

Geometric Constraints on CFT-Like Data at Genus 0, 1, and 2

Thesis by
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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

The Caltech logo, featuring the word "Caltech" in a bold, orange, sans-serif font.

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2026

(Defended May 8, 2026)

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ACKNOWLEDGEMENTS

This thesis marks the end of my PhD, one of the most transformative journeys in my life. I am grateful to the many people who made it possible. I have benefited greatly from the guidance, encouragement, and generosity of mentors, collaborators, colleagues, friends, and family throughout my time in graduate school. Their support not only shaped the work presented here, but also the way I think, learn, and approach challenges.

First and foremost, I am deeply grateful to my advisor, David Simmons-Duffin, for sharing with me not only his knowledge of science, but also the pleasure of scientific research. Discussions with David have always been enjoyable. I have been continually inspired by his breadth of knowledge, clarity of thought, and technical strength in bringing creative ideas to life. Apart from his academic guidance, I am also thankful for his kindness, which carried me through some of the most difficult stretches of my graduate journey. His mentorship has not only shaped my scientific thinking, but also encouraged me to strive toward becoming a better human being.

I owe special thanks to my collaborator Sridip Pal for generously sharing his knowledge of physics with me and for the many discussions we have had, some in the office, many in restaurants, and others simply on the streets. A few of these conversations stretched from evening into the early morning hours. Sridip's dedication and passion for physics have influenced me deeply and reminded me of what I truly value.

In addition to Sridip, I also had the pleasure of working with Nathan Benjamin, Elliott Gesteau, and Jaeha Lee on various projects that made this thesis possible. I learned a great deal from our discussions, from the process of developing ideas together, and from the many calculations, drafts, and revisions we shared along the way.

I would also like to thank my fellow graduate students in Downs-Lauritsen, who

have been an essential part of my graduate experience. Many thanks to Jaeha Lee and Maria Derda for always remembering everyone's birthday and for organizing social activities that made graduate school feel less lonely, and to Jerry Zhang for his photography skills and for the laughter he brought to many offices on the 4th floor through jokes and brain teasers. I am especially grateful to my officemate Maria Derda, whose companionship made many days in the office brighter. I have cherished our small conversations, our shared reflections on the struggles and uncertainties of graduate school, and the kindness she showed me when I was feeling low. I am thankful for our walk at the Huntington Library, for the wonderful puzzle book she gave me on my birthday, and for teaching me Polish. I am also grateful to the organizers and participants of our journal clubs, Joonhwi Kim, Cyuan-Han Chang, Yakov Landau, Maria Derda, Mrunmay Jagadale, and Jaeha Lee, for many stimulating discussions. Finally, thank you to Mrunmay, Yufeng, Yanky, Maria, Jerry, and Jaeha for the board game nights; I very much enjoyed our battles over Catan.

I am grateful to Prof. Hiroshi Ooguri, Prof. Yifan Wang, and Prof. Matilde Marcolli for serving on my PhD committee, and for their time, guidance, and thoughtful feedback throughout this process. Many thanks to Carol Silberstein for her help with many administrative matters over the years, and to Nam Ung and Darrell Peterson for providing support along the way. I would also like to thank the therapists and staff at Caltech SWS, especially my therapist, whose kindness and warmth grounded me in a crucial period of graduate school. Their care has meant more to me than I can adequately express.

I am also deeply grateful to my friends outside Caltech, many of whom have known me since middle school or college. Although we now live in different parts of the world, their friendship has remained a steady source of comfort and strength. They have been there for me when I needed them, across time zones and distances, reminding me that I am cared for, that my well-being matters, and that I am worthy of love and

kindness. I am grateful for their generosity and the many years of shared memories that continue to sustain me.

Finally, I owe my deepest gratitude to my parents and my family. Their love and support have been the foundation beneath everything I have done. I am especially grateful to my grandmother, whose love and influence have stayed with me throughout my life. She always enjoyed a good math puzzle, and her patience and determination in working through confusion have remained a source of inspiration for me. Whenever I am stuck, I think of her; her love, curiosity, and quiet strength will stay with me forever.

ABSTRACT

This thesis studies how self-consistency constraints on conformal field theory like observables can be used to extract information about CFT or geometric data. The observables considered are associated with genus-0, genus-1, and genus-2 geometries: the four-punctured sphere, the torus $S^1 \times S^{d-1}$, and the genus-2 manifold $(S^1 \times S^{d-1})\sharp(S^1 \times S^{d-1})$. Although these observables differ in geometric and analytic complexity, they are united by a common principle: the same observable admits multiple equivalent decompositions, and the requirement that these descriptions agree imposes nontrivial constraints on the underlying spectrum and operator product expansion.

The first part develops a hyperbolic bootstrap for compact hyperbolic spin surfaces. Using an analogy between harmonic analysis on hyperbolic manifolds and the conformal bootstrap, we derive rigorous bounds on Laplacian and Dirac spectral gaps by combining representation-theoretic methods with semidefinite programming. The second part studies thermal partition functions with angular twists. We show that, even in the absence of full modular invariance in dimensions greater than two, the high-temperature behavior of these observables is controlled by thermal effective field theory, leading to universal relations among twisted partition functions. The third part studies the genus-2 partition function in conformal field theory. By equating different decompositions of this observable, we derive a genus-2 crossing equation and use it to relate thermal one-point data to heavy operator product coefficients.

PUBLISHED CONTENT AND CONTRIBUTIONS

- [Ben+24] Nathan Benjamin et al. “Angular fractals in thermal QFT”. In: *JHEP* 11 (2024), p. 134. DOI: 10.1007/JHEP11(2024)134. URL: [https://doi.org/10.1007/JHEP11\(2024\)134](https://doi.org/10.1007/JHEP11(2024)134).
Included as Chapter 2. I performed the computation presented in section 9, appendix B and appendix D. I also contributed to the writing of these sections. For the remaining part of the paper, I participated in discussions that shaped the development of the results.
- [Ges+25] Elliott Gesteau et al. “Bounds on spectral gaps of hyperbolic spin surfaces”. In: *Journal of the Association for Mathematical Research* 3.1 (2025). DOI: 10.56994/JAMR.003.001.003. URL: <https://doi.org/10.56994/JAMR.003.001.003>.
Included as Chapter 1. I conducted the numerical bootstrap analysis and developed a Mathematica package for computing spectra of hyperbolic surfaces using Selberg trace formula. I also contributed to the writing of section 3, 4 and appendix B.
- [SX25] David Simmons-Duffin and Yixin Xu. *A genus-2 crossing equation in $d \geq 2$* . Manuscript draft, CALT-TH 2025-036. 2025.
Included as Chapter 3. I performed all the computations presented in this paper and wrote the initial draft. The final version of the draft benefited from discussions with David Simmons-Duffin.

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SUMMARY INTRODUCTION

0.1 Why CFT?

Conformal field theory (CFT) occupies a central place in modern theoretical physics. In statistical mechanics, CFTs describe the universal long-distance behavior of systems at continuous phase transitions [Pol70; BPZ84; DMS97; Ryc17; Sim16]. In high-energy theory, through the AdS/CFT correspondence, CFTs provide nonperturbative descriptions of certain gravitational theories in anti-de Sitter space [Mal98; GKP98; Wit98]. More broadly, CFTs appear as fixed points of RG flows in the space of quantum field theories. Local operators describe directions in the neighborhood of such fixed points, and in special cases deformations by various operators can lead to rich structures such as conformal manifolds, integrable massive theories, or RG flows between distinct fixed points [LS95; Zam89; Zam87]. Monotonicity theorems in various dimensions further constrain this global picture by restricting which fixed points can be connected by unitary RG flows [Zam86; KS11; CHM11].

Part of the appeal of CFT is that it sits at a useful point between rigidity, richness, and accessibility. On the one hand, conformal symmetry is much larger than the Poincare symmetry of an ordinary relativistic quantum field theory. This enlarged symmetry strongly constrains correlation functions and makes many nonperturbative questions accessible. On the other hand, CFTs are not a narrow class of theories: they include free theories and strongly interacting theories; they can be integrable or chaotic. Moreover, CFTs are not merely formal objects. Their predictions can be compared with Monte Carlo simulations, Hamiltonian and fuzzy-sphere regularizations, and experiments on systems near criticality [CVP15; YZ90; Ana+17; HHZ24; HM76]. Thus CFTs are constrained enough to be studied systematically, broad enough to capture many physical phenomena, and concrete enough to make contact with numerical and experimental data.

In this thesis, we explore several ways in which rigidity can be converted into concrete constraints on a CFT.

0.2 What Specifies a CFT?

Before turning to the specific problems studied in this thesis, let us recall what it means to specify a conformal field theory. A useful guiding principle, in the spirit of Wigner’s classification of particles by unitary irreducible representations of spacetime symmetries, is that the degrees of freedom of a quantum theory should be organized by the appropriate symmetry algebra [Wig39]. The CFT data are intrinsic to the theory and do not depend on a choice of background. In practice, these data are often described through local observables on flat space, or S^d . The conformal transformations preserving this structure generate the conformal algebra: in dimensions $d > 2$, this is $\mathfrak{so}(d+1, 1)$ in Euclidean signature and $\mathfrak{so}(d, 2)$ in Lorentzian signature [Ryc17; Sim16]. Two dimensions are special: local conformal transformations enhance the global conformal algebra to two copies of the Virasoro algebra [BPZ84].

Local operators are therefore organized into representations of the conformal algebra, labeled by quantum numbers such as scaling dimension, spin, and possible internal symmetry charges. In radial quantization, a local operator inserted at the origin creates a state on S^{d-1} . Primary operators are those whose corresponding states are highest-weight states of the conformal algebra: they are eigenstates of the dilatation operator and are annihilated by the special conformal transformations. Descendant operators are obtained by acting with translations. The insertion of an operator at a general point is obtained by translating the operator at the origin,

$$\mathcal{O}(x) = e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P}.$$

Thus the local operator spectrum decomposes into conformal multiplets, each consisting of a primary operator and its descendants [Ryc17; Sim16].

In addition to the labeling of each individual local operator, another set of the CFT data determines, loosely speaking, the way in which local operators “fuse” into each other. These are the operator product expansion coefficients. There are two different ways to introduce the appearance of such coefficients into the subject. First, conformal symmetry fixes the form of the three-point function between scalar primary operators up to a single normalization constant:

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle = \frac{C_{ijk}}{(x_1 - x_2)^{\Delta_1 + \Delta_2 - \Delta_3} (x_1 - x_3)^{\Delta_1 + \Delta_3 - \Delta_2} (x_2 - x_3)^{\Delta_2 + \Delta_3 - \Delta_1}}.$$

The normalization constant C_{ijk} is precisely the OPE coefficient associated with the three scalar primary operators. A second, more conceptual way to understand these coefficients comes from radial quantization and the state-operator correspondence. In a CFT, the path integral over a ball B^d prepares a state on the boundary sphere S^{d-1} . Local operator insertions inside the ball therefore determine states in the Hilbert space on the sphere. Moreover, by the state-operator correspondence, this Hilbert space is organized by states created by local operators inserted at the origin. Thus, the state prepared by adding two insertions inside a ball can be re-expanded in this operator-state basis. This expansion is the operator product expansion. Schematically,

$$\mathcal{O}_i(x) \mathcal{O}_j(0) = \sum_k C_{ijk}(x, \partial) \mathcal{O}_k(0),$$

where $C_{ijk}(x, \partial)$ is a differential operator fixed by symmetry, up to an OPE coefficient.

We have introduced two sets of CFT data: the spectrum $\{(\Delta_i, \rho_i)\}$ and the OPE coefficients $\{C_{ijk}\}$ between them. This set of data allows us, in principle, to reconstruct many CFT observables such as multi-point correlation functions, torus partition functions, and, perhaps more exotically, the genus-2 partition function. Taking the four-point function as an example, successive operator products allow us to eventually write the correlator in the form

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}} G_{\mathcal{O}}(x_i),$$

where $G_{\mathcal{O}}(x_i)$ is a function completely fixed by symmetry. The same logic extends beyond four-point functions. Higher-point correlators may be decomposed by applying the OPE successively, producing a sum over intermediate conformal multiplets associated with the internal edges of an OPE diagram. For example, a five-point function involves two exchanged primaries and is weighted by products of three OPE coefficients, while a six-point function can already be decomposed in inequivalent topologies, such as the comb channel and the snowflake channel [Ros19; FMS19; FMS20]. In general, the theory-dependent input remains the same CFT data: the spectrum specifies which conformal multiplets may propagate along the internal lines, and the OPE coefficients specify how these multiplets are joined at the vertices. The remaining dependence on the insertion points is captured by higher-point conformal blocks, which are fixed by conformal symmetry once the external and exchanged representations are specified [DO04; Sim16; Ros19; FMS19; FMS20].

Another interesting class of observables is thermal observables, namely correlation functions defined on the thermal background $S^1_{\beta} \times S^{d-1}$. These are of particular interest because they probe the CFT in a finite-temperature state, and are therefore much closer to observables accessible in experiments on critical systems. In particular, thermal one-point functions on $S^1_{\beta} \times S^{d-1}$ admit a natural spectral expansion,

$$\langle \mathcal{O} \rangle_{\beta} = \frac{1}{Z(\beta)} \text{Tr} (e^{-\beta H} \mathcal{O}),$$

where the trace is taken over the Hilbert space on S^{d-1} . By the state-operator correspondence, this Hilbert space is identified with local operators of the CFT, so the trace becomes a Boltzmann-weighted sum over conformal families, while the matrix elements $\langle i | \mathcal{O} | i \rangle$ are determined by the three-point functions $\langle \mathcal{O}_i^{\dagger} \mathcal{O} \mathcal{O}_i \rangle$, or equivalently by the OPE coefficients. Thus thermal one-point functions on $S^1_{\beta} \times S^{d-1}$ are controlled by the ordinary CFT data, namely the spectrum together with the OPE coefficients. In the high-temperature limit in which $S^1_{\beta} \times S^{d-1}$ degenerates to $S^1_{\beta} \times \mathbb{R}^{d-1}$, transla-

tion invariance implies that the one-point function of, for example, a scalar primary is position-independent. Dimensional analysis then fixes its dependence on the thermal scale to take the form

$$\langle \mathcal{O} \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}.$$

It is therefore often natural in this noncompact thermal geometry to package the thermal data into the coefficient $b_{\mathcal{O}}$. Although $b_{\mathcal{O}}$ is, strictly speaking, not independent of the underlying OPE data, it provides a particularly convenient rewrite of that information [Ili+18; Gob+19].

At this stage, it is worth emphasizing that the inclusion of genuinely non-local operators can introduce new data beyond the local CFT data discussed above. A standard example is provided by conformal defects: in the presence of a defect, correlation functions are no longer controlled solely by the spectrum of local operators and their OPE coefficients, but also by additional data describing how bulk operators couple to the defect, together with the operator content living intrinsically on the defect itself. Thus, once one enlarges the class of observables beyond local operator insertions, the data required to characterize the theory become correspondingly richer. Although such extended observables have been studied extensively in the literature, they will not play a central role in this thesis [Bil+16; LM20].

0.3 Rigidity of CFT Data

The CFT data are not arbitrary. Their consistency is tested by the fact that the same observable can often be assembled in more than one way. Take a four-point correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$ as an example: by applying the operator product expansion to different pairs of operators, one may decompose the same correlator into conformal blocks in different channels. For instance, one may first fuse $\mathcal{O}_1$ with $\mathcal{O}_2$, or instead $\mathcal{O}_1$ with $\mathcal{O}_4$. These two decompositions correspond to the s - and t -channel expansions, and consistency requires them to agree. This is the basic mechanism of the conformal

bootstrap [FGG73; Pol74; Rat+08].

Simple as it is, the conformal bootstrap has produced striking successes on both numerical and analytic fronts. Numerically, crossing symmetry and unitarity can be recast as a convex optimization problem. This approach has led to rigorous exclusion bounds on operator dimensions and OPE coefficients, and in favorable cases to small “islands” in the space of CFT data. A celebrated example is the three-dimensional Ising CFT, whose leading scaling dimensions and OPE coefficients have been determined to very high precision using mixed-correlator bootstrap methods [KPS14; Kos+16]. More generally, similar techniques have been applied to $O(N)$ vector models and other universality classes, giving nonperturbative access to strongly coupled fixed points. Analytically, crossing symmetry also reveals universal structures in appropriate kinematic limits. In particular, the lightcone bootstrap implies the existence of large-spin double-twist families, whose twists approach sums of lower twists, and large-spin perturbation theory organizes their anomalous dimensions and OPE coefficients in a controlled expansion [Fit+13; KZ13; ABL15].

The same bootstrap philosophy also applies to partition functions. In two-dimensional CFTs, modular invariance of the torus partition function expresses the equality of two descriptions of the same path integral, and therefore imposes constraints on the spectrum of states. These constraints lead both to bounds on the low-lying spectrum and to universal Cardy-type asymptotics for the density of high-energy states [Car86; Hel11; HKS14].

The work in this thesis is motivated by this broader view of bootstrap consistency. In recent years, a striking parallel has emerged between the harmonic analysis of hyperbolic geometry and the conformal bootstrap of one-dimensional CFTs [KMP24; BMP23; BH20]. In this analogy, spectral data of differential operators on hyperbolic manifolds play a role analogous to operator dimensions, while certain overlap integrals

of eigenfunctions play a role analogous to OPE coefficients. The associativity of pointwise multiplication of functions then leads to identities structurally similar to crossing equations for four-point functions. Utilizing this parallel, one can adapt familiar numerical bootstrap techniques to obtain rigorous bounds on spectral gaps of various geometric differential operators. This is the main subject of Chapter 1, where we apply this hyperbolic bootstrap philosophy to compact hyperbolic spin surfaces and derive bounds on Laplacian and Dirac spectra [Ges+25].

Chapters 2 and 3 return to conformal field theories, with an emphasis on dimensions greater than two. In two-dimensional CFTs, modular invariance of the torus partition function famously relates the low-temperature and high-temperature behavior of the theory [Car86]. More generally, it implies hidden relations among partition functions with different twists or choices of quantization [Ben+19]. At first sight, one might expect such relations to be special to two dimensions, where the modular group acts geometrically on the torus. However, analogous relations appear in a variety of higher-dimensional examples, and even in more general QFTs. Chapter 2 explains the origin of this phenomenon for thermal partition functions with angular twists. The key idea is that, although full modular invariance is absent in higher dimensions, the high-temperature expansion is still constrained by thermal effective field theory. This effective description reveals universal structures in the angular dependence of the free energy [Ben+24].

Finally, Chapter 3 studies a more exotic CFT observable: the genus-2 partition function. This observable provides a natural higher-genus analogue of the usual bootstrap equations. Just as a four-point function can be decomposed in different OPE channels, the genus-2 partition function can be assembled by cutting and gluing the underlying geometry in inequivalent ways. Equating these different decompositions gives a genus-2 crossing equation valid in $d \geq 2$. In an appropriate asymptotic regime, this crossing equation relates thermal one-point coefficients to averaged heavy OPE coef-

ficients. Thus Chapter 3 provides another instance of the same organizing principle: CFT data are constrained because the same path integral admits multiple equivalent decompositions [SX25].

0.4 Thermal EFT

Hydrodynamic effective theory describes the long-distance dynamics of finite-temperature QFTs near local thermal equilibrium, using slowly varying variables such as temperature, velocity, and chemical potentials. In exact equilibrium, these fields are time-independent with respect to a timelike Killing flow, and dissipative terms vanish. Wick rotating along this time direction and imposing thermal periodicity gives a Euclidean theory on $S^1_\beta \times \Sigma^{d-1}$. The resulting thermal effective action is a local functional of the spatial geometry and background sources, organized in a derivative expansion compared to the thermal scale [Ban+12; Jen+12; Ben+24].

For a CFT, this structure simplifies because the temperature is the only dimensionful scale. Scale invariance therefore fixes how each term in the thermal effective action depends on temperature, leaving dimensionless Wilson coefficients that encode theory-dependent thermal data. On $S^1_\beta \times S^{d-1}$, the high-temperature partition function can then be organized as an expansion in powers of β/R , where R is the radius of the sphere. The leading term is controlled by the flat-space thermal free-energy coefficient, while subleading terms describe the response of the thermal state to the curvature of the spatial sphere and to other possible background sources. Although these Wilson coefficients are not themselves the usual vacuum CFT data, they control asymptotic features of the spectrum and, in the case of the genus-2 partition function, averaged heavy OPE coefficients. This perspective will be useful in Chapter 2, where angularly twisted thermal partition functions reveal universal spin-refined structures in higher-dimensional CFTs [Ben+24], and again in Chapter 3, where thermal one-point data enter a genus-2 crossing equation and are related to averaged heavy OPE

coefficients [SX25].

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BOUNDS ON SPECTRAL GAPS OF HYPERBOLIC SPIN SURFACES

- [Ges+25] Elliott Gesteau et al. “Bounds on spectral gaps of hyperbolic spin surfaces”. In: *Journal of the Association for Mathematical Research* 3.1 (2025). DOI: 10.56994/JAMR.003.001.003. URL: <https://doi.org/10.56994/JAMR.003.001.003>.

1.1 Introduction

In this paper, we derive new upper bounds on the first nonzero eigenvalue of Laplace and Dirac operators on compact orientable hyperbolic surfaces and orbifolds with spin. Our approach is inspired by a technique known in high energy physics as the conformal bootstrap [Rat+08; Sim17; Kos+16; PS16; PRV19]. Bootstrap methods have already been successfully applied to bound the first nonzero eigenvalue of the Laplace operator on hyperbolic surfaces [KMP24; Bon22a; Bon22b] and hyperbolic 3-manifolds [BMP23], see also [BH19; BH20]. A remarkable achievement of [KMP24; Bon22a] is that for several low genera, it improves on the Yang–Yau bounds [YY80a] and its successors [IE83; ROS22; KV22], as well as the bounds of [Hub80], which collectively had been the strongest bounds available for decades. (In the large genus limit, [Hub80] gives the optimal bound, as was recently shown in [HM23].)

Recently, [BP23] used linear programming methods and the Selberg trace formula (STF) to obtain new bounds on Laplacian eigenvalues and other geometric quantities for hyperbolic surfaces. These bounds are stronger than the ones of [KMP24] for genus $g > 6$. However, the technique of [BP23] cannot immediately be extended to the Dirac operator due to a lack of positivity in the STF with spin. Furthermore, [BP23] treats orbifolds and manifolds on a separate footing, while [KMP24] treats

them uniformly. In this paper, we follow the methods of [KMP24].

The basic idea of [KMP24] is to leverage the fact that the orthonormal frame bundle of any compact orientable hyperbolic surface or orbifold can be written as $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$, where Γ is a cocompact Fuchsian group. Constraints on spectra come from combining the associativity of pointwise multiplication of functions on $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ with the representation theory of $\mathrm{PSL}(2, \mathbb{R})$. By decomposing functions into irreducible representations of $\mathrm{PSL}(2, \mathbb{R})$, and imposing associativity of their pointwise products, one obtains an infinite set of constraint equations on the eigenvalues of the Laplace operator. Linear programming techniques applied to these equations yield bounds on the spectrum.

We adapt this method to study surfaces and orbifolds with spin structure as follows. Two operators are relevant in this context: the Laplace operator, which is insensitive to the spin structure, and the Dirac operator, which depends on the spin structure. In group-theoretic language, adding a spin structure amounts to considering a quotient of the form $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$, where $\tilde{\Gamma}$ is a lift of the cocompact Fuchsian group Γ to $\mathrm{SL}(2, \mathbb{R})$. Each lift corresponds to a spin structure. We can then study the interplay between associativity of pointwise multiplication of functions on $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$ and the representation theory of $\mathrm{SL}(2, \mathbb{R})$. By decomposing functions into irreducible representations of $\mathrm{SL}(2, \mathbb{R})$, and imposing associativity of their pointwise products, we derive spectral identities, which are amenable to linear/semidefinite programming.

Our bounds depend on the number of linearly-independent modular forms of various weights possessed by a surface. For forms of weight 2 or higher, the Riemann–Roch theorem determines their number from purely topological information: the genus and the orders of orbifold points. The situation is different for weight 1 forms (which correspond to harmonic spinors): the number of weight 1 forms can be different for surfaces with the same topology but different geometries. For example, except in

genus 4 and 6, a surface carries the maximal possible number of weight 1 modular forms if and only if it is hyperelliptic. This allows us to obtain stronger bounds for hyperelliptic surfaces.

Among our results, we prove the following theorems:

Theorem 1.1.1 (Universal upper bound on Laplacian eigenvalue: I). *Given a compact orientable hyperbolic spin orbifold X , the first non-zero eigenvalue of the Laplacian operator, $\lambda_1^{(0)}(X)$ satisfies*

$$\lambda_1^{(0)}(X) < 12.137980.$$

Remark 1.1.2. The above bound is nearly saturated by $[0; 3, 3, 5]$, a genus 0 orbifold with three orbifold singularities of order 3, 3 and 5. We have $\lambda_1^{(0)}([0; 3, 3, 5]) \approx 12.13623$.¹

Theorem 1.1.3 (Universal upper bound on Laplacian eigenvalue: II). *Given a compact orientable hyperbolic spin orbifold X admitting a harmonic spinor, the first non-zero eigenvalue of the Laplacian operator, $\lambda_1^{(0)}(X)$ satisfies*

$$\lambda_1^{(0)}(X) < 4.763782.$$

Remark 1.1.4. The above bound is nearly saturated by the most symmetric point (see fig. 1.1) of the moduli space of $[1; 3]$ i.e a torus with one orbifold singularity of order 3. Let us call the orbifold $[1; 3]_{sym}$. Using **FreeFEM++**, we learn that $\lambda_1^{(0)}([1; 3]_{sym}) \approx 4.7609$.

More generally, we obtain two types of bounds. The first type are *exclusion plots* in a two-dimensional parameter space, labelled by the first nonzero eigenvalues of

¹See Appendix 1.8 for the estimate of the interval $[a_1, b_1]$, which contains $\lambda_1^{(0)}([0; 3, 3, 5])$.

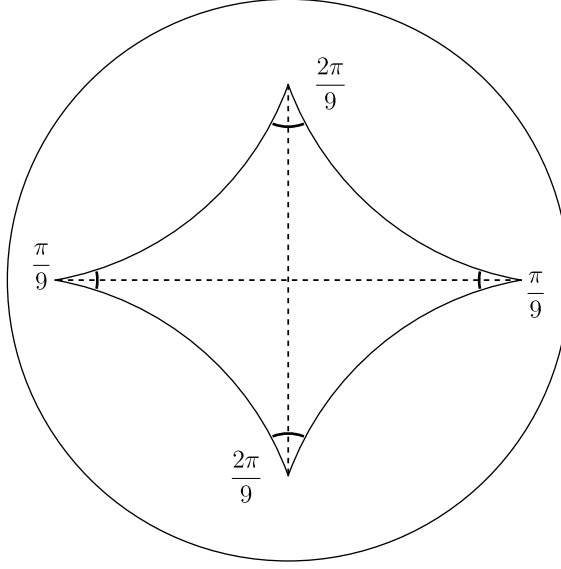


Figure 1.1: A fundamental domain of Γ drawn on the Poincaré disc where Γ is a subgroup of $\text{PSL}(2, \mathbb{R})$ isomorphic to the fundamental group of the most symmetric point in the moduli space of $[1; 3]$. This fundamental domain is a hyperbolic quadrilateral symmetric under reflections against the two dashed lines.

the Laplace and Dirac operators. We use $\lambda_1^{(0)}$ to denote the first non-zero Laplacian eigenvalue. For the first non-zero Dirac eigenvalue $t_1^{(1/2)}$, we use the variable $\lambda_1^{(1/2)} := 1/4 + (t_1^{(1/2)})^2$ in the plots. Examples of such plots are given on Figures 1.2, 1.8 and 1.10. Let us focus on Figure 1.2. For any hyperbolic spin orbifold X , equipped with a spin structure such that there is no harmonic spinor, the pair $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ must lie in the union of the pink-shaded and yellow-shaded regions. If the hyperbolic spin orbifold X has genus 1 or more and is equipped with a spin structure admitting one or more harmonic spinors, then $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ must lie in the pink shaded region. The yellow shaded region as well as the pink one has a kink at rightmost corner. Exclusion plots with kinks are abundant in the conformal bootstrap literature. Often, one can identify the object that lives near the kink. Here a similar phenomenon is true. Indeed, the corner point in the yellow-shaded region is very close to $(\lambda_1^{(0)}([0; 3, 3, 5]), \lambda_1^{(1/2)}([0; 3, 3, 5]))$. See Figure 1.7 for a zoomed-in version of the exclusion plot. For the pink shaded region, the corner point

is close to $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$, where X is the most symmetric point of the moduli space of $[1; 3]$, equipped with the odd spin structure. See Table 1.3 in Appendix 1.8 for the numerical estimates of the eigenvalues corresponding to these nearly saturating examples.

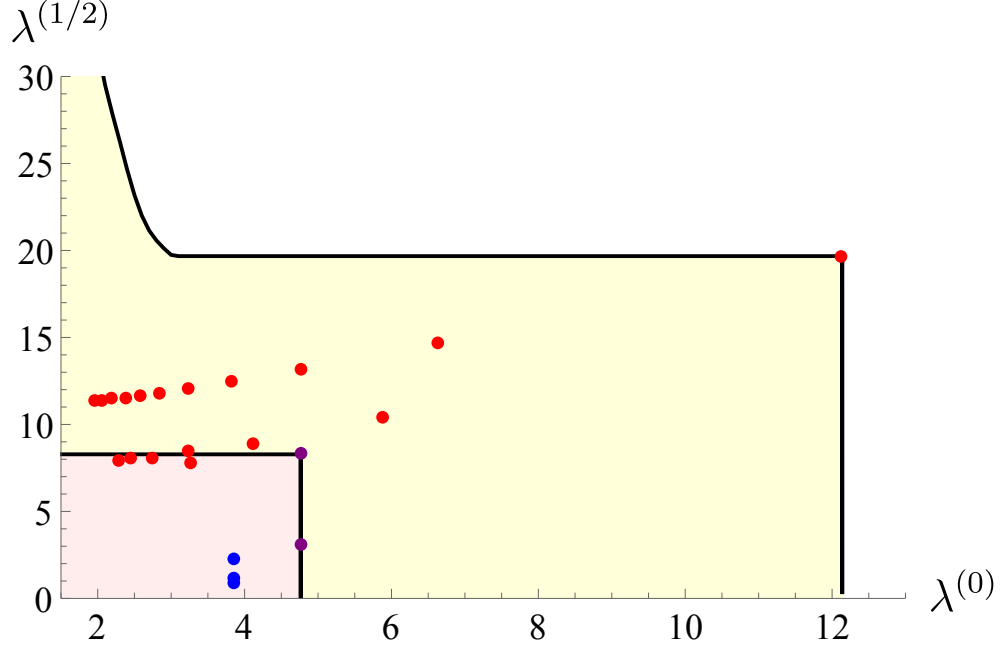


Figure 1.2: For any compact orientable hyperbolic spin orbifold X , equipped with a spin structure such that there is no harmonic spinor, $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ must lie in the union of the pink-shaded and yellow-shaded regions. Note that, even though it is not shown explicitly, we have $\lambda_1^{1/2} > 1/4$. If the hyperbolic spin orbifold X has genus 1 or more and is equipped with a spin structure admitting one or more harmonic spinors, then $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ must lie in the pink shaded region. The red dot in the corner corresponds to $(\lambda_1^{(0)}([0; 3, 3, 5]), \lambda_1^{(1/2)}([0; 3, 3, 5]))$. The other red dots come from the eigenvalues corresponding to various hyperbolic triangles while the blue dots come from the eigenvalues corresponding to the Bolza surface, equipped with various spin structures. The purple dot in the corner of the pink shaded region corresponds to $(\lambda_1^{(0)}([1; 3]_{sym}), \lambda_1^{(1/2)}([1; 3]_{sym}))$, where $[1; 3]_{sym}$ is equipped with an odd spin structure. See the remark 1.1.4. The other purple dot lying on the boundary of the pink shaded region corresponds to $[1; 3]_{sym}$, equipped with an even spin structure. See the Table 1.3 in Appendix 1.8 for the numerical estimates of these eigenvalues. Here the origin is at $(1.5, 0)$.

We can derive more specific theorems from the exclusion plot. For example, we prove

Theorem 1.1.5 (Conditional upper bound on Dirac eigenvalue). *Given any compact*

orientable hyperbolic spin orbifold X such that the first non-zero eigenvalue of the Laplacian operator satisfies $\lambda_1^{(0)}(X) > 1.5$, we have

$$\lambda_1^{(1/2)}(X) < 55.9.$$

Here $\lambda_1^{(1/2)}(X) = 1/4 + t^2$ and $|t|$ is the lowest non-zero positive eigenvalue of Dirac operator on X .

To estimate the Laplace and Dirac eigenvalues of a given surface and compare them with our bounds, we use a numerical method based on the STF, described in [LL22]. Using this method, we can populate the exclusion plots as shown on Figure 1.2. We also derive more specific exclusion plots, for example Figures 1.10 and 1.11, which describe the bound applicable to all compact orientable hyperbolic genus 2 spin orbifolds. As we can see, $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$, where X is the Bolza surface with odd spin structure, sit very close to the boundary, see Table 1.3 in Appendix 1.8 for the numerical estimates.

The second type of bound is obtained from the first type, by noticing that the first nonzero eigenvalue of the Dirac operator corresponds to $\lambda_1^{(1/2)}(X) > 1/4$. This leads to upper bounds on the first nonzero eigenvalue of the Laplace operator for a given choice of spin structure, admitting a given number of weight 1 modular form. These upper bounds are collected in Table 1.1. Interestingly, when the spin structure allows for enough harmonic spinors, the upper bounds become more stringent. In particular, this leads to improved upper bounds on the first Laplace eigenvalue for hyperelliptic surfaces compared to [KMP24]. As a warmup, we can use a small number of spectral identities to derive a very simple analytical bound for genus g hyperelliptic surfaces by using the fact that they can be equipped with a spin structure, admitting $\lfloor \frac{g+1}{2} \rfloor$ harmonic spinors.

g	$\ell_{1/2}$	$\lambda_1^{(0)}(g, \ell_{1/2})$	KMP [KMP24]	FP [BP23]
2	0,1	3.8388976481	3.8388976481	4.625307
3	0,1 2	2.6784823893 1.9497673318*	2.6784823893	2.816427
4	0,1 2	2.1545041334 1.9497673318*	2.1545041334	2.173806
5	0,1,2 3	1.8526509456 1.18275751*	1.8526509456	1.836766
6	0,1,2 3	1.654468363 1.386265630*	1.654468363	1.625596
7	0,1,2 3 4	1.51326783 1.38626563* 0.9160143*	1.51326783	1.480008
8	0,1,2 3 4	1.40690466 1.38626563* 1.0443114*	1.40690466	1.372804
9	0,1,2,3 4 5	1.32348160 1.148493243* 0.78690*	1.32348160	1.289024
10	0,1,2,3 4 5	1.25602193 1.148493243* 0.85292*	1.25602193	1.222189

Table 1.1: Table of upper bounds on the Scalar-Laplacian gap as a function of genus and the number of harmonic spinors. For comparison, we also tabulate the spinor-independent bounds from [KMP24] and [BP23]. We note that bounds from [BP23] are only applicable to manifolds while the bounds from [KMP24] and the bounds in this paper are applicable to both manifolds and orbifolds. For $2 \leq g \leq 10$, the bounds here coincide with the ones from [KMP24], except when, given the genus g , the surface admits a spin structure such that number of harmonic spinors is strictly more than $\lceil \ell_{Max}/2 \rceil$, where $\ell_{Max} = \lfloor (g+1)/2 \rfloor$ is the maximal number of harmonic spinors allowed for that genus. In that case, the bounds are more restrictive; we mark them with *.

Theorem 1.1.6. *Given a genus $g \geq 3$ compact hyperbolic hyperelliptic surface X , we must have*

$$\lambda_1^{(0)}(X) \leq \frac{\lfloor \frac{g+1}{2} \rfloor}{\lfloor \frac{g+1}{2} \rfloor - 1}. \quad (1.1)$$

Remark 1.1.7. For a given numerical value of g , it is possible to achieve a much stronger bound using computer assistance. This is evident from table 1.1, upon using

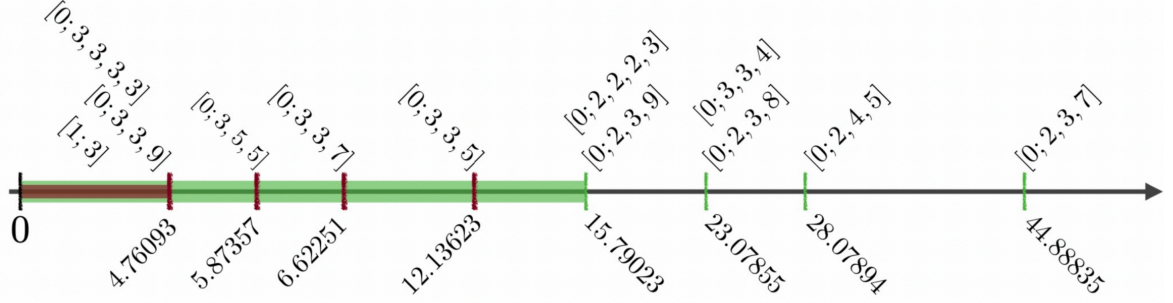


Figure 1.3: Pictorial description of Conjecture 1.1.8: The green color denotes the set of $\lambda_1^{(0)}(X)$, as X runs over all compact orientable hyperbolic orbifolds. The red color denotes the set of $\lambda_1^{(0)}(X)$, as X runs over all compact orientable hyperbolic orbifolds equipped with a spin structure. The red continuum comes from the moduli space of $[1; 3]$ and $[0; 3, 3, 3, 3]$ while the green continuum comes from the moduli space of $[0; 2, 2, 2, 3]$. This conjecture is a refinement of Conjecture 4.2 in [KMP24] as one can see that the red continuum along with 3 discrete red points lie entirely within the green continuum.

the fact that the hyperelliptic surfaces can be equipped with a spin structure which admits $\ell_{1/2} = \lfloor \frac{g+1}{2} \rfloor$.

Using the STF, we can compute the location of $\lambda_1^{(0)}(X)$ for many orbifolds. Combining these numerical data, we also put forward the following conjecture:

Conjecture 1.1.8. *Let $\lambda_1^{(0)}(X)$ be the first non-zero eigenvalue of the Laplacian on an hyperbolic orbifold X . Let $E_{\text{spin}} \subset \mathbb{R}_{>0}$ be the set of $\lambda_1^{(0)}(X)$ as X runs over all compact hyperbolic spin orbifolds. Then E_{spin} is the union of the interval $(0, \lambda_1^{[3,3,9]})$ with the finite set $\{\lambda_1^{[3,3,5]}\} \cup \{\lambda_1^{[3,5,5]}\} \cup \{\lambda_1^{[3,3,7]}\}$. When X is an orbifold with signature $[0; k_1, k_2, k_3]$, we write $\lambda_1^{[k_1, k_2, k_3]}$ to denote $\lambda_1^{(0)}([0; k_1, k_2, k_3])$. See Figure 1.3.*

Remark 1.1.9. The above conjecture is a refinement of Conjecture 4.2 in [KMP24], where the relevant set $E \subset \mathbb{R}_{>0}$ was the set of $\lambda_1^{(0)}(X)$ as X runs over all orbifolds. We believe that this conjecture can partially be proven by bootstrapping the spectral identities as done for Conjecture 4.2 in [KMP24].

To put the above results in context, let us make some historical remarks. The Dirac operator (in particular, the twisted Dirac operator, i.e. Dirac operator in presence of a gauge connection) plays numerous interesting roles in physics. An upper bound on the spectral gap of such a twisted Dirac operator was found for the first time in the context of QCD in a paper by Vafa and Witten [VW84], followed by a paper by Atiyah [Ati85]. These two papers prove a *uniform upper bound* (uniform with respect to the gauge connection) on the n th nontrivial eigenvalue of the twisted Dirac operator on a compact Riemannian spin manifold in any dimension D . Similar methods were used to arrive at further specialized bounds in [Bau91] in terms of a smooth map from the manifold to the sphere. They further derived bounds for surfaces with positive sectional curvature. Ref.[AF99] derived extrinsic bounds on Dirac eigenvalues for isometrically immersed surfaces inside $\mathbb{R}^3$ in terms of mean curvature and a smooth function from the surface to $\mathbb{R}$, as well as intrinsic bounds for genus 0 and genus 1 surfaces. Finally, lower bounds for the first nonzero Dirac eigenvalue have also been studied for various kinds of spin manifolds [Bär92], for general manifolds of positive curvature [Fri80; BD04], as well as in the case of surfaces depending on some geometric parameters including the choice of spin structure [Bär99; AB02]. For a general discussion on the spectrum of the Dirac operator on various kinds of Riemannian manifolds, we refer the readers to [FN12] for a review, or [Bär00] for details on the hyperbolic case.

Our bounds have a different flavor from these results, and it is not straightforward to directly compare them. In particular, our method encodes interesting interplays between eigenvalues of Laplacian operator and eigenvalues of Dirac operator on a hyperbolic spin orbifold, for example, see Theorem 1.1.5 or Figure 1.2.

Organization of the paper

The rest of this paper is organized as follows. In Section 1.2, we introduce the basic notions of spectral geometry that we will need. In Section 1.3, we derive our constraint equations from associativity and harmonic analysis. In Section 1.4, we use the Selberg trace formula to develop a numerical technique to estimate the Dirac and Laplace spectra of an orbifold of interest. In Section 1.5, we derive our main bounds and exclusion plots. In Section 1.6, we discuss a few possible future directions that our results suggest. In Appendix 1.7, we relate the number of harmonic spinors a surface of low genus can carry to its geometrical properties. In Appendix 1.8 we tabulate the numerical estimates of $\lambda_1^{(0)}$ and $\lambda_1^{(1/2)}$ for various surfaces and orbifolds.

We emphasize that while the search for functionals in our linear/semidefinite programming algorithm necessitates computer assistance, the output can be rationalized in such a way that one can explicitly exhibit the required properties of the rationalized functionals leading to the bounds (such verifications of bounds coming from linear/semidefinite programming using *rational arithmetic* appear in many contexts, for example in sphere packing [CE03] and in ref. [KMP24]). Consequently, the proofs of our theorems are completely rigorous and the computer is only used as an oracle. It is only in Section 1.4 that our individual estimates, which are independent from the theorems, are numerical and not proven at a mathematical level, although it should also be possible to do so.

Technical note: Unless specified otherwise, all surfaces or orbifolds considered in this paper are connected, compact and orientable.

1.2 Spectral geometry on a spin-surface

Dirac operator and Automorphic Laplacian

Let us consider a d -dimensional orientable connected Riemannian manifold (X, g) and the orthonormal frame bundle over X , $(Q, \pi, X; SO(d))$; here Q is the total space, $\pi : Q \rightarrow X$ is the projection map. We will often suppress π and write the shorthand $Q(X)$ to mean the orthonormal frame bundle over X . A spin structure on this principal bundle Q is a pair (P, Λ) such that P is a $\text{Spin}(d)$ -principal bundle and there exists a principal bundle morphism $\Lambda : P \rightarrow Q$, that is equivariant with respect to the double covering $\text{Spin}(d) \rightarrow SO(d)$. A spin structure on Q exists if and only if the second Stiefel-Whitney class of the $SO(d)$ principal bundle vanishes.

To set the stage, we restrict our attention to the upper half-plane $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. The metric g is given by the Poincaré metric, i.e.,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \quad (1.2)$$

There exists a unique spin structure on the orthonormal frame bundle over $\mathbb{H}$ (which is an $SO(2)$ principal bundle). The corresponding $\text{Spin}(2)$ principal bundle is trivial and given by $P = \mathbb{H} \times \text{Spin}(2)$. If ρ is the representation of $\text{Spin}(2)$ on its Clifford module $\mathbb{C}^2$, then the bundle S associated with ρ ,

$$S := P \times_{\rho} \mathbb{C}^2, \quad (1.3)$$

is the spinor bundle over the upper half-plane. The sections of this spinor bundle are spinor fields defined on $\mathbb{H}$. In particular, we will be interested in $\mathfrak{C}^{\infty}(S)$, the space of smooth sections of the spinor bundle S .

Definition 1.2.1 (Dirac operator on $\mathbb{H}$). The Dirac operator $\not{D} : \mathfrak{C}^{\infty}(S) \rightarrow \mathfrak{C}^{\infty}(S)$

is given by²

$$\mathcal{D} := i \begin{bmatrix} 0 & iy\partial_x + y\partial_y - \frac{1}{2} \\ -iy\partial_x + y\partial_y - \frac{1}{2} & 0 \end{bmatrix}. \quad (1.4)$$

Remark 1.2.2. By restricting the domain of $\mathcal{D}$ to $\mathfrak{C}_c^\infty(S)$, the space of smooth compactly supported sections of the spinor bundle S , one can construct a pre-Hilbert space, which can then be promoted to a Hilbert space in the standard way: first, we define the inner product of $f, g \in \mathfrak{C}_c^\infty(S)$ by

$$\langle f, g \rangle := \int_{\mathbb{H}} \langle f(z), g(z) \rangle_{\mathbb{C}^2} d\mu(z), \quad (1.5)$$

where μ is the hyperbolic measure on $\mathbb{H}$ and then complete $\mathfrak{C}_c^\infty(S)$ into a Hilbert space $\mathfrak{H}(S)$ using the induced norm. The space $\mathfrak{C}_c^\infty(S)$ is dense inside $\mathfrak{H}(S)$.

Remark 1.2.3. The Dirac operator is elliptic and essentially self-adjoint on $\mathfrak{C}_c^\infty(S)$, which in particular implies that its spectrum is real [BS06].

A compact connected orientable hyperbolic surface Σ can be thought of as $\Gamma \backslash \mathbb{H}$, where Γ is a cocompact Fuchsian group, a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. More precisely, the cocompact Fuchsian group Γ acts properly discontinuously on $\mathbb{H}$. If $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the action is defined by

$$\begin{aligned} \Gamma \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (\gamma, z) &\longmapsto \frac{az+b}{cz+d}. \end{aligned} \quad (1.6)$$

The above induces an action on the orthonormal frame bundle $Q(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$, which is simply the action by left multiplication:

$$\begin{aligned} \Gamma \times \mathrm{PSL}(2, \mathbb{R}) &\longrightarrow \mathrm{PSL}(2, \mathbb{R}) \\ (\gamma, g) &\longmapsto \gamma \cdot g. \end{aligned} \quad (1.7)$$

The double cover of the bundle Q is $P(\mathbb{H}) \cong \mathrm{SL}(2, \mathbb{R})$.

²For more details on defining the Dirac operator on a d -dimensional oriented connected Riemannian manifold (X, g) , see Section 3 of [BGR07].

Definition 1.2.4. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a cocompact Fuchsian group. Let $p : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ be the canonical projection. We define the $\bar{\Gamma} := p^{-1}(\Gamma)$.

In order to describe spin structures more directly in terms of the action of Γ , it is convenient to apply the following result to our context:

Proposition 1.2.5 (Ref.[Mor99]). *Spin structures on Σ are in one-to-one correspondence with lifts to $\mathrm{SL}(2, \mathbb{R})$ of the action by left multiplication of Γ on $\mathrm{PSL}(2, \mathbb{R})$. In other words, the spin structures on $\Gamma \backslash \mathbb{H}$ are in one-to-one correspondence with the possible right splittings of the short exact sequence:*

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \bar{\Gamma} \xrightarrow{p} \Gamma \longrightarrow 1. \quad (1.8)$$

$\nwarrow \text{---} \chi \text{---} \swarrow$ $\nwarrow \text{---} \rho \text{---} \swarrow$

The number of right splittings ρ (i.e. ρ for which $p \circ \rho = \mathrm{Id}$) of this exact sequence is equal to the number of left splittings χ , for which $\chi \circ \iota = \mathrm{Id}$, which amounts to imposing the constraint $\chi(-\mathrm{Id}) = -1$. A natural way to establish a one-to-one correspondence between the two is to characterize χ by $\mathrm{Ker} \chi = \mathrm{Im} \rho$. As a result, we obtain:

Proposition 1.2.6. *Spin structures on a compact orientable smooth Riemann surface Σ can be labelled by homomorphisms $\chi : \bar{\Gamma} \rightarrow \mathbb{Z}_2$ satisfying $\chi(-\mathrm{Id}) = -1$.*

Remark 1.2.7. In the language of automorphic forms, χ is a multiplier which allows the existence of nontrivial odd weight automorphic forms.

Remark 1.2.8. There is also a geometric interpretation of the morphism χ . If $\tilde{\gamma} \in \bar{\Gamma}$, $\chi(\tilde{\gamma})$ gives the holonomy of the spin bundle along the corresponding closed geodesic.

This new characterization of spin structure in terms of a homomorphism $\chi : \bar{\Gamma} \rightarrow \mathbb{Z}_2$ such that $\chi(-\mathrm{Id}) = -1$ has the very convenient feature of easily generalizing to the case in which the cocompact Fuchsian group Γ has some elliptic elements, in

particular, to the case of orbifolds which will be of particular interest to us. We hence define:

Definition 1.2.9. Let Γ be a general cocompact Fuchsian group. A spin structure on $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ is a group homomorphism $\chi : \bar{\Gamma} \rightarrow \mathbb{Z}_2$ such that $\chi(-Id) = -1$.

Remark 1.2.10. Unlike Riemann surfaces, not every orbifold admits a spin structure. In particular, if there exists an element of order 2 in Γ , the injectivity of ρ implies that this element must be mapped to $-\mathrm{Id} \in \bar{\Gamma}$, which lies in the image of ι , hence the kernel of p . This is inconsistent with the fact that $p \circ \rho = \mathrm{Id}$, thus there does not exist any right splitting of the exact sequence (1.8), which, by the splitting lemma, implies that there does not exist any left splitting χ . Hence even order orbifold points are obstructions to the existence of a spin structure.

Remark 1.2.11. The geometric interpretation of χ naturally extends to the case of orbifolds. In the case of elliptic elements, it gives the holonomy around orbifold singularities.

Proposition 1.2.12 (Ref. [GG12], Theorem 3). *On a smooth Riemann surface of genus g , there are 2^{2g} spin structures. An orbifold admits a spin structure if and only if all its orbifold points have odd order, and then the number of spin structures is also given by 2^{2g} .*

To define the Dirac operator for Σ , we need to restrict the domain of the $\not{D}$, the Dirac operator on the upper half-plane to special sets of sections (since the bundle is trivial, they are actually functions on $\mathbb{H}$ to $\mathbb{C}^2$) that satisfy the right automorphic properties under $\bar{\Gamma}$.

Definition 1.2.13 (Automorphy factor). For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}$, and $2k \in \mathbb{N}$, let us

define the automorphy factor $j_\gamma : \mathbb{H} \times \frac{1}{2}\mathbb{N} \rightarrow \mathbb{C}$ and $J_\gamma : \mathbb{H} \times \frac{1}{2}\mathbb{N} \rightarrow \mathbb{C}^2$ as

$$j_\gamma(z, k) := \frac{(cz + d)^k}{(c\bar{z} + d)^k}, \quad J_\gamma(z, k) := \begin{pmatrix} j_\gamma(z, k) & 0 \\ 0 & j_\gamma(z, k-1) \end{pmatrix}, \quad (1.9)$$

where the choice of square roots in the definitions of the numerator and denominator of $j_\gamma(z, k)$ are chosen to be complex conjugate.

Definition 1.2.14. Let Γ be a cocompact Fuchsian group, and let χ be a multiplier of weight $2k \in \mathbb{N}$ on $\bar{\Gamma}$ ($\chi(-\text{Id}) = (-1)^{2k}$). We define the space $\mathfrak{F}(\Gamma, k, \chi)$ as the space of functions $\Psi : \mathbb{H} \rightarrow \mathbb{C}^2$ that satisfy

$$\Psi(\gamma z) = \chi(\gamma) J_\gamma(z, k) \Psi(z), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}. \quad (1.10)$$

If these functions are asked to be smooth (resp. L^2 over the fundamental domain of Γ), the corresponding space is denoted $\mathfrak{C}^\infty(\Gamma, k, \chi)$ (resp. $\mathfrak{L}^2(\Gamma, k, \chi)$).

It can be checked that $\not{D}$ leaves the space $\mathfrak{C}^\infty(\Gamma, 1/2, \chi)$ invariant and thus we have

Definition 1.2.15 (Dirac operator on Σ). The Dirac operator associated to Σ is defined to be the restriction of $\not{D}$ to $\mathfrak{C}^\infty(\Gamma, 1/2, \chi)$.

Under a slight abuse of notation, we will continue calling $\not{D}$ this Dirac operator with restricted domain. $\mathfrak{C}^\infty(\Gamma, 1/2, \chi)$ is dense inside $\mathfrak{L}^2(\Gamma, 1/2, \chi)$. Compactness of the surface implies that the spectrum of $\not{D}$ is discrete.

Definition 1.2.16. Let Γ be a cocompact Fuchsian group, and let χ be a multiplier of weight $2k$ on $\bar{\Gamma}$ ($\chi(-\text{Id}) = (-1)^{2k}$). We define the space $\mathcal{F}(\Gamma, k, \chi)$ as the space of functions $\psi : \mathbb{H} \rightarrow \mathbb{C}$ that satisfy

$$\psi(\gamma z) = \chi(\gamma) j_\gamma(z, k) \psi(z), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}. \quad (1.11)$$

If these functions are asked to be smooth (resp. L^2 over the fundamental domain of Γ), the corresponding space is denoted $\mathcal{C}^\infty(\Gamma, k, \chi)$ (resp. $L^2(\Gamma, k, \chi)$).

Definition 1.2.17. Define the weight $2k$ automorphic Laplacian $\Delta_k : \mathcal{C}^\infty(\Gamma, k, \chi) \rightarrow \mathcal{C}^\infty(\Gamma, k, \chi)$ as

$$\Delta_k := y^2(\partial_x^2 + \partial_y^2) - 2iky\partial_x. \quad (1.12)$$

Remark 1.2.18. The square of the Dirac operator, $\mathcal{D}^2 : \mathfrak{C}^\infty(\Gamma, 1/2, \chi) \rightarrow \mathfrak{C}^\infty(\Gamma, 1/2, \chi)$ is given by

$$\mathcal{D}^2 = \begin{pmatrix} -\Delta_{1/2} - \frac{1}{4} & 0 \\ 0 & -\Delta_{-1/2} - \frac{1}{4} \end{pmatrix}. \quad (1.13)$$

$\Delta_{1/2}$ is also called 1-Laplacian.

Now the following key proposition relates the spectrum of $\mathcal{D}$ to that of $\Delta_{1/2}$.

Proposition 1.2.19. *If $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is an eigenform of $-\mathcal{D}$ with eigenvalue t , then ψ_1 is an eigenform of $-\Delta_{1/2}$ with eigenvalue $\lambda = 1/4 + t^2$. Conversely, if ψ is an eigenform of $-\Delta_{1/2}$ with eigenvalue $\lambda = 1/4 + t^2$, then $\Psi = \begin{pmatrix} t\psi \\ i(iy\partial_x - y\partial_y + \frac{1}{2})\psi \end{pmatrix}$ is an eigenform of $-\mathcal{D}$ with eigenvalue t .*

Proof. See Proposition 1 of [BS06] and its proof. □

Thus bounding the Dirac spectrum associated to a given surface Σ amounts to bounding the spectra of $\Delta_{\pm 1/2}$ on Σ . Another important feature of the Dirac spectrum is that it has two important symmetries. First, the spectrum of the Dirac operator is symmetric with respect to the origin:

Proposition 1.2.20. *If $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is an eigenvector of $\mathcal{D}$ with eigenvalue t , then $\begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}$ is an eigenvector of $\mathcal{D}$ with eigenvalue $-t$. In particular, the spectrum of $\mathcal{D}$ is symmetric with respect to the origin.*

The Dirac operator enjoys one more symmetry leading to what is called *Kramers degeneracy* (degeneracy is a synonym of multiplicity in the physics literature). This implies that all non-zero eigenvalues of the Dirac operator are at least doubly degenerate,

Proposition 1.2.21 (Kramers degeneracy). *Let C be the complex conjugation operator, and $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. If Ψ is an eigenvector of $\mathcal{D}$ with eigenvalue t , then so is $(i\sigma_2 C)\Psi$.*

In quantum field theory, the operator implementing Kramers degeneracy is usually interpreted as the time reversal operator. In Section 1.4, we will use Kramers degeneracy in our algorithm to estimate the spectra of Dirac operator on orbifolds and surfaces. More precisely, we will note that since all the non-zero eigenvalues of the Dirac operator are degenerate, we can use Proposition 1.4.5 instead of Proposition 1.4.4, and look for the first nonzero value of t for which $I_r(\lambda) = 2$, instead of the first nonzero value of t for which $I_r(\lambda) = 1$.

Holomorphic Modular forms and spin structure

A spin structure determines the definition of holomorphic modular forms of odd weight on the surface. Let us recall that for even weights, holomorphic modular forms on a Riemann surface Σ are defined independently from any choice of spin structure. More precisely:

Definition 1.2.22. Let Γ be a cocompact Fuchsian group. A holomorphic Γ -modular form of weight $2k \in 2\mathbb{N}$ is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$, such that for all $\gamma : z \mapsto \frac{az+b}{cz+d}$ in Γ ,

$$f(\gamma(z)) = (cz + d)^{2k} f(z), \tag{1.14}$$

Note that since $2k$ is an even number, $(cz+d)^{2k}$ has the same value on the two possible lifts of γ inside $\mathrm{SL}(2, \mathbb{R})$, which ensures that the above definition is unambiguous. This changes for odd powers of $cz+d$, because the automorphy factor differs by a sign depending on the choice of lift. Instead, one must use the spin structure χ as an input to define a modular form of odd weight:

Definition 1.2.23. Let $\chi : \bar{\Gamma} \rightarrow \mathbb{Z}_2$ be a group homomorphism (or equivalently, a choice of spin structure on Σ). A holomorphic modular form of weight k for $\bar{\Gamma}$ is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$, such that for all $\gamma : z \mapsto \frac{az+b}{cz+d}$ in Γ ,

$$f(\gamma(z)) = \chi(\gamma)^k (cz+d)^k f(z). \quad (1.15)$$

In the case of k even, this definition reduces to the previous one and does not depend on the choice of χ , but this is no longer true for k odd.

Our approach will use the number of linearly independent holomorphic modular forms of a given (even or odd) weight as an input to the set of consistency conditions, i.e. *bootstrap* constraints. Therefore, it is important to understand what moduli space of surfaces and orbifolds supports a given number of linearly independent holomorphic modular forms of a given (even or odd) weight. Apart from the case of modular forms of weight 1, this is determined by purely topological data, as shown by the Riemann–Roch theorem, which we now review.

The Riemann–Roch theorem comes from algebraic geometry, and allows to count the number of linearly independent holomorphic modular forms of weight ≥ 2 on Σ only from the information about its topology.

Theorem 1.2.24 (Riemann–Roch for orbifolds). *Let $n \geq 2$ and $n \in \mathbb{Z}$. If an orbifold of genus g with elliptic points $p_1, \dots, p_r$ of orders $k_1, \dots, k_r$ admits a spin structure, then it possesses*

$$\ell_{n/2} = (n-1)(g-1) + \sum_{i=1}^r \left\lfloor \frac{n}{2} \left(1 - \frac{1}{k_i} \right) \right\rfloor + \delta_{n,2} \quad (1.16)$$

holomorphic modular forms of weight n (regardless of the choice of multiplier), and in particular g holomorphic modular forms of weight 2.

Remark 1.2.25. Note that when n is odd, there can be cases in which there is no spin structure on the space under consideration (if it is an orbifold). The Riemann–Roch theorem requires the existence of a spin structure for odd n .

Remark 1.2.26. The Riemann–Roch theorem will be the central tool that we will use in order to apply our bootstrap techniques. The number of modular forms of a certain weight will be used as an input in our symmetry constraints, and the Riemann–Roch theorem will enable us to characterize the class of Σ supporting enough modular forms in order for our constraints to apply.

Harmonic spinors

Calculating the number of modular forms of a given weight is straightforward thanks to the Riemann–Roch formula for $n \geq 2$: it is a topological invariant of the surfaces or orbifolds under consideration. However, the case $n = 1$ is particularly tricky. In particular, for compact orientable smooth surfaces of genus 2 or more, this number is not topologically invariant, and actually carries some nontrivial information about the surface’s geometry and the spin structure on it.

Remark 1.2.27. The fact that the $n = 1$ case behaves differently is most easily understandable in terms of sheaf cohomology. In this language, the Riemann–Roch formula involves the dimension of the zeroth and first cohomology groups of the sheaf under consideration, which is a power of the canonical line bundle determined by n (see for example Theorem 16.9 of [FG99]). It is the dimension of the zeroth cohomology group that gives the number of linearly independent holomorphic modular forms of weight n . If $n > 1$, the first sheaf cohomology group automatically vanishes, which allows to directly compute the number of linearly independent holomorphic modular forms from the Riemann–Roch formula. It is no longer right to ignore the

term coming from the first sheaf cohomology group for $n = 1$, which explains why the situation is more complicated.

What remains true in the $n = 1$ case is that the genus gives an upper bound on the number of harmonic spinors:

Proposition 1.2.28 (Ref. [Hit74]). *The number $\ell_{1/2}$ of harmonic spinors on a compact orientable smooth Riemann surface of genus g with a spin structure satisfies*

$$\ell_{1/2} \leq \left\lfloor \frac{g+1}{2} \right\rfloor. \quad (1.17)$$

The natural next step is to ask when this upper bound is saturated. Here the answer is interesting, and requires the following definition:

Definition 1.2.29. Let Σ be a (smooth) Riemann surface. Σ is *hyperelliptic* if it can be realized as a branched double cover $\Sigma \rightarrow \mathbb{CP}^1$.

Remark 1.2.30. Hyperelliptic surfaces enjoy particularly nice properties, for example, they can always be described by an algebraic equation of the form

$$y^2 = F(x), \quad (1.18)$$

where F is a polynomial with distinct roots. At genus 1 and 2, every surface is hyperelliptic. However, at genus higher than or equal to 3, the moduli space of hyperelliptic surfaces is not the entire moduli space and hence is interesting to study.

For our purposes, the following result is relevant:

Proposition 1.2.31 (Ref. [Mar68]). *Given Σ , a smooth surface of genus different from 4 or 6, there exists a spin structure on Σ for which the bound (1.17) is saturated if and only if Σ is hyperelliptic.*

This shows that apart from the case of genus 4 and the case of genus 6,³ we can use the existence of a maximal number of harmonic spinors to write down the set of consistency conditions, leading to bounds on Laplacian spectra for a genus g hyperelliptic surface. These bounds are refined (and consequently, as we will see later, they are stronger as well), in contrast with the bounds obtained in [KMP24], which are valid for all surfaces of genus g .

The upper bound on the gap of the Laplacian on some appropriately chosen moduli space of surfaces is likely to be saturated on surfaces with a large automorphism group (i.e. a locally maximal automorphism group). The classification of hyperelliptic surfaces with large automorphism groups is achieved in Table 1 of [MP17]. More generally, it is interesting to ask for a geometric description of the space of surfaces of a given genus that can carry a given number of harmonic spinors. As we saw, all surfaces of genus 1 and 2 are hyperelliptic so such an endeavor is trivial there, but from genus 3 onwards, the answer starts to give nontrivial information about the geometry of the surfaces. The classification of Riemann surfaces in terms of the possible dimensions of their spaces of harmonic spinors is still open in general, however, for genus $3 \leq g \leq 6$, precise results are available. In Appendix 1.7, we summarize these explicit classifications in low genus. The results collected in this appendix are condensed in Table 1.2.

1.3 Spectral identities for hyperbolic surfaces from associativity

The aim of this section is to derive the spectral identities which we will eventually *bootstrap* to bound the Laplacian and Dirac spectra on a compact connected orientable hyperbolic surfaces and orbifolds. A compact connected orientable hyperbolic orbifold⁴ can be thought of as $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R}) / \mathrm{PSO}(2)$, where Γ is a cocompact Fuchsian

³It is intriguing to note that these genera are precisely the ones for which bootstrap bounds [KMP24] for Laplace spectra are weaker than the Yang–Yau bounds [YY80b].

⁴We are following the convention that the manifolds are orbifolds with no orbifold-singularity unless otherwise mentioned.

g	$\ell_{1/2}^{Max}$	Moduli space	Most symmetric surface	Automorphism group
2	1	All	Bolza surface	$GL(2, \mathbb{F}_3)$
3	1	Non-hyperelliptic	Klein quartic	$PSL(2, \mathbb{F}_7)$
	2	Hyperelliptic	$y^2 = x^8 + 14x^4 + 1$	$\mathbb{Z}_2 \times S_4$
4	1	Non-hyperelliptic with smooth canonical quadric	Bring surface	S_5
	2	Hyperelliptic or non-hyperelliptic with singular canonical quadric	$z^3 y^2 = x(x^4 + y^4)$	Order 72
5	1 or 2	Non-hyperelliptic	$x_1^2 + x_2^2 + 2x_4^2 = x_4^2(x_2^2 + x_4^2) - x_3^4 = 0$	Order 192
	3	Hyperelliptic	$y^2 = x(x^{10} + 11x^5 - 1)$	$\mathbb{Z}_2 \times A_5$
6	1 or 2	Non-hyperelliptic and not a smooth quintic	Wiman sextic	S_5
	3	Hyperelliptic or smooth quintic	Fermat quintic	Order 150

Table 1.2: Table of the moduli spaces of surfaces, satisfying each genus/spin constraint, and of the most symmetric surfaces in each of these moduli spaces. Here g is the genus and $\ell_{1/2}^{Max}$ is the maximal number of harmonic spinors that a surface can carry.

group. For a spin orbifold with a given spin structure, there exists $\bar{\Gamma}$, a subgroup of $SL(2, \mathbb{R})$, such that $\Gamma \simeq \bar{\Gamma}/\mathbb{Z}_2$. This means that Γ can be consistently embedded inside $\bar{\Gamma}$ such that the embedding does not have $-Id$. We will denote this embedding as $\tilde{\Gamma}$. As a set, we have

$$\bar{\Gamma} = \tilde{\Gamma} \sqcup (-\tilde{\Gamma}), \quad (1.19)$$

and $\tilde{\Gamma}$ is isomorphic to Γ . Proposition 1.2.6 implies that $\chi(\tilde{\gamma}) = \pm 1$ for $\tilde{\gamma} \in \pm \tilde{\Gamma}$ respectively. In other words, different spin structures correspond to different homomorphisms $\chi : \bar{\Gamma} \rightarrow \mathbb{Z}_2$, which amounts to saying that (1.19), and in particular $\tilde{\Gamma}$, depends on the spin structure. We stress that even though Γ is isomorphic to $\tilde{\Gamma}$, by choosing the embedding $\tilde{\Gamma}$, we have already committed to a spin structure. Thus it is important to distinguish between Γ and $\tilde{\Gamma}$ even though they are isomorphic.

The upshot of the above is that a compact connected orientable hyperbolic spin orbifold X , with a given spin structure can be thought of as $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R}) / \mathrm{Spin}(2)$. The total space for the spin bundle, i.e. $P(X)$, is given by $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$. We formulate the spectral problem for the Laplacian and Dirac operator on these hyperbolic spin orbifolds with a spin structure using the harmonic analysis on $L^2(\tilde{\Gamma} \backslash G)$, where $G := \mathrm{SL}(2, \mathbb{R})$. Evidently, a key role is played by irreducible unitary representations of $\mathrm{SL}(2, \mathbb{R})$.

Representation theory of $\mathrm{SL}(2, \mathbb{R})$

Theorem 1.3.1 (See [Kna01]). *The unitary irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ are given up to equivalence by:*

1. *the trivial representation,*
2. *the complementary series $\mathcal{C}_s$ for $s \in (0, \frac{1}{2})$,*
3. *the principal series $\mathcal{P}_{i\nu}^\pm$ for $\nu \in \mathbb{R}$, except for $\mathcal{P}_0^-$,*
4. *the holomorphic discrete series $\mathcal{D}_{n/2}$ and anti-holomorphic discrete series $\overline{\mathcal{D}}_{n/2}$,
 $n \geq 2, n \in \mathbb{Z}$,*
5. *the limits of discrete series $\mathcal{D}_{\frac{1}{2}}, \overline{\mathcal{D}}_{\frac{1}{2}}$.*

The only equivalence between the representations listed above is the following: $\mathcal{P}_{i\nu}^\pm \simeq \mathcal{P}_{-i\nu}^\pm$.

Proposition 1.3.1. *The complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by $L_0, L_{\pm 1}$ and we have $[L_m, L_n] = (m - n)L_{m+n}$. For unitary representations, we have $L_n^\dagger = L_{-n}$.*

Proposition 1.3.2. *The quadratic Casimir C_2 takes the form*

$$C_2 := L_0^2 - \frac{1}{2} (L_{-1}L_1 + L_1L_{-1}) .$$

To explicitly construct the irreps, we find it convenient to recall that $\mathrm{SL}(2, \mathbb{R})$ can be mapped to $SU(1, 1)$ by conjugation within $\mathrm{SL}(2, \mathbb{C})$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, it gets mapped to $u \in SU(1, 1)$; explicitly we have

$$u = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(1, 1) \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \quad (1.20)$$

Under the Möbius transformation induced by the matrix,

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

the upper half-plane gets mapped to the unit disk $\mathbb{D} = \{z : |z| < 1\}$. In what follows, we denote the elements of $SU(1, 1)$ by u , where

$$u := \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha\bar{\alpha} - \beta\bar{\beta} = 1, \quad \alpha, \beta \in \mathbb{C}. \quad (1.21)$$

Then, $SU(1, 1)$ acts on $z \in \mathbb{C}$ as

$$u \cdot z := \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad (1.22)$$

Remark 1.3.3. The irreps of $\mathrm{PSL}(2, \mathbb{R})$ are already reviewed in detail in [KMP24]. For $n \in \mathbb{Z}$, $\mathcal{D}_n$ can be realized as antiholomorphic functions living on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ or holomorphic functions on $\mathbb{D}' := \{z : |z| > 1\} \cup \{\infty\}$. $\bar{\mathcal{D}}_n$ can be realized as holomorphic functions living on the unit disk $\mathbb{D}$. The complementary series $\mathcal{C}_s$ and the principal series $P_{i\nu}^+$ can be realized as functions living on $\partial\mathbb{D} = \{z : |z| = 1\}$. All these functions belong to an L^2 space with respect to an appropriate G -invariant measure. The new ingredients that we are going to use are $\mathcal{P}_{i\nu}^-$ and $\mathcal{D}_n, \bar{\mathcal{D}}_n$ with $n \in 1/2 + \mathbb{Z}$. Hence, we will be brief in explaining the construction of irreps, often referring the readers to section 3 of [KMP24] for details, and only highlight the parts that are new to this paper.

Definition 1.3.4 (Antiholomorphic discrete series $\overline{\mathcal{D}}_{n/2}$). The explicit realization of antiholomorphic discrete series $\overline{\mathcal{D}}_{n/2}$ ($n \in \mathbb{Z}_+$) is achieved by holomorphic functions $f(z)$ on $\mathbb{D}$ such that the action of $u \in SU(1, 1)$ is given by

$$u \cdot f(z) = (-\overline{\beta}z + \alpha)^{-n} f(u^{-1} \cdot z) , \quad (1.23)$$

In particular, the nontrivial element of the center of $SL(2, \mathbb{R})$, given as in (1.21) by $\alpha = -1, \beta = 0$, acts nontrivially iff n is odd. The norm of a vector f inside $\overline{\mathcal{D}}_{n/2}$ is given by

$$\begin{aligned} \|f\|_{\overline{\mathcal{D}}_{1/2}} &= \sup_{0 \leq r < 1} \int_0^{2\pi} d\theta |f(re^{i\theta})|^2 , \\ \|f\|_{\overline{\mathcal{D}}_{n/2}} &= \int_{\mathbb{D}} dz (1 - |z|^2)^{n-1} |f(z)|^2 , \quad n > 1 . \end{aligned} \quad (1.24)$$

Remark 1.3.5. We note that

$$\overline{\mathcal{D}}_{n/2} \simeq \bigoplus_{k \in -n/2 - \mathbb{Z}_{\geq 0}} V_k ,$$

where the V_k are the one-dimensional irreducible representations of $Spin(2)$. This amounts to saying that the L_0 spectrum of such an irrep is given by $\{-n/2 - k : k \in \mathbb{Z}_{\geq 0}\}$. The eigenvalue of the quadratic Casimir corresponding to the irrep $\overline{\mathcal{D}}_{n/2}$ is $-n/2(1 - n/2)$.

Definition 1.3.6 (Holomorphic discrete series $\mathcal{D}_{n/2}$). The holomorphic discrete series $\mathcal{D}_{n/2}$ with $n \in \mathbb{Z}_+$ is realized by the space of antiholomorphic functions f on $\mathbb{D}$ such that the action of $u \in SU(1, 1)$ is given by

$$u \cdot f(\overline{z}) = (-\beta\overline{z} + \overline{\alpha})^{-n} f(u^{-1} \cdot \overline{z}) , \quad (1.25)$$

In particular, the nontrivial element of the center of $SL(2, \mathbb{R})$, given as in (1.21) by $\alpha = -1, \beta = 0$, acts nontrivially iff n is odd. The norm of a vector f inside $\mathcal{D}_{n/2}$ is given by

$$\begin{aligned} \|f\|_{0, \mathcal{D}_{1/2}} &= \sup_{0 \leq r < 1} \int_0^{2\pi} d\theta |f(re^{i\theta})|^2 , \\ \|f\|_{0, \mathcal{D}_{n/2}} &= \int_{\mathbb{D}} dz (1 - |z|^2)^{n-1} |f(z)|^2 , \quad n > 1 . \end{aligned} \quad (1.26)$$

However as explicitly worked out in [KMP24], it is easier to consider another realization by holomorphic functions F on $\mathbb{D}' = \{z : |z| > 1\} \cup \{\infty\}$, where

$$F(z) := z^{-n} f(\bar{z}^{-1}), \quad (1.27)$$

and the norm of F is given by

$$\|F\|_{\mathcal{D}_{n/2}} = \|f\|_{0, \mathcal{D}_{n/2}}. \quad (1.28)$$

The functions F transform exactly in the same way as given by eq. (1.25). Again the action of the center is nontrivial iff n is odd.

Remark 1.3.7. We note that

$$\mathcal{D}_{n/2} \simeq \bigoplus_{k \in n/2 + \mathbb{Z}_{\geq 0}} V_k,$$

where the V_k are the one-dimensional irreducible representations of $\text{Spin}(2)$. This amounts to saying that the L_0 spectrum of such an irrep is given by $\{n/2 + k : k \in \mathbb{Z}_{\geq 0}\}$. The eigenvalue of the quadratic Casimir corresponding to the irrep $\mathcal{D}_{n/2}$ is $-n/2(1 - n/2)$.

Definition 1.3.8 (Principal series $\mathcal{P}_{i\nu}^+$). They are realized in the space of equivalence classes of square integrable functions on $\partial\mathbb{D} = \{z : |z| = 1\}$ such that the action of $u \in SU(1, 1)$ is given by

$$(u \cdot f)(z) = |-\beta\bar{z} + \bar{\alpha}|^{-1-2i\nu} f(u^{-1} \cdot z). \quad (1.29)$$

In particular, the nontrivial element of the center of $\text{SL}(2, \mathbb{R})$, given as in (1.21) by $\alpha = -1, \beta = 0$, acts trivially. The norm of f is given by

$$\|f\|_{\mathcal{P}_{i\nu}^+}^2 = \int_0^{2\pi} d\theta |f(e^{i\theta})|^2. \quad (1.30)$$

Remark 1.3.9. We have

$$\mathcal{P}_{i\nu}^+ \simeq \bigoplus_{k \in \mathbb{Z}} V_k,$$

where the V_k are the one-dimensional irreducible representations of $\text{Spin}(2)$. The spectrum of L_0 for vectors inside $\mathcal{P}_{i\nu}^+$ consists of all integers and each integer appear exactly once. A basis for functions living inside $\mathcal{P}_{i\nu}^+$ is given by $f_j(z) = z^j$ with $j \in \mathbb{Z}$. The eigenvalue of the quadratic Casimir corresponding to the irrep $\mathcal{P}_{i\nu}^+$ is $-(1/4 + \nu^2)$.

Definition 1.3.10 (Principal series $\mathcal{P}_{i\nu}^-$). They are realized in the space of equivalence classes of square integrable functions f on $\partial\mathbb{D}$ such that the action of $u \in SU(1, 1)$ is given by

$$(u \cdot f)(z) = \text{sgn}(-\bar{\beta}z + \alpha) |-\beta\bar{z} + \bar{\alpha}|^{-1-2i\nu} f(u^{-1} \cdot z). \quad (1.31)$$

In particular, the nontrivial element of the center of $\text{SL}(2, \mathbb{R})$, given as in (1.21) by $\alpha = -1, \beta = 0$, acts nontrivially and gives a minus sign. The norm of f is given by ($z = e^{i\theta}$)

$$\|f\|_{\mathcal{P}_{i\nu}^-}^2 = \int_0^{2\pi} d\theta |f(e^{i\theta})|^2. \quad (1.32)$$

Remark 1.3.11. We have

$$\mathcal{P}_{i\nu}^- \simeq \bigoplus_{k \in 1/2 + \mathbb{Z}} V_k,$$

where V_k are the one-dimensional irreducible representations of $\text{Spin}(2)$. The spectrum of L_0 for vectors inside $\mathcal{P}_{i\nu}^-$ consists of all half-integers, and each half-integer appears exactly once. A basis for functions inside $\mathcal{P}_{i\nu}^-$ is given by $f_j(z) = z^j$ with $j \in \mathbb{Z}$ (see sections 8.2 and 8.3 of [Lib12] for a pedagogical exposure). The eigenvalue of the quadratic Casimir corresponding to the irrep $\mathcal{P}_{i\nu}^-$ is $-(1/4 + \nu^2)$.

Definition 1.3.12 (Complementary series $\mathcal{C}_s$). They are realized in the space of equivalence classes of square integrable functions on $\partial\mathbb{D}$ such that the action of $u \in SU(1, 1)$ is given by

$$(u \cdot f)(z) = |-\beta\bar{z} + \bar{\alpha}|^{-1-2s} f(u^{-1} \cdot z). \quad (1.33)$$

In particular, the nontrivial element of the center of $\mathrm{SL}(2, \mathbb{R})$, given as in (1.21) by $\alpha = -1$, $\beta = 0$, acts trivially. The norm of f is given by ($z = e^{i\theta}$)

$$\|f\|_{\mathcal{C}_s} = \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \frac{|f(e^{i\theta})|^2}{|e^{i\theta} - e^{i\phi}|^{2-2\Delta}}, \quad \Delta = 1/2 + s. \quad (1.34)$$

Remark 1.3.13. We have

$$\mathcal{C}_s \simeq \bigoplus_{k \in \mathbb{Z}} V_k,$$

where V_k are the one-dimensional irreducible representations of $\mathrm{Spin}(2)$. The spectrum of L_0 for vectors inside $\mathcal{C}_s$ consists of all integers and each integer appears exactly once. The eigenvalue of the quadratic Casimir corresponding to the irrep $\mathcal{C}_s$ is $-1/4 + s^2$.

Coherent States

We now construct the coherent states, these are functions living inside the irrep $\mathcal{D}_{n/2}$ or $\overline{\mathcal{D}}_{n/2}$. The construction of these states mimics the one presented in [KMP24], extended in the present paper to the case when n is an odd integer.

Definition 1.3.14. The coherent state $\mathcal{O}_{n/2}(z)$ for $z \in \mathbb{D}$ is identified with a holomorphic function f on $\mathbb{D}'$, i.e.

$$\begin{aligned} \mathcal{O}_{1/2}(z)(w) &:= \sqrt{\frac{1}{2\pi}} (z - w)^{-1}, \\ \mathcal{O}_{n/2}(z)(w) &= \sqrt{\frac{n-1}{\pi}} (z - w)^{-n}, \quad n > 1, \end{aligned} \quad (1.35)$$

where $w \in \mathbb{D}'$.

Definition 1.3.15. The coherent state $\tilde{\mathcal{O}}_{n/2}(z)$ for $z \in \mathbb{D}'$ is identified with a holomorphic function f on $\mathbb{D}$. Explicitly we have, for $z \in \mathbb{D}'$ and $w \in \mathbb{D}$,

$$\begin{aligned} \tilde{\mathcal{O}}_{1/2}(z)(w) &:= \sqrt{\frac{1}{2\pi}} (z - w)^{-1}, \\ \tilde{\mathcal{O}}_{n/2}(z)(w) &= \sqrt{\frac{n-1}{\pi}} (z - w)^{-n}, \quad n > 1. \end{aligned} \quad (1.36)$$

Proposition 1.3.16. *The inner products of coherent states are given by*

$$\begin{aligned} (\mathcal{O}_{n/2}(z_1), \mathcal{O}_{n/2}(z_2))_{\mathcal{D}_{n/2}} &= (1 - \bar{z}_1 z_2)^{-n}, \\ (\tilde{\mathcal{O}}_{n/2}(z_1), \tilde{\mathcal{O}}_{n/2}(z_2))_{\bar{\mathcal{D}}_{n/2}} &= (\bar{z}_1 z_2 - 1)^{-n}. \end{aligned} \quad (1.37)$$

Proof. See [KMP24] for the case when n is an even integer. The extended version for $n \in \mathbb{Z}_+$ follows from the definition of the inner product for $\mathcal{D}_{n/2}$ and $\bar{\mathcal{D}}_{n/2}$, $\square$

For a unitary representation R of G we write R^∞ for its Fréchet space of smooth vectors. We borrow the following propositions from [KMP24], which will be useful for our purposes:

Proposition 1.3.17 (Proposition 3.3 [KMP24]). *The coherent states $\mathcal{O}$ and $\tilde{\mathcal{O}}$ take values in $\mathcal{D}_{n/2}^\infty$ and $\bar{\mathcal{D}}_{n/2}^\infty$ respectively. They are holomorphic as functions $\mathbb{D} \rightarrow \mathcal{D}_{n/2}^\infty$ and $\mathbb{D}' \rightarrow \bar{\mathcal{D}}_{n/2}^\infty$. The span of the coherent states $\mathcal{O}(z) \in \mathcal{D}_{n/2}$ for $z \in \mathbb{D}$ is dense in $\mathcal{D}_{n/2}$. The span of the coherent states $\tilde{\mathcal{O}}(z) \in \bar{\mathcal{D}}_{n/2}$ for $z \in \mathbb{D}'$ is dense in $\bar{\mathcal{D}}_{n/2}$.*

Remark 1.3.18. (See the proof of Proposition 3.3 of [KMP24] and the references therein.) The reproducing kernel for $\bar{\mathcal{D}}_{n/2}$ is given by

$$K_w(z) := \begin{cases} \frac{n-1}{\pi} (1 - \bar{w}z)^{-n}, & n > 1 \\ \frac{1}{2\pi} (1 - \bar{w}z)^{-1}, & n = 1 \end{cases},$$

i.e., for $w \in \mathbb{D}$ and $f \in \bar{\mathcal{D}}_{n/2}$, we have

$$(K_w, f) = f(w)$$

Thus, the coherent states are related to the reproducing kernel

$$K_w = \begin{cases} \sqrt{\frac{n-1}{\pi}} (\bar{w})^{-n} \widetilde{\mathcal{O}_{n/2}((\bar{w})^{-1})}, & n > 1 \\ \sqrt{\frac{1}{2\pi}} (\bar{w})^{-1} \widetilde{\mathcal{O}_{n/2}((\bar{w})^{-1})}, & n = 1 \end{cases},$$

and $\bar{\mathcal{D}}_{n/2}$ is a reproducing kernel Hilbert space. Similar statements apply to $\mathcal{D}_{n/2}$.

Proposition 1.3.19 (Proposition 3.4 [KMP24]). *We have $L^2(\tilde{\Gamma} \backslash G)^\infty = C^\infty(\tilde{\Gamma} \backslash G)$.*

Finally we define coherent states inside continuous series irreps.

Definition 1.3.20 (See section 3.4 of [KMP24]). The coherent states inside R_k where $R_k = \mathcal{P}_{i\nu}^+$ with $\lambda_k^{(0)} = 1/4 + \nu^2$ or $R_k = \mathcal{C}_s$ with $\lambda_k^{(0)} = 1/4 - s^2$ are defined as R_k^∞ -valued distributions on $C^\infty(\partial\mathbb{D})$ by

$$\mathcal{O}(f) := N_{\Delta_k}^{(0)} f, \quad f \in C^\infty(\partial\mathbb{D}), \quad (1.38)$$

where $N_{\Delta_k}^{(0)} > 0$ is chosen to ensure $N_{\Delta_k}^{(0)} \|1\|_{R_k} = 1$.

Definition 1.3.21. The coherent states inside $\mathcal{P}_{i\nu}^-$ with $\lambda_k^{(1/2)} = 1/4 + \nu^2$ are defined as R_k^∞ -valued distributions on $C^\infty(\partial\mathbb{D})$ by

$$\mathcal{O}(f) = N_{\Delta_k}^{(1/2)} f, \quad f \in C^\infty(\partial\mathbb{D}), \quad (1.39)$$

where $N_{\Delta_k}^{(1/2)} > 0$ is chosen to ensure $N_{\Delta_k}^{(1/2)} \|z\|_{R_k} = 1$.

Spectrum of $L^2(\tilde{\Gamma} \backslash G)$

The space $\tilde{\Gamma} \backslash G$ is the spin bundle over the compact orbifold X . We would like to study the space $L^2(\tilde{\Gamma} \backslash G)$, consisting of equivalence class of square integrable functions $F : G \rightarrow \mathbb{C}$ such that $F(\tilde{\gamma}g) = F(g)$ for all $\tilde{\gamma} \in \tilde{\Gamma}$ and $g \in G$. We normalize the Haar measure on G in a way such that $\mu(\tilde{\Gamma} \backslash G) = 1$. Subsequently, we define the inner product as

$$(F_1, F_2) = \int_{\tilde{\Gamma} \backslash G} dg \overline{F_1(g)} F_2(g). \quad (1.40)$$

The inner product induces the following norm on $F \in L^2(\tilde{\Gamma} \backslash G)$:

$$\|F\|^2 := (F, F). \quad (1.41)$$

We can turn $L^2(\tilde{\Gamma} \backslash G)$ into a representation of $\mathrm{SL}(2, \mathbb{R})$ by defining the following G -action: $\tilde{g} \in G$ acts on elements of $L^2(\tilde{\Gamma} \backslash G)$ as

$$[\tilde{g} \cdot F](g) := F(g\tilde{g}). \quad (1.42)$$

It is easy to verify that the norm of F , $\|F\| \equiv (F, F)$ is G invariant and thus $L^2(\tilde{\Gamma} \backslash G)$ indeed becomes a representation of G . Recalling the NAK decomposition of $\mathrm{SL}(2, \mathbb{R})$:

$$g(x, y, \theta) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},$$

we will often write the functions $F(g)$ as $F(x, y, \theta)$.

Proposition 1.3.22. *The elements of the complexified Lie algebra corresponding to $\mathrm{SL}(2, \mathbb{R})$ act on $F \in L^2(\tilde{\Gamma} \backslash G)$ as follows:*

$$\begin{aligned} L_0 F(x, y, \theta) &= i \partial_\theta F(x, y, \theta), \\ L_1 F(x, y, \theta) &= e^{-i\theta} [y(\partial_x + i \partial_y) + \partial_\theta] F(x, y, \theta), \\ L_{-1} F(x, y, \theta) &= -e^{i\theta} [y(\partial_x - i \partial_y) + \partial_\theta] F(x, y, \theta). \end{aligned} \tag{1.43}$$

The quadratic Casimir $C_2 := L_0^2 - \frac{1}{2}(L_{-1}L_1 + L_1L_{-1})$ acts as

$$C_2 F(x, y, \theta) = [y^2(\partial_x^2 + \partial_y^2) - 2y\partial_x\partial_\theta] F(x, y, \theta). \tag{1.44}$$

Proof. The above follows the transformation law, given by eq. (1.42) and NAK decomposition of $\mathrm{SL}(2, \mathbb{R})$. $\square$

Proposition 1.3.23. *For a cocompact $\tilde{\Gamma} \subset \mathrm{SL}(2, \mathbb{R})$, such that $-Id \notin \tilde{\Gamma}$, we have a discrete decomposition of $L^2(\tilde{\Gamma} \backslash G)$ in terms of irreducible representations. In particular we have*

$$L^2(\tilde{\Gamma} \backslash G) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} (\mathcal{D}_{n/2} \oplus \overline{\mathcal{D}}_{n/2}) \oplus \bigoplus_{k=1}^{\infty} \mathcal{C}_{\lambda_k^{(0)}} \oplus \bigoplus_{k=1}^{\infty} \mathcal{P}_{\lambda_k^{(1/2)}}^{-}. \tag{1.45}$$

Remark 1.3.24. The group action is transitive, hence the trivial irrep $\mathbb{C}$ appears exactly once. This corresponds to the constant functions on X .

Remark 1.3.25. $\mathcal{D}_{n/2}$ is unitarily isomorphic to $\mathbb{C}^{\ell_{n/2}} \otimes \mathcal{D}_{n/2}$, where $\ell_{n/2}$ is the number of times $\mathcal{D}_{n/2}$ appears inside $L^2(\tilde{\Gamma} \backslash G)$. The $\overline{\mathcal{D}}_{n/2}$ is unitarily isomorphic to $\mathbb{C}^{\ell_{n/2}} \otimes \overline{\mathcal{D}}_{n/2}$, where $\ell_{n/2}$ is the number of times $\overline{\mathcal{D}}_{n/2}$ appears inside $L^2(\tilde{\Gamma} \backslash G)$.

Proposition 1.3.26. $\ell_{n/2}$ is the number of independent normalized holomorphic modular forms of weight n . The number of independent normalized antiholomorphic modular forms of weight n is also given by $\ell_{n/2}$.

Proof. Consider a lowest weight vector $F_{n/2,a}$ inside $\mathcal{D}_{n/2}$. Clearly, $F_{n/2,a} \in V_{n/2}$ and $e^{in\theta/2}F_{n/2,a}$ is independent of θ . Hence, one can define

$$h_{n,a}(x, y) := y^{-n/2}e^{in\theta/2}F_{n/2,a}(x, y, \theta). \quad (1.46)$$

The $\tilde{\Gamma}$ invariance of F implies that $\tilde{h}_{n,a}(z) := h_{n,a}(\operatorname{Re}(z), \operatorname{Im}(z))$ transforms like a holomorphic modular form of weight n for $\gamma \in \tilde{\Gamma}$. Recalling eq. (1.19) and Proposition 1.2.6, this action can be naturally extended to $\bar{\Gamma}$ by using χ . The fact that $F_{n/2,a}$ is a lowest weight vector translates to $L_1 F_{n/2,a} = 0$, which implies $\tilde{h}_{n,a}(z)$ is holomorphic. A similar proof holds for $\bar{\mathcal{D}}_{n/2}$ by considering the highest weight vector, belonging to $V_{-n/2}$. Hence the proposition follows. $\square$

Remark 1.3.27. $\mathcal{C}_{\lambda_k^{(0)}}$ is unitarily isomorphic to

1. $\mathbb{C}^{d_k} \otimes \mathcal{C}_s$ with $s := \sqrt{1/4 - \lambda_k^{(0)}}$ if $\lambda_k^{(0)} < 1/4$,
2. $\mathbb{C}^{d_k} \otimes \mathcal{P}_{i\nu}^+$ with $\nu := \sqrt{\lambda_k^{(0)} - 1/4}$ if $\lambda_k^{(0)} \geq 1/4$,

where d_k is the number of times $\mathcal{C}_s$ or $\mathcal{P}_{i\nu}^+$ appears inside $L^2(\tilde{\Gamma} \backslash G)$.

Proposition 1.3.28. d_k is the degeneracy of $\lambda_k^{(0)}$, an eigenvalue of the Laplace operator on X .

Proof. Recall

$$\mathcal{C}_{\lambda_k^{(0)}} \simeq \mathbb{C}^{d_k} \otimes \bigoplus_{k \in \mathbb{Z}} V_k.$$

Consider a vector inside V_0 , it gets mapped to a vector F inside $\mathcal{C}_{\lambda_k^{(0)}}$. Since $i\partial_\theta F = L_0 F = 0$, we can define $\phi(x, y) := F(x, y, \theta)$ and it follows $\phi(x, y)$ is a $\tilde{\Gamma}$ invariant

function on X . Furthermore, we have

$$C_2\phi(x, y) = y^2(\partial_x^2 + \partial_y^2)\phi(x, y). \quad (1.47)$$

On the other hand, C_2 of $\mathcal{C}_{\lambda_k^{(0)}}$ is $\lambda_k^{(0)}$. Hence, $\phi(x, y)$ is a $\tilde{\Gamma}$ invariant eigenfunction of the Laplace operator, i.e. $y^2(\partial_x^2 + \partial_y^2)$, with eigenvalue $\lambda_k^{(0)}$, and the proposition follows. $\square$

Remark 1.3.29. $\mathcal{P}_{\lambda_k^{(1/2)}}^-$ is unitarily isomorphic to $\mathbb{C}^{d'_k} \otimes \mathcal{P}_{i\nu}^-$ with $\nu := \sqrt{\lambda_k^{(1/2)} - 1/4}$ and $\lambda_k^{(1/2)} > 1/4$, where d'_k is the number of times $\mathcal{P}_{i\nu}^-$ appears inside $L^2(\tilde{\Gamma} \backslash G)$.

Proposition 1.3.30. d'_k is the degeneracy of $\lambda_k^{(1/2)}$, an eigenvalue of the weight-1 automorphic Laplace operator (see defn. 1.2.17) on X .

Proof. Recall

$$\mathcal{P}_{\lambda_k^{(1/2)}}^- \simeq \mathbb{C}^{d'_k} \otimes \bigoplus_{k \in 1/2 + \mathbb{Z}} V_k.$$

Consider a vector inside $V_{1/2}$, it gets mapped to a vector $\Psi(x, y, \theta)$ inside $\mathcal{P}_{\lambda_k^{(1/2)}}^-$. Define $\psi(x, y) := e^{i\theta/2}\Psi(x, y, \theta)$. Since $i\partial_\theta\Psi = L_0\Psi = 1/2$, it follows that $\psi(x, y)$ is a $\tilde{\Gamma}$ -equivariant form on X . Furthermore, we have

$$C_2\psi(x, y) = [y^2(\partial_x^2 + \partial_y^2) - iy\partial_x]\psi(x, y). \quad (1.48)$$

On the other hand, C_2 of $\mathcal{P}_{\lambda_k^{(1/2)}}^-$ is $\lambda_k^{(1/2)}$. Hence, $\psi(x, y)$ is a $\tilde{\Gamma}$ -equivariant eigenform of the weight-1 automorphic Laplacian operator, i.e. $y^2(\partial_x^2 + \partial_y^2) - iy\partial_x$. Therefore, the proposition follows. $\square$

Now let us identify the coherent states inside $L^2(\tilde{\Gamma} \backslash G)$.

Definition 1.3.31 (Coherent States inside $\mathcal{D}_{n/2}^\infty$ and $\overline{\mathcal{D}_{n/2}^\infty}$). Recall Proposition(1.3.23) and Remark 1.3.25 that $\mathcal{D}_{n/2}$ and $\overline{\mathcal{D}_{n/2}}$ are unitarily isomorphic to $\mathbb{C}^{\ell_{n/2}} \otimes \mathcal{D}_{n/2}$ and $\mathbb{C}^{\ell_{n/2}} \otimes \overline{\mathcal{D}_{n/2}}$ respectively.

Following [KMP24], we choose an isomorphism $\tau_{n/2} : \mathbb{C}^{\ell_{n/2}} \otimes \mathcal{D}_{n/2} \rightarrow \mathcal{D}_{n/2}$ and define

$$\mathcal{O}_{n/2,a}(z) := \tau_{n/2}(e_a \otimes \mathcal{O}(z)) \in \mathcal{D}_{n/2}^\infty \subset \mathcal{C}^\infty(\tilde{\Gamma} \backslash G). \quad (1.49)$$

Here e_a for $a = 1, 2, \dots, \ell_{n/2}$ is the standard basis for $\mathbb{C}^{\ell_{n/2}}$. We further define $\bar{\tau}_{n/2} : \mathbb{C}^{\ell_{n/2}} \otimes \bar{\mathcal{D}}_{n/2} \rightarrow \bar{\mathcal{D}}_{n/2}$ as

$$\bar{\tau}_{n/2}(v \otimes f) = \overline{\tau_{n/2}(\bar{v} \otimes \bar{f})}, \quad \bar{f}(z) := z^{-n} \overline{f(\bar{z}^{-1})}, \quad (1.50)$$

and

$$\tilde{\mathcal{O}}_{n/2,a}(z) := \bar{\tau}_{n/2}(e_a \otimes \tilde{\mathcal{O}}(z)) \in \bar{\mathcal{D}}_{n/2}^\infty \subset \mathcal{C}^\infty(\tilde{\Gamma} \backslash G), \quad (1.51)$$

Proposition 1.3.32 ([KMP24]). *The generators of the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ act on coherent states as follows:*

$$\begin{aligned} L_{-1} \cdot \mathcal{O}_{n/2,a}(z) &= \partial_z \mathcal{O}_{n/2,a}(z), & L_{-1} \cdot \tilde{\mathcal{O}}_{n/2,a}(z) &= \partial_z \tilde{\mathcal{O}}_{n/2,a}(z), \\ L_0 \cdot \mathcal{O}_{n/2,a}(z) &= (z\partial_z + n/2) \mathcal{O}_{n/2,a}(z), & L_0 \cdot \tilde{\mathcal{O}}_{n/2,a}(z) &= (z\partial_z + n/2) \tilde{\mathcal{O}}_{n/2,a}(z), \\ L_1 \cdot \mathcal{O}_{n/2,a}(z) &= (z^2\partial_z + nz) \mathcal{O}_{n/2,a}(z), & L_1 \cdot \tilde{\mathcal{O}}_{n/2,a}(z) &= (z^2\partial_z + nz) \tilde{\mathcal{O}}_{n/2,a}(z). \end{aligned} \quad (1.52)$$

Proposition 1.3.33 ([KMP24]). *We have the following identity:*

$$\overline{(\mathcal{O}_{n/2,a}(z))} = (\bar{z})^{-n} \tilde{\mathcal{O}}_{n/2,a}((\bar{z})^{-1}).$$

Correlators of smooth functions on G

Definition 1.3.34. We consider smooth functions $F_1, F_2, \dots, F_n$ on $\tilde{\Gamma} \backslash G$. Their correlator is defined as

$$\langle F_1 F_2 \cdots F_n \rangle := \int_{\tilde{\Gamma} \backslash G} dg \, F_1(g) F_2(g) \cdots F_n(g). \quad (1.53)$$

Since the functions F_i are smooth and $\tilde{\Gamma} \backslash G$ is compact, the correlator is well-defined and finite.

Proposition 1.3.35. *The correlator $\langle \dots \rangle : \left(\mathcal{C}^\infty(\tilde{\Gamma} \backslash G) \right)^N \rightarrow \mathbb{C}$ is a G invariant functional.*

Proposition 1.3.36. *The two-function correlator is given by*

$$\langle \mathcal{O}_{m,i}(z_1) \tilde{\mathcal{O}}_{n,j}(z_2) \rangle = \frac{\delta_{m,n} \delta_{i,j}}{(z_2 - z_1)^{2n}}, \quad \text{where } 2m, 2n \in \mathbb{Z}_+ \quad (1.54)$$

Proof. Using Proposition 1.3.33 alongside with orthogonality of the decomposition given in Proposition 1.3.23, we find

$$\langle \mathcal{O}_{m,i}(z_1) \tilde{\mathcal{O}}_{n,j}(z_2) \rangle = z_2^{-2n} \left(\mathcal{O}_{n,j}(\bar{z}_2^{-1}), \mathcal{O}_{m,i}(z_1) \right) = \delta_{m,n} \delta_{i,j} z_2^{-2n} \left(\mathcal{O}_n(\bar{z}_2^{-1}), \mathcal{O}_n(z_1) \right),$$

and then we use Proposition 1.3.16. □

Proposition 1.3.37. *Define $z_{ij} = z_i - z_j$. The three-function correlator between vectors inside $\mathcal{D}_k, \mathcal{D}_l$ and $\overline{\mathcal{D}}_m$ with $2k, 2l, 2m \in \mathbb{Z}_+$ is given by*

$$\left\langle \mathcal{O}_{k,a}(z_1) \mathcal{O}_{l,b}(z_2) \tilde{\mathcal{O}}_{m,c}(z_3) \right\rangle = \frac{f_{(k;a)(l;b)}^{m;c}}{z_{21}^{k+l-m} z_{31}^{k+m-l} z_{32}^{l+m-k}}, \quad (1.55)$$

for some constants $f_{(k;a)(l;b)}^{m;c}$.

Proof. The result follows from using the same steps involving G -invariance to prove a similar result in [KMP24] for $k, l, m \in \mathbb{Z}$. □

Remark 1.3.38. We recall that

$$\mathcal{O}_{k,a}(0) = F_{k,a}, \quad \mathcal{O}_{l,b}(0) = F_{l,b}, \quad \lim_{z \rightarrow \infty} z^{2m} \mathcal{O}_{m,c}(z) = \overline{F}_{m,c}. \quad (1.56)$$

Thus we have

$$f_{(k;a)(l;b)}^{k+l,c} = \int_{\tilde{\Gamma} \backslash G} dg \, F_{k,a} F_{l,b} \overline{F}_{k+l,c} = \frac{1}{\text{vol}(X)} \int_X dx \, dy \, y^{2(k+l-1)} h_{k,a} h_{l,b} \overline{h}_{m,c}. \quad (1.57)$$

Here the second equality follows from the definition of h , as given by eq. (1.46).

Definition 1.3.39. The G -invariant cross-ratio (denoted as z) for four points z_1, z_2, z_3, z_4 is given by

$$z := \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad z_{ij} := z_i - z_j. \quad (1.58)$$

In what follows, we will often suppress the degeneracy index a in $\mathcal{O}_{(n,a)}$ and write it as $\mathcal{O}_n$ for notational brevity.

Proposition 1.3.40. *Let $2n_1, 2n_2, 2n_3, 2n_4 \in \mathbb{Z}_+$.*

$$\langle \mathcal{O}_{n_1}(z_1) \mathcal{O}_{n_2}(z_2) \tilde{\mathcal{O}}_{n_3}(z_3) \tilde{\mathcal{O}}_{n_4}(z_4) \rangle = \frac{1}{z_{12}^{n_1+n_2} z_{34}^{n_3+n_4}} \left(\frac{z_{24}}{z_{14}} \right)^{n_1-n_2} \left(\frac{z_{14}}{z_{13}} \right)^{n_3-n_4} g_{\{n_1, n_2, n_3, n_4\}}(z), \quad (1.59)$$

for some function $g_{\{n_1, n_2, n_3, n_4\}}$ of the cross-ratio z .

Proof. The proof mimics the one appearing in [KMP24] for $n_1 = n_2 = n \in \mathbb{Z}$. The main idea is that G -invariance implies that the 4-function correlator can be written in terms of a single variable function of the cross-ratio z . $\square$

Product expansion

$$F_1 F_2 = P_{\mathbb{C}}(F_1 F_2) + \sum_{n=1}^{\infty} \left[P_{\mathcal{D}_{n/2}}(F_1 F_2) + P_{\overline{\mathcal{D}}_{n/2}}(F_1 F_2) \right] + \sum_{k=1}^{\infty} P_{\mathcal{E}_{\lambda_k^{(0)}}}(F_1 F_2) + \sum_{k=1}^{\infty} P_{\mathcal{D}_{\lambda_k^{(1/2)}}}^-(F_1 F_2). \quad (1.60)$$

Here P_H refers to the orthogonal projection onto the irrep H .

We choose F_i from the irrep H_i . Now the key point is that given an irrep H_m , G -invariance constrains the dependence of $P_{H_m}(F_1 F_2)$ on F_1 and F_2 upto finitely many constants. And those finitely many constants are related to triple product integrals between elements of H_1, H_2 and H_m . The spectral identities come from the associativity constraints i.e.

$$((F_1 F_2) F_3) = (F_1 (F_2 F_3)). \quad (1.61)$$

In what follows we will take $F_1 = \mathcal{O}_{n_1}$, $F_2 = \mathcal{O}_{n_2}$, $F_3 = \tilde{\mathcal{O}}_{n_3}$. A convenient way to encode these constraints is to consider 4 point functions of the form $\langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \tilde{\mathcal{O}}_{n_3} \tilde{\mathcal{O}}_{n_4} \rangle$.

Now we introduce some lemmas which are relevant to the product expansion of coherent states.

Lemma 1.3.2. $P_H(\mathcal{O}_{n_1}(z_1) \mathcal{O}_{n_2}(z_2)) = 0$ unless $H = \mathcal{D}_p$ with $p - n_1 - n_2 \in \mathbb{Z}_{\geq 0}$. If $\mathcal{O}_{n_1} = \mathcal{O}_{n_2}$ with $n_1 = n_2$, we further have $p - 2n = 0 \pmod{2}$.

Proof. The proof is similar to that of Lemma 3.9 of [KMP24] with minor modifications to include the cases when n_1 or n_2 are half-integers and possibly different. $\square$

Proposition 1.3.41.

$$\mathcal{O}_{n_1, a_1}(z_1) \mathcal{O}_{n_2, a_2}(z_2) = \sum_{p - n_1 - n_2 \in \mathbb{Z}_{\geq 0}} \sum_{a=1}^{\ell_p} f_{(n_1; a_1)(n_2; a_2)}^{p; a} \tau_p(e_a \otimes C_p(z_1, z_2)). \quad (1.62)$$

Here $C_p(z_1, z_2) \in \mathcal{D}_p$ is defined as

$$C_p(z_1, z_2)(z_3) := \sqrt{\frac{2p-1}{\pi}} \frac{1}{z_{12}^{n_1+n_2-p} z_{13}^{n_1+p-n_2} z_{23}^{n_2+p-n_1}}, \quad (1.63)$$

and $f_{(n_1; a_1)(n_2; a_2)}^{p; a}$ are some constants. If $n_1 = n_2 = n$, $f_{(n_1; a_1)(n_2; a_2)}^{p; a}$ vanishes for $p = 2n + 1$.

Proof. The proof is similar to that of Lemma 3.10 of [KMP24]. The key step is to use the fact that $\mathcal{D}_p$ is a reproducing kernel Hilbert space to relate $C_p(z_1, z_2)(z)$ with the three function correlator $\langle \mathcal{O}_{n_1, a_1}(z_1) \mathcal{O}_{n_2, a_2}(z_2) \tilde{\mathcal{O}}_{p, a}(z) \rangle$, followed by the use of proposition 1.3.37. $\square$

Lemma 1.3.3. $P_H(\mathcal{O}_n(z_1) \tilde{\mathcal{O}}_n(z_2)) = 0$ unless $H = \mathbb{C}$ or $H = \mathcal{C}_{\lambda_k^0}$.

Proof. The proof is similar to that of Lemma 3.11 of [KMP24] with minor modifications. $\mathcal{P}_{i\nu}^-$ does not appear since the center of $\text{SL}(2, \mathbb{R})$ acts trivially on $\mathcal{O}_n(z_1) \tilde{\mathcal{O}}_n(z_2)$ while the center acts nontrivially on vectors inside $\mathcal{P}_{i\nu}^-$. $\square$

Lemma 1.3.4. *Let $n_1 > n_2$ such that $2(n_1+n_2) = 0 \pmod{2}$. $P_H \left(\mathcal{O}_{n_1}(z_1) \tilde{\mathcal{O}}_{n_2}(z_2) \right) = 0$ unless $H = \mathcal{C}_{\lambda_k^{(0)}}$ or $H = \mathcal{D}_m$ with $1 \leq m \leq n_1 - n_2$ and $m \in \mathbb{Z}$.*

Proof. The proof is similar to that of Lemma 3.11 of [KMP24], with minor modifications like in the previous lemma. $\square$

Lemma 1.3.5. *Let $n_1 < n_2$ such that $2(n_1+n_2) = 0 \pmod{2}$. $P_H \left(\mathcal{O}_{n_1}(z_1) \tilde{\mathcal{O}}_{n_2}(z_2) \right) = 0$ unless $H = \mathcal{C}_{\lambda_k^{(0)}}$ or $H = \overline{\mathcal{D}}_m$ with $1 \leq m \leq n_2 - n_1$ and $m \in \mathbb{Z}$.*

Proof. The proof is similar to the above. $\square$

Proposition 1.3.42 (See Lemma 3.12 of [KMP24]). *Let $n_2 - n_1 \in \mathbb{Z}$ and $n_1 + n_2 \in \mathbb{Z}$.*

We have

$$P_{\mathcal{C}_{\lambda_k^{(0)}}} \left(\mathcal{O}_{n_1,a}(z_1) \tilde{\mathcal{O}}_{n_2,b}(z_2) \right) = \sum_{r=1}^{d_k} c_{(n_1;a)(n_2;b)}^{k;r} \kappa_k^{(0)} \left(e_r \otimes \tilde{C}_k^{(0)}(z_1, z_2) \right), \quad (1.64)$$

for some constants $c_{(n_1;a)(n_2;b)}^{k;r}$. Here $\kappa_k^{(0)}$ is the unitary isomorphism between $\mathbb{C}^{d_k} \otimes R_k$ and $\mathcal{C}_{\lambda_k^{(0)}}$, where $R_k = \mathcal{P}_{i\nu}^+$ with $\lambda_k^{(0)} = 1/4 + \nu^2$ or $R_k = \mathcal{C}_s$ with $\lambda_k^{(0)} = 1/4 - s^2$. Furthermore, $\tilde{C}_k^{(0)}(z_1, z_2) \in R_k$, and is given by

$$\tilde{C}_k^{(0)}(z_1, z_2)(z_0) := \frac{N_k z_2^{-2n_2} z_0^{n_2-n_1}}{(1 - z_1 z_2^{-1})^{n_1+n_2-\Delta_k} (1 - z_1 z_0^{-1})^{n_1-n_2+\Delta_k} (1 - z_2^{-1} z_0)^{n_2-n_1+\Delta_k}}, \quad |z_0| = 1.$$

Here $N_k \in \mathbb{C}$ is a constant such that the constant function equal to N_k has unit norm in R_k .

Proposition 1.3.43. *Let $n_1 > n_2$ such that $2(n_1 + n_2) = 0 \pmod{2}$. We have for $p \leq n_1 - n_2$ and $p \in \mathbb{Z}_{\geq 0}$,*

$$P_{\mathcal{D}_p} \left(\mathcal{O}_{n_1,a_1}(z_1) \tilde{\mathcal{O}}_{n_2,a_2}(z_2) \right) = \sum_{a=1}^{\ell_p} f_{(n_1;a_1)(p;a)}^{n_2;a_2} \tau_p \left(e_a \otimes \tilde{C}_p(z_1, z_2) \right). \quad (1.65)$$

Here $\tilde{C}_p(z_1, z_2) \in \mathcal{D}_p$ is defined as

$$\tilde{C}_p(z_1, z_2)(z_3) := \sqrt{\frac{2p-1}{\pi}} \frac{1}{z_{13}^{n_1+n_2-p} z_{12}^{n_1+p-n_2} z_{32}^{n_2+p-n_1}}. \quad (1.66)$$

Remark 1.3.44. The structure constant $f_{(n_1;a_1)(p;a)}^{n_2;a_2}$ appearing here is the same as the one appearing in $\langle \mathcal{O}_{n_1,a_1}(z_1) \mathcal{O}_{p,a}(z_3) \tilde{\mathcal{O}}_{n_2,a_2}(z_2) \rangle$.

Remark 1.3.45. A similar statement can be made for $n_1 < n_2$.

Lemma 1.3.6. *Let $n_1 \in \mathbb{N}/2$ and $n_2 \in \mathbb{N}/2$. Further assume that $n_1 < n_2$ (or $n_1 > n_2$) such that $2(n_1 + n_2) = 1(\text{mod } 2)$. $P_H \left(\mathcal{O}_{n_1}(z_1) \tilde{\mathcal{O}}_{n_2}(z_2) \right) = 0$ unless $H = \mathcal{P}_{\lambda_k^{(1/2)}}^-$ or $H = \overline{\mathcal{D}}_m$ ($\mathcal{D}_m$) with $1/2 \leq m \leq |n_2 - n_1|$ and $m \in 1/2 + \mathbb{Z}$.*

Proof. $\mathbb{C}$ and $\mathcal{C}_{\lambda_k^{(0)}}$ do not appear because the center acts nontrivially on $\mathcal{O}_{n_1}(z_1) \tilde{\mathcal{O}}_{n_2}(z_2)$ for $2(n_1 + n_2) = 1(\text{mod } 2)$. The rest of the proof is similar to that of Lemma 3.11 of [KMP24]. $\square$

Proposition 1.3.46. *Let $n_1 \in \mathbb{N}/2$ and $n_2 \in \mathbb{N}/2$ be such that $2(n_1 + n_2) = 1(\text{mod } 2)$.*

$$P_{\mathcal{P}_{\lambda_k^{(1/2)}}^-} \left(\mathcal{O}_{n_1,a}(z_1) \tilde{\mathcal{O}}_{n_2,b}(z_2) \right) = \sum_{r=1}^{d'_k} s_{(n_1;a)(n_2;b)}^{k;r} \kappa_k^{(1/2)} \left(e_r \otimes \tilde{C}_k^{(1/2)}(z_1, z_2) \right), \quad (1.67)$$

for some constants $s_{(n_1;a)(n_2;b)}^{k;r}$. Here $\kappa_k^{(1/2)}$ is the unitary isomorphism between $\mathbb{C}^{d'_k} \otimes \mathcal{P}_{i\nu}^-$ and $\mathcal{P}_{\lambda_k^{(1/2)}}^-$, where $\lambda_k^{(1/2)} = 1/4 + \nu^2$ and $\tilde{C}_k^{(1/2)}(z_1, z_2) \in \mathcal{P}_{i\nu}^-$ is given by

$$\tilde{C}_k^{(1/2)}(z_1, z_2)(z_0) := \frac{N_k^{(1/2)} z_2^{-2n_2} z_0^{n_2-n_1+1/2}}{(1 - z_1 z_2^{-1})^{n_1+n_2-\Delta_k} (1 - z_1 z_0^{-1})^{n_1-n_2+\Delta_k} (1 - z_2^{-1} z_0)^{n_2-n_1+\Delta_k}}. \quad (1.68)$$

Here $N_k^{(1/2)} \in \mathbb{C}$ is a constant such the constant function equal to $N_k^{(1/2)}$ has unit norm in $\mathcal{P}_{i\nu}^-$.

Proof. The proof is similar to that of Lemma 3.12 of [KMP24]: here we need to perform projections onto the $\mathcal{P}_{i\nu}^-$ rather than the $\mathcal{P}_{i\nu}^+$ and the $\mathcal{C}_s$, but the techniques are exactly the same. $\square$

Proposition 1.3.47. *Let $n_1 > n_2$ such that $2(n_1 + n_2) = 1(\text{mod } 2)$. We have for $p \leq n_1 - n_2$ and $p \in 1/2 + \mathbb{Z}_{\geq 0}$*

$$P_{\mathcal{D}_p} \left(\mathcal{O}_{n_1,a_1}(z_1) \tilde{\mathcal{O}}_{n_2,a_2}(z_2) \right) = \sum_{a=1}^{\ell_p} f_{(n_1;a_1)(p;a)}^{n_2;a_2} \tau_p \left(e_a \otimes \tilde{C}_p(z_1, z_2) \right). \quad (1.69)$$

for some constants $f_{(n_1; a_1)(p; a)}^{n_2; a_2}$. Here $\tilde{C}_p(z_1, z_2) \in \mathcal{D}_p$ is defined as

$$\begin{aligned}\tilde{C}_p(z_1, z_2)(z_3) &:= \sqrt{\frac{2p-1}{\pi}} \frac{1}{z_{13}^{n_1+n_2-p} z_{12}^{n_1+p-n_2} z_{32}^{n_2+p-n_1}}, \quad p > 1/2 \\ \tilde{C}_{1/2}(z_1, z_2)(z_3) &:= \sqrt{\frac{1}{2\pi}} \frac{1}{z_{13}^{n_1+n_2-1/2} z_{12}^{n_1+1/2-n_2} z_{32}^{n_2+1/2-n_1}}.\end{aligned}\tag{1.70}$$

Equipped with the above lemmas, we can study correlators of the form $\langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \tilde{\mathcal{O}}_{n_2} \tilde{\mathcal{O}}_{n_1} \rangle$.

They are studied in [KMP24] for $n \in \mathbb{Z}_+$. They can easily be extended for half-integer n_i . In particular we have

Theorem 1.3.48 (Theorem 2.3 [KMP24]: Extended). *Let $g_0(z) := g_{\{n_1, n_2, n_2, n_1\}}(z)$ be as in eq. (1.59) and $n_{ij} := n_i - n_j$.*

1. $g_0(z)$ has the following expansion, known as the s -channel expansion:

$$g_0(z) = \sum_{p=n_1+n_2}^{\infty} \sum_{a=1}^{\ell_p} |f_{n_1, n_2}^{p, a}|^2 \mathcal{G}_p(n_1, n_2, n_2, n_1; z), \tag{1.71}$$

where

$$\mathcal{G}_p(n_1, n_2, n_3, n_4; z) := z^p {}_2F_1(p - n_{12}, p + n_{34}, 2p, z). \tag{1.72}$$

The s -channel sum and its derivatives converge uniformly on compact subsets of $\mathbb{C} \setminus (1, \infty)$. Furthermore, $f_{n_1, n_2}^{p, a} \in \mathbb{C}$ and satisfies

$$f_{n, n}^p = 0, \text{ if } p - 2n = 1 \pmod{2}. \tag{1.73}$$

2. $g_0(z)$ has the following expansion, known as the t -channel expansion:

$$g_0(z) = \frac{z^{n_1+n_2}}{(1-z)^{2n_2}} \left(1 + \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} c_{n_1, n_1}^{k; a} c_{n_2, n_2}^{k; a} \mathcal{H}_{\Delta_k^{(0)}}(n_1, n_2, n_2, n_1; z) \right), \tag{1.74}$$

where

$$\mathcal{H}_{\Delta}(n_1, n_2, n_2, n_1; z) := {}_2F_1\left(\Delta, 1 - \Delta, 1, \frac{z}{z-1}\right), \tag{1.75}$$

and $c_{n_i, n_i}^{k; a} \in \mathbb{R}$, $\Delta_k^{(0)} = \frac{1}{2} + i\sqrt{\lambda_k^{(0)} - 1/4}$. The t -channel sum and its derivatives converge uniformly on compact subsets of $\mathbb{C} \setminus (1, \infty)$.

Proof. The proof of Theorem 1.3.48 goes exactly the same way as that of Theorem 2.3 in [KMP24], by allowing n_1 and n_2 to take different values. $\square$

Theorem 1.3.49 (Theorem 2.3 [KMP24]: Extended). *Let $g_1(z) := g_{\{n_1, n_2, n_1, n_2\}}(z)$ be as in eq. (1.59) with $n_1 > n_2$ and $n_1 \in \mathbb{Z}_+$, $2n_2 = 1 \pmod{2}$. Define $n_{ij} := n_i - n_j$.*

1. $g_1(z)$ has the following expansion, known as the *s-channel expansion*:

$$g_1(z) = \sum_{p=n_1+n_2}^{\infty} \sum_{a=1}^{\ell_p} (-1)^{p-n_1-n_2} |f_{n_1, n_2}^{p, a}|^2 \mathcal{G}_p(n_1, n_2, n_1, n_2; z), \quad (1.76)$$

where $\mathcal{G}_p(n_1, n_2, n_3, n_4; z)$ is defined in (1.72).

The *s-channel sum* and its derivatives converge uniformly on compact subsets of $\mathbb{C} \setminus (1, \infty)$. Furthermore, $f_{n_1, n_2}^{p, a} \in \mathbb{C}$ and satisfies

$$f_{n, n}^p = 0, \text{ if } p - 2n = 1 \pmod{2}. \quad (1.77)$$

2. $g_1(z)$ has the following expansion, known as the *t-channel expansion*:

$$\begin{aligned} g_1(z) = & \frac{z^{n_1+n_2}}{(1-z)^{n_1+n_2}} \left(\sum_{k=1}^{\infty} \sum_{a=1}^{d'_k} |s_{n_1, n_2}^{k; a}|^2 \mathcal{H}_{\Delta_k^{(1/2)}}(n_1, n_2, n_1, n_2; z) + \right. \\ & \left. + \sum_{0 < m \leq n_1 - n_2} \sum_{a=1}^{\ell_m} |f_{n_2, (m; a)}^{n_1}|^2 \mathcal{H}_m(n_1, n_2, n_1, n_2; z) \right), \end{aligned} \quad (1.78)$$

where

$$\mathcal{H}_{\Delta}(n_1, n_2, n_1, n_2; z) := (1-z)^{n_{12}} {}_2F_1 \left(\Delta + n_{21}, 1 - \Delta + n_{21}, 1, \frac{z}{z-1} \right), \quad (1.79)$$

and $s_{n_1, n_2}^{k; a}, f_{n_2, (m; a)}^{n_1} \in \mathbb{C}$, $\Delta_k^{(1/2)} = \frac{1}{2} + i\sqrt{\lambda_k^{(1/2)} - 1/4}$. The *t-channel sum* and its derivatives converge uniformly on compact subsets of $\mathbb{C} \setminus (1, \infty)$.

Proof. The proof of Theorem 1.3.49 is similar to that of Theorem 2.3 in [KMP24] with some modifications.

For the part (1), we derive

$$g_1(z) = \sum_{p=n_1+n_2}^{\infty} \sum_{a=1}^{\ell_p} f_{n_1,n_2}^{p,a} \overline{f_{n_2,n_1}^{p,a}} \mathcal{G}_p(n_1, n_2, n_1, n_2; z),$$

in the same way we derive (1.71). In other words, we can derive the above by swapping the labels n_1 and n_2 in the discrete antiholomorphic coherent states in the correlator appearing in the theorem 1.3.48, leading to a similar swap in the R.H.S of (1.71). We then use Proposition 3.37 to deduce

$$f_{n_2,n_1}^{p,a} = (-1)^{p-n_1-n_2} \overline{f_{n_1,n_2}^{p,a}},$$

to arrive at (1.76).

For the part (2), we note that the lemma 1.3.6 implies that in the R.H.S of (1.78), $\mathcal{P}_{i\nu}^-$ (instead of $\mathcal{P}_{i\nu}^+$) and the discrete series representations appear. The rest of the proof involves deriving the expression for the t channel conformal block $\mathcal{H}$, the derivation goes in a similar way as in the proof of the theorem 1.3.48 with appropriate swapping of the labels. $\square$

Spectral Identities and Linear Programming

Definition 1.3.50. Define $r := 1 - m - n_1 - n_2$, $\tilde{n}_i := 1 - n_i$, and $\Delta := 1/2 + i\sqrt{\lambda - 1/4}$. We further define

$$\mathcal{F}_m^{(n_1, n_2, n_3, n_4)}(\lambda) := \oint_{z=0} \frac{dz}{2\pi i} z^{-2} \mathcal{G}_r(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4; z) \frac{z^{n_1+n_2}}{(1-z)^{n_2+n_3}} \mathcal{H}_{\Delta}(n_1, n_2, n_3, n_4; z). \quad (1.80)$$

Definition 1.3.51. We define $S_{p;(n_1, n_2)} := \sum_{a=1}^{\ell_p} |f_{n_1, n_2}^{p,a}|^2$.

Each four function correlator gives rise to a consistency condition by equating the s and t channel expansions. We can then obtain infinitely many conditions by Taylor expanding the original consistency condition around $z = 0$ and collecting the coefficient of z^m for each $m \in \mathbb{Z}_{\geq 0}$.

Proposition 1.3.52. *Let $m \in \mathbb{Z}_{\geq 0}$. The consistency conditions, one for each m , coming from the four function correlator, $\langle \mathcal{O}_{n_1}(z_1) \mathcal{O}_{n_2}(z_2) \tilde{\mathcal{O}}_{n_2}(z_3) \tilde{\mathcal{O}}_{n_1}(z_4) \rangle$ are given by*

$$S_{n_1+n_2+m;(n_1,n_2)} = \left(\mathcal{F}_m^{(n_1,n_2,n_2,n_1)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} c_{n_1,n_1}^{k;a} c_{n_2,n_2}^{k;a} \right] \mathcal{F}_m^{(n_1,n_2,n_2,n_1)}(\lambda_k^{(0)}) \right). \quad (1.81)$$

Proof. We recall Theorem 1.3.48 and equate the s -channel and t -channel expansions and integrate against $\mathcal{G}_r$ introduced in 1.3.50, followed by exchanging the sum and integral using uniform convergence of the expansion. The proposition will follow from the orthogonality condition

$$\oint_{z=0} \frac{dz}{2\pi i} z^{-2} \mathcal{G}_{1-m_1-n_1-n_2}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4; z) \mathcal{G}_{m_2+n_1+n_2}(n_1, n_2, n_3, n_4) = \delta_{m_1, m_2}, \quad (1.82)$$

where $\tilde{n}_i = 1 - n_i$. To prove (1.82), first note that $\mathcal{G}_p(n_1, n_2, n_3, n_4)$ satisfies the eigenvalue equation

$$D_{\{n_i\}} \mathcal{G}_p(n_1, n_2, n_3, n_4; z) = p(1-p) \mathcal{G}_p(n_1, n_2, n_3, n_4; z), \quad (1.83)$$

where the differential operator $D_{\{n_i\}}$ is given by

$$D_{\{n_i\}} = z^2(z-1) \frac{\partial^2}{\partial z^2} + (1 - n_{12} + n_{34}) z^2 \frac{\partial}{\partial z} - n_{12} n_{34} z. \quad (1.84)$$

By integrating by parts, one can show that $D_{\{n_i\}}$ satisfies

$$\oint_{z=0} \frac{dz}{2\pi i} z^{-2} F(z) (\mathcal{D}_{\{n_i\}} G(z)) = \oint_{z=0} \frac{dz}{2\pi i} z^{-2} (\mathcal{D}_{\{\tilde{n}_i\}} F(z)) G(z) \quad (1.85)$$

for any functions $F(z)$ and $G(z)$ holomorphic in an annulus around $z = 0$. Thus, for the left-hand side of (1.82) to be nonzero, the eigenvalues of $\mathcal{G}_{1-m_1-n_1-n_2}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4; z)$ and $\mathcal{G}_{m_2+n_1+n_2}(n_1, n_2, n_3, n_4)$ with respect to $D_{\{\tilde{n}_i\}}$ and $D_{\{n_i\}}$, respectively, must be equal, which implies $m_1 = m_2$. Finally, when $m_1 = m_2$, the left-hand side of (1.82) can be evaluated by residues, picking up the contribution of the simple pole at $z = 0$ with residue 1. $\square$

Proposition 1.3.53. *Let $m \in \mathbb{Z}_{\geq 0}$, $n_1 \in \mathbb{N}$ and $2n_2 = 1 \pmod{2}$. The consistency condition coming from $\langle \mathcal{O}_{n_1}(z_1) \mathcal{O}_{n_2}(z_2) \tilde{\mathcal{O}}_{n_1}(z_3) \tilde{\mathcal{O}}_{n_2}(z_4) \rangle$ is given by*

$$\begin{aligned}
 (-1)^m S_{n_1+n_2+m; (n_1, n_2)} = & \left(\sum_{k=1}^{\infty} \left[\sum_{a=1}^{d'_k} |s_{n_1, n_2}^{k; a}|^2 \right] \mathcal{F}_m^{(n_1, n_2, n_1, n_2)}(\lambda_k^{(1/2)}) + \right. \\
 & \left. + \sum_{\substack{1/2 \leq q \leq n_1 - n_2 \\ q \in 1/2 + \mathbb{Z}_{\geq 0}}} \sum_{a=1}^{\ell_q} |f_{n_2, (q; a)}^{n_1}|^2 \mathcal{F}_m^{(n_1, n_2, n_1, n_2)}(q(1-q)) \right). \quad (1.86)
 \end{aligned}$$

Proof. It follows from Theorem 1.3.49. The proof is similar to that of Proposition 1.3.52. □

In what follows, we consider a hyperbolic spin orbifold supporting holomorphic modular forms of weight n and $2n$ such that $n = 1 \pmod{2}$. If $\ell_{n/2} > 1$ and/or $\ell_n > 1$, we choose a particular $\mathcal{O}_n$ and a particular $\mathcal{O}_{n/2}$ such that $f_{n/2, n/2}^n \neq 0$. One can always consider a bigger system correlator using explicitly the value of ℓ_n and $\ell_{n/2}$, however, in order to find out a universal bound irrespective of ℓ_n and $\ell_{n/2}$, we choose not to do so. This leads us to consider the system of following correlators $\langle \mathcal{O}_n \mathcal{O}_n \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_n \rangle$, $\langle \mathcal{O}_{\frac{n}{2}} \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle$, $\langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \rangle$ and $\langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle$, where we have suppressed the degeneracy index a in (n, a) and so on. The explicit crossing equations read

Proposition 1.3.54. *Let $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{N}$. We have*

$$\begin{aligned}
S_{2n+2m;(n,n)} &= \left(\mathcal{F}_{2m}^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n,n}^{k;a})^2 \right] \mathcal{F}_{2m}^{(n,n,n,n)}(\lambda_k^{(0)}) \right), \\
0 &= \left(\mathcal{F}_{2m+1}^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n,n}^{k;a})^2 \right] \mathcal{F}_{2m+1}^{(n,n,n,n)}(\lambda_k^{(0)}) \right), \\
S_{n+2m;(n/2,n/2)} &= \left(\mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n/2,n/2}^{k;a})^2 \right] \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \right), \\
0 &= \left(\mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n/2,n/2}^{k;a})^2 \right] \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \right), \\
S_{3n/2+m;(n,n/2)} &= \left(\mathcal{F}_m^{(n,n/2,n/2,n)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} c_{n/2,n/2}^{k;a} c_{n,n}^{k;a} \right] \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) \right), \\
(-1)^m S_{3n/2+m;(n,n/2)} &= \left(\sum_{k=1}^{\infty} \left[\sum_{a=1}^{d'_k} |s_{n,n/2}^{k;a}|^2 \right] \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) + \right. \\
&\quad \left. + \sum_{0 < q \leq n/2} \sum_{a=1}^{\ell_q} |f_{n/2,(q;a)}^n|^2 \mathcal{F}_m^{(n,n/2,n,n/2)}(q(1-q)) \right).
\end{aligned}$$

Proof. This follows from Propositions 1.3.52 and 1.3.53 by choosing appropriate values of n_1 and n_2 . $\square$

Based on the above crossing equations, let us formulate the following linear programming problems.

Notation: Given a hyperbolic spin orbifold X , we consider the Laplacian spectra $\{0\} \cup \{\lambda_k^{(0)}(X) : \lambda_k^{(0)}(X) > 0, k \in \mathbb{N}\}$, and the weight-1 automorphic Laplacian spectrum (not including the eigenvalue corresponding to a harmonic spinor if there is any) $\{\lambda_k^{(1/2)}(X) : \lambda_k^{(1/2)}(X) > 1/4, k \in \mathbb{N}\}$. Furthermore, without loss of generality, let us assume $\lambda_{k+1}^{(0)}(x) > \lambda_k^{(0)}(X)$ and $\lambda_{k+1}^{(1/2)}(X) > \lambda_k^{(1/2)}(X)$ for all $k \in \mathbb{N}$.

Linear Program 1 (Spectral bound on Laplacian). Given $\Lambda \in \mathbb{N}$, $\lambda_*^{(0)} \in \mathbb{R}_+$ and a hyperbolic spin orbifold with a holomorphic modular form of weight n such that

$n = 1 \pmod{2}$, if there exists $\alpha \in \mathbb{R}^{\Lambda+1}$ with such that

$$1. \quad \alpha_{2m} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0},$$

$$2. \quad \sum_{m=0}^{\Lambda} \alpha_m \mathcal{F}_m^{(n/2, n/2, n/2, n/2)}(0) = 1,$$

$$3. \quad \forall \lambda_k^{(0)} \geq \lambda_*^{(0)} \text{ we have}$$

$$\sum_{m=0}^{\Lambda} \alpha_m \mathcal{F}_m^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) \geq 0, \quad (1.87)$$

we must have $\lambda_1^{(0)} < \lambda_*^{(0)}$.

Proof. We consider the third and the fourth equation of the ones appearing in the proposition 1.3.54 and write them below.

$$\begin{aligned} S_{n+2m; (n/2, n/2)} &= \left(\mathcal{F}_{2m}^{(n/2, n/2, n/2, n/2)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} \left(c_{n/2, n/2}^{k; a} \right)^2 \right] \mathcal{F}_{2m}^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) \right), \\ 0 &= \left(\mathcal{F}_{2p+1}^{(n/2, n/2, n/2, n/2)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} \left(c_{n/2, n/2}^{k; a} \right)^2 \right] \mathcal{F}_{2p+1}^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) \right). \end{aligned}$$

Here we renamed the variable m to p to write down the second equation.

For a given $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{\geq 0}$, we multiply the first equation above with α_{2m} , the second equation above with α_{2p+1} , and add them together. Now we sum over m and p such that $2m \leq \Lambda$ and $2p+1 \leq \Lambda$. Then we use the trivial identity

$$\begin{aligned} &\sum_{0 \leq 2m \leq \Lambda} \alpha_{2m} \mathcal{F}_{2m}^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) + \sum_{0 \leq 2p+1 \leq \Lambda} \alpha_{2p+1} \mathcal{F}_{2p+1}^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) \\ &= \sum_{q=0}^{\Lambda} \alpha_q \mathcal{F}_q^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}), \end{aligned}$$

and the property (2) of α to finally obtain the following:

$$\sum_{k=1}^{\infty} \left(\left[\sum_{a=1}^{d_k} \left(c_{n/2, n/2}^{k; a} \right)^2 \right] \sum_{q=0}^{\Lambda} \alpha_q \mathcal{F}_q^{(n/2, n/2, n/2, n/2)}(\lambda_k^{(0)}) \right) = -1 + \sum_{m=0}^{\lfloor \frac{\Lambda}{2} \rfloor} \alpha_{2m} S_{n+2m; (n/2, n/2)}.$$

Now using the properties of α , we show that the R.H.S of the above is strictly negative. Therefore the L.H.S is strictly negative. Using (1.87), the proposition follows. $\square$

Semidefinite Program 1 (Spectral bound on Laplacian and Dirac: I). Let us consider a hyperbolic spin orbifold, for which, $\exists n \in \mathbb{N}$ with $n = 1 \pmod{2}$ such that the orbifold does not support any holomorphic modular form of weight strictly below n and does support a (at least one) holomorphic modular form of weight n .

Given $\Lambda \in \mathbb{N}$, $\lambda_*^{(0)}, \lambda_*^{(1/2)} \in \mathbb{R}_+$, if there exists $\alpha_i \in \mathbb{R}^{\Lambda+1}$ with $i = 1, 2, 3, 4$ such that

$$1. \alpha_{1;2m} \leq 0, \forall m \in \mathbb{Z}_{\geq 0},$$

$$2. \sum_{m=0}^{\Lambda} \left(\alpha_{1;m} \mathcal{F}_m^{(n,n,n,n)}(0) + \alpha_{2;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(0) \right) = 1,$$

$$3. \alpha_{2;2m} \leq 0, \forall m \in \mathbb{Z}_{\geq 0},$$

$$4. \alpha_{3;m} + (-1)^m \alpha_{4;m} \leq 0, \forall m \in \mathbb{Z}_{\geq 0},$$

$$5. \sum_{m'=0}^{\Lambda} \alpha_{4;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) \geq 0, \forall m \in \mathbb{Z}_{\geq 0},$$

$$6. \forall \lambda_k^{(1/2)} \geq \lambda_*^{(1/2)}, \text{ we have}$$

$$\sum_{m=0}^{\Lambda} \alpha_{4;m} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) \geq 0, \quad (1.88)$$

$$7. \forall \lambda_k^{(0)} \geq \lambda_*^{(0)} \text{ we have}$$

$$M(\lambda_k^{(0)}) := \begin{pmatrix} \sum_{m=0}^{\Lambda} \alpha_{1;m} \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) & \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) \\ \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) & \sum_{m=0}^{\Lambda} \alpha_{2;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \end{pmatrix} \succeq 0, \quad (1.89)$$

we must have either $\lambda_1^{(0)} < \lambda_*^{(0)}$ or $\lambda_1^{(1/2)} < \lambda_*^{(1/2)}$.

Proof. The proof is similar to the proof of the Linear Program 1 and involves taking appropriate linear combination of equations appearing in (1.3.54). To be concrete, let us define

$$S_{2n+2m+1;(n,n)} = 0, \quad S_{2n+2m+1;(n/2,n/2)} = 0, \quad m \in \mathbb{Z}_{\geq 0} \quad (1.90)$$

and rewrite the equations appearing in proposition (1.3.54) in the following form:

$$\begin{aligned} S_{2n+m;(n,n)} &= \left(\mathcal{F}_m^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n,n}^{k;a})^2 \right] \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) \right), \\ S_{n+m;(n/2,n/2)} &= \left(\mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} (c_{n/2,n/2}^{k;a})^2 \right] \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \right), \\ S_{3n/2+m;(n,n/2)} &= \left(\mathcal{F}_m^{(n,n/2,n/2,n)}(0) + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d_k} c_{n/2,n/2}^{k;a} c_{n,n}^{k;a} \right] \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) \right), \\ (-1)^m S_{3n/2+m;(n,n/2)} &= \left(\sum_{k=1}^{\infty} \left[\sum_{a=1}^{d'_k} |s_{n,n/2}^{k;a}|^2 \right] \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) + \right. \\ &\quad \left. + \sum_{0 < q \leq n/2} \sum_{a=1}^{\ell_q} |f_{n/2,(q;a)}^n|^2 \mathcal{F}_m^{(n,n/2,n,n/2)}(q(1-q)) \right), \end{aligned}$$

For the orbifold under consideration, there does not exist a holomorphic modular form of weight strictly below n . Hence, the sum appearing in the last line of the fourth equation appearing above collapses to a single term corresponding to $q = n/2$.

For each value of m , there are four crossing equations above. We multiply the i -th one with $\alpha_{i;m}$. Then we sum over the indices i and m such that $1 \leq i \leq 4$ and

$0 \leq m \leq \Lambda$. Finally, we use (1.90) and property (2) of α to obtain:

$$\begin{aligned}
& \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} \left(c_{n,n}^{k;a}, c_{n/2,n/2}^{k;a} \right) M(\lambda_k^{(0)}) \begin{pmatrix} c_{n,n}^{k;a} \\ c_{n/2,n/2}^{k;a} \end{pmatrix} + \sum_{k=1}^{\infty} \left[\sum_{a=1}^{d'_k} |s_{n,n/2}^{k;a}|^2 \right] \sum_{m=0}^{\Lambda} \alpha_{4;m} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) \\
&= -1 + \sum_{m=0}^{\lfloor \frac{\Lambda}{2} \rfloor} \left(\alpha_{1,2m} S_{2n+2m;(n,n)} + \alpha_{2,2m} S_{n+2m;(n/2,n/2)} \right) - \\
& - \left[\sum_{a=1}^{\ell_{n/2}} |f_{n/2,(n/2;a)}^n|^2 \right] \sum_{m'=0}^{\Lambda} \alpha_{4;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) + \\
& + \sum_{m'=0}^{\Lambda} \left(\alpha_{3;m'} + (-1)^{m'} \alpha_{4;m'} \right) S_{3n/2+m';(n,n/2)} .
\end{aligned}$$

Now using the properties of α , we show that the R.H.S of the above is strictly negative.

Therefore the L.H.S is strictly negative. Using (1.88) and (1.89), the proposition follows. $\square$

Remark 1.3.55. When $\ell_n = \ell_{n/2} = 1$, the structure constant $\sum_{a=1}^{\ell_{n/2}} |f_{n/2,(n/2;a)}^n|^2$ appearing in the t -channel expansion in the last equation of (1.3.54) is exactly same as $S_{n;(n/2,n/2)}$. Thus we can formulate a stronger SDP problem, which we state below.

Semidefinite Program 2 (Spectral bound on Laplacian and Dirac: II). Let us consider a hyperbolic spin orbifold, for which, $\exists n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$ such that the orbifold does not support any holomorphic modular form of weight strictly below n . Finally, assume that the multiplicity of holomorphic modular forms of weight n and $2n$ are respectively given by $\ell_{n/2} = 1$ and $\ell_n = 1$.

Given $\Lambda \in \mathbb{N}$, $\lambda_*^{(0)}, \lambda_*^{(1/2)} \in \mathbb{R}_+$, if there exists $\alpha_i \in \mathbb{R}^{\Lambda+1}$ with $i = 1, 2, 3, 4$ such that

$$1. \quad \alpha_{1,2m} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}$$

$$2. \quad \sum_{m=0}^{\Lambda} \left(\alpha_{1;m} \mathcal{F}_m^{(n,n,n,n)}(0) + \alpha_{2;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(0) \right) = 1$$

$$3. \alpha_{2;2m} - \delta_{2m,0} \sum_{m'=0}^{\Lambda} \alpha_{4;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0},$$

$$4. \alpha_{3;m} + (-1)^m \alpha_{4;m} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0},$$

$$5. \forall \lambda_k^{(1/2)} \geq \lambda_*^{(1/2)}, \text{ we have}$$

$$\sum_{m=0}^{\Lambda} \alpha_{4;m} \mathcal{F}_m^{(n,n/2,n,n/2)} (\lambda_k^{(1/2)}) \geq 0, \quad (1.91)$$

$$6. \forall \lambda_k^{(0)} \geq \lambda_*^{(0)} \text{ we have}$$

$$M(\lambda_k^{(0)}) := \begin{pmatrix} \sum_{m=0}^{\Lambda} \alpha_{1;m} \mathcal{F}_m^{(n,n,n,n)} (\lambda_k^{(0)}) & \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)} (\lambda_k^{(0)}) \\ \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{3;m} \mathcal{F}_m^{(n,n/2,n/2,n)} (\lambda_k^{(0)}) & \sum_{m=0}^{\Lambda} \alpha_{2;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)} (\lambda_k^{(0)}) \end{pmatrix} \succeq 0, \quad (1.92)$$

we must have either $\lambda_1^{(0)} < \lambda_*^{(0)}$ or $\lambda_1^{(1/2)} < \lambda_*^{(1/2)}$.

Proof. The proof involves taking the same linear combinations of equations appearing in (1.3.54), as done in the proof of Semidefinite Program 1 and further using the fact $\ell_{n/2} = 1$, leading to

$$\sum_{a=1}^{\ell_{n/2}} |f_{n/2,(n/2;a)}^n|^2 = S_{n;(n/2,n/2)}.$$

In this way, we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{a=1}^{d_k} \left(c_{n,n}^{k;a}, c_{n/2,n/2}^{k;a} \right) M(\lambda_k^{(0)}) \begin{pmatrix} c_{n,n}^{k;a} \\ c_{n/2,n/2}^{k;a} \end{pmatrix} + \sum_{k=1}^{\infty} \sum_{a=1}^{d'_k} |s_{n,n/2}^{k;a}|^2 \sum_{m=0}^{\Lambda} \alpha_{4;m} \mathcal{F}_m^{(n,n/2,n,n/2)} (\lambda_k^{(1/2)}) \\ &= -1 + \sum_{m=0}^{\lfloor \Lambda/2 \rfloor} \left(\alpha_{1,2m} S_{2n+2m;(n,n)} + \left(\alpha_{2;2m} - \delta_{2m,0} \sum_{m'=0}^{\Lambda} \alpha_{4;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) \right) \times \right. \\ & \quad \left. \times S_{n+2m;(n/2,n/2)} \right) + \sum_{m=0}^{\Lambda} (\alpha_{3;m} + (-1)^m \alpha_{4;m}) S_{3n/2+m;(n,n/2)}. \end{aligned}$$

Now using the properties of α , we show that the R.H.S of the above is strictly negative.

Therefore the L.H.S is strictly negative. Using (1.91) and (1.92), the proposition follows. $\square$

In what follows, we will use $\mathcal{O}_n$ and $\mathcal{O}_{n/2}$ and assume that there is no holomorphic modular form of weight strictly below n on the orbifold under consideration. This is clearly true for $n = 1$. Furthermore, we will use a clever trick introduced in [KMP24] to reduce the size of the problem. The idea is to consider the correlators and do a Haar integral over $U(\ell_n)$ and $U(\ell_{n/2})$.

Proposition 1.3.56. *For a unitary group $U(\ell)$, we have*

$$\begin{aligned} \int_{U(\ell)} dU \ U_{a_1}^{a'_1} (\overline{U})_{a_2}^{a'_2} &= \frac{1}{\ell} \delta_{a_1, a_2} \delta_{a'_1, a'_2}, \\ \int_{U(\ell)} dU \ U_{a_1}^{a'_1} U_{a_2}^{a'_2} (\overline{U})_{a_3}^{a'_3} (\overline{U})_{a_4}^{a'_4} &= \frac{1}{\ell^2 - 1} \left(\delta_{a_1, a_3} \delta_{a_2, a_4} \delta^{a'_1, a'_3} \delta^{a'_2, a'_4} + \delta_{a_1, a_4} \delta_{a_2, a_3} \delta^{a'_1, a'_4} \delta^{a'_2, a'_3} \right) \\ &\quad - \frac{1}{\ell(\ell^2 - 1)} \left(\delta_{a_1, a_3} \delta_{a_2, a_4} \delta^{a'_1, a'_4} \delta^{a'_2, a'_3} + \delta_{a_1, a_4} \delta_{a_2, a_3} \delta^{a'_1, a'_3} \delta^{a'_2, a'_4} \right). \end{aligned}$$

Next the idea is to consider the symmetrized set of correlators:

Definition 1.3.57. The symmetrized correlators are defined by

$$\begin{aligned} \langle \mathcal{O}_n \mathcal{O}_n \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_n \rangle_{\text{SYM}} &:= \int_{U(\ell_n)} dU \ U_{a_1}^{a'_1} U_{a_2}^{a'_2} (\overline{U})_{a_3}^{a'_3} (\overline{U})_{a_4}^{a'_4} \langle \mathcal{O}_{n, a_1} \mathcal{O}_{n, a_2} \tilde{\mathcal{O}}_{n, a_3} \tilde{\mathcal{O}}_{n, a_4} \rangle, \\ \langle \mathcal{O}_{\frac{n}{2}} \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle_{\text{SYM}} &:= \int_{U(\ell_{n/2})} dU \ U_{b_1}^{b'_1} U_{b_2}^{b'_2} (\overline{U})_{b_3}^{b'_3} (\overline{U})_{b_4}^{b'_4} \langle \mathcal{O}_{\frac{n}{2}, b_1} \mathcal{O}_{\frac{n}{2}, b_2} \tilde{\mathcal{O}}_{\frac{n}{2}, b_3} \tilde{\mathcal{O}}_{\frac{n}{2}, b_4} \rangle, \\ \langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \rangle_{\text{SYM}} &:= \int_{U(\ell_n)} dU \ U_{a_1}^{a'_1} (\overline{U})_{a_2}^{a'_2} \int_{U(\ell_{n/2})} dU \ U_{b_1}^{b'_1} (\overline{U})_{b_2}^{b'_2} \langle \mathcal{O}_{n, a_1} \mathcal{O}_{\frac{n}{2}, b_1} \tilde{\mathcal{O}}_{\frac{n}{2}, b_2} \tilde{\mathcal{O}}_{n, a_2} \rangle, \\ \langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle_{\text{SYM}} &:= \int_{U(\ell_n)} dU \ U_{a_1}^{a'_1} (\overline{U})_{a_2}^{a'_2} \int_{U(\ell_{n/2})} dU \ U_{b_1}^{b'_1} (\overline{U})_{b_2}^{b'_2} \langle \mathcal{O}_{n, a_1} \mathcal{O}_{\frac{n}{2}, b_1} \tilde{\mathcal{O}}_{n, a_2} \tilde{\mathcal{O}}_{\frac{n}{2}, b_2} \rangle. \end{aligned} \tag{1.93}$$

Definition 1.3.58. Let us define the following quantities:

$$\begin{aligned}
\mathbf{S}_{p;n} &:= \frac{1}{\ell_n(\ell_n + 1)} \sum_{i=1}^{\ell_p} \sum_{a_1, a_2=1}^{\ell_n} \frac{1}{4} \left| f_{(n;a_1),(n;a_2)}^{(p;i)} + f_{(n;a_2),(n;a_1)}^{(p;i)} \right|^2, \\
\mathbf{A}_{p;n} &:= \frac{1}{\ell_n(\ell_n - 1)} \sum_{i=1}^{\ell_p} \sum_{a_1, a_2=1}^{\ell_n} \frac{1}{4} \left| f_{(n;a_1),(n;a_2)}^{(p;i)} - f_{(n;a_2),(n;a_1)}^{(p;i)} \right|^2, \\
\mathbf{T}_{k;n} &:= \frac{1}{\ell_n^2 - 1} \sum_{i=1}^{d_k} \sum_{a_1, a_2=1}^{\ell_n} \left[c_{(n;a_1),(n;a_2)}^{(k;i)} \overline{c_{(n;a_1),(n;a_2)}^{(k;i)}} - \frac{1}{\ell_n} c_{(n;a_1),(n;a_1)}^{(k;i)} \overline{c_{(n;a_2),(n;a_2)}^{(k;i)}} \right], \\
\mathbf{Q}_{k;n} &:= \frac{1}{\ell_n} \sum_{i=1}^{d_k} c_{(n;a),(n;a)}^{(k;i)}, \\
\mathbf{B}_{p;(n_1, n_2)} &:= \frac{1}{\ell_{n_1} \ell_{n_2}} \sum_{i=1}^{\ell_p} \sum_{a=1}^{\ell_{n_1}} \sum_{b=1}^{\ell_{n_2}} \left| f_{(n_1;a),(n_2;b)}^{(p;i)} \right|^2, \\
\mathbf{P}_{k;(n_1, n_2)} &:= \frac{1}{\ell_{n_1} \ell_{n_2}} \sum_{i=1}^{d_k} \sum_{a=1}^{\ell_{n_1}} \sum_{b=1}^{\ell_{n_2}} \left| s_{(n_1;a),(n_2;b)}^{(k;i)} \right|^2.
\end{aligned} \tag{1.94}$$

Let ρ_n denote the fundamental representations of $U(\ell_n)$ and ρ_n^* its complex conjugate. The quantities $\mathbf{S}_{p;n}, \mathbf{A}_{p;n}, \mathbf{T}_{k;n}$ and $\mathbf{Q}_{k;n}$ label different trivial representations in the decomposition of $(\rho_n^{\otimes 2}) \otimes (\rho_n^*)^{\otimes 2}$. $\mathbf{S}_{p;n}$ is the element in the trivial representation which arises from $\text{Sym}^2(\rho_n) \otimes \text{Sym}^2(\rho_n^*)$ and $\mathbf{A}_{p;n}$ is from $\wedge^2(\rho_n) \otimes \wedge^2(\rho_n^*)$. The tensor product $\rho_n \otimes \rho_n^*$ decomposes into a symmetric traceless representation and a trivial representation. $\mathbf{T}_{k;n}$ and $(\mathbf{Q}_{k;n})^2$ come from the tensor product of these two representations respectively. Similarly, $\mathbf{B}_{p;(n_1, n_2)}$ and $\mathbf{P}_{k;(n_1, n_2)}$ are $U(\ell_{n_1}) \times U(\ell_{n_2})$ invariants from $(\rho_{n_1} \times \rho_{n_2}) \otimes (\rho_{n_1}^* \times \rho_{n_2}^*)$.

Remark 1.3.59. Except for $\mathbf{Q}_{k;n}$, all quantities defined above are positive semi-definite combinations of the OPE coefficients.

Proposition 1.3.60. *Given a hyperbolic spin orbifold, for which, $\exists n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$ such that the orbifold does not support holomorphic modular forms with weight strictly smaller than n , and it supports $\ell_n \neq 0$ different holomorphic modular*

forms with weight $2n$ and $\ell_{n/2} \neq 0$ different holomorphic modular forms with weight n , the spectral identities derived from the set of symmetrized correlators defined above are given by:

$$\forall m \in \mathbb{Z}_{\geq 0} :$$

$$\begin{aligned} \mathbf{S}_{2m+2n;n} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n} \mathcal{F}_{2m}^{(n,n,n,n)}(\lambda_k^{(0)}), \\ -\mathbf{A}_{2m+1+2n;n} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n} \mathcal{F}_{2m+1}^{(n,n,n,n)}(\lambda_k^{(0)}), \\ \mathbf{S}_{2m+2n;n} &= \mathcal{F}_{2m}^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n}^2 - \frac{1}{\ell_n} \mathbf{T}_{k;n} \right) \mathcal{F}_{2m}^{(n,n,n,n)}(\lambda_k^{(0)}), \\ \mathbf{A}_{2m+1+2n;n} &= \mathcal{F}_{2m+1}^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n}^2 - \frac{1}{\ell_n} \mathbf{T}_{k;n} \right) \mathcal{F}_{2m+1}^{(n,n,n,n)}(\lambda_k^{(0)}), \\ \mathbf{S}_{2m+n;n/2} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n/2} \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\ -\mathbf{A}_{2m+1+n;n/2} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n/2} \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\ \mathbf{S}_{2m+n;n/2} &= \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \frac{1}{\ell_{n/2}} \mathbf{T}_{k;n/2} \right) \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\ \mathbf{A}_{2m+1+n;n/2} &= \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \frac{1}{\ell_{n/2}} \mathbf{T}_{k;n/2} \right) \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\ \mathbf{B}_{m+3n/2;(n,n/2)} &= \mathcal{F}_m^{(n,n/2,n/2,n)}(0) + \sum_{k=1}^{\infty} \mathbf{Q}_{k;n} \mathbf{Q}_{k;n/2} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}), \\ (-1)^m \mathbf{B}_{m+3n/2;(n,n/2)} &= \sum_{k=1}^{\infty} \mathbf{P}_{k;(n,n/2)} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) \\ &\quad + \frac{\ell_{n/2} + 1}{\ell_n} \mathbf{S}_{n;n/2} \mathcal{F}_m^{(n,n/2,n,n/2)}\left(\frac{n}{2} \left(1 - \frac{n}{2}\right)\right). \end{aligned}$$

Proof. The first four equations above come from $\langle \mathcal{O}_n \mathcal{O}_n \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_n \rangle_{\text{SYM}}$. The idea is to expand the integrand, appearing in the definition (1.93), in s and t -channel; then exchange the sum and the Haar integral, followed by performing the Haar inte-

gral using the second equation in proposition 1.3.56. These are exactly same as the ones in [KMP24]. The 5th to the 8th equations are derived from considering $\langle \mathcal{O}_{n/2} \mathcal{O}_{n/2} \tilde{\mathcal{O}}_{n/2} \tilde{\mathcal{O}}_{n/2} \rangle_{\text{SYM}}$ and following similar steps. The 9th and the 10th equations are coming from considering $\langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \rangle_{\text{SYM}}$ and $\langle \mathcal{O}_n \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_n \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle_{\text{SYM}}$ respectively. In short, the unsymmetrized consistency conditions imply the symmetrized consistency conditions just by averaging over all possible bases of the relevant eigenspaces. $\square$

Semidefinite Program 3 (Spectral bound on Laplacian and Dirac: III). Let us consider a hyperbolic spin orbifold which supports $\ell_n \neq 0$ different holomorphic modular forms with weight $2n$ and $\ell_{n/2} \neq 0$ different holomorphic modular forms with weight n . Assume that the orbifold does not support any holomorphic modular form with weight strictly smaller than n .

Given $\Lambda \in \mathbb{N}$, $\lambda_*^{(0)}, \lambda_*^{(1/2)} \in \mathbb{R}_+$, if there exist $\alpha_i \in \mathbb{R}^{\Lambda+1}$ with $i = 1, 2, 3, 4, 5, 6$ such that

$$1. \sum_{m=0}^{\Lambda} \alpha_{2;m} \mathcal{F}_m^{(n,n,n,n)}(0) + \alpha_{4;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \alpha_{5;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(0) = 1,$$

$$2. \alpha_{1;2m} + \alpha_{2;2m} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}, 2m \leq \Lambda,$$

$$3. \alpha_{3;2m} + \alpha_{4;2m} - \delta_{m,0} \frac{\ell_{n/2}+1}{\ell_n} \sum_{m'=0}^{\Lambda} \alpha_{6;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) \leq 0, \\ \forall m \in \mathbb{Z}_{\geq 0}, 2m \leq \Lambda,$$

$$4. \alpha_{2;2m+1} - \alpha_{1;2m+1} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}, 2m+1 \leq \Lambda,$$

$$5. \alpha_{4;2m+1} - \alpha_{3;2m+1} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}, 2m+1 \leq \Lambda,$$

$$6. \alpha_{5;m} + (-1)^m \alpha_{6;m} \leq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}, m \leq \Lambda,$$

7.

$$\sum_{m=0}^{\Lambda} \left(\alpha_{1;m} - \frac{1}{\ell_n} \alpha_{2;m} \right) \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) \geq 0, \quad \forall \lambda_k^{(0)} \geq \lambda_*^{(0)}, \quad (1.95)$$

8.

$$\sum_{m=0}^{\Lambda} \left(\alpha_{3;m} - \frac{1}{\ell_{n/2}} \alpha_{4;m} \right) \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \geq 0, \quad \forall \lambda_k^{(0)} \geq \lambda_*^{(0)}, \quad (1.96)$$

9.

$$\sum_{m=0}^{\Lambda} \alpha_{6;m} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) \geq 0, \quad \forall \lambda_k^{(1/2)} \geq \lambda_*^{(1/2)}, \quad (1.97)$$

$$10. \forall \lambda_k^{(0)} \geq \lambda_*^{(0)},$$

$$M'(\lambda_k^{(0)}) := \begin{pmatrix} \sum_{m=0}^{\Lambda} \alpha_{2;m} \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) & \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{5;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) \\ \frac{1}{2} \sum_{m=0}^{\Lambda} \alpha_{5;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}) & \sum_{m=0}^{\Lambda} \alpha_{4;m} \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) \end{pmatrix} \succeq 0, \quad (1.98)$$

then at least one of the following statements must be true:

$$1. \lambda_1^{(0)} < \lambda_*^{(0)},$$

$$2. \lambda_1^{(1/2)} < \lambda_*^{(1/2)}.$$

Proof. The proof is similar in nature to the ones appearing before for the semidefinite linear programs. Let us define

$$\begin{aligned} V_{1;m} &:= \begin{cases} \mathbf{S}_{m+2n;n}, & m \equiv 0 \pmod{2} \\ -\mathbf{A}_{m+2n;n}, & m \equiv 1 \pmod{2} \end{cases}, & V_{2;m} &:= (-1)^m V_{1;m} \\ V_{3;m} &:= \begin{cases} \mathbf{S}_{m+2n;n/2}, & m \equiv 0 \pmod{2} \\ -\mathbf{A}_{m+2n;n/2}, & m \equiv 1 \pmod{2} \end{cases}, & V_{4;m} &:= (-1)^m V_{3;m} \\ V_{5;m} &:= \mathbf{B}_{m+3n/2;(n,n/2)}, & V_{6;m} &:= (-1)^m V_{5;m}. \end{aligned} \quad (1.99)$$

Using these, we rewrite the equations appearing in (1.3.60):

$$\begin{aligned} V_{1;m} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n} \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}), \\ V_{2;m} &= \mathcal{F}_m^{(n,n,n,n)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n}^2 - \frac{1}{\ell_n} \mathbf{T}_{k;n} \right) \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}), \\ V_{3;m} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n/2} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \\ V_{4;m} &= \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \frac{1}{\ell_{n/2}} \mathbf{T}_{k;n/2} \right) \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\ V_{5;m} &= \mathcal{F}_m^{(n,n/2,n/2,n)}(0) + \sum_{k=1}^{\infty} \mathbf{Q}_{k;n} \mathbf{Q}_{k;n/2} \mathcal{F}_m^{(n,n/2,n/2,n)}(\lambda_k^{(0)}), \\ V_{6;m} &= \sum_{k=1}^{\infty} \mathbf{P}_{k;(n,n/2)} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) \\ &\quad + \frac{\ell_{n/2} + 1}{\ell_n} \mathbf{S}_{n;n/2} \mathcal{F}_m^{(n,n/2,n,n/2)}\left(\frac{n}{2} \left(1 - \frac{n}{2}\right)\right). \end{aligned}$$

For a given $m \in \mathbb{Z}_{\geq 0}$, there are six crossing equations above. We multiply the i -th equation by $\alpha_{i;m}$ and sum them up. Then we take linear combination by summing over m such that $m \leq \Lambda$. At this point, we write the variable V in terms of the

original variable $\mathbf{S}$, $\mathbf{A}$, $\mathbf{B}$ to obtain:

$$\begin{aligned}
& \sum_{m=0}^{\lfloor \Lambda/2 \rfloor} \left[(\alpha_{1;2m} + \alpha_{2;2m}) \mathbf{S}_{2m+2n;n} \right. \\
& \quad + \left(\alpha_{3;2m} + \alpha_{4;2m} - \delta_{2m,0} \frac{\ell_{n/2} + 1}{\ell_n} \times \sum_{m'=0}^{\Lambda} \alpha_{6;m'} \mathcal{F}_{m'}^{(n,n/2,n,n/2)} \left(\frac{n}{2} \left(1 - \frac{n}{2} \right) \right) \right) \mathbf{S}_{2m+n;n/2} \Big] \\
& \quad + \sum_{m=0}^{\lfloor \frac{\Lambda-1}{2} \rfloor} \left[(\alpha_{2;2m+1} - \alpha_{1;2m+1}) \mathbf{A}_{2m+1+2n;n} + (\alpha_{4;2m+1} - \alpha_{3;2m+1}) \mathbf{A}_{2m+1+n;n/2} \right] \\
& \quad + \sum_{m=0}^{\Lambda} (\alpha_{5;m} + (-1)^m \alpha_{6;m}) \mathbf{B}_{m+3n/2;(n,n/2)} \\
& \quad - \sum_{m=0}^{\Lambda} \left(\alpha_{2;m} \mathcal{F}_m^{(n,n,n,n)}(0) + \alpha_{4;m} \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(0) + \alpha_{5;m} \mathcal{F}_m^{(n,n/2,n/2,n)}(0) \right) \\
& \quad = \sum_{m=0}^{\Lambda} \sum_{k=1}^{\infty} \left[\mathbf{T}_{k,n} \left(\alpha_{1;m} - \frac{1}{\ell_n} \alpha_{2;m} \right) \mathcal{F}_m^{(n,n,n,n)}(\lambda_k^{(0)}) \right. \\
& \quad + \mathbf{T}_{k;n/2} \left(\alpha_{3;m} - \frac{1}{\ell_{n/2}} \alpha_{4;m} \right) \mathcal{F}_m^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}) \\
& \quad \left. + \mathbf{P}_{k;(n,n/2)} \alpha_{6;m} \mathcal{F}_m^{(n,n/2,n,n/2)}(\lambda_k^{(1/2)}) + \begin{pmatrix} \mathbf{Q}_{k;n} & \mathbf{Q}_{k;n/2} \end{pmatrix} M'(\lambda_k^{(0)}) \begin{pmatrix} \mathbf{Q}_{k;n} \\ \mathbf{Q}_{k;n/2} \end{pmatrix} \right].
\end{aligned}$$

Now using the properties of α , the L.H.S of the above identity is strictly negative. Therefore the R.H.S is strictly negative. Using (1.95), (1.96), (1.97), (1.98), the proposition follows. $\square$

Remark 1.3.61. The Semidefinite Program 3 and its proof explicitly show that analyzing the symmetrized correlators is sufficient to derive a bound.

1.4 Estimates of spectral gaps from the Selberg trace formula

In this section, we use the Selberg trace formula to estimate the spectral gap of a given surface X . The standard approach for numerically computing the operator spectrum on a hyperbolic surface is the finite element method. Thanks to the **FreeFEM++** package, the spectrum of Laplacian operator, with periodic and twisted periodic boundary conditions (see supplementary material of [MR21]), is known for many surfaces. For our purposes, we look at a different approach to compute the gaps of the Laplacian

and the Dirac operator of explicit orbifolds and surfaces. This will allow us to compare these spectra to the previously established bootstrap bounds. The key idea is to use the Selberg trace formula, with the geodesic spectrum as an input, to rule out points on the real line that are not in the eigenvalue spectrum. Our algorithm can be divided into three steps:

- Find a Dirichlet domain for the surface of interest and explicit generators for the fundamental group.
- Use this Dirichlet domain to produce a complete non-redundant list of conjugacy classes of geodesics on the surface whose length is bounded by a given value L .
- Choose a compactly supported test function that yields an estimate of the first eigenvalue of the Laplace/Dirac spectrum when substituted into the Selberg trace formula.

Note that the first two steps of this algorithm work in the same way in order to produce the Laplace and Dirac spectra. Only the explicit form of the Selberg trace formula used in the last step differs.

Step 1: Constructing the Dirichlet domain

We start by reviewing some basic definitions.

Definition 1.4.1. (Dirichlet domain of a Fuchsian group) Let Γ be a Fuchsian group and let $x \in \mathbb{H}$ be a point not fixed by any element of Γ other than the identity. A *Dirichlet domain with base point x* is the set

$$D(x) := \{p \in \mathbb{H} \mid \forall \gamma \in \Gamma \setminus \{e\}, d(x, p) < d(x, \gamma p)\}. \quad (1.100)$$

For a cocompact Fuchsian group, its Dirichlet domain is a hyperbolic polygon with finite area. The boundary of this hyperbolic polygon consists of finitely many geodesic

segments. We will denote this set of geodesic segments by $S(D)$ and it satisfies the property [Voi09] that for every $s \in S(D)$, there exists a unique $s' \in S(D)$ and a unique $\gamma_s \in \Gamma$ such that $s' = \gamma_s s$. This is known as the *side-pairing* property. The segment s is the perpendicular bisector of the geodesic between x and $\gamma_s x$. The set $G(S) := \{\gamma_s | s \in S(D)\}$ is a generating set for Γ and we will refer to its elements as *side-pairing generators*. See [Voi09] for a systematic algorithm computing $G(S)$ and its application to arithmetic Fuchsian groups.

Definition 1.4.2. (Maximin edge distance of the Dirichlet domain) Let Γ be a co-compact Fuchsian group with Dirichlet domain $D(x)$ where x is the base point. Let $S(D)$ be the set of geodesic segments forming the boundary of $D(x)$. The *maximin edge distance* $R(D)$ is defined as the maximum over all the domain's edges of the minimum distance from the edge to the base-point x :

$$R(D) := \max_{s \in S(D)} \left(\inf_{p \in s} d(x, p) \right). \quad (1.101)$$

We briefly summarize the Dirichlet domain and the corresponding side-pairing generators for surfaces and orbifolds relevant to us:

1. Hyperbolic triangles:

As shown on Figure 1.4, the Dirichlet domain of a genus-0 hyperbolic triangle $[0; p, q, r]$ consists of two geodesic triangles $\Delta(p, q, r)$ with the four vertices located at $\widehat{s}(1/p, 1/q, 1/r; 0)$, $\widehat{s}(1/p, 1/q, 1/r; 1)$ and $\pm \widehat{s}(1/p, 1/q, 1/r; \infty)$, where $\widehat{s}(1/p, 1/q, 1/r; z)$ is the rescaled Schwarz function [Har05]

$$\widehat{s}(\alpha, \beta, \gamma; z) = \nu s(\alpha, \beta, \gamma; z), \quad s(\alpha, \beta, \gamma; z) = z^\alpha \frac{{}_2F_1(a', b', c'; z)}{{}_2F_1(a, b, c; z)}, \quad (1.102)$$

$$\nu = \sqrt{\frac{\cos(\pi\alpha + \pi\beta) + \cos(\pi\gamma)}{\cos(\pi\alpha - \pi\beta) + \cos(\pi\gamma)} \cdot \frac{\cos(\pi\alpha - \pi\beta - \pi\gamma) + 1}{\cos(\pi\alpha + \pi\beta + \pi\gamma) + 1} \cdot \frac{\Gamma(a')\Gamma(b')}{\Gamma(c')}} \cdot \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}, \quad (1.103)$$

with

$$a = \frac{1 - \alpha - \beta - \gamma}{2}, \quad b = \frac{1 - \alpha + \beta - \gamma}{2}, \quad c = 1 - \alpha, \quad (1.104)$$

and

$$a' = \frac{1 + \alpha - \beta - \gamma}{2}, \quad b' = \frac{1 + \alpha + \beta - \gamma}{2}, \quad c' = 1 + \alpha. \quad (1.105)$$

2. The Bolza surface:

The Dirichlet domain of the Bolza surface is a regular octagon. The side-pairing generators are $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}, \gamma_4^{-1}$ with

$$\gamma_k = \pm \begin{pmatrix} \cosh(l/2) & e^{\frac{i\pi k}{4}} \sinh(l/2) \\ e^{-\frac{i\pi k}{4}} \sinh(l/2) & \cosh(l/2) \end{pmatrix}, \quad l = 2 \cosh^{-1} \left(\cot \left(\frac{\pi}{8} \right) \right). \quad (1.106)$$

Here we used “ $\pm$ ” to stress the fact that γ_k ’s are elements in $\text{PSU}(1, 1)$. The sign ambiguity will be removed once the surface is given a spin structure.

3. The most symmetric point in the moduli space with signature [1;3]:

The Dirichlet domain for the one-punctured torus is an octagon with side-pairing generators being $\gamma_1, \gamma_2, \gamma_1\gamma_2, \gamma_2\gamma_1, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_1^{-1}\gamma_2^{-1}, \gamma_2^{-1}\gamma_1^{-1}$ where

$$\gamma_1 = \begin{pmatrix} \frac{1-x^2y^2}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} & \frac{-x(1-y^2)+i(1-x^2)y}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} \\ \frac{-x(1-y^2)-i(1-x^2)y}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} & \frac{1-x^2y^2}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} \end{pmatrix}, \quad (1.107)$$

$$\gamma_2 = \begin{pmatrix} \frac{1-x^2y^2}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} & \frac{x(1-y^2)+i(1-x^2)y}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} \\ \frac{x(1-y^2)-i(1-x^2)y}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} & \frac{1-x^2y^2}{\sqrt{(1-x^2)(1-y^2)(x^2y^2+1)}} \end{pmatrix}, \quad (1.108)$$

with

$$x = \tanh \left[\frac{1}{2} \cosh^{-1} \left(\frac{\cos(\pi/9)}{\sin(\pi/18)} \right) \right], \quad y = \tanh \left[\frac{1}{2} \cosh^{-1} \left(\frac{\cos(\pi/18)}{\sin(\pi/9)} \right) \right]. \quad (1.109)$$

We conclude this subsection with a few comments on generalizations to hyperbolic surfaces with higher genus. We will sketch a general method for constructing a fundamental domain and to look for explicit expressions of the generators. The surfaces or orbifolds that are most likely to saturate the numerical bounds are conjectured to always possess a large number of symmetries, as we already observed in genus 1 and genus 2. For these special surfaces, the following result holds:

Proposition 1.4.3 ([MP17]). *Let Σ be a compact hyperbolic Riemann surface of genus $g > 1$. Σ realizes a local maximum of the number of automorphisms on the moduli space of Riemann surfaces of genus g if and only if it is isomorphic to a quotient of the upper half-plane by a torsion-free normal subgroup of a cocompact Fuchsian triangle group.*

This result means that in the relevant cases for our purposes, all fundamental domains can be constructed from gluing geodesic triangles $\Delta(p, q, r)$ ⁵ together. Hence, the very general problem of finding an explicit fundamental domain for a given Riemann surface maps to a much simpler one in our case: the problem of

1. Constructing a tiling of the hyperbolic plane by triangles of given angles, and
2. Gluing geodesic triangles to obtain fundamental domains of the surface/orbifold of interest.

⁵The geodesic triangles are also the fundamental domains of the triangle group $T(p, q, r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle$. The generators a, b, c are reflections against the three edges.

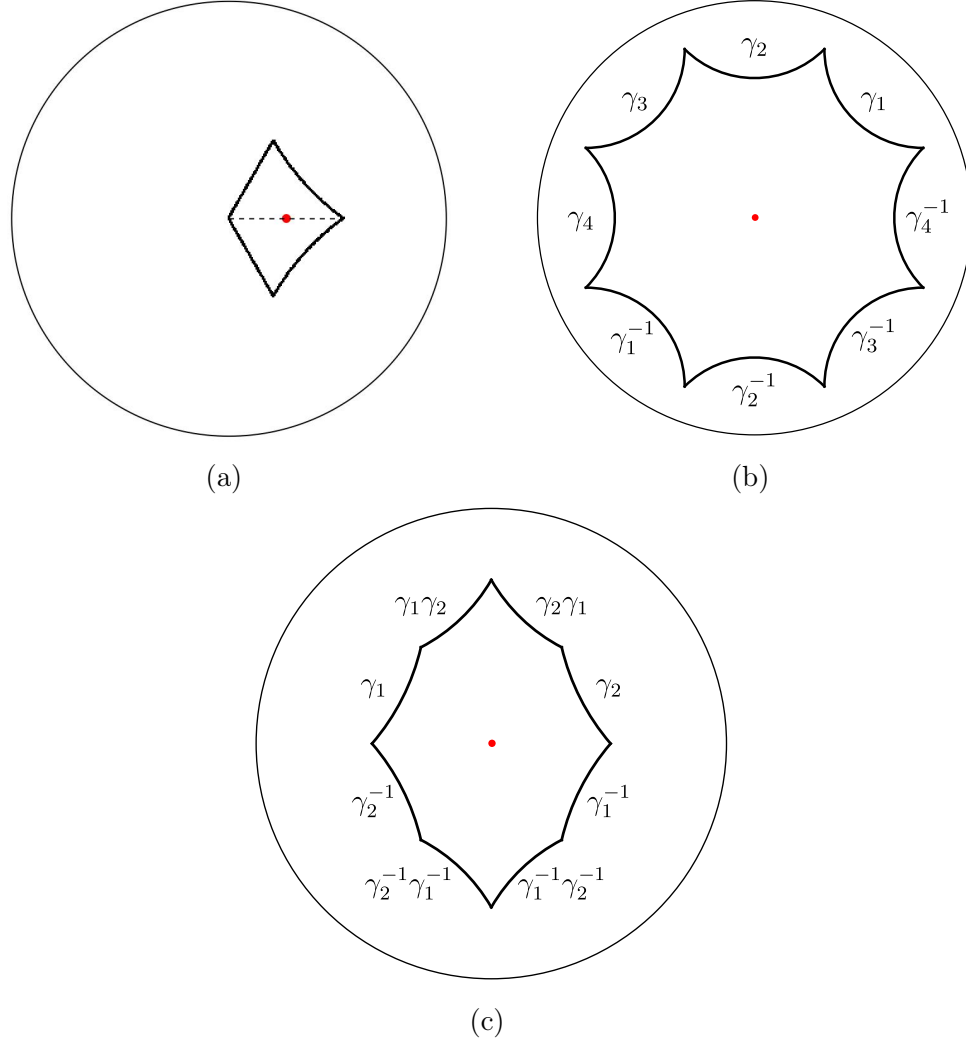


Figure 1.4: (A) The Dirichlet domain for the hyperbolic triangle $[0; 3, 3, 5]$ consists of two geodesic triangles $\Delta(3, 3, 5)$ (B) The Dirichlet domain for the Bolza surface. Each side of the Dirichlet domain is labeled by the corresponding side-pairing generator. (C) The Dirichlet domain of the most symmetric point in the moduli space of one punctured torus with signature $[1; 3]$. Each side of the Dirichlet domain is labeled by the corresponding side-pairing generator. The red dots in the figures are the corresponding base points.

To find explicit expressions for the generators, the simplest method would be to directly search for a presentation of a normal subgroup N of the triangle group $T(p, q, r)$ such that

$$\text{Aut}(\Sigma) \cong T(p, q, r)/N. \quad (1.110)$$

This can in principle be done using the GAP software, although there could be cases in

which there is more than one possible choice. We reserve a more careful examination of this problem to future work.

Once this subgroup is found, one can construct a fundamental domain by singling out one representative triangle per coset in T/N and taking the union of all such representatives. Note that the obtained fundamental domain is made of exactly $|\text{Aut}(\Sigma)|$ triangles.

Step 2: Enumerating the closed geodesics of length bounded by L

For this second step, we will mostly adapt the method proposed in [HW94] to our case. Before stating the algorithm, we first summarize the ideas behind it:

- In every hyperbolic conjugacy class, there is at least one representative whose axis⁶ passes within a distance $R(D)$ of the basepoint, where $R(D)$ is the maximum edge distance of the Dirichlet domain. (See Propositions 3.2 and 3.3 from [HW94].)
- Working with such representatives, it is intuitive to see that we only need to tile the Poincaré disk $\mathbb{D}$ up to a certain radius in order to exhaust all hyperbolic conjugacy classes below a certain length cutoff. This radius depends both on the length cutoff and the size of the Dirichlet domain D . To be more precise, it suffices to find all translates γD such that $d(x, \gamma x) \leq 2 \cosh^{-1}(\cosh R(D) \cosh \frac{L}{2})$. (see Proposition 3.5 of [HW94].)
- If two hyperbolic elements with geodesic length L are conjugate to each other and both of their axes pass within a distance $R(D)$ from the base point, then the conjugating element γ must satisfy $d(x, \gamma x) \leq 2 \cosh^{-1}(\cosh R(D) \cosh \frac{L}{4})$ (see Proposition 3.7 of [HW94].)

⁶Recall that a hyperbolic element γ can be specified by the geodesic it fixes. This geodesic is known as the axis of γ .

With these basic ideas in mind, we state the algorithm:

1. (Initialize) Set T and T_h to be two empty lists. Given a spin structure, write all side-pairing generators $\tilde{\gamma}_i$ of $\tilde{\Gamma}$ as 2×2 matrices .
2. (Enumerate translates) Let T_n be the list of irreducible n -letter words formed by the $\tilde{\gamma}_i$. Remove from T_n all elements $\tilde{\tau}$ such that $d(x, \tilde{\tau}x) > 2 \cosh^{-1}(\cosh R(D) \cosh \frac{L}{2})$. Here x is the base point of the Dirichlet domain. If T_n becomes an empty list, terminate this step. Otherwise append T_n to T . To generate T_{n+1} , multiply every element in T_n by side-pairing generators $\tilde{\gamma}_i$ and remove the redundant matrices that are already in T .⁷ This procedure terminates after a finite number of steps.
3. (Count hyperbolic conjugacy classes) Given the list T , let T_h be the subset of hyperbolic elements whose axis passes within a distance $R(D)$ from the base point. Group T_h by geodesic length (or equivalently, absolute value of the traces of the corresponding 2×2 matrices):

$$T_h = \bigsqcup_l T_h^{(l)}, \quad T_h^{(l)} = \{\tilde{\gamma} \in T_h \mid 2 \cosh^{-1}(|\text{Tr}(\tilde{\gamma})|/2) = l\}. \quad (1.111)$$

For all sublists of T_h sharing the same geodesic length l , select all $\tilde{\tau} \in T$ such that $d(x, \tilde{\tau}x) \leq 2 \cosh^{-1}(\cosh R(D) \cosh \frac{l}{4})$:

$$C_l := \left\{ \tilde{\tau} \in T \mid d(x, \tilde{\tau}x) \leq 2 \cosh^{-1} \left(\cosh R \cosh \frac{l}{4} \right) \right\}, \quad (1.112)$$

and use these elements to partition the sublist into conjugacy classes:

$$\text{Orb}_l(\tilde{\gamma}) := \{\tilde{\gamma}' \in T_h^{(l)} \mid \exists \tilde{\tau} \in C_l, \tilde{\tau}\tilde{\gamma}\tilde{\tau}^{-1} = \pm\tilde{\gamma}'\}, \quad T_h^{(l)} = \bigsqcup_i \text{Orb}_l(\tilde{\gamma}_i). \quad (1.113)$$

Note that if the orbifold $\Gamma \backslash \mathbb{H}$ is spin, the traces of all matrices in the same conjugacy class should share the same sign. This sign is exactly the multiplier

⁷The enumeration procedure would fail if there exist some translate γD such that γx is closer to x than all of its neighboring tiles, but this won't happen if we work with Dirichlet domain instead of an arbitrary fundamental domain. See Proposition 3.1 of [HW94].

χ that enters the sum over hyperbolic elements in the Selberg trace formula for the 1-Laplacian.

4. (Compute the winding) Compute the winding number of each conjugacy class by checking whether it could be powers of other classes.

Step 3: Applying the Selberg trace formula to a well-chosen test function

Now that our algorithm can find all the geodesics up to a certain length, we can plug them into the geometric side of the Selberg trace formula. The Selberg trace formulae for the Laplacian and the squared Dirac operator (which is related to 1-Laplacian, see the remark 1.2.18) are respectively (see Theorem 5.1 of Chapter 3 of [Hej06a] and Equation 6.56 of Chapter 9 of [Hej06b]):

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{h}(t_n^{(0)}) &= \frac{V(\Sigma)}{4\pi} \int_{-\infty}^{\infty} t \widehat{h}(t) \tanh(\pi t) dt + \sum_{\{\gamma_h\}} \frac{l(\gamma_{h;0})}{2 \sinh(l(\gamma_h)/2)} h(l(\gamma_h)) \\ &\quad + \sum_{\{\gamma_e\}} \frac{ie^{-i\theta}}{4M_{\gamma_e} \sin \theta} \int_{-\infty}^{\infty} \frac{h(u) e^{-\frac{u}{2}} (e^u - e^{2i\theta})}{\cosh u - \cos(2\theta)} du, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{h}(t_n^{(1/2)}) &= \frac{V(\Sigma)}{4\pi} \int_{-\infty}^{\infty} t \widehat{h}(t) \frac{\sinh(2\pi t)}{\cosh(2\pi t) - 1} dt + \sum_{\{\gamma_h\}, \text{Tr } \gamma_h > 2} \frac{\chi(\gamma_h) l(\gamma_{h;0})}{2 \sinh(l(\gamma_h)/2)} h(l(\gamma_h)) \\ &\quad + \sum_{\{\gamma_e\}, \pi > \theta(\gamma_e) > 0} \frac{i\chi(\gamma_e)}{4M_{\gamma_e} \sin \theta} \int_{-\infty}^{\infty} \frac{h(u) (e^u - e^{2i\theta})}{\cosh u - \cos(2\theta)} du. \end{aligned}$$

Let us explain the notations in these formulas:

- h is an acceptable even compactly supported test function such that $\widehat{h}$ is even.
- The $t_n^{(0)}$ and $t_n^{(1/2)}$ are equal to the $\sqrt{\lambda_n^{(0)} - 1/4}$ and $\sqrt{\lambda_n^{(1/2)} - 1/4}$, where the $\lambda_n^{(0)}$ and $\lambda_n^{(1/2)}$ are the eigenvalues of the Laplacian and 1-Laplacian, respectively.
- $V(\Sigma)$ is the volume of Σ .
- The sum over $\{\gamma_h\}$ is a sum over conjugacy classes of hyperbolic elements in $\pi_1(\Sigma)$, or of its double cover in $\text{SL}(2, \mathbb{R})$ in the case of the 1-Laplacian.

- For a given hyperbolic element γ_h , $l(\gamma_h) = 2 \cosh^{-1}(|\text{Tr}(\gamma_h)|/2)$ is the length of the corresponding geodesic.
- For a given hyperbolic element γ_h , $\gamma_{h;0}$ denotes the primitive hyperbolic element associated with γ_h , that is, $\gamma_{h;0}$ is the element with shortest geodesic length such that γ_h can be expressed as a power of $\gamma_{h;0}$.
- The sum over $\{\gamma_e\}$ is a sum over conjugacy classes of elliptic elements in $\pi_1(\Sigma)$, or of its double cover in $\text{SL}(2, \mathbb{R})$ in the case of the 1-Laplacian.
- For a given elliptic element γ_e , M_{γ_e} is the order of γ_e .
- χ is a multiplier on the double cover of $\pi_1(\Sigma)$ in $\text{SL}(2, \mathbb{R})$ implementing a choice of spin structure.
- $\theta(\gamma_e)$ is the angle of the rotation matrix γ_e .
- A hat denotes a Fourier transform.

Our strategy will closely follow the reference [LL22]. The idea is to apply the Selberg trace formula to functions whose Fourier transform that appears on the geometric side of the Selberg trace formula has compact support - so that only a finite number of terms contribute to the sum over hyperbolic elements, and that we can find all of them using the procedure outlined in the previous subsection.

Let $h_1, \dots, h_n$ be such test functions with support in $[-r, r]$ and λ_i be the eigenvalues of the operator of interest (it can be the Laplacian or the squared Dirac operator). It will also be useful to introduce extra parameters t_i , related to the eigenvalues through

$$\lambda_i := \frac{1}{4} + t_i^2. \quad (1.114)$$

Define a matrix A whose entries are given by

$$A_{ab} = \sum_j \widehat{h}_a(t_j) \widehat{h}_b(t_j). \quad (1.115)$$

If c_t is the column vector

$$c_t = \begin{bmatrix} \widehat{h}_1(t) \\ \vdots \\ \widehat{h}_n(t) \end{bmatrix}, \quad (1.116)$$

let

$$I_r(\lambda) := \inf_{\langle c_t, x \rangle = 1} \langle Ax, x \rangle. \quad (1.117)$$

$I_r(\lambda)$ can be calculated by introducing a Lagrange multiplier, and is equal to

$$I_r(\lambda) = \frac{1}{\langle A^{-1}c_t, c_t \rangle}. \quad (1.118)$$

This quantity can be evaluated explicitly using the geometric side of the Selberg trace formula applied to the functions $h_a * h_b$. One can then prove (see [LL22]):

Proposition 1.4.4. *If $I_r(\lambda) < 1$, then λ is not an eigenvalue of the operator of interest.*

Proof. A test function h with support on $[-r, r]$ is called *admissible* if $\widehat{h} \geq 0$ and the Selberg trace formula of interest can be applied to it. For $t \in [-r, r]$, define the quantity

$$I_r^\bullet(\lambda) := \inf_{\substack{h \text{ admissible} \\ \text{supp } h \subset [-r, r] \\ \widehat{h}(t)=1}} \sum_j \widehat{h}(t_j). \quad (1.119)$$

Then, if there exists j_0 such that $t = t_{j_0}$, then for any test function $\widehat{h}$, we have

$$1 = \widehat{h}(t) \leq \sum_j \widehat{h}(t_j). \quad (1.120)$$

This is just because a sum is larger than or equal to all of its summands if the summands are non-negative. Hence,

$$I_r^\bullet(\lambda) \geq 1. \quad (1.121)$$

By contrapositive, we deduce that if $I_r^\bullet(\lambda) < 1$, then t cannot be one of the t_j . All that now remains to be shown is that for all t , $I_r^\bullet(\lambda) \leq I_r(\lambda)$. Now, $I_r(\lambda)$ can itself be reexpressed as an infimum: let

$$S := \left\{ h * h, \text{supp } h \subset \left[-\frac{r}{2}, \frac{r}{2}\right], h = \sum x_i h_i \text{ for } x_i \in \mathbb{R} \right\}. \quad (1.122)$$

We have

$$\inf_{\substack{H=h*h \in S \\ \widehat{h}(t)=1}} \sum_j \widehat{H}(t_j) = I_r(\lambda). \quad (1.123)$$

But now, $I_r^\bullet(\lambda)$ is also an infimum, over a larger set of functions. We deduce

$$I_r^\bullet(\lambda) \leq I_r(\lambda), \quad (1.124)$$

which completes the proof of the proposition. $\square$

There is a straightforward generalization of the proof of the above result to the case in which the eigenvalue of the operator of interest is degenerate:

Proposition 1.4.5. *Let $k \in \mathbb{N}$. If $I_r(\lambda) < k$, then λ is not an eigenvalue of multiplicity k or higher of the operator of interest.*

These results provide a method to numerically estimate a lower bound on the gap of the Laplace and Dirac operators on a hyperbolic surface or orbifold. The method works as follows:

1. Calculate the matrix A_{ab} for some compactly supported test functions h_a and h_b using the Selberg trace formula and the geodesic spectrum generated in the previous step.
2. Deduce the value of $I_r(\lambda)$ as a function of $t = \sqrt{\lambda - \frac{1}{4}}$.

3. For Laplace operators, find the first interval $[a_1, b_1] \subset \mathbb{R}_+$ such that $I_r(\lambda) \geq 1$ when $\lambda \in [a_1, b_1]$. This interval $[a_1, b_1]$ provides an estimate for $\lambda_1^{(0)}$ of the surface or orbifold under consideration. In the case of the 1-Laplacian, all eigenvalues above $1/4$ must have even multiplicity due to Kramers degeneracy, see Proposition 1.2.21. Hence, the interval containing $\lambda_1^{(1/2)}$ is the first interval $[c_1, d_1] \subset \mathbb{R}_+$ such that $I_r(\lambda) \geq 2$ when $\lambda \in [c_1, d_1]$.

The accuracy of our method is dependent on the choice of test functions h_k . In order to get a good bound, we need to slightly modify the choices of test functions made in [LL22]. We now explain in more detail how we choose these test functions.

For our application, we define our test functions h_a in terms of three parameters : n , m and L . Let

$$\delta = \frac{L}{2m + 2n}, \quad (1.125)$$

introduce

$$h := \left(\frac{1}{2\delta} \mathbb{1}_{[-\delta, \delta]} \right)^{*m}. \quad (1.126)$$

Then our test functions h_a are defined for $a = 1, \dots, n$ by

$$h_a(x) := \frac{1}{2}(h(x + a\delta) + h(x - a\delta)). \quad (1.127)$$

In this formula:

- L is the length of the longest geodesic that appears on the right hand side of the Selberg trace formula. Numerically, we observe that when L gets larger, $I_r(\lambda)$ gets more and more relatively sharp peaks, that is, there are more intervals $[a_i, b_i]$ and $[c_i, d_i]$ such that $b_i - a_i < \epsilon$, $d_i - c_i < \epsilon$ where ϵ is a parameter characterising the precision of the estimation. The more information we have on the geodesic spectrum, the more eigenvalues we can estimate with a reasonably high precision.

- n is the size of the A matrix. Numerically, we observe that increasing n makes the peaks of $I_r(\lambda)$ sharper. However, the integral following $V(\Sigma)/4\pi$ on the geometric side of the Selberg trace formulae needs to be evaluated numerically. A very large value of n produces very fast oscillations in the integrand, making the numerical integration difficult to handle.
- m is the number of convolutions. Increasing it also makes the peaks of $I_r(\lambda)$ sharper, but we face a similar problem related to fast oscillations.

Figure 1.6 shows the results given by this method when applied to the case of the $[0; 3, 3, 5]$ orbifold. In Appendix 1.8 we tabulate the numerical estimates of $\lambda_1^{(0)}$ and $\lambda_1^{(1/2)}$ for various surfaces and orbifolds.

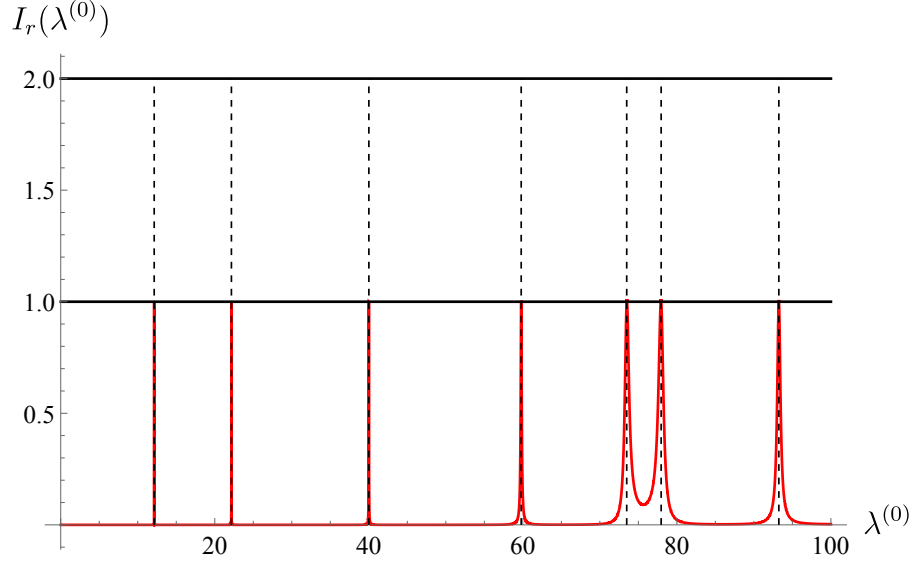


Figure 1.5: The spectral exclusion plot for the Laplacian spectrum of the hyperbolic triangle $[0; 3, 3, 5]$ with $m = 4$, $n = 24$ and $L = 10.9$. The dashed vertical lines correspond to the first few eigenvalues computed using the finite element method.

Remark 1.4.6. The rigorous result that Figure 1.5 and similar plots can show is a *lower* bound on the first eigenvalue. $I_r(\lambda^{(0)})$ remaining lower than 1 on a certain interval starting at the origin shows rigorously that there is no eigenvalue in that interval. Our algorithm does not show rigorously that $I_r(\lambda^{(0)})$ being larger than 1

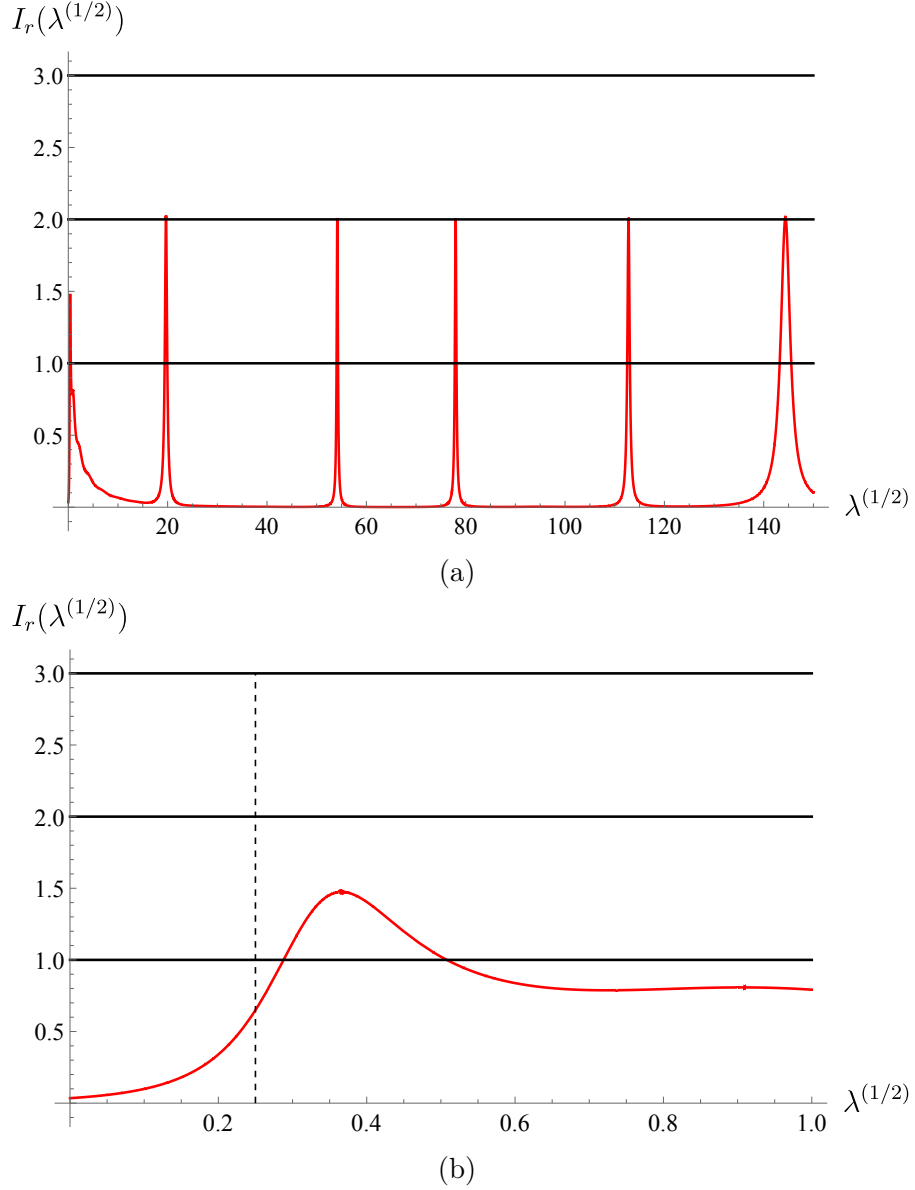


Figure 1.6: (A) Spectral exclusion plot for the 1-Laplacian spectrum of the hyperbolic triangle $[0;3,3,5]$ with $m = 4, n = 24$ and $L = 10.9$. The leftmost peak does not imply the existence of any eigenvalues. The curve is below 1 at $\lambda^{(1/2)} = 1/4$ therefore it does not indicate the existence of a harmonic spinor. The peak is also below 2 and by Kramers degeneracy it does not represent a non-harmonic eigenfunction. (B) Spectral exclusion plot for the 1-Laplacian spectrum of the hyperbolic triangle $[0;3,3,5]$ zoomed into the interval from 0 to 1. The dashed vertical line is located at $\lambda^{(1/2)} = 1/4$.

implies the presence of a new eigenvalue. However, we observe that the curve in the figure 1.5 and similar plots are well converged numerically, i.e., the maximum value of $I_r(x)$ with x in that interval stays larger than 1, even with the inclusion of more geodesic-length data. This indicates that the interval should contain an eigenvalue. It is possible to make this claim completely rigorous by using a similar algorithm (see section 5.4 of [BMP23] and the reference therein), which certifies that a given interval contains an eigenvalue. We do not pursue that here because our goal is to saturate the upper bounds that we obtain from the bootstrap equations. Therefore what matters to us is to find a lower bound for a given candidate orbifold that is close to the upper bound provided by the bootstrap. Similar statements apply to $I_r(\lambda_1^{(1/2)})$ as well.

1.5 Results: bounds from semi-definite programming

In this section we present the proofs of the theorems mentioned earlier, as well as write down further bounds that we obtain using linear and semidefinite programming.

Proofs of theorems using Semi definite programming

Proof of Theorem 1.1.1. For genus 1 or more, the above bound follows the results in [KMP24]. For genus 0 hyperbolic spin orbifold, we use the fact $\ell_{1/2} = 0$. Since it is a spin orbifold, the orbifold orders are odd i.e $k_i \geq 3$ and there has to be at least 3 orbifold points for the surface to be spin. Using Riemann-Roch, we then deduce $\ell_3 \geq 1$ and $\ell_{3/2} \geq 1$. Furthermore, we have $\lambda^{(1/2)} \geq 1/4$ for all spin orbifolds. Now we leverage $\text{SDP}(1)$ with $n = 3$ and choosing $\lambda_*^{(1/2)} = 1/4$. We have verified the bound using rational arithmetic as done in [KMP24]. $\square$

Proof of Theorem 1.1.3. We note that for a hyperbolic spin orbifold X admitting harmonic spinor, we have $\ell_{1/2} \geq 1$ and hence $\ell_1 \geq 1$. Furthermore, we have $\lambda^{(1/2)} \geq 1/4$ for all spin orbifolds. Now we leverage $\text{SDP}(1)$ with $n = 1$ and choosing $\lambda_*^{(1/2)} =$

1/4. We have verified the bound using rational arithmetic as done in [KMP24]. $\square$

Proof of Theorem 1.1.5. We leverage $\text{SDP}(1)$ with $n = 3$ and choosing $\lambda_*^{(0)} = 3/2$. $\square$

Proof of Theorem 1.1.6. Given a compact orientable hyperbolic hyperelliptic spin manifold X of genus $g > 2$, we have $\ell_{1/2} = \lfloor (g+1)/2 \rfloor > 1$. We consider the spectral identities derived from $\langle \mathcal{O}_{\frac{n}{2}} \mathcal{O}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \tilde{\mathcal{O}}_{\frac{n}{2}} \rangle_{\text{SYM}}$ with $n = 1$. They are given by (see Proposition 1.3.60)

$$\begin{aligned}
\mathbf{S}_{2m+n;n/2} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n/2} \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\
-\mathbf{A}_{2m+1+n;n/2} &= \sum_{k=1}^{\infty} \mathbf{T}_{k;n/2} \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\
\mathbf{S}_{2m+n;n/2} &= \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \frac{1}{\ell_{n/2}} \mathbf{T}_{k;n/2} \right) \mathcal{F}_{2m}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}), \\
\mathbf{A}_{2m+1+n;n/2} &= \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \frac{1}{\ell_{n/2}} \mathbf{T}_{k;n/2} \right) \mathcal{F}_{2m+1}^{(n/2,n/2,n/2,n/2)}(\lambda_k^{(0)}),
\end{aligned} \tag{1.128}$$

To proceed, we consider the set of identities coming from $m = 0$, from which we can derive

$$\begin{aligned}
0 &= \mathcal{F}_0^{(1/2,1/2,1/2,1/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 - \left(1 + \frac{1}{\ell_{1/2}} \right) \mathbf{T}_{k;1/2} \right) \mathcal{F}_0^{(1/2,1/2,1/2,1/2)}(\lambda_k^{(0)}) \\
0 &= \mathcal{F}_1^{(1/2,1/2,1/2,1/2)}(0) + \sum_{k=1}^{\infty} \left(\mathbf{Q}_{k;n/2}^2 + \left(1 - \frac{1}{\ell_{1/2}} \right) \mathbf{T}_{k;1/2} \right) \mathcal{F}_1^{(1/2,1/2,1/2,1/2)}(\lambda_k^{(0)})
\end{aligned} \tag{1.129}$$

Plugging in the values of $\mathcal{F}$, we obtain from the above

$$0 = \sum_{k=1}^{\infty} \left[\lambda_k^{(0)} \mathbf{Q}_{k;n/2}^2 + \left(\lambda_k^{(0)} \frac{(\ell_{1/2} - 1)}{\ell_{1/2}} - 1 \right) \mathbf{T}_{k;1/2} \right]. \tag{1.130}$$

The above implies that

$$\lambda_1^{(0)} < \frac{\ell_{1/2}}{\ell_{1/2} - 1}. \tag{1.131}$$

Plugging in $\ell_{1/2} = \lfloor (g+1)/2 \rfloor$, the theorem follows. $\square$

Theorem 1.5.1. *Given a compact orientable hyperbolic spin 2-orbifold X of genus g admitting $\ell_{1/2} > 0$ harmonic spinors, the first non-zero eigenvalue of the Laplacian operator, $\lambda_1^{(0)}(X)$ satisfies the bound, recorded in Table 1.1.*

Proof of theorem 1.5.1. On a hyperbolic spin orbifold X of genus g , we have $\ell_1 = g$ and $\ell_{1/2} \leq \lfloor (g+1)/2 \rfloor$. We use the fact that $\lambda^{(1/2)} \geq 1/4$ for all spin orbifolds. We use SDP (3) with $n = 1$ and $\lambda_*^{(1/2)} = 1/4$. We have verified these bounds using rational arithmetic as done in [KMP24]. $\square$

Exclusion plots

As mentioned in the introduction, an *exclusion plot* refers to a region $D \subset (0, \infty) \times (1/4, \infty)$ such that $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X)) \notin D$, where X can be any compact connected orientable hyperbolic spin surface or orbifold. We may also choose to consider choices of X with prescribed additional properties like genus, number of harmonic spinors it can carry etc. Here $\lambda_1^{(0)}(X)$ is the first nontrivial eigenvalue of the Laplace operator on the compact hyperbolic spin orbifold, while $\lambda_1^{(1/2)}(X)$ is the first nontrivial eigenvalue of the weight-1 automorphic Laplacian. In order to obtain the exclusion plots, we perform the following algorithm:

1. Use SDP (1) or (2), pick a value for $\lambda_*^{(1/2)}$ and search for a functional with the desired properties, leading to a bound $\lambda_1^{(0)} < \lambda_*^{(0)}$ subject to the condition $\lambda_1^{(1/2)} > \lambda_*^{(1/2)}$.
2. Repeat the above step for various values of $\lambda_*^{(1/2)}$.

or the following one:

1. Use SDP (1) or (2), pick a value for $\lambda_*^{(0)}$ and search for a functional with the desired properties, leading to a bound $\lambda_1^{(1/2)} < \lambda_*^{(1/2)}$ subject to the condition $\lambda_1^{(0)} > \lambda_*^{(0)}$.

2. Repeat the above step for various values of $\lambda_*^{(0)}$.

Using the aforementioned algorithm involving **SDP**, we find the disallowed region D , which is everything except the shaded region (yellow or pink) in the *exclusion* plots. We obtain Figure 1.2 by using **SDP** (1) with $n = 3$ (for the union of the pink and yellow shaded region) and $n = 1$ (for the pink shaded region). The corner point of the yellow shaded region is approximately at $(12.13629, 19.67)$, while the corner point of the pink-shaded region is at $(4.7611, 8.28)$, both these points being disallowed. Now, using the Selberg trace formula, we can compute $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ for $X = [0; 3, 3, 5]$ (this orbifold has $\ell_{3/2} = \ell_3 = 1$). It is given by $(12.1362327 \pm 10^{-7}, 19.669 \pm 0.03)$. This point is depicted in red on Figure 1.2 and evidently, lies very close to the kink. Figure 1.7 zooms onto the kink and shows that the red dot is in the allowed region (up to uncertainty, predicted by the Selberg trace formula). Similarly, using the Selberg trace formula, we can compute $(\lambda_1^{(0)}(X), \lambda_1^{(1/2)}(X))$ for $X = [1; 3]_{sym}$, the most symmetric point in the moduli space of $[1; 3]$, equipped with an odd spin structure such that $\ell_{1/2} = \ell_1 = 1$. In particular we find that $\lambda_1^{(0)} \in (4.7609, 4.7654)$ and $\lambda_1^{(1/2)} \in (8.255, 8.298)$. See the table in Appendix 1.8 for a refined interval. For the Laplacian eigenvalue, we can use **FreeFEM++** as well, this provides the estimate $\lambda_1^{(0)}([1; 3]_{sym}) \approx 4.7609$.

Figure 1.8 is obtained by using **SDP** (2) with $n = 3$. Here we restrict to the case where $\ell_3 = \ell_{3/2} = 1$, as opposed to using **SDP** (1) with $n = 3$, where $\ell_3 \geq 1, \ell_{3/2} \geq 1$. While Figure 1.8 looks almost similar to Figure 1.7, zooming near the kink reveals the differences. Figure 1.9 is Figure 1.8, zoomed in near the kink. The corner point is approximately at $(12.13623353125, 19.673)$ and disallowed.

Finally, Figure 1.10 is obtained by using a simplified version of **SDP** (3) with $n = 1$, $\ell_1 = 2$, but $\ell_{1/2} = 1$. The red dot here corresponds to the Bolza surface, equipped with an odd spin structure. See the table in Appendix 1.8 for the precise coordinates

of the red dot. Figure 1.11 is the zoomed version of the above.

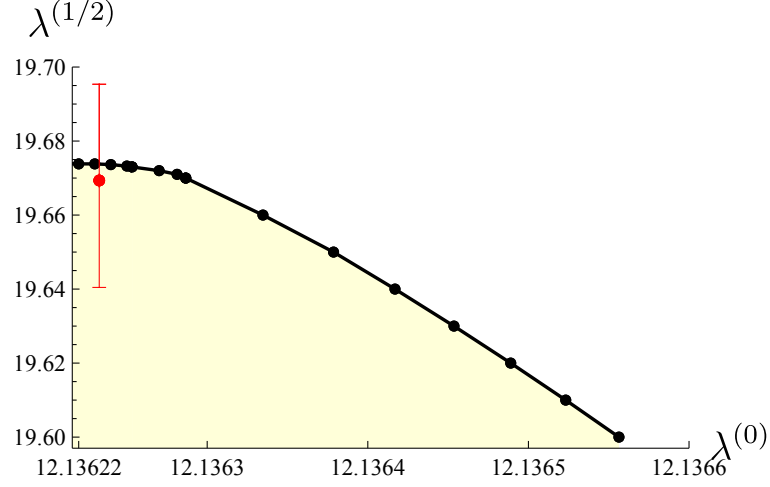


Figure 1.7: Bounds from the $\ell_{3/2} \geq 1$, $\ell_3 \geq 1$ system; zooming onto the kink. Everything except the shaded region is disallowed. The red dot near the corner corresponds to $[0; 3, 3, 5]$.

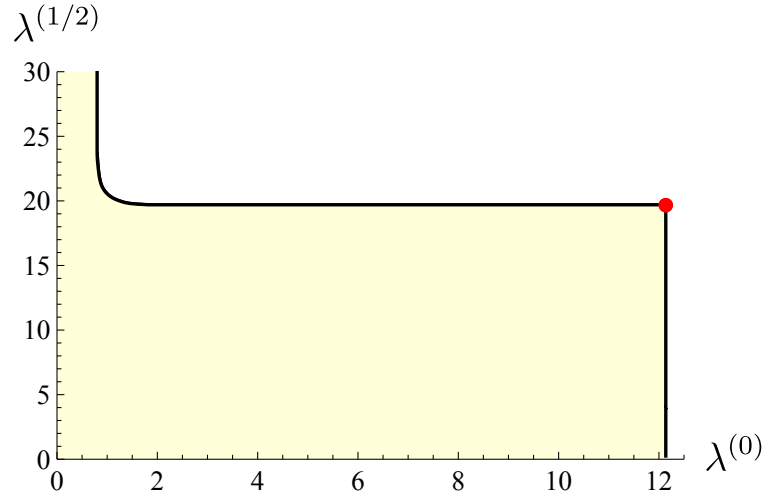


Figure 1.8: Bounds from the $\ell_{3/2} = \ell_3 = 1$ system. Everything except the yellow shaded region is disallowed for hyperbolic spin orbifolds with $\ell_{3/2} = \ell_3 = 1$. The red dot in the corner corresponds to $[0; 3, 3, 5]$.

1.6 Discussion

In this paper, we applied linear programming techniques to constrain the first nonzero eigenvalue of the Laplace and Dirac operators on a compact connected orientable hyperbolic surface or orbifold with a spin structure. The essential tool is the spectral identities coming from the associativity of the product of functions on $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$. It

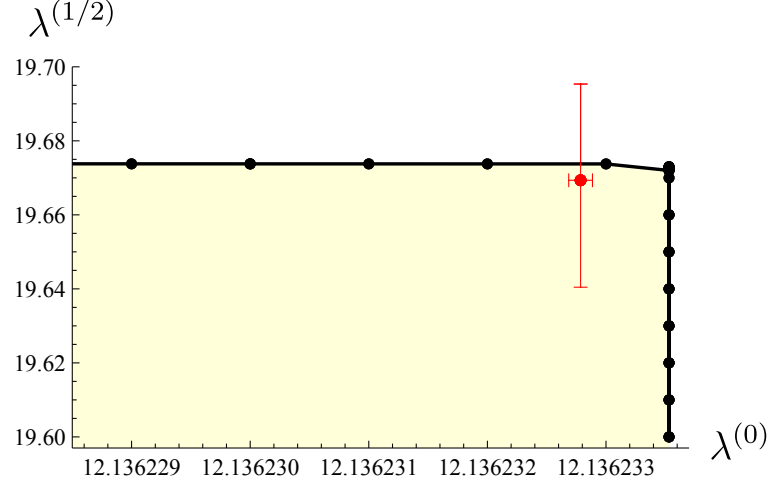


Figure 1.9: Bounds from the $\ell_{3/2} = \ell_3 = 1$ system; zooming onto the kink. Everything except the shaded region is disallowed. The red dot near the corner corresponds to $[0; 3, 3, 5]$.

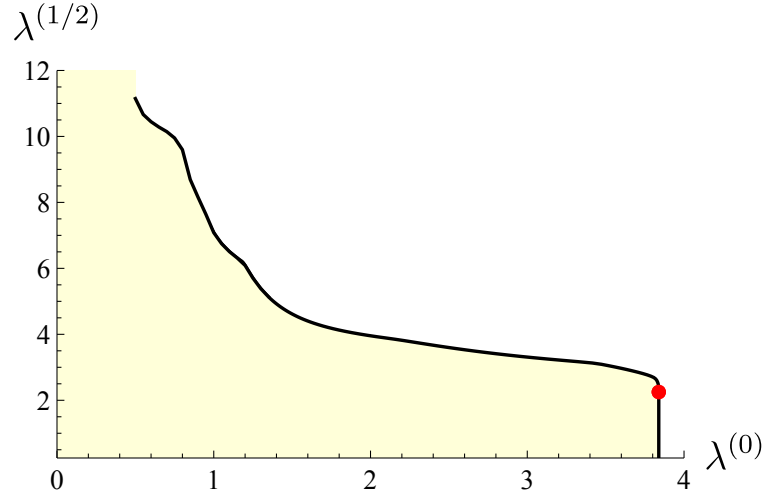


Figure 1.10: Bounds from the $\ell_{1/2} = 1, \ell_1 = 2$ system. Everything except the yellow-shaded region is disallowed. The red dot in the corner corresponds to Bolza surface with odd spin structure.

is the analog of the associativity of the operator product expansion in conformal field theory, which lies at the heart of the conformal bootstrap program.

Our approach is closely related to that of Bernstein and Reznikov [BR10], who investigated the consistency constraints arising from the spectral decomposition of integrals involving a quadruple product of functions within the principal series in $L^2(\Gamma \backslash G)$. See also [Sar94; MV10; Nel21]. In our current study, we have demonstrated that inte-

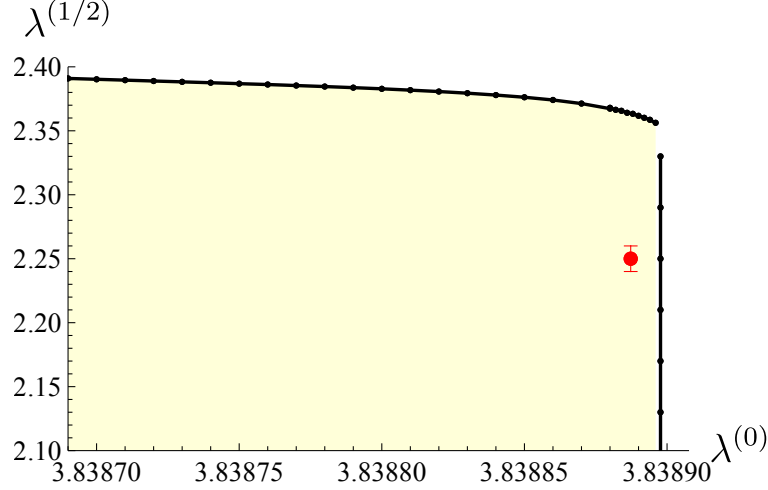


Figure 1.11: Bounds from $\ell_{1/2} = 1, \ell_1 = 2$ system, zooming onto the kink. Everything except the shaded region is disallowed. The red dot corresponds to Bolza surface with odd spin structure.

grating these consistency constraints with linear programming techniques, especially when applied to the discrete series, can yield almost optimal bounds on Laplace and Dirac spectra. There is a close analogy of the above with the analytical bootstrap in CFT, in particular, the approach of Bernstein and Reznikov is akin to asymptotic analysis involving Tauberian theorems, leading to universal behavior of various physical quantities in CFTs [QR17; MZ19; MP20; PQ24].

Several points could be analysed further. In particular:

1. It is interesting to note that the bounds we obtain, for example on Laplace eigenvalues, do not always become more stringent as the number of harmonic spinors or genus increases. For example, Table 1.1 shows that the bound at genus 5 with 3 harmonic spinors is less restrictive than the bound at genus 6 with 3 harmonic spinors. We contrast this scenario with that of [KMP24] where the bound for orbifolds with genus g_1 is not weaker compared to that of genus g_2 whenever $g_1 > g_2$. The origin of monotonicity in the context of [KMP24] is the fact that the functional used in linear programming for genus g_2 can easily be promoted to a functional that is appropriate for genus g_1 . However, unlike the

spectral identities of [KMP24], the spectral identities that we bootstrap in this paper have discrete series irreps appearing in the t channel. This constitutes an obstruction to the previous argument. It would be interesting to understand the origin of this non-monotonicity directly and analytically at the level of the spectral identities.

2. In terms of the Dirac spectrum, it is curious that on the exclusion plot of Figure 1.8, the allowed values of $\lambda^{(1/2)}$ seem to blow up quickly as $\lambda^{(0)}$ approaches zero. This raises the question whether our methods could be improved to obtain better bounds in this region of parameter space, and maybe even a universal upper bound on $\lambda^{(1/2)}$.
3. It would also be interesting to study Dirac spectra in higher dimensions using similar spectra identities originating from associativity. In particular the three-dimensional case is quite accessible, extending the work of [BMP23].

One can see the results of this paper as a new contribution to an emerging program, started in [KMP24], to apply techniques originating from conformal field theory to questions relevant to analytic number theory and hyperbolic geometry. Already in our case, as well as in the three-dimensional case [BMP23], these techniques have been very fruitful. Natural extensions of this work on spinors would be to allow fractional spin structures, or to investigate the case of super-Riemann surfaces. More generally, it would be very interesting to push this program towards quotients of Lie groups of higher rank (for example, $\mathrm{SL}(3, \mathbb{R})$ seems to be a natural next step), as well as the study of non-Archimedean geometries over $\mathbb{Q}_p$.

Perhaps the most striking feature of our findings is that at least in the cases displayed on Figures 1.7 and 1.10, the most symmetric surfaces (i.e. the $[0; 3, 3, 5]$ orbifold, $[1; 3]_{sym}$ and the Bolza surface respectively) sit almost exactly at the boundary, or even the kink, of the allowed region in parameter space. The fact that hyperbolic

surfaces seem to know about the boundary of the exclusion plot, which purely comes from the constraint equations, seems to suggest that the space of solutions to the constraint equations is closely related to the space of quotients of the form $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$, where Γ is a cocompact lattice. There are well-known results establishing a one-to-one correspondence between abelian algebras and various kinds of geometric spaces. For example, the Gelfand–Naimark theorem [GN43] shows that each commutative C^* -algebra can be thought of as an algebra of continuous functions on a locally compact Hausdorff space. There is an analog of this theorem for von Neumann algebras and measured spaces [Tak07; Pav22], and more recently, Connes showed in his reconstruction theorem [Con13] that for every spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A}$ commutative, and a few extra conditions, one can realize $\mathcal{A}$ as $C^\infty(X)$, where X is a smooth oriented compact spin^c manifold, and that all such manifolds can be described by such a spectral triple. It would be interesting to see whether such reconstruction theorems hold for commutative algebras on which various Lie groups act. The existence of such theorems would likely help understand how closely the space of solutions to the constraint equations described here, and the space of 2-dimensional compact orientable spin-orbifolds, are related.

Finally, we remark on the relevance of our work in context of physics, in particular, conformal field theory. In the present paper, the key role is played by the positivity of the measure of $\tilde{\Gamma} \backslash \mathrm{SL}(2, \mathbb{R})$. On a similar note we expect that in the context of path integral formulation of conformal field theories, the existence of a $\mathrm{SO}(1, d)$ invariant positive measure along with associativity might lead to the discovery of new constraints on such theories. Additionally, our framework provides a systematic approach for analyzing the de Sitter bootstrap numerically as suggested recently in [Sun21; LQS23; Lop+23; PSS23], and for extending this program to include fermions.

Appendix Material

1.7 Dimension of the space of harmonic spinors in low genus

In this work, we obtained bounds on the Laplace and Dirac spectra on a surface or orbifold based on the number of modular forms of various weights on this geometric object. A particularly interesting case has been that of harmonic spinors (i.e. modular forms of weight one), as they carry more than only topological information at genus $g \geq 3$. In this appendix, we summarize the state of the art in the classification of Riemann surfaces of genus $g \geq 3$ in terms of the number of harmonic spinors they can carry. A key role is played by *hyperelliptic surfaces*, which are branched double covers of the Riemann sphere. Note that in genus up to 2, every Riemann surface is hyperelliptic, which informs why the space of harmonic spinors can only retain nontrivial geometric information at genus $g \geq 3$. This appendix cross-references the general results of [Bur98] with explicit examples of surfaces with many automorphisms from the literature.

Genus 3

In genus 3, the situation is particularly nice: either the surface is hyperelliptic, or it admits a representation as a regular quartic in $\mathbb{CP}^2$.

The two cases are defined as follows:

- If the surface is hyperelliptic, there exists a choice of spin structure for which the upper bound of two independent harmonic spinors is saturated.
- If the surface is a regular quartic, then the maximum number of harmonic spinors is 1, and it is realized by exactly the odd spin structures on the surface.

In genus 3, the automorphism group of a hyperelliptic surface that has the largest order is $\mathbb{Z}_2 \times S_4$. The corresponding surface, given in [MP17], has affine equation

$$y^2 = x^8 + 14x^4 + 1. \tag{1.132}$$

The other candidate special surfaces have automorphism groups U_6 , V_8 (following the notations of [Sha13]), and $\mathbb{Z}_{14}$, see [MP17] for more details.

On the other hand, the automorphism group of a non-hyperelliptic surface of genus 3 can be much larger [Bar05]. In particular, there exists a non-hyperelliptic surface of genus 3 with isometry group $\mathrm{PSL}(2, \mathbb{F}_7)$, that has 168 elements. This surface is called the *Klein quartic*, and has equation

$$z^3x + x^3y + y^3z = 0. \quad (1.133)$$

The complete classification of possible automorphism groups of genus 3 surfaces is given in [Bar05].

Genus 4

In order to understand the situation in genus 4, we need to introduce the notion of *canonical embedding*.

Definition 1.7.1. Let Σ be a Riemann surface, and let $\{\omega_1, \dots, \omega_g\}$ be a basis of holomorphic differentials on Σ . The canonical embedding of Σ into $\mathbb{CP}^{g-1}$ is the map

$$\sigma \longmapsto [\omega_1(\sigma), \dots, \omega_g(\sigma)]. \quad (1.134)$$

Note that the canonical embedding depends on the choice of basis of holomorphic differentials, however, a change of basis simply amounts to a projective transformation in $\mathbb{CP}^{g-1}$. We have the following general result:

Proposition 1.7.2 ([Nar92]). *The canonical embedding of Σ is injective if and only if Σ is not hyperelliptic.*

In genus 4, we have the additional fact:

Proposition 1.7.3 ([Bur98]). *The canonical model of a non-hyperelliptic surface of genus 4 is the intersection of a unique quadric and a cubic in $\mathbb{CP}^3$. Moreover, the quadric is either smooth or a cone.*

It turns out [Bur98] that this classification in terms of canonical models is the right one to consider to classify the possible dimensions of harmonic spinors. More precisely:

- If the surface is hyperelliptic, there exist spin structures for which the surface admits 2 harmonic spinors.
- If the surface is not hyperelliptic and the quadric supporting its canonical embedding is a cone, then it admits a unique spin structure with 2 harmonic spinors.
- If the surface is not hyperelliptic and the quadric supporting its canonical embedding is a smooth, then the maximum number of harmonic spinors is 1, and it is realized by exactly the odd spin structures on the surface.

In genus 4, still with the notations of [Sha13], the automorphism group of a hyperelliptic surface that has the largest order is V_{10} . The corresponding surface, given in [MP17], has affine equation

$$y^2 = x^{10} - 1. \quad (1.135)$$

The other candidate special surfaces have automorphism groups $\mathrm{SL}(2, \mathbb{F}_3)$, U_8 , and $\mathbb{Z}_{18}$, see [MP17] for more details.

In a similar fashion to the case of genus 3, there exist non-hyperelliptic surfaces of genus 4 with much larger automorphism groups. These surfaces are enumerated in [DBD22]. The surface with the largest automorphism group is *Bring's surface*, with 120 automorphisms. However, the quartic associated to Bring's surface is smooth.

If we want to impose that the canonical embedding of the surface lies on a cone, the largest possible automorphism group has order 72. The corresponding surface has equation

$$z^3 y^2 = x(x^4 + y^4). \quad (1.136)$$

A large number of other large groups (but of smaller order) arise, they are described in detail in [DBD22].

Genus 5

In genus 5, a surface can carry a maximal number of harmonic spinors if and only if it is hyperelliptic. The automorphism group of a hyperelliptic surface that has the largest order (120) is $\mathbb{Z}_2 \times A_5$. The corresponding surface, given in [MP17], has affine equation

$$y^2 = x(x^{10} + 11x^5 - 1). \quad (1.137)$$

The other candidate special surfaces have automorphism groups W_2 , U_{10} , V_{12} and $\mathbb{Z}_{22}$ [Sha13], see [MP17] for more details.

If one drops the requirement of hyperellipticity, there is a surface that has more automorphisms: its order is 192. This surface is described in [Con+11; DBD22].

Genus 6

In the case of genus 6, there are once again special surfaces with the maximal (3) number of harmonic spinors but that are not hyperelliptic. More precisely,

- If the surface is hyperelliptic, there exist spin structures for which the surface admits 3 harmonic spinors.
- If the surface is a smooth plane quintic, then it admits a unique spin structure with 3 harmonic spinors.
- If the surface is neither hyperelliptic nor a plane quintic, then there are at most 2 harmonic spinors per spin structure.

Note that the classification of the remaining cases is subtle. It is known, however [Bur98], that trigonal surfaces cannot have more than 1 harmonic spinor per spin

structure.

In genus 6, the automorphism group of a hyperelliptic surface that has the largest order is V_{14} (following the notations of [Sha13]). The corresponding surface, given in [MP17], has affine equation

$$y^2 = x^{14} - 1. \quad (1.138)$$

The other candidate special surfaces have automorphism groups U_{12} , $GL(2, \mathbb{F}_3)$ and $\mathbb{Z}_{26}$, see [MP17] for more details.

Once again, there exist non-hyperelliptic surfaces of genus 6 with much larger automorphism groups. These surfaces are enumerated in [Bur98]. The surface with the largest automorphism group is *Fermat's quintic* [Bur98], with 150 automorphisms. It is a smooth plane quintic of equation

$$x^5 + y^5 = z^5, \quad (1.139)$$

so interestingly, it also can carry three harmonic spinors.

The surface of genus 6 that *cannot* carry 3 harmonic spinors and has the largest automorphism group is the Wiman sextic, described for example in [DBD22]. Its automorphism group is isomorphic to S_5 , which means that it has 120 automorphisms.

1.8 Numerical estimate of $\lambda_1^{(0)}$ and $\lambda_1^{(1/2)}$ for various orbifolds and surfaces

In this appendix, we present the table 1.3, showing the intervals $[a_1, b_1]$ and $[c_1, d_1]$ containing the $\lambda_1^{(0)}$ and the $\lambda_1^{(1/2)}$ respectively. Note that we can rigorously only show that the intervals $[0, a_1)$ and $[0, c_1)$ does not contain $\lambda_1^{(0)}$ and $\lambda_1^{(1/2)}$ respectively. So the intervals presented in the table 1.3 are the ones, potentially containing the first eigenvalue. However, from the numerics we observe that these intervals contain the first non-trivial eigenvalue, see the remark 1.4.6.

Signature	Interval containing $\lambda_1^{(0)}; [a_1, b_1]$	Interval containing $\lambda_1^{(1/2)}; [c_1, d_1]$
[0;3,3,5]	[12.13623266082, 12.13623279684]	[19.62850299650, 19.70606979308]
[0;3,3,7]	[6.622512981830, 6.62251303689]	[14.58137931985, 14.70308279925]
[0;3,3,9]	[4.760935531772, 4.760935540974]	[13.04476106136, 13.20743304307]
[0;3,3,11]	[3.817638624612, 3.817638645376]	[12.31511817207, 12.56227388521]
[0;3,3,13]	[3.243870176473, 3.243870434758]	[11.89945757571, 12.22641050708]
[0;3,3,15]	[2.856060123567, 2.856061719363]	[11.66435497067, 12.00545171009]
[0;3,3,17]	[2.575237981382, 2.575242811203]	[11.50074150836, 11.87999680844]
[0;3,3,19]	[2.361748257939, 2.372476650392]	[11.23045200383, 11.84361592566]
[0;3,3,21]	[2.193498651071, 2.195285976813]	[11.19103656533, 11.78223049525]
[0;3,3,23]	[2.057131576807, 2.057705367329]	[11.11887135826, 11.73877352328]
[0;3,3,25]	[1.944123693427, 1.950822253982]	[11.06202801296, 11.70678314765]
[0;3,5,5]	[5.873575959007, 5.873576043568]	[10.26823286914, 10.46485970804]
[0;3,5,7]	[4.105916028717, 4.105919429281]	[8.807322253657, 9.080333571881]
[0;3,5,9]	[3.240661780680, 3.240671025102]	[8.255608604384, 8.583931161697]
[0;3,5,11]	[2.734102690099, 2.735649379604]	[7.907206673453, 8.351477142335]
[0;3,5,13]	[2.400941404204, 2.475085745560]	[7.726023042016, 8.311960209309]
[0;3,5,15]	[2.163736742483, 2.379780806385]	[7.585768436685, 8.286345064939]
[0;3,7,7]	[3.253194157760, 3.263120231408]	[7.649577360853, 8.068585239517]
[1;3] _{sym}	[4.760933182368, 4.765358782461]	[8.255418967910, 8.297252909660]* [3.108229958351, 3.190576244169]
Bolza surface	[3.838886940769, 3.842772834639]	[2.246498128260, 2.259880718024]* [1.188383272199, 1.192003537383] [0.8970133121437, 0.8975387540243]

Table 1.3: $\lambda_1^{(0)}$ and $\lambda_1^{(1/2)}$ for various hyperbolic spin surfaces and spin orbifolds. Orbifolds with signature [1;3] have 4 different spin structures but for the most symmetric point on the moduli space, there are only two different sets of spectra for the Dirac operator. Similarly, the Bolza surface has 16 different spin structures but there are only three different Dirac spectra. The intervals marked with * correspond to spin structures that support a harmonic spinor.

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ANGULAR FRACTALS IN THERMAL QUANTUM FIELD THEORY

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2.1 Introduction

Many aspects of conformal field theories (CFTs) are universal at high energies. A famous example is Cardy’s formula, which states that the entropy of local operators at sufficiently high energies takes a universal form in all unitary, compact 2d CFTs [Car86] (see [MZ19; PS20b; MP20] for a precise formulation). Equivalently, the partition function of a 2d CFT

$$\mathrm{Tr} \left[e^{-\beta H + i\theta J} \right] \quad (2.1)$$

is universal in the high temperature regime $\beta \rightarrow 0$ with $\theta \sim O(\beta)$.

The derivation of Cardy’s formula uses invariance of the torus partition function under the modular transformation $S : \tau \mapsto -1/\tau$. By instead using the full modular group $\mathrm{PSL}(2, \mathbb{Z})$, one finds similar universal behavior as $\beta \rightarrow 0$, near *any rational angle* $\theta = \frac{2\pi p}{q}$, see e.g. [Ben+19]. This leads to universal “spin-refined” versions of the density of states. For example, in the case $\frac{p}{q} = \frac{1}{2}$, the modular transformation $\tau \mapsto \frac{-\tau}{2\tau-1}$ gives the universal behavior of

$$\mathrm{Tr} \left[e^{-\beta(H - i\Omega J)} (-1)^J \right] = \mathrm{Tr} \left[e^{-\beta H + i\theta J} \right]_{\theta = \pi + \beta\Omega}, \quad (2.2)$$

in the regime $\beta \rightarrow 0$ with $\Omega \sim O(1)$. For any given 2d CFT, the logarithm of (2.2) is 1/4 the logarithm of (2.1) at high temperature, leading to a universal result for the

difference between densities of even- and odd-spin operators in 2d CFTs.¹

While modular invariance is not available on $S^{d-1} \times S^1$ in higher dimensions, higher dimensional CFTs still display forms of universality at high energies, both in their density of states [Ver00; KL01; Bha+08; Sha16b; Sha16a; Ben+24; AS24], and OPE coefficients [Del20; Ben+24]. A central insight from [Bha+08; Jen+12; Ban+12] is that the high temperature behavior of a CFT can be captured by a “thermal Effective Field Theory (EFT)” that efficiently encodes the constraints of conformal symmetry and locality. In [Sha16b; Sha16a; Ben+24; AS24], thermal EFT plays the role of a surrogate for the modular S -transformation (as well as modular transformations on genus-2 surfaces).

In this work, we will be interested in “spin-refined” information about the CFT density of states in general dimensions. In particular, we will study the partition function (2.1) with high temperature ($\beta \ll 1$) and finite angles $\vec{\theta}$. (In higher dimensions we promote $\vec{\theta}$ and $\vec{J}$ to vectors with $[d/2]$ components coming from the rank of $SO(d)$.) The regime $\vec{\theta} = \beta \vec{\Omega}$ with fixed $\vec{\Omega}$ is captured by thermal EFT as discussed in [Bha+08; Jen+12; Ban+12; Sha16a; Ben+24]. However, when $\vec{\theta}$ does not scale to zero as $\beta \rightarrow 0$, the naïve EFT description breaks down.

A simple example of a partition function with finite $\vec{\theta}$ is (2.2): the relative density of even-spin and odd-spin operators with respect to some particular Cartan generator J of the rotation group. This observable is naïvely outside the regime of validity of the thermal EFT, since θ remains finite as $\beta \rightarrow 0$.

More generally, we can consider a partition function that includes a rotation by finite rational angles in each of the Cartan directions:²

$$\text{Tr} \left[e^{-\beta(H - i\vec{\Omega} \cdot \vec{J})} R \right], \quad \text{where} \quad R = e^{2\pi i \left(\frac{p_1}{q_1} J_1 + \dots + \frac{p_n}{q_n} J_n \right)}. \quad (2.3)$$

¹Modular invariance on higher genus surfaces also leads to universal results for OPE coefficients in 2d CFTs, as derived in [KM17; CMM17; DDP18; BDD18; HKT18], and unified in [Col+20].

²In parity-invariant theories, we can also include reflections.

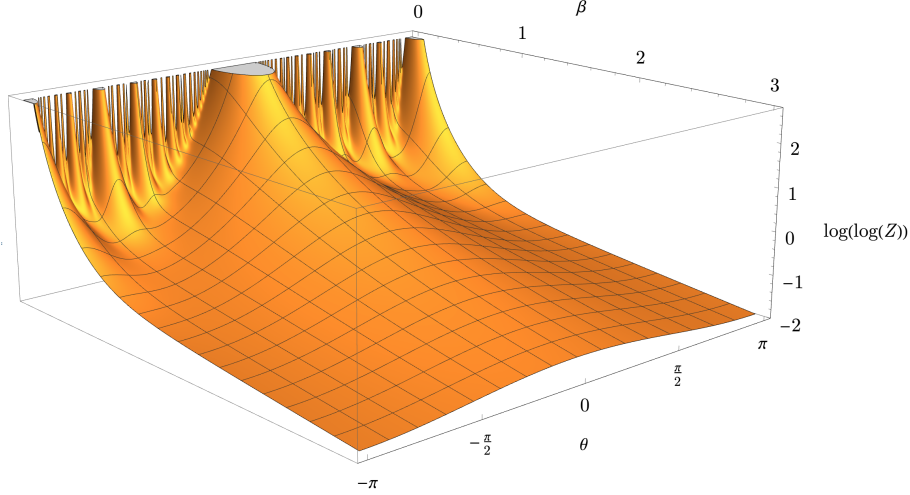


Figure 2.1: A qualitative picture of $\log(\log(Z))$ in the 3d Ising CFT, where $Z = \text{Tr}(e^{-\beta H + i\theta J})$ is the $S^2 \times S^1$ partition function. To construct this picture, we took the leading terms in the EFT description around each rational angle (up to denominator 15), and combined them with a root-mean-square. We give more detail in appendix 2.11.

Using a trick that was applied in [AM21] to study superconformal indices near roots of unity, we will find a *different* EFT description for this partition function, in terms of the thermal EFT on a background geometry with inverse temperature $q\beta$ and spatial manifold $S^{d-1}/\mathbb{Z}_q$, where $q = \text{lcm}(q_1, \dots, q_n)$. This determines the small- β expansion of (2.3) in terms of the usual Wilson coefficients of thermal EFT, up to new subleading contributions from “Kaluza-Klein vortices” that we classify. For example, the effective free energy density of (2.3), coming from the leading term in the thermal effective action, is smaller than the usual free energy density by a factor of $1/q^d$. In particular, the effective free energy density of even-spin minus odd-spin operators described by (2.2) is smaller by $1/2^d$. (This generalizes the factor of $1/4$ in 2d.)³

The EFT descriptions around each rational angle patch together to create fractal-like behavior in the high-temperature partition function — see figure 2.1 for an illustration in the 3d Ising CFT. It is remarkable that effective field theory constrains

³Note that simply taking the density of states computed in [Ben+24] and inserting the phase R into the trace will *not* give the correct answer to the partition function. For a demonstration of this in 2d, see appendix B of [Ben+19].

the asymptotics of the partition function in such an intricate way, even in higher dimensions.

Kaluza-Klein vortices appear whenever the rational rotation R does not act freely on the sphere S^{d-1} . Each vortex creates a defect in the thermal EFT, whose action can be written systematically in a derivative expansion in background fields. By contrast, when R generates a group that acts freely, no vortex defects are present, and the complete perturbative expansion of (2.3) in β is determined in terms of thermal EFT Wilson coefficients, with no new undetermined parameters.

While most of our discussion and examples are focused on CFTs, our formalism also applies to general QFTs. In particular, using thermal effective field theory, we derive a relation between the partition function at temperature T with a discrete isometry of order q inserted, to the partition function with no insertion at temperature T/q , in the thermodynamic limit.⁴ For example, we have

$$-\log \text{Tr}_{\mathcal{H}(\mathcal{M}_L)} [e^{-\beta H} R] \sim -\frac{1}{q} \log \text{Tr}_{\mathcal{H}(\mathcal{M}_L)} [e^{-q\beta H}] + \text{topological} + \text{KK defects} \\ (\text{as } L \rightarrow \infty). \quad (2.4)$$

Here, $\mathcal{M}_L$ is a spatial manifold of characteristic size L , with associated Hilbert space $\mathcal{H}(\mathcal{M}_L)$, R is a discrete isometry of order q , and “ $\sim$ ” denotes agreement to all perturbative orders in $1/L$. The relation (2.4) holds whenever the theory is gapped at inverse temperature $q\beta$. We write the most general relation in (2.32), which we check in both massive and massless examples. An interesting consequence of this simple formula is that twists by discrete isometries can be sensitive to lower-temperature phases of the theory. For example, the partition function of QCD at temperature $T > \Lambda_{\text{QCD}}$, twisted by a discrete isometry with order q , becomes sensitive to physics below the confinement scale when $T/q < \Lambda_{\text{QCD}}$.

⁴We are extremely grateful to Luca Delacretaz for emphasizing the general QFT case to us.

This universality of partition functions with spacetime symmetry insertions is in contrast to the case for global symmetry insertions. The insertion of a global symmetry generator operator is equivalent to turning on a new background field in the thermal EFT. The dependence of the effective action on this background field introduces new Wilson coefficients that are not necessarily related in a simple way to the Wilson coefficients without the global symmetry background, see e.g. [PS20a; HO22; KLO23].

The paper is organized as follows. In section 2.2, we present a derivation of our main result: a systematic study of the high temperature expansion of the partition function of any quantum field theory with the insertion of a discrete isometry. In section 2.3, we look in more detail at the Kaluza-Klein vortices that appear on S^{d-1} when the discrete isometry (which is a rational rotation in this case) has fixed points. In section 2.4, we discuss subtleties that appear for fermionic theories. In section 2.5, we give several examples in free theories that illustrate our general results. In section 2.6, we consider thermal effective actions with topological terms. In section 2.7, we apply our results to holographic CFTs. In section 2.8, we look at irrational θ . In section 2.9, we discuss non-perturbative corrections in temperature. Finally in section 2.10, we conclude and discuss future directions.

2.2 Folding and unfolding the partition function

Thermal effective action and finite velocities

Equilibrium correlators of generic interacting QFTs at finite temperature are expected to have a finite correlation length. Equivalently, the dimensional reduction of a generic interacting QFT on a Euclidean circle is expected to be gapped. When this is the case, long-distance finite-temperature observables of the QFT can be captured by a local “thermal effective action” of background fields [Bha+08; Jen+12; Ban+12]. For example, consider the partition function of a QFT_d on $\mathcal{M}_L \times S^1_\beta$, where the spatial

$d-1$ -manifold $\mathcal{M}_L$ has size L . In the thermodynamic limit of large L , we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}(\mathcal{M}_L)}[e^{-\beta H_L}] &= Z_{\mathrm{QFT}}[\mathcal{M}_L \times S^1_\beta] \\ &= Z_{\mathrm{gapped}}[\mathcal{M}_L] \\ &\sim e^{-S_{\mathrm{th}}[g,A,\phi]} + \text{nonperturbative in } 1/L \quad (L \rightarrow \infty),\end{aligned}\tag{2.5}$$

where $\mathcal{H}(\mathcal{M}_L)$ is the Hilbert space of states on $\mathcal{M}_L$, and H_L is the Hamiltonian. Here, the thermal effective action S_{th} depends on a $d-1$ -dimensional metric g_{ij} , a Kaluza-Klein gauge field A_i , and a dilaton ϕ , which can be obtained by placing the d -dimensional metric in Kaluza-Klein (KK) form

$$G_{\mu\nu}dx^\mu dx^\nu = g_{ij}(\vec{x})dx^i dx^j + e^{2\phi(\vec{x})}(d\tau + A_i(\vec{x}))^2,\tag{2.6}$$

where $\tau \sim \tau + \beta$ is a periodic coordinate along the thermal circle. The derivative expansion for S_{th} becomes an expansion in inverse powers of the length L .

If the spatial manifold $\mathcal{M}_L$ possesses a continuous isometry ξ , then we can additionally twist the partition function by the corresponding charge Q_ξ :

$$\mathrm{Tr}_{\mathcal{H}(\mathcal{M}_L)} \left[e^{-\beta(H_L - i\alpha Q_\xi)} \right].\tag{2.7}$$

Geometrically, this twist corresponds to a deformation of the background fields g, A, ϕ that depends on $\alpha\xi$. In the thermodynamic limit, we can describe (2.7) using the thermal effective action, provided that the background fields g, A, ϕ remain finite as $L \rightarrow \infty$. In particular, the combination $\alpha\xi$ must remain finite as $L \rightarrow \infty$. The physical reason is that $i\alpha$ represents the velocity of the system in the direction of ξ in the canonical ensemble. This velocity must remain finite in order to have a good thermodynamic limit.

By contrast, suppose that $\mathcal{M}_L$ possesses a nontrivial discrete isometry R with finite order $R^q = 1$. If we twist the partition function by R ,

$$\mathrm{Tr}_{\mathcal{H}(\mathcal{M}_L)} \left[e^{-\beta H} R \right],\tag{2.8}$$

then physically this corresponds to a system whose “velocity” is of order L . The background fields g, A, ϕ naïvely do not have a good thermodynamic limit, and we cannot apply the thermal effective action in an obvious way.

Example: CFT partition function

An important example for us is the partition function of a CFT_d on $S^{d-1} \times S^1_\beta$. Conformal invariance dictates that

$$\text{Tr} \left[e^{-\beta(H - i\vec{\Omega} \cdot \vec{J})} \right] = \text{Tr}_{\mathcal{H}(S^{d-1}_L)} \left[e^{-L\beta(H_L - i\vec{\Omega} \cdot \vec{J}_L)} \right]. \quad (2.9)$$

On the left-hand side, we have the usual partition function of the CFT on a sphere of radius 1. On the right-hand side, H_L denotes the Hamiltonian on a sphere S^{d-1}_L of radius L , and $\vec{J}_L$ are generators of isometries of the sphere, normalized so that the corresponding Killing vectors are finite in the flat-space limit $L \rightarrow \infty$. (For example, for a rotation of the sphere by an angle ϕ , a Killing vector with a finite flat-space limit is $\frac{1}{L} \frac{\partial}{\partial \phi}$.)

When β is small, we can set $L = O(1/\beta)$ on the right-hand side and try to apply the thermal effective action (2.5). We find that in order to have a good thermodynamic limit as $\beta \rightarrow 0$, the angular potentials $\vec{\Omega}$ must remain finite. Phrased in terms of the rotation angle $\vec{\theta} = \beta \vec{\Omega}$, we find that $\vec{\theta}$ must scale to zero as $\beta \rightarrow 0$. Provided this is the case, the $1/L$ expansion of the thermal effective action gives an expansion in small β for the CFT partition function.

We can also understand condition $\vec{\theta} \rightarrow 0$ more explicitly from a direct computation using the thermal effective action. In a CFT, the thermal effective action is constrained by d -dimensional Weyl invariance. The most general coordinate- and Weyl-invariant

action takes the form^{5,6}

$$S_{\text{th}} = \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}} \sqrt{\widehat{g}} \left(-f + c_1 \beta^2 \widehat{R} + c_2 \beta^2 F^2 + \dots \right) + S_{\text{anom}}. \quad (2.10)$$

Here $\widehat{g} = e^{-2\phi} g$, $\widehat{R}$ is the Ricci scalar built from $\widehat{g}$, F^2 is a Maxwell term, etc. The term S_{anom} accounts for Weyl anomalies (which are not important for the present discussion).

On the geometry $S^{d-1} \times S^1_\beta$, we can easily determine g , A , ϕ and evaluate S_{th} [Ben+24]:

$$S_{\text{th}} = \frac{\text{vol } S^{d-1}}{\prod_{i=1}^n (1 + \Omega_i^2)} \left[-f T^{d-1} + (d-2) \left((d-1)c_1 + \left(2c_1 + \frac{8}{d}c_2 \right) \sum_{i=1}^n \Omega_i^2 \right) T^{d-3} + \dots \right]. \quad (2.11)$$

We see that terms of order $T^{d-1-k} = \beta^{k-d+1}$ in the high-temperature expansion of S_{th} are multiplied by a polynomial in the angular potentials Ω_i of degree k (see e.g. examples in [BHR23]). Consequently, $\theta_i \rightarrow 0$ as $\beta \rightarrow 0$ is necessary for the high-temperature expansion to be well-behaved.

To summarize, the thermal effective action can describe “small” angles $\theta \sim \beta\Omega$, where the angular velocity remains finite in the thermodynamic limit. However, results from the thermal effective action like (2.11) break down outside this regime. How can we access more general angles?

Spin-refined partition functions: warm-up in 2d CFT

As a warm-up, in 2d CFT, we can compute partition functions at more general angles using modular invariance. Let us review how this works and derive some example results. For convenience, we write the partition function as:

$$Z(\tau, \bar{\tau}) = \text{Tr} \left[e^{2\pi i \tau (L_0 - \frac{c}{24}) - 2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right], \quad (2.12)$$

⁵For simplicity, here we assume that the theory is free of gravitational anomalies.

⁶Note that [Ben+24] worked in conventions where τ has periodicity 1, and β is absorbed into the field ϕ . In this paper, we instead use conventions where τ has dimensionful periodicity β (later we will also have other periodicities) so that explicit powers of β appear in the action (2.10), as required by dimensional analysis. To convert from the conventions of [Ben+24] to the conventions in this work, one shifts the dilaton by $\phi \rightarrow \phi + \log \beta$.

where $\tau = \frac{i\beta}{2\pi} + \frac{\theta}{2\pi}$ and $\bar{\tau} = \tau^*$.⁷ The high temperature behavior of $Z(\tau, \bar{\tau})$ at small angles can be obtained by performing the modular transformation $\tau \rightarrow -1/\tau$ (similarly for $\bar{\tau}$) and approximating by the contribution of the vacuum state. The result agrees with the thermal effective action:

$$\text{Tr} [e^{-\beta(H-i\Omega J)}] \sim e^{-S_{\text{th}}} = \exp \left[\frac{\text{vol } S^1}{(1+\Omega^2)} \frac{f}{\beta} \right] = \exp \left[\frac{4\pi^2}{\beta(1+\Omega^2)} \frac{c}{12} \right] \quad (\text{CFT}_2), \quad (2.13)$$

where $f = \frac{2\pi c}{12}$. Here, we assume $c_L = c_R$ for simplicity. In this case, only the cosmological constant term appears in the thermal effective action in 2d.

Now let us instead assume that $\frac{\theta}{2\pi}$ is close to a nonzero rational angle $\frac{p}{q}$, so that $\tau, \bar{\tau}$ are very close to $\frac{p}{q}$. Following [Ben+19], we can perform a different modular transformation to map $(\tau, \bar{\tau})$ close to $\pm i\infty$ and approximate the partition function by the vacuum state in the new channel. For example, let us study the partition function with an insertion of $(-1)^J$ given in (2.2). In this case, we have

$$\begin{aligned} \tau &= \frac{1}{2} + \frac{\beta\Omega}{2\pi} + \frac{i\beta}{2\pi} \\ \bar{\tau} &= \frac{1}{2} + \frac{\beta\Omega}{2\pi} - \frac{i\beta}{2\pi}, \quad \beta \ll 1, \quad \Omega \sim O(1). \end{aligned} \quad (2.14)$$

Modular invariance is the statement

$$Z(\gamma \circ \tau, \gamma \circ \bar{\tau}) = Z(\tau, \bar{\tau}), \quad \gamma \in \text{PSL}(2, \mathbb{Z}). \quad (2.15)$$

An appropriate transformation in this case is

$$\gamma = \pm \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}), \quad (2.16)$$

which leads to

$$\text{Tr} [e^{-\beta(H-i\Omega J)} (-1)^J] = \text{Tr} \left[e^{2\pi i \tilde{\tau} (L_0 - \frac{c}{24}) - 2\pi i \tilde{\bar{\tau}} (\bar{L}_0 - \frac{c}{24})} \right] \sim \exp \left[\frac{1}{4} \frac{4\pi^2}{\beta(1+\Omega^2)} \frac{c}{12} \right],$$

⁷Note that in this section, τ denotes the modular parameter of the torus, while in other sections τ denotes Euclidean time. We hope this will not cause confusion.

$$\text{where } \tilde{\tau} = -\frac{1}{2} + \frac{\pi i}{2\beta(1-i\Omega)}, \quad \bar{\tilde{\tau}} = \tilde{\tau}^*. \quad (2.17)$$

On the right-hand side, we approximated the trace by the contribution of the vacuum state in the $\beta \rightarrow 0$ limit. We find that the partition function weighted by $(-1)^J$ grows exponentially in $1/\beta$, with an exponent that is $1/4$ of the un-weighted case (2.13).

For a general angle $\frac{\theta}{2\pi}$ close to $\frac{p}{q}$, we repeat the same logic above but with a more complicated modular transformation, namely

$$\gamma = \pm \begin{pmatrix} -(p^{-1})_q & b \\ q & -p \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}), \quad (2.18)$$

where $(p^{-1})_q$ is the inverse of p modulo q , and b is chosen so the matrix has determinant 1. We get

$$\text{Tr} \left[e^{-\beta(H-i\Omega J)} e^{2\pi i \frac{p}{q} J} \right] \sim \exp \left[\frac{1}{q^2} \frac{4\pi^2}{\beta(1+\Omega^2)} \frac{c}{12} \right]. \quad (2.19)$$

In general, we find that the partition function of a 2d CFT weighted by $e^{2\pi i \frac{p}{q} J}$ grows exponentially in $1/\beta$, with an exponent that is $1/q^2$ of the un-weighted case (2.13).

Because modular invariance is not available in higher dimensions, it will be useful to rederive (2.19) in a different way. We now describe two (related) approaches that can generalize to higher dimensions.

Folding and unfolding

Thermal EFT naively breaks down in spin-refined partition functions like (2.2) because the large spacetime symmetry $(-1)^J$ moves us outside the thermodynamic limit. One way to recover an EFT description is to perform a change of coordinates that makes $(-1)^J$ look more like a global symmetry.

For example, consider a spin-refined partition function of a 2d QFT (not necessarily conformal) on $S_L^1 \times S_\beta^1$,

$$\text{Tr} \left[e^{-\beta H} (-1)^J \right], \quad (2.20)$$

where $(-1)^J$ denotes a rotation of the spatial circle S_L^1 by π . We can reinterpret *one* copy of the QFT on $S_L^1 \times S_\beta^1$ as *two* copies of the QFT on $(S_L^1/\mathbb{Z}_2) \times S_\beta^1$, with topological defects that glue the two copies to each other, see the middle of figure 2.2. In this picture, the operator $(-1)^J$ becomes a topological defect that simply permutes the two copies of the QFT as we move along the time direction. If we begin in one copy of the QFT and move by β in Euclidean time, we pass once through the $(-1)^J$ defect and go to the other copy. Moving by β again, we pass through the $(-1)^J$ defect again and end up in the first copy. Thus, inserting $(-1)^J$ into the partition function creates a new effective thermal circle of length 2β .

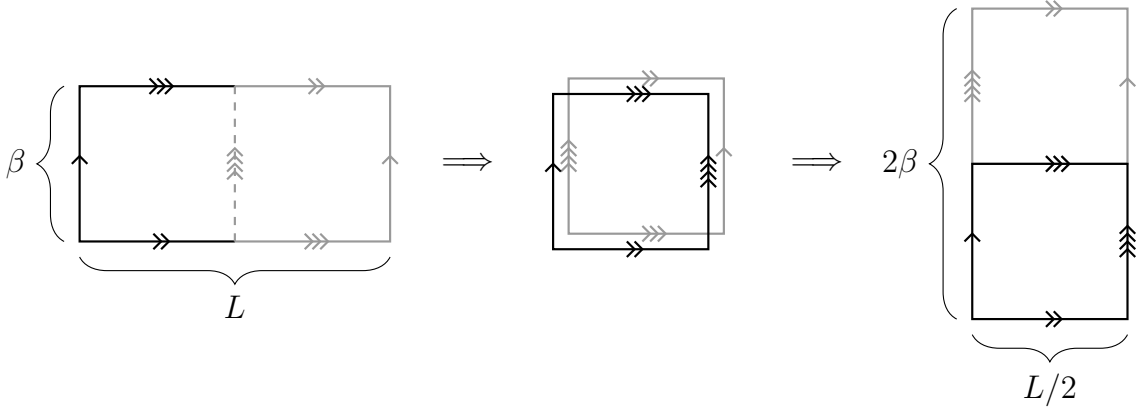


Figure 2.2: **Left:** The torus partition function with spatial cycle of length L , inverse temperature β , and an insertion of $(-1)^J$. The $(-1)^J$ insertion means we must glue the top and bottom of the figure with a half shift around the spatial circle. We split the figure into a left and right half using the trivial defect (vertical dashed line), and for convenience we color the right half grey. **Middle:** Placing the black and grey rectangles on top of each other, we can interpret this same observable as the partition function of two copies of the QFT (black and grey) on an $(L/2) \times \beta$ rectangle, with boundary conditions inherited from the left figure. **Right:** Finally, we can re-stack the two copies of the QFT, resulting in a single copy of the QFT with a new spatial circle of length $L/2$ and an effective thermal circle of length 2β . Note that the effective thermal circle is nontrivially fibered over the new spatial circle.

This reinterpretation of the path integral with a $(-1)^J$ insertion is illustrated in figure 2.2. One wrinkle (that is clear in the figure) is that the effective thermal $S_{2\beta}^1$ is nontrivially fibered over the spatial circle $S_L^1/\mathbb{Z}_2$: when we go once around the new spatial circle, the $S_{2\beta}^1$ shifts by β .

So far, we have considered a rotation angle of π . However, it is straightforward to study nearby rotation angles of the form $\theta = \pi + \beta\Omega$. On the left-hand side of figure 2.2, we simply insert an additional topological operator along the spatial cycle that implements the small rotation $e^{i\beta\Omega J}$. Following the manipulations in the figure, we end up with a product of two such operators on the new spatial cycle $S_L^1/\mathbb{Z}_2$, which together implement a rotation of $2\beta\Omega$.

The advantage of this rewriting of the path integral is that we can now smoothly take the thermodynamic limit $L \rightarrow \infty$ and use the thermal effective action. The effective inverse temperature is 2β , the rotation angle is $2\beta\Omega$, and the effective spatial cycle is $S_L^1/\mathbb{Z}_2$.

In fact, the above construction is straightforward to generalize to twists by any rational angle:

$$\text{Tr} \left[e^{-\beta(H - i\Omega J) + 2\pi i \frac{p}{q} J} \right]. \quad (2.21)$$

We interpret (2.21) as the partition function of q copies of the QFT on the space $S_L^1/\mathbb{Z}_q$, with appropriate topological defects that glue the copies together. The operator $e^{2\pi i \frac{p}{q} J}$ becomes a topological defect that permutes the copies of the QFT as we move around the Euclidean time circle. This creates an effective thermal circle $S_{q\beta}^1$, which is fibered over $S_L^1/\mathbb{Z}_q$. We can now apply thermal EFT on $S_L^1/\mathbb{Z}_q$.

Example: 2d CFT

As an example application, we can recover our previous answer for the spin-refined partition function of a 2d CFT. For a twist by $(-1)^J$, we find

$$\text{Tr} \left[e^{-\beta(H - i\Omega J)} (-1)^J \right] \sim e^{-S_{\text{th}}[S^1/\mathbb{Z}_2 \times S_{2\beta}^1]} = \exp \left[\frac{2\pi c}{12} \frac{\text{vol}(S^1/\mathbb{Z}_2)}{2\beta(1 + \Omega^2)} \right] = \exp \left[\frac{1}{4} \frac{4\pi^2}{\beta(1 + \Omega^2)} \frac{c}{12} \right]. \quad (2.22)$$

In the action, we obtain one factor of $\frac{1}{2}$ from the smaller spatial cycle $S^1/\mathbb{Z}_2$, and another factor of $\frac{1}{2}$ from the larger thermal circle, resulting in an overall factor of $\frac{1}{4}$

that agrees with the result from modular invariance (2.17).⁸

More generally, for a twist by $\frac{2\pi p}{q}$, the thermal effective action gives

$$\mathrm{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{2\pi i p}{q} J} \right] \sim e^{-S_{\mathrm{th}}[S^1/\mathbb{Z}_q \times S^1_{q\beta}]} = \exp \left[\frac{2\pi c}{12} \frac{\mathrm{vol}(S^1/\mathbb{Z}_q)}{q\beta(1+\Omega^2)} \right] = \exp \left[\frac{1}{q^2} \frac{4\pi^2}{\beta(1+\Omega^2)} \frac{c}{12} \right]. \quad (2.23)$$

We find that the effective free energy at high temperature for the spin-refined partition function (2.21) is down by a factor of q^2 , in agreement with (2.19). Note that the precise permutation of the copies of the CFT implemented by $e^{2\pi i \frac{p}{q} J}$ depends on p , but the length of the resulting thermal circle does not. Consequently the partition function is independent of p , up to nonperturbative corrections as $\beta \rightarrow 0$.⁹

Higher dimensions

The above construction works for $d > 2$ as well, and on more general geometries. Consider a QFT $_d$ on any $(d-1)$ -dimensional spatial manifold $\mathcal{M}_L$ with a discrete isometry R of finite order $R^q = 1$. Again, we can reinterpret one copy of the QFT on $\mathcal{M}_L \times S^1_\beta$ as q copies of the QFT on $(\mathcal{M}_L/\mathbb{Z}_q) \times S^1_\beta$, with topological defects that glue the copies to each other. In this picture, R is represented as a topological defect that simply permutes the q copies of the QFT as we move along the time direction, creating an effective inverse temperature $q\beta$.

The EFT bundle

Before exploring further consequences of this idea, it will be helpful to adopt a more abstract, geometrical perspective on this construction. Consider again a d -

⁸The fact that the thermal circle is nontrivially fibered plays no role here because the thermal effective action is the integral of a local coordinate-invariant quantity that does not detect global features of the bundle. In a theory with a gravitational anomaly, the thermal effective action would contain an additional 1-dimensional Chern-Simons for the Kaluza-Klein gauge field, which can detect the nontrivial topological structure of the thermal circle bundle, see section 2.6. The nontrivial topology also enters into nonperturbative corrections, see section 2.9.

⁹There is p -dependence if the theory is fermionic (see section 2.4) or has a gravitational anomaly (see section 2.6).

dimensional QFT with spatial manifold $\mathcal{M}_L$. Given an isometry $U \in \text{Iso}(\mathcal{M}_L)$, the partition function twisted by U ¹⁰

$$\text{Tr}_{\mathcal{H}(\mathcal{M}_L)} [e^{-\beta H} U]. \quad (2.24)$$

is computed by the path integral of the CFT on the mapping torus

$$M_{\beta,U} \equiv (\mathcal{M}_L \times \mathbb{R})/\mathbb{Z}, \quad (2.25)$$

where $\mathbb{Z} = \langle h \rangle$ is generated by

$$\begin{aligned} h : \mathcal{M}_L \times \mathbb{R} &\rightarrow \mathcal{M}_L \times \mathbb{R}, \\ h : (\vec{x}, \tau) &\mapsto (U\vec{x}, \tau + \beta), \end{aligned} \quad (2.26)$$

where $\vec{x}$ is a coordinate on $\mathcal{M}_L$.

Now let us specialize to $U = R$, where R has order q . In this case, the q -th power of h acts very simply: it leaves $\mathcal{M}_L$ invariant, and shifts τ by $q\beta$:

$$h^q : (\vec{x}, \tau) \mapsto (\vec{x}, \tau + q\beta). \quad (2.27)$$

Consequently, it is useful to decompose $\mathbb{Z} \cong q\mathbb{Z} \times \mathbb{Z}_q = \langle h^q \rangle \times (\langle h \rangle / \langle h^q \rangle)$, and obtain the mapping torus $M_{\beta,R}$ via two successive quotients. We first quotient by $q\mathbb{Z} \cong \langle h^q \rangle$ (which turns $\mathbb{R}$ into $S_{q\beta}^1$), and then quotient by $\mathbb{Z}_q = \langle h \rangle / \langle h^q \rangle$:

$$M_{\beta,R} = ((\mathcal{M}_L \times \mathbb{R})/q\mathbb{Z})/\mathbb{Z}_q = (\mathcal{M}_L \times S_{q\beta}^1)/\mathbb{Z}_q. \quad (2.28)$$

The quotient $(\mathcal{M}_L \times S_{q\beta}^1)/\mathbb{Z}_q$ on the right-hand side of (2.28) can be viewed as a bundle in two different ways. Firstly, it is a $\mathcal{M}_L$ -bundle over $S_{q\beta}^1/\mathbb{Z}_q \cong S_{\beta}^1$. This is the usual point of view of the trace as a spatial manifold evolving over Euclidean time β . However, we can alternatively view $(\mathcal{M}_L \times S_{q\beta}^1)/\mathbb{Z}_q$ as an $S_{q\beta}^1$ bundle over $\mathcal{M}_L/\mathbb{Z}_q$.

We call this latter description the ‘‘EFT bundle.’’ In section (2.2), the EFT bundle was

¹⁰Here, we abuse notation and write U for both the isometry and the operator implementing its action on the Hilbert space $\mathcal{H}(\mathcal{M}_L)$.

a nontrivial $S^1_{q\beta}$ bundle over $S^1_L/\mathbb{Z}_q$. As we saw, the virtue of the EFT bundle is that the thermodynamic limit $L \rightarrow \infty$ is straightforward: we can dimensionally reduce along the effective thermal circle $S^1_{q\beta}$ without leaving the thermodynamic limit. The theory is then described by thermal EFT with effective inverse temperature $q\beta$ and spatial cycle $\mathcal{M}_L/\mathbb{Z}_q$.

Suppose for the moment that the action of R on $\mathcal{M}_L$ is free, so that $\mathcal{M}_L/\mathbb{Z}_q$ is smooth. (This is the case, for example, for a rational rotation of the spatial circle in 2d.) For any term in the thermal effective action that is the integral of a local density, the effect of the quotient by $\mathbb{Z}_q$ is simply to multiply its contribution by $1/q$. Thus, we conclude

$$-\log \text{Tr} [e^{-\beta H} R] \sim -\frac{1}{q} \log \text{Tr} [e^{-q\beta H}] + \text{topological} \quad (\text{if the } R \text{ action is free}). \quad (2.29)$$

Here, “ $\sim$ ” denotes agreement to all perturbative orders in the $1/L$ expansion. The term “topological” indicates potential contributions from a finite number of terms capable of detecting the topology of the EFT bundle, which cannot be written as the integral of a local gauge/coordinate-invariant density. We discuss such terms in section 2.6.

Let us pause to note that the result (2.29) really only requires that the theory be gapped at inverse temperature $q\beta$ (not necessarily at inverse temperature β), since we only use locality of the thermal effective action on the right-hand side.

Adding “small” isometries

Just as before, we can also consider inserting into the trace an additional “small” isometry $U = e^{i\beta(\alpha Q_\xi)}$, where ξ is a Killing vector on $\mathcal{M}_L$, Q_ξ is its corresponding charge, and α is the corresponding thermodynamic potential. We will be mainly interested in the case where U commutes with the discrete isometry R , so we assume

this henceforth. The insertion of U can be thought of as a topological defect that wraps $\mathcal{M}_L$. Consequently, the defect wraps q times around the base of the EFT bundle $\mathcal{M}_L/\mathbb{Z}_q$, resulting in an effective rotation U^q . We conclude that

$$-\log \text{Tr} [gR] \sim -\frac{1}{q} \log \text{Tr} [g^q] + \text{topological} \quad (\text{if the } R \text{ action is free}), \quad (2.30)$$

where $g = e^{-\beta H} U$.

In fact, this argument applies to any global symmetry element V as well, so (2.30) holds when g is multiplied by a global symmetry group element: $g = e^{-\beta H} UV$. We can think of V as implementing a nontrivial flat connection for a background gauge field coupled to the global symmetry. In this case, the “topological” terms in (2.30) could include contributions from nontrivial topology of this connection.

We can also understand the insertion of “small” isometries geometrically. Again, the idea is to view the mapping torus $M_{\beta, UR}$ as the result of two successive quotients

$$M_{\beta, UR} = ((\mathcal{M}_L \times \mathbb{R}) / \langle h^q \rangle) / \mathbb{Z}_q = M_{q\beta, U^q} / \mathbb{Z}_q, \quad (2.31)$$

where $h : (\vec{x}, \tau) \mapsto (UR\vec{x}, \tau + \beta)$,

where $\mathbb{Z}_q = \langle h \rangle / \langle h^q \rangle$, and we have used $(UR)^q = U^q$. On the right-hand side, we have the mapping torus $M_{q\beta, U^q}$ which is described by the thermal effective action at inverse temperature $q\beta$, with small isometries U^q turned on. The effect of the $\mathbb{Z}_q$ quotient is to multiply the contribution of any integral of a local density by $1/q$. This again leads to (2.30).¹¹

The work [AM21] uses similar ideas to characterize superconformal indices of 4d CFTs near roots of unity. Our novel contribution is to apply these ideas in not-necessarily-supersymmetric, not-necessarily-conformal theories, on general spatial geometries, and also to describe the effects of Kaluza-Klein vortices (see below), which do not appear in superconformal indices.

¹¹When U and R don’t commute, the same logic works but we have $M_{q\beta, (UR)^q} / \mathbb{Z}_q$ on the right-hand side of (2.31). We can still use thermal EFT, since $(UR)^q$ is $O(\beta)$ close to the identity.

Non-free actions and Kaluza-Klein vortex defects

What happens if the action of R is not free? For example, in a 3d QFT on $S^2 \times S^1_\beta$, the action of $(-1)^J$ (where J is the Cartan generator of the rotation group) has fixed points at the north and south poles of S^2 . In this case, the EFT bundle degenerates at the fixed loci of nontrivial elements of $\mathbb{Z}_q$, namely $R, \dots, R^{q-1} \in \mathbb{Z}_q$. After dimensional reduction, these degeneration loci becomes defects $\mathfrak{D}_i$ (with i labelling the set of defects) in the $d - 1$ dimensional thermal effective theory. We call them “Kaluza-Klein vortex defects” because the KK gauge field A has nontrivial holonomy around them, as we explain in section 2.3.

Each defect $\mathfrak{D}_i$ contributes to the partition function a coordinate-invariant effective action $S_{\mathfrak{D}_i}$ of the background fields g, A, ϕ *in the infinitesimal neighborhood of $\mathfrak{D}_i$* . We then have the more general result

$$-\log \text{Tr} [gR] \sim -\frac{1}{q} \log \text{Tr} [g^q] + \text{topological} + \sum_{\mathfrak{D}_i} S_{\mathfrak{D}_i}, \quad (2.32)$$

We conjecture that for generic interacting QFTs, the KK vortex defects will be gapped. (In fact, in this work, we will study several examples of free theories where the appropriate defects are still gapped.) In this case, each $S_{\mathfrak{D}_i}$ will be a local functional of g, A, ϕ .

In CFTs, the defect actions $S_{\mathfrak{D}_i}$ are additionally constrained by Weyl-invariance, just like the bulk terms in the thermal effective action. We will determine the explicit form of $S_{\mathfrak{D}} := \sum_{\mathfrak{D}_i} S_{\mathfrak{D}_i}$ in CFTs later in section 2.3. For now, we simply note that the leading term in the derivative expansion of $S_{\mathfrak{D}_i}$ in a CFT is a cosmological constant localized on $\mathfrak{D}_i$:

$$S_{\mathfrak{D}_i} = a_{\mathfrak{D}_i} \int_{\mathfrak{D}_i} \frac{d^{n_i} y}{(q\beta)^{n_i}} \sqrt{\widehat{g}|_{\mathfrak{D}_i}} + \text{higher derivatives}. \quad (2.33)$$

Here, we assume that $\mathfrak{D}_i$ is n_i -dimensional, y are coordinates on the defect, and $\widehat{g}|_{\mathfrak{D}_i}$ denotes the pullback of $\widehat{g} = e^{-2\phi} g$ to $\mathfrak{D}_i$. This term behaves like β^{-n_i} as $\beta \rightarrow 0$. In

the case $n_i = 0$, i.e. when $\mathfrak{D}_i$ is point-like (for example the north/south poles of S^2), the “cosmological constant” becomes simply a constant.

Example: CFT in general d

As an example application, consider a d -dimensional CFT on $S^{d-1} \times S^1_\beta$. Although our discussion so far has been somewhat abstract, and we have used only basic geometry and principles of EFT, our conclusion (2.32) makes powerful predictions about CFT spectra. For example, to leading order as $\beta \rightarrow 0$, the defect term $S_{\mathfrak{D}}$ does not contribute, so very generally we obtain a higher dimensional generalization of (2.23),

$$-\log \text{Tr}[e^{-\beta(H - i\vec{\Omega} \cdot \vec{J})} R] \sim \frac{1}{q^d} \frac{\text{vol} S^{d-1}}{\prod_{i=1}^n (1 + \Omega_i^2)} \frac{f}{\beta^{d-1}} + \dots, \quad (2.34)$$

valid for any element R of the Cartan subgroup of $\text{SO}(d)$ with order q . For example, the relative density of even- and odd-spin operators (with respect to any Cartan generator) grows exponentially at a rate precisely $1/2^d$ times the rate for the unweighted density of states.

Unlike in $d = 2$, the thermal effective action in $d > 2$ can have more than just a cosmological constant term. Consequently, the “...” in (2.34) includes higher-derivative corrections (in addition to possible vortex defect contributions). However, these higher-derivative corrections can be predicted in the same way: they differ from the un-spin-refined case by replacing $\beta \rightarrow q\beta$ and multiplying by $1/q$ to account for the smaller spatial manifold.

The results (2.34) and (2.32) display an important difference between partition functions weighted by spacetime symmetries and partition functions weighted by global symmetries. If we replace R with a global symmetry element, this corresponds to turning on new background gauge fields in the thermal effective action, whose contributions are captured by Wilson coefficients that are not active when the global symmetry generators are turned off. For example, the density of states weighted by a

global symmetry generator U is controlled by a U -dependent free energy density f_U with no (obvious) relation to f when $U \neq 1$ (see e.g. [HO22; KLO23]). By contrast, the density of states weighted by different discrete *spacetime* symmetries are all controlled by the same f (and the same higher Wilson coefficients like $c_1, c_2, \dots$), in a predictable way.

Finally, let us describe the possible discrete rotations R for which (2.34) applies. Let us write $R = e^{i\vec{\theta} \cdot \vec{J}}$. In order for R to have finite order q , we must have $\vec{\theta} = 2\pi(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$, where the p_i/q_i are rational numbers (which we assume are in reduced form, so that p_i and q_i are relatively prime). The order of R is $q = \text{lcm}(q_1, \dots, q_n)$.

When d is even, the action of $\mathbb{Z}_q$ is free if all $q_i = q$. In this case, the quotient $S^{d-1}/\mathbb{Z}_q$ is a lens space $L(q; p_1, \dots, p_n)$, and there are no vortex defects $\mathfrak{D}$. If instead there exists at least one $q_i \neq q$, then the group element R^{q_i} will have a fixed locus S^{2k-1} , where k is the number of q_j 's such that $q_j | q_i$, and there will be a corresponding defect $\mathfrak{D}$ at this location (or rather its image after quotienting by $\mathbb{Z}_q$). Note that it is possible for fixed loci to intersect, creating higher codimension defects. For example, if $\vec{\theta} = 2\pi(1, \frac{1}{2}, \frac{1}{3})$, the element R^2 has a fixed S^3 , the element R^3 has its own fixed S^3 , and the two S^3 's intersect along an S^1 . Quotienting by $\mathbb{Z}_6$, we obtain a defect localized on $S^3/\mathbb{Z}_3 \subset S^5/\mathbb{Z}_6$, a defect localized on $S^3/\mathbb{Z}_2 \subset S^5/\mathbb{Z}_6$, and they intersect along an $S^1 \in S^5/\mathbb{Z}_6$. In this case, the thermal effective action will include terms localized on the defects and their intersection. When d is odd, any element of $\text{SO}(d)$ necessarily has a nontrivial fixed locus, since there is a direction left invariant by the Cartan generators.

If the theory has a reflection symmetry, then we can more generally consider $R \in O(d)$. The above arguments continue to hold, essentially unmodified. When R includes a reflection, the base of the EFT bundle $S^{d-1}/\mathbb{Z}_q$ will be non-orientable. For example, if we take R to be the parity operator $R : \vec{n} \rightarrow -\vec{n}$, then $S^{d-1}/\mathbb{Z}_2 = \mathbb{RP}^{d-1}$. Note

that the parity operator acts freely, so in this case we can apply (2.30).

2.3 Kaluza-Klein vortex defects

In this section, we explore the form of the defect action $S_{\mathfrak{D}}$ that contributes whenever the group generated by the discrete rotation R does not act freely. For simplicity, we will restrict our attention to CFT's in d -dimensions on a spatial sphere S^{d-1} .

Background fields and EFT gauge

As before, we wish to compute the partition function of a CFT on the geometry $M_{q\beta, U^q}/\mathbb{Z}_q$, where the mapping torus in the numerator is $M_{q\beta, U^q} = (S^{d-1} \times \mathbb{R})/\langle h^q \rangle$, the group in the denominator is $\mathbb{Z}_q = \langle h \rangle/\langle h^q \rangle$, and the action of h is given by

$$h : (\vec{n}, \tau) \mapsto (UR\vec{n}, \tau + \beta) = (e^{i(\frac{2\pi p_a}{q_a} + \beta\Omega_a)J^a} \vec{n}, \tau + \beta). \quad (2.35)$$

First, let us be more precise about the form of the background fields in this geometry. Following [Ben+24], we use radius-angle coordinates on the sphere S^{d-1} . These are given by a pair of radius and angle $\{r_a, \theta_a\}$ for each orthogonal 2-plane ($a = 1, \dots, n = \lfloor \frac{d}{2} \rfloor$). If d is odd, we have an additional radial coordinate r_{n+1} . Together, the radii satisfy the constraint $\sum_{a=1}^{n+\epsilon} r_a^2 = 1$, where $\epsilon = 0$ in even d and $\epsilon = 1$ in odd d .

To write the metric on $M_{q\beta, U^q}$ in Kaluza-Klein form, we switch to co-rotating coordinates

$$\varphi_a \equiv \theta_a - \Omega_a \tau, \quad (2.36)$$

where τ is the coordinate on $\mathbb{R}$. In co-rotating coordinates, the action of h simplifies to

$$h : (r_a, \varphi_a, \tau) \mapsto (r_a, \varphi_a + \frac{2\pi p_a}{q_a}, \tau + \beta). \quad (2.37)$$

In particular h^q , becomes simply a shift $h^q : \tau \mapsto \tau + q\beta$. Thus, quotienting by $\langle h^q \rangle$ to obtain $M_{q\beta, U^q}$ makes τ periodic with period $q\beta$.

The metric of $M_{q\beta, U^q}$ in co-rotating coordinates takes the Kaluza-Klein form

$$ds^2 = g + e^{2\phi}(d\tau + A)^2, \quad (2.38)$$

where the fields g, A, ϕ are given by [Ben+24]

$$e^{2\phi} = 1 + \sum_{a=1}^n r_a^2 \Omega_a^2, \quad (2.39)$$

$$A = \sum_{a=1}^n \frac{r_a^2 \Omega_a}{1 + \sum_b r_b^2 \Omega_b^2} d\varphi_a, \quad (2.40)$$

$$g = \sum_{a=1}^{n+\epsilon} dr_a^2 + \sum_{a,b=1}^n \left(r_a r_b \delta_{ab} - \frac{r_a^2 r_b^2 \Omega_a \Omega_b}{1 + \sum_{c=1}^n r_c^2 \Omega_c^2} \right) d\varphi_a d\varphi_b. \quad (2.41)$$

The metric of the EFT bundle $M_{q\beta, U^q}/\mathbb{Z}_q$ is locally the same as (2.38). Consequently, we can choose a local trivialization of the EFT bundle such that the fields g, A, ϕ are identical to (2.39) in each patch. However, such a local trivialization will have nontrivial transition functions between patches that contribute to holonomies of the Kaluza-Klein connection along various cycles (including around the defect locus).

If we like, we can perform a gauge transformation that makes the transition functions trivial, at the cost of introducing new contributions to A . We refer to such a gauge as “EFT gauge” because it will be convenient for discussing the EFT limit of the CFT on this geometry. In EFT gauge, the curvature $F = dA$ has δ -function type singularities at the fixed-loci of R whose coefficients reflect the topology of the EFT bundle.

Example: 2d CFT

Let us illustrate these ideas with an example. Consider a $2d$ CFT, where the action of h is given by $h : (\varphi, \tau) \mapsto (\varphi + \frac{2\pi p}{q}, \tau + \beta)$. The metric on $M_{q\beta, U^q}$ is

$$\begin{aligned} ds^2 &= d\tau^2 + d\theta^2 = d\tau^2 + (d\varphi + \Omega d\tau)^2 \\ &= \underbrace{\left(1 - \frac{\Omega^2}{1+\Omega^2}\right)d\varphi^2}_g + \underbrace{(1 + \Omega^2)}_{e^{2\phi}} \underbrace{\left(d\tau + \frac{\Omega}{1+\Omega^2}d\varphi\right)^2}_A. \end{aligned} \quad (2.42)$$

To choose a local trivialization of the EFT bundle, we first specify two intervals in the φ coordinate:

$$I_1 = \{\varphi : 0 < \varphi < \frac{2\pi}{q}\}, \quad I_2 = \{\varphi : -\epsilon < \varphi < \epsilon\}, \quad (2.43)$$

with $0 < \epsilon < \frac{\pi}{q}$. We denote their images in $S^1/\mathbb{Z}_q$ by U_1 and U_2 , respectively. Together U_1 and U_2 cover the quotient space $S^1/\mathbb{Z}_q$, see figure 2.3.

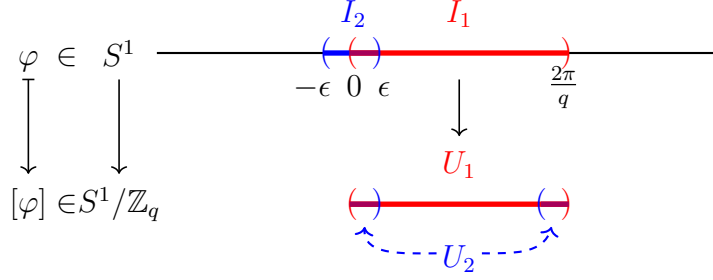


Figure 2.3: The open intervals I_1 and I_2 are subsets of S^1 . Their images under the quotient map $S^1 \rightarrow S^1/\mathbb{Z}_q$ are U_1 and U_2 , respectively, which together cover $S^1/\mathbb{Z}_q$. We can choose a gauge where the KK fields g, A, ϕ are given by (2.42) in each of U_1 and U_2 . However, in this gauge, there will be a nontrivial transition function between U_1 and U_2 .

The bundle projection $\pi : M_{q\beta, U^q}/\langle h \rangle \rightarrow S^1/\mathbb{Z}_q$ acts by $\pi : [(\varphi, \tau)] \mapsto [\varphi]$, where $[(\varphi, \tau)]$ denotes an equivalence class modulo the action of h , and $[\varphi]$ denotes an equivalence class modulo $\frac{2\pi p}{q}$. Over each open set U_1, U_2 , we must define trivialization maps

$$\phi_{U_i} : \pi^{-1}(U_i) \rightarrow U_i \times S^1_{q\beta}. \quad (2.44)$$

We choose them as follows. Given $p \in \pi^{-1}(U_i)$, thought of as an equivalence class modulo $\langle h \rangle$, let (φ, τ) be a representative of the equivalence class such that φ is contained in I_i . Then we define

$$\phi_{U_i}(p) = ([\varphi], \tau). \quad (2.45)$$

Note that τ is well-defined modulo $q\beta$ because the only elements of $\langle h \rangle$ that map the I_i to themselves are powers of h^q . With this local trivialization, the fields g, A, ϕ are given by (2.42) in each patch. In particular, we have $A = \frac{\Omega}{1+\Omega^2} d\varphi$ in both patches.

However, the data of the Kaluza-Klein connection includes both the value of A in each patch, as well as the transition functions between patches. We must also determine these transition functions.

There are two overlap regions to consider. The first is $(0, \epsilon)$. In this region, the transition function is trivial. The second overlap region is the image of $(-\epsilon, 0) \subset I_2$ in $S^1/\mathbb{Z}_q$, which coincides with the image of $(\frac{2\pi}{q} - \epsilon, \frac{2\pi}{q}) \subset I_1$ in $S^1/\mathbb{Z}_q$. Note that (φ, τ) for $\varphi \in (-\epsilon, 0)$ is equivalent modulo $\langle h \rangle$ to $(\varphi + \frac{2\pi}{q}, \tau + (p^{-1})_q \beta)$, where $\varphi + \frac{2\pi}{q} \in (\frac{2\pi}{q} - \epsilon, \frac{2\pi}{q})$. Here, $(p^{-1})_q$ denotes the inverse of $p \bmod q$, i.e. it satisfies $p(p^{-1})_q = qn + 1$ for some integer n . Thus, the transition function in this second overlap region is

$$\phi_{U_2} \circ \phi_{U_1}^{-1} : ([\varphi], \tau) \mapsto ([\varphi], \tau - (p^{-1})_q \beta). \quad (2.46)$$

The holonomy of the connection gets a nontrivial contribution from the transition functions:¹²

$$-\oint A = -\frac{2\pi}{q} \frac{\Omega}{1 + \Omega^2} - (p^{-1})_q \beta. \quad (2.47)$$

(The holonomy is an example of the “topological” terms discussed in section 2.6 which can contribute in the thermal effective action, but are not the integral of a local gauge/coordinate invariant density.)

To go to EFT gauge, we perform a gauge transformation (i.e. a φ -dependent redefinition of τ) that trivializes the transition functions. One possible choice is

$$\tau' = \tau - (p^{-1})_q \beta \left\lfloor \frac{\varphi}{2\pi/q} \right\rfloor. \quad (2.48)$$

Note that the function $\left\lfloor \frac{\varphi}{2\pi/q} \right\rfloor$ is multi-valued on the entire circle S^1 , but there is no problem defining it inside the intervals I_1, I_2 where we perform the gauge transformation. In terms of τ' , the transition functions are now trivial in both overlap regions.

¹²The parallel transport equation is $d\tau + A = 0$, so the holonomy of τ is computed by $-\oint A$.

(A quick way to see why is to note that τ' is invariant under the h -action (2.37).)

The gauge field becomes

$$A' = A + d\tau - d\tau' = \frac{\Omega}{1 + \Omega^2} d\varphi + (p^{-1})_q \beta \delta(\varphi) d\varphi \quad (\text{EFT gauge}). \quad (2.49)$$

The holonomy $-\oint A'$ is still given by (2.47), but that is now manifest in the local expression for the gauge field (2.49).

Example: 3d CFT

Now consider the same setup in a 3d CFT. The metric on S^2 is

$$ds_{S^2}^2 = dr_1^2 + dr_2^2 + r_1^2 d\varphi^2 \quad (r_1^2 + r_2^2 = 1). \quad (2.50)$$

Essentially all of the above discussion goes through un-modified, with the radii r_1, r_2 coming along for the ride. We can again go to EFT gauge, and the gauge field (2.49) now gets interpreted as a gauge field on S^2 . This time, A has δ -function-localized curvature at the north and south poles:

$$dA = \mp (p^{-1})_q \beta \delta(\vec{n}, \vec{n}_\pm) d^2 \vec{n} + \text{nonsingular}, \quad (2.51)$$

where $\delta(\vec{n}, \vec{n}')$ represents a δ -function on S^2 , and $\vec{n}_\pm$ are the north/south poles.

EFT gauge in general

More generally, we go to EFT gauge as follows. First choose a fundamental domain F for the quotient map $S^{d-1} \rightarrow S^{d-1}/\mathbb{Z}_q$. Define an integer valued function $k(\vec{n})$ by

$$\begin{aligned} k(\vec{n}) &= 0 \text{ if } \vec{n} \in F, \\ k(R\vec{n}) &= 1 + k(\vec{n}). \end{aligned} \quad (2.52)$$

In words, $k(\vec{n})$ counts the power of R needed to move from somewhere in F to $\vec{n}$. Again, $k(\vec{n})$ is multi-valued if we try to define it on the entire sphere, but we only

need to define it inside a collection of open sets that cover $S^{d-1}/\mathbb{Z}_q$. For example, in the 2d case considered above, we had $k(\varphi) = (p^{-1})_q \lfloor \frac{\varphi}{2\pi/q} \rfloor$.

Finally, we define

$$\tau' \equiv \tau - \beta k(\vec{n}(r_a, \varphi_a)). \quad (2.53)$$

In the coordinates (r_a, φ_a, τ') , h acts simply by shifting angles φ_a :

$$h : (r_a, \varphi_a, \tau') \mapsto (r_a, \varphi_a + \frac{2\pi p_a}{q_a}, \tau'). \quad (2.54)$$

Consequently, a local trivialization of the EFT bundle defined using the τ' coordinate has trivial transition functions. The gauge field is given by

$$A' = A + d\tau - d\tau' = \sum_{a=1}^n \frac{r_a^2 \Omega_a}{1 + \sum_b r_b^2 \Omega_b^2} d\varphi_a + \beta dk(\vec{n}(r_a, \varphi_a)). \quad (2.55)$$

The curvature dA' has δ -function contributions $\beta d^2 k(\vec{n})$ at the fixed loci of powers of R .

Effective action

Following the logic of the thermal effective action, let us now equip the EFT bundle with a more general metric G and try to write down a local action of G . We will demand that G satisfy the following conditions:

- It possesses a circle isometry, so that it can be written in Kaluza-Klein form (2.38).
- In EFT gauge, the curvature dA is a sum of δ -function singularities of the form $\beta d^2 k(\vec{n})$, plus something smooth on $S^{d-1}/\mathbb{Z}_q$. (This ensures that the Kaluza-Klein bundle has the same topology as $M_{q\beta, V^q}/\mathbb{Z}_q$.)
- g and $e^{2\phi}$ should be smooth on $S^{d-1}/\mathbb{Z}_q$.

Here, a field is “smooth on $S^{d-1}/\mathbb{Z}_q$ ” if it lifts to a smooth $\mathbb{Z}_q$ -invariant field on S^{d-1} .

In the limit $\beta \rightarrow 0$, we can separate each of the background fields into a long-wavelength part, with wavelengths much longer than β , and a short-wavelength part, with wavelengths comparable to (or smaller than) β . The long-wavelength parts become background fields for the thermal EFT. Meanwhile, the short-wavelength parts become operator insertions in that EFT.

In our case, the δ -function curvature singularities $dA \sim \beta d^2 k(\vec{n})$ are short-wavelength. They determine the insertion of an operator in the thermal EFT, which is described by the defect action $S_{\mathfrak{D}}$. This action is a functional of the long-wavelength parts of g, A, ϕ . As mentioned in section 2.2, we will assume that the defect is gapped, so that the action functional is local and can be organized in a derivative expansion. To construct it, we should compute curvatures and other invariants of g, A, ϕ , and *throw away δ -function singularities*. Since g and ϕ are smooth, this effectively amounts to the replacement

$$dA \rightarrow d\tilde{A} \equiv dA - \beta d^2 k(\vec{n}). \quad (2.56)$$

Henceforth, we leave this replacement implicit. In other words, when we write dA in the defect action, we mean its long-wavelength part $d\tilde{A}$, with δ -functions thrown away.

The defects live at singularities in the quotient space $S^{d-1}/\mathbb{Z}_q$. How should we write an action for long wavelength fields near these singularities? Recall that the long-wavelength parts of g, A, ϕ lift to smooth $\mathbb{Z}_q$ -invariant fields on S^{d-1} . We will write $S_{\mathfrak{D}}$ as a functional of these $\mathbb{Z}_q$ -invariant lifts, integrated over the preimage of the defect locus modulo $\mathbb{Z}_q$, which we denote by $\tilde{\mathfrak{D}}$. We also conventionally divide by q , which ensures that Wilson coefficients of defects living at singularities with the same local structure (but possibly different global structure) are the same.

The action should be invariant under gauge/coordinate transformations that preserve the defect locus. For now, we ignore the possibility of nontrivial Weyl anomalies on the defect $\mathfrak{D}$, and we impose that $S_{\mathfrak{D}}$ be Weyl-invariant as well. Consequently, it will be a functional of A and the Weyl-invariant combination $\widehat{g} = e^{-2\phi}g$.

Consider an n -dimensional defect $\mathfrak{D}$ whose preimage $\widetilde{\mathfrak{D}}$ is the fixed locus of an element $R^l \in \langle R \rangle$ with order m . Given a point p on $\widetilde{\mathfrak{D}}$, we can choose a vielbein $\widehat{e}_i^a$ at p satisfying $\delta_{ab}\widehat{e}_i^a\widehat{e}_j^b = \widehat{g}_{ij}$, where a, b are indices for the local rotation group $\text{SO}(d-1)$. The group $\langle R^l \rangle \cong \mathbb{Z}_m$ acts as a subgroup of the local rotation group $\text{SO}(d-1)$, so the $\widehat{e}_i^a$ can be classified into representations of this $\mathbb{Z}_m$. Singlets under $\mathbb{Z}_m$ represent directions parallel to the defect. They are acted upon by an $\text{SO}(n) \subset \text{SO}(d-1)$ that commutes with $\mathbb{Z}_m$. Hence, altogether the $\widehat{e}_i^a$ can be classified into representations of $\mathbb{Z}_m \times \text{SO}(n)$.

To build the defect action, we enumerate curvature tensors built from $\widehat{g}$ and A , in a derivative expansion, and contract them with $\widehat{e}_i^a$ to build $\mathbb{Z}_m \times \text{SO}(n)$ invariants I_i with d_i derivatives. The defect action is then

$$S_{\mathfrak{D}} = \frac{1}{q} \int_{\widetilde{\mathfrak{D}}} d^n y \sqrt{\widehat{g}|_{\widetilde{\mathfrak{D}}}} \left(\sum_i a_i (q\beta)^{d_i-n} I_i \right), \quad (2.57)$$

where $\widehat{g}|_{\widetilde{\mathfrak{D}}}$ denotes the pullback of $\widehat{g}$ to $\widetilde{\mathfrak{D}}$, and y are coordinates on the defect. The factors of $q\beta$ are supplied using dimensional analysis.

Finally, to evaluate the defect action on $M_{q\beta, V^q}/\mathbb{Z}_q$, we simply plug in the expressions (2.39), (2.40), (2.41), which are precisely the $\mathbb{Z}_q$ -lifts to S^{d-1} of the long-wavelength parts of g, A, ϕ .

In what follows, we will sometimes use the notation $\mathfrak{D}$ to refer to *both* a defect on $S^{d-1}/\mathbb{Z}_q$ and the lift $\widetilde{\mathfrak{D}}$ of the defect locus to S^{d-1} . We hope this will not cause confusion.

Example: point-like vortex defects in 3d CFTs

As an example, consider a 3d CFT, where R acts by the discrete rotation $\varphi \rightarrow \varphi + \frac{2\pi p}{q}$. The action of R fixes the north and south poles of S^2 . Consequently, there are two point-like vortex defects: $\mathfrak{D}_{p/q}$ located at the north pole, and its orientation reversal $\mathfrak{D}_{-p/q}$ located at the south pole. Let us focus on $\mathfrak{D}_{p/q}$.

Classifying the vielbein at the north pole into representations of $\mathbb{Z}_q$, we have basis elements $\widehat{e}_+^i, \widehat{e}_-^i$ with charges $+p$ and $-p$, respectively. We normalize them so that $\widehat{e}_\pm \cdot (\widehat{e}_\pm)^* = 1$. To build basic $\mathbb{Z}_q$ -invariant curvatures, we begin with tensors $\widehat{\nabla}_i \cdots \widehat{\nabla}_j \widehat{R}$ and $\widehat{\nabla}_i \cdots \widehat{\nabla}_j F_{kl}$, where $F = dA$, $\widehat{R}$ is the curvature scalar built from $\widehat{g}$, and $\widehat{\nabla}$ denotes a covariant derivative with respect to $\widehat{g}$. We then contract their indices with $\widehat{e}_\pm^i$ in such a way that the total $\mathbb{Z}_q$ charge vanishes. The action at each order in a derivative expansion is a polynomial in these basic $\mathbb{Z}_q$ -invariants.

Note that we cannot build valid terms in the action by multiplying two $\mathbb{Z}_q$ -charged objects to obtain a $\mathbb{Z}_q$ -singlet. For example, $(\widehat{e}_+^i \widehat{\nabla}_i \widehat{R})(\widehat{e}_-^j \widehat{\nabla}_j \widehat{R})$ is not admissible. The reason is that $\widehat{e}_+^i \widehat{\nabla}_i \widehat{R}$ individually vanishes, due to $\mathbb{Z}_q$ -invariance.

Proceeding in this way, the leading invariants in a derivative expansion are

$$S_{\mathfrak{D}_{p/q}} \ni 1, \widehat{\star}F, \widehat{R}, (\widehat{\star}F)^2, \dots, \quad (2.58)$$

where $\widehat{\star}F = i\widehat{e}_+^k \widehat{e}_-^l F_{kl}$ is the Hodge star of F in the metric $\widehat{g}$. Concretely, the action is

$$S_{\mathfrak{D}_{p/q}} = \frac{1}{q} \left(a_{0,p/q} + (q\beta) a_{1,p/q} \widehat{\star}F + (q\beta)^2 \left(a_{2,p/q} \widehat{R} + a_{3,p/q} (\widehat{\star}F)^2 \right) + \dots \Big|_{\mathfrak{D}_{p/q}} \right), \quad (2.59)$$

where $(\dots)|_{\mathfrak{D}_{p/q}}$ denotes evaluation at the preimage of the defect on S^2 — in this case the north pole. We have written the Wilson coefficients as $a_{i,p/q}$ to emphasize that they depend on the rotation fraction p/q . In bosonic theories, the $a_{i,x}$ are periodic in x with period 1, while in fermionic theories, they are periodic in x with period 2.

At higher orders in derivatives, we can also include laplacians $\widehat{\nabla}^2$, as well as q -th powers of charged derivatives $(\widehat{e}_\pm \cdot \nabla)^q$. However, note that the background fields on $M_{q\beta, U^q}$ given in (2.39), (2.40), and (2.41) are invariant not only under $\mathbb{Z}_q$, but under the full maximal torus $\text{SO}(2)$. Consequently, terms involving charged derivatives $(\widehat{e}_\pm \cdot \nabla)^q$ will actually vanish on $M_{q\beta, U^q}/\mathbb{Z}_q$, and in practice we only need to keep polynomials in $\widehat{\nabla}^{2k}\widehat{\star}F$ and $\widehat{\nabla}^{2k}\widehat{R}$.

Plugging in the fields on $M_{q\beta, U^q}$, we find

$$\begin{aligned}\widehat{\star}F|_\pm &= \pm 2\Omega, \\ \widehat{R}|_\pm &= 2 + 10\Omega^2,\end{aligned}\tag{2.60}$$

where $(\cdots)|_\pm$ denotes the north/south poles of S^2 . Thus, summing up the contributions from the north and south poles, the total defect contribution to $\text{Tr}[e^{-\beta H}UR]$ is

$$\begin{aligned}S_{\mathfrak{D}} &= \frac{a_{0,p/q} + a_{0,-p/q}}{q} + 2\beta\Omega(a_{1,p/q} - a_{1,-p/q}) \\ &\quad + \frac{(q\beta)^2}{q}((2 + 10\Omega^2)(a_{2,p/q} + a_{2,-p/q}) + 4\Omega^2(a_{3,p/q} + a_{3,-p/q})) + \dots\end{aligned}\tag{2.61}$$

In general, the point-like defect action at each order β^k is a polynomial in Ω that is even if k is even and odd if k is odd. We will verify this structure in several examples below.

There is an important distinction between the terms (2.61) arising in the defect action $S_{\mathfrak{D}}$ and the “bulk” terms (2.11). Note that the bulk terms contain poles at $\Omega_a = \pm i$. Physically, such poles arise because a great circle $r_a = 1$ of the spinning S^{d-1} approaches the speed of light as $\Omega_a \rightarrow \pm i$. The measure $\sqrt{\widehat{g}}$ becomes singular at the great circle, and the integral over r_a cannot be deformed away from the singularity because it is at an endpoint of the integration contour. By contrast, the defects $\mathfrak{D}_{\pm p/q}$ are located at the north and south poles of S^2 , where this phenomenon does not occur, and thus their contributions do not have poles at $\Omega = \pm i$. In general, the action of

a defect $\mathfrak{D}$ on $M_{q\beta, V^q}/\mathbb{Z}_q$ can have poles at $\Omega_a = \pm i$ if and only if the support of $\mathfrak{D}$ intersects the great circle $r_a = 1$. We will see an example in the next subsection.

Example: vortex defects in 4d CFTs

Consider now a 4d CFT, where R acts by discrete rotations on each of the angles $\varphi_1 \rightarrow \varphi_1 + \frac{2\pi p_1}{q_1}$ and $\varphi_2 \rightarrow \varphi_2 + \frac{2\pi p_2}{q_2}$. If $q_1 \neq q_2$, we have two 1-dimensional vortex defects $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$. The first defect $\mathfrak{D}^{(1)}$ is located at the fixed locus of R^{q_1} , which is given by $(r_1, r_2) = (1, 0)$ with $\varphi_1 \in [0, \frac{2\pi}{q})$, where $q := \text{lcm}(q_1, q_2)$. The second defect $\mathfrak{D}^{(2)}$ is located at the fixed locus of R^{q_2} , which is given by $(r_1, r_2) = (0, 1)$ with $\varphi_2 \in [0, \frac{2\pi}{q})$.

Let us focus on $\mathfrak{D}^{(1)}$ for now. On $\mathfrak{D}^{(1)}$, the leading term in the effective action is a cosmological constant $\int d\phi_1 \sqrt{\hat{g}|_{\mathfrak{D}^{(1)}}}$, as usual. At the first subleading order in a derivative expansion, we have the term $\int d\varphi_1 \sqrt{\hat{g}|_{\mathfrak{D}^{(1)}}} i\hat{e}_+^k \hat{e}_-^l F_{kl}$, which can be written more simply as $\int \hat{\star} F$.

The Wilson coefficients of a defect depend only on the geometry of the singularity where the defect lives. To describe this geometry, it is helpful to introduce the co-prime integers

$$\begin{aligned} P_1 &:= \frac{p_1 q_2}{(q_1, q_2)}, & Q_1 &:= \frac{q_1}{(q_1, q_2)}, \\ P_2 &:= \frac{p_2 q_1}{(q_1, q_2)}, & Q_2 &:= \frac{q_2}{(q_1, q_2)}. \end{aligned} \tag{2.62}$$

where (q_1, q_2) is the greatest common divisor of q_1, q_2 . The structure of the singularity at $\mathfrak{D}^{(1)}$ is determined by the action of R^{q_1} , which is

$$R^{q_1} : \varphi_2 \mapsto \varphi_2 + \frac{2\pi P_2}{Q_2}. \tag{2.63}$$

Thus, the Wilson coefficients of $\mathfrak{D}^{(1)}$ should depend only on P_2/Q_2 . However, there is a subtlety in fermionic theories: Note that R^{q_1} implements a rotation by $2\pi p_1$, which is $(-1)^{p_1 F}$ in fermionic theories. Thus, the Wilson coefficients of $\mathfrak{D}^{(1)}$ can additionally

depend on $(-1)^{p_1}$ in that case. Consequently, we will write the Wilson coefficients of $\mathfrak{D}^{(1)}$ as $a_{i,P_2/Q_2,(-1)^{p_1}}$ to emphasize the data they depend on. (We will see subtleties of a similar flavor in section 2.4.)

Putting everything together, the action $S_{\mathfrak{D}^{(1)}}$ takes the form

$$\begin{aligned} S_{\mathfrak{D}^{(1)}} &= \frac{1}{q} \left(\frac{a_{0,P_2/Q_2,(-1)^{p_1}}}{q\beta} \int d\varphi_1 \sqrt{\widehat{g}|_{\mathfrak{D}}} + a_{1,P_2/Q_2,(-1)^{p_1}} \int \widehat{\star} F + \dots \right) \\ &= \frac{2\pi}{q(1 + \Omega_1^2)} \left(\frac{a_{0,P_2/Q_2,(-1)^{p_1}}}{q\beta} + 2a_{1,P_2/Q_2,(-1)^{p_1}} \Omega_2 + \dots \right), \end{aligned} \quad (2.64)$$

where in the second line, we evaluated the action in the background $M_{q\beta,U^q}/\mathbb{Z}_q$. Note that because $\mathfrak{D}^{(1)}$ lives on the great circle $r_1 = 1$, its action has poles at $\Omega_1 = \pm i$.

Adding similar terms for $\mathfrak{D}^{(2)}$, the total defect contribution to $\text{Tr}[e^{-\beta H} U R]$ is

$$\begin{aligned} S_{\mathfrak{D}} &= \frac{2\pi}{q^2\beta} \left(\frac{a_{0,P_2/Q_2,(-1)^{p_1}}}{1 + \Omega_1^2} + \frac{a_{0,P_1/Q_1,(-1)^{p_2}}}{1 + \Omega_2^2} \right) \\ &\quad + \frac{4\pi}{q} \left(a_{1,P_2/Q_2,(-1)^{p_1}} \frac{\Omega_2}{1 + \Omega_1^2} + a_{1,P_1/Q_1,(-1)^{p_2}} \frac{\Omega_1}{1 + \Omega_2^2} \right) + \dots, \end{aligned} \quad (2.65)$$

where “...” represents higher-order terms in β coming from higher dimension operators in the defect action.

2.4 Fermionic theories

In this section, we describe some subtleties associated with partition functions of fermionic theories. Again, for simplicity we mostly restrict our discussion to CFT_d on a spatial sphere S^{d-1} , though the final conclusion (2.84) holds in a general QFT. In short, the results (2.30) and (2.32) work in fermionic CFTs as well, but we must take care to keep track of the spin structure of the manifold (in particular whether we have periodic or antiperiodic boundary conditions for fermions around S_β^1 and $S_{q\beta}^1$), and we must consider the rotation R as an element of $\text{Spin}(d)$.

Review of 2d

Let us first review fermionic CFTs in $2d$. In $2d$, we need to specify the boundary conditions of the fermions around both the space and time circles. This defines four

different fermion partition functions:

$$\begin{aligned}
Z_{R,+}(\tau, \bar{\tau}) &:= \text{Tr}_R \left(e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right) \\
Z_{R,-}(\tau, \bar{\tau}) &:= \text{Tr}_R \left((-1)^F e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right) \\
Z_{NS,+}(\tau, \bar{\tau}) &:= \text{Tr}_{NS} \left(e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right) \\
Z_{NS,-}(\tau, \bar{\tau}) &:= \text{Tr}_{NS} \left((-1)^F e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right). \tag{2.66}
\end{aligned}$$

The partition functions in (2.66) are not independent. The partition functions $Z_{R,+}$, $Z_{NS,+}$, and $Z_{NS,-}$ are invariant under different subgroups of $SL(2, \mathbb{Z})$ and can transform into each other. More precisely, $Z_{R,-}$ is invariant under all of $SL(2, \mathbb{Z})$; and $Z_{R,+}$, $Z_{NS,+}$, and $Z_{NS,-}$ are invariant under the congruence subgroups $\Gamma_0(2)$, Γ_θ , and $\Gamma^0(2)$ respectively, which are defined as:

$$\begin{aligned}
\Gamma_0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad c \text{ even} \right\} \\
\Gamma_\theta &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad a+b \text{ odd}, c+d \text{ odd} \right\} \\
\Gamma^0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad b \text{ even} \right\}. \tag{2.67}
\end{aligned}$$

Finally they transformation into each other as:

$$\begin{aligned}
Z_{R,+}(-1/\tau, -1/\bar{\tau}) &= Z_{NS,-}(\tau, \bar{\tau}), \\
Z_{NS,+}(\tau + 1, \bar{\tau} + 1) &= Z_{NS,-}(\tau, \bar{\tau}). \tag{2.68}
\end{aligned}$$

The NS sector partition function (with or without a $(-1)^F$ insertion) at low temperature is well-approximated by the vacuum state (which is a bosonic state), with Casimir energy $-\frac{c}{12}$:

$$\text{Tr}_{NS}(e^{-\beta(\Delta - \frac{c}{12})}) \sim e^{\frac{\beta c}{12}}, \quad \beta \gg 1. \tag{2.69}$$

The Ramond sector ground state, in contrast, has a Casimir energy of $E_{gs} - \frac{c}{12}$ where E_{gs} is the Ramond ground-state energy, a non-negative number that is theory-dependent.¹³ Finally, the Ramond ground-state may not necessarily be unique, so we call the degeneracy $N_{gs} \in \mathbb{N}$.

$$\text{Tr}_R(e^{-\beta(\Delta - \frac{c}{12})}) \sim N_{gs} e^{-\beta(E_{gs} - \frac{c}{12})}, \quad \beta \gg 1. \quad (2.70)$$

To study the high temperature behavior of the NS-sector partition function with an arbitrary phase $e^{2\pi i p/q J}$ inserted (with $0 \leq p/q < 2$ and p, q coprime)

$$\text{Tr}_{NS}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}), \quad \beta \ll 1 \quad (2.71)$$

we can use an $SL(2, \mathbb{Z})$ transform. In particular we would like to apply a modular transformation of the form

$$\begin{pmatrix} a & b \\ q & -p \end{pmatrix} \quad (2.72)$$

to (2.71). The result crucially depends on the parity of p, q . If $p + q$ is odd, then we can choose (2.72) to be in Γ_θ and map the partition function to the NS sector at low temperature. However, if $p + q$ is even, we map the partition function to the R sector at low temperature instead. We therefore get, for $\beta \ll 1$ and $0 \leq p/q < 2$:

$$\begin{aligned} \text{Tr}_{NS}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)} \frac{c}{12}} & p + q \text{ odd, } \beta \ll 1, \\ \text{Tr}_{NS}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim N_{gs} e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)} (\frac{c}{12} - E_{gs})} & p + q \text{ even, } \beta \ll 1. \end{aligned} \quad (2.73)$$

Equivalently we can always take $0 \leq \frac{p}{q} < 1$ with the insertion of a $(-1)^F$:

$$\begin{aligned} \text{Tr}_{NS}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)} \frac{c}{12}} & p + q \text{ odd, } \beta \ll 1, \\ \text{Tr}_{NS}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim N_{gs} e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)} (\frac{c}{12} - E_{gs})} & p + q \text{ even, } \beta \ll 1, \\ \text{Tr}_{NS}((-1)^F e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)} \frac{c}{12}} & p \text{ odd, } \beta \ll 1, \end{aligned}$$

¹³For supersymmetric theories, $E_{gs} = \frac{c}{12}$, but for generic fermionic theories, E_{gs} can be above or below or equal to $\frac{c}{12}$.

$$\mathrm{Tr}_{\mathrm{NS}}((-1)^F e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) \sim N_{gs} e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)}(\frac{c}{12} - E_{gs})} \quad p \text{ even, } \beta \ll 1. \quad (2.74)$$

We see that in (2.74), there are two real numbers that can determine the behavior of fermionic partition functions: the central charge c and the Ramond ground state energy E_{gs} . Moreover, which of the two high temperature behaviors we get ($\frac{c}{12}$ or $\frac{c}{12} - E_{gs}$ multiplying temperature in the free energy) depends on the parity of p and q . We will see this exact same behavior repeat itself for fermionic theories in higher dimensions in section 2.4.

In 2d, we can also analyze the behavior of the partition function in the Ramond sector. This does not have a direct analog as far as we are aware in higher dimension, but we include it for completeness. By using the same modular transformation properties discussed earlier, we find:

$$\begin{aligned} \mathrm{Tr}_{\mathrm{R}}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)}\frac{c}{12}} & q \text{ even, } \beta \ll 1, \\ \mathrm{Tr}_{\mathrm{R}}(e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\sim N_{gs} e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)}(\frac{c}{12} - E_{gs})} & q \text{ odd, } \beta \ll 1, \\ \mathrm{Tr}_{\mathrm{R}}((-1)^F e^{-\beta[(\Delta - \frac{c}{12}) - i\Omega J]} e^{2\pi i J p/q}) &\lesssim N_{gs} e^{\frac{4\pi^2}{q^2\beta(1+\Omega^2)}(\frac{c}{12} - E_{gs})} & \beta \ll 1. \end{aligned} \quad (2.75)$$

Because the final spin structure (where the fermion is periodic in both space and time directions) is invariant under the full modular group, it has the same universal behavior at high temperature regardless of the phase. In general this spin structure cannot be directly derived from the other three (although there are potentially powerful constraints coming from unitarity, and knowledge of $Z_{\mathrm{R},+}$ [BL20]). We write $\lesssim$ rather than $\sim$ in the last line of (2.75) due to possible cancellations between fermionic and bosonic Ramond ground states, which would effectively reduce N_{gs} (by an even number) for the final spin structure.

Higher d

Let us now consider fermionic CFTs in $d > 2$. For simplicity let us first consider turning on only a single spin and consider:

$$\text{Tr}[e^{-\beta(H-i\Omega J)} e^{2\pi i \frac{p}{q} J}], \quad (2.76)$$

with $\beta \ll 1$, $\Omega \sim O(1)$, $0 \leq p/q < 2$. As discussed before, this can be computed from a path integral of $S^{d-1} \times S^1_\beta$, where we insert a defect that rotates the sphere by $2\pi p/q$. Moreover, due to the fermions in the theory, we need to specify a spin structure on this geometry. In (2.76) we compute the path integral with anti-periodic boundary conditions for the fermion around the S^1_β . We now imagine the setup as in figure 2.2 to reduce the setup into a geometry we can obtain a thermal EFT. In particular we need to stack q copies of the CFT, and perform a $2\pi p$ rotation in the spatial direction. The final geometry we get – in addition to the spatial sphere being modded by $\mathbb{Z}_q$ and the thermal circle increasing by a factor of q – has different periodicity for the fermions about the time circle depending on the parity of $p + q$. If $p + q$ is odd, the fermions remain antiperiodic, and we can use the original EFT description for fermionic CFTs on a long, thin cylinder. If $p + q$ is even, however, we need a *new* EFT, for fermionic CFTs with periodic boundary conditions on a long, thin cylinder.

We now see there are *two* different thermal EFTs we consider, depending on the periodicity of the fermions around S^1_β . Each EFT comes with its own set of Wilson coefficients. We write them as:

$$\begin{aligned} Z_{\text{CFT}}[S^{d-1} \times S^1_{\beta, \text{ anti-periodic}}] &= Z_{\text{gapped}}[S^{d-1}] \sim e^{-S_{\text{th}}[g_{ij}, A_i, \phi]}, \\ Z_{\text{CFT}}[S^{d-1} \times S^1_{\beta, \text{ periodic}}] &= \tilde{Z}_{\text{gapped}}[S^{d-1}] \sim e^{-\tilde{S}_{\text{th}}[g_{ij}, A_i, \phi]}. \end{aligned} \quad (2.77)$$

The two expressions $e^{-S_{\text{th}}}$ and $e^{-\tilde{S}_{\text{th}}}$ respectively compute the partition function with and without the insertion of $(-1)^F$:

$$\text{Tr} [e^{-\beta(H-i\Omega J)}] \sim e^{-S_{\text{th}}},$$

$$\mathrm{Tr} \left[(-1)^F e^{-\beta(H-i\Omega J)} \right] \sim e^{-\tilde{S}_{\mathrm{th}}}. \quad (2.78)$$

Let us write the two thermal actions as:

$$\begin{aligned} S_{\mathrm{th}} &= \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}} \sqrt{\widehat{g}} \left(-f + c_1 \beta^2 \widehat{R} + c_2 \beta^2 F^2 + \dots \right) + S_{\mathrm{anom}}, \\ \tilde{S}_{\mathrm{th}} &= \int \frac{d^{d-1}\vec{x}}{\beta^{d-1}} \sqrt{\widehat{g}} \left(-\tilde{f} + \tilde{c}_1 \beta^2 \widehat{R} + \tilde{c}_2 \beta^2 F^2 + \dots \right) + S_{\mathrm{anom}}. \end{aligned} \quad (2.79)$$

The most general leading term behavior of (2.76) then goes as

$$\log \left[\mathrm{Tr} \left(e^{-\beta(H-i\Omega J)} e^{2\pi i \frac{p}{q} J} \right) \right] \sim \begin{cases} \frac{1}{q^d} \frac{f \, \mathrm{vol} \, S^{d-1}}{\beta^{d-1}(1+\Omega^2)} + \dots, & p+q \text{ odd}, \\ \frac{1}{q^d} \frac{\tilde{f} \, \mathrm{vol} \, S^{d-1}}{\beta^{d-1}(1+\Omega^2)} + \dots, & p+q \text{ even}. \end{cases} \quad (2.80)$$

Note that if we specialize (2.80) to $d = 2$, this is precisely the behavior we get for (2.73), if we identify:

$$\begin{aligned} f &= \frac{2\pi c}{12}, \\ \tilde{f} &= 2\pi \left(\frac{c}{12} - E_{gs} \right). \end{aligned} \quad (2.81)$$

We can think of the difference between the two sets of Wilson coefficients in (2.79) as the higher-dimensional analog of the ‘‘Ramond ground state energy’’.

We can generalize (2.76) to turning on n spins and consider

$$\mathrm{Tr} \left[e^{-\beta(H-i(\Omega_1 J_1 + \dots + \Omega_n J_n))} e^{2\pi i \frac{p_1}{q_1} J_1} \dots e^{2\pi i \frac{p_n}{q_n} J_n} \right] \quad (2.82)$$

for $\beta \ll 1$, $\Omega_i \sim O(1)$, $0 \leq \frac{p_i}{q_i} < 2$. If we define

$$q := \mathrm{lcm}(q_1, \dots, q_n), \quad (2.83)$$

then in our EFT, we will go around the time circle q times and go around the spatial sphere $\sum_i \frac{p_i q}{q_i}$ times. Thus depending on if $q + \sum_i \frac{p_i q}{q_i}$ is odd or even, we get the thermal EFT described by S_{th} or $\tilde{S}_{\mathrm{th}}$.

Finally, all of these results are consistent with (2.32) if we simply interpret $e^{i\beta\Omega \cdot J} R$ as an element of $\mathrm{Spin}(d)$ and interpret Tr as imposing periodic boundary conditions for

both bosonic and fermionic variable. With this understanding, (2.32) is the general recipe. If we instead interpret Tr as imposing periodic boundary conditions for bosons and antiperiodic boundary conditions for fermions, (2.32) should be modified to

$$-\log \text{Tr} [gR] \sim -\frac{1}{q} \log \text{Tr} [(-1)^{(q-1)F} g^q] + \text{topological} + \sum_{\mathfrak{D}_i} S_{\mathfrak{D}_i}. \quad (2.84)$$

We will see explicit examples of this in section 2.5.

2.5 Free theories

We now present several examples involving free fields to show explicitly that (2.30) and (2.32) hold (along with their appropriate generalizations to fermionic theories). We begin in section 2.5 by checking a *massive* quantum field theory, namely a massive free boson in 2d. We then consider free CFTs in 3d and 4d. Our main tool for computing partition functions in free CFTs is the *plethystic exponential*, which we review in appendix 2.12 (see e.g. [FHH07; MP21; Hen+17]).

In section 2.5, we consider various examples involving free scalar fields and free fermions in 3 dimensions. In section 2.5, we explore few more examples in 4 dimension. We present additional 4d and 6d examples in appendix 2.13. Lastly, in section 2.6, we consider 2d CFTs with a local gravitational anomaly. Such theories can include Chern-Simons term in their thermal effective action, which are not integrals of local gauge/coordinate invariant densities. These furnish examples of the “topological” terms in (2.30) and (2.32).

Massive free boson in 2d

As a first check on our formalism (and to illustrate that the basic ideas do not require conformal symmetry), let us study the partition function of a free scalar with mass m in 2d. We begin by computing the partition function on a rectangular torus $S_L^1 \times S_\beta^1$:

$$\log Z = -\frac{1}{2} \text{Tr} \log(m^2 + \Delta) - S_{\text{ct}}$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_{(r,s) \in \mathbb{Z}^2} \log \left(m^2 + \left(\frac{2\pi r}{L} \right)^2 + \left(\frac{2\pi s}{\beta} \right)^2 \right) - S_{\text{ct}} \\
&= - \sum_{r \in \mathbb{Z}} \log 2 \sinh \left(\frac{\beta \sqrt{m^2 + \left(\frac{2\pi r}{L} \right)^2}}{2} \right) - S_{\text{ct}}, \tag{2.85}
\end{aligned}$$

where S_{ct} is a cosmological constant counterterm, and in the third line we performed the sum over r and threw away a constant using the fact that $\sum_{s \in \mathbb{Z}} 1 = 0$ in ζ -function regularization.

Because the summand is slowly-varying in r , we can take the thermodynamic limit $L \rightarrow \infty$ by replacing the sum over r with an integral over momentum $k = \frac{2\pi r}{L}$:

$$\log Z \sim -\frac{L}{\pi} \int_0^\infty dk \left(\log \left(1 - e^{-\beta \sqrt{m^2 + k^2}} \right) - \frac{\beta \sqrt{m^2 + k^2}}{2} \right) - S_{\text{ct}} \quad (L \rightarrow \infty). \tag{2.86}$$

The second term in the integrand is UV-divergent, but proportional to $L\beta$. Hence, it takes the form of a cosmological constant and can be removed with an appropriate choice of S_{ct} . We will choose S_{ct} to simply subtract this contribution, which is equivalent to setting the free energy density to zero in flat $\mathbb{R}^2$. We find

$$\log Z \sim \frac{L}{\beta} f(\beta m) \quad (L \rightarrow \infty), \tag{2.87}$$

where minus the effective free energy density is

$$f(y) = -\frac{y}{\pi} \int_0^\infty dx \log \left(1 - e^{-y \sqrt{1+x^2}} \right) \sim \begin{cases} \frac{\pi}{6} & y \ll 1, \\ \sqrt{\frac{y}{2\pi}} e^{-y} & y \gg 1. \end{cases} \tag{2.88}$$

Note that the limit of $f(y)$ as $y \rightarrow 0$ is consistent with $f = \frac{2\pi c}{12}$ with $c = 1$ for the massless free boson in 2d.

Before continuing to the twisted partition function, let us make two observations. Firstly, the partition function Z obeys a form of modular invariance. To formulate it, let us write $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$, where the lattice Λ is spanned by basis vectors $\vec{e}_1, \vec{e}_2$. We can

arrange the basis vectors into a matrix $E \in \mathbb{R}^{2 \times 2}$ whose columns are $\vec{e}_1$ and $\vec{e}_2$, and consider the partition function as a function of E . (Above, we studied the case where $\vec{e}_1 = (L, 0)$ and $\vec{e}_2 = (0, \beta)$, and thus $E = \begin{pmatrix} L & 0 \\ 0 & \beta \end{pmatrix}$.) Rotational invariance implies that $Z(E)$ is invariant under left-multiplication by an orthogonal group element $g \in \text{SO}(2)$. Modular invariance is the statement that $Z(E)$ is also invariant under an integer change of basis of the lattice Λ , which is equivalent to right-multiplication by $\gamma \in \text{SL}(2, \mathbb{Z})$:

$$Z(E) = Z(gE\gamma). \quad (2.89)$$

In other words, Z is a function on the moduli space $\text{SO}(2) \backslash \mathbb{R}^{2,2} / \text{SL}(2, \mathbb{Z})$.

The second key observation is that in the thermodynamic limit, Z is unchanged under a small shift of the basis vector that grows as $L \rightarrow \infty$: $\vec{e}_1 \rightarrow \vec{e}_1 + \alpha \vec{e}_2$. To see why, note that the corresponding dual basis shifts by $\hat{e}_2 \rightarrow \hat{e}_2 - \alpha \hat{e}_1$ (with $\hat{e}_1$ unchanged). Thus, the sum (2.85) changes by shifting $r \rightarrow r - s\alpha$:

$$\begin{aligned} \log Z \left(\begin{pmatrix} L & 0 \\ \alpha\beta & \beta \end{pmatrix} \right) &= -\frac{1}{2} \sum_{(r,s) \in \mathbb{Z}^2} \log \left(m^2 + \left(\frac{2\pi(r-s\alpha)}{L} \right)^2 + \left(\frac{2\pi s}{\beta} \right)^2 \right) - S_{\text{ct}} \\ &\sim \frac{L}{\beta} f(\beta m) \quad (L \rightarrow \infty). \end{aligned} \quad (2.90)$$

In the continuum limit, we can identify the momentum $k = \frac{2\pi(r-s\alpha)}{L}$ and rewrite the sum as an integral over k . The shift $r \rightarrow r - s\alpha$ becomes immaterial in this approximation.

Now let us finally consider the partition function with the insertion of a discrete rotation of the spatial circle S_L^1 by an angle $2\pi p/q$. This corresponds to the matrix

$$E' = \begin{pmatrix} L & \frac{pL}{q} \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \frac{L}{q} & 0 \\ (p^{-1})_q \beta & q\beta \end{pmatrix} \gamma', \quad \text{where} \quad \gamma' = \begin{pmatrix} q & p \\ -(p^{-1})_q & -b \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (2.91)$$

As before $(p^{-1})_q$ denotes the inverse of p modulo q , and b is chosen so that γ' has determinant 1. Applying modular invariance (2.89) and the result (2.90), we find

$$\log Z(E') = \log Z\left(\begin{smallmatrix} \frac{L}{(p^{-1})_q} & 0 \\ (p^{-1})_q \beta & q\beta \end{smallmatrix}\right) \sim \frac{L}{q^2 \beta} f(q\beta m) \quad (L \rightarrow \infty), \quad (2.92)$$

consistent with (2.32).

Using slightly fancier technology (see e.g. [Don+21]), we can be more precise about what information is thrown away in the “ $\sim$ ” in equation (2.92). We rewrite the partition function in terms of a spectral zeta function

$$\log Z(E) = \frac{1}{2} \zeta'_E(0) - S_{\text{ct}}, \quad (2.93)$$

where

$$\begin{aligned} \zeta_E(s) &= \text{Tr} [(m^2 + \Delta)^{-s}] = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \text{Tr} [e^{-t(m^2 + \Delta)}] \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-tm^2} \sum_{\vec{r} \in \mathbb{Z}^2} e^{-t(2\pi E^{-T} \vec{r})^2} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-tm^2} \frac{\det E}{4\pi t} \sum_{\lambda \in \Lambda} e^{-\frac{\lambda^2}{4t}}. \end{aligned} \quad (2.94)$$

In the last line, we performed Poisson resummation to obtain a sum over lattice points $\lambda \in \Lambda$. (Recall that Λ is the lattice spanned by the columns of E .) We can now perform the integral over t and plug the result into (2.93). The term $\lambda = (0, 0)$ precisely cancels against S_{ct} , and we find

$$\log Z(E) = \frac{m \det E}{2\pi} \sum_{\lambda \in \Lambda - (0,0)} \frac{K_1(m|\lambda|)}{|\lambda|}, \quad (2.95)$$

where $K_1(x)$ is a modified Bessel function.

Now we can see clearly that in the thermodynamic limit, the lattice vectors $\lambda \in \Lambda$ that become longer give an exponentially suppressed contribution to $\log Z$ (since $K_1(r) \sim e^{-r}$ for large r). Indeed, we have

$$\log Z(E) = \frac{m \det E}{2\pi} \sum_{\lambda \in \Lambda_{\text{short}} - (0,0)} \frac{K_1(m|\lambda|)}{|\lambda|} + O(e^{-Lm}) \quad (L \rightarrow \infty), \quad (2.96)$$

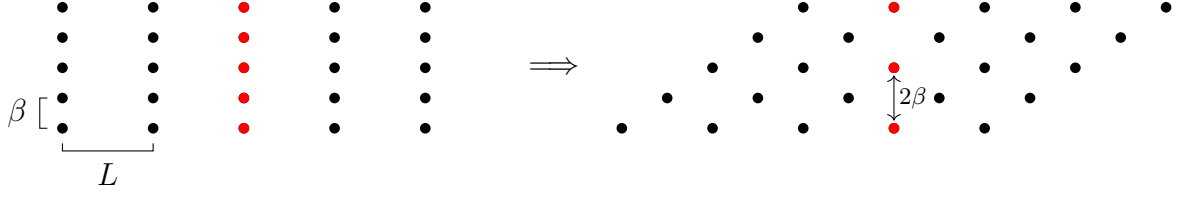


Figure 2.4: **Left:** the lattice Λ for a rectangular torus $S_L^1 \times S_\beta^1$. In the thermodynamic limit $L \rightarrow \infty$, the sum (2.95) is dominated by $\lambda \in \Lambda_{\text{short}}$ (depicted in red), which are spaced apart by β . **Right:** the lattice after twisting by a spatial rotation by π . In the limit $L \rightarrow \infty$, the partition function is dominated by $\lambda \in \Lambda'_{\text{short}}$, which are now spaced apart by 2β .

where Λ_{short} are the lattice vectors that do not get longer in the thermodynamic limit $L \rightarrow \infty$, see figure 2.4. Applying the result (2.96) to the rectangular torus $S_L^1 \times S_\beta^1$, we find another expression for the effective free energy density:

$$f(y) = \frac{y}{\pi} \sum_{\ell=1}^{\infty} \frac{1}{\ell} K_1(y\ell), \quad (2.97)$$

which agrees with (2.88), as we can see by expanding $\log(1 - e^{-y\sqrt{1+x^2}}) = -\sum_{\ell} \frac{1}{\ell} e^{-\ell y\sqrt{1+x^2}}$ and integrating term-by term. The result (2.96) also immediately implies (2.90). Combining this with modular invariance, we find that (2.92) holds up to exponential corrections of the form e^{-mL} . Note that this conclusion relies on finiteness of m in the thermodynamic limit. If instead we take $m \rightarrow 0$, then the thermal theory has a gapless sector, and we do not expect (2.32) to hold.

We can also understand (2.92) more directly from (2.96) as follows. When the partition function has a twist by a rational angle $\frac{2\pi p}{q}$, then a new Λ'_{short} emerges, as depicted in figure 2.4. The emergent Λ'_{short} looks like Λ_{short} in the un-twisted case, but with the replacement $\beta \rightarrow q\beta$.

It is also straightforward to generalize this analysis to a massive free scalar in d dimensions. The torus partition function is¹⁴

$$\log Z(E) = \det E \left(\frac{m}{2\pi} \right)^{d/2} \sum_{\lambda \in \Lambda} \frac{K_{d/2}(m|\lambda|)}{|\lambda|^{d/2}}. \quad (2.98)$$

¹⁴This is an example of an Epstein ζ -function, see e.g. [Dun08].

This is again consistent with (2.32) (with vanishing topological and defect contributions) through the mechanism depicted in figure 2.4.

3d CFTs

Free scalar

We now turn our attention to partition functions of higher dimensional CFTs on a spatial S^{d-1} . Let us begin by studying the partition function of a free scalar in 3d, with various discrete rotations inserted. The usual KK reduction of a free scalar on a circle possesses a zero mode. However, in order to apply thermal EFT, the KK reduced theory must be gapped. Thus, in order to avoid zero modes, we will study a *complex* scalar charged under a $\mathbb{Z}_k$ flavor symmetry. We will turn on the $\mathbb{Z}_k$ fugacity as appropriate to eliminate zero modes, and study the thermal EFT description of the resulting twisted partition functions. Note that in a generic interacting CFT (as opposed to a free theory) we would not expect to have zero modes, and it would not be necessary to consider flavor symmetry.

Let $R(\theta) \in \text{SO}(3)$ denote a rotation around the z -axis, and let Ψ denote a reflection in the z direction. We will consider insertions of $R(\theta, s) = R(\theta) \circ \Psi^s \in O(3)$, where $\theta = \frac{2\pi p}{q}$. Note that the reflection potential s takes values in $\{0, 1\}$, since $\Psi^2 = 1$. The partition function is given by

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H - i\Omega J)} e^{\frac{2\pi i \ell Q}{k}} R(\theta, s) \right] \\ &= \sum_n \frac{1}{n} \left(e^{\frac{2\pi i \ell n}{k}} + e^{-\frac{2\pi i \ell n}{k}} \right) \frac{e^{-n\beta/2} (1 - e^{-2n\beta})}{(1 - (-1)^{ns} e^{-n\beta}) (1 - e^{-n\beta} x^n) (1 - e^{-n\beta} x^{-n})}, \end{aligned} \quad (2.99)$$

where $x = e^{2\pi i p/q + i\beta\Omega}$, and Q is the charge of the scalar under $U(1)$ (of which $\mathbb{Z}_k$ is a subgroup).

When we turn on a big rotation $R(\theta, s)$ with order q and include the global symmetry operator $V = e^{\frac{2\pi i \ell Q}{k}}$, a zero mode will be present on the EFT bundle whenever $V^q = 1$,

since V wraps q times around the base of the EFT bundle — see the discussion around (2.30).

Let us consider the case $p/q = 1/2$, corresponding to a rotation by an angle π , which fixes the north and south poles of S^2 . If the flavor group were $\mathbb{Z}_2$, then we would necessarily have a zero mode on the EFT bundle. Hence we instead consider $\mathbb{Z}_3$ flavor symmetry.

The thermal partition function of a free scalar field with non-zero $\mathbb{Z}_3$ flavor and small rotation-fugacity has the following high temperature expansion

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{4\pi i Q}{3}} \right] \\ &= -\frac{16\zeta(3)}{9(1+\Omega^2)\beta^2} - \frac{(1+2\Omega^2)\log 3}{12(1+\Omega^2)} + \frac{(21+4\Omega^2-24\Omega^4)\beta^2}{4320(1+\Omega^2)} + O(\beta^4). \end{aligned} \quad (2.100)$$

Now we would like to turn on the fugacity for the big rotation with or without a reflection, leading to absence or presence of non trivial defect action, respectively.

• **Free action: without defect**

Let us first consider an insertion of $R(\pi, 1) = R(\pi) \circ \Psi$. This is a parity transformation $\vec{n} \mapsto -\vec{n}$ on S^2 , and it does not have any fixed points. Thus, the thermal effective action will be free of defects. Concretely, we find

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{2\pi i Q}{3}} R(\pi, 1) \right] \\ &= -\frac{2\zeta(3)}{9(1+\Omega^2)\beta^2} - \frac{(1+2\Omega^2)\log 3}{24(1+\Omega^2)} + \frac{(21+4\Omega^2-24\Omega^4)\beta^2}{2160(1+\Omega^2)} + O(\beta^4). \end{aligned} \quad (2.101)$$

Comparing (2.100) with (2.101) we find

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{2\pi i Q}{3}} R(\pi, 1) \right] \sim \frac{1}{2} \log \text{Tr} \left[e^{-2\beta(H-i\Omega J)} e^{\frac{4\pi i Q}{3}} \right], \quad (2.102)$$

so (2.30) holds.

• **Non-free action: defect**

Without the reflection fugacity, the action of R has fixed points and thermal EFT predicts a non-trivial $S_{\mathfrak{D}}$ as in (2.32). We find

$$\begin{aligned} & \log \operatorname{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{4\pi i Q}{3}} R(\pi, 0) \right] \\ &= -\frac{2\zeta(3)}{9(1+\Omega^2)\beta^2} - \frac{(7+8\Omega^2)\log 3}{24(1+\Omega^2)} + \frac{(-69+94\Omega^2+156\Omega^4)\beta^2}{2160(1+\Omega^2)} + O(\beta^4). \end{aligned} \quad (2.103)$$

Comparing with (2.100), we obtain

$$-\log \operatorname{Tr} \left[e^{-\beta(H-i\Omega J)} e^{\frac{2\pi i Q}{3}} R(\pi, 0) \right] \sim -\frac{1}{2} \log \operatorname{Tr} \left[e^{-2\beta(H-i\Omega J)} e^{\frac{4\pi i Q}{3}} \right] + S_{\mathfrak{D}}, \quad (2.104)$$

where the total defect action $S_{\mathfrak{D}}(\beta, \Omega)$ is given by

$$S_{\mathfrak{D}}(\beta, \Omega) = -\frac{\log 3}{4} + \frac{2\Omega^2 - 1}{24}\beta^2 + O(\beta^4). \quad (2.105)$$

Note that $S_{\mathfrak{D}}$ has precisely the form predicted in (2.61), with

$$a_{0,1/2} = -\frac{\log 3}{4}, \quad a_{2,1/2} = -\frac{1}{192}, \quad a_{3,1/2} = \frac{7}{384}. \quad (2.106)$$

Furthermore, the linear term in β vanishes because $a_{1,1/2} = a_{1,-1/2}$ in bosonic theories.

More generally, we can consider a rotation $R(\frac{2\pi p}{q}, 0)$, where p and q are coprime and q is not divisible by 3 (so that there is no zero mode upon dimensional reduction).

The leading defect Wilson coefficients in this case satisfy:

$$\begin{aligned} a_{0,p/q} + a_{0,-p/q} &= -\frac{1}{3} \sum_{k=1}^{q-1} \frac{1}{\sin(\frac{\pi kp}{3})^2} \left[\cos\left(\frac{2\pi(k+q)}{3}\right) \left(\psi\left(\frac{k}{3q} + \frac{1}{3}\right) - \psi\left(\frac{k}{3q}\right) \right) \right. \\ &\quad \left. + \cos\left(\frac{2\pi(k+2q)}{3}\right) \left(\psi\left(\frac{k}{3q} + \frac{2}{3}\right) - \psi\left(\frac{k}{3q}\right) \right) \right], \end{aligned} \quad (2.107)$$

$$a_{1,p/q} - a_{1,-p/q} = \frac{1}{6} \sum_{k=1}^{q-1} \frac{\cos(\frac{\pi kp}{3})}{\sin(\frac{\pi kp}{3})^3} \left[\cos\left(\frac{2\pi(k+q)}{3}\right) + 2 \cos\left(\frac{2\pi(k+2q)}{3}\right) \right], \quad (2.108)$$

where $\psi(z)$ is the digamma function.

Free fermion

We can perform a similar exercise with free Dirac fermions. The partition function is given by

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta H} e^{\frac{2\pi i \ell Q}{k}} R(\theta, 0) (-1)^{Fw} \right] \\ &= \sum_n (-1)^{nw} \frac{(-1)^{n+1}}{n} \left(e^{\frac{2\pi i \ell n}{k}} + e^{-\frac{2\pi i \ell n}{k}} \right) \frac{e^{-n\beta} \left[\left((\sqrt{x})^n + \frac{1}{(\sqrt{x})^n} \right) - e^{-n\beta} \left((\sqrt{x})^n + \frac{1}{(\sqrt{x})^n} \right) \right]}{(1 - e^{-n\beta})(1 - e^{-n\beta}(\sqrt{x})^{2n})(1 - e^{-n\beta}(\sqrt{x})^{-2n})}. \end{aligned} \quad (2.109)$$

Fermions are antiperiodic in the time direction at finite temperature. Hence, when no fugacities corresponding to a big rotation or fermion number are turned on, there is no zero mode contribution to the thermal EFT description of the partition function. In this case, we have the following high temperature behavior:

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} \right] = \frac{3\zeta(3)}{(1+\Omega^2)\beta^2} - \frac{(2+\Omega^2)\log 2}{6(1+\Omega^2)} + \frac{(24-4\Omega^2-21\Omega^4)\beta^2}{5760(1+\Omega^2)} + O(\beta^4). \quad (2.110)$$

However, when $(-1)^F$ and big rotations are turned on, we must be careful about zero modes. If the big rotation is given by $e^{2\pi i p/q}$, then we have various cases depending on the parity of p, q and presence or absence of $(-1)^F$. We consider all possible cases in the Table 2.1.

Let us consider the case when q is even. We take $q = 2$ for simplicity. According to table 2.1, we do not need a flavor twist to remove zero modes. We compute

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i\Omega J)} R(\pm\pi, 0) \right] \\ &= \frac{3\zeta(3)}{8(1+\Omega^2)\beta^2} - \frac{(2+\Omega^2)\log 2}{12(1+\Omega^2)} \mp \frac{\Omega\beta}{4} + \frac{(24-4\Omega^2-21\Omega^4)\beta^2}{2880(1+\Omega^2)} + O(\beta^3). \end{aligned} \quad (2.111)$$

Note that $\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} (-1)^F R(\pi, 0) \right] = \log \text{Tr} \left[e^{-\beta(H-i\Omega J)} R(-\pi, 0) \right]$.

Comparing (2.110) and (2.111), we verify the appropriate generalizations of (2.32) to fermionic CFT. In particular, see (2.84) and note that as an element of spin group,

$(-1)^F?$	Rotation p/q	Boundary Condition $(-1)^{q(w+1)}(-1)^p$	Flavor?	Equation
No ($w = 0$)	1/2	$(-1)^2(-1)^1 = -1$	No	(2.111), (2.112)
Yes ($w = 1$)	1/2	$(-1)^4(-1)^1 = -1$	No	(2.111), (2.112)
No ($w = 0$)	1/3	$(-1)^3(-1)^1 = +1$	$\mathbb{Z}_2$	(2.115), (2.116)
Yes ($w = 1$)	1/3	$(-1)^6(-1)^1 = -1$	No	(2.115), (2.116)
No ($w = 0$)	2/3	$(-1)^3(-1)^2 = -1$	No	(2.118), (2.119)
Yes ($w = 1$)	2/3	$(-1)^6(-1)^2 = +1$	$\mathbb{Z}_2$	(2.118), (2.119)

Table 2.1: Turning on fugacities corresponding to flavor, rotation and fermionic number. $w = 0, 1$ refers to turning off and on the fugacity corresponding to $(-1)^F$ respectively. The fugacity corresponding to big rotation is $e^{2\pi ip/q}$. The third column lists the effective spin structure on the EFT bundle, ± 1 in this column refers to periodic and antiperiodic boundary condition respectively. The fourth column lists whether we need a flavor twist to make sure that there is no zero mode. Note that antiperiodic boundary condition rules out the presence of zero modes. The final column refers to relevant equations in the main text.

we have $R^2(\pm\pi, 0) = (-1)^F$. We find

$$-\log \text{Tr} [e^{-\beta(H-i\Omega J)} R(\pm\pi, 0)] \sim -\frac{1}{2} \log \text{Tr} [e^{-2\beta(H-i\Omega J)}] + S_{\mathfrak{D}_{\pm}}, \quad (2.112)$$

where $S_{\mathfrak{D}_{\pm}}$ is given by

$$S_{\mathfrak{D}_{\pm}} = \mp \frac{\Omega\beta}{4} + O(\beta^3). \quad (2.113)$$

This has precisely the form predicted in (2.61), with

$$a_{1,1/2} - a_{1,-1/2} = \frac{1}{8}, \quad a_{0,1/2} + a_{0,-1/2} = 0. \quad (2.114)$$

Next we consider the case when q is odd. Now p can be either even or odd. For example, let us consider $q = 3$ and $p = 1$ or $p = 2$. We introduce a flavor twist according to Table 2.1. Note that, operationally the insertion of $(-1)^F$ is same as introducing a $\mathbb{Z}_2$ flavor twist. The insertion of both amounts to inserting nothing. In what follows, we choose to insert $(-1)^F$ to eliminate the zero mode.

For $p/q = 1/3$, we find that

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i\Omega J)} (-1)^F R(2\pi/3, 0) \right] \\ & \sim \frac{\zeta(3)}{9(1+\Omega^2)\beta^2} - \frac{(10+9\Omega^2)\log 2}{18(1+\Omega^2)} + \frac{5\Omega\beta}{3\sqrt{3}} + \frac{(-568+1028\Omega^2+1617\Omega^4)\beta^2}{5760(1+\Omega^2)} + O(\beta^3). \end{aligned} \quad (2.115)$$

Comparing (2.110) and (2.115) we find that

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} (-1)^F R(2\pi/3, 0) \right] \sim \frac{1}{3} \log \text{Tr} \left[e^{-3\beta(H-i\Omega J)} \right] + S_{\mathfrak{D}}(\beta, \Omega), \quad (2.116)$$

where the total defect action $S_{\mathfrak{D}}(\beta, \Omega)$ takes the form predicted in (2.61):

$$S_{\mathfrak{D}}(\beta, \Omega) \sim -\frac{4\log 2}{9} + \frac{5\Omega\beta}{3\sqrt{3}} - \frac{(8-21\Omega^2)\beta^2}{72} + O(\beta^3), \quad (2.117)$$

The $p/q = 2/3$ case can be easily be done using the following identity

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} R(4\pi/3, 0) \right] = \log \text{Tr} \left[e^{-\beta(H+i\Omega J)} (-1)^F R(2\pi/3, 0) \right], \quad (2.118)$$

from which it follows that

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega J)} R(4\pi/3, 0) \right] \sim \frac{1}{3} \log \text{Tr} \left[e^{-3\beta(H-i\Omega J)} \right] + S_{\mathfrak{D}}(\beta, -\Omega), \quad (2.119)$$

where $S_{\mathfrak{D}}(\beta, \Omega)$ is given by (2.117).

More examples with a defect action: 4d CFTs

In 4d, the rotation group has two Cartan generators, which we call J_1 and J_2 . Let us consider inserting $R = \exp(2\pi i \frac{p_1}{q_1} J_1 + 2\pi i \frac{p_2}{q_2} J_2)$ into the trace. When $q_1 = q_2$, the action of R is free and there will be no vortex defects. However, when $q_1 \neq q_2$, we will have two 1-dimensional vortex defects $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$, whose combined action is given by (2.65).

As a concrete example, consider the free Dirac fermion in 4d. Using plethystic exponentials, we find

$$-\log \text{Tr} \left[e^{-\beta(H-i\vec{\Omega} \cdot \vec{J})} e^{2\pi i \frac{p_1}{q_1} J_1 + 2\pi i \frac{p_2}{q_2} J_2} \right] \sim -\frac{1}{q} \log \text{Tr} \left[e^{-q\beta(H-i\vec{\Omega} \cdot \vec{J})} \right] + S_{\mathfrak{D}}. \quad (2.120)$$

Here we impose that $q \left(1 + \frac{p_1}{q_1} + \frac{p_2}{q_2}\right)$ is odd. This ensures absence of the $(-1)^F$ insertion in the right hand side above. Thus the zero mode is absent and the thermal EFT applies.

Furthermore, $S_{\mathfrak{D}}$ has precisely the form predicted by (2.65) with the leading Wilson coefficients given by the following function of $(-1)^p$ and coprime integers P, Q :

$$a_{0,P/Q,(-1)^p} = \sum_{k=1}^{Q-1} \frac{(-1)^{pk} \cos(\pi k \frac{P}{Q})}{8\pi \sin^2(\pi k \frac{P}{Q})} \left(\psi' \left(\frac{1}{2} + \frac{k}{2Q} \right) - \psi' \left(\frac{k}{2Q} \right) \right), \quad (2.121)$$

where $\psi'(z)$ is the derivative of the digamma function. This is a highly nontrivial check of (2.65). Matching (2.65) requires not just reproducing the correct function of $\beta, \Omega_1, \Omega_2$, but also the fact that the Wilson coefficient of $\mathfrak{D}^{(1)}$ depends only on P_2/Q_2 and $(-1)^{p_1}$ (and analogously for $\mathfrak{D}^{(2)}$).

Note that when $q_1 = q_2$, we have $Q_1 = Q_2 = 1$, hence the sum is non-existent, leading to $a_{0,P_1/Q_1,(-1)^{p_2}} = a_{0,P_2/Q_2,(-1)^{p_1}} = 0$, which is consistent with the fact that the action of R becomes free and $S_{\mathfrak{D}}$ should vanish.

2.6 Topological terms: example in 2d CFT

So far, we have focused on cases where the thermal effective action can be written as the integral of a local gauge- and coordinate-invariant density. As discussed in section 2.3, we can choose a gauge ("EFT gauge") where the background fields g, A, ϕ on the EFT bundle look locally the same as on $S^{d-1} \times S^1_{\beta}$ (possibly with "small" angular twists turned on). Curvature invariants built out of these fields are then also the same as on $S^{d-1} \times S^1_{\beta}$, and their integral is not sensitive to global properties of the EFT bundle.

By contrast, when d is even, the thermal effective action can also include Chern-Simons-type terms that are not integrals of local curvature invariants, and hence can be sensitive to global properties of the EFT bundle. The contributions of such terms were called "topological" in (2.30) and (2.32). The coefficients of such Chern-Simons

terms can be determined systematically from the anomaly polynomial of the CFT [LS12; JLY13; Log13; JLY14a; JLY14b; NS15]. The simplest case is when $d = 2$, where the thermal effective action contains a 1d Chern Simons term whose coefficient is proportional to the local gravitational anomaly $c_L - c_R$.

In more detail, consider such a 2d CFT with a local gravitational anomaly, $c_L \neq c_R$. We assume that $c_L - c_R = 24k$ with $k \in \mathbb{Z}$. From modular invariance we have a high-temperature expansion of the partition function as:¹⁵

$$\log \left(\text{Tr} \left[e^{-\beta(H-i\Omega J)} e^{2\pi i \frac{p}{q} J} \right] \right) \sim \frac{4\pi^2(c_L + c_R - 24ik\Omega)}{24q^2(1 + \Omega^2)\beta} - \frac{2\pi ik(p^{-1})_q}{q}, \quad (2.122)$$

where (2.122) is accurate to all orders in perturbation theory in β . This generalizes (2.19) to theories with $c_L \neq c_R$.

In the thermal effective action, we can reproduce the terms in (2.122) with a Chern-Simons term from the KK gauge field in the action. In particular, we add a term of the form

$$-\frac{2\pi ik}{q\beta} \oint A \quad (2.123)$$

to the thermal effective action. From (2.47), we see that (2.123) precisely reproduces the additional terms in (2.122). Note that (2.123) is properly quantized precisely when k is an integer, since A is a connection on a circle bundle where the fiber has circumference $q\beta$.

2.7 Holographic theories

In this section we consider CFTs dual to semiclassical Einstein gravity in AdS. The high temperature behavior of the thermal partition function, $\text{Tr}[e^{-\beta H + i\vec{\theta} \cdot \vec{J}}]$ around $\theta_i = 0$, of such holographic CFTs in d dimensions are captured by the thermodynamics of black hole solutions in AdS_{d+1} . We would like to understand the bulk solution that

¹⁵To derive (2.122), we apply the modular transform (2.18) to the vacuum state $e^{-\frac{2\pi i \tau c_L - 2\pi i \bar{\tau} c_R}{24}}$ with $\tau = \frac{i\beta}{2\pi} + \frac{\beta\Omega}{2\pi} + \frac{p}{q}$ and expand in small β .

captures the high temperature behavior of $\text{Tr}[e^{-\beta H + i\vec{\theta} \cdot \vec{J}}]$ near $\theta_i = 2\pi p_i/q_i$, where at least one of the $p_i \neq 0$ and $q_i \geq 2$.

First, let us consider a $d = 2$ dimensional CFT in the context of $\text{AdS}_3/\text{CFT}_2$ duality.

The relevant bulk solution is given by the rotating BTZ black hole with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi - \frac{r_+ r_-}{r^2} dt \right)^2, \quad f(r) := \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}, \quad (2.124)$$

where $r_{\pm}$ are radius of outer and inner horizon respectively. The BH temperature β^{-1} and angular potential $\Omega = \theta/\beta$ are given by

$$\Omega = \frac{r_-}{r_+}, \quad \beta^{-1} = \frac{r_+^2 - r_-^2}{2\pi r_+}. \quad (2.125)$$

Asymptotically, the metric is Weyl equivalent to $-dt^2 + d\phi^2$. In Euclidean signature, $t_E = it$ and we have $(\phi, t_E) \sim (\phi - \beta\Omega, t_E + \beta)$. The Euclidean action evaluated on this bulk saddle reproduces the high temperature behavior of $\text{Tr}[e^{-\beta(H - i\Omega J)}]$.

Now let us compute $\text{Tr}[e^{-\beta(H - i\Omega J) + 2\pi i p/q J}]$ for a holographic 2D CFT. As explained in section 2.2, this can be computed by doing a path integral over $M_{q\beta, \Omega}/\mathbb{Z}_q$,¹⁶ where the action of $\mathbb{Z}_q = \langle h \rangle$ is given by

$$h : (\phi, t_E) \mapsto (\phi + 2\pi p/q - \beta\Omega, t_E + \beta). \quad (2.126)$$

and $M_{q\beta, \Omega}$ is obtained from $S^1 \times \mathbb{R}$ by quotienting by $\langle h^q \rangle$.

Note that the action of $\mathbb{Z}_q$ has a natural extension into the AdS bulk, where the radial direction goes along for the ride

$$h : (r, \phi, t_E) \mapsto (r, \phi + 2\pi p/q - \beta\Omega, t_E + \beta). \quad (2.127)$$

We can use this natural extension to build a bulk dual solution. We start with a BTZ black hole $\Sigma_{q\beta, \Omega}$ with parameters $\tilde{r}_{\pm} = r_{\pm}/q$, such that the black hole is at inverse

¹⁶In this section, we use the compressed notation $M_{\beta, e^{iJ\Omega}} \rightarrow M_{\beta, \Omega}$.

temperature $\tilde{\beta} = q\beta$. The Euclidean metric is given by

$$ds^2 = \tilde{f}(r) dt_E^2 + \frac{dr^2}{\tilde{f}(r)} + r^2 \left(d\phi + i \frac{\tilde{r}_+ \tilde{r}_-}{r^2} dt_E \right)^2, \quad \tilde{f}(r) := \frac{(r^2 - \tilde{r}_+^2)(r^2 - \tilde{r}_-^2)}{r^2}, \quad (2.128)$$

Note that $(\phi, t_E) \sim (\phi - q\beta\Omega, t_E + q\beta)$. We can then quotient $\Sigma_{q\beta, \Omega}$ by the action of $\mathbb{Z}_q = \langle h \rangle$, given by (2.127). The manifold $\Sigma_{q\beta, \Omega}/\mathbb{Z}_q$ is smooth because the action of h in the bulk (2.127) is free.

Recall that on the boundary, the quotient $M_{q\beta, \Omega}/\mathbb{Z}_q$ is a nontrivial bundle over $S^1/\mathbb{Z}_q$. The discussion in 2.3 applies here, with the radial direction of AdS going along for the ride. In short, we consider open sets in the total space and consider the bulk region that asymptotes to this set. Locally in this bulk region, the metric is precisely given by (2.128). However, there are non-trivial transition functions when we go from one open patch to a different one.

By construction, the quotient $\Sigma_{q\beta, \Omega}/\mathbb{Z}_q$ solves Einstein's equations with the appropriate asymptotic geometry. Compared to a rotating BTZ black hole at temperature $q\beta$, the above quotient geometry has its angular variable restricted to $0 < \phi < 2\pi/q$. (Note that this almost covers the full manifold except the locus $\phi = 0$. This locus is covered by another open patch and we have nontrivial transition function between these two patches.) Thus the evaluation of the bulk action amounts to an integral over the angular variable in the range $0 < \phi < 2\pi/q$ producing a factor of $1/q$, and a replacement $\beta \rightarrow q\beta$ which produces another factor of $1/q$ compared to the evaluation of Euclidean action on BTZ black hole at temperature β . Thus, the Euclidean action evaluated on the saddle $\Sigma_{q\beta, \Omega}/\mathbb{Z}_q$ reproduces

$$\log \text{Tr}[e^{-\beta(H - i\Omega J) + 2\pi i p/q J}] \sim \log Z_{\text{grav}}[\Sigma_{q\beta, \Omega}/\mathbb{Z}_q] = \frac{1}{q} \log Z_{\text{grav}}[\Sigma_{q\beta, \Omega}] \sim \frac{1}{q} \log \text{Tr}[e^{-q\beta(H - i\Omega J)}], \quad (2.129)$$

where $Z_{\text{grav}}[\Sigma]$ is the gravitational path integral evaluated on the saddle Σ .

Overall, we have a quotient of a BTZ black hole whose boundary has a temporal cycle

of length $\beta_L = q\beta(1 - i\Omega)$ and a spatial cycle of length $L = \frac{2\pi}{q}$. This naively leads to the modular parameter

$$\tilde{\tau} \text{ “=” } iL/\beta_L = \frac{2\pi i}{q^2\beta(1 - i\Omega)}. \quad (2.130)$$

The above is almost correct, but we must amend it by recalling the presence of nontrivial transition functions. This leads to the following identification

$$\tilde{\tau} = \frac{2\pi i}{q^2\beta(1 - i\Omega)} - \frac{(p^{-1})_q}{q}, \quad \tilde{\tilde{\tau}} = -\frac{2\pi i}{q^2\beta(1 + i\Omega)} - \frac{(p^{-1})_q}{q}, \quad (2.131)$$

which precisely matches the modular parameter $\tilde{\tau}$ obtained after applying a modular transformation, as explained around (2.17). Note that we have

$$\frac{1}{2} \left(\tilde{\tau} + \tilde{\tilde{\tau}} \right) = -\frac{(p^{-1})_q}{q} - \frac{2\pi\Omega}{q^2\beta(1 + \Omega^2)}, \quad (2.132)$$

which is consistent with the holonomy of A derived in (2.47). We conclude that $\Sigma_{q\beta,\Omega}/\mathbb{Z}_q$ is simply a modular transformation of the BTZ black hole geometry.

In dimensions greater than two, we follow the same prescription. Let $\Sigma_{\beta,\vec{\Omega}}$ be a black hole solution that captures the high temperature behavior of the thermal partition function with small angular fugacity — i.e. the AdS-Kerr black hole with inverse temperature β and angular velocities $i\vec{\Omega}$. We claim that the insertion of a big angular fugacity R with $R^q = 1$ is captured by the bulk manifold $\Sigma_{q\beta,\vec{\Omega}}/\mathbb{Z}_q$, where $\mathbb{Z}_q$ is the natural extension of the boundary $\mathbb{Z}_q$ into the bulk. To be precise, the AdS-Kerr solution possesses a time-translation isometry, which we parametrize by t_E . It furthermore possesses isometries under the Cartan subgroup of the rotation group, which we parametrize by ϕ_a . Then the $\mathbb{Z}_q = \langle h \rangle / \langle h^q \rangle$ group is generated by

$$h : (r, r_a, \phi_a, t_E) \mapsto (r, r_a, \phi_a + \frac{2\pi p_a}{q_a}, t_E + \beta), \quad (2.133)$$

where r_a and r are the remaining bulk coordinates.

When the bulk action of h is free, $\Sigma_{q\beta,\vec{\Omega}}/\mathbb{Z}_q$ is a smooth solution to Einstein's equations with the correct asymptotic geometry to describe the partition function with

an insertion of R . At very high temperatures, the gravitational path integral on this geometry matches the field theory prediction (2.30) because the quotient by $\mathbb{Z}_q$ just divides the semiclassical gravitational action by q . Thus we expect $\Sigma_{q\beta, \tilde{\Omega}}/\mathbb{Z}_q$ is the dominant solution at high temperatures. As we move away from high temperatures, we conjecture that $\Sigma_{q\beta, \tilde{\Omega}}/\mathbb{Z}_q$ remains the dominant solution down to a finite temperature.

Unlike in 2d, in higher dimensions it is possible for the bulk $\mathbb{Z}_q$ action (2.133) to have a fixed locus. Such a fixed locus must occur at the horizon of the black hole (where the Euclidean time circle degenerates), and at a location on the sphere S^{d-1} where the boundary action of h has a fixed point as well. For example, consider a 3d CFT, with a 4-dimensional bulk dual, and consider the case $p_1/q_1 = 1/2$. The bulk action of h rotates the ϕ circle by π , and also shifts $t_E \rightarrow t_E + \beta$ (which is equivalent to rotating the thermal circle by π). Fixed points of h occur at the north and south poles of the horizon S^2 . For example, near the north pole, we can choose coordinates so that the metric locally takes the form

$$dr_1^2 + r_1^2 d\phi^2 + dy^2 + y^2 d\psi^2, \quad (2.134)$$

where $\psi = \frac{2\pi}{\beta} t_E$, and $y \propto \sqrt{r - r_H}$ (with r_H the location of the horizon). In these coordinates, the $\mathbb{Z}_2$ action rotates both angles ϕ, ψ by π . Quotienting by this $\mathbb{Z}_2$, we obtain an orbifold singularity of the form $\mathbb{C}^2/\mathbb{Z}_2$.

When $\Sigma_{q\beta, \tilde{\Omega}}/\mathbb{Z}_q$ contains an orbifold singularity, it no longer furnishes a smooth solution to Einstein's equations. To understand the correct bulk dual of the R -twisted partition function, we must understand the fate of the orbifold singularity in the bulk theory. The physics of the singularity (or its resolution) essentially determines the defect action $S_{\mathfrak{D}}$ in holographic CFTs. Note that in a spacetime with vanishing cosmological constant, a $\mathbb{C}^2/\mathbb{Z}_2$ singularity can be resolved by the "gravitational instanton" described by the Eguchi-Hansen metric. Perhaps orbifold singularities occurring in

the $\Sigma_{q\beta,\vec{\Omega}}/\mathbb{Z}_q$ can be resolved similarly. Or perhaps the orbifold singularity is resolved by stringy effects. We leave these questions to future work.

For fermionic theories dual to Einstein gravity (which all known examples are), there is another set of Wilson coefficients $\tilde{f}, \tilde{c}_1, \dots$ that can be defined (see (2.79)) coming from periodic boundary conditions for the fermion. The behavior of the partition function with $(-1)^F$ inserted was explored in [CT24], which found a black hole solution that lead to an exponentially subleading (in temperature) contribution in the large N limit. Since generically we expect $\tilde{f}$ to be nonvanishing, this means the black hole solution found in [CT24] should not be the dominant contribution in the $T \gg N \gg 1$ limit. It would be interesting to explore further if there are universal results or constraints on the Wilson coefficients $\tilde{f}, \tilde{c}_1, \dots$ in (fermionic) holographic CFTs (for instance by looking at the theories studied in [BHR23]).

Finally, let us note that our construction works also in the case when the boundary theory possesses a reflection symmetry and the group element R includes a reflection. In this case, the bulk dual solution $\Sigma_{q\beta,\vec{\Omega}}/\mathbb{Z}_q$ is non-orientable. This is allowed because the boundary reflection symmetry must be gauged in the bulk, which means we must include contributions from non-orientable geometries. It so happens that a non-orientable geometry dominates in this case.¹⁷

2.8 Journey to $\beta = 0$

So far we have described the structure of CFT partition functions $Z(\beta, \vec{\theta})$ at high temperature near rational angles $\frac{\vec{\theta}}{2\pi} \in \mathbb{Q}^n$. In this section we will use these results to sketch the behavior of the partition function at high temperature around any (possibly irrational) angle. For simplicity let us consider turning on only one angle θ , and study the partition function

$$Z(\beta, \theta) = \text{Tr} [e^{-\beta H + iJ\theta}] \quad (2.135)$$

¹⁷See [HN23] for a recent discussion of gauging spacetime symmetries in the bulk.

at small β with θ *fixed*. Suppose the angle has the following continued fraction expansion:

$$\frac{\theta}{2\pi} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \quad (2.136)$$

where $a_i \in \mathbb{N}$. We also define the fractions $\frac{p_i}{q_i}$ as truncations of the continued fraction, i.e.

$$\frac{p_i}{q_i} := a_0 + \frac{1}{a_1 + \dots + \frac{1}{a_i}}. \quad (2.137)$$

For a generic angle, the probability of $a_i = n$ scales as $\frac{1}{n^2}$ for large n , which means that the distribution of a_i 's for a generic angle has infinite mean.¹⁸

Suppose we have an angle θ where $a_i \gg 1$ for some i . How does the partition function behave at high temperature? Since we assume $a_i \gg 1$, we have $q_i \approx a_i q_{i-1} \gg q_{i-1}$. So for

$$q_{i-1} \ll T \ll q_i, \quad (2.138)$$

we expect

$$\log Z(T) \sim \text{vol} S^{d-1} f T^{d-1} q_{i-1}^{-d}. \quad (2.139)$$

However, when

$$T \sim q_i \quad (2.140)$$

the constant or proportionality in (2.139) suddenly shrinks.

In figure 2.5, we illustrate this explicitly for two example CFTs. We plot both the Klein invariant j function (which is the partition function of some 2d CFTs at central charge 24) and the 3d free boson partition function as a function of temperature, with chemical potential $\frac{\theta}{2\pi} = \frac{3-\pi}{7\pi-22}$. This has the continued fraction expansion:

$$\frac{\theta}{2\pi} = \frac{3-\pi}{7\pi-22} = 15 + \frac{1}{292 + \frac{1}{1+\dots}}. \quad (2.141)$$

¹⁸One way to see this scaling is, for a random real number between 0 and 1, the probability that $a_1 = n$ is roughly $\frac{1}{n} - \frac{1}{n+1} \sim \frac{1}{n^2}$.

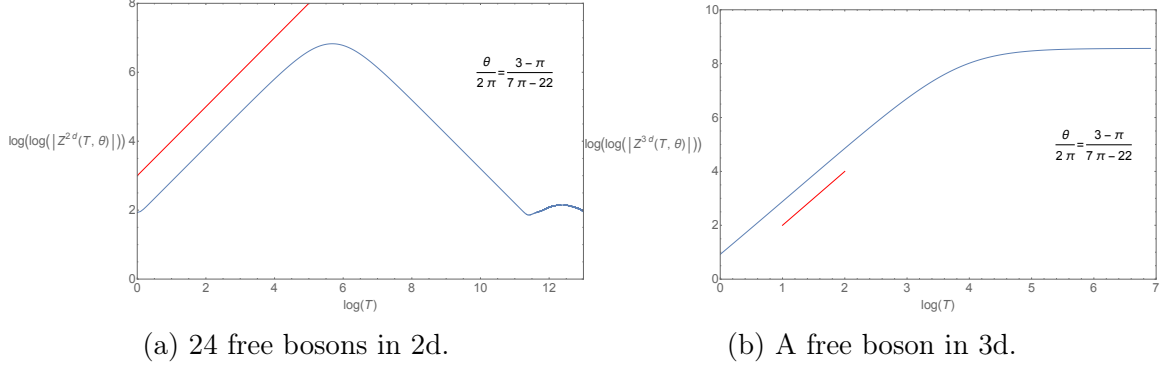


Figure 2.5: In blue, the log of the free energy of two CFTs evaluated at chemical potential $\frac{\theta}{2\pi} = \frac{3-\pi}{7\pi-22}$ vs. $\log T$. In red, a line with slope $d-1$. (a): $d=2$. (b): $d=3$. We see that at some temperature, there is a region where the slope matches $d-1$, meaning the effective field theory is a good description. If we continue this plot to higher and higher temperatures, there will be infinitely many times the slope matches $d-1$.

(Although any generic real number would serve to illustrate the partition function's behavior, we choose (2.141) for convenience because of the large 292 showing up immediately.) We see that there indeed is a region of T where the free energies scale as we predict from the effective field theory; but when we increase T further, the scaling breaks down. At large T for generic chemical potential, there will be an infinite number of times the plot in figure 2.5 has slope $d-1$, which is when the continued fraction approximation is well-approximated by a rational number (i.e. whenever $a_i \gg a_{i-1}$ in the continued fraction).

Finally we note that if we fine-tune the angle θ so that none of the a_i 's ever become large, we can make there be no regime where the partition function obviously grows as T^{d-1} . For example, choosing the angle to be the golden ratio

$$\frac{\theta}{2\pi} = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\dots} \quad (2.142)$$

gives an angle where there's never a large enough regime in T trust our effective field theory. In figure 2.6, we again plot the Klein invariant j function and the free boson partition functions as a function of temperature, but this time with the chemical potential set to $\frac{\theta}{2\pi} = \frac{1+\sqrt{5}}{2}$. We see that the slope of $\log \log |Z|$ against $\log T$ never

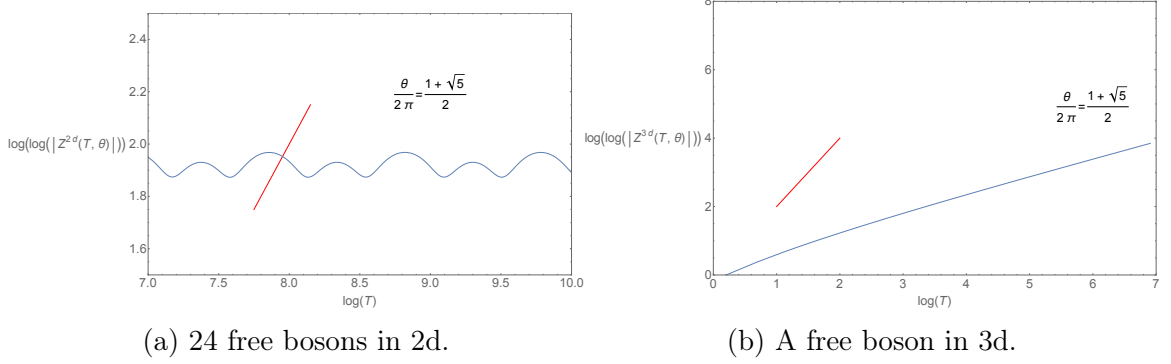


Figure 2.6: In blue, the log of the free energy of two CFTs evaluated at chemical potential $\frac{\theta}{2\pi} = \frac{1+\sqrt{5}}{2}$ vs. $\log T$. In red, a line with slope $d - 1$. (a): $d = 2$. (b): $d = 3$. We see that at no temperature is there a region where the slope matches $d - 1$, meaning the effective field theory is never a good description. Since we have fine-tuned the chemical potential to be a real number whose continued fraction has no large numbers in it, we do not guarantee any region in temperature where we have a good EFT description of our theory.

matches $d - 1$ in a large region, so there is no good EFT description of our system.

2.9 Nonperturbative corrections

In this section, we consider nonperturbative corrections to the thermodynamic limit $L \rightarrow \infty$ at finite β . For concreteness, we focus on a CFT_d and its dimensional reduction on S^1_β to a $d-1$ -dimensional gapped theory. By conformal symmetry, the thermodynamic limit is equivalent to $\beta \rightarrow 0$ (with a fixed-size spatial manifold). For simplicity, we will not turn on “small” angular twists $\beta\vec{\Omega}$, though it would be straightforward to incorporate them.

The thermal effective action essentially captures the dynamics of the ground state of the $d-1$ dimensional gapped theory, while nonperturbative corrections come from particle excitations. On the geometry $\mathbb{R}^{d-1} \times S^1_\beta$, the excitations can be classified into irreps of the $d-1$ -dimensional Poincaré group and the Kaluza-Klein $U(1)$ that rotates the S^1_β . Irreps of the Poincaré group are labeled by a mass and a little group representation — for simplicity, we will focus on scalars. Thus, each excitation of interest is labeled by a mass m_i and a KK charge $\mathbf{q}_i \in \mathbb{Z}$. The lightest mass $m_{\mathbf{q}}$ for

each KK charge $\mathbf{q}$ is sometimes called the “ $\mathbf{q}$ -th screening mass”, while the lightest nonzero mass overall is the “thermal mass” of the theory. Note that when $d = 2$, the spectrum of masses $(m_i, \mathbf{q}_i)$ are simply $(\frac{2\pi}{\beta} \Delta_i, \ell_i)$, where (Δ_i, ℓ_i) are scaling dimensions and spins of local operators. However, in higher dimensions, the masses m_i are not related in an obvious way to the local operator spectrum.

In the partition function $Z_{\text{CFT}}[\mathcal{M}_{d-1} \times S^1_\beta]$, the leading nonperturbative effects at small β are expected to come from “worldline instantons” associated with particles of mass m_i propagating along geodesics of $\mathcal{M}_{d-1}$, see e.g. [Don+21; GKT21; HO21; Hel21; CKW24]. Such contributions can be computed from the worldline path integral

$$\log Z \Big|_{\text{single-particle}} = \sum_{(m_i, \mathbf{q}_i)} \int D x^\mu(\tau) \exp \left(-m_i \int d\tau \sqrt{\dot{x}_\mu \dot{x}^\mu} + \frac{2\pi i \mathbf{q}_i}{\beta} \oint A \right). \quad (2.143)$$

Here, for each particle, we have included a length term $-m_i \int ds$ proportional to the mass, along with a coupling $\frac{2\pi i \mathbf{q}_i}{\beta} \oint A$ to the background KK gauge field. (Note that we include a factor of $\frac{2\pi}{\beta}$ because in our conventions, A is a connection on a circle bundle where the fiber has circumference β , instead of the usual 2π .)

In appendix 2.14, we compute the wordline path integral (2.143) on some geometries of interest. For example, by computing (2.143) on S^{d-1} , we find that the leading nonperturbative terms in the partition function $Z_{\text{CFT}}[S^{d-1} \times S^1_\beta]$ have the form

$$\log(\text{Tr}_{\mathcal{H}_{S^{d-1}}} [e^{-\beta H}]) \supset \sum_{m_i} e^{-2\pi m_i} \frac{(\pm i)^{d-2} m_i^{d-2}}{\Gamma(d-1)} \left(1 + O\left(\frac{1}{m_i}\right) \right). \quad (2.144)$$

Note that the effect of each particle is exponential in the mass $e^{-2\pi m_i}$, where 2π is the length of a great circle on S^{d-1} . By dimensional analysis, the masses are proportional to $1/\beta$, and hence these are indeed nonperturbative corrections in β . In addition to the exponential dependence, the worldline path integral makes an unambiguous prediction for the leading coefficient of the exponential, coming from a gaussian determinant. We immediately see that the interpretation of (2.144) is subtle because the coefficient

becomes *imaginary* when d is odd (even when the partition function must be real). We discuss this phenomenon and its interpretation in appendix 2.14.

Specializing to 4d, we can similarly compute leading nonperturbative corrections to a partition function on a lens space $L(q; 1)$, coming from a “short” geodesic of length $2\pi/q$:

$$\log(\text{Tr}_{\mathcal{H}_{L(q;1)}}[e^{-\beta H}]) \supset \begin{cases} \sum_{m_i} e^{-\frac{2\pi m_i}{q}} \frac{m_i}{2\sin(\frac{2\pi}{q})} \left(1 + O\left(\frac{1}{m_i}\right)\right) & (q \neq 2), \\ \sum_{m_i} e^{-\pi m_i} \frac{m_i^2}{2} \left(1 + O\left(\frac{1}{m_i}\right)\right) & (q = 2). \end{cases} \quad (2.145)$$

By our discussion of the EFT bundle, this result is closely related to the partition function on S^3 , with a twist by a rational angle of order q . To obtain the latter, we must replace $\beta \rightarrow q\beta$, and account for the presence of a nontrivial holonomy for the KK gauge field (since $S_{q\beta}^1$ is nontrivially fibered over $L(q; 1)$). We find

$$\log(\text{Tr}_{\mathcal{H}_{S^3}}[e^{-\beta H - \frac{2\pi i}{q} J_{12} - \frac{2\pi i}{q} J_{34}}]) \supset \begin{cases} \sum_{(m_i, q_i)} e^{-\frac{2\pi m_i}{q^2} + \frac{2\pi i q_i}{q}} \frac{m_i}{2q \sin(\frac{2\pi}{q})} \left(1 + O\left(\frac{1}{m_i}\right)\right) & (q \neq 2), \\ \sum_{(m_i, q_i)} e^{-\frac{\pi m_i}{2} + \pi i q_i} \frac{m_i^2}{8} \left(1 + O\left(\frac{1}{m_i}\right)\right) & (q = 2). \end{cases} \quad (2.146)$$

In particular, nonperturbative corrections to the twisted partition function go like $e^{-2\pi m_i/q^2}$, as opposed to $e^{-2\pi m_i}$ in the un-twisted case.

More generally, in any dimension d , when the quotient by $\mathbb{Z}_q$ creates a short geodesic of length $\ell_{\text{short}} = 2\pi/q$, the leading nonperturbative corrections will behave like $e^{-(m_i/q)\ell_{\text{short}}} = e^{-2\pi m_i/q^2}$, where m_i/q comes from the replacement $\beta \rightarrow q\beta$. Note that this matches the result from modular invariance in 2d. In 2d, after applying a modular transformation, the twisted partition function becomes $\text{Tr} \left[e^{2\pi i \tilde{\tau} (L_0 - \frac{c}{24}) - 2\pi i \tilde{\tau} (\bar{L}_0 - \frac{c}{24})} \right]$, where $\tilde{\tau} = \text{const} + \frac{2\pi i}{q^2 \beta}$. Thus, the leading β -dependence of the contribution of excited states to the twisted partition function is $e^{-4\pi^2 \Delta_i / (q^2 \beta)} = e^{-2\pi m_i/q^2}$, where $m_i = \frac{2\pi \Delta_i}{\beta}$.

Note that the action (2.143) is only the leading approximation to the effective action of a worldline instanton in the small β limit. In particular, there can be power-law corrections in β coming from higher curvature terms. Thus, while the tree-level

and 1-loop terms (2.144), (2.145), and (2.146) in the worldline path integral are universal, the subleading corrections in $1/m_i$ are not necessarily universal, since they get contributions both from (computable) loops and from higher curvature terms.¹⁹

In appendix 2.14, we derive (2.144), (2.145) and (2.146) by performing the worldline path integral explicitly. We also verify the universal leading terms in several examples from free field theory.

2.10 Discussion and future directions

In this work, we found that the high-temperature partition function, twisted by a finite-order discrete rotation R , is captured by the same thermal EFT as the untwisted partition function. One consequence is that “spin-refined” densities of states (like the difference between the density of even-spin and odd-spin operators) are determined by the same Wilson coefficients as the usual density of states, up to subleading contributions from Kaluza-Klein vortices. Furthermore, the partition function $Z(\beta, \vec{\theta})$ itself has an intricate fractal-like structure as a function of angle at small β , with the same universal asymptotics controlling the neighborhood of every rational angle.

These results follow entirely from effective field theory, together with the assumption that generic CFTs develop a mass gap when dimensionally-reduced. It would be interesting to revisit this assumption and understand how our results are modified in the presence of potential gapless modes. It would also be interesting to investigate whether Kaluza-Klein vortex defects can support gapless excitations, contribute to Weyl anomalies, and/or have nontrivial topological terms in their effective action.

Our central construction is simple: it is essentially the observation that it is useful to construct a mapping torus $M_{\beta,R} = (\mathcal{M}_L \times \mathbb{R})/\mathbb{Z}$ from two successive quotients:

¹⁹Furthermore, our choice of ζ -function regularization does not in general respect coordinate-invariance of the worldline path integral. To restore coordinate invariance one must add non-coordinate-invariant counterterms like $g^{\mu\nu}g^{\rho\tau}g_{\alpha\beta}\Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\tau}^{\beta}$ with the appropriate coefficients. See [BN06] for discussion.

first quotienting by $q\mathbb{Z}$, and then by $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. This idea is applicable on other geometries besides $S^{d-1} \times \mathbb{R}$, and it would be interesting to explore its implications for other types of partition functions. For example, one could explore “spin-refined” statistics of OPE coefficients by studying the behavior of discrete spacetime symmetries on the “genus-2” geometry of [Ben+24], or spin-refined lens-space partition functions [RW15], or the interaction of discrete spacetime symmetries with other forms of higher-dimensional “modular invariance” [Sha16b; Sha17; LW23; AS24]. Supersymmetric partition functions have been studied on a wide variety of geometries, see e.g. [Pes+17]. It is an enduring challenge to understand observables of non-supersymmetric (potentially nonperturbative) theories on these geometries.

One can also consider applying thermal EFT to BCFTs to study the asymptotic spectra of boundary operators. In 2 dimensions, this boils down to studying the partition function on a finite cylinder in the $\beta \rightarrow 0$ limit and writing down an EFT on a finite interval with two end points. The end points will become point-like defects in the thermal effective action. In higher dimensions, by introducing defects one may break the $SO(d)$ invariance down to some subgroup H . We can imagine turning on a rotation belonging to H and applying thermal EFT ideas in this context.

So far, our main tool for computing partition functions has been thermal EFT, which is organized in an expansion in small β . This expansion is likely asymptotic in general. In fact we can see its asymptotic nature explicitly in odd-dimensional free theories. It is an important question whether one can obtain more precise results about high temperature partition functions, potentially including convergent expansions and/or numerical bounds (as is possible in 2d using the modular bootstrap [Hel11]).

One possible approach is through a better understanding of resurgence in the small- β expansion. In particular, it would be nice to better understand the structure of nonperturbative terms beyond the worldline instantons discussed in section 2.9. We

expect that in an interacting theory, there should also be contributions from an infinite sum of “instanton graphs,” representing massive particles propagating and interacting. An old example of instanton graphs are Lüscher corrections in field theories on torii [Lus86a; Lus86b]. However, to our knowledge, the rules for general instanton graphs on general geometries in general massive QFTs have not been spelled out.

Relatedly, modular invariance in 2d CFTs constrains some of the nonperturbative behavior of the partition function. Given some input light spectra, modular invariance highly constrains the resulting spectra, which in effect forces the partition functions with any phase inserted to behave in a certain way. Techniques such as Poincare series and Rademacher series have been used to complete the light spectrum of a 2d CFT (see e.g. [Dij+00; MW10; KM15; AB20]), which roughly take the form of a (convergent) sum over rational angles. It would be interesting if there were related techniques in higher dimensions, resumming all rational angle insertions in the partition function trace.

Another potential avenue to making the high-temperature expansion precise is using Tauberian techniques, which have been applied successfully to correlation functions [Pap+12; QR17; DKP21; PQR23] and torus partition functions in 2d [PS20b; MP20; PS20a]. An essential ingredient in Tauberian methods is positivity,²⁰ which has not yet played an important role in applications of thermal EFT.

Acknowledgements

We thank Arash Ardehali, Luca Delacretaz, Jake McNamara, Hiroshi Ooguri, Edgar Shaghoulian, Joaquin Turiaci, and Yifan Wang for helpful discussions. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0011632. In addition, NB

²⁰See [MMP23] where, even if OPE coefficient can become negative, Tauberian theorems were used along with some boundedness conditions from below, to predict asymptotics of OPE coefficient averaged over a large microcanonical window. However, it is not clear how to extend the result rigorously for an order one window. The same theorem is used in [Pal20] as well.

and SP are supported in part by the Sherman Fairchild Foundation. DSD, JL, and YX are supported in part by Simons Foundation grant 488657 (Simons Collaboration on the Nonperturbative Bootstrap).

Appendix Material

2.11 Qualitative picture of the 3d Ising partition function

In this appendix, we explain our procedure for producing figure 2.1. Let $S_{p/q}$ denote the leading contribution to the thermal effective action around the angle $\frac{2\pi p}{q}$ in 3d:

$$-S_{p/q} = \frac{1}{q^3} \frac{f \text{vol } S^2}{\beta^2 + (\theta - \frac{2\pi p}{q})^2}. \quad (2.147)$$

We expect $-S_{p/q}$ to be a good estimate for the action when it is large. Thus, to patch together different EFT descriptions, we roughly want to choose the action $-S_{p/q}$ that is largest for each (β, θ) . This would lead to the approximation $-S \approx \max_{p,q} -S_{p/q}$, where p, q runs over co-prime pairs of integers. However, such an estimate would not be smooth, so we instead combine the different actions with a root-mean-square:

$$-S \approx \left(\sum_{p,q} (-S_{p/q})^2 \right)^{1/2}. \quad (2.148)$$

In figure 1, we plot $\log \log Z = \log(-S)$, for S given in (2.148), with coprime pairs of integers up to denominator 15. We use the value of minus the free energy density $f \approx 0.153$ determined from Monte-Carlo simulations [Vas+09; KL96; Kre97].

Note that the approximation (2.148) has some unrealistic features. Firstly, its expansion around each rational angle contains subleading corrections in β that do not conform with the expectation from thermal EFT (2.32). Secondly, it does not incorporate the nonperturbative effects discussed in section 2.9. Our goal with this approximation is simply to give a qualitative picture of the partition function. It is interesting to ask whether there is a natural basis of functions for Z that naturally incorporates these constraints.

In principle, the qualitative features of figure 2.1 can be checked. For instance, one can explicitly build the partition function of the 3d Ising model with a phase (e.g. $(-1)^J$) inserted, by simply using the operator scaling dimensions that have been estimated from the conformal bootstrap (or other methods). One can then plot it

as a function of β, Ω , and check that, for instance, the leading Wilson coefficient is approximately $\frac{0.153}{8}$ (similar to what was done in appendix A of [Ili+18] and appendix D of [Ben+24]). An important technical obstacle we run into when attempting this is: when computing the partition function with the phase $(-1)^J$, the EFT is valid when $2\beta \ll 1$ (rather than $\beta \ll 1$), so in effect, one needs to keep more operators in the partition function to get a trustworthy estimate. It would be interesting if other techniques to estimate the scaling dimensions of the 3d Ising model know about enough high energy operators to see explicitly the qualitative features of figure 2.1.

2.12 Review of plethystic sums

In this appendix, we review some facts about the high temperature expansion of plethystic sums. Plethystic sums can be written as derivatives of various spectral zeta functions and to compute the high temperature expansion, one “resums” the zeta functions following a procedure very similar to (2.94). The same resummation technique is also used to compute the high temperature expansion of massive free field free energy, as we discussed in the section 2.14.

Plethystic sums and spectral zeta functions We start with some general discussion. Let $f(\beta) = \sum_k d_k e^{-\beta\lambda_k}$ be a generating function. We define a few related quantities:

- Plethystic sum:

$$\log(\text{PE}[f]) := \sum_{n=1} \frac{1}{n} f(n\beta). \quad (2.149)$$

- Spectral zeta function:

$$\zeta(s; f) := \sum_{n \in \mathbb{Z}} \sum_k d_k \left[\left(\frac{2\pi n}{\beta} \right)^2 + \lambda_k^2 \right]^{-s}. \quad (2.150)$$

- Heat trace:

$$H(t; f) := \sum_k d_k e^{-t\lambda_k^2}. \quad (2.151)$$

The plethystic sum is related to the spectral zeta function in a simple way:

$$\log(\text{PE}[f]) - \frac{\beta}{2} \sum_k d_k \lambda_k = \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta(s; f). \quad (2.152)$$

where the second term on the left hand side is the zeta function regularized Casimir energy.²¹

To derive (2.152), we start by using the Schwinger parametrization to obtain:

$$\begin{aligned} \zeta(s; f) &= \sum_{n \in \mathbb{Z}} \sum_k d_k \left[\left(\frac{2\pi n}{\beta} \right)^2 + \lambda_k^2 \right]^{-s} = \frac{1}{\Gamma(s)} \sum_{n \in \mathbb{Z}} \sum_k d_k \int_0^\infty \frac{dt}{t} t^s e^{-t(\frac{2\pi n}{\beta})^2 - t\lambda_k^2} \\ &= \frac{1}{\Gamma(s)} \frac{\beta}{2\sqrt{\pi}} \sum_{m \in \mathbb{Z}} \sum_k d_k \int_0^\infty \frac{dt}{t} t^{s-\frac{1}{2}} e^{\frac{m^2 \beta^2}{4t} - t\lambda_k^2}, \end{aligned} \quad (2.153)$$

where in the last line we've Poisson resummed with respect to n . Now we split the summation over m into an "excited part" with $m \neq 0$ and a "vacuum part" where $m = 0$:

$$\begin{aligned} \zeta_{\text{excited}}(s; f) &= \frac{1}{\Gamma(s)} \frac{\beta}{\sqrt{\pi}} \sum_{m=1}^\infty \sum_k d_k \int_0^\infty \frac{dt}{t} t^{s-\frac{1}{2}} e^{\frac{m^2 \beta^2}{4t} - t\lambda_k^2}, \\ \zeta_{\text{vacuum}}(s; f) &= \frac{1}{\Gamma(s)} \frac{\beta}{\sqrt{\pi}} \sum_k d_k \int_0^\infty \frac{dt}{t} t^{s-\frac{1}{2}} e^{-t\lambda_k^2}. \end{aligned} \quad (2.154)$$

Performing the integral over t , we get:

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \zeta_{\text{excited}}(s; f) &= \frac{\beta}{\sqrt{\pi}} \sum_{m=1}^\infty \sum_k d_k \int_0^\infty \frac{dt}{t} t^{-1/2} e^{-\frac{m^2 \beta^2}{4t} - t\lambda_k^2} \\ &= 2 \sum_{m=1}^\infty \frac{1}{m} \sum_k d_k e^{-\beta m \lambda_k} = 2 \log(\text{PE}[f]). \end{aligned} \quad (2.155)$$

²¹Note that the Casimir energy computed using zeta function regularization is NOT the same as the Casimir energy included in appendix 2.13. See section 3.1 in [Ben+24] for a discussion of schemes and [HH13] for the difference with zeta function regularization.

For $\zeta_{\text{vacuum}}(s; f)$, we invert the Schwinger parametrization:

$$\left. \frac{d}{ds} \right|_{s=0} \zeta_{\text{vacuum}}(s; f) = \frac{\beta}{2\sqrt{\pi}} \Gamma(-1/2) \sum_k d_k \lambda_k = -\beta \sum_k d_k \lambda_k. \quad (2.156)$$

Putting everything together we arrive at (2.152).

To compute partition functions with flavor twists and rational angular fugacities we need to consider plethystic sums taking the following form:

$$\sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n} F(n\beta, \omega_q^n), \quad \omega_q = \exp\left(\frac{2\pi i}{q}\right). \quad (2.157)$$

Some formal manipulation shows that when the “twisted generating function” $F(\beta, \omega_q)$ satisfies $F(\beta, \omega_q) = F(\beta, \omega_q^{-1})$, the following relation holds:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{n} F(n\beta, \omega_q^n) + \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} F(n\beta, \omega_q^n) &= \frac{1}{q} \left. \frac{d}{ds} \right|_{s=0} \zeta[q\theta, s; F(q\beta, 1)] + \frac{\beta}{q} \sum_k d_k \lambda_k \\ &+ \frac{1}{2q} \sum_{\ell=1}^{q-1} \left. \frac{d}{ds} \right|_{s=0} \left[L\left(q\theta, \frac{\ell}{q}, s; F(q\beta, \omega_q^\ell)\right) + L\left(-q\theta, \frac{\ell}{q}, s; F(q\beta, \omega_q^\ell)\right) \right]. \end{aligned} \quad (2.158)$$

where:

- In the second sum on the right hand side, d_k and λ_k are supposed to be read from $F(q\beta, 1)$, that is, $F(q\beta, 1) = \sum_{k=0} d_k e^{-\beta \lambda_k}$

•

$$\zeta(\theta, s; f) = \sum_{n \in \mathbb{Z}} \sum_k d_k \left[\left(\frac{2\pi n}{\beta} + \frac{\theta}{\beta} \right)^2 + \lambda_k^2 \right]^{-s}, \quad \text{with } f(\beta) = \sum_{k=0} d_k e^{-\beta \lambda_k}. \quad (2.159)$$

•

$$L(\theta, a, s; f) = \sum_{n \in \mathbb{Z}} \sum_k d_k \frac{e^{2\pi i n a}}{\left[\left(\frac{2\pi n}{\beta} + \frac{\theta}{\beta} \right)^2 + \lambda_k^2 \right]^s}, \quad \text{with } f(\beta) = \sum_{k=0} d_k e^{-\beta \lambda_k}. \quad (2.160)$$

Now that we've related different plethystic sums to various spectral zeta functions, the high temperature expansion can be computed by “resumming” all these zeta functions. For concreteness, we look at some examples:

Example: free scalar on $S^1 \times S^3$ The relevant generating function is²²

$$f(\beta) = \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - e^{-\beta})^4} = \sum_{k=1}^{\infty} k^2 e^{-\beta k}. \quad (2.161)$$

The spectral zeta function is:

$$\zeta(s; f) = \frac{1}{\Gamma(s)} \sum_{n \in \mathbb{Z}} \int_0^{\infty} \frac{dt}{t} t^s e^{-t(\frac{2\pi n}{\beta})^2} H(t; f). \quad (2.162)$$

with

$$H(t; f) = \sum_{k=1}^{\infty} k^2 e^{-tk^2} = \frac{\sqrt{\pi}}{4} t^{-3/2} + \sqrt{\pi} \sum_{\ell=1}^{\infty} e^{-\ell^2 \pi^2 / t} \left(\frac{1}{2} t^{-3/2} - \ell^2 \pi^2 t^{-5/2} \right).$$

Here we performed a Poisson resummation with respect to k and separated the piece with $\ell = 0$ from $\ell \neq 0$.

- The term with $n = 0$ in the sum (2.162) is the contribution of zero mode:

$$\zeta(s; f) \Big|_{n=0} = \sum_{k=1}^{\infty} d_k \lambda_k^{-s} = \sum_{k=1}^{\infty} k^{2-2s} = \zeta_R(2s - 2). \quad (2.163)$$

- Terms with $n \neq 0$ and $\ell = 0$ generate the perturbative series:

$$\zeta(s; f) \Big|_{n \neq 0, \ell=0} = \frac{\sqrt{\pi}}{4\Gamma(s)} \sum_{n \in \mathbb{Z}, n \neq 0} \int_0^{\infty} \frac{dt}{t} t^s e^{-t(\frac{2\pi n}{\beta})^2} t^{-3/2} = \sqrt{\pi} \frac{\Gamma(s - \frac{3}{2})}{2\Gamma(s)} \left(\frac{2\pi}{\beta} \right)^{3-2s} \zeta_R(2s - 3). \quad (2.164)$$

- Terms with $n \neq 0$ and $\ell \neq 0$ generate the non-perturbative corrections:

$$\zeta(s; f) \Big|_{n \neq 0, \ell \neq 0} = \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \left[2^{\frac{5}{2}-s} \left(\frac{n}{\ell\beta} \right)^{\frac{3}{2}-s} K_{-\frac{3}{2}+s} \left(\frac{4\pi^2 \ell n}{\beta} \right) - 2^{\frac{9}{2}-s} \pi^2 \ell^2 \left(\frac{n}{\ell\beta} \right)^{\frac{5}{2}-s} K_{-\frac{5}{2}+s} \left(\frac{4\pi^2 \ell n}{\beta} \right) \right]. \quad (2.165)$$

²²This is the spectral generating function of the operator $\Delta + \frac{(d-2)}{4(d-1)} R$ with Δ being the Laplacian on S^3 and $d = 4$

Following (2.152), we find:

$$\log(\text{PE}[f]) - \frac{\beta}{240} = \frac{\pi^4}{45\beta^3} - \frac{\zeta(3)}{4\pi^2} - \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{4\pi^2 n\ell}{\beta}} \left(\frac{1}{\ell^3} + \frac{4n\pi^2}{\ell^2\beta} + \frac{8n^2\pi^4}{\ell\beta^2} \right). \quad (2.166)$$

With the same d_k and λ_k , we can construct a slightly different zeta function:

$$\zeta_m(s) = \sum_{k=0}^{\infty} \frac{d_k}{(\lambda_k^2 + m^2)^s}. \quad (2.167)$$

this spectral zeta function satisfies

$$\frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta_m(s) = \log(Z_{\text{massive}}[S^3]) \quad (2.168)$$

where $Z_{\text{massive}}[S^3]$ is given in eqn(2.216). Applying the same resummation procedure, we get (2.217). The lens space partition functions $Z_{\text{massive}}[L(q; 1)]$ can be computed in an almost identical way. One simply replace $d_k = k^2$ by [Aso+13]:

$$d_k = \begin{cases} k \left[\frac{k}{q} \right], & k - q \left[\frac{k}{q} \right] \in 2\mathbb{Z}, \\ k \left(\left[\frac{k}{q} \right] + 1 \right), & k - q \left[\frac{k}{q} \right] \in 2\mathbb{Z} + 1, \end{cases} \quad (q \in 2\mathbb{Z} + 1),$$

$$d_k = \begin{cases} 0, & k \in 2\mathbb{Z}, \\ k \left(2 \left[\frac{k}{q} \right] + 1 \right), & k \in 2\mathbb{Z} + 1, \end{cases} \quad (q \in 2\mathbb{Z}). \quad (2.169)$$

which are the degeneracy for $\Delta + \frac{(d-2)}{4(d-1)}R$ with Δ being the Laplacian in $L(q; 1)$ and $d = 4$. The same calculations will yield (2.221) and (2.222).

Example: free scalar on $S^1 \times S^5$

The relevant generating function is:

$$f(\beta) = \frac{q^2(1 - q^2)}{(1 - q)^6} = \sum_{k=1}^{\infty} \frac{k^2(k^2 - 1)}{12} e^{-\beta k}.$$

and the heat trace is:

$$H(t; f) = \frac{1}{24} \sum_{k \in \mathbb{Z}} k^2(k^2 - 1) e^{-tk^2}$$

$$= \sqrt{\pi} \left(\frac{1}{32t^{5/2}} - \frac{1}{48t^{3/2}} \right) + \sqrt{\pi} \sum_{l=1}^{\infty} e^{-\frac{l^2\pi^2}{t}} \left[\frac{\pi^4 l^4}{12t^{9/2}} - \frac{\pi^2 l^2}{4t^{7/2}} + \frac{1}{t^{5/2}} \left(\frac{\pi^2 l^2}{12} + \frac{1}{16} \right) - \frac{1}{24t^{3/2}} \right]. \quad (2.170)$$

Following the same procedure as in the example above, we obtain:

$$\log(\text{PE}[f]) - \frac{31}{60480}\beta = \frac{2\pi^6}{945\beta^5} - \frac{\pi^4}{540\beta^3} + \frac{\zeta(5)}{16\pi^4} + \frac{\zeta(3)}{48\pi^2} \quad (2.171)$$

$$+ \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{4\pi^2 n \ell}{\beta}} \left[\frac{4\pi^4 n^4}{3\beta^4 \ell} + \frac{4\pi^2 n^3}{3\beta \ell^2} + \frac{1}{\beta^2} \left(\frac{n^2}{\ell^3} + \frac{\pi^2 n^2}{3\ell} \right) + \frac{1}{\beta} \left(\frac{n}{2\pi^2 \ell^4} + \frac{n}{6\ell^2} \right) + \frac{1}{8\pi^4 \ell^5} + \frac{1}{24\pi^2 \ell^3} \right]. \quad (2.172)$$

if we replace $\zeta(s; f)$ by $\zeta_m(s)$, we reproduce (2.218).

Example: free scalar on $S^1 \times S^3$ with $\mathbb{Z}_2$ flavor twist

The plethystic sum in interest is:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(n\beta), \quad f(\beta) = \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - e^{-\beta})^4} = \sum_{k=1}^{\infty} k^2 e^{-\beta k}. \quad (2.173)$$

Setting $q = 1$ and $\theta = \pi$ in (2.158), we find that:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(n\beta) = \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \zeta(\pi, s; f) + \frac{\beta}{2} \sum_k d_k \lambda_k. \quad (2.174)$$

where $\zeta(\pi, s; f)$ is defined in (2.159). After resumming the zeta function, we find:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} f(n\beta) - \frac{\beta}{240} = -\frac{7\pi^4}{360\beta^3} - \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{(4n-2)\ell\pi^2}{\beta}} \left(\frac{(2n-1)^2\pi^2}{\ell\beta^2} + \frac{2n-1}{\ell^2\beta} + \frac{1}{\ell^3\pi^2} \right). \quad (2.175)$$

Finally, we discuss twisted partition function with rational angular fugacity. These are the main objects of interest in this paper. Even though the techniques outlined in this appendix can be very easily generalized to higher dimensions and to more complicated combinations of angles, we will focus on 4d scalar partition functions twisted by $e^{-\frac{2\pi i}{q} J_{12} - \frac{2\pi i}{q} J_{34}}$ for simplicity. As explained in section 2.14, the subtle difference between the lens space partition function and the twisted partition function is related to the non-trivial topology of the EFT bundle.

Example: free scalar in $S^1 \times S^3$ with $\mathbb{Z}_q$ rotation, q odd

We focus on the twisted partition function $\log \left(\text{Tr} [e^{-\beta H - \frac{2\pi i}{q} J_{12} - \frac{2\pi i}{q} J_{34}}] \right)$ and restrict to the case where q is odd. The plethystic sum in interest is:

$$\sum_{n=1}^{\infty} \frac{1}{n} F(n\beta, \omega_q^n), \quad F(\beta, \omega_q) = \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - \omega_q e^{-\beta})^2 (1 - \omega_q^{-1} e^{-\beta})^2}. \quad (2.176)$$

It is easy to verify:

$$F(q\beta, \omega_q^\ell) = \begin{cases} \sum_{k=1}^{\infty} k^2 e^{-q\beta k}, & \ell = 0 \\ \sum_{k=1}^{\infty} \frac{k \sin(\frac{2\pi k \ell}{q})}{\sin(\frac{2\pi \ell}{q})} e^{-q\beta k}, & \ell \neq 0 \end{cases} \quad (2.177)$$

Setting $\theta = 0$ in (2.158), we find:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} F(n\beta, \omega_q^n) &= \frac{1}{2q} \frac{d}{ds} \Big|_{s=0} \zeta[0, s; F(q\beta, 1)] + \frac{\beta}{q} \sum_k d_k \lambda_k \\ &\quad + \frac{1}{2q} \sum_{\ell=1}^{q-1} \frac{d}{ds} \Big|_{s=0} L \left(q\theta, \frac{\ell}{q}, s; F(q\beta, \omega_q^\ell) \right). \end{aligned} \quad (2.178)$$

where $d_k = k^2$, $\lambda_k = qk$ for the second sum on the right hand side. After resumming all the zeta functions, we find:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-n\beta}(1 - e^{-2n\beta})}{(1 - \omega_q^n e^{-n\beta})^2 (1 - \omega_q^{-n} e^{-n\beta})^2} &= \frac{\pi^4}{45\beta^3 q^4} + \frac{\beta}{240} - \frac{\zeta(3)}{4\pi^2 q} \\ &\quad - \frac{1}{q} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{4\pi^2 \ell n}{\beta q}} \left(\frac{1}{2\pi^2 \ell^3} + \frac{2n}{\beta \ell^2 q} + \frac{4\pi^2 n^2}{\beta^2 \ell q^2} \right) \\ &\quad + \sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} \frac{\cos(\frac{2\pi n \ell}{q})}{\sin(\frac{2\pi \ell}{q})} e^{-\frac{4\pi^2 n \ell}{q^2 \beta}} \left(\frac{2\pi n}{\ell q \beta} + \frac{q}{2\ell^2 \pi} \right) \\ &\quad + \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} \frac{1}{4\pi \ell^2} \frac{1}{\sin(\frac{2\pi \ell}{q})}. \end{aligned} \quad (2.179)$$

Example: free scalar in $S^1 \times S^3$ with $\mathbb{Z}_q$ rotation and $\mathbb{Z}_2$ flavor twist, q =odd

The relevant plethystic sum is:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} F(n\beta, \omega_q^n), \quad F(\beta, \omega_q) = \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - \omega_q e^{-\beta})^2 (1 - \omega_q^{-1} e^{-\beta})^2}. \quad (2.180)$$

Setting $\theta = \pi$ in (2.158) and resum all the zeta functions, we get:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{e^{-n\beta}(1 - e^{-2n\beta})}{(1 - \omega_q^n e^{-n\beta})^2 (1 - \omega_q^{-n} e^{-n\beta})^2} = -\frac{7\pi^4}{360\beta^3 q^4} + \frac{\beta}{240} \\
& - \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{2\ell(2n-1)\pi^2}{q\beta}} \left(\frac{(2n-1)^2 \pi^2}{\ell q^3 \beta^2} + \frac{2n-1}{\beta \ell^2 q^2} + \frac{1}{2\ell^3 q \pi^2} \right) \\
& + \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} \sum_{n=0}^{\infty} \frac{\cos(\frac{2\pi\ell}{q}(\frac{q-1}{2} - n))}{\sin(\frac{2\pi\ell}{q})} e^{-\frac{2\ell(2n+1)\pi^2}{q^2\beta}} \left(\frac{(2n+1)\pi}{\ell q \beta} + \frac{q}{2\ell^2 \pi} \right). \tag{2.181}
\end{aligned}$$

2.13 More examples for 4d and 6d CFTs

In this appendix, we compute a few more examples of the high temperature expansion of CFTs with a large rotation inserted. We will focus on insertions R without fixed points (so that $S_{\mathcal{D}}$ vanishes). In odd d , $SO(d)$ necessarily has a nontrivial fixed locus, so we focus on even d . In even d , if we insert $R = \exp\left(2\pi i \left(\frac{p_1}{q} J_1 + \dots + \frac{p_n}{q} J_n\right)\right)$ (namely all the denominators are equal, with $\gcd(p_i, q) = 1$), then R has no fixed points and there is no defect action. Previously in section 2.5, we described 4d examples with a defect coming from $q_1 \neq q_2$; in this example we will consider the (simpler) case with no defect action.

We note that the formulas in this section are accurate to all orders in perturbation theory in β ; this is because the free energy of free theories in even dimensions truncates at $O(\beta)$. This is an accident of free theories in even dimensions and does not generalize.

Finally, we note that in even dimensions, the Hamiltonian $H = D + \varepsilon_0$ includes a contribution from the Casimir energy, which is not accounted for in the sum over characters using plethystic exponentials (see section 3.1 of [Ben+24] for details on this scheme, [HH13] for a calculation of the Casimir energy, and e.g. [GK15] for values of the a anomaly). We have included this contribution to the final expression (although terms linear in β are invariant when acting on with (2.32)).

Example: 4d $\mathbb{Z}_2$ -twisted complex scalar

Here we consider a 4d free complex scalar. We insert a $\mathbb{Z}_2$ twist (which we denote as $e^{\pi i Q}$) where we identify the field ϕ with $-\phi$ as we go around the thermal circle, in order to remove the gapless sector. Without any insertion of a rotation R , we get the following free energy:

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H - i\Omega_1 J_1 - i\Omega_2 J_2)} e^{\pi i Q} \right] \\ & \sim \frac{1}{\prod_{i=1}^2 (1 + \Omega_i^2)} \left(-\frac{7\pi^4}{180\beta^3} - \frac{\pi^2 \sum_{i=1}^2 \Omega_i^2}{36\beta} + \frac{(3 + 6 \sum_{i=1}^2 \Omega_i^2 - 10\Omega_1^2 \Omega_2^2 + 6 \sum_{i=1}^2 \Omega_i^4) \beta}{720} \right). \end{aligned} \quad (2.182)$$

Now let us consider a rotation by $2\pi/3$ in each Cartan direction. As expected, when we insert the large rotation, we find:

$$\log \text{Tr} \left[e^{-\beta(H - i\Omega_1 J_1 - i\Omega_2 J_2)} e^{\pi i Q} e^{2\pi i \left(\frac{J_1}{3} + \frac{J_2}{3} \right)} \right] \sim \frac{1}{3} \log \text{Tr} \left[e^{-3\beta(H - i\Omega_1 J_1 - i\Omega_2 J_2)} e^{3\pi i Q} \right]. \quad (2.183)$$

Example: 4d Dirac fermion

As discussed in section 2.4, for fermionic theories we need to compute the partition function without and with a $(-1)^F$ insertion:

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H - i\Omega_1 J_1 - i\Omega_2 J_2)} \right] \\ & \sim \frac{1}{\prod_{i=1}^2 (1 + \Omega_i^2)} \left(\frac{7\pi^4}{90\beta^3} - \frac{\pi^2 (3 + \sum_{i=1}^2 \Omega_i^2)}{36\beta} + \frac{(18 + 6 \sum_{i=1}^2 \Omega_i^2 + 10\Omega_1^2 \Omega_2^2 - 21 \sum_{i=1}^2 \Omega_i^4) \beta}{1440} \right) \end{aligned} \quad (2.184)$$

$$\begin{aligned} & \log \text{Tr} \left[(-1)^F e^{-\beta(H - i\Omega_1 J_1 - i\Omega_2 J_2)} e^{4\pi i Q/3} \right] \\ & \sim \frac{1}{\prod_{i=1}^2 (1 + \Omega_i^2)} \left(\frac{52\pi^4}{1215\beta^3} - \frac{\pi^2 (3 + \sum_{i=1}^2 \Omega_i^2)}{54\beta} + \frac{(18 + 6 \sum_{i=1}^2 \Omega_i^2 + 10\Omega_1^2 \Omega_2^2 - 21 \sum_{i=1}^2 \Omega_i^4) \beta}{1440} \right). \end{aligned} \quad (2.185)$$

Fermions carry $U(1)$ charge. We use a convention where the particle carries charge $+1$ and anti-particle carries charge -1 . To make sure that the zero mode does not contribute in the thermal effective field theory description, we turned on the fugacity corresponding to $U(1)$ and set it to $e^{4\pi i/3}$, along with a $(-1)^F$ insertion. This

amounted to turning on $\mathbb{Z}_3 \subset U(1)$ flavor symmetry for the partition function with a $(-1)^F$ insertion. After turning on a large rotation $e^{2\pi i J_1 p_1/q} e^{2\pi i J_2 p_2/q}$, we get the following results (exactly as predicted in section 2.4):

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega_1 J_1 - i\Omega_2 J_2)} e^{2\pi i \left(\frac{J_1}{3} + \frac{J_2}{3} \right)} \right] \sim \frac{1}{3} \log \text{Tr} \left[e^{-3\beta(H-i\Omega_1 J_1 - i\Omega_2 J_2)} \right] \quad (2.186)$$

$$\log \text{Tr} \left[e^{-\beta(H-i\Omega_1 J_1 - i\Omega_2 J_2)} e^{2\pi i Q/3} e^{2\pi i \left(\frac{J_1}{2} + \frac{J_2}{2} \right)} \right] \sim \frac{1}{2} \log \text{Tr} \left[(-1)^F e^{-2\beta(H-i\Omega_1 J_1 - i\Omega_2 J_2)} e^{4\pi i Q/3} \right]. \quad (2.187)$$

Example: 6d twisted complex scalar

Here we consider a 6d free scalar with a $\mathbb{Z}_2$ twist:

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} e^{\pi i Q} \right] \\ & \sim \frac{1}{\prod_{i=1}^3 (1 + \Omega_i^2)} \left(-\frac{31\pi^6}{7560\beta^5} + \frac{7\pi^4(1 - \sum_{i=1}^3 \Omega_i^2)}{2160\beta^3} + \frac{\pi^2(8 \sum_{i=1}^3 \Omega_i^2 - 5 \sum_{cyc} \Omega_1^2 \Omega_2^2 - 3 \sum_{i=1}^3 \Omega_i^4)}{2160\beta} \right. \\ & \quad \left. - \frac{(37 + 62 \sum_{i=1}^3 \Omega_i^2 - 154 \sum_{cyc} \Omega_1^2 \Omega_2^2 - 104 \sum_{i=1}^3 \Omega_i^4 + 70 \Omega_1^2 \Omega_2^2 \Omega_3^2 + 42 \sum_{i=1}^3 \Omega_i^2 \sum_{i=1}^3 \Omega_i^4 - 22 \sum_{i=1}^3 \Omega_i^6) \beta}{60480} \right). \end{aligned} \quad (2.188)$$

With the insertion of a large rotation, we get:

$$\log \text{Tr} \left[e^{-\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} e^{\pi i Q} e^{2\pi i \left(\frac{J_1}{3} + \frac{J_2}{3} + \frac{J_3}{3} \right)} \right] \sim \frac{1}{3} \log \text{Tr} \left[e^{-3\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} e^{3\pi i Q} \right] \quad (2.189)$$

Example: 6d Dirac fermion

Finally we consider a 6d free Dirac fermion. We get:

$$\begin{aligned} & \log \text{Tr} \left[e^{-\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} \right] \\ & \sim \frac{1}{\prod_{i=1}^3 (1 + \Omega_i^2)} \left(\frac{31\pi^6}{1890\beta^5} - \frac{7\pi^4(5 + \sum_{i=1}^3 \Omega_i^2)}{1080\beta^3} + \frac{\pi^2(135 + 26 \sum_{i=1}^3 \Omega_i^2 + 10 \sum_{cyc} \Omega_1^2 \Omega_2^2 - 21 \sum_{i=1}^3 \Omega_i^4)}{4320\beta} \right. \\ & \quad \left. + \frac{(-880 + 55 \sum_{i=1}^3 \Omega_i^2 - 14 \sum_{cyc} \Omega_1^2 \Omega_2^2 - 743 \sum_{i=1}^3 \Omega_i^4 - 70 \Omega_1^2 \Omega_2^2 \Omega_3^2 + 147 \sum_{i=1}^3 \Omega_i^2 \sum_{i=1}^3 \Omega_i^4 - 302 \sum_{i=1}^3 \Omega_i^6) \beta}{120960} \right). \end{aligned} \quad (2.190)$$

With an insertion of a rotation R , we get the following:

$$\log \text{Tr} \left[e^{-\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} e^{2\pi i \left(\frac{J_1}{2} + \frac{J_2}{2} + \frac{J_3}{2} \right)} \right] \sim \frac{1}{2} \log \text{Tr} \left[e^{-2\beta(H-i \sum_{k=1}^3 \Omega_k J_k)} \right] \quad (2.191)$$

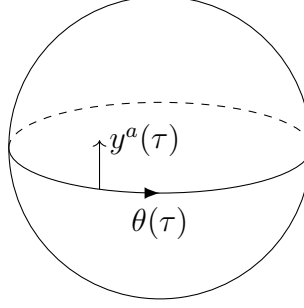


Figure 2.7: Coordinates on S^{d-1} . The classical path is $\theta(\tau) = 2\pi\tau$ and perpendicular fluctuations $y^a(\tau)$ are integrated over.

2.14 More on nonperturbative terms

In this appendix, we discuss more details on the nonperturbative parts of the partition function outlined in section 2.9. In appendix 2.14, we derive equations (2.144), (2.145) and (2.146) by performing the worldline path integral. In appendices 2.14, 2.14, and 2.14, we verify the universal leading terms in several examples from free field theory.

Worldline path integral

Worldline path integral on the sphere

Let us begin by computing the worldline path integral on S^{d-1} . Let θ denote a coordinate along a great circle and let y^i ($i = 1, \dots, d-2$) parametrize the perpendicular directions, see figure 2.7. The metric on S^{d-1} takes the form:

$$ds^2 = dy^2 + \cos^2 |y| d\theta^2. \quad (2.192)$$

In the absence of a background gauge field, a single worldline instanton contribution is given by

$$\begin{aligned} \int Dx^\mu(\tau) e^{-m_i \int ds} &= \int Dy^i(\tau) \exp \left[-m_i \int_0^1 d\tau \sqrt{\cos^2 |y| \left(\frac{d\theta}{d\tau} \right)^2 + \left(\frac{dy}{d\tau} \right)^2} \right] \\ &\sim e^{-2\pi m_i} \int Dy^i(\tau) \exp \left[-m_i \int_0^1 d\tau \left(\frac{\dot{y}^2}{4\pi} - \pi y^2 \right) \right], \end{aligned} \quad (2.193)$$

where in the last line we Taylor-expanded the square root in fluctuations around the great circle.²³ We also used reparameterization invariance of the Nambu-Goto action to set $\theta(\tau) = 2\pi\tau$. In the path integral, we sum over periodic paths $y^i(\tau)$ satisfying $y^i(0) = y^i(1)$, with the following mode expansion:

$$y^i(\tau) = \sum_{n=1}^{\infty} a_n^i \sin(2\pi n\tau) + \sum_{n=1}^{\infty} b_n^i \cos(2\pi n\tau) + \frac{b_0^i}{\sqrt{2}}. \quad (2.194)$$

In writing this mode expansion, we have defined the mode numbers and coefficients a_n^i, b_n^i so that the measure on the moduli space of geodesics is locally $\prod_i da_1^i db_1^i$. We furthermore imposed that each mode should be unit-normalized under the inner product $\langle y, y \rangle = 2 \int d\tau |y|^2$. The path integral can then be reduced to a product of ordinary integrals over a_n^i, b_n^i :²⁴

$$\begin{aligned} & \int Dy^i(\tau) \exp \left[-m_i \int_0^1 d\tau \left(\frac{\dot{y}^2}{4\pi} - \pi y^2 \right) \right] \\ &= \prod_{i=1}^{d-2} \left\{ \prod_{n=1}^{\infty} \int da_n^i db_n^i \exp \left[-\frac{m_i \pi}{2} (n^2 - 1) ((a_n^i)^2 + (b_n^i)^2) \right] \times \int db_0^i \exp \left[\frac{m_i \pi}{2} (b_0^i)^2 \right] \right\}. \end{aligned} \quad (2.195)$$

Modes with $n > 1$ are massive and correspond to oscillations around the geodesic. The modes with $n = 0$ are tachyonic. After adding these tachyonic fluctuations, the trajectory of the particle will deviate from the great circle and shrink towards a point, reducing its length (see e.g. figure 4 and the surrounding discussions in [Don+21]). Consequently, the classical trajectory we are expanding around is an *unstable* saddle point of the Nambu-Goto action. To define the integral over the tachyonic directions, we need to make sense of integral of the form $\int_{-\infty}^{\infty} dx e^{\alpha^2 x^2}$ where $\text{Re}(\alpha^2) > 0$. We do

²³This approximation is allowed as long as the thermal wavelength $\sqrt{2\pi/m}$ is much smaller than the size of the sphere.

²⁴The path integral measure $Dy(\tau)$ is induced from a choice of inner product. This inner product must be local — i.e. an integral over τ — in order for the path integral to satisfy proper cutting and gluing rules. However, the overall coefficient of the inner product does not matter because $\det(\text{const}) = \prod_{n \in \mathbb{Z}} \text{const} = 1$ in ζ -function regularization. As long as the modes are all normalized the same way with respect to the inner product, the measure factorizes into an integral over each mode coefficient $Dy = \prod_i db_0^i \prod_{n,i} da_n^i db_n^i$.

so by analytically continuing from the region where $\text{Re}(\alpha^2) < 0$ and defining:

$$\int_{-\infty}^{\infty} dx e^{\alpha^2 x^2} := (\pm i) \frac{\sqrt{\pi}}{|\alpha|}, \quad \text{Re } \alpha^2 > 0. \quad (2.196)$$

Furthermore, we make the same choice of sign in (2.196) for integrals over all such tachyonic modes. This prescription, however, does not remove all ambiguities. Since there are $d-2$ perpendicular directions and each of them contributes a single tachyonic mode, the overall phase factor is $(\pm i)^{(d-2)}$. When d is even, both choices of sign give the same result. But when d is odd, the ambiguity remains. As we will see in appendix 2.14, this remaining two-fold ambiguity is related to a contour ambiguity in performing a Borel resummation.

The zero modes with $n = 1$ represent rotations of the great circle. The corresponding mode coefficients a_1^i, b_1^i are local coordinates on the moduli space of geodesics. Every geodesic on S^{d-1} is an intersection of S^{d-1} with a 2-plane in $\mathbb{R}^d$ containing the origin, so this moduli space is the same as the Grassmannian $\text{Gr}(2, \mathbb{R}^d)$. Hence, the zero mode integral is the volume of this Grassmannian:

$$\prod_{i=1}^{d-2} \int da_1^i db_1^i = \text{vol}(\text{Gr}(2, \mathbb{R}^d)) = \frac{\text{vol}(O(d))}{\text{vol}(O(2)) \times \text{vol}(O(d-2))} = \frac{\text{vol}(S^{d-1}) \times \text{vol}(S^{d-2})}{2\text{vol}(S^1)}. \quad (2.197)$$

We can finally explicitly calculate the path-integral over $y^i(\tau)$:

$$\begin{aligned} & \int Dy^i(\tau) \exp \left[-m_i \int_0^1 d\tau \left(\frac{\dot{y}^2}{4\pi} - \pi y^2 \right) \right] \\ &= \text{vol}(\text{Gr}(2, \mathbb{R}^d)) \prod_{i=1}^{d-2} \left[(\pm i) \sqrt{\frac{2}{m_i}} \times \prod_{n=1}^{\infty}{}' \frac{2}{m_i} \prod_{n=1}^{\infty}{}' \frac{1}{n^2 - 1} \right] \\ &= \frac{(\pm i)^{d-2} m_i^{d-2}}{\Gamma(d-1)}, \end{aligned} \quad (2.198)$$

where $\prod_n{}'$ means we skip the mode $n = 1$. We computed the infinite product above using zeta function regularization [All20]. In particular, the following formulas are useful for such computations:

$$\prod_{n=1}^{\infty} n = \sqrt{2\pi}, \quad \text{and} \quad \prod_{n=1}^{\infty} a = a^{-1/2} \quad (a > 0). \quad (2.199)$$

Putting everything together, we obtain (2.144).

Worldline path integral on lens space

Three dimensional homogeneous lens space $L(q; 1)$ can be defined as the quotient of the three sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ by the equivalence relation $(z_1, z_2) \sim (e^{2\pi i/q} z_1, e^{2\pi i/q} z_2)$. Geodesics in $L(q; 1)$ comes in two types: the contractable “long geodesics” with length 2π , and the non-contractable “short geodesics” with length $2\pi/q$. If $q \neq 2$, each short geodesic is an intersection of a complex line $\alpha z_1 + \beta z_2 = 0$ with S^3 , so the corresponding moduli space is $\text{Gr}(1, \mathbb{C}^2)$. When $q = 2$, every real two-plane $\alpha z_1 + \beta z_2 + \gamma \bar{z}_1 + \delta \bar{z}_2$ in $\mathbb{R}^4$ intersects S^3 at a short geodesic, so the corresponding moduli space is $\text{Gr}(2, \mathbb{R}^4)$.

To compute worldline instanton contributions to the lens space partition function, we choose the classical trajectory $\theta(\tau)$ to wind around a short geodesic and integrate over perpendicular fluctuations obeying a slightly different boundary condition:

$$\begin{pmatrix} y^1(1) \\ y^2(1) \end{pmatrix} = \begin{pmatrix} \cos(\frac{2\pi}{q}) & -\sin(\frac{2\pi}{q}) \\ \sin(\frac{2\pi}{q}) & \cos(\frac{2\pi}{q}) \end{pmatrix} \begin{pmatrix} y^1(0) \\ y^2(0) \end{pmatrix}. \quad (2.200)$$

Introducing the complex variable $z(\tau) = y^1(\tau) + iy^2(\tau)$, we have the following mode expansion:

$$z(\tau) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i(n+1/q)\tau}. \quad (2.201)$$

Plugging in $\theta(\tau) = \frac{2\pi\tau}{q}$, we find the worldline path integral

$$\begin{aligned} & \int Dz(\tau) \exp \left[-m_i \int_0^1 d\tau \sqrt{(1 - |z|^2) \left(\frac{d\theta}{d\tau} \right)^2 + \left| \frac{dz}{d\tau} \right|^2} \right] \\ &= e^{-\frac{2m_i\pi}{q}} \prod_{n \in \mathbb{Z}} \int dc_n d\bar{c}_n \exp \left[-m_i\pi (n^2 q + 2n) |c_n|^2 \right]. \end{aligned} \quad (2.202)$$

When $q \neq 2$, the only zero mode is $c_0 e^{(2\pi i/q)\tau}$. In this case, following the treatment of zero modes outlined above, we find

$$\prod_{n \in \mathbb{Z}} \int dc_n d\bar{c}_n \exp [-m_i \pi (n^2 q + 2n) |c_n|^2] = \text{vol}(\text{Gr}(1, \mathbb{C}^2)) \prod_{n \in \mathbb{Z} \neq 0} \frac{1}{m_i} \frac{1}{n(2+nq)} = \frac{m_i}{2 \sin(\frac{2\pi}{q})}, \quad (2.203)$$

where we've used

$$\text{vol}(\text{Gr}(k, \mathbb{C}^d)) = \frac{\text{vol}(U(d))}{\text{vol}(U(k)) \times \text{vol}(U(d-k))} = \frac{\pi^{k(d-k)} G(k+1) G(d-k+1)}{G(d+1)}, \quad (2.204)$$

where $G(z)$ is the Barnes G-function.

When $q = 2$, there are two zero modes $c_0 e^{\pi i \tau}$ and $c_{-1} e^{-\pi i \tau}$. Thus,

$$\prod_{n \in \mathbb{Z}} \int dc_n d\bar{c}_n \exp [-m_i \pi (2n^2 + 2n) |c_n|^2] = \int dc_0 d\bar{c}_0 dc_{-1} d\bar{c}_{-1} \prod_{n \in \mathbb{Z} \neq 0, -1} \frac{1}{2m_i} \frac{1}{n(1+n)}. \quad (2.205)$$

The integral over zero mode coefficients c_0 and c_{-1} is not exactly the same as the volume of $\text{Gr}(2, \mathbb{R}^4)$: the correct local coordinates on $\text{Gr}(2, \mathbb{R}^4)$ should be the coefficient in front of $\cos(\pi\tau)$, $i \cos(\pi\tau) \sin(\pi\tau)$ and $i \sin(\pi\tau)$, therefore:

$$d\text{Vol}(\text{Gr}(2, \mathbb{R}^4)) = 4 dc_0 d\bar{c}_0 dc_{-1} d\bar{c}_{-1}. \quad (2.206)$$

Combing zero modes with massive modes, we get:

$$\prod_{n \in \mathbb{Z}} \int dc_n d\bar{c}_n \exp [-m_i \pi (2n^2 + 2n) |c_n|^2] = \frac{m_i^2}{2}. \quad (2.207)$$

Notice that tachyonic modes never appeared in the computations above. This is consistent with the fact that short geodesics are non-contractable!

World line contribution to spin-refined partition function in 4d

We would like to compute single-particle non-perturbative corrections for a spin-refined sphere partition function in 4d. As explained in section 2.2, the corresponding

EFT bundle is non-trivial. By winding around the fundamental cycle in $L(q; 1)$, one also advances by β along the thermal circle whose total length is $q\beta$. Thus, after Kaluza-Klein reduction, we have a flat connection on an $S^1_{q\beta}$ bundle over $L(q; 1)$ whose holonomy along a short geodesic is given by $\oint A = \beta$. The corresponding phase in the worldline action is thus $\exp(\frac{2\pi i q_i}{q\beta}\beta) = \exp(\frac{2\pi i q_i}{q})$. Since the connection is flat, the classical equations of motion will not be affected, and we can compute the “Nambu-Goto part” of the worldline path integral in exactly the same way as for the lens space partition function. Putting everything together, we arrive at (2.146).

A closer look at free field theory

In this subsection, we discuss non-perturbative corrections in free field partition functions. In particular, we will show that they indeed have the structure predicted in section 2.9. Actually, thanks to the Fock space structure of the Hilbert space, we can easily predict the leading terms in all non-perturbative corrections to free theory partition functions using worldline instantons. The “free-theory upgraded predictions” in (2.211), (2.213), and (2.214) can also be checked explicitly, as we will do in section 2.14.

In the language of section 2.14, there is a worldline instanton correction for each Poincaré irrep in the $d-1$ dimensional massive theory on $\mathbb{R}^{d-1}$. In a free theory, the Hilbert space furthermore has the structure of a Fock space, so that for any symmetry generator g , we have

$$\mathrm{Tr}_{\mathcal{H}}[g] = \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n}[g] = \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathrm{Sym}^n(\mathcal{H}_1)}[g] = \exp\left(\sum_{\ell=1}^{\infty} \frac{1}{\ell} \mathrm{Tr}_{\mathcal{H}_1}[g^{\ell}]\right), \quad (2.208)$$

where $\mathcal{H}_1$ is the single-particle Hilbert space.

We can apply this result to a free massive theory on S^{d-1} as follows. Consider the sphere S^{d-1} in “angular” quantization, where we choose an angle $\phi \in [0, 2\pi)$ and slice the path integral on slices of constant ϕ . Each spatial slice is a $d-2$ -dimensional hemisphere, with some boundary conditions at the equator of the hemisphere. (Our

arguments will be heuristic because we will be vague about the nature of these boundary conditions, see [AJ22] for a recent discussion.) Let the corresponding Hamiltonian be H . The sphere partition function is then

$$\mathrm{Tr}_{\mathcal{H}}(e^{-2\pi H}) = \exp\left(\sum_{\ell=1}^{\infty} \frac{1}{\ell} \mathrm{Tr}_{\mathcal{H}_1}[e^{-2\pi\ell H}]\right). \quad (2.209)$$

Each single-particle worldline instanton discussed in section 2.14 computes the leading contribution to $\mathrm{Tr}_{\mathcal{H}_1}[e^{-2\pi H}]$. To instead compute $\mathrm{Tr}_{\mathcal{H}_1}[e^{-2\pi\ell H}]$, we should expand around a classical trajectory $\theta(\tau) = 2\pi\ell\tau$ that winds ℓ times around the great circle. Expanding the worldline path integral around this trajectory, we find:

$$\mathrm{Tr}_{\mathcal{H}_1}[e^{-2\pi\ell H}] = \sum_m e^{-2\pi\ell m} \frac{(\pm i)^{(d-2)(2\ell-1)} m^{d-2}}{\Gamma(d-1)} \left(1 + O\left(\frac{1}{m}\right)\right). \quad (2.210)$$

For a free scalar CFT on $S^{d-1} \times S^1_{\beta}$, the single-particle states are KK modes, so that we expect

$$\log(Z_{\mathrm{free}}[S^{d-1} \times S^1_{\beta}]) \sim \sum_m \sum_{\ell=1}^{\infty} \frac{1}{\ell} \mathrm{Tr}_{\mathcal{H}_1} e^{-2\pi\ell H}, \quad (2.211)$$

where each $\mathrm{Tr}_{\mathcal{H}_1} e^{-2\pi\ell H}$ is given by (2.210), and m runs over the spectrum of KK masses. Here, “ $\sim$ ” means that the right-hand side displays the nonperturbative terms in the left-hand side.

We can derive similar results for a free massive theory on the lens space $L(q; 1)$. If we consider an instanton on the locus $|z_1| = 1$, wrapping ℓ times around the “short geodesic,” then the lens space worldline path integral evaluates $\mathrm{Tr}_{\mathcal{H}_1}[e^{-\frac{2\pi\ell}{q}H - \frac{2\pi\ell i}{q}J}]$, where J generates a rotation in the z_2 plane. Hence for the free scalar on $L(q; 1) \times S^1_{\beta}$, when q is odd:

$$\begin{aligned} \log(Z_{\mathrm{free}}[L(q; 1) \times S^1_{\beta}]) &= \sum_m \sum_{\ell=1}^{\infty} \frac{1}{\ell} \mathrm{Tr}_{\mathcal{H}_1}[e^{-\frac{2\pi\ell}{q}H - \frac{2\pi\ell i}{q}J}] \\ &= \sum_m \sum_{\ell=1}^{\infty} \frac{1}{q\ell} \mathrm{Tr}_{\mathcal{H}_1}[e^{-2\pi\ell H}] + \sum_m \sum_{\ell \notin q\mathbb{Z}} \frac{1}{\ell} \mathrm{Tr}_{\mathcal{H}_1}[e^{-\frac{2\pi\ell}{q}H - \frac{2\pi\ell i}{q}J}]. \end{aligned} \quad (2.212)$$

In the last line, we singled out terms where $\ell \in q\mathbb{Z}$. Up to a $1/q$ factor, the first sum is exactly the same as the nonperturbative terms in $\log(Z_{\text{free}}[S^{d-1} \times S^1_\beta])$. Indeed, since the short geodesic generates $\pi_1(L(q; 1)) \simeq \mathbb{Z}_q$, after winding around q times, the loop becomes contractable. Notice that each contractable long geodesic in $L(q; 1)$ will be lifted to q different geodesics in S^3 , so the moduli space of a long geodesic in $L(q; 1)$ should have $1/q$ -th the volume of $\text{Gr}(2, \mathbb{R}^4)$ and this is the origin of the $1/q$ factor.

Combining everything, the leading order terms are:

$$\log(Z_{\text{free}}[L(q; 1) \times S^1_\beta]) \sim \sum_m \left(-\frac{1}{q} \sum_{\ell=1}^{\infty} e^{-2\pi\ell m} \frac{m^2}{2\ell} + \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} e^{-\frac{2\pi\ell m}{q}} \frac{m}{2\ell \sin(\frac{2\pi\ell}{q})} \right) \quad (\text{odd } q), \quad (2.213)$$

where “ $\sim$ ” means we show nonperturbative corrections. Similarly when q is even, we find

$$\begin{aligned} & \log(Z_{\text{free}}[L(q; 1) \times S^1_\beta]) \\ & \sim \sum_m \left(-\frac{1}{q} \sum_{\ell=1}^{\infty} e^{-2\pi\ell m} \frac{m^2}{2\ell} + \sum_{\substack{\ell=1 \\ \ell \notin (q/2)\mathbb{Z}}}^{\infty} e^{-\frac{2\pi\ell m}{q}} \frac{m}{2\ell \sin(\frac{2\pi\ell}{q})} + \frac{1}{q} \sum_{\ell=1}^{\infty} e^{-\pi(2\ell-1)m} \frac{m^2}{2\ell-1} \right) \quad (\text{even } q). \end{aligned} \quad (2.214)$$

Partition function from functional determinants

In this section, we use the methods of [Aso+13] to compute the partition function of free scalar theories by explicitly summing over the Laplacian spectrum. We will find agreement with (2.211), (2.213), and (2.214). In particular, this will verify the general predictions of section 2.14 for free theories.

Consider a massive scalar field on S^{d-1} with the action

$$S_{\text{free}} = \frac{1}{2} \int d^{d-1}x \sqrt{g} \left[g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + m^2 \phi^2 + \frac{1}{4} \frac{(d-2)}{(d-1)} R \phi^2 \right], \quad (2.215)$$

with $R = (d-1)(d-2)$. Here, we have chosen the curvature coupling $\frac{1}{2}\xi R\phi^2$ to have the form appropriate for a conformally-coupled scalar in d -dimensions, dimensionally

reduced to $d-1$ dimensions. However, this coupling will not affect the leading form of nonperturbative corrections that are our main focus. The partition function is the functional determinant

$$Z_{\text{massive}}[S^{d-1}] = \det \left[\Delta + m^2 + \frac{(d-2)^2}{4} \right]^{-1/2}, \quad (2.216)$$

where Δ is the Laplace operator on S^{d-1} , which has eigenvalues $\lambda_k = k(k+d-2)$ with degeneracies $d_k = \frac{(d-3+k)!(d-2+2k)}{k!(d-2)!}$. Let us focus on $d=4$ and $d=6$ i.e the sphere partition function of a massive theory on S^3 and S^5 .

Using zeta function regularization, we find

$$\log(Z_{\text{massive}}[S^3]) = \frac{\pi m^3}{6} - \sum_{\ell} \frac{e^{-2\pi\ell m}}{2\ell} \left(m^2 + \frac{m}{\pi\ell^2} + \frac{1}{2\pi^2\ell^3} \right), \quad (2.217)$$

$$\begin{aligned} \log(Z_{\text{massive}}[S^5]) = & -\frac{\pi}{120}m^5 - \frac{\pi}{72}m^3 + \sum_{\ell=1}^{\infty} \frac{e^{-2\pi\ell m}}{24\ell} \left(m^4 + \frac{2m^3}{\pi\ell} + m^2 \left(1 + \frac{3}{\pi^2\ell^2} \right) \right. \\ & \left. + m \left(\frac{1}{\pi\ell} + \frac{3}{\pi^2\ell^2} + \frac{3}{2\pi^4\ell^4} + \frac{1}{2\pi^2\ell^2} \right) \right). \end{aligned} \quad (2.218)$$

We now need to sum over the KK masses to obtain $Z_{\text{free}}[S^{d-1} \times S^1_{\beta}]$. Let us consider $\mathbb{Z}_2$ -twisted free scalar fields as an example, where the $\mathbb{Z}_2$ twist is introduced to remove a zero mode upon dimensional reduction. The KK mass spectrum is $m = |(2n-1)/\beta|$ for $n \in \mathbb{Z}$. Thus, the free energy in $d=4$ and $d=6$ is

$$\log(Z_{\text{free}}[S^3 \times S^1_{\beta}]) \sim - \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{2\pi^2(2n-1)\ell}{\beta}} \frac{1}{\ell} \left(\frac{(2n-1)^2\pi^2}{\beta^2} + \frac{2n-1}{\beta\ell^2} + \frac{1}{2\pi^2\ell^3} \right), \quad (2.219)$$

$$\begin{aligned} & \log(Z_{\text{free}}[S^5 \times S^1_{\beta}]) \\ & \sim \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{e^{-2\pi^2(2n-1)\ell/\beta}}{12\ell} \left(\frac{\pi^4(2n-1)^4}{\beta^4} + \frac{2\pi^2(2n-1)^3}{\ell\beta^3} \right. \\ & \quad \left. + \frac{\pi^2(2n-1)^2}{\beta^2} \left(1 + \frac{3}{\pi^2\ell^2} \right) + \frac{\pi(2n-1)}{\beta} \left(\frac{1}{\pi\ell} + \frac{3}{\pi^2\ell^2} + \frac{3}{2\pi^4\ell^4} + \frac{1}{2\pi^2\ell^2} \right) \right). \end{aligned} \quad (2.220)$$

where “ $\sim$ ” means we only show non-perturbative corrections. These results are in agreement with the general form (2.211) predicted from worldline instantons, together with the Fock-space structure of the free theory.

The same approach can be generalized to compute a lens space partition function in 4d. We display the result here for a more direct comparison with the twisted partition function:

$$\begin{aligned} \log(Z_{\text{massive}}[L(q; 1)]) &= \frac{\pi m^3}{6q} - \frac{1}{q} \sum_{\ell}^{\infty} \frac{e^{-2\pi\ell m}}{2\ell} \left(m^2 + \frac{m}{\pi\ell} + \frac{1}{2\pi^2\ell^2} \right) \\ &\quad + \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} \frac{e^{-\frac{2\pi\ell m}{q}}}{\sin\left(\frac{2\pi\ell}{q}\right)} \left(\frac{m}{2\ell} + \frac{q}{4\pi\ell^2} \right) \quad (q \in 2\mathbb{Z} + 1), \end{aligned} \quad (2.221)$$

$$\begin{aligned} \log(Z_{\text{massive}}[L(q; 1)]) &= \frac{\pi m^3}{6q} - \frac{1}{q} \sum_{\ell}^{\infty} \frac{e^{-2\pi\ell m}}{2\ell} \left(m^2 + \frac{m}{\pi\ell} + \frac{1}{2\pi^2\ell^2} \right) \\ &\quad + \frac{1}{q} \sum_{\ell}^{\infty} \frac{e^{-\pi(2\ell-1)m}}{2\ell-1} \left(m^2 + \frac{2m}{\pi(2\ell-1)} + \frac{2}{\pi^2(2\ell-1)^2} \right) \\ &\quad + \sum_{\substack{\ell=1 \\ \ell \notin \frac{q}{2}\mathbb{Z}}}^{\infty} \frac{e^{-\frac{2\pi\ell m}{q}}}{\sin\left(\frac{2\pi\ell}{q}\right)} \left(\frac{m}{2\ell} + \frac{q}{4\pi\ell^2} \right) \quad (q \in 2\mathbb{Z}). \end{aligned} \quad (2.222)$$

In appendix 2.12, we computed the high temperature expansion of a free scalar twisted by $e^{-\frac{2\pi i}{q} J_{12} - \frac{2\pi i}{q} J_{34}}$. Restricting to odd q and inserting a $\mathbb{Z}_2$ twist to remove the zero mode, we find

$$\begin{aligned} &\log \left(\text{Tr} \left[e^{-\beta H - \frac{2\pi i}{q} J_{12} - \frac{2\pi i}{q} J_{34}} (-1)^N \right] \right) \\ &\sim - \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} e^{-\frac{2\ell(2n-1)\pi^2}{q\beta}} \left(\frac{(2n-1)^2\pi^2}{\ell q^3\beta^2} + \frac{2n-1}{\beta\ell^2 q^2} + \frac{1}{2\ell^3 q\pi^2} \right) \\ &\quad + \sum_{\substack{\ell=1 \\ \ell \notin q\mathbb{Z}}}^{\infty} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{2\pi\ell}{q}\left(\frac{q-1}{2} - n\right)\right)}{\sin\left(\frac{2\pi\ell}{q}\right)} e^{-\frac{2\ell(2n+1)\pi^2}{q^2\beta}} \left(\frac{(2n+1)\pi}{\ell q\beta} + \frac{q}{2\ell^2\pi} \right), \end{aligned} \quad (2.223)$$

where “ $\sim$ ” means we only show non-perturbative corrections and N is the ϕ -number operator, so that $(-1)^N$ implements the $\mathbb{Z}_2$ twist. Note that this is not quite identical to the lens space partition function (2.221) summed over the mass spectrum $m =$

$\frac{|2n+1|\pi}{q\beta}$. To obtain the twisted partition function (2.223), we must additionally modify the lens space result to account for the nontrivial background gauge field. To do so, we multiply each term in the summation over short geodesics by a phase $\exp(\frac{2\pi i \ell q_i}{q})$. For a $\mathbb{Z}_2$ twisted free scalar field, the KK charge spectrum is $\mathbf{q} = n + \frac{1}{2} + \frac{q}{2}$. After putting in all the phases, we recover (2.223). Finally, note that this result is consistent with the general prediction (2.146) from worldline instantons.

Free theories in odd dimension

The examples presented in section 2.14 were all in even d . In odd d , we face the puzzle that the contribution of a worldline instanton (2.144) is imaginary (even when the partition function should be real), and furthermore its phase depends on how we choose the integration contours for tachyonic modes. The proper interpretation of these kinds of contributions is explained for example in [Don+21]. They can be understood as characterizing singularities in the Borel plane when computing the partition function via Borel resummation.

In this appendix, we provide a quick summary of the discussion from [Don+21] in the example of a massive free scalar on S^2 . (We can think of this theory as the contribution of a single KK mode to the partition function of the 3d free scalar on $S^2 \times S^1_\beta$.) We can compute the partition function in terms of the heat trace, following appendix 2.12. The heat trace on S^2 is:

$$\begin{aligned} \text{Tr} \left[e^{-t(\Delta + (\frac{d-2}{2})^2)} \right] &= \sum_{k=0}^{\infty} (2k+1) e^{-t(k+\frac{1}{2})^2} = \sum_{k \in \mathbb{Z}} |k + \frac{1}{2}| e^{-t(k+\frac{1}{2})^2} \\ &= \sum_{\ell \in \mathbb{Z}} \left(\frac{1}{t} - \frac{2\pi \ell}{t^{3/2}} F\left(\frac{\pi \ell}{\sqrt{t}}\right) \right), \end{aligned} \quad (2.224)$$

where $F(z)$ is Dawson's function which admits the following asymptotic expansion near $z = \infty$:

$$F(z) \sim \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^{n+1}} \left(\frac{1}{z}\right)^{2n+1}. \quad (2.225)$$

The heat trace therefore has the following expansion near $t = 0$:

$$\mathrm{Tr} \left[e^{-t(\Delta + (\frac{d-2}{2})^2)} \right] \sim \sum_{n=0}^{\infty} a_n t^{n-1}, \quad a_n = \frac{(-1)^{n+1}(1 - 2^{1-2n})}{n!} B_{2n}, \quad (2.226)$$

where B_{2n} are the Bernoulli numbers.

Let us now perform a Borel resummation of the series $\Phi \equiv \sum_{n=0}^{\infty} a_n t^n$:

$$\mathcal{S}[\Phi](t) = \frac{2}{\sqrt{t}} \int_0^{\infty} d\zeta e^{-\zeta^2/t} \mathcal{B}\Phi(\zeta), \quad (2.227)$$

where the Borel transformed series $\mathcal{B}\Phi(\zeta)$ is:

$$\mathcal{B}\Phi(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n + \frac{1}{2})} \zeta^{2n} = \frac{1}{\sqrt{\pi}} \frac{\zeta}{\sin(\zeta)}. \quad (2.228)$$

With this expression for $\mathcal{B}\Phi$, the integral in (2.227) is apparently ill-defined since there are poles located at $\zeta = \ell\pi$, $\ell \in \mathbb{Z}$ on the positive real axis. A contour prescription is needed to define the integral.

One natural option is to deform the integration contour to pass over (we will refer to the corresponding contour as $\mathcal{C}_+$) or under (we will refer to the corresponding contour as $\mathcal{C}_-$) the poles. However, these two choices are not equivalent. Their difference is the sum of residues at the poles:

$$\frac{2}{\sqrt{t}} \int_{\mathcal{C}_+ - \mathcal{C}_-} d\zeta e^{-\zeta^2/t} \mathcal{B}\Phi(\zeta) = 2it \left(\frac{\pi}{t} \right)^{3/2} \sum_{\ell \neq 0} (-1)^{\ell+1} |\ell| e^{-\ell^2 \pi^2/t}. \quad (2.229)$$

The ambiguity in integration contour leads to an ambiguity in the Borel-resummed heat trace:

$$\mathrm{Tr} \left[e^{-t(\Delta + (\frac{d-2}{2})^2)} \right] = \frac{2}{\sqrt{\pi} t^{3/2}} \int_{\mathcal{C}_{\pm}} d\zeta \frac{\zeta e^{-\zeta^2/t}}{\sin \zeta} + 2i \left(\frac{\pi}{t} \right)^{3/2} \sum_{\ell \neq 0} \sigma_{\ell}^{\pm} (-1)^{\ell+1} |\ell| e^{-\frac{\ell^2 \pi^2}{t}}, \quad (2.230)$$

where $\sigma_{\ell}^{\pm}$ are arbitrary coefficients which jump as we switch from $\mathcal{C}_+$ to $\mathcal{C}_-$. If we require the heat trace to be real when $t \in \mathbb{R}_+$, then the $\sigma_{\ell}^{\pm}$ are fixed to be $\pm \frac{1}{2}$. This is equivalent to a principal value prescription for the Borel integral:

$$\mathrm{Tr} \left[e^{-t(\Delta + (\frac{d-2}{2})^2)} \right] = \frac{2}{\sqrt{\pi} t^{3/2}} \int_0^{\infty} d\zeta \mathcal{P} \left[\frac{\zeta e^{-\zeta^2/t}}{\sin \zeta} \right]. \quad (2.231)$$

To compute $\log Z$, we must supply a factor of e^{-tm^2} and integrate $\int dt/t$. If we start with (2.231), we get a valid integral representation for the partition function. However, if we start with (2.230) and perform the t -integral term by term, we obtain the series of nonperturbative corrections

$$\log(Z_{\text{free}}[S^2]) \Big|_{\text{non-perturbative}} = \pm i \sum_{\ell=1}^{\infty} \frac{(-1)^\ell e^{-2\pi m \ell}}{\ell} \left(m + \frac{1}{2\pi \ell} \right), \quad (2.232)$$

whose leading terms in m agree precisely with the worldline instanton predictions (2.210) and (2.211) when $d = 3$.

To summarize, worldline instantons encode residues of certain singularities in the Borel plane. We conjecture that this remains true in interacting theories. In general, the thermal effective action gives an asymptotic expansion in β . When we Borel-resum this expansion, we encounter singularities in the Borel plane coming from worldline instantons (together with other nonperturbative effects like instanton graphs).

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A GENUS-2 CROSSING EQUATION IN $d \geq 2$

[SX25] David Simmons-Duffin and Yixin Xu. *A genus-2 crossing equation in $d \geq 2$* . Manuscript draft, CALT-TH 2025-036. 2025.

3.1 Introduction

Requiring a CFT to be consistently defined on different geometries places strong constraints on its dynamical data. In 2d, modular invariance of the torus partition function is a well-studied example. In higher dimensions, the best-studied probe of these constraints is the four-punctured sphere. The analysis (both analytical and numerical) of four-point crossing has proved extremely fruitful. However, the information in a given finite set of four-point correlation functions may not provide a full picture of a theory. This observation has motivated the investigation of larger, more intricate systems of four-point correlators, see e.g. [Cha+25], as well as recent investigations of higher-point correlators [Bur+21a; Bur+22; Bur+21b; Ant+24; Kav+23; PPT23; PPT24; Har+24; AHK25; Har25; PPT25; Ant+21].

In this paper, we explore a crossing equation that encapsulates infinitely many four-point functions in a different way. Our main focus is the *genus-2 partition function*, which was explored from a bootstrap perspective in 2d CFTs in [KMZ18; CCY19; CMM17], and more recently in $d > 2$ in [Ben+24b]. In $d > 2$, we define the *genus-2 manifold* M_2 as the connected sum of two copies of $S^1 \times S^{d-1}$. The manifold M_2 can be built by gluing punctured spheres in multiple ways. Each gluing yields different decompositions for its partition function into sums over states. We will focus on two such decompositions: the sunrise channel, where we sum over OPE coefficients multiplied by the *genus-2 block* from [Ben+24b], and the dumbbell channel, where

we sum over squares of one-point functions on $S^1 \times S^{d-1}$. Schematically, equality between these two channels takes the form:

$$\sum_{123} c_{123}^2 B_{123} = \sum_{1'2'3'} c_{1'1'2'} c_{2'3'3'} B'_{1'2'3'}, \quad (3.1)$$

where B_{123} is the sunrise block, and $B'_{1'2'3'}$ is the dumbbell block. This equation can also be understood as a consequence of taking the usual four-point crossing equation and summing over external operators, with weights that depend on the moduli of M_2 , see figure 3.1.

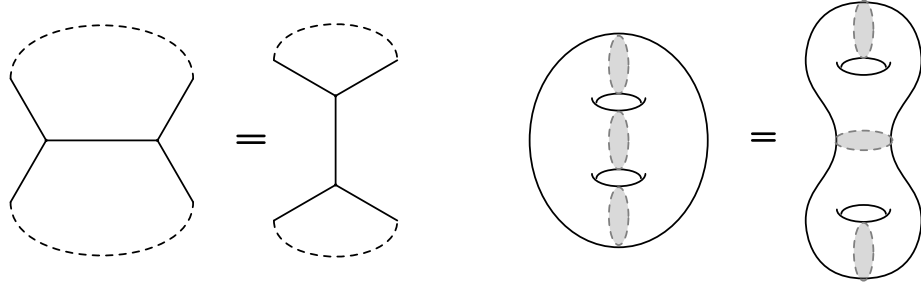


Figure 3.1: Left: the genus-2 crossing equation obtained by starting with the four-point crossing equation (solid lines), and contracting the external operators (indicated by dashed lines). The left-hand side of the resulting equation is the “sunrise” channel (since the diagram looks like a sunrise Feynman diagram), and the right-hand side is the “dumbbell” channel. Right: the resulting crossing equation in terms of geometry, drawn in 2d for simplicity. A resolution of unity is inserted at each gray circle, giving a sum over genus-2 blocks in the sunrise channel and a sum over squared torus one-point functions in the dumbbell channel.

In section 3.2, we review the moduli space of flat conformal structures on M_2 , introducing coordinates suited for each channel. The dimension of the moduli space is $(d+1)(d+2)/2$, much larger than that of the four-punctured sphere. This counting reflects the fact that even though the genus-2 crossing equation follows ultimately from the usual four-point crossing, it encodes more information than any finite set of four-point functions. After explaining the matching between dumbbell and sunrise coordinates, we comment on why the genus-2 crossing equation (3.1) ensures mapping class group invariance of the partition function in 3d.

To further understand the implications of the crossing equation (3.1), we restrict to an interesting one-dimensional locus within the aforementioned moduli space. Starting from section 3.3, we focus on the *thermal flat limit*, where the $S^1 \times S^{d-1}$'s blow up into copies of thermal flat space $S^1 \times \mathbb{R}^{d-1}$. The genus 2 manifold M_2 then becomes two copies of thermal flat space $S^1 \times \mathbb{R}^{d-1}$, connected by a cylinder, see figure 3.2. A special case within the thermal flat limit was investigated in [Ben+24b], producing an asymptotic expression for thermal one-point coefficients. Relatedly, in the crossing equation considered in this work, the following function naturally arises in the dumbbell channel decomposition:

$$h(z) = \sum_{\mathcal{O}} b_{\mathcal{O}}^2 q_J z^{\Delta_{\mathcal{O}}}, \quad (3.2)$$

where z is the coordinate along the thermal flat locus, q_J is a J -dependent constant, and $b_{\mathcal{O}}$ is the thermal one-point function coefficient of $\mathcal{O}$, see e.g. [Ili+18]. This function simultaneously encodes all the thermal one-point functions of a CFT, and may have interesting applications to the thermal bootstrap.

As a concrete application of the thermal flat limit of the genus 2 crossing equation, in section 3.4 we derive an asymptotic relation between (averaged) heavy-heavy-heavy OPE coefficients and thermal one-point coefficients. Analogous relations between thermal one-point data and heavy-heavy-light OPE coefficients $c_{\phi\mathcal{O}\mathcal{O}}$ are known [Gob+19; Bur+25], obtained by inverting the block decomposition of one-point functions on $S^1 \times S^{d-1}$ [Bur+24]. The OPE asymptotics we present in this work, however, hold in a qualitatively different regime. For example, for scalars in 3d, we find that, schematically:

$$\rho_1 \rho_2 \rho_3 (c_{123})^2 \sim h(z_*) \exp \left(3(f\pi)^{1/3} (\Delta_1^{2/3} + \Delta_2^{2/3}) - 16\pi c_1 \right) \times \text{“one-loop factors”}, \quad (3.3)$$

where ρ_i are the density of states for representations $\pi_i = (\Delta_i, \lambda_i)$, h is given in (3.2),

$$z_* = \left(\frac{\Delta_1^2 \Delta_2^2}{\pi f \Delta_3^3} \right)^{\frac{2}{3}}, \quad (3.4)$$

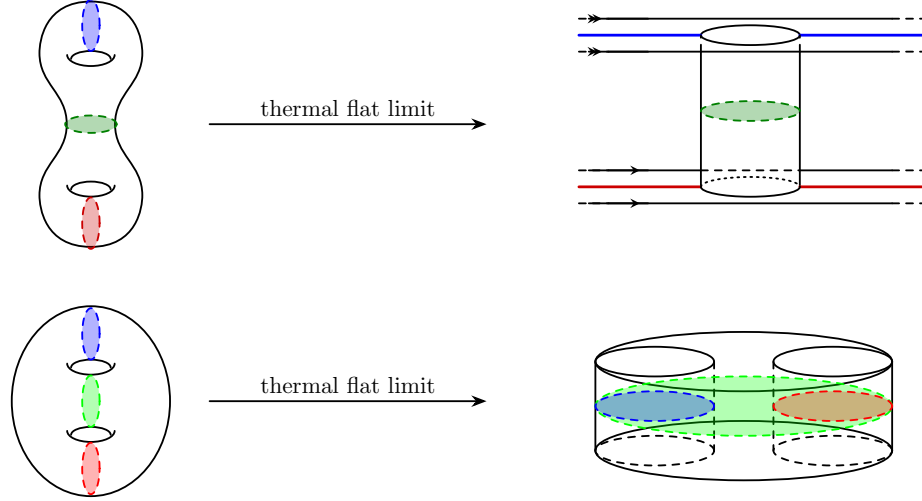


Figure 3.2: Cartoon illustrating the thermal flat limit of the genus-2 manifold with colored circles and lines representing different d -dimensional balls. In the upper graph, the two strips in the right diagram represent two copies of $S^1 \times \mathbb{R}^{d-1}$ where the lines in the front and back are identified as suggested by the arrows. The lower graph represents the dumbbell channel geometry. In the thermal flat limit, the two smaller balls are tangential to the larger one but they are not tangent to each other.

(The “one-loop factors” are also important at large Δ , but we have not included them in (3.3) for brevity. The precise formula is given below in (3.142).) The quantities f and c_1 are theory-dependent Wilson coefficients in the thermal effective action [Bha+08; Ban+12; KLO23; Ben+24b; Ben+24a; Jen+12]. The formula (3.3) holds for large $\Delta_1, \Delta_2, \Delta_3$, with z_* held fixed — in other words, for Δ_3 parametrically larger than Δ_1 and Δ_2 . (This hierarchy reflects an asymmetry among the three cutting spheres in the sunrise channel in the thermal flat limit.) We sometimes refer to this regime as the “heavy-heavy-heavier” limit. A similar result holds for spinning operators, as long as their spins are held fixed as $\Delta_i \rightarrow \infty$, see (3.142) below. Unlike the OPE coefficient asymptotics in [Ben+24b], our results do not depend on the “hot spot conjecture” of [Ben+24b], although they are fully consistent with it.

Equation (3.3) follows from taking an inverse Laplace transform of the genus-2 partition function in the thermal flat limit. A crucial step in this calculation is to determine the quantum numbers dominating the sunrise channel decomposition. This can be

done by computing expectation values of various Casimir operators, as we explain below. The expectation values of Casimirs tell us which scaling limit to take of the genus-2 block. We can then compute the genus-2 block in this scaling limit using the shadow formalism and saddle point approximations. The integral of interest has multiple saddle points, corresponding to the genus-2 block and seven different shadow blocks. The identification of the correct saddle point can be non-trivial — for example in [Ben+24b] it was done by numerically following the correct low-temperature saddle to a high temperature limit. Here, we find a more direct method to identify the correct saddle using expectation values of conformal generators.

Finally, we devote space in appendices to further discussion of genus 2 blocks. Similarly to four-point conformal blocks, the genus-2 blocks satisfy a set of Dolan-Osborn-like Casimir equations [DO01; DO04; DO11]. We derive these equations for genus-2 sunrise and dumbbell blocks in appendix 3.7.

3.2 The genus-2 geometry

In this section, we review the parameterization of flat conformal structures on M_2 . We introduce two sets of coordinates: dumbbell coordinates and sunrise coordinates, and discuss the translation between them. After writing down the genus-2 crossing equation, we comment on why mapping class group invariance in 3d is trivialized by this crossing equation.

Genus-2 geometry in the sunrise channel

The genus-2 manifold M_2 can be built by taking two copies of the plane $\mathbb{R}^d$, drilling out three balls (B_1, B_2, B_3) and (B'_1, B'_2, B'_3) from each plane and connecting their boundaries by three cylinders C_1, C_2, C_3 . Using the Weyl-equivalence between a cylinder and an annulus, one can map each cylinder C_i either to the interior of ∂B_i or the interior of $\partial B'_i$, introducing two sets of coordinates x and x' inside C_i . A flat conformal structure on M_2 can be specified by giving transition maps between these coordinates

in each cylinder:

$$x = g_i x', \quad g_i \in G^-, \quad (\text{inside } C_i), \quad (3.5)$$

where G^- denotes the orientation reversing¹ component of $O(d+1, 1)$. Since conformal transformations performed on each plane $\mathbb{R}^d$ alter the coordinates x, x' without changing the conformal structure of M_2 , the moduli (covering)² space of the flat conformal structure is given by the following double quotient:

$$\mathcal{M} = G \backslash (G^-)^3 / G. \quad (3.6)$$

More concretely, let us enumerate the geometric parameters relevant to this construction and understand how they are related by conformal transformations. We have four types of parameters to start with:

- The positions of the punctures (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) on both planes.
- The radii of the six removed balls r_i and r'_i .
- The lengths β_i of the three cylinders.
- The angular twists h_i along the three cylinders.

By performing individual conformal transformations on each plane, we can bring the punctures to standard locations, say, $0, e, \infty$ on each plane, leaving an $\text{SO}(d-1) \times \text{SO}(d-1)$ subgroup that preserves this configuration. The radii of the removed balls can be converted into the length of the cylinders: to glue an S^{d-1} of radius r'_i with an S^{d-1} with radius r_i via a cylinder of length β_i , the transition map we write down is:

$$x' = e^{-(\beta_i - \log r_i - \log r'_i)D} I \cdot x. \quad (3.7)$$

¹Even though we are gluing by orientation-reversing conformal transformations, the manifold itself is orientable: along any closed loop on M_2 , an even number of orientation reversals are applied, creating no obstruction for the existence of a global orientation.

²The actual moduli space is the quotient of $\mathcal{M}$ by the action of the mapping class group.

This is equivalent to the map which glues two unit sphere with a cylinder of length $\beta_i - \log r_i - \log r'_i$. Intuitively, one can use the plane-cylinder map to “flatten” part of the cylinder onto the plane or “squeeze” part of the plane onto the cylinder. Therefore, the triplet (r_i, r'_i, β_i) contributes collectively as a single modulus. Because of the remaining stabilizer subgroup after fixing the centers of the removed balls, the angular twists (h_1, h_2, h_3) are subject to the following equivalence relation:

$$(h_1, h_2, h_3) \sim (kh_1k', kh_2k', kh_3k'), \quad k, k' \in \text{SO}(d-1). \quad (3.8)$$

Together, the number of independent moduli is:

$$3 + 3 \dim \text{SO}(d) - 2 \dim \text{SO}(d-1) = \frac{(d+1)(d+2)}{2}. \quad (3.9)$$

For doing calculations, it is often useful to choose a *conformal frame*, that is, to fix a subset of geometric parameters so that variations of the rest lead to a truly different flat conformal structure. However, completely fixing the conformal frame comes at the cost of breaking certain symmetries. Therefore, we will “partially gauge fix” and leave the $\text{SO}(d-1) \times \text{SO}(d-1)$ redundancy untouched.

One possible conformal frame for describing the sunrise channel is described in [Ben+24b]. That conformal frame captures the “high-temperature” limit described in that work, but it is not well-suited to the “thermal flat limit” that will be our eventual focus. Instead, we will work with the following frame:

$$\begin{aligned} x_1 &= (-a_1 - 1, 0, \dots, 0), & x_2 &= (a_2 + 1, 0, \dots, 0), & x_3 &= \infty, \\ r_1 &= r'_1 = 1 - a_1, & r_2 &= r'_2 = 1 - a_2, & r_3 &= r'_3 = 2, & \beta_1 &= \beta_2 = 0, \end{aligned} \quad (3.10)$$

where $0 < a_1 < 1, 0 < a_2 < 1$. The corresponding gluing maps are:

$$\begin{aligned} g_1 &= e^{-(1+a_1)P^1} e^{2 \log(1-a_1)D} h_1 I e^{(1+a_1)P^1}, \\ g_2 &= e^{(1+a_2)P^1} e^{2 \log(1-a_2)D} h_2 I e^{-(1+a_2)P^1}, \\ g_3 &= e^{(\beta_3 + 2 \log(2))D} h_3 I, \end{aligned} \quad (3.11)$$

and the angular twists h_i are given by:

$$h_i = \exp \left(\sum_a i \alpha_i^a M_{1,a} \right) \exp \left(\sum_{a,b=2}^d i \Phi_i^{a,b} M_{a,b} \right), \quad i = 1, 2, 3, \quad (3.12)$$

where $M_{i,j}$ are (hermitian) rotation generators. In this parameterization of the angular twists we’ve chosen to separate:

- The generators $M_{a,b}$ with $2 \leq a, b \leq d$ that generate the subgroup $\text{SO}(d-1)$ preserving the x^1 -axis an x^1 -preserving $\text{SO}(d-1)$, from
- The generators $M_{1,a}$ with $2 \leq a \leq d$ that transform in the vector representation under the $\text{SO}(d-1)$.

The essential difference between this frame and the one in [Ben+24b] is as follows. In [Ben+24b], the three spheres ∂B_i (and similarly $\partial B'_i$) were mutually tangent. This made it easy to probe the limit where three “hot spots” form by tuning the lengths of all three cylinders to zero. By contrast, the thermal flat limit includes only two hot spots. Our new conformal frame makes it easier to probe this regime by choosing ∂B_1 and ∂B_2 to be tangent to ∂B_3 , but not to each other (unless $a_1 = a_2 = 0$).

Note that our choice of conformal frame gives rise to a piecewise differentiable metric on M_2 , given by the flat metric on the planes $\mathbb{R}^d$ and the conventional metric on the cylinders C_i connecting them. We will refer to this as the “sunrise metric.”

Sunrise channel decomposition of the partition function

The above construction of M_2 leads to the decomposition of $Z(M_2)$ as a sum of squared OPE coefficients, weighted by a purely kinematic function — the genus-2 sunrise block:

$$\begin{aligned}
& Z(M_2)|_{\text{sunrise}} \\
&= \left(\frac{|Z_{\text{glue}}(2)|}{|Z_{\text{glue}}(1-a_1)||Z_{\text{glue}}(1-a_2)|} \right)^2 \times e^{-\varepsilon_0 \beta_3} \\
&\times \sum_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} \left(e^{-\beta_3 \Delta_3} \right. \\
&\quad \langle \mathcal{O}_1^{a'}(-(a_1+1)e) \mathcal{O}_2^{b'}((a_2+1)e) \mathcal{O}_3^{c'}(\infty e) \rangle^* \langle h_1 \cdot \mathcal{O}_1^a(-(a_1+1)e) h_2 \cdot \mathcal{O}_2^b((a_2+1)e) h_3 \cdot \mathcal{O}_3^c(\infty e) \rangle \\
&\quad \times \left({}_{1-a_1} \langle \mathcal{O}_1^a | \mathcal{O}_1^{a'} \rangle_{1-a_1} \right)^{-1} \left({}_{1-a_2} \langle \mathcal{O}_2^b | \mathcal{O}_2^{b'} \rangle_{1-a_2} \right)^{-1} \left({}_2 \langle \mathcal{O}_1^a | \mathcal{O}_1^{a'} \rangle_2 \right)^{-1} \\
&\quad \left. + \text{descendants} \right). \tag{3.13}
\end{aligned}$$

In odd dimensions, where there is no Weyl anomaly, $Z(M_2)$ will depend only in the flat conformal structure on M_2 . In the presence of a Weyl anomaly, there is some residual dependence on the metric. The expression (3.13) for $Z(M_2)$ holds in the sunrise metric described in section 3.2.

A detailed derivation of (3.13) can be found in [Ben+24b]. For now, we briefly unpack the notation:

- The inner product ${}_r \langle \mathcal{O} | \mathcal{O}' \rangle_r$ is taken over the Hilbert space associated with a sphere S_r^{d-1} with radius r . It is related the usual inner product over $\mathcal{H}_{S_1^{d-1}}$ by a numerical factor:

$${}_r \langle \mathcal{O} | \mathcal{O}' \rangle_r = r^{2\Delta_{\mathcal{O}}} \langle \mathcal{O} | \mathcal{O}' \rangle. \tag{3.14}$$

- The three-point functions can be further expanded with respect to a basis of conformally-invariant three-point structures:

$$\langle \mathcal{O}_1^{a'}(-(a_1+1)e) \mathcal{O}_2^{b'}((a_2+1)e) \mathcal{O}_3^{c'}(\infty e) \rangle = (a_1 + a_2 + 2)^{\Delta_3 - \Delta_1 - \Delta_2} c_{123}^s V^{s;abc}(0, e, \infty). \tag{3.15}$$

Separating the OPE coefficients from the kinematic structures as shown above, we can write each term in the summand as :

$$(c_{123}^{s'})^* c_{123}^s B_{123}^{s',s}, \quad (3.16)$$

where the function $B_{123}^{s',s}$ encapsulates the sum of contracted three-point structures over a triplet of conformal multiplets.

- Whenever a cylinder is glued onto a plane, the metric has delta-function-type curvature localized at the junction. To account for this curvature, a “junction factor” is associated with the spheres $\partial B_i, \partial B'_i$. The exact value of the junction factor depends on the radius of the sphere and the “type” of the junction. There are two types of junctions in the geometry discussed above: an open junction where the exterior of the sphere ∂B_i is filled and a closed junction where the interior of the sphere ∂B_i is filled. An open junction factor is associated with $\partial B_1, \partial B_2, \partial B'_1, \partial B'_2$ and a closed junction factor is associated with ∂B_3 and $\partial B'_3$.

To compute these junction factors, first note that the open junction factor should be the inverse of the closed function factor, since the plane is Weyl equivalent to an open junction placed infinitely close to a closed junction. This relation holds exactly even in the presence of a nontrivial Weyl anomaly, since the cylinder connecting an open and closed junction can be made infinitesimally short. One can then use the Weyl equivalence between a sphere S^d and a capped cylinder to determine the closed junction factor. We denote the closed junction factor by Z_{glue} and it is given by:

$$Z_{\text{glue}}(r) = Z(S^d)^{1/2} \times f_{\text{Weyl}}(r, \beta) \times e^{\varepsilon_0 \beta/2}, \quad (3.17)$$

where ε_0 is the Casimir energy in $\mathcal{H}_{S^{d-1}}$, β is the length of the capped cylinder and f_{Weyl} accounts for the Weyl anomaly associated with the transformation between a unit sphere and the capped cylinder. A detailed computation of f_{Weyl} can be found in [Ben+24b].

Our eventual focus will be on 3d, where Weyl factors and Casimir energies don't contribute, and the gluing factor is just $Z(S^d)^{1/2}$.

Genus-2 geometry in the dumbbell channel

In the dumbbell channel, we use a different cutting and gluing scheme to construct M_2 . We start by building the two “dumbbells” as mapping tori of conformal group elements $g_L, g_R \in \text{SO}(d+1, 1)$. We then remove a ball from each of the $S^1 \times S^{d-1}$'s and glue their boundary components via an orientation reversing conformal map $g_M \in G^-$ (“ M ” for “middle”). The conformal structure is determined by $(g_L, g_M, g_R) \in G \times G^- \times G$, up to the gauge redundancy:

$$(g_L, g_M, g_R) \sim (gg_L g^{-1}, gg_M g'^{-1}, g' g_R g'^{-1}), \quad g, g' \in \text{SO}(d+1, 1). \quad (3.18)$$

We choose the following representative geometry (see figure 3.3): take two pieces of $\mathbb{R}^d$ with flat coordinates x_L and x_R . We can build the left dumbbell A_L by gluing an S^{d-1} of radius $e^{-\beta_L/2r} r$ with another S^{d-1} with

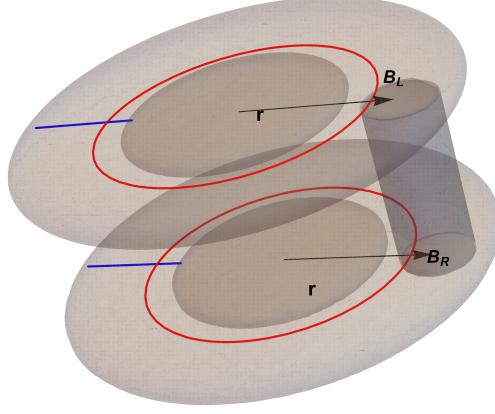


Figure 3.3: In the dumbbell channel, we obtain M_2 by removing two unit balls B_L and B_R from two pieces of $S^1 \times S^{d-1}$ and glue their boundaries with a cylinder of inverse temperature β . The two blue lines represent two non-contractable loops with length $2r \sinh(\beta_L/2r)$ and $2r \sinh(\beta_R/2r)$. In $d > 2$, the two red loops can be shrunk towards poles of S^{d-1} .

radius $e^{\beta_L/2r}r$ after applying an angular twist $h_L \in \text{SO}(d)$. The right dumbbell A_R can be built similarly via a gluing map $g_R \equiv e^{-(\beta_R/r_R)D}h_R$.

From the two annuli A_L and A_R , we remove unit balls B_L and B_R centered at $x_L = (-r, 0, \dots, 0)$ and $x_R = (-r, 0, \dots, 0)$, respectively³. Let x_M, x'_M be coordinate systems with origins at the centers of B_L, B_R and let ∂B_L and ∂B_R be located at $|x_M| = 1$ and $|x'_M| = 1$. Using Weyl equivalence between the cylinder and the annulus, we can extend the coordinates x_M and x'_M to cover a cylinder connecting ∂B_L and ∂B_R . At any point on such a cylinder, the transition map between x_M and x'_M is given by an orientation reversing conformal group element:

$$x_M = e^{-\beta D} h_M I x'_M, \quad h_M \in \text{SO}(d). \quad (3.19)$$

Combined with the transition map between x_L, x_M and x_R, x'_M :

$$x_M = e^{rP^1} x_L, \quad x'_M = e^{rP^1} x_R, \quad (3.20)$$

this leads to the transition map between x_L and x_R :

$$x_L = e^{-rP^1} e^{-\beta D} h_M I e^{rP^1} x_R. \quad (3.21)$$

In this conformal frame, the flat conformal structure on M_2 is therefore specified by the following gluing maps:

$$\begin{aligned} g_L &= e^{-(\beta_L/r)D} h_L, \\ g_M &= e^{-rP^1} e^{-\beta D} h_M I e^{rP^1}, \\ g_R &= e^{-(\beta_R/r)D} h_R, \end{aligned} \quad (3.22)$$

³For this operation to make sense geometrically, we need $\beta_L, \beta_R > 2r \log(\frac{r}{r-1})$.

subject to the equivalence relation:

$$(h_L, h_M, h_R) \sim (k_L h_L k_L^{-1}, k_L h_M k_R^{-1}, k_R h_R k_R^{-1}), \quad k_L, k_R \in \text{SO}(d-1), \quad (3.23)$$

and

$$(r, \beta) \sim (e^{-\delta} r, \beta + 2\delta), \quad \delta \in \mathbb{R}. \quad (3.24)$$

The number of independent parameters is again $(d+1)(d+2)/2$. One can in principle fix the gauge redundancy in (3.24) but we will keep r and β both as free parameters since it makes the definition of the thermal flat limit more convenient.

Dumbbell channel decomposition of the partition function

Now we are ready to derive the dumbbell channel decomposition of $Z(M_2)$ in detail. The basic idea is to insert resolutions of the identity at ∂B_L and ∂B_R . More precisely, we have:

$$\begin{aligned} & Z(M_2)|_{\text{dumbbell}} \\ &= \frac{1}{|Z_{\text{glue}}(1)|^2} \sum_{\mathcal{O}_L, \mathcal{O}_R} Z(A_L \setminus B_L)(|\mathcal{O}_L^a\rangle) \langle \mathcal{O}_L^a | \mathcal{O}_L^{a'} \rangle^{-1} \langle \mathcal{O}_L^{a'} | e^{-\beta D} h_M | \mathcal{O}_R^{b'} \rangle \langle \mathcal{O}_R^b | \mathcal{O}_R^{b'} \rangle^{-1} (Z(A_R \setminus B_R)(|\mathcal{O}_R^b\rangle))^* \\ &+ \text{descendants}, \end{aligned} \quad (3.25)$$

In the equation above, the indices a, a', b, b' are implicitly summed over. $\langle \mathcal{O}_L^a | \mathcal{O}_L^{a'} \rangle$ is the inner product in $\mathcal{H}_{S_{r=1}^{d-1}}$ and it is equivalently computed by the following two point function:

$$\langle \mathcal{O}_L^a(0) [\mathcal{O}_L^{a'}(0)]^\dagger \rangle, \quad (3.26)$$

where $[\dots]^\dagger$ denotes BPZ conjugation. Since $\partial(A_L \setminus B_L) = -\partial B_L$, we can view $Z(A_L \setminus B_L)$ as a map $\mathcal{H}_{S_{r=1}^{d-1}} \rightarrow \mathbb{C}$. When acting on $|\mathcal{O}^a\rangle$, it evaluates an (unnormalized) one-point function on A_L with $\mathcal{O}^a$ inserted at the center of B_L :

$$Z(A_L \setminus B_L)(|\mathcal{O}^a\rangle) = \langle \mathcal{O}^a(r, \hat{n}) \rangle_{A_L} \times \langle 1 \rangle_{A_L}. \quad (3.27)$$

Using Weyl equivalence between an annulus and the cylinder, we have:

$$\langle 1 \rangle_{A_L} = Z(S_{\beta_L}^1 \times S_r^{d-1}), \quad \langle \mathcal{O}^a(r, \hat{n}) \rangle_{A_L} = e^{-\beta_L \Delta_{\mathcal{O}}/2r} \langle \mathcal{O}^a(\beta_L/2, \hat{n}) \rangle_{S_{\beta_L}^1 \times S_r^{d-1}}. \quad (3.28)$$

Combining everything, we get:

$$\begin{aligned} & Z(M_2)|_{\text{dumbbell}} \\ &= \frac{1}{|Z_{\text{glue}}(1)|^2} Z(S_{\beta_L}^1 \times S_r^{d-1}) Z(S_{\beta_R}^1 \times S_r^{d-1}) \\ &\quad \times e^{-\beta \varepsilon_0} \sum_{\mathcal{O}} \left(\langle \mathcal{O}^a(\beta_L/2, \hat{n}) \rangle_{S_{\beta_L}^1 \times S_r^{d-1}} \langle (h_M \cdot \mathcal{O})^{a'}(\beta_R/2, \hat{n}) \rangle_{S_{\beta_R}^1 \times S_r^{d-1}} \langle \mathcal{O}^a | \mathcal{O}^{a'} \rangle^{-1} e^{-(\beta + \frac{\beta_L + \beta_R}{2r}) \Delta_{\mathcal{O}}} \right. \\ &\quad \left. + \text{descendants} \right), \end{aligned} \quad (3.29)$$

where ε_0 is the Casimir energy on S^{d-1} and for an operator in the conformal representation (Δ, λ) , h_M acts by:

$$h_M \cdot \mathcal{O}^a = \lambda(h_M^{-1})^a_b \mathcal{O}^b. \quad (3.30)$$

This expression for $Z(M_2)$ is valid in the metric associated to the geometry described in section 3.2.

Let us review some basic features of the thermal one-point function on S^{d-1} [Bur+24], which appears in (3.29). When the angular twists h_R and h_L are turned off, any correlation function on $S_\beta^1 \times S_r^{d-1}$ must be invariant under the $SO(2) \times SO(d)$ isometry group of the background geometry. Therefore, the one-point function $\langle \mathcal{O}^a(\tau, \hat{n}) \rangle_{S_\beta^1 \times S_r^{d-1}}$ cannot have τ or $\hat{n}$ dependence. In this case, one-point functions of descendants vanish, and the sum (3.29) truncates to only the primary contributions. Furthermore, only even-spin symmetric traceless tensors can have non-vanishing one-point functions. Their form is fixed by symmetry to be

$$\langle \mathcal{O}^{\mu_1 \dots \mu_\ell}(\tau, \hat{n}) \rangle_{S_\beta^1 \times S_r^{d-1}} = \frac{b_{\mathcal{O}}}{\beta \Delta_{\mathcal{O}}} \times f_{\mathcal{O}}(\beta/r) (e^{\mu_1} \dots e^{\mu_\ell} - \text{traces}), \quad (3.31)$$

where e^μ denotes the unit vector along the τ -direction. The function $f_{\mathcal{O}}(\beta/r)$ satisfies the boundary condition $f_{\mathcal{O}}(0) = 1$. Odd-spin one-point functions are forbidden by a π rotation that fixes the operator insertion, flips the direction of S^1 , and reflects the S^{d-1} .

When angular fugacities are turned on, $SO(d)$ invariance is generically broken to the Cartan subgroup, allowing new coordinates to enter the one-point function. For example, when $d = 3$, in the presence of an angular twist $e^{i\theta M_{12}}$, a one-point function can also depend on $\hat{n}_1^2 + \hat{n}_2^2 = 1 - \hat{n}_3^2$. In this case, descendant operators can have non-trivial one-point functions. For more discussion in 3d, see [Bur+24].

The large- r asymptotics of the partition functions $Z(S_{\beta_L}^1 \times S_r^{d-1})$ and $Z(S_{\beta_R}^1 \times S_r^{d-1})$ are captured by the thermal effective action [Ben+24b]. At leading order:

$$Z(\beta_L) \sim \exp \left(\frac{f r^{d-1} \text{vol} S^{d-1}}{\beta_L^{d-1} \prod_{i=1}^n (1 + \Omega_i^2)} \right), \quad Z(\beta_R) \sim \exp \left(\frac{f r^{d-1} \text{vol} S^{d-1}}{\beta_R^{d-1} \prod_{i=1}^n (1 + \Omega_i^2)} \right), \quad (3.32)$$

where $n = \lfloor d/2 \rfloor$ is the rank of $SO(d)$ and $\beta_L \Omega_i / r$ and $\beta_R \Omega_i / r$ are angular fugacities in h_L and h_R , respectively. Here, f is a theory-dependent coefficient, and is proportional to the central charge in 2d. In general dimensions, f is proportional to the free energy density of the CFT at finite temperature, the Casimir energy of the CFT on $S_\beta^1 \times \mathbb{R}^{d-2}$ and the thermal one-point function of the stress energy tensor on $S^1 \times \mathbb{R}^{d-1}$.

Matching between dumbbell and sunrise channels

Now that we've introduced two sets of coordinates on the moduli space of flat conformal structures $\mathcal{M}$, let us discuss how to match them. The basic idea is the following: parameterizations of flat conformal structures in both channels give two homomorphisms from the fundamental group of M_2 to the conformal group $\varphi_D, \varphi_S : \pi_1(M_2) \rightarrow G$. For example, in the sunrise channel, the elements a, b illustrated in figure 3.4 are mapped to

$$\varphi_S(a) = g_1^{-1} g_3, \quad \varphi_S(b) = g_3^{-1} g_2, \quad (3.33)$$

while in the dumbbell channel:

$$\varphi_D(a) = g_M^{-1} g_L^{-1} g_M, \quad \varphi_D(b) = g_R. \quad (3.34)$$

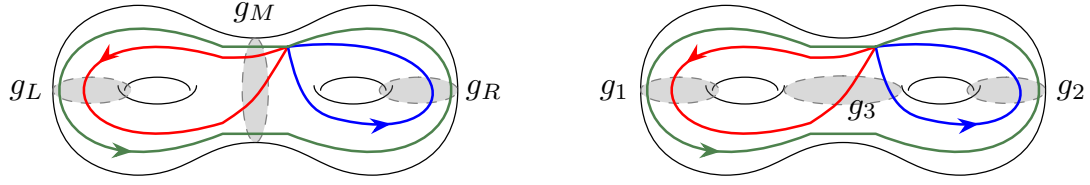


Figure 3.4: The correspondence of loops and conformal group elements in different channels. In the sunrise channel (right) the a -loop (red) is mapped to the group element $g_1^{-1}g_3$, the loop b (blue) is mapped to $g_3^{-1}g_2$ and the loop $a \cdot b$ (green) is mapped to $g_1^{-1}g_2$. In the dumbbell channel (left), the loop a is mapped to $g_M^{-1}g_L^{-1}g_M$, the loop b is mapped to g_R and the loop $a \cdot b$ is mapped to $g_M^{-1}g_L^{-1}g_Mg_R$.

If (g_L, g_M, g_R) and (g_1, g_2, g_3) correspond to the same conformal structure, then for any element $\ell \in \pi_1(M_2) \simeq \mathbb{Z} * \mathbb{Z}$, the group element $\varphi_D(\ell)$ should be conjugate to $\varphi_S(\ell)$ in G . In practice, we will choose a set of loops $\{\ell_i\}$ and a particular representation ρ of the conformal group G , then match the dumbbell coordinates and the sunrise coordinates via the identity:

$$\chi_\rho(\varphi_D(\ell_i)) = \chi_\rho(\varphi_S(\ell_i)), \quad (3.35)$$

where $\chi_\rho(g)$ denotes the character of g in the representation ρ .

As a simple example, let us look at the case $d = 1$. (The genus-2 manifold M_2 does not make sense in this case, but the essential group theory still works. The computations below also apply when the angular twists are turned off, so that all gluing group elements lie in an $SO(1, 2)$ subgroup.) Let us choose ρ to be the vector representation of $SO(1, 2)$ and $\{\ell_1, \ell_2, \ell_3\} = \{a, b, a \cdot b\}$. We can match the three dumbbell coordinates β_L, β_R, β with the three sunrise coordinates by solving the following equations:

$$\begin{aligned} \frac{a_1^2 (4e^{-\beta_3} - 1) - 6a_1 + 4e^{\beta_3} - 1}{(a_1 - 1)^2} &= e^{-\beta_L/r} + e^{\beta_L/r} + 1, \\ \frac{a_2^2 (4e^{-\beta_3} - 1) - 6a_2 + 4e^{\beta_3} - 1}{(a_2 - 1)^2} &= e^{-\beta_R/r} + e^{\beta_R/r} + 1, \\ \frac{(7a_2 + a_1(a_2 + 7) + 1)(5a_2 + a_1(3a_2 + 5) + 3)}{(a_1 - 1)^2(a_2 - 1)^2} &= e^{-\frac{\beta_L + \beta_R}{r}} \left[e^{2\beta} r^4 (e^{\beta_L/r} - 1)^2 (e^{\beta_R/r} - 1)^2 \right. \\ &\quad \left. - 2e^\beta r^2 (e^{\beta_L/r} - 1)(e^{\beta_R/r} - 1) \left(e^{\frac{(\beta_L + \beta_R)}{r}} + 1 \right) + e^{\frac{\beta_L + \beta_R}{r}} + e^{\frac{2(\beta_L + \beta_R)}{r}} + 1 \right]. \end{aligned} \quad (3.36)$$

We will come back to these equations later and solve them in the large- r limit. A priori, it might seem like this matching method can provide infinitely many “loop relations.” However, only finitely many loops give rise to independent identities. For example, when $d = 1$ and $\dim \rho = 3$, the identity coming from a^2 is redundant since:

$$\chi_\rho(\varphi_{S,D}(a^2)) = \chi_\rho(\varphi_{S,D}(a))^2 - 2\chi_\rho(\varphi_{S,D}(a)). \quad (3.37)$$

When working with vector representations of $SO(d+1, 1)$ in higher dimension, we find the following set of loops to give a sufficient set of independent relations:

$$\begin{aligned}
d = 2 : & a, a^2, b, b^2, ab, abab \\
d = 3 : & a, a^2, b, b^2, ab, abab, aba, bab, ab^{-1}, ab^{-1}ab \\
d = 4 : & a, a^2, a^3, b, b^2, b^3, ab, (ab)^2, (ab)^3, aba, bab, ab^{-1}, ab^{-1}ab, b^{-2}a, (ab)^2a \\
d = 5 : & a, a^2, a^3, b, b^2, b^3, ab, (ab)^2, (ab)^3, aba, bab, ab^{-1}, ab^{-1}ab, b^{-2}a, (ab)^2a \\
& (ab)^3a, (ab)^3b, b^{-3}a, a^3b^{-1}, a^3b, a^4b \\
d = 6 : & a, a^2, a^3, a^4, b, b^2, b^3, b^4, ab, (ab)^2, (ab)^3, (ab)^4, aba, bab, ab^{-1}, ab^{-1}ab, b^{-2}a, (ab)^2a \\
& (ab)^3a, (ab)^3b, b^{-3}a, a^3b^{-1}, a^3b, a^4b, b^{-3}a^{-1}, b^{-2}(ab)^{-2}, b^{-1}a(ab)^{-2}, a^2(ab)^{-2}.
\end{aligned} \tag{3.38}$$

The number of independent loop relations is always equal to $(d+1)(d+2)/2$. In fact, $\chi_\rho(\ell_i)$ furnishes another set of coordinates on $\mathcal{M}$. In a later section we will write down Casimir operators acting on the genus-2 block using these coordinates.

Given these identifications between sunrise and dumbbell coordinates, the genus-2 crossing equation is

$$Z(M_2)|_{\text{sunrise}} = e^{S_{\text{Weyl}}} Z(M_2)|_{\text{dumbbell}}, \tag{3.39}$$

where $Z(M_2)|_{\text{sunrise}}$ is given in (3.13), $Z(M_2)|_{\text{dumbbell}}$ is given in (3.29), and $e^{S_{\text{Weyl}}}$ is the contribution from the Weyl anomaly in changing from the the dumbbell metric (section 3.2) to the sunrise metric (section 3.2). We will not attempt to evaluate this contribution in general. However our eventual interest will be in 3d CFTs, where $e^{S_{\text{Weyl}}}$ is 1.

Mapping class group of $(S^1 \times S^2) \# (S^1 \times S^2)$

The mapping class group $\text{MCG}(M_2)$ of a manifold is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms. Apart from satisfying Ward identities, CFT correlation functions should be invariant under the mapping class group. In this section, we discuss the mapping class group of $(S^1 \times S^2) \#^2$ and explain why our genus 2 crossing equation trivializes MCG invariance. Roughly speaking, the generators of $\text{MCG}(M_2)$ come in two types: the ones that act non-trivially on the fundamental group $\pi(M_2) \simeq \mathbb{Z} * \mathbb{Z}$ and those that act trivially. The mapping class group elements that act non-trivially on $\pi_1(M_2)$ form the group $\text{Out}(F_2)$.⁴ Those that act trivially form $(\mathbb{Z}/2\mathbb{Z})^2$ — they consist of sphere twists around the core S^2 's in each $S^1 \times S^2$.⁵ The precise statement is [BBP23]:

$$\text{MCG}(M_2) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \text{Out}(F_2). \tag{3.40}$$

⁴Recall that the outer automorphism group of a group G is defined as $\text{Aut}(G)/\text{Inn}(G)$ where the inner automorphism group $\text{Inn}(G)$ is given by conjugation action of a group element. The reason $\text{Out}(\pi_1)$ appears is because the action of the inner automorphisms $\text{Inn}(\pi_1)$ simply alters the starting points of the loops.

⁵Given a sphere S^2 and a tubular neighborhood $S^2 \times [0, 1]$ around it, to implement a sphere twist, keep everything outside of the tubular neighborhood unchanged but implement a full turn from $S^2 \times \{0\}$ to $S^2 \times \{1\}$. This is very much like a higher dimensional version of Dehn twist in 2d surfaces but unlike in $2d$, the sphere twist has order (at most) two — the configuration of the sphere inside the cylinder $S^2 \times [0, 1]$ traces out a loop in the group $SO(3)$ and $\pi_1(SO(3)) \simeq \mathbb{Z}/2\mathbb{Z}$.

To analyze MCG invariance of the CFT partition function, we must understand how to translate MCG generators to operators acting on Hilbert space. Generators of $(\mathbb{Z}/2\mathbb{Z})^2$ correspond to insertions of a 2π rotation in g_2, g_3 or g_L, g_R . For bosonic theories, this has no effect, and so the partition function is invariant under this operation. In fermionic theories, this corresponds to an insertion of $(-1)^F$ and hence relates partition functions with different spin-structures. The partition functions with different spin structures can be collected into a “partition vector,” and insertions of $(-1)^F$ have a linear action on this vector — we can say that the partition vector is MCG covariant.

The $\text{Out}(F_2)$ part of the mapping class group is generated by three generators σ, ι, ρ with the following actions on a, b :

$$\begin{aligned}\sigma &: (a, b) \rightarrow (b^{-1}, a^{-1}), \\ \iota &: (a, b) \rightarrow (a, b^{-1}), \\ \rho &: (a, b) \rightarrow (ab, b^{-1}),\end{aligned}\tag{3.41}$$

The action σ and ρ can be easily understood in the sunrise channel: they correspond to exchanging g_1, g_2 and g_2, g_3 . From the sunrise channel expansion of $Z(M_2)$, it is clear that the partition function is invariant under such exchanges since invariance of three-point functions under permutation implies:

$$\sum_{s, s'} P_{123}^{s, s'} B_{123}^{s', s}(g_1, g_2, g_3) = \sum_{s, s'} P_{213}^{s, s'} B_{213}^{s', s}(g_2, g_1, g_3) = \sum_{s, s'} P_{132}^{s, s'} B_{132}^{s', s}(g_1, g_3, g_2).\tag{3.42}$$

The action of ι , however, is better understood from the dumbbell channel. Under ι , an extra π -rotation is performed along the “neck” of the dumbbell so $g_M \rightarrow e^{i\pi M_{12}} g_M$. We call this operation a “dumbbell flip.” In A_R , ι flips the sign of a spatial direction and a time direction, therefore:

$$g_R = e^{-(\beta_R/r)D} h_R I \rightarrow e^{-(\beta_R/r)D} \mathcal{R} h_R^{-1} \mathcal{R} I = (I\mathcal{R}) g_R^{-1} (I\mathcal{R}),\tag{3.43}$$

where $\mathcal{R}$ is a reflection along, say x_2 . When viewed as an operator acting on $\mathcal{H}_{S_r^{d-1}}$, the change in g_R can be implemented as conjugation by $J_\Omega \equiv \text{CRT}$. Recall that the operator J_Ω is an anti-unitary operator that commutes with the Hamiltonian. Thus:

$$J_\Omega e^{-(\beta_R/r)D} J_\Omega^{-1} = e^{-(\beta_R/r)D}, \quad J_\Omega e^{i\vec{M} \cdot \vec{\theta}} J_\Omega^{-1} = \mathcal{R} e^{-i\vec{M} \cdot \vec{\theta}} \mathcal{R}^{-1},\tag{3.44}$$

Similarly the change in $(h_M \cdot \mathcal{O})_a$ under ι is implemented as:

$$(e^{i\pi M_{12}} h_M \cdot \mathcal{O})_a(x) = J_\Omega (h_M \cdot \mathcal{O})_a(x) J_\Omega^{-1}.\tag{3.45}$$

Invariance under ι then follows naturally in the dumbbell channel from cyclicity of the trace.

The argument for invariance under the dumbbell flip ι is even simpler when angular fugacities are turned off in A_R . In that case, as discussed in section 3.2, only even-spin operators have non-vanishing one-point functions on $S^1 \times S^{d-1}$. The rotation $e^{i\pi M_{12}}$ flips the direction of e^μ (the tangent vector in the S^1 direction), and even-spin one-point functions (3.31) are invariant under this operation. It follows that the partition function is invariant under the dumbbell flip.

We conclude this section with a side remark. The mapping class group of the n -fold connected sum $M_n \equiv (S^1 \times S^2)^{\sharp n}$ is known to be [BBP23]:

$$(S^1 \times S^2)^{\sharp n} = H^1(M_n, \mathbb{Z}/2\mathbb{Z}) \rtimes \text{Out}(F_n), \quad (3.46)$$

where $H^1(M_n, \mathbb{Z}/2\mathbb{Z})$, as a group, is just the group of sphere twists. Restricting to the case $n = 1$, we find that the mapping class group of $(S^1 \times S^2)$ is $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. One of the generators is a sphere twist along S^1 . In bosonic theories, the partition function is invariant under this action, while in fermionic theories the spin-structure will be changed. The other $\mathbb{Z}/2\mathbb{Z}$ is generated by the π -rotation discussed in section 3.2 and invariance under this subgroup forbids odd spin operators from having non-zero thermal one-point functions.

3.3 The thermal flat limit

In this section, we focus on a simplifying limit of the genus-2 crossing equation (3.39). After defining this limit in terms of the dumbbell coordinates, we translate it into sunrise coordinates via the matching procedure described previously. In section 3.4, we use our simplified crossing equation to derive asymptotic relations between OPE coefficients and thermal one-point coefficients. As a prerequisite, we must pinpoint the dominant quantum numbers in the genus-2 block decomposition of $Z(M_2)$. We explain how to do this by evaluating the expectation value of Casimir operators in subsection 3.3. Similar techniques can be used to locate saddle points in the partial wave integral — an integral essential for the computation of the genus-2 sunrise block. We discuss this in subsection 3.3.

Thermal flat limit of the genus-2 geometry

Thermal flat limit in dumbbell coordinates

We define the thermal flat limit as the limit where r scales towards infinity and β_L, β_R, β are all held fixed. Physically, this means that the radii of the annuli A_L, A_R get large, while their temperatures are held fixed. For simplicity, let us simultaneously turn off the angular twists:

$$\text{thermal flat limit : } h_M = h_L = h_R = 1, \quad r \rightarrow \infty. \quad (3.47)$$

As discussed in subsection 3.2, we work in a conformal frame where the “dumbbell” A_L (respectively A_R) is built by gluing a sphere S^{d-1} of radius $e^{-\beta_L/2r}r$ (resp. $e^{-\beta_R/2r}r$) with a larger sphere of radius $e^{\beta_L/2r}r$ (respectively $e^{\beta_R/2r}r$). As $r \rightarrow \infty$, the spheres blow up to flat space $\mathbb{R}^{d-1}$ while the distance between them stays fixed.

In the thermal flat limit, the genus-2 partition function $Z(M_2)$ becomes infinite, but this singularity is entirely captured by the factors $Z(S_{\beta_L}^1 \times S_r^{d-1})Z(S_{\beta_R}^1 \times S_r^{d-1})$. The quotient by these factors remains finite. From (3.29), we find:

$$\lim_{r \rightarrow \infty} \frac{Z(M_2)}{Z(S_{\beta_L}^1 \times S_r^{d-1})Z(S_{\beta_R}^1 \times S_r^{d-1})} = \frac{e^{-\beta \varepsilon_0}}{|Z_{\text{glue}}(1)|^2} \sum_{\mathcal{O}_i, \mathcal{O}_j} \langle \mathcal{O}_i^a \rangle_{S_{\beta_L}^1 \times \mathbb{R}^{d-1}} \langle \mathcal{O}_j^b \rangle_{S_{\beta_R}^1 \times \mathbb{R}^{d-1}} \langle \mathcal{O}_i^a \mathcal{O}_j^b \rangle^{-1} e^{-\beta \Delta_{\mathcal{O}_i}}. \quad (3.48)$$

The sum on the right hand side is over even-spin STT primaries. We used superscripts a, b as spin indices, while subscripts i, j indicate multiplicities. Using the identity:⁶

$$(x^{\mu_1} \cdots x^{\mu_J} - \text{traces})(y_{\mu_1} \cdots y_{\mu_J} - \text{traces}) = q_J \mathcal{P}_J \left(\frac{x \cdot y}{|x||y|} \right) |x|^J |y|^J, \quad (3.50)$$

where $\mathcal{P}_J(z) = {}_2F_1(-J, d-2+J, \frac{d-1}{2}; \frac{1-z}{2})$ is a Gegenbauer polynomial and the constant q_J is:

$$q_J = \frac{\Gamma(\frac{d-2}{2})\Gamma(J+d-2)}{2^J \Gamma(d-2)\Gamma(J+\frac{d-2}{2})}, \quad (3.51)$$

the sum above simplifies to:

$$\frac{e^{-\beta\epsilon_0}}{|Z_{\text{glue}}(1)|^2} \sum_{\mathcal{O}} b_{\mathcal{O}}^2 c_{\mathcal{O}}^{-1} q_{J_{\mathcal{O}}} \left(\frac{e^{-\beta}}{\beta_L \beta_R} \right)^{\Delta_{\mathcal{O}}} \equiv \frac{e^{-\beta\epsilon_0}}{|Z_{\text{glue}}(1)|^2} h \left(\frac{e^{-\beta}}{\beta_L \beta_R} \right), \quad (3.52)$$

where the coefficient $c_{\mathcal{O}}$ is the normalization constant entering the two point function $\langle \mathcal{O}^a \mathcal{O}^b \rangle$. The function $h(z)$ is therefore independent of normalization conventions.

Note that the normalized genus-2 partition function now depends on a single combination of the moduli: $z \equiv e^{-\beta}/(\beta_L \beta_R)$. By sending r to infinity, we've scaled towards a one-dimensional locus in $\mathcal{M}$. This can alternatively be seen by examining the characters $\chi_{\rho}(\varphi_D(a))$, $\chi_{\rho}(\varphi_D(b))$, and $\chi_{\rho}(\varphi_D(ab))$. In 1d, if $\rho = \square$, the characters read:

$$\begin{aligned} \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(a)) &= 3, \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(b)) &= 3, \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(a \cdot b)) &= 3 + 4\beta_L \beta_R e^{\beta} - (\beta_L \beta_R e^{\beta})^2. \end{aligned} \quad (3.53)$$

As expected, the only free parameter that appears is $z \equiv e^{-\beta}/\beta_L \beta_R$. This pattern holds in general dimensions. For example, choosing $d = 3$ and $\rho = \square$, we similarly have:

$$\begin{aligned} \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(a)) &= \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(a^2)) = \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(b)) = \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(b^2)) = 5 \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(ab)) &= 5 - 4z^{-1} + z^{-2}, \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D((ab)^2)) &= z^{-4} - 8z^{-3} + 20z^{-2} - 16z^{-1} + 5 \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(aba)) &= \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(bab)) = 4z^{-2} - 8z^{-1} + 5, \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(ab^{-1})) &= z^{-2} + 4z^{-1} + 5, \\ \lim_{r \rightarrow \infty} \chi_{\rho}(\varphi_D(ab^{-1}ab)) &= z^{-4} - 4z^{-2} + 5. \end{aligned} \quad (3.54)$$

Though we will mostly focus on the special case $h_M = h_L = h_R = 1$, let us make some comments on angular twists. We distinguish different cases:

⁶This follows because homogeneity in x, y dictates the contraction takes the form $|x|^J |y|^J f \left(\frac{x \cdot y}{|x||y|} \right)$. Meanwhile, tracelessness translates to the differential equation:

$$\delta^{\mu\nu} \frac{d}{dy^{\mu}} \frac{d}{dy^{\nu}} \left(|x|^J |y|^J f \left(\frac{x \cdot y}{|x||y|} \right) \right) = 0 \quad (3.49)$$

Together with regularity at $\frac{x \cdot y}{|x||y|} = 1$, these constraints force f to be a Gegenbauer polynomial.

- If $h_L = h_R = 1$ and we only turn on h_M , then the two thermal one-point functions are still fixed by conformal symmetry. If h_M lies in the $\text{SO}(d-1)$ subgroup that preserves the x^1 axis, the geometry is unaffected by h_M . (This can also be checked explicitly by computing all the characters and observe that they are not affected.) More generically, suppose h_M takes the following form:

$$h_M = \exp(i\alpha^i M_{1i}) \exp(i\phi^{ij} M_{ij}), \quad i, j = 2, \dots, d. \quad (3.55)$$

We can use the redundancy (3.23) to rotate the $(d-1)$ -dimensional vector $\vec{\alpha}$ onto the x^2 -axis so that the only effective parameter is $\theta \equiv \sqrt{\sum_{i=2}^d (\alpha^i)^2}$. Thus, the flat locus becomes two-dimensional. We can directly verify this claim by computing the “loop coordinates” in the presence of a non-trivial h_M defined in (3.55). Again, working with the vector representation in 3d, we find that in the $r \rightarrow \infty$ limit:

$$\begin{aligned} \chi_\rho(\varphi_D(a)) &= \chi_\rho(\varphi_D(a^2)) = \chi_\rho(\varphi_D(b)) = \chi_\rho(\varphi_D(b^2)) = 5 \\ \chi_\rho(\varphi_D(ab)) &= 5 - 4z^{-1} \cos(\theta) + z^{-2}, \\ \chi_\rho(\varphi_D((ab)^2)) &= 5 - 16z^{-1} \cos(\theta) + (16 + 4 \cos(2\theta))z^{-2} - 8 \cos(\theta)z^{-3} + z^{-4} \\ \chi_\rho(\varphi_D(aba)) &= \chi_\rho(\varphi_D(bab)) = 5 - 8 \cos(\theta)z^{-1} + 4z^{-2} \\ \chi_\rho(\varphi_D(ab^{-1})) &= 5 + 4z^{-1} \cos(\theta) + z^{-2}, \\ \chi_\rho(\varphi_D(ab^{-1}ab)) &= 5 - 4 \cos(2\theta)z^{-2} + z^{-4}. \end{aligned} \quad (3.56)$$

The function $h(z)$ gets replaced by

$$h(z) \rightarrow \sum_{\mathcal{O}} b_{\mathcal{O}}^2 c_{\mathcal{O}}^{-1} q_{J_{\mathcal{O}}} \mathcal{P}_{J_{\mathcal{O}}}(\cos \theta) z^{\Delta_{\mathcal{O}}}. \quad (3.57)$$

- If angular fugacities entering h_L and h_R are infinitesimally small, that is, we assume h_L, h_R takes the following form:

$$h_{L/R} = \exp \left(i \left(\frac{\vec{s}_{L/R}}{r} \right) M_{1i} \right) \exp \left(i \left(\frac{\vec{y}_{L/R}}{r} \right) M_{ij} \right), \quad i, j = 2, \dots, d, \quad (3.58)$$

the flat locus remains two-dimensional with the two moduli being z_{eff} and θ_{eff} , where:

$$z_{\text{eff}} = \frac{e^{-\beta}}{\beta_{L/\text{eff}}^{\text{eff}} \beta_{R/\text{eff}}^{\text{eff}}}, \quad \beta_{L/R}^{\text{eff}} = \sqrt{\beta_{L/R}^2 + \vec{s}_{L/R} \cdot \vec{s}_{L/R}}, \quad (3.59)$$

and θ_{eff} is the angle between the two d -dimensional vectors $(\beta_L, \vec{s}_L)$ and $h_M \cdot (\beta_R, \vec{s}_R)$. The normalized genus-2 partition function takes the form:

$$\sum_{\mathcal{O}, J \in 2\mathbb{Z}} b_{\mathcal{O}_i} b_{\mathcal{O}_j} c_{\mathcal{O}_i \mathcal{O}_j}^{-1} q_J \mathcal{P}_J(\cos \theta_{\text{eff}}) \left(\frac{e^{-\beta}}{\beta_{L, \text{eff}} \beta_{R, \text{eff}}} \right)^{\Delta_{\mathcal{O}}}. \quad (3.60)$$

In this case, every point on the inner sphere of A_L and A_R is still identified with another point finite distance away, even as $r \rightarrow \infty$. If we zoom in to the neighborhood of the two operator insertions, located at the center of B_L and B_R , the net effect of h_L and h_R is to tilt the unit vector along the Euclidean time direction from $(1, \vec{0})$ to $e'_{L/R} = (\cos(|\vec{s}_{L/R}|/r), \hat{s}_{L/R} \sin(|\vec{s}_{L/R}|/r))$. Since the point located at point $re^{-\beta_{L/R}/2r}(1, \vec{0})$ is now identified with the point at $re^{\beta_{L/R}/2r}e'_{L/R}$, the effective length of the thermal circle becomes $\beta_{L/R}^{\text{eff}}$. We checked that in the presence of (3.58), (3.56) still holds as long as z, θ is replaced with $z_{\text{eff}}, \theta_{\text{eff}}$!

- When angular fugacities in h_L and h_R are finite, points on A_L and A_R will be identified with another point infinitely far away as $r \rightarrow \infty$. For this reason, thermal effective theory breaks down and in general, it is not clear if the $r \rightarrow \infty$ limit makes sense.⁷ However, if only $y_{L/R}^{ij}/r$ become $O(1)$ and $s_{L/R}^i/r$ remains small, the operator insertion at the center of B_L and B_R are still identified with points finite distance away. Orbifold singularities develop on the quotient $S^{d-1}/(h_{L/R}|_{s_{L/R}^i=0})$. We expect that the $r \rightarrow \infty$ limit still exists and it should only depend on the geometry in the vicinity of the two orbifold points.

We leave the systematic exploration of angular twists to future work and focus on the zero-angle geometry for the rest of this paper.

Thermal flat limit in sunrise coordinates

In this subsection, we match coordinates between the two channels in the thermal flat limit. Solving (3.36) order by order in $1/r$, we find:

$$\begin{aligned}
 a_1 &= \frac{\beta_L - 2\sqrt{z}\beta_R}{\beta_L + 2\sqrt{z}\beta_R} - \frac{\sqrt{z}(-3\beta_L^3\beta_R - \beta_L\beta_R^3 + 24z\beta_L^2\beta_R^2 + 24z\beta_L\beta_R^3)}{24r^2(\beta_L + 2\sqrt{z}\beta_R)^2} + O(r^{-4}), \\
 a_2 &= \frac{\beta_R - 2\sqrt{z}\beta_L}{\beta_R + 2\sqrt{z}\beta_L} - \frac{\sqrt{z}(-\beta_L^3\beta_R - 3\beta_L\beta_R^3 + 24z\beta_L^3\beta_R + 24z\beta_L^2\beta_R^2)}{24r^2(\beta_R + 2\sqrt{z}\beta_L)^2} + O(r^{-4}), \\
 \beta_3 &= \frac{1}{2r^2}\sqrt{z}\beta_L\beta_R + \frac{1}{192r^4}\sqrt{z}\beta_L\beta_R(-\beta_L^2 + 24z\beta_L\beta_R - \beta_R^2) + O(r^{-6}).
 \end{aligned} \tag{3.61}$$

Here, it suffices to use the 1d matching equations, since all gluing elements lie inside the 1d conformal group.

In the sunrise channel, it is useful to define the “relative inverse temperature” β_{ij} and “relative angles” θ_{ij} defined using eigenvalues of the product $g_i^{-1}g_j$:

$$g_i^{-1}g_j \xrightarrow{\text{conjugate to}} \exp(-\beta_{ij}D + i\theta_{ij}M_{12} + \dots). \tag{3.62}$$

In the thermal flat limit, all the relative angles vanish and the relative inverse temperatures become:

$$\beta_{13} = \beta_L/r, \quad \beta_{23} = \beta_R/r, \quad \beta_{12} = \cosh^{-1}\left(\frac{1-4z}{2z^2} + 1\right) + O(r^{-2}). \tag{3.63}$$

The three relative temperatures can be understood as the lengths of the three colored loops indicated in figure 3.5. For generic z , β_{12} remains finite as $r \rightarrow \infty$, while $\beta_{13}, \beta_{23} \rightarrow 0$.

Let us pause and discuss the range of various parameters. In order for β_{12} to be real, z must satisfy $0 < z < 1/4$. For any fixed z within this interval, to ensure a_1, a_2, β are in their appropriate range and

⁷Even though when the angles are rational, $Z(S^1 \times S^{d-1})$ can still be computed in combination with the “folding trick” [Ben+24a].

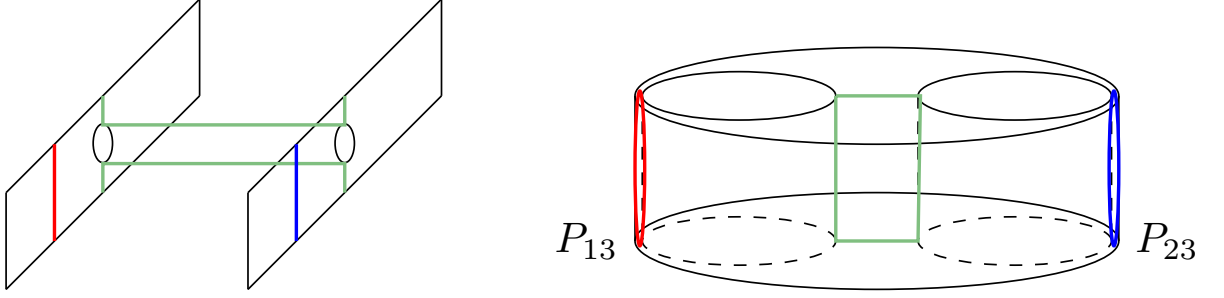


Figure 3.5: Matching of loops between the two channels. In the sunrise channel geometry (right picture), the red, blue and green circles have length β_{13}, β_{23} and β_{12} . The corresponding loops in the dumbbell channel geometry are drawn in the left picture. The two rectangular strips represent two pieces of $S^1 \times \mathbb{R}^{d-1}$ with the vertical directions being the S^1 direction. The upper edges of the two strips are all identified with the lower edges, as indicated by the arrow. By carefully tracing the identifications, one can check that the green lines in the left picture form a closed loop. The height of the cylinder in the right picture becomes infinitesimally small in the thermal flat limit, but with a generic value of z , the two horizontal segments of the green loop will always have a finite length. The blue, red circle will shrink to size zero and two hot spots will develop at P_{13}, P_{23} . In the dumbbell channel, the two hot spots get blown up to two pieces of flat space.

that the representative geometry discussed in subsection 3.2 makes sense, β_L, β_R must satisfy the following conditions, see figure 3.6:

$$\beta_L, \beta_R \geq 2, \quad 2\sqrt{z} \leq \beta_L/\beta_R \leq (2\sqrt{z})^{-1}, \quad \beta_L\beta_R \leq z^{-1}. \quad (3.64)$$

When $z = 1/4$, the only allowed region in the β_L, β_R plane is the point $(2, 2)$. In the dumbbell channel, the “neck” of the dumbbell becomes infinitesimally short and B_L, B_R become tangent to themselves since they wrap around the thermal circle. In the sunrise channel, $a_1 = a_2 = O(r^{-1})$, and therefore B_1, B_2 become tangent to each other, and all three relative inverse temperatures become parametrically small. This is the geometry studied in [Ben+24b] to derive an asymptotic formula for thermal one-point coefficients.

Another important limit is when $z \rightarrow 0$. In this limit, $\beta \rightarrow \infty$ and the neck of the dumbbell becomes infinitely long. Up to gluing factors and Casimir energy contributions, the genus-2 partition function factorizes into two $S^1 \times S^{d-1}$ partition functions. Meanwhile, a_1 and a_2 approach 1, and therefore the radii of the two balls B_1 and B_2 shrink to zero. From the sunrise channel perspective, this is a mixture of an infinite temperature limit and a zero temperature limit since $\beta_{12} \rightarrow \infty$ while $\beta_{13}, \beta_{23} \rightarrow 0$.

Comment on the hotspot conjecture. In [Ben+24b], the “hot spot conjecture” is used to estimate $Z(M_2)$ in an appropriate “high-temperature” limit. It states that the leading contribution to the partition function in the limit comes from regions where the geometry looks like a circle fibration with a large local temperature. Contributions from these regions can be well-approximated by the thermal effective

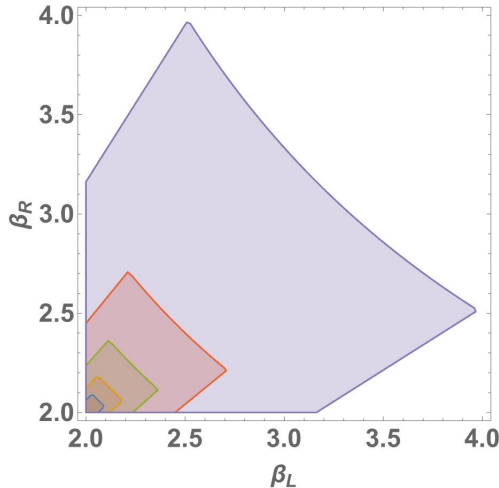


Figure 3.6: The allowed region on the β_L, β_R plane for different of z . We chose z to be $4/17, 2/9, 1/5, 1/6, 1/10$. As z approaches $1/4$, the allowed region shrinks to the left corner $(2, 2)$.

action. Though the hot spot conjecture is not needed in this work, we note that the dumbbell channel decomposition is consistent with it: in the thermal flat limit, two hot spots develop with local temperatures $1/\beta_{13}$ and $1/\beta_{23}$. Since $\beta_{13} = \beta_L/r$ and $\beta_{23} = \beta_R/r$, the contribution from these hot spots exactly reproduces the product of partition functions $Z(S_L^1 \times S_r^{d-1})Z(S_R^1 \times S_r^{d-1})$. Indeed, $Z(S_L^1 \times S_r^{d-1})$ and $Z(S_R^1 \times S_r^{d-1})$ encompasses all the singular terms as $r \rightarrow \infty$.

The function $h(z)$

The function $h(z)$ is the partition function of a CFT on two copies of thermal flat space connected by a cylinder (normalized by the square of the thermal partition function). This is an interesting function about which little has been written in the literature.

As a first example, it is interesting to compute $h(z)$ in 2d CFTs. One way to access $h(z)$ would be to determine the Weyl rescaling from the sphere to a pair of infinite cylinders connected by a finite cylinder (i.e. an infinitely tall “H”), and then use the Weyl anomaly. This shows in particular that $h(z)$ depends only on the central charge in 2d and takes the form $h(z) = e^{cf(z)}$ for some universal $f(z)$. We will not attempt this route in this work, but instead we will compute $h(z)$ to the first few orders in z by summing over low-lying states.

Due to Weyl equivalence between the cylinder and the plane, only quasi-primaries in the conformal family of the identity operator can have non-zero cylinder one-point function. For simplicity, we only include the

holomorphic sector. The first few quasi-primary states are given as follows:

$$\begin{aligned}
|\mathbb{Q}_2\rangle &\equiv L_{-2}|0\rangle, \\
|\mathbb{Q}_4\rangle &\equiv (L_{-4} - \frac{5}{3}L_{-2}L_{-2})|0\rangle, \\
|\mathbb{Q}_6^1\rangle &\equiv \left(L_{-6} + \frac{21}{4}L_{-3}^2 - \frac{14}{3}L_{-2}^3\right)|0\rangle, \quad |\mathbb{Q}_6^2\rangle \equiv \left(L_{-4}L_{-2} - \frac{5}{2}L_{-3}^2 + \frac{5}{3}L_{-2}^3\right)|0\rangle \\
&\dots
\end{aligned} \tag{3.65}$$

Under the plane-cylinder map $z_{\text{pl.}} = \exp(2\pi w_{\text{cyl.}}/\beta)$, these quasi-primaries admit the following non-zero one-point functions [Gab94]:

$$\begin{aligned}
\langle \mathbb{Q}_2 \rangle_{\text{cyl.}} &= -\frac{\pi^2 c}{6} \frac{1}{\beta^2}, \quad \langle \mathbb{Q}_4 \rangle_{\text{cyl.}} = -\frac{\pi^4 c(5c+22)}{108\beta^4}, \\
\langle \mathbb{Q}_6^1 \rangle_{\text{cyl.}} &= \frac{\pi^6 c(7c(5c+66) + 2062)}{1620\beta^6}, \quad \langle \mathbb{Q}_6^2 \rangle_{\text{cyl.}} = -\frac{\pi^6 c(5c(5c+66) + 1712)}{3240\beta^6}.
\end{aligned} \tag{3.66}$$

Combining both the holomorphic and the anti-holomorphic sectors, we find the first few terms in $h(z)$ are:

$$h(z) = 1 + \frac{1}{9}\pi^4 c z^2 + \frac{1}{810}\pi^8 c(5c+11)z^4 + \frac{32\pi^{12}c(175c^2 + 1155c + 1322)}{382725}z^6 + \dots, \tag{3.67}$$

where we've multiplied products of cylinder one-point functions with inverse two-point coefficients. Note that $h(z)$ can indeed be written in the form $h(z) = e^{cf(z)}$ with

$$f(z) = \frac{\pi^4 z^2}{9} + \frac{11\pi^8 z^4}{810} + \frac{661\pi^{12} z^6}{382725} + \dots \tag{3.68}$$

In 2 dimensions, the genus-2 crossing equation also includes a nontrivial contribution from the Weyl anomaly which itself has the form $e^{S_{\text{Weyl}}} = e^{cg(z)}$ for some $g(z)$. So $f(z)$ is only a piece of the structure of the thermal flat limit in 2d.

It is also interesting to estimate $h(z)$ in a 3d CFT, such as the 3d Ising model. In this case, only a few thermal one-point coefficients are known, see table 3.1. We can use this data to estimate $h(z)$ up to order $\sim z^5$. However, some thermal one-point functions of higher dimension operators have been estimated as well, and it is reasonable to include these in our computation of $h(z)$, since all contributions are sign-definite. We plot the resulting estimates in figure 3.7.

While summing over operators gives an estimate for $h(z)$ at small z , the behavior of $h(z)$ near $z = 1/4$ (the maximum value) can be computed using the hot-spot conjecture and thermal effective action [Ben+24b]. It is given by

$$h(z) \sim \exp\left(\frac{f \text{vol } S^2}{\beta_{\text{rel}}^2} + \text{nonsingular}\right) \quad (\text{near } z = 1/4), \tag{3.69}$$

$$\beta_{\text{rel}} = \cosh^{-1}\left(1 - \frac{2}{z} + \frac{1}{2z^2}\right). \tag{3.70}$$

In the 3d Ising CFT, we have $f \approx 0.153$. The “nonsingular” contributions come from higher-order terms in the thermal effective action, such as $c^{(1)}$, but also parts of the geometry away from the hotspot whose contributions we don't know how to estimate. In figure 3.7, we show the behavior (3.69) with different multiplicative contributions representing the unknown effects of the “nonsingular” terms.

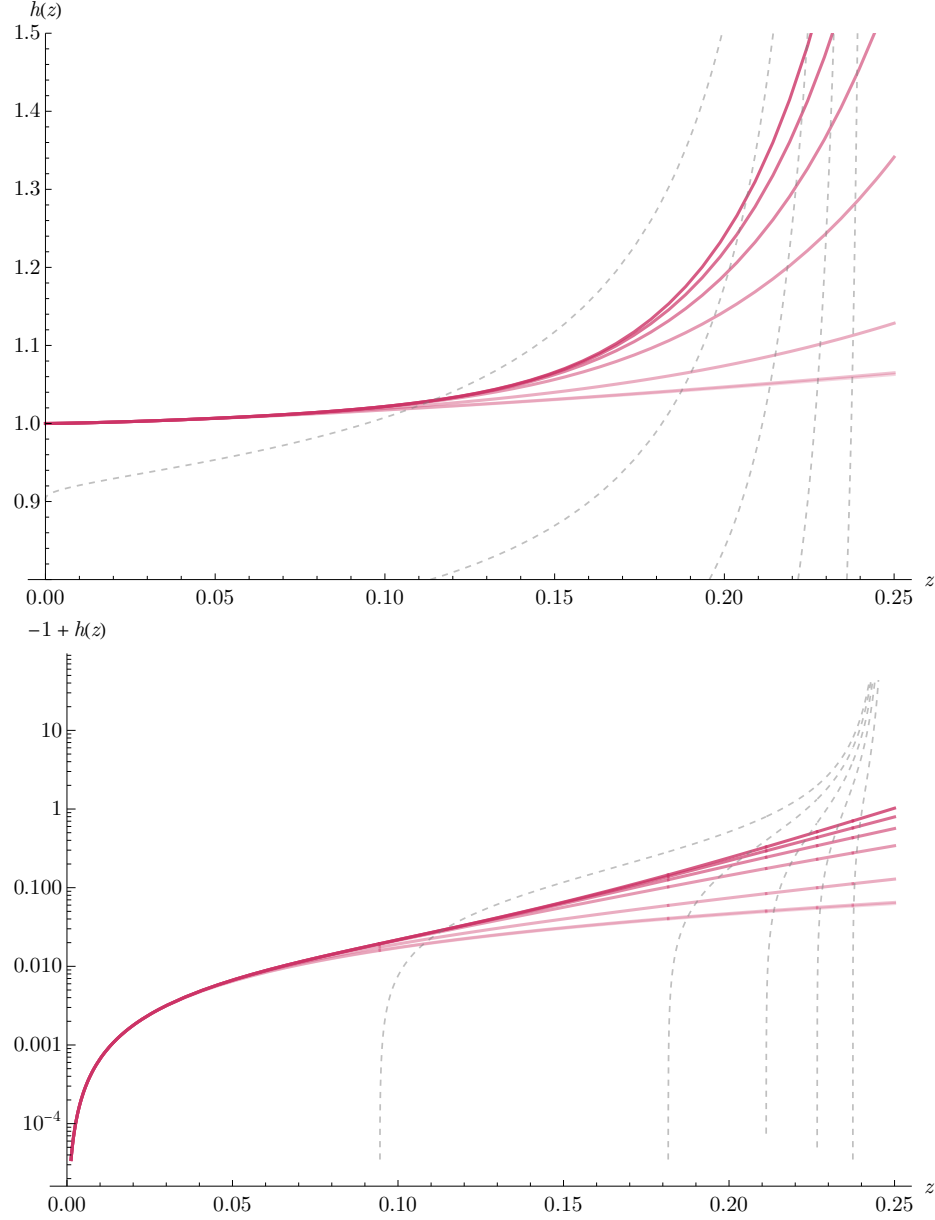


Figure 3.7: An estimate of $h(z)$ for the 3d Ising CFT, using data in table 3.1. On top, we plot $h(z)$, while below we show $h(z) - 1$ on a log scale. The red curves are the result of summing contributions from known thermal one-point functions, including more and more terms, with the highest curve including the contributions of $1, \epsilon, T, \epsilon', C_4, C_6, C_8, C_{10}$. Unknown contributions from $T', C'_4, \epsilon'', \dots$, are not included, so this is an estimated lower bound on $h(z)$. The gray dashed curves show the expected singular behavior (3.69) near $z = 1/4$, with different curves corresponding to different overall multiplicative factors.

$\mathcal{O}$	Δ	spin	$b_{\mathcal{O}}$
ϵ	1.41262528(29) [Cha+25]	0	0.667(3) [Has]
T	3	2	$-0.459(3)$ [Vas+09; KL96; Kre97]
ϵ'	3.82951(61) [Ree22]	0	3.6 ± 1.1 [Bar+25]; 3.2 ± 0.4 [Vas+09; KL96; Kre97; IKS19]
C_4	5.022665(28) [Sim17]	4	31(?) [IKS19]
T'	5.499(17) [Ree22]	2	?
C'_4	6.42065(65) [Sim17]	4	?
ϵ''	6.8956(43) [Sim17]	0	?
C_6	7.028488(16) [Sim17]	6	234(?) [IKS19]
C_8	9.031023(30) [Sim17]	8	1770(?) [IKS19]
C_{10}	11.0324141(99) [Sim17]	10	13500(?) [IKS19]

Table 3.1: Low-dimension operators in the 3d Ising CFT and their thermal one-point functions. Dimensions are taken from [Cha+25; Ree22; Sim17]; thermal one-point coefficients are taken from [Has; Vas+09; KL96; Kre97; IKS19; Bar+25]. Statistical, rigorous, and approximation-based error bars retain the conventions used in those references. When no reliable estimate is available, we mark the entry with a question mark. The higher-spin coefficients shown for $C_6, C_8, \dots$ are included because they play an important role in the thermal bootstrap for the two-point function of σ . All operators in this table are $\mathbb{Z}_2$ and parity even.

Saddles from moments

From a detailed analysis in the dumbbell channel we've determined the large- r asymptotics of $Z(M_2)$. We now turn to the sunrise channel decomposition and ask what kind of distribution over the triplet $((\Delta_1, \lambda_1), (\Delta_2, \lambda_2), (\Delta_3, \lambda_3))$ can reproduce such asymptotics. We will assume that in the thermal flat limit, the integral over the scaling dimensions Δ_i as well as the sum over $SO(d)$ representations λ_i are dominated by a single saddle point $((\Delta_1^*, \lambda_1^*), (\Delta_2^*, \lambda_2^*), (\Delta_3^*, \lambda_3^*))$. We will identify the position of this saddle via expectation values of Casimirs.

Furthermore, the genus-2 block $B_{123}^{s,s'}$ can be viewed as a sum over states in the conformal multiplet $\mathcal{R}_{(\Delta_1, \lambda_1)} \otimes \mathcal{R}_{(\Delta_2, \lambda_2)} \otimes \mathcal{R}_{(\Delta_3, \lambda_3)}$. Within the dominating multiplet $\mathcal{R}_{(\Delta_1^*, \lambda_1^*)} \otimes \mathcal{R}_{(\Delta_2^*, \lambda_2^*)} \otimes \mathcal{R}_{(\Delta_3^*, \lambda_3^*)}$, the sum also localizes to a coherent state labeled by a triplet of coordinates (x_1^*, x_2^*, x_3^*) . Such a coherent state can be identified from moments of the conformal generators. To keep the presentation simple, we will illustrate this technique mostly in 1d and 2d throughout the remainder of the section, while noting that the method generalizes straightforwardly to higher dimensions.

Saddles in Δ_i

The genus-2 blocks are simultaneous eigenfunctions of the three Casimirs, namely:

$$C_2^{(i)} B_{123}^{s,s'} = (\Delta_i(\Delta_i - d) + C_2(\lambda_i)) B_{123}^{s,s'}, \quad (3.71)$$

where $C_2(\lambda_i)$ denotes the quadratic Casimir eigenvalue of the $SO(d)$ irrep λ_i . (For symmetric traceless tensors, it is given by $J(J + d - 2)$.) Acting with $C_2^{(i)}$ on both sides of the sunrise channel decomposition,

we find:

$$\begin{aligned} \langle \mathcal{C}_2^{(i)} \rangle &\equiv \frac{\mathcal{C}_2^{(i)} Z([g_1, g_2, g_3])}{Z([g_1, g_2, g_3])} = Z^{-1} \sum_{\lambda_1, \lambda_2, \lambda_3} \int d\Delta_1 d\Delta_2 d\Delta_3 C_2(\Delta_i, \lambda_i) P_{123}^{s, s'} B_{123}^{s', s}([g_1, g_2, g_3]) \\ &\sim C_2(\Delta_i^*, \lambda_i^*), \end{aligned} \quad (3.72)$$

where we've assumed that the action of the Casimir operators do not shift the position of the saddle.

Now let us derive an expression for the Casimir operator $\mathcal{C}_2^{(i)}$. We will first explain how this works in 1d. We write the partition function and the block as functions of the “loop coordinates” $\chi_\rho(a), \chi_\rho(b), \chi_\rho(ab)$ introduced earlier. To simplify notation, we call them $\tau_{13}, \tau_{32}, \tau_{12}$, respectively, from now on. The conformal generators at each site are given by the following differential operators:

$$\mathcal{L}_{AB}^{(i)} = \sum_{jk} (\mathcal{L}_{AB}^{(i)} \tau_{jk}) \frac{\partial}{\partial \tau_{jk}}, \quad \mathcal{L}_{AB}^{(i)} \tau_{jk} = \begin{cases} -\text{tr}(g_j^{-1} L_{AB} g_k), & i = j \\ \text{tr}(g_j^{-1} L_{AB} g_k), & i = k \\ 0, & \text{otherwise.} \end{cases} \quad (3.73)$$

Combining the generators into quadratic Casimirs, we find:

$$\begin{aligned} \mathcal{C}_2^{(i)} &= -\frac{1}{2} \mathcal{L}_{AB}^{(i)} \mathcal{L}^{AB, (i)} = -\frac{1}{2} (\mathcal{L}_{AB}^{(i)} \tau_{jk}) (\mathcal{L}^{AB, (i)} \tau_{\ell m}) \frac{\partial^2}{\partial \tau_{jk} \partial \tau_{\ell m}} - \frac{1}{2} (\mathcal{L}_{AB}^{(i)} \mathcal{L}^{AB, (i)} \tau_{jk}) \frac{\partial}{\partial \tau_{\ell m}}, \\ (\mathcal{L}_{AB}^{(i)} \mathcal{L}^{AB, (i)} \tau_{jk}) &= \begin{cases} \text{tr}(g_j^{-1} L_{AB} L^{AB} g_k) = -2C_2(\rho) \tau_{jk}, & i \in \{j, k\} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.74)$$

The term $(\mathcal{L}_{AB}^{(i)} \tau_{jk}) (\mathcal{L}^{AB, (i)} \tau_{\ell m})$ can be further simplified into polynomials of τ_{ij} . We present the detailed forms of various Casimir operators in Appendix 3.7.

For now, let us make a simplifying observation: when acting on a function of the form $\exp(f(\tau_{ij}))$ with $f(\tau_{ij}) \rightarrow \infty$ in the relevant kinematic limit, the first term on the right hand side of (3.74) dominates over the second. Furthermore:

$$\frac{\partial^2 \exp(f)}{\partial \tau_{jk} \partial \tau_{\ell m}} \sim \exp(f) \frac{\partial f}{\partial \tau_{ij}} \frac{\partial f}{\partial \tau_{\ell k}}. \quad (3.75)$$

This suggests that at leading order, we can write the expectation value of Casimirs as polynomials in expectation values of the conformal generators $\langle \mathcal{L}_{AB}^{(i)} \rangle$:

$$\langle \mathcal{C}_2^{(i)} \rangle \sim -\frac{1}{2} \langle \mathcal{L}_{AB}^{(i)} \rangle \langle \mathcal{L}^{AB, (i)} \rangle. \quad (3.76)$$

In other words, the conformal generators can be treated classically and replaced with their moments. The same argument also applies to higher Casimirs.

An example in 3d: Making the choice $\rho = \square$ and turning off angular fugacities, the moments of the generators $\vec{P}, \vec{K}, D, J_{12}$ are given as follows:

$$\langle \mathcal{P}_2^{(i)} \rangle = \langle \mathcal{K}_2^{(i)} \rangle = \langle \mathcal{P}_3^{(i)} \rangle = \langle \mathcal{K}_3^{(i)} \rangle = \langle \mathcal{J}_{12}^{(i)} \rangle = \langle \mathcal{J}_{13}^{(i)} \rangle = \langle \mathcal{J}_{23}^{(i)} \rangle = 0, \quad (3.77)$$

$$\langle \mathcal{P}_1^{(1)} \rangle = -\frac{\tilde{f}_{13}}{\beta_{13}^3 \beta_{23} \sqrt{z}} + \frac{(\beta_{13}^2 + 24\beta_{23}\beta_{13}(z + \sqrt{z}) + \beta_{23}^2(24z - 1)) \tilde{f}_{13}}{96\beta_{13}^3 \beta_{23} \sqrt{z}} + O(r), \quad (3.78)$$

$$\langle \mathcal{K}_1^{(1)} \rangle = -\frac{4\tilde{f}_{13}}{\beta_{13}^3\beta_{23}\sqrt{z}} + \frac{(\beta_{13}^2 + 24\beta_{23}\beta_{13}(z - \sqrt{z}) + \beta_{23}^2(24z - 1))\tilde{f}_{13}}{24\beta_{13}^3\beta_{23}\sqrt{z}} + O(r), \quad (3.79)$$

$$\langle \mathcal{D}^{(1)} \rangle = \frac{2\tilde{f}_{13}}{\beta_{13}^3\beta_{23}\sqrt{z}} - \frac{(\beta_{13} - \beta_{23})(\beta_{13} + \beta_{23} + 24\beta_{23}z)\tilde{f}_{13}}{48\beta_{13}^3\beta_{23}\sqrt{z}} + O(r), \quad (3.80)$$

$$\langle \mathcal{P}_1^{(2)} \rangle = \frac{\tilde{f}_{23}}{\beta_{13}\beta_{23}^3\sqrt{z}} - \frac{(\beta_{23}^2 + \beta_{13}^2(24z - 1) + 24\beta_{23}\beta_{13}(z + \sqrt{z}))\tilde{f}_{23}}{96\beta_{13}\beta_{23}^3\sqrt{z}} + O(r), \quad (3.81)$$

$$\langle \mathcal{K}_1^{(2)} \rangle = \frac{4\tilde{f}_{23}}{\beta_{13}\beta_{23}^3\sqrt{z}} - \frac{(\beta_{23}^2 + \beta_{13}^2(24z - 1) + 24\beta_{23}\beta_{13}(z - \sqrt{z}))\tilde{f}_{23}}{24\beta_{13}\beta_{23}^3\sqrt{z}} + O(r), \quad (3.82)$$

$$\langle \mathcal{D}^{(2)} \rangle = \frac{2\tilde{f}_{23}}{\beta_{13}\beta_{23}^3\sqrt{z}} + \frac{(\beta_{13} - \beta_{23})(\beta_{13} + \beta_{23} + 24\beta_{13}z)\tilde{f}_{23}}{48\beta_{13}\beta_{23}^3\sqrt{z}} + O(r), \quad (3.83)$$

along with

$$\begin{aligned} \langle \mathcal{P}_1^{(3)} \rangle &= \frac{\beta_{23}^2\tilde{f}_{13} - \beta_{13}^2\tilde{f}_{23}}{\beta_{13}^3\beta_{23}^3\sqrt{z}} \\ &+ \frac{\beta_{13}^2(\beta_{23}^2 + \beta_{13}^2(24z - 1) + 24\beta_{23}\beta_{13}(z + \sqrt{z}))\tilde{f}_{23} - \beta_{23}^2(\beta_{13}^2 + 24\beta_{23}\beta_{13}(z + \sqrt{z}) + \beta_{23}^2(24z - 1))\tilde{f}_{13}}{96\beta_{13}^3\beta_{23}^3\sqrt{z}} \\ &+ O(r), \end{aligned} \quad (3.84)$$

$$\begin{aligned} \langle \mathcal{K}_1^{(3)} \rangle &= \frac{4\beta_{23}^2\tilde{f}_{13} - 4\beta_{13}^2\tilde{f}_{23}}{\beta_{13}^3\beta_{23}^3\sqrt{z}} \\ &+ \frac{\beta_{23}^2(-\beta_{13}^2 - 24\beta_{23}\beta_{13}(z - \sqrt{z}) + \beta_{23}^2(1 - 24z))\tilde{f}_{13} + \beta_{13}^2(\beta_{23}^2 + \beta_{13}^2(24z - 1) + 24\beta_{23}\beta_{13}(z - \sqrt{z}))\tilde{f}_{23}}{24\beta_{13}^3\beta_{23}^3\sqrt{z}} \\ &+ O(r), \end{aligned} \quad (3.85)$$

$$\begin{aligned} \langle \mathcal{D}^{(3)} \rangle &= -\frac{2(\beta_{23}^2\tilde{f}_{13} + \beta_{13}^2\tilde{f}_{23})}{\beta_{13}^3\beta_{23}^3\sqrt{z}} + \frac{(\beta_{13} - \beta_{23})(\beta_{23}^2(\beta_{13} + \beta_{23} + 24\beta_{23}z)\tilde{f}_{13} - \beta_{13}^2(\beta_{13} + \beta_{23} + 24\beta_{13}z)\tilde{f}_{23})}{48\beta_{13}^3\beta_{23}^3\sqrt{z}} \\ &+ O(r). \end{aligned} \quad (3.86)$$

Notice that

$$\sum_{i=1}^3 \langle \mathcal{P}_1^{(i)} \rangle = \sum_{i=1}^3 \langle \mathcal{K}_1^{(i)} \rangle = \sum_{i=1}^3 \langle \mathcal{D}^{(i)} \rangle = 0, \quad (3.87)$$

which is expected since the partition function is invariant under simultaneous left action of any conformal group element.

In computing the moments listed above, we used the following asymptotic formula for $Z(M_2)$:

$$Z(M_2) = h(z) \exp \left(\frac{\tilde{f}_{13}}{\beta_{13}^2} + \frac{\tilde{f}_{32}}{\beta_{32}^2} \right), \quad \tilde{f}_{ij} \equiv f_{ij} \text{vol} S^2, \quad (3.88)$$

where for generality, we made the free energy densities f_{ij} at the two hot spots different. (This could happen if topological defects are inserted onto M_2 .) As long as

z is away from $1/4$, the function $h(z) = \sum_{\mathcal{O}} b_{\mathcal{O}}^2 q_{J_{\mathcal{O}}} z^{\Delta_{\mathcal{O}}}$ remains finite as $r \rightarrow \infty$. As a consequence, the dependence on $h(z)$ only appears at subleading order.

Combining generators into various Casimir invariants, we find:

$$\langle \mathcal{C}_2^{(1)} \rangle \sim \frac{4\tilde{f}_{13}^2}{\beta_{13}^6}, \quad \langle \mathcal{C}_2^{(2)} \rangle \sim \frac{4\tilde{f}_{32}^2}{\beta_{32}^6}, \quad \langle \mathcal{C}_2^{(3)} \rangle \sim \frac{16\tilde{f}_{13}\tilde{f}_{32}}{\beta_{13}^4\beta_{23}^4z}, \quad (3.89)$$

$$\langle \mathcal{C}_4^{(1)} \rangle \sim \frac{16\tilde{f}_{13}^4}{\beta_{13}^{12}}, \quad \langle \mathcal{C}_4^{(2)} \rangle \sim \frac{16\tilde{f}_{32}^4}{\beta_{32}^{12}}, \quad \langle \mathcal{C}_4^{(3)} \rangle \sim \frac{256\tilde{f}_{13}^2\tilde{f}_{32}^2}{\beta_{13}^8\beta_{23}^8z^2}. \quad (3.90)$$

From these values, we can determine the saddles in the quantum numbers:

$$\Delta_1^* \sim \frac{2\tilde{f}_{13}}{\beta_{13}^3}, \quad \Delta_2^* \sim \frac{2\tilde{f}_{32}}{\beta_{32}^3}, \quad \Delta_3^* \sim \frac{4\tilde{f}_{32}^{1/2}\tilde{f}_{13}^{1/2}}{\beta_{32}^2\beta_{13}^2\sqrt{z}}, \quad J_i^* = 0. \quad (3.91)$$

Note that the saddle values of the spins J_i vanish because the moments of the quartic Casimirs are squares of the moments of the quadratic Casimirs.

This calculation easily generalizes to other dimensions d . As long as the angular fugacities are turned off, only the generators $\mathcal{P}_1^{(i)}, \mathcal{K}_1^{(i)}, \mathcal{D}^{(i)}$ can have non-zero moments. This is because when all angular twists are absent, the geometry is invariant under the action of the x^1 -preserving $\text{SO}(d-1)$ subgroup. However, under its adjoint action, the generators K_a, P_a, J_{1a} with $a, b > 1$ transform according to the vector representation and the generators J_{ab} transform according to the adjoint representation. Their expectation values are thus forced to vanish. More explicitly, let g be any element in the $\text{SO}(1, 2)$ subgroup generated by P_1, K_1, D . We have:

$$\text{tr}(P_a g) = \text{tr}(h P_a g h^{-1}) = \pi(h)_a^b \text{tr}(P_b h g h^{-1}) = \pi(h)_a^b \text{tr}(P_b g), \quad \forall h \in \text{SO}(d-1), \quad (3.92)$$

which immediately implies $\text{tr}(P_a g) = 0$. Similarly, $\text{tr}(K_a g) = \text{tr}(J_{1a} g) = \text{tr}(M_{ab} g) = 0$. The calculation of $\langle \mathcal{C}_2^{(i)} \rangle$ then reduces to an effective 1d computation. Ultimately, for generic d , we find:

$$\Delta_1^* \sim \frac{(d-1)\tilde{f}_{13}}{\beta_{13}^d}, \quad \Delta_2^* \sim \frac{(d-1)\tilde{f}_{32}}{\beta_{32}^d}, \quad \Delta_3^* \sim \frac{2(d-1)}{\sqrt{z}} \left(\frac{\tilde{f}_{13}\tilde{f}_{32}}{\beta_{13}^{d+1}\beta_{32}^{d+1}} \right)^{1/2}, \quad C_2(\lambda_i^*) = 0. \quad (3.93)$$

Recall from (3.63) that β_{13} and β_{23} become small in the thermal flat limit. Thus, we see that in terms of quantum numbers in the sunrise channel, this corresponds to $\Delta_1, \Delta_2, \Delta_3$ becoming large, with the ratio $\Delta_1 \Delta_2 / \Delta_3^{\frac{2d}{d+1}}$ held fixed. We refer to this as the “heavy-heavy-heavier” regime.

Saddles in the shadow integral

The expectation values (3.77)-(3.84) not only fix the dominant quantum numbers in the sunrise channel; they also locate the saddle of the shadow integral for the sunrise block. Recall that when the Δ_i ’s are on the principal series $\frac{d}{2} + i\mathbb{R}$, the shadow integral evaluates the genus-2 partial wave, which can be expressed as a linear combination of the genus-2 block and seven shadow blocks. We refer to the coefficients multiplying these blocks as *triple shadow coefficients*. When the scaling dimensions are parametrically large, the integrand of the shadow integral becomes rapidly oscillatory, and the genus-2 partial wave is well-approximated by a saddle-point approximation. The shadow integrand typically has multiple saddle points. To isolate the contribution of the genus-2 block from the shadow blocks, we need to identify the appropriate saddle point.

The physical genus-2 block is then obtained by analytically continuing all Δ_i from $\frac{d}{2} + i\mathbb{R}$ to $\mathbb{R}$. Further details are deferred to section 3.4. For now, we write schematically:

$$\begin{aligned}
& B_{\Delta_1^*, \Delta_2^*, \Delta_3^*}([g_1, g_2, g_3]) \\
& \sim (S_{\Delta_1^*, \Delta_2^*, \Delta_3^*}^3)^{-1} V_{d-\Delta_1^*, d-\Delta_2^*, d-\Delta_3^*}(x_1, x_2, x_3) g_1 g_2 g_3 \cdot V_{\Delta_1^*, \Delta_2^*, \Delta_3^*}(x_1, x_2, x_3) \Bigg|_{(x_1^*, x_2^*, x_3^*)},
\end{aligned} \tag{3.94}$$

here $B_{\Delta_1^*, \Delta_2^*, \Delta_3^*}$ denotes a scalar genus-2 block, $S_{\Delta_1^*, \Delta_2^*, \Delta_3^*}^3$ is the corresponding triple shadow coefficient and (x_1^*, x_2^*, x_3^*) denotes the saddle point corresponding to the genus-2 block. At leading order, the expectation values of the conformal generators

are given by:

$$\begin{aligned} \langle \mathcal{L}_{AB}^{(i)} \rangle &= \frac{\mathcal{L}_{AB}^{(i)} Z([g_1, g_2, g_3])}{Z([g_1, g_2, g_3])} \\ &\sim \frac{\mathcal{L}_{AB}^{(i)} B_{\Delta_1^*, \Delta_2^*, \Delta_3^*}([g_1, g_2, g_3])}{B_{\Delta_1^*, \Delta_2^*, \Delta_3^*}([g_1, g_2, g_3])} \sim \frac{\mathcal{L}_{AB}^{(i)} (g_1 g_2 g_3 \cdot V_{\Delta_1^* \Delta_2^* \Delta_3^*}(x_1, x_2, x_3))}{g_1 g_2 g_3 \cdot V_{\Delta_1^* \Delta_2^* \Delta_3^*}(x_1, x_2, x_3)} \Big|_{(x_1^*, x_2^*, x_3^*)}. \end{aligned} \quad (3.95)$$

To evaluate the rightmost expression, we no longer treat $\mathcal{L}_{AB}^{(i)}$ as a differential operator. Instead, we multiply the corresponding g_i on the left by $L_{AB}^{(i)}$, yielding a rational function of (x_1^*, x_2^*, x_3^*) . Matching them with the previously computed moments $\langle \mathcal{L}_{AB}^{(i)} \rangle$ determines the saddle values.

Let us illustrate this procedure for $d = 2$. By $SO(d-1)$ invariance, the saddle points lie on the x^1 axis, which greatly simplifies the rational functions. We show the action of $\mathcal{P}_1^{(1)}$ for the purpose of illustration. The expression for the rest of the conformal generators are similar in flavor:

$$\frac{\mathcal{P}_1^{(1)} (g_1 g_2 g_3 \cdot V_{\Delta_1^* \Delta_2^* \Delta_3^*}(x_1, x_2, x_3))}{g_1 g_2 g_3 \cdot V_{\Delta_1^* \Delta_2^* \Delta_3^*}(x_1, x_2, x_3)} \Big|_{x_1^2=x_2^2=x_3^2=0} = \frac{N}{D} + O(r^2), \quad (3.96)$$

where

$$N = \sqrt{\tilde{f}_{13} \tilde{f}_{23}} \left(\beta_{13} (x_1^1 + 2) + 2\beta_{23} x_1^1 \sqrt{z} \right)^2 \left(\beta_{32} (x_2^1 - 2) (x_3^1 - 2) + 4\beta_{13} (x_3^1 - x_2^1) \sqrt{z} \right), \quad (3.97)$$

$$\begin{aligned} D &= \beta_{13}^2 \beta_{23}^2 \sqrt{z} \left(\beta_{13} (x_1^1 + 2) (x_3^1 + 2) + 4\beta_{23} \sqrt{z} (x_1^1 - x_3^1) \right) \\ &\quad \times \left(\beta_{13} \beta_{23} (x_1^1 + 2) (x_2^1 - 2) + 4\beta_{13} \beta_{23} z (x_1^1 - x_2^1) \right. \\ &\quad \left. + \sqrt{z} (\beta_{13}^2 (x_1^1 + 2) (x_2^1 + 2) + \beta_{23}^2 (x_1^1 - 2) (x_2^1 - 2)) \right). \end{aligned} \quad (3.98)$$

Equating $\mathcal{L}_{AB}^{(i)} (g_1 g_2 g_3 \cdot V_{\Delta_1^*, \Delta_2^*, \Delta_3^*}(x_1, x_2, x_3)) / (g_1 g_2 g_3 \cdot V_{\Delta_1^*, \Delta_2^*, \Delta_3^*}(x_1, x_2, x_3))|_{(x_1^*, x_2^*, x_3^*)}$ to the corresponding expectation values (3.78)-(3.84), we find the following solution for $x_1^{1*}, x_2^{1*}, x_3^{1*}$:

$$(x_1^{1*}, x_2^{1*}, x_3^{1*}) = \left(-2, 2, -2 \frac{(\beta_{23}^2 \tilde{f}_{13})^{1/2} - (\beta_{13}^2 \tilde{f}_{32})^{1/2}}{(\beta_{13}^2 \tilde{f}_{32})^{1/2} + (\beta_{23}^2 \tilde{f}_{13})^{1/2}} \right). \quad (3.99)$$

As we will show later, the saddle-point equations of the shadow integral can be solved order by order in the thermal flat limit, with $\Delta_1 \sim r^d$, $\Delta_2 \sim r^d$, $\Delta_3 \sim r^{d+1}$. To leading order, the saddle point associated with the genus-2 block is:

$$\left(-2, 2, 2\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\right). \quad (3.100)$$

Combining (3.91) and (3.120), one verifies that (3.99) indeed yields the correct saddle value.

3.4 OPE asymptotics

In this section, we invert the genus-2 sunrise block expansion of the partition function and relate the heavy-heavy-heavier OPE coefficients to thermal one-point coefficients. The relevant background is explained in section 7 of [Ben+24b]. For the purpose of self-consistency, however, we start this section with a lightning review.

A lightning review of the genus-2 sunrise block

Given a triplet of conformal multiplets $\pi_i = (\Delta_i, \lambda_i)$, $i = 1, 2, 3$ and a pair of three-point structures s, s' , the genus-2 block is a simultaneous eigenfunction of the three Casimirs:

$$\mathcal{C}_2^{(i)} B_{123}^{s,s'}(g_1, g_2, g_3) = C_2(\pi_i) B_{123}^{s,s'}(g_1, g_2, g_3), \quad i = 1, 2, 3. \quad (3.101)$$

It satisfies a boundary condition consistent with its definition as a sum over states (see the discussion around (3.13)). To compute $B_{123}^{s,s'}$ in the large quantum-number limit, we can focus on a closely related object — the genus-2 partial wave:

$$\Psi_{123}^{s,s'}(g_1, g_2, g_3) = \int d^d x_1 d^d x_2 d^d x_3 V_{abc}^{s,*}(x_1, x_2, x_3) g_1 g_2 g_3 \cdot V^{s,abc}(x_1, x_2, x_3), \quad (3.102)$$

where a, b, c are spin indices and

$$g_1 g_2 g_3 \cdot V^{s,a_1 a_2 a_3}(x_1, x_2, x_3) = \left[\prod_{i=1}^3 \Omega(x'_i)^{\Delta_i} (\lambda_i)^{a_i}_{b_i} (h_i^{-1}(x'_i)) \right] V^{s,b_1 b_2 b_3}(x'_1, x'_2, x'_3), \quad (3.103)$$

with $x'_i = g_i \cdot x_i$. By construction, $\Psi_{123}^{s,s'}$ solves the same set of Casimir equations as the genus-2 block. However, it is different from $B_{123}^{s,s'}$ in that

- The integral for $\Psi_{123}^{s,s'}$ is defined for principal series representations where $\Delta_i = \frac{d}{2} + i\mathbb{R}$, while we want to compute $B_{123}^{s,s'}$ for physical representations (real Δ above the unitarity bound).
- $\Psi_{123}^{s,s'}$ satisfies different boundary conditions from $B_{123}^{s,s'}$.

Analogous to how four-point conformal blocks are related to four-point partial waves, the following identity is expected to hold:

$$\Psi_{123}^{s',s} = (I^{-3}S_{\bar{1}\bar{2}\bar{3}}^3)^{s'} {}_{t'}B_{123}^{t's} + 7 \text{ shadow blocks.} \quad (3.104)$$

where:

- For a principal series representation $\pi_i = (\frac{d}{2} + i\nu_i, \lambda_i)$, its shadow representation $\tilde{\pi}_i$ is defined as $(\frac{d}{2} - i\nu_i, \lambda_i^R)$ where λ_i^R denotes the reflected representation of λ_i . The shadow blocks are obtained from $B_{123}^{s,s'}$ by making the replacement $\pi_i \rightarrow \tilde{\pi}_i$.
- Let us write $[\pi_1 \otimes \pi_2 \otimes \pi_3]$ for the space of solutions to the conformal Ward identities for three-point functions of those representations. An element in $[\pi_1 \otimes \pi_2 \otimes \pi_3]$ is $V^{s,abc}(x_1, x_2, x_3) \equiv \langle \mathcal{O}^a(x_1) \mathcal{O}^b(x_2) \mathcal{O}^c(x_3) \rangle^{(s)}$ where $\mathcal{O}^a(x_1), \mathcal{O}^b(x_2), \mathcal{O}^c(x_3)$ transform according to π_1, π_2, π_3 respectively and s labels an element in $(\lambda_1 \otimes \lambda_2 \otimes \lambda_3)^{\text{SO}(d-1)}$.⁸

In (3.104), the coefficients $(I^{-3}S_{\bar{1}\bar{2}\bar{3}}^3)^{s'} {}_{t'}B_{123}^{t's}$ are components of the following map:

$$(I^{-3}S_{\bar{1}\bar{2}\bar{3}}^3) : [\pi_1^* \otimes \pi_2^* \otimes \pi_3^*] \rightarrow [\tilde{\pi}_1^\dagger \otimes \tilde{\pi}_2^\dagger \otimes \tilde{\pi}_3^\dagger], \quad (3.105)$$

⁸We use the following notation from [KKS19]: if an operator $\mathcal{O}^a(x)$ transforms in the representation $\pi = (\Delta, \lambda)$, we denote its shadow in the representation $\tilde{\pi} = (\tilde{\Delta}, \lambda^R)$ by $\tilde{\mathcal{O}}^{\bar{a}}$, where the barred index indicates the reflected $\text{SO}(d)$ representation. We denote an operator that transforms according to $\tilde{\pi}^\dagger \equiv (\tilde{\Delta}, \lambda^*)$ by $\tilde{\mathcal{O}}_a^\dagger(x)$ and an operator that transforms according to $\pi^\dagger \equiv (\Delta, (\lambda^*)^R)$ by $\mathcal{O}_a^\dagger$.

where $\pi_i^* \equiv (\Delta_i, \lambda_i^*)$, $\tilde{\pi}_i^\dagger \equiv (d - \Delta_i, \lambda_i^*)$ with λ_i^* being the dual representation of λ_i . $S_{1^\dagger 2^\dagger 3^\dagger}^3 : [\otimes_{i=1}^3 \pi_i^\dagger] \rightarrow [\otimes_{i=1}^3 \tilde{\pi}_i^\dagger]$ is the composition of three “shadow maps”:

$$(S_{1^\dagger 2^\dagger 3^\dagger}^3) = S([\tilde{\pi}_1^\dagger] \tilde{\pi}_2^\dagger \tilde{\pi}_3^\dagger) \circ S(\pi_1^\dagger [\tilde{\pi}_2^\dagger] \tilde{\pi}_3^\dagger) \circ S(\pi_1^\dagger \pi_2^\dagger [\tilde{\pi}_3^\dagger]^\dagger), \quad (3.106)$$

where $S(\pi_1 \pi_2 [\pi_3]) : [\pi_1 \otimes \pi_2 \otimes \tilde{\pi}_3] \rightarrow [\pi_1 \otimes \pi_2 \otimes \pi_3]$ is defined using a shadow transform:

$$\int d^d y \langle \tilde{\mathcal{O}}^\dagger(x_3) \tilde{\mathcal{O}}^\dagger(y) \rangle \langle \mathcal{O}^a(x_1) \mathcal{O}^b(x_2) \mathcal{O}^c(y) \rangle^{(s)} \equiv S(\pi_1 \pi_2 [\pi_3])_t^s \langle \mathcal{O}^a(x_1) \mathcal{O}^b(x_2) \tilde{\mathcal{O}}^\dagger(x_3) \rangle^{(t)}. \quad (3.107)$$

Given an element in $[\otimes_{i=1}^3 \pi_i^*]$, contracting the three spin indices simultaneously with the inverse inversion tensor I^{-1} will reflect the $\text{SO}(d)$ representations λ_i^* , yielding an element in the desired space $[\otimes_{i=1}^3 \pi_i^\dagger]$.

When the Δ_i ’s are large, the integrand in (3.102) develops multiple saddle points. Expanding around different saddle points and contracting with the (inverse of) the triple shadow coefficients, we get leading asymptotics of different blocks. Even though $\Psi_{123}^{s',s}$ is initially defined for principal series representations, after multiplying by the inverse triple shadow coefficient $((I^{-3} S_{1^\dagger 2^\dagger 3^\dagger}^3)^{-1})^{s'}_{t'}$ we should be able to analytically continue the resulting expression onto the real Δ -axis.

An important feature of the genus-2 sunrise block is the following orthogonality relation:

$$\oint d\mu B_{\pi_1 \pi_2 \pi_3}^{s's} B_{\tilde{\pi}_1^\dagger \tilde{\pi}_2^\dagger \tilde{\pi}_3^\dagger}^{t'^* t^*} = T^{ts} T^{s't'} \prod_{i=1}^3 2\pi \delta(\Delta_i - \Delta'_i) \frac{2^d \text{vol SO}(d)}{\dim \lambda_i} \delta_{\lambda_i, \lambda'_i}, \quad (3.108)$$

The three-point pairing matrix T^{ts} is defined as:

$$T^{ts} = \frac{V^t(0, e, \infty)^* V^s(0, e, \infty)}{2^d \text{vol SO}(d-1)}, \quad (3.109)$$

where $V^s(0, e, \infty)$ is a tensor transforming under $\lambda_1 \otimes \lambda_2 \otimes \lambda_3$ and $V^{t*}(0, e, \infty)$ transforms under $\lambda_1^* \otimes \lambda_2^* \otimes \lambda_3^*$. The spin indices of V^{t*} and V^s are implicitly contracted.

This orthogonality relation is derived by deforming the integration contour into the “low temperature region” where all three cylinders C_i become long, and using the low temperature limit of the block.

The integration measure $d\mu$ in (3.108) is the natural quotient measure on the moduli space $\mathcal{M} = G \backslash (G^-)^3 / G$. In practice, however, we want to work with a partially gauge-fixed measure $da_1 da_2 d\beta_3 d\vec{\alpha} d\vec{\Phi}$ which is better suited for our parameterization of the genus-2 sunrise geometry. To relate the measures, we go through the Faddeev-Popov procedure and find:

$$d\mu = Q(a_1, a_2, \beta_3, \vec{\alpha}_i, \vec{\Phi}_i) da_1 da_2 d\beta_3 d\vec{\alpha}_i d\vec{\Phi}_i, \quad (3.110)$$

where

$$Q(a_1, a_2, \beta_3, \vec{\alpha}_i, \vec{\Phi}_i) = \frac{2^{3d+5}(1+a_1)^{2d-1}(1+a_2)^{2d-1}(a_1+a_2-2)^{d-1}\beta_3}{(\text{vol SO}(d-1))^2(1-a_1)^{2d+1}(1-a_2)^{2d+1}} + O(\alpha, \Phi, \beta_3^2). \quad (3.111)$$

For more details, we refer the reader to [Ben+24b].

Equipped with the orthogonality relation (3.108) and FP determinant (3.110), we can invert the genus-2 block expansion of the partition function:

$$Z = \sum_{\lambda_1, \lambda_2, \lambda_3} \int d\Delta_1 d\Delta_2 d\Delta_3 P_{123}^{ss'} B_{123}^{s's}, \quad (3.112)$$

to get the genus-2 block coefficient $P_{123}^{ss'}$:

$$P_{123}^{ss'} = (T^{-1})^{st}(T^{-1})^{t's'} \frac{1}{(2\pi)^3} \prod_{i=1}^3 \left(\frac{\dim \lambda_i}{2^d \text{vol SO}(d)} \right) \mathcal{I}_{123}^{tt'}, \quad (3.113)$$

$$\mathcal{I}_{123}^{tt'} = \int da_1 da_2 d\beta_3 \prod_{i=1}^3 d\vec{\alpha}_i d\vec{\Phi}_i Q Z B_{1\uparrow 2\uparrow 3\uparrow}^{t't}. \quad (3.114)$$

The majority of this section will be devoted to the computation of $\mathcal{I}_{123}^{tt'}$.

Here is an outline of the computation:

1. Compute the asymptotics of $B_{123}^{s,s'}$ and $B_{1\bar{2}\bar{3}}^{s,s'}$ in the heavy-heavy-heavier regime. As explained before, we do this via saddle point approximation of the partial wave integral, namely:

$$B_{123}^{s,s'} = ((I^{-3}S_{1\bar{1}\bar{2}\bar{3}}^3)^{-1})^s {}_t\Psi_{123}^{t,s'} \Big|_{\text{correct saddle}}. \quad (3.115)$$

We divide this computation into three steps:

- Set $\Delta_i = \frac{d}{2} + i\nu_i$ and locate saddle points of the partial-wave integrand in the limit where $\nu_1 \sim r^d, \nu_2 \sim r^d, \nu_3 \sim r^{d+1}$, with $\nu_1\nu_2/\nu_3^{2d/(d+1)}$ held fixed. Note that the positions of the saddle points, to leading order at large ν_i , depend only on the part of the integrand that is exponential in the ν_i (the “rapidly-varying part”). Thus, for example, we can replace $d - \Delta_i \rightarrow -\Delta_i$ in the saddle point equations. This ensures the equations are linear in the Δ_i , and thus their solutions only depend on ratios of the Δ_i .
- Identify the correct saddle point by matching $\vec{q}(\Delta_i^*)$ with (3.99). The “dominant quantum numbers” Δ_i^* are given in (3.93), not to be confused with the complex conjugates of Δ_i .
- Compute the leading asymptotics of the triple shadow coefficients $(I^{-3}S_{1\bar{1}\bar{2}\bar{3}}^3)^s {}_t\Psi_{123}^{t,s'}$ in the appropriate large ν_i limit, then analytically continue

$$((I^{-3}S_{1\bar{1}\bar{2}\bar{3}}^3)^{-1})^s {}_t\Psi_{123}^{t,s'} \Big|_{\text{correct saddle}}$$

from $\Delta_i \in \frac{d}{2} + i\mathbb{R}$ to $\Delta_i \in \mathbb{R}$ by simultaneously rotating the Δ_i ’s to the real axis.

We can then obtain the shadow blocks by replacing $\pi_i \rightarrow \tilde{\pi}_i$, or alternatively by repeating the above computation with a different saddle.

2. Compute the large- r asymptotics of $Z(M_2)$ using the thermal effective action and re-write the asymptotic expansion in terms of the sunrise channel coordinates $a_1, a_2, \beta_3, \vec{\alpha}, \vec{\Phi}$.

3. Perform the integral $\mathcal{I}_{123}^{tt'}$ using results from the previous steps. This integral is again well-approximated via a saddle point.

Saddle point approximation of the block

To find the saddle points of the partial wave integral, we start with the limit $\beta_3 = 0$ and $\vec{\alpha}_i = 0, \vec{\Phi}_i = 0$, then solve perturbatively in small β_3 and small α, Φ . When all the angles are set to zero, $\text{SO}(d-1)$ invariance of the partial wave integrand forces the saddle points to lie on the x^1 axis. Collecting the unknowns together into a triplet $\vec{q} = (x_1^1, x_2^1, x_3^1)$, we find the following four solutions:

$$\vec{q}_1 = \left(-2, 2, \frac{2(\gamma_2 - \gamma_1)}{\gamma_1 + \gamma_2} \right) + O(1/r), \quad (3.116)$$

$$\vec{q}_2 = \left(-2, 2, \frac{2(\gamma_2 + \gamma_1)}{\gamma_1 - \gamma_2} \right) + O(1/r), \quad (3.117)$$

$$\vec{q}_3 = \left(-2a_1, 2a_2, \frac{2(a_1a_2 - 1 - \sqrt{(1-a_1^2)(1-a_2^2)})}{a_1 - a_2} \right) + O(1/r), \quad (3.118)$$

$$\vec{q}_4 = \left(-2a_1, 2a_2, \frac{2(a_1a_2 - 1 + \sqrt{(1-a_1^2)(1-a_2^2)})}{a_1 - a_2} \right) + O(1/r), \quad (3.119)$$

where for brevity and future convenience, we define

$$\begin{aligned} \gamma_1 &\equiv \sqrt{(1-a_2)(4(a_1+1)\Delta_1^2 + (1-a_1)\beta_3\Delta_3^2)}, \\ \gamma_2 &\equiv \sqrt{(1-a_1)(4(a_2+1)\Delta_2^2 + (1-a_2)\beta_3\Delta_3^2)}, \\ \gamma_3 &\equiv (1-a_1)(1-a_2)\beta_3\Delta_3^2. \end{aligned} \quad (3.120)$$

Due to shadow symmetry of the partial wave integral, the saddle points transform among themselves under the action of a $\mathbb{Z}_2^3$ group, described in [Ben+24b]. Let's denote the generators by $\sigma_1, \sigma_2, \sigma_3$, with σ_i related to a shadow transform at site i . It can be checked explicitly that the four saddle points above are each fixed by σ_1 and σ_2 , while σ_3 exchanges $\vec{q}_1$ with $\vec{q}_2$ and $\vec{q}_3$ with $\vec{q}_4$.

There are four additional saddle points located at:

$$\begin{aligned}
\vec{q}_5 &= \left(-2, \frac{-2(a_2+1)^2\gamma_3 + 2(a_2-1)^2\gamma_2^2 - 4(a_2^2-1)\gamma_2\sqrt{\gamma_2^2-\gamma_3}}{(a_2-1)(a_2+3)\gamma_2^2 - (a_2+1)^2\gamma_3}, -2 \right) + \dots \\
\vec{q}_6 &= \left(-2, \frac{-2(a_2+1)^2\gamma_3 + 2(a_2-1)^2\gamma_2^2 + 4(a_2^2-1)\gamma_2\sqrt{\gamma_2^2-\gamma_3}}{(a_2-1)(a_2+3)\gamma_2^2 - (a_2+1)^2\gamma_3}, -2 \right) + \dots \\
\vec{q}_7 &= \left(\frac{2(a_1+1)^2\gamma_3 - 2(a_1-1)^2\gamma_1^2 - 4(a_1^2-1)\gamma_1\sqrt{\gamma_1^2-\gamma_3}}{(a_1-1)(a_1+3)\gamma_1^2 - (a_1+1)^2\gamma_3}, 2, 2 \right) + \dots \\
\vec{q}_8 &= \left(\frac{2(a_1+1)^2\gamma_3 - 2(a_1-1)^2\gamma_1^2 + 4(a_1^2-1)\gamma_1\sqrt{\gamma_1^2-\gamma_3}}{(a_1-1)(a_1+3)\gamma_1^2 - (a_1+1)^2\gamma_3}, 2, 2 \right) + \dots
\end{aligned} \tag{3.121}$$

These saddle points scale towards the singularities of the saddle point equations and cannot be found via naive expansion at generic values of $\vec{q}$. The saddle points $\vec{q}_5$ and $\vec{q}_6$ are exchanged by σ_2 , while they are invariant under σ_1 and σ_3 . Similarly, $\vec{q}_7$ and $\vec{q}_8$ are exchanged by σ_1 and invariant under σ_2, σ_3 .

We will make the assumption that the large-quantum-number block corresponds to a single saddle point in the partial wave integral. This is verified analytically at low temperatures, and supported numerically at high temperatures in [Ben+24b], and we assume that it continues to be true in the regime of interest here. To determine which saddle point corresponds to the block, we can take the infinite temperature limit within our thermal flat limit and match it to the infinite temperature limit within the high-temperature limit of [Ben+24b]. Recall that in [Ben+24b], the correct high-temperature saddle $\vec{q}_0 = \left(\frac{2\Delta_3}{2\Delta_2-\Delta_3}, -\frac{2\Delta_3}{2\Delta_1-\Delta_3}, \frac{2(\Delta_2-\Delta_1)}{\Delta_1+\Delta_2} \right)$ was found by numerically tracking the correct low temperature saddle to high temperature. In the regime where Δ_3 is parametrically larger than Δ_1, Δ_2 , we can re-expand $\vec{q}_0$ to be:

$$\left(-2, 2, \frac{2(\Delta_2 - \Delta_1)}{\Delta_1 + \Delta_2} \right) + \dots, \tag{3.122}$$

which matches with $\vec{q}_1$ when $a_1 = a_2 = \beta_3 = 0$. As a separate check, one can verify that $\vec{q}_1$ indeed agrees with (3.99) when the Δ_i 's are taken to be their saddle values.

Thus, we identify $\vec{q}_1$ as the saddle point corresponding to the block in the thermal flat limit, and we focus on it henceforth.

Starting from the leading order expression for $\vec{q}_1$ in (3.116), it is straightforward to compute corrections in β_3 order by order. To evaluate the partial wave integral, we divide the integrand into a rapidly-varying piece and a slowly-varying piece. The slowly-varying piece is evaluated directly at the saddle point while the exponent of the rapidly-varying piece is expanded to second order around the saddle point, giving rise to a gaussian integral with an associated one-loop determinant. Combining the partial wave integral with the inverse triple shadow coefficient and continuing the Δ_i from principal axis to the real axis, we find the following asymptotic expression for the genus-2 sunrise block in the scaling regime appropriate to the thermal flat limit:

$$\begin{aligned}
& B_{123}^{s's} \Big|_{h_i=1} \\
&= \pi^{-d} 2^{2(\Delta_1+\Delta_2+\Delta_3)-\frac{7d}{2}} \left(\frac{(1-a_1)(1-a_2)\sqrt{\gamma_1\gamma_2}}{\sqrt{(a_1+1)(a_2+1)(\gamma_1^2(2\gamma_2^2-\gamma_3)-\gamma_2^2\gamma_3)}} \right)^d \\
&\quad \times (-\Delta_1 - \Delta_2 + \Delta_3)^{-\Delta_1-\Delta_2+\Delta_3} (\Delta_1 - \Delta_2 + \Delta_3)^{-\Delta_1+\Delta_2-\Delta_3} (-\Delta_1 + \Delta_2 + \Delta_3)^{\Delta_1-\Delta_2-\Delta_3} \\
&\quad \times (\Delta_1 + \Delta_2 + \Delta_3)^{-\Delta_1-\Delta_2-\Delta_3} \Delta_1^{2\Delta_1-\frac{d}{2}} \Delta_2^{2\Delta_2-\frac{d}{2}} \Delta_3^{2(d+\Delta_3)} \times \exp \left(-\sqrt{\frac{\beta_3}{\gamma_3}} \frac{\gamma_1\gamma_2}{\sqrt{(a_1-1)(a_2-1)}} \right) \\
&\quad \times V^{s'}(0, e, \infty)^* \cdot V^s(0, e, \infty). \tag{3.123}
\end{aligned}$$

We stress that the factors $(-\Delta_1 - \Delta_2 + \Delta_3)^{-\Delta_1-\Delta_2+\Delta_3}$, $(\Delta_1 - \Delta_2 + \Delta_3)^{-\Delta_1+\Delta_2-\Delta_3}$, and $(-\Delta_1 + \Delta_2 + \Delta_3)^{\Delta_1-\Delta_2-\Delta_3}$ are all positive, since we are working in a scaling limit where Δ_3 is parametrically larger than Δ_1, Δ_2 .

The expression above is valid when all angular fugacities are turned off. However, in order to compute the inversion integral, we also need the angular dependence of the “classical piece” $\exp(\dots)$. Once the angular fugacities are turned on, the saddle

points move away from the x^1 axis. To leading order in the angles, we find:

$$\vec{x}_1^\perp = \frac{\Delta_1 \Delta_3 (1 - a_1) (1 - a_2) ((1 - a_1) \gamma_1^2 \vec{\alpha}_1 - (1 - a_2) \gamma_2^2 \vec{\alpha}_2 - 2 (\gamma_1^2 - \gamma_2^2) \vec{\alpha}_3)}{\gamma_1^3 \gamma_2} + O(\alpha^2, \Phi^2) \quad (3.124)$$

$$\vec{x}_2^\perp = \frac{\Delta_2 \Delta_3 (1 - a_1) (1 - a_2) ((1 - a_1) \gamma_1^2 \vec{\alpha}_1 - (1 - a_2) \gamma_2^2 \vec{\alpha}_2 - 2 (\gamma_1^2 - \gamma_2^2) \vec{\alpha}_3)}{\gamma_1 \gamma_2^3} + O(\alpha^2, \Phi^2) \quad (3.125)$$

$$\vec{x}_3^\perp = -\frac{\Delta_3^2 (1 - a_1) (1 - a_2) ((1 - a_1) \gamma_1^2 \vec{\alpha}_1 - (1 - a_2) \gamma_2^2 \vec{\alpha}_2 - 2 (\gamma_1^2 - \gamma_2^2) \vec{\alpha}_3)}{\gamma_1 \gamma_2 (\gamma_1 + \gamma_2)^2} + O(\alpha^2, \Phi^2). \quad (3.126)$$

where $\vec{x}_i^\perp$ and $\vec{\alpha}_i$ are $(d-1)$ -dimensional vectors and $\vec{\alpha}_i$ are defined in (3.12). Plugging in the expansion of $\vec{q}_1$ up to second order in $\vec{\alpha}$ and $\vec{\Phi}$, we find the following correction term in the classical piece of the block:

$$B_{123}^{s's} \Big|_{\text{exp}(\dots)} = \exp \left(-\sqrt{\frac{\beta_3}{\gamma_3}} \frac{\gamma_1 \gamma_2}{\sqrt{(a_1 - 1)(a_2 - 1)}} + \sum_{\substack{i,j=1 \\ i \leq j}}^3 C_{i,j} \vec{\alpha}_i \cdot \vec{\alpha}_j \right), \quad (3.127)$$

where $C_{i,j}$ are Δ -dependent coefficients that we write explicitly in Appendix 3.6.

The inversion integral

In the thermal flat limit, the partition function takes the form:

$$Z = h(z) \exp(-S_{13} - S_{23}), \quad (3.128)$$

where $h(z) = \sum_{\mathcal{O}} b_{\mathcal{O}} c_{\mathcal{O}}^{-1} q_{J_{\mathcal{O}}} z^{\Delta_{\mathcal{O}}}$ and S_{13}, S_{23} are the thermal effective actions of the two dumbbells. We can write them in terms of the relative coordinates:

$$S_{ij} = -\frac{\text{vol } S^{d-1}}{\prod_{a=1}^n (1 + (\Omega_{ij}^a)^2)} \left[\frac{f_{ij}}{\beta_{ij}^{d-1}} - \frac{d-2}{\beta_{ij}^{d-3}} \left((d-1)c_{ij}^{(1)} + (2c_{ij}^{(1)} + \frac{8}{d}c_{ij}^{(2)}) \sum_{a=1}^n (\Omega_{ij}^a)^2 \right) + \dots \right], \quad (3.129)$$

where $n = \lfloor d/2 \rfloor$ is the rank of the orthogonal group $SO(d)$. Here, f_{ij} , $c_{ij}^{(1)}$, and $c_{ij}^{(2)}$ are Wilson coefficients associated to the cosmological constant term, the Einstein term $\widehat{R}$, and Maxwell term F^2 in the thermal effective action. For generality, we assume that the Wilson coefficients can be different in the two dumbbells, as would be the case if different topological defects were inserted on the dumbbells.

It is useful to re-write the thermal effective actions in terms of the coordinates $a_1, a_2, \beta_3, \vec{\alpha}$ and Φ . For example, the cosmological constant term reads

$$S_{13}\Big|_{\text{leading}} = -f_{13}\text{vol}S^{d-1}\left(\frac{1-a_1}{4\beta_3(1+a_1)}\right)^{\frac{d-1}{2}} \times \left(1 - \frac{(1-a_1)(\vec{\Phi}_1 - \vec{\Phi}_3)^2}{4(1+a_1)\beta_3} - \Gamma_{\Lambda}^{1,1}|\vec{\alpha}_1|^2 - \Gamma_{\Lambda}^{3,3}|\vec{\alpha}_3|^2 - \Gamma_{\Lambda}^{1,3}(\vec{\alpha}_1 \cdot \vec{\alpha}_3)\right), \quad (3.130)$$

where the coefficients $\Gamma_{\Lambda}^{1,1}, \Gamma_{\Lambda}^{3,3}, \Gamma_{\Lambda}^{1,3}$ are given by

$$\begin{aligned} \Gamma_{\Lambda}^{1,1} &= (d+1)\frac{(1-a_1)^2}{32\beta_3^2} - \frac{(1-a_1)}{384(1+a_1)\beta_3}(a_1^2(d^2-8d-1) - 4a_1(d^2+4d-1) + d^2+16d-73) + O(1), \\ \Gamma_{\Lambda}^{3,3} &= \frac{d+1}{8\beta_3^2} - \frac{(a_1^2-4a_1+1)(d^2+4d-13)}{96(1-a_1^2)\beta_3} + O(1), \\ \Gamma_{\Lambda}^{1,3} &= -(d+1)\frac{(1-a_1)}{8\beta_3^2} + \frac{a_1^2(d^2+4d-13) - 4a_1(d^2+d-4) + d^2+16d-49}{96(a_1+1)\beta_3} + O(1). \end{aligned} \quad (3.131)$$

The cosmological constant term in S_{23} has a similar expansion, with $f_{13}, a_1, \vec{\alpha}_1, \vec{\Phi}_1$ replaced by $f_{23}, a_2, \vec{\alpha}_2, \vec{\Phi}_2$. Recall that when $a_1 = a_2 = 0$,⁹ the geometry of the genus-2 manifold reduces to the one discussed in [Ben+24b], where all three balls are tangent to each other. In this limit, $\beta_3 = \beta_{13,0}^2/4 = \beta_{23,0}^2/4$, and $S_{13}\Big|_{\text{leading}}$ becomes:

$$-\frac{f_{13}\text{vol}S^{d-1}}{\beta_{13,0}^{d-1}}\left(1 - \frac{(\vec{\Phi}_1 - \vec{\Phi}_3)^2}{\beta_{13,0}^2} - 8(d+1)\frac{(\frac{1}{4}\vec{\alpha}_1 - \frac{1}{2}\vec{\alpha}_3)^2}{\beta_{13,0}^4} + \dots\right), \quad (3.132)$$

this agrees with the hotspot formula in the high-temperature limit.

The Einstein and Maxwell terms are similarly given by:

$$S_{13}\Big|_R = c_{13}^{(1)}(d-1)(d-2)\text{vol}S^{d-1}\left(\frac{1-a_1}{4\beta_3(a_1+1)}\right)^{\frac{d-3}{2}} \times \left(1 - \frac{(1-a_1)(d-3)(\vec{\Phi}_1 - \vec{\Phi}_3)^2}{4(1+a_1)(d-1)\beta_3} - \Gamma_R^{1,1}|\vec{\alpha}_1|^2 - \Gamma_R^{3,3}|\vec{\alpha}_3|^2 - \Gamma_R^{1,3}(\vec{\alpha}_1 \cdot \vec{\alpha}_3)\right), \quad (3.133)$$

$$S_{13}\Big|_{F^2} = \frac{8(d-2)c_{13}^{(2)}}{d}\text{vol}S^{d-1}\left(\frac{1-a_1}{4\beta_3(a_1+1)}\right)^{\frac{d-1}{2}}$$

⁹This is equivalent to setting $\beta_{13,0} = \beta_{23,0}$, $z = 1/4$.

$$\times \left((\vec{\Phi}_1 - \vec{\Phi}_3)^2 - \Gamma_{F^2}^{1,1} |\vec{\alpha}_1|^2 - \Gamma_{F^2}^{3,3} |\vec{\alpha}_3|^2 - \Gamma_{F^2}^{1,3} (\vec{\alpha}_1 \cdot \vec{\alpha}_3) \right), \quad (3.134)$$

with

$$\begin{aligned} \Gamma_R^{1,1} &= \frac{(a_1 - 1)^2 (d - 3)(d + 1)}{32(d - 1)\beta_3^2} \\ &\quad + \frac{(a_1 - 1)(d - 3)(a_1^2(d^2 - 10d - 3) - 4a_1(d^2 + 2d - 3) + d^2 + 14d - 75)}{384(d - 1)(a_1 + 1)\beta_3} + O(1), \\ \Gamma_R^{3,3} &= -\frac{(d - 3)(d + 1)}{8(1 - d)\beta_3^2} + \frac{(a_1^2 - 4a_1 + 1)(d + 5)(d - 3)^2}{96(a_1 - 1)(a_1 + 1)(d - 1)\beta_3} + O(1), \\ \Gamma_R^{1,3} &= \frac{(a_1 - 1)(d - 3)(d + 1)}{8(d - 1)\beta_3^2} + \frac{(d - 3)^2(a_1^2(d + 5) - 4a_1(d + 2) + d + 17)}{96(a_1 + 1)(d - 1)\beta_3} + O(1), \end{aligned} \quad (3.135)$$

and

$$\Gamma_{F^2}^{1,1} = \frac{a_1^2 - 1}{4\beta_3} + O(1), \quad \Gamma_{F^2}^{3,3} = \frac{a_1 + 1}{(a_1 - 1)\beta_3} + O(1), \quad \Gamma_{F^2}^{1,3} = \frac{a_1 + 1}{\beta_3} + O(1). \quad (3.136)$$

At leading order in the $1/r$ expansion, the saddle points of the inversion integral and the associated one-loop factors are determined by the cosmological constant terms. However, the contributions of the subleading terms are comparable to the contribution of $h(z)$, so we will keep them in the final formula.

Finally, we have all the ingredients needed to evaluate the inversion integral $\mathcal{I}_{123}^{s,s'}$. We specialize to $d = 3$ in what follows, though the computation generalizes straightforwardly to other dimensions. We will also now assume $f_{13} = f_{23} = f$.

We first perform the integral over the angles $\vec{\alpha}$ and $\vec{\Phi}_i$ via saddle point approximation. Using the small-angle expansion of Z and $B_{123}^{s,s'}$, we find the one-loop factors to be:

$$\text{one-loop factor in } \Phi_i = \text{vol SO}(2) \times \frac{4\pi^{4/3} f^{1/3}}{\Delta_1^{2/3} \Delta_2^{2/3}}. \quad (3.137)$$

The Φ part of the integrand only depends on the differences $\vec{\Phi}_1 - \vec{\Phi}_3$ and $\vec{\Phi}_2 - \vec{\Phi}_3$. This is a consequence of the $SO(d - 1)$ gauge symmetry. To evaluate its one-loop factor, we first gauge fix $\vec{\Phi}_3$ to be zero and evaluate the hessian with respect to $\vec{\Phi}_1, \vec{\Phi}_2$, then multiply the final result by $\text{vol SO}(d - 1)$.

For the α integral, we find:

one-loop factor in α_i

$$= \frac{64f^{1/3}\pi^{10/3} \left(\Delta_1^{2/3} + \Delta_2^{2/3}\right) \left(2\Delta_2^{1/3}\Delta_1 + (\pi f)^{1/3}\Delta_3\right)^2 \left(2\Delta_1^{1/3}\Delta_2 + (\pi f)^{1/3}\Delta_3\right)^2}{\Delta_1^{7/3}\Delta_2^{7/3}\Delta_3^4 \left(\Delta_1^{1/3} - \Delta_2^{1/3}\right)^2}. \quad (3.138)$$

Note that at leading order, the hessian in the α integral has vanishing determinant if $\Delta_1 = \Delta_2$. Something must regulate this singularity when $\Delta_1 = \Delta_2$. We leave this question for the future, and for now we assume $\Delta_1 \neq \Delta_2$.

For the integral over a_1, a_2, β_3 , the saddle point is located at

$$\begin{aligned} a_1^* &= 1 - \frac{4\Delta_1\Delta_2^{1/3}}{2\Delta_2^{1/3}\Delta_1 + (\pi f)^{1/3}\Delta_3} + O(r^{-2}), \\ a_2^* &= 1 - \frac{4\Delta_1^{1/3}\Delta_2}{2\Delta_1^{1/3}\Delta_2 + (\pi f)^{1/3}\Delta_3} + O(r^{-2}), \\ \beta_3^* &= \frac{2(\pi f\Delta_1\Delta_2)^{1/3}}{\Delta_3} + O(r^{-4}), \end{aligned} \quad (3.139)$$

and the corresponding one-loop factor is:

$$\text{one loop factor in } a_1, a_2, \beta_3 = \frac{64\sqrt{f}\pi^2\Delta_1\Delta_2\Delta_3\sqrt{\Delta_1^{2/3} + \Delta_2^{2/3}}}{3(2\Delta_2^{1/3}\Delta_1 + (\pi f)^{1/3}\Delta_3)^2(2\Delta_1^{1/3}\Delta_2 + (\pi f)^{1/3}\Delta_3)^2}. \quad (3.140)$$

To ensure $a_1^* > 0$ and $a_2^* > 0$, the following inequality should be satisfied:

$$\frac{\Delta_1^2\Delta_2^2}{\Delta_3^3} \max\left(\frac{\Delta_2}{\Delta_1}, \frac{\Delta_1}{\Delta_2}\right) < \frac{\pi f}{8}. \quad (3.141)$$

Combining everything together, we find the following asymptotic formula for the

genus-2 sunrise block coefficient in 3d, assuming $\Delta_1 \neq \Delta_2$:

$$\begin{aligned}
& P_{123}^{s,s'} \\
& \sim (T^{-1})^{s,s'} \prod_{i=1}^3 \left(\frac{\dim \lambda_i}{\text{volSO}(3)} \right) \left[\sum_{\mathcal{O}} b_{\mathcal{O}}^2 q_{J_{\mathcal{O}}} \left(\frac{\Delta_1^2 \Delta_2^2}{\pi f \Delta_3^3} \right)^{\frac{2\Delta_{\mathcal{O}}}{3}} \right] \exp \left((f\pi)^{1/3} (\Delta_1^{2/3} + \Delta_2^{2/3}) - 16\pi c^{(1)} \right) \\
& \times \pi^{17/6} f^{10/3} 2^{-2\Delta_1 - 2\Delta_2 - 2\Delta_3 + 15} \Delta_1^{-2\Delta_1 - \frac{23}{6}} \Delta_2^{-2\Delta_2 - \frac{23}{6}} \Delta_3^{\frac{27}{2} - 2\Delta_3} (-\Delta_1 - \Delta_2 + \Delta_3)^{\Delta_1 + \Delta_2 - \Delta_3 - 3} \\
& \times (\Delta_1 - \Delta_2 + \Delta_3)^{\Delta_1 - \Delta_2 + \Delta_3 - 3} (-\Delta_1 + \Delta_2 + \Delta_3)^{-\Delta_1 + \Delta_2 + \Delta_3 - 3} (\Delta_1 + \Delta_2 + \Delta_3)^{\Delta_1 + \Delta_2 + \Delta_3 - 9} \\
& \times \frac{\left(4\Delta_1 \Delta_2 + (\pi f)^{1/3} \Delta_3 (\Delta_2^{2/3} + \Delta_1^{2/3}) \right)^2}{3 \left(\Delta_1^{1/3} - \Delta_2^{1/3} \right)^2}.
\end{aligned} \tag{3.142}$$

Again, this formula is valid in the thermal flat limit $\Delta_1, \Delta_2, \Delta_3$ large with fixed ratio $\Delta_1^2 \Delta_2^2 / \Delta_3^3$, while the spins λ_i are kept finite.

3.5 Discussion

In this work, we initiated the study of a genus-2 crossing equation in $d \geq 2$. This equation can be understood as arising from the usual four-point crossing equation after contracting pairs of external states. We described natural coordinates for both channels (dumbbell and sunrise) and the relationship between them, described the conformal blocks that appear on both sides, and discussed mapping class group invariance of the genus-2 partition function in 3d. In the thermal flat limit, the function $h(z)$ encoding squares of thermal one-point coefficients arises naturally on the dumbbell side of the crossing equation. This leads to a relation between asymptotic heavy-heavy-heavier OPE coefficients and squared thermal one-point functions in 3d CFTs.

Our formula for OPE asymptotics joins others in the literature for different regimes of quantum numbers in higher dimensional CFTs. For example, [Ben+24b] studies a heavy-heavy-heavy limit where all operator dimensions become large at the same rate (with fixed spins). It is natural to expect a direct interpolation between the heavy-

heavy-heavy limit (more concretely, see equation (7.51) in [Ben+24b]) and the heavy-heavy-heavier limit studied in this work. The formula for leading OPE asymptotics don't seem to be connected in a simple way. However, as we discussed in section 3.4, saddle point positions in the partial wave integral as well as the expansion of the genus-2 partition function do interpolate nicely. It would be interesting to systematically compute subleading corrections and understand how OPE asymptotics in the two scaling limits are connected. Relatedly, it would be nice to develop more systematic methods for solving the saddle point equations.

Meanwhile, [Del20] uses hydrodynamics to make predictions for OPE coefficients $c_{LHH'}$ where H and H' are potentially different heavy operators with similar energies. Our derivation of OPE asymptotics was based on applying a kind of inversion formula to the genus-2 partition function, and looking for saddle points in the inversion integral. In general, such methods lead to rough asymptotics but not precise statements. To be more precise, one could try to characterize the behavior of averages over "windows" in the space of states, analogous to [PS20; MP20; PQR25] for the modular bootstrap in 2d, or [PQR23] for the lightcone bootstrap. We expect that gaining control over small windows in the space of operator energies might be important for making contact with hydrodynamics [Del20]. Reference [Ano+22] also makes predictions for OPE statistics using higher-point crossing.

It would be nice to check these predictions in free theories by explicitly constructing and analyzing three-point coefficients. It is also interesting to ask whether one can make connections with holography. Here the situation in higher dimensions seems somewhat different from the situation for Virasoro primaries in 2d. In the 2d context, the statistics of Virasoro primaries seems to be well captured by holographic wormhole calculations [Col+20; Aba+23]. On the other hand, the statistics of OPE coefficients of global primaries and Virasoro primaries are quite different [Ben+24b]. Naively, we might expect wormhole calculations in higher dimensions to capture something more

like the statistics of “Virasoro-primary-like” states rather than global primaries. We do not know the correct definition of “Virasoro-primary-like,” but perhaps it means states that do not include a gas of gravitons around them (which would not modify the density of states, but could have nontrivial effects on three-point coefficients).

Let us mention some other possible directions for future exploration.

Kinematics of the angular twist

In the thermal flat limit studied in this work, all the angular fugacities in g_L, g_R, g_M were set to zero. An immediate next step is to generalize to geometries with nontrivial h_L, h_R and h_M . In this case, the $SO(d-1)$ symmetry that fixes $\langle J_i \rangle$ to be zero will be broken and we expect the sum over $\{(\Delta_1, \lambda_1), (\Delta_2, \lambda_2), (\Delta_3, \lambda_3)\}$ to be dominated by a representation involving parametrically large spin. What would the saddle points of the shadow integral look like in this case? How do they depend on the three-point tensor structures? What would be the asymptotic behavior of the corresponding OPE coefficients?

As noted in section 3.3, when h_L, h_R are not infinitesimally close to the identity, thermal EFT typically breaks down. However, in [Ben+24a], it was pointed out that in the case of rational angular fugacity, thermal EFT can still produce the correct high temperature asymptotics using the “folding trick,” which relates rational angular twists to physics at a lower temperature. Does the “folding trick” provide an obvious relation between $\text{tr}_{\mathcal{H}}(\mathcal{O}(\tau, \hat{n})e^{-\beta D})$ and $\text{tr}_{\mathcal{H}}(\mathcal{O}(\tau, \hat{n})e^{-\beta D}e^{i\pi M_{ij}})$, or their normalized versions? For generic angles, it is natural to expect that $\langle \mathcal{O}(x) \rangle_h$ only depends on the geometry in the vicinity of the operator insertion. The presence of angular twists single out special subspaces in $\mathbb{R}^d$. The basis vectors of these subspaces, when combined with the unit vector along the thermal circle, provide extra kinematic structures compatible with the symmetry of the geometric background. Can we systematically count and enumerate them? In dimensions greater than three, what are the nor-

malized thermal one-point functions of mixed-symmetry tensors in the presence of angular twists?

Understanding the genus-2 block

In Appendix 3.7, we discussed the Casimir equations solved by the genus-2 block. It would be nice to solve the Casimir equations recursively and to work out weight-shifting relations for the genus-2 blocks. It might also be interesting to derive Zamolodchikov-type recursion relations for genus-2 blocks, analogous to those derived for four-point blocks [HR13; KPS14b; KPS14a; PTY16; EIK19].

Comments on the thermal bootstrap

In the thermal flat limit, the genus-2 crossing equation simplifies to:

$$\sum_{\mathcal{O}, J} b_{\mathcal{O} q_J}^2 z^{\Delta_{\mathcal{O}}} = Z^{-1}(S_{\beta_{13}}^1 \times S^{d-1}) Z^{-1}(S_{\beta_{23}}^1 \times S^{d-1}) \sum_{123} c_{123}^s c_{123}^{s'} B_{123}^{s, s'}. \quad (3.143)$$

One might hope to use this crossing equation to “complete the square” of the thermal bootstrap [Ili+18], which on its own lacks the positivity conditions required for numerical bootstrap techniques. Here we offer a non-rigorous argument against the practicality of this idea: from our analysis in section 3.4, we see that when r is very large, the product $B_{1^\dagger 2^\dagger 3^\dagger}^{s, s'} Z(\beta_{13}) Z(\beta_{23})$ develops a saddle point with the corresponding one-loop factor scaling like r^{-5} . In the $r \rightarrow \infty$ limit, this saddle point will become a δ -distribution. For this reason, we expect that the quotient $B_{123}^{s, s'} / Z(\beta_{13}) Z(\beta_{23})$, as a function (more precisely: distribution) over the thermal flat locus, will be proportional to $\delta\left(z - \left(\frac{\Delta_1^2 \Delta_2^2}{\pi f \Delta_3^2}\right)^{2/3}\right)$. Thus, when $r = \infty$ the right hand side of the crossing equation will be proportional to:

$$\int d\Delta_1 d\Delta_2 c_{\Delta_1, \Delta_2, \Delta_3(z, \Delta_1, \Delta_2)}^2 \mu(\Delta_1, \Delta_2, \Delta_3(z, \Delta_1, \Delta_2)), \quad (3.144)$$

with $\mu(\Delta_1, \Delta_2, \Delta_3(z, \Delta_1, \Delta_2))$ being a positive function and $\Delta_3(z, \Delta_1, \Delta_2) = \frac{\Delta_1^{2/3} \Delta_2^{2/3}}{(\pi f)^{1/3} \sqrt{z}}$. Not only does the measure μ depend on z , but z also determines which combinations

of $(\Delta_1, \Delta_2, \Delta_3)$ contribute in the sum. It is not clear what one can learn from this crossing equation except for the fact that the left hand side should be positive for any given value of z . It is interesting to explore what happens if we turn on angular fugacities in the $r \rightarrow \infty$ limit.

Towards the genus-2 numerical bootstrap

For finite r , the genus-2 crossing equation (3.1) has all the necessary positivity conditions on both sides to make the standard numerical bootstrap approach [Rat+08] possible. An advantage compared to the usual numerical bootstrap for four-point functions is that this equation “knows” about all of the CFT data at once, as opposed to just the data appearing in a fixed set of four-point functions. On the other hand, it may be technically challenging to numerically bootstrap this equation due to the large number of cross-ratios, and the multitude of quantum numbers conjugate to those cross ratios. Perhaps, to begin, one could study a lower-dimensional slice of the genus 2 moduli space. A numerical study of genus-2 crossing in 2d was performed in [CCY19], where they expanded around a $\mathbb{Z}_3$ symmetric point in the genus-2 moduli space. We do not know of an analogous expansion point for the equation (3.1), so it is not immediately obvious what the optimal expansion locus might be. This is an interesting question for future work.

Genus-2 lightcone bootstrap and Lorentzian physics

In the context of the four-point bootstrap, a particularly interesting limit is the lightcone limit, which leads to the lightcone bootstrap and large-spin perturbation theory [KZ13; Fit+13]. What is the analog of the lightcone limit for the genus-2 geometry? More generally, what kind of physics do we encounter on the genus-2 moduli space after analytic continuation away from Euclidean signature? Are there analogs of dispersive functionals [Maz16; MP19a; MP19b; MRZ21; PSZ20; Car+21b; Car+21a] for the genus-2 crossing equation?

Acknowledgements

We thank Ilija Buric for initial collaboration on this work, and for helpful discussions. We thank Tom Hartman, Murat Koloğlu, Sridip Pal and Francesco Russo for discussions, and Nathan Benjamin, Sridip Pal, and Yifan Wang for comments on the draft. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0011632. DSD and YX are supported in part by Simons Foundation grant 488657 (Simons Collaboration on the Nonperturbative Bootstrap).

Appendix Material

3.6 The angular expansion coefficients $C_{i,j}$

In this appendix, we record the angular expansion coefficients $C_{i,j}$ appearing in (3.127).

$$\begin{aligned}
 C_{1,1} = & -\frac{(a_1-1)^2 \gamma_1 \gamma_3 \Delta_3}{32 \beta_3 \gamma_2^3} + \frac{\Delta_3}{256 (a_1+1)^2 \gamma_1^3 \gamma_2^7} \left((a_1-1)(a_1+1)(\gamma_2^2 - \gamma_3) \gamma_1^6 (2(a_1^2 + 8a_1 + 1) \gamma_2^2 - 5(a_1+1)^2 \gamma_3) \right. \\
 & + (a_1-1) \gamma_2^2 \gamma_1^4 (2(a_1^3 + 9a_1^2 + a_1 - 7) \gamma_2^4 + 6(a_1-1)(a_1^2 - 1) \gamma_3 \gamma_2^2 - 3(a_1-1)^2 (a_1+1) \gamma_3^2) \\
 & + (a_1-1) \gamma_2^4 \gamma_3 \gamma_1^2 ((a_1+1)(a_1-1)^2 \gamma_3 + (a_1^3 + 7a_1^2 + 23a_1 + 17) \gamma_2^2) + (a_1+1)^2 (a_1^2 - 1) \gamma_2^6 \gamma_3^2 \Big) \\
 & + O(r^d).
 \end{aligned} \tag{3.145}$$

$$\begin{aligned}
 C_{2,2} = & -\frac{(a_1-1)^2 \gamma_2 \gamma_3 \Delta_3}{32 \beta_3 \gamma_1^3} + \frac{\Delta_3}{256 (a_1+1)^2 \gamma_1^7 \gamma_2^3} \left((a_1^2 - 1) \gamma_1^6 (2(a_1^2 + 8a_1 - 7) \gamma_2^4 + (a_1^2 + 6a_1 + 17) \gamma_3 \gamma_2^2 + (a_1+1)^2 \gamma_3^2) \right. \\
 & + (a_1-1) \gamma_2^2 \gamma_1^4 (2(a_1+1)(a_1^2 + 8a_1 + 1) \gamma_2^4 + 6(a_1-1)(a_1^2 - 1) \gamma_3 \gamma_2^2 + (a_1-1)^2 (a_1+1) \gamma_3^2) \\
 & - (a_1-1)(a_1+1) \gamma_2^4 \gamma_3 \gamma_1^2 (3(a_1-1)^2 \gamma_3 + (7a_1^2 + 26a_1 + 7) \gamma_2^2) + 5(a_1+1)^2 (a_1^2 - 1) \gamma_2^6 \gamma_3^2 \Big) \\
 & + O(r^d).
 \end{aligned} \tag{3.146}$$

$$\begin{aligned}
 C_{3,3} = & -\frac{(\gamma_1^2 - \gamma_2^2)^2 \gamma_3 \Delta_3}{8 \beta_3 \gamma_1^3 \gamma_2^3} + \frac{(\gamma_1^2 - \gamma_2^2)^2 \Delta_3}{64 (a_1^2 - 1) \gamma_1^7 \gamma_2^7} \left((\gamma_2^2 - \gamma_3) \gamma_1^6 (2(a_1^2 + 8a_1 + 1) \gamma_2^2 - 5(a_1+1)^2 \gamma_3) \right. \\
 & + \gamma_1^4 (2(a_1^2 + 8a_1 + 1) \gamma_2^6 - 2(a_1^2 + 14a_1 + 1) \gamma_3 \gamma_2^4 + (a_1^2 + 14a_1 + 1) \gamma_3^2 \gamma_2^2) \\
 & + \gamma_2^4 \gamma_3 \gamma_1^2 ((a_1^2 + 14a_1 + 1) \gamma_3 - (7a_1^2 + 26a_1 + 7) \gamma_2^2) + 5(a_1+1)^2 \gamma_2^6 \gamma_3^2 \Big) \\
 & + O(r^d).
 \end{aligned} \tag{3.147}$$

$$\begin{aligned}
 C_{1,2} = & \frac{(a_1-1)^2 \gamma_3 \Delta_3}{16 \beta_3 \gamma_1 \gamma_2} + \frac{\Delta_3}{128 (a_1+1)^2 \gamma_1^5 \gamma_2^5} \\
 & \times \left((a_1-1) \gamma_1^6 (2(a_1^3 - 3a_1^2 + a_1 + 5) \gamma_2^4 + (3a_1^3 + 13a_1^2 + 5a_1 - 5) \gamma_3 \gamma_2^2 - 3(a_1+1)^3 \gamma_3^2) \right. \\
 & + (a_1-1) \gamma_2^2 \gamma_1^4 (2(a_1^3 - 3a_1^2 + a_1 + 5) \gamma_2^4 - 6(a_1-1)^2 (a_1+1) \gamma_3 \gamma_2^2 + (a_1-1)^2 (a_1+1) \gamma_3^2) \\
 & + (a_1-1) \gamma_2^4 \gamma_3 \gamma_1^2 ((a_1+1)(a_1-1)^2 \gamma_3 + (3a_1^3 + 13a_1^2 + 5a_1 - 5) \gamma_2^2) - 3(a_1+1)^2 (a_1^2 - 1) \gamma_2^6 \gamma_3^2 \Big) \\
 & + O(r^d).
 \end{aligned} \tag{3.148}$$

$$\begin{aligned}
 C_{1,3} = & -\frac{(a_1-1)(\gamma_1^2 - \gamma_2^2) \gamma_3 \Delta_3}{8 \beta_3 \gamma_1 \gamma_2^3} + \frac{(\gamma_1^2 - \gamma_2^2) \Delta_3}{64 (a_1+1) \gamma_1^5 \gamma_2^7} \left((\gamma_2^2 - \gamma_3) \gamma_1^6 (2(a_1^2 + 8a_1 + 1) \gamma_2^2 - 5(a_1+1)^2 \gamma_3) \right. \\
 & - \gamma_1^4 (2(a_1^2 - 4a_1 + 5) \gamma_2^6 - 2(a_1^2 - 10a_1 + 1) \gamma_3 \gamma_2^4 + (a_1^2 - 10a_1 + 1) \gamma_3^2 \gamma_2^2) \\
 & + \gamma_2^4 \gamma_3 \gamma_1^2 ((-3a_1^2 - 10a_1 + 5) \gamma_2^2 + (a_1^2 + 6a_1 + 1) \gamma_3) + 3(a_1+1)^2 \gamma_2^6 \gamma_3^2 \Big) \\
 & + O(r^d).
 \end{aligned} \tag{3.149}$$

$$\begin{aligned}
C_{2,3} = & \frac{(a_1 - 1)(\gamma_1^2 - \gamma_2^2)\gamma_3\Delta_3}{8\beta_3\gamma_1^3\gamma_2} + \frac{(\gamma_1^2 - \gamma_2^2)\Delta_3}{64(a_1 + 1)\gamma_1^7\gamma_2^5} \left(\gamma_1^6(2(a_1^2 - 4a_1 + 5)\gamma_2^4 + (3a_1^2 + 10a_1 - 5)\gamma_3\gamma_2^2 - 3(a_1 + 1)^2\gamma_3^2) \right. \\
& - \gamma_1^4(2(a_1^2 + 8a_1 + 1)\gamma_2^6 + 2(a_1^2 - 10a_1 + 1)\gamma_3\gamma_2^4 + (a_1^2 + 6a_1 + 1)\gamma_3^2\gamma_2^2) \\
& + \gamma_2^4\gamma_3\gamma_1^2((7a_1^2 + 26a_1 + 7)\gamma_2^2 + (a_1^2 - 10a_1 + 1)\gamma_3) - 5(a_1 + 1)^2\gamma_2^6\gamma_3^2) \\
& \left. + O(r^d) \right)
\end{aligned} \tag{3.150}$$

3.7 Details on Casimir equations

Notation: For a string $I = (i_1, j_1, i_2, j_2, \dots)$, the symbol τ_I^ρ is defined as:

$$\tau_I^\rho = \text{Tr}(\rho(g_{i_1})^{-1}\rho(g_{j_1})\rho(g_{i_2})^{-1}\rho(g_{j_2})\cdots), \tag{3.151}$$

where ρ is a representation of G . For example, $\tau_{1213}^\rho \equiv \text{Tr}(\rho_{g_1}^{-1}\rho(g_2)\rho_{g_1}^{-1}\rho(g_3))$. The choice of ρ is usually obvious from context so to lighten the notation, we will often times just use τ_I and write:

$$\tau_I = \text{Tr}(g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots). \tag{3.152}$$

$d = 1$, vector representation

In 1d, the moduli space of the flat conformal structure is three dimensional and we will choose the following set of invariants as coordinates on this space:

$$\{\tau_{12}, \tau_{23}, \tau_{13}\}. \tag{3.153}$$

For functions on this moduli space, infinitesimal left actions are realized as a differential operator in terms of $\tau_{12}, \tau_{23}, \tau_{31}$. For example,

$$\begin{aligned}
\mathcal{L}_{AB}^{(1)}B_{123}(g_1, g_2, g_3) & \equiv \left. \frac{d}{dt} \right|_{t=0} B_{123}(e^{tL_{AB}}g_1, g_2, g_3) \\
& = \left. \frac{d}{dt} \right|_{t=0} \text{Tr}((e^{tL_{AB}}g_1)^{-1}g_2)\frac{\partial B_{123}}{\partial \tau_{12}} + \left. \frac{d}{dt} \right|_{t=0} \text{Tr}((e^{tL_{AB}}g_1)^{-1}g_3)\frac{\partial B_{123}}{\partial \tau_{13}} \\
& = -\text{Tr}(g_1^{-1}L_{AB}g_2)\frac{\partial B_{123}}{\partial \tau_{12}} - \text{Tr}(g_1^{-1}L_{AB}g_3)\frac{\partial B_{123}}{\partial \tau_{13}},
\end{aligned} \tag{3.154}$$

where it is understood that in expressions such as $\text{Tr}(g_1^{-1}L_{AB}g_2)$, g_1, L_{AB}, g_2 denote the corresponding three dimensional matrices in the vector representation of $SO(1, 2)$

We can then combine the generators $\mathcal{L}_{AB}$ into the quadratic Casimir:

$$\begin{aligned} \mathcal{C}_2^{(1)} \equiv -\frac{1}{2}\mathcal{L}_{AB}^{(1)}\mathcal{L}^{AB,(1)} = & -\frac{1}{2}\text{Tr}(g_1^{-1}L_{AB}g_2)\text{Tr}(g_1^{-1}L^{AB}g_2)\frac{\partial^2}{\partial\tau_{12}^2} - \frac{1}{2}\text{Tr}(g_1^{-1}L_{AB}g_3)\text{Tr}(g_1^{-1}L^{AB}g_3)\frac{\partial^2}{\partial\tau_{13}^2} \\ & - \text{Tr}(g_1^{-1}L_{AB}g_2)\text{Tr}(g_1^{-1}L^{AB}g_3)\frac{\partial^2}{\partial\tau_{12}\partial\tau_{13}} - \frac{1}{2}\text{Tr}(g_1^{-1}L_{AB}L^{AB}g_2)\frac{\partial}{\partial\tau_{12}} - \frac{1}{2}\text{Tr}(g_1^{-1}L_{AB}L^{AB}g_3)\frac{\partial}{\partial\tau_{13}}. \end{aligned} \quad (3.155)$$

In vector representations, matrix elements of the generators are given by:

$$(L_{AB})^I_J = \delta^I_A\eta_{BJ} - \delta^I_B\eta_{AJ}, \quad (3.156)$$

therefore,

$$\begin{aligned} \text{Tr}(L_{AB}h_1L^{AB}h_2) &= 2\text{Tr}(h_1^{-1}h_2) - 2\text{Tr}(h_1)\text{Tr}(h_2), \\ \text{Tr}(L_{AB}h_1)\text{Tr}(L^{AB}h_2) &= 2\text{Tr}(h_1^{-1}h_2) - 2\text{Tr}(h_1h_2), \end{aligned} \quad (3.157)$$

thus,

$$\text{Tr}(g_1^{-1}L_{AB}g_2)\text{Tr}(g_1^{-1}L^{AB}g_2) = 6 - 2\tau_{1212}, \quad \text{Tr}(g_1^{-1}L_{AB}g_2)\text{Tr}(g_1^{-1}L^{AB}g_3) = 2\tau_{23} - 2\tau_{1213}, \quad (3.158)$$

$$\text{Tr}(g_1^{-1}L_{AB}L^{AB}g_2) = -4\tau_{12}, \quad (3.159)$$

where we've used the fact that the Casimir operator of $SO(1, d+1)$ vector representations is always $(d+1)\mathbb{I}$. Invariants such as $\tau_{1212}, \tau_{1313}, \tau_{1213}$ are related to $\tau_{12}, \tau_{23}, \tau_{13}$ as:

$$\begin{aligned} \tau_{1212} &= \tau_{12}^2 - 2\tau_{12}, \quad \tau_{1313} = \tau_{13}^2 - 2\tau_{13}, \\ \tau_{1213} &= \tau_{12}\tau_{13} + \tau_{12} + \tau_{13} + \tau_{23} + 1 \pm \sqrt{(\tau_{12} + 1)(\tau_{13} + 1)(\tau_{23} + 1)}. \end{aligned} \quad (3.160)$$

The first two identities above follow from the Cayley-Hamilton theorem, that is, the matrices $g_i^{-1}g_j$ satisfy the following matrix identity:

$$(g_i^{-1}g_j)^3 - \text{Tr}(g_i^{-1}g_j)(g_i^{-1}g_j)^2 + \text{Tr}(g_i^{-1}g_j)(g_i^{-1}g_j) - \mathbb{I} = 0, \quad (3.161)$$

the last identity in (3.160) looks complicated and the $\pm$ sign introduces ambiguity. To understand it, we need to consider the spinor representation of $SO(1, 2)$. Since

the vector representation is just the symmetric product of the spinor representation, we have:

$$\tau_I^{\text{vector}} = (\tau_I^{\text{spinor}})^2 - 1. \quad (3.162)$$

The $SL(2, \mathbb{R})$ matrices satisfy the following identities:

$$g^2 - \text{Tr}(g)g + \mathbb{I} = 0, \quad (3.163)$$

therefore:

$$\tau_{1213}^{\text{spinor}} = \tau_{12}^{\text{spinor}} \tau_{13}^{\text{spinor}} - \tau_{23}^{\text{spinor}}. \quad (3.164)$$

This explains the last identity in (3.160) and the sign in front of $\sqrt{\cdots}$ is determined by $\text{sgn}(\tau_{12}^{\text{spinor}} \tau_{13}^{\text{spinor}} \tau_{23}^{\text{spinor}})$.

$d = 1$, spinor representation

From the discussion above, it is clear that characters in the spinor representation provide a better set of coordinates. We write down the complete expression of the Casimir operator in terms of τ_I^{spinor} in this subsection:

$$\begin{aligned} \mathcal{C}_2^{(1)} &= \left(\frac{1}{4} \tau_{12}^2 - 1 \right) \partial_{\tau_{12}}^2 + \left(\frac{1}{4} \tau_{13}^2 - 1 \right) \partial_{\tau_{13}}^2 + \left(\frac{1}{2} \tau_{12} \tau_{13} - \tau_{23} \right) \partial_{\tau_{13}} \partial_{\tau_{12}} \\ &\quad + \frac{3}{4} \tau_{12} \partial_{\tau_{12}} + \frac{3}{4} \tau_{13} \partial_{\tau_{13}}, \\ \mathcal{C}_2^{(2)} &= \left(\frac{1}{4} \tau_{12}^2 - 1 \right) \partial_{\tau_{12}}^2 + \left(\frac{1}{4} \tau_{23}^2 - 1 \right) \partial_{\tau_{23}}^2 + \left(\frac{1}{2} \tau_{12} \tau_{23} - \tau_{13} \right) \partial_{\tau_{13}} \partial_{\tau_{23}} \\ &\quad + \frac{3}{4} \tau_{12} \partial_{\tau_{12}} + \frac{3}{4} \tau_{23} \partial_{\tau_{23}}, \\ \mathcal{C}_2^{(3)} &= \left(\frac{1}{4} \tau_{13}^2 - 1 \right) \partial_{\tau_{13}}^2 + \left(\frac{1}{4} \tau_{23}^2 - 1 \right) \partial_{\tau_{23}}^2 + \left(\frac{1}{2} \tau_{13} \tau_{23} - \tau_{12} \right) \partial_{\tau_{13}} \partial_{\tau_{23}} \\ &\quad + \frac{3}{4} \tau_{23} \partial_{\tau_{23}} + \frac{3}{4} \tau_{13} \partial_{\tau_{13}}. \end{aligned} \quad (3.165)$$

Solving the 1d Casimir differential equations order by order in the low temperature limit¹⁰, we find the following expansion for the 1d genus-2 sunrise block:

$$B_{\Delta_1, \Delta_2, \Delta_3}$$

¹⁰By “low temperature limit”, we mean when $e^{-\beta_{ij}} \sim \epsilon$ and $\epsilon \rightarrow 0$. Here $\beta_{ij} = 2 \cosh^{-1}(\tau_{ij}/2)$.

$$\begin{aligned}
&= y_{12}^{\Delta_1+\Delta_2-\Delta_3} y_{13}^{\Delta_1-\Delta_2+\Delta_3} y_{23}^{-\Delta_1+\Delta_2+\Delta_3} \\
&\times \left(1 + \frac{(\Delta_1 + \Delta_2 - \Delta_3)(-\Delta_1 + \Delta_2 + \Delta_3)}{2\Delta_2} \frac{y_{12}y_{23}}{y_{13}} \right. \\
&\quad + \frac{(\Delta_1 - \Delta_2 + \Delta_3)(-\Delta_1 + \Delta_2 + \Delta_3)}{2\Delta_3} \frac{y_{13}y_{23}}{y_{12}} \\
&\quad + \frac{(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 - \Delta_2 + \Delta_3)}{2\Delta_1} \frac{y_{13}y_{12}}{y_{23}} \\
&\quad + \frac{(\Delta_1 - \Delta_2 - \Delta_3)^2(\Delta_1 + \Delta_2 - \Delta_3 - 1)(\Delta_1 - \Delta_2 + \Delta_3 - 1)}{4\Delta_2\Delta_3} y_{23}^2 \\
&\quad - \frac{(\Delta_1 - \Delta_2 - \Delta_3 + 1)(\Delta_1 + \Delta_2 - \Delta_3 - 1)(\Delta_1 - \Delta_2 + \Delta_3)^2}{4\Delta_1\Delta_3} y_{13}^2 \\
&\quad - \frac{(\Delta_1 - \Delta_2 - \Delta_3 + 1)(\Delta_1 + \Delta_2 - \Delta_3)^2(\Delta_1 - \Delta_2 + \Delta_3 - 1)}{4\Delta_1\Delta_2} y_{12}^2 \\
&\quad + \frac{(\Delta_1 - \Delta_2 - \Delta_3 - 1)(\Delta_1 - \Delta_2 - \Delta_3)(\Delta_1 - \Delta_2 + \Delta_3)(\Delta_1 - \Delta_2 + \Delta_3 + 1)}{4\Delta_3(2\Delta_3 + 1)} \frac{y_{13}^2 y_{23}^2}{y_{12}^2} \\
&\quad + \frac{(\Delta_1 - \Delta_2 - \Delta_3 - 1)(\Delta_1 - \Delta_2 - \Delta_3)(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 + \Delta_2 - \Delta_3 + 1)}{4\Delta_2(2\Delta_2 + 1)} \frac{y_{12}^2 y_{23}^2}{y_{13}^2} \\
&\quad + \frac{(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 + \Delta_2 - \Delta_3 + 1)(\Delta_1 - \Delta_2 + \Delta_3)(\Delta_1 - \Delta_2 + \Delta_3 + 1)}{4\Delta_1(2\Delta_1 + 1)} \frac{y_{12}^2 y_{13}^2}{y_{23}^2} \\
&\quad \left. + \dots \right), \tag{3.166}
\end{aligned}$$

where

$$y_{ij} = e^{-\beta_{ij}/2}, \quad \tau_{ij} = 2 \cosh \left(\frac{\beta_{ij}}{2} \right). \tag{3.167}$$

Each term in this expansion comes from a descendant state in the multiplet $\mathcal{R}_{\Delta_1} \otimes \mathcal{R}_{\Delta_2} \otimes \mathcal{R}_{\Delta_3}$:

$$y_{12}^{n_1+n_2-n_3} y_{13}^{n_1+n_3-n_2} y_{23}^{n_2+n_3-n_1} \leftrightarrow (P^{n_1}|\Delta_1\rangle) \otimes (P^{n_2}|\Delta_2\rangle) \otimes (P^{n_3}|\Delta_3\rangle) \tag{3.168}$$

Observe that:

$$\lim_{\Delta_3 \rightarrow 0} \lim_{\Delta_1, \Delta_2 \rightarrow \Delta} B_{\Delta_1, \Delta_2, \Delta_3}(y_{12}, y_{13}, y_{23}) = y_{12}^{2\Delta} (1 + y_{12}^2 + \dots), \tag{3.169}$$

which is the low-temperature expansion of the character $e^{-\beta_{12}\Delta}/(1 - e^{-\beta_{12}})$. Similarly:

$$\lim_{\Delta_1 \rightarrow 0} \lim_{\Delta_2, \Delta_3 \rightarrow \Delta} B_{\Delta_1, \Delta_2, \Delta_3}(y_{12}, y_{13}, y_{23}) = y_{23}^{2\Delta} (1 + y_{23}^2 + \dots), \tag{3.170}$$

$$\lim_{\Delta_2 \rightarrow 0} \lim_{\Delta_1, \Delta_3 \rightarrow \Delta} B_{\Delta_1, \Delta_2, \Delta_3}(y_{12}, y_{13}, y_{23}) = y_{13}^{2\Delta} (1 + y_{13}^2 + \dots). \tag{3.171}$$

$d = 3$, spinor representation

Given our experience with 1d Casimir equation, we focus on the spinor representation.

We will be using the following set of invariants as coordinates on the moduli space:

$$\mathcal{S} = \{\tau_{12}, \tau_{23}, \tau_{13}, \tau_{1212}, \tau_{1313}, \tau_{2323}, \tau_{1213}, \tau_{1232}, \tau_{1323}, \tau_{121323}\}. \quad (3.172)$$

Following a similar derivation as in the 1d case, we find:

$$\begin{aligned} \mathcal{C}_2^{(1)} = & -2(\tau_{12} - \tau_{121212})\partial_{\tau_{12}}\partial_{\tau_{1212}} + \frac{1}{2}(\tau_{1212} - 4)\partial_{\tau_{12}}^2 + 2(\tau_{121213} - \tau_{13})\partial_{\tau_{12}}\partial_{\tau_{1213}} \\ & + 2(\tau_{12121323} - \tau_{1323})\partial_{\tau_{12}}\partial_{\tau_{121323}} + (\tau_{121232} - \tau_{23})\partial_{\tau_{12}}\partial_{\tau_{1232}} - 2(\tau_{12} - \tau_{121313})\partial_{\tau_{1213}}\partial_{\tau_{13}} \\ & + (\tau_{1213} - \tau_{23})\partial_{\tau_{12}}\partial_{\tau_{13}} + 2(\tau_{121313} - \tau_{1323})\partial_{\tau_{12}}\partial_{\tau_{1313}} - 2(\tau_{12} - \tau_{1213231323})\partial_{\tau_{121323}}\partial_{\tau_{1323}} \\ & + (\tau_{121323} - \tau_{2323})\partial_{\tau_{12}}\partial_{\tau_{1323}} - 2(\tau_{12} - \tau_{13131323})\partial_{\tau_{1313}}\partial_{\tau_{1323}} + 2(\tau_{12121212} - 4)\partial_{\tau_{1212}}^2 \\ & + 4(\tau_{12121213} - \tau_{23})\partial_{\tau_{1212}}\partial_{\tau_{1213}} + 4(\tau_{1212121323} - \tau_{2323})\partial_{\tau_{1212}}\partial_{\tau_{121323}} + 2(\tau_{12121232} - \tau_{13})\partial_{\tau_{1212}}\partial_{\tau_{1232}} \\ & + 2(\tau_{121213} - \tau_{1232})\partial_{\tau_{1212}}\partial_{\tau_{13}} + 4(\tau_{12121313} - \tau_{123132})\partial_{\tau_{1212}}\partial_{\tau_{1313}} + 2(\tau_{12121323} - \tau_{123232})\partial_{\tau_{1212}}\partial_{\tau_{1323}} \\ & + (\tau_{12121313} + \tau_{12131213} - \tau_{123123} - 4)\partial_{\tau_{1213}}^2 \\ & + (\tau_{1212131323} + \tau_{1212132313} + \tau_{1213121323} - \tau_{12312323} - \tau_{12323123} - 2\tau_{23})\partial_{\tau_{1213}}\partial_{\tau_{121323}} \\ & + (\tau_{121213231323} + \tau_{121323121323} - \tau_{1232312323} - 4)\partial_{\tau_{121323}}^2 + (\tau_{12123213} + \tau_{12131232} - 2\tau_{1323})\partial_{\tau_{1213}}\partial_{\tau_{1232}} \\ & + (\tau_{1212321323} + \tau_{1213231232} - 2\tau_{132323})\partial_{\tau_{121323}}\partial_{\tau_{1232}} + 4(\tau_{12131313} - \tau_{23})\partial_{\tau_{1213}}\partial_{\tau_{1313}} \\ & + (\tau_{12131323} + \tau_{12132313} - 2\tau_{1232})\partial_{\tau_{1213}}\partial_{\tau_{1323}} + 2(\tau_{1213131323} + \tau_{1213231313} - 2\tau_{123123})\partial_{\tau_{121323}}\partial_{\tau_{1313}} \\ & + (\tau_{12131323} + \tau_{12132313} - 2\tau_{13})\partial_{\tau_{121323}}\partial_{\tau_{13}} + \frac{1}{2}(\tau_{12321232} - 4)\partial_{\tau_{1232}}^2 + (\tau_{123213} - \tau_{2323})\partial_{\tau_{13}}\partial_{\tau_{1232}} \\ & + 2(\tau_{12321313} - \tau_{132323})\partial_{\tau_{1313}}\partial_{\tau_{1232}} + (\tau_{12321323} - \tau_{232323})\partial_{\tau_{1323}}\partial_{\tau_{1232}} - 2(\tau_{13} - \tau_{131313})\partial_{\tau_{13}}\partial_{\tau_{1313}} \\ & + \frac{1}{2}(\tau_{1313} - 4)\partial_{\tau_{13}}^2 + (\tau_{131323} - \tau_{23})\partial_{\tau_{1323}}\partial_{\tau_{13}} + 2(\tau_{13131313} - 4)\partial_{\tau_{1313}}^2 + \frac{1}{2}(\tau_{13231323} - 4)\partial_{\tau_{1323}}^2 \\ & + \frac{5}{2}\tau_{12}\partial_{\tau_{12}} + \frac{1}{2}(5\tau_{1212} + \tau_{12}^2 + 4)\partial_{\tau_{1212}} + \frac{5}{2}\tau_{31}\partial_{\tau_{31}} + \frac{1}{2}(5\tau_{3131} + \tau_{31}^2 + 4)\partial_{\tau_{3131}} \\ & + (5\tau_{1213} + \tau_{12}\tau_{31} + \tau_{23})\partial_{\tau_{1213}} + \frac{5}{2}\tau_{1323}\partial_{\tau_{1323}} + \frac{5}{2}\tau_{1232}\partial_{\tau_{1232}} + (5\tau_{121323} + \tau_{12}\tau_{1323} + \tau_{2323})\partial_{\tau_{121323}}. \end{aligned} \quad (3.173)$$

The expressions for $\mathcal{C}_2^{(2)}, \mathcal{C}_2^{(3)}$ take analogous forms and we omit them for brevity. The derivation of the 3d Casimir operator is almost identical as the 1d version except that one needs to be careful about multiple occurrence of the same letter. Let's look at an example:

$$\begin{aligned} & -\frac{1}{2}(\mathcal{L}_{AB}\tau_{1213})\left(\mathcal{L}^{AB}\tau_{121323}\right) \\ & = -\frac{1}{2}\left(\text{Tr}(g_1^{-1}L_{AB}g_2g_1^{-1}g_3) + \text{Tr}(g_1^{-1}g_2g_1^{-1}L_{AB}g_3)\right)\left(\text{Tr}(g_1^{-1}L^{AB}g_2g_1^{-1}g_3g_2^{-1}g_3) + \text{Tr}(g_1^{-1}g_2g_1^{-1}L^{AB}g_3g_2^{-1}g_3)\right) \\ & = (\tau_{1212131323} + \tau_{1212132313} + 2\tau_{1213121323} - \tau_{12312323} - \tau_{12323123} - 2\tau_{23}), \end{aligned} \quad (3.174)$$

where we've used the fact that in spinor representation:

$$\text{Tr}(L_{AB}h_1)\text{Tr}(L^{AB}h_2) = \text{Tr}(h_2^{-1}h_1) - \text{Tr}(h_1h_2). \quad (3.175)$$

In (3.173), there are a few invariants which do not belong to the set of 10 coordintas:

$$\begin{aligned} &\tau_{121212}, \tau_{12121212}, \tau_{121313}, \tau_{1213231323}, \tau_{1213131323}, \tau_{1213231313}, \tau_{123123}, \tau_{12121313}, \tau_{12131213}, \tau_{123132}, \tau_{121213}, \\ &\tau_{12131323}, \tau_{12132313}, \tau_{12321232}, \tau_{121213231323}, \tau_{121323121323}, \tau_{1232312323}, \tau_{12121323}, \tau_{123232}, \tau_{12121232}, \tau_{131313}, \\ &\tau_{13131313}, \tau_{13131323}, \tau_{12123213}, \tau_{12131232}, \tau_{13231323}, \tau_{12321313}, \tau_{132323}, \tau_{1212321323}, \tau_{1213231232}, \tau_{12121213}, \tau_{12131313}, \\ &\tau_{121232}, \tau_{131323}, \tau_{1212131323}, \tau_{1212132313}, \tau_{1213121323}, \tau_{12312323}, \tau_{12323123}, \tau_{1212121323}, \tau_{123213}, \tau_{12321323}, \tau_{232323}. \end{aligned} \quad (3.176)$$

Many of them can be reduced to a polynomial of the 10 coordinates using the following Cayley-Hamilton identity:

$$g^4 - \text{Tr}(g)g^3 + \frac{1}{2}(\text{Tr}(g)^2 - \text{Tr}(g^2))g^2 - \text{Tr}(g)g^3 + \mathbb{I} = 0, \quad (3.177)$$

for example:

$$\begin{aligned} \tau_{12(13)^2} &= -\frac{1}{2}\tau_{12}(\tau_{13}^2 - \tau_{1313}) + \tau_{1213}\tau_{13} + \tau_{13}\tau_{23} - \tau_{1323}, \\ \tau_{(12)^3} &= \frac{1}{2}(3\tau_{12}\tau_{1212} - \tau_{12}^3 + 6\tau_{12}), \\ \tau_{(12)^4} &= \frac{1}{2}(2\tau_{12}^2(\tau_{1212} + 4) - \tau_{12}^4 + \tau_{1212}^2 - 8), \\ \tau_{(12)^332} &= \frac{1}{2}(\tau_{12}((\tau_{1212} + 2)\tau_{23} - 2\tau_{1213}) + \tau_{12}^2(\tau_{1232} + 2\tau_{13}) - \tau_{12}^3\tau_{23} + \tau_{1212}\tau_{1232} - 2\tau_{13}). \end{aligned} \quad (3.178)$$

However, there are some other invariants which we don't know how to express in terms of the ten coordinates. One such example is τ_{123123} . By numerically computing the Jacobian between the invariants and group parameters, we verify that τ_{123123} is not independent from the ones listed in 3.172. From experience in 1d, one expect to find a polynomial relation between τ_{123123} and those in $\mathcal{S}$. We numerically searched over the following set of monomials:

$$\left\{ \prod_{I \in \mathcal{S}'} \tau_I^{n_I} \mid \sum_I n_I \leq 4 \right\}, \quad \mathcal{S}' = \mathcal{S} \cup \{\tau_{123123}\}. \quad (3.179)$$

but no linear relation was found. As a first sanity check, note that when acting on a function which only depends on τ_{12} and τ_{1212} , $\mathcal{C}_2^{(1)}$ reduces to:

$$\begin{aligned} \mathcal{C}_2^{(1)} f(\tau_{12}, \tau_{1212}) &= 4(\tau_{12121212} - 4) \partial_{\tau_{1212}}^2 f + 2(\tau_{121212} - \tau_{12}) \partial_{\tau_{12}} \partial_{\tau_{1212}} f + 2\left(\frac{\tau_{1212}}{4} - 1\right) \partial_{\tau_{12}}^2 f \\ &\quad + 2\left(\frac{\tau_{12}^2}{2} + \frac{5\tau_{1212}}{2} + 2\right) \partial_{\tau_{1212}} f + \frac{5}{2} \tau_{12} \partial_{\tau_{12}} f \end{aligned} \quad (3.180)$$

Switching to the coordiantes y_{12}, x_{12} defined using the relation:

$$y_{12} = \exp(-\beta_{12}/2), \quad x_{12} = \exp(i\theta_{12}/2) \quad (3.181)$$

$$\tau_{12} = x_{12}y_{12} + \frac{1}{x_{12}y_{12}} + \frac{x_{12}}{y_{12}} + \frac{y_{12}}{x_{12}}, \quad \tau_{1212} = x_{12}^2y_{12}^2 + \frac{1}{x_{12}^2y_{12}^2} + \frac{x_{12}^2}{y_{12}^2} + \frac{y_{12}^2}{x_{12}^2} \quad (3.182)$$

we get:

$$\begin{aligned} \mathcal{C}_2^{(1)} f(x_{12}, y_{12}) &= \frac{1}{4} \left(x_{12} \partial_{x_{12}}^2 + y_{12} \partial_{y_{12}}^2 \right. \\ &\quad + \frac{y_{12} (7x_{12}^2y_{12}^6 - 3x_{12}^4y_{12}^4 - 3x_{12}^2y_{12}^4 - x_{12}^4y_{12}^2 - x_{12}^2y_{12}^2 + 5x_{12}^2 - 3y_{12}^4 - y_{12}^2)}{(y_{12} - 1)(y_{12} + 1)(y_{12} - x_{12})(x_{12} + y_{12})(x_{12}y_{12} - 1)(x_{12}y_{12} + 1)} \partial_{y_{12}} \\ &\quad \left. + \frac{x_{12} (7x_{12}^6y_{12}^2 - 3x_{12}^4y_{12}^4 - 3x_{12}^4y_{12}^2 - x_{12}^2y_{12}^4 - x_{12}^2y_{12}^2 - 3x_{12}^4 - x_{12}^2 + 5y_{12}^2)}{(x_{12} - 1)(x_{12} + 1)(x_{12} - y_{12})(x_{12} + y_{12})(x_{12}y_{12} - 1)(x_{12}y_{12} + 1)} \partial_{x_{12}} \right) f(x_{12}, y_{12}). \end{aligned} \quad (3.183)$$

It is then easy to check that the conformal character is indeed an eigenfunction of $\mathcal{C}_2^{(1)}$:

$$\begin{aligned} \mathcal{C}_2^{(1)} \chi_{\Delta,j}(x_{12}, y_{12}) &= [\Delta(\Delta - 3) + j(j + 1)] \chi_{\Delta,j}(x_{12}, y_{12}), \\ \chi_{\Delta,j}(x_{12}, y_{12}) &= \frac{y_{12}^{2\Delta}}{(1 - y_{12}^2)(1 - x_{12}^2y_{12}^2)(1 - x_{12}^{-2}y_{12}^2)} \frac{x_{12}^{2j+1} - x_{12}^{-2j-1}}{x_{12}^2 - x_{12}^{-2}}. \end{aligned} \quad (3.184)$$

Casimir equation for the dumbbell block

For completeness, we include Casimir equations for the dumbbell block in. The dumbbell block $B_{L,R,M}$ should satisfy the follwoing three Casimir equations simultaneously:

$$\begin{aligned} \mathcal{C}_2^{(L)} B_{L,R,M} &= C_2(\pi_L) B_{L,R,M}, \\ \mathcal{C}_2^{(R)} B_{L,R,M} &= C_2(\pi_R) B_{L,R,M}, \\ \mathcal{C}_2^{(M)} B_{L,R,M} &= C_2(\pi_M) B_{L,R,M}. \end{aligned} \quad (3.185)$$

In 1d, following our convention in section 3.3, $\tau_{12}, \tau_{13}, \tau_{23}$ can be written as:

$$\tau_{13} = \text{Tr}(g_L^{-1}), \quad \tau_{23} = \text{Tr}(g_R), \quad \tau_{12} = \text{Tr}(g_M^{-1} g_L^{-1} g_M g_R). \quad (3.186)$$

Working with the spinor representatio, we find that $\mathcal{C}_2^{(L)} = \mathcal{C}_2^{(1)}$, $\mathcal{C}_2^{(R)} = \mathcal{C}_2^{(2)}$ but $\mathcal{C}_2^{(M)}$ is different from $\mathcal{C}_2^{(3)}$.

Since τ_{12} is the only variable sensitive to g_M , we have:

$$\begin{aligned} \mathcal{C}_2^{(M)} &= -\frac{1}{2} \text{Tr}(g_M^{-1} [g_L^{-1}, L_{AB}] g_M g_R) \text{Tr}(g_M^{-1} [g_L^{-1}, L^{AB}] g_M g_R) \frac{\partial^2}{\partial \tau_{12}^2} \\ &\quad - \frac{1}{2} \text{Tr}(g_M^{-1} [[g_L^{-1}, L_{AB}], L^{AB}] g_M g_R) \frac{\partial}{\partial \tau_{12}}, \end{aligned} \quad (3.187)$$

with

$$\begin{aligned} -\frac{1}{2}\text{Tr}(g_M^{-1}[g_L^{-1}, L_{AB}]g_M g_R)\text{Tr}(g_M^{-1}[g_L^{-1}, L^{AB}]g_M g_R) &= -\tau_{12}\tau_{13}\tau_{23} + \tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2 - 4, \\ -\frac{1}{2}\text{Tr}(g_M^{-1}[[g_L^{-1}, L_{AB}], L^{AB}]g_M g_R) &= 2\tau_{12} - \tau_{13}\tau_{23}, \end{aligned} \quad (3.188)$$

where we've used the following identities for matrices in the 2-dimensional representation of $Spin(2, 1)$:

$$\begin{aligned} \text{Tr}(L_{AB}h_1)\text{Tr}(L^{AB}h_2) &= \frac{1}{2}\text{Tr}(h_1)\text{Tr}(h_2) - \text{Tr}(h_1h_2), \\ \text{Tr}(L_{AB}h_1L^{AB}h_2) &= \frac{1}{2}\text{Tr}(h_1h_2) - \text{Tr}(h_1)\text{Tr}(h_2). \end{aligned} \quad (3.189)$$

The 1d dumbbell block has the following low-temperature expansion:

$$\begin{aligned} B_{\Delta_L, \Delta_R, \Delta_M}(y_{12}, y_{13}, y_{23}) &= y_{13}^{2\Delta_L} y_{23}^{2\Delta_R} \left(1 - \frac{y_{13}y_{23}(\Delta_M - 1)\Delta_M}{y_{12}} \right. \\ &+ \frac{y_{13}^2(2\Delta_L((\Delta_M - 1)\Delta_M + 1) + (\Delta_M - 1)\Delta_M)}{2\Delta_L} + \frac{y_{23}^2(\Delta_M^2(2\Delta_R + 1) - \Delta_M(2\Delta_R + 1) + 2\Delta_R)}{2\Delta_R} \\ &\left. + \frac{y_{13}y_{23}^2(\Delta_M - 2)(\Delta_M - 1)\Delta_M(\Delta_M + 1)}{4y_{12}^2} + \dots \right). \end{aligned} \quad (3.190)$$

Similarly in 3d, the ten coordinates in $\mathcal{S}$ can be written as:

$$\begin{aligned} \tau_{13} &= \text{Tr}(g_L^{-1}), \quad \tau_{23} = \text{Tr}(g_R), \quad \tau_{12} = \text{Tr}(g_M^{-1}g_L^{-1}g_M g_R), \\ \tau_{1313} &= \text{Tr}(g_L^{-2}), \quad \tau_{2323} = \text{Tr}(g_R^2), \quad \tau_{1212} = \text{Tr}((g_M^{-1}g_L^{-1}g_M g_R)^2), \\ \tau_{1213} &= \text{Tr}(g_M^{-1}g_L^{-2}g_M g_R), \quad \tau_{1232} = \text{Tr}(g_M^{-1}g_L^{-1}g_M g_R^2), \quad \tau_{1323} = \text{Tr}(g_M^{-1}g_L^{-1}g_M g_R^{-1}) \\ \tau_{121323} &= \text{Tr}(g_M^{-1}g_L^{-1}g_M g_R^{-1}g_M^{-1}g_L^{-1}g_M g_R). \end{aligned} \quad (3.191)$$

Working with the spinor representation, we can again check that $\mathcal{C}_2^{(L)} = \mathcal{C}_2^{(1)}$, $\mathcal{C}_2^{(R)} = \mathcal{C}_2^{(2)}$ while $\mathcal{C}_2^{(M)}$ is given by:

$$\begin{aligned} \mathcal{C}_2^{(M)} &= (\tau_{1212} + \tau_{123123} - \tau_{123213} - 4)\partial_{\tau_{12}}^2 \\ &+ 2(-2\tau_{12} + 2\tau_{121212} + \tau_{12123123} - \tau_{12123213} - \tau_{12131232} + \tau_{12132132})\partial_{\tau_{12}}\partial_{\tau_{1212}} \\ &+ 4(\tau_{12121212} + \tau_{1212312123} - \tau_{1212321213} - 4)\partial_{\tau_{1212}}^2 \\ &+ (2\tau_{121213} + \tau_{12312313} + \tau_{12313123} - 2\tau_{12321313} - 2\tau_{13})\partial_{\tau_{12}}\partial_{\tau_{1213}} \\ &+ 2(\tau_{121323} + \tau_{123132} - \tau_{1313} - \tau_{2323})\partial_{\tau_{12}}\partial_{\tau_{1323}} \\ &+ (4\tau_{12121323} - 2\tau_{121313} + 2\tau_{12132123} + \tau_{1231231323} + \tau_{1231323123} - \tau_{1232131323} - \tau_{1232132313} - 4\tau_{1323})\partial_{\tau_{12}}\partial_{\tau_{121323}} \\ &+ (4\tau_{12121213} + 2\tau_{1212313123} - 2\tau_{1212321313} - 2\tau_{1213131232} + 2\tau_{1213213132} - 4\tau_{23})\partial_{\tau_{1212}}\partial_{\tau_{1213}} \\ &+ 4(\tau_{12121323} + \tau_{12123132} - \tau_{121313} - \tau_{123232})\partial_{\tau_{1212}}\partial_{\tau_{1323}} \\ &+ 2(4\tau_{1212121323} + 2\tau_{1212312132} + \tau_{121231323123} - \tau_{121232132313} - 2\tau_{12131213} - \tau_{121313231232} + \tau_{121321323132} - 4\tau_{2323}) \\ &\quad \times \partial_{\tau_{1212}}\partial_{\tau_{121323}} \\ &+ (\tau_{12131213} + \tau_{1231312313} - \tau_{1232131313} - 4)\partial_{\tau_{1213}}^2 \\ &+ (\tau_{12131323} + \tau_{12132313} + 2\tau_{12313213} - 2\tau_{1232} - 2\tau_{131313})\partial_{\tau_{1213}}\partial_{\tau_{1323}} \end{aligned}$$

$$\begin{aligned}
& + \left(\tau_{1212131323} + \tau_{1212132313} + 2\tau_{1213121323} - 2\tau_{12131313} + 2\tau_{1213213123} - \tau_{12312323} \right. \\
& \quad \left. + \tau_{123131231323} + \tau_{123132312313} - \tau_{123213131323} - \tau_{123213231313} - \tau_{12323123} - 2\tau_{23} \right) \partial_{\tau_{1213}} \partial_{\tau_{121323}} \\
& + (\tau_{123123} - \tau_{13132323} + \tau_{13231323} - 4) \partial_{\tau_{1323}}^2 \\
& + (-4\tau_{12} + \tau_{12123123} - \tau_{1213132323} + \tau_{12132132} + 4\tau_{1213231323} - \tau_{1213232313} + 2\tau_{1231321323} - 2\tau_{13131323}) \\
& \quad \times \partial_{\tau_{121323}} \partial_{\tau_{1323}} \\
& + \left(2\tau_{121213231323} + \tau_{1212312123} - \tau_{121312132323} - 2\tau_{1213132313} + 2\tau_{121321323123} + 2\tau_{121323121323} \right. \\
& \quad \left. + \tau_{12313231231323} - \tau_{12321323131323} - 2\tau_{1232312323} - 8 \right) \partial_{\tau_{121323}}^2 \\
& + (2\tau_{121232} + \tau_{12312323} + \tau_{12323123} - 2\tau_{12323213} - 2\tau_{23}) \partial_{\tau_{12}} \partial_{\tau_{1232}} \\
& + (4\tau_{12121232} + 2\tau_{1212312323} - 2\tau_{1212323213} - 2\tau_{1213123232} + 2\tau_{1213232132} - 4\tau_{13}) \partial_{\tau_{1212}} \partial_{\tau_{1232}} \\
& + (-2\tau_{1213} + \tau_{12313232} + 2\tau_{12321323} + \tau_{12323132} - 2\tau_{232323}) \partial_{\tau_{1232}} \partial_{\tau_{1323}} \\
& + \left(-2\tau_{121213} + 2\tau_{1212321323} + \tau_{1213212323} + 2\tau_{1213231232} + \tau_{1213232123} + \tau_{123123231323} + \tau_{123231323123} \right. \\
& \quad \left. - \tau_{123232131323} - \tau_{123232132313} - 4\tau_{132323} \right) \partial_{\tau_{121323}} \partial_{\tau_{1232}} \\
& + (\tau_{12321232} + \tau_{1232312323} - \tau_{1232323213} - 4) \partial_{\tau_{1232}}^2 \\
& + (5\tau_{12} - \tau_{13}\tau_{23} - \tau_{1323}) \partial_{\tau_{12}} + (2\tau_{12}^2 + 10\tau_{1212} - 2\tau_{1213}\tau_{23} - 2\tau_{121323} - 2\tau_{123132} - 2\tau_{1232}\tau_{13} + 8) \partial_{\tau_{1212}} \\
& + (5\tau_{1213} - \tau_{1313}\tau_{23} - \tau_{131323}) \partial_{\tau_{1213}} + (-\tau_{12} - \tau_{13}\tau_{23} + 5\tau_{1323}) \partial_{\tau_{1323}} \\
& + (5\tau_{1232} - \tau_{13}\tau_{2323} - \tau_{132323}) \partial_{\tau_{1232}} \\
& + (2\tau_{12}\tau_{1323} - \tau_{1212} - \tau_{1213}\tau_{23} + 10\tau_{121323} - 2\tau_{123123} - 2\tau_{13}^2 - \tau_{131323}\tau_{23} - \tau_{13231323} + 2\tau_{2323}) \partial_{\tau_{121323}}
\end{aligned} \tag{3.192}$$

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