

SERRATED APERTURES:
STATISTICAL DIFFRACTION THEORY AND EXPERIMENTS

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George Michael Morris

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ABSTRACT

Diffraction of monochromatic light by rough apertures is analyzed. Correlation functions are derived for the electric field and for the intensity in the optical transform plane. The expectations calculated are over an ensemble of edges; their roughness is described in terms of a second order density and associated characteristic function. It is shown that in-plane roughness causes a speckle pattern. The analytical details are markedly different from the more usual case in which speckle is caused by longitudinal phase delay across an extended aperture. Detailed solutions are presented for serrated edges and gaps. Both space and wavelength dependences are included and solutions for cross-correlations of electric field and intensity are obtained. These are valid for arbitrary roughness and correlation coefficient. Experiments are described contrasting the optical transforms of serrated and sharp edges. The serration causes a damping of the major spike in the edge transform and it leads to considerable scattering of the radiation. It is shown that edge statistics may be determined by processing the transform plane intensity measurements. Edge roughness (or standard deviation) is established from transform plane measurements along the central spike. For edges and gaps, intensity measurements of the spread of energy perpendicular to the central spike are Fourier transformed to determine

the functional form for the correlation coefficient of the edge roughness. Two classes of serrated edges have been fabricated to verify the theory. One group is generated by plotting computer-generated number sets, with independently controlled roughness and correlation length, on microfiche. A metallic edge is then made from the film images using photo-etching techniques. These edges have been particularly useful in verifying the theory.

TABLE OF CONTENTS

	Page
I INTRODUCTION	1-6
II THEORY FOR THE SERRATED APERTURE	7-66
2.1 Introduction	7
2.2 General Formulation	9
2.2.1 The Diffracted Field	9
2.2.2 Classification of Statistical Problem in Diffraction Theory	13
2.2.3 Statistics for the Serration	16
2.2.4 Moments of the Electric Field	19
2.3 The Serrated Edge	22
2.3.1 Second-Order Moment of the Field, $\langle E_1 E_2^* \rangle$	22
2.3.2 Fourth-Order Moment of the Field, $\langle E_1 E_1^* E_2 E_2^* \rangle$	31
2.3.3 Discussion of Solution	48
2.4 The Serrated Gap	56
III REMOTE SENSING OF EDGE STATISTICS	67-86
3.1 Introduction	67
3.2 The Serrated Edge	69
3.2.1 Second Order Moment of the Electric Field	69
3.2.2 Reversion of Series	76
3.3 The Serrated Gap	79
IV EXPERIMENTS	87-139
4.1 Introduction	87
4.2 The Experimental Arrangement	88
4.2.1 Description of System	88
4.2.2 Photographs of Optical Transform Patterns	90

	Page
4.3 Computer-Generated Edges	95
4.3.1 Gaussian Random Processes in a Space-Invariant Linear System	95
4.3.2 The Generation Scheme	102
4.4 Fiber Optic Measurement of Intensity	114
4.5 Remote Sensing of Correlation Coefficient-- Computer-Generated Edges	119
4.5.1 Fiber Optic Measurement of Intensity	119
4.5.2 Processing the Intensity Data	124
4.6 Remote Sensing of Correlation Coefficient-- Computer-Generated Gap	132
V SUMMARY AND CONCLUSIONS	136-139
APPENDICES	
A. Fourier Transform of a Unit Step Function	140
B. Expansion of the Fourth Order Correlation Function	145
C. Terms Involved in the Calculation of $\langle E_1 E_1^* E_2 E_2^* \rangle$	147
D. Computer Program to Generate a Number set with Controlled Statistics	151
E. Computer Program for Processing Transform Plane Intensity Measurements	155
F. Transform Plane Intensity Data for the Case $\sigma_{\eta_1} > 1$	159

CHAPTER I

INTRODUCTION

In the theory of diffraction, as in other areas of physics, one often encounters problems that are best described by a statistical formulation. In such a formulation, one is not interested in the detailed analytic structure of a particular occurrence. Rather, the objective is to describe the media or the collection of objects in terms of overall properties, such as the expected value or correlation of a given quantity. Within this statistical mode of description, the problem of diffraction of an electromagnetic wave by a planar aperture with rough edges represents a new class of diffraction theory problems.

For roughness which is large on a wavelength scale, the problem is readily formulated using Sommerfeld's integral equation, which expresses the electric field in the right half space in terms of the electric field at the aperture plane. The aperture field is approximated in the usual way by letting it equal the incident wave perfectly blocked by the aperture function, i.e., Kirchhoff's approximation. From this expression, second and fourth order correlation functions of the electric field can be calculated, the expectations being over an ensemble of edge

roughness functions assuming monochromatic illumination.

As is demonstrated below (see Section 4.4), the edge diffraction causes a speckle pattern. However, since the random process lies only on the edge boundary, its effect is very different from the usual speckle problem in which longitudinal roughness causes rapid phase variations across the entire aperture.⁽¹⁾ In this context it is noted also that the calculation of the radiation from a partially coherent extended source^(2,3) or illumination field⁽⁴⁾ differs from that for the serrated aperture, analyzed herein.

A computer simulation of diffraction by an edge with a triangular serration has been done by Peters.⁽⁵⁾ A more closely related statistical analysis is the Fresnel diffraction at n -half planes with nonuniform edges published by Polischuk.^(6,7,8) In this work a series of blocking screens of different heights with superimposed roughness was chosen to model decimeter and centimeter wave propagation in mountainous regions. An interesting classification of statistical problems in diffraction theory is given by Kratsov, Rytov, and Tatarskii.⁽⁹⁾ Various formulations of problems in the statistical theory of diffraction and wave propagation were also discussed.

A major portion of the analysis presented herein is devoted to calculation of the optical transform (or far-field) radiation pattern. The motivation for this emphasis is two-fold. One, due to the very special structure of the

field, namely its Fourier transform nature, many expressions are greatly simplified thus facilitating the physical interpretation. Secondly, the optical transform plane is of practical importance in optical processing schemes,^(10,11) such as automatic pattern recognition.^(12,13) Related to the present treatment, film grain noise and other image quality degradation can give rise to characteristic features in their optical transforms which are readily sampled and recognized. Also, edge finish on aperture and field stops, as well as lenses, can give rise to scattered light or noise in optical systems.⁽¹⁴⁾

In Chapter II the general formulation of diffraction by serrated apertures is presented. In the formulation a scheme for classifying statistical problems in diffraction theory, based on a second order moment of the electric field, is discussed. This classification clearly shows the similarities and differences between diffraction by serrated apertures and other statistical diffraction problems. In addition to the general formulation two specific aperture shapes, namely the serrated edge and serrated gap, are analyzed in detail using a Gaussian probability density function of roughness. For each case an exact evaluation of the second order moment of the field is obtained. An approximate evaluation of the cross-correlation of intensity for a serrated edge, a fourth order moment of the electric field, is also discussed.

Chapter III addresses the question: How can the statistical properties of the serration be determined from measurements in the optical transform plane? Here, an expression for the expected value of intensity in the optical transform plane, produced by an edge of Gaussian roughness, is expressed as a power series whose terms involve the correlation coefficient of the serration. By the technique of reversion of series, the equation is inverted; yielding the correlation coefficient expressed as a series of terms involving the expected value of the intensity.

In Chapter IV remote sensing experiments using computer-generated edges are described. The computer-generated edges allowed independent control of edge statistics, i.e., the standard deviation and the correlation coefficient; consequently were useful in verifying the theory.

A recapulation of the major results of the research is contained in Chapter V.

CHAPTER I

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CHAPTER II

THEORY FOR THE SERRATED APERTURE

2.1 Introduction

In this chapter a theory for the diffraction of monochromatic light by a serrated aperture is presented. The problem is formulated within the framework of Rayleigh-Sommerfeld diffraction theory; with the aperture (or input) field being approximated as the incident field, perfectly blocked by the aperture function (Kirchhoff's approximation). It is of particular interest to examine how the edge statistics in the aperture plane map into the diffracted field; and the effect of these statistics on the spatial and wavelength dependences of the pattern.

Section 2.2 contains a general formulation of the problem in terms of an arbitrary serrated aperture function, A_r , together with a scheme for classifying various statistical problems in diffraction theory.

In Sections 2.3 and 2.4 the theory for particular geometries, i.e., the serrated edge and serrated gap is given. For each case an exact, closed-form, algebraic solution for the second order central moment of the electric field is obtained for Gaussian distributed roughness, see Eqs. (2.34) and (2.100). An approximate solution for the fourth-

order central moment of the field, for a serrated edge, is discussed in detail in Section 2.3.2B, and summarized by Eqs. (2.75) through (2.78).

For the serrated edge, we prefer the singularities in the solution attendant to the idealization of an infinite aperture rather than added notational complexity which tends to obscure the main theme. The effect of a finite aperture in suppressing the delta-function singularity is well-known, and published results for the optical transform of an edge with a finite aperture can be reviewed for comparison. (1,2)

2.2 General Formulation

2.2.1 The Diffracted Field

A serrated aperture $A_r(\xi, \gamma)$, located at the $z=0$ plane, is illuminated by a plane-polarized, monochromatic wave traveling in the positive z -direction, as in Fig. 2.1. The coordinates (ξ, γ) denote a point in the aperture plane and (x, y) denote the coordinates of a point in the observation plane.

We represent a tangential scalar component of the incident electric field as a complex function of position, $E_{inc}(\xi, \gamma)$. $E_{inc}(\xi, \gamma)$ is related to the real electric field variable, $E_{inc}^{Re}(\xi, \gamma)$, as follows:

$$E_{inc}^{Re}(\xi, \gamma; t) = \text{Re} \left(E_{inc}(\xi, \gamma) e^{-i\omega t} \right) . \quad (2.1)$$

The corresponding complex component of the diffracted field, $E(x, y)$, can be expressed as an integration over the (ξ, γ) plane in the well-known form⁽²⁾:

$$E(x, y) = \iint_{-\infty}^{\infty} d\xi d\gamma A_r(\xi, \gamma) t(\xi, \gamma) E_{inc}(\xi, \gamma) S(x, y; \xi, \gamma) \quad (2.2)$$

where the field component exciting the (ξ, γ) plane is written as a product of three terms: the incident field $E_{inc}(\xi, \gamma)$, the blocking function $A_r(\xi, \gamma)$ defined by

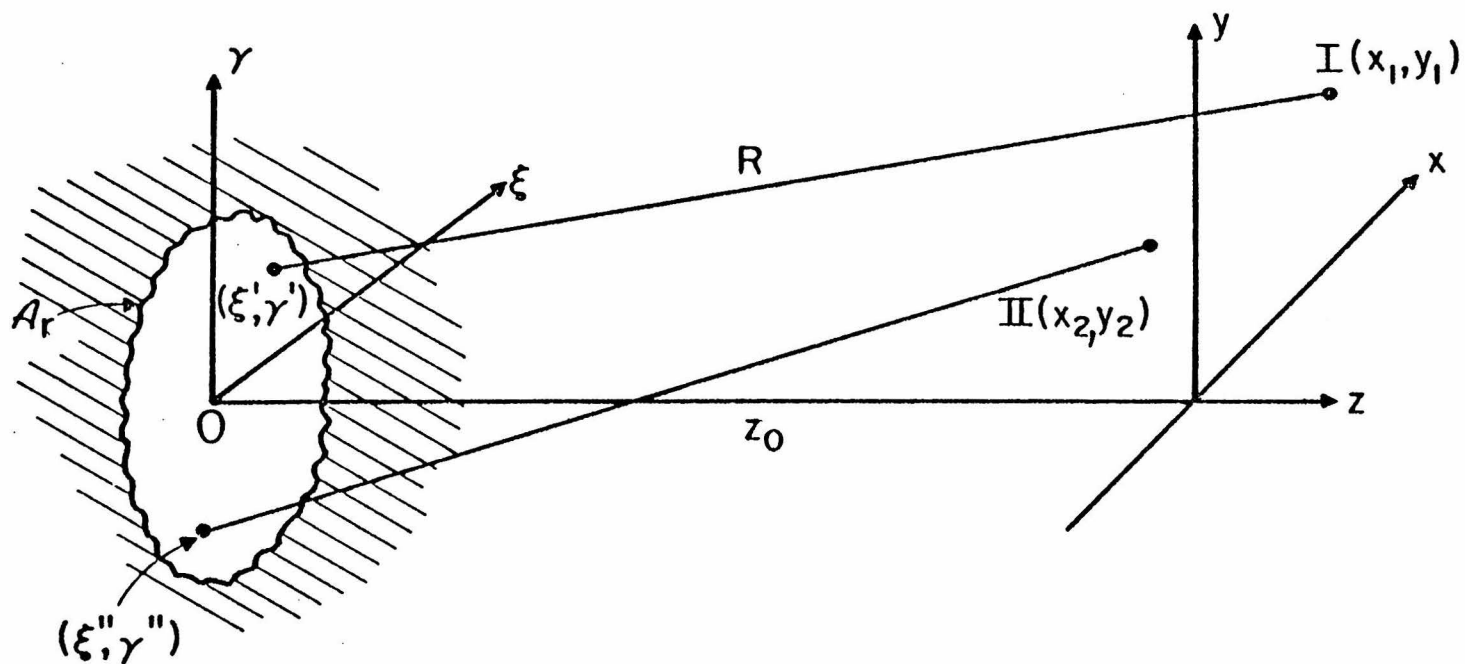


Fig. 2.1 Coordinate system for the serrated aperture defined by the curve $A_r(\xi, \gamma)$.

$$A_r(\xi, \gamma) \equiv \begin{cases} 1, & \text{for } (\xi, \gamma) \text{ inside the curve } A_r \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

and a transmission function $t(\xi, \gamma)$ whose magnitude lies between zero and one,

$$0 \leq |t(\xi, \gamma)| \leq 1. \quad (2.4)$$

$t(\xi, \gamma)$ allows for the possibility of structure inside the curve A_r . With $\exp(-i\omega t)$ suppressed, the exact form and the paraxial Fresnel approximation for $S(x, y; \xi, \gamma)$ are given by Eqs. (2.5a) and (2.5b), respectively. R equals $[(x-\xi)^2 + (y-\gamma)^2 + z_0^2]^{\frac{1}{2}}$, k equals $2\pi/\lambda$ - the wave number (λ = wavelength of light) and z_0 is the distance between the aperture plane (ξ, γ) and the observation plane (x, y) .

$$S(x, y; \xi, \gamma) = - \frac{\exp(ikR)}{2\pi R} \frac{z_0}{R} \left(ik - \frac{1}{R} \right) \quad (2.5a)$$

$$S(x, y; \xi, \gamma) = \frac{-ik \exp(ikz_0)}{2\pi z_0} \exp\left(\frac{ik}{2z_0} [(x-\xi)^2 + (y-\gamma)^2]\right) \quad (2.5b)$$

In later sections, we will be concerned with the calculation of both second and fourth order moments of the electric field. The general form for the second order term is writted from Eq. (2.2) as follows:

$$\begin{aligned}
 E(x_1, y_1) E^*(x_2, y_2) = & \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\gamma' d\gamma'' A_r(\xi', \gamma') \\
 & \cdot A_r^*(\xi'', \gamma'') t(\xi', \gamma') t^*(\xi'', \gamma'') E_{inc}(\xi', \gamma') E_{inc}^*(\xi'', \gamma'') \\
 & \cdot S(x_1, y_1; \xi', \gamma') S^*(x_2, y_2; \xi'', \gamma''). \quad (2.6)
 \end{aligned}$$

The expression for the electric field in the optical transform plane is obtained by placing the observation (x, y) plane at the focal plane of a converging spherical wave. The paraxial form of $E_{inc}(\xi, \gamma)$, representing a converging wave which focuses at $z = z_0$, is given by

$$E_{inc}(\xi, \gamma) = \exp[-ik(\xi^2 + \gamma^2)/(2z_0)]. \quad (2.7)$$

Equations (2.5b) and (2.7) are then substituted into Eq. (2.2) yielding the electric field in the optical transform plane,

$$E(x, y) = A \iint_{-\infty}^{\infty} d\xi d\gamma A_r(\xi, \gamma) t(\xi, \gamma) \exp[-i(\kappa\xi + \eta\gamma)] \quad (2.8)$$

where A is a complex quantity defined as $A \equiv [-ik/(2\pi z_0)] \cdot \exp[i\kappa z_0 + ik(x^2 + y^2)/(2z_0)]$; and the generalized coordinates κ and η are defined as

$$\kappa \equiv kx/z_0$$

$$\eta \equiv ky/z_0. \quad (2.9)$$

2.2.2 Classification of Statistical Problems in Diffraction Theory

In the literature a large variety of problems has been published by researchers in the field of statistical optics. It is thus worthwhile at the outset to classify various statistical problems that arise in diffraction theory.⁽³⁾ Our scheme for classification is based on Eqs. (2.2) and (2.6). We group problems into four categories depending on whether the ensemble average is over the aperture, the transmissive structure, the illumination field, or the medium between the two planes. These four cases call for ensemble averages, denoted by brackets $\langle \dots \rangle$, as follows:

$$\begin{aligned} \langle A_r' A_r''^* \rangle & \text{aperture boundary}, \quad \langle t' t''^* \rangle \text{diffuser} \\ \langle E_{inc}' E_{inc}''^* \rangle & \text{field coherence} \quad \text{and} \quad \langle S' S''^* \rangle \text{medium} \end{aligned}$$

The prime and double prime superscripts have been used to denote the arguments in the functions corresponding in an

obvious way to Eq. (2.6); but note that A_r has the same functional form as A_r' , A_r'' , etc.

As can be seen from Eq. (2.6), the cross-correlation $\langle E(x_1, y_1) E^*(x_2, y_2) \rangle$ is directly coupled with the correlation function of the random process. Although each category represents a distinct physical situation, it is noted that the cross-correlation of the diffracted field produced from the different cases can often be quite similar - if not identical. As an illustration of this similarity, consider the speckle case $\langle t(\xi', \gamma') t^*(\xi'', \gamma'') \rangle$, in the limit as the correlation area in the aperture plane approaches zero, i.e.,

$$\langle t(\xi', \gamma') t^*(\xi' + \xi_1, \gamma' + \gamma_1) \rangle \rightarrow \langle t(\xi', \gamma') t^*(\xi', \gamma') \rangle$$

(a zero mean, stationary process has been assumed). In this limit we see the field correlation $\langle E(x_1, y_1) E^*(x_2, y_2) \rangle$ from Eq. (2.6), depends on the transmitted intensity at the aperture plane. This observation is directly analogous to the Van Cittert-Zernike theorem⁽⁴⁾ in the theory of partial coherence.

Note each category, represented above, can be thought of as being a statistically pure random process. Mixed problems may also be considered. For example, one could consider the diffraction of partially coherent light by a serrated aperture; or the propagation of partially coherent

light in a turbulent medium⁽⁵⁾, or any other possible combination of the pure categories.

It is also noted this is not an exhaustive listing since the temporal statistical case has not been included. Formally, this is treated using the correlation function $\langle e(\underline{r}_1, t_1) e^*(\underline{r}_2, t_2) \rangle$ where temporal fluctuations are involved in the time varying electric field $e(\underline{r}, t)$. In this instance, Eq. (2.2) is appropriate for the calculation of the cross-spectral intensity P of the correlation function provided that Eq. (2.2) is interpreted as the temporal Fourier transform of the time truncated field $e(\underline{r}, t)$, i.e.

$$E_T(\underline{r}, \omega) = \int_{-T}^T dt e(\underline{r}, t) \exp(i\omega t). \quad (2.10a)$$

It is the time-truncated field that must be considered since, in general, $\int_{-\infty}^{\infty} |e(\underline{r}, t)| dt$ does not exist. Interpretation of Eq. (2.2) in the sense of Eq. (2.10a), together with the following theorem from the theory of spectral analysis⁽⁶⁾ allows us to calculate $\langle e(\underline{r}_1, t) e(\underline{r}_2, t + \tau) \rangle$.

$$P(\underline{r}_1, \underline{r}_2; \omega) = \lim_{T \rightarrow \infty} \left\langle \frac{E_T(\underline{r}_1, \omega) E_T^*(\underline{r}_2, \omega)}{2T} \right\rangle$$

$$\langle e(\underline{r}_1, t) e^*(\underline{r}_2, t + \tau) \rangle = (1/2\pi) \int_{-\infty}^{\infty} d\omega P(\underline{r}_1, \underline{r}_2; \omega) \exp(-i\omega \tau). \quad (2.10b)$$

2.2.3 Statistics for the Serration

We limit our treatment to ensemble averages over the aperture function $A_r(\xi, \gamma)$. Hence, hereafter, the brackets, $\langle \dots \rangle$, will be used exclusively to denote the expected value over an ensemble of apertures.

In our treatment of specific aperture shapes, two assumptions are made about the serration along the aperture boundary. One, the serration is assumed to be a spatially stationary process, at least in the wide sense.⁽⁷⁾ Secondly, we assume the random process is a function of only one of the coordinates in the aperture plane. For example, in the case of a rough edge (Section 2.3), the serration is represented by the curve $y_r(\xi)$, i.e., a function dependent only on the ξ coordinate - independent of γ .

While the first assumption is standard in statistical optics, the second produces an interesting ramification in the theory - different from that normally encountered, say, in a diffuser problem. Taking the random process dependent only on one coordinate in the aperture plane, allows an explicit (or deterministic) integration of the independent coordinate in Eq. (2.2). The effect is a mapping of the one-dimensional random process at the aperture plane into a two-dimensional radiation pattern. In optical transform calculations, this reduction in the dimensionality of the random process produces a spatial non-stationarity in the resulting intensity pattern. One consequence of this non-

stationarity is to allow determination of edge statistics from transform plane measurements of the expected value of intensity - a second order moment (see Chapter IV). In contrast, a two-dimensional random process produces an intensity pattern that is approximately stationary. In order to obtain information about the statistics of the process in this case, one must consider the fourth-order correlation function $\langle E(x_1, y_1) E^*(x_1, y_1) E(x_2, y_2) E^*(x_2, y_2) \rangle$. (8,9,10)

In Sections 2.3 and 2.4, an additional assumption is made concerning the correlation coefficient, ρ , of the serration. We assume the correlation coefficient is non-zero only over a finite interval, e.g., for a serrated edge,

$$\begin{aligned} \rho(\xi'' - \xi') &\equiv \frac{\langle y_r(\xi'') y_r(\xi') \rangle}{\sigma^2} \\ &= \begin{cases} \text{arbitrary, when } |\xi'' - \xi'| \leq \epsilon_0 \\ 0, \text{ otherwise} \end{cases} \end{aligned} \quad (2.11)$$

This assumption permits exact evaluation of second order moments appearing in the theory. For the analysis in Chapter III, the last assumption, concerning truncation of the correlation coefficient, is not used.

In the discussion that follows, we will adopt the following notation. First and second order density functions for the random process will be denoted by ψ and ϕ ,

Table 2.1. Density and characteristic function notation⁽¹¹⁾
and selected moment expansions.^(12,13)

Probability density function	Characteristic function
------------------------------	-------------------------

1. General

$$\psi_a(y)$$

$$\Psi_a(\eta) = \langle \exp(i\eta y_a) \rangle$$

$$\phi_a[y', y''; \rho(\xi'' - \xi')]$$

$$\Phi_a[\eta_1, \eta_2; \rho] = \langle \exp[i\eta_1 y_a(\xi') + i\eta_2 y_a(\xi'')] \rangle$$

2. Gaussian

$$\exp[-y^2/(2\sigma^2)]/(2\pi\sigma^2)$$

$$\exp[-\sigma^2 \eta^2/2]$$

$$\frac{\exp\left[-\frac{y'^2 - 2r_{12}y'y'' + y''^2}{2\sigma^2(1-r_{12}^2)}\right]}{[2\pi\sigma^2(1-r_{12}^2)]^{1/2}}$$

$$\exp\left[-\frac{1}{2}\sigma^2(\eta_1^2 + 2r_{12}\eta_1\eta_2 + \eta_2^2)\right]$$

$$\frac{\sqrt{|M|}}{(2\pi)^2} \exp\left(-\underline{\tilde{y}} M \underline{\tilde{y}}/2\right),$$

$$\exp\left(-\underline{\tilde{\eta}} M^{-1} \underline{\tilde{\eta}}/2\right),$$

$$\underline{\tilde{y}} = \begin{bmatrix} y' \\ y'' \\ y''' \\ y'''' \end{bmatrix}, \quad \underline{\tilde{\eta}} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}$$

$$M^{-1} = \sigma^2 \begin{bmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{12} & 1 & r_{23} & r_{24} \\ r_{13} & r_{23} & 1 & r_{34} \\ r_{14} & r_{24} & r_{34} & 1 \end{bmatrix},$$

in which $r_{12} = \rho(\xi' - \xi'')$, etc.

3. Moment theorems for Gaussian processes

a) Real random variables, $x_1, \dots, x_4$:

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle$$

b) Complex random process with $v_m = v(\xi_m)$:

$$\langle v_1 v_2^* v_3 v_4^* \rangle = \langle v_1 v_2^* \rangle \langle v_3 v_4^* \rangle + \langle v_1 v_4^* \rangle \langle v_2^* v_3 \rangle$$

respectively. Corresponding characteristic functions will be represented by Ψ and Φ , where the Fourier transform kernel is taken with a plus sign, i.e.,

$$\Psi(\eta) = \int_{-\infty}^{\infty} dy_r \psi(y_r(\xi)) \exp(+i\eta y_r(\xi)).$$

The notation used is summarized in Table 2.1⁽¹¹⁾, together with the functional forms for first, second and fourth order Gaussian processes. A statement of the Gaussian moment theorem for both real and complex variables is also included in Table 2.1.

2.2.4 Moments of the Electric Field

What are the appropriate correlation functions to calculate? In our study we are interested in investigating the spatial and wavelength dependences of the diffracted radiation pattern. The basic correlation function of intensity u , including wavelength dependence, is given by

$$R_u = \langle u(\underline{r}_1, k_1) u(\underline{r}_2, k_2) \rangle; \quad (2.12)$$

noting $u = EE^*$, we write

$$R_u = \langle E(x_1, y_1; k_1) E^*(x_1, y_1; k_1) E(x_2, y_2; k_2) E^*(x_2, y_2; k_2) \rangle. \quad (2.13)$$

Another important correlation function to consider is

the second order moment $\langle E_1 E_2^* \rangle$. Thus, the ensemble average of Eq. (2.6), denoted by

$$\langle E_1 E_2^* \rangle = \langle E(x_1, y_1; k_1) E^*(x_2, y_2; k_2) \rangle, \quad (2.14)$$

is written

$$\begin{aligned} \langle E_1 E_2^* \rangle = & \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\gamma' d\gamma'' \langle A_r(\xi', \gamma') A_r^*(\xi'', \gamma'') \rangle \\ & \cdot t(\xi', \gamma') t^*(\xi'', \gamma'') E_{\text{inc}}(\xi', \gamma') E_{\text{inc}}^*(\xi'', \gamma'') \\ & \cdot S(x_1, y_1; \xi', \gamma') S^*(x_2, y_2; \xi'', \gamma'') \end{aligned} \quad (2.15)$$

in which state variables are $I(x_1, y_1; k_1)$ and $II(x_2, y_2; k_2)$. This quantity is useful for a couple of reasons. One, if we let $(x_2, y_2; k_2) \rightarrow (x_1, y_1; k_1)$, $\langle E_1 E_2^* \rangle$ becomes $\langle E_1 E_1^* \rangle$, which is the expected value of the intensity. Two, the calculation of $\langle E_1 E_2^* \rangle$ serves as a fundamental building block in the calculation of the cross-correlation of intensity, Eq. (2.13).

As will be seen in later sections, the quantity $\langle E_1 E_1^* \rangle$ is identified with the envelope function for the diffracted energy. To obtain information about fine-scale fluctuations in the pattern, we must consider the higher order moment, $\langle E_1 E_1^* E_2 E_2^* \rangle$. For a visualization of the

envelope and fine-scale fluctuations in the optical transform plane intensity, see Figs. (4.2) and (4.12).

2.3 The Serrated Edge

2.3.1 Second-Order Moment of the Field, $\langle E_1 E_2^* \rangle$

Consider an opaque serrated edge (Fig. 2.2) with the boundary function $\gamma = y_r(\xi)$ such that unity transmission of a convergent wave $E_{inc}(\xi, \gamma)$ is obtained when $\gamma \geq y_r(\xi)$. The index sub- r denotes the r^{th} member of the ensemble. We represent the aperture function, $A_r(\xi, \gamma)$, by

$$A_r(\xi, \gamma) = u[\gamma - y_r(\xi)], \quad (2.16)$$

where $u[\gamma - y_r(\xi)]$ is the unit step function defined as

$$u[\gamma - y_r(\xi)] = \begin{cases} 1, & \gamma \geq y_r(\xi) \\ 0, & \text{otherwise.} \end{cases} \quad (2.17)$$

Combining Eqs. (2.8) and (2.16), we find the optical transform field at $I(x_1, y_1; k_1)$ is given by

$$E(x_1, y_1; k_1) = A_1 \iint_{-\infty}^{\infty} d\xi d\gamma u[\gamma - y_r(\xi)] \exp[-i(\kappa_1 \xi + \eta_1 \gamma)] \quad (2.18)$$

where

$$\begin{aligned} A_1 &= [-ik_1/(2\pi z_0)] \exp[ik_1 z_0 + ik_1(x_1^2 + y_1^2)/(2z_0)] \\ \kappa_1 &= k_1 x_1 / z_0 \\ \eta_1 &= k_1 y_1 / z_0 \end{aligned} \quad (2.19)$$

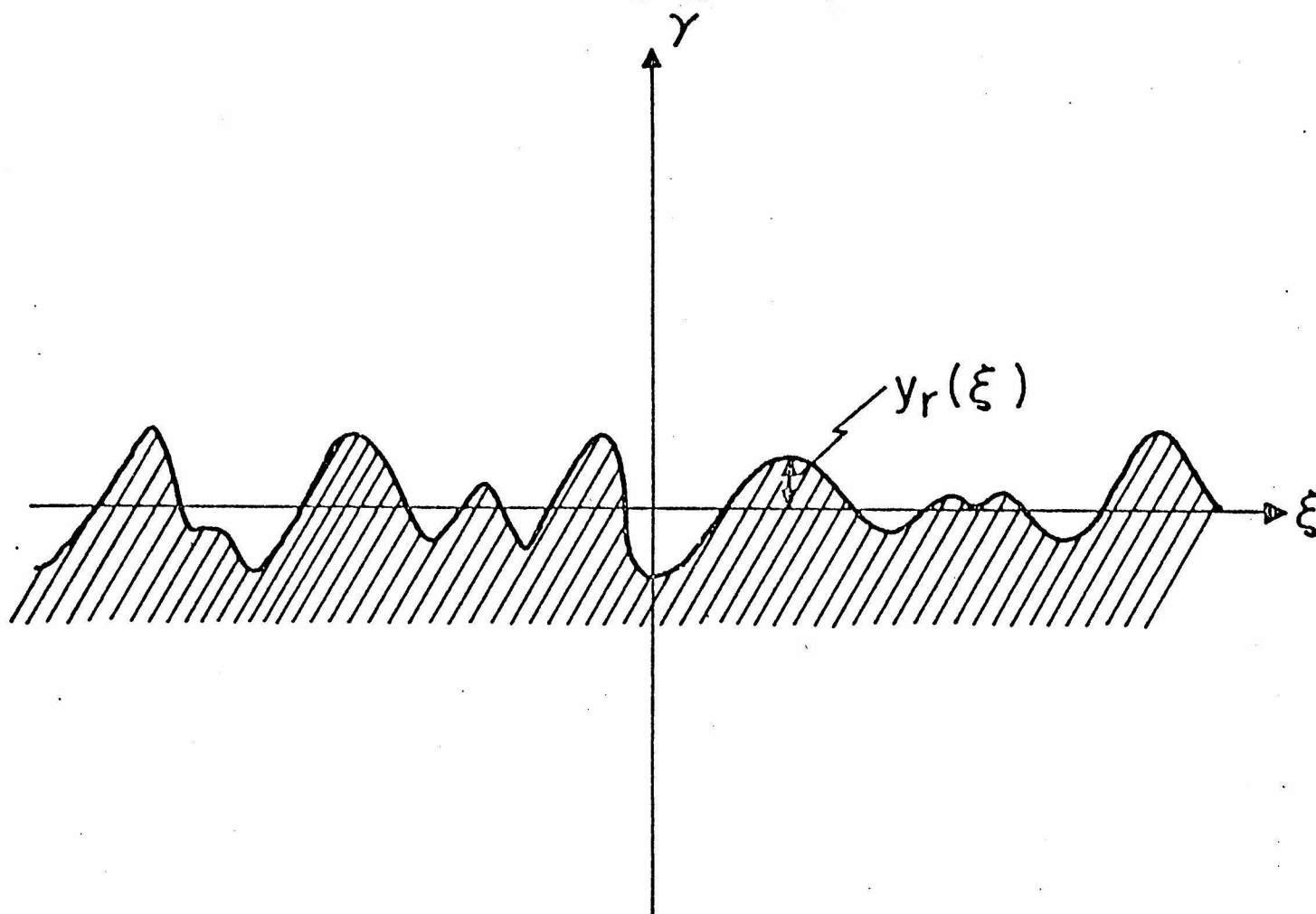


Fig. 2.2 Serrated edge with roughness function, $y_r(\xi)$.

and the transmission function, $t(\xi, \gamma)$, in Eq. (2.8) has been taken to be unity above the edge, $\gamma \geq y_r(\xi)$.

The integral over γ in Eq. (2.18) is recognized as the Fourier transform of a unit step function. Using the shift theorem of Fourier analysis⁽¹⁴⁾ together with Eq. (A14), we write the electric field as follows:

$$E(x_1, y_1; k_1) = 2\pi^2 A_1 \delta(\kappa_1, \eta_1) - (iA_1/\eta_1) \int_{-\infty}^{\infty} d\xi \exp[-i(\kappa_1 \xi + \eta_1 y_r(\xi))]. \quad (2.20)$$

Two fine points are noted. The use of an infinite aperture leads to the delta function, $\delta(\kappa_1, \eta_1)$, rather than an Airy disc. The effect of a large circular aperture, a product term in the integrand of Eq. (2.18), would be a convolution of an Airy disc with E_1 of Eq. (2.20). While one does not want to ignore the finite spread of the Airy disc pattern, still neglecting the cumbersome convolution does lead to helpful simplification in the notation.

Second, the Fourier transform of a unit step function has been taken as $\pi\delta(\eta_1) - (i/\eta_1)$, which implicitly means the principal value should be used for any subsequent integration involving the $(-i/\eta_1)$ term. This condition is denoted P.V. $(-i/\eta_1)$ by some authors,⁽¹⁵⁾ but we find this adds needless complication to our notation herein. The

details of this point are discussed in Appendix A.

The second order moment $\langle E_1 E_2^* \rangle$, defined in Eq. (2.14), can be calculated directly from Eq. (2.20); and in Chapter III this will be our starting point for the discussion of remote sensing of aperture statistics. However, at present we note that central moments, such as $\langle (E_1 - \langle E_1 \rangle)(E_2^* - \langle E_2^* \rangle) \rangle$, give a systematic grouping which leads to considerably simpler algebra than directly forming $\langle E_1 E_2^* \rangle$. The central field quantities E_1 and E_2 are defined as

$$E_1 \equiv E_1 - \langle E_1 \rangle \quad \text{and} \quad E_2 \equiv E_2 - \langle E_2 \rangle. \quad (2.21)$$

Hence, the central moment form of the cross-correlation of fields, $\langle E_1 E_2^* \rangle = \langle (E_1 - \langle E_1 \rangle)(E_2^* - \langle E_2^* \rangle) \rangle$ is expanded and regrouped. The following equation gives the cross-correlation grouped in the preferred order of calculation:

$$\langle E_1 E_2^* \rangle = \langle E_1 E_2^* \rangle + \langle E_1 \rangle \langle E_2^* \rangle. \quad (2.22)$$

The second term on the right of Eq. (2.22) is calculated by forming the product of the expected values from Eq. (2.20). The result is

$$\begin{aligned} \langle E_1 \rangle \langle E_2^* \rangle = & 4\pi^2 A_1 A_2^* \delta(\kappa_1) \delta(\kappa_2) [\pi \delta(\eta_1) - (i/\eta_1) \Psi(-\eta_1)] \\ & \cdot [\pi \delta(\eta_2) + (i/\eta_2) \Psi(\eta_2)]. \end{aligned} \quad (2.23)$$

In Eq. (2.23) Ψ is the first order characteristic for the random process $y_r(\xi)$. For the calculation of $\langle E_1 E_2^* \rangle$ both first and second order characteristic functions are needed. The notation is summarized in Table 2.1.

The calculation of $\langle E_1 E_2^* \rangle$ proceeds by substituting Eq. (2.20) into Eq. (2.21), the resulting expression is

$$\begin{aligned} \langle E_1 E_2^* \rangle = [A_1 A_2^* / \eta_1 \eta_2] & \iint_{-\infty}^{\infty} d\xi' d\xi'' \exp(i\kappa_2 \xi'' - i\kappa_1 \xi') \left[\langle \exp(in_2 y_r(\xi'') - in_1 y_r(\xi')) \rangle \right. \\ & \left. - \langle \exp[in_2 y_r(\xi'')] \rangle \langle \exp[-in_1 y_r(\xi')] \rangle \right]. \quad (2.24) \end{aligned}$$

The rationale for computing central moments can now be seen. It is evident from Eq. (2.24) that the main contribution to $\langle E_1 E_2^* \rangle$ comes from the interval which exhibits significant correlation between the points ξ' and ξ'' . This is clarified by the following evaluation of Eq. (2.24).

For the evaluation we adopt the assumption discussed in Section 2.2.3, concerning a correlation coefficient of finite length, see Eq. (2.11). Using this assumption we can rewrite Eq. (2.24); noting that the integrand is identically zero outside the interval $|\xi'' - \xi'| \leq \varepsilon_0$. Adopting the notation of Table 2.1 gives

$$\begin{aligned} \langle E_1 E_2^* \rangle = & \frac{A_1 A_2^*}{\eta_1 \eta_2} \int_{-\infty}^{\infty} d\xi' \exp(-i\kappa_1 \xi') \int_{\xi' - \epsilon_0}^{\xi' + \epsilon_0} d\xi'' \exp(i\kappa_2 \xi'') \\ & \cdot \left[\Phi(-\eta_1; \eta_2; \rho(\xi'' - \xi')) - \Psi(-\eta_1) \Psi(\eta_2) \right]. \end{aligned} \quad (2.25)$$

Letting $\xi_1 = \xi'' - \xi'$ in Eq. (2.25) we obtain the result:

$$\begin{aligned} \langle E_1 E_2^* \rangle = & \frac{2\pi A_1 A_2^*}{\eta_1 \eta_2} \delta(\kappa_2 - \kappa_1) \int_{-\epsilon_0}^{\epsilon_0} d\xi_1 \exp(i\kappa_2 \xi_1) \\ & \cdot \left[\Phi(-\eta_1, \eta_2; \rho(\xi_1)) - \Psi(-\eta_1) \Psi(\eta_2) \right]. \end{aligned} \quad (2.26)$$

Thus, using Eq. (2.22), we find the general expression for the cross-correlation function of the electric field is given by summing Eqs. (2.26) and (2.23). While a finite correlation length $\pm\epsilon_0$ has been assumed, no assumption has been made on the r-m-s roughness, and the functional form for the correlation coefficient is general within the specification of Eq. (2.11).

The integral appearing in Eq. (2.26) is fundamental in the theory of the serrated edge and serrated gap; and will appear repeatedly in later sections. For these reasons, we assign it a label, $G(-\eta_1, \eta_2; \kappa_2)$.

$$G(-\eta_1, \eta_2; \kappa_2) \equiv \int_{-\varepsilon_0}^{\varepsilon_0} d\xi_1 \exp(i\kappa_2 \xi_1) \left[\Phi(-\eta_1, \eta_2; \rho(\xi_1)) - \Psi(-\eta_1) \Psi(\eta_2) \right] \quad (2.27)$$

Thus, Eq. (2.25) can be expressed as

$$\langle E_1 E_2^* \rangle = \frac{2\pi A_1 A_2^*}{\eta_1 \eta_2} \delta(\kappa_2 - \kappa_1) G(-\eta_1, \eta_2; \kappa_2). \quad (2.28)$$

A. Evaluation for Gaussian Density of Roughness

For the serration, assume that the probability density function for heights is jointly normal. From Table 2.1, we write

$$\begin{aligned} \Phi(-\eta_1, \eta_2; \rho(\xi)) &= \exp(-\sigma^2(\eta_2 - \eta_1)^2/2) \\ &\cdot \exp(-\sigma^2 \eta_1 \eta_2 [1 - \rho(\xi)]) \end{aligned} \quad (2.29)$$

$$\Psi(\eta_2) = \exp(-\sigma^2 \eta_2^2/2). \quad (2.30)$$

For a Gaussian edge $y_r(\xi)$, using Eqs. (2.29) and (2.30), we rewrite Eq. (2.27) as

$$G(-\eta_1, \eta_2; \kappa_2) = \int_{-\varepsilon_0}^{\varepsilon_0} d\xi \exp(i\kappa_2 \xi) \left\{ \left[\exp(-\sigma^2(\eta_2 - \eta_1)^2/2) \cdot \exp(-\sigma^2 \eta_1 \eta_2 [1 - \rho(\xi)]) \right] - \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \right\}. \quad (2.31)$$

Equation (2.31) can be integrated for a wide variety of correlation coefficients ρ . As an example of the evaluation, we assume the correlation coefficient is given by the triangle function $\Lambda(\xi/\varepsilon_0)$, i.e.,

$$\rho(\xi) = \Lambda(\xi/\varepsilon_0) \equiv \begin{cases} 1 - |\xi|/\varepsilon_0, & \text{when } |\xi| \leq \varepsilon_0 \\ 0 & , \text{otherwise} . \end{cases} \quad (2.32)$$

Substituting Eq. (2.32) into Eq. (2.31) and integrating using Eq. GR-2.663-3⁽¹⁶⁾, we obtain the following result:

$$G(-\eta_1, \eta_2; \kappa_2) = \frac{2 \exp[-\sigma^2(\eta_2 - \eta_1)^2/2]}{(\sigma^2 \eta_1 \eta_2 / \varepsilon_0)^2 + (\kappa_2)^2} \left\{ \frac{\sigma^2 \eta_1 \eta_2}{\varepsilon_0} + \exp(-\sigma^2 \eta_1 \eta_2) \left[-\frac{\sigma^2 \eta_1 \eta_2}{\varepsilon_0} \cos \kappa_2 \varepsilon_0 + \kappa_2 \sin \kappa_2 \varepsilon_0 \right] \right\} - 2 \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] [\sin(\kappa_2 \varepsilon_0) / \kappa_2]. \quad (2.33)$$

The expression for the central moment of the serrated edge with Gaussian roughness of arbitrary σ and ε_0 is obtained

by writing Eq. (2.28) with G given by Eq. (2.38), and $A_{1,2}$, $\kappa_{1,2}$ and $\eta_{1,2}$ defined in Eq. (2.19):

$$\langle E_1 E_2^* \rangle = \frac{2\pi A_1 A_2^*}{\eta_1 \eta_2} \delta(\kappa_2 - \kappa_1) G(-\eta_1, \eta_2; \kappa_2). \quad (2.34)$$

As detailed in Appendix B, the central moment term corresponding to intensity is written from Eq. (B5):

$$\langle E_1 E_1^* \rangle = [A_1 A_1^* / (\eta_1)^2] 2L_x G(-\eta_1, \eta_1; \kappa_1) \quad (2.35)$$

in which the large scale length ($2L_x$) is used as a limit form for the x -dimension, arising from $\int_{-L_x}^{L_x} d\xi' \exp[i(\kappa_2 - \kappa_1)\xi']$ as $\kappa_2 \rightarrow \kappa_1$. $G(-\eta_1, \eta_1; \kappa_1)$ is obtained from indicated variable changes in Eq. (2.27). Expressed in the characteristic function notation, the result is

$$G(-\eta_1, \eta_1; \kappa_1) = \int_{-\epsilon_0}^{\epsilon_0} d\xi \exp(i\kappa_1 \xi) [\Phi[-\eta_1, \eta_1; \rho(\xi)] - \Psi(-\eta_1) \Psi(\eta_1)]. \quad (2.36)$$

For the Gaussian density, we obtain

$$G(-\eta_1, \eta_1; \kappa_1) = \int_{-\epsilon_0}^{\epsilon_0} d\xi \exp(i\kappa_1 \xi) [\exp(-\sigma^2 \eta_1^2 [1 - \rho(\xi)]) - \exp(-\sigma^2 \eta_1^2)], \quad (2.37)$$

and analogously to Eq. (2.33), we find

$$G(-\eta_1, \eta_1; \kappa_1) = \frac{2}{(\sigma^2 \eta_1^2 / \epsilon_0)^2 + \kappa_1^2} \left\{ \frac{\sigma^2 \eta_1^2}{\epsilon_0} + \left[\frac{\sigma^2 \eta_1^2}{\epsilon_0} \cos(\kappa_1 \epsilon_0) + \kappa_1 \sin(\kappa_1 \epsilon_0) \right] \cdot \exp(-\sigma^2 \eta_1^2) \right\} - 2 \exp(-\sigma^2 \eta_1^2) [\sin(\kappa_1 \epsilon_0) / \kappa_1]. \quad (2.38)$$

2.3.2 Fourth-order Moment of the Field, $\langle E_1 E_1^* E_2 E_2^* \rangle$

We now wish to consider the spatial and wavelength dependence of the fine-scale intensity fluctuations in the optical transform pattern, e.g., see Fig. (4.12). To this end, it is instructive to calculate the central moment of fourth order $\langle E_1 E_1^* E_2 E_2^* \rangle$ and see the dominant role played by the second order central moment, $\langle E_1 E_2^* \rangle$, already computed. Other terms involved in $R_u = \langle u_1 u_2 \rangle$ are listed in Appendix B. Using Eqs. (2.20) and (2.21), we write a general expression:

$$\begin{aligned} \langle E_1 E_1^* E_2 E_2^* \rangle &= [B/(\eta_1 \eta_2)^2] \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp[i(-\kappa_1 \xi' + \kappa_1 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''')] \\ &\cdot [\exp(-i\eta_1 y_r') - \Psi(-\eta_1)] [\exp(i\eta_1 y_r'') - \Psi(\eta_1)] \\ &\cdot [\exp(-i\eta_2 y_r''') - \Psi(-\eta_2)] [\exp(i\eta_2 y_r''') - \Psi(\eta_2)] \end{aligned} \quad (2.39)$$

where $y_r' \equiv y_r(\xi')$, $y_r'' = y_r(\xi'')$, etc. Note, the primes denote different positions along the same edge y_r - not different members of the ensemble. The multiplier B defined as $B \equiv A_1 A_1^* A_2 A_2^*$ is written using Eq. (2.19) as

$$B = 1/(\lambda_1 \lambda_2 z_0^2)^2. \quad (2.40)$$

The exponentials in Eq. (2.19), including time dependences, exactly cancel.

An approximate solution of Eq. (2.39) is written in Sec. 2.3.2-A for the case of small roughness and normal density of heights, y_r . These assumptions permit us to use an expansion of the fourth order expectation in the integrand in terms of products of second order. These integrals are evaluated and a useful solution is readily obtained. In Sec. 2.3.2-B a general solution is derived for arbitrary roughness. Direct integration of the fourth order moment leads to a solution that is seen to contain three dominant terms. In Sec. 2.3.3 the general solution is shown to reduce to the prior solution for the special case of small roughness. A limiting form for the solution for large roughness is also discussed.

A. Evaluation of fourth order central moment for small roughness

From the form of Eq. (2.39), the essential statistical calculation involves an expectation of a product of four terms. Let z_1 denote one of these:

$$z_1 = \exp(-i\eta_1 y_r') - \Psi(-\eta_1) \quad (2.41)$$

where $\Psi(-\eta_1) = \langle \exp(-i\eta_1 y_r') \rangle$ and $\sigma^2 = \langle y_r'^2 \rangle$. For small values of $\sigma\eta_1$, the form of z_1 is approximately

$$z_1 = 1 + i\eta_1 y_r' + \dots - \Psi(\eta_1)$$

$$z_1 \approx i\eta_1 y_r' . \quad (2.42)$$

Thus z_1 is normal for small $\sigma\eta_1$ provided that y_r is normally distributed. Substituting Eq. (2.42) together with analogous terms for z_1^* , z_2 , z_2^* into Eq. (2.39) gives

$$\begin{aligned} \langle E_1 E_1^* E_2 E_2^* \rangle = & B \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp \left[i(-\kappa_1 \xi' + \kappa_1 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''') \right] \\ & \cdot \langle y_r' y_r'' y_r''' y_r'''' \rangle \text{rect}[\xi'/(2L_x)] \text{rect}[\xi''/(2L_x)] \text{rect}[\xi'''/(2L_x)] \text{rect}[\xi''''/(2L_x)]. \end{aligned} \quad (2.43)$$

In Eq. (2.43) a rect function formalism is used to indicate the finite aperture along the ξ -direction. The function $\text{rect}[\xi'/(2L_x)]$ is defined as

$$\text{rect}[\xi'/(2L_x)] \equiv \begin{cases} 1, & |\xi| \leq L_x \\ 0, & \text{otherwise.} \end{cases} \quad (2.44)$$

This will greatly simplify the interpretation of the speckle pattern for fine-scale variations and wavelength dependence. Equation (2.43) is evaluated by direct application of the Gaussian moment theorem for real variables listed in Table 2.1. This results in three similar integrals, which are denoted by D_1 , D_2 , D_3 , respectively.

As an example, consider D_2 resulting from the moment

product $\langle y_r' y_r'' \rangle \langle y_r'' y_r''' \rangle$ and written explicitly as follows:

$$D_2 = \sigma^4 B \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp \left[i(-\kappa_1 \xi' + \kappa_2 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''') \right] \\ \cdot \rho(\xi'''' - \xi') \rho(\xi''' - \xi'') \text{rect}[\xi' / (2L_x)] \text{rect}[\xi'' / (2L_x)] \text{rect}[\xi''' / (2L_x)] \\ \cdot \text{rect}[\xi'''' / (2L_x)]. \quad (2.45)$$

Equation (2.45) separates into the product of two double integrals. Then, it can be integrated substituting from Eq. (2.32) for ρ and using Fourier transform theorems as follows:

$$D_2 = 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} \left(\frac{\sin(\kappa_1 \epsilon_0 / 2)}{(\kappa_1 \epsilon_0 / 2)} \right)^2 * \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]_{\kappa_1} \\ \cdot \left[\frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} \left(\frac{\sin(\kappa_2 \epsilon_0 / 2)}{(\kappa_2 \epsilon_0 / 2)} \right)^2 * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2}. \quad (2.46)$$

The asterisk denotes convolution with respect to the variable indicated, κ_1 and κ_2 , respectively.

In Eq. (2.46) the convolution by $2L_x \sin(\kappa_2 L_x) / (\kappa_2 L_x)$ causes only a slight spreading of the other function when L_x is very large. However, the multiplier terms of the

form $\sin[(\kappa_2 \pm \kappa_1)L_x]/[(\kappa_2 \pm \kappa_1)L_x]$ exhibit oscillatory or interference pattern behavior. This latter term describes the speckle pattern, e.g., the fine scale space and wavelength dependence. Hence, these terms are retained, and in our approximate solution the convolution can be suppressed or neglected. D_1 and D_3 are evaluated in a similar manner. The solution for the correlation function in Eq. (2.43) follows:

$$\langle E_1 E_1^* E_2 E_2^* \rangle = D_1 + D_2 + D_3 \quad (2.47)$$

where

$$D_1 \cong 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin(\kappa_1 \epsilon_0/2)}{(\kappa_1 \epsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2 \epsilon_0/2)}{(\kappa_2 \epsilon_0/2)} \right]^2, \quad (2.48)$$

$$D_2 \cong 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} \right]^2 \left[\frac{\sin(\kappa_1 \epsilon_0/2)}{(\kappa_1 \epsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2 \epsilon_0/2)}{(\kappa_2 \epsilon_0/2)} \right]^2, \quad (2.49)$$

$$D_3 \cong 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin[(\kappa_2 + \kappa_1)L_x]}{(\kappa_2 + \kappa_1)L_x} \right]^2 \left[\frac{\sin(\kappa_1 \epsilon_0/2)}{(\kappa_1 \epsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2 \epsilon_0/2)}{(\kappa_2 \epsilon_0/2)} \right]^2. \quad (2.50)$$

B. Fourth order moment for arbitrary roughness

Equation (2.39) can be solved for arbitrary roughness, σ . In the evaluation we assume jointly normal statistics for the serration $y_r(\xi)$, as in Table 2.1; and

the ratio $\varepsilon_0/L_x \ll 1$. The details of the calculation are contained in Sections 2.3.2-B1 to B5 below. In Section 2.3.2-B5 we evaluate the dominant terms of $\langle E_1 E_1^* E_2 E_2^* \rangle$. The solution is given in Eqs. (2.75), (2.76) to (2.78).

B1. Counting method for evaluation of $\langle E_1 E_1^* E_2 E_2^* \rangle$

We evaluate Eq. (2.39) assuming the roughness of the edge can be represented by the fourth order normal density as in Table 2.1. Consideration is limited to the case for which the correlation interval, ε_0 in Eq. (2.32), is finite and small in comparison to the length of the edge, i.e., $\varepsilon_0 \ll L_x$. The general equation for the fourth order central moment, Eq. (2.39), is rewritten with finite integration along the edge explicitly characterized by the rect functions as follows:

$$\begin{aligned} \langle E_1 E_1^* E_2 E_2^* \rangle = & [B/(\eta_1 \eta_2)^2] \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp \left[i(-\kappa_1 \xi' + \kappa_1 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''') \right] \\ & \cdot \langle Z_1(\xi') Z_1^*(\xi'') Z_2(\xi''') Z_2^*(\xi''') \rangle \text{rect}[\xi'/(2L_x)] \text{rect}[\xi''/(2L_x)] \text{rect}[\xi'''/(2L_x)] \\ & \cdot \text{rect}[\xi''''/(2L_x)], \end{aligned} \quad (2.51)$$

where

$$z_1(\xi') = \exp[-in_1 y_r(\xi')] - \Psi(-\eta_1), \text{ etc.}, \quad (2.52)$$

and $\text{rect}[\xi'/(2L_x)]$ is defined in Eq. (2.44). For arbitrary

σ , direct evaluation of the fourth order expectation and integration is required for Eq. (2.51), since in general z_1 and z_2 are not normally distributed and expansions in terms of second order moments are not known to us.

In the four-dimensional integration over $(\xi', \xi'', \xi''', \xi'''')$, there are only a few regions contributing since $\langle z_1 z_1^* z_2 z_2^* \rangle$ is zero except in these regions. As shown below, the assertion of finite correlation length and the form of z_1 and z_2 , in which the characteristic function is subtracted from $\exp[-iny_r]$, are responsible for this. Define P by

$$P = \langle z_1(\xi') z_1^*(\xi'') z_2(\xi''') z_2^*(\xi''') \rangle. \quad (2.53)$$

Consider writing P as the sum of several mutually exclusive terms using "indicator" functions in the variables $(\xi', \xi'', \xi''', \xi''')$ so that only the appropriate term contributes at any point. The integration in Eq. (2.39) then becomes a sum of separate integrals over the regions defined by the indicator function.

- 1) Let ξ' be distinct from ξ'', ξ''', ξ''' by more than the correlation interval. For the normally distributed random process, this assures the independence⁽¹⁷⁾ of $y_r(\xi')$ and Eq. (2.53) becomes

$$P = \langle z_1(\xi') \rangle \langle z_1^*(\xi'') z_2(\xi''') z_2^*(\xi''') \rangle = 0, \quad (2.54)$$

since $\langle z_1(\xi') \rangle = 0$. The function P is zero of course when any ξ of the four is distinct from the others by more than ϵ_0 .

- 2) Correlation in pairs, excluding mutual correlation between the pairs.

$$\begin{aligned}
 L_1: \quad & |\xi'' - \xi'| \leq \epsilon_0 & |\xi''' - \xi'| > \epsilon_0 \\
 & |\xi''' - \xi''| \leq \epsilon_0 & |\xi''' - \xi''| > \epsilon_0 \\
 & & |\xi''' - \xi'| > \epsilon_0 \\
 & & |\xi''' - \xi''| > \epsilon_0
 \end{aligned} \tag{2.55}$$

$$\begin{aligned}
 L_2: \quad & |\xi''' - \xi'| \leq \epsilon_0 & |\xi'' - \xi'| > \epsilon_0 \\
 & |\xi''' - \xi''| \leq \epsilon_0 & |\xi''' - \xi'| > \epsilon_0 \\
 & & |\xi''' - \xi''| > \epsilon_0 \\
 & & |\xi''' - \xi''| > \epsilon_0
 \end{aligned} \tag{2.56}$$

$$\begin{aligned}
 L_3: \quad & |\xi''' - \xi'| \leq \epsilon_0 & |\xi'' - \xi'| > \epsilon_0 \\
 & |\xi''' - \xi''| \leq \epsilon_0 & |\xi''' - \xi'| > \epsilon_0 \\
 & & |\xi''' - \xi''| > \epsilon_0 \\
 & & |\xi''' - \xi''| > \epsilon_0
 \end{aligned} \tag{2.57}$$

- 3) Correlation in pairs with mutual correlation between the pairs. These terms are formed by negating any exclusion term (right-hand member inequalities) in Eqs. (2.55, 2.56, 2.57). As an example, one negation of Eq. (2.55) is written

as Eq. (2.58):

$$\begin{aligned}
 L_4: \quad & |\xi'' - \xi'| \leq \epsilon_0 & |\xi''' - \xi'| > \epsilon_0 \\
 & |\xi''' - \xi''| \leq \epsilon_0 & |\xi''' - \xi''| > \epsilon_0 \quad (2.58) \\
 & |\xi''' - \xi''| \leq \epsilon_0 & |\xi''' - \xi'| > \epsilon_0.
 \end{aligned}$$

In all there are 34 additional terms, L_4 through L_{37} . A detailed listing of this group of integrals is given in Appendix C. The appendix contains a listing of 12 terms of the form in Eq. (2.58), 15 terms with four correlation inequalities (left-member) and two exclusion inequalities (right-member), 6 terms with five correlations and one exclusion, and 1 term with six correlations. An order of magnitude calculation for members of this group is discussed in Sec. 2.3.2-B4. The terms L_1 through L_3 are shown to dominate those in this category.

B2. Indicator function notation

The integration in Eq. (2.51) is written as the sum

$$\langle E_1 E_1^* E_2 E_2^* \rangle = L_1 + L_2 + L_3 + \sum_{m=4}^{37} L_m \quad (2.59)$$

in which the contributions are defined with reference to Eqs. (2.55-2.58). For example, from Eqs. (2.53) and (2.55), the associated integral L_1 is given by

$$\begin{aligned}
 L_1 = & [B/(\eta_1 \eta_2)^2] \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp \left[i(-\kappa_1 \xi' + \kappa_1 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''') \right] \\
 & \cdot \langle z_1(\xi') z_1^*(\xi'') \rangle \langle z_2(\xi''') z_2^*(\xi''') \rangle \text{rect}[\xi'/(2L_x)] \text{rect}[\xi''/(2L_x)] \\
 & \cdot \text{rect}[\xi'''/(2L_x)] \text{rect}[\xi''''/(2L_x)] \text{rect}[(\xi'' - \xi')/(2\epsilon_0)] \text{rect}[(\xi'''' - \xi''')/(2\epsilon_0)] \\
 & \cdot (1 - \text{rect}[(\xi'''' - \xi')/(2\epsilon_0)]) (1 - \text{rect}[(\xi'''' - \xi'')/(2\epsilon_0)]) (1 - \text{rect}[(\xi'''' - \xi')/(2\epsilon_0)]) \\
 & \cdot (1 - \text{rect}[(\xi'''' - \xi'')/(2\epsilon_0)]). \tag{2.60}
 \end{aligned}$$

The second order expectation $\langle z_1(\xi') z_1^*(\xi'') \rangle$ can be written from Table 2.1 as follows:

$$\langle z_1(\xi') z_1^*(\xi'') \rangle = \Phi(-\eta_1, \eta_1; \rho(\xi'' - \xi')) - \Psi(-\eta_1) \Psi(\eta_1), \tag{2.61}$$

and this reduces to

$$\langle z_1(\xi') z_1^*(\xi'') \rangle = \exp[-\sigma^2 \eta_1^2] \left\{ \exp(\sigma^2 \eta_1^2 \rho[\xi'' - \xi']) - 1 \right\}, \tag{2.62}$$

in which the correlation coefficient is given by Eq. (2.32).

In Eq. (2.60) the prescription for the indicator function is obtained from Eq. (2.55) and consists of $\text{rect}[(\xi'' - \xi')/(2\epsilon_0)]$ terms for the correlation (left-members) and

$(1 - \text{rect}[(\xi''' - \xi')/(2\varepsilon_0)])$ terms for suppression where correlation is excluded.

B3. Evaluation of term L_1

In Eq. (2.60), for L_1 , the four suppression terms of the indicator function can be expanded. The first term is unity, followed by sums of the form $\text{rect}[(\xi''' - \xi')/(2\varepsilon_0)]$ and higher order products. We will denote the contribution to L_1 from the various terms of the expansion as follows:

- $L_{1,0}$: contribution from the unity term;
 - $L_{1,1}$: contribution from the sum of rect function (four members in the sum);
 - $L_{1,2}$: contribution from the sum of all possible combinations of pairs of rect function (six members in the sum);
 - $L_{1,3}$: contribution from the sum of all combinations of triple products of rect function (four members in the sum);
 - $L_{1,4}$: contribution from the product of the four rect function (1 member).
- (2.63)

Thus, the integration of L_1 , in Eq. (2.60), is written as the sum

$$L_1 = L_{1,0} + L_{1,1} + L_{1,2} + L_{1,3} + L_{1,4} \quad (2.64)$$

A calculation of the upper bound for the terms in Eq. (2.64) is given in Sec. 2.3.2-B4. Here, we evaluate $L_{1,0}$, the dominant contribution to L_1 . Equations (2.62) and (2.32) are substituted in Eq. (2.60) and integrated to obtain the following form for $L_{1,0}$:

$$L_{1,0} = 4B[L_x/(\eta_1\eta_2)]^2 \left[G(-\eta_1, \eta_1; \kappa_1)^* \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]_{\kappa_1} \cdot \left[G(-\eta_2, \eta_2; \kappa_2)^* \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \quad (2.65)$$

where G is given in Eq. (2.33).

B4. Upper bound for terms in $\langle E_1 E_1^* E_2 E_2^* \rangle$

Upper bounds for terms $L_{1,m}$, $m = 0, 1, \dots, 4$ are established in the following manner. From Eq. (2.62), it is evident that

$$\langle z_1 z_1^* \rangle \leq 1 - \exp(-\sigma^2 \eta_1^2) \quad (2.66)$$

since the correlation coefficient, ρ , has a maximum value of one. Substitution of Eq. (2.66) together with an analogous term for $\langle z_2 z_2^* \rangle$ into Eq. (2.60), gives

$$L_1 \leq [B/(\eta_1 \eta_2)^2][1 - \exp(-\sigma^2 \eta_1^2)][1 - \exp(-\sigma^2 \eta_2^2)]$$

$$\begin{aligned} & \cdot \iiint_{-\infty}^{\infty} d\xi' d\xi'' d\xi''' d\xi'''' \exp \left[i(-\kappa_1 \xi' + \kappa_1 \xi'' - \kappa_2 \xi''' + \kappa_2 \xi''') \right] \\ & \cdot \text{rect}[\xi'/(2L_x)] \text{rect}[\xi''/(2L_x)] \text{rect}[\xi'''/(2L_x)] \text{rect}[\xi''''/(2L_x)] \\ & \cdot \text{rect}[(\xi'' - \xi')/(2\epsilon_0)] \text{rect}[(\xi''' - \xi'')/(2\epsilon_0)] (1 - \text{rect}[(\xi''' - \xi')/(2\epsilon_0)]) \\ & \cdot (1 - \text{rect}[(\xi''' - \xi'')/(2\epsilon_0)]) (1 - \text{rect}[(\xi'''' - \xi')/(2\epsilon_0)]) \\ & \cdot (1 - \text{rect}[(\xi'''' - \xi'')/(2\epsilon_0)]). \end{aligned} \quad (2.67)$$

From Eq. (2.67) an upper bound for each term in Eq. (2.64) can be established.

For example we obtain an upper bound for $L_{1,0}$ by integrating Eq. (2.68).

$$\begin{aligned} L_{1,0} & \leq [B/(\eta_1 \eta_2)^2][1 - \exp(-\sigma^2 \eta_1^2)][1 - \exp(-\sigma^2 \eta_2^2)] \\ & \cdot \int_{-\infty}^{\infty} d\xi' \exp(-i\kappa_1 \xi') \text{rect}[\xi'/(2L_x)] \int_{-\infty}^{\infty} d\xi'' \exp(i\kappa_1 \xi'') \text{rect}[\xi''/(2L_x)] \\ & \quad \cdot \text{rect}[(\xi'' - \xi')/(2\epsilon_0)] \\ & \cdot \int_{-\infty}^{\infty} d\xi''' \exp(-i\kappa_2 \xi''') \text{rect}[\xi'''/(2L_x)] \int_{-\infty}^{\infty} d\xi'''' \exp(i\kappa_2 \xi''') \\ & \quad \cdot \text{rect}[\xi''''/(2L_x)] \text{rect}[(\xi'''' - \xi''')/(2\epsilon_0)] \end{aligned} \quad (2.68)$$

The ξ''' -integration, in Eq. (2.68), is recognized as a Fourier transform of the product $\text{rect}[\xi'''/(2L_x)] \cdot \text{rect}[(\xi''' - \xi'')/(2\varepsilon_0)]$. Application of the similarity and convolution theorems of Fourier analysis⁽¹⁸⁾ permits evaluation of the ξ''' -integration. Then, by interchanging the order of convolution and ξ''' -integration, the ξ''' -integration may be performed. Evaluating the ξ', ξ'' -integration in the same way gives,

$$L_{1,0} \leq 16L_x^2 \varepsilon_0^2 [B/(\eta_1 \eta_2)^2] [1 - \exp(-\sigma^2 \eta_1^2)] [1 - \exp(-\sigma^2 \eta_2^2)]$$

$$\left[\frac{\sin(\kappa_1 \varepsilon_0)}{\kappa_1 \varepsilon_0} * \frac{2L_x \sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right]_{\kappa_1} \left[\frac{\sin(\kappa_2 \varepsilon_0)}{\kappa_2 \varepsilon_0} * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \quad (2.69)$$

Noting the $\sin(\kappa_1 \varepsilon_0)/(\kappa_1 \varepsilon_0)$ terms are always less than or equal to one and the convolution merely causes a slight spreading, we write the bound for $L_{1,0}$ as

$$L_{1,0} \leq 16L_x^2 \varepsilon_0^2 [B/(\eta_1 \eta_2)]^2 [1 - \exp(-\sigma^2 \eta_1^2)] [1 - \exp(-\sigma^2 \eta_2^2)]$$

$$\leq 16L_x^2 \varepsilon_0^2 [B/(\eta_1 \eta_2)^2]. \quad (2.70)$$

TABLE 2.2

Order of Magnitude of Terms in $\langle E_1 E_1^* E_2 E_2^* \rangle$

Term	Order of Magnitude*
$L_{2,0}, L_{3,0}$	$L_{1,0}$
$L_{1,1}, L_{2,1}, L_{3,1}, L_4$ to L_{15}	$(\epsilon_0/L_x)L_{1,0}$
$L_{1,2}, L_{2,2}, L_{3,2},$ etc.	$(\epsilon_0/L_x)^2 L_{1,0}$

*Note: The order of magnitude of $L_{1,0}$ is given in Eq. (2.70).

The bounds for succeeding terms decrease in magnitude by the ratio ε_0/L_x , e.g.,

$$L_{1,1} \leq 16L_x \varepsilon_0^3 [B/(\eta_1 \eta_2)^2]. \quad (2.71)$$

Upper bounds for terms L_2 to L_{37} , in Eq. (2.59), are obtained by the same procedure as above. In Table 2.2 we list some of these terms and their order of magnitude relative to the term $L_{1,0}$ in Eq. (2.70).

B5. Dominant Terms

As discussed in the preceding sections, the four suppression terms of the indicator function in Eq. (2.60), for L_1 , can be expanded. Consideration of the integral term by term establishes that the dominant contribution, $L_{1,0}$, results from the first (or unity) term. Neglecting succeeding terms results in a fractional error on the order of ε_0/L_x . Thus the approximate form for L_1 is given by $L_{1,0}$, in Eq. (2.65),

$$L_1 \approx 4B[L_x/(\eta_1 \eta_2)]^2 \left[G(-\eta_1, \eta_1; \kappa_1) * \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]_{\kappa_1} \cdot \left[G(-\eta_2, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \quad (2.72)$$

where, again, G is given in Eq. (2.33).

Integrals for L_2 , L_3 , L_4 can be written in the form of Eq. (2.60) using Eqs. (2.56), (2.57), (2.58), respectively to express the indicator function. In an analogous manner to the derivation of Eq. (2.65), these integrals are evaluated. The order of magnitude for these terms relative to $L_{1,0}$ is listed in Table 2.2. The approximate solutions for L_2 and L_3 are as follows:

$$L_2 \cong 4B[L_x/(\eta_1\eta_2)]^2 \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} G(-\eta_1, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \\ \cdot \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} G(\eta_1, -\eta_2; \kappa_1) * \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]_{\kappa_1} \quad (2.73)$$

$$L_3 \cong 4B[L_x/(\eta_1\eta_2)]^2 \left[\frac{\sin(\kappa_2 + \kappa_1)L_x}{(\kappa_2 + \kappa_1)L_x} G(\eta_1, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \\ \cdot \left[\frac{\sin(\kappa_2 + \kappa_1)L_x}{(\kappa_2 + \kappa_1)L_x} G(-\eta_1, -\eta_2; \kappa_1) * \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]_{\kappa_1} \quad (2.74)$$

Thus, by Eq. (2.59) the solution of Eq. (2.51) has been found as the sum $L_1 + L_2 + L_3$ given by Eqs. (2.72) to

(2.74). This is written in Eqs. (2.75) to (2.78) below in the limit of large L_x using the same simplification of notation explained to obtain Eq. (2.49) from Eq. (2.46).

$$\langle E_1 E_1^* E_2 E_2^* \rangle \cong L_1 + L_2 + L_3, \quad (2.75)$$

in which

$$L_1 \cong 4B[L_x/(\eta_1 \eta_2)]^2 G(-\eta_1, \eta_1; \kappa_1) G(-\eta_2, \eta_2; \kappa_2) \quad (2.76)$$

$$L_2 \cong 4B[L_x/(\eta_1 \eta_2)]^2 \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} \right]^2 G(-\eta_1, \eta_2; \kappa_2) G(\eta_1, -\eta_2; \kappa_1) \quad (2.77)$$

$$L_3 \cong 4B[L_x(\eta_1 \eta_2)]^2 \left[\frac{\sin(\kappa_2 + \kappa_1)L_x}{(\kappa_2 + \kappa_1)L_x} \right]^2 G(\eta_1, \eta_2; \kappa_2) G(-\eta_1, -\eta_2; \kappa_1) \quad (2.78)$$

G is defined in Eq. (2.27) and evaluated in Eq. (2.33).

2.3.3 Discussion of Solution

A. Limiting forms of the general solution

The fourth order central moment can be rewritten for small roughness σ . First, from Eq. (2.31), we rewrite G for small values of $\sigma\eta$. Expanding the exponentials in a power series gives

$$G(-\eta_1, \eta_2; \kappa_2) = \int_{-\varepsilon_0}^{\varepsilon_0} d\xi \exp(i\kappa_2 \xi) \left\{ \left[1 - (\sigma^2(\eta_2 - \eta_1)^2/2 + \sigma^2\eta_1\eta_2[1-\rho(\xi)]) \right] + \dots \right\} - \left[1 - \sigma^2(\eta_1^2 + \eta_2^2)/2 + \dots \right] \}, \quad (2.79)$$

and by collecting terms Eq. (2.79) is written as

$$G(-\eta_1, \eta_2; \kappa_2) \cong \sigma^2\eta_1\eta_2 \int_{-\varepsilon_0}^{\varepsilon_0} d\xi \rho(\xi) \exp(i\kappa_2 \xi). \quad (2.80)$$

Equation (2.80) is integrated for the correlation coefficient in Eq. (2.32) giving

$$G(-\eta_1, \eta_2; \kappa_2) \cong \sigma^2\eta_1\eta_2\varepsilon_0 [\sin(\kappa_2\varepsilon_0/2)/(\kappa_2\varepsilon_0/2)]^2. \quad (2.81)$$

In the general solution the limiting form for small $\sigma\eta$ is obtained by substitution of Eq. (2.81) into Eqs. (2.75) to (2.78). The resulting expression is

$$\langle E_1 E_1^* E_2 E_2^* \rangle \cong L_1 + L_2 + L_3, \quad (2.82)$$

where

$$L_1 \cong 4B(\sigma^2 L_{x\varepsilon_0})^2 \left[\frac{\sin(\kappa_1\varepsilon_0/2)}{(\kappa_1\varepsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2\varepsilon_0/2)}{(\kappa_2\varepsilon_0/2)} \right]^2, \quad (2.83)$$

$$L_2 \cong 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} \right]^2 \left[\frac{\sin(\kappa_1 \epsilon_0/2)}{(\kappa_1 \epsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2 \epsilon_0/2)}{(\kappa_2 \epsilon_0/2)} \right]^2, \quad (2.84)$$

$$L_3 \cong 4B(\sigma^2 L_x \epsilon_0)^2 \left[\frac{\sin(\kappa_2 + \kappa_1)L_x}{(\kappa_2 + \kappa_1)L_x} \right]^2 \left[\frac{\sin(\kappa_1 \epsilon_0/2)}{(\kappa_1 \epsilon_0/2)} \right]^2 \left[\frac{\sin(\kappa_2 \epsilon_0/2)}{(\kappa_2 \epsilon_0/2)} \right]^2. \quad (2.85)$$

Comparing Eqs. (2.82) to (2.85) with the solution in Eqs. (2.47) to (2.50) that was obtained in an alternative manner, agreement is readily seen.

In Eqs. (2.75) to (2.78) a polar symmetry is evident. For example, if point I(κ_1, η_1) is fixed, the function $\langle E_1 E_1^* E_2 E_2^* \rangle$ is large as point II(κ_2, η_2) goes either to (κ_1, η_1) or $(-\kappa_1, -\eta_1)$. For large $|\sigma \eta_1|$, the central moment is dominated by L_1 and L_2 as $II \rightarrow I$ and by L_1 and L_3 as $II \rightarrow (-\kappa_1, -\eta_1)$.

For small $|\sigma \eta_1|$, the central moment in Eqs. (2.82) to (2.85) is seen to be independent of the η -coordinate. Hence, the polar symmetry exhibited by the general solution becomes reflection symmetry about the y -axis in this case. Because of this symmetry, both L_2 and L_3 must be retained in the solution even in the limiting cases.

B. Computations of the second order moments

In the experiments, described in Chapter IV, it is desired to verify, qualitatively, the analytical form for the expected value of the transform plane intensity pattern as produced by a serrated edge. This expression is obtained by combining Eqs. (2.23) and (2.28) using Eq. (2.22). We find

$$\begin{aligned} \langle E_1 E_1^* \rangle = & A_1 A_1^* \left| 2\pi^2 \delta(\kappa_1, \eta_1) + (i2L_x / \eta_1) [\sin(\kappa_1 L_x) / (\kappa_1 L_x)] \exp(-\sigma^2 \eta_1^2 / 2) \right|^2 \\ & + (A_1 A_1^* / \eta_1^2) 2L_x G(-\eta_1, \eta_1; \kappa_1) \end{aligned} \quad (2.86)$$

where $G(-\eta_1, \eta_1; \kappa_1)$ is given in Eq. (2.38) and the scale factor $2L_x$ is written in the second member. In Eq. (2.86) for the expected value of intensity, the first term is essentially the Airy disc while the second term shows the effect of a straight edge modified by the exponential due to the serration. The third term containing the function $G(-\eta_1, \eta_1; \kappa_1)$ is entirely due to the serration and can be seen to vanish in the limit $\sigma \rightarrow 0$. Equation (2.86) is valid for all values of σ , ε_0 , κ_1 , η_1 .

Consider the x -dependence of $\langle E_1 E_1^* \rangle$. At small values of κ_1 , the second term dominates the G -term, particularly when σ is small. However, the second term is extremely spike-like as a function of κ_1 or x due to the δ -function

nature of $2L_x \sin(\kappa_1 L_x)/(\kappa_1 L_x)$. The x -dependence of the third member is shown by curves of $G(\eta_0, -\eta_0; \kappa_1)$ vs. x_1 in Figs. (2.3) to (2.5). In Fig. 2.3, G is plotted as a function of the transform plane coordinate x_1 and the correlation length ϵ_0 , with the parameter $\sigma\eta_1$ fixed at 0.5. In Figs. 2.4 and 2.5, G is plotted as a function of x_1 , and $\sigma\eta_1$, with ϵ_0 fixed at 300 μm and 50 μm , respectively. Note, the general shape of G versus x_1 is unaffected by the parameter $\sigma\eta_1$. Rather, the effect of $\sigma\eta_1$ is to control the amplitude of G versus x_1 - not its shape. The width of these curves is seen to be largely controlled by the correlation length ϵ_0 , of course increasing in spread of energy as ϵ_0 decreases.

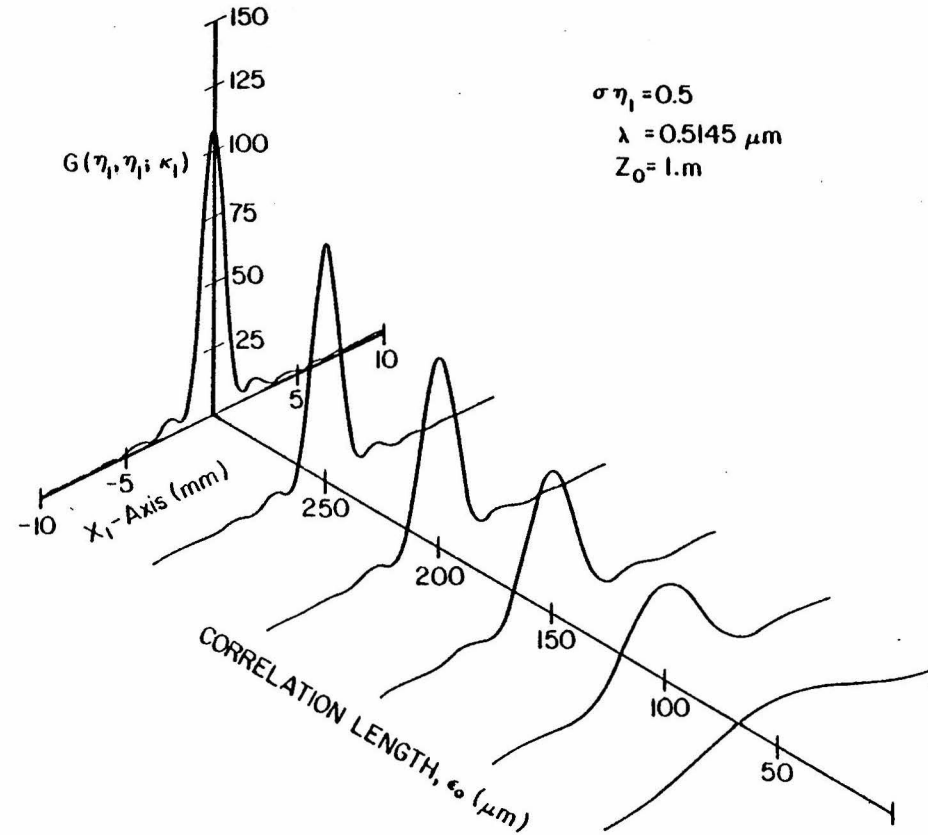


Fig. 2.3 The function $G(-\eta_1, \eta_1; \kappa_1)$, in Eq. (2.38), plotted vs. x_1 and ϵ_0 . G vs. x_1 corresponds to the envelope for the spread of energy perpendicular to the central spike. Other parameters were: $\sigma \eta_1 = 0.5$, $\lambda = 0.5145 \mu\text{m}$, $z_0 = 1.0 \text{ m}$. G is expressed in micrometers.

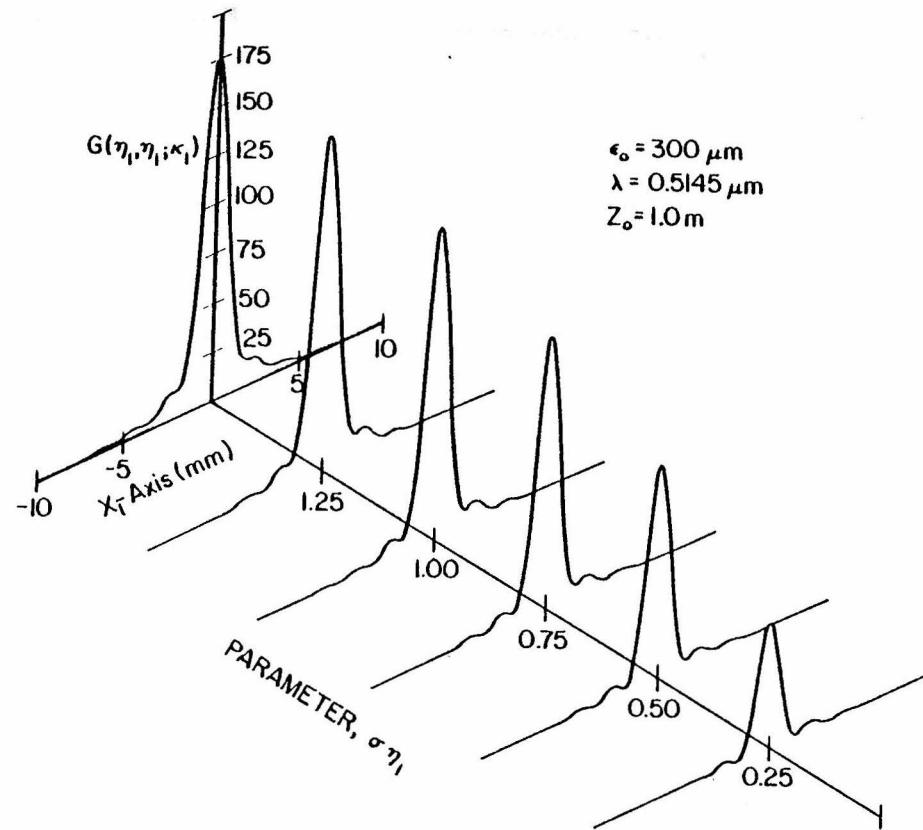


Fig. 2.4 The function $G(-\eta_1, \eta_1; \kappa_1)$, Eq. (2.38), plotted vs. x_1 and $\sigma \eta_1$. Note the general shape of G vs. x_1 is essentially independent of the parameter $\sigma \eta_1$. Other parameters were: $\epsilon_0 = 300 \mu\text{m}$, $\lambda = 0.5145 \mu\text{m}$ and $z_0 = 1.0 \text{ m}$. G is expressed in micrometers.

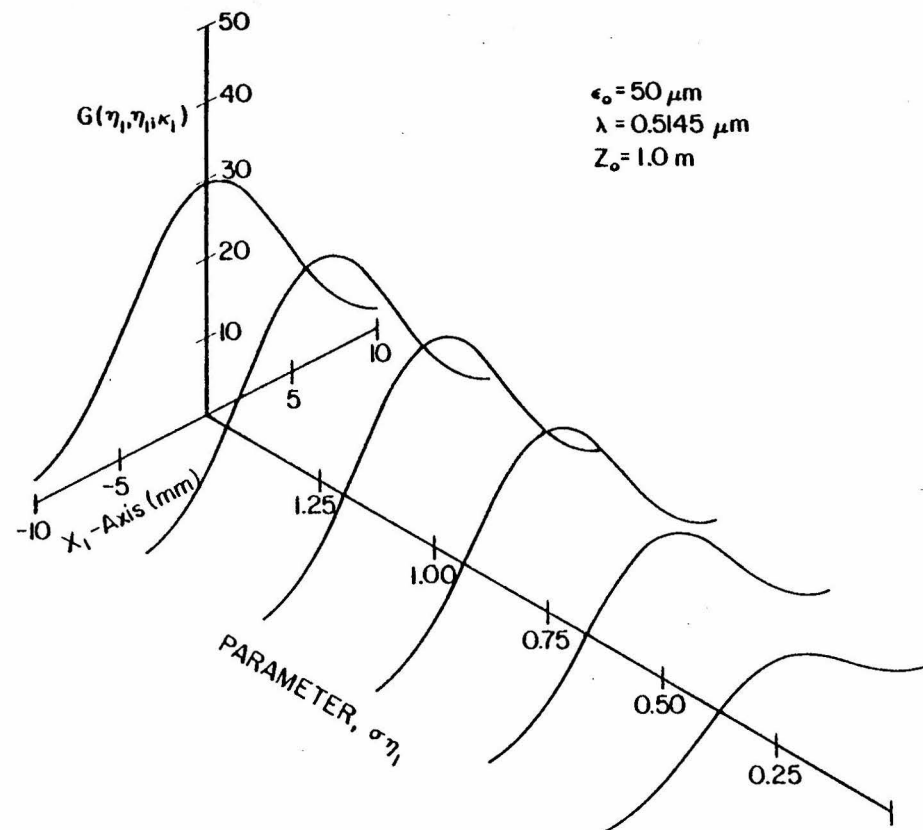


Fig. 2.5 $G(-\eta_1, \eta_1; \kappa_1)$, Eq. (2.38), plotted vs. x_1 and $\sigma\eta_1$. Other parameters were:
 $\epsilon_0 = 50 \mu\text{m}$, $\lambda = 0.5145 \mu\text{m}$, $z_0 = 1.0 \text{ m}$.
 G is expressed in micrometers.

2.4 The Serrated Gap

Consider a long thin gap with rough edges (Fig. 2.6) such that at any ξ , the open portion of unity transmission is given by

$$-w/2 + y_{ra}(\xi) \leq \gamma \leq w/2 + y_{rb}(\xi). \quad (2.87)$$

Hence, the gap has an average width w and roughness functions y_{ra} and y_{rb} along the lower and upper edges, respectively. The index sub- r denotes the r -th member of the ensemble, and we take the length, $2L_x$, of the gap along ξ to be nearly infinite. As in the analysis for a serrated edge, we adopt the assumptions discussed in Section 2.2.3. Briefly summarizing these are:

- (1) The serrations $y_{ra}(\xi)$ and $y_{rb}(\xi)$ are stationary, at least in the wide sense,
- (2) The random processes, $y_{ra}(\xi)$ and $y_{rb}(\xi)$, are dependent only on the ξ coordinate in the aperture plane,
- (3) All correlation coefficients, $\rho_{a,b}(\xi''-\xi')$, associated with y_{ra} and y_{rb} are zero when $|\xi''-\xi'| \geq \epsilon_0$.

Assumption (2) permits direct evaluation of the γ -integration in Eq. (2.8). Performing the integration with the limits in Eq. (2.87) and taking $t(\xi, \gamma)$ equal to unity,

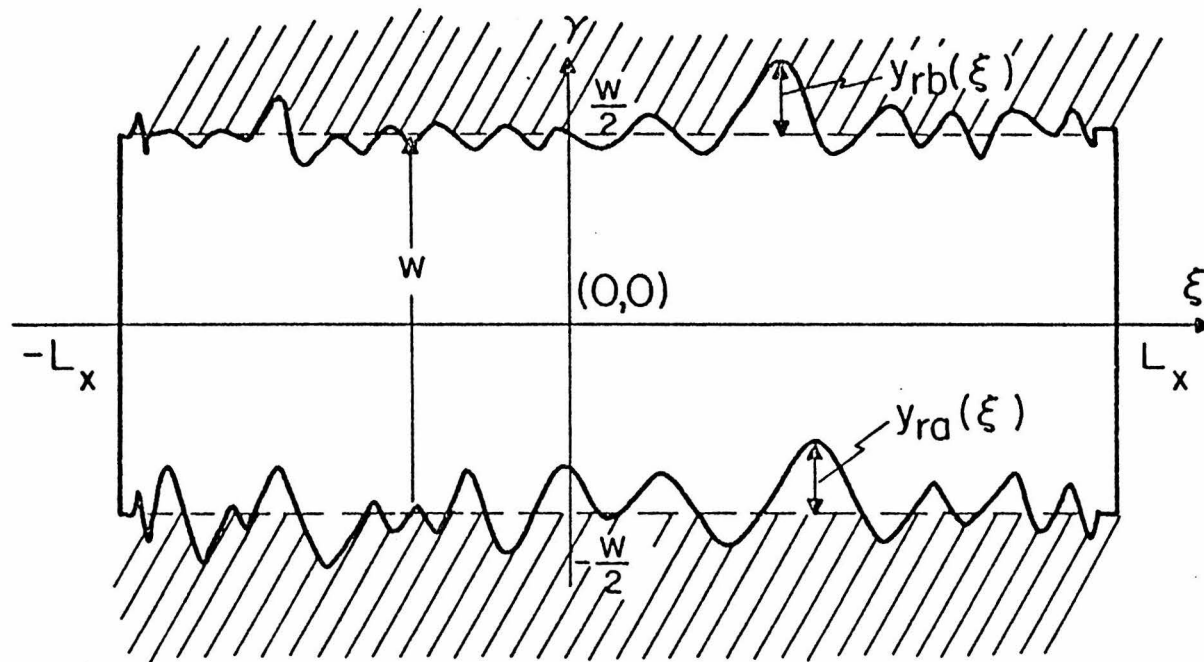


Fig. 2.6 Serrated gap with rough edges $y_{ra}(\xi)$ and $y_{rb}(\xi)$ and average width W .

the electric field, $E(x_1, y_1; k_1)$, takes the form

$$E(x_1, y_1; k_1) = (iA_1/\eta_1) \int_{-L_x}^{L_x} d\xi' \exp(-i\kappa_1 \xi') \left(\exp[-i\eta_1 w/2 - i\eta_1 y_{rb}(\xi')] \right. \\ \left. - \exp[i\eta_1 w/2 - i\eta_1 y_{ra}(\xi')] \right) \quad (2.88)$$

where A_1 , κ_1 and η_1 are defined in Eq. (2.19).

The expected value $\langle E_1 \rangle$ is computed from Eq. (2.88) using the notation defined in Table 2.1:

$$\langle E_1 \rangle = (iA_1/\eta_1) \int_{-L_x}^{L_x} d\xi' \exp(-i\kappa_1 \xi') \left(\exp(-i\eta_1 w/2) \psi_b(-\eta_1) \right. \\ \left. - \exp(i\eta_1 w/2) \psi_a(-\eta_1) \right) \quad (2.89)$$

or

$$\langle E_1 \rangle = (iA_1/\eta_1) \left[2L_x \sin(\kappa_1 L_x) / (\kappa_1 L_x) \right] \left[\exp(-i\eta_1 w/2) \psi_b(-\eta_1) \right. \\ \left. - \exp(i\eta_1 w/2) \psi_a(-\eta_1) \right] . \quad (2.90)$$

Equation (2.89) is subtracted from Eq. (2.88) giving the central field term $E_1 = E_1 - \langle E_1 \rangle$, as follows:

$$\begin{aligned}
 E_1 = & (iA_1/\eta_1) \int_{-L_x}^{L_x} d\xi' \exp(-i\kappa_1 \xi') \left\{ \exp(-i\eta_1 w/2) \right. \\
 & \cdot [\exp(-i\eta_1 y_{rb}(\xi')) - \Psi_b(-\eta_1)] - \exp(i\eta_1 w/2) \\
 & \cdot [\exp(-i\eta_1 y_{ra}(\xi')) - \Psi_a(-\eta_1)] \left. \right\}. \quad (2.91)
 \end{aligned}$$

The central moment $\langle E_1 E_2^* \rangle$ is formed from Eq. (2.91) for output points $I(x_1, y_1; k_1)$ and $II(x_2, y_2; k_2)$ in Eq. (2.92).

$$\begin{aligned}
 \langle E_1 E_2^* \rangle = & [A_1 A_2^* / (\eta_1 \eta_2)] \iint_{-L_x}^{L_x} d\xi' d\xi'' \exp(i\kappa_2 \xi'' - i\kappa_1 \xi') \\
 & \cdot \left\{ \exp(-i\eta_1 w/2) \left(\exp[-i\eta_1 y_{rb}(\xi')] - \Psi_b(-\eta_1) \right) - \exp(i\eta_1 w/2) \right. \\
 & \cdot \left(\exp[-i\eta_1 y_{ra}(\xi')] - \Psi_a(-\eta_1) \right) \left. \right\} \\
 & \cdot \left\{ \exp(i\eta_2 w/2) \left(\exp[i\eta_2 y_{rb}(\xi'')] - \Psi_b(\eta_2) \right) - \exp(-i\eta_2 w/2) \right. \\
 & \cdot \left(\exp[i\eta_2 y_{ra}(\xi'')] - \Psi_a(\eta_2) \right) \left. \right\} \quad (2.92)
 \end{aligned}$$

The second order moment $\langle E_1 E_2^* \rangle$ can be formed as prescribed in Eq. (2.22), using Eq. (2.90) with an analogous expression for $\langle E^*(x_2, y_2; k_2) \rangle$ and Eq. (2.92).

The calculation of $\langle E_1 E_2^* \rangle$, in Eq. (2.92), proceeds in a straightforward way. First, we expand and regroup the

terms in brackets. This gives

$$\langle E_1 E_2^* \rangle = [A_1 A_2^* / (\eta_1 \eta_2)] \iint_{-L_x}^{L_x} d\xi' d\xi'' \exp(i\kappa_2 \xi'' - i\kappa_1 \xi') [T_1 + T_2] \quad (2.93)$$

where T_1 and T_2 are defined by

$$\begin{aligned} T_1 = & \exp[i(\eta_2 - \eta_1)w/2] \langle [\exp(-i\eta_1 y'_{rb}) - \Psi_b(-\eta_1)] [\exp(i\eta_2 y''_{rb}) - \Psi_b(\eta_2)] \rangle \\ & + \exp[-i(\eta_2 - \eta_1)w/2] \langle [\exp(-i\eta_1 y'_{ra}) - \Psi_a(-\eta_1)] [\exp(i\eta_2 y''_{ra}) - \Psi_a(\eta_2)] \rangle; \end{aligned} \quad (2.94)$$

$$\begin{aligned} T_2 = & -\exp[i(\eta_1 + \eta_2)w/2] \langle [\exp(-i\eta_1 y'_{ra}) - \Psi_a(-\eta_1)] [\exp(i\eta_2 y''_{rb}) - \Psi_b(\eta_2)] \rangle \\ & - \exp[-i(\eta_1 + \eta_2)w/2] \langle [\exp(-i\eta_1 y'_{rb}) - \Psi_b(-\eta_1)] [\exp(i\eta_2 y''_{ra}) - \Psi_a(\eta_2)] \rangle. \end{aligned} \quad (2.95)$$

T_1 can be seen to consist of a sum of two expectations where one contains only upper edge terms and the other only lower or sub-a terms. T_1 can be evaluated independently of correlations which may exist between y_{ra} and y_{rb} . T_2 contains mixed y_b and y_a terms and so can be expected to depend markedly on the degree of correlation between y_{ra} and y_{rb} . Note the expectations in T_1 and T_2 involve terms of the type $\langle z_1 z_2 \rangle$ where $z_{1,2}$ have zero mean. If the correlation interval between z_1 and z_2 is short, then $\langle z_1 z_2 \rangle$ has a sharply peaked value contributed entirely in the

correlation interval.

In the evaluation of Eq. (2.93), let us make an additional simplifying assumption:

- (4) The roughness functions, $y_{ra}(\xi)$ and $y_{rb}(\xi)$, for the upper and lower edge are statistically independent.

With assumption (4), we see from Eq. (2.95) that $T_2 \equiv 0$.

Using this simplification and regrouping T_1 , we write

$$\begin{aligned} \langle E_1 E_2^* \rangle = & [A_1 A_2^* / (\eta_1 \eta_2)] \int_{-\infty}^{\infty} d\xi' \exp(-i\kappa_1 \xi') \text{rect}[\xi' / (2L_x)] \int_{-\infty}^{\infty} d\xi'' \exp(i\kappa_2 \xi'') \\ & \cdot \text{rect}[\xi'' / (2L_x)] \left\{ \exp[i(\eta_2 - \eta_1)w/2] \left[\Phi_b[-\eta_1, \eta_2; \rho(\xi'' - \xi')] - \Psi_b(-\eta_1) \Psi_b(\eta_2) \right] \right. \\ & \left. + \exp[-i(\eta_2 - \eta_1)w/2] \left[\Phi_a[-\eta_1, \eta_2; \rho(\xi'' - \xi')] - \Psi_a(-\eta_1) \Psi_a(\eta_2) \right] \right\} \end{aligned} \quad (2.96)$$

where rect functions have been used to denote the finite limits of integration, $-L_x$ to L_x . In Eq. (2.96), two points are noted. The first is that the form of Eq. (2.96) is array-like with respect to the upper and lower contributions. Second, the ξ'' -integration may be regarded as product of the function $\text{rect}[\xi'' / (2L_x)]$ with the contribution from both the upper and lower serrations. We represent one component by the integral $G_a(-\eta_1, \eta_2; \kappa_2)$, where

$$G_a(-\eta_1, \eta_2; \kappa_2) = \int_{-\infty}^{\infty} d\xi \exp(i\kappa_2 \xi) \left[\Phi_a[-\eta_1, \eta_2; \rho(\xi)] - \Psi_a(-\eta_1) \Psi_a(\eta_2) \right]. \quad (2.97)$$

Using Eq. (2.97) and the convolution theorem of Fourier analysis, we write $\langle E_1 E_2^* \rangle$ in the form:

$$\begin{aligned} \langle E_1 E_2^* \rangle = & [A_1 A_2^* / (\eta_1 \eta_2)] \int_{-\infty}^{\infty} d\xi' \exp(-i \kappa_1 \xi') \text{rect}[\xi' / (2L_x)] \\ & \cdot \left\{ \exp[i(\eta_2 - \eta_1)w/2] \left[2L_x \sin(\kappa_2 L_x) / (\kappa_2 L_x) * e^{i\kappa_2 \xi'} G_b(-\eta_1, \eta_2; \kappa_2) \right]_{\kappa_2} \right. \\ & \left. + \exp[-i(\eta_2 - \eta_1)w/2] \left[2L_x \sin(\kappa_2 L_x) / (\kappa_2 L_x) * e^{i\kappa_2 \xi'} G_2(-\eta_1, \eta_2; \kappa_2) \right]_{\kappa_2} \right\} \end{aligned} \quad (2.98)$$

in which the asterisk denotes a convolution with respect to κ_2 . Interchanging the order of ξ' -integration and convolution in Eq. (2.98), we find the general expression for the central moment of the electric field in the optical transform plane:

$$\begin{aligned} \langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} & \left\{ \exp[i(\eta_2 - \eta_1)w/2] \left[\frac{2L_x \sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} G_b(-\eta_1, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{(\kappa_2 L_x)} \right]_{\kappa_2} \right. \\ & \left. + \exp[-i(\eta_2 - \eta_1)w/2] \left[\frac{2L_x \sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} G_a(-\eta_1, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{(\kappa_2 L_x)} \right]_{\kappa_2} \right\}. \end{aligned} \quad (2.99)$$

A Gaussian density of roughness has not been assumed in deriving Eq. (2.99), nor has any assumption been made of finite correlation length ϵ_0 .

Noting the convolution with $2L_x \sin(\kappa_2 L_x)/(\kappa_2 L_x)$ results in only a slight spreading of the first members, we write $\langle E_1 E_2^* \rangle$ neglecting or suppressing the convolution below:

$$\begin{aligned} \langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} 2L_x & \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} \right] \left[\exp[i(\eta_2 - \eta_1)w/2] G_b(-\eta_1, \eta_2; \kappa_2) \right. \\ & \left. + \exp[-i(\eta_2 - \eta_1)w/2] G_a(-\eta_1, \eta_2; \kappa_2) \right]. \end{aligned} \quad (2.100)$$

A. Evaluation for normally distributed roughness

For the serrated gap, assume that the density function for heights is jointly normal and each serration has the same standard deviation, σ , and correlation coefficient, $\rho(\xi)$. From Table 2.1, we write

$$\Phi_a(-\eta_1, \eta_2; \rho(\xi)) = \exp \left[-\sigma^2 (\eta_2 - \eta_1)^2 / 2 - \sigma^2 \eta_1 \eta_2 [1 - \rho(\xi)] \right]$$

$$\Psi_a = \exp(-\sigma^2 \eta_1^2 / 2). \quad (2.101)$$

Equation (2.97) can be integrated exactly for a wide variety of correlation coefficients ρ . As an example of the evaluation of Eq. (2.97), we assume the correlation coefficient is given by the triangle function, $\Lambda(\xi/\epsilon_0)$, of Eq. (2.32). Substituting Eq. (2.32) into (2.97) and integrating gives Eq. (2.33). The following result is obtained for the central moment of the serrated gap with Gaussian roughness of arbitrary σ and ϵ_0 by rewriting Eq. (2.100) with G given by Eq. (2.33) and $\eta_{1,2}$ and $\kappa_{1,2}$ defined in Eq. (2.9):

$$\langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} \frac{2L_x \sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} 2G \cos[(\eta_2 - \eta_1)w/2]. \quad (2.102)$$

CHAPTER II

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CHAPTER III

REMOTE SENSING OF EDGE STATISTICS

3.1 Introduction

Diffraction of laser light by serrated apertures produces a speckle pattern. In the literature there are numerous treatments emphasizing the usefulness of speckle to sense surface structure⁽¹⁻⁴⁾. In addition to the spatial characteristics of speckle, George⁽⁵⁾ et al. have derived expressions for the wavelength sensitivity of speckle produced by scattering polychromatic radiation at frequencies ν_1 and ν_2 from large roughened spheres. The wavelength dependence of the pattern was shown useful in sensing general large-scale features of objects.

In this chapter we derive an expression for the cross-correlation function $\langle E(x_1, y_1; k_1) E^*(x_2, y_2; k_2) \rangle$ for the field in the optical transform plane of a serrated edge and a serrated gap. The solution, expressed as a power series, is particularly useful for sensing edge statistics. The theory for the serrated edge and gap is presented in Sections 3.2 and 3.3, respectively. In each section the expression for the expected value of intensity is represented as a power series in which the terms involve

the Fourier transform of the edge's correlation coefficient, Eqs. (3.19) and (3.37). Using Fourier transform analysis and the technique of reversion of series, the equation is inverted; yielding the correlation coefficient expressed as a series of terms involving the Fourier transform of the expected value of intensity, see Eqs. (3.26) and (3.40). It is found that both edge roughness, σ , and the correlation coefficient, $\rho(\xi'' - \xi')$, can be determined from the expected value of intensity.

3.2 The Serrated Edge

3.2.1 Second Order Moment of the Electric Field

From Eq. (2.20) the expression for the electric field in the optical transform plane of a serrated edge is

$$E(x_1, y_1; k_1) = 2\pi^2 A_1 \delta(\kappa_1, \eta_1) - (iA_1/\eta_1) \int_{-\infty}^{\infty} d\xi \exp[-i(\kappa_1 \xi + \eta_1 y_r(\xi))] \text{rect}[\xi/(2L_x)] \quad (3.1)$$

where

$$A_1 = [-ik_1/(2\pi z_0)] \exp[ik_1 z_0 + ik_1(x_1^2 + y_1^2)/(2z_0)]$$

$$\kappa_1 = k_1 x_1 / z_0 \quad (3.2)$$

$$\eta_1 = k_1 y_1 / z_0$$

and $y_r(\xi)$ is shown in Fig. 3.1. As in Chapter II we again adopt the rect function formalism for the ξ -integration.

The second moment of the field, including both spatial and wavelength dependences is given by Eq. (2.22) and rewritten Eq. (3.3):

$$\langle E_1 E_2^* \rangle = \langle E_1 E_2^* \rangle + \langle E_1 \rangle \langle E_2 \rangle, \quad (3.3)$$

in which,

$$E_1 \equiv E_1 - \langle E_1 \rangle \quad \text{and} \quad E_2 \equiv E_2 - \langle E_2 \rangle. \quad (3.4)$$

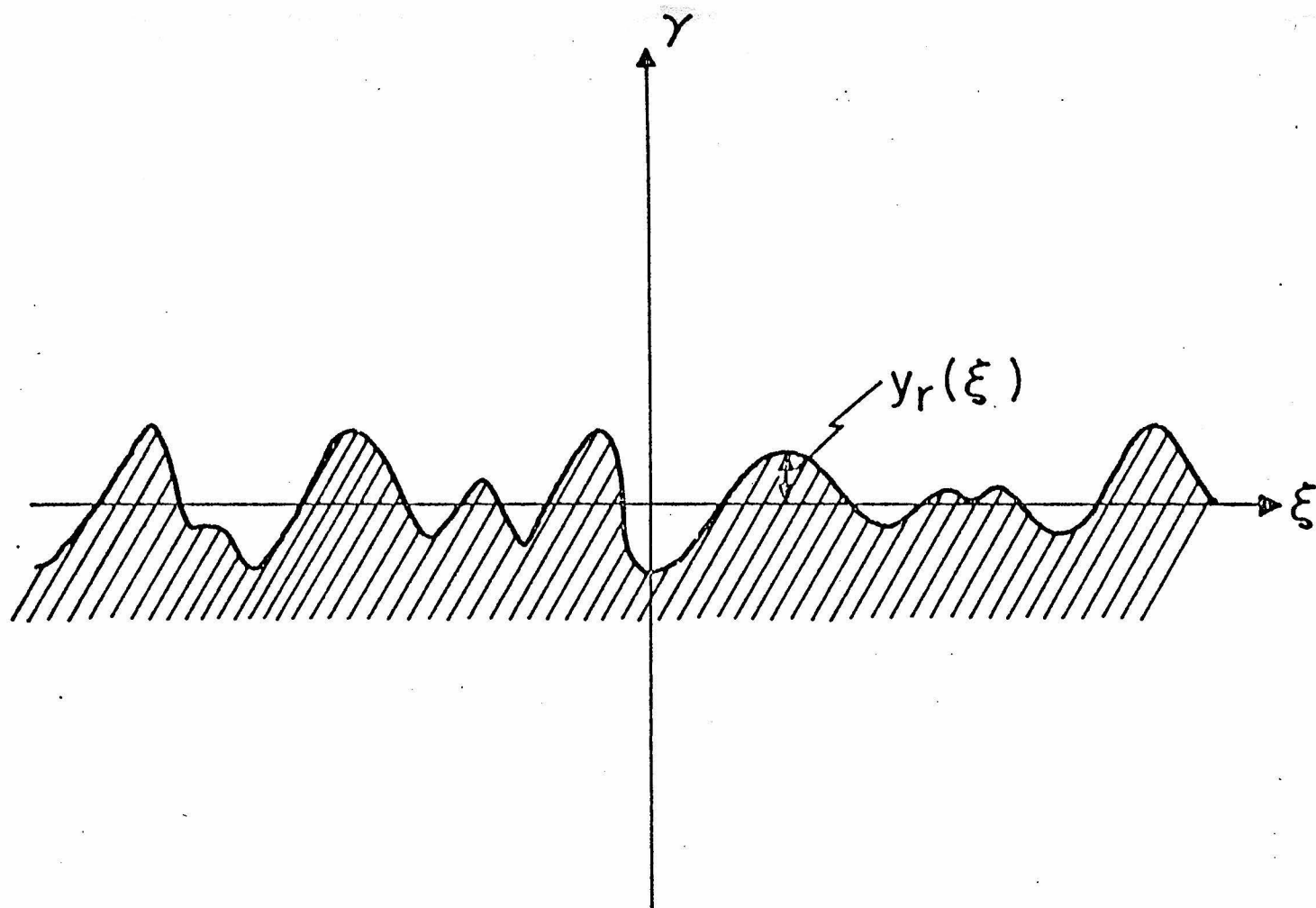


Fig. 3.1 Serrated edge with roughness function, $y_r(\xi)$.

Taking the expected value of the field gives

$$\begin{aligned} \langle E_1 \rangle = & 2\pi^2 A_1 \delta(\kappa_1, \eta_1) - (iA_1/\eta_1) \int_{-\infty}^{\infty} d\xi \exp(i\kappa_1 \xi) \langle \exp(-i\eta_1 y_r(\xi)) \rangle \\ & \cdot \text{rect}[\xi/(2L_x)]. \end{aligned} \quad (3.5)$$

Evaluating the integral in Eq. (3.5) and adopting the notation in Table 2.1 yields $\langle E_1 \rangle$ in the following form:

$$\langle E_1 \rangle = 2\pi^2 A_1 \delta(\kappa_1, \eta_1) - (iA_1/\eta_1) \frac{2L_x \sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \Psi(-\eta_1). \quad (3.6)$$

The second term on the right in Eq. (3.3) is formed by taking the product of terms of the form in Eq. (3.6). The result is

$$\begin{aligned} \langle E_1 \rangle \langle E_2^* \rangle = & A_1 A_2^* \left[2\pi^2 \delta(\kappa_1, \eta_1) - (i/\eta_1) \frac{2L_x \sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \Psi(-\eta_1) \right] \\ & \cdot \left[2\pi^2 \delta(\kappa_2, \eta_2) + (i/\eta_2) \frac{2L_x \sin(\kappa_2 L_x)}{(\kappa_2 L_x)} \Psi(\eta_2) \right]. \end{aligned} \quad (3.7)$$

The first term on the right side of Eq. (3.3) is formed by substituting Eqs. (3.1), (3.5) and (3.4) into $\langle E_1 E_2^* \rangle$.

This is written as

$$\begin{aligned} \langle E_1 E_2^* \rangle &= [A_1 A_2^* / (\eta_1 \eta_2)] \iint_{-\infty}^{\infty} d\xi' d\xi'' \exp[i\kappa_2 \xi'' - i\kappa_1 \xi'] \text{rect}[\xi' / (2L_x)] \text{rect}[\xi'' / (2L_x)] \\ &\quad \cdot \langle \exp[-i\eta_1 y_r(\xi')]] - \Psi(-\eta_1) \rangle \left(\exp[i\eta_2 y_r(\xi'')]] - \Psi(\eta_2) \right) \rangle . \end{aligned} \quad (3.8)$$

Multiplying the terms inside the expected value symbol, $\langle \dots \rangle$, and using the characteristic function notation of Table 2.1 yields the central moment of the electric field

$$\begin{aligned} \langle E_1 E_2^* \rangle &= [A_1 A_2^* / (\eta_1 \eta_2)] \iint_{-\infty}^{\infty} d\xi' d\xi'' \exp[i\kappa_2 \xi'' - i\kappa_1 \xi'] \text{rect}[\xi' / (2L_x)] \text{rect}[\xi'' / (2L_x)] \\ &\quad \cdot \left[\Phi[-\eta_1, \eta_2; \rho(\xi'' - \xi')]] - \Psi(-\eta_1) \Psi(\eta_2) \right] , \end{aligned} \quad (3.9)$$

and for normally distributed roughness, we write

$$\begin{aligned} \langle E_1 E_2^* \rangle &= [A_1 A_2^* / (\eta_1 \eta_2)] \iint_{-\infty}^{\infty} d\xi' d\xi'' \exp[i\kappa_2 \xi'' - i\kappa_1 \xi'] \text{rect}[\xi' / (2L_x)] \text{rect}[\xi'' / (2L_x)] \\ &\quad \cdot \exp \left[-\sigma^2 (\eta_1^2 + \eta_2^2) / 2 \right] \left(\exp[\sigma^2 \eta_1 \eta_2 \rho(\xi'' - \xi')]] - 1 \right) . \end{aligned} \quad (3.10)$$

The desired form of $\langle E_1 E_2^* \rangle$ for our calculations is obtained by noting the ξ'' -integration of Eq. (3.10) can be regarded as a Fourier transform of a product of $\text{rect } \xi'' / (2L_x)$ times the exponential function representing the serration.

Applying the convolution theorem to this product, then interchanging the order of ξ' -integration and convolution

we obtain:

$$\langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} 2L_x \left[\frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} G(-\eta_1, \eta_2; \kappa_2) * \frac{2L_x \sin(\kappa_2 L_x)}{\kappa_2 L_x} \right]_{\kappa_2} \quad (3.11)$$

where the asterisk denotes a convolution with respect to κ_2 . The function $G(-\eta_1, \eta_2; \kappa_2)$ is written below in Eq. (3.12).

$$G(-\eta_1, \eta_2; \kappa_2) = \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \int_{-\infty}^{\infty} d\xi_1 \exp(i\kappa_2 \xi_1) \cdot \left(\exp[\sigma^2 \eta_1 \eta_2 \rho(\xi)] - 1 \right). \quad (3.12)$$

As explained in obtaining Eq. (2.49) from Eq. (2.46) the convolution in Eq. (3.11) can be neglected or suppressed, and we write:

$$\langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} 2L_x \frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} G(-\eta_1, \eta_2; \kappa_2). \quad (3.13)$$

The main results of this chapter are obtained by using Eq. (3.12). First, the term $\exp[\sigma^2 \eta_1 \eta_2 \rho(\xi)]$ is expanded in a Taylor series,

$$\exp[\sigma^2 \eta_1 \eta_2 \rho(\xi)] = \sum_{n=0}^{\infty} [\sigma^2 \eta_1 \eta_2 \rho(\xi)]^n / n!. \quad (3.14)$$

Substituting Eq. (3.14) into Eq. (3.12) and interchanging the order of integration and summation gives

$$G(-\eta_1, \eta_2; \kappa_2) = \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \sum_{n=1}^{\infty} (\sigma^2 \eta_1 \eta_2)^n \cdot \int_{-\infty}^{\infty} d\xi_1 \exp(i\kappa_2 \xi_1) \rho^n(\xi_1). \quad (3.15)$$

The integral in Eq. (3.15) is seen to be the Fourier transform of the correlation coefficient to the n^{th} power, $\rho^n(\xi_1)$. Denoting this transform by,

$$F_{\kappa_2}(\rho^n(\xi)) \equiv \int_{-\infty}^{\infty} d\xi_1 \exp(i\kappa_2 \xi_1) \rho^n(\xi_1), \quad (3.16)$$

we write G as follows:

$$G = \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \sum_{n=1}^{\infty} (\sigma^2 \eta_1 \eta_2)^n F_{\kappa_2}(\rho^n(\xi_1)). \quad (3.17)$$

The desired expression for the central moment of the electric field is given by substituting Eq. (3.17) in Eq.(3.13). This is written as

$$\langle E_1 E_2^* \rangle = \frac{A_1 A_2^*}{\eta_1 \eta_2} 2L_x \frac{\sin[(\kappa_2 - \kappa_1)L_x]}{(\kappa_2 - \kappa_1)L_x} \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \sum_{n=1}^{\infty} (\sigma^2 \eta_1 \eta_2)^n F_{\kappa_2}(\rho^n(\xi_1)). \quad (3.18)$$

The correlation function of the electric field, $\langle E_1 E_2^* \rangle$, can be found by combining Eqs. (3.18) and (3.7) using Eq. (3.3).

As seen below, information about both the roughness, σ , and the correlation coefficient, $\rho(\xi_1)$, can be obtained from consideration of the expected value of intensity. This expression is a special case of the correlation function $\langle E_1 E_2^* \rangle$; and is obtained by letting the generalized frequency coordinates $(\kappa_2, \eta_2) \rightarrow (\kappa_1, \eta_1)$. The resulting expression for $\langle E_1 E_1^* \rangle$ is

$$\begin{aligned} \langle E_1 E_1^* \rangle = & A_1 A_1^* \left| 2\pi^2 \delta(\kappa_1, \eta_1) + (i/\eta_1) 2L_x \frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \Psi(\eta_1) \right|^2 \\ & + \frac{A_1 A_1^*}{\eta_1^2} 2L_x \exp[-\sigma^2 \eta_1^2] \sum_{n=1}^{\infty} (\sigma \eta_1)^{2n} F_{\kappa_1}(\rho^n(\xi_1)). \end{aligned} \quad (3.19)$$

In Eq. (3.19) for the expected value of the intensity, the first term is essentially the Airy disc while the second term shows the effect of a straight edge modified by the first order characteristic function of the serration. The third term containing the series is entirely due to the serration, and can be seen to vanish as $\sigma \rightarrow 0$.

3.2.2 Reversion of Series

Equation (3.19) expresses the expected value of intensity in the transform plane as an Airy disc, a modified straight edge and a series of terms which involve the statistics of the serration. However, for remote sensing applications it is desirable to represent the statistics of the edge, e.g. the correlation coefficient, as a series in which the terms involve the transform plane intensity. This "inverse" relation can be obtained by applying the technique of reversion of series as follows. From Abramowitz and Stegun⁽⁶⁾ - 3.6.25, if we are given

$$q = ap + bp^2 + cp^3 + dp^4 + \dots, \quad (3.20)$$

then p can be expressed in terms of q as

$$p = Aq + Bq^2 + Cq^3 + Dq^4 + \dots \quad (3.21)$$

where

$$aA = 1$$

$$a^3B = -b \quad (3.22)$$

$$a^5C = 2b^2 - ac$$

$$a^7D = 5abc - a^2d - 5b^3.$$

Note, to get Eq. (3.19) into the form of Eq. (3.20), we must first subtract the absolute value term; then Fourier transform both sides with respect to κ_1 . Doing this we get

$$F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) = (A_1 A_1^* / \eta_1^2) 2L_x \exp(-\sigma^2 \eta_1^2) \sum_{n=1}^{\infty} (\sigma \eta_1)^{2n} \rho^n(\xi_1), \quad (3.23)$$

in which,

$$\langle E_1 \rangle \langle E_1^* \rangle = A_1 A_1^* \left| 2\pi^2 \delta(\kappa_1, \eta_1) + (i/\eta_1) \frac{2L_x \sin(\kappa_1 L_x)}{\kappa_1 L_x} \Psi(\eta_1) \right|^2.$$

In Eq. (3.23) we make the following identifications. Let

$$q = F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) \quad (3.24)$$

$$p = \rho(\xi_1)$$

and the constants,

$$a = A_1 A_1^* 2L_x \sigma^2 \exp(-\sigma^2 \eta_1^2) \quad (3.25)$$

$$b = A_1 A_1^* 2L_x \sigma^4 \eta_1^2 \exp(-\sigma^2 \eta_1^2), \text{ etc.}$$

Using Eqs. (3.24) and (3.25) in Eqs. (3.21), (3.22), we find the correlation function, $\rho(\xi_1)$, for the edge can be

represented in the form:

$$\rho(\xi_1) = \frac{\exp(\sigma^2 \eta_1^2)}{2L_x \sigma^2 A_1 A_1^*} F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) - \frac{\eta_1^2 \exp(2\sigma^2 \eta_1^2)}{(A_1 A_1^* 2L_x \sigma)^2} \left[F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) \right]^2 \quad (3.26)$$

+ higher order terms.

In Eq. (3.26) the symbols are defined in Eqs. (3.2), (3.16) and Table 2.1.

3.3 The Serrated Gap

The analysis for a gap proceeds in an analogous way to that of the serrated edge discussed in the previous section. For a serrated gap, Fig. 3.2, the electric field in optical transform plane is given in Eq. (2.88) as

$$E(x_1, y_1; k_1) = (iA_1/\eta_1) \int_{-L_x}^{L_x} d\xi' \exp(i\kappa_1 \xi') \left(\exp[-i\eta_1 W/2 - i\eta_1 y_{rb}(\xi')] \right. \\ \left. - \exp[i\eta_1 W/2 - i\eta_1 y_{ra}(\xi')] \right) \quad (3.27)$$

in which A_1 , κ_1 , and η_1 are defined in Eq. (3.2). The expected value of the electric field depends on the first order characteristic function of the serrations, and is written in Eqs. (3.28) and (3.29) using the notation of Table 2.1.

$$\langle E_1 \rangle = (iA_1/\eta_1) \int_{-L_x}^{L_x} d\xi' \exp(-i\kappa_1 \xi') \left(\exp[-i\eta_1 W/2] \Psi_b(-\eta_1) \right. \\ \left. - \exp[i\eta_1 W/2] \Psi_a(-\eta_1) \right) \quad (3.28)$$

or

$$\langle E_1 \rangle = (iA_1/\eta_1) [2L_x \sin(\kappa_1 L_x) / (\kappa_1 L_x)] \left(\exp[-i\eta_1 W/2] \Psi_b(-\eta_1) \right. \\ \left. - \exp[i\eta_1 W/2] \Psi_a(-\eta_1) \right) . \quad (3.29)$$

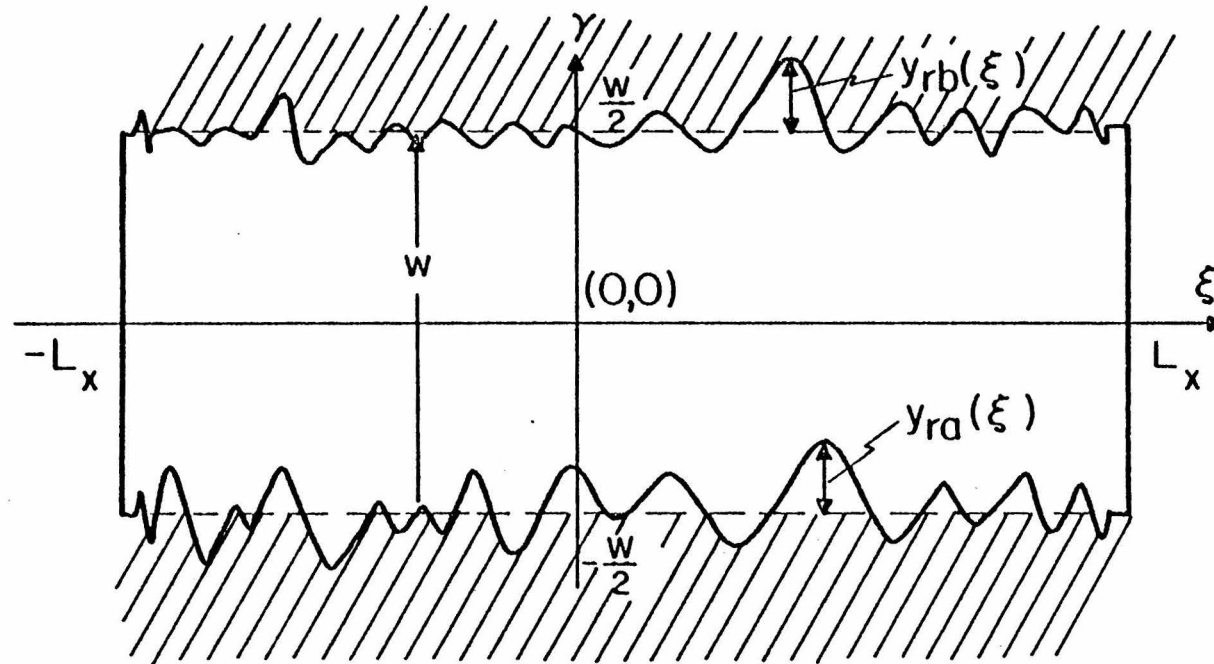


Fig. 3.2 Serrated gap with rough edges $y_{ra}(\xi)$ and $y_{rb}(\xi)$ and average width W .

Central field terms, Eq. (3.4), can be formed by subtracting Eq. (3.28) from Eq. (3.27). Using these terms, we can form the central moment of the electric field, as in Eqs. (2.93), (2.94), and (2.95). Rewriting we have

$$\langle E_1 E_2^* \rangle = [A_1 A_2^* / (\eta_1 \eta_2)] \int_{-L_x}^{L_x} d\xi' d\xi'' \exp[i(\kappa_2 \xi'' - \kappa_1 \xi')] [T_1 + T_2] \quad (3.30)$$

where

$$T_1 = \exp[i(\eta_2 - \eta_1)W/2] \left[\Phi_b[-\eta_1, \eta_2; \rho(\xi'' - \xi')] - \Psi_b(-\eta_1) \Psi_b(\eta_2) \right] \\ + \exp[-i(\eta_2 - \eta_1)W/2] \left[\Phi_a[-\eta_1, \eta_2; \rho(\xi'' - \xi')] - \Psi_a(-\eta_1) \Psi_a(\eta_2) \right] \quad (3.31)$$

and

$$T_2 = -\exp[i(\eta_1 + \eta_2)W/2] \left[\langle \exp[-i\eta_1 y'_{ra} + i\eta_2 y''_{rb}] \rangle - \Psi_a(-\eta_1) \Psi_b(\eta_2) \right] \\ - \exp[-i(\eta_1 + \eta_2)W/2] \left[\langle \exp[-i\eta_1 y'_{rb} + i\eta_2 y''_{ra}] \rangle - \Psi_b(-\eta_1) \Psi_a(\eta_2) \right] \quad (3.32)$$

T_1 is a sum of two terms; one of which depends only on the upper serration, $y_{rb}(\xi)$, and the other depends only on the lower, $y_{ra}(\xi)$. T_2 contains a sum of two cross-terms between the upper and lower serrations. This term accounts for any correlation that might exist between $y_{ra}(\xi')$ and $y_{rb}(\xi'')$.

In the analysis that follows, we make the assumptions that edges $y_{ra}(\xi)$ and $y_{rb}(\xi)$ are jointly normal and that there is no correlation between the upper and lower edges, i.e., $T_2 \equiv 0$. For simplicity we also take the edges to have identical second order statistics, that is

$$\sigma = \sigma_a = \sigma_b$$

and

$$\rho(\xi'' - \xi') = \rho_a(\xi'' - \xi') = \rho_b(\xi'' - \xi') . \quad (3.33)$$

With these simplifications we can write the central moment $\langle E_1 E_2^* \rangle$ as

$$\begin{aligned} \langle E_1 E_2^* \rangle &= [2A_1 A_2^* / (\eta_1 \eta_2)] \cos[(\eta_2 - \eta_1)W/2] \iint_{-L_x}^{L_x} d\xi' d\xi'' \exp[i(\kappa_2 \xi'' - \kappa_1 \xi')] \\ &\cdot \left[\Phi[-\eta_1, \eta_2; \rho(\xi'' - \xi')] - \Psi(-\eta_1) \Psi(\eta_2) \right] . \end{aligned} \quad (3.34)$$

The double integral in Eq. (3.34) is evaluated as in Eqs. (3.9)-(3.13), thus we write

$$\begin{aligned} \langle E_1 E_2^* \rangle &= \frac{A_1 A_2^*}{\eta_1 \eta_2} 4L_x \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} \right] \cos[(\eta_2 - \eta_1)W/2] \\ &\cdot G(-\eta_1, \eta_2; \kappa_2) \end{aligned} \quad (3.35)$$

where G is defined in Eq. (3.12).

The power series representation of Eq. (3.35) is obtained by substituting Eq. (3.17) for $G(-\eta_1, \eta_2; \kappa_2)$. It follows

$$\begin{aligned} \langle E_1 E_2^* \rangle = & \frac{A_1 A_2^*}{\eta_1 \eta_2} 4L_x \left[\frac{\sin(\kappa_2 - \kappa_1)L_x}{(\kappa_2 - \kappa_1)L_x} \right] \cos[(\eta_2 - \eta_1)W/2] \\ & \cdot \exp[-\sigma^2(\eta_1^2 + \eta_2^2)/2] \sum_{n=1}^{\infty} (\sigma^2 \eta_1 \eta_2)^n F_{\kappa_2} \rho^n(\xi_1), \quad (3.36) \end{aligned}$$

where $F_{\kappa_2}(\rho^n(\xi_1))$ is defined by Eq. (3.16). Thus for a serrated gap, as for an edge, the central moment of the optical transform plane electric field can be represented as a series in which the terms depend on the Fourier transform of the edge correlation coefficient. The first term is proportional to the Fourier transform of $\rho(\xi)$; the second proportional to the convolution of $F(\rho(\xi_1))$ with itself, etc.

The expected value of intensity, $\langle E_1 E_1^* \rangle$, is obtained from Eqs. (3.3), (3.29) and (3.36) by letting $(\kappa_2, \eta_2) \rightarrow (\kappa_1, \eta_1)$. This gives

$$\begin{aligned} \langle E_1 E_1^* \rangle = & 16L_x^2 \frac{A_1 A_1^*}{\eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right]^2 \cos^2[\eta_1 W/2] \exp(-\sigma^2 \eta_1^2) \\ & + \frac{4L_x A_1 A_1^*}{\eta_1^2} \exp[-\sigma^2 \eta_1^2] \sum_{n=1}^{\infty} (\sigma \eta_1)^{2n} F_{\kappa_1}(\rho^n(\xi_1)) . \end{aligned} \quad (3.37)$$

It is interesting to compare forms for the expected value of intensity for a serrated edge and gap. Note the only difference between the series terms in Eqs. (3.19) and (3.37) is a factor of 2.

The inversion of Eq. (3.37) proceeds in a straightforward way. We subtract the first term from both sides, then Fourier transform with respect to κ_1 . Equations (3.20) to (3.22) are then applied by letting

$$\begin{aligned} q &= F_{\kappa_1} (\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle) \\ p &= \rho(\xi_1) \end{aligned} \quad (3.38)$$

and the constants,

$$\begin{aligned} a &= A_1 A_1^* 4L_x \sigma^2 \exp(-\sigma^2 \eta_1^2) \\ b &= A_1 A_1^* 4L_x \sigma^4 \eta_1^2 \exp(-\sigma^2 \eta_1^2), \text{ etc.}, \end{aligned} \quad (3.39A)$$

where in Eq. (3.38), we have used

$$\langle E_1 \rangle \langle E_1^* \rangle = 16 L_x^2 \frac{A_1 A_1^*}{\eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right] \cos^2 \eta_1 W/2 \exp(-\sigma^2 \eta_1^2). \quad (3.39B)$$

The result follows:

$$\begin{aligned} \rho(\xi_1) = & \frac{\exp(\sigma^2 \eta_1^2)}{4 L_x^2 \sigma^2 A_1 A_1^*} F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) \\ & - \frac{\eta_1^2 \exp(2\sigma^2 \eta_1^2)}{(A_1 A_1^* 4 L_x \sigma)^2} \left[F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \langle E_1 \rangle \langle E_1^* \rangle \right) \right]^2 \\ & + \text{higher order terms.} \end{aligned} \quad (3.40)$$

CHAPTER III

REFERENCES

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CHAPTER IV

EXPERIMENTS

4.1 Introduction

To illustrate the theory, an experimental investigation of the optical transform pattern produced by rough edges and gaps was performed. In this study two classes of edges were considered. One class consisted of edges roughened by sandpaper of various grades; and the second consisted of edges produced from computer-generated curves. The second group proved to be extremely useful, since the roughness, σ , and the correlation length, ϵ_0 , could be controlled independently.

In Section 4.2 the experimental system is described and photographs of the optical transform pattern for straight and roughened edges are shown. Section 4.3 contains a discussion about random process in a space-invariant linear system, together with a description of the edge generation scheme. In Section 4.4 we show measurements of the intensity pattern recorded using a fiber optic probe. Intensity measurements along the central spike and perpendicular to the spike are included. In Section 4.5 we test the theory of Chapter 3. Using Eqs. (3.26) and (3.40) we compare the actual correlation coefficient of the edge with our estimate obtained from transform plane intensity measurements.

4.2 The Experimental Arrangement

4.2.1 Description of System

A diagram of the experimental layout is shown in Fig. 4.1. The illumination source consisted of a Spectra Physics Model 165 argon laser which in turn pumped a Spectra Physics Model 370 Rhodamine 6G dye laser with a linewidth of less than $0.5\text{\AA}$. The beam was directed into a spatial filter, which had a 20X microscope objective and $12\text{ }\mu\text{m}$ pinhole. The expanding beam passed through a Jaegers, No. 9E1158 Type 1, converging achromatic lens and was subsequently focused into the detector plane. The roughened aperture was placed between the transform lens and the detector at a distance z_0 away from the optical transform plane.

The main elements of the detection system consisted of a Gamma Scientific scanning eyepiece, Model 700-10-26A in which the detector, a $50\text{ }\mu\text{m}$ fiber, was connected to an RCA 4463 photomultiplier tube via a fiber optic bundle. The output of the photo tube was fed into a Type 116 pre-amplifier of a Princeton Applied Research Model 124A lock-in amplifier. The lock-in was operated in a band-pass mode with a reference supplied by an incandescent light source/photodiode combination on a mechanical chopper operating at 1.2KHz. The output signal was fed into a Fluke Model 8000A digital voltmeter which was interfaced to a Teletype terminal, where data were punched on paper tape.

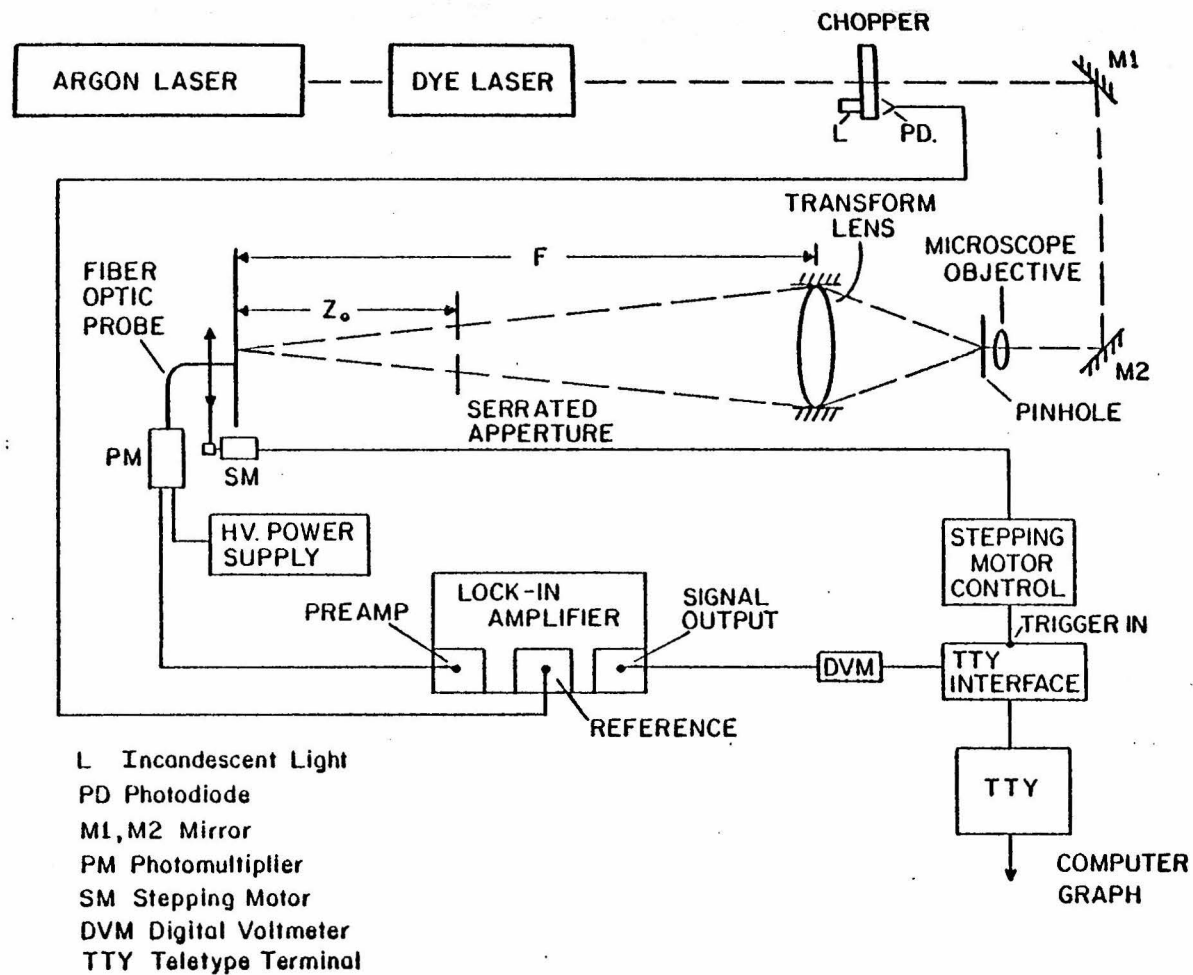


Fig. 4.1 Experimental arrangement for recording intensity in the optical transform plane.

Later the data were read into a PDP11/34 computer system and stored on a floppy disk. A link between the PDP11/34 and the University of Rochester's IBM 3032 computer system permitted the data to be transferred, thus making readily available the graphics capabilities of the larger system. A Slo-syn driving motor, type SS25-1011, connected to a custom built stepping motor controller permitted accurate positioning of the fiber optic probe with a minimum step size of 3.2 micrometers.

In the measurements described herein, the laser was plane-polarized parallel to the edge or gap, i.e. along the ξ axis. The distance, F , between the lens and detector plane was 175.5 cm; and z_0 , the distance between the serrated aperture and detector plane, was chosen to be 25.0 cm. The pupil diameter, D , at the lens was 2.54 cm, unless specified otherwise. The wavelength, λ , was fixed at $5,800\text{\AA}$, thus producing a speckle size, $\lambda F/D$, of approximately $40\text{ }\mu\text{m}$. The speckle size was chosen less than the probe diameter in order to average some of the cyclic structure in the pattern. This provided a better estimate of the envelope function for the intensity.

4.2.2 Photographs of Optical Transform Patterns

Photographs of the patterns were obtained by replacing the detector with a film plate, located in the focal plane, see Fig. 4.1. The photographs of the optical transform

pattern of a straight and serrated edge are shown in Figs. 4.2(a) and 4.2(b), respectively. The inset in the lower right-hand corner is an approximate representation of the object transmission function where the circular portion illustrates the finite size of the beam. The characteristic feature of the transform of a straight edge is the intense spike of energy occurring perpendicular to the edge. When the edge is serrated, the effect is a damping of the central spike accompanied by a spread of energy in the direction parallel to the edge.

For the photographs the distances F and z_0 , of Fig. (4.1), were 400 cm and 200 cm, respectively. Thus the Airy disc size was approximately 100 μm . This size corresponds to the smallest structure visible in the patterns.

The complicated, circular interference pattern in the center is produced by the particular arrangement of the setup. Its structure is influenced by two parameters. One, it is extremely sensitive to edge position within the circular beam. This is illustrated by raising the edge above center, as in Figs. 4.3(a) and 4.3(b), such that the major portion of the beam is blocked. Secondly, the pattern is affected by the position of the edge relative to the entrance pupil located at the transform lens. Placing the edge at the entrance pupil produces a symmetric pattern irrespective of the edge position within the circular aperture.⁽¹⁾ However, as the edge is moved away from the

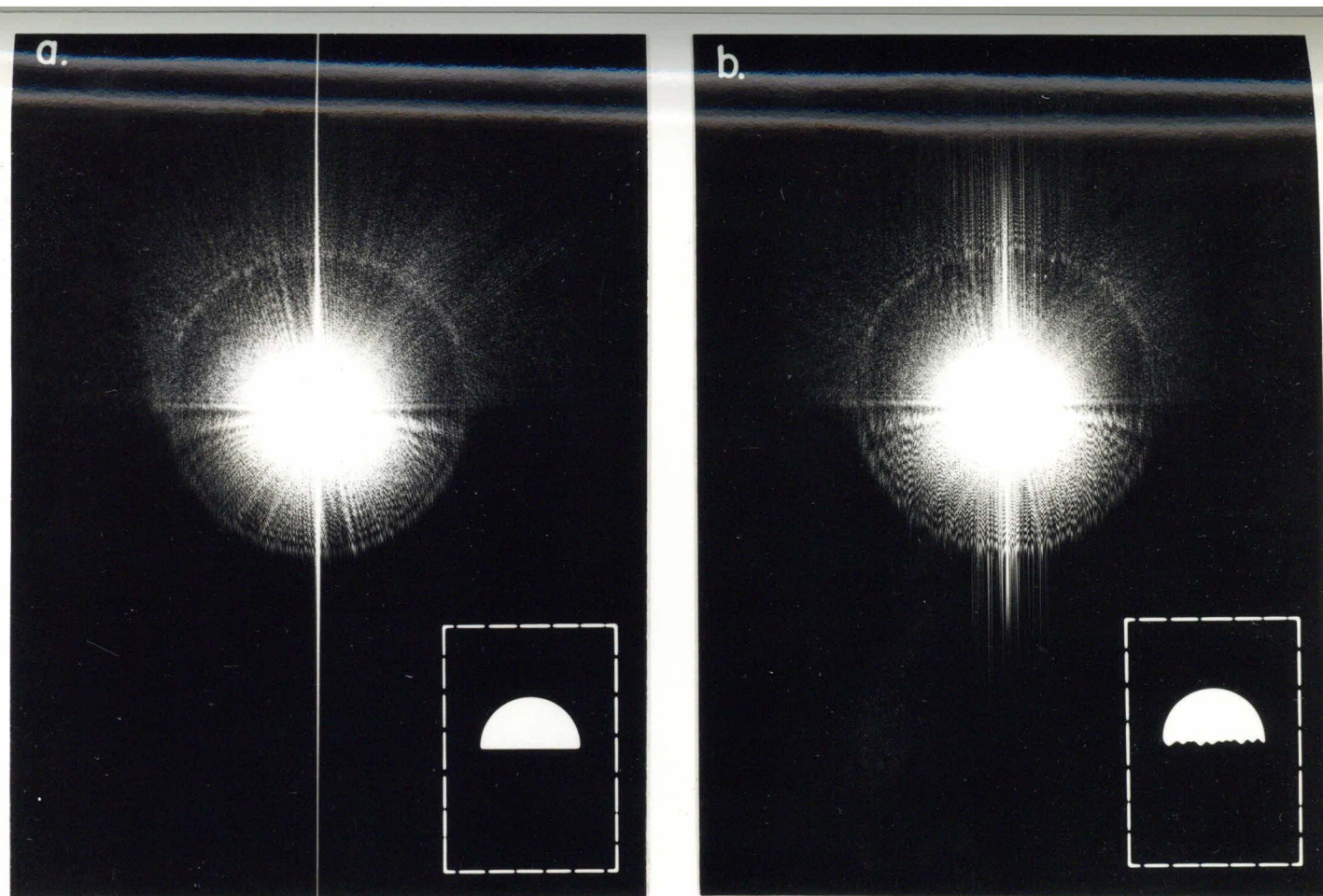


Fig. 4.2 Optical transform patterns for (a) straight edge and (b) serrated edge with $\sigma = 15 \mu\text{m}$ (shown inset). Illumination is by a convergent monochromatic wave with $z_0 = 200\text{cm}$ and $\lambda = 5,145\text{\AA}$. Note the appreciable decrease in energy in the central spike for the roughened edge. The radius of the illuminated sector is 6mm so the speckle or resolution scale size is on the order, $\lambda z_0/\rho$, of $200 \mu\text{m}$. The pattern subtends an angle $\pm 1/40$ radian.

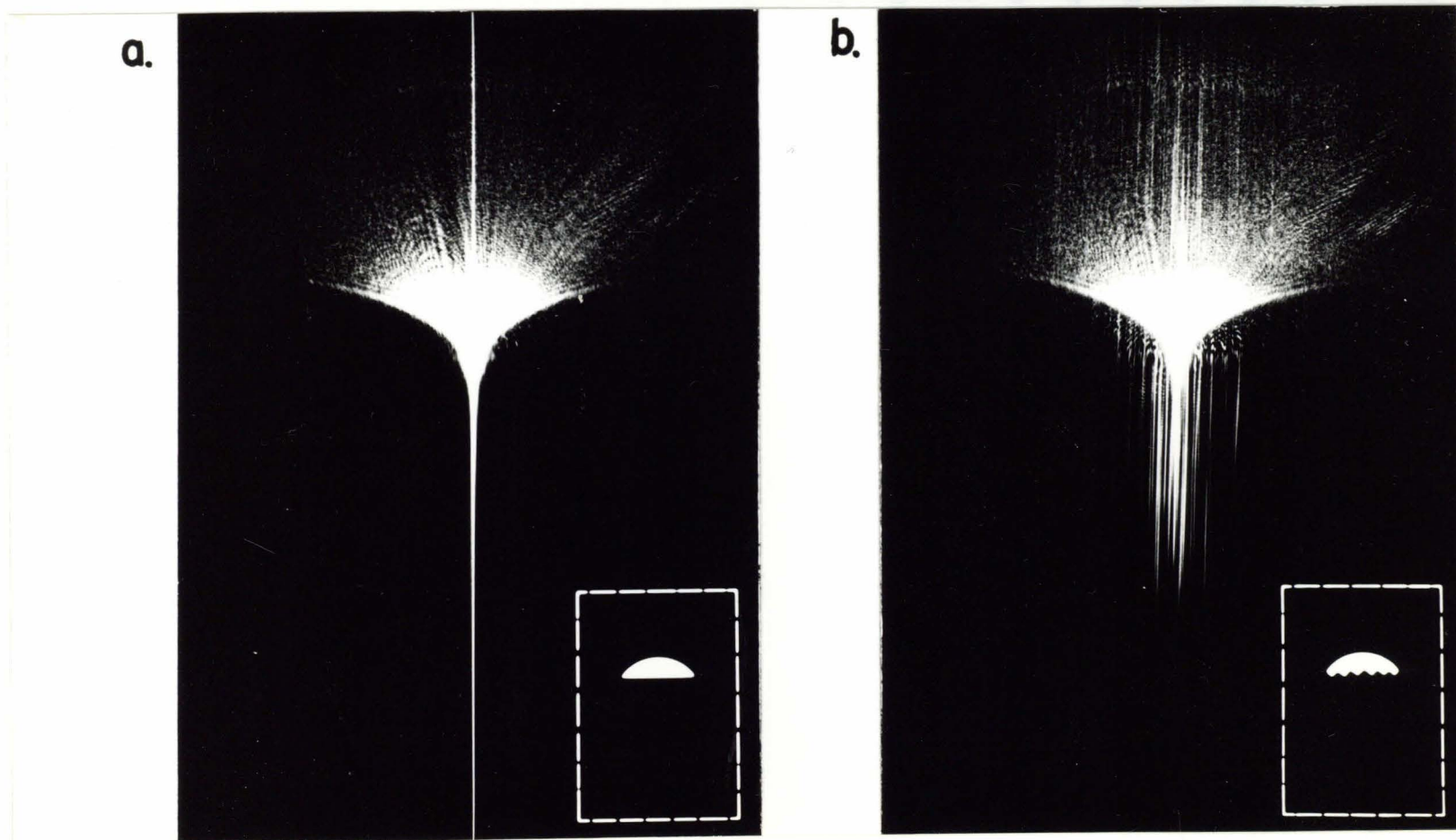


Fig. 4.3 Optical transform patterns for an off-centered (a) straight edge and (b) serrated edge with $\sigma = 15 \mu\text{m}$ (shown inset). Other parameters were the same as those in Fig. 4.2.

entrance pupil, the symmetry vanishes. This lack of symmetry is enhanced with increased distance between the edge and entrance pupil.

In the experiments described below, our interest is in the damping of the central spike and the spread of energy perpendicular to the spike; the circular interference pattern is regarded as noise. It is noted the spike and associated side lobe structure is insensitive, within limits, to edge position in the circular beam. Hence, in all fiber optic measurements, the edges were positioned as in Fig. 4.3. Probe scans down the spike and perpendicular to the central spike were made below the pattern's center.

4.3 Computer-Generated Edges

When an optical system is affected by the statistical properties of an element within that system, such as a diffuser or serrated edge, one would like to insert sample elements, which possess known statistics, in order to observe their effect. Here, we describe the process used to obtain serrated edges with desired statistics.

In Section 4.3.1 a linear system approach is used to describe a Gaussian random function with known second order statistics. In Section 4.3.2, we describe the generation scheme for obtaining metallic edges with controlled statistics. Here, we applied the linear system theory developed in Section 4.3.1 and created number sets which possessed the desired statistics. These sets were then plotted on microfiche, blacking the area under the curve, thereby creating an edge. Bi-metal masks were made from the film data using photo-etching techniques.

4.3.1 Gaussian Random Processes in a Space-Invariant Linear System

Consider a stationary, ergodic, random function $\Psi(x)$, where x is a deterministic argument, as an input to a space-invariant linear system possessing a window function $W(x)$, as illustrated in Fig. 4.4. The output of the system is denoted by the random function $\Phi(x)$.

Although random functions with many possible densities may be generated, we illustrate the theory using a Gaussian

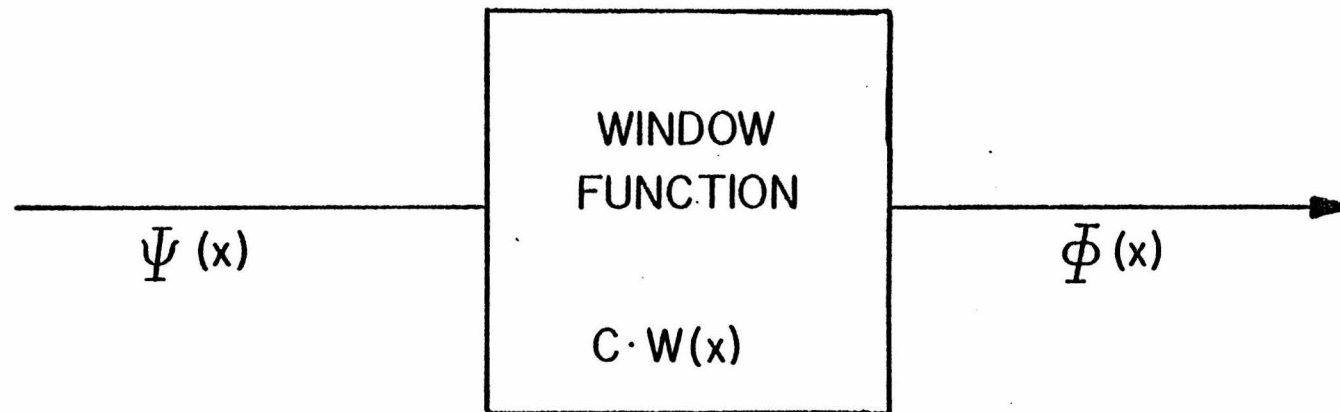


Fig. 4.4 Linear system model used to describe second-order Gaussian statistics.
C is a multiplicative constant.

density for $\Psi(x)$. Making a linear transformation of a Gaussian random process gives a Gaussian process, thus $\Phi(x)$ is Gaussian also. We write the joint normal density for $\Phi(x)$ as:

$$p(\phi_1, \phi_2; x_1, x_2) = \frac{1}{d\phi_1 d\phi_2} p \left[\phi_1 \leq \Phi(x_1) \leq \phi_1 + d\phi_1, \right. \\ \left. \phi_2 \leq \Phi(x_2) \leq \phi_2 + d\phi_2 \right], \quad (4.1)$$

$$p(\phi_1, \phi_2; x_1, x_2) = \left(\frac{1}{2\pi\sigma^2 \sqrt{1-\rho^2}} \right) \exp \left\{ -\frac{1}{2\sigma^2(1-\rho^2)} \right. \\ \left. \cdot [\phi_1^2 - 2\rho\phi_1\phi_2 + \phi_2^2] \right\}. \quad (4.2)$$

In Eq. (4.2) we have assumed the random function has a mean value equal to zero, and variance equal to σ^2 . The correlation coefficient, $\rho(x_2-x_1)$, is given by

$$\rho(x_2-x_1) = \langle \Phi(x_1)\Phi(x_2) \rangle / \sigma^2 \quad (4.3)$$

where the brackets, $\langle \dots \rangle$, denote an ensemble average. When $\rho(x_2-x_1) = 0$, $p(\phi_1, \phi_2; x_1, x_2)$ reduces to a product of two one-dimensional Gaussian densities; therefore $\Phi(x_1)$ and $\Phi(x_2)$ become two independent random functions. If $\rho(x_2-x_1)$ decreases to some specified value in a distance $\epsilon_0 = |x_2-x_1|$, we say the curve has a correlation length, ϵ_0 , associated with it. This is one of the parameters we would like to control.

For the input function $\Psi(x)$, we take the mean value,

$$\langle \Psi(x) \rangle = 0, \quad (4.4)$$

and the variance,

$$\langle \Psi^2(x) \rangle = \sigma_\Psi^2 . \quad (4.5)$$

For the present, we assume no specific form for the correlation coefficient of $\Psi(x)$. We merely denote it as follows:

$$r_\Psi(x_2-x_1) = \langle \Psi(x_1)\Psi(x_2) \rangle / \sigma_\Psi^2 . \quad (4.6)$$

The output function Φ is defined by the convolution relation

$$\Phi(x) = C \int_{-\infty}^{\infty} \Psi(\xi) W(x-\xi) d\xi \quad (4.7)$$

where C is a constant.

The question we now address is: Given the input statistics for $\Psi(x)$, i.e., σ_Ψ^2 and $r_\Psi(x_2-x_1)$, and the window function $W(x)$ with the constant C , what is the variance, σ^2 , and correlation coefficient, $\rho(x_2-x_1)$, for the random function $\Phi(x)$?

Both σ^2 and $\rho(x_2-x_1)$ can be determined by considering the autocorrelation function of $\Phi(x)$. The autocorrelation function, R_{12} , is defined as⁽²⁾

$$R_{12} = \langle \Phi(x_1)\Phi(x_2) \rangle . \quad (4.8)$$

Substituting Eq. (4.7) into Eq. (4.8) gives

$$R_{12} = \langle C^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi' d\xi'' \Psi(\xi') \Psi(\xi'') W(x_1-\xi') W(x_2-\xi'') \rangle \quad (4.9)$$

Since the integrals in Eq. (4.9) are essentially sums, we may interchange integration and averaging to obtain

$$R_{12} = c^2 \iint_{-\infty}^{\infty} d\xi' d\xi'' \langle \Psi(\xi') \Psi(\xi'') \rangle W(x_1 - \xi') W(x_2 - \xi'') \quad (4.10)$$

The term in triangular brackets is identified as the autocorrelation function of the input random function Ψ .

Using Eqs. (4.5) and (4.6), the autocorrelation function of Ψ is written in the following form:

$$\langle \Psi(\xi') \Psi(\xi'') \rangle = \sigma_\psi^2 r_\psi(\xi'' - \xi') \quad (4.11)$$

Substituting Eq. (4.11) into (4.10) gives

$$R_{12} = c^2 \sigma_\psi^2 \iint_{-\infty}^{\infty} d\xi' d\xi'' r_\psi(\xi'' - \xi') W(x_1 - \xi') W(x_2 - \xi''). \quad (4.12)$$

In Eq. (4.12) consider the ξ'' integration,

$$B_{rW} = \int_{-\infty}^{\infty} d\xi'' r_\psi(\xi'' - \xi') W(x_2 - \xi''). \quad (4.13)$$

Making the substitution, $\eta = \xi'' - \xi'$, gives

$$B_{rW}(x_2 - \xi') = \int_{-\infty}^{\infty} d\eta r_\psi(\eta) W[(x_2 - \xi') - \eta] \quad , \quad (4.14)$$

where B_{rW} is recognized as a convolution of the window function $W(x)$ and the correlation coefficient of Ψ , r_ψ .

Using the notation of Eq. (4.14) in Eq. (4.12), we write

$$R_{12} = c^2 \sigma_\psi^2 \int_{-\infty}^{\infty} d\xi' W(x_1 - \xi') B_{rW}(x_2 - \xi') \quad . \quad (4.15)$$

Making the substitution, $\xi = x_1 - \xi'$ in Eq. (4.15) we write the autocorrelation function, R_{12} , in the following form:

$$R_{12}(x_2 - x_1) = c^2 \sigma_\psi^2 \left[W(x) * B_{rW}(x) \right] \quad (4.16)$$

where

$$W(x) * B_{rW}(x) = \int_{-\infty}^{\infty} d\xi W(\xi) B_{rW} \left[(x_2 - x_1) + \xi \right] \quad (4.17)$$

and B_{rW} is given in Eq. (4.14).

The variance of $\Phi(x)$, $\sigma^2 = \langle \Phi^2(x) \rangle$, is obtained from R_{12} when $x_2 = x_1$, i.e.

$$\begin{aligned} \sigma^2 &= R_{12}(0) \\ &= c^2 \sigma_\psi^2 \int_{-\infty}^{\infty} d\xi W(\xi) B_{rW}(\xi) . \end{aligned} \quad (4.18)$$

With Eqs. (4.3), (4.8), (4.16), (4.18), the correlation coefficient of $\Phi(x)$, $\rho(x_2 - x_1)$, is found to be

$$\begin{aligned} \rho(x_2 - x_1) &= R_{12}(x_2 - x_1) / R_{12}(0) \\ &= W(x) * B_{rW}(x) / \int_{-\infty}^{\infty} d\xi W(\xi) B_{rW}(\xi) \end{aligned} \quad (4.19)$$

with symbols as previously defined in Eqs. (4.14) and (4.17).

We now consider a special case of the general equations for σ^2 and $\rho(x_2 - x_1)$. If we associate with the window function, $W(x)$, a length L such that $W(x)$ is significant

only in the interval $|x| \leq L$; and require Δx , the associated correlation length of $\Psi(x)$, to be much less than L , then σ^2 and $\rho(x_2-x_1)$ assume a simple form. With the condition $\Delta x \ll \epsilon_0$, we note the convolution in Eq. (4.14) results in only a slight spreading of the window function $W(x)$, thus we have

$$B_{rW}(x_2-\xi') \simeq \Delta x W(x_2-\xi') \quad (4.20)$$

or equivalently,

$$B_{rW}(x) \simeq \Delta x W(x) \quad (4.21)$$

Substituting Eq. (4.21) into Eqs. (4.18) and (4.19) yields σ^2 and $\rho(x_2-x_1)$, respectively:

$$\sigma^2 \simeq C^2 \sigma_\psi^2 \Delta x \int_{-\infty}^{\infty} dx W^2(x) \quad (4.22)$$

and

$$\rho(x_2-x_1) \simeq W(x) * W(x) / \int_{-\infty}^{\infty} dx W^2(x) \quad (4.23)$$

where the asterisk denotes an autocorrelation. Note, the variance σ^2 is easily adjusted by selecting different values of the constant C . For example, if we wanted $\sigma^2 = \sigma_\psi^2$, then

$$C^2 = 1 / \left[\Delta x \int_{-\infty}^{\infty} dx W^2(x) \right] \quad (4.24)$$

Likewise, the desired correlation coefficient, $\rho(x_2-x_1)$, is readily obtained by an appropriate choice for $W(x)$.

4.3.2 The Generation Scheme

With the aid of a digital computer, the results from the linear system model described above are used to generate number sets with controlled statistics. The input function $\Psi(x)$ is obtained using the computer system's random number generator. Standard routines are available in which one may specify the mean and the standard deviation; however, the numbers are, in general, uncorrelated. In fact, the number set obtained usually decorrelates at the first point of shift in the autocorrelation function. Hence, assuming that the correlation length of $\Psi(x)$, Δx , is much less than ϵ_0 of the window function, in the derivation of Eqs. (4.22) and (4.23), is a good approximation for computer generated processes.

The set from the random number generator is then convolved with a window function. Note, this is equivalent to a sliding average of the window function with the random number set.

In Fig. 4.5(a)-(d) we show random curves obtained using Eqs. (4.7) and (4.24); and their corresponding autocorrelation functions. For each curve we used the same set of random numbers. The difference lies in the choice for the window function. In Fig. 4.5(a), the window function was taken as a rect,⁽³⁾

$$\begin{aligned} W(x) &= \text{rect}(x/100.) \\ &= \begin{cases} 1; & |x/100.| \leq 1/2 \\ 0; & \text{otherwise;} \end{cases} \end{aligned}$$

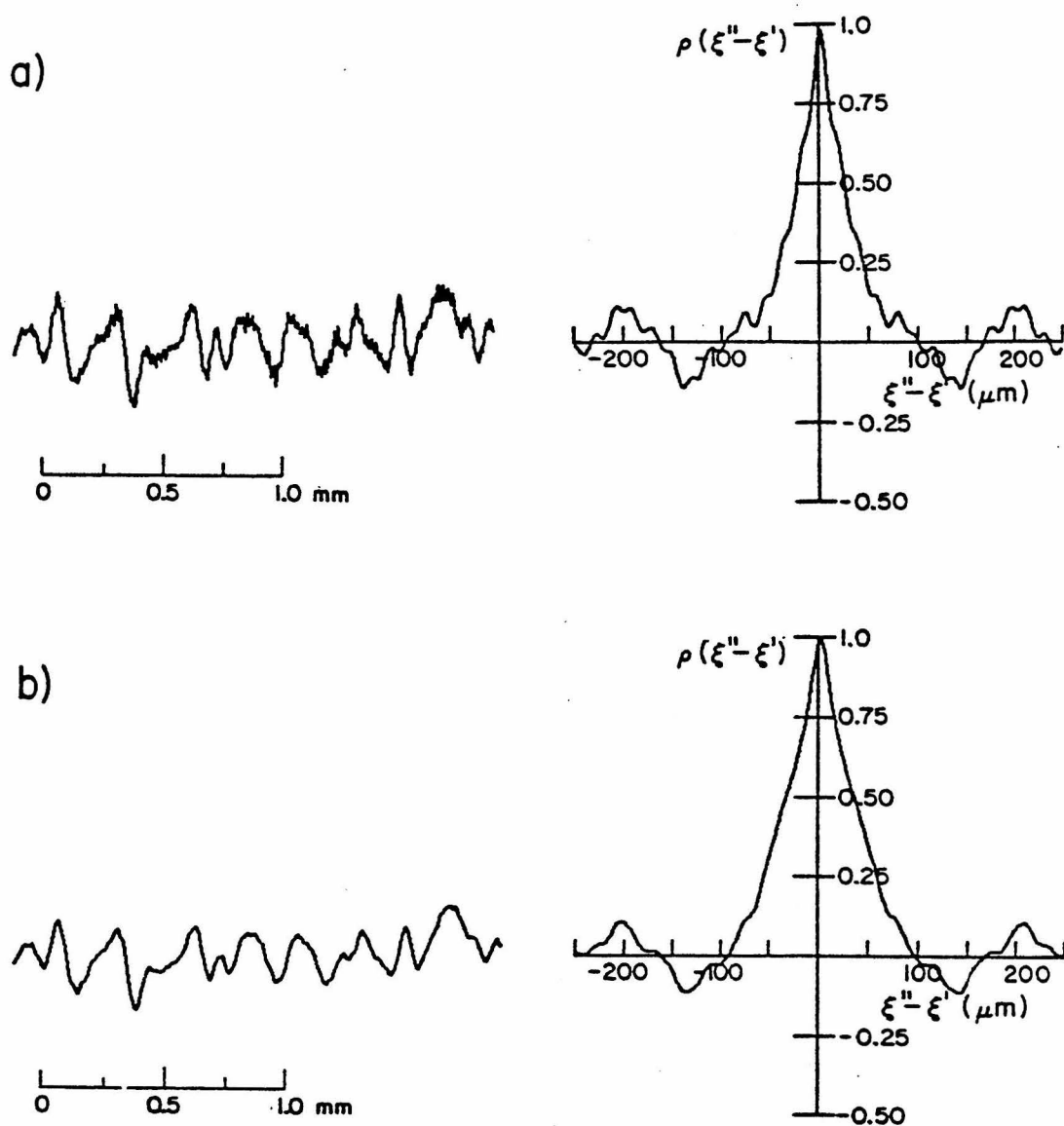
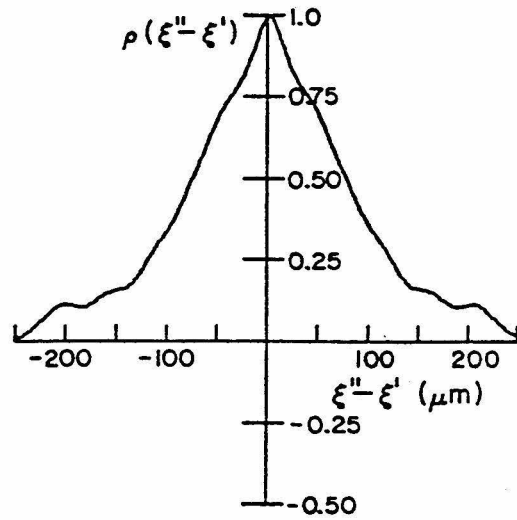
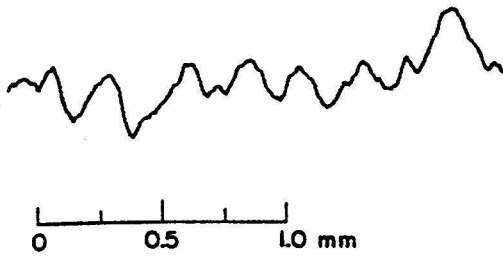


Fig. 4.5 Curves and their associated correlation coefficient, $\rho(x_2 - x_1)$, obtained using the linear system model in Eq. (4.7). The input to the system was generated using a random number generator. This was then convolved with a window function $W(x)$: (a) $W(x) = \text{rect}(x/100)$; (b) $W(x) = \Lambda(x/50)$; (continued on next page)

c)



d)

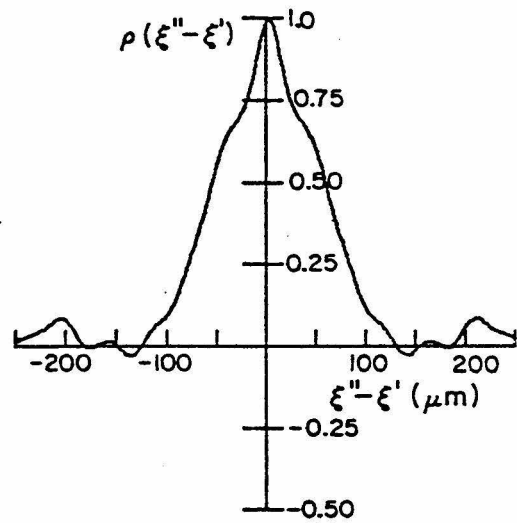
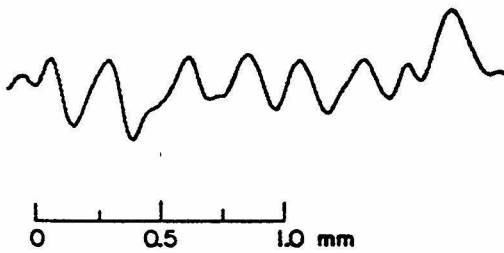


Fig. 4.5 Continued. (c) $W(x) = \exp(-|x|/50)$; (d) $W(x) = \exp[-(x/50)^2]$. An analytical expression for the correlation coefficient is given in Eq. (4.24).

in Fig. 4.5(b),

$$\begin{aligned} W(x) &= \Lambda(x/50) \\ &= \begin{cases} 1 - |x/50|; & |x/50| \leq 1 \\ 0 & ; \text{ otherwise;} \end{cases} \end{aligned}$$

in Fig. 4.5(c), $W(x)$ is equal to $\exp(-|x|/50.)$; and in Fig. 4.5(d), $W(x) = \exp[-(x/50.)^2]$. The units of x are in micrometers.

As an example of the calculations, consider the window of Fig. 4.5(a):

$$W(x) = \text{rect}(x/100.). \quad (4.25)$$

The correlation of the resulting number set $\phi(x)$ in Eq. (4.7) is given by Eq. (4.23) as

$$\begin{aligned} \rho(x_2-x_1) &= W(x) * W(x) / \int_{-\infty}^{\infty} dx W^2(x) \\ &= \Lambda[(x_2-x_1)/100.] . \end{aligned} \quad (4.26)$$

Note, the agreement with Fig. 4.5(a) is good.

The curves and autocorrelation functions were obtained using fast Fourier transform (FFT) techniques. The program used to generate Fig. 4.5(a) is contained in Appendix D. In each case the constant C , see Eq. (4.24), was chosen such that $\sigma = \sigma_{\psi} = 50 \mu\text{m}$. The actual computed standard deviation for the curves in Fig. 4.5(a)-(d) were $48.6 \mu\text{m}$, $43.8 \mu\text{m}$, $56.9 \mu\text{m}$, $55.6 \mu\text{m}$, respectively. Finally, it is interesting to note the decrease in high frequency detail in going from (a) to (d).

A flow chart indicating the steps involved to create a metallic edge from a number set as described above is shown in Fig. 4.6. For these edges, the number set from the random number generator was convolved with a rectangular window function of various widths. The convolved sets were then plotted using a Calcomp 1675 microfiche plotter. The film used was Kodak Recordak Dacomatic E film with a resolution of 320 lines/mm. The area under the curve is exposed by directing the beam to the point on the curve, then drawing a line down to the bottom of the frame.

The film plots were then sent to Photo Sciences, Inc., a custom photo-etching business located in Gardena, California. There a one-to-one copy of the film plot was made on a bimetallic substrate using photo-etching techniques.

Although we have good control of the process digitally, we do not have the same degree of control of the filter function for the microfiche plotter or the photo-etching process. The general trend was a blurring or smoothing of the number set by both the plotter and the etching.

As a result of this smoothing, we could not rely on our original autocorrelation functions to accurately describe the metallic serration. Therefore, in order to compare the estimates of the correlation coefficient obtained using transform plane intensity measurements, an independent determination of $\rho(x_2-x_1)$ was performed; the steps of

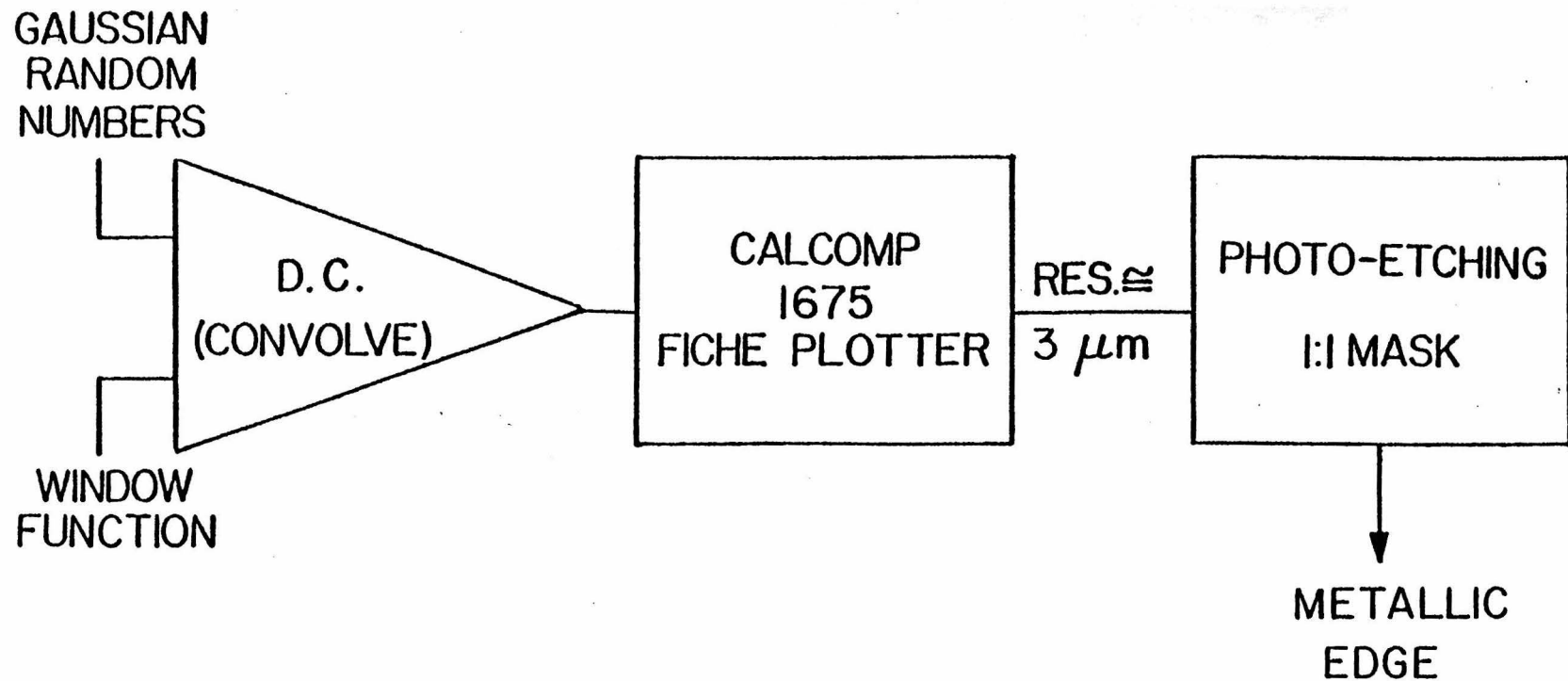


Fig. 4.6 Flow chart showing the processing steps used to obtain a metallic edge with controlled statistics. A number set from a computer random number generator is convolved with a window function. The resulting number set is plotted on microfiche. The 1:1 metallic edge is obtained from the microfilm by photo-etching techniques.

which are shown on the left-hand side of Fig. 4.7. The serration was placed under a Leitz Ortholux microscope, using a 2.5X or 10X objective depending on the structure size, together with a 5X eyepiece. The magnified image was photographed and an 8" x 10" print was made of the edge. The curve was digitized using a Hewlett-Packard 9830 calculator/curve digitizer with a resolution, in edge units, of 13.5 μm and 3.5 μm for the 2.5X and 10X objectives, respectively. The digitized data were read into a computer, where they were autocorrelated, normalized and plotted.

Photographs of the edges used in the experiments together with the absolute value of their respective correlation coefficients obtained by this "photo-digitization" route are shown in Figs. 4.8 - 4.10.

In the alternate processing route, on the right-hand side of Fig. 4.7, an estimate of the absolute value of the correlation coefficient was obtained from transform plane intensity measurements using the remote sensing techniques described in Chapter 3. The need to take the absolute value of the correlation coefficient arises because the intensity data do not satisfy the stringent symmetry requirements of the FFT needed to obtain a real transform. As a result of this lack in symmetry, there is a significant imaginary component in the transform data, therefore we take the absolute value.

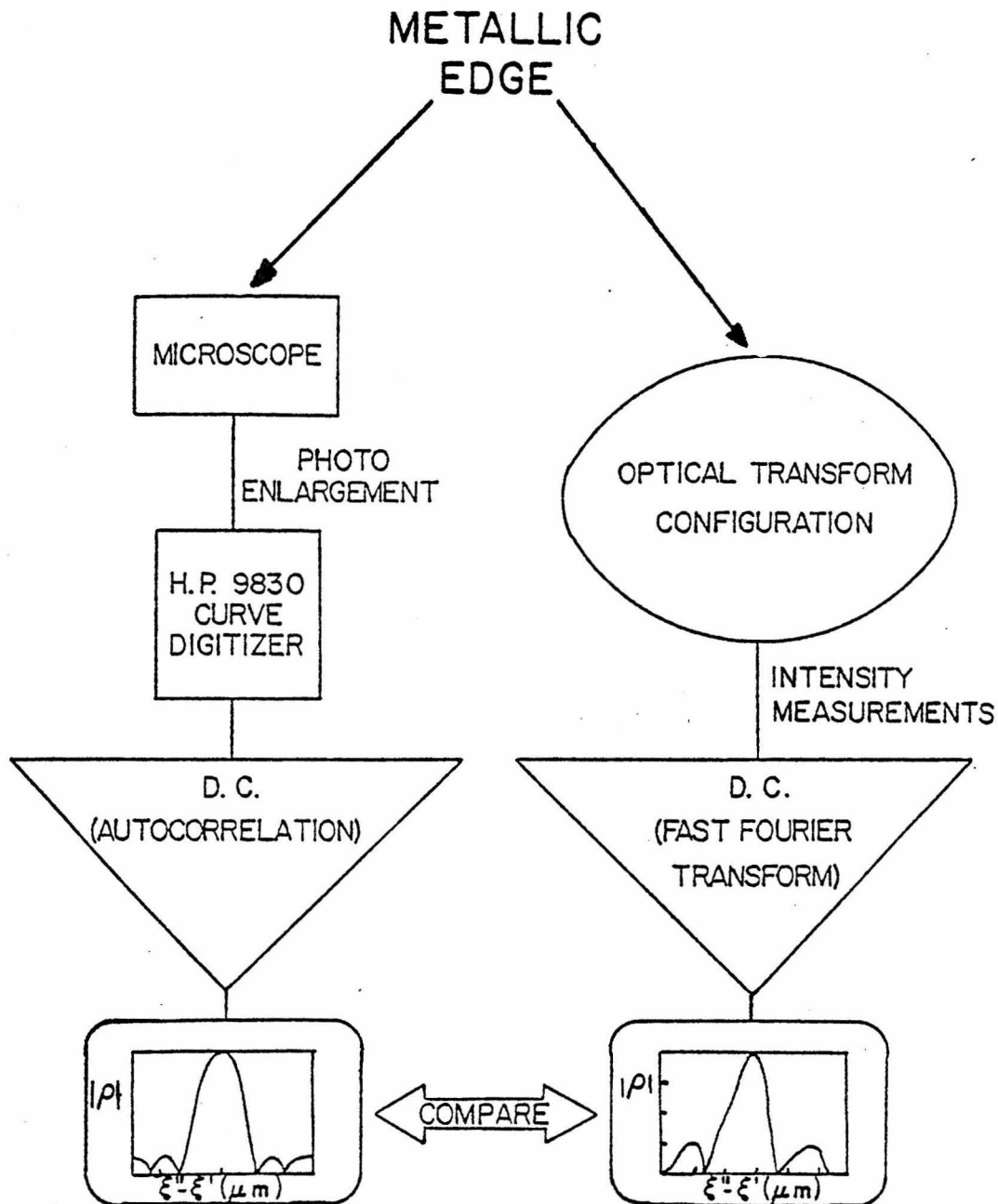


Fig. 4.7 Independent processing routes used to measure the absolute value of the correlation coefficient, $|\rho(\xi'' - \xi')|$, of the rough edge.

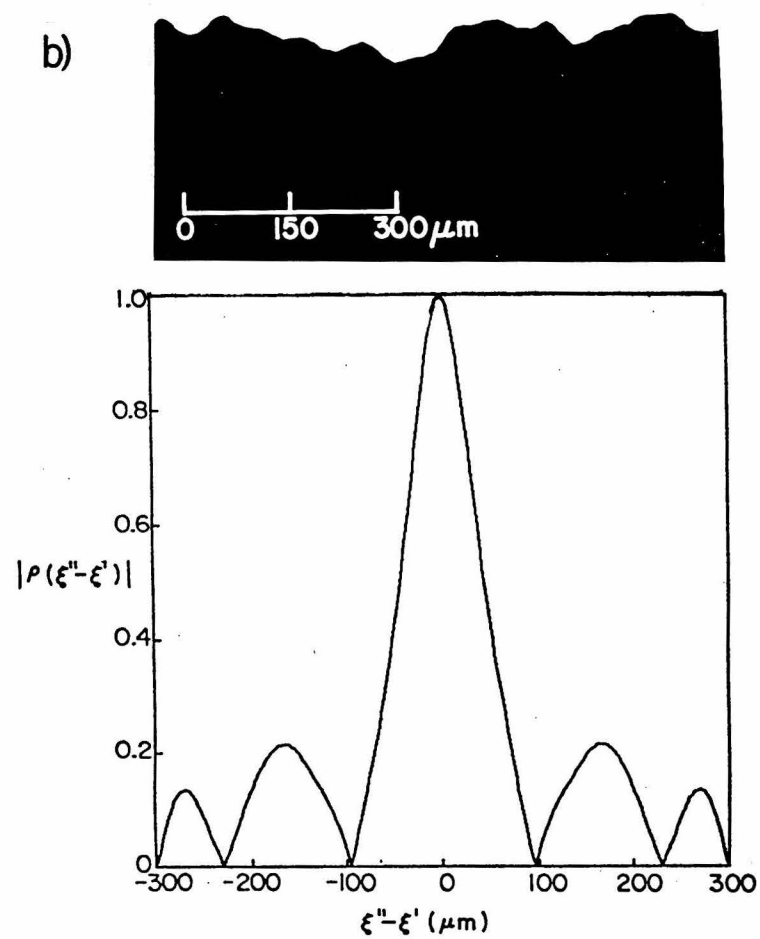
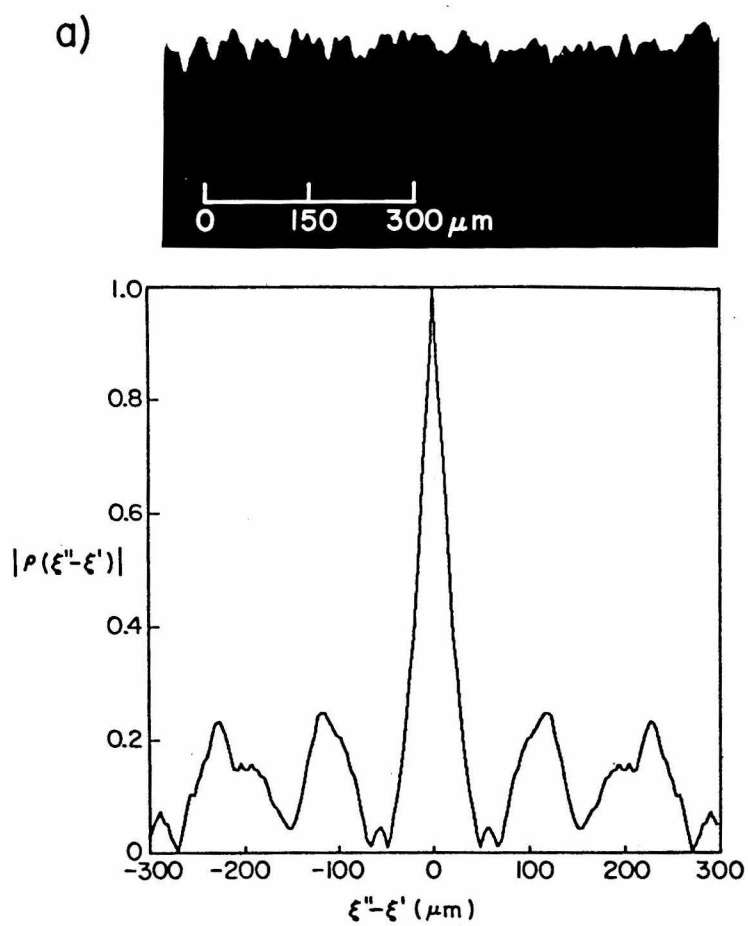


Fig. 4.8 Microscope photographs of computer-generated edges (a) and (b) with their associated correlation coefficients below. The standard deviation is (a) $\sigma = 29 \mu\text{m}$ and (b) $\sigma = 57 \mu\text{m}$.

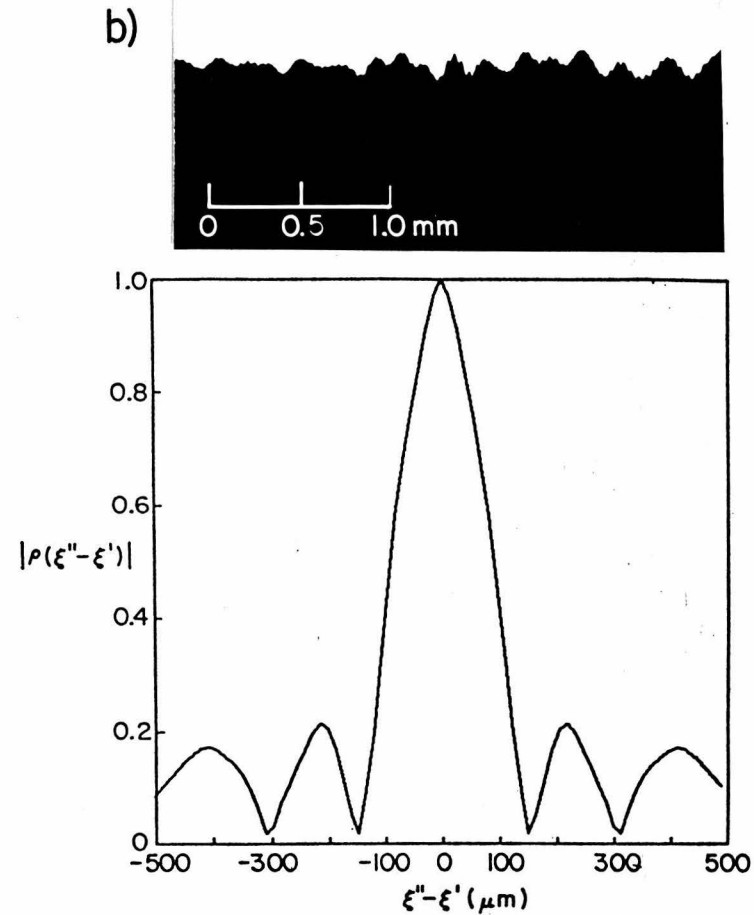
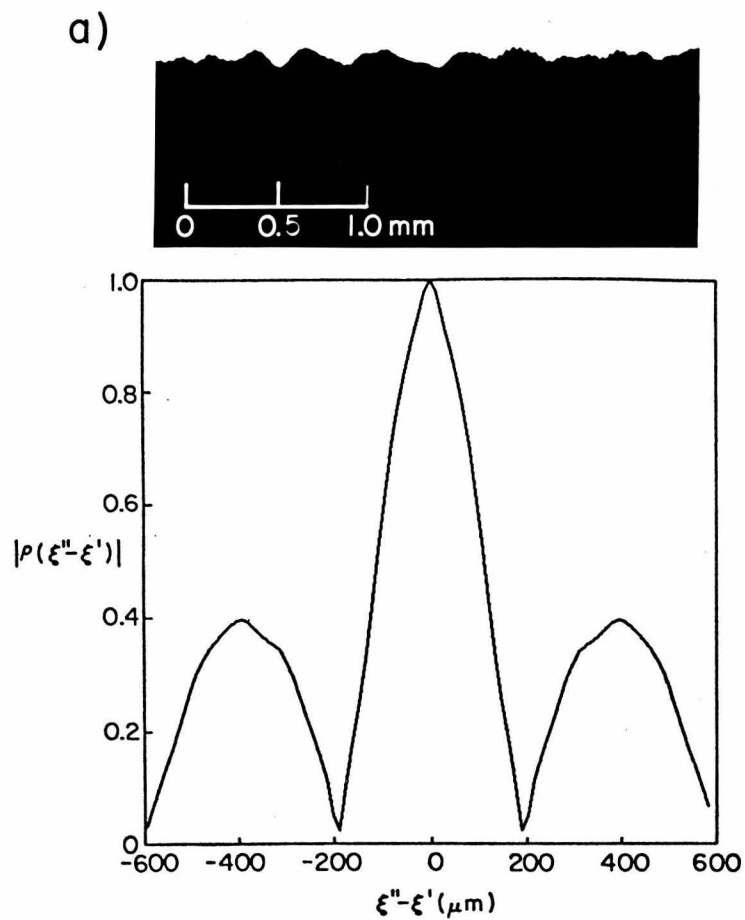


Fig. 4.9 Microscope photographs of computer-generated edges (a) and (b) with their associated correlation coefficients below. The standard deviation is (a) $\sigma = 57 \mu\text{m}$ and (b) $\sigma = 81 \mu\text{m}$.

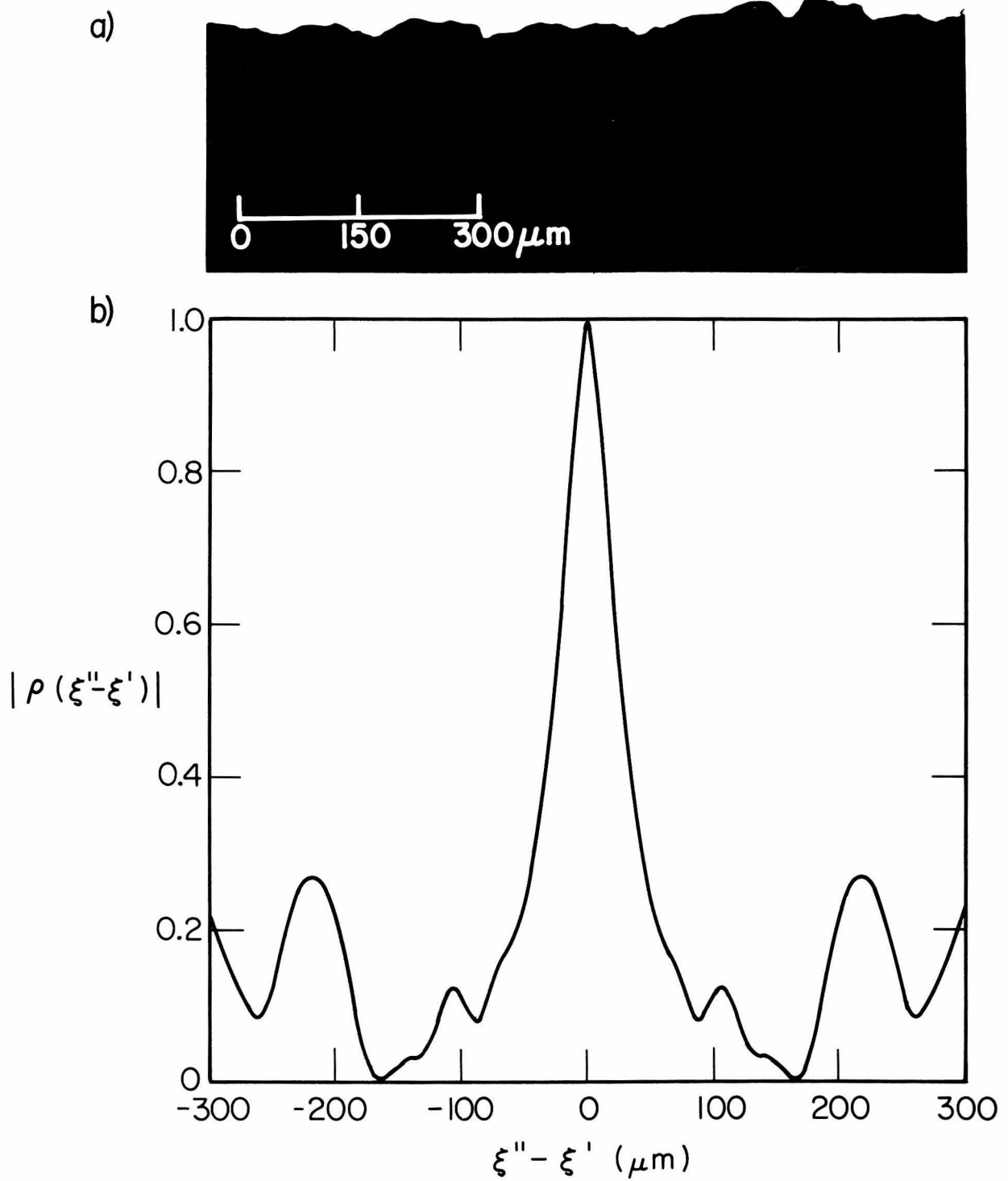


Fig. 4.10 (a) Microscope photograph of serrated edge generated by No. 60 grit sandpaper and (b) its correlation coefficient.

Correlation coefficients associated with processing the transform plane intensity data will be labeled "fiber optic measurement". This processing will be described in detail in Section 4.5.

4.4 Fiber Optic Measurement of Intensity

Two series of measurements are described in this section. In these, intensity data along the spike, $u(0,y)$, and perpendicular to the spike, $u(x,y = \text{constant})$, are recorded by a fiber optic probe and photomultiplier, so that accurate square law detection is assured, see Fig. 4.1.

The serrated edges used in this section were not the computer-generated edges described previously, instead these were generated by scraping razor blades over different grades of aluminum oxide sandpaper. One was generated using No. 60 (coarse) grit sandpaper; the other, using No. 150 (fine) grit. A photograph of the coarse edge and its associated correlation coefficient are shown in Figs. 4.10(a) and (b). The roughness (or standard deviation) is approximately $15 \mu\text{m}$ for this edge.

For the measurements along the central spike, Fig. 4.11, edges of roughness $\sigma = 6 \mu\text{m}$ and $15 \mu\text{m}$ are seen to exhibit the severe damping of the central spike as predicted by Eq. (2.86). A fine-scale interference pattern is observed visually along the y-axis. This is not resolved in the optical fiber data since the probe diameter was chosen to average a few cycles of the pattern.

For the second series, we study the spread of energy along x, perpendicular to the central spike. The edges are positioned as in Fig. 4.2. Figures 4.12(a) and (b) show fiber optic probe data for the small and large σ ,

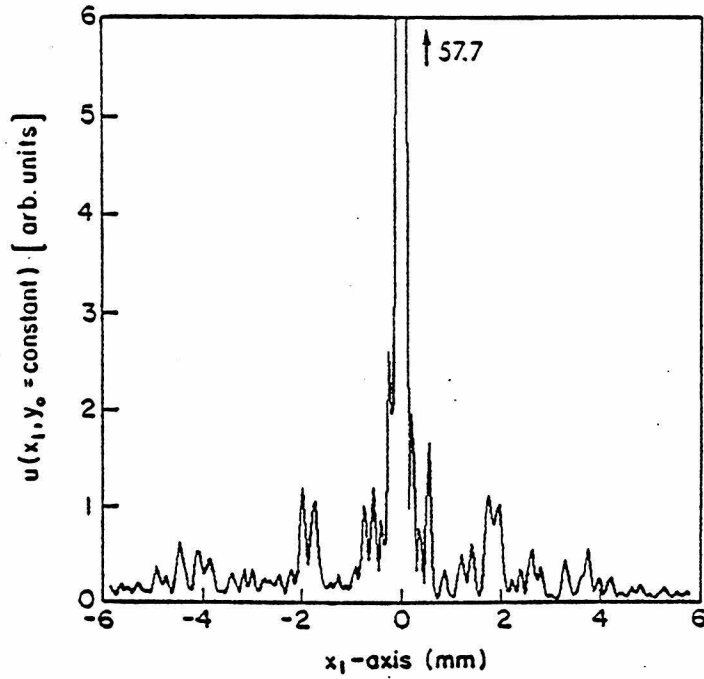
respectively. Good qualitative agreement with Eq. (2.86) is seen. For $x \approx 0$, there is a large central spike with the ratio 57.7 to 6 corresponding to the ratio $6 \mu\text{m}$ to $15 \mu\text{m}$ in roughness. The damping of the second member is due to the coefficient $\exp(-\sigma^2 \eta_1^2)$.

To illustrate the speckle in the optical transform, the Airy disc size relative to the fiber optic probe was increased for the data in Fig. 4.12. The ensemble average $\langle E_1 E_1^* \rangle$ would not be expected to contain the fluctuation inherent in data taken for one member of the ensemble. Notice however that the average value of the data for correlation lengths ε_0 of $300 \mu\text{m}$ and $160 \mu\text{m}$ is in good agreement with the curves of G shown in Fig. 4.13. It is the envelope of data that one should compare to $\langle E_1 E_1^* \rangle$. The prediction of the fine scale structure in intensity is obtained from the spatial dependence of $\langle u_1 u_2 \rangle$ and is given by Eqs. (2.75)-(2.78). From Eq. (2.77), the speckle size is given by $\Delta x \approx \lambda z_0 / (2L_x)$, and this is also in good agreement with the data in Fig. 4.12.

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a)



b)

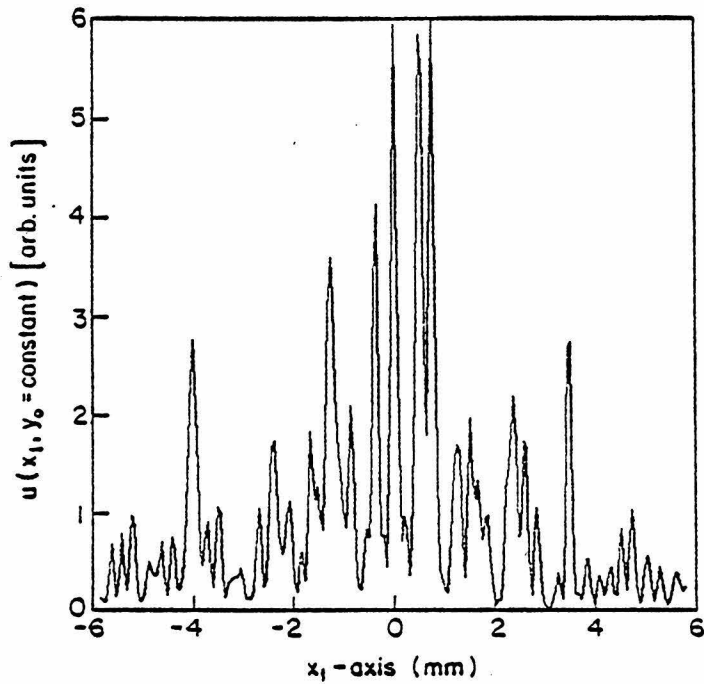
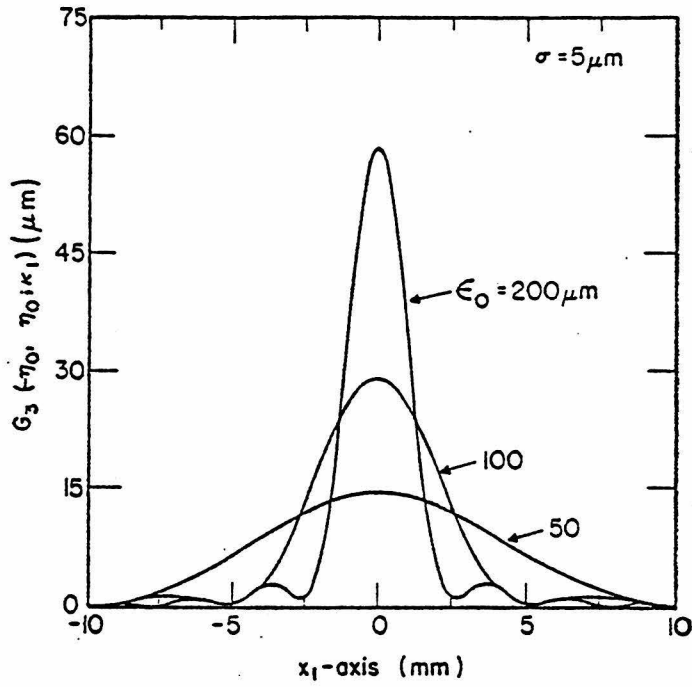


Fig. 4.12 Intensity measurements of the spread of energy perpendicular to the spike for edges with (a) $\sigma \sim 6 \mu\text{m}$ and $\varepsilon_0 \sim 300 \mu\text{m}$ (No. 150 grit sandpaper), (b) $\sigma = 15 \mu\text{m}$ and $\varepsilon_0 = 160 \mu\text{m}$ (No. 60 grit sandpaper). Other parameters were: $y = 15 \text{ mm}$, $F = 190 \text{ cm}$, $z_0 = 120 \text{ cm}$, $\lambda = 5145\text{\AA}$.

a)



b)

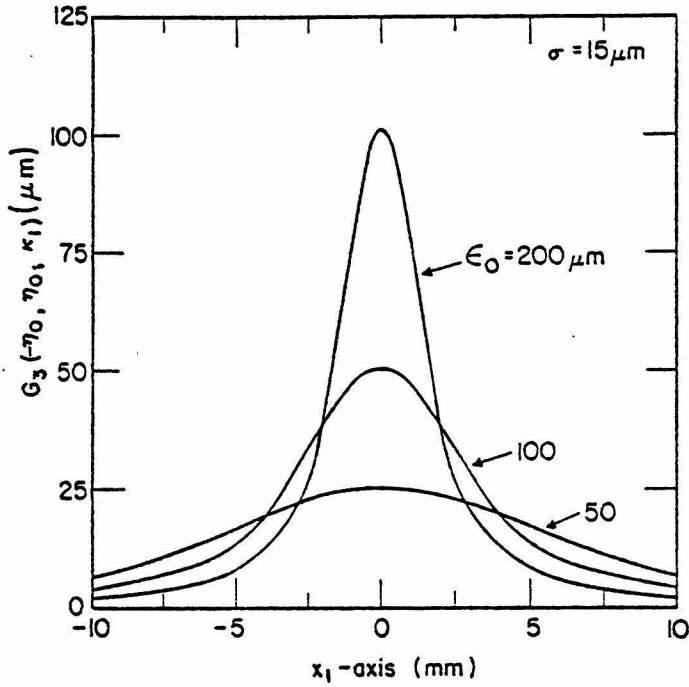


Fig. 4.13 The function $G(-\eta_0, \eta_0; \kappa_1)$ vs. x_1 plotted from Eq. (2.33) for (a) $\sigma = 5 \mu\text{m}$ and (b) $\sigma = 15 \mu\text{m}$ and for correlation lengths $\epsilon_0 = 50, 100$ and $200 \mu\text{m}$. Wavelength $\lambda = 5, 145 \text{\AA}$, $y_1 = 10 \text{ mm}$, $z_0 = 1 \text{ m}$.

4.5 Remote Sensing of Correlation Coefficient-- Computer-Generated Edges

4.5.1 Fiber Optic Measurement of Intensity

In this section we compare intensity measurements in the optical transform plane with the power series representation, derived in Chapter 3, for the expected value of intensity $\langle E_1 E_1^* \rangle$, in Eq. (3.19). For convenience, this expression is rewritten below in Eq. (4.27):

$$\begin{aligned} \langle E_1 E_1^* \rangle = & A_1 A_1^* \left| 2\pi^2 \delta(\kappa_1, \eta_1) + (i/\eta_1) 2L_x \frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \exp(-\sigma^2 \eta_1^2 / 2) \right|^2 \\ & + \frac{A_1 A_1^*}{\eta_1^2} 2L_x \exp(-\sigma^2 \eta_1^2) \sum_{n=1}^{\infty} (\sigma \eta_1)^{2n} F_{\kappa_1}(\rho^n(\xi_1)) ; \end{aligned} \quad (4.27)$$

where the notation is

$$F_{\kappa_1}(\rho^n(\xi)) = \int_{-\infty}^{\infty} d\xi \exp(i\kappa_1 \xi) \rho^n(\xi)$$

$$\kappa_1 = kx/z_0 \quad ; \quad k = 2\pi/\lambda \quad (4.28)$$

$$\eta_1 = ky/z_0$$

$$A_1 A_1^* = 1/(\lambda z_0)^2 .$$

In Eq. (4.27), the first term inside the absolute value symbols is essentially the Airy disc while the second term is due to a straight edge but damped exponentially by the roughness, σ . The third term is entirely due to the serration and contains all the information about the correlation coefficient, $\rho(\xi)$.

The expansion parameter used to obtain Eq. (4.27) is $\sigma\eta_1$. Note, for the limiting case $\sigma\eta_1 \ll 1$, the expression is greatly simplified. It is in this limit of small $\sigma\eta_1$ that we make comparisons between theory and experiment. In addition to the requirement $\sigma\eta_1 \ll 1$, we will also take $|\eta_1|$ to be several Airy disc units away from the pattern's center. The latter condition allows us to neglect the $\delta(\kappa_1, \eta_1)$ term in Eq. (4.27). With these restrictions Eq. (4.27) takes the form

$$\begin{aligned} \langle E_1 E_1^* \rangle = & \frac{4L_x^2}{(\lambda z_0)^2 \eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right]^2 \exp(-\sigma^2 \eta_1^2) \\ & + \frac{2L_x \sigma^2}{(\lambda z_0)^2} \exp(-\sigma^2 \eta_1^2) F_{\kappa_1}(\rho(\xi)) . \end{aligned} \quad (4.29)$$

It is seen that in the limit of small, but nonzero, $\sigma\eta_1$, the expected value of intensity has two terms. The first is the central spike--damped exponentially with σ . The spread of energy in the x-direction due to this term, however, is limited to only a few Airy disc units. The main contribution to the spread of energy perpendicular to the spike is represented by the second term, which says the spread is proportional to the Fourier transform of the correlation coefficient. In Figs. 4.14, 4.15 and 4.16, we show intensity measurements vs. the x_1 -coordinate for the edges in Figs. 4.8(a), 4.8(b) and 4.9(a), respectively.

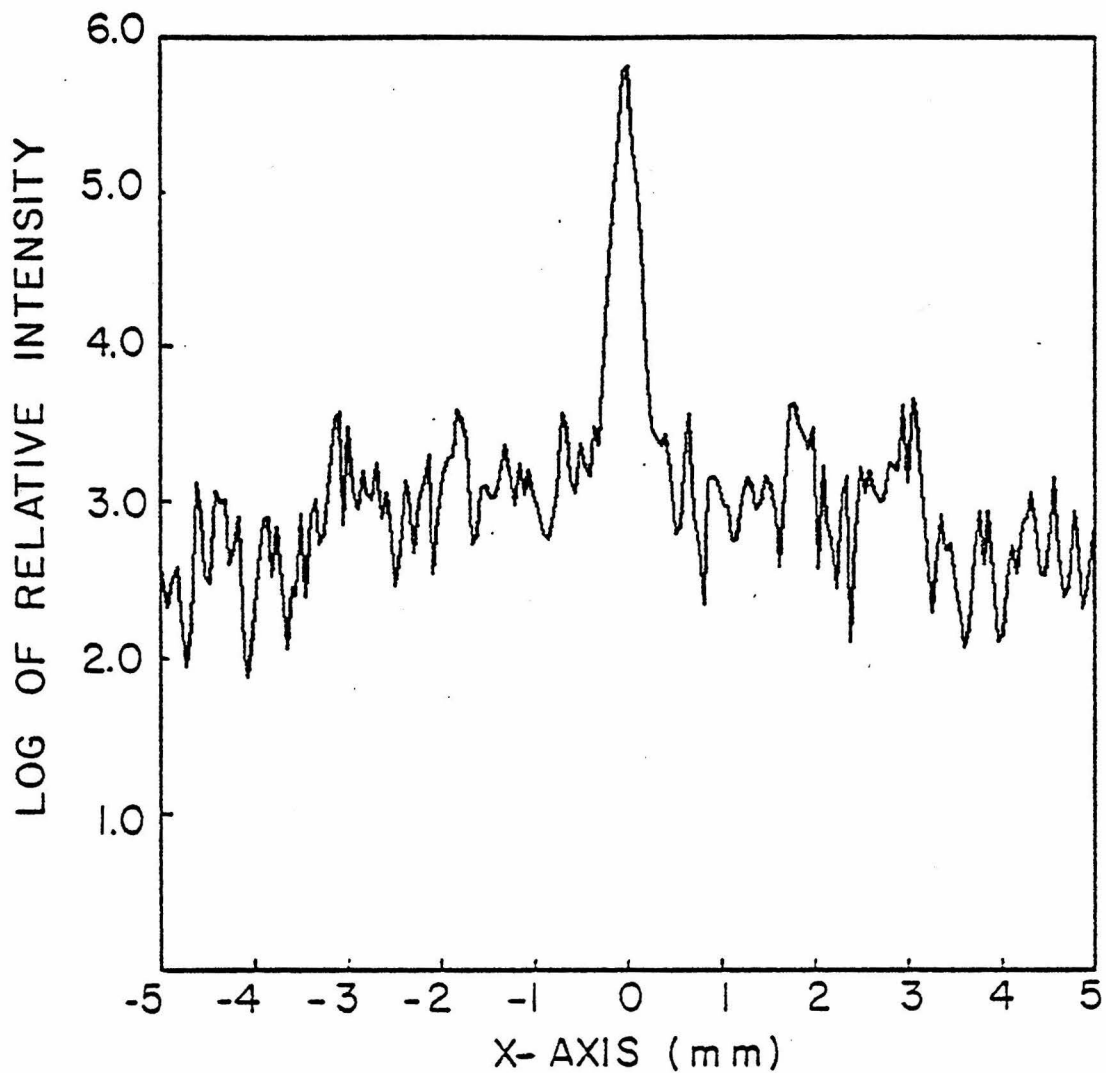


Fig. 4.14 Intensity measurements of the spread of energy perpendicular to the central spike for the edge in Fig. 4.8 (a). The edge statistics are: $\sigma = .29 \mu\text{m}$ and $\epsilon_0 = 50 \mu\text{m}$.

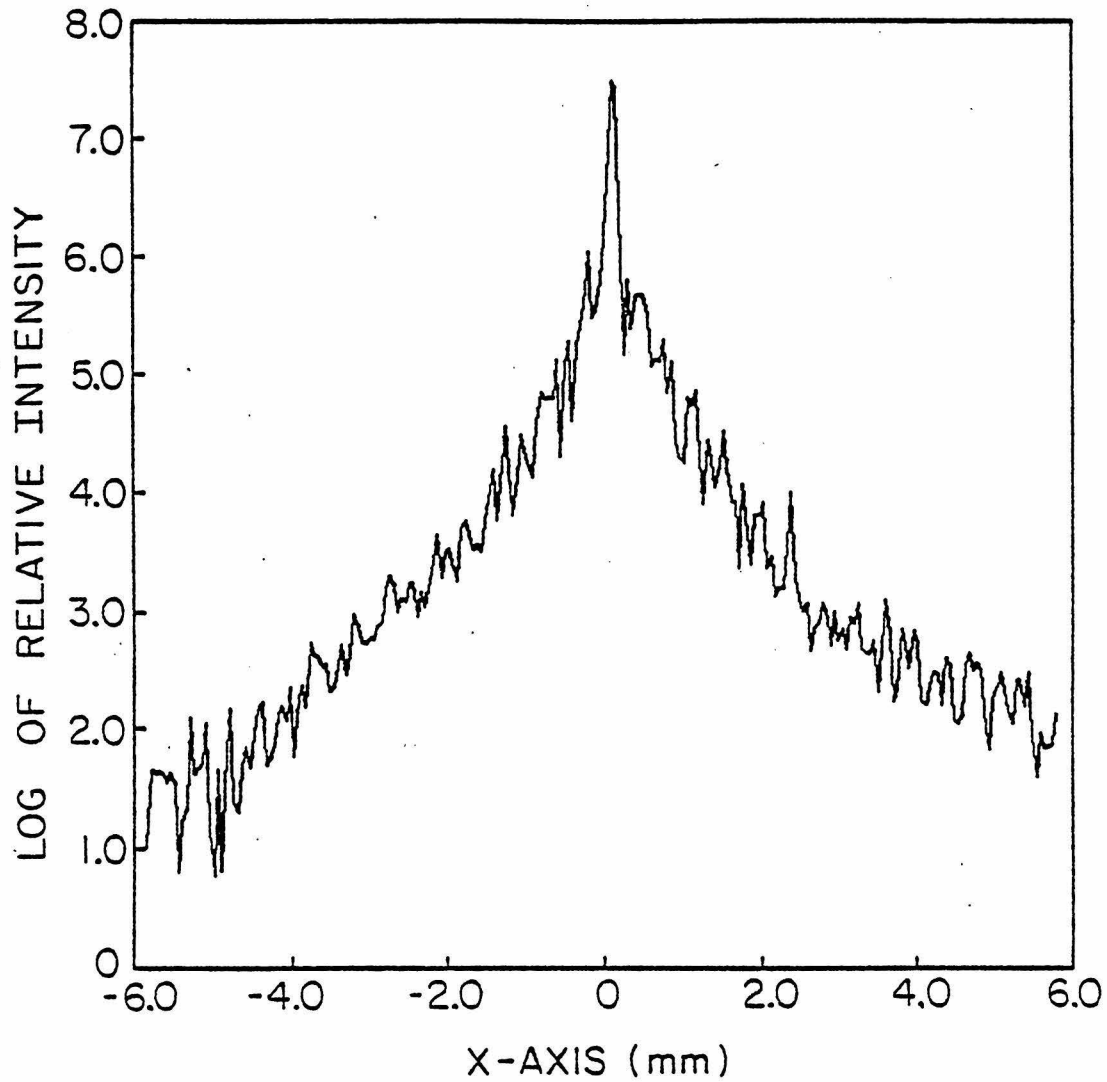


Fig. 4.15 Intensity measurements of the spread of energy perpendicular to the central spike for the edge in Fig. 4.8 (b). The edge statistics are: $\sigma = 57 \mu\text{m}$ and $\epsilon_0 = 100 \mu\text{m}$.

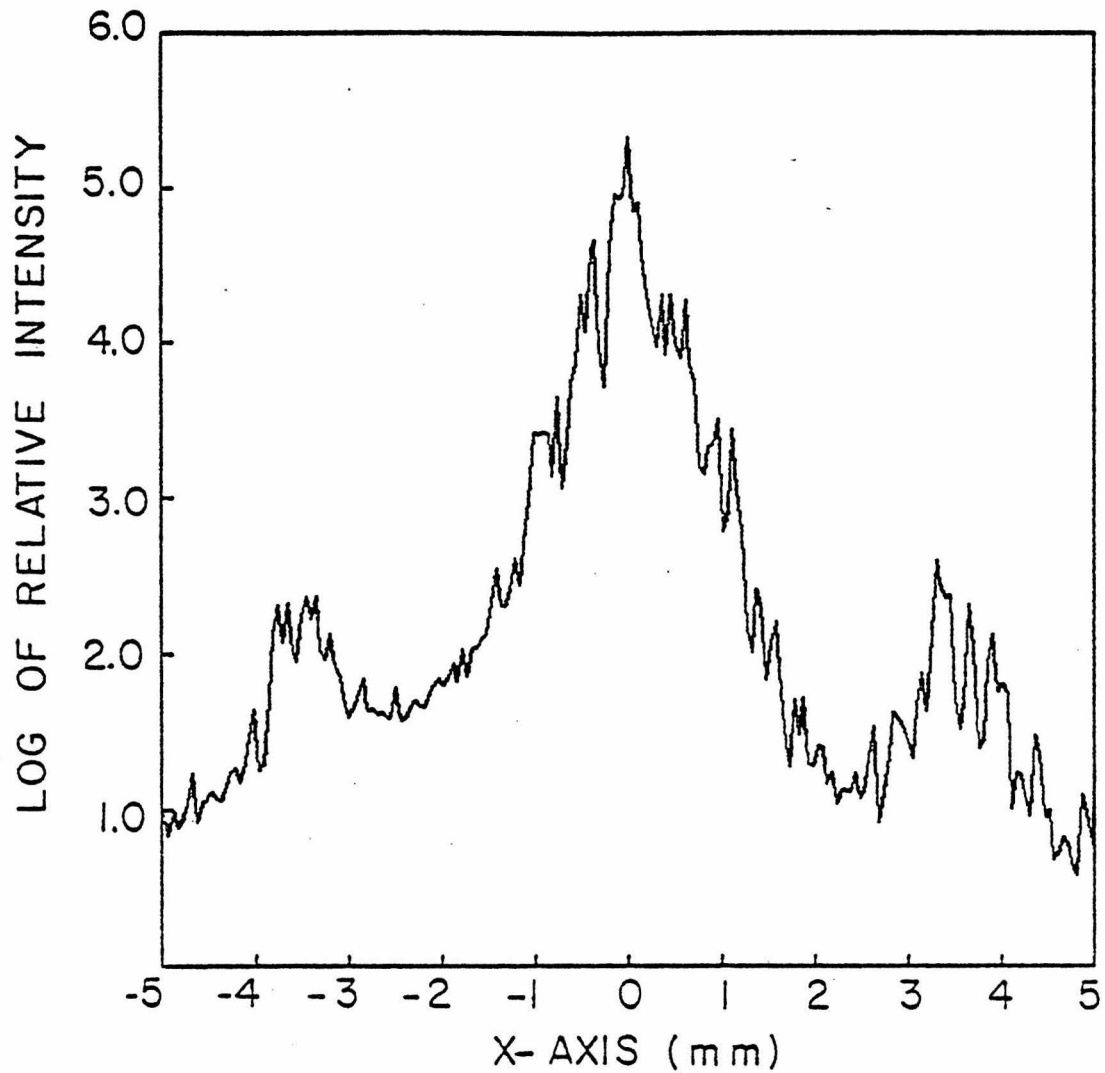


Fig. 4.16 Intensity measurements of the spread of energy perpendicular to the central spike for the edge in Fig. 4.9 (a). The edge statistics are: $\sigma = 57 \mu\text{m}$ and $\varepsilon_0 = 200 \mu\text{m}$.

Of course, the finer the structure in the edge plane, the greater the spread of energy in the transform plane. Note the Airy disc size has been reduced from that in Fig. 4.12 in order to get a better estimate for the envelope function for $\langle E_1 E_1^* \rangle$. In this series of measurements the parameters were: $y_1 = .38$ mm, $z_0 = 250$ mm, $F = 1755$ mm, $\lambda = .58$ μ m, $L_x \approx 3.5$ mm. Using Eq. (4.28), we find $\eta_1 = 1.65 \times 10^{-2} \mu\text{m}^{-1}$. For comparison, some experimental curves for the case where $\sigma\eta_1 > 1$ are shown in Appendix F.

4.5.2 Processing the Intensity Data

From the form of Eq. (4.29), it is evident that with the appropriate processing we can obtain information about the correlation coefficient, $\rho(\xi)$. Using Eq. (4.29), $\rho(\xi)$ can be expressed as follows:

$$\rho(\xi) = \frac{(\lambda z_0)^2 \exp(\sigma^2 \eta_1^2)}{2L_x \sigma^2} F_{\kappa_1} \left(\langle E_1 E_1^* \rangle - \frac{4L_x^2}{\eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right]^2 \cdot \exp(-\sigma^2 \eta_1^2) \right) \quad (4.30)$$

As seen by Eq. (4.30), the processing steps needed to find $\rho(\xi)$ from transform plane intensity measurements are:

- (A) Subtract the contribution produced by the central spike,
- (B) Fourier transform the resulting data,
- (C) Normalize the transform data to a maximum value of one.

Step A is accomplished by deleting the central spike present in Figs. 4.14-4.16. The main difficulties associated with this step are: One, determining how much of the spike to cut off; and secondly, the manner in which the spike is truncated. In Step B the intensity data minus the central spike are Fourier transformed. However, since we are considering only one member of the ensemble, the fine scale structure is present in the data. As a result of these fluctuations, the rigid symmetry requirements, imposed by the FFT routine, which are needed to produce real transform data are not satisfied. Therefore we take the absolute value of the transformed data. Step C is easily accomplished by normalizing the transformed data by the maximum value of the array. The computer program used to process the data is listed in Appendix E.

In Fig. 4.17(a)-(e) we illustrate the effect of chopping the spike at various heights for the data in Fig. 4.14. In each case we show where the data were chopped and what the estimate would be for $|\rho(\xi''-\xi')|$. In Fig. 4.18 we compare the curve for $|\rho(\xi''-\xi')|$ obtained by "fiber optic measurement" (Fig. 4.17(e)) with the curve obtained by "photo-digitization" of the edge. The latter method is illustrated in flow chart form on the left-hand side of Fig. 4.7.

In Figs. 4.19(a) and 4.20(a) we show the truncated spike data for the edges in Figs. 4.15 and 4.16, respectively.

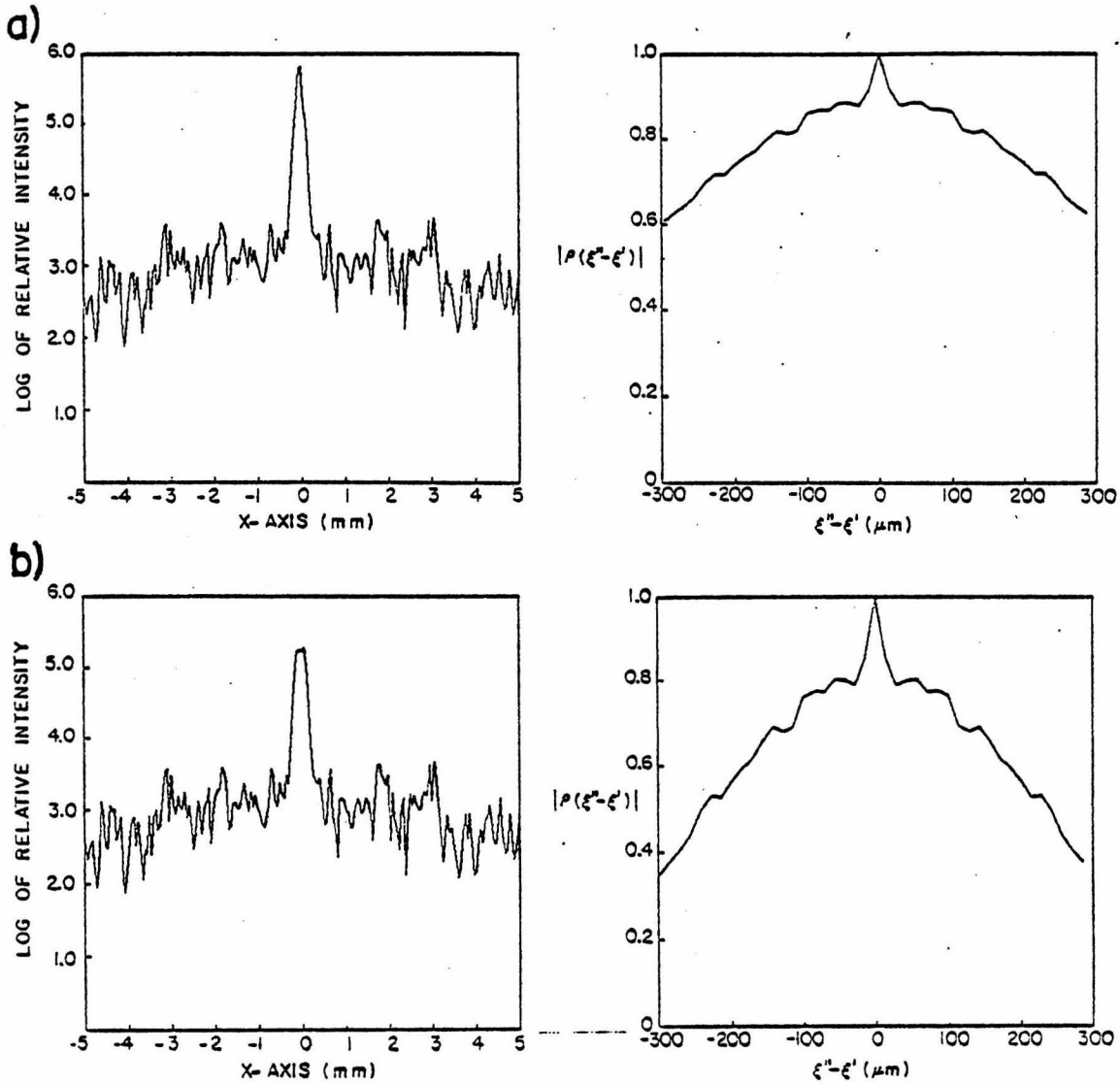


Fig. 4.17 Sequence showing the effect of truncating the central spike on our estimate for $|\rho(\xi'' - \xi')|$. In (a)-(e) the intensity data (left-hand member) is Fourier transformed (FFT) and normalized to obtain the estimate for $|\rho(\xi'' - \xi')|$ (right-hand member).

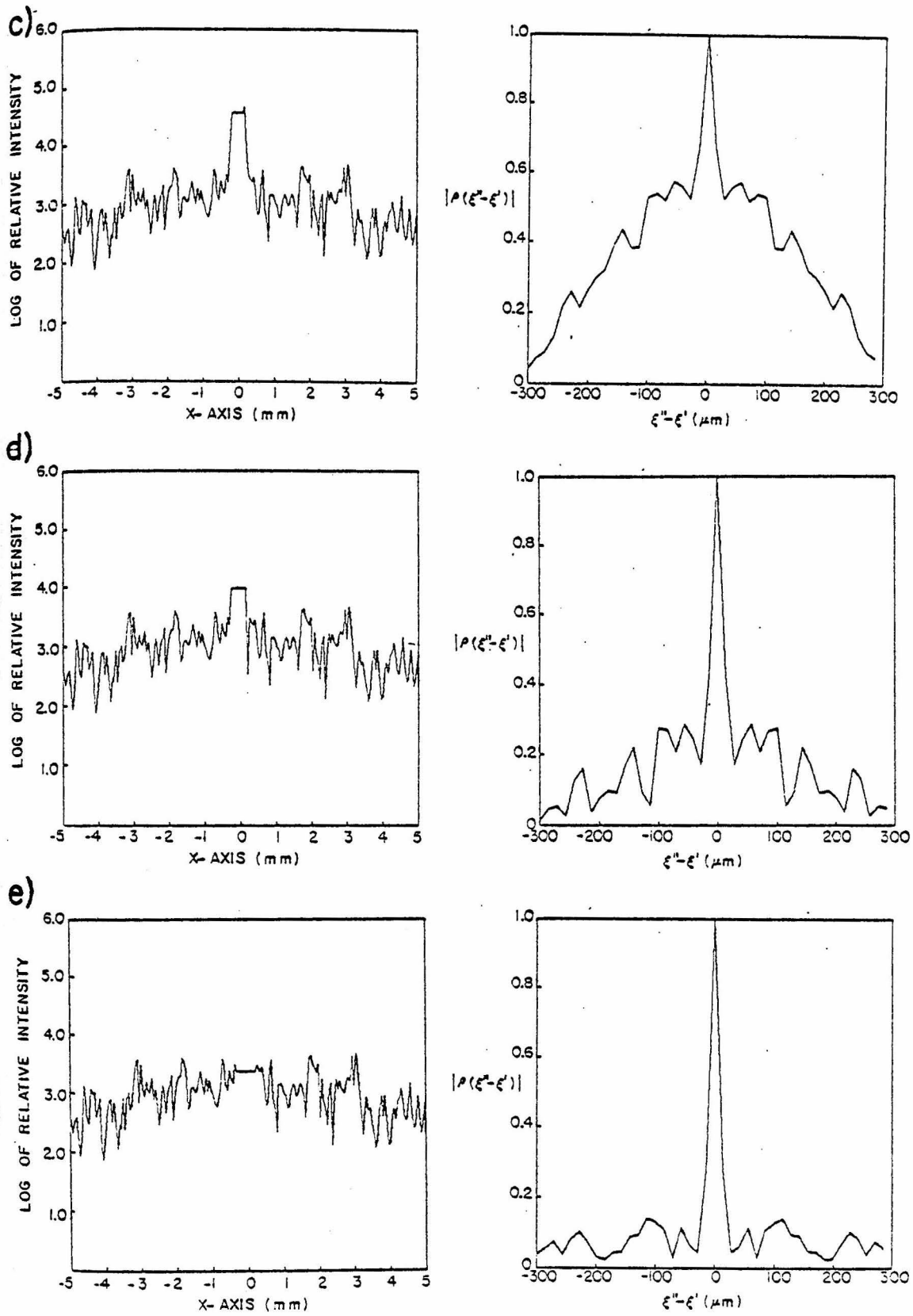


Fig. 4.17 Continued.

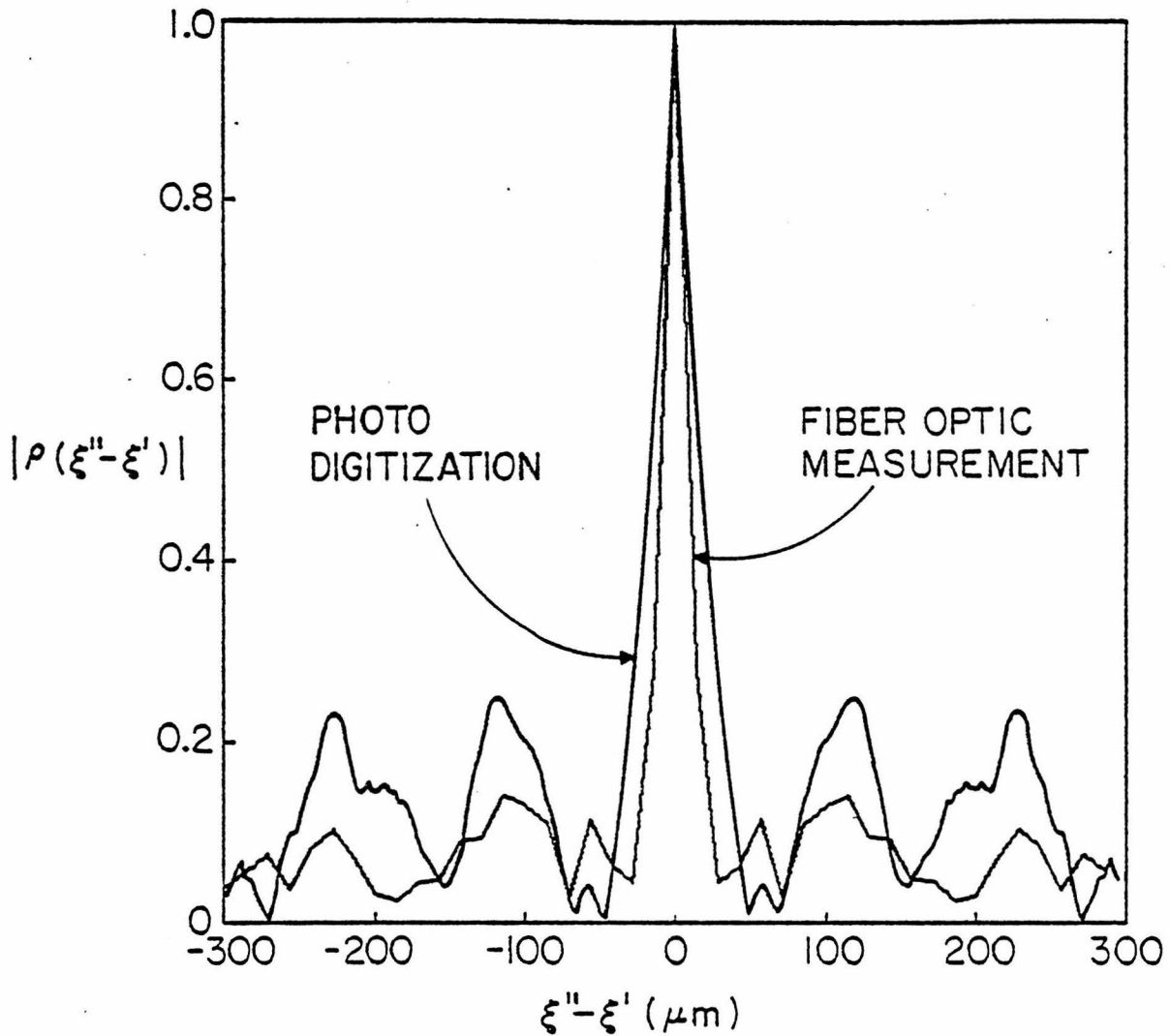


Fig. 4.18 The absolute value of the correlation coefficient, $|\rho(\xi'' - \xi')|$, obtained by the two independent processing methods for the edge in Fig. 4.8 (a). The processing routes are illustrated in flowchart form in Fig. 4.7.

In Figs. 4.19(b) and 4.20(b), a comparison is made between the "photo-digitization" method for determining $|\rho(\xi''-\xi')|$ (Figs. 4.15 and 4.16), and the results obtained by "fiber optic measurement" of intensity. It is noted that the two independent methods for determining $|\rho(\xi''-\xi')|$ are in good agreement. While each method establishes a measure of the fineness of structure for the serration, it is the fiber optic method that is capable of real-time operation.

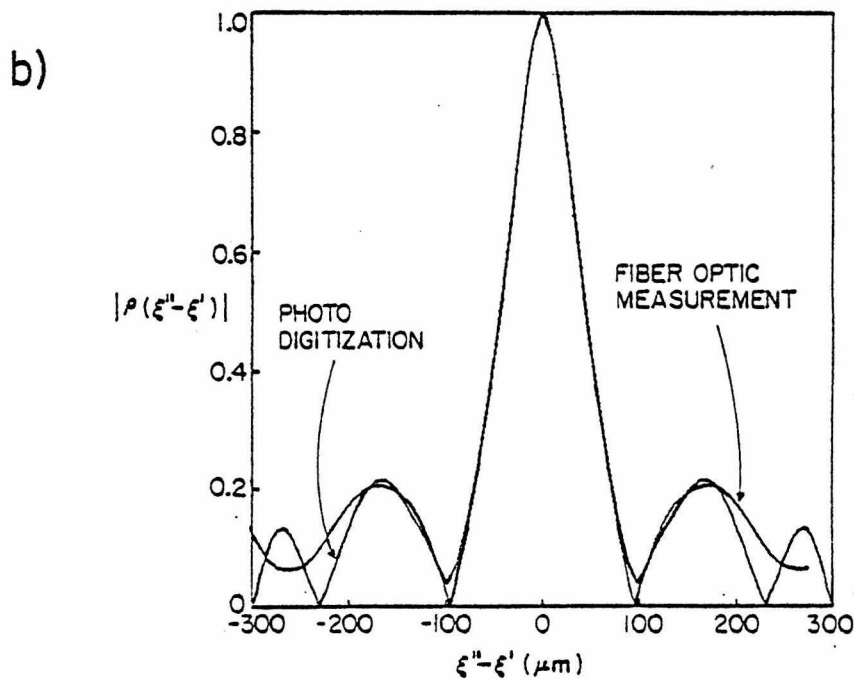
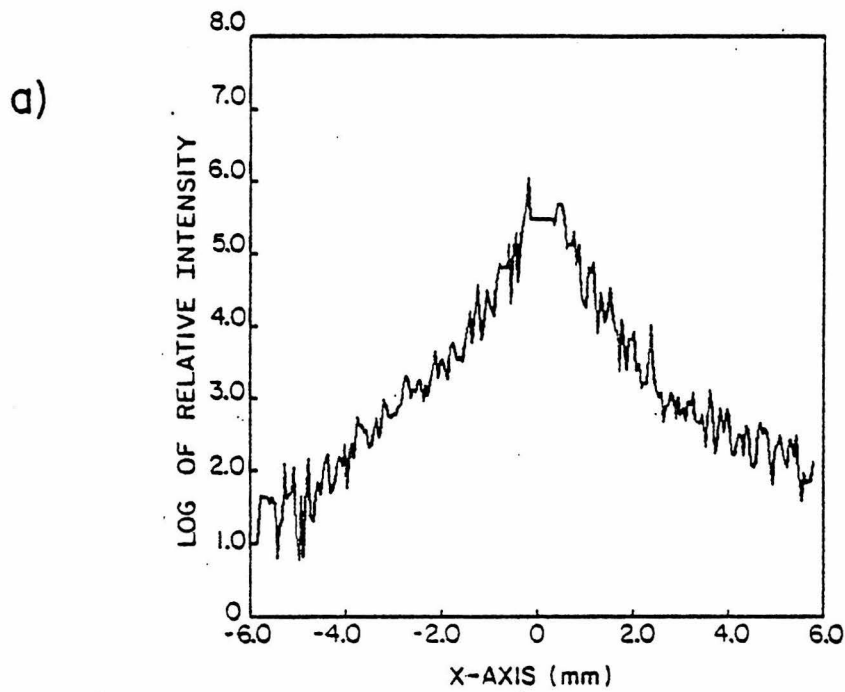


Fig. 4.19 (a) Truncation of the central spike for intensity measurements in Fig. 4.15 and (b) $|\rho(\xi'' - \xi')|$ obtained by the two processing methods.

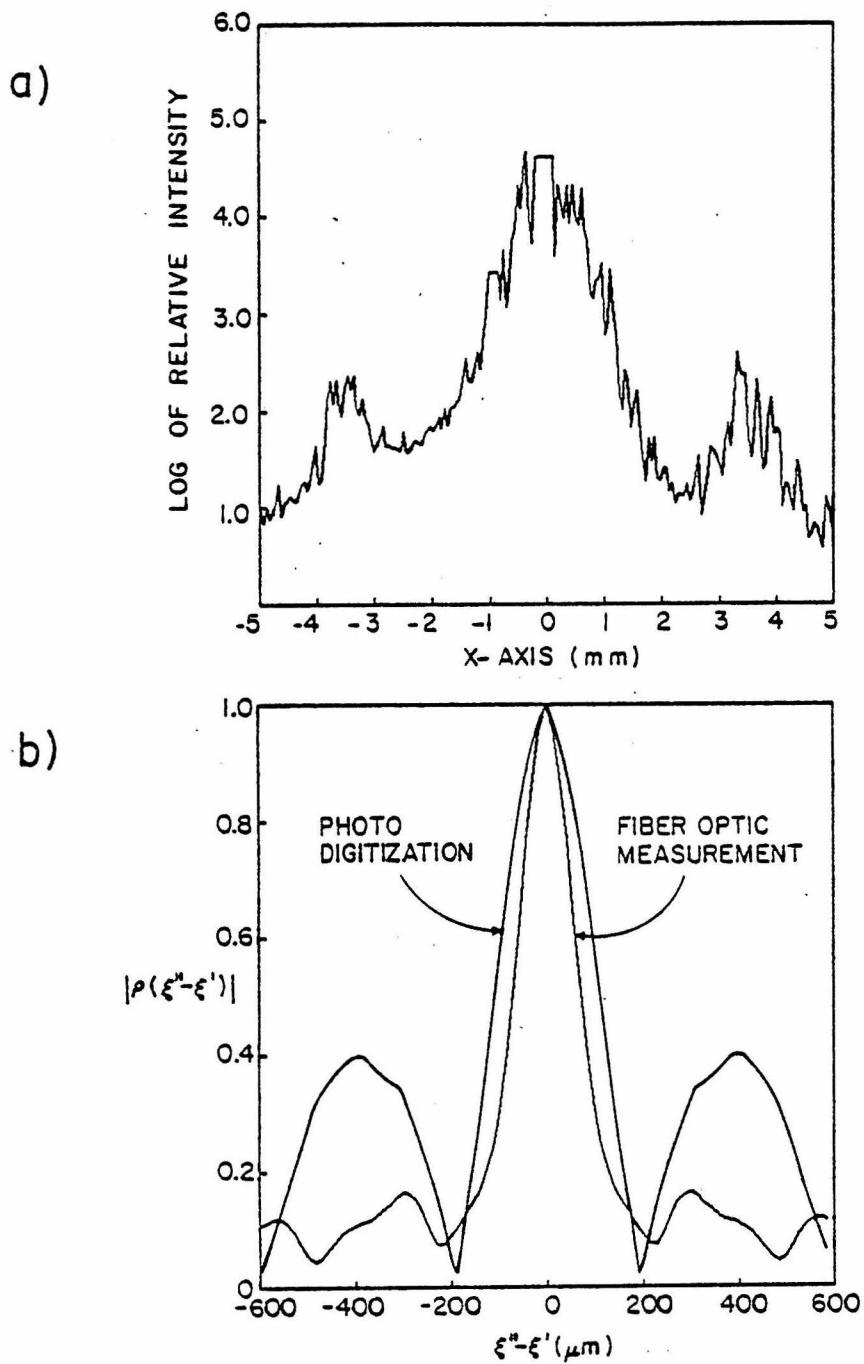


Fig. 4.20 (a) Truncation of the central spike for intensity measurements in Fig. 4.16 and (b) $|\rho(\xi'' - \xi')|$ obtained by the two processing methods.

4.6 Remote Sensing of Correlation Coefficient-- Computer-Generated Gap

The power series representation for the expected value of intensity in the optical transform plane of a serrated gap is given by Eq. (3.37), and for convenience, is rewritten in Eq. (4.31).

$$\begin{aligned} \langle E_1 E_1^* \rangle = & \frac{16L_x^2 A_1 A_1^*}{\eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{(\kappa_1 L_x)} \right]^2 \cos^2 \left[\eta_1 W/2 \right] \exp(-\sigma^2 \eta_1^2) \\ & + \frac{4L_x A_1 A_1^*}{\eta_1^2} \exp \left[-\sigma^2 \eta_1^2 \right] \sum_{n=1}^{\infty} (\sigma \eta_1)^{2n} F_{\kappa_1} \left(\rho^n(\xi_1) \right) . \end{aligned} \quad (4.31)$$

For small $\sigma \eta_1$ we keep only the first term ($n=1$) of the series, thus we have

$$\begin{aligned} \langle E_1 E_1^* \rangle = & \frac{16L_x^2 A_1 A_1^*}{\eta_1^2} \left[\frac{\sin(\kappa_1 L_x)}{\kappa_1 L_x} \right]^2 \cos^2 \left[\eta_1 W/2 \right] \exp(-\sigma^2 \eta_1^2) \\ & + 4L_x A_1 A_1^* \sigma^2 F_{\kappa_1} \left(\rho(\xi_1) \right) . \end{aligned} \quad (4.32)$$

The first term in Eq. (4.32) is the intensity pattern produced by a slit of width W , but modified by the exponential damping due to the roughness. As for a straight edge the spread of energy in the x -direction from this term extends only a few Airy disc units. It is the second term that dictates the spread of energy perpendicular to the central

spike; and as for a rough edge, see Eq. (4.29), the spread goes as the Fourier transform of the correlation coefficient.

The processing for a serrated gap is identical to that described above. From the transform plane intensity data, we subtract the spike produced by the slit, then Fourier transform the remaining data and normalize.

Figure 4.21(a) shows the intensity pattern, without the spike, as produced by a serrated gap in which the correlation coefficient for each edge of the gap is shown in Fig. 4.9(b). In Fig. 4.21(b) the comparison between the "photo digitization" method, Fig. 4.9(b), and "fiber optic measurement" of intensity is shown. Again, the experimental results are in good agreement with the theoretical predictions of Chapter 3.

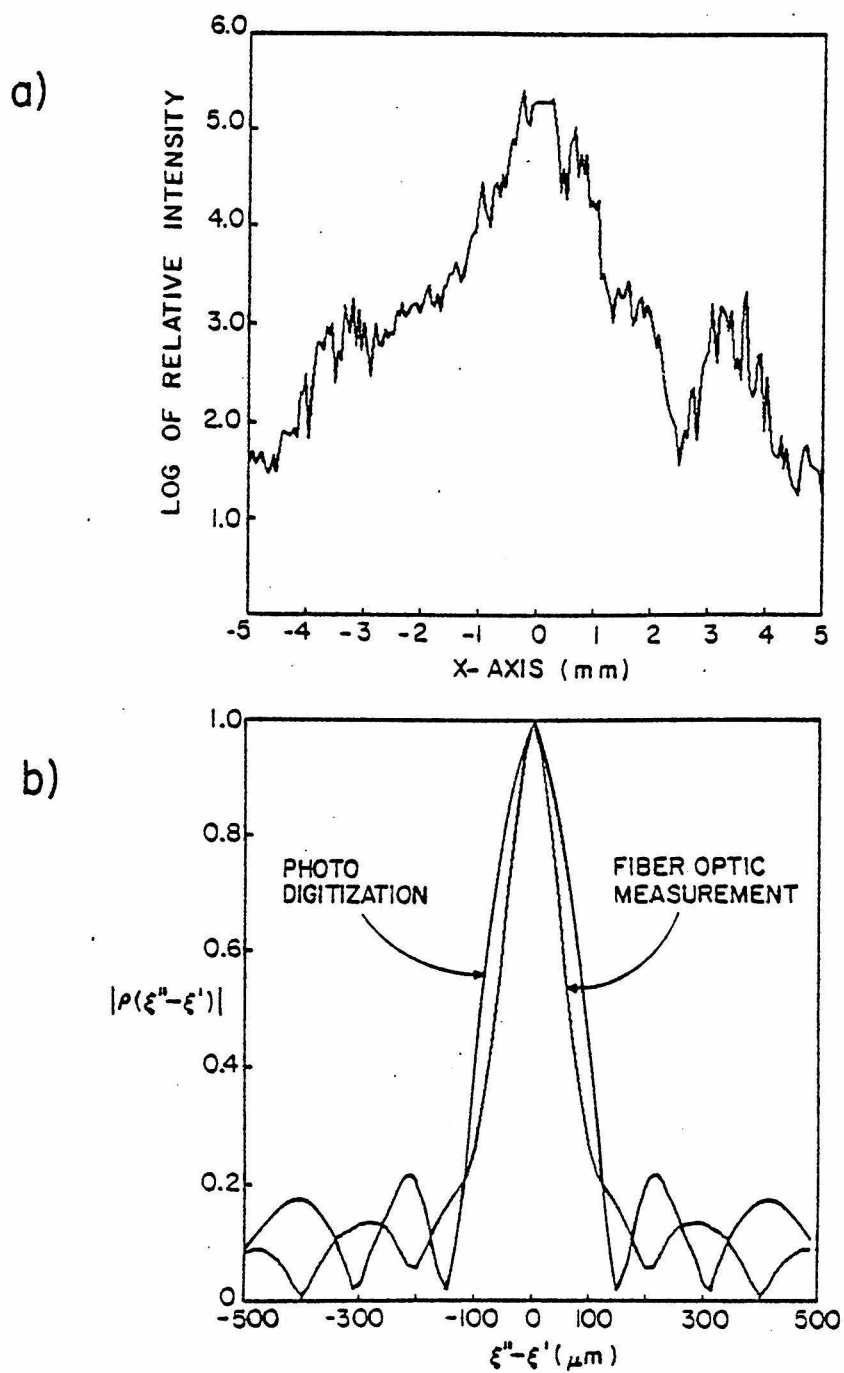


Fig. 4.21 (a) Truncated intensity measurements for the spread of energy for a serrated gap and (b) $|\rho(\xi'' - \xi')|$ obtained by the two independent processing methods. Each edge of the gap has the same statistics as the edge in Fig. 4.9 (b).

CHAPTER IV

REFERENCES

1. A.C. Livanos and N. George, "Edge Diffraction of a Convergent Wave," Appl. Opt. 14, 608-613 (1975).
2. A. Papoulis, Probability, Random Variables, and Stochastic Processes (McGraw-Hill, New York, 1965), p. 282.
3. For another application using the rectangular window function, see H. Fujii, J. Uozumi, and T. Asakura, "Computer simulation study of image speckle patterns with relation to object surface profile," J. Opt. Soc. Am. 66, p. 1222-1236 (1976).

CHAPTER V

SUMMARY AND CONCLUSIONS

Diffraction by apertures having rough boundaries can be analyzed by Eq. (2.15) in the Fresnel zone and by Eqs. (2.8) and (2.14) in the optical transform plane. Under monochromatic illumination, we have shown that a speckle pattern results. Both spatial and wavelength dependences are studied by calculation of $\langle E_1 E_2^* \rangle$ and $\langle E_1 E_1^* E_2 E_2^* \rangle$. In the usual speckle pattern due to longitudinal phase fluctuations, the expected value of the intensity is stationary; and one goes to the fourth order moment of the electric field to study the scale of the fluctuation. However, the optical transform of a rough aperture gives rise to a non-stationary pattern so useful information about the edge can be obtained from $\langle E_1 E_1^* \rangle$. For the scale size of the speckle, again the fourth order moment, in which $\langle E_1 E_2^* \rangle$ is the essential building block, is the useful correlation function.

In the theory discussed in Chapter II, exact calculations of several complicated integrals are greatly simplified by making the reasonable idealization that the correlation coefficient of edge roughness is exactly zero outside of some finite limit, $\pm \epsilon_0$ in Eq. (2.11).

Illustrative calculation of $\langle E_1 E_2^* \rangle$ are presented for the serrated edge. The second order central moment $\langle E_1 E_2^* \rangle$, is given in general by Eq. (2.26). In Eq. (2.27) we define a function, $G(-\eta_1, \eta_2; \kappa_2)$, that is evaluated for normally distributed roughness of arbitrary σ and triangular correlation coefficient in Eq. (2.33). $G(-\eta_1, \eta_2; \kappa_2)$ is seen to be important in understanding the spatial and wavelength dependences of the speckle for both edges and gaps. The total intensity $\langle E_1 E_1^* \rangle$ is given in Eq. (2.86) together with curves in Figs. 2.3, 2.4, and 2.5 illustrating the spread of energy perpendicular to the central spike due to G.

For a serrated gap the second order central moment $\langle E_1 E_2^* \rangle$, is given in general by Eqs. (2.93) through (2.95), where correlations are included between the two rough edges. Assuming joint normal statistics for the edge roughness and dropping the correlation between the upper and lower edge, we find a useful result, in Eq. (2.100).

For our study of the fourth order correlation function, we preferred to analyze a roughened edge since it is analytically less formidable than the gap while still exhibiting the same type of functional dependence. The fourth order central moment, $\langle E_1 E_1^* E_2 E_2^* \rangle$ in Eq. (2.39), is calculated from first principles for arbitrary roughness and correlation length. The solution is given in Eqs. (2.75) to (2.78) for jointly normal statistics and for $\epsilon_0 \ll L_x$. This

solution is compared in the limit of small roughness, $|\sigma\eta| \ll 1$, to a separate evaluation in Sec. 2.3.2A derived using moment expansion theorems but valid only for small roughness. The agreement between these solutions is verified in Sec. 2.3.3A.

In Chapter III power series representations for the second order central moment, $\langle E_1 E_2^* \rangle$ are derived for rough edges, in Eq. (3.18), and serrated gaps, in Eq. (3.36). These expressions are found to be particularly useful for remote sensing of edge statistics from transform plane intensity measurements. Also, using the technique of reversion of series, these expressions have been inverted yielding the correlation coefficient of roughness, $\rho(\xi'' - \xi')$, expressed in terms of the transform plane intensity, Eqs. (3.26) and (3.40).

Experiments using roughened edges and gaps are presented in Chapter IV to show the main effects predicted. In the transform plane the central spike for a smooth edge is greatly damped and energy is scattered throughout the plane (Fig. 4.2). The damping of this spike is shown in Fig. 4.11. The spread of energy perpendicular to the central spike is shown in Figs. 4.8 and 4.14 through 4.16-- illustrating the dependence of the spread of energy on edge correlation length, ϵ_0 . The transform plane intensity pattern is found to contain useful information about the edge statistics. Remote sensing experiments of the edge

correlation coefficient were performed using computer generated edges with controlled statistics (Section 4.5). The estimates for $|\rho(\xi''-\xi')|$ obtained from transform plane intensity measurements, see Figs. 4.18, 4.19(b), 4.20(b), are in good agreement with those found by an independent method.

APPENDIX A

FOURIER TRANSFORM OF A UNIT STEP FUNCTION

The "Fourier integral formula" is given by the expression

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx f(x) \exp[-i\omega(x-t)] \quad (A1)$$

where in the derivation⁽¹⁾ of Eq. (A1) the following assumptions were made:

(A) In every finite interval $f(x)$ is piecewise continuous and possesses a piecewise continuous first derivative.

(B) The integral $\int_{-\infty}^{\infty} |f(x)| dx = C$ exists.

We wish to consider the Fourier transform of a unit step function, written as

$$F\{u(x)\} = U(\omega) = \int_{-\infty}^{\infty} dx u(x) \exp(-i\omega x) \quad (A2)$$

where

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (A3)$$

Note, we immediately encounter difficulties, as $u(x)$ violates assumption (B). To get around this difficulty, we introduce a slight amount of damping, i.e., Eq. (A2)

is to be interpreted in the following sense:

$$U(\omega) = \lim_{\sigma \rightarrow 0} \int_{-\infty}^{\infty} dx \, u(x) \exp(-\sigma x) \exp(-i\omega x). \quad (A4)$$

Using Eq. (A3) and integrating, Eq. (A4) gives

$$U(\omega) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma + i\omega}, \quad (A5)$$

and separating into real and imaginary parts $U(\omega)$ takes the form

$$U(\omega) = \lim_{\sigma \rightarrow 0} \left[\frac{\sigma}{\omega^2 + \sigma^2} - i \frac{\omega}{\omega^2 + \sigma^2} \right]. \quad (A6)$$

We now show that

$$\lim_{\sigma \rightarrow 0} \frac{\sigma}{\omega^2 + \sigma^2} = \pi \delta(\omega) \quad (A7)$$

and

$$-i \lim_{\sigma \rightarrow 0} \frac{\omega}{\omega^2 + \sigma^2} = -i \text{P.V.} \left(\frac{1}{\omega} \right), \quad (A8)$$

where $\text{P.V.} \left(\frac{1}{\omega} \right)$ is called the principle value of $1/\omega$.⁽²⁾

It behaves like $1/\omega$ where $\omega \neq 0$, and vanishes for $\omega = 0$.

This has the meaning that when multiplied by a function $g(\omega)$ and integrated over ω the effect is to delete the integration over the small interval $-\epsilon$ to ϵ , i.e.,

$$\int_{-a}^{+b} d\omega g(\omega) \text{P.V.}(1/\omega) = \int_{-a}^{-\epsilon} d\omega g(\omega)(1/\omega) + \int_{\epsilon}^{+b} d\omega g(\omega)(1/\omega), (\epsilon \rightarrow 0). \quad (\text{A9})$$

First let's verify Eq. (A7). Recall the delta function is defined⁽³⁾ by

$$\int_{-a}^{+b} d\omega g(\omega) \delta(\omega) = g(0). \quad (\text{A10})$$

We substitute Eq. (A7) into Eq. (A10), and see if Eq. (A10) is satisfied. Making this substitution gives

$$\lim_{\sigma \rightarrow 0} (1/\pi) \int_{-a}^{+b} d\omega g(\omega) \left[\sigma / (\omega^2 + \sigma^2) \right] = \lim_{\sigma \rightarrow 0} (1/\pi) \int_{-a}^{+b} d\omega g(\omega) (1/\sigma) \left[1 / [(\omega/\sigma)^2 + 1] \right]. \quad (\text{A11})$$

On the right-hand side of Eq. (A11), make the change of variables $(\omega/\sigma) \rightarrow y$. This gives

$$\begin{aligned} \lim_{\sigma \rightarrow 0} (1/\pi) \int_{-a}^{+b} d\omega g(\omega) \left[\sigma / (\omega^2 + \sigma^2) \right] &= \lim_{\sigma \rightarrow 0} (1/\pi) \int_{-a/\sigma}^{+b/\sigma} dy g(\sigma y) \left[1 / (1 + y^2) \right] \\ &= (1/\pi) g(0) \int_{-\infty}^{\infty} dy [1 / (y^2 + 1)] \\ &= g(0). \end{aligned} \quad (\text{A12})$$

From Eq. (A12) we see that $(1/\pi) \lim_{\sigma \rightarrow 0} [\sigma / (\omega^2 + \sigma^2)]$ does indeed behave as $\delta(\omega)$, thus establishing Eq. (A7).

From Eqs. (A8) and (A9), we see that Eq. (A8) is

satisfied only if

$$-i \lim_{\sigma \rightarrow 0} \int_{-e}^e d\omega g(\omega) [\omega / (\omega^2 + \sigma^2)]$$

is identically zero for any e -arbitrarily small, yet non-zero. To show this we, again, make the change of variables $(\omega/\sigma) \rightarrow y$, yielding

$$\begin{aligned} -i \lim_{\sigma \rightarrow 0} \int_{-e}^e d\omega g(\omega) [\omega / (\omega^2 + \sigma^2)] &= \lim_{\sigma \rightarrow 0} \int_{-e/\sigma}^{e/\sigma} dy g(\sigma y) [y / (y^2 + 1)] \\ &= g(0) \int_{-\infty}^{\infty} dy [y / (y^2 + 1)] \\ &= 0 \end{aligned} \tag{A13}$$

since the integrand is an odd function.

Thus, from Eqs. (A2), (A6), (A7) and (A8), the Fourier transform of a unit step function is

$$U(\omega) = \pi \delta(\omega) - i \text{ P.V.}(1/\omega) . \tag{A14}$$

In writing Eq. (2.20), the principal value notation in Eq. (A14) is dropped so as to avoid needless complication of the notation in the main text. In the text it is implicit that any subsequent integration involving the $(-i/\eta_1)$ term is taken in the sense of Eq. (A9).

APPENDIX A
REFERENCES

1. See, for example, R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience, New York, 1953), p. 77-79.
2. We adopt the notation used in the discussion of the "zeta" function, W. Heitler, The Quantum Theory of Radiation (Oxford University Press, London, 1954), p. 69.
3. See, for example, B. Friedman, Principles and Techniques of Applied Mathematics (Wiley, New York, 1956), p. 135.

APPENDIX B

EXPANSION OF THE FOURTH ORDER CORRELATION FUNCTION

The intensity cross-correlation function for states $I(x_1, y_1; k_1)$ and $II(x_2, y_2; k_2)$ is defined in Eqs. (2.12) and (2.13) by

$$R_u = \langle E_1 E_1^* E_2 E_2^* \rangle . \quad (B1)$$

In terms of central field quantities (zero mean) defined by $E_n = E_n - \langle E_n \rangle$, it is readily shown that

$$\langle E_1 E_1^* E_2 E_2^* \rangle = \langle E_1 E_1^* E_2 E_2^* \rangle + \text{terms in (B3) below} \quad (B2)$$

$$+ b_1 \langle E_1^* E_2 E_2^* \rangle - b_1 b_1^* \langle E_2 E_2^* \rangle - b_1 b_2^* \langle E_1^* E_2 \rangle$$

$$+ b_1^* \langle E_1 E_2 E_2^* \rangle - b_1 b_2 \langle E_1^* E_2^* \rangle - b_1^* b_2 \langle E_1 E_2^* \rangle$$

$$+ b_2 \langle E_1 E_1^* E_2^* \rangle - b_1^* b_2^* \langle E_1 E_2 \rangle$$

$$+ b_2^* \langle E_1 E_1^* E_2 \rangle - b_2 b_2^* \langle E_1 E_1^* \rangle + 3b_1 b_1^* b_2 b_2^* \quad (B3)$$

In (B3) the notation is $b_1 = \langle E_1 \rangle$ and $b_2 = \langle E_2 \rangle$. An alternate form for the difference $R_u - \langle E_1 E_1^* E_2 E_2^* \rangle$ in (B3) can be found in terms of central moments, as follows:

$$\begin{aligned}
 & + 2\text{Re}\{b_1\langle E_1^*E_2E_2^*\rangle + b_2\langle E_1E_1^*E_2^*\rangle + b_1b_2\langle E_1^*E_2^*\rangle + b_1^*b_2\langle E_1E_2^*\rangle\} \\
 & + b_1b_1^*\langle E_2E_2^*\rangle + b_2b_2^*\langle E_1E_1^*\rangle + b_1b_1^*b_2b_2^* .
 \end{aligned} \tag{B4}$$

For the serrated edge, any of the second order moments in (B3) can be calculated from Eq. (2.24) by properly assigning subscripts. For example $\langle E_1E_1^*\rangle$ is obtained from Eq. (2.24) as follows:

$$\begin{aligned}
 \langle E_1E_1^*\rangle &= [A_1A_1^*/(\eta_1^2)] \iint_{-\infty}^{\infty} d\xi' d\xi'' \exp(i\kappa_1\xi'' - i\kappa_1\xi') \left[\langle \exp(in_1y_r(\xi'') - in_1y_r(\xi')) \rangle \right. \\
 &\quad \left. - \langle \exp[in_1y_r(\xi'')] \rangle \langle \exp[-in_1y_r(\xi')] \rangle \right].
 \end{aligned} \tag{B5}$$

This is rewritten in the text in Eq. (2.35) using the function $G(-\eta_1, \eta_1; \kappa_1)$ summarized in Eqs. (2.36) to (2.38). Note that the L_x which appears in Eq. (2.35) is introduced naturally in the integration of Eq. (B5) establishing the order of the singularity appearing in the delta function of Eq. (2.34).

We will concentrate on the calculation of the central moment $\langle E_1E_1^*E_2E_2^*\rangle$. The third order case is simpler, but for the sake of brevity will not be included. However, it is noted that the formalism for $\langle E_1E_1^*E_2E_2^*\rangle$ developed in Sec. 2.3.2 is equally valid in the third order calculation.

APPENDIX C

TERMS INVOLVED IN THE CALCULATION OF $\langle E_1 E_1^* E_2 E_2^* \rangle$

From the form of matrix M^{-1} in Table 2.1, we see there are six independent correlation coefficients, r_{ij} , connecting the elements of the point set $(\xi', \xi'', \xi''', \xi''')$ in pairs. We group terms L_1 to L_{37} , in Eq. (2.59), into categories characterized by the number of non-zero correlation coefficients. For example, in Sec. 2.2.3-B1 we listed three terms, L_1 , L_2 , L_3 , Eqs. (2.55) to (2.57), in which there were two non-zero correlation coefficients and four correlation coefficients identically zero. Recall, it is the condition in Eq. (2.11) that allows us to take correlation coefficients identically zero. The terms are separated into the categories as shown in Table C1.

TABLE C1
Listing of Terms L_1 - L_{37}

Term	Non-zero Correlation Coefficients	Zero Correlation Coefficients
I. Two non-zero correlations		
L_1	r_{12}, r_{34}	$r_{13}, r_{23}, r_{14}, r_{24}$
L_2	r_{14}, r_{23}	$r_{12}, r_{13}, r_{34}, r_{24}$
L_3	r_{13}, r_{24}	$r_{12}, r_{14}, r_{34}, r_{23}$
II. Three non-zero correlations		
L_4	r_{12}, r_{34}, r_{24}	r_{13}, r_{23}, r_{14}
L_5	r_{12}, r_{34}, r_{14}	r_{13}, r_{23}, r_{24}
L_6	r_{12}, r_{34}, r_{23}	r_{13}, r_{14}, r_{24}
L_7	r_{12}, r_{34}, r_{13}	r_{23}, r_{14}, r_{24}
L_8	r_{14}, r_{23}, r_{24}	r_{12}, r_{13}, r_{34}
L_9	r_{14}, r_{23}, r_{12}	r_{13}, r_{24}, r_{34}
L_{10}	r_{14}, r_{23}, r_{34}	r_{12}, r_{13}, r_{24}
L_{11}	r_{14}, r_{23}, r_{13}	r_{12}, r_{24}, r_{34}
L_{12}	r_{13}, r_{24}, r_{23}	r_{12}, r_{14}, r_{34}
L_{13}	r_{13}, r_{24}, r_{12}	r_{14}, r_{23}, r_{34}
L_{14}	r_{13}, r_{24}, r_{34}	r_{12}, r_{14}, r_{23}
L_{15}	r_{13}, r_{24}, r_{14}	r_{12}, r_{23}, r_{34}

TABLE C1
(Continued)

Term	Non-zero Correlation Coefficients	Zero Correlation Coefficients
III. Four non-zero correlations		
L_{16}	$r_{12}, r_{34}, r_{14}, r_{24}$	r_{13}, r_{23}
L_{17}	$r_{12}, r_{34}, r_{14}, r_{13}$	r_{23}, r_{24}
L_{18}	$r_{12}, r_{34}, r_{14}, r_{23}$	r_{13}, r_{24}
L_{19}	$r_{12}, r_{34}, r_{24}, r_{13}$	r_{14}, r_{23}
L_{20}	$r_{12}, r_{34}, r_{24}, r_{23}$	r_{13}, r_{14}
L_{21}	$r_{12}, r_{34}, r_{13}, r_{23}$	r_{14}, r_{24}
L_{22}	$r_{13}, r_{24}, r_{12}, r_{23}$	r_{14}, r_{34}
L_{23}	$r_{13}, r_{24}, r_{12}, r_{14}$	r_{23}, r_{34}
L_{24}	$r_{13}, r_{24}, r_{23}, r_{14}$	r_{12}, r_{34}
L_{25}	$r_{13}, r_{24}, r_{23}, r_{34}$	r_{12}, r_{14}
L_{26}	$r_{13}, r_{24}, r_{14}, r_{34}$	r_{12}, r_{23}
L_{27}	$r_{14}, r_{23}, r_{12}, r_{24}$	r_{13}, r_{34}
L_{28}	$r_{14}, r_{23}, r_{12}, r_{13}$	r_{24}, r_{34}
L_{29}	$r_{14}, r_{23}, r_{24}, r_{34}$	r_{12}, r_{13}
L_{30}	$r_{14}, r_{23}, r_{34}, r_{13}$	r_{12}, r_{24}

TABLE C1
(Continued)

Term	Non-zero Correlation Coefficients	Zero Correlation Coefficients
IV. Five non-zero correlations		
L_{31}	$r_{12}, r_{34}, r_{14}, r_{24}, r_{13}$	r_{23}
L_{32}	$r_{12}, r_{34}, r_{14}, r_{24}, r_{23}$	r_{13}
L_{33}	$r_{12}, r_{34}, r_{24}, r_{13}, r_{23}$	r_{14}
L_{34}	$r_{12}, r_{34}, r_{13}, r_{23}, r_{14}$	r_{24}
L_{35}	$r_{13}, r_{24}, r_{12}, r_{23}, r_{14}$	r_{34}
L_{36}	$r_{13}, r_{24}, r_{23}, r_{14}, r_{34}$	r_{12}
V. Six non-zero correlations		
L_{37}	$r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$	

APPENDIX D

COMPUTER PROGRAM TO GENERATE A NUMBER

SET WITH CONTROLLED STATISTICS

```
//EDGEPLT JOB G0576, 'MORRIS, G.H.'
//*OPTIONS LINES=5K, TIME=(1,30), ROUTE=TAY, CLASS=B
//*SETUP TAPE9, EDGES, HL, RING
//*SETUP PLOT TAPE
//STEPA EXEC FORTCLG, REGION.G0=520K
//FORT.SYSIN DD *
    DIMENSION RANFUN(16384), ARRAY(16384), X(512)
    CALL SETPLT (2.0)

C
C
C   THE PROGRAM EDGEPLT GENERATES A GAUSSIAN RANDOM CURVE
C   WITH ANY DESIRED MEAN, STANDARD DEVIATION AND CORRELATION
C   LENGTH. THE USER MUST SPECIFY:
C
C       AVG= THE MEAN VALUE OF DESIRED DATA SET.
C       SIGMA= THE STANDARD DEVIATION.
C       NWINDO= THE NUMBER OF POINTS OVER WHICH THE
C               DATA IS TO REMAIN CORRELATED. THE BASE OF THE
C               AUTOCORRELATION FUNCTION HAS A LENGTH OF 2*NWINDO.
C       N= THE TOTAL NUMBER OF POINTS TO BE GENERATED.
C
C
C   N=16384
C   AVG=0.0
C   SIGMA=25.0
C   NWINDO=50
C   NTWO=2*N
C   WRITE (6,100)

C
C   GENERATE X-AXIS FOR PLOTS.
C
C   DO 60 I=1,512
60  X(I)=-256+I
    CALL RANDOM (RANFUN,N,AVG,SIGMA)
    WRITE (6,100)
    CALL STAT (RANFUN,N,TAVG,TSIGMA)
    WRITE (6,100)
    CALL CONCOR (RANFUN,ARRAY,N,1,1.0,1)
    CALL CONCOR (RANFUN,ARRAY,N,NWINDO,1.0,-1)
    CALL PLOTXY (X,ARRAY,512,1)
    WRITE (6,101)
    CALL STAT (ARRAY,N,TAVG,TSIGMA)
    WRITE (6,101)
    CALL CONCOR (ARRAY,ARRAY,N,1,1.0,1)
    CALL PLOTXY (X,ARRAY,512,2)
    CALL ENDPLT
100  FORMAT (55X, 'RANDOM NUMBER GENERATOR'/)
101  FORMAT (49X, 'CONVOLVED ARRAY- (RANDOM WITH PECT)'/)
    STOP
    END
    SUBROUTINE RANDOM (P,N,AVG,SIGMA)

C
C
C   THIS SUBROUTINE GENERATES A RANDOM ARRAY OF NUMBERS WITH
C   A MEAN "AVG" AND A STANDARD DEVIATION OF "SIGMA".
C
C
C   DIMENSION P(1)
C   NSEED=317956811
C   DO 10 I=1,N
```

```

10 P(I)=RAND (15,NSEED,AVG,SIGMA)
   WRITE (6,22) (P(I),I=1,512)
   WRITE (6,3)
22 FORMAT (1X,10E13.5)
3  FORMAT (//)
   RETURN
   END
   SUBROUTINE STAT(Y,N,TAVG,TSIGMA)

```

C
C
C
C
C

THIS SUBROUTINE CALCULATES THE ACTUAL MEAN "TAVG" AND THE ACTUAL STANDARD DEVIATION "TSIGMA" OF THE ARRAY "Y".

```

   DIMENSION Y(1)
   RN=N
   SUMSQ=0.0
   SUM=0.0
   DO 15 I=1,N
   SUM=SUM+Y(I)
15  SUMSQ=SUMSQ+Y(I)*Y(I)
   TAVG=SUM/RN
   HOLD=SUMSQ/RN
   TEMP=HOLD+(TAVG*TAVG)
   TSIGMA=SQRT(TEMP)
   WRITE (6,103) TAVG,TSIGMA
102 FORMAT (57X,'COMPUTED STATISTICS')
103 FORMAT (44X,'MEAN:',F10.3,5X,'STANDARD DEVIATION:',F10.3,/)
   RETURN
   END
   SUBROUTINE CONCOR(B,Q,N,MU,STEP,IFLAG)

```

C
C
C
C
C
C
C
C
C
C

THIS SUBROUTINE EITHER PERFORMS AN AUTOCORRELATION OF THE DATA OR CONVOLVES IT WITH A RECT FUNCTION OF WIDTH L.

IF IFLAG IS LESS THAN ZERO: DATA "B" IS CONVOLVED WITH RECT(X/L).

IF IFLAG IS GREATER THAN ZERO: DATA "B" IS AUTOCORRELATED.

```

   DIMENSION B(1),Q(1),W(16384),WORK(32768),OUT(32768)
   COMPLEX OUTPUT(16384)
   EQUIVALENCE(OUTPUT(1),OUT(1))
   DATA PI/3.1415927/
   RN=N
   RNW=RNW
   RWSQRT=SQRT(RNW)
   NHALF=N/2
   NTWO=2*N
   RNSQ=N*N
   DO 5 I=1,N
5  Q(I)=B(I)
   IF (IFLAG.GT.0) GO TO 40

```

C
C
C
C
C
C

GENERATE FOURIER TRANSFORM OF WINDOW FUNCTION: $L \cdot \text{SINC}(L \cdot \text{FREQ.})$; $L = \text{MU} \cdot \text{STEP}$
 $W(\text{NHALF}+1) = L \cdot C$. 'C' IS USED TO OBTAIN THE DESIRED VARIANCE.
 FOR MORE INFORMATION ABOUT THIS NORMALIZATION, SEE SECTION 4.3.1;
 GAUSSIAN RANDOM PROCESSES IN A SPACE-INVARIANT LINEAR SYSTEM

```

   CALL HOUNFL
   ARG=RNW*PI/2.
   W(NHALF+1)=RWSQRT
   W(1)=W(NHALF+1)*SIN(ARG)/ARG
   DO 20 I=2,NHALF

```

```

ARG=RNW*PI*(1-1)/RN
W(1+NHALF)=W(NHALF+1)*SIN(ARG)/ARG
IF (W(1+NHALF).LT.1.0E-10)W(1+NHALF)=0.0
20 W(NHALF+2-1)=W(1+NHALF)
CALL UNFLOV
40 CONTINUE

C
C   CREATE INPUT FUNCTION FOR FFT ROUTINE.
C
DO 25 I=1,N
25 OUTPUT(I)=Q(I)

C
C   TAKE THE FOURIER TRANSFORM OF THE FUNCTION: OUTPUT.
C
CALL FFT (OUTPUT,N,1,WORK)
IF (IFLAG.GT.0) GO TO 45

C
C   MULTIPLY WINDOW FUNCTION, 'W', WITH 'OUTPUT' OR 'OUT'.
C
DO 30 I=1,N
OUT (2*I-1)=W(I)*OUT(2*I-1)
30 OUT(2*I)=W(I)*OUT(2*I)
GO TO 50
45 CONTINUE

C
C   COMPUTE: OUTPUT(I)*COMPLEX CONJUGATE (OUTPUT(I))
C
DO 55 I=2,NTWO,2
OUT(I-1)=OUT(I-1)*OUT(I-1)
OUT(I)=OUT(I)*OUT(I)
OUT(I-1)=OUT(I)+OUT(I-1)
55 OUT(I)=0.0
50 CONTINUE

C
C   INVERSE FOURIER TRANSFORM: OUTPUT.
C
CALL FFT (OUTPUT,N,-1,WORK)
CALL SHIFT (OUTPUT,N)
IF (IFLAG.GT.0) GO TO 80

C
C   IN ORDER TO GET CORRECT CONVOLUTION DIVIDE FFT DATA BY RN AND MULTIPLY
C   BY (-1)**I
C
J=-1
DO 90 I=1,N
OUT(2*I-1)=J*OUT(2*I-1)/RN
OUT(2*I)=J*OUT(2*I)/RN
Q(I)=OUT(2*I-1)
90 J=-J
GO TO 95
80 CONTINUE

C
C   IN ORDER TO GET CORRECT AUTOCORRELATION DIVIDE FFT DATA BY RN**2.
C
DO 35 I=1,N
OUT(2*I-1)=OUT(2*I-1)/RNSQ
OUT(2*I)=OUT(2*I)/RNSQ
35 Q(I)=OUT(2*I-1)
IF (IFLAG.GT.0) GO TO 70
95 CONTINUE
WRITE (6,4)

C
C   WRITE Q(I) ON A MAGNETIC TAPE
C
WRITE(9,9) (Q(I),I=1,N)
ENDFILE 9
GO TO 75
70 CONTINUE

```

```

WRITE (6,6)
75 CONTINUE
DO 100 I=1,512
100 Q(I)=Q(7935+I)
WRITE (6,2) (Q(I),I=1,512)
WRITE (6,3)
2 FORMAT (1X,10E13.5)
3 FORMAT (//)
4 FORMAT (31X,'CONVOLUTION OF RANDOM FUNCTION WITH WINDOW FUNCTION:
1 REAL PART OF ARRAY'//)
6 FORMAT (44X,'AUTOCORRELATION FUNCTION: REAL PART OF ARRAY'//)
9 FORMAT(1X,6E13.5)
RETURN
END
SUBROUTINE SHIFT(VSHIFT,H)

C
C
C THIS SUBROUTINE REARRANGES THE OUTPUT OF THE FFT ROUTINE TO OBTAIN A
C CONVENTIONAL PLOT OF THE FOURIER TRANSFORM, I.E. A PLOT FROM -PI TO PI,
C INSTEAD OF 0 TO 2PI.
C
C
COMPLEX VSHIFT(16384)
NHALF=N/2
DO 10 I=1,NHALF
TEMP=VSHIFT(NHALF+I)
VSHIFT(NHALF+I)=VSHIFT(I)
10 VSHIFT(I)=TEMP
RETURN
END
SUBROUTINE PLOTXY (P,Q,P,K)
DIMENSION P(1),Q(1),R(512),S(512)
DO 500 I=1,N
R(I)=P(I)
500 S(I)=Q(I)
CALL SCALE (R,N,15.,R'IN,RX,1,0)
CALL SCALE (S,N,10.,S'IN,SX,1,0)
CALL AXIS (0.0,0.0,6HX-AXIS,-6,15.0,0.0,R'IN,RX)
IF (K.EQ.2) GO TO 60
CALL AXIS (0.0,0.0,4HEDGE,4,10.,90.0,S'IN,SX)
GO TO 70
60 CONTINUE
CALL AXIS (0.0,0.0,15HAUTOCORRELATION,15,10.,90.,S'IN,SX)
70 CONTINUE
CALL LINE (R,S,I',1,0,1)
CALL PLOT (18.,0.0,-3)
RETURN
END
//GO.FT09F001 DD UNIT=TAPE9,VOL=SER=EDGES,DISP=(NEW,PASS),
// DCB=(RECFI=FB,LRECL=80,BLKSIZE=3200),LABEL=(6,HL)
//GO.PLOTTAPE DD DSN=SYSPL0T,UNIT=TAPE7,DISP=OLD
//

```

APPENDIX E

COMPUTER PROGRAM FOR PROCESSING TRANSFORM

PLANE INTENSITY MEASUREMENTS

```
//PROBE JOB GOS76, 'MORRIS, G.M.'
//*OPTIONS TIME=(1,30)
//*SETUP PLOT, MORRIS, SL, RING
//*MESSAGE .5 IN PEN
//STEPA EXEC FORTCLG, LEVEL=G1, REGION.G0=100K
//FORT.SYSIN DD *
```

```
C
C
C THIS PROGRAM READS IN A DATA SET Y(X), WRITES AND PLOTS
C THE LOGRITHM OF Y(X), COMPUTES THE ABSOLUTE VALUE OF ITS
C FOURIER TRANSFORM, THEN NORMALIZES BY THE MAXIMUM VALUE
C MAGNIFICATION OF THE CENTER PORTION OF THE FOURIER TRANSFORM IS
C OPTIONAL. SEE THE "MAG" PARAMETER.
```

```
C
C TO PLOT THE INPUT DATA:
```

1. SPECIFY THE INCREMENT, DELX, FOR THE X-COORDINATE.
2. N= NUMBER OF DATA POINTS.
3. M=NUMBER OF DATA SETS.

```
C
C IN THE TRANSFORM PLOTS THE FREQUENCY COORDINATE IS FREQ= X/LAMDA/Z.
C TO PLOT THE TRANSFORM ARRAY THE USER MUST SPECIFY:
```

1. LAMDA= WAVELENGTH OF LIGHT (IN MICRONS).
2. Z= DISTANCE BETWEEN THE TRANSFORM PLANE AND THE OBJECT
PLANE (IN IN).
(A CONVERGING GEOMETRY HAS BEEN ASSUMED).
3. MAG = NUMBER OF POINTS FOR THE MAGNIFIED FOURIER
TRANSFORM PLOT (CENTER PORTION).
IF MAG IS LESS THAN ZERO, A MAGNIFIED FOURIER TRANSFORM
IS NOT PLOTTED.

```
C
C
C DIMENSION Y(200), A(200), RY(200), RX(200), ALOGY(200)
C DIMENSION WORK(450), RY*MAG(200), RX*MAG(200), NDATA(4)
C CALL SETPLT (2.0)
C INTEGER A
C REAL LAMDA
C COMPLEX Y, HOLD
C DATA STEP/0.002/, LAMDA/0.58/, Z/250./
C DATA NDATA/200, 200, 200/
C M=1
C DELX=STEP*25.4
C RX*MIN=-.2*25.4
C FMIN=-0.5/DELX
C DO 400 KK=1, M
```

```
C
C READ AND WRITE INTENSITY VALUES.
```

```
C
C N=NDATA(KK)
C RN=N
C READ (5,*) (A(I), I=1, N)
C WRITE (6,4)
C WRITE (6,1) (A(I), I=1, N)
C WRITE (6,2)
```

```
C
C CORRECT FOR ADJUSTMENTS MADE IN THE SENSITIVITY OF THE LOCK-IN AMPLIFIER
```

```

DO 900 I=1,35
900 RY(I)=A(I)/50.
DO 901 I=36,44
901 RY(I)=A(I)/10.
DO 902 I=45,64
902 RY(I)=A(I)
DO 903 I=65,89
903 RY(I)=A(I)*4.
DO 904 I=90,97
904 RY(I)=A(I)*40.
DO 905 I=98,101
905 RY(I)=RY(97)
DO 907 I=102,103
907 RY(I)=A(I)*100.
DO 908 I=104,111
908 RY(I)=A(I)*40.
DO 909 I=112,117
909 RY(I)=A(I)*4.
DO 910 I=118,129
910 RY(I)=A(I)*10.0
DO 911 I=130,147
911 RY(I)=A(I)
DO 912 I=148,158
912 RY(I)=A(I)/10.
DO 913 I=159,200
913 RY(I)=A(I)/50.

```

C
C

```

DO 100 I=1,N
Y(I)=RY(I)
ALOGY(I)=ALOG10(RY(I))
100 RX(I)=(I-1)*DELX+RXMIN
WRITE (6,5)
WRITE (6,3) (RY(I),I=1,N)
WRITE (6,2)
WRITE (6,9)
WRITE (6,10) (ALOGY(I),I=1,N)
WRITE (6,2)
WRITE (6,6)
WRITE (6,3) (RX(I),I=1,N)
WRITE (6,2)
N3=N-3

```

C
C
C

THE FIRST X - VALUE IS CHOSEN TO BE -5.0 MM

```

DO 125 I=1,N3
RX(I)=RX(I+2)
RY(I)=ALOGY(I+2)
125 CONTINUE

```

C
C
C

PLOT LOGRITHM OF INTENSITY VS. X-COORDINATE.

CALL PLOTX (RX,RY,N3,10,6,-5.,6.0)

C
C
C

FOURIER TRANSFORM DATA.

```

CALL FFT (Y,N,1,WORK)
WRITE (6,11)
WRITE (6,3) (Y(I),I=1,N)
WRITE (6,2)
DO 200 J=1,N
HOLD=Y(J)
RY(J)=CABS(HOLD)
FREQ=(J-1)/RN/DELX+FMIN
200 RX(J)=LAMBDA*Z*FREQ
CALL SHIFT(RY,N)
RYMAX=RY(1)
DO 250 I=2,N
IF (RY(I).GT.RYMAX) RYMAX=RY(I)

```


CCC

```

PTICK=PSIZE/NPT
XZERO=MPT*PTICK/2.
QTICK=QSIZE/HQT
DO 500 I=1,M
  R(I)=P(I)*PFACT
500 S(I)=Q(I)*QFACT
  CALL PLOT (0.0,0.0,3)
  CALL PLOT(PSIZE,0.0,2)
  CALL PLOT (PSIZE,QTOP,2)
  CALL PLOT (0.0,QTOP,2)
  CALL PLOT (0.0,QMIN,2)
  DO 600 I=1,MPT1
    TICK=PTICK*I
    CALL PLOT (TICK,0.0,3)
    CALL PLOT (TICK,0.0625,2)
600 CONTINUE
  DO 700 I=1,HQT1
    TICK=QTICK*I+QMIN
    CALL PLOT (0.0,TICK,3)
    CALL PLOT (0.0625,TICK,2)
700 CONTINUE
  CALL PLOT (XZERO,0.0,-3)
  CALL LINE (R,S,M,1,0,1)
  CALL PLOT (10.0,0.0,-3)
  RETURN
  END
  SUBROUTINE SHIFT(VSHIFT,N)

```

THIS SUBROUTINE REARRANGES THE OUTPUT OF THE FFT ROUTINE TO OBTAIN A CONVENTIONAL PLOT OF THE FOURIER TRANSFORM, I.E. A PLOT FROM $-\pi$ TO π , INSTEAD OF 0 TO 2π .

```

DIMENSION VSHIFT(512)
NHALF=N/2
DO 10 I=1,NHALF
  TEMP=VSHIFT(NHALF+I)
  VSHIFT(NHALF+I)=VSHIFT(I)
10 VSHIFT(I)=TEMP
  RETURN
  END

```

```

//GO.PLOTTAPE DD DSN=SYSPL0T,UNIT=TAPE7,DISP=OLD,VOL=SER=HOBBS
//

```

APPENDIX F

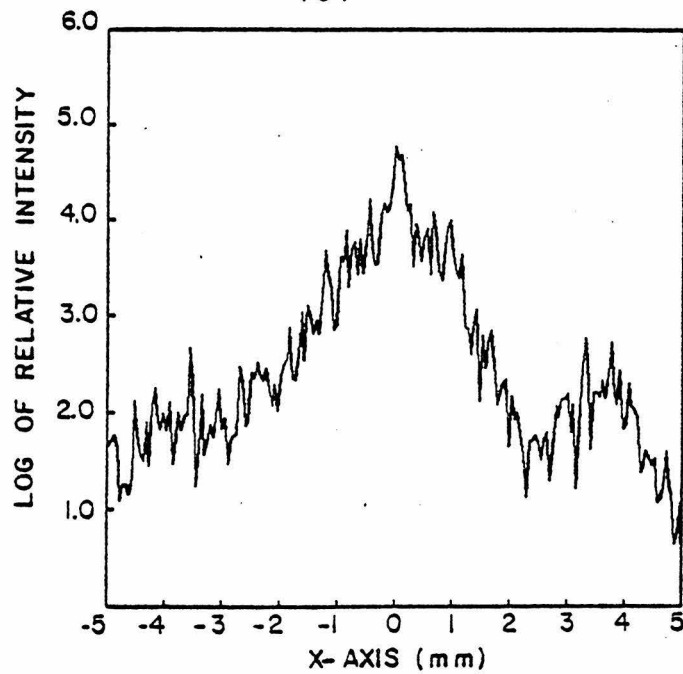
TRANSFORM PLANE INTENSITY DATA FOR THE CASE $\sigma\eta_1 > 1$

In Section 4.5 of the main text we discuss the measurement and processing of transform plane intensity data when the parameter $\sigma\eta_1 < 1$. It is found that information about the absolute value of the correlation coefficient, $|\rho(\xi''-\xi')|$, could be attained by processing the intensity data. The processing involved subtracting or deleting the central spike due to the edge, Fourier transforming (FFT) the resulting data, then normalizing. The steps of the process are discussed in detail in Section 4.5.2.

Here, we illustrate the effect of this processing when using an edge in which $\sigma\eta_1 > 1$. The correlation coefficient for this edge is shown in Fig. 4.9(b). Figure F1 contains the transform plane intensity data produced, and the estimate for $|\rho(\xi''-\xi')|$ using the entire data set, i.e., with the spike. For these measurements $\sigma\eta = 1.34$. Note, for larger $\sigma\eta_1$ the spike does not dominate the pattern as in the case of small $\sigma\eta_1$ (Figs. 4.14-4.16). This makes the decision of where to truncate the spike more difficult. In Fig. F2 the spike is truncated in (a) and the associated estimate for $|\rho(\xi''-\xi')|$ is shown in (b). Although there is some narrowing of $|\rho(\xi''-\xi')|$ in Fig. F2(b), it is not particularly significant.

Also, comparing $F_2(b)$ with $|\rho(\xi''-\xi')|$, in Fig. 4.9(b), obtained by "photo-digitization," one finds the agreement is not as good as for the cases when $\sigma\eta_1 < 1$. This is to be expected since for $\sigma\eta_1 > 1$, more than one term from the series in Eq. (4.28) should be retained.

a)



b)

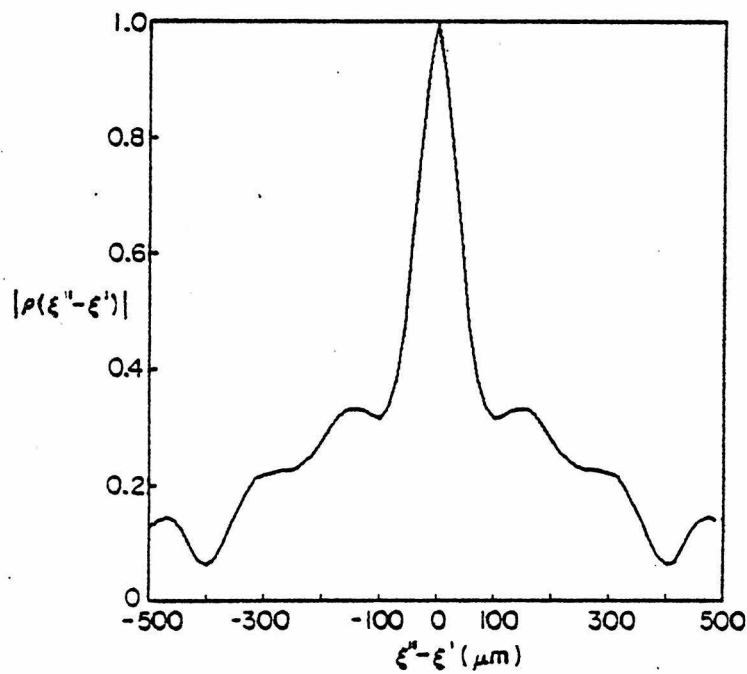
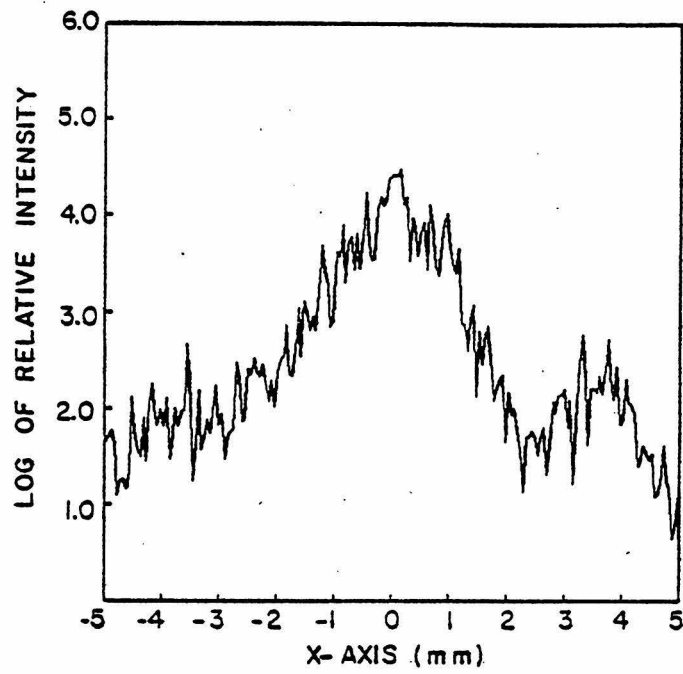


Fig. F1 (a) Intensity measurements of the spread of energy perpendicular to the central spike for the edge in Fig. 4.9 (b) in which $\sigma_{\eta_1} > 1$ and (b) the associated estimate for $|\rho(\xi'' - \xi')|$. The central spike has not been truncated.

a)



b)

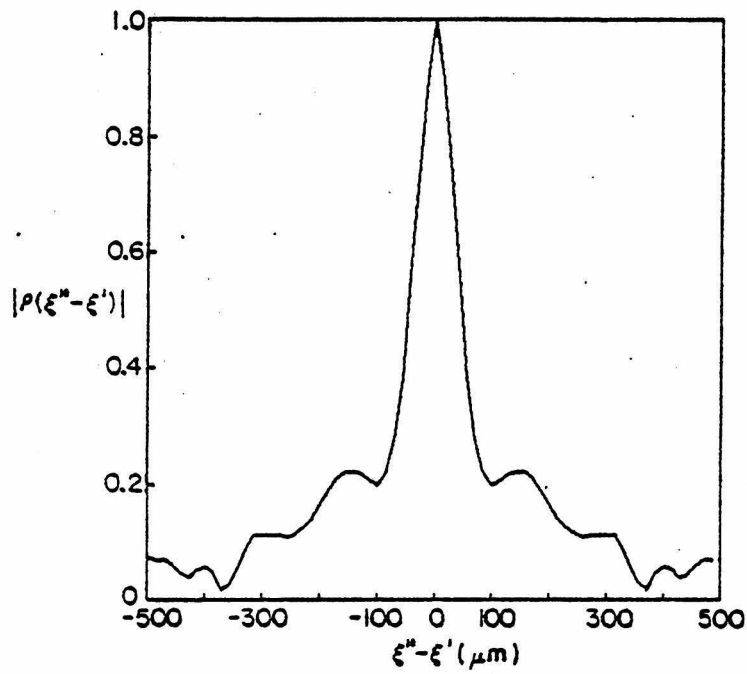


Fig. F2 (a) Truncation of the spike for the intensity measurements in Fig. F1(a) and (b) the associated estimate for $|\rho(\xi'' - \xi')|$.