilde{\mathcal{M}}(m, r, \underline{a})$. At a point represented by $[\overline{C}]$ for $\overline{C} \in \delta_J$, it is:

$$\mathcal{O}(\delta_J)_{[\overline{C}]} = T_{C_1, P_1} \otimes T_{C_2, Q_1}$$

Recall that $\tilde{I}$ acts as $d \in \mathbb{Z}/m\mathbb{Z}$ on C_1 and fixes C_2 . Hence, we just have to compute the weight on $T_{C_1, P_1} \cong (\mathfrak{m}_{P_1}/\mathfrak{m}_{P_1}^2)^\vee$.

Let $e = \frac{m}{\gcd(m, a(r_1))}$ and $b = \frac{a(r)}{\gcd(m, a(r_1))}$. By Section 3.5, we can choose a uniformizer of C_1 at P_1 as $g = \frac{y^s}{(x-x_1)^{\lfloor \frac{sb}{e} \rfloor}}$. Then, $d\mathbb{Z}/m\mathbb{Z}$ acts on g by weight sd . To get the action on T_{C_1, P_1} we simply take its dual, so the weight is $-sd$. This finishes the proof of Proposition 5.2.6. $\square$

With the formula of $w(\tilde{I})_{\lambda|_{\delta_J}}$ and $w(\tilde{I})_{\mathcal{O}(\delta_J)|_{\delta_J}}$, we can compute

$$a_J = \frac{w(\tilde{I})_{\lambda|_{\delta_J}}}{w(\tilde{I})_{\mathcal{O}(\delta_J)|_{\delta_J}}} = \left(\sum_{i=1}^{e-1} \left(\sum_{k=1}^{d-1} f(ke + i) \right) \cdot i \right) \cdot (-b) \pmod{e}$$

in $\mathbb{Z}/e\mathbb{Z}$. Let k_0 be the minimal power such that λ^{k_0} has global section. Then, $k_0 a_J$ is the coefficient of the torsion part $\mathcal{O}(\delta_J)$ in λ^{k_0}

5.3 Hodge class when $r = 4$

We apply the Sections 5.1 and 5.2 to compute Hodge line bundle explicitly when $r = 4$.

When $r = 4$, we have that $\overline{M}_{0,r} \cong \mathbb{P}^1$. Therefore, $q^*\mathcal{L} = q^*\mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \in \mathbb{Z}$.

Fix a boundary $\delta_J \in \widetilde{\mathcal{M}}(m, r, \underline{a})$. Suppose δ_J is of type $(e, \dots, e)$. That is, $\frac{m}{\gcd(m, a(r_1))} = e$. By Lemma 5.1.3, we see that $q^*\mathcal{O}(1) = \mathcal{O}(e\delta_J)$. In particular, $\delta_J = \frac{1}{e_J}q^*\mathcal{O}(1)$.

Suppose $\lambda = \mathcal{O}(\sum b_J \delta_J)$ up to 12 torsion and $q^*\mathcal{O}(n) \otimes \mathcal{O}(\sum a_J \delta_J)$. Substitute δ_J by $\frac{1}{e_J}q^*\mathcal{O}(1)$, we get that

$$n + \sum_J \frac{a_J}{e_J} = \sum_J \frac{b_J}{e_J}$$

We get that $n = \sum_J \frac{b_J - a_J}{e_J}$.

We sum over $J \subseteq \{1, 2, \dots, r\}$ such that $2 \leq |J| \leq r - 2$. When $r = 4$, we have $J = \{12\}, \{13\}, \{14\}$, corresponding to the three degeneration types on $\widetilde{\mathcal{M}}(m, r, \underline{a})$, also corresponding to the 3 boundary divisors $\partial\overline{M}_{0,4} = \{0, 1, \infty\}$. Recall from Notation 5.0.1 that δ_J consists of $\kappa(C_1 \times C_2)$ where C_1 has inertia type constructed from J and C_2 has inertia type constructed from the complement J^c .

When the coefficients at all the boundary divisors are the same, we use δ to denote the sum of all the boundary divisors.

Below is the table of Hodge class for Moonen's special family for $r = 4$. We compute the minimal descending power k_0 such that $k_0 \sum if(i) \equiv 0 \pmod{m}$, and express λ^{k_0} in terms of $q^*\mathcal{O}(n) \otimes \mathcal{O}(\sum k_0 a_J \delta_J)$.

Family	m	a	k_0	λ^{k_0}
1 (non-compact)	2	(1, 1, 1, 1)	2	$q^*O(1)$
3 (non-compact)	3	(1, 1, 2, 2)	1	$O(2\delta_{(12)})$
4	4	(1, 2, 2, 3)	1	$q^*O(-1) \otimes O(3\delta_{(12)} + 3\delta_{(13)})$
5	6	(2, 3, 3, 4)	1	$q^*O(-1) \otimes O(5\delta_{(6)})$
7	4	(1, 1, 1, 1)	1	$q^*O(-1) \otimes O(\delta)$
9	6	(1, 3, 4, 4)	3	$q^*O(1) \otimes O(3\delta_{(12)} + 3\delta_{(13)})$
11	5	(1, 3, 3, 3)	5	$q^*O(2)$
12	6	(1, 1, 1, 3)	6	$q^*O(3)$
13	6	(1, 1, 2, 2)	3	$q^*O(1) \otimes O(\delta_{(13)} + \delta_{(14)})$
15	8	(2, 4, 5, 5)	2	$q^*O(1)$
17	7	(2, 4, 4, 4)	7	$q^*O(4)$
18	10	(3, 5, 6, 6)	1	$q^*O(-1) \otimes O(2\delta_{(12)} + 6\delta_{(13)} + 6\delta_{(14)})$
19	9	(3, 5, 5, 5)	1	$O(2\delta)$
20	12	(4, 6, 7, 7)	12	$q^*O(8)$

5.4 Hodge class when $r = 5$

When $r = 5$, we use the description of the divisors on $\overline{M}_{0,5}$. We know that $\text{Pic}(\overline{M}_{0,5})$ is generated by the 10 boundary divisors Δ_J . Since $r = 5$, J has 10 possibilities:

$$J \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$$

Each corresponding to a boundary configuration on $\widetilde{\mathcal{M}}(m, r, \underline{a})$. Then, we can write $\lambda = q^*O(\sum_J n_J \Delta_J) \otimes O(\sum a_J \delta_J) = O(\sum b_J \delta_J)$. By Lemma 5.1.3, we have that $q^*O(\Delta_J) = e_J \delta_J$. Substitute Δ_J , we get that:

$$n_J = \frac{b_J - a_J}{e_J}$$

Below is the table of Hodge class for Moonen's special family for $r = 5$. Similar to the case for $r = 4$, we compute the minimal descending power k_0 such that $k_0 \sum if(i) \equiv 0 \pmod{m}$, and express λ^{k_0} in terms of $q^*\mathcal{L} \otimes O(\sum k_0 a_J \delta_J)$ for $\mathcal{L} \in \text{Pic}(\overline{M}_{0,5})$. Recall in Section 3.3, we identify $\overline{M}_{0,5} \cong \text{Bl}_4(\mathbb{P}^2)$, and then $\text{Pic}(\overline{M}_{0,5})$ is generated by $H, E_{P_1}, E_{P_2}, E_{P_3}, E_{P_4}$, and the anti-canonical divisor $-K = 3H - \sum_{i=1}^4 E_{P_i}$. Hence, we may express $\mathcal{L}$ as linear combination of H, E_{P_i} and possibly $-K$.

Family	m	a	k_0	λ^{k_0}
6 (non-compact)	3	(1, 1, 1, 1, 2)	3	$q^* \mathcal{O}_{\overline{M}_{0,5}}(2H)$
8	4	(1, 1, 2, 2, 2)	4	$q^* \mathcal{O}_{\overline{M}_{0,5}}(-2K - 2E_{P_1})$
14	6	(2, 2, 2, 3, 3)	6	$q^* \mathcal{O}_{\overline{M}_{0,5}}(-2K + 2E_{P_1})$
16	5	(1, 1, 1, 1, 1)	1	$\mathcal{O}(\delta)$

5.5 Global section of Hodge line bundle on $\widetilde{\mathcal{M}}[16]$

In this section, we show the structure of the ring of global sections of Hodge line bundle on $\widetilde{\mathcal{M}}[16]$. The main results are Proposition 5.5.2 and Proposition 5.5.5.

Notation 5.5.1. We use the following notations. Notice that some of these notations are from Section 3.3.

- λ : the Hodge line bundle on $\widetilde{\mathcal{M}}[16]$. In Example 5.1.7, we show that

$$\lambda = \delta = \sum_J \delta_J$$

- Δ : The sum of the 10 boundary divisors in $\overline{M}_{0,5}$. Recall that in Section 3.3, we identify $\overline{M}_{0,5} \cong \text{Bl}_4(\mathbb{P}^2)$, where blow up of $\mathbb{P}^2$ are at $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$, $P_4 = [1 : 1 : 1]$. Under this identification,

$$\begin{aligned} \Delta = & L_{P_1 P_2} + L_{P_1 P_3} + L_{P_1 P_4} + L_{P_2 P_3} + L_{P_2 P_4} + L_{P_3 P_4} \\ & + E_{P_1} + E_{P_2} + E_{P_3} + E_{P_4} \end{aligned}$$

Where $L_{P_i P_j}$ is the strict transformation of the line connecting P_i, P_j , and E_{P_i} is the exceptional divisor at the point P_i .

In this section, we will drop the alphabet P and use L_{ij} for $L_{P_i P_j}$, and E_i for E_{P_i} , for $1 \leq i, j \leq 4$.

- Y : the degree 5 Del-Pezzo surface in $\mathbb{P}^5$.
- Σ : The sum of the 10 lines in Y .
- $-K_{\text{Bl}_4(\mathbb{P}^2)}$: the anti-canonical divisor on $\text{Bl}_4(\mathbb{P}^2)$. Recall in Section 3.3 we have that $2(-K_{\text{Bl}_4(\mathbb{P}^2)}) = \Delta$, and the anti-canonical embedding

$$|-K_{\text{Bl}_4(\mathbb{P}^2)}| : \text{Bl}_4(\mathbb{P}^2) \cong Y \subseteq \mathbb{P}^5$$

$$\Delta \cong \Sigma$$

- $-K_Y$: the anti-canonical divisor on Y . We have $-K_Y \cong \mathcal{O}_Y(1)$.
- s_{ij} , $1 \leq i \neq j \leq 3$: these are the 6 homogenous coordinate on $\mathbb{P}^5$. They can also be identified with a basis of $H^0(Y, \mathcal{O}_Y(-K_Y))$, and notice $s_{ij} \neq s_{ji}$.
- $\Gamma_0(\lambda)$: $\Gamma_0(\lambda) = \bigoplus_m H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m})$.
- K : The field $K = \mathbb{Q}(\zeta_5)$. This is the field of definition of $\widetilde{\mathcal{M}}[16]$, and the coefficient of the ring of global sections $\Gamma_0(\lambda)$.

Using results from Section 5.1, we first compute the vector space of the sections of $m\lambda$.

Proposition 5.5.2. *The global section of the m fold tensor of λ on $\widetilde{\mathcal{M}}[16]$ is*

$$H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m}) \cong H^0(\overline{M}_{0,5}, \mathcal{O}(\lfloor \frac{m}{5} \rfloor \Delta)) = H^0(Y, \mathcal{O}(2\lfloor \frac{m}{5} \rfloor (-K_Y)))$$

Proof. Consider the maps:

$$\widetilde{\mathcal{M}}[16] \xrightarrow{q} \overline{M}_{0,5} \xrightarrow{|-K_{\text{Bl}_4(\mathbb{P}^2)}|} Y$$

Where q is the coarse moduli space map, $|-K_{\text{Bl}_4(\mathbb{P}^2)}|$ is the anti-canonical embedding of $\overline{M}_{0,5}$ into $\mathbb{P}^5$, Y is the degree 5 Del-Pezzo surface.

Since q is surjective onto the image, $H^0(\widetilde{\mathcal{M}}, \lambda^{\otimes m}) = H^0(\overline{M}_{0,5}, q_* \lambda^{\otimes m})$. By previous sections, $q_* \lambda^{\otimes m} = q_* \mathcal{O}(m\delta)$. We have the following two cases:

- If $5 \mid m$: Then, say $m = 5n$. We have that $m\delta = 5n\delta = nq^*\Delta$. Hence,

$$q_* \mathcal{O}(m\delta) = q_* q^* \mathcal{O}(n\Delta) = \mathcal{O}(n\Delta) \otimes q_* \mathcal{O}_{\widetilde{\mathcal{M}}} = \mathcal{O}(n\Delta)$$

We have $q_* \mathcal{O}_{\widetilde{\mathcal{M}}[16]} = \mathcal{O}_{\overline{M}_{0,5}}$ because $\overline{M}_{0,5}$ is the coarse moduli space of $\widetilde{\mathcal{M}}[16]$.

In this case, we have that

$$H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m}) \cong H^0(\overline{M}_{0,5}, \mathcal{O}(n\Delta))$$

Since $\Delta \equiv -2K$, when we use $\mathcal{O}(-K)$ to embed $\overline{M}_{0,5}$ into $\mathbb{P}^5$ we further have that:

$$H^0(\overline{M}_{0,5}, \mathcal{O}(n\Delta)) \cong H^0(Y, \mathcal{O}_Y(2n))$$

Hence, the weight m modular form is given by $H^0(Y, \mathcal{O}_Y(2n))$.

- If $5 \nmid m$. Then, we claim that

$$H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m}) = H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes(m-1)})$$

Indeed, we consider the exact sequence:

$$0 \rightarrow \mathcal{O}((m-1)\delta) \rightarrow \mathcal{O}(m\delta) \rightarrow \mathcal{O}(m\delta) |_{\delta} \rightarrow 0$$

Then, locally around δ , the sections of $\mathcal{O}(m\delta)$ are $\{f(z) \mid \text{ord}_0(f(z)) \geq -m\}$. However, on δ , $f(z)$ has the extra automorphism $z \rightarrow \zeta_5 z$. Hence, $f(z) = f(\zeta_5 z)$, so $f(z) = g(z^5)$ for some g . So $\text{ord}_0(f(z))$ must be a multiple of 5. So, the condition that f has pole with order at most m is the same as f has pole with order at most $5\lfloor \frac{m}{5} \rfloor$. Therefore, $H^0(\widetilde{\mathcal{M}}[16], \mathcal{O}((m-1)\delta)) = H^0(\widetilde{\mathcal{M}}[16], \mathcal{O}(m\delta))$

Combining the above two cases, we get that $H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m}) = H^0(\overline{\mathcal{M}}_{0,5}, \mathcal{O}(\lfloor \frac{m}{5} \rfloor \Delta)) \cong H^0(Y, \mathcal{O}_Y(2\lfloor \frac{m}{5} \rfloor))$ $\square$

Now that we get each vector space $H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m})$, we want to get the graded ring $\Gamma_0(\lambda) := \bigoplus_{m \geq 0} H^0(\widetilde{\mathcal{M}}[16], \lambda^{\otimes m})$. By the above proposition, this is related $H^0(Y, \mathcal{O}_Y(n))$ for Del-Pezzo surface Y . We cite the following result for the homogeneous coordinate ring of the Del-Pezzo surface and the $\mathfrak{S}_5$ action on it.

Proposition 5.5.3. (*Bauer and Catanese, 2021, Theorem 3.3*) Let $R_Y = \bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(n))$ be the homogeneous coordinate ring of the Del-Pezzo surface. Let $s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}$ be the coordinates on $\mathbb{P}^5$. Then we have the following:

1. Let $Q_{ij} = s_{ij} \cdot \sigma_{ij}$, where

$$\begin{aligned} \sigma_{ij} &= \epsilon_{ij} \cdot (s_{ik} - s_{ki} - s_{jk}) \\ \epsilon_{ij} &:= \epsilon \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \end{aligned}$$

Then the ideal of Y in $\mathbb{P}^5$ is given by

$$\mathcal{I}_Y = (2Q_{12} - 2Q_{21}, 2Q_{12} - 2Q_{13}, 2Q_{12} - 2Q_{31}, 2Q_{12} - 2Q_{23}, 2Q_{12} - 2Q_{32})$$

Hence, the homogenous coordinate ring of Y is given by:

$$R_Y = \mathbb{Q}[s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}] / (Q_{12} - Q_{21}, Q_{12} - Q_{23}, Q_{12} - Q_{32}, Q_{12} - Q_{31})$$

2. The $\mathfrak{S}_5$ -action on R_Y is given as follows:

- For $\tau \in \mathfrak{S}_3 \subseteq \mathfrak{S}_5$:

$$\tau(s_{ij}) = \epsilon(\tau)s_{\tau(i)\tau(j)}$$

$$\tau(\sigma_{ij}) = \sigma_{\tau(i)\tau(j)}$$

$$\tau(Q_{ij}) = \epsilon(\tau)Q_{ij}$$

- For $\tau := (34)$ acts as

$$- \tau(s_{12}) = \sigma_{31};$$

$$- \tau(s_{13}) = s_{23};$$

$$- \tau(s_{21}) = -\sigma_{32};$$

$$- \tau(s_{23}) = s_{13};$$

$$- \tau(s_{31}) = -\sigma_{12};$$

$$- \tau(s_{32}) = \sigma_{21}.$$

- For $\tau = (45)$ acts as

$$\tau(s_{ij}) = s_{ik}$$

In particular, we know that $\mathfrak{S}_5$ acts transitively on the set $S = \{\pm s_{ij}, \pm \sigma_{ij}\}$, and the stabilizer of $s_{21} = \langle (12345) \rangle \cong \mathbb{Z}/5\mathbb{Z}$. The stabilizers of s_{ij} are then the conjugate of $\langle (12345) \rangle$.

Remark 5.5.4. In (Bauer and Catanese, 2021), the homogeneous coordinate ring R_Y is defined over $\mathbb{C}$. To obtain the defining equations in $\mathcal{I}_Y$, they consider the complete linear series $V = |3H - \sum_{i=1}^4 E_{P_i}|$ on $\text{Bl}_4(\mathbb{P}^2)$. They notice that $V \cong \{f \in \mathbb{C}[x_1, x_2, x_3]_3 : f(P_i) = 0, 1 \leq i \leq 4\}$, and V has a basis given by $\{x_i x_j (x_j - x_k) \mid \{i, j, k\} = \{1, 2, 3\}\}$, where x_i are the pulled back of the hyperplane sections on $\mathbb{P}^2$. The defining relations of $\mathcal{I}$ are the generators of the kernel of the ring homomorphism $s_{ij} \rightarrow x_i x_j (x_j - x_k)$.

Notice that the ideal $\mathcal{I}_Y$, when considered over $\mathbb{Q}$, also gives the defining equation of the image of $\overline{M}_{0,5}$ in $\mathbb{P}^5$. Indeed, the moduli space $\overline{M}_{0,5}$ is defined over $\mathbb{Q}$, and the blow up $\text{Bl}_4(\mathbb{P}^2)$ is an arithmetic model of $\overline{M}_{0,5}$ over $\mathbb{Q}$. The set of sections $\{x_i x_j (x_j - x_k) \mid \{i, j, k\} = \{1, 2, 3\}\}$ are rational sections of the line bundle $3H - \sum_{i=1}^4 E_{P_i}$. They span the $\mathbb{Q}$ -vector space $\{f \in \mathbb{Q}[x_1, x_2, x_3]_3 : f(P_i) = 0, 1 \leq i \leq 4\}$. Hence, the anti-canonical embedding $|-K_{\text{Bl}_4(\mathbb{P}^2)}| : \text{Bl}_4(\mathbb{P}^2) \rightarrow \mathbb{P}^5$ is defined over $\mathbb{Q}$.

The ideal $\mathcal{I}_Y$ with coefficient in $\mathbb{Q}$ is also the ideal of the image of $\text{Bl}_4(\mathbb{P}^2)$ in $\mathbb{P}^5$. For that, we need to show that, for any homogeneous polynomials $f \in \mathbb{Q}[s_{ij}]$, if f restricts to 0 to Y , then f must be polynomials of $(2Q_{12} - 2Q_{ij})$ with rational coefficient. However, if f restricts to 0 in Y , then f is some $\mathbb{C}$ combinations of products of the generators $2Q_{12} - 2Q_{ij}$, and $2Q_{12} - 2Q_{ij}$ are invariant under the $\text{Gal}(\mathbb{C}/\mathbb{Q})$ action. By inducting on the number of terms in f , we see that f is $\mathbb{Q}$ combinations of products of the generators in $\mathcal{I}_Y$. Hence, $\mathcal{I}_Y$ is the ideal of $\text{Bl}_4(\mathbb{P}^2)$ in $\mathbb{Q}[s_{ij}]$.

Then, for any finite field extension $K/\mathbb{Q}$, we have $\overline{M}_{0,5}/K \cong \text{Bl}_4(\mathbb{P}^2)/K$, and $R_Y \otimes K$ is the coordinate ring of the image of $\overline{M}_{0,5}/K$ in $\mathbb{P}^5/K$. For the rest of the paper, we will take $K = \mathbb{Q}(\zeta_5)$ so that $\widetilde{\mathcal{M}}[16]$ is defined over K . By abuse of notation, we write

$$R_Y = K[s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}]/\mathcal{I}_Y$$

R_Y is now the homogeneous coordinate ring of the degree 5 Del-Pezzo surface in $\mathbb{P}^5$ over K .

Combining Proposition 5.5.2 and the $\mathfrak{S}_5$ action on s_{ij} , we compute the ring structure of $\Gamma_0(\lambda)$ and the $\mathfrak{S}_5$ action on it. In the following, recall that $K = \mathbb{Q}(\zeta_5)$. Also, for R a graded ring, we denote by R_{ev} the even graded subring in R .

Proposition 5.5.5. $\Gamma_0(\lambda) \cong R_{ev}$, where

$$R = K[s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}, \varepsilon]/\mathcal{I}$$

$$\mathcal{I} = (\varepsilon^5 - Q_{12}, Q_{12} - Q_{21}, Q_{12} - Q_{23}, Q_{12} - Q_{32}, Q_{12} - Q_{13}, Q_{12} - Q_{13})$$

Here:

1. R is a graded ring with s_{ij} at degree 1 and weight $\frac{5}{2}$, ε at degree $\frac{2}{5}$ and weight 1.
2. The generators of R_{ev} are ε at weight 1 and degree $\frac{2}{5}$, and $s_{ij}s_{i'j'}$ at weight 5 and degree 10.
3. We have the $\mathfrak{S}_5$ action on R , hence on R_{ev} . The actions of $\mathfrak{S}_5$ on s_{ij} are given as in Proposition 5.5.3. The action on ε is given as

$$\tau(\varepsilon) = \text{sign}(\tau)\varepsilon$$

Proof. We will prove the ring structure and the action of $\mathfrak{S}_5$ on ε .

To illustrate the proof, let's keep the following commutative diagram in mind:

$$\begin{array}{ccccccc}
 q_*\mathcal{O}(0\delta) & \hookrightarrow & q_*\mathcal{O}(5\delta) & \hookrightarrow & q_*\mathcal{O}(10\delta) & \hookrightarrow & \dots \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 \mathcal{O}_Y(0\Sigma) & \hookrightarrow & \mathcal{O}_Y(\Sigma) & \hookrightarrow & \mathcal{O}_Y(2\Sigma) & \hookrightarrow & \dots \\
 \downarrow f_0 & & \downarrow f_2 & & \downarrow f_4 & & \\
 \mathcal{O}_Y & \xrightarrow{Q_{12}} & \mathcal{O}_Y(2) & \xrightarrow{Q_{12}} & \mathcal{O}_Y(4) & \xrightarrow{Q_{12}} & \dots
 \end{array}$$

First we explain how is the element ε is chosen. Notice that we have the natural embeddings

$$q_*\mathcal{O}(0\delta) \hookrightarrow q_*\mathcal{O}(\delta) \hookrightarrow q_*\mathcal{O}(2\delta) \hookrightarrow q_*\mathcal{O}(3\delta) \hookrightarrow q_*\mathcal{O}(4\delta)$$

Since $H^0(\widetilde{\mathcal{M}}, \mathcal{O}) = K$, we let $\alpha \in H^0(\widetilde{\mathcal{M}}, \mathcal{O}(\delta))$ to be the image of 1 under this natural embedding. Equivalently, α is the global section of $\mathcal{O}(\delta)$ whose divisor of zero is exactly δ . Then, by proposition 5.5.2, we have that for $0 \leq j \leq 4$,

$$\cdot\alpha : H^0(\widetilde{\mathcal{M}}, \mathcal{O}(j\delta)) \rightarrow H^0(\widetilde{\mathcal{M}}, \mathcal{O}((j+1)\delta))$$

are isomorphism. That is, $\Gamma_0(\lambda)_j = \alpha^j \Gamma_0(\lambda)_0$.

On the other hand, we have the commutative diagram

$$\begin{array}{ccc}
 \mathcal{O}_Y(0\Sigma) & \hookrightarrow & \mathcal{O}_Y(\Sigma) \\
 \downarrow = & & \downarrow f_2 \\
 \mathcal{O}_Y & \xrightarrow{Q_{12}} & \mathcal{O}_Y(2)
 \end{array}$$

Here, we identify

$$H^0(Y, \mathcal{O}_Y(\Sigma)) = \{f \mid f \in k(Y)^\times, \text{div}(f) + \Sigma \geq 0\}$$

Recall that the divisor Σ is the sum of the 10 lines in Y , and $\text{div}(Q_{12}) = \Sigma$. Then, for $f \in H^0(Y, \mathcal{O}_Y(\Sigma))$, $f \cdot Q_{12}$ are degree 2 rational functions effective on Y . That is, they are in $R_Y(2) = H^0(Y, \mathcal{O}_Y(2))$. Hence, the isomorphism f_2 is given explicitly as $\cdot Q_{12}$. Similarly, the isomorphisms $f_{2k} : \mathcal{O}_Y(k\Sigma) \rightarrow \mathcal{O}_Y(2k)$ are given explicitly by $\cdot Q_{12}^k$.

Now we consider the f_{2k} all at once:

$$f : \oplus H^0(\widetilde{\mathcal{M}}, \mathcal{O}(5n\delta)) \rightarrow \oplus H^0(Y, \mathcal{O}_Y(2n)) = R_{Y, ev}$$

Let $\varepsilon := f(\alpha)$. Then $\varepsilon^5 = Q_{12}$, since in the commutative diagram, both of $f(\alpha^5)$ and Q_{12} are the image of 1 in $\mathcal{O}_Y(2)$. This explains what is ε and the relation $\varepsilon^5 = Q_{12}$.

Now We want to show that

$$\begin{aligned} f : \Gamma_0(\lambda) &\cong (R_Y[\varepsilon]/\varepsilon^5 - Q_{12})_{ev} \\ \Gamma_0(\lambda)_m &\mapsto (R_Y[\varepsilon]/\varepsilon^5 - Q_{12})_{\frac{2m}{5}} \end{aligned}$$

is an isomorphism. Here $(R_Y[\varepsilon]/\varepsilon^5 - Q_{12})_{ev}$ is the even graded piece of the ring $(R_Y[\varepsilon]/\varepsilon^5 - Q_{12})$. Notice that $\deg(\varepsilon) = \frac{2}{5}$, so $(R_Y[\varepsilon]/\varepsilon^5 - Q_{12})_{ev} = R_{Y, ev}[\varepsilon]/\varepsilon^5 - Q_{12}$, where $R_{Y, ev} = \oplus R_Y(2n)$.

Since both sides are graded rings, we only need to show that f is isomorphism as K vector space on the corresponding graded pieces. Write $m = 5n + j$, where $0 \leq j \leq 4$. Then, $\Gamma_0(\lambda)_m = \Gamma_0(\lambda)_{5n+j}$. By Proposition 5.5.2, $\Gamma_0(\lambda)_{5n+j} \cong \Gamma_0(\lambda)_{5n} = R_Y(2n)$. On the other hand, $\varepsilon^j \Gamma_0(\lambda)_{5n} \subseteq \Gamma_0(\lambda)_{5n+j}$. So by comparing the dimension, we see that $\Gamma_0(\lambda)_{5n+j} = \varepsilon^j \Gamma_0(\lambda)_{5n} = \varepsilon^j R_Y(2n)$.

On the other hand, $(R_{Y, ev}[\varepsilon]/\varepsilon^5 - Q_{12})_{\frac{2m}{5}}$ has K -basis of the form $\varepsilon^a f(s_{ij})$, where $f(s_{ij})$ is homogenous of degree b . We can assume that $0 \leq a \leq 4$ because of the relation $\varepsilon^5 = Q_{12}$. Then, a, b satisfies that $\frac{2a}{5} + 5b = \frac{2}{5}(5n + j)$. So, $a = j$ and $b = 2n$. That is, $(R_{Y, ev}[\varepsilon]/\varepsilon^5 - Q_{12})_{\frac{2m}{5}} = \varepsilon^j R_Y(2n)$. Hence, f is an isomorphism of the corresponding graded pieces.

Finally, we want to show that $\mathfrak{S}_5$ acts on ε as sign permutation. Take $\tau \in \mathfrak{S}_5$. Notice that $\varepsilon^5 = Q_{12}$, so $\tau(\varepsilon^5) = \text{sign}(\tau)\varepsilon^5$. Then we can define the group homomorphism:

$$f : \mathfrak{S}_5 \rightarrow \zeta_5^{\mathbb{Z}/5\mathbb{Z}} : f(\tau) = \frac{\tau(\varepsilon)}{\text{sign}(\tau)\varepsilon}$$

And the kernel is a normal subgroup of $\mathfrak{S}_5$ of order 120 or 24. But there isn't a normal subgroup of $\mathfrak{S}_5$ of order 24, so the kernel must be of size 120. That is, $\tau(\varepsilon) = \text{sign}(\tau)\varepsilon$. $\square$

Chapter 6

COMPUTING THE LEVEL

Recall that in Corollary 4.1.4, we have that if $\mathcal{M}(m, r, \underline{a})$ is special and its ambient Shimura variety $\mathrm{Sh}(\mathcal{D})$ is compact, then $T : \overline{\mathcal{M}}(m, r, \underline{a}) \rightarrow \mathrm{Sh}(\mathcal{D})$ is an isomorphism. In Definition 3.1.9, we have constructed a finite covering $\widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \overline{\mathcal{M}}(m, r, \underline{a})$ that arises by putting a marking on the ramification points on C . Ideally, there should be an isomorphism $\widetilde{T} : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \mathrm{Sh}_\Gamma(\mathcal{D})$ lifting T , where $\Gamma \subseteq \mathcal{G}(\hat{\mathbb{Z}})$ is a level subgroup, and $\mathrm{Sh}(\Gamma) \rightarrow \mathrm{Sh}(\mathcal{D})$ is a Shimura cover with corresponding level Γ .

How should we interpret this Γ ? Notice that $(C, \iota, \eta) \in \widetilde{\mathcal{M}}(m, r, \underline{a})(k)$ parameterizes C together with the r marked ramification points $P_{11}, P_{21}, \dots, P_{r1}$. If these marked points generate some subgroup in the torsion of $J(C)$, then Γ is the subgroup of $\mathcal{G}(\hat{\mathbb{Z}})$ that trivializes this level structure. So, in order to compute Γ , we need to understand whether the marked ramification points $\{P_{i1}\}_{i=1}^r$ generates a torsion subgroup, and what is the torsion subgroup generated by them. Then, we can compute the corresponding Γ from this description.

In this section, we will proceed as follows. In Subsection 6.1.1, we constructed divisors $\langle Q_i \rangle_{i=1}^{r-1} \subseteq J(C)[m]$ and computed the group K spanned by them. In Subsection 6.1.2, we identify K with the μ_m -invariant subgroup of $J(C)[m]$. In Subsection 6.1.3, we also compute the Weil pairing restricted to this subgroup. Then, in Subsection 6.3.1, we specialize to the case where m is an odd prime, and show that there is a factorization of the Torelli morphism $T : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \mathrm{Sh}(\mathcal{D})$:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{\widetilde{T}} & \mathrm{Sh}_{\Gamma_m}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \mathrm{Sh}(\mathcal{D}) \end{array}$$

Where $\Gamma_m \subseteq \mathcal{G}(\mathbb{Z})$ is the subgroup defined the principal level $1 - \zeta_m$ structure in Definition 4.2.1.

In Section 6.2, we explore the situation where m is not a prime. Under certain restrictions on the monodromy datum $(m, r, \underline{a})$, we constructed a larger torsion

group $\tilde{K} = \langle Q_{ij} \rangle$. We then exhibits some examples for which it defines a finer level than K in Subsection 6.3.2.

Throughout these sections, we do not make any assumption like $\mathcal{M}$ being special or $\text{Sh}(\mathcal{D})$ being compact. All the assumptions are on the monodromy datum $(m, r, \underline{a})$.

In this section we use the following notations. These build on Notation 5.0.1.

Notation 6.0.1. In this section, we focus on $[C \rightarrow P] \in \widetilde{\mathcal{M}}(m, r, \underline{a})$. We explicitly write out the affine equation of $C : y^m = \prod_{i=1}^r (x - x_i)^{a(i)}$. Recall in Notation 5.0.1 we have the quantities

- $d(i) = \gcd(m, a(i))$
- $e(i) = \frac{m}{d(i)}$
- $b(i) = \frac{a(i)}{d(i)}$

We put restriction on the inertia type $\underline{a}$ so that $\gcd(a(r), m) = 1$ and we move the r^{th} branching point x_r to be $\infty \in \mathbb{P}^1$. Hence, $x_r = \infty$, $d(r) = 1$, $e(r) = m$. There is a unique preimage $P_\infty \in C$ above $\infty \in \mathbb{P}^1$. This will be the assumption throughout this section unless specified otherwise. The purpose is so that we can define the Abel-Jacobi map $C \rightarrow J(C)$ explicitly.

We have the following notation: On the base curve $P \cong \mathbb{P}^1$:

- B_i : The i^{th} branching point on P . For $1 \leq i \leq r - 1$, we have $B_i = [x(i) : 1]$ and $B_r = [1 : 0]$ is the point at infinity on $\mathbb{P}^1$.

On the top curve C :

- P_{ij} : Here $1 \leq i \leq r - 1$ and $1 \leq j \leq d(i)$. It is the j^{th} preimage of the i^{th} branching point.
- $P_i = \sum_{j=1}^{d(i)} P_{ij}$. Here $1 \leq i \leq r - 1$.
If $\gcd(m, a(i)) = 1$, or more generally if m is a prime. Then we will drop j and use P_i to denote the unique preimage.
- P_∞ : The unique preimage on C of $B_r = \infty$.

We define the Abel-Jacobi embedding:

$$\begin{aligned} AJ : C &\rightarrow J(C) \\ P &\rightarrow P - P_\infty \end{aligned}$$

On the Jacobian $J(C)$:

- $Q_{ij} = P_{ij} - P_\infty$. This is the image P_{ij} in $J(C)$
- $Q_i = \sum_{j=1}^{d(i)} Q_{ij}$. We will show that $Q_i \in J(C)[m]$.

On $H^0(C, \mathcal{O}_C)$:

- $v_i = \frac{y^i}{\prod_{j=1}^{r-1} (x-x_j)^{\lfloor \frac{ia(j)}{m} \rfloor}}$. Here $1 \leq i \leq r-1$. Notice that this expression is consistent with (Lin, Mantovan, and Singhal, 2024, Notation 5.2), where $v_i = \frac{y^i}{(x-x_1)^{\lfloor \frac{ia(1)}{m} \rfloor} \dots (x-x_r)^{\lfloor \frac{ia(r)}{m} \rfloor}}$. Indeed, we can specialize to $x_r = \infty$ and get the expression.

When m is a prime, in $\mathbb{Z}[\zeta_m]$:

- ω is the ideal $(1 - \zeta_m)$.

Remark 6.0.2. Notice that the requirement on the inertia type $\underline{a}$ that $x_r = \infty$ and $\gcd(a(r), m) = 1$ is equivalent to asking for $\gcd(a(i_0), m) = 1$ for some $1 \leq i_0 \leq r$. Indeed, if such i_0 exists, then a fractional linear transformation sending $x_{i_0} \rightarrow \infty$ extends to an isomorphism sending $[C \rightarrow P] \rightarrow [C' \rightarrow P']$, where C' ramified above ∞ with the unique preimage P_∞ . So, starting from any family $\overline{\mathcal{M}}(m, r, \underline{a})$ with some $\gcd(a(i_0), m) = 1$, we can first multiply $\underline{a}$ by $g \in \mathbb{Z}/m\mathbb{Z}$, so $g \cdot a(i_0) = 1 \pmod{m}$. Then, $\underline{a} \sim \underline{a}'$ where $\underline{a}' = g \cdot \underline{a}$, and $a'(i_0) = 1$. Then, we apply the isomorphism that sends $x_i \rightarrow \infty$ and $\widetilde{\mathcal{M}}(m, r, \underline{a}')$ satisfies the requirement.

6.1 The level structure when m is a prime

6.1.1 The group $K = \langle Q_i \rangle_{i=1}^{r-1}$

In this section, our goal is to derive Proposition 6.1.4. We focus on the group $K = \langle Q_i \rangle_{i=1}^{r-1}$ in $J(C)$ and compute its size.

We first consider the possible relations on $J(C)$ provided by the divisor of the rational functions. On $C : y^m = \prod_{i=1}^{r-1} (x - x_i)^{a(i)}$, there are the natural rational functions y and $x - x_i$. The following proposition computes their divisors.

Proposition 6.1.1. *We have the following computation on the divisors:*

$$\begin{aligned} \operatorname{div}(y) &= \sum_{i=1}^{r-1} \sum_{j=1}^{d(i)} b(i) P_{ij} - \left(\sum_{i=1}^{r-1} a(i) \right) P_{\infty} = \sum_{i=1}^{r-1} b(i) Q_i \\ \operatorname{div}(x - x_i) &= \sum_{j=1}^{d(i)} e(i) P_{i,j} - m P_{\infty} = e(i) Q_i \end{aligned}$$

Proof. The functions y and $x - x_i$ only has zeros and poles at P_{ij} and P_{∞} . Let's compute their order at P_{ij} .

Let C' be the scheme cut out by $y^m = \prod (x - x_i)^{a(i)}$ before normalization, so C is the normalization of C' . Recall from Section 3.5 that the local ring at $(x_i, 0)$ is as follows: $\mathcal{O}_{C', B_i} = R' = \prod_{j \in \mathbb{Z}/d(i)\mathbb{Z}} R'_j$, where $R'_j = \mathcal{O}_{\mathbb{P}^1, B_i}[y] / (y^{e(i)} - \zeta_{d(i)}^j (x - x_i)^{b(i)})$. Let R_j be the normalization of R'_j in its function field. Since $\gcd(e(i), b(i)) = 1$, let s be such that $s \cdot b(i) = 1 + l \cdot e(i)$. We also denote $\zeta_{d(i)}^j$ by u_j .

Then, for each $j \in \mathbb{Z}/d(i)\mathbb{Z}$, we can take $g_j = y^s (x - x_i)^{-\lfloor \frac{sb(i)}{e(i)} \rfloor} \in R_j$, and g_j is the uniformizer at the local ring $R_j = \mathcal{O}_{C, P_{ij}}$. We have that $g_j^{e(i)} = u_j^s (x - x_i)$ and $y = u_j^{-l} g_j^{b(i)}$. Therefore, we get that:

$$\begin{aligned} \operatorname{ord}_{P_{ij}}(y) &= b(i) \\ \operatorname{ord}_{P_{ij}}(x - x_i) &= e(i) \end{aligned}$$

Finally, since $\deg(\operatorname{div}(y)) = 0$ and $\deg(\operatorname{div}(x - x_i)) = 0$, we get the desired expression. For the sake of completeness, we include a separate computation of the order at P_{∞} for y and $x - x_i$. Since $\sum_{i=1}^r a(i) \equiv 0 \pmod{m}$, say $\sum_{i=1}^r a(i) = km$, so $\sum_{i=1}^{r-1} a(i) = km - a(r)$. Apply the change of coordinate $u = \frac{1}{x}, v = \frac{y}{x^k}$. Now the affine equation of C become $v^m = \prod_{i=1}^{r-1} (1 - x_i u)^{a(i)} u^{a(r)}$, and $u(P_{\infty}) = 0, v(P_{\infty}) = 0$. Now apply the same computation as above, we get:

$$\begin{aligned} \operatorname{ord}_{P_{\infty}}(v) &= b(r) = 1 \\ \operatorname{ord}_{P_{\infty}}(u) &= e(r) = m \end{aligned}$$

From which we get that:

$$\begin{aligned} \operatorname{ord}_{P_{\infty}}(x - x_i) &= -\operatorname{ord}_{P_{\infty}}(u) = -m \\ \operatorname{ord}_{P_{\infty}}(y) &= \operatorname{ord}_{P_{\infty}}(v) - k \cdot \operatorname{ord}_{P_{\infty}}(u) = -(km - 1) \end{aligned}$$

Which is the desired result. $\square$

Remark 6.1.2. Notice that the above computation still holds if $\gcd(a(r), m) \neq 1$. In this case we have

$$\begin{aligned} \operatorname{div}(y) &= \sum_{i=1}^{r-1} \sum_{j=1}^{d(i)} b(i) P_{ij} - \sum_{j=1}^{d(r)} (e(r) - b(r)) P_{\infty j} = \sum_{i=1}^{r-1} b(i) P_i - (ke(r) - b(r)) P_{\infty} \\ \operatorname{div}(x - x_i) &= \sum_{j=1}^{d(i)} e(i) P_{ij} - \sum_{j=1}^{d(r)} e(r) P_{\infty j} = e(i) P_i - e(r) P_{\infty} \end{aligned}$$

but not the relation with Q_i , since we can't construct a Q_i in this case.

Now that we find some relations among the divisors on C , we want to use them to characterize the subgroup of $J(C)$ generated by Q_i . Let $K = \langle Q_1, \dots, Q_{r-1} \rangle$. Our next proposition shows that the relations $e(i)Q_i = 0$, $\sum_{i=1}^{r-1} b(i)Q_i = 0$ are actually the only relations in K .

Proposition 6.1.3. *The relations generated by Proposition 6.1.1 are the only relations in K .*

Proof. Suppose we have another relation $\sum_{i=1}^{r-1} t_i Q_i = 0$. Since $e(r-1)Q_{r-1} = 0$ and $\gcd(e(r-1), b(r-1)) = 1$, we can find u such that $ub(r-1) \equiv t_{r-1} \pmod{e(r-1)}$. Subtracting the relation $\sum_{i=1}^{r-1} ub(i)Q_i = 0$, we can cancel out the coefficient t_{r-1} . So we get that $\sum_{i=1}^{r-1} t'_i Q_i = 0$. We can further adjust by multiple of e_i to obtain a relation

$$\sum_{i=1}^{r-2} t_i Q_i = 0$$

With $0 \leq t_i < e(i)$. Let $t_r = -\sum_{i=1}^{r-2} t_i$. Then, there exists a function $h \in k(C)^\times$ such that

$$\operatorname{div}(h) = \sum_{i=1}^{r-2} t_i P_i + t_r P_r$$

where $0 \leq t_i < e_i$ for $1 \leq i \leq r-2$, and $t_r < 0$. Here $P_i = \sum_{j=1}^{d(i)} P_{ij}$ is the sum of the primages of B_i .

We want to show that h is a product of y and $(x - x_i)$. This will give us that the relation of h is still among the span of the relation given by y and $(x - x_i)$. Since $t_i \geq 0$, h does not have a pole except for at P_{∞} . Following the basis for cohomology

in (Lin, Mantovan, and Singhal, 2024), let $U = \pi^{-1}(\mathbb{P}^1 - \infty) = C - P_\infty$. Then $h \in \mathcal{O}_C(U) = \oplus_{i=1}^{m-1} k[x]v_i$. So, we can write $h = \sum_{i=1}^{m-1} f_i(x)v_i$, where $v_0 = 1$.

Now, notice that the divisors $P_1, \dots, P_\infty$ are all fixed by action of μ_m . Indeed, let σ be the generator ζ_m in μ_m , then σ acts on P_{ij} such that $\zeta_m \cdot P_{ij} = P_{ij+1}$. Hence, h^σ and h has the same divisors, which means that they are constant multiple of each other. On the other hand, recall that σ fixes x and sends v_i to $\zeta_m^i v_i$. Therefore, in order for h and h^σ to be constant multiple of each other, only one of the v_i has non-zero coefficient. So we can write $h = f_{i_0}(x)v_{i_0}$. Since the zeros of $f_{i_0}(x)$ must be included in the zeros of h , so $f_{i_0}(x)$ is just some products of $(x - x_i)$. Hence, h is some products of y and $(x - x_i)$, and so $\text{div}(h)$ is just some $\mathbb{Z}$ combinations of $\text{div}(x - x(P_i))$ and $\text{div}(y)$, which does not provide us with new relations. Hence, the relations in proposition 6.1.1 are the only ones in $J(C)$. $\square$

From Proposition 6.1.1 and 6.1.3, we can classify K . Consider

$$\begin{aligned} \varphi : \mathbb{Z}/m\mathbb{Z} &\rightarrow \prod_{i=1}^{r-1} \mathbb{Z}/e_i\mathbb{Z} \\ t &\mapsto (tb(1), \dots, tb(r-1)) \end{aligned}$$

Notice that φ is injective. Indeed, if $\varphi(t) = 0$, then we have for $1 \leq i \leq r-1$,

$$e(i) \mid tb(i) \implies e(i) \mid t \implies m \mid td(i)$$

So $m \mid t \cdot \gcd(d(1), \dots, d(r-1))$. Now, let $d = \gcd(m, d(1), \dots, d(r-1))$, then $d \mid \gcd(m, \sum_{i=1}^{r-1} a(i))$, so $d \mid \gcd(m, a(r))$. By our assumption, $\gcd(m, a(r)) = 1$, so we get that $d = 1$. Since $\gcd(m, d(1), \dots, d(r-1)) = 1$ and $m \mid t \cdot \gcd(d(1), \dots, d(r-1))$, we get that $m \mid t$, so $t = 0$.

By proposition 6.1.3, we get the main proposition of this subsection:

Proposition 6.1.4. *For $1 \leq i \leq r-1$, let w_i be the element in $\prod_{i=1}^{r-1} \mathbb{Z}/e_i\mathbb{Z}$ that is 1 in the i^{th} coordinate and 0 elsewhere. Then we have the isomorphism:*

$$\begin{aligned} f : \frac{\prod_{i=1}^{r-1} \mathbb{Z}/e_i\mathbb{Z}}{\text{Im}(\varphi)} &\rightarrow K \\ w_i &\mapsto Q_i \end{aligned}$$

Consequently, $\#K = \frac{\prod_{i=1}^{r-1} e_i}{m}$.

In the case that m is a prime, we immediately have that $e(i) = m$ and $b(i) = 1$. So, K is generated by $\langle Q_1, \dots, Q_{r-2} \rangle$. Then Proposition 6.1.4 simplifies to the following:

Corollary 6.1.5. *If m is a prime, then we have that*

$$K \cong (\mathbb{Z}/m\mathbb{Z})^{r-2}$$

With basis of K given by $\langle Q_1, \dots, Q_{r-1} \rangle$.

Remark 6.1.6. If we remove the assumption that $\gcd(a(r), m) = 1$, then Proposition 6.1.1 and Proposition 6.1.4 still hold, but we are not able to define the map φ using Q_i . In this case, let $P_i = \sum_{j=1}^{d(i)} P_{ij}$ for $1 \leq i \leq r$, and consider $K = \text{Pic}^0(C) \cap \langle P_1, \dots, P_r \rangle$. We instead classify the structure of K as follows. We consider

$$\begin{aligned} 0 \rightarrow \mathbb{Z}/m\mathbb{Z} &\xrightarrow{\varphi} \prod_{i=1}^r \mathbb{Z}/e_i\mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}/m\mathbb{Z} \rightarrow 0 \\ \varphi(t) &= (tb(1), \dots, tb(r)) \\ \Sigma(y_1, \dots, y_r) &= \sum_{i=1}^r d(i)y_i \end{aligned}$$

We define the morphism $f : \ker(\Sigma) \rightarrow K$ as follows. For $(y_1, \dots, y_r) \in \ker(\Sigma)$, we take a lift of y_i in $\mathbb{Z}$ and suppose $\sum d(i)y_i = km$. Then, we define $f(y_1, \dots, y_r) = \sum_{i=1}^r y_i P_i - ke(r)P_\infty$. From Remark 6.1.2, we see that f defines an isomorphism $f : \frac{\ker(\Sigma)}{\text{Im}(\varphi)} \cong K$. Consequently in this general case, $\#K = \frac{\prod_{i=1}^r e(i)}{m^2}$.

6.1.2 Identifying K with μ_m -invariant subgroup of $J(C)[m]$

From the above section, we see that when $\gcd(m, a(r)) = 1$, the marked points on the curve C give us $r - 1$ m -torsion points $\{Q_i\}_{i=1}^{r-1}$, where $Q_i = \sum_{j=1}^{d(i)} (P_{i,j} - P_\infty)$. We computed that the group $K = \langle Q_1, \dots, Q_{r-1} \rangle$ has cardinality $\#K = \frac{\prod_{i=1}^{r-1} e(i)}{m}$. In particular, notice that $K \subseteq J(C)[m]^{\mu_m}$, since each of the Q_i is invariant under the μ_m action. In this section, our goal is to show the following Proposition:

Proposition 6.1.7.

$$\#J(C)[m]^{\mu_m} \leq \frac{\prod_{i=1}^{r-1} e_i}{m}$$

Since $\#K = \frac{\prod_{i=1}^{r-1} e_i}{m}$, we obtain the following corollary:

Corollary 6.1.8.

$$K = J(C)[m]^{\mu_m}$$

Let $[\pi : C \rightarrow \mathbb{P}^1] \in \mathcal{M}(m, r, \underline{a})(k)$ be a μ_m -cover over an algebraically closed field k . Let R be the ramification locus on C and $i : R \hookrightarrow C$ be the closed immersion. Let $U = C - R$ and $j : U \hookrightarrow C$ be the open immersion. Let $B = \pi(R)$ be the branching locus on $\mathbb{P}^1$ and $U_0 = \mathbb{P}^1 - B$ be its complement. In our setting, $\#R = r' = \sum_{i=1}^r d(i)$ and $\#B = r$. In this proof, there is a locally constant sheaf $\mu_m \in \text{Sh}(C_{et})$ and there is the group μ_m acting on C . To differentiate them, we use G to denote the covering group $C \rightarrow \mathbb{P}^1$ and μ_m for the constant sheaf, so G is the finite group μ_m through this section.

By the proof of Section 03RN, Lemma 59.69.3 we have the following exact sequence

$$0 \rightarrow J(C)[m] \rightarrow H^1(U, \mu_m) \rightarrow (\mathbb{Z}/m\mathbb{Z})^{r'} \xrightarrow{\Sigma} \mathbb{Z}/m\mathbb{Z} \rightarrow 0 \quad (6.1)$$

where $r' = \#R = \sum_{i=1}^r d(i)$. Apply $G = \mu_m$ -invariant to this SES, we get the left exact sequence:

$$0 \rightarrow J(C)[m]^G \rightarrow H^1(U, \mu_m)^G \rightarrow (\mathbb{Z}/m\mathbb{Z})^r \xrightarrow{\Sigma} \mathbb{Z}/m\mathbb{Z} \quad (6.2)$$

To determine $J(C)[m]^G$, we want to determine $H^1(U, \mu_m)^G$ and understand the map $H^1(U, \mu_m) \rightarrow (\mathbb{Z}/m\mathbb{Z})^{r'}$.

First determine $H^1(U, \mu_m)^G$. By definition it is the same as $H^0(G, H^1(U, \mu_m))$. We use the Hochschild-Serre spectral sequence (Milne, 2012, Theorem 14.9):

$$H^p(G, H^q(U, \mu_m)) \rightarrow H^{p+q}(U_0, \mu_m)$$

This spectral sequence converges at the E^2 page. From Section 03RN, Lemma 59.69.3, we further have that

- $H^0(U, \mu_m) = \mu_m(k)$
- $H^1(U, \mu_m) = (\mathbb{Z}/m\mathbb{Z})^{2g+r'-1}$. Here g is the genus of C .
- $H^q(U, \mu_m) = 0$ for $q \geq 2$.

Hence we have that $E_2^{p,q} = 0$ for $q \geq 2$.

The seven term LES associated to the spectral sequence gives:

$$0 \rightarrow E_2^{1,0} \rightarrow E^1 \rightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \rightarrow E_1^2$$

Where $E_1^2 = \ker(E^2 \rightarrow E_2^{0,2})$. Since $E^2 = H^2(U_0, \mu_m) = 0$, we obtain the exact sequence:

$$0 \rightarrow H^1(G, \mu_m(k)) \rightarrow H^1(U_0, \mu_m) \rightarrow H^1(U, \mu_m)^G \xrightarrow{d_2^{0,1}} H^2(G, \mu_m(k)) \rightarrow 0 \quad (6.3)$$

Since k is algebraically closed of characteristic not dividing m , $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ as a group scheme. Hence $\mu_m(k) \cong \mathbb{Z}/m\mathbb{Z}$. From Group cohomology of finite cyclic groups, we have that

- $H^1(G, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}[m] = \mathbb{Z}/m\mathbb{Z}$
- $H^2(G, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}/(m\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$.
- $H^1(U_0, \mu_m) = (\mathbb{Z}/m\mathbb{Z})^{r-1}$

Plug the above result into equation 6.3, we see that $H^1(U, \mu_m)^G$ has size equal to $(\mathbb{Z}/m\mathbb{Z})^{r-1}$.

Now we focus on $f : H^1(U, \mu_m)^G \rightarrow (\mathbb{Z}/m\mathbb{Z})^r$. Notice that $J(C)[m]^G = \ker(f)$ since Equation 6.2 is left exact. Hence, to show that $\#\ker(f) \leq \frac{\prod_{i=1}^{r-1} e_i}{m}$, it suffices to show that $|\text{Im}(f)| \geq m \prod_{i=1}^{r-1} d_i$.

Proposition 6.1.9. *The image of the map $f : H^1(U, \mu_m)^G \rightarrow (\mathbb{Z}/m\mathbb{Z})^r$ contains a group of size $m \prod_{i=1}^{r-1} d_i$.*

Proof. To show this proposition, we need an alternative description of the map. Let Z be the group of all divisors supported on R . By Section 03RN, The group $H^1(U, \mu_m) \cong \{(\mathcal{L}, D, \alpha) \mid \mathcal{L} \in \text{Pic}^0(C), D \in Z, \alpha : \mathcal{L}^m \cong \mathcal{O}_C(D)\} / \{(\mathcal{O}_C(D_1), mD_1, id)\}$, where $D_1 \in Z$. Moreover, the map $\tilde{f}$ can be described as:

$$\begin{aligned} \tilde{f} : H^1(U, \mu_m) &\rightarrow (\mathbb{Z}/m\mathbb{Z})^{r'} \\ (\mathcal{L}, D, \alpha) &\rightarrow ((a_{ij})_{j=1}^{d(i)})_{i=1}^r \end{aligned}$$

Where a_{ij} is the coefficient of P_{ij} in D . This in terms give an description for $f : H^0(U, \mu_m)^G \rightarrow (\mathbb{Z}/m\mathbb{Z})^r$.

Now, for $1 \leq i \leq r-1$, consider the $H_i = \langle (\mathcal{L}, D_i, \alpha) \rangle \subset H^1(U, \mu_m)^G$, where $\mathcal{L} = \mathcal{O}_C$, $D_i = e_i(\sum_{j=1}^{d(i)} P_{ij}) - mP_\infty$, and α is given by $\text{div}(x - x_i)$. Indeed, by Proposition 6.1.1, D_i exactly the divisor of $(x - x_i)$, and this tuple is none trivial in the group.

Now, we write the vector $v_i \in (\mathbb{Z}/m\mathbb{Z})^{r'}$ such that v_i has coordinate 1 at the place corresponds to $\{P_{ij}\}_{j=1}^{d(i)}$ and 0 otherwise. In other words, $v_i = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$. Then, $f(H_i) = \langle e(i)v_i - mv_r \rangle = \langle e(i)v_i \rangle \subset (\mathbb{Z}/m\mathbb{Z})^{r'}$ and $f(H_i) \cong \mathbb{Z}/d(i)\mathbb{Z}$.

In addition, consider $H_y = \langle (\mathcal{L}, D, \alpha) \rangle \subset H^1(U, \mu_m)^G$, where $\mathcal{L} = \mathcal{O}_C$, $D = \sum_{i=1}^r b(i) \sum_{j=1}^{d(i)} (P_{ij} - P_\infty)$, and α is given by $\text{div}(y)$. Indeed, by Proposition 6.1.1, D is the divisor of y . Then, $f(H_y) = \langle \sum_{i=1}^{r-1} b(i)v_i - (\sum_{i=1}^{r-1} a(i))v_r \rangle \subset (\mathbb{Z}/m\mathbb{Z})^{r'}$. Since $\gcd(b_1, \dots, b_{r-1}) = m$, we have that $f(H_y) \cong \mathbb{Z}/m\mathbb{Z}$.

Now, let $H = \langle H_1, \dots, H_{r-1}, H_y \rangle \subset (\mathbb{Z}/m\mathbb{Z})^{r'}$. We want to show that following:

Lemma 6.1.10. *Define the morphism*

$$g : \bigoplus_{i=1}^{r-1} \mathbb{Z}/d(i)\mathbb{Z} \bigoplus \mathbb{Z}/m\mathbb{Z} \rightarrow H$$

$$g(w_i) \rightarrow e(i)v_i$$

$$g(w) \rightarrow \sum_{i=1}^{r-1} b(i)v_i - \left(\sum_{i=1}^{r-1} a(i) \right) v_r$$

Here,

- For $1 \leq i \leq r-1$, w_i be the vector that is 1 in $\mathbb{Z}/e_i\mathbb{Z}$ and 0 elsewhere.
- w is the the vector with 1 in $\mathbb{Z}/m\mathbb{Z}$ and 0 elsewhere.

Then g is injective.

Proof. Suppose v is in the kernel of g . Write

$$v = \sum_{i=1}^{r-1} y_i w_i + y w$$

Here, $y_i \in \mathbb{Z}/d(i)\mathbb{Z}$ and $y \in \mathbb{Z}/m\mathbb{Z}$. Since $g(v) = 0 \in (\mathbb{Z}/m\mathbb{Z})^{r'}$, by examining each coordinate of $g(v)$, we get that $y_i e_i + y b_i \equiv 0 \pmod{m}$ for $1 \leq i \leq r-1$. Hence, $y b_i \equiv 0 \pmod{e(i)}$. Since $\gcd(b(i), e(i)) = 1$, we get that $e(i) \mid y$ for $1 \leq i \leq r-1$. Since $\gcd(e(1), \dots, e(r-1)) = m$, we get that $m \mid y$, so $y = 0$. But then $y_i e(i) \equiv 0 \pmod{m}$, which implies that $y_i \equiv 0 \pmod{m}$, so $y_i = 0$ for all the y_i . Therefore, g is injective. $\square$

Consequently, we see that $H \subset \text{Im}(f)$ and $\#H \geq \frac{\prod_{i=1}^{r-1} d(i)}{m}$. This finishes the proof of Proposition 6.1.9 and hence proposition 6.1.7. $\square$

6.1.3 The Weil pairing on $J(C)[m]^{\mu_m}$

In this section, we compute the Weil pairing on $J(C)[m]^{\mu_m}$ using the explicit formulation of Weil pairing from (Howe, 1996, Theorem 1). Our goal is to show that the Weil pairing restricted to $K = J(C)[m]^{\mu_m}$ is trivial when m is an odd prime, and as a corollary derives Proposition 6.3.3.

Let C be a smooth projective curve over k . Let

$$e_m : J(C)[m] \times J(C)[m] \rightarrow \mu_m$$

be the m -torsion Weil pairing. Then e_m has a formula explicitly in terms of divisors, as follow:

Theorem 6.1.11. (Howe, 1996, Theorem 1) Let $[D], [E] \in J(C)[m]$ be an m -torsion divisor. Let $f, g \in k(C)$ be such that $\text{div}(f) = mD$ and $\text{div}(g) = mE$. Then we can define the Weil pairing as:

$$e_m(D, E) = \prod_{P \in R} (-1)^{m \text{ord}_P(D) \text{ord}_P(E)} \frac{g^{\text{ord}_P(D)}}{f^{\text{ord}_P(E)}}(P)$$

Where R is the union of the support of D and E .

We use this description to compute $e_m(Q_i, Q_j)$ for $Q_i = \sum_{k=1}^{d(i)} (P_{ik} - P_\infty)$, $Q_j = \sum_{k=1}^{d(j)} (P_{jk} - P_\infty)$. By Proposition 6.1.1, we can take $f_i = (x - x_i)^{d(i)}$, so that $\text{div}(f_i) = mQ_i$. Similarly we take $f_j = (x - x_j)^{d(j)}$, so $\text{div}(f_j) = mQ_j$.

Notice that $\text{ord}_{P_{ik}}(Q_i) = 1$, $\text{ord}_{P_{ik}}(Q_j) = 0$, $\text{ord}_{P_\infty}(Q_i) = -d(i)$.

Then,

$$\begin{aligned} e_m(Q_i, Q_j) &= \prod_{k=1}^{d(i)} (-1)^{m \text{ord}_{P_{ik}}(Q_i) \text{ord}_{P_{ik}}(Q_j)} \frac{(x - x_j)^{d(j) \text{ord}_{P_{ik}}(Q_i)}}{(x - x_i)^{d(i) \text{ord}_{P_{ik}}(Q_j)}}(P_{ik}) \\ &\quad \times \prod_{k=1}^{d(j)} (-1)^{m \text{ord}_{P_{jk}}(Q_i) \text{ord}_{P_{jk}}(Q_j)} \frac{(x - x_j)^{d(j) \text{ord}_{P_{jk}}(Q_i)}}{(x - x_i)^{d(i) \text{ord}_{P_{jk}}(Q_j)}}(P_{j,k}) \\ &\quad \times (-1)^{m \text{ord}_{P_\infty}(Q_i) \text{ord}_{P_\infty}(Q_j)} \frac{(x - x_j)^{d(j) \text{ord}_{P_\infty}(Q_i)}}{(x - x_i)^{d(i) \text{ord}_{P_\infty}(Q_j)}}(P_\infty) \\ &= (x_i - x_j)^{d(j)d(i)} \times \frac{1}{(x_j - x_i)^{d(i)d(j)}} \times (-1)^{m(-d(i)(-d(j)))} \frac{(1)^{-d(j)d(i)}}{(1)^{-d(i)d(j)}} \\ &= (-1)^{(m-1)(d(i)d(j))} \end{aligned}$$

In particular, when m is odd, then $m - 1$ is even, so the Weil pairing is always trivial, and the subgroup $J(C)[m]^{\mu_m}$ is totally anisotropic under Weil pairing. When m is even, $(m - 1)$ is odd, then the Weil pairing is trivial when either $d(i)$ or $d(j)$ is even.

6.2 The level structure when m is composite

This section explores the question of smallest level $\tilde{\Gamma}$ provided by $\widetilde{\mathcal{M}}(m, r, \underline{a})$, and is independent from the rest of the paper. We focus on $[C \rightarrow \mathbb{P}^1] \in \mathcal{M}(m, r, \underline{a})$ where $\gcd(m, a(r)) = 1$, so P_∞ is the unique preimage of $\infty \in \mathbb{P}^1$. We further assume that $a(r) = 1$. Indeed, $\underline{a}' \sim \underline{a}$ if there exists $b \in (\mathbb{Z}/m\mathbb{Z})^\times$ where $\underline{a}' = b\underline{a}$, and we can take b such that $ba(r) \equiv 1 \pmod{m}$.

In the previous sections, we computed the group $K = \langle Q_i \rangle_{i=1}^{r-1}$, and find out that the basis $\langle Q_i \rangle_{i=1}^{r-1}$ of $J(C)[\omega]$ provide us with a principal level $(1 - \zeta_m)$ structure. A natural question to ask is: Is Γ_m the smallest level, for which the factorization $\tilde{T} : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \text{Sh}_{\Gamma_m}(\mathcal{D})$ exists? If not, what is the smallest level $\tilde{\Gamma}$ for which such factorization exists?

To be more precise, notice that $Q_i = \sum_{j=1}^{r-1} Q_{ij}$, where $Q_{ij} = P_{ij} - P_\infty$. Is it possible that Q_{ij} themselves are m -torsions, so that the data of $\widetilde{\mathcal{M}}(m, r, \underline{a})$ actually gives basis of a larger subgroup $\tilde{K} = \langle Q_{ij} \rangle \subseteq J(C)[m]$, which corresponds to a smaller level $\tilde{\Gamma} \subseteq \mathcal{G}(\mathbb{Z}_m)$?

In this section, we explore the condition for Q_{ij} to be m torsions, and under some constraints on $(m, r, \underline{a})$, we computed the group $\tilde{K} = \langle Q_{ij} \rangle$ and identify $\tilde{K}$ as certain invariant subgroup in $J(C)[m]$ (Proposition 6.2.9 and 6.2.7). We also compute the Weil pairing restricted to $\tilde{K}$.

6.2.1 The group $\tilde{K} = \langle Q_{ij} \rangle$

Recall that we find out Q_i are $e(i)$ torsions by computing the divisors of $x - x_i$. To find functions whose divisor is a multiple of Q_{ij} , a natural thing to do is to consider functions of the form $v_i - c$ and see if its divisor is multiple of Q_{ij} . Here, c is a constant that we will determine later, and $\langle v_i \rangle_{i=0}^m$ are as in Notation 6.0.1.

Recall the signature $f : \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*) \rightarrow \mathbb{Z}$ defined in Proposition 3.1.10 Equation 4. To simplify the notation, we denote $f(i) := f(\tau_i)$ for $\tau_i \in \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*)$ such that $\tau_i(1) = \zeta_m^i$.

The following three lemmas aim at computing the divisor of v_i .

Lemma 6.2.1. *Let $(m, r, \underline{a})$ be a monodromy data such that $a(r) = 1$. Let v_i be as in Notation 6.0.1. Then, $\text{ord}_{P_\infty}(v_i) = i - m(1 + f(i^*))$*

Proof. Recall that $v_i = \frac{y^i}{\prod_{j=1}^{r-1} (x-x_j)^{\lfloor \frac{ia(j)}{m} \rfloor}}$. By Proposition 6.1.1, we have that

$$\begin{aligned} \text{ord}_{P_\infty}(y) &= - \sum_{i=1}^{r-1} a(i) \\ \text{ord}_{P_\infty}(x - x_i) &= -m \\ \implies \text{ord}_{P_\infty}(v_i) &= i \left(\sum_{j=1}^{r-1} a(j) \right) - \sum_{j=1}^{r-1} \lfloor \frac{ia(j)}{m} \rfloor (-m) \end{aligned}$$

Now, let $\{k_j\}_{j=1}^r$ be such that $ia(j) = k_j m + r_j$. In particular, $k_r = 0$ and $r_r = i$. Then we have that

$$\text{ord}_{P_\infty}(v_i) = \sum_{i=1}^{r-1} k_j m - ia(j) = \sum_{i=1}^{r-1} -r_j$$

On the other hand, $f(i^*) = -1 + \sum_{j=1}^r \langle \frac{ia(j)}{m} \rangle = -1 + \sum_{j=1}^r \frac{r_j}{m}$. Hence,

$$\text{ord}_{P_\infty}(v_i) = -m(f(i^*) + 1) + r_r = i - m(f(i^*) + 1)$$

□

Lemma 6.2.2. For $1 \leq t \leq m-1$ and $1 \leq j \leq m-1$, consider the coefficient of Q_j in $\text{div}(v_t)$:

- If $\gcd(m, a(j)) = 1$, then Q_j appears in $\text{div}(v_t)$ with positive coefficient.
- If $\gcd(m, a(j)) = d(j) > 1$, then Q_j appears in $\text{div}(v_t) \implies e(t) \nmid i$.
That is, v_t only vanishes at P_j where $e(j) \nmid t$.

Proof. Let $d_t = \gcd(m, t)$. Let $m_0 = \frac{m}{d_t}$ and $t_0 = \frac{t}{d_t}$. We also let k_t, r_t be such that $ta(j) = k_j m + r_j$, so $k_j = \lfloor \frac{ta(j)}{m} \rfloor$ and $r_j = m \langle \frac{ta(j)}{m} \rangle$. In particular, $k_r = 0$ and $r_r = t$.

$$\begin{aligned} v_t^{\frac{m}{d_t}} &= \frac{\prod_{j=1}^{r-1} (x - x_j)^{a(j)t_0}}{\prod_{j=1}^{r-1} (x - x_j)^{k_j m_0}} \\ \text{div}(v_t^{\frac{m}{d_t}}) &= \sum_{j=1}^{r-1} (a(j)t_0 - k_j m_0) e(j) Q_j \\ &= \frac{1}{d_t} \sum_{j=1}^{r-1} (a(j)t - k_j m) e(j) Q_j \end{aligned}$$

Notice that $ta(j) = mk_j \iff m \mid ta(j)$.

Now, if $\gcd(m, a(j)) = 1$. Then, $m \mid ta(j) \iff m \mid t$, which is impossible since $1 \leq i \leq m-1$. Hence, Q_j has positive coefficient in $\text{div}(v_t)$.

If $\gcd(m, a(j)) = d(j) > 1$. Then, $m \mid ta(j)$ iff $e(j) \mid t$. Hence, Q_j has positive coefficient in $\text{div}(v_t)$ iff $e(j) \nmid t$. $\square$

Lemma 6.2.3. *For $1 \leq t \leq m-1$, we have:*

$$\deg_x(v_t^{\frac{m}{d_t}}) = \frac{m(1 + f(t^*)) - t}{d_t}$$

Here, $d_t = \gcd(m, t)$. Notice that this is different notation from $d(i) = \gcd(m, a(i))$.

Proof. We compute this directly. On the affine coordinate, we have $y^m = \prod_{i=1}^{r-1} (x - x_i)^{a(i)}$. Hence, $(y^t)^{\frac{m}{d_t}} = (y^m)^{\frac{t}{d_t}}$. We denote $t_0 = \frac{t}{d_t}$ and $m_0 = \frac{m}{d_t}$. We also let k_t, r_t be such that $ta(j) = k_j m + r_j$, so $k_j = \lfloor \frac{ta(j)}{m} \rfloor$ and $r_j = m \langle \frac{ta(j)}{m} \rangle$. In particular, $k_r = 0$ and $r_r = t$.

Then,

$$\begin{aligned} v_t^{\frac{m}{d_t}} &= \frac{\prod_{j=1}^{r-1} (x - x_j)^{a(j)t_0}}{\prod_{j=1}^{r-1} (x - x_j)^{k_j m_0}} \\ \deg_x(v_t^{\frac{m}{d_t}}) &= \sum_{j=1}^{r-1} (a(j)t_0 - k_j m_0) = \frac{1}{d_t} \sum_{j=1}^{r-1} (a(j)t - k_j m) \\ &= \frac{1}{d_t} \sum_{j=1}^{r-1} r_j \end{aligned}$$

Recall that $m(f(t^*) + 1) = \sum_{j=1}^r r_j$. Hence, the degree in x of $v_t^{\frac{m}{d_t}}$ is $\frac{m(1+f(t^*)) - t}{d_t}$. $\square$

Using these two computation, we identify a condition for Q_{ij} to be $e(i)$ -torsion (Hence m -torsion). Notice that if $d(i) = 1$, then $Q_{ij} = Q_i$, and it is already proven that these are m -torsion. So, the condition is when $\gcd(a(i), m) \neq 1$.

Proposition 6.2.4. *Let $1 \leq i \leq r-1$ be such that $\gcd(m, a(i)) \neq 1$. If $f(e(i)) = 0$, then Q_{ij} are $e(i)$ -torsion.*

Proof. We take $t = m - e(i)$. By Lemma 6.2.1 and Lemma 6.2.3 we have:

$$\begin{aligned} \text{ord}_{P_\infty}(v_t^{d(i)}) &= -m \\ \deg_x(v_t^{d(i)}) &= 1 \end{aligned}$$

Here $d_t = \gcd(m, t) = e(i)$ and $m/d_t = d(i)$. In particular, we get that $v_t = x - c$ for some constant c .

Now let $c_1 = (x_i - c)^{\frac{1}{d(i)}}$. Notice that $c_1 \neq 0$. Indeed, if $c_1 = 0$, then $v_t(P_{ij}) = x_i - c = 0$ for $\{P_{ij}\}_{j=1}^{d(i)}$. On the other hand, by Proposition 6.1.1

$$\begin{aligned} \text{ord}_{P_{ij}}(v_t) &= t \text{ord}_{P_{ij}}(y) - \sum_{j=1}^{r-1} \left\lfloor \frac{ta(j)}{m} \right\rfloor \text{ord}_{P_{ij}}(x - x_j) \\ &= tb(i) - \left\lfloor \frac{tb(i)}{e(i)} \right\rfloor e(i) = tb(i) \pmod{e(i)} \end{aligned}$$

So, for $t = m - e(i)$, $\text{ord}_{P_{ij}} v_t = 0$, so $v_t(P_{ij}) \neq 0$. Hence $c_1 \neq 0$.

Now consider the function $g_t = v_t^{d(i)} - (x_i - c) = v_t^{d(i)} - c_1^{d(i)}$.

On one hand, $\{P_{ij}\}_{j=1}^{d(i)}$ are the only zeros of g_t , since $g_t(P) = x(P) - x_i$ and only vanish at P_{ij} . Moreover, g_t vanishes to order $e(i)$ on each of the P_{ij} . This is because g_t is a polynomial in x , so it is μ_m invariant. So, its divisors must also be fixed under the action of μ_m . So, $\text{div}(g_t) = y \cdot \sum_{j=1}^{d(i)} P_{ij} - mP_\infty$. Set the degree to be zero we obtain that $y = e(i)$.

On the other hand, we have the factorization

$$\begin{aligned} g_t &= v_t^{d(i)} - c_1^{d(i)} \\ &= \prod_{j=1}^{d(i)} (v_t - \zeta_{d(i)}^j c_1) = \prod_{j=1}^{d(i)} g_{t_j} \end{aligned}$$

Where we set $g_{t_j} = v_t - \zeta_{d(i)}^j c_1$. We claim that $\text{div}(g_{t_j}) = e_i Q_{ij}$. Since we know that $\text{ord}_{P_\infty}(v_t) = -e(i)$, we want to show that g_{t_j} only vanishes on P_{ij} and not at any other $P_{ij'}$ where $j' \neq j$.

We consider the map

$$v_t : \{P_{ij}\}_{j=1}^{d(i)} \rightarrow \{\zeta_{d(i)}^j c_1\}_{j=1}^{d(i)}$$

and we need to show that this map is injective. Since both sets are of cardinality d , it suffices to show that v_t is surjective. However, if some $\zeta_{d(i)}^j c_1$ is not in the image, then g_{t_j} does not vanish on any of the P_{ij} , which contradicts the fact that $g_{t_j} \mid g_t$. Therefore, we conclude that $\text{div}(g_{t_j}) = e_i Q_{ij}$, which shows that Q_{ij} are $e(i)$ -torsion. $\square$

Remark 6.2.5. From the above proof, we see that if $\gcd(m, a(i)) \neq 1$ and $f(e(i)) = 0$, then $v_{m-e(i)}$ is the coordinate that distinguish the $d(i)$ ramification points $\{P_{ij}\}_{j=1}^{d(i)}$ above x_i .

It turns out that there are several ways one can distinguish P_{ij} explicitly in the coordinate ring of C . Recall that in Section 3.5, we have defined the uniformizer $g_j = (1, \dots, \frac{y^s}{t^{\lfloor \frac{sb(i)}{e(i)} \rfloor}}, \dots, 1) \in \bigoplus_{j=0}^d R_j$. Here, $R = \bigoplus_{j=0}^d R_j$ is the local ring at x_i , and $t = x - x_i$. These g_j are exactly the uniformizer at P_{ij} .

In general, when $d(i) \neq 1$ and $f(e(i)) \neq 0$, One can consider the following function: let $t = x - x_i$ and $u(t) = \prod_{j \neq i} (x - x_j)^{a(j)}$.

$$\frac{y^{e(i)}}{t^{b(i)}} : \{P_{ik}\}_{k=1}^{d(i)} \rightarrow \{\zeta_{d(i)}^k (u(0))^{\frac{1}{d(i)}}\}_{k=1}^{d(i)}$$

By similar reasoning, one can show that $\frac{y^{e(i)}}{t^{b(i)}}$ is well defined and injective on the set $\{P_{ik}\}_{k=1}^{d(i)}$. So, the functions $\frac{y^{e(i)}}{t^{b(i)}}$ evaluate to different value on P_{ik} . However, when we remove the condition that $f(e(i)) = 0$, it is not guarantee that P_{ik} are the only zeros of the function $\frac{y^{e(i)}}{t^{b(i)}}$, so the conclusion of the above statement does not generalize beyond $f(e(i)) = 0$.

We now deal with the simplest case. That is, we put constraints on the monodromy data $(m, r, \underline{a})$, such that there is only one value of $e(i)$ among $1 \leq i \leq m-1$, other than $e(i) = m$. That is, $\{e(i)\}_{i=1}^r = \{e, m\}$. Notice that still more than one i can have $e(i) < m$.

Lemma 6.2.6. *Let $(m, r, \underline{a})$ be a monodromy datum such that there is only 1 value of $e(i)$ with $e(i) < m$. Denote this value as e , and denote $\frac{m}{e}$ as d . Suppose furthermore that $f(e) = 0$.*

Without loss of generality, suppose $x_1, \dots, x_s$ has ramification index $e(i) = e$, and $x_{s+1}, \dots, x_r$ has ramification index $e(i) = m$. Recall that we have assume $a(r) = 1$.

Then, we have $r = s + 2$. That is, $\tilde{K} \cong (\mathbb{Z}/e\mathbb{Z})^{d(r-2)}$.

Proof. Since $f(e) = 0$, we have

$$\sum_{j=1}^r \left\langle \frac{-ea(j)}{m} \right\rangle = 1$$

For $1 \leq j \leq s$, $\gcd(a(j), m) = d$. Hence, $m \mid ea(j)$, so $\sum_{j=1}^s \left\langle \frac{-ea(j)}{m} \right\rangle = 0$.

For $s + 1 \leq j \leq r - 1$, we have $\gcd(a(j), m) = 1$. Hence, $\langle \frac{-ea(j)}{m} \rangle \geq \frac{e}{m}$. Hence, $\sum_{j=s+1}^{r-1} \langle \frac{-ea(j)}{m} \rangle \geq (r - s - 1) \frac{e}{m}$.

Finally, for $j = r$, $a_r = 1$. Hence, $\langle \frac{-ea_r}{m} \rangle = \frac{m-e}{m}$.

Hence, $0 = f(e) \geq (r - s - 1) \frac{e}{m} + 1 - \frac{e}{m} - 1 = (r - s - 2) \frac{e}{m}$. Thus, $r - s - 2 \leq 0$, so $r \leq s + 2$. If $r = s + 1$, then $\sum_{j=1}^r a(j) \equiv 1 \pmod{d}$, contradicts the fact that $\sum_{j=1}^r a(j) \equiv 0 \pmod{m}$. Hence, $r = s + 2$. $\square$

Proposition 6.2.7. *Let $(m, r, \underline{a})$ be an inertia type that satisfies the assumption of Lemma 6.2.6. Let $s = r - 2$, so that $e(i) = e$ for $1 \leq i \leq s$.*

Let $\tilde{K}$ be the group in $J(C)[m]$ spanned by $Q_{ij} = P_{ij} - P_\infty$. Notice that for $1 \leq i \leq s$, there are $d(i) = \frac{m}{e}$ number of Q_{ij} , and for $s + 1 \leq i \leq r$ there is a unique Q_{i1} , so we drop the index j and write Q_i for Q_{i1} .

Then, $\tilde{K} = \langle Q_{ij} \rangle$ are subject to the following relations:

- $eQ_{ij} = 0$ for $1 \leq i \leq s, 1 \leq j \leq d$.
- $mQ_i = 0$ for $s + 1 \leq i \leq r - 1$
- $\sum_{i=1}^{r-1} b(i) \sum_{j=1}^{d(i)} Q_{ij} = 0$

And these are the only relations among them. Consequently,

$$\tilde{K} \cong (\mathbb{Z}/e\mathbb{Z})^{d(r-2)}$$

Proof. The relations come from Proposition 6.2.4 and Proposition 6.1.1. So, we only need to show that there is no more relations. Notice that whenever $e(i) = m$, we have $d(i) = \gcd(a(i), m) = 1$ and $b(i) = a(i)$. Hence, we can always write Q_{r-1} as linear combination of $\langle Q_{ij} \rangle_{i=1}^{r-2}$ by the third relation. So we want to show there are no other relations among $\langle Q_{ij} \rangle_{i=1}^{r-2}$.

For the sake of contradiction, suppose that $f \in k(C)^\times$ is such that

$$\begin{aligned} \operatorname{div}(f) &= \sum_{i=1}^s \sum_{j=1}^{d(i)} y_{ij} Q_{ij} + \sum_{i=s+1}^{r-2} y_i Q_i \\ f &= \sum_{i=0}^{m-1} f_i(x) v_i \end{aligned}$$

Let σ be the generator of μ_m . All the Q_{ij} are fixed by σ^d . Hence, $\sigma^d(f) = cf$ for some constant c . Since $\sigma^d(v_i) = \zeta_m^{id} v_i = \zeta_e^i v_i$, the index of i that appears in f should have the same residue r modulo e . That is, f must have the following form:

$$f = \sum_{s=1}^k v_{se+r} f_{se+r}$$

for some $r \in \mathbb{Z}/e\mathbb{Z}$.

Moreover, $r = 0$. Indeed, if all the v_i appearing in f has $i > 0$, then by Lemma 6.2.2, f would vanish at P_{r-1} , which contradicts our assumption on $\text{div}(f)$ that the coefficient of Q_{r-1} is 0. Hence, f is of the form:

$$f = \sum_{s=0}^{d-1} v_{se} f_{se} = \sum_{t=1}^d v_{m-te} f_{m-te}$$

Now, by Lemma 6.2.3, we have that $(v_{m-e})^d = x - c$ for some constant c . Moreover,

$$v_{m-te} = \frac{y^{m-te}}{\prod_{j=1}^{r-1} (x-x_j)^{\lfloor \frac{(m-te)a(j)}{m} \rfloor}}, \text{ so}$$

$$\begin{aligned} v_{m-e}^t &= \frac{y^{tm-te}}{\prod_{j=1}^{r-1} (x-x_j)^{\lfloor \frac{(m-e)a_j}{m} \rfloor t}} \\ &= \left(\prod_{j=1}^{r-1} (x-x_j)^{a(j)} \right)^{(t-1)} \frac{y^{m-te}}{\prod_{j=1}^{r-1} (x-x_j)^{\lfloor \frac{(m-e)a_j}{m} \rfloor t}} \\ \implies v_{m-te} &= v_{m-e}^t \prod_{j=1}^{r-1} (x-x_j)^{t \lfloor \frac{(m-e)a(j)}{m} \rfloor - (t-1)a(j) - \lfloor \frac{(m-te)a(j)}{m} \rfloor} \\ &= v_{m-e}^t \prod_{j=1}^{r-1} (x-x_j)^{s_j} \end{aligned}$$

Here, $s_j \in \mathbb{Z}$. Hence, we can replace v_{m-te} and x by polynomials in v_{m-e} and write f as a rational function in v_{m-e} . Now, we can factor f such that

$$\begin{aligned} f &= \prod_k (v_{m-e} - \alpha_k)^{t_k} \\ \text{div}(f) &= \sum_k t_k \text{div}(v_{m-e} - \alpha_k) \end{aligned}$$

Consider each factor $\text{div}(v_{m-e} - \alpha_k)$. We want to show that $\alpha_k \in \{v_{m-e}(P_{ij})\}$. First, the pole of $v_{m-e} - \alpha_k$ is only P_∞ . For its zeros, notice that for $k \neq k'$, the zero of $v_{m-e} - \alpha_k$ and the zeros of $v_{m-e} - \alpha_{k'}$ are disjoint. Hence, the zeros of $v_{m-e} - \alpha_k$ is

contained in the zeros of f , which is among $\{P_{ij}\}$. Hence, each factor of f is of the form $v_{m-e} - v_{m-e}(P_{ij}) = v_{m-e} - \zeta_d^j c_1$ by the proof of Proposition 6.2.4. So there is no relation provided by f . $\square$

Below is a counter example where the conditions in Proposition 6.2.7 are violated.

Example 6.2.8. Consider the family $\mathcal{M}[15] = \mathcal{M}(8, 4, (4, 2, 1, 1))$. The signautre is $(2, 1, 1, 0, 1, 0, 0)$. In particular, $f(e(4)) = f(2) = 1$, so it violates the conditions in Proposition 6.2.4.

Write a curve $[C] \in \mathcal{M}[15]$ as $C : y^8 = x^4(x-1)^2(x-t)$. We find certain curve in the family C_t where Q_{1j} are not torsion/ not $e(1)$ -torsion.

- We show that when $t = -81$, Q_{11} (and equivalently Q_{1j} for $1 \leq j \leq 4$) is not a torsion point.

Recall the ring of regular functions on C is generated as a vector space over $k[x]$ by $y, v_2, \dots, v_7$. In particular, $v_4 = \frac{y^4}{x^2(x-1)}$ and $v_2 = \frac{y^2}{x}$. Hence, we have that $v_4^2 = x - t$ and $v_2^2 = v_4(x + 1)$. Therefore, v_2, v_4 satisfies the relation $v_2^2 = v_4(v_4^2 + t + 1)$. The point P_{11} has coordinate $v_4(P_{11}) = (-t)^{\frac{1}{2}}$ and $v_2(P_{11}) = (-t)^{\frac{1}{4}}$.

Consider the morphism

$$\varphi : C \rightarrow C'$$

$$\varphi(x, y, v_2, \dots, v_7) = (v_4, v_2)$$

φ is surjective, and the image C' is an elliptic curve cut out by the equation $v_2^2 = v_4(v_4^2 + t + 1)$.

Now, suppose $Q_{11} = P_{11} - P_\infty$ is k -torsion. Let $f \in \mathcal{O}_C$ such that $\text{div}(f) = k(P_{11} - P_\infty)$. We consider the push-forward of f . We have:

$$\begin{aligned} \varphi_*(\text{div}(f)) &= k(\varphi_*(P_{11}) - k(\varphi_*(P_\infty))) \\ &= \text{div}(\text{Nm}_{C/C'}(f)) \end{aligned}$$

By the definition of push-forward, $\varphi_*(P_{11})$ is given by coordinate $v_2 = (-t)^{\frac{1}{4}}$ and $v_4 = (-t)^{\frac{1}{2}}$. By the group law of elliptic curve, saying that $(\varphi_*(P_{11}) - \varphi_*(P_\infty))$ is k -torsion in $J(C')$ is the same as saying that $\varphi_*(P_{11})$ is k -torsion in the group law of elliptic curve.

We specialize to $-t = 81$. Then $\varphi_*(P_{11})$ is given by $(3, 9)$, and C' has equation $y^2 = x(x^2 - 80)$. Then, $\text{disc}(C') = -16 \cdot 4 \cdot 26^3$, so $y^2 \nmid \text{disc}(C')$. By Nagell

Lutz theorem, $(x, y) = (3, 9)$ can not be a torsion point of C' . Hence, Q_{11} under $t = -81$ is not a torsion in $J(C)$.

- We show that when $t = 1 \pm \frac{\sqrt{3}}{2}$, then Q_{11} (equivalently Q_{1j} for $1 \leq j \leq 4$) is 6-torsion but not 2-torsion. Notice that $e(1) = 2$, so $Q_1 = \sum_{j=1}^4 Q_{1j}$ is 2-torsion.

Again consider the morphism $\varphi(x, y, v_2, \dots, v_7) = (v_2, v_4)$. Notice that if $\varphi(Q_{ij})$ is a k -torsion, then Q_{ij} is a $2k$ -torsion. Indeed, if there exists $f \in k(C')$ where $\text{div}_{C'}(f) = k\varphi(Q_{ij})$, then since φ is a finite degree 2 morphism, we have $\text{div}_C(\varphi^*(f)) = 2k(Q_{ij})$.

The coordinate of $\varphi_*(P_{11})$ is given by $((-t)^{\frac{1}{2}}, i(-t)^{\frac{1}{4}})$. We can find values of t such that it is 3-torsion on C' . By computation, we get $t = 1 \pm \frac{\sqrt{3}}{2}$. We specialize to these value of t , so Q_{11} is a 6-torsion.

Now, if Q_{11} is actually a 2-torsion, then $\varphi_*(P_{11})$ would be a 1-torsion (since it is both 2-torsion and 3-torsion) on C' , which means it coincide with point at infinity. But $((-t)^{\frac{1}{2}}, i(-t)^{\frac{1}{4}})$ for $t = 1 \pm \frac{\sqrt{3}}{2}$ is not the point at infinity. Therefore, Q_{11} is not a 2-torsion.

6.2.2 Identifying $\tilde{K}$ with μ_e -invariant subgroup of $J(C)[e]$

Now that we computed the basis of $\tilde{K}$, similar to subsection 6.1.2, we want to identify $\tilde{K}$ with some canonical subgroup of $J(C)[m]$. We first show a lemma that states that under our restriction, $\tilde{K}$ is in fact in the e -torsion subgroup.

Proposition 6.2.9. *Following the notation in Subsection 6.1.2, we let $G = \mu_m$ acting on C . Let σ be the generator of G and let $H = \langle \sigma^d \rangle$. Notice that $H \cong \mu_e$.*

With the same assumption on $(m, r, \underline{a})$ as Proposition 6.2.7, we have

$$\tilde{K} = J(C)[e]^H$$

Proof of Prop 6.2.9. We already have that $\#\tilde{K} = e^{d(r-2)}$ and $\tilde{K} \subseteq J(C)[e]^{\langle \sigma^d \rangle}$. We want to show that $\#J(C)[e]^{\langle \sigma^d \rangle} \leq e^{d(r-2)}$.

Take $[C \rightarrow \mathbb{P}^1] \in \mathcal{M}(m, r, \underline{a})(k)$. Let C_H be the quotient curve by the action of subgroup H . We have the covers:

$$C \rightarrow C_H \rightarrow \mathbb{P}^1$$

Where $C_H \rightarrow \mathbb{P}^1$ is a $G/H \cong \mathbb{Z}/d\mathbb{Z}$ cover. Let $U = C - R$ and $U_0 = \mathbb{P}^1 - B$. We also have $U_H = C_H - R_H$, where R_H is the image of R under the covering map.

We claim that $g(C_H) = 1$, and C_H ramifies only above x_{r-1} and x_r with ramification index d , and $x_1, \dots, x_{r-1}$ has ramification index 1 in C_H . To see this, we can compute the monodromy datum $(m', r', \underline{a}')$ of C_H following (Lin, Mantovan, and Singhal, 2024, Example 3.4). Consider $\tau_e \in \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{C}^*)$ such that $\tau_e(i) = \zeta_m^{ie}$. Then, $C_H = C/\ker(\tau_e)$. The Galois group of $C_H \rightarrow \mathbb{P}^1$ is given by $\tau_e(G) \cong \mathbb{Z}/d\mathbb{Z}$, and the inertia type $\underline{a}'(i)$ is computed by $\tau_e(a(i)) = \zeta_d^{a'(i)}$. Since $\tau_e(a(i)) = \zeta_m^{ea(i)} = \zeta_d^{a(i)}$, we can take $\underline{a}' = \underline{a}$, but now treated as elements in $\mathbb{Z}/d\mathbb{Z}$. For $1 \leq i \leq r-2$, we have $\gcd(m, a(i)) = d$, so $d \mid a(i)$, which means x_i are not branching points. For $r-2 \leq i \leq r$, we have $\gcd(m, a(i)) = 1$, so the ramification index is now $\frac{d}{\gcd(d, a(i))} = d$. Hence, the monodromy datum for C_H is $(d, 2, (0 \dots, 0, a(r-1), a(r)))$. The genus of C_H is then computed by the Riemann-Hurwitz formula applying to $C_H \rightarrow \mathbb{P}^1$:

$$\begin{aligned} 2g(C_H) - 2 &= d(-2) + \sum_{r-1 \leq i \leq r} (d-1) \\ \implies g(C_H) &= 0 \end{aligned}$$

Similar to how we obtain Equation 6.2, we apply the H -invariant to the LES from the localization sequence and get

$$0 \rightarrow J(C)[e]^H \rightarrow H^1(U, \mu_e)^H \xrightarrow{f} (\mathbb{Z}/e\mathbb{Z})^{r'} \xrightarrow{\Sigma} \mathbb{Z}/e\mathbb{Z} \quad (6.4)$$

Here, $r' = \sum_{i=1}^r d(i) = d(r-2) + 2$. To compute $H^1(U, \mu_e)^H$, we recall the Larrey spectral sequence $H^p(H, H^q(U, \mu_e)) \rightarrow H^{p+q}(U_H, \mu_e)$. Similar to how we obtain Equation 6.3, we get:

$$0 \rightarrow H^1(H, \mu_e(k)) \rightarrow H^1(U_H, \mu_e) \rightarrow H^1(U, \mu_e)^H \xrightarrow{d_2^{0,1}} H^2(H, \mu_e(k)) \rightarrow 0 \quad (6.5)$$

Since $H \cong \mathbb{Z}/e\mathbb{Z}$ and C_H has genus 0, we get:

- $H^1(H, \mu_e(k)) = \mathbb{Z}/e\mathbb{Z}[e] = \mathbb{Z}/e\mathbb{Z}$
- $H^1(U_H, \mu_e) = (\mathbb{Z}/e\mathbb{Z})^{d(r-2)+1}$
- $H^2(H, \mu_e) = \mathbb{Z}/e\mathbb{Z}/e(\mathbb{Z}/e\mathbb{Z}) = \mathbb{Z}/e\mathbb{Z}$

Hence, $H^1(U, \mu_e)^H$ has size equal to $(\mathbb{Z}/e\mathbb{Z})^{d(r-2)+1}$.

Now we focus on Equation 6.4. Recall that the map f comes from $\tilde{f}$ described in the proof of Proposition 6.1.9. To show that $|J(C)[e]^H| \leq e^{d(r-2)}$, by Equation 6.4, we want to show that $\text{Im}(f)$ contains a subgroup of size e .

We consider $(\mathcal{L}, D, \alpha) \in H^1(U, \mu_e)$, where $\mathcal{L} = \mathcal{O}_C$, $D = \sum_{i=1}^{r-1} b(i) \sum_{j=1}^{d(i)} Q_{ij}$, and $\alpha : \mathcal{L}^e \cong \mathcal{O}_C(D)$ is given by $\text{div}(y)$. Notice that $(\mathcal{L}, D, \alpha) \in H^1(U, \mu_e)^H$ since each of Q_{ij} is fixed under H .

We claim that $f(D) \in (\mathbb{Z}/e\mathbb{Z})^{r'}$ is of order e . Indeed, $f(D)_{ij} = b(i) \pmod{e}$, and $xf(D) = 0 \implies xb(i) \equiv 0 \pmod{e(i)}$ for all $1 \leq i \leq r-1$, which implies that $e \mid x$. Hence, $\text{Im}(f)$ contains a group of size e , which gives us $|J(C)[e]^H| \leq e^{d(r-2)}$ as desired.

□

6.2.3 The Weil pairing on $J(C)[e]^{\mu_e}$

Proposition 6.2.10. *The Weil pairing in $\tilde{K} = J(C)[e]^H \cong (\mathbb{Z}/e\mathbb{Z})^{d(r-2)}$ is trivial iff e is odd.*

Proof. We recall the formulation of m -torsion Weil pairing from (Howe, 1996, Theorem 1), cited in Subsection 6.1.3.

Let $D = Q_{ik}$ and $E = Q_{js}$. By Proposition 6.2.7 we have that $v_{m-e}^d = x - c$. Let $c_i = (x_i - c)^{\frac{1}{d}}$.

$$\text{div}(v_{m-e} - \zeta_d^k c_i) = eQ_{ik}$$

$$\text{div}(v_{m-e} - \zeta_d^s c_j) = eQ_{js}$$

Hence, we can take $f_{ik} = (v_{m-e} - \zeta_d^k c_i)$, $f_{js} = (v_{m-e} - \zeta_d^s c_j)$. Notice that $\text{ord}_{P_{ik}}(Q_{ik}) = 1$, $\text{ord}_{P_{\infty}}(Q_{ik}) = -1$.

We have

$$\begin{aligned} e_e(Q_{ik}, Q_{js}) &= (-1)^{e \text{ord}_{P_{ik}}(Q_{ik}) \text{ord}_{P_{ik}}(Q_{js})} \frac{(v_{m-e} - \zeta_d^s c_j)^{\text{ord}_{P_{ik}}(Q_{ik})}}{(v_{m-e} - \zeta_d^k c_i)^{\text{ord}_{P_{ik}}(Q_{js})}} (P_{ik}) \\ &\quad \times (-1)^{e \text{ord}_{P_{js}}(Q_{ik}) \text{ord}_{P_{js}}(Q_{js})} \frac{(v_{m-e} - \zeta_d^s c_j)^{\text{ord}_{P_{js}}(Q_{ik})}}{(v_{m-e} - \zeta_d^k c_i)^{\text{ord}_{P_{js}}(Q_{js})}} (P_{js}) \\ &\quad \times (-1)^{e \text{ord}_{P_{\infty}}(Q_{ik}) \text{ord}_{P_{\infty}}(Q_{js})} \frac{(v_{m-e} - \zeta_d^s c_j)^{\text{ord}_{P_{js}}(Q_{ik})}}{(v_{m-e} - \zeta_d^k c_i)^{\text{ord}_{P_{js}}(Q_{js})}} (P_{\infty}) \\ &= \frac{(\zeta_d^k c_i - \zeta_d^s c_j)}{(\zeta_d^s c_j - \zeta_d^k c_i)} (-1)^{e(-1)(-1)} \\ &= (-1)^{(e+1)} \end{aligned}$$

In particular, the pairing is trivial iff e is odd.

□

6.3 The subgroup corresponding to level structure

In Section 6.1, we compute $K = \langle Q_i \rangle_{i=1}^{r-2} = J(C)[m]^{\mu_m}$. In Section 6.2, we compute $\tilde{K} = \langle \langle Q_{ij} \rangle_{j=1}^d \rangle_{i=1}^{r-2} = J(C)[e]^{\mu_e}$, when $(m, r, \underline{a})$ satisfies conditions in 6.2.6. In the following two subsections, we will explore the trivialization of torsion points given by K and $\tilde{K}$. We will interpret them in terms of the factorization of the diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, r, \underline{a}) & & \\ \downarrow & \searrow & \\ \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \text{Sh}(\mathcal{D}) \end{array}$$

6.3.1 When m is prime

When m is a prime, we see that $K = \tilde{K}$. Also, $\langle Q_i \rangle_{i=1}^{r-2}$ provides a $\mathbb{Z}/m\mathbb{Z}$ basis of the group $J(C)[m]^{\mu_m}$.

At the first sight, it seems that $\langle Q_i \rangle_{i=1}^{r-2}$ gives a trivialization of the subgroup $J(C)[m]^{\mu_m}$ via

$$\begin{aligned} (\mathbb{Z}/m\mathbb{Z})^{r-2} &\rightarrow J(C)[m]^{\mu_m} \\ e_i &\rightarrow Q_i \end{aligned}$$

In particular, we obtain a full flag of $J(C)[m]^{\mu_m}$ via

$$\{1\} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_{r-2} = K \subseteq J(C)[m]$$

where $K_i = \langle Q_j \rangle_{j=1}^i$ and $\langle Q_i \rangle = K_i/K_{i-1}$. We denote the corresponding subgroup in $\mathcal{G}(\hat{\mathbb{Z}})$ as $\Gamma_1^{\mu_m}(m)$. Using the identification that $\mathcal{G} = \text{GSp}_{2g}(\mathbb{Z}) \cap \text{GL}_B(V)$, we see that $\Gamma_1^{\mu_m}(m) \subseteq \mathcal{G}(\mathbb{Z})$ are the matrices that have an upper triangular block of size $r-2$ with 1 on the diagonal when reduced mod m .

However, it is hard for us to obtain an actual matrix representation of the elements of $\Gamma_1^{\mu_m}(m)$ from this description. We claim that in this case, K actually gives a full level structure on certain ω torsion group $J(C)[\omega]$. Consequently, the Shimura cover $\text{Sh}(\Gamma)$ corresponding to $\widetilde{\mathcal{M}}(m, r, \underline{a})$ has level group $\Gamma_m \mathcal{G}(\hat{\mathbb{Z}}^m)$, where Γ_m is as defined in the relative moduli problem in Definition 4.2.2.

We first present the following lemma that identifies $J(C)[e]^{\mu_e}$ with $J(C)[(1-\sigma^d)]$, where $(1-\sigma^d)$ is an ideal in $\mathbb{Z}[\mu_m] \subseteq \text{End}(A)$. Notice that we do not require m to be a prime for the following lemma.

Lemma 6.3.1. *Let $\text{Sh}(\mathcal{D})$ be the smallest PEL Shimura subvariety of $\mathcal{A}_g$ containing $\mathcal{M}(m, r, \underline{a})$. Let $A \in \text{Sh}(\mathcal{D})$ be an abelian variety in this family.*

Let $\sigma \in \mu_m$ be the image of 1 under the isomorphism $\mathbb{Z}/m\mathbb{Z} \rightarrow \mu_m$. Let e be a divisor of m that $e \mid m, e \neq 1$. Consider the subgroup $H \subseteq \mu_m$ such that $H = \langle \sigma^d \rangle$, so $H \cong \mu_e$. Then, we have the identification

$$A[e]^H \cong A[(1 - \sigma^d)]$$

Proof. By the explanation at the start of Section 2.4, We can take a curve $C \in \mathcal{M}(m, r, \underline{a})$ and take $L = H^1(C(\mathbb{C}), \mathbb{Z})$. The group algebra $\mathbb{Z}[\mu_m] = \mathbb{Z}[\sigma]$ acts on L , so $\mathbb{Z}[\sigma] \subseteq \text{End}(A)$. Hence, we can define $A[(1 - \sigma^d)]$ to be the finite flat group scheme whose points are the $(1 - \sigma^d)$ -torsion of A .

We consider the $\mathbb{C}$ points. We have $A[e]^H = (\frac{1}{e}L/L)^H = (\frac{1}{e}L/L)^{\sigma^d}$ and $A[(1 - \sigma^d)] = \frac{1}{1 - \sigma^d}L/L$. We define

$$g : \frac{1}{1 - \sigma^d}L \rightarrow (\frac{1}{e}L/L)^H$$

$$\frac{1}{1 - \sigma^d}\lambda \mapsto \frac{a\lambda}{e} \pmod{L}$$

Where $a = \frac{e}{1 - \sigma^d}$. Since $(\sigma^d)^e = \sigma^m = 1$, we have $a \in \mathbb{Z}[\sigma]$.

Notice that $\frac{a\lambda}{e} \in (\frac{1}{e}L/L)^H$, since $(1 - \sigma^d) \cdot \frac{a\lambda}{e} = \lambda \in L$. Also notice that this map is surjective. Indeed, for $\frac{\lambda_1}{e} \in (\frac{1}{e}L/L)^H$, since it is σ^d invariant, $\frac{(1 - \sigma^d)\lambda_1}{e} = \lambda_0$ for some $\lambda_0 \in L$. So, $\lambda_0 = a\lambda_1$, and hence $g(\frac{\lambda_0}{1 - \sigma^d}) = \frac{\lambda_1}{e}$. Finally, the kernel of g is exactly L , since $\frac{a\lambda}{e} = \lambda_1$ iff $\lambda = (1 - \sigma^d)\lambda_1$. So, g is an isomorphism of $\mathbb{Z}[\sigma]$ module. $\square$

For a general m , it is hard to interpret what are the $(1 - \sigma^d)$ -torsion of A . A priori, $\mathbb{Z}[\sigma]/(1 - \sigma^d)$ is not a field, so $A[1 - \sigma^d]$ is not a vector space, so it does not make sense to talk about its dimension. However, when m is an odd prime and $d = 1$, we have a nice interpretation of $A[1 - \sigma^d]$

Corollary 6.3.2. *Under the setting of Lemma 6.3.1, in addition assuming that m is an odd prime, we have*

$$A[m]^{\mu_m} \cong A[\omega]$$

Where $\omega = 1 - \zeta_m$.

Proof. We take $d = 1$ and $e = m$. Hence $A[m]^{\mu_m} = A[(1 - \sigma)]$.

When m is an odd prime, we see that $L = H^0(C(\mathbb{C}), \mathbb{Z})$ is a $\mathbb{Z}[\mu_m] = \mathbb{Z} \oplus \mathbb{Z}[\zeta_m]$ module. By the formula of signature type in Proposition 3.1.10 Equation (4), we see that the $\mathbb{Z}$ component does not act on L . In other words, $\lambda = \sum_{i=1}^{m-1} \lambda_{\tau_i}$, where $\sigma \cdot \lambda_{\tau_i} = \zeta_m^i \lambda_{\tau_i}$. For any $1 \leq i \leq m-1$, $(1 - \zeta_m^i)$ and $(1 - \zeta_m)$ are associates in $\mathbb{Z}[\zeta_m]$, so a point in $A(\bar{k})$ is a $(1 - \sigma)$ torsion if and only if it is an $(1 - \zeta_m)$ -torsion. $\square$

Now recall the definition of integral level $(1 - \zeta_m)$ structure in Definition 4.2.1, and $\mathcal{M}_{\Gamma_m}$ in Definition 4.2.2.

We want to show that, when m is an odd prime, there exists a factorization $\tilde{T}$ of the Torelli morphism T :

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{\tilde{T}} & \text{Sh}_{\Gamma_m}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \text{Sh}(\mathcal{D}) \end{array}$$

Where $\tilde{T} : \widetilde{\mathcal{M}}(m, r, \underline{a}) \rightarrow \text{Sh}_{\Gamma_m}(\mathcal{D})$ comes from the m -torsions $\langle Q_i \rangle_{i=1}^{r-2}$, and $\text{Sh}_{\Gamma_m}(\mathcal{D})$ is defined by the moduli problem $\mathcal{M}_{\Gamma_m}$ as in Definition 4.2.2.

To show that $\langle Q_i \rangle_{i=1}^{r-2}$ gives a principal level $(1 - \zeta_m)$ structure, we just need to verify that the isomorphism provided by $\langle Q_i \rangle_{i=1}^{r-2}$ is compatible of the pairing Ψ_m on L and the Weil pairing on $J(C)$.

More precisely, recall that $\text{Sh}(\mathcal{D})$ is defined by the integral PEL datum $(\mathbb{Z}[\mu_m], *, L, \Psi, h_0)$, where $L = H^1(C, \mathbb{Z})$ and $\Psi : L \times L \rightarrow \mathbb{Z}$ by Θ divisor. Ψ restricted to $\Psi_m : L/\omega L \times L/\omega L \rightarrow \mu_m$.

On the other hand, we have the universal family of principally polarized abelian scheme $(\mathcal{A}, \lambda, i) \rightarrow \text{Sh}(\mathcal{D})$ extending $\mathcal{J}(C)$, and λ provides us the Weil pairing e_λ on A .

Notice that L is a $\mathbb{Z}[\zeta_m]$ module with $\mathbb{Z}$ -rank $2g = (r-2)(m-1)$ by Equation (3) in 3.1.10. Hence, $L/\omega L$ is a $\mathbb{Z}[\zeta_m]/(1 - \zeta_m) \cong \mathbb{F}_m$ -vector space of dimension $(r-2)$. Let $e_1, \dots, e_{r-2}$ be the standard basis of $L/\omega L$. Then, we need to show that the isomorphism

$$\eta : L/\omega L \rightarrow A[\omega]$$

$$\eta(e_i) = Q_i$$

commutes with Ψ_m on the left and e_λ on the right. However, we have shown that e_λ restricted to $A[\omega]$ is trivial, and so as Ψ_m since it is defined from $J(C)$. So, η automatically respect the pairing on both sides.

As conclusion, we can state the following Proposition:

Proposition 6.3.3. *When m is an odd prime, The basis $\langle Q_i \rangle_{i=1}^{r-2}$ defines a principal level $(1 - \zeta_m)$ structure in the sense of Definition 4.2.1. Consequently, we get the factorization $\tilde{T}$ of the Torelli morphism:*

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{\tilde{T}} & \text{Sh}_{\Gamma_m}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \text{Sh}(\mathcal{D}) \end{array}$$

Where $\text{Sh}_{\Gamma_m}(\mathcal{D})$ is the Shimura cover of $\text{Sh}(\mathcal{D})$ defined in Definition 4.2.2.

Remark 6.3.4. The trivialization of full level ω structure gives a nice matrix description of elements in $\Gamma_1^{\mu_m}(m) = \Gamma_m$. If we in addition know that $L \cong \mathbb{Z}[\zeta_m]^{r-2}$ (This is the case if $\mathbb{Z}(\zeta_m)$ has class number 1), then $\mathcal{G}_1(\mathbb{Z}) = \text{U}(O_K^{r-2})$ then we can explicitly write $g \in \Gamma_m \cap \mathcal{G}_1(\mathbb{Z})$ as $\text{U}(\omega) = \{g \in \text{U}(O_K^{r-2}) \mid g \equiv \text{Id} \pmod{\omega}\}$.

Remark 6.3.5. Notice that the above proposition does not require $\mathcal{M}(m, r, \underline{a})$ to be special or $\text{Sh}(\mathcal{D})$ to be compact. These two assumptions are imposed when we want to show that this diagram is **exact**.

6.3.2 When m is composite

When m is a composite, we still get the partial level structure $\Gamma_1^{\mu_m}(m)$ from the flag by $K = \langle Q_i \rangle_{i=1}^{r-2}$. However, we are not able to realize it as a full level structure modulo ω , as we no longer have the identification $A[m]^{\mu_m} = A[\omega]$.

When m is a composite and the monodromy datum $(m, r, \underline{a})$ satisfies conditions in Lemma 6.2.6, we get that $K = \langle Q_i \rangle_{i=1}^{r-2} \cong (\mathbb{Z}/e\mathbb{Z})^{r-2}$ and $\tilde{K} = \langle Q_{ij} \rangle \cong (\mathbb{Z}/e\mathbb{Z})^{d(r-2)}$. Similar to how K gives a full flag on $J(C)[m]^{\mu_m}$, $\tilde{K}$ gives a full flag on $J(C)[e]^{\mu_e}$.

Hence, we have the following factorization:

$$\begin{array}{ccc}
 & & \mathrm{Sh}_{\Gamma_1^{\mu_e}(e)}(\mathcal{D}) \\
 & \nearrow \tilde{T}' & \downarrow \\
 \widetilde{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{\tilde{T}} & \mathrm{Sh}_{\Gamma_1^{\mu_m}(m)}(\mathcal{D}) \\
 \downarrow & & \downarrow \\
 \overline{\mathcal{M}}(m, r, \underline{a}) & \xrightarrow{T} & \mathrm{Sh}(\mathcal{D})
 \end{array}$$

In general, we are not able to realize $\Gamma_1^{\mu_e}(e)$ as a full level $1 - \zeta_e$ level structure. However, this is possible for some family with specific signature. Among Moonen's 20 special families, we have identified $\mathcal{M}[4]$, $\mathcal{M}[8]$ and $\mathcal{M}[13]$ whose monodromy datum satisfies the condition of Lemma 6.2.6. In these cases, we are able to write $\Gamma_1^{\mu_e}(e)$ as a full level $1 - \zeta_e$ structure.

We first need an analogue of Corollary 6.3.2 for $\mathcal{M}[13]$. Recall that $\mathcal{M}[13] = \mathcal{M}(6, 4, (1, 1, 2, 2))$, so $m = 6$, $d = 2$ and $e = 3$. The signature $\underline{f} = (2, 1, 0, 1, 0)$. In particular, it satisfies the conditions of Lemma 6.2.6.

Corollary 6.3.6. *Following the notation of Lemma 6.3.1, let $\sigma \in \mu_m$ be the generator corresponding to $1 \in \mathbb{Z}/m\mathbb{Z}$, and let $H = \langle \sigma^2 \rangle$, so $H \cong \mu_3$.*

For $A \in \mathrm{Sh}[13]$ where $\mathrm{Sh}[13]$ the Deligne-Mostow Shimura variety associated with $\mathcal{M}[13]$, we have:

$$A[3]^H \cong A[(1 - \zeta_3)]$$

Proof. By Lemma 6.3.1, it suffices to show that $1 - \sigma^2$ -torsion is the same as $1 - \zeta_3$ -torsion. Since $\mathcal{M}[13]$ has signature type $(2, 1, 0, 1, 0)$, we see that σ acts on $L = H_1(A, \mathbb{Z})$ via multiplication by ζ_6, ζ_6^2 and ζ_6^4 . Hence, σ^2 acts on L via scalar multiplication by ζ_3 or ζ_3^2 . Since $(1 - \zeta_3)$ and $(1 - \zeta_3^2)$ are associates in $\mathbb{Z}[\zeta_3]$, the $(1 - \sigma^2)$ torsion is the same as $(1 - \zeta_3)$ torsion. Hence, $A[3]^H = A[1 - \sigma^2] = A[(1 - \zeta_3)]$. $\square$

Corollary 6.3.7. *$\tilde{K}$ gives a trivialization of the full level 2 structure on $J(C)$ for $C \in \mathcal{M}[4], \mathcal{M}[8]$ and of the full level $(1 - \zeta_3)$ structure on $J(C)$ for $C \in \mathcal{M}[13]$.*

Proof. Consider the families $\mathcal{M}[4] = \mathcal{M}(4, 4, (1, 2, 2, 3))$, $\mathcal{M}[8] = \mathcal{M}(4, 5, (1, 1, 2, 2, 2))$, $\mathcal{M}[13] = \mathcal{M}(6, 4, (1, 1, 2, 2))$.

Family	m	$\underline{a}$	e	f
$\mathcal{M}[4]$	4	(1,2,2,3)	2	(1,0,1)
$\mathcal{M}[8]$	4	(1,1,2,2,2)	2	(2,0,1)
$\mathcal{M}[13]$	6	(1,1,2,2)	3	(2,1,0,1,0)

Hence, these three families satisfy the condition in Lemma 6.2.6. Hence, we can apply Proposition 6.2.7 and get its $\tilde{K} = J(C)[e]^H = (\mathbb{Z}/e\mathbb{Z})^{d(r-2)}$. In particular we have the following

Family	m	g	$J(C)[e]$	$\tilde{K} = J(C)[e]^H$
$\mathcal{M}[4]$	4	2	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^4$
$\mathcal{M}[8]$	4	3	$(\mathbb{Z}/2\mathbb{Z})^6$	$(\mathbb{Z}/2\mathbb{Z})^6$
$\mathcal{M}[13]$	6	4	$(\mathbb{Z}/3\mathbb{Z})^8$	$(\mathbb{Z}/3\mathbb{Z})^4$

In the case of $\mathcal{M}[4]$ and $\mathcal{M}[8]$, in particular, the Q_{ij} give a basis of the entire $J(C)[2]$. In the case of $\mathcal{M}[13]$, by Corollary , $\tilde{K} = \langle Q_{ij} \rangle = J(C)[1 - \zeta_3]$. Hence, Q_{ij} provides a trivialization of the full $1 - \zeta_3$ level structure.

By (Li, Mantovan, and Pries, 2021, Corollary 7.5), in the case of $\mathcal{M}[4]$, $\mathcal{M}[8]$, the lattice L are $\mathcal{O}_F^2, \mathcal{O}_F^3$ respectively, where $F = \mathbb{Q}(i)$. Hence, the corresponding level subgroup $\Gamma(2) \subseteq \mathcal{G}(\mathbb{Z})$ is the principal congruence subgroup at level 2. In the case of $\mathcal{M}[13]$, since $\mathbb{Z}[\zeta_3]$ is a principal ideal domain and L is a torsion free $\mathbb{Z}[\zeta_3]$ module, we know that $L \cong \mathbb{Z}[\zeta_3]^4$ as $\mathbb{Z}[\zeta_3]$ module. Hence, if we write $\mathcal{G}(\mathbb{Z}) = \text{GU}(\mathcal{O}_K^4) \cap \text{GL}_{\mathbb{Q}[\mu_6]}(V)$, then the corresponding level subgroup $\Gamma(1 - \zeta_3)$ is $\text{GU}(1 - \zeta_3) \cap \text{GL}_{\mathbb{Q}[\mu_6]}(V)$. That is, they are the elements congruent to identity modulo $(1 - \zeta_3)$ in $\mathcal{G}(\mathbb{Z})$. A more explicit matrix presentation would be difficult since we don't know the structure of the lattice L as a $\mathbb{Z}[\mu_6]$ module. $\square$

Chapter 7

COMPUTING THE RING OF MODULAR FORMS

7.1 The quotient of level subgroups

From the computation in Chapter 6, we see that there is a factorization $\tilde{T}$ of the Torelli isomorphism T :

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[16] & \xrightarrow{\tilde{T}} & \text{Sh}_{\Gamma_5}[16] \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}[16] & \xrightarrow{T} & \text{Sh}[16] \end{array}$$

The aim is to show that this square is exact for $\mathcal{M}[16]$.

By (Achter and Pries, 2007, Lemma 2.3), we see that the forgetful morphism $\widetilde{\mathcal{M}}[16] \rightarrow \overline{\mathcal{M}}[16]$ is étale and Galois, with Galois group $\{\sigma \in \mathfrak{S}_r \mid \sigma \cdot \underline{a} = \underline{a}\}$. For $\mathcal{M}[16]$, its inertia type is $\underline{a} = (1, 1, 1, 1, 1)$. Hence, the Galois group for $\widetilde{\mathcal{M}}[16] \rightarrow \overline{\mathcal{M}}[16]$ is the entire $\mathfrak{S}_5$.

Hence, we want to compute the degree of the Shimura cover $\text{Sh}_{\Gamma_5}[16] \rightarrow \text{Sh}[16]$. Recall that in Section 2.4, we have $\text{Sh}[16]$ by the following integral PEL datum. Let $K = \mathbb{Q}(\zeta_5)$, $K_0 = \mathbb{Q}(\sqrt{5})$, $L = \mathcal{O}_K^3$. Let $H : L \times L \rightarrow \mathcal{O}_K$ be defined by

$$H(x, y) = x^* B y, \text{ where } B = \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})} \text{ and } \alpha = \zeta_5 + \zeta_5^{-1}. \text{ From the}$$

Hermitian pairing we define the unitary group associated to H

$$\mathcal{G}_1 = \text{Res}_{\mathbb{Z}}^{\mathcal{O}_{K_0}} U(L, H)$$

And the unitary similitude $\mathcal{G}$ that sits in the exact sequence

$$1 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G} \xrightarrow{c} \mathbb{G}_{m, \mathbb{Z}} \rightarrow 1$$

Recall from Definition 4.2.2 and Notation 4.2.3 that $\text{Sh}_{\Gamma_5}[16]$ has level subgroup $\mathcal{K} = \Gamma_5 \mathcal{G}(\hat{\mathbb{Z}}^5)$, where $\Gamma_5 = \{g \in \mathcal{G}(\mathbb{Z}_5) \mid g \equiv \text{I} \pmod{(1 - \zeta_5)L_5}\}$. Since $\text{Sh}[16]$ is induced from $\mathcal{M}[16]$, whose corresponding lattice $L \cong \mathbb{Z}[\zeta_5]^3$, we have the description that

$$\mathcal{G}(\mathbb{Z}) = \{g \in \text{GL}_3(\mathcal{O}_K) \mid \bar{g}^T H g = c H, c \in \mathbb{Z}^\times\}$$

$$\Gamma_5(\mathbb{Z}) = \{g \in \mathcal{G}(\mathbb{Z}) \mid g \equiv I_3 \pmod{1 - \zeta_5}\}$$

We will make use of the following notations in this section

Notation 7.1.1.

$$\begin{aligned} K &= \mathbb{Q}(\zeta_5), K_0 = \mathbb{Q}(\sqrt{5}), \mathcal{O}_K = \mathbb{Z}[\zeta_5] \\ \omega &= ((1 - \zeta_5)) \subseteq \mathcal{O}_K \\ U(\mathcal{O}_K^3, H) &:= \mathcal{G}_1(\mathbb{Z}) = \{g \in \mathrm{GL}_3(\mathcal{O}_K) \mid \bar{g}^T B g = r B, r \in \mathbb{Z}^\times\} \\ \mathrm{PU}(\mathcal{O}_{K_\omega}^3, H) &:= \mathcal{G}_1^{\mathrm{ad}}(\mathbb{Z}) = \mathcal{G}_1(\mathbb{Z}) / \{c \mathrm{Id} \in \mathcal{G}_1(\mathbb{Z})\} \\ U(\omega) &:= \Gamma_5(\mathbb{Z}) \cap \mathcal{G}_1(\mathbb{Z}) = \{g \in \mathcal{G}_1(\mathbb{Z}) \mid g \equiv \mathrm{Id} \pmod{\omega}\} \\ \mathrm{PU}(\omega) &:= U(\omega) / \{c \mathrm{Id} \in U(\omega)\} \\ U(\mathbb{F}_5^3, H) &:= \mathcal{G}_1(\mathbb{F}_5) = \{g \in \mathrm{GL}_3(\mathbb{F}_5) \mid g^T \bar{B} g = r \bar{B}\} \\ \mathrm{PU}(\mathbb{F}_5^3, H) &:= \mathcal{G}_1(\mathbb{F}_5) / \{c \mathrm{Id} \in \mathcal{G}_1(\mathbb{F}_5)\} \\ G &= \mathcal{G}_{\mathbb{Q}}, G_1 = \mathcal{G}_{1, \mathbb{Q}} \end{aligned}$$

Where

$$\begin{aligned} \bullet \quad B &= \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})} \text{ and } \alpha = \zeta_5 + \zeta_5^{-1}.. \\ \bullet \quad \bar{B} &= \frac{B}{\beta} \pmod{(1 - \zeta_5)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

To compute the index $\mathrm{Sh}_{\Gamma_5}[16] \rightarrow \mathrm{Sh}[16]$, it is useful to know some facts of the reductive group G and G_1 . These are the generic fibre over $\mathbb{Q}$ of $\mathcal{G}$ and $\mathcal{G}_1$, respectively.

Lemma 7.1.2. *We have the following facts of the algebraic group G and G_1 :*

1. G and G_1 are connected algebraic groups.
2. $G^{\mathrm{der}} = G_1^{\mathrm{der}} = \mathrm{Res}_{\mathbb{Q}}^{K_0} \mathrm{SU}(V, H)$
3. $G^{\mathrm{ad}} = G_1^{\mathrm{ad}} = \mathrm{Res}_{\mathbb{Q}}^{K_0} \mathrm{PU}(V, H)$. In particular,

$$G^{\mathrm{ad}}(\mathbb{R}) = \mathrm{PU}(3, 0) \times \mathrm{PU}(2, 1)$$

Proof. These facts are mostly standard, but we include the proof here for the sake of completeness. These proof can be generalized to Deligne-Mostow Shimura varieties of unitary type.

1. We just need to show the base changed to $\overline{\mathbb{Q}}$ are connected. We claim that $G_{1,\overline{\mathbb{Q}}} \cong \mathrm{GL}_3 \times \mathrm{GL}_3$ and $G_{\overline{\mathbb{Q}}} \cong \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathbb{G}_m$.

By definition, $G_1(\overline{\mathbb{Q}}) := \mathrm{U}(V, H)(K_0 \otimes \overline{\mathbb{Q}})$.

Consider $\mathrm{Hom}(K, \overline{\mathbb{Q}}) = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ where $\tau_i(\zeta_5) = \zeta_5^i$. In particular, $\overline{\tau_1} = \tau_4, \overline{\tau_2} = \tau_3$. We choose the CM type $\mathrm{Hom}(K_0, \mathbb{Q}) = \{\tau_1, \tau_2\}$. Then, we have

$$\begin{aligned} K \otimes \overline{\mathbb{Q}} &\cong \bigoplus_{\tau_i \in \mathrm{Hom}(K, \overline{\mathbb{Q}})} \overline{\mathbb{Q}}_{\tau_i} \\ G_1(\overline{\mathbb{Q}}) &= \mathrm{U}(V, H)(K_0 \otimes \overline{\mathbb{Q}}) \\ &= \{g \in \mathrm{GL}_{K \otimes \overline{\mathbb{Q}}}(V \otimes \overline{\mathbb{Q}}) \mid \overline{g^T} H g = H\} \\ &= \{(g_{\tau_i}) \in \prod_{\tau_i \in \mathrm{Hom}(K, \overline{\mathbb{Q}})} \mathrm{GL}_3(\overline{\mathbb{Q}}_{\tau_i}) \mid \overline{g_{\tau_i}^T} H_{\tau_i} g_{\tau_i} = H_{\tau_i}\} \end{aligned}$$

Since $H \in \mathrm{GL}_3(K_0)$, we have $H_{\tau_1} = H_{\tau_4}, H_{\tau_2} = H_{\tau_3}$. Also, complex conjugation acts by swopping the factor at τ_i , so $\overline{g_{\tau_1}} = g_{\tau_4}, \overline{g_{\tau_2}} = g_{\tau_3}$. Hence, for any $g_{\tau_1} \in \mathrm{GL}_3(\overline{\mathbb{Q}}_{\tau_1})$, there is a unique g_{τ_4} satisfying $\overline{g_{\tau_1}^T} H_{\tau_1} g_{\tau_1} = H_{\tau_1}$, so g_{τ_4} is uniquely determined by g_{τ_1} , g_{τ_3} is uniquely determined by g_{τ_2} . Hence, $G_{1,\overline{\mathbb{Q}}} = \mathrm{GL}_3 \times \mathrm{GL}_3$.

Under the same isomorphism,

$$G(\overline{\mathbb{Q}}) = \{(g_{\tau_1}, g_{\tau_2}, c) : g_{\tau_i} \in \mathrm{GL}_3(\overline{\mathbb{Q}}), c \in \mathbb{G}_m(\overline{\mathbb{Q}})\}$$

And g_{τ_4} is determined by $\overline{g_{\tau_1}^T} H_{\tau_1} g_{\tau_1} = c H_{\tau_1}$ and so is g_{τ_3} . Hence, $G_{\overline{\mathbb{Q}}} \cong \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathbb{G}_m$.

2. We have the exact sequence

$$1 \rightarrow G_1 \rightarrow G \xrightarrow{c} \mathbb{G}_m \rightarrow 1$$

Since G^{der} is the commutator subgroup of G , $c(G^{\mathrm{der}}) = 1$. Hence, $G^{\mathrm{der}} \subseteq G_1$, so $G^{\mathrm{der}} = G_1^{\mathrm{der}}$.

Now consider the $\mathbb{Q}$ -point of the derived sequence:

$$1 \rightarrow G_1^{\mathrm{der}}(\mathbb{Q}) \rightarrow \mathrm{U}(V, H)(K_0) \xrightarrow{\det} T \rightarrow 1$$

The maximal abelian quotient of G is the image of $\det : G \rightarrow \mathbb{G}_m$. Hence, $G^{\text{der}}(\mathbb{Q}) = \ker(\det) = \text{SU}(V, H)$. Hence, $G^{\text{der}} = G_1^{\text{der}} = \text{Res}_{\mathbb{Q}}^{K_0} \text{SU}(V, H)$.

3. Since G and G_1 are connected, the map $G^{\text{der}} \rightarrow G^{\text{ad}}$ is surjective. Hence, $G^{\text{ad}} = G_1^{\text{ad}} = (G^{\text{der}})^{\text{ad}} = \text{Res}_{\mathbb{Q}}^{K_0} \text{PU}(V, H)$

□

The covering index is closely related to $G^{\text{ad}}(\mathbb{Q}) = \text{PU}(\mathcal{O}_K^3, H)$. Notice that there is a morphism

$$\begin{aligned} f : \text{PU}(\mathcal{O}_K^3, H) &\rightarrow \text{PU}(\mathbb{F}_5^3, H) \\ g &\mapsto g \pmod{\omega} \end{aligned}$$

And $\ker(f) = \text{PU}(\omega)$. We first want to show that the quotient $\text{PU}(\mathbb{F}_5^3, H) \cong \mathfrak{S}_5$:

Proposition 7.1.3. *There is an isomorphism*

$$\rho : \mathfrak{S}_5 \cong \text{PU}(\mathbb{F}_5^3, H)$$

Proof. First observe that $\text{PU}(\mathbb{F}_5^3, H) \cong \text{PO}_3(\mathbb{F}_5)$ where $\text{PO}_3(\mathbb{F}_5)$ is the projective orthogonal group over $\mathbb{F}_5$. Indeed, when reduced mod ω , $\overline{B}$ is just a symmetric bilinear form, so the group preserving it is an orthogonal group. By Isomorphism of groups, $\mathfrak{S}_5$ is abstractly isomorphic to $\text{PO}_3(\mathbb{F}_5)$. This gives us the conclusion of the proposition.

We give the isomorphism ρ explicitly in this case. For $[C] \in \widetilde{\mathcal{M}}[16]$, let Q_1, Q_2, Q_3 be as defined by Notation 6.0.1. We first define a map $g : \mathfrak{S}_5 \rightarrow \text{GL}_3(\mathbb{F}_5)$ as follows. it suffices to define g on the generators of $\mathfrak{S}_5$. Given $\sigma \in \mathfrak{S}_5$, it acts on Q_i by permutations.

If $\sigma \in \mathfrak{S}_3$, then $\sigma(Q_i) = Q_{\sigma_i}$, from which we get the permutation matrix for (12), (13). If $\sigma = (34)$, we see that σ fixes Q_1, Q_2 and sends Q_3 to $P_4 - P_5 =$

$-Q_1 - Q_2 - Q_3$. Hence, the matrix of (34) is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$. If $\sigma = (45)$, we see

that $\sigma(Q_1) = 2Q_1 + Q_2 + Q_3$, similarly for Q_2, Q_3 . Hence, the matrix of (45) is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Now notice that each of these matrices is in $U(\mathbb{F}_5^3, H')$ where H' is defined by

$$\overline{B}' = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}. \text{ If we take } P = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \text{ then notice that } P^T \overline{B} P = \overline{B}'. \text{ Hence, for}$$

$g \in \text{PU}(\mathbb{F}_5^3, H'), PgP^{-1} \in \text{PU}(\mathbb{F}_5^3, H)$. Hence, the composite is the following map defined on the generator:

$\sigma \in \mathfrak{S}_5$	$g(\sigma) \in \text{PU}(\mathbb{F}_5^3, H')$	$Pg(\sigma)P^{-1} \in \text{PU}(\mathbb{F}_5^3, H)$
(12)	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
(13)	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$
(34)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 4 \\ 3 & 2 & 0 \end{bmatrix}$
(45)	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

This gives us the desired isomorphism ρ . □

Next, we show that this reduction map is surjective as expected:

Proposition 7.1.4. *The reduction map*

$$f : \text{PU}(\mathcal{O}_K^3, H) / \text{PU}(\omega) \rightarrow \text{PU}(\mathbb{F}_5^3, H)$$

is surjective.

Proof. We just need to find a lift for each of the generator in $\text{PU}(\mathbb{F}_5^3, H)$. We directly compute the lifts. Here $\alpha = \zeta_5 + \zeta_5^{-1}$.

$\mathfrak{S}_5$	$\text{PU}(\mathbb{F}_5^3, H)$	$\text{PU}(\mathcal{O}_K^3, H)$
(12)	$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -\alpha & 0 & \alpha \\ 0 & 1 & 0 \\ 1 & 0 & \alpha \end{bmatrix}$
(13)	$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2-2\alpha & \alpha & -2+3\alpha \\ \alpha & -\alpha & 1-\alpha \\ 1-2\alpha & \alpha & -1+3\alpha \end{bmatrix}$
(34)	$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 4 \\ 3 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} -1+2\alpha & 2-2\alpha & -2+4\alpha \\ 2-2\alpha & -1+2\alpha & 2-4\alpha \\ 2-2\alpha & -2+2\alpha & 3-4\alpha \end{bmatrix}$
(45)	$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$	$\begin{bmatrix} -\alpha & -2+2\alpha & 2-3\alpha \\ \alpha & -\alpha & -1+\alpha \\ -\alpha & -1+2\alpha & 1-3\alpha \end{bmatrix}$

□

We are now ready to compute the covering index of $\text{Sh}_{\Gamma_5}[16] \rightarrow \text{Sh}[16]$. Since we are computing the modular forms on the connected components, we want the index of the connected components, which can be obtained from the corresponding level subgroup $\mathcal{K}$.

For $\mathcal{K} \subseteq \mathcal{G}(\hat{\mathbb{Z}})$, the corresponding Shimura variety has the double coset representation:

$$\text{Sh}_{\mathcal{K}}(\mathcal{D}) = \mathcal{G}(\mathbb{Q}) \backslash (\mathcal{G}(\mathbb{A}_f) / \mathcal{K}) \times X$$

$\text{Sh}(\mathcal{D})$ is when $\mathcal{K} = \mathcal{G}(\hat{\mathbb{Z}})$ and $\text{Sh}(\Gamma_m)$ is when $\mathcal{K} = \mathcal{G}(\hat{\mathbb{Z}}^m) \Gamma_m$. The identity component of $\text{Sh}_{\mathcal{K}}(\mathcal{D})$ can be written as:

$$\text{Sh}_{\mathcal{K}}(\mathcal{D}) = \Gamma_0 \backslash X^+$$

Where Γ_0 is defined as follows: Let $G = \mathcal{G}_{\mathbb{Q}}$. Let $\rho : G \rightarrow G^{\text{ad}}$ be the map modulo its center. Let $G^{\text{ad}+}(\mathbb{R})$ be the identity component of $G^{\text{ad}}(\mathbb{R})$ and let $G(\mathbb{Q})_+$ be the part of $G(\mathbb{Q})$ mapped to $G^{\text{ad}+}(\mathbb{R})$. Then, $\Gamma_0 = \mathcal{K} \cap G(\mathbb{Q})_+$.

By Lemma 7.1.2, $G^{\text{ad}}(\mathbb{R}) = G_1^{\text{ad}}(\mathbb{R}) = \text{PU}(3, 0) \times \text{PU}(2, 1)$ which is connected. Therefore, Γ_0 is just the image of $\mathcal{K} \cap \mathcal{G}(\mathbb{Q})$ in $G^{\text{ad}}(\mathbb{Q})$. For $\text{Sh}[16]$, its Γ_0 is $\text{PU}(\mathcal{O}_K^3, H)$. For $\text{Sh}_{\Gamma_5}[16]$ its Γ_0 is $\text{PU}(\omega)$. Hence, the degree of $\text{Sh}_{\Gamma_5}[16] \rightarrow \text{Sh}[16]$ is $[\text{PU}(\omega) : \text{PU}(\mathcal{O}_K^3, H)]$. By proposition 7.1.4 and 7.1.3, we see that this index is the same as $\#\text{PU}(\mathbb{F}_5^3, H) = \#\mathfrak{S}_5$. Hence, we get the following corollary:

Corollary 7.1.5. *The following square is exact:*

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[16] & \xrightarrow{\tilde{T}} & \mathrm{Sh}_{\Gamma_5}[16] \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}[16] & \xrightarrow{T} & \mathrm{Sh}[16] \end{array}$$

Hence, $\tilde{T} : \widetilde{\mathcal{M}}[16] \rightarrow \mathrm{Sh}_{\Gamma_5}[16]$ is an isomorphism.

Recall that in Section 5.5, We computed the ring of global sections of Hodge line bundle on $\widetilde{\mathcal{M}}[16]$ as $\Gamma_0(\lambda)$ and identify it explicitly as some polynomial ring.

Since $\tilde{T}$ is an isomorphism and $\tilde{T}^* \lambda_{\mathrm{Sh}_{\Gamma_5}[16]} = \lambda_{\widetilde{\mathcal{M}}[16]}$, we see that $\Gamma_0(\lambda)$ is the same as $M(\mathrm{Sh}_{\Gamma_5}[16])$. We get Theorem 4.3.1 as a corollary.

7.2 Computing the ring of modular forms

The goal of this section is to compute the ring of modular form of $\mathrm{Sh}[16]$. Recall from Theorem 4.3.1 that the ring of modular form of $\mathrm{Sh}_{\Gamma_5}[16]$ is R_{ev} , and $\mathrm{Sh}_{\Gamma_5}[16] \rightarrow \mathrm{Sh}[16]$ has covering group isomorphic to $\mathfrak{S}_5$. Then, the ring for $\mathrm{Sh}[16]$ is the invariant subring of R_{ev} under the $\mathfrak{S}_5$ action, denoted as $R_{\mathrm{ev}}^{\mathfrak{S}_5}$.

The goal of this section is to prove Corollary 7.2.7, which will prove Theorem 4.3.2. That is, we want to compute $R_{\mathrm{ev}}^{\mathfrak{S}_5}$.

In this section, we will use the following notation. Some of them are borrowed from Proposition 5.5.3.

Notation 7.2.1. In the ring R_{ev} :

- s_{ij} are the coordinates on $\mathbb{P}^5$. Here $ij \in \{12, 21, 13, 31, 23, 32\}$
- $\sigma_{ij} = \epsilon_{ij}(s_{ik} - s_{ki} - s_{jk})$, $\epsilon_{ij} = \mathrm{sign}\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$
- $Q_{ij} = s_{ij} \cdot \sigma_{ij}$.
- $f_{2n} = \sum_{ij \in \{12, 21, 13, 31, 23, 32\}} s_{ij}^{2n} + \sigma_{ij}^{2n}$. Notice that by the $\mathfrak{S}_5$ action on R_1 described by Proposition 5.5.3, f_{2n} are invariant under the $\mathfrak{S}_5$ action.

Rings related to R_{ev} :

- $K = \mathbb{Q}(\zeta_5)$ is the base field of the polynomial rings $\Gamma_0(\lambda)$, R_Y , R , R_{ev} .
- $R = K[s_{ij}, \varepsilon]/\varepsilon^5 - Q_{12}, Q_{12} - Q_{ij}$.
- $R_Y = K[s_{ij}]/Q_{12} - Q_{ij}$. Hence, $R = R_Y[\varepsilon]/\varepsilon^5 - Q_{12}$.
- S' is the sub-algebra of R generated by $f_2, f_4, f_6, \varepsilon^2$.
- S_Y is the sub-algebra of R_Y generated by f_2, f_4, f_6

We first show that S' has the right transcendence degree. In the below, for an integral ring R , we use $k(R)$ to denote its field of fraction.

Proposition 7.2.2. *Let S' be the subalgebra of R generated by $f_2, f_4, f_6, \varepsilon^2$. Then*

$$k(S') = k(R^{\mathfrak{S}_5})$$

In particular, $\text{trans}(S') = \text{trans}(R^{\mathfrak{S}_5}) = 3$. Here, trans means the transcendence degree of the ring over K , and $k(S')$ means the fraction field of S' .

The proof of this proposition is long and technical, and is postponed in the Appendix A.

Proposition 7.2.3. *Let $R_Y = K[s_{ij}]/Q_{12} - Q_{ij}$. Let S_Y be the sub-algebra of R_Y generated by f_2, f_4, f_6 . Then*

$$S_Y = K[f_2, f_4, f_6]$$

Consequently, $\text{trans}_K(S_Y) = 3$.

Proof. Consider the map

$$\begin{aligned} \varphi : K[y_1, y_2, y_3] &\rightarrow S_Y \\ y_i &\mapsto f_{2i} \end{aligned}$$

φ is by definition surjective. If $f(y_1, y_2, y_3) \neq 0 \in \ker \varphi$, then $\text{trans}(S_Y) \leq 2$. On the other hand, by Lemma 7.2.2, $\text{trans}(S') = 3$. Since S' is an algebraic extension of S_Y of degree 2, we have that $\text{trans}(S_Y) = 3$. Hence, $\ker(\varphi) = 0$, so $S_Y = K[f_2, f_4, f_6]$. $\square$

Lemma 7.2.4. *We have the following relation in R_Y :*

$$Q_{ij}^2 = \frac{1}{52}(4f_4 - f_2^2)$$

Consequently, $\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4) = 0$ in R .

Proof. It suffices to consider the case where $ij = 12$. By (Bauer and Catanese, 2021, Theorem 3.3), $\langle Q_{12} - Q_{ij} \rangle$ is the kernel of homomorphism $s_{ij} \rightarrow x_i \cdot x_j \cdot (x_j - x_k)$. Hence, mod out by the ideal $\mathcal{I} = \langle Q_{12} - Q_{ij} \rangle$ in R_Y is the same as substituting s_{ij} by $x_i \cdot x_j \cdot (x_j - x_k)$. We directly check that $Q_{12}^2(x_i) = \frac{1}{52}(4f_4(x_i) - f_2^2(x_i))$. $\square$

Proposition 7.2.5. *We define the extension of φ to $K[y_1, y_2, y_3, y]$ as follows:*

$$\begin{aligned} g : K[y_1, y_2, y_3][y] &\rightarrow S' \\ y &\mapsto \varepsilon^2 \end{aligned}$$

Then, $\ker(g) = \langle y^5 + \frac{1}{52}(y_1^2 - 4y_2) \rangle$. Consequently,

$$S' \cong K[f_2, f_4, f_6, \varepsilon^2] / (\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4))$$

Proof. Since $K[y_1, y_2, y_3][y]$ is a principal ideal domain, $\ker(g)$ is generated by a polynomial $f(y)$, and $f(y)$ is the element with lowest degree with respect to y in $\ker(g)$. Suppose $f(y) = y^k g_k + \dots + g_0$, $g_i \in K[y_1, y_2, y_3]$. Here $g_k \neq 0$.

By Proposition 5.5.5, we see that S' is the graded ring where $\deg(\varepsilon^2) = \frac{2}{5}$ and $\deg(f_{2i}) = 2i$. Hence, for g to be a homomorphism of graded ring, we must have $\deg(y_i) = 2i$ and $\deg(y) = \frac{2}{5}$. Hence, $\deg(y^k g_k) \equiv \frac{2}{5}k \pmod{2\mathbb{Z}}$.

Since $g(f(y)) = 0$, We have $\sum_{j=0}^k \varepsilon^{2j} g_j(f_2, f_4, f_6) = 0$ in R . In particular, all the monomials in $\varepsilon^{2k} g_k(f_2, f_4, f_6)$ must be canceled with a monomial in $\varepsilon^{2i} g_i$ for some $i < k$. Hence, there exists some $0 \leq i < k$, such that $\frac{4}{5}k \equiv \frac{4}{5}i \pmod{2\mathbb{Z}}$. This implies that $i \equiv k \pmod{5}$, so $f(y)$ has degree at least 5. On the other hand, by Lemma 7.2.4, we have that $y^5 + \frac{1}{52}(y_1^2 - 4y_2) \in \ker(g)$. Hence, $f(y) = y^5 + \frac{1}{52}(y_1^2 - 4y_2)$. $\square$

Proposition 7.2.6. *S' is an integrally closed domain.*

Proof. For the sake of convenience, we write $S' = K[y_1, y_2, y_3, y] / y^5 + \frac{1}{52}(y_1^2 - 4y_2)$. Then $S_Y = K[y_1, y_2, y_3]$.

We already know that S' is a domain. To show that it is integrally closed, let $z \in k(S')$ be integral over S' . Since $k(S')/k(S_Y)$ is the separable extension defined by $y^5 + \frac{1}{52}(y_1^2 - 4y_2)$, we can write z as

$$z = y^4 + \frac{p_3}{q_3}y^3 + \frac{p_2}{q_2}y^2 + \frac{p_1}{q_1}y + \frac{p_0}{q_0}, p_i, q_i \in K[y_1, y_2, y_3]$$

We have the automorphism of $k(S')/k(S_Y)$ given by $\tau : y \rightarrow \zeta_5 y$ and τ fixes $K[y_1, y_2, y_3]$. We have $\tau^i(z)$ integral over S_Y for $0 \leq i \leq 4$. On the other hand, we

have

$$\frac{p_k}{q_k} y^k = \frac{1}{5} \sum_{i=0}^4 \zeta_5^{-ik} \tau^i(z)$$

for $0 \leq k \leq 4$. So $\frac{p_k}{q_k} y^k$ are integral over S_Y . Since y is also integral over S_Y , we have that $\frac{p_k}{q_k}$ integral over S_Y . Since $S_Y = K[y_1, y_2, y_3]$ is integrally closed, so $\frac{p_k}{q_k} \in S_Y$. Hence, $z \in S'$, so S' is integrally closed. $\square$

Corollary 7.2.7. $R_{\text{ev}}^{\mathfrak{S}_5} = S' = K[f_2, f_4, f_6, \varepsilon^2]/\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4)$

Proof. By definition, $S' \subset R_{\text{ev}}^{\mathfrak{S}_5}$ and $R_{\text{ev}}^{\mathfrak{S}_5}$ is integral over S' . By Proposition 7.2.2, $k(S') = k(R_{\text{ev}}^{\mathfrak{S}_5})$.

Now suppose $y \in R_{\text{ev}}^{\mathfrak{S}_5}$, then y is integral over S' and $y \in k(R_{\text{ev}}^{\mathfrak{S}_5}) = k(S')$. By Proposition 7.2.6, S' is integrally closed, so $y \in S'$. Hence, $R_{\text{ev}}^{\mathfrak{S}_5} = S' = K[f_2, f_4, f_6, \varepsilon^2]/(\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4))$. $\square$

This finishes the computation for the ring of modular form of Sh[16] and hence the proof of Theorem 4.3.2.

7.3 Application: Ring of modular forms on a Shimura curve

In this section, we relate the results of Theorem 4.3.2 to the modular forms of the Shimura curve arising from $\mathcal{M}[11]$. While the ring structure of $\mathcal{M}[11]$ is already known, we obtain the ring of modular forms explicitly as a quotient of $M(\text{Sh}[16])$. Using the strategy established described in Section 4.3 and Theorem 4.3.1, we obtain the analogue of Theorem 4.3.2 and Theorem 4.3.1 for this Shimura curve. We also obtain the canonical j function on $\mathcal{M}[11]$ in terms of $f_2, f_4, f_6, \varepsilon^2$.

Notation 7.3.1. We use the following notations in this section:

For the number fields:

- $K = \mathbb{Q}(\zeta_5), K_0 = \mathbb{Q}(\sqrt{5}), O_K = \mathbb{Z}[\zeta_5]$
- $\alpha = \zeta_5 + \zeta_5^{-1}, \omega = ((1 - \zeta_5)) \subseteq O_K$

For the moduli spaces:

- $\mathcal{M}[11] = \mathcal{M}(5, 4, (1, 1, 1, 2))$. This is the 11th of Moonen's 20 special family. It is one-dimensional family parameterizing curves of $g = 4$.

- $\widetilde{\mathcal{M}}[11] = \widetilde{\mathcal{M}}(5, 4, (1, 1, 1, 2))$. This is the finite cover $\widetilde{\mathcal{M}}[11] \rightarrow \overline{\mathcal{M}}[11]$ obtained by fixing $P_i \in \mathcal{C}$ above x_i for $1 \leq i \leq r$.
- $\text{Sh}[11]$. This is the smallest PEL subvariety of $\mathcal{A}_4$ containing $T(\mathcal{M}[11])$. It has signature $\underline{f} = (2, 1, 1, 0)$. It is compact, parameterizing abelian varieties of dimension 4.
- $\text{Sh}_{\Gamma_5}[11]$. This is the finite cover $\text{Sh}_{\Gamma_5}[11] \rightarrow \text{Sh}[11]$ defined by fixing an integral $1 - \zeta_5$ structure on $(\mathcal{A}, \lambda, i)$ as defined in Definition 4.2.2.
- $M(\text{Sh}[16])$ and $M(\text{Sh}[16])$: The ring of modular forms for $\text{Sh}[16]$ and $\text{Sh}[11]$, respectively.
- $M(\text{Sh}_{\Gamma_5}[11])$: The ring of modular forms on $\text{Sh}_{\Gamma_5}[11]$.

For the groups:

$$\begin{aligned}
 \text{U}(\mathcal{O}_K^2, H_1) &:= \mathcal{G}_{1, B_1}(\mathbb{Z}) = \{g \in \text{GL}_2(\mathcal{O}_K) \mid \overline{g}^T B_1 g = B_1\} \\
 \text{PU}(\mathcal{O}_K^2, H_1) &:= \mathcal{G}_{1, B_1}^{\text{ad}}(\mathbb{Z}) = \text{U}(\mathcal{O}_K^2, H_1) / \{c \text{Id} \in \text{U}(\mathcal{O}_K^2, H_1)\} \\
 \text{U}(\omega, H_1) &:= \Gamma_{m, 1}(\mathbb{Z}) \cap \mathcal{G}_{1, B_1}(\mathbb{Z}) = \{g \in \mathcal{G}_{1, B_1}(\mathbb{Z}) \mid g \equiv \text{Id} \pmod{\omega}\} \\
 \text{PU}(\omega, H_1) &:= \text{U}(\omega, H_1) / \{c \text{Id} \in \text{U}(\omega, H_1)\} \\
 \text{U}(\mathbb{F}_5^2, H_1) &:= \mathcal{G}_{1, B_1}(\mathbb{F}_5) = \{g \in \text{GL}_2(\mathbb{F}_5) \mid g^T \overline{B_1} g = \overline{B_1}\} \\
 \text{PU}(\mathbb{F}_5^2, H_1) &:= \text{U}(\mathbb{F}_5^2, H_1) / \{c \text{Id} \in \text{U}(\mathbb{F}_5^2, H_1)\} \\
 G_{B_1} &= \mathcal{G}_{B_1, \mathbb{Q}}, G_{1, B_1} = \mathcal{G}_{1, B_1, \mathbb{Q}}
 \end{aligned}$$

Where

$$\begin{aligned}
 \bullet \quad B_1 &= \beta \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})} \text{ and } \alpha = \zeta_5 + \zeta_5^{-1} \dots \\
 \bullet \quad \overline{B_1} &= \frac{B_1}{\beta} \pmod{(1 - \zeta_5)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.
 \end{aligned}$$

In terms of the Notation 5.0.1, we see that there is a unique boundary component of $\overline{\mathcal{M}}[16] = \overline{\mathcal{M}}(5, 5, (1, 1, 1, 1, 1))$, given by the clutching morphism

$$\kappa(\overline{\mathcal{M}}(5, 4, (1, 1, 1, 2)) \times \overline{\mathcal{M}}(5, 3, (3, 1, 1))) \hookrightarrow \overline{\mathcal{M}}(5, 5, (1, 1, 1, 1, 1))$$

$\overline{\mathcal{M}}(5, 3, (3, 1, 1))$ is a 0-dimensional family, whereas $\overline{\mathcal{M}}(5, 4, (1, 1, 1, 2))$ is a one-dimensional family of curves. In fact, $\overline{\mathcal{M}}(5, 4, (1, 1, 1, 2))$ is the special family

$\overline{\mathcal{M}}[11]$. Let $\text{Sh}[11]$ be the smallest PEL type Shimura subvariety of $\mathcal{A}_4$ containing $\mathcal{M}[11]$ as in Definition 4.1.2. Its signature type, by (4) of Proposition 3.1.10, is given by $(2, 1, 1, 0)$. In particular, $\text{Sh}[11]$ is also compact, and hence the Torelli morphism is also an isomorphism.

For the later propositions, we list lattice L_1 and the Hermitian pairing H_{B_1} of $\text{Sh}[11]$.

Proposition 7.3.2 (Integral datum of $\text{Sh}[11]$). *Let $\mathcal{D} = (O, *, L, \Psi, h_0)$ be the integral Shimura datum for $\text{Sh}[11]$. Then we have:*

1. $O = O_K$

2. $L = O_K^2$

3. $\Psi : L \times L \rightarrow \mathbb{Z}$ is given by $\Psi(x, y) = \text{Tr}_{\mathbb{Q}}^K(\bar{x}^T B_1 y)$, where $B_1 = \frac{1}{\beta} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$

where $\beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})}$ and $\alpha = \zeta_5 + \zeta_5^{-1}$.

The associated group schemes are $\mathcal{G}_{1, B_1}$ and $\mathcal{G}_{B_1}$ with the descriptions:

$$\begin{aligned} \mathcal{G}_{1, B_1} &= \text{Res}_{\mathbb{Z}}^{O_{K_0}} U(L_1, H_1) \\ \mathcal{G}_{1, B_1}(\mathbb{Z}) &= \{g \in GL_2(O_K) \mid \bar{g}^T B_1 g = B_1\} \\ \mathcal{G}_{B_1}(\mathbb{Z}) &= \{g \in GL_2(O_K) \mid \bar{g}^T B_1 g = r B_1 \text{ for } r \in \mathbb{Z}^\times\} \end{aligned}$$

Proof. We follow (Li, Mantovan, and Pries, 2021) to determine L_1 and H_{B_1} . The approach is as summarized in Example 4.1.5.

Notice that $g(\mathcal{M}[11]) = 4$. So $L_1 = H^1(C(\mathbb{C}), \mathbb{Z})$ is a $\mathbb{Z}[\zeta_5]$ module of $\mathbb{Z}$ -rank $2g = 8$. Since $\mathbb{Z}[\zeta_5] = O_K$ is a principal ideal domain, we see that $L = \mathbb{Z}[\zeta_5]^2 = O_K^2$.

To determine B_1 , pick $P \in \partial \overline{\mathcal{M}}[11]$. C_P is the image of clutching morphism of C_1, C_2 , where $C_1 \in \mathcal{M}(5, 3, (1, 1, 3))$ and $C_2 \in \mathcal{M}(5, 3, (2, 1, 2))$. $J(C_i) \cong \mathbb{C}^n / \Phi_i(O_K)$ where Φ_i is the CM-type of $J(C_i)$. Recall that $\beta_0 = \frac{5}{\zeta_5^3 - \zeta_5^2}$ generates the codifferent ideal $\mathcal{D}_{K/\mathbb{Q}}$. Similar to Example 4.1.5, we need to find u_1, u_2 such that for $\tau \in \Phi_j$, $\Im(\tau(u_j \beta_0)) < 0$. We made the following choice of u_j for $j = 1, 2$:

$\mathcal{M}$	$\underline{f}$	Φ	u
$\mathcal{M}_1 = \mathcal{M}(5, 3, (1, 1, 3))$	$\underline{f}_1 = (1, 1, 0, 0)$	$\Phi_1 = \{\tau_1, \tau_2\}$	$u_1 = \frac{1-\sqrt{5}}{2}$
$\mathcal{M}_2 = \mathcal{M}(5, 3, (2, 1, 2))$	$\underline{f}_2 = (1, 0, 1, 0)$	$\Phi_2 = \{\tau_1, \tau_3\}$	$u_2 = -1$

Hence, $B_1 = \text{diag}[u_1^{-1}\beta_0^{-1}, u_2^{-1}\beta_0^{-1}] = \frac{1}{\beta}\text{diag}[1, \alpha]$ with $\beta = \frac{\zeta_5^3 - \zeta_5^2}{5(\zeta_5 + \zeta_5^{-1})}$ and $\alpha = \zeta_5 + \zeta_5^{-1}$. The group scheme $\mathcal{G}_{1, B_1}$ and $\mathcal{G}_{B_1}$ are as defined in Remark 2.2.3. $\square$

We have the following exact diagram of Torelli isomorphisms:

$$\begin{array}{ccc} \overline{\mathcal{M}}[11] & \xrightarrow{T} & \text{Sh}[11] \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}[16] & \xrightarrow{T} & \text{Sh}[16] \end{array}$$

Notice that $\text{Sh}[11] \cong \mathbb{P}^1$. Hence, $M(\text{Sh}[11])$ is known to be abstractly isomorphic to $K[X, Y]$. But the above diagram suggests that we should be able to obtain $M(\text{Sh}[11])$ as an explicit quotient of $M(\text{Sh}[16])$. By using the geometry of $\overline{\mathcal{M}}[16]$, we can obtain $M(\text{Sh}[11])$ as a quotient of $M(\text{Sh}[16])$ as an immediate corollary of Theorem 4.3.2.

Theorem 7.3.3. *The ring of modular forms of $\text{Sh}[11]$ is given by setting $\varepsilon^2 = 0$. That is,*

$$M(\text{Sh}[16]) = K[f_2, f_6]$$

and on $\text{Sh}[11]$, $f_4 = \frac{1}{4}f_2^2$.

Proof. By the proof of Proposition 5.5.5, ε is chosen exact so that its divisor of zero is exactly δ . On the other hand, $\overline{\mathcal{M}}[11]$ is the the divisor δ in $\overline{\mathcal{M}}[16]$. Therefore, ε restricted to 0 on $\text{Sh}[11]$.

By Theorem 4.3.2, we have $M(\text{Sh}[16]) = K[f_2, f_4, f_6, \varepsilon^2]/(\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4))$. Setting $\varepsilon^2 = 0$, we obtain that $f_2^2 = 4f_4$, and $M(\text{Sh}[11]) = K[f_2, f_6]$. $\square$

On $\mathcal{M}[11]$, since $m = 5$ is a prime, by Proposition 6.3.3, we have the following factorization:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[11] & \xrightarrow{\tilde{T}} & \text{Sh}_{\Gamma_5}[11] \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}[11] & \xrightarrow{T} & \text{Sh}[11] \end{array}$$

It turns out that we also have that this diagram is exact. That is, we have the following analogue of Corollary 7.1.5:

Proposition 7.3.4. $\widetilde{T} : \widetilde{\mathcal{M}}[11] \rightarrow \text{Sh}_{\Gamma_5}[11]$ is an isomorphism.

Proof. We follow the steps outlined in Section 7.1. The Galois group of $\widetilde{\mathcal{M}}[11] \rightarrow \overline{\mathcal{M}}[11]$ is given by $\{\sigma \in \mathfrak{S}_4 \mid \sigma \underline{a} = \underline{a}\}$. Since $\underline{a} = (1, 1, 1, 2)$, we have that $\widetilde{\mathcal{M}}[11] \rightarrow \overline{\mathcal{M}}[11]$ is a $\mathfrak{S}_3$ -covering. Hence, we want to show that $\text{Sh}_{\Gamma_5}[11] \rightarrow \text{Sh}[11]$ has degree 6.

1. Show that $\mathfrak{S}_3 \cong \text{PU}(\mathbb{F}_5^2, H_1)$: Notice that $B_1 = \text{diag}[1, 2]$, automorphism of $\mathbb{F}_5^2$ respecting B_1 forms the orthogonal group. Since 2 is not a quadratic residue in $\mathbb{F}_5$, B_1 defined the non-split orthogonal group. Hence, $\text{U}(\mathbb{F}_5^2, H_1) \cong \text{O}_2^-(\mathbb{F}_5)$ and $\#\text{O}_2^-(\mathbb{F}_5) = 12$. Moreover, if $c\text{Id} \in \text{O}_2^-(\mathbb{F}_5)$, then $c^2 = 1$. In $\mathbb{F}_5$, only $c = \pm 1$ squared to 1. Hence, $\#\text{PO}_2^-(\mathbb{F}_5) = 6$.

We explicitly gives the map from $\mathfrak{S}_3 \rightarrow \text{PU}(\mathbb{F}_5^2, H_1)$ on generators

$\mathfrak{S}_3$	$\text{PU}(\mathbb{F}_5^2, H_1)$
(123)	$r = \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix}$
(12)	$s = \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}$

Notice that $r^3 = \text{I} = s^2$ and $sr s^{-1} = r^{-1}$. Hence, this is a well defined isomorphism.

2. Show that $\text{PU}(\mathcal{O}_K^2, H_1)/\text{PU}(\omega, H_1) \cong \text{PU}(\mathbb{F}_5^2, H_1)$: The reduction map is defined by reduce the entries modulo ω so the kernel is $\text{PU}(\omega, H_1)$. Hence, we just need to show that each generator in $\text{PU}(\mathbb{F}_5^2, H_1)$ can be lifted to $\text{PU}(\mathcal{O}_K^2, H_1)$. We give the lifts:

$\text{PU}(\mathbb{F}_5^2, H_1)$	Lift in $\text{PU}(\mathcal{O}_K^2, H_1)$
$r = \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix}$	$\tilde{r} = \begin{vmatrix} -\alpha & -\alpha \\ 1 & -\alpha \end{vmatrix}$
$s = \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}$	$\tilde{s} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

Hence, the reduction map is an isomorphism.

3. Conclude that $\text{Sh}_{\Gamma_5}[11] \rightarrow \text{Sh}[11]$ has degree 6: Let $G_{B_1} = \mathcal{G}_{B_1, \mathbb{Q}}$ and similarly for G_{1, B_1} . Following the proof of Lemma 7.1.2, we see that $G_{B_1, \overline{\mathbb{Q}}} \cong \text{GL}_2 \times \text{GL}_2 \times \mathbb{G}_m$ and $G_{1, B_1, \overline{\mathbb{Q}}} \cong \text{GL}_2 \times \text{GL}_2$. Hence, G_{B_1} and G_{1, B_1} are

connected. From the determinant map, we see that $G_{B_1}^{\text{ad}} = G_{1,B_1}^{\text{ad}} = \text{PU}(2, 0) \times \text{PU}(1, 1)$, which is connected. Hence, the Γ_0 for $\text{Sh}[11]$ and $\text{Sh}_{\Gamma_5}[11]$ is just $(\mathcal{K} \cap G_{B_1}(\mathbb{Q}))^{\text{ad}}$, which are $\text{PU}(\mathcal{O}_K^2, H_1)$ and $\text{PU}(\omega, H_1)$, respectively. Hence, the index of $\text{Sh}_{\Gamma_5}[11] \rightarrow \text{Sh}[11]$ is

$$[\text{PU}(\omega, H_1) : \text{PU}(\mathcal{O}_K^2, H_1)] = \#\text{PU}(\mathbb{F}_5^2, H_1) = \#\mathfrak{S}_3 = 6$$

From the steps, we see that the diagram is exact. Since T is an isomorphism, we have $\widetilde{T}$ is an isomorphism. $\square$

As a consequence of $\widetilde{T}$ being an isomorphism, we have that $M(\text{Sh}_{\Gamma_5}[11]) \cong \Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]})$, where $\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]}) := \bigoplus_{m \geq 0} H^0(\widetilde{\mathcal{M}}[11], \lambda^{\otimes m})$.

Now, notice that since $\text{Sh}[11] \cong \mathbb{P}^1$, $M(\text{Sh}_{\Gamma_5}[11])$ is abstractly some finite extension of $K[X, Y]$. We want to express these rings as some quotient $M(\text{Sh}_{\Gamma_5}[16])$.

At the level of $\widetilde{\mathcal{M}}$ we have the following diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[11] & \xrightarrow{q} & \overline{M}_{0,4} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}[16] & \xrightarrow{q} & \overline{M}_{0,5} \xrightarrow{|-K_{\text{Bl}_4(\mathbb{P}^2)}|} Y \subseteq \mathbb{P}^5 \end{array}$$

So $\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[16]})$ surjects onto $\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]})$.

There are 10-isomorphic copies of $\widetilde{\mathcal{M}}[11]$ in $\widetilde{\mathcal{M}}[16]$, which is in one to one correspondence of the 10 isomorphic copies of $\overline{M}_{0,4}$ in $\overline{M}_{0,5}$. Recall that in Section 3.3, we have listed all divisors of $\overline{M}_{0,5}$ under the isomorphism $\overline{M}_{0,5} \cong \text{Bl}_4(\mathbb{P}^2)$. So to specify $\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]})$ as a quotient of $\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[16]})$, we need to specify the isomorphic copy of $\widetilde{\mathcal{M}}[11]$. For the sake of convenience, we consider the copy of $\widetilde{\mathcal{M}}[11]$ maps to L_{P_1, P_2} in $\text{Bl}_4(\mathbb{P}^2)$. Here L_{P_1, P_2} is as in (10) of Proposition 3.3.4.

Theorem 7.3.5. *Identifying $\widetilde{\mathcal{M}}[11]$ as the pre-image of L_{P_1, P_2} under q , then*

$$M(\text{Sh}_{\Gamma_5}[11]) \cong K[s_{12}^2, s_{21}^2, s_{12}s_{21}] \cong K[X, Y, Z]/XY - Z^2$$

In particular, it is generated by weight 5 modular forms $s_{12}^2, s_{21}^2, s_{12}s_{21}$.

Proof. We compute the restriction of $\mathcal{M}(\text{Sh}_{\Gamma_5}[16])$ to $\text{Sh}_{\Gamma_5}[11]$.

We use (x_1, x_2, x_3) for coordinate on $\mathbb{P}^2$. Let $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [0 : 0 : 1]$, $P_4 = [1 : 1 : 1]$, and the $\text{Bl}_4(\mathbb{P}^2)$ is the blow up of $\mathbb{P}^2$ at these four points.

Let $V' = \{f \in K[x_1, x_2, x_3]_3 \mid f(P_1) = f(P_2) = f(P_3) = f(P_4) = 0\}$. By (Bauer and Catanese, 2021, Proposition 1.2), the anticanonical embedding $|-K_{\text{Bl}_4(\mathbb{P}^2)}|$ is defined by the basis of V' given by

$$s_{ij} = x_i x_j (x_j - x_k), \{i, j, k\} = \{1, 2, 3\}$$

This is how we obtain the 6 coordinates s_{ij} of $\mathbb{P}^5$.

Since L_{P_1, P_2} is the strict transformation of the line in $\mathbb{P}^2$ connecting P_1, P_2 , we have that $s_{ij} \mid_{L_{P_1, P_2}}$ sets $x_3 = 0$. In particular, if $3 \in \{i, j\}$, then $s_{ij} = 0$. This sets $s_{13}, s_{31}, s_{23}, s_{32}$ to become 0. Also, $\varepsilon = 0$ since $\text{div}(\varepsilon) = \delta$ and $\widetilde{\mathcal{M}}[11] \subseteq \delta$. The sub-algebra of $\mathcal{M}(\text{Sh}_{\Gamma_5}[16])$ generated s_{12}, s_{21} is then $K[s_{12}^2, s_{21}^2, s_{12}s_{21}] \cong K[X, Y, Z]/XY - Z^2$. Hence, there is a surjection $f : K[X, Y, Z]/XY - Z^2 \rightarrow \mathcal{M}(\text{Sh}_{\Gamma_5}[11])$

Now we just need to verify that there is no other relations in $\mathcal{M}(\text{Sh}_{\Gamma_5}[11])$. Notice that $\mathcal{M}(\text{Sh}_{\Gamma_5}[11])$ has transcendental degree 2. Then, suppose we have some non-trivial $g(X, Y, Z) \in \ker(f)$, then we can write $g(X, Y, Z) = g_0(X, Y) + g_1(X, Y)Z$. But then this would imply that $g_1(X, Y)^2 XY = g_0(X, Y)^2$. This is a non-trivial algebraic relation of X, Y since XY is not a square. Then $\mathcal{M}(\text{Sh}_{\Gamma_5}[11])$ has transcendental degree 1, which is not possible since $\text{Sh}_{\Gamma_5}[11]$ has dimension 1. Hence, f is an isomorphism. $\square$

Remark 7.3.6. We can also compute the abstract form of $\mathcal{M}(\text{Sh}_{\Gamma_5}[11]) \cong \Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]})$ using the table we obtain in Section 5.2. It has that $\lambda_{\widetilde{\mathcal{M}}[11]} = q^* \mathcal{O}(-\frac{4}{5}) \otimes \mathcal{O}(\delta)$. By Remark 3.2.2, the minimal $k > 0$ such that λ^k has global section is 5. Hence,

$$\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]}) = \bigoplus_{m \geq 0} H^0(\widetilde{\mathcal{M}}[11], \lambda^{5m})$$

$\lambda^5 = q^* \mathcal{O}(-4) \otimes \mathcal{O}(10\delta)$. Notice that $\widetilde{\mathcal{M}}[11]$ has 3 boundary divisors $\delta_{(12)}, \delta_{(13)}, \delta_{(14)}$ mapped to $[0], [1], [\infty] \in \overline{\mathcal{M}}_{0,4}$. By Lemma 5.1.3, for $J = (12), (13), (14)$, We have $\mathcal{O}(5\delta_J) = q^* \mathcal{O}(1)$. Hence, $\mathcal{O}(10\delta) = \mathcal{O}(10(\delta_{(12)} + \delta_{(13)} + \delta_{(14)})) = q^* \mathcal{O}(6)$. Hence, $\lambda^5 = q^* \mathcal{O}(2)$, so

$$\Gamma_0(\lambda_{\widetilde{\mathcal{M}}[11]}) = \bigoplus_{m \geq 0} H^0(\widetilde{\mathcal{M}}[11], \lambda^{5m}) = \bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_{0,4}, \mathcal{O}(2m)) = K[X^2, Y^2, XY]$$

And $X^2 \rightarrow s_{12}^2, Y^2 \rightarrow s_{21}^2, XY \rightarrow s_{12}s_{21}$ realizes this abstract ring as the quotient of $M(\text{Sh}_{\Gamma_5}[16])$.

Now, notice that $\overline{\mathcal{M}}[11]$ is one dimensional and is isomorphic to $\mathbb{P}^1$, and its field of rational function is generated by the canonical j -function.

Let us elaborate on this. Let $t \in \mathbb{P}^1$ be the local parameter, then we can consider the following family of curves:

$$C_t : y^5 = x(x-1)(x-t)$$

$C_t \rightarrow \mathbb{P}_t^1$ is a family of curves with inertia type $(5, 4, (1, 1, 1, 2))$, so it has a morphism to the universal family $C \rightarrow \overline{\mathcal{M}}[11]$:

$$\begin{array}{ccc} C_t & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathbb{P}_t^1 & \longrightarrow & \mathbb{P}_j^1 = \overline{\mathcal{M}}[11] \end{array}$$

And the corresponding map on function field $\mathbb{C}(j) \rightarrow \mathbb{C}(t)$ is given by

$$j(t) = \frac{(t^2 - t + 1)^3}{t^2(t-1)^2} \quad (7.1)$$

By (Li, Mantovan, Pries, and Tang, 2025, Lemma 2.2), $C_{t'} \cong C_t$ iff $j(t') = j(t)$, so indeed $j(t)$ is the canonical generator of the field of rational functions on $\overline{\mathcal{M}}[11]$. Hence, j can be expressed as a rational function of f_2 and f_6 .

Proposition 7.3.7. *Let j be as given in Equation 7.1. Then*

$$j = \frac{3}{2} \cdot \frac{f_2^3}{16f_6 - f_2^3}$$

Proof. We consider j as in $\Gamma_0(\lambda)$, where the coordinates are given as s_{ij} on $\mathbb{P}^5$. At the level of $\widetilde{\mathcal{M}}$ we have

$$\begin{array}{ccc} \widetilde{\mathcal{M}}[11] & \xrightarrow{q} & \overline{M}_{0,4} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}[16] & \xrightarrow{q} & \overline{M}_{0,5} \xrightarrow{|-K_{\text{Bl}_4(\mathbb{P}^2)}|} Y \subseteq \mathbb{P}^5 \end{array}$$

Since f_2, f_6 are all $\mathfrak{S}_5$ invariant, $\frac{3}{2} \cdot \frac{f_2^3}{16f_6 - f_2^3}$ restricted to any of the 10 copies of $\mathbb{P}^1$ in $\overline{M}_{0,5} \cong \text{Bl}_4(\mathbb{P}^2)$ will give us the same result, so we just need to verify that its restriction to L_{P_1, P_2} agrees with j . Here L_{P_1, P_2} is as in (10) of Proposition 3.3.4.

We follow the notations as in the proof of Theorem 7.3.5 and use (x_1, x_2, x_3) for coordinate on $\mathbb{P}^2$. Let $P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0], P_3 = [0 : 0 : 1], P_4 = [1 : 1 : 1]$, and the $\text{Bl}_4(\mathbb{P}^2)$ is the blow up of $\mathbb{P}^2$ at these four points. L_{P_1, P_2} is the strict transformation of the line connecting P_1, P_2 . Also recall that the basis we use for the anti-canonical embedding is

$$s_{ij} = x_i x_j (x_j - x_k), \{i, j, k\} = \{1, 2, 3\}$$

This is how we obtain the 6 coordinates s_{ij} of $\mathbb{P}^5$.

Now, let t be the local coordinate of $L_{P_1, P_2} \cong \mathbb{P}^1$. Since L_{P_1, P_2} is the strict transformation of the line in $\mathbb{P}^2$ connecting P_1, P_2 , we have that $s_{ij}|_{L_{P_1, P_2}}$ is obtained by setting $[x_1 : x_2 : x_3] \rightarrow [1 : t : 0]$. In particular, we get:

$$\{s_{12}, s_{21}, s_{13}, s_{31}, s_{23}, s_{32}\} = \{t^2, t, 0, 0, 0, 0\}$$

$$\{\sigma_{12}, \sigma_{21}, \sigma_{13}, \sigma_{31}, \sigma_{23}, \sigma_{32}\} = \{0, 0, t(1-t), t, t(1-t), t^2\}$$

Hence, we can compute the restriction of f_2, f_6 to L_{P_1, P_2} :

$$\begin{aligned} f_2 &= \sum s_{ij}^2 + \sigma_{ij}^2 = 4t^2(t^2 - t + 1) \\ f_2^3 &= 64t^6(t^2 - t + 1)^3 \\ f_6 &= \sum s_{ij}^6 + \sigma_{ij}^6 = 2t^6(t^6 + 1 + (t-1)^6) \end{aligned}$$

Hence,

$$\frac{f_6}{f_2^3} = \frac{1}{32} \frac{t^6 + 1 + (t-1)^6}{(t^2 - t + 1)^3}$$

Now, recall $j(t) = \frac{(t^2-t+1)^3}{t^2(t-1)^2}$, so $\frac{1}{j(t)} = \frac{t^2(t-1)^2}{(t^2-t+1)^3}$. Let $B = t(t-1)$, we have

$$\begin{aligned} \frac{f_6}{f_2^3} &= \frac{1}{32} \frac{t^6 + 1 + (t-1)^6}{(t^2 - t + 1)^3} \\ &= \frac{1}{32} \frac{(2B+1)(B^2+4B+1)+1}{(B+1)^3} \\ \frac{1}{j} &= \frac{B^2}{(B+1)^3} \\ \implies 16 \frac{f_6}{f_2^3} - 1 &= \frac{(2B+1)(B^2+4B+1)+1-2(B+1)^3}{2(B+1)^3} = \frac{3}{2j} \end{aligned}$$

Hence we get that $j = \frac{3}{2} \cdot \frac{f_2^3}{16f_6 - f_2^3}$ on $\overline{M}[11]$, as desired. $\square$

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Appendix A

APPENDIX

Below is the complete proof of the following proposition:

Proposition 7.2.2. *Let S' be the subalgebra of R generated by $f_2, f_4, f_6, \varepsilon^2$. Then*

$$k(S') = k(R_{\text{ev}}^{\mathfrak{S}_5})$$

Proof. First notice that S' and R_{ev} are integral domains, so their fraction field make sense. Also, by definition, f_{2n} and ε^2 are fixed by $\mathfrak{S}_5$, so $k(S') \subseteq k(R_{\text{ev}}^{\mathfrak{S}_5})$. Since $[k(R_{\text{ev}}^{\mathfrak{S}_5}) : k(R_{\text{ev}})] = \#\mathfrak{S}_5 = 120$ and $k(S') \subseteq k(R_{\text{ev}})$, we want to show that $[k(R_{\text{ev}}^{\mathfrak{S}_5}) : k(R_{\text{ev}})] \geq 120$.

We decompose the field extension into several monogenic field extension:

$$k(S') \xrightarrow{f_1} k(S')[\varepsilon] \xrightarrow{f_2} k(S')[\varepsilon, s_{12}^2] \xrightarrow{f_3} k(S')[\varepsilon, s_{12}^2, s_{31}^2] \xrightarrow{f_4} k(S')[\varepsilon, s_{12}^2, s_{31}^2, s_{23}^2] \rightarrow k(R_{\text{ev}})$$

For each of the field extension, we want to construct the minimal polynomial $f_i(x)$ of the newly added element, and show that $f_i(x)$ has coefficient in the previous field. Finally, we show that $k(R_{\text{ev}}) = k(S')[\varepsilon, s_{12}^2, s_{31}^2, s_{23}^2]$.

$\mathfrak{A}$ First notice that $\mathfrak{S}_5$ has a unique normal subgroup $\mathfrak{A}_5$. If it is the case that $k(R_{\text{ev}})/k(S')$ is a $\mathfrak{S}_5$ -extension, then there is a unique Galois subfield fixed by $\mathfrak{A}_5$. Notice that the stabilizer of ε , denoted as $\text{Stab}(\varepsilon)$, is $\mathfrak{A}_5 \subseteq \mathfrak{S}_5$. Hence, we start with $k(S')[\varepsilon]$.

1. $k(S') \xrightarrow{f_1} k(S')[\varepsilon]$: Notice that the $\mathfrak{S}_5$ -orbit of ε is $\{\varepsilon, -\varepsilon\}$. Hence, we can construct $f_1(x) = x^2 - \varepsilon^2$. $f_1(x)$ the minimal polynomial of ε with coefficient in $k(S')$. Hence, $k(S') \xrightarrow{f_1} k(S')[\varepsilon]$ is a degree 2 extension.
2. $k(S')[\varepsilon] \xrightarrow{f_2} k(S')[\varepsilon, s_{12}^2]$: Notice the $\text{Stab}(s_{12}^2) = \langle (12345), (13)(24) \rangle \cong D_{10} \subseteq \mathfrak{A}_5$. The $\mathfrak{A}_5$ -orbit of s_{12}^2 is $O = \{s_{12}^2, s_{23}^2, s_{31}^2, \sigma_{21}^2, \sigma_{32}^2, \sigma_{13}^2\}$. Its complement orbit $O^c = \{\sigma_{12}^2, \sigma_{23}^2, \sigma_{31}^2, s_{21}^2, s_{32}^2, s_{13}^2\}$. Hence, we can construct $f_2(x) = \prod_{\beta \in O} (x - \beta)$. $f_2(x)$ is of degree 6 and is the minimal polynomial s_{12}^2 .

We want to show that $f_2(x) \in k(S')[\varepsilon][x]$. Write

$$f_2(x) = x^6 + e_1x^5 + e_2x^4 + e_3x^3 + e_4x^2 + e_5x + e_6$$

Here, $e_i = \sum_{S \subseteq O, \#S=i} \prod_{s \in S} s$. Then we want to show that $\{e_1 \dots e_6\} \in K[f_2, f_4, f_6, \varepsilon]$.

To do this, we write $g(i) = \sum_{s \in O} s^i$ and $g'(i) = \sum_{s \in O^c} s^i$. Then e_i can be written as rational combinations of $g(1), \dots, g(i)$, and so $e_i \in k(S')[\varepsilon]$ is equivalent to $g(i) \in k(S')[\varepsilon]$.

- $e_1 \in k(S')[\varepsilon]$: Notice that $g(1) + g'(1) = f_2$. On the other hand, $g(1) - g'(1)$ is of degree 2 in s_{ij} and $\mathfrak{S}_5$ acts on it via sign. Hence, $g(1) - g'(1) = c \cdot \varepsilon^5$ for some $c \in \mathbb{C}$. By plugging in $s_{ij} = x_i x_j (x_j - x_k)$ and comparing the two expressions we have that $g(1) - g'(1) = 6\varepsilon^5$. Hence $g(1) = \frac{1}{2}(f_2 + 6\varepsilon^5) \in k(S')[\varepsilon]$.
- $e_2 \in k(S')[\varepsilon]$: $g(2) + g'(2) = f_4$. We have $g(2) - g'(2) = \varepsilon^5 F(\varepsilon, f_2, f_4, f_6)$, and F should have degree 2. The only monomials of degree 2 is f_2 . By comparing the coefficient we get that $g(2) - g'(2) = 5\varepsilon^5 f_2$. Hence $g(2) = \frac{1}{2}(f_4 + 5\varepsilon^5 f_2)$.
- $e_3 \in k(S')[\varepsilon]$: $g(3) + g'(3) = f_6$. We have $g(3) - g'(3) = \varepsilon^5 F(\varepsilon, f_2, f_4, f_6)$, and F should have degree 4. The only degree 4 monomials are f_2^2 and ε^{10} . By plugging in x_i and comparing coefficients we get that $g(3) - g'(3) = \frac{1}{8}(21f_2^2 \varepsilon^5 + 252\varepsilon^{15})$.
- $e_6 \in k(S')[\varepsilon]$: By plugging in x_i we see that $e_6 = \prod_{s \in O} s = \varepsilon^{30}$.
- $e_5 \in k(S')[\varepsilon]$: $e_5 = \sum_{s \in O} \frac{e_6}{s} = \sum_{s \in O^c} \frac{s}{\varepsilon^5} \cdot e_6 = \frac{g'(1)e_6}{\varepsilon^5}$. Since $g'(1) \in k(S')[\varepsilon]$ we get that $e_5 \in k(S')[\varepsilon]$.
- $e_4 \in k(S')[\varepsilon]$: $e_4 = \sum_{s_i, s_j \in O} \frac{e_6}{s_i s_j} = \varepsilon^{10} \cdot e'_2$. Here $e'_2 = \sum_{S \subseteq O^c, \#S=2} \prod_{s \in S} s$. By (2), since $g'(1), g'(2) \in k(S')[\varepsilon]$, we get that $e'_2 \in k(S')[\varepsilon]$, which implies that $e_4 \in k(S')[\varepsilon]$.

This finishes the proof that $f_2(x) \in k(S')[\varepsilon]$.

3. $k(S')[\varepsilon, s_{12}^2] \xrightarrow{f_3} k(S)[\varepsilon, s_{12}^2, s_{31}^2]$: Notice that $\text{Stab}(s_{31}^2) = \langle (15), (34) \rangle \subseteq D_{10}$, where D_{10} is the Dihedral group of order 10. The D_{10} -orbit of s_{31}^2 is $X_{31} = \{s_{23}^2, s_{31}^2, \sigma_{21}^2, \sigma_{32}^2, \sigma_{13}^2\}$. Hence, we construct $f_3(x) = \prod_{s \in X_{31}} (x - s)$. $f_3(x)$ is of degree 5, and we only need to show that coefficients of $f_3(x) \in k(S')[\varepsilon, s_{12}^2]$. But notice that $f_3(x)(x - s_{21}^2) = f_2(x)$, so the coefficients of $f_3(x)$ are rational combination of s_{21}^2 and coefficients of $f_2(x)$.

4. $k(S')[\varepsilon, s_{12}^2, s_{31}^2] \xrightarrow{f_4} k(S')[\varepsilon, s_{12}^2, s_{31}^2, s_{23}^2]$: The $\langle(15), (34)\rangle$ -orbit of s_{23}^2 is $X_{23} = \{s_{23}^2, \sigma_{21}^2\}$. Hence, we let $f_4(x) = (x - s_{23}^2)(x - \sigma_{21}^2)$. $f_4(x)$ is of degree 2 and we want to show that $a_1 = s_{23}^2 + \sigma_{21}^2, b_1 = s_{23}^2 \cdot \sigma_{21}^2 \in k(S')[\varepsilon, s_{12}^2, s_{31}^2]$.

There is not a particularly good way to find this other than brute force, so we explain how we program the search here.

Notice that a_1 is of degree 2 in s_{ij} and so of degree 6 in x_i . We write

$$V_n = K[f_2(x_i), f_6(x_i), \varepsilon^5(x_i), s_{12}^2(x_i), s_{31}^2(x_i), \sigma_{12}^2(x_i), \sigma_{31}^2(x_i)]_{6n}$$

Notice that $\varepsilon^5(x_i) = \Sigma = x_1 x_2 x_3 (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$. To find a_1 , our goal is to find some n , such that $a_1 V_n \cap V_{n+1} \neq \{0\}$. If such n exists, we take $v \in a_1 V_n \cap V_{n+1}$, and $a_1 = \frac{v_n}{v}$ for some $v_n \in V_n$.

We enumerate through such n . For each n , we compute a generating set of vector for V_n , compute the rank of $a_1 V_n \cap V_{n+1}$, and if it is non-zero then obtain the first vector in the basis of $a_1 V_n \cap V_{n+1}$. All of these can be done by SageMath. Let

$$\begin{aligned} T &= s_{12}^2 + s_{31}^2, U = \sigma_{12}^2 + \sigma_{31}^2 \\ P &= s_{12}^2 s_{31}^2, Q = \sigma_{12}^2 \sigma_{31}^2 \end{aligned}$$

- For a_1 : We found that when $n = 2$, $\dim a_1 V_2 \cap V_3 \neq 0$. We can write $a_1 = \frac{v_3}{v_2}$, where

$$\begin{aligned} v_3 &= -\frac{25}{6}f_6 + \frac{25}{96}f_2^3 + \frac{25}{16}f_2^2 T \\ &+ f_2 \left(\frac{725}{8}\Sigma^2 + \frac{25}{4}\Sigma T - \frac{25}{2}T^2 + \frac{29}{2}P + \frac{25}{2}Q \right) \\ &+ \Sigma(-17\Sigma^2 - \frac{55}{4}\Sigma T - 121\Sigma U - 71T^2 \\ &+ 21TU - 25U^2 + 71P - 25Q) \\ &+ 25T^3 - 54PT \end{aligned}$$

$$\begin{aligned} v_2 &= \frac{29}{16}f_2^2 + f_2 \left(\frac{21}{4}\Sigma - \frac{29}{2}(T + U) \right) + \frac{145}{4}\Sigma^2 - 75\Sigma T \\ &+ 12\Sigma U + 29(T^2 + TU + U^2) - 25P - 4Q \end{aligned}$$

- For b_1 : We find that when $n = 2$, $\dim(b_1 V_2 \cap V_4) \neq 0$. We can write $b_1 = \frac{w_4}{w_2}$, where

$$\begin{aligned}
w_2 &= -\frac{35}{16}f_2^2 + f_2 \left(\frac{5}{4}\Sigma + \frac{39}{4}T + 21U \right) \\
&\quad - \frac{251}{4}\Sigma^2 + \frac{131}{2}\Sigma T - 103\Sigma U \\
&\quad - 4T^2 - TU - 49U^2 + 34P + 39Q \\
w_4 &= \frac{7}{32}f_2^3 (\sigma_{12}^2 - s_{31}^2) \\
&\quad + f_2^2 \left(-\frac{7}{8}\Sigma^2 + \frac{1}{8}\Sigma (5s_{31}^2 - 19\sigma_{12}^2) + \frac{7}{4}s_{31}^2 T \right. \\
&\quad \quad \left. - \frac{1}{8}\sigma_{12}^2 (5s_{31}^2 + 9\sigma_{12}^2 + 7\sigma_{31}^2) \right) \\
&\quad + f_2 \left(6\Sigma^3 + \frac{1}{8}\Sigma^2 (-6s_{12}^2 + 47s_{31}^2 + 117\sigma_{12}^2 + 34\sigma_{31}^2) \right. \\
&\quad \quad + \frac{1}{2}\Sigma (-3P - 3s_{31}^4 - 5s_{31}^2\sigma_{12}^2 + 40\sigma_{12}^4 + 19Q) \\
&\quad \quad \left. - \frac{7}{2}s_{31}^2 (s_{12}^4 + P + s_{31}^4) + 5s_{31}^4\sigma_{12}^2 - \frac{1}{2}\sigma_{12}^4 (3\sigma_{12}^2 + 10\sigma_{31}^2) \right) \\
&\quad + \frac{1}{2}\Sigma^3 (-111\Sigma - 115s_{12}^2 + 5s_{31}^2 - 5\sigma_{12}^2 + 55\sigma_{31}^2) \\
&\quad + \Sigma^2 \left(17s_{12}^4 - 35P - 31s_{12}^2\sigma_{31}^2 - 65s_{31}^4 - \frac{87}{2}s_{31}^2\sigma_{12}^2 \right. \\
&\quad \quad \left. - \frac{91}{2}\sigma_{12}^4 - \frac{25}{2}Q + 14\sigma_{31}^4 \right) \\
&\quad + 6\Sigma \left(4s_{31}^2 (s_{12}^4 + P + s_{31}^4) + 5s_{31}^4\sigma_{12}^2 + \sigma_{12}^4 (\sigma_{12}^2 + 5\sigma_{31}^2) \right) \\
&\quad + 10 \left(-s_{31}^6\sigma_{12}^2 + \sigma_{12}^8 + \sigma_{12}^6\sigma_{31}^2 \right)
\end{aligned}$$

These shows that $a_1, b_1 \in k(S')[\varepsilon, s_{12}^2, s_{31}^2]$.

5. $k(S')[\varepsilon, s_{12}^2, s_{31}^2, s_{23}^2] = k(R_{\text{ev}})$: We show that we can generate all the combinations $s_{ij}s_{i'j'}$ as rational combinations with coefficients in $f_2, f_6, \Sigma, s_{12}^2, s_{31}^2, s_{23}^2, \sigma_{12}^2, \sigma_{31}^2, \sigma_{23}^2$. By adjoining s_{23}^2 , we also get s_{21}^2, σ_{21}^2 . For this part we ask Chatgpt to directly obtain these linear combinations.

- $s_{12}s_{31}$:

$$s_{12}s_{31} = \frac{n(s_{12}s_{31})}{d(s_{12}s_{31})}$$

$$n(s_{12}s_{31}) = -\left(s_{12}^4 s_{31}^2 \sigma_{21}^2 - s_{12}^2 s_{31}^2 \sigma_{21}^4 + s_{12}^2 s_{31}^4 \sigma_{21}^2 + 2s_{12}^2 s_{31}^2 \sigma_{21}^2 \Sigma - s_{12}^2 s_{31}^2 \Sigma^2 + s_{12}^2 \sigma_{21}^2 \Sigma^2 + s_{31}^2 \sigma_{21}^2 \Sigma^2 - \Sigma^4\right),$$

$$d(s_{12}s_{31}) = 2\left(s_{12}^2 s_{31}^2 \sigma_{21}^2 + s_{12}^2 \sigma_{21}^2 \Sigma + s_{31}^2 \sigma_{21}^2 \Sigma - \Sigma^3\right).$$

- $s_{12}s_{21}$:

$$s_{12}s_{21} = \frac{n(s_{12}s_{21})}{s_{12}s_{21}}$$

$$n(s_{12}s_{21}) = \Sigma\left(s_{12}^4 s_{31}^2 \sigma_{21}^2 - s_{12}^2 s_{31}^4 \sigma_{21}^2 + s_{12}^2 s_{31}^2 \sigma_{21}^4 + 2s_{12}^2 s_{31}^2 \sigma_{21}^2 \Sigma + s_{12}^2 s_{31}^2 \Sigma^2 - s_{12}^2 \sigma_{21}^2 \Sigma^2 + s_{31}^2 \sigma_{21}^2 \Sigma^2 - \Sigma^4\right),$$

$$d(s_{12}s_{21}) = 2\sigma_{21}^2\left(s_{12}^2 s_{31}^2 \sigma_{21}^2 + s_{12}^2 s_{31}^2 \Sigma + s_{31}^2 \sigma_{21}^2 \Sigma - \Sigma^3\right).$$

- $s_{21}s_{31} = \frac{(s_{12}s_{21})(s_{12}s_{31})}{s_{12}^2}$

- $s_{12}s_{23} = \frac{\Sigma s_{12}^2}{s_{12}s_{21} - s_{12}^2 - s_{12}s_{31}}$

- $s_{21}s_{23} = \frac{\Sigma(s_{12}s_{21})}{s_{12}s_{21} - s_{12}^2 - s_{12}s_{31}}$

- $s_{31}s_{23} = \frac{\Sigma(s_{12}s_{31})}{s_{12}s_{21} - s_{12}^2 - s_{12}s_{31}}$

- $s_{12}s_{13} = \Sigma + s_{12}s_{31} + s_{12}s_{23}$

- $s_{21}s_{13} = \frac{(s_{12}s_{21})(s_{12}s_{13})}{s_{12}^2}$

- $s_{31}s_{13} = \frac{(s_{12}s_{31})(s_{12}s_{13})}{s_{12}^2}$

- $s_{23}s_{13} = \frac{(s_{12}s_{23})(s_{12}s_{13})}{s_{12}^2}$

- $s_{21}s_{32} = \Sigma - s_{21}s_{13} + s_{21}s_{23}$

- $s_{12}s_{32} = \frac{(s_{12}s_{21})(s_{21}s_{32})}{s_{21}^2}$

- $s_{31}s_{32} = \frac{(s_{21}s_{31})(s_{21}s_{32})}{s_{21}^2}$

- $s_{23}s_{32} = \frac{(s_{21}s_{23})(s_{21}s_{32})}{s_{21}^2}$

- $s_{13}s_{32} = \frac{(s_{21}s_{13})(s_{21}s_{32})}{s_{21}^2}$

- s_{13}^2 :

$$s_{13}^2 = \frac{\Sigma^2 + s_{12}^2 s_{31}^2 + s_{12}^2 s_{23}^2 + 2\Sigma(s_{12}s_{31}) + 2\Sigma(s_{12}s_{23}) + 2s_{12}^2(s_{31}s_{23})}{s_{12}^2},$$

• s_{32}^2 :

$$s_{32}^2 = \frac{\Sigma^2 + s_{21}^2 s_{13}^2 + s_{21}^2 s_{23}^2 - 2\Sigma(s_{21}s_{13}) + 2\Sigma(s_{21}s_{23}) - 2s_{21}^2(s_{13}s_{23})}{s_{21}^2}.$$

□

