

Probing and Harnessing Entanglement, Noise, and Dissipation across Quantum Metrology and Computing

Thesis by
Su Direkci

In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy



CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2027
Defended September 1, 2026

© 2027

Su Direkci

ORCID: 0009-0002-3867-4337

All rights reserved

*Ayaklarıyla ezip fıçıya mı bastılar seni
Nefti kasnaklı bir fıçıya,
Aldırma, kara üzüm!
Sen, o Kırmızı şarabına doğru
İçten içe
Harıl harıl
Çalışmana bak, iki gözüüm!*

Can Yücel

ACKNOWLEDGEMENTS

I want to thank my wonderful advisor, Yanbei Chen. When I first came to Caltech, I was having a difficult time finding an advisor due to my background in electrical engineering. Yanbei took me into his group with no hesitation. I still remember what he told me when I shared my anxious thoughts about research in theoretical physics with him: "It's just math, you'll figure it out". Throughout our five-year journey of me figuring it out, he never underestimated any of my potentially naive ideas. He attacked every problem with the same excitement and endless love for physics. His group has also been remarkable in terms of gender equality, which is a commendable feat in theoretical physics. Outside of academia, he has made my PhD experience incredibly joyful through his great sense of humor and his endless positivity. He's been more than a mentor to me, and I will always be incredibly grateful to him for everything.

During my junior years, there weren't so many people working on quantum metrology at Caltech. I still remember the first time that Yanbei mentioned that there was a postdoc in John Preskill's group, Tuvia Gefen, with whom we could collaborate. We have indeed been collaborating ever since. Tuvia has been an amazing mentor and a dear friend for all of these years. He single-handedly introduced me to the field of quantum metrology, and to the concepts that helped me get my PhD today. With his warmth, kindness, and patience, he made our meetings special. Outside of work, he was a steady source of support during a formative period of my life, during which I was probably a bit dramatic sometimes. I want to thank him for memories that I remember with a smile on my face.

I got to collaborate with some of the brightest minds in their field here, some of whom were my committee members: I want to thank Manuel Endres, Lee McCuller, and John Preskill for all of their guidance during my thesis journey. I'd like to thank Manuel especially for our collaboration over the years, during which his ideas gave rise to so many interesting discussions. I also want to express my appreciation for our discussions with Lee, as I always left them with a new understanding of concepts related to the field, and a great number of remarkable paper references. Finally, I want to thank John for having very inclusive meetings that foster great collaborations. Other bright minds that I got to collaborate with were: Markus Aspelmeyer, Rana Adhikari, Nelson Darkwah Oppong, Soonwon Choi, Klemens Hammerer, and Vivishek Sudhir. I want to thank Markus for hosting me in Vienna,

and all of these people for our discussions, which were extremely thought-provoking.

I did an internship at the AWS Center for Quantum Computing during the last year of my PhD, where I got to do research in a whole new field with a lot of interesting problems to tackle. I want to thank Simon Lieu, Kyungjoo Noh, and Connor Hann for our great discussions and their guidance during this time.

During the initial years of my PhD, I had the chance to collaborate with and especially learn from my amazing peers Klemens Winkler and Corentin Gut. I want to thank them for their patience while I was learning the basics of academia and also for hosting me in Vienna, where I had a great time. I also want to thank James Gardner, who is technically my academic little brother, but has provided me with guidance and company for several years while we were trying to find our place in the field of quantum metrology. During this time, he probably said "quantum Fisher information" a thousand times. Finally, during my last year, I got to collaborate with Zachary Mann and Shruti Maliakal: I want to express my appreciation for all of our discussions that gave me a new perspective on displacement sensing.

I want to thank all of my friends in Turkey who had to give up their Saturday evenings to talk with me. They welcomed me with open arms every summer when I visited Turkey, and never held a grudge because of me leaving and missing a large part of their lives: Sila, Beste, Elif, Zeren, Öykü, and many more. Also, I want to express my gratitude for my extended family and family friends who would always host me with the worry that I hadn't eaten any good food in the United States.

Caltech would only be a collection of buildings for me if it weren't for the amazing people that I met here. I've grown with some of these people: we've seen each other at our best and worst, laughed, cried, and got drunk too many times. I'm happy to know that I have someone that I call a friend in most corners of the world now. I want to thank all of them for something that I appreciate about them: Cat for always seeing the good in everyone, Elina for being the free-spirited extrovert that she is, Eyyüp for understanding all of my chronically online jokes and being ready to dance at all times, Alkım for teaching me how to play tennis and his creative cursing skills, Volkan for being an incredible sport against my offensive jokes, Bilge for taking me to amazing concerts, Kemal for his maturity beyond his years and expertise at the gym, Damla for her photography skills, Andrew for his love of cats and solidarity, Brian for his authenticity, Colin for being a nihilist who is good at impressions, and Kadircan for his refined tastes in life.

I also want to acknowledge the people who made this place into a community for me. I had very fun conversations with them, which were like medicine after endless calculations and failed simulations. Here's a list with some of them: Utku, Mert, Arda D., Elvira, Hannah, Gyohei, Honami, Richard, Nadine, Solène, Théo, Fatma, Maryam, Ecem, Rahaf, Dylan, Huma, Madison, Taylor, Steven, Jeff, Yufeng, Annika, and Ian.

I want to thank my parents for always doing the best that they could. Growing up in less favorable conditions than the ones I got to enjoy, they sometimes had to put their aspirations aside for the benefit of their children, which I'm always grateful for. Thanks to their sacrifices, I was able to take the risk of moving abroad, and build a life that they once dreamed of for themselves. I also want to shout out my sister Duru for tolerating, even as a teenager, my despotic regime in the house whenever I visit. I'm in general grateful to my family for always being supportive, even when I'm hours deep in a derivation (during which I can be grumpy apparently—who knew?).

Finally, I want to thank my partner Elie. With his infinite patience, love and wisdom, he's been an incredible support for me during this time. With him and our cat Soliş, we were able to build a home away from both of our homes. I'm grateful to all of the small coincidences of life that led us to this version.

ABSTRACT

Modern quantum systems have two facets: they are resources we would like to harness, and they contain physics we would like to probe. This thesis studies both through several case studies in quantum metrology and computing, unified by the interplay between entanglement, noise, and dissipation. We first probe entanglement in an optomechanical setting, where a mechanical oscillator is continuously driven and monitored by a light field. Assuming Gaussian dynamics and arbitrary, possibly non-Markovian noise, we derive necessary and sufficient conditions for entanglement. For a vacuum input, the entangling-disentangling transition is set by the environmental noise alone and is independent of the optomechanical coupling strength. Furthermore, frequency-dependent squeezing cannot entangle a system that is separable when driven by vacuum. In the Advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO), we find that the present level of classical noise, dominated by low-frequency seismic noise, makes significant stationary entanglement unlikely. We then turn to quantum frequency estimation and present two schemes. The first scheme harnesses entanglement as a metrological resource, approaching Heisenberg-limited precision up to a constant factor using blocks of Greenberger-Horne-Zeilinger (GHZ) states prepared with logarithmic-depth circuits and read out with local, adaptive measurements. The second scheme addresses the precision-bandwidth trade-off through a weak-measurement Ramsey protocol that recovers the classical precision limit over an extended dynamic range. Finally, we harness dissipation by introducing the 0 - n Fock qubit, a noise-biased bosonic encoding stabilized by frequency-selective dissipation, with bit-flip errors exponentially suppressed in n (the highest Fock level). Used as a bias-preserving ancilla for cat-qubit architectures, it enables noise-biased controlled-X gates and low logical error rates with experimentally realistic parameters. Together, these results explore the regimes where entanglement and quantum advantage can survive noise, and show how measurement and dissipation can be engineered to protect them.

PUBLISHED CONTENT AND CONTRIBUTIONS

- [1] Su Direkci, Manuel Endres, and Tuvia Gefen. “Extending the dynamic range in quantum frequency estimation with sequential weak measurements”. In: *Phys. Rev. Res.* 8 (2 2026), p. L022046. doi: [10.1103/2hyn-cqlq](https://doi.org/10.1103/2hyn-cqlq).
Su Direkci co-developed the theoretical framework and carried out the analytical and numerical calculations.
- [2] Su Direkci, Simon Lieu, Kyungjoo Noh, and Connor T. Hann. *Dissipatively Stabilized 0-n Fock Qubits for Noise-Biased Quantum Computing*. 2026. arXiv: [2608.25959](https://arxiv.org/abs/2608.25959) [[quant-ph](#)].
Su Direkci developed the numerical model and performed the analytical and numerical calculations.
- [3] Su Direkci, Ran Finkelstein, Manuel Endres, and Tuvia Gefen. “Heisenberg-limited Bayesian phase estimation with low-depth digital quantum circuits”. In: *npj Quantum Information* 12.1 (Jan. 2026). issn: 2056-6387. doi: [10.1038/s41534-025-01177-9](https://doi.org/10.1038/s41534-025-01177-9).
Su Direkci co-developed the theoretical framework and carried out the analytical and numerical calculations.
- [4] Su Direkci, Klemens Winkler, Corentin Gut, Markus Aspelmeyer, and Yanbei Chen. “Characterizing stationary optomechanical entanglement in the presence of non-Markovian noise”. In: *Phys. Rev. A* 112 (4 Oct. 2025), p. 043512. doi: [10.1103/gysx-m9p2](https://doi.org/10.1103/gysx-m9p2).
Su Direkci co-developed the theoretical framework and performed the analytical and numerical calculations.
- [5] Su Direkci, Klemens Winkler, Corentin Gut, Markus Aspelmeyer, and Yanbei Chen. “Universality of Stationary Entanglement in an Optomechanical System Driven by Non-Markovian Noise and Squeezed Light”. In: *Phys. Rev. Lett.* 135 (15 Oct. 2025), p. 153601. doi: [10.1103/vzhk-cdc3](https://doi.org/10.1103/vzhk-cdc3).
Su Direkci co-developed the theoretical framework and performed the analytical and numerical calculations.
- [6] S. Direkci, K. Winkler, C. Gut, K. Hammerer, M. Aspelmeyer, and Y. Chen. “Macroscopic quantum entanglement between an optomechanical cavity and a continuous field in presence of non-Markovian noise”. In: *Phys. Rev. Res.* 6 (1 Feb. 2024), p. 013175. doi: [10.1103/PhysRevResearch.6.013175](https://doi.org/10.1103/PhysRevResearch.6.013175).
Su Direkci developed the numerical model and performed the analytical and numerical calculations.

TABLE OF CONTENTS

Acknowledgements	iv
Abstract	vii
Published Content and Contributions	viii
Table of Contents	viii
List of Illustrations	xi
List of Tables	xxviii
Chapter I: Introduction	1
Chapter II: Theoretical Background	3
2.1 Quantum parameter estimation	3
2.2 Open quantum systems and non-Markovian noise	7
2.3 Gaussian states and continuous-variable entanglement	14
2.4 Encoding qubits in bosonic modes	18
2.5 Cavity optomechanics and input–output relations	18
2.6 Probe preparation versus continuous monitoring	21
Chapter III: Universality of Stationary Optomechanical Entanglement	23
3.1 Introduction	23
3.2 System Dynamics	25
3.3 Structure of the Covariance Matrix	28
3.4 Entanglement Criterion	30
3.5 Universal Entanglement	32
3.6 Generalization to arbitrary squeezing	46
3.7 Discussion for Frequency-Dependent Squeezing	53
3.8 Conclusions	55
Chapter IV: Macroscopic Quantum Entanglement in LIGO	57
4.1 Introduction	57
4.2 System Dynamics	60
4.3 Entanglement Criteria and Partitions	63
4.4 Entanglement in the Presence of Non-Markovian Noises	64
4.5 Noise Model of aLIGO	68
4.6 Entanglement in aLIGO	69
4.7 Discussion and Conclusions	73
Chapter V: Bayesian phase estimation with blocks of GHZ states	75
5.1 Introduction	75
5.2 Results	79
5.3 Phase estimation for atomic clocks	80
5.4 Methods	102
5.5 Discussion	105
Chapter VI: Extending the dynamic range in quantum frequency estimation with sequential weak measurements	107

6.1	Introduction	107
6.2	Formulation	109
6.3	Ramsey Interferometry with Weak Measurements	110
6.4	Cramér-Rao bounds	113
6.5	Saturability of the CRB and the threshold effect	117
6.6	Conclusions and outlook	129
Chapter VII: Dissipatively Stabilized 0-n Fock Qubits for Noise-Biased Quantum Computing		130
7.1	Introduction	130
7.2	Properties of the 0-n qubit	132
7.3	Proposed Experimental Implementation	137
7.4	0-n qubit as an ancilla	143
7.5	Conclusions	151
Chapter VIII: Conclusions		153
Bibliography		155
Appendix A: Supplementary Information for Chapter III		178
A.1	The Wiener-Hopf method	178
A.2	Universal entanglement in the presence of frequency-independent squeezing	180
A.3	General symplectic transformations on the light field	182
A.4	Realization of frequency-dependent squeezing	183
A.5	Passive losses	184
Appendix B: Supplementary Information for Chapter IV		186
B.1	Structure of the Covariance Matrix	186
B.2	Numerical Implementation for aLIGO's Noise	189
Appendix C: Supplementary Information for Chapter V		196
C.1	Allan deviation for various clocks	196
C.2	Optimality of the sine state	197
C.3	Overhead for three blocks with optimal measurements	199
C.4	Optimization of the adaptive measurement	201
C.5	Proof of claim I (OQI and CSS Plateaus)	202
C.6	Impact of noise on the proposed scheme	204
Appendix D: Supplementary Information for Chapter VI		211
D.1	Imperfect Weak Measurements	211
D.2	Weak Measurements with Light	215
Appendix E: Supplementary Information for Chapter VII		218
E.1	Filter optimization	218
E.2	Independence of dissipators	218
E.3	CX bit-flip probability	225
E.4	Bit-flip floor	229
E.5	Analytical solution to bit-flip probability after CX gates with decay	232
E.6	Logical error rates with a dephasing budget	235

LIST OF ILLUSTRATIONS

<i>Number</i>	<i>Page</i>
2.1 The two pictures of parameter estimation: the prepare-and-measure picture (left), where the parameter is imprinted on the probe and is measured directly, vs the continuous monitoring picture, where a meter interacts with the probe and is measured, causing back action on the probe.	22
3.1 Schematic of the optomechanical system. The mechanical mode $\hat{b}$ interacts with a light field and reaches a steady state. We refer to the modes of the input light field as $\hat{u}$, and the reflected (output) light field as $\hat{v}$, respectively. The schematic shows the equivalence between the spatial and the temporal modes of the light field: for example, the temporal modes of the input field located at the light-mechanics interface for $t > 0$ can be thought of as the spatial modes at $t = 0$ that have yet to interact with the mirror. Similarly, the temporal modes of the output field for $t < 0$ can be thought of as the spatial modes at $t = 0$ that have already reflected off the mirror. We label the duration of the temporal modes as dt in the schematic, however, we are interested in the continuum limit, where $dt \rightarrow 0$. Lastly, the mechanical mode is subjected to a force noise $\hat{n}_F$, and the output field is subjected to a sensing noise $\hat{n}_S$	26

- 3.2 Logarithmic negativity for the case where the input light field is the vacuum state, for a photodetector efficiency of $\eta = 0.9$ (plotted in yellow), and $\eta = 1$ (plotted in blue), and white environmental noise sources. We fix the force noise level with $\Omega_F/(2\pi) = 10$ Hz, and vary the sensing noise level in the range of $\Omega_S/(2\pi) \in [1, 5000]$ Hz. The parameters of the mechanical oscillator are chosen as $\omega_m/(2\pi) = 1$ Hz and $\gamma_m/(2\pi) = 0.01$ Hz. We observe that for a unity photodetector efficiency ($\eta = 1$), the entangling transition is characterized by $\Omega_F \approx \Omega_S$, as observed numerically in Ref. [65]. For a non-unity photodetector efficiency ($\eta < 1$), the entangling transition is not universal with respect to the interaction strength Ω_q . For this case, the transition is characterized by $\sqrt{\Omega_F^2 + (1 - \eta)\Omega_q^2/2} \approx \Omega_S$, drawn by the red vertical lines in the inset. 44
- 3.3 Impact of frequency-dependent squeezing on the entangling-disentangling transition. a) Amplitude spectra $\sqrt{S(\Omega)}$ normalized by the interaction strength Ω_q , as a function of different interaction strengths and squeezing levels. Spectra of the force and the sensing noises are shown in dashed lines. Although Markovian at the input, the force noise is transduced by the mechanical oscillator, causing it to decrease as Ω^2 in amplitude at the output. b) The entangling-disentangling transition with respect to the force and the sensing noises in the system, for different squeezing configurations. The optomechanical state is entangled for the noise configurations below their respective curves. When the input light is in the vacuum state (blue curve), the transition is independent of the interaction strength. We observe that as Ω_q increases, the transitions of the other configurations approach that of the vacuum case, in terms of the parameters for which they occur. 53

3.4	Logarithmic negativity for the case where the input light field is frequency-dependently squeezed with $r = 1$ (c.f. Eq. (3.89)), and for the vacuum input. We further consider the cases where we compute the entanglement between the mechanical oscillator at time $t = 0$ and the light field, where the field is considered to be i) the outgoing field that has left the oscillator during $t < 0$, ii) the joint field combining the outgoing field that has left the oscillator during $t < 0$, and the ingoing field that has yet to interact with the oscillator, but will be reflected from the oscillator during $t > 0$. These cases (for a squeezed input) are plotted in yellow and blue curves, respectively. For the vacuum input plotted in red, we only consider case i). We observe that for a squeezed input, entanglement is harder to achieve for case i), and the entangling transition is not universal with respect to the interaction strength Ω_q . However, for case ii), universality is restored.	54
4.1	Figurative representation of the two different partitions that are used while testing for entanglement, which are defined by tracing over (top row) and <i>not</i> tracing over (bottom row) the cavity. Note that the system configuration is not changed, i.e. cavity is still present for both partitions.	59
4.2	Logarithmic negativity between the mechanical oscillator and the outgoing light field in the free-mass limit as a function of Ω_X/Ω_F for various Ω_q/Ω_F . We plot the results for Markovian and non-Markovian force and sensing noise with dashed and plain lines, respectively. Note that entanglement does not exist for $\Omega_X/\Omega_F \lesssim 1$ with Markovian and for $\Omega_X/\Omega_F \lesssim 2.6$ with non-Markovian noise sources, for $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, $\phi = 0.05$, and a cutoff frequency of $\Omega_c/(2\pi) = 0.05$ Hz.	67
4.3	aLIGO noise budget obtained from pygwinc. Only the dominant classical noise sources are plotted, along with the quantum noise. The total force noise in the system is the sum of the seismic noise and the suspension thermal noise, which is effective at low frequencies. The coating Brownian thermal noise is taken as the main constituent of the sensing noise. As can be seen from the figure, quantum noise dominates over the sensing noise by a large margin at high frequencies.	68

4.4	Force (top row) and sensing noise (bottom row) spectra parametrized by α_{F_1} , α_{F_2} , α_{X_1} , and α_{X_2} . The effect of α_{F_1} is to raise and lower the nominal noise strength below the cutoff frequency and α_{F_2} shifts the cutoff frequency. Similarly, α_{X_1} shifts the cutoff frequency where the sensing noise starts increasing as Ω^2 , whereas α_{X_2} shifts the nominal noise level.	70
4.5	The effect of the force noise spectrum on the logarithmic negativity when we do not trace over the cavity. Note that the force noise levels increase toward the bottom-right of the figure and our fit of aLIGO's operation regime is for $\alpha_{F_1} = \alpha_{F_2} = 1$	71
4.6	The effect of resonant modes on the logarithmic negativity, $E_{\mathcal{N}}$, for both partitions. We see that the negativity reduces with increasing α_{F_1} or with the introduction of resonant modes. Note that $\alpha_{F_2} = 1$	73
5.1	Summary of contributions. We propose a scheme that, given the prior width $\delta\phi$ and the total number of qubits N , computes the optimal partition over blocks of GHZ states, as well as optimal local, adaptive measurements. Partitions of blocks of GHZ states can be used to approximate the optimal initial states for phase estimation. If optimal measurements are available, blocks of GHZ states approximately reach the benchmark sensitivity for all prior widths. If local, adaptive measurements are used instead, we obtain Heisenberg scaling up to a constant overhead. Furthermore, optimal partitions can be rescaled to include slow atoms (atoms that accumulate phases of $\phi/2, \phi/4, \dots$), which enable an extended dynamic range beyond $\delta\phi \gg \pi$. We show that the proposed scheme surpasses other phase unwinding protocols in the literature that extend the dynamic range.	76

5.2 Operation of an optical atomic clock in the Bayesian context. (a) The fluctuating laser frequency $\omega_L(t)$ causes the atoms to accumulate a phase ϕ . The phase is measured at the end of the interrogation cycle, and is used to estimate the laser frequency. (b) Allan deviation $\sigma_y(\tau)$, normalized by $\gamma_{\text{LO}}/\omega_A N^{1/2}$, as a function of the total interrogation time τ , for various interrogation schemes, and $N = 200$ atoms. The vertical dashed line represents the coherence time limit due to local oscillator noise [235], which determines the maximum Ramsey time T . The Allan deviation decreases as $\tau^{1/2}$ after this point for the uncorrelated and GHZ interrogation. If phase slip errors are suppressed, it is possible for the stability to scale as τ^{-1} , as seen from the stabilities of the best classical scheme and the optimal quantum clock (OQC). The fundamental limit of sensitivity is set by the individual particle decoherence (Eq. (5.6)), which is plotted with a gray, dashed line. (c) Allan deviation $\sigma_y(\tau)$, normalized by $\gamma_{\text{LO}}/\omega_A N^{1/2}$, as a function of the total number of atoms N , for $\tau = 10^{-3}\gamma_{\text{LO}}^{-1}$ (top) and $\tau = 10\gamma_{\text{LO}}^{-1}$ (bottom). Note that the legend for the plots is shown in b. For $\tau\gamma_{\text{LO}} < 1$, the stabilities of all the clocks scale as $N^{1/2}$. Protocols that show HS can increase the clock stability by an additional factor of $N^{1/2}$. For $\tau\gamma_{\text{LO}} > 1$, T takes its largest value of $T \approx \gamma_{\text{LO}}^{-1}$ for the uncorrelated interrogation, causing the stability to scale as $\tau^{1/2}$. Extending the Ramsey time with the best classical scheme enables to obtain a stability that scales as τ , and using protocols that show HS, such as the OQC, can further boost the stability by a factor of $N^{1/2}$. . 82

- 5.3 Summary of the introduced protocol for $N = 4$ qubits, and a prior width of $\delta\phi = \pi/3 \approx 1.05$ rad. (a) The possible partitions of the blocks of GHZ states for $N = 4$ qubits are given in the Table, and the MSE as a function of ϕ is plotted. For this prior width, the best-performing partition is plotted in yellow, highlighted in the Table. (b) Optimization over local, adaptive measurements. The measurement consists of successive $\hat{Z}(\Phi_i)$ operations and projective measurements. The $\hat{Z}(\Phi_i)$ operation corresponds to a rotation around the z -axis by Φ_i in the Bloch sphere picture, such that $\phi - \Phi_i$ is measured. Based on the measurement outcome, the next rotation angle is determined. During the optimization, m phases are sampled uniformly from the interval $[-6\delta\phi, 6\delta\phi]$, which are then used to compute the BMSE analytically. Gradient descent is used to tune the single-qubit rotations Φ_i in the adaptive measurement. This process is iterated until the BMSE converges to a minimum. (c) Numerical values (in units of radians) of the single-qubit rotations Φ_i at the end of the optimization. 86

- 5.4 Performance of the proposed scheme for $N = 21$ qubits. (a) RBMSE $\Delta\tilde{\phi}$ as a function of the prior width, $\delta\phi$, for $N = 21$. The performance of the phase estimation with a 21-qubit GHZ state, 21-qubit CSS, the OQI, and the proposed scheme is plotted in blue, yellow, gray, and red, respectively. The different shapes of the red markers correspond to different ways of partitioning the qubits into blocks of GHZ states. (b) Optimal partitions of 21 qubits into blocks of GHZ states for different prior widths. For a small prior width the optimal strategy is to use the GHZ state with the maximum number of atoms possible. The remaining atoms are distributed to smaller GHZ states. In contrast, for a large prior width, it is more advantageous to use GHZ states with smaller numbers of atoms and distribute them equally. (c) The amplitudes of states with n_{up} excitations, $0 \leq n_{\text{up}} \leq 21$, for the optimal initial states of the OQI and the initial states obtained with blocks of GHZ states, for different prior widths. $n_{\text{up}} = 0$ ($n_{\text{up}} = 21$) corresponds to the state $|0\rangle^{\otimes N}(|1\rangle^{\otimes N})$. We see that for $\delta\phi \approx 1$ rad, the optimal initial state of the OQI resembles a sine state. (d) MSE of the OQI, CSS, and the proposed scheme for $N = 21$ and $\delta\phi = 0.7$ rad, plotted in purple, yellow, and red, respectively. The proposed scheme achieves a metrological gain of $g = 2.29$ (3.59 dB). 88
- 5.5 RBMSE as a function of number of qubits N , normalized by the prior width, for $\delta\phi = 0.7$ rad. We see that the CSS and the OQI constitute the SQL and HL, respectively. For larger qubit numbers, the BMSE of all of the schemes converges to a constant independent of N . We also plot the π -corrected HL in the absence of phase slip errors, which touches the OQI curve at $N \approx 70$. We observe that all of the schemes that use blocks of GHZ states show sub-SQL sensitivity. Furthermore, the scheme with a varying block size and the proposed scheme obtain HS. For all N , the proposed scheme obtains the maximal sensitivity, among existing schemes. *Inset.* We plot the RBMSE of the OQI, CSS, and the proposed scheme in the range $40 \leq N \leq 100$, with gray, yellow, and red, respectively. We see that the proposed scheme shows HS with an overhead of ≈ 1.56 , shown with the light green line. 91

- 5.6 RBMSE $\Delta\tilde{\phi}$, normalized by 2π , as a function of the total qubit number N , for different prior widths. For small enough prior widths ($\delta\phi = 0.7$ rad), no slow atoms are needed. The RBMSE of the proposed scheme in this regime is plotted in gray. For larger prior widths, the number of atoms needed to extend the dynamic range depends on the *phase unwinding protocol*. We define non-adaptive and adaptive phase unwinding, also propose a more efficient phase unwinding protocol that extends the proposed scheme to large prior widths. Furthermore for the proposed scheme, in the limit of $N \gg 1$, the RBMSE for different prior widths converges to the RBMSE for $\delta\phi = 0.7$ rad. Therefore, in this limit, we can obtain a clock stability that scales as the total interrogation time τ even in the regime where $\tau\gamma_{\text{LO}} > 1$. *Inset.* MSE for the proposed phase unwinding scheme for $N = 16$ atoms and a prior width of $\delta\phi = 2.8$ rad. We see that the MSE is close to that of the OQI, and that the proposed scheme has a dynamic range of $\approx 8\pi$ 94
- 5.7 RBMSE normalized by 2π , as a function of the number of qubits N , for the optimal partition of the blocks of GHZ states, and the OQI. The OQI constitutes a benchmark for Bayesian phase estimation, and we assume that arbitrarily complex optimal measurements are available for the optimal partitions. The comparison to the OQI shows that the BMSE of the optimal partitions performs almost optimally for all prior widths. We also plot the π -corrected HL, defined as $\Delta\tilde{\phi} = \pi/N$ in the limit of $N \gg 1$ [125], with a gray line. The BMSE of the OQI (as well as the optimal partition) converges to the HL for all prior widths except $\delta\phi \approx 1.05$ rad, due to phase slip errors. 103

- 6.1 Dynamic range problem in frequency estimation: a) Circuit representation of the standard Ramsey experiment, where a strong measurement is applied after a phase accumulation time T . b) We consider a uniform prior distribution of the unknown frequency $\mathcal{P}_{\delta\omega}(\omega)$ in the interval $[0, \delta\omega]$. c) The square-root of the Bayesian mean squared error (BMSE) $\Delta\tilde{\omega}$, normalized by the prior width $\delta\omega$ and the number of qubits $N = 64$. The solid, red line corresponds to the measurement-optimized BMSE, which suffers from phase slips beyond $\delta\omega T = \pi$, as shown in the Bloch sphere picture. The dashed-blue line corresponds to the quantum Fisher information bound of $(\Delta\omega)^{-2} = I_Q = 4NT^2$. d) Circuit representation of the approach studied here: Ramsey experiment with sequential weak measurements. 108
- 6.2 Average classical Fisher information (CFI) $\bar{I}_C$, normalized by the QFI, $I_Q = 4NT^2$, for the weak-only protocol (plotted in yellow), and weak-with-strong protocol (plotted in green). We fix the measurement strength as $g = 0.1$, measurement period as $\tau = 0.1$ s, and vary the interrogation time $T \in [1, 100]$ s. In the weak back action regime, i.e. $g^2T/\tau \ll 1$, the CFI for the weak-with-strong protocol saturates I_Q . The CFI for the weak-only protocols is much smaller, yet it increases rapidly as T^3 . Eventually, in the strong back action regime, $g^2T/\tau \gg 1$, the two protocols obtain the same CFI of $4NT\tau/g^2$ 115

- 6.3 Performance of the weak-only protocol (left) and weak-with-strong protocol (right) with respect to the interrogation time T . We plot the inverse of the MSE scaled by the QFI I_Q , which quantifies the amount of information gained from the estimation. We fix the measurement strength as $g = 0.1$, and sample the unknown frequency from the uniform prior distribution in $[0, \delta\omega]$, where $\delta\omega = 5\pi$, such that the measurement period is chosen as $\tau = \pi/2\delta\omega = 0.1$ s. Finally, we plot the CFI of both protocols, I_C^w and I_C^{ws} , normalized by the QFI, as the benchmark. The CFIs are about $\omega = 5$ rad/s. The information gained from the estimation does not saturate the CFI for small SNR (i.e. small T) due to the threshold effect. As the SNR is proportional to the number of qubits N , we can saturate to the CFI faster (for smaller T) with a larger number of qubits. In order to obtain HS, this saturation needs to occur in the weak back action regime: the transition between the two regimes is marked by a green dashed line in the plot on the left. 127

- 6.4 Performance of the proposed protocol with respect to the interrogation time T and the threshold effect. The frequency ω is sampled from a uniform distribution in $[0, \delta\omega]$, which determines the measurement period as $\tau = \pi/2\delta\omega$. a) We fix $\delta\omega = 5\pi$ (i.e. $\tau = 0.1$), $T = 10$, and plot the inverse of BMSE $(\Delta\tilde{\omega})^2$ scaled by the QFI I_Q for the two weak measurement protocols. $(\Delta\tilde{\omega})^2 \cdot I_Q = 1$ signifies saturating the QFI, and $\bar{I}_C/I_Q$ is plotted with gray, plain lines. We observe a threshold effect such that the attainability of the CFI $\bar{I}_C^w, \bar{I}_C^{ws}$ requires large enough SNR. The green, dashed, vertical line corresponds to $T = \sqrt{3/2}\tau/g^2$, which is a transition point between the weak and strong back action regimes. Achieving HS requires the threshold to occur before this line (which requires $N > 20$ qubits for this case). b) We fix $N = 64$, $\delta\omega = \pi$ ($\tau = 0.5$ s) and plot the square-root of the BMSE $\Delta\tilde{\omega}$ normalized by the prior width $\delta\omega$, and scaled by the square-root of the number of qubits N . The vertical green line corresponds to $\delta\omega T = \pi$: for larger interrogation times, phase slips degrade the sensitivity of standard protocols. We compare the performance of: the OCI [167], the proposed weak-with-strong protocol, and the cascaded protocol [224]. The latter two demonstrate an extended dynamic range, however the proposed protocol outperforms the cascaded protocol and achieves HS, with an overhead of 1.18 in $\Delta\tilde{\omega}$. The measurement strength g is optimized for each T for the proposed protocol. 128
- 7.1 a) Schematic of the 0-n Fock qubit. We encode the logical states $\{|0_L\rangle, |1_L\rangle\}$ into the ground state $|0\rangle$ and the n-th excited state $|n\rangle$ of a Fock qubit. b) Smallest non-zero eigenvalue (in absolute magnitude) $\lambda_{\min}$ of the Lindbladian describing the evolution of the 0-n Fock qubit. We define $r_{\uparrow,\downarrow} := \kappa_i^{\uparrow,\downarrow}/\kappa_s \ll 1$ as the ratio of the rates between the intrinsic processes and the engineered dissipation. The numerical values are plotted with the markers, whereas the analytical fit given by Eq. (7.9) is plotted with the dashed lines. 133

- 7.2 a) Schematic of the implementation for the 0-n Fock qubit. The qubit (system, drawn in purple) is connected to a multimode filter with coupling rates g_{bs} and g_{tm} for the beam splitter and the two-mode squeezing interaction, respectively. The excitations travel along the filter and are dissipated to a bath, with a rate κ_f . b) Effective rates κ_{eff} of engineered processes between adjacent levels of a 0-n Fock qubit, with $n = 5$. Decay and gain are engineered via the beam splitter and two-mode squeezing coupling with the filter. The rates are plotted as a function of the detuning Δ , normalized with the filter bandwidth B . We observe that $\frac{\kappa_{eff}}{g^2/B} \approx 2\pi$, where g is the coupling rate for the corresponding interaction. For an implementation of the Fock qubit with the Kerr Hamiltonian, the difference in detuning between adjacent transitions corresponds to the anharmonicity η 139
- 7.3 Use of the 0-n Fock qubit as an ancilla for noise-biased CX gates on stabilized cat qubits. a) Schematic of the protocol. The ancilla is prepared in the $|+_{0n}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$ state using X pulses, then dissipatively stabilized (indicated by $\mathcal{S}_{fock}$) using the implementation in Sec. 7.3. Two CX gates are applied sequentially to two stabilized cat qubits (the stabilization is indicated by $\mathcal{S}_{cat}$) coupled dispersively to the ancilla, after which the cat qubits are restabilized and the ancilla is measured in the X basis. b) Logical bit-flip probability p_{bf} of the CX gate as a function of the stabilization time t_s , for $n = 7$, $|\alpha|^2 = 10$, and a filter with $N_f = 7$ filter modes. We assume perfect CX gates and consider two intrinsic noise configurations ($\tau_i^\downarrow = 50 \mu s$, $\tau_i^\uparrow = 5$ ms, and $\tau_i^\downarrow = 250 \mu s$, $\tau_i^\uparrow = 250$ ms) and pulse error probabilities ($p_e = 10^{-4}$, $p_e = 10^{-3}$). p_{bf} decays exponentially with stabilization time before converging to the bit-flip floor. *Inset*: dependence of p_{bf} on N_f , showing that ~ 7 modes suffice to suppress spurious engineered dissipation. c) p_{bf} and ancilla phase-flip probability p_{pf} as a function of the number of levels $N = n + 1$ in the Fock qubit. We consider the intrinsic single-photon transitions during the CX gates. The gate time is assumed as $t_g = 200$ ns, $p_e = 10^{-3}$, and $|\alpha|^2 = 10$. The dashed lines correspond to analytic predictions. Bit-flip probabilities below 10^{-8} can be obtained with $N \geq 10$ levels. 144

- 7.4 Average logical error per round p_L^* when the 0-n Fock qubit is used as an ancilla in the repetition code, paired with cat data qubits. The contours for $p_L^* \in \{10^{-4}, 10^{-5}, 10^{-6}\}$ are shown as a function of $\kappa_1 T_{\text{cycle}}$ and the code distance d , where κ_1 is the single-photon loss rate of the cat qubits, and T_{cycle} is the cycle time for a single round of error correction. We work with a phenomenological noise model, where Z errors are applied on the ancilla and the data qubits at the start of each error correction round. The ancilla Z errors occur only due to single-photon events ($\tau_i^\downarrow = 250 \mu\text{s}$, $\tau_i^\uparrow = 250 \text{ ms}$) during the stabilization ($t_s = 500 \text{ ns}$), or the CX gates ($t_g = 200 \text{ ns}$): we assume large enough E_J/E_C such that dephasing due to charge noise is sufficiently suppressed. Lastly, the cat qubits have a mean photon number of $|\alpha|^2 = 10$ 149
- B.1 $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ for $\alpha_{F_1} = 10^{-15}$ and $\alpha_{F_2} = 15$ [top row, (a)]; $\alpha_{F_1} = 10^{-8}$, $\alpha_{F_2} = 10$, and $\alpha_{X_2} = 10^3$ [bottom row, (b)]. We set $T = 0.1 \text{ s}$, and $dt \in [0.1, 10] \text{ ms}$. Note that dt decreases to the right of the plots. In Figure a, entanglement exists for both partitions, and $\lambda_{\mathcal{B}}(\lambda_{\mathcal{N}})$ is re-plotted in the inset on the left(right) for $dt \in [0.1, 0.5] \text{ ms}$. In Figure b, the state is separable for both partitions, and $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ are re-plotted in the inset for $dt \in [0.1, 0.5] \text{ ms}$ 192
- B.2 The real and imaginary parts of the eigenvector for the negative eigenvalue of $\mathbf{V}_{\text{pt}} + \frac{1}{2}\mathbf{K}$ for $\alpha_{F_1} = 10^{-15}$, $\alpha_{F_2} = 15$, denoted by e_1 and e_2 , which correspond to the first and second halves of the entire eigenvector, respectively. The reason behind this slicing is the block-matrix structure of the light-field sector of the covariance matrix. Furthermore, since the light field modes are continuous in time, e_1 and e_2 are also functions of time. 193
- B.3 Contour plot depicting the effect of the force noise spectrum on our numerical precision for both partitions. The force noise levels increase toward the bottom-right of the figure. The black and the red dashed lines separate the regions where numerics converge from the regions where numerics fail for the partition where we do and do not trace over the cavity, respectively. The region where numerics converge for both partitions is marked in gray, whereas numerics fail for both partitions toward the bottom right of the figure, past the red dashed line, in the yellow region. 194

B.4	Examples of force noise spectra along the boundary where we lose numerical precision (defined with $ \lambda_{\mathcal{N}} = 100 \lambda_{\mathcal{B}} $) for the partition where we do not trace over the cavity, plotted in black. aLIGO noise curves are also plotted for comparison. The black curves are parametrized with (from left to right) $\alpha_{F_1} = 1$ and $\alpha_{F_2} = 200$, $\alpha_{F_1} = 10^{-3}$ and $\alpha_{F_2} = 14$, $\alpha_{F_1} = 10^{-10}$ and $\alpha_{F_2} = 1$, and $\alpha_{F_1} = 10^{-15}$ and $\alpha_{F_2} = 0.029$	194
C.1	RBMSE $\Delta\tilde{\phi}$ as a function of the prior width $\delta\phi$, normalized by $\delta\phi$, for $N = 12$ and in the presence of amplitude damping with a probability of $p_a = 10^{-2}$. The benchmark is the noiseless OQI, plotted in purple. The 12-qubit CSS for $p_a = 10^{-2}$ is also shown in the plot in blue. The minimum RBMSE obtained from optimal partitioning into blocks of GHZ states—i.e., the optimal initial state (OIS) for blocks of GHZ states—for $p_a = 10^{-2}$, assuming that optimal measurements are available, is plotted in green. We notice that this curve is very close to the noiseless OQI. Finally, the RBMSE given optimal partitioning into blocks of GHZ states and local, adaptive measurements is plotted with yellow rectangles. We see that in the presence of amplitude damping, even though the optimal partitions perform very close to the OQI when optimal measurements are available, there is a large gap between the minimum achievable sensitivity and the sensitivity achieved when local, adaptive measurements are used.	206

- C.2 Impact of noise on the proposed scheme. (a) RBMSE $\Delta\tilde{\phi}$, as a function of the prior width, $\delta\phi$, for $N = 21$ and various probabilities of decay p_a . As before, the RBMSE is normalized by the prior width. We plot the RBMSE of the 21-qubit CSS and the proposed scheme (PS on the Figure) for $p_a = 10^{-3}$ and $p_a = 10^{-2}$. Our benchmark is the noiseless OQI, plotted in gray. Increase in the RBMSE due to amplitude damping is most prominent around the prior width of $\delta\phi = 0.7$ rad. (b) The metrological gain as a function of the effective fidelity per qubit F_0 , or the bit-flip measurement error probability p_e . We plot the performances of the optimal partitions for qubit numbers of $N = 9, 15, 18, 21$. The partitions are shown in the plot in the form of (x, y, z) , where x, y , and z are the number of 4-qubit, 2-qubit, and 1-qubit GHZ states, respectively. We fit lines with different slopes to the performances of the scheme for different N , signifying an exponential decay in the metrological gain with decreasing F_0 , or increasing p_e . We observe a correlation between the decay rate and the number of copies of the 4-qubit GHZ state: the partitions with $N = 21$ and $N = 18$ both have three 4-qubit GHZ states, and they have similar decay rates. 209
- D.1 Effect of imperfect weak measurements on the CFI, with respect to the measurement strength g . We plot the CFI I_C normalized by the QFI I_Q as a function of g and for different error probabilities p_e . The left plot corresponds to the weak-only protocol, and the right one to the weak-with-strong protocol. We fix the interrogation time, the measurement period, and the frequency-to-be-estimated as $T = 10$ s, $\tau = 0.1$ s, and $\omega = 5$ rad, respectively. Note that in the weak-with-strong protocol (plotted on the right) we assume that the strong projective measurement is noiseless. 212

- D.2 Effect of imperfect weak measurements on the CFI, with respect to the interrogation time T . We plot the CFI I_C normalized by the number of qubits N for the weak-only protocol and the weak-with-strong protocol with yellow and purple lines, respectively. We fix the measurement strength and period as $g = 0.1$ and $\tau = 0.1$ s. Furthermore, we fix the frequency to be estimated as $\omega = 5$ rad/s. We observe that in the weak back action limit where $g^2 T / \tau \ll 1$, the imperfect measurement effectively scales the measurement strength as $g \rightarrow g(1 - 2p_e)$, but the scaling with respect to T is approximately unaffected such that the yellow dashed and plain curves are parallel to each other. In the strong back action limit, the CFI for the protocols with imperfect measurements (plain lines) converge to the classical T scaling faster compared to the protocols with perfect measurements (dashed lines). We observed numerically that for larger p_e , the T scaling gets modified to T^α , $\alpha < 1$ 214
- E.1 Bit-flip probability p_{bf} of the CX gate on cat qubits with mean photon number $|\alpha|^2$, when the 0-n Fock qubit is used as the control qubit. a) On relatively short timescales $t \sim 1/\kappa_s$, the 0-n Fock qubit stabilizes and p_{bf} converges to a fixed value referred to as the bit-flip floor, inversely proportional to n . b) Bit-flip floor as a function of the number of levels $N = n + 1$ of the Fock qubit. The floor decays exponentially with n for small n , however the scaling is modified to $1/n^2$ for $n \gg 1$ (see Eq. (E.56)). We set $r_\downarrow = \frac{\kappa_\downarrow}{\kappa_s}$ to 10^{-3} (plotted with circles) and 10^{-4} (plotted with stars), for $|\alpha|^2 = 5$ (plotted in blue) and $|\alpha|^2 = 10$ (plotted in yellow). The analytical fit for large n is plotted with the respective colors, with plain and dashed lines for $r_\downarrow = 10^{-3}$ and $r_\downarrow = 10^{-4}$, respectively. 230

- E.2 Average logical error per round p_L^* when the 0-n Fock qubit is used as an ancilla in the repetition code, paired with cat data qubits, with a dephasing budget on the Fock qubit. The contours for p_L^* are shown as a function of $\kappa_1 T_{\text{cycle}}$ and the code distance d , where κ_1 is the single-photon loss rate of the cat qubits, and T_{cycle} is the cycle time for a single round of error correction. Z errors are applied on the ancilla and the data qubits at the start of each error correction round. We assume single-photon events on the ancilla with $\tau_i^\downarrow = 250 \text{ } \mu\text{s}$, $\tau_i^\uparrow = 250 \text{ ms}$, and dephasing such that the overall Z error probability p_Z^{ancilla} is doubled. Lastly, we assume a stabilization time $t_s = 500 \text{ ns}$, CX gate time $t_g = 200 \text{ ns}$, and cat mean photon number $|\alpha|^2 = 10$ 235

LIST OF TABLES

<i>Number</i>	<i>Page</i>
3.1 Classification of the noise sources affecting the dynamics, given in Eqs. (3.1). The noise sources fall under two orthogonal categories, demonstrated by the x and y-axes of the Table. The first category (x-axis) arises due to the nature of the noise: we refer to the fundamental noise that occurs due to the continuous monitoring of the oscillator as quantum noise. Conversely, any noise arising due to the environment (e.g. seismic noise, coating thermal noise, etc.) is referred to as environmental noise. The second category (y-axis) describes how the noise source affects the dynamics: a force noise introduces stochastic momentum kicks on the oscillator, whereas a sensing noise is sensed with the position of the oscillator and does not affect its dynamics. . .	27
B.1 Classical Noise Model Parameters	189
B.2 aLIGO Parameters	189
B.3 Resonant Mode Parameters	190

Chapter 1

INTRODUCTION

Modern quantum systems operate in a regime of two facets: first, they constitute resources that we'd like to harness, and second, they contain underlying physics that we'd like to probe. For the first facet, concepts like squeezing and entanglement allow us to build instruments that achieve unprecedented precision and computational power. Examples of these instruments are atomic clocks, gravitational-wave detectors, and ultimately, quantum computers. However, being able to fundamentally describe the mechanics of these devices requires probing the underlying physics, which constitutes the second facet. Here, we aim to probe the same quantum properties of these devices that we'd like to harness: for example, the ability to become entangled. This thesis examines some case studies related to both facets of quantum mechanics across various settings in quantum metrology and computing.

Entanglement is the most fundamental property of a system that we'd like to probe, as it decides whether the system is *quantum* or *classical*. For example, whether a massive, macroscopic object can become entangled with the field that drives it is directly related to the question of whether quantum mechanics governs macroscopic degrees of freedom. Furthermore, entanglement is a metrological resource that allows an ensemble of probes to estimate a parameter with a precision that improves with the probe number more rapidly than any classical strategy permits, which establishes the Heisenberg limit.

The quantum properties of the system are highly fragile against *noise*. Environmental noise destroys the correlations required for macroscopic entanglement, reduces quantum-enhanced sensitivity toward its classical value, and corrupts the states on which computation depends. However, related irreversible processes can be engineered to become resources: for example, *dissipation* can be exploited to manipulate the system's internal dynamics to create useful states for quantum computation. The interplay between entanglement, noise, and dissipation constitutes the unifying theme of this thesis: we probe entanglement where it serves as a signature of quantum behavior, and harness entanglement and dissipation where they serve as resources for sensing and computation. The theoretical framework common to these problems, which comprises quantum estimation theory, noisy and dissipative

dynamics, Gaussian systems, cavity optomechanics, and basic elements of quantum error correction, is reviewed in Chapter 2.

The rest of this thesis is structured as follows. In Chapters 3 and 4, we investigate the conditions under which a mechanical oscillator becomes entangled with the light field that continuously monitors and drives it. We first establish in Chapter 3 that the boundary between entangled and unentangled configurations is determined by the environmental noise alone, and is independent of the coupling strength between the oscillator and the light field. Entanglement also cannot be induced by squeezing the input field in a frequency-dependent manner when the vacuum-driven system is not entangled. Then, in Chapter 4, we take the Advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) as a case study, and develop a numerical framework that tackles the non-Markovian noise of heavy, low-frequency oscillators to investigate the existence of entanglement in these systems.

The next two chapters concern Bayesian quantum frequency estimation. Chapter 5 describes a scheme that uses entanglement as a metrological resource to approximate optimal probe states with blocks of Greenberger-Horne-Zeilinger (GHZ) states. Combined with local, adaptive measurements, this strategy allows for approaching Heisenberg-limited precision, up to a constant factor. Chapter 6 approaches this problem from a different angle and tackles the precision-bandwidth problem, without being concerned with Heisenberg-limited precision. We propose a scheme to recover the classical precision limit for an extended bandwidth using sequential weak measurements, each of which extracts partial information about the phase while disturbing the system through measurement back action.

The final chapter concerns the use of dissipation as a resource. Chapter 7 introduces the $0-n$ Fock qubit, a noise-biased bosonic encoding that is constructed with a multilevel anharmonic oscillator. We propose an experimental implementation of this qubit through dissipative stabilization realizable with a multimode filter. The resulting noise bias makes the qubit an effective ancilla for bias-preserving error correction with cat qubits, achieving low logical error rates under experimentally realistic conditions. Finally, we summarize the principal results of all of these chapters and outline directions for future work in Chapter 8.

Chapter 2

THEORETICAL BACKGROUND

2.1 Quantum parameter estimation

In this Section, we review the fundamental limits of single-parameter estimation. The parameter to-be-measured is encoded in an input state $\hat{\rho}$ that undergoes a completely positive and trace-preserving channel $\Lambda_\theta(\cdot)$, parametrized by θ . The output $\hat{\rho}_\theta = \Lambda_\theta(\hat{\rho})$ is then subjected to some positive operator-valued measure (POVM) $\{E_x\}$ in order to infer the unknown parameter θ . Let us denote the measurement results with $\{(x, p(x|\theta))\}$, where $p(x|\theta) = \text{Tr}[\hat{\rho}_\theta E_x]$.

Given the measurement results $x_1, x_2, \dots$, estimating θ is a classical problem, for which the sensitivity is given by the classical Fisher information (CFI). However, one can also optimize over the measurement strategy to obtain the ultimate precision for an input state $\hat{\rho}$, which is the quantum Fisher information (QFI). We derive these quantities in the next sections below.

Classical estimation and the Cramér–Rao bound

Let us set up the classical problem of estimating a parameter θ from measuring x . We can define a probability distribution $p(x|\theta)$ that describes how the outcome of x depends on the unknown parameter θ . We also define an estimator $\hat{\theta}(x)$ of this unknown parameter. We assume that the estimator is unbiased, with

$$E[\hat{\theta}(x) - \theta] = 0. \quad (2.1)$$

Or, equivalently, $\int \hat{\theta}(x) p(x|\theta) dx = \theta$. Differentiating this identity with respect to θ , we obtain

$$\int \hat{\theta}(x) \frac{\partial p(x|\theta)}{\partial \theta} dx = E \left[\hat{\theta}(x) \frac{\partial \ln(p(x|\theta))}{\partial \theta} \right] = 1 \quad (2.2)$$

where we used $\partial_\theta p = p \partial_\theta \ln p$. Also, notice that $\int \frac{\partial p(x|\theta)}{\partial \theta} dx = E \left[\frac{\partial \ln(p(x|\theta))}{\partial \theta} \right] = 0$. Then, we can rewrite Eq. (2.2) as

$$E \left[\frac{\partial \ln(p(x|\theta))}{\partial \theta} (\hat{\theta}(x) - \theta) \right] = 1 \quad (2.3)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$1 = E \left[\partial_\theta \ln p(x|\theta) (\hat{\theta}(x) - \theta) \right]^2 \leq E[(\hat{\theta}(x) - \theta)^2] E[(\partial_\theta \ln p(x|\theta))^2], \quad (2.4)$$

which yields the classical Cramér–Rao bound (CRB) [53, 215, 133]

$$\text{var}(\hat{\theta}) \geq \frac{1}{\nu F_C(\theta)}, \quad F_C(\theta) = E[(\partial_\theta \ln p(x|\theta))^2] = -E[\partial_\theta^2 \ln p(x|\theta)], \quad (2.5)$$

where the factor ν accounts for ν independent repetitions of the measurement, and $F_C(\theta)$ is the Fisher information. The bound is saturated asymptotically by the maximum-likelihood estimator.

Quantum Fisher information and the quantum Cramér–Rao bound

In the quantum case the parameter is encoded in a state $\hat{\rho}_\theta$, and a measurement is a POVM $\{E_x\}$ with $p(x|\theta) = \text{Tr}[\hat{\rho}_\theta E_x]$. Maximizing the Fisher information (2.5) over all POVMs results in the quantum Fisher information (QFI). We can derive this quantity by first defining the symmetric logarithmic derivative (SLD) $\hat{L}_\theta$ through

$$\partial_\theta \hat{\rho}_\theta = \frac{1}{2}(\hat{L}_\theta \hat{\rho}_\theta + \hat{\rho}_\theta \hat{L}_\theta). \quad (2.6)$$

Then, similar to the classical case, let us define a Hermitian observable $\hat{X}$, with $\text{Tr}[\hat{X}\hat{\rho}_\theta] = \theta$. We can write $\partial_\theta \text{Tr}[\hat{X}\hat{\rho}_\theta] = 1$, and $\text{Tr}[\partial_\theta \hat{\rho}_\theta] = 0$, which result in

$$\text{Tr}[(\hat{X} - \theta \hat{I})\partial_\theta \hat{\rho}_\theta] = 1. \quad (2.7)$$

Using the definition of the SLD, we write

$$\frac{1}{2}\text{Tr}[(\hat{X} - \theta \hat{I})(\hat{L}_\theta \hat{\rho}_\theta + \hat{\rho}_\theta \hat{L}_\theta)] = 1 \quad (2.8)$$

where $\hat{I}$ is the identity operator. Since $\hat{\rho}_\theta$ is Hermitian and positive semi-definite, we can find a square root $\sqrt{\hat{\rho}_\theta}$, and rewrite this equation as

$$\text{Re}[\text{Tr}[\sqrt{\hat{\rho}_\theta}(\hat{X} - \theta \hat{I})\hat{L}_\theta\sqrt{\hat{\rho}_\theta}]] = 1. \quad (2.9)$$

Then, we can again apply the Cauchy-Schwarz inequality to find

$$1 \leq |\text{Tr}[\sqrt{\hat{\rho}_\theta}(\hat{X} - \theta \hat{I})\hat{L}_\theta\sqrt{\hat{\rho}_\theta}]|^2 \leq \text{Tr}[(\hat{X} - \theta \hat{I})^2 \hat{\rho}_\theta] \text{Tr}[\hat{L}_\theta^2 \hat{\rho}_\theta] \quad (2.10)$$

Then, the quantum Cramér–Rao bound (QCRB) reads [33]

$$\text{var}(\hat{X}) \geq \frac{1}{\nu F_C(\theta)} \geq \frac{1}{\nu F_Q(\hat{\rho}_\theta)}, \quad (2.11)$$

with $F_Q(\hat{\rho}_\theta) = \text{Tr}[\hat{L}_\theta^2 \hat{\rho}_\theta]$, $\text{var}(\hat{X}) := \text{Tr}[\hat{X}^2 \hat{\rho}_\theta] - \theta^2$, and ν is the number of independent repetitions of the experiment. The optimal measurement strategy is thus to

project the state $\hat{\rho}_\theta$ into the eigenbasis of $\hat{L}_\theta$. However, this POVM is often hard to implement experimentally.

For a pure state subject to unitary encoding $|\psi_\theta\rangle = e^{-i\theta\hat{G}}|\psi_0\rangle$ with a Hermitian generator $\hat{G}$, the QFI simplifies as follows. First, for $\hat{\rho}_\theta = |\psi_\theta\rangle\langle\psi_\theta|$, notice that $\partial_\theta\hat{\rho}_\theta = -i[\hat{G}, \hat{\rho}_\theta]$. Then, since $\hat{\rho}_\theta^2 = \hat{\rho}_\theta$, the SLD is $\hat{L}_\theta = -2i[\hat{G}, \hat{\rho}_\theta]$, and the QFI is found as

$$F_Q = 4 \left(\langle \hat{G}^2 \rangle - \langle \hat{G} \rangle^2 \right) = 4 \text{var}(\hat{G}), \quad (2.12)$$

with the expectation values taken with respect to the initial state, i.e. $\langle \hat{X} \rangle := \langle \psi | \hat{X} | \psi \rangle$. Given this observation, it is straightforward to maximize the QFI with respect to the input state $|\psi\rangle$: The result is given by

$$F_Q^{\max} = (\lambda_{\max} - \lambda_{\min})^2, \quad (2.13)$$

with $\lambda_{\max, \min}$ being the largest and smallest eigenvalues of the generator $\hat{G}$, respectively. Thus, the ultimate sensitivity is bounded by the spread of the spectra of the generator.

Equation (2.12) displays quantum advantage in sensing: consider N probes with collective generator $\hat{G} = \sum_{i=1}^N \hat{g}_i$. If the initial state is a product state, $\text{var}(\hat{G}) = N \text{var}(\hat{g})$, and we obtain the standard-quantum-limit scaling $\Delta\theta \geq 1/\sqrt{N}$, where $\Delta\theta$ is the standard deviation in estimating θ . In contrast, if the initial state is maximally entangled (e.g. a Greenberger-Horne-Zeilinger state), the generator eigenvalues can span a range $\propto N$, so that $\text{var}(\hat{G}) \propto N^2$ and the Heisenberg scaling $\Delta\theta \geq 1/(\sqrt{N})$ is reached [84]. Saturating these bounds in the presence of noise is generally not possible, and a range of tighter limits has been developed [69, 61, 145, 90].

For a mixed input state $\hat{\rho}$, the QFI can be found through the spectral decomposition $\hat{\rho} = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$:

$$F_Q(\hat{\rho}_\theta) := 2 \sum_{i,j} \frac{|\langle \phi_i | \partial_\theta \hat{\rho}_\theta | \phi_j \rangle|^2}{\lambda_i + \lambda_j} \quad (2.14)$$

where the sum excludes any pair λ_i, λ_j such that $\lambda_i + \lambda_j = 0$. Similarly, the SLD $\hat{L}_\theta$ is found as

$$\hat{L}_\theta := 2 \sum_{i,j} \frac{\langle \phi_i | \partial_\theta \hat{\rho}_\theta | \phi_j \rangle}{\lambda_i + \lambda_j} |\phi_i\rangle\langle\phi_j|. \quad (2.15)$$

Finally, if the parameter encoding is through the Hermitian operator $\hat{G}$ for the mixed input state case, the QFI takes a special form

$$F_Q(\hat{\rho}_\theta) = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle \phi_k | \hat{G} | \phi_l \rangle|^2. \quad (2.16)$$

Again, the sum excludes $\lambda_k + \lambda_l = 0$. The QFI has a convexity property, where

$$F_Q(p\hat{\rho}_1 + (1-p)\hat{\rho}_2) \leq pF_Q(\hat{\rho}_1) + (1-p)F_Q(\hat{\rho}_2). \quad (2.17)$$

The convexity implies that the QFI is maximized by a pure input state, i.e.

$$F_Q\left(\sum_i \lambda_i \Lambda_\theta(|\phi_i\rangle\langle\phi_i|)\right) \leq \sum_i \lambda_i F_Q(\Lambda_\theta(|\phi_i\rangle\langle\phi_i|)) \quad (2.18)$$

Bayesian estimation

The CRB is used in the context of frequentist estimation, where the unknown parameter is assumed as fixed. Another approach is the Bayesian estimation, where the unknown parameter θ is assumed to be distributed according to $P(\theta)$. Since the CRB is saturable only asymptotically, or in the limit of many independent experiments, the Bayesian treatment is useful when few measurements are available, or when prior knowledge is significant. In this context, assuming a parameter encoding $\hat{\rho}_\theta = \Lambda_\theta(\hat{\rho})$, the figure of merit is the Bayesian mean-squared error (BMSE) [167]

$$(\Delta\tilde{\theta})^2 = \int d\theta \int dx P(\theta) \text{Tr}(\hat{\rho}_\theta \Pi_x) (\hat{\theta}_x - \theta)^2, \quad (2.19)$$

where $\hat{\theta}_x$ is the estimator of θ given an outcome x , and Π_x is the corresponding measurement operator, which can be restricted to be a projective measurement in the form of $\Pi_x = |x\rangle\langle x|$. The optimization over these operators results in

$$(\Delta\tilde{\theta})^2 = (\delta\theta)^2 - \text{Tr}[\bar{\rho}\hat{L}^2], \quad \frac{1}{2}\{\hat{L}, \bar{\rho}\} = \bar{\rho}', \quad (2.20)$$

where $\hat{L}$ is the Bayesian SLD, $\bar{\rho} := \int d\theta P(\theta) \hat{\rho}_\theta$, $\bar{\rho}' := \int d\theta \theta P(\theta) \hat{\rho}_\theta$, and $(\delta\theta)^2$ is the variance of the prior distribution $P(\theta)$. $(\Delta\tilde{\theta})^2$ quantifies the minimum achievable variance in estimating θ with the input probe state $\hat{\rho}$. This equation signifies that the decrease in the variance after a measurement is $\text{Tr}[\bar{\rho}\hat{L}^2]$. In the case where the measurement does not provide any information, the posterior variance $(\Delta\tilde{\theta})^2$ equals to the prior variance $(\delta\theta)^2$.

Similar to the frequentist case, it is valuable to find the minimum possible BMSE, which would result in the ultimate limit of sensitivity for the Bayesian case. This

optimization is in general not convex, however, efficient algorithms have been suggested. We outline the algorithm of Ref. [167], where the optimization is performed iteratively. For a given initial state $\hat{\rho}$, one first finds the optimal $\hat{L}$ (i.e. the measurement strategy) from $\{L, \bar{\rho}\} = 2\bar{\rho}'$. For any $\hat{L}$, the optimal probe state $\hat{\rho}$ is the eigenstate of $\Lambda_{\theta}^{\dagger}(\int d\theta P(\theta)(L^2 - 2\theta L))$ with the maximal eigenvalue. Therefore, starting from a random initial state, BMSE can be achieved by iteratively calculating (i) the optimal $\hat{L}$ given $\hat{\rho}$, (ii) the optimal $\hat{\rho}$ given $\hat{L}$. The iteration stops after the desired precision and convergence is achieved. It can be proved that this algorithm converges to the global maximum when, for each iteration, $\hat{\rho}$ and $\hat{L}$ are unique.

Lastly, a Bayesian analogue of (2.5), the van Trees inequality, bounds the BMSE $(\Delta\tilde{\theta})^2$ below by $1/(\bar{F}_C + F_{\text{prior}})$, where $F_{\text{prior}} = \int d\theta P(\theta)[\partial_{\theta} \ln P(\theta)]^2$ is the prior information, and $\bar{F}_C(\theta) = \int d\theta P(\theta)F_C(\theta)$ is the CFI averaged over the prior distribution [259, 116, 167, 77].

2.2 Open quantum systems and non-Markovian noise

System–bath dynamics and the quantum Langevin equation

A ubiquitous number of physical scenarios can be modeled as a system mode coupled to a heat bath, which is an assembly of harmonic oscillators. The total Hamiltonian is in the form of $\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{\text{int}}$, where $\hat{H}_S$ is the system Hamiltonian, $\hat{H}_B = \sum_n \frac{\hat{p}_n^2}{2m_n} + \frac{k_n \hat{q}_n^2}{2}$ is the bath Hamiltonian (with $\hat{q}_n, \hat{p}_n$ being the position and momentum operators of the n^{th} mode), and $\hat{H}_{\text{int}}$ is the interaction Hamiltonian. The system mode is coupled linearly to the bath. The bath modes can be eliminated by means of solving the equations of motion in terms of the system mode variables and substitution.

For the case where the system mode is also a harmonic oscillator of mass m and frequency ω_m , we obtain the generalized quantum Langevin equation

$$m\ddot{x}(t) + m \int_{-\infty}^t \gamma(t-t') \dot{x}(t') dt' + m\omega_m^2 \hat{x}(t) = \hat{F}_{\text{th}}(t) + F_{\text{ext}}(t), \quad (2.21)$$

where $\gamma(t)$ is a memory function, $\hat{F}_{\text{th}}$ is the thermal force arising from the coupling to the bath, and F_{ext} is any other external force acting on the system. For example, this may be the force causing the test masses in gravitational wave detectors to displace. The Markovian limit corresponds to a memoryless kernel $\gamma(t) = 2\gamma \delta(t)$, where $\delta(t)$ is the Dirac delta function. In contrast, any other memory function with non-zero temporal correlations for $t > 0$ that satisfies $\lim_{t \rightarrow \infty} \gamma(t) = 0$ describes non-Markovian dynamics with finite memory [34, 267, 76].

Linear response theory

Let us perturb a system in equilibrium with Hamiltonian $\hat{H}_0$ with a classical, time-dependent force $f(t)$, and examine its first-order response to this perturbation. For this purpose, let us define interaction picture operators $\hat{A}_{\text{int}}(t) := e^{i\hat{H}_0 t/\hbar} \hat{A}(t) e^{-i\hat{H}_0 t/\hbar}$. Then, the Hamiltonian in this picture can be written as

$$\hat{H}_{\text{int}}(t) = -f(t)\hat{x}_{\text{int}}(t) \quad (2.22)$$

where $\hat{x}_{\text{int}}$ is the position operator of the system. The density matrix in this picture evolves as

$$i\hbar \frac{d\hat{\rho}_{\text{int}}}{dt} = [\hat{H}_{\text{int}}(t), \hat{\rho}_{\text{int}}(t)] = -f(t)[\hat{x}_{\text{int}}(t), \hat{\rho}_{\text{int}}(t)] \quad (2.23)$$

The solution of this evolution is given by

$$\begin{aligned} \hat{\rho}_{\text{int}}(t) &= \hat{\rho}(0) - \frac{1}{i\hbar} \int_{-\infty}^t d\tau f(\tau) [\hat{x}_{\text{int}}(\tau), \hat{\rho}_{\text{int}}(\tau)] \\ &\approx \hat{\rho}(0) - \frac{1}{i\hbar} \int_{-\infty}^t d\tau f(\tau) [\hat{x}_{\text{int}}(\tau), \hat{\rho}(0)]. \end{aligned} \quad (2.24)$$

This approximation holds up to first order in $f(t)$ and is used assuming that the perturbation is weak. Let us use it to find the expectation value of the position of the system as a function of time. This value is frame independent, and is defined as $\langle \hat{x} \rangle(t) := \text{Tr}(\hat{x}_{\text{int}}(t)\hat{\rho}_{\text{int}}(t))$. Let us also define the expectation value of any operator $\hat{A}(t)$ with respect to the initial state $\hat{\rho}(0)$ as $\langle \hat{A}(t) \rangle_0 := \text{Tr}(\hat{A}(t)\hat{\rho}(0))$. Using Eq. (2.24), we find:

$$\langle \hat{x} \rangle(t) = \text{Tr}(\hat{x}_{\text{int}}(t)\hat{\rho}(0)) - \frac{1}{i\hbar} \int_{-\infty}^t d\tau f(\tau) \text{Tr}(\hat{x}_{\text{int}}(t) [\hat{x}_{\text{int}}(\tau), \hat{\rho}(0)]) \quad (2.25)$$

The first term is the expectation value of this operator when no perturbation is applied: since the system starts from equilibrium, $\hat{\rho}(0)$ commutes with $\hat{H}_0$, and $\text{Tr}(\hat{x}_{\text{int}}(t)\hat{\rho}(0)) = \text{Tr}(e^{i\hat{H}_0 t/\hbar} \hat{x} e^{-i\hat{H}_0 t/\hbar} \hat{\rho}(0)) = \text{Tr}(\hat{x}\hat{\rho}(0))$. We refer to this term as $\langle \hat{x} \rangle_0$. In contrast, the second term arises from the response of the system to the perturbation. Using the cyclic property of the trace, we can simplify the trace inside the integral as $\text{Tr}(\hat{x}_{\text{int}}(t) [\hat{x}_{\text{int}}(\tau), \hat{\rho}(0)]) = \text{Tr}(\hat{\rho}(0) [\hat{x}_{\text{int}}(t), \hat{x}_{\text{int}}(\tau)]) = \langle [\hat{x}_{\text{int}}(t), \hat{x}_{\text{int}}(\tau)] \rangle_0$. Lastly, we define $\delta \langle \hat{x}(t) \rangle = \langle \hat{x}(t) \rangle - \langle \hat{x} \rangle_0$ to capture the change in the expectation value of this operator due to the perturbation. From Eq. (2.25), we have

$$\delta \langle \hat{x}(t) \rangle = \frac{i}{\hbar} \int_{-\infty}^t d\tau f(\tau) \langle [\hat{x}_{\text{int}}(t), \hat{x}_{\text{int}}(\tau)] \rangle_0 \quad (2.26)$$

Then, we can see that the *response function* to the perturbation is

$$\chi(t) = \frac{i}{\hbar} \theta(t) \langle [\hat{x}_{\text{int}}(t), \hat{x}] \rangle_0 \quad (2.27)$$

where $\theta(t)$ is the Heaviside theta, and the response can be written in the form of

$$\delta \langle \hat{x}(t) \rangle = \int_{-\infty}^{\infty} d\tau f(\tau) \chi(t - \tau). \quad (2.28)$$

Fluctuation–dissipation theorem and noise spectra

The fluctuation-dissipation theorem (FDT) [40, 150] relates the thermal noise experienced by the system in thermal equilibrium to the temperature of the bath, as well as the dissipation experienced by the system. We provide a proof for this theorem below.

We first define a symmetrized power spectral density (PSD) of operators $\hat{a}$, $\hat{b}$ as $\bar{S}_{ab}(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle \{\hat{a}(t), \hat{b}(0)\} \rangle$. We also define the Fourier transform of a function $f(t)$ as $\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$. First, from taking the Fourier transform of Eq. (2.27),

$$\text{Im} [\tilde{\chi}(\omega)] = \frac{1}{2\hbar} \int_{-\infty}^{+\infty} \langle \hat{x}_{\text{int}}(t) \hat{x}(0) - \hat{x}(0) \hat{x}_{\text{int}}(t) \rangle_0 e^{i\omega t} dt. \quad (2.29)$$

To find the expectation value of the commutator, let us define the functions

$$\tilde{C}^>(\omega) = \int dt e^{i\omega t} \langle \hat{x}_{\text{int}}(t) \hat{x}(0) \rangle_0, \quad \tilde{C}^<(\omega) = \int dt e^{i\omega t} \langle \hat{x}(0) \hat{x}_{\text{int}}(t) \rangle_0 \quad (2.30)$$

Since the system starts out in thermal equilibrium, the initial state will be given by $\hat{\rho}(0) = e^{-\beta \hat{H}_0} / Z$, where $Z = \text{Tr}(e^{-\beta \hat{H}_0})$ is the partition function, and $\beta = (k_B T)^{-1}$. Plugging this into the expectation value, we first find that $\langle \hat{x}(0) \hat{x}_{\text{int}}(t) \rangle_0 = \frac{1}{Z} \text{Tr}(e^{-\beta \hat{H}_0} \hat{x}(0) \hat{x}_{\text{int}}(t))$. Plugging in the definition of $\hat{x}_{\text{int}}(t)$ and using the cyclicity of the trace, this expression simplifies into $\frac{1}{Z} \text{Tr}(e^{-\beta \hat{H}_0} e^{i\hat{H}_0(t-i\hbar\beta)/\hbar} \hat{x}(0) e^{-i\hat{H}_0(t-i\hbar\beta)/\hbar} \hat{x}(0)) = \langle \hat{x}_{\text{int}}(t-i\hbar\beta) \hat{x}(0) \rangle_0$, where we grouped the β -dependent term into an imaginary time evolution. Plugging this expression into Eq. (2.30) and integrating over t , we find

$$\tilde{C}^<(\omega) = e^{-\beta \hbar \omega} \tilde{C}^>(\omega) \quad (2.31)$$

Thus, the imaginary part of the Fourier transform of the response function can be expressed as

$$\text{Im} [\tilde{\chi}(\omega)] = \frac{1}{2\hbar} (\tilde{C}^>(\omega) - \tilde{C}^<(\omega)) = \frac{\tilde{C}^>(\omega)}{2\hbar} (1 - e^{-\beta \hbar \omega}) \quad (2.32)$$

Similarly, the symmetrized PSD of the position operator can be written as

$$\begin{aligned}\bar{S}_{xx}(\omega) &= \frac{1}{2} (\tilde{C}^>(\omega) + \tilde{C}^<(\omega)) \\ &= \hbar \coth\left(\frac{\beta\hbar\omega}{2}\right) \text{Im} [\tilde{\chi}(\omega)]\end{aligned}\quad (2.33)$$

This is the statement of the FDT: equilibrium fluctuations and dissipation are linked by the above equation.

As an example, we can consider the harmonic oscillator coupled to a thermal bath, described by the quantum Langevin equation in Eq. (2.21). The response function of this system can be obtained from the response to an external force, in the form of

$$\tilde{x}(\omega) = \tilde{\chi}(\omega) \tilde{F}_{\text{ext}}(\omega). \quad (2.34)$$

For the case of Eq. (2.21), we find the response function (also referred to as the mechanical susceptibility) as

$$\tilde{\chi}(\omega) = \frac{1}{m(\omega_m^2 - \omega^2) - i\omega m \tilde{\gamma}(\omega)}. \quad (2.35)$$

Then, Eq. (2.33) gives

$$\bar{S}_{xx}(\omega) = \frac{\hbar\omega \text{Re}[\tilde{\gamma}(\omega)]}{m[(\omega_m^2 - \omega^2 + \omega \text{Im}[\tilde{\gamma}(\omega)])^2 + \omega^2 \text{Re}[\tilde{\gamma}(\omega)]^2]} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \quad (2.36)$$

The frequency dependence of the damping $\gamma(\omega)$ encodes the noise model. Viscous (Markovian) damping has $\tilde{\gamma}(\omega) = \gamma_m$ constant, giving a white force-noise spectrum (in the limit of $k_B T \gg \hbar\omega$). By contrast, structural damping—dominant in the suspensions and coatings of gravitational-wave-scale oscillators—is described by a constant loss angle ϕ , $\tilde{\chi}^{-1}(\omega) = m[\omega_m^2(1 + i\phi) - \omega^2]$, producing a force noise that scales as $1/\omega$ and is intrinsically non-Markovian [227, 87, 108].

Standard quantum limit and the Heisenberg limit

The FDT is intimately tied to the sensitivity achievable in the quantum regime. When a probe is coupled to a system, the same coupling that lets the probe extract information also subjects the system to the probe's fluctuations and imposes dissipation. The FDT states that the fluctuations and the dissipation are related closely, and one cannot be eliminated without the other. Hence, the sensitivity in probing a system is limited on a fundamental level. To see this better, consider the following: let us probe a system that is a harmonic oscillator with mass m and resonance frequency ω_m , as in the previous subsections. We refer to the output observable with

units of position as $\hat{o}(t)$. The fluctuations in $\hat{o}$ will result from (i) the fluctuations in the system itself, and (ii) measurement imprecision due to the probe. For example, in gravitational wave interferometers, the second effect arises due to the shot noise (or statistical uncertainty) of the photons. We can formally write this as

$$\hat{o}(t) = \hat{o}_{\text{imp}}(t) + \hat{x}(t) \quad (2.37)$$

where $\hat{o}_{\text{imp}}(t)$ is the added measurement imprecision, and $\hat{x}(t)$ is the position of the sensing device. Furthermore, we have derived from the linear response theory that the fluctuations in $\hat{x}$ will depend on the spectra of the external force probing the system, in the form of $\tilde{\hat{x}}(\omega) = \tilde{\chi}(\omega)\tilde{\hat{f}}(\omega)$. We can then write the total spectrum of the output variable as

$$\begin{aligned} \bar{S}_{oo} &= \bar{S}_{xx}(\omega) + \bar{S}_{o_{\text{imp}}o_{\text{imp}}}(\omega) + 2\text{Re}[\bar{S}_{o_{\text{imp}}x}(\omega)] \\ &= |\tilde{\chi}(\omega)|^2 \bar{S}_{ff}(\omega) + \bar{S}_{o_{\text{imp}}o_{\text{imp}}}(\omega) + 2\text{Re}[\tilde{\chi}(\omega)\bar{S}_{o_{\text{imp}}f}(\omega)] \end{aligned} \quad (2.38)$$

We refer to $\bar{S}_{ff}(\omega)$ as the back action force spectrum. The imprecision noise and the back action, as they arise from the same measurement, satisfy the Heisenberg uncertainty principle. This principle can be stated as

$$\bar{S}_{o_{\text{imp}}o_{\text{imp}}}(\omega)\bar{S}_{ff}(\omega) - |\bar{S}_{o_{\text{imp}}f}(\omega)|^2 \geq \frac{\hbar^2}{4} \quad (2.39)$$

If these noise sources are uncorrelated, $|\bar{S}_{o_{\text{imp}}f}(\omega)|^2 = 0$. Furthermore, for an ideal detector that saturates the bound, $\bar{S}_{o_{\text{imp}}o_{\text{imp}}}(\omega)\bar{S}_{ff}(\omega) = \hbar^2/4$. Then, the total spectrum of the output variable is $\bar{S}_{oo} = |\tilde{\chi}(\omega)|^2 \bar{S}_{ff}(\omega) + \frac{\hbar^2}{4\bar{S}_{ff}(\omega)}$, which is minimized for $\bar{S}_{ff}(\omega) = \hbar/(2|\tilde{\chi}(\omega)|)$. Then, the minimal spectra at the output (ignoring thermal fluctuations, i.e. setting $\gamma_m = 0$) is found as

$$\begin{aligned} \bar{S}_{oo} &= \hbar|\tilde{\chi}(\omega)| \\ &= \frac{\hbar}{m|\omega_m^2 - \omega^2|} \end{aligned} \quad (2.40)$$

This results in the standard quantum limit (SQL) $S_{\text{SQL}}^x = \hbar/(m\omega^2)$, in the limit $\omega \gg \omega_m$. One can get around the SQL by noticing that the measurement imprecision and the back action can be correlated, i.e. by having $|\bar{S}_{o_{\text{imp}}f}(\omega)|^2 \neq 0$ in Eq. (2.38) and (2.39). In particular, we notice that $\bar{S}_{oo}$ can be decreased by tuning the phases of the response function $\tilde{\chi}(\omega)$ and the cross spectra $\bar{S}_{o_{\text{imp}}f}(\omega)$. Doing so, one can achieve unlimited precision for a given frequency in theory—however, experimentally, one is usually constrained by the amount of power in the optical

cavities, i.e. the number of photons. After the constrained optimization, we recover the Heisenberg limit (HL).

In an undetuned Michelson interferometer like LIGO, the measurement process can be understood more clearly from the input-output relations [47]

$$\tilde{v}_1(\omega) = e^{2i\beta} \tilde{u}_1(\omega) \quad (2.41a)$$

$$\tilde{v}_2(\omega) = e^{2i\beta} (\tilde{u}_2(\omega) - \mathcal{K} \tilde{u}_1(\omega)) + e^{i\beta} \left| \frac{\mathcal{K}}{S_{\text{SQL}}^x} \right|^{1/2} x_G(\omega) \quad (2.41b)$$

where $\tilde{v}_{1,2}$ are the quadratures of the outgoing light field at the dark port (corresponding to the antisymmetric mode), $\tilde{u}_{1,2}$ are the quadratures of the ingoing vacuum, and $x_G(\omega)$ is the displacement due to the signal, which we try to estimate. The other parameters are dependent on the physical parameters of the interferometer, with

$$e^{2i\beta} = \frac{\omega - i\gamma}{\omega + i\gamma}, \quad \mathcal{K} = \frac{4G^2\gamma}{m(\omega^2 - \omega_m^2)(\omega^2 + \gamma^2)} \quad (2.42)$$

γ is the linewidth of the effective cavity corresponding to the antisymmetric mode, and $G = \sqrt{2\omega_0 I_c / (Lc)}$ where I_c is the circulating power in the common mode, ω_0 is the optical pumping frequency, L is the length of the interferometer arms, and c is the speed of light. Then, the output observable sensed can be defined as $\tilde{o}(\omega) := \left| \frac{\mathcal{K}}{S_{\text{SQL}}^x} \right|^{-1/2} \tilde{v}_2(\omega)$, in units of displacement. As the ingoing light field is the vacuum, $\bar{S}_{u_1 u_1} = \bar{S}_{u_2 u_2} = 1/2$, $\bar{S}_{u_1 u_2} = 0$, and $\bar{S}_{oo} = \frac{S_{\text{SQL}}^x}{2} (\mathcal{K} + 1/\mathcal{K}) \geq S_{\text{SQL}}^x$. Thus, the sensitivity is limited by the SQL. However, if the ingoing light field is squeezed in a frequency-dependent way, a sensitivity of $\bar{S}_{oo} = \frac{S_{\text{SQL}}^x}{2} (\mathcal{K} + 1/\mathcal{K}) e^{-2R}$ can be achieved. Here, R is the squeezing factor, and we describe in Appendix A.4 a strategy to achieve this type of squeezing.

Markovian master equations and decoherence

When the correlation time with the bath is short compared with all system timescales, the reduced dynamics of the system are Markovian, and are governed by the following Lindblad master equation

$$\dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \hat{\rho} \} \right), \quad (2.43)$$

with jump operators $\hat{L}_k$. Through these operators, the system–environment coupling drives decoherence and the emergence of classical behaviour [230, 282, 127], which is what makes the persistence of entanglement in a macroscopic, noisy system a nontrivial question.

Engineered dissipation and dissipative stabilization

The Lindblad master equation describes the interaction of the system with its environment. This interaction can be unintentional, resulting in decoherence, or *engineered*, resulting in the relaxation of the input state to a target state. The latter is referred to as dissipative stabilization. Writing (2.43) with the dissipator $\mathcal{D}[\hat{L}]\hat{\rho} := \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}$,

$$\dot{\hat{\rho}} = \mathcal{L}(\hat{\rho}) = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_k \kappa_k \mathcal{D}[\hat{L}_k]\hat{\rho}. \quad (2.44)$$

The Liouvillian $\mathcal{L}$ generates a CPTP semigroup, so its eigenvalues obey $\text{Re } \lambda_j \leq 0$. Every eigenoperator of the Liouvillian with a non-zero eigenvalue is traceless. This can straightforwardly be shown from taking the trace of Eq. (2.44). Using the cyclicity of the trace, we obtain

$$\text{Tr}[\mathcal{L}(\hat{X})] = \lambda \text{Tr}[\hat{X}] = 0 \quad (2.45)$$

Then, $\text{Tr}[\hat{X}] = 0$ if $\lambda \neq 0$. Let us assume that the Liouvillian is diagonalizable. Then, the time evolution of any density matrix $\hat{\rho}(t)$ can be decomposed into the eigenbasis of the Liouvillian as $\hat{\rho}(t) = \sum c_j e^{\lambda_j t} \hat{X}_j$, where $\{\lambda_j\}$ are the eigenvalues, and $\{\hat{X}_j\}$ are the corresponding eigenoperators. In the limit of $t \rightarrow \infty$, the density matrix admits a steady state [34] $\mathcal{L}(\hat{\rho}_{\text{ss}}) = 0$ in the linear subspace of the eigenoperators $\hat{X}_j$ for which $\lambda_j = 0$. Reservoir engineering chooses $\hat{H}, \{\hat{L}_k\}$ so that $\hat{\rho}_{\text{ss}}$ is the desired (ideally unique) state. A sufficient construction is a *dark state*: if

$$\hat{L}_k |\psi\rangle = 0 \quad \forall k, \quad \hat{H} |\psi\rangle = E |\psi\rangle, \quad (2.46)$$

then $|\psi\rangle\langle\psi|$ is an eigenoperator of $\mathcal{L}$ with a zero eigenvalue. A shared dark subspace of the $\hat{L}_k$ is then a protected code space.

A prominent example of dissipative stabilization is the two-photon dissipation, where cat qubits [49, 155] are stabilized in the $\{|\pm\alpha\rangle\}$ manifold [181, 152],

$$\hat{L} = \sqrt{\kappa_2} (\hat{a}^2 - \alpha^2), \quad \hat{a}^2 |\pm\alpha\rangle = \alpha^2 |\pm\alpha\rangle. \quad (2.47)$$

Here, $|\alpha\rangle$ is the coherent state with a mean photon number of $|\alpha|^2$. Relaxation toward $\hat{\rho}_{\text{ss}}$ occurs at the *Liouvillian gap* $\Delta = \min_{\lambda_j \neq 0} |\text{Re } \lambda_j|$: fast modes decay at the engineered rate $\sim \kappa_2$, while residual processes competing with the engineered dissipation set a small gap governing slow leakage. Such stabilization has been realized in trapped ions and superconducting circuits [209, 186, 136, 83].

2.3 Gaussian states and continuous-variable entanglement

Commutation relations and symplectic operators

In this section, we present a short review of Gaussian states and Gaussian entanglement. Let us assume an N -mode system with canonical quadrature operators $\hat{x}_i, \hat{p}_i$, $i = 1, 2, \dots, N$, where $\hat{x}_i$ ($\hat{p}_i$) is the position (momentum) operator of the i^{th} mode. Furthermore, $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, where δ_{ij} is the Kronecker delta (we set $\hbar = 1$). The commutation relations can be summarized in the form of

$$[\hat{\mathbf{r}}, \hat{\mathbf{r}}^T] = i\Omega, \quad \Omega := \bigoplus_{i=1}^N \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \quad (2.48)$$

where we defined $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N)^T$. Ω is referred to as the symplectic form. It is a $2N \times 2N$ antisymmetric matrix, with $\Omega^2 = -\mathbb{I}_{2N}$, where $\mathbb{I}_{2N}$ is the identity matrix. We also define the symplectic matrix S , which is a real valued matrix that preserves the symplectic form when acting by congruence, i.e.

$$S\Omega S^T = \Omega. \quad (2.49)$$

The group of matrices S is referred to as the symplectic group, $\text{Sp}(2N, \mathbb{R})$.

Moments of a Gaussian state

The first and second order moments of a Gaussian state, $\bar{\mathbf{r}}$, and σ , can be written as

$$\bar{\mathbf{r}} := \langle \hat{\mathbf{r}} \rangle, \quad \sigma := \langle \{ \hat{\mathbf{r}}, \hat{\mathbf{r}}^T \} \rangle - 2\langle \hat{\mathbf{r}} \rangle \langle \hat{\mathbf{r}}^T \rangle \quad (2.50)$$

The Robertson-Schrödinger uncertainty relation dictates that for σ to be the covariance matrix of a physical quantum state, it needs to satisfy $\sigma + i\Omega \geq 0$ (where $A \geq 0$ denotes that A is a positive semi-definite matrix). For any covariance matrix σ , Williamson's theorem states that there is a decomposition

$$\sigma = S \left(\bigoplus_{j=1}^N \nu_j \mathbb{I}_2 \right) S^T, \quad (2.51)$$

where ν_j , $j = 1, 2, \dots, N$ are the symplectic eigenvalues of σ , S is a symplectic matrix, and $\mathbb{I}_2$ is the 2×2 identity matrix. From the uncertainty principle, we have that $\nu_j \geq 1 \forall j$.

Gaussian Hamiltonians

A generic Gaussian Hamiltonian will be of quadratic order in terms of the system quadratures $\hat{\mathbf{r}}$. Therefore, it can be written in the form of

$$\hat{H} = \frac{1}{2} \hat{\mathbf{r}}^T H \hat{\mathbf{r}} + \hat{\mathbf{r}}^T \mathbf{d} \quad (2.52)$$

where H is the Hamiltonian matrix. Note that here, H is a symmetric $2N \times 2N$ matrix. The action of $\hat{H}$ on the quadratures $\hat{\mathbf{r}}$ can be found from the Heisenberg equations of motion, with [239]

$$\dot{\hat{\mathbf{r}}} = \Omega(H\hat{\mathbf{r}} + \mathbf{d}) \quad (2.53)$$

For a strictly quadratic Hamiltonian, $\mathbf{d} = \mathbf{0}$, and this unitary action transforms the statistical moments of the state as $\bar{\mathbf{r}} \rightarrow S\bar{\mathbf{r}}$, and $\sigma \rightarrow S\sigma S^T$, where $S = e^{\Omega H t}$ is a symplectic transformation, as it preserves the symplectic form.

In general, a completely positive, deterministic Gaussian map can be described through its effect on the statistical moments as

$$\bar{\mathbf{r}} \rightarrow X\bar{\mathbf{r}} \quad (2.54a)$$

$$\sigma \rightarrow X\sigma X^T + Y \quad (2.54b)$$

with the constraint that $Y + i\Omega - iX\Omega X^T \geq 0$, with real matrices X and Y . Here, Y represents additive noise that makes sure that the final state satisfies the uncertainty relations. If $Y = 0$, the inequality $i\Omega - iX\Omega X^T \geq 0$ forces X to be a symplectic matrix, which reduces to the dynamics under a quadratic Hamiltonian described above. Such a map entails coupling to an environment (or an ancillary system) through a quadratic Gaussian Hamiltonian, which is traced out at the end. It does not describe any measurement operation on the system itself. For example, $X = 0$, $Y \geq 0$ represents classical mixing. Any Gaussian state can be written as classical mixing applied to a pure Gaussian state [239].

Some examples of unitary Gaussian symplectic transformations are single-mode phase shifts $R(\phi)$, single-mode squeezing $S(r)$, and two-mode squeezing $S_2(r)$. These operations can be implemented with the following unitary transformations respectively: $\hat{R}(\phi) = \exp(i\phi \hat{a}^\dagger \hat{a})$, $\hat{S}(r) = \exp(\frac{ir}{2}((\hat{a}^\dagger)^2 + \hat{a}^2))$, and $\hat{S}_2(r) = \exp(r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b}))$, when implemented on modes $\hat{a}$ and $\hat{b}$. These are the annihilation operators that act in the Fock basis $\{|m\rangle\}$ as $\hat{a}|m\rangle = \sqrt{m}|m-1\rangle$, $\hat{a}^\dagger|m\rangle = \sqrt{m+1}|m+1\rangle$. The corresponding symplectic matrices that act on the first and

second moments of the state are

$$R(\phi) := \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}, \quad (2.55a)$$

$$S(r) := \begin{bmatrix} \cosh(r) & \sinh(r) \\ \sinh(r) & \cosh(r) \end{bmatrix}, \quad (2.55b)$$

$$S_2(r) := \begin{bmatrix} \cosh(r) \mathbb{I}_2 & \sinh(r) \sigma_z \\ \sinh(r) \sigma_z & \cosh(r) \mathbb{I}_2 \end{bmatrix} \quad (2.55c)$$

with $\sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ as the Pauli Z matrix.

Gaussian states

Until this part, we have discussed the properties of the moments of the Gaussian states. Let us now connect these moments to the density matrix of the state. First, any Gaussian state can be defined by a Hermitian operator $\hat{H}$ as defined in Eq. (2.52) and $\beta \in \mathbb{R}^+$ as

$$\hat{\rho}_G = \frac{e^{-\beta \hat{H}}}{\text{Tr}[e^{-\beta \hat{H}}]}. \quad (2.56)$$

The limit of $\beta \rightarrow \infty$ describes pure states. Given the Hamiltonian in Eq. (2.52), let us write this explicitly. First, up to an irrelevant constant, we can write this operator as $\hat{H} = \frac{1}{2}(\hat{\mathbf{r}} + \mathbf{d}')^T H(\hat{\mathbf{r}} + \mathbf{d}')$, with $\mathbf{d}' := \hat{H}^{-1} \mathbf{d}$. Then, we define the unitary displacement operator $\hat{D}(\mathbf{d}) = e^{i\mathbf{d}^T \Omega \hat{\mathbf{r}}}$, which has the following action on the vector of first moments: $\hat{D}(\mathbf{d}) \hat{\mathbf{r}} \hat{D}^\dagger(\mathbf{d}) = \hat{\mathbf{r}} + \mathbf{d}$. Then, given this unitary, we can rewrite the Hamiltonian as $\hat{H} = \hat{D}(\mathbf{d}')(\frac{1}{2} \hat{\mathbf{r}}^T H \hat{\mathbf{r}}) \hat{D}^\dagger(\mathbf{d}')$. Next, we can use Williamson's theorem to decompose the Hamiltonian matrix as $H = S^T (\oplus_{i=1}^N \omega_i \mathbb{I}_2) S$, where $\mathbb{I}_2$ is the 2x2 identity matrix, and S is a symplectic matrix. Furthermore, for every symplectic S , there exists a unitary transformation $\hat{S}$, such that $S \hat{\mathbf{r}} = \hat{S} \hat{\mathbf{r}} \hat{S}^\dagger$ [239]. Using this property, we can write the Hamiltonian in terms of its normal modes, in the form of

$$\begin{aligned} \hat{H} &= \hat{D}(\mathbf{d}') \hat{\mathbf{r}}^T S^T \left(\oplus_{i=1}^N \frac{\omega_i}{2} \mathbb{I}_2 \right) S \hat{\mathbf{r}} \hat{D}^\dagger(\mathbf{d}') \\ &= \hat{D}(\mathbf{d}') \hat{S} \left(\sum_{i=1}^N \frac{\omega_i}{2} (\hat{x}_i^2 + \hat{p}_i^2) \right) \hat{S}^\dagger \hat{D}^\dagger(\mathbf{d}') \end{aligned} \quad (2.57)$$

Let us refer to the normal mode decomposition Hamiltonian as $\hat{H}_N = \sum_{i=1}^N \frac{\omega_i}{2} (\hat{x}_i^2 + \hat{p}_i^2)$. Due to the unitarity of the operators $\hat{S}$, $\hat{D}(\mathbf{d}')$, plugging this expression into

Eq. (2.56) results in

$$\hat{\rho}_G = \hat{D}(\mathbf{d}') \hat{S} \frac{e^{-\beta \hat{H}_N}}{\text{Tr}[e^{-\beta \hat{H}_N}]} \hat{S}^\dagger \hat{D}^\dagger(\mathbf{d}'). \quad (2.58)$$

This expression can be understood as the following: any N -mode Gaussian state can be obtained from decoupled, free harmonic oscillators, after the application of a symplectic operation, and displacing all of the modes. Finally, let us express the density matrix of the free harmonic oscillators in the Fock basis:

$$\frac{e^{-\beta \hat{H}_N}}{\text{Tr}[e^{-\beta \hat{H}_N}]} = \left[\prod_{i=1}^N (1 - e^{-\beta \omega_i}) \right] \left(\bigotimes_{i=1}^N \sum_{m=0}^{\infty} e^{-\beta \omega_i m} |m\rangle_i \langle m|_i \right) \quad (2.59)$$

Entanglement properties

In order to quantify the amount of entanglement in an N -mode system, we define the logarithmic negativity E_N , which is an entanglement monotone that takes a simple form for the Gaussian case [261, 208]:

$$E_N = \sum_j \max\{0, -\log \tilde{\nu}_j\}. \quad (2.60)$$

Here $\tilde{\nu}_j$, $j = 1, 2, \dots, N$ are the symplectic eigenvalues of the partially transposed covariance matrix, $\tilde{\sigma}$, and $E_N > 0$ indicates entanglement. For a bipartite system, partial transposition with respect to one of the modes amounts to performing the transformation $\hat{p}_i \rightarrow -\hat{p}_i$ on the corresponding mode. If $\tilde{\sigma} + i\Omega \geq 0$, the positivity of the partial transpose (PPT) criterion states that the state is separable [202, 243], which is a necessary and sufficient condition for this case.

For bipartite states of two-modes, the covariance matrix can be decomposed in the form of $\sigma = S \Lambda S^T$, with a local symplectic matrix $S = S_1 \oplus S_2$ [67]:

$$\sigma = \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} a & & c_+ \\ & a & c_- \\ c_+ & & b \\ & c_- & b \end{bmatrix} \begin{bmatrix} S_1^T & \\ & S_2^T \end{bmatrix} \quad (2.61)$$

where $a, b \geq 1$, and we can choose $c_+ \geq c_-$ without loss of generality. Here, Λ is referred to as the standard form. Any state whose standard form fulfills $a = b$ is said to be symmetric. Furthermore, for a pure, symmetric, two-mode squeezed state, $a = b = \cosh(2r)$, and $c_+ = -c_- = \sinh(2r)$, $r \geq 0$.

The symplectic eigenvalues of the partial transpose of the covariance matrix given in Eq. (2.61) can be written as

$$\tilde{\nu}_{\pm}^2 = \frac{\tilde{\Delta} \pm \sqrt{\tilde{\Delta}^2 - 4\text{Det}(\sigma)}}{2} \quad (2.62)$$

where $\tilde{\Delta} := a^2 + b^2 - 2c_+c_-$, and $\text{Det}(\sigma) = (ab - c_+^2)(ab - c_-^2)$. Then, the logarithmic negativity is found as $E_N = \max\{0, -\log \tilde{\nu}_-\}$. For the pure, symmetric, two-mode squeezed state with $a = \cosh(2r)$, the symplectic eigenvalues of the partially transposed matrix are calculated as $\tilde{\nu}_{\pm} = e^{\pm 2r}$ [4]. Therefore, the logarithmic negativity is $E_N = 2r \geq 0$. For a review on Gaussian continuous variables, see Refs. [239, 5, 72].

2.4 Encoding qubits in bosonic modes

The Gaussian states of Section 2.3 are efficiently simulable classically and cannot support quantum error correction; fault-tolerant bosonic computation requires *non-Gaussian* encodings. A bosonic code stores one logical qubit in a single oscillator. The logical states of the qubit, $|0_L\rangle$ and $|1_L\rangle$ are encoded in the infinite dimensional space $\text{span}\{|n\rangle\}_{n=0}^{\infty}$, with $\langle 0_L | 1_L \rangle = 0$. This redundancy may provide an advantage for error correction: some of the prominent examples are cat codes [49, 181, 155], Kerr-cat codes [211, 98], binomial codes [178], and Gottesman-Kitaev-Preskill (GKP) codes [93]. In these codes, the dominant error is usually the single-photon loss, $\hat{a}$, since they are implemented optically. Combined with the engineered dissipation of Sec. 2.2, cat codes exhibit an error asymmetry: bit flips suppressed exponentially in the mean photon number $|\alpha|^2$, whereas phase flips are enhanced only linearly. This provides a hardware-level *noise bias* best exploited by the bias-tailored codes.

An error set $\{\hat{E}_a\}$ is correctable if and only if it satisfies the Knill–Laflamme conditions [142]

$$\langle i_L | \hat{E}_a^\dagger \hat{E}_b | j_L \rangle = c_{ab} \delta_{ij}, \quad (2.63)$$

with c_{ab} independent of i, j : no correctable error distinguishes the logical states, and distinct errors map to orthogonal syndrome spaces.

2.5 Cavity optomechanics and input–output relations

The optomechanical Hamiltonian and its linearization

Cavity optomechanics describes the interaction between a light field in an optical cavity, and a mechanical mode whose motion creates a moving boundary for the

light field. Modern interferometric gravitational-wave detectors such as LIGO can be understood within this framework. Specifically, in LIGO, the mechanical coordinate of the differential motion of the Fabry-Perot arm cavities produced by the motion of the mechanical test masses under the effect of gravitational waves is sensed by the circulating laser field. The radiation pressure due to the photons bouncing off the mechanical mode, hence transferring momentum, creates phase modulation on the output light, as evidenced by Eq. (2.41). Let us present a brief review of cavity optomechanics.

We define a driven optical cavity mode $\hat{a}$ with resonance frequency ω_c , coupled to a mechanical mode $\hat{b}$ with frequency ω_m . In the frame rotating at the drive frequency ω_L , the optomechanical Hamiltonian can be written as

$$\hat{H} = -\hbar\Delta \hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger), \quad (2.64)$$

where $\Delta = \omega_L - \omega_c$ is the detuning, and g_0 the vacuum optomechanical coupling strength [11, 81, 47]. One can linearize this Hamiltonian by considering the cavity field as $\hat{a} = \alpha + \delta\hat{a}$, where $\alpha := \langle \hat{a} \rangle$ is the average coherent amplitude of the field, and $\delta\hat{a}$ are the small fluctuations around this value. Then, $|\alpha|^2 = n_c$ is the intracavity photon number. The linearization results in

$$\hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger) \approx (|\alpha|^2 + \alpha^* \delta\hat{a} + \alpha \delta\hat{a}^\dagger) (\hat{b} + \hat{b}^\dagger). \quad (2.65)$$

The constant term (proportional to $|\alpha|^2$) shifts the mechanical equilibrium, and we neglected the term quadratic in $\delta\hat{a}$ as $|\delta\hat{a}| \ll |\alpha|$. Without loss of generality, we can choose α to be real valued, which gives an interaction term $-\hbar g (\delta\hat{a} + \delta\hat{a}^\dagger) (\hat{b} + \hat{b}^\dagger)$, where $g = g_0 |\alpha| = g_0 \sqrt{n_c}$ is the optomechanical coupling strength.

The interaction term gives rise to two separate dynamics,

$$-\hbar g (\delta\hat{a} + \delta\hat{a}^\dagger) (\hat{b} + \hat{b}^\dagger) = \underbrace{-\hbar g (\delta\hat{a} \hat{b}^\dagger + \delta\hat{a}^\dagger \hat{b})}_{\text{beam splitter}} \underbrace{-\hbar g (\delta\hat{a} \hat{b} + \delta\hat{a}^\dagger \hat{b}^\dagger)}_{\text{two-mode squeezing}}, \quad (2.66)$$

The beam-splitter interaction is resonant for a red-detuned drive ($\Delta = -\omega_m$), whereas the two-mode-squeezing interaction is resonant for a blue-detuned drive ($\Delta = \omega_m$). In contrast, in the resonant case of $\Delta = 0$, the radiation pressure is independent of the mechanical motion up to first order, which is ideal for position measurements as the mechanical dynamics are not affected by the optical field.

Langevin equations and input–output relations

The optomechanical cavity will inevitably be connected to some external field: this external field is the drive, which keeps the field circulating inside the cavity at a steady level. Let us refer to the fluctuations in this external field as $\hat{a}_{\text{in}}$, and the coupling rate as κ_{ext} . Similarly, the mechanical object is coupled to some thermal bath at rate γ_m , and will be driven by external fluctuations $\hat{b}_{\text{in}}$. The linearized quantum Langevin equations read

$$\delta\dot{\hat{a}} = \left(i\Delta - \frac{\kappa}{2}\right)\delta\hat{a} + ig(\hat{b} + \hat{b}^\dagger) - \sqrt{\kappa_{\text{ext}}} \hat{a}_{\text{in}}, \quad (2.67)$$

$$\dot{\hat{b}} = \left(-i\omega_m - \frac{\gamma_m}{2}\right)\hat{b} + ig(\delta\hat{a} + \delta\hat{a}^\dagger) - \sqrt{\gamma_m} \hat{b}_{\text{in}}, \quad (2.68)$$

The intracavity field is connected to the propagating output field by the input–output relation [76, 269]

$$\hat{a}_{\text{out}}(t) = \hat{a}_{\text{in}}(t) + \sqrt{\kappa} \delta\hat{a}(t). \quad (2.69)$$

The linearized equations can be solved in the frequency domain by taking a Fourier transform. Note that this field obeys the continuum commutation relations

$$[\hat{a}_{\text{out}}(t), \hat{a}_{\text{out}}^\dagger(t')] = \delta(t - t'). \quad (2.70)$$

A canonical description is recovered by projecting the field onto a normalised temporal envelope $f(t)$, with $\int dt |f(t)|^2 = 1$, defining the *temporal mode*

$$\hat{A}_f := \int dt f^*(t) \hat{a}_{\text{out}}(t). \quad (2.71)$$

From Eq. (2.70), $[\hat{A}_f, \hat{A}_f^\dagger] = \int dt |f(t)|^2 = 1$, so $\hat{A}_f$ is a canonical bosonic mode. Choosing an orthonormal family $\{f_k(t)\}$ with $\int dt f_j^*(t) f_k(t) = \delta_{jk}$ decomposes the entire output field into a countable set of independent modes $\hat{A}_k$ obeying $[\hat{A}_j, \hat{A}_k^\dagger] = \delta_{jk}$.

Continuous measurement

Homodyne detection of the output field (2.69) realizes a continuous measurement. With balanced homodyne detection, we can measure a given quadrature of a temporal mode. This is done by interfering the output field on a balanced beam splitter with a strong coherent local oscillator (LO) of phase ϕ and normalised envelope $L(t)$, $\int dt |L(t)|^2 = 1$. Reading out the photocurrent $\hat{i}(t)$ that is the difference of the beam splitter outputs results in

$$\int dt \hat{i}(t) \propto \hat{X}_\phi^{(L)}, \quad \hat{X}_\phi^{(L)} = \frac{\hat{A}_L e^{-i\phi} + \hat{A}_L^\dagger e^{i\phi}}{\sqrt{2}}, \quad (2.72)$$

where $\hat{A}_L = \int dt L^*(t) \hat{a}_{\text{out}}(t)$ is the temporal mode selected by the LO. The quadrature at the readout can be modified by changing the LO phase.

2.6 Probe preparation versus continuous monitoring

The estimation theory of Section 2.1 (prepare-and-measure picture) and the continuous measurement picture of Section 2.5 describe two operationally distinct ways of learning an unknown parameter θ . The essential difference is what the experimenter is free to prepare.

In the first, *prepare-and-measure* picture, one prepares the probe. The parameter is imprinted on a freshly prepared input state $\hat{\rho}$ through the encoding channel Λ_θ , and the figure of merit, i.e. the QFI of Eq. (2.12), is maximized over that input state. The sensitivity gain obtainable from preparing an entangled state, in the absence of any losses during encoding, arises from the spread of the generator spectrum. As the probe is destructively measured at the end of the interaction, one aims to maximize the sensitivity for each interrogation by preparing a new state.

In contrast, in the second, *continuous-monitoring* picture, the probe (the system that interacts with the parameter) is not directly accessible. We define a meter, an auxiliary system that interacts with the probe to measure the imprinted parameter. While doing so, it inevitably disturbs the probe system, a phenomenon that we had referred to as back action in the previous sections. Then, the meter also drives the probe through this effect: once the meter is measured at the output, the conditional state of the probe is altered. In LIGO, the interaction coupling the meter and the probe is the radiation pressure. In this picture, there arises a trade-off for the interaction strength between the meter and the probe: a larger interaction strength results in more information extracted, however also a larger disturbance that makes it harder to extract the same information in the later rounds. This trade-off brings about the SQL. Obtaining an enhanced sensitivity thus requires back action evasion through meter preparation, as we cannot initialize a probe state directly.

These settings are not opposed but mark the two endpoints relevant to this thesis: the metrology chapters work in the prepare-and-measure setting, where the resource is an engineered probe state, while the optomechanics chapters work in the continuous setting, where the resource is a sustained interaction with a system one can monitor but not prepare. However, the common point of these settings is that the sensitivity is maximized for some fixed resource. In the first picture, this resource is the number of probes N , where the sensitivity grows as N for an optimal preparation. Whereas

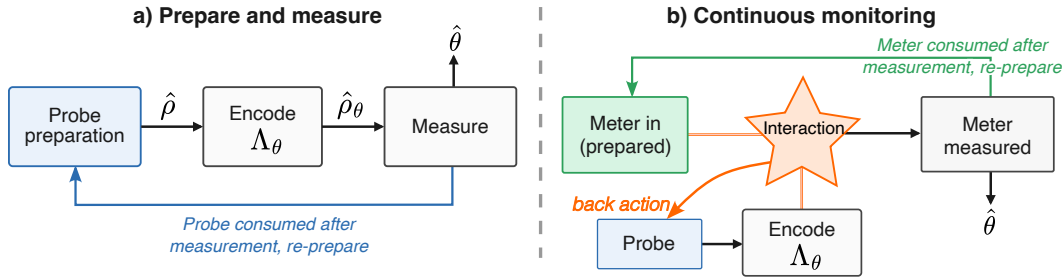


Figure 2.1: The two pictures of parameter estimation: the prepare-and-measure picture (left), where the parameter is imprinted on the probe and is measured directly, vs the continuous monitoring picture, where a meter interacts with the probe and is measured, causing back action on the probe.

in the second case, if one wants to maximize the sensitivity for a given frequency, the limited resources are the interaction strength and the energy of the meter state. The interaction strength determines the baseline of sensitivity for a classical input meter state (i.e. the vacuum state). For example for LIGO, the interaction strength is limited by the optical power at the bright port. Then, quantum advantage is obtained by using the limited meter energy to prepare a back action evading state: for LIGO, this is a frequency-dependently squeezed state, where the energy of the meter is set by the maximum squeezing level R (in the absence of optical losses).

Both pictures admit a further resource: *control*, the freedom to interleave the encoding with arbitrary operations on the probe or meter and on ancillary systems. With such control the relevant bound is no longer the state QFI maximized over inputs but the channel QFI maximized over the full adaptive strategy [61]. For a noiseless, unitary encoding, the two coincide and one cannot benefit from an increased sensitivity while doing local estimation (i.e. when the relevant metric is the QFI). However, for global estimation, where the relevant metric is the BMSE, control can enhance the precision even for the noiseless case. Perhaps the most well-known example of this is the dynamic range enhancement in atomic clocks [134].

In the presence of noise, ancilla-assisted error correction can restore Heisenberg scaling whenever the generator lies outside the span of the noise operators of Eq. (2.43), and cannot otherwise [281, 237, 60]. In the continuous picture the same control extends to the interaction itself: ancillary modes can synthesize an effective quantum-non-demolition or back-action-evading coupling [256] that breaks the imprecision-back-action trade off (2.39) without squeezing the meter. However, if the meter is lossy, it limits the sensitivity enhancement obtainable from control.

Chapter 3

UNIVERSALITY OF STATIONARY OPTOMECHANICAL ENTANGLEMENT

3.1 Introduction

The transition from quantum to classical behavior has been an intriguing question in the past decades¹. Quantum systems interact with the environment and develop correlations with environmental degrees of freedom, which in turn prohibit observing the quantum nature of the system when viewed by an independent external observer, a phenomenon that is referred to as *decoherence* [282, 229]. Above some rate of decoherence, it is expected that a system starts displaying classical behavior [7].

Optomechanical systems are examples of quantum systems, where a mechanical oscillator interacts with an optical field in the simplest case. Due to the high quality factor of the mechanical oscillator, these devices ensure strong isolation from the environment and maintain quantum coherence [11]. Furthermore, the momentum exchange between the mechanical and electromagnetic degrees of freedom, a phenomenon that is referred to as radiation pressure [81, 11], gives rise to the entanglement between these two components. Optomechanical entanglement has applications in quantum sensing [166], quantum control [114, 86, 45], testing fundamental physics [177, 274, 263], and new quantum technologies [17, 240]. Stationary entanglement, which arises when the object is measured continuously by a continuum of light modes coupled to the mechanical object, has been studied theoretically and proposed for experimental demonstration [81, 175, 177, 103, 65, 201, 262].

Optomechanical systems suffer from decoherence due to environmental noise sources, which can destroy or diminish the amount of generated optomechanical entanglement [175, 2, 65]. Examples of such noises are Brownian thermal noise [117, 148], suspension thermal noise [105, 107], and other technical noises. In this chapter, we classify environmental noise sources based on their effect on the mechanical

¹This chapter is adapted from: S. Direkci et al., "Universality of Stationary Entanglement in an Optomechanical System Driven by Non-Markovian Noise and Squeezed Light," *Phys. Rev. Lett.* **135**, 153601 (2025), DOI: 10.1103/vzhk-cdc3; and S. Direkci et al., "Characterizing stationary optomechanical entanglement in the presence of non-Markovian noise," *Phys. Rev. A* **112**, 043512 (2025), DOI: 10.1103/gysx-m9p2. Both articles are published by the American Physical Society under a Creative Commons Attribution 4.0 International (CC BY 4.0) license.

oscillator: we refer to noise sources that get transduced by the mechanical oscillator before being read out by the probe (i.e. the light field) as force noise, and noise sources affecting the probe only as sensing noise.

In [65], the behavior of optomechanical entanglement was studied assuming a linearized interaction and Gaussian dynamics between an oscillator and the vacuum fluctuations in the input light field [11]. Given this system, it has been observed numerically that optomechanical entanglement vanishes above a certain threshold concerning the environmental noise sources only, and cannot be recovered by increasing the interaction strength between the oscillator and the input light field. In other words, the entangling-disentangling transition is *universal* with respect to the interaction strength. Furthermore, this phenomenon has been observed for environmental noises that are not necessarily Markovian, due to the presence of, for example, structural damping [227].

Given this observation, in this chapter, we answer a general question about optomechanical entanglement: what determines the presence of stationary light-mass entanglement? Is it (i) the quality of isolation from the environment, (ii) the optomechanical interaction strength, and/or (iii) the state of the incoming light field? These factors are intimately tied to the sensitivity of the optomechanical system to weak forces acting on the mass: (i) determines the *environmental noise* of the device, while (ii) and (iii) determine its *quantum noise*, i.e. the unavoidable, fundamental measurement noise of the light measuring the position of the mass. In fact, our work has been directly motivated by the achievement of quantum noise below the *standard quantum limit* [58, 47] by the Laser Interferometer Gravitational-Wave Observatory [52], via the injection of optical squeezed states [278, 172]. Our short answer is that factor (i) plays a decisive role, while (ii) and (iii), as important as they are for sensing weak forces, cannot be used to bring optomechanical entanglement into existence. However, once (i) allows for entanglement, (ii) and (iii) can be used to tune the level of entanglement.

While answering these questions, we analytically prove the universality of optomechanical entanglement in the presence of general, not necessarily Markovian, environmental noise sources, when the input field is the vacuum state (or is frequency-independently squeezed). Furthermore, we show that if the system is separable when the input field is the vacuum state, entanglement cannot be achieved with arbitrary frequency-dependent squeezing in the input light field. The universality of the optomechanical entanglement can be interpreted as a quantum-to-classical transition,

where the system appears to be "classical" due to the environment, i.e. it cannot get entangled with the light field since it is strongly correlated with the environment. Calculating the corresponding threshold for environmental noise sources is crucial for upcoming quantum technologies with optomechanical devices, and observing quantum behavior in the macroscopic regime.

This chapter is organized as follows. We state the system dynamics, the covariance matrix, and the entanglement criterion in Sec. 3.2. Then, we demonstrate the universality of the entangling-disentangling transition for arbitrary environmental noise sources in Sec. 3.5, assuming that the input light field is in the vacuum state. We relax this assumption and consider arbitrary frequency-dependent squeezing in the input light field in Sec. 3.6.

3.2 System Dynamics

The system consists of a mechanical oscillator driven by a coherent, input optical field. For a schematic, see Fig. 3.1. We work in a suitably displaced frame where the first moments of the dynamical variables are zero. We denote the position fluctuations of the center of mass of the mechanical oscillator with $\hat{b}_1$, its associated momentum fluctuations with $\hat{b}_2$, the quantum fluctuations in the amplitude (phase) quadratures of the input (output) optical fields with $\hat{u}_1(t)$, $\hat{u}_2(t)$ ($\hat{v}_1(t)$, $\hat{v}_2(t)$), respectively. The dimensionless, linearized equations of motion in the Heisenberg picture are [183]

$$\hat{v}_1(t) = \hat{u}_1(t), \quad (3.1a)$$

$$\hat{v}_2(t) = \hat{u}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}}(\hat{b}_1(t) + \hat{n}_S(t)), \quad (3.1b)$$

$$\dot{\hat{b}}_2(t) = -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q}{\sqrt{\omega_m}} \hat{u}_1(t) + \hat{n}_F(t), \quad (3.1c)$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t), \quad (3.1d)$$

where Ω_q is the characteristic light–mechanics interaction strength, or the coherent coupling², ω_m is the mechanical resonance frequency, γ_m is the damping rate of the oscillator, and $\hat{n}_F(t)$ and $\hat{n}_S(t)$ are the force and sensing noises, respectively. These linearized equations are valid when the input light is highly excited (e.g. for a high

²In terms of system parameters, $\Omega_q = \sqrt{\frac{I_{\text{probe}} \omega_{\text{probe}}}{mc^2}}$, where I_{probe} and ω_{probe} are the intensity and carrier frequency of the light probing the mechanics, c is the speed of light in vacuum, and m is the mass of the oscillator.

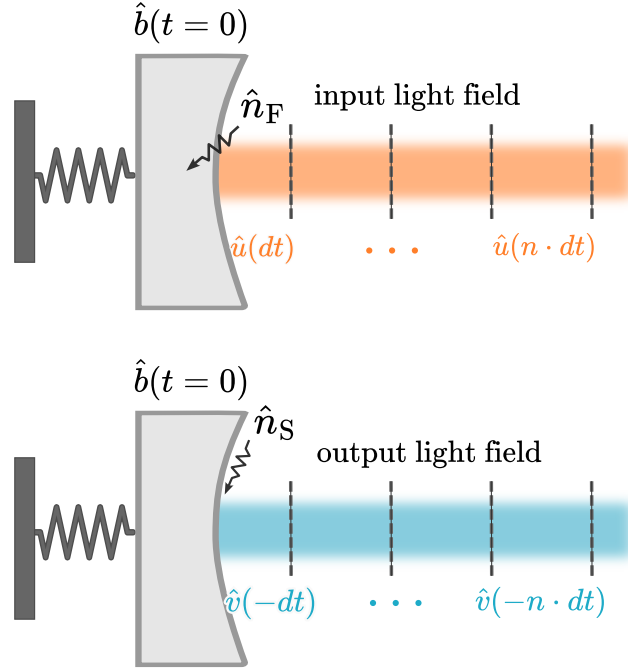


Figure 3.1: Schematic of the optomechanical system. The mechanical mode $\hat{b}$ interacts with a light field and reaches a steady state. We refer to the modes of the input light field as $\hat{u}$, and the reflected (output) light field as $\hat{v}$, respectively. The schematic shows the equivalence between the spatial and the temporal modes of the light field: for example, the temporal modes of the input field located at the light-mechanics interface for $t > 0$ can be thought of as the spatial modes at $t = 0$ that have yet to interact with the mirror. Similarly, the temporal modes of the output field for $t < 0$ can be thought of as the spatial modes at $t = 0$ that have already reflected off the mirror. We label the duration of the temporal modes as dt in the schematic, however, we are interested in the continuum limit, where $dt \rightarrow 0$. Lastly, the mechanical mode is subjected to a force noise $\hat{n}_F$, and the output field is subjected to a sensing noise $\hat{n}_S$.

intensity laser $|\alpha\rangle$, with $\alpha \gg 1$), and the commutation relation for the same-time quadrature operators are $[\hat{b}_1(t), \hat{b}_2(t)] = 2i^3$.

Instead of considering a thermal bath, where the spectrum of $\hat{n}_F$ or $\hat{n}_S$ is given by the fluctuation-dissipation theorem [150, 40], we consider phenomenologically general, non-Markovian, and Gaussian stationary noises with a power spectral density in the form of

$$S_{n_E}(\Omega) := \int_{-\infty}^{\infty} dt \langle \hat{n}_E(t) \hat{n}_E(0) \rangle e^{i\Omega t} = P(\Omega)/Q(\Omega) \quad (3.2)$$

³We choose to normalize the quadratures of the mechanical oscillator such that the entries of the commutator matrix, defined in Eq. (3.10), do not have a factor of 1/2.

	Force noise	Sensing noise
Quantum (measurement) noise	$\hat{u}_1$	$\hat{u}_2$
Environmental noise	$\hat{n}_F$	$\hat{n}_S$

Table 3.1: Classification of the noise sources affecting the dynamics, given in Eqs. (3.1). The noise sources fall under two orthogonal categories, demonstrated by the x and y-axes of the Table. The first category (x-axis) arises due to the nature of the noise: we refer to the fundamental noise that occurs due to the continuous monitoring of the oscillator as quantum noise. Conversely, any noise arising due to the environment (e.g. seismic noise, coating thermal noise, etc.) is referred to as environmental noise. The second category (y-axis) describes how the noise source affects the dynamics: a force noise introduces stochastic momentum kicks on the oscillator, whereas a sensing noise is sensed with the position of the oscillator and does not affect its dynamics.

with $E \in \{S, F\}$, and P and Q are arbitrary polynomials in Ω such that $\deg(Q) \geq \deg(P)$ due to stability requirements.

First, we assume that the driving light (in the displaced frame) is in the vacuum state, with temporally uncorrelated fluctuations. This assumption will be relaxed at the end of the chapter, where an arbitrarily frequency-dependent squeezed input light field will be considered. The commutation relations for the quadratures of the input light field are given as $[\hat{u}_1(t), \hat{u}_2(t')] = 2i \delta(t - t')$, $[\hat{u}_i(t), \hat{u}_i(t')] = 0$, $i = 1, 2$. The same relations hold for the output light field.

The noise sources driving the mechanical oscillator, $\hat{u}_{1,2}$, $\hat{n}_F$, $\hat{n}_S$, can be categorized in two manners, summarized in Table 3.1. First, we refer to the fundamental measurement noise occurring due to the continuous monitoring of the oscillator position as *quantum noise*. This noise is sourced by $\hat{u}_{1,2}$ in Eqs. (3.1). Conversely, any additional noise in the system that can in principle be eliminated is referred to as *environmental noise*. This noise is sourced by $\hat{n}_F$ and $\hat{n}_S$ in Eqs. (3.1). An orthogonal characterization of the noise sources can be obtained by considering their effect on the system. We refer to the noise sources that produce stochastic momentum kicks to the oscillator as *force noise*, sourced by $\hat{u}_1$ and $\hat{n}_F$. Lastly, the noises that do not affect the dynamics of the oscillator, but are sensed simultaneously with the mechanical position are referred to as *sensing noise*. They are sourced by $\hat{u}_2$, and $\hat{n}_S$.

Examples of $\hat{n}_F$ are seismic noise [255, 124], suspension thermal noise [87, 56],

and structural damping [227, 189, 88] in systems with suspended oscillators such as gravitational wave detectors. Further examples include thermal excitations of a substrate in contact with the mechanical oscillator [36], and collision with gas molecules or photon scattering in levitated nanoparticles [123, 92, 89]. Additionally, examples to $\hat{n}_S$ are, thermal coating noise for coated mirrors [108, 97], and additional mechanical modes contributing to the outgoing light in membranes or levitated systems [276]. Note that the sensing noise here couples to the detection channel with a rate proportional to the interaction strength Ω_q (c.f. Eqs. (3.1)). There can be other types of sensing noises that do not couple with this rate, e.g. passive or photon losses, and detector dark noise [112]. We only consider the former type of sensing noise in $\hat{n}_S$, as in the presence of the latter, the universality of optomechanical entanglement does not hold (see Sec. 3.5).

Eqs. (3.1) describe a system where the mechanical oscillator is driven by the input light field and a force noise, represented by the terms $\frac{\Omega_q}{\sqrt{\omega_m}}\hat{u}_1(t)$ and $\hat{n}_F(t)$ in Eq. (3.1c), respectively. The system is unconditionally stable and reaches a steady state. Taking the Fourier transform⁴ of Eqs. (3.1c) and (3.1d), the steady state solution for the mechanical oscillator is written as $\hat{b}_1(\Omega) = \chi(\Omega)F_{ext}(\Omega)$, where $F_{ext}(\Omega) = \Omega_q\sqrt{\omega_m}\hat{u}_1(\Omega) + \omega_m\hat{n}_F(\Omega)$ is the Fourier transform of the sum of external forces driving the oscillator, and $\chi(\Omega) = (-\Omega^2 - i\gamma_m\Omega + \omega_m^2)^{-1}$ is the mechanical susceptibility. We refer to the inverse Fourier transform of the mechanical susceptibility as $\chi(t)$, which is a real, causal function of time.

3.3 Structure of the Covariance Matrix

We are interested in the optomechanical entanglement between the traveling light field and the oscillator's motion. At any given time T , as shown in Fig. 3.1, we have three parties: the oscillator $\{\hat{b}_{1,2}(T)\}$; the output light field $\{\hat{v}_{1,2}(t) : t < T\}$ that *has* been reflected during $t < T$ (shown in blue); and input light field $\{\hat{u}_{1,2}(t) : t > T\}$ that *will* be reflected during $t > T$ (shown in orange). Stationary dynamics are invariant under time translations, therefore we set $T = 0$.

The dynamics in Eqs. (3.1) are linear and stable. Since the driving noises are Gaussian, the system reaches a Gaussian steady state completely characterized by the covariance matrix of its quadratures. As we are interested in the bipartite entanglement between the single mechanical mode $\hat{b}_j(0)$ and the output field $\{\hat{v}_{1,2}(t), t < 0\}$,

⁴We use the convention $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt$

their covariance matrix $\mathbf{V}$ can be written in the following block matrix form

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}^{bb} & \mathbf{V}^{bv} \\ \mathbf{V}^{vb} & \mathbf{V}^{vv} \end{bmatrix}, \quad \mathbf{V}^{mn} = \begin{bmatrix} V_{11}^{mn} & V_{12}^{mn} \\ V_{21}^{mn} & V_{22}^{mn} \end{bmatrix} \quad (3.3)$$

where $m, n \in \{v, b\}$, $\mathbf{V}^{bb}$ and $\mathbf{V}^{vv}$ are the covariance matrices of the mechanical mode and the light field respectively, and $\mathbf{V}^{bv}$, $\mathbf{V}^{vb}$ contain the cross-correlations between the mechanical mode and the modes of the optical field, with $\mathbf{V}^{vb} = (\mathbf{V}^{bv})^T$ (T denotes the transpose). $\mathbf{V}^{bb}$ is a 2×2 matrix of real numbers, as the mechanical mode is only relevant at the instant of $t = 0$. On the other hand, we are interested in the output light field for $t < 0$, where the continuous index t labels this infinite set of modes. $\mathbf{V}^{vv}$ is thus a 2-by-2 matrix of bounded operators on $\mathcal{L}^2(0, \infty)$, the Hilbert space of square-integrable real functions on the half real line. Similarly, the elements of the 2-by-2 matrix $\mathbf{V}^{vb}$ ($\mathbf{V}^{bv}$) are functions in $\mathcal{L}^2(0, \infty)$. For example, $V_{12}^{bv}(t)$ is a function denoting the correlations between $\hat{b}_1(t=0)$ and $\hat{v}_2(t)$ for $t < 0$. To compute the elements of $\mathbf{V}$, we take the inverse Fourier transform of cross spectra between quadratures, as

$$\begin{aligned} \mathbf{V}_{kl}^{mn} &= \frac{1}{2} \langle \{\hat{m}_k(t_1), \hat{n}_l(t_2)\} \rangle \\ &= \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_{m_k n_l}(\Omega) e^{-i\Omega(t_1 - t_2)}, \end{aligned} \quad (3.4)$$

where $m, n \in \{v, b\}$, $k, l \in \{1, 2\}$, $t_{1(2)} = 0$ if $m(n) = b$, $t_{1(2)} < 0$ otherwise, and $S_{m_k n_l}$ is the double-sided cross spectrum between $\hat{m}_k$ and $\hat{n}_l$, given by

$$S_{m_k n_l}(\Omega) \delta(\Omega - \Omega') = \frac{1}{4\pi} \langle \hat{m}_k(\Omega) \hat{n}_l^\dagger(\Omega') + \hat{n}_l^\dagger(\Omega') \hat{m}_k(\Omega) \rangle \quad (3.5)$$

Note that since the input light field is the vacuum, $S_{u_1 u_1} = S_{u_2 u_2} = 1$, and $S_{u_1 u_2} = 0$. Furthermore, we denote the spectra of $\hat{n}_F(t)$ and $\hat{n}_S(t)$ with $S_{\hat{n}_F}(\Omega)$ and $S_{\hat{n}_S}(\Omega)$, respectively.

First, let us compute $\mathbf{V}^{vv}$. The operators in $\mathbf{V}^{vv}$ are indexed by $\tau \in (-\infty, \infty)$, where τ increases (decreases) with increasing columns (rows) of the matrix of the operator. $\mathbf{V}^{vv}$ is given as

$$\mathbf{V}^{vv} = \begin{bmatrix} \delta(\tau) & \Omega_q^2 \chi(-\tau) \\ \Omega_q^2 \chi(\tau) & \delta(\tau) + \Omega_q^2 \chi_{\text{EN}}(\tau) + \Omega_q^4 \Theta(\tau) \end{bmatrix}, \quad (3.6)$$

where $\Theta(\Omega) = |\chi(\Omega)|^2$ and we define a "force noise susceptibility" $\chi_F(\tau)$, with $\chi_F(\Omega) = \omega_m |\chi(\Omega)|^2 S_{\hat{n}_F}(\Omega)$, and a "sensing noise susceptibility" $\chi_S(\tau)$, with

$\chi_S(\Omega) = S_{\hat{n}_S}(\Omega)/\omega_m$ so that $\chi_{EN}(\tau) := \chi_F(\tau) + \chi_S(\tau)$ summarizes the susceptibility of the environmental noises. The matrix representations of $V_{ij}^{vv}, i, j = 1, 2$ are in the Toeplitz form, as their elements depend only on the difference between their respective row and column indices.

Next, let us compute $\mathbf{V}^{bv}$, which will be a matrix of functions indexed by $t, t > 0$. It contains the correlations between the mechanical oscillator and the light field:

$$\mathbf{V}^{bv} = \sqrt{\omega_m} \Omega_q \begin{bmatrix} \chi(t) & \chi_F(t) + \Omega_q^2 \Theta(t) \\ \frac{1}{\omega_m} \dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}_F(t) + \frac{\Omega_q^2}{\omega_m} \dot{\Theta}(t) \end{bmatrix}, \quad (3.7)$$

where t increases with increasing columns. Since $\hat{b}_2(t) = \hat{b}_1(t)/\omega_m$ from Eq. (3.1d), the cross-correlations involving $\hat{b}_2(t)$ are derivatives of the cross-correlations involving $\hat{b}_1(t)$, normalized by ω_m . The terms containing Θ in the second columns correspond to the amplitude fluctuations of the input light, transduced twice by the mechanical oscillator (recall that Θ is $|\chi|^2$ in the frequency domain). These fluctuations emerge as noise in the phase quadrature of the output light field. This phenomenon is referred to as *back action*.

Lastly, let us compute $\mathbf{V}^{bb}$, which is a matrix of real numbers:

$$\mathbf{V}^{bb} = \begin{bmatrix} V_{11}^{bb} & 0 \\ 0 & V_{22}^{bb} \end{bmatrix} \quad (3.8a)$$

$$V_{11}^{bb} = \omega_m \left(\chi_F(t=0) + \Omega_q^2 \Theta(t=0) \right) \quad (3.8b)$$

$$V_{22}^{bb} = \frac{-1}{\omega_m} \left(\ddot{\chi}_F(t=0) + \Omega_q^2 \ddot{\Theta}(t=0) \right) \quad (3.8c)$$

This matrix is diagonal due to vanishing cross-correlations between $\hat{b}_1(t)$ and $\hat{b}_2(t)$ at equal times.

3.4 Entanglement Criterion

We are interested in the bipartite entanglement between the mechanical oscillator and the continuum of temporal modes of the output light. Since we assume Gaussian dynamics and noise sources, the state of the mechanical mode and the temporal modes of the light are Gaussian. The mechanical oscillator represents a single mode, while there are $N \rightarrow \infty$ countable modes of the output field. In this $1 \times N$ configuration, the PPT criterion is necessary and sufficient for verifying the presence of bipartite entanglement [202, 243, 67, 268]. For continuous variables, the criterion asserts that a state is separable iff $\mathbf{V}_{pt} + i\mathbf{K} \geq 0$; $\mathbf{V}_{pt}$ denotes the partial transpose

of $\mathbf{V}$ with respect to one of the parties [243, 4], and $\mathbf{K}$ is the commutator matrix, defined as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^b & \\ & \mathbf{K}^v \end{bmatrix}, \quad (3.9)$$

with

$$\mathbf{K}^b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{K}^v = \begin{bmatrix} 0 & \delta(t) \\ -\delta(t) & 0 \end{bmatrix}. \quad (3.10)$$

In our configuration, the partial transpose operation amounts to reversing the sign of the momentum of the mechanical mode: $\hat{b}_2(t) \rightarrow -\hat{b}_2(t)$, which also puts a minus sign on the rows and columns of $\mathbf{V}$ associated with $\hat{b}_2(t)$. For example,

$$\mathbf{V}_{\text{pt}}^{bv} = \begin{bmatrix} V_{11}^{bv}(t) & V_{12}^{bv}(t) \\ -V_{21}^{bv}(t) & -V_{22}^{bv}(t) \end{bmatrix}. \quad (3.11)$$

In the $1 \times N$ configuration, $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ can have at most one negative eigenvalue [239], and we have the following necessary and sufficient test of optomechanical entanglement:

$$\det(\mathbf{V}_{\text{pt}} + i\mathbf{K}) < 0 \iff \text{entanglement}. \quad (3.12)$$

Standard matrix-determinant properties allow to express Eq. (3.12) as a product of determinants, factoring out $\det(\mathbf{V}^{vv} + i\mathbf{K}^v)$, which is positive for $\Omega_q > 0$ ⁵. We can then conclude that optomechanical entanglement exists if the following condition holds:

$$\det[\mathbf{V}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot (\mathbf{V}^{vv} + i\mathbf{K}^v)^{-1} \cdot \mathbf{V}_{\text{pt}}^{vb}] < 0. \quad (3.13)$$

Note that the dot product $\cdot$ is defined as, for an n -by- n matrix of operators $\mathbf{A}$, an n -by- m (m -by- n) matrix of functions $\mathbf{f}(\mathbf{g})$,

$$\begin{aligned} (\mathbf{g} \cdot \mathbf{A} \cdot \mathbf{f})_{kl} \\ = \sum_{i,j} \int_0^\infty \int_0^\infty dt dt' g_{ki}(t') A_{ij}(t-t') f_{jl}(t). \end{aligned} \quad (3.14)$$

From Eq. (3.13), the existence of optomechanical entanglement depends on a complex trade-off between the strength of the force noise, the sensing noise, and the

⁵We disregard the trivial case where $\Omega_q = 0$: the joint state will be separable, as there is no optomechanical interaction.

quantum noise due to vacuum fluctuations. We explore this relation in depth in the following sections.

Lastly, entanglement is quantified with the *logarithmic negativity*, defined as [261, 208]

$$E_{\mathcal{N}} = \sum_j \max\{0, -\ln(\lambda_j)\} \quad (3.15)$$

where the sum is over the symplectic eigenvalues λ_j of the partially transposed covariance matrix, and a non-zero logarithmic negativity indicates the presence of optomechanical entanglement.

3.5 Universal Entanglement

In this Section, we first simplify the condition to observe entanglement in Eq. (3.13). Subsequently, we demonstrate the universality of the entangling-disentangling transition. Lastly, we give an example of how this transition constrains the amount of environmental noise allowable in order to observe optomechanical entanglement.

Entanglement Indicator

To simplify Eq. (3.13), we first evaluate $(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1}$. Using standard properties of the 2-by-2 block-matrices, we write

$$(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1} = \mathbf{\Sigma} \cdot \mathbf{D}_M \cdot \mathbf{\Sigma}^\dagger + \mathbf{\Delta}, \quad (3.16a)$$

$$\mathbf{\Delta} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.16b)$$

$$\mathbf{\Sigma} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & \delta(\tau) \end{bmatrix} \cdot \begin{bmatrix} V_{12}^{vv}(\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (3.16c)$$

$$\mathbf{D}_M = \begin{bmatrix} M^{-1}(\tau) & 0 \\ 0 & M^{-1}(\tau) \end{bmatrix}, \quad (3.16d)$$

with⁶

$$M(\tau) = V_{22}^{vv}(\tau) - [-i\delta(\tau) + V_{21}^{vv}(\tau)] \cdot V_{11}^{vv}(\tau)^{-1} \cdot [i\delta(\tau) + V_{12}^{vv}(\tau)] \quad (3.17)$$

and $M^{-1}(\tau)$ is an operator calculated with the Wiener-Hopf method; see Appendix A.1 for a summary. Plugging in $V_{ij}^{vv}(t)$, $i, j = 1, 2$ from Eq. (3.6), and using the dot

⁶Note that in general, the operator M is a function of two time indices, in the form of $M(t, t')$. Here, assume $M(\tau) = M(t - t')$ for simplicity, which is true when the input light field is the vacuum state.

product convention in Eq. (3.14), we obtain

$$(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1} = \mathbf{\Sigma} \cdot \mathbf{D}_M \cdot \mathbf{\Sigma}^\dagger + \begin{bmatrix} \delta(\tau) & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.18a)$$

$$\mathbf{\Sigma} = \begin{bmatrix} \Omega_q^2 \chi(-\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (3.18b)$$

$$M(\tau) = -i\Omega_q^2 (\chi(\tau) - \chi(-\tau)) + \Omega_q^2 \chi_{\text{EN}}(\tau). \quad (3.18c)$$

When the second operator in Eq. (3.18a) operates on $\mathbf{V}_{\text{pt}}^{bv}$ and $\mathbf{V}_{\text{pt}}^{vb}$ in Eq. (3.13), we obtain

$$\int_0^\infty dt \begin{bmatrix} V_{11}^{bv}(t)^2 & -V_{11}^{bv}(t)V_{21}^{bv}(t) \\ -V_{21}^{bv}(t)V_{11}^{bv}(t) & V_{21}^{bv}(t)^2 \end{bmatrix} \quad (3.19a)$$

$$= \int_0^\infty dt \Omega_q^2 \begin{bmatrix} \omega_m \chi(t)^2 & -\chi(t)\dot{\chi}(t) \\ -\chi(t)\dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}(t)^2 \end{bmatrix} \quad (3.19b)$$

$$= \int_{-\infty}^\infty dt \Omega_q^2 \begin{bmatrix} \omega_m \chi(t)^2 & -\chi(t)\dot{\chi}(t) \\ -\chi(t)\dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}(t)^2 \end{bmatrix} \quad (3.19c)$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty d\Omega \Omega_q^2 \begin{bmatrix} \omega_m |\chi(\Omega)|^2 & 0 \\ 0 & \frac{\Omega^2}{\omega_m} |\chi(\Omega)|^2 \end{bmatrix} \quad (3.19d)$$

$$= \Omega_q^2 \begin{bmatrix} \omega_m \Theta(0) & 0 \\ 0 & \frac{-1}{\omega_m} \ddot{\Theta}(0) \end{bmatrix}, \quad (3.19e)$$

since $\int_0^\infty dt \chi(t)\dot{\chi}(t) = -\chi(0)^2/2 = 0$. We were able to extend the lower integration limit to $-\infty$ in Eq. (3.19c) due to the causality of $\chi(t)$. Then, we used the Plancherel theorem to go from the time domain to the frequency domain to calculate the diagonal elements in Eq. (3.19d). Lastly, we expressed the diagonal elements as inverse Fourier transforms at time $t = 0$ in Eq. (3.19e). We notice that this expression is contained in $\mathbf{V}^{bb}$ [Eq. (3.8)], therefore it cancels out of the determinant in Eq. (3.13), which becomes

$$\det \left[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot \mathbf{\Sigma} \cdot \mathbf{D}_M \cdot \mathbf{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T \right] < 0, \\ \tilde{\mathbf{V}}^{bb} := \begin{bmatrix} \omega_m \chi_F(0) & 0 \\ 0 & \frac{-1}{\omega_m} \ddot{\chi}_F(0) \end{bmatrix} \quad (3.20)$$

where we rewrote $\mathbf{V}_{\text{pt}}^{vb}$ as $(\mathbf{V}_{\text{pt}}^{bv})^T$. We refer to this simplification as the *vacuum cancellation*, since the variances of the quadratures of the mechanical oscillator due to the vacuum fluctuations factor out of the inequality in Eq. (3.13).

To calculate $\mathbf{V}_{\text{pt}}^{bv} \cdot \boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$, we first operate with $\boldsymbol{\Sigma}$ on $\mathbf{V}_{\text{pt}}^{bv}$ from the right,

$$\begin{bmatrix} V_{11}^{bv}(t) & V_{12}^{bv}(t) \\ -V_{21}^{bv}(t) & -V_{22}^{bv}(t) \end{bmatrix} \cdot \begin{bmatrix} \Omega_q^2 \chi(-\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}. \quad (3.21)$$

Here, we use the index $t > 0$ to indicate functions, and $\tau \in (-\infty, \infty)$ to indicate operators. We use this convention throughout the rest of the chapter. Performing the matrix product, we obtain

$$\begin{bmatrix} iV_{11}^{bv}(t) + \Omega_q^2 V_{11}^{bv}(t) \cdot \chi(-\tau) - V_{12}^{bv}(t) & 0 \\ -iV_{21}^{bv}(t) - \Omega_q^2 V_{21}^{bv}(t) \cdot \chi(-\tau) + V_{22}^{bv}(t) & 0 \end{bmatrix} \quad (3.22)$$

where we used the fact that taking the dot product with the identity operator $\delta(\tau)$ does not modify the functions. Taking the dot product between $V_{11}^{bv}(t)$ and $\chi(-\tau)$ and going to the frequency domain, we have

$$\begin{aligned} \left(V_{11}^{bv}(t) \cdot \chi(-\tau) \right) (\Omega) &= \mathcal{F} \left\{ \int_0^\infty dt' \chi(t' - t) V_{11}^{bv}(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_{-\infty}^\infty dt' \chi(t' - t) \chi(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} \chi(\Omega)^\dagger \chi(\Omega) \\ &= \Omega_q \sqrt{\omega_m} |\chi(\Omega)|^2 \end{aligned} \quad (3.23)$$

where we used the definition of V_{11}^{bv} given in Eq. (3.7), and we extended the integration boundary using that $\chi(t) = 0$ for $t < 0$. Then, we find that

$$\begin{aligned} V_{11}^{bv}(t) \cdot \chi(-\tau) &= \sqrt{\omega_m} \Omega_q \mathcal{F}^{-1} \{ |\chi(\Omega)|^2 \} \\ &= \sqrt{\omega_m} \Omega_q \Theta(t) \end{aligned} \quad (3.24)$$

Similarly, we can also write $V_{21}^{bv}(t) \cdot \chi(-\tau) = \sqrt{\omega_m} \Omega_q \dot{\Theta}(t)$. Substituting these expressions back in Eq. (3.22), we obtain

$$\begin{bmatrix} iV_{11}^{bv} - \tilde{V}_{12}^{bv} & 0 \\ -iV_{21}^{bv} + \tilde{V}_{22}^{bv} & 0 \end{bmatrix} \quad (3.25)$$

where we have defined

$$\tilde{V}_{12}^{bv}(t) = \sqrt{\omega_m} \Omega_q \chi_F(t), \quad \tilde{V}_{22}^{bv}(t) = \frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}_F(t) \quad (3.26)$$

i.e. we redefine $\tilde{V}_{i2}^{bv}(t)$, $i = 1, 2$ such that they only contain the correlations due to the force noise, and not the back action.

Similar calculations allow to re-express $\mathbf{V}_{\text{pt}}^{bv} \cdot \boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$ as

$$\tilde{\mathbf{V}}_{\text{pt}}^{bv}(t') \cdot \mathbf{V}_M(\tau = t - t') \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv}(t))^T \quad (3.27)$$

with

$$\tilde{\mathbf{V}}_{\text{pt}}^{bv} := \begin{bmatrix} V_{11}^{bv} & \tilde{V}_{12}^{bv} \\ -V_{21}^{bv} & -\tilde{V}_{22}^{bv} \end{bmatrix}, \mathbf{V}_M := \begin{bmatrix} M^{-1} & -iM^{-1} \\ iM^{-1} & M^{-1} \end{bmatrix}. \quad (3.28)$$

Therefore, we can rewrite Eq. (3.20) and define the following *entanglement indicator*

$$\det \left[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b - \tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T \right] < 0 \\ \iff \text{optomechanical entanglement.} \quad (3.29)$$

This inequality is our starting point for the subsequent considerations.

Demonstrating Universality

Given the entanglement indicator in Eq. (3.29), we explore the existence of entanglement for different configurations of environmental noise sources. For example, we first assume a very large sensing noise, i.e. the limit of $S_{\hat{n}_s} \rightarrow \infty$. In this limit, $\chi_S(t) \rightarrow \infty$, therefore $M \rightarrow \infty$, and consequently $M^{-1} \rightarrow 0$. However, the elements of $\mathbf{V}_{\text{pt}}^{bv}$ are finite, since they do not contain the correlations due to the sensing noise. Then, the rightmost term in Eq. (3.29) goes to zero, and the entanglement indicator is modified to

$$\det \left[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b \right] < 0 \iff \text{entanglement.} \quad (3.30)$$

$\tilde{\mathbf{V}}^{bb}$ contains the fluctuations in the quadratures of the mechanical oscillator due to the force noise only. Therefore, this matrix can be thought of as the covariance matrix of an oscillator in the absence of any optomechanical interaction, subjected to force noise. Since this is a physical covariance matrix, $\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b \geq 0$ from the Heisenberg uncertainty principle. Therefore, Eq. (3.30) cannot be realized for any physical force noise spectrum, and there can be no optomechanical entanglement in this limit.

Next, we investigate entanglement in the absence of sensing noise. Let us parametrize the spectrum of the force noise in the system as $\alpha_F S_{\hat{n}_F}$, where α_F is a dimensionless constant scaling the overall strength of the noise; and $\alpha_F = 0$ indicates that there is no force noise. If the system is entangled for α_F , it will also be entangled for $\alpha_F - \epsilon$, since the entanglement criterion gets modified as $\det(\mathbf{V}_{\text{pt}} + i\mathbf{K} - \epsilon\mathbf{V}') < 0$, where

$\det(\mathbf{V}_{\text{pt}} + i\mathbf{K}) < 0$, and $-\epsilon\mathbf{V}'$ is negative semi-definite. Then, the criterion will be satisfied for $\alpha_F - \epsilon$. Therefore, we set $S_{\hat{n}_s}(\Omega) = 0$ for the rest of the Subsection and characterize the force noise as $\alpha_F S_{\hat{n}_F}$, assuming that $\alpha_F \gg 1$. Due to the discussion above, if the system is entangled in this limit, it will be entangled for all force noise spectra, regardless of the amplitude.

In this limit, from Eq. (3.18c),

$$\begin{aligned} M(\Omega) &= \Omega_q^2 |\chi(\Omega)|^2 (\omega_m \alpha_F S_{\hat{n}_F}(\Omega) + 2\gamma_m \Omega) \\ &= M_R(\Omega) + M_I(\Omega) \end{aligned} \quad (3.31)$$

where

$$\begin{aligned} M_R(\Omega) &= \Omega_q^2 |\chi(\Omega)|^2 \omega_m \alpha_F S_{\hat{n}_F}(\Omega) = \Omega_q^2 \chi_F(\Omega), \\ M_I(\Omega) &= 2\Omega_q^2 |\chi(\Omega)|^2 \gamma_m \Omega = 2\Omega_q^2 \Theta(\Omega) \gamma_m \Omega \end{aligned} \quad (3.32)$$

Note that in the time domain, $M_R(t) = \Omega_q^2 \chi_F(t)$, $M_I(t) = 2i\Omega_q^2 \gamma_m \dot{\Theta}(t)$. Since $M_R(\Omega)$ is a real and even function of Ω , $M_R(t)$ is a real and symmetric function of time. Similarly, since $M_I(\Omega)$ is a real and odd function of Ω , $M_I(t)$ is a purely imaginary and anti-symmetric function. Now, we want to express the inverse of the operator M as a function of the operators M_R and M_I , where M_R is real and symmetric, and M_I is imaginary and Hermitian (as $M_I(\tau) = -M_I(-\tau)$). In the limit of $\alpha_F \gg 1$, we have

$$\begin{aligned} M^{-1} &= M_R^{-1} \cdot \left(1 + M_I \cdot M_R^{-1}\right)^{-1} \\ &\approx M_R^{-1} \cdot \left(1 - M_I \cdot M_R^{-1}\right) \\ &= M_R^{-1} - M_R^{-1} \cdot M_I \cdot M_R^{-1} \end{aligned} \quad (3.33)$$

where the approximation used that $M_R^{-1} \cdot M_I \propto 1/\alpha_F \ll 1$.

With this approximation of M^{-1} , we evaluate Eq. (3.29). In order to calculate the dot products with M_R^{-1} using the Wiener-Hopf method, we write M_R^{-1} as a product of its causal and anti-causal parts. Note that we omit writing α_F for the rest of the subsection. First, we express $\chi_F(\Omega)$ as a product of its causal and anti-causal parts:

$$\begin{aligned} \chi_F(\Omega) &= \chi_F(\Omega)_+ \chi_F(\Omega)_-, \\ \chi_F(\Omega)_+ &= \sqrt{\omega_m} \chi(\Omega) S_{\hat{n}_F}(\Omega)_+, \end{aligned} \quad (3.34)$$

and $\chi_F(\Omega)_- = \chi_F(\Omega)_+^\dagger$. Here, $S_{\hat{n}_F} = S_{\hat{n}_F}(\Omega)_+ S_{\hat{n}_F}(\Omega)_-$, where $S_{\hat{n}_F}(\Omega)_+$ contains the causal roots and zeros of $S_{\hat{n}_F}(\Omega)$. This decomposition is possible for any

power noise spectrum $S_{\hat{n}_F}(\Omega)$ that is a rational function of Ω . Then, we have: $M_R(\Omega)_\pm = \Omega_q \chi_F(\Omega)_\pm$.

We now want to calculate $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ in Eq. (3.29). For this purpose, we can define $f \star g = f \cdot M^{-1} \cdot g$ for functions f and g . Using this notation, $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ is given by (we drop the superscripts bv)

$$\begin{aligned} & \begin{bmatrix} V_{11} \star V_{11}^T & -V_{11} \star V_{21}^T \\ -V_{21} \star V_{11}^T & V_{21} \star V_{21}^T \end{bmatrix} + \begin{bmatrix} \tilde{V}_{12} \star \tilde{V}_{12}^T & -\tilde{V}_{12} \star \tilde{V}_{22}^T \\ -\tilde{V}_{22} \star \tilde{V}_{12}^T & \tilde{V}_{22} \star \tilde{V}_{22}^T \end{bmatrix} \\ & - i \begin{bmatrix} V_{11} \star \tilde{V}_{12}^T - \tilde{V}_{12} \star V_{11}^T & \tilde{V}_{12} \star V_{21}^T - V_{11} \star \tilde{V}_{22}^T \\ \tilde{V}_{22} \star V_{11}^T - V_{21} \star \tilde{V}_{12}^T & V_{21} \star \tilde{V}_{22}^T - \tilde{V}_{22} \star V_{21}^T \end{bmatrix} \end{aligned} \quad (3.35)$$

We first calculate Eq. (3.35) by substituting M^{-1} with M_R^{-1} from Eq. (3.33). We notice that

$$\begin{aligned} \tilde{V}_{12}^{bv}(\Omega) &= \sqrt{\omega_m \Omega_q} \chi_F(\Omega)_+ \chi_F(\Omega)_-, \\ \tilde{V}_{22}^{bv}(\Omega) &= -i\Omega \frac{\Omega_q}{\sqrt{\omega_m}} \chi_F(\Omega)_+ \chi_F(\Omega)_-. \end{aligned} \quad (3.36)$$

We proceed with the Wiener-Hopf method to calculate $\tilde{V}_{12}^{bv} \cdot M_R^{-1}$. In the frequency domain, we have

$$(\tilde{V}_{12}^{bv} \cdot M_R^{-1})(\Omega) = \frac{\sqrt{\omega_m \Omega_q}}{\Omega_q^2} \frac{1}{\chi_F(\Omega)_+} \left[\frac{\chi_F(\Omega)_+ \chi_F(\Omega)_-}{\chi_F(\Omega)_-} \right]_+ = \frac{\sqrt{\omega_m}}{\Omega_q} \quad (3.37)$$

therefore, in the time domain,

$$\tilde{V}_{12}^{bv} \cdot M_R^{-1} = 2\delta(t) \sqrt{\omega_m} / \Omega_q \quad (3.38)$$

for $t > 0$. The factor of 2 arises from the causality of the delta function. Then, the inner product $\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T$ is given by

$$\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T = \frac{2\sqrt{\omega_m}}{\Omega_q} \int_0^\infty \tilde{V}_{12}^{bv}(t) \delta(t) dt = \omega_m \chi_F(0) = (\tilde{\mathbf{V}}^{bb})_{11} \quad (3.39)$$

Similarly,

$$\tilde{V}_{22}^{bv} \cdot M_R^{-1} = \frac{2\dot{\delta}(t)}{\sqrt{\omega_m \Omega_q}} \quad (3.40)$$

which implies that the second term in Eq. (3.35) is given by

$$\begin{bmatrix} \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T & -\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \\ -\tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T & \tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \end{bmatrix} = \begin{bmatrix} \omega_m \chi_F(0) & 0 \\ 0 & \frac{1}{\omega_m} \ddot{\chi}_F(0) \end{bmatrix} = \tilde{\mathbf{V}}^{bb} \quad (3.41)$$

which cancels out with $\tilde{\mathbf{V}}^{bb}$ in Eq. (3.29). Next, let us evaluate the third term in Eq. (3.35). From Eq. (3.38) and Eq. (3.40), we have

$$\begin{aligned} & \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T - \tilde{V}_{11}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \\ &= \left(\frac{\sqrt{\omega_m}}{\Omega_q} V_{21}^{bv}(0^+) + \frac{\dot{V}_{11}^{bv}(0^+)}{\sqrt{\omega_m} \Omega_q} \right) = 2\dot{\chi}(0^+) = 2 \end{aligned} \quad (3.42)$$

and the entries in the diagonals are zero, since M_R is a real and symmetric operator. Then, the third term in Eq. (3.35) will be

$$\begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix} \quad (3.43)$$

Finally, we want to calculate the first term in Eq. (3.35). However, one cannot directly take the dot products of M_R^{-1} with V_{11}^{bv} and V_{21}^{bv} : indeed if the force noise spectrum decays faster than $1/\Omega^2$, the expressions are not well-defined. However, (quantum) physically, we expect that any (bosonic) continuous variable force noise will remain finite; as we shall see at the end of this subsection. Then, we work in the limit of $M_R(t) = \epsilon + \Omega_q^2 \chi_F(t)$, $\epsilon \ll 1$. Similar to above, we need to write $M_R(t)$ as a product of a causal and anticausal function in the Fourier domain. For this purpose, notice that the spectrum of M_R is—up to some constant—the spectrum of the force noise, $S_{\hat{n}_F}(\Omega)$, multiplied by the absolute value squared of the mechanical susceptibility, $|\chi(\Omega)|^2$, due to the response of the harmonic oscillator to the external forces caused by the force noise. The force noise spectra can be written as a product of rational functions of Ω^2 as it is a real and even function of Ω . We assume that $S_{\hat{n}_F}(\Omega)$ has z zeros and p poles, $k = p - z \geq 0$, k even. First, notice that we can always write $S_{\hat{n}_F}(\Omega)$ in the form of

$$S_{\hat{n}_F}(\Omega) \propto \frac{A(\Omega)_+ A(\Omega)_-}{B(\Omega)_+ B(\Omega)_-} = \frac{|A(\Omega)_+|^2}{|B(\Omega)_+|^2} \quad (3.44)$$

since $S_{\hat{n}_F}(\Omega)$ is real. We set the coefficient multiplying Ω^2 to the largest power in $B(\Omega)_+ B(\Omega)_-$ to unity, since any coefficient can be absorbed into $A(\Omega)_\pm$. Furthermore, we define $D(\Omega)_+ = \Omega^2 + i\gamma_m \Omega - \omega_m^2 = -1/\chi(\Omega)$, $D(\Omega)_- = D(\Omega)_+^\dagger$, which contain the poles of the mechanical susceptibility. Combining the new definitions, we write

$$\begin{aligned} M_R(\Omega) &= \epsilon + \frac{1}{D(\Omega)_+ D(\Omega)_-} \frac{A(\Omega)_+ A(\Omega)_-}{B(\Omega)_+ B(\Omega)_-} \\ &\approx \frac{\epsilon \Omega^{4+p} + A(\Omega)_+ A(\Omega)_-}{D(\Omega)_+ D(\Omega)_- B(\Omega)_+ B(\Omega)_-} \end{aligned} \quad (3.45)$$

where the approximation holds in the limit of $\epsilon \ll 1$. Now, in order to find the roots of the numerator, notice that in the limit $\epsilon \rightarrow 0$, the numerator goes to $A(\Omega)_+ A(\Omega)_-$. Then, we can write it as a product of $A(\Omega)_+ A(\Omega)_-$ and the rest of the roots, of which there are $4 + k$. To find the remaining roots, notice that they will be solutions to

$$\epsilon \Omega^{4+p} + L \Omega^z = 0 \quad (3.46)$$

where L is a real number. since the roots will be at large Ω in the limit of $\epsilon \rightarrow 0$, and only the term $L \Omega^z$ in $A(\Omega)_+ A(\Omega)_-$ will be of importance. The roots of Eq. (3.46) are $\Omega_n = L \epsilon^{-1/(4+k)} \exp((i2\pi n + i\pi)/(4+k))$, $n = 0, 1, \dots, 3+k$. Then,

$$\begin{aligned} M_R(\Omega) &\approx \frac{A(\Omega)_+ A(\Omega)_- \prod_{n=0}^{3+k} (\epsilon^{1/(4+k)} \Omega - L e^{\frac{i\pi}{4+k} + \frac{i2\pi n}{4+k}})}{D(\Omega)_+ D(\Omega)_- B(\Omega)_+ B(\Omega)_-} \\ M_R(\Omega)_\pm &\approx \frac{A(\Omega)_\pm \prod_{n=0}^{1+k/2} (\epsilon^{1/(4+k)} \Omega \pm L e^{\frac{i\pi}{4+k} + \frac{i2\pi n}{4+k}})}{D(\Omega)_\pm B(\Omega)_\pm} \end{aligned} \quad (3.47)$$

While using the Wiener-Hopf method, one needs to calculate $1/M_R(\Omega)_+$. We see that thanks to the terms with ϵ , the number of poles and zeros of $1/M_R(\Omega)_+$, in the limit of $\epsilon \rightarrow 0$, are equal. Therefore, using Eq. (A.13), dot products with M_R^{-1} are guaranteed to be bounded.

We now return to the evaluation of the first term in Eq. (3.35). In taking the dot products of M_R^{-1} with V_{11}^{bv} and V_{21}^{bv} one encounters expressions such as

$$V_{11}^{bv} \cdot M_R^{-1} \cdot V_{11}^{bv} = \int \frac{d\Omega}{2\pi} \frac{V_{11}^{bv}(\Omega)^\dagger}{M_R(\Omega)_+} \left[\frac{V_{11}^{bv}(\Omega)}{M_R(\Omega)_-} \right]_+ \quad (3.48)$$

Since the number of poles and zeros of $M_R(\Omega)_+$ in the limit of $\epsilon \rightarrow 0$ are equal from the discussion, and $V_{11}^{bv}(\Omega)$ has two poles, we can write this as the dot product of two bounded, causal functions of time. Or, in the frequency domain, as

$$\int \frac{d\Omega}{2\pi} \left[\frac{V_{11}^{bv}(\Omega)^\dagger}{M_R(\Omega)_+} \right]_- \left[\frac{V_{11}^{bv}(\Omega)}{M_R(\Omega)_-} \right]_+ \quad (3.49)$$

Then, we only need the residues of $1/M_R(\Omega)_-$ at the poles of the mechanical susceptibility to compute the dot products in the limit of $\epsilon \rightarrow 0$. In the end, the first term in Eq. (3.35) is

$$\begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix} \quad (3.50)$$

Now that we have finished discussing the terms with M_R^{-1} , we want to calculate $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot \tilde{\mathbf{V}}_{\text{pt}}^{vb}$ (Eq. (3.35)) again, substituting M^{-1} with $M_R^{-1} \cdot M_I \cdot M_R^{-1}$; cf. Eq. (3.33). First, note that after this substitution, the first term in Eq. (3.35) will be proportional to $1/\alpha_F^2 \ll 1$, therefore we can ignore this term. From Eqs. (3.38), (3.40), the second term will be given by

$$\frac{1}{\Omega_q^2} \begin{bmatrix} \omega_m M_I(0) & \dot{M}_I(0) \\ -\dot{M}_I(0) & \frac{-1}{\omega_m} \ddot{M}_I(0) \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (3.51)$$

since $M_I(0) = \ddot{M}_I(0) = 0$. In order to calculate the third term in Eq. (3.35), we notice that

$$\begin{aligned} \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot M_I &= \delta(t) \cdot M_I \frac{\sqrt{\omega_m}}{\Omega_q} = M_I(t) \frac{\sqrt{\omega_m}}{\Omega_q} \\ \tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot M_I &= \dot{\delta}(t) \cdot M_I \frac{1}{\sqrt{\omega_m} \Omega_q} = \frac{\dot{M}_I(t)}{\sqrt{\omega_m} \Omega_q} \end{aligned} \quad (3.52)$$

for $t > 0$. Then, the third term in Eq. (3.35) will involve inner products with $M_I(t)$ and $\dot{M}_I(t)$. Since these products will have support only on the half real line (i.e. the integration limits will be from 0 to $+\infty$), we can replace $M_I(t)$ and $\dot{M}_I(t)$ with their causal counterparts. Notice that

$$M_I(t) \frac{\sqrt{\omega_m}}{\Omega_q} = -i \Omega_q \sqrt{\omega_m} \chi(t) = -i V_{11}^{bv}(t) \quad (3.53)$$

for $t > 0$. Similarly, $\dot{M}_I(t)/\sqrt{\omega_m} \Omega_q = -i V_{21}^{bv}(t)$ for $t > 0$. Therefore, the third term in Eq. (3.35) can be written as

$$2 \begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix} \quad (3.54)$$

Therefore, summing all of the matrices above, the condition to be entangled in the absence of sensing noise is given by

$$\det \left\{ \begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix} + \begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} \right\} < 0 \quad (3.55)$$

using the Wiener-Hopf method and the expressions for V_{11}^{bv} , V_{21}^{bv} , and M_R , where we calculate the following elegant expression:

$$\begin{aligned} & (V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T)(V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{12}^{bv})^T) \\ & - (V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T)^2 = \frac{4\gamma_m^2}{|S_{\hat{n}_F}(\Omega_*)|^4} \end{aligned} \quad (3.56)$$

where $\Omega_* = -i\gamma_m/2 \pm \sqrt{\omega_m^2 - \gamma_m^2/4}$ is one of the poles of the mechanical susceptibility. From Eqs. (3.55) and (3.56), the condition for optomechanical entanglement reduces to

$$|S_{\hat{n}_F-}(\Omega_*)|^2 > \gamma_m \iff \text{entanglement.} \quad (3.57)$$

There exists a lower limit to $|S_{\hat{n}_F-}(\Omega_*)|$ from the fluctuation-dissipation theorem [40, 150], since $S_{\hat{n}_F}(\Omega)$ includes the thermal-fluctuations that give rise to velocity damping on the mechanical oscillator, with a damping rate γ_m . Then, assuming that there is no additional force noise in the system, $S_{\hat{n}_F}(\Omega)$ is connected to the mechanical susceptibility with

$$S_{\hat{n}_F}(\Omega)|\chi(\Omega)|^2 = \frac{2\gamma_m\Omega}{\omega_m}|\chi(\Omega)|^2 \coth\left(\frac{\hbar\Omega}{2k_B T}\right) \quad (3.58)$$

where T is the temperature of the bath, and k_B is the Boltzmann constant. Eliminating the mechanical susceptibility from each side, and writing $S_{\hat{n}_F}$ as $S_{\hat{n}_F}(\Omega) = S_{\hat{n}_F}(\Omega)_+ S_{\hat{n}_F}(\Omega)_-$, the minimum value of $|S_{\hat{n}_F-}(\Omega_*)|^2$ is given by

$$|S_{\hat{n}_F-}(\Omega_*)|^2 \geq 2\gamma_m \quad (3.59)$$

We therefore see that all physical force noise spectra ensure optomechanical entanglement, from Eq. (3.57). Therefore, the mechanical oscillator is entangled with the output light field for any force noise spectrum in the absence of sensing noise.

We propose the following heuristic physical interpretation to explain this phenomenon: entanglement is generated by the two-mode-squeezing interaction included in the optomechanical coupling [11], and two parties—one in a pure Gaussian state, the other in an arbitrary thermal state—become entangled after any two-mode-squeezing interaction [170]. In the stationary regime, the force noise (and its associated bath) essentially sets the state of the mechanical party (similar to thermalization), while the probing light is in a pure Gaussian state. Therefore, it can be expected that the joint optomechanical state after interaction is entangled for any finite force noise. This heuristic argument disregards the possibilities of the mechanical bath and the measuring light being non-Markovian.

Summarizing the two scenarios discussed above, for a given force noise spectrum, the existence of entanglement depends on the amount of sensing noise present in the system, signifying that there exists an entangling-disentangling transition at which optomechanical entanglement emerges. Furthermore, this transition is unique for a given force noise and sensing noise spectrum. To show this, let us assume a force

and sensing noise spectrum $\alpha_F S_{\hat{n}_F}(\Omega)$, $\beta_S S_{\hat{n}_S}(\Omega)$, respectively, where α_F and β_S are real, positive constants. We want to show that given α_F , the value of β_S for which the entangling-disentangling transition occurs is unique. In other words, if the system is not entangled for a pair (α_F, β_S) , entanglement cannot be achieved by increasing β_S . Note that when we increase β_S , we introduce more sensing noise to the system, however, we do not change the functional form of the spectrum (i.e. $S_{\hat{n}_S}(\Omega)$).

We write the covariance matrix as $\mathbf{V}$, and add some additional sensing noise to the system, modifying the covariance matrix in the form of $\mathbf{V} \rightarrow \mathbf{V} + r\mathbf{V}'$, where $r \in \mathbb{R}$, and $\mathbf{V}'$ is positive semi-definite as it is a covariance matrix. $\mathbf{V}'$ will be non-zero only in the sector of the covariance matrix containing the variance of $\hat{v}_2$ due to the structure of the equations of motion, given in Eq. (3.1). This modification corresponds to modifying β_S , such that $\beta_S \rightarrow \beta_S + r$.

We recall from Sec. 3.4 that optomechanical entanglement exists when $\det(\mathbf{V}_{\text{pt}} + i\mathbf{K}) < 0$, where $\mathbf{V}_{\text{pt}}$ is the covariance matrix of the system. If the system is separable for a given β_S , it will remain separable for $\beta_S + r$, $r > 0$. This is because the entanglement criterion becomes $\det(\mathbf{V}_{\text{pt}} + r\mathbf{V}' + i\mathbf{K}) < 0$, where $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ is positive semi-definite, and $\mathbf{V}'$ is positive semi-definite (since it is a covariance matrix). Therefore, the criterion is not satisfied, and optomechanical entanglement does not exist. Conversely, if optomechanical entanglement exists for β_S , it will also exist for $\beta_S - r$, $r > 0$, since the criterion gets modified as $\mathbf{V}_{\text{pt}} + i\mathbf{K} - r\mathbf{V}' < 0$, where $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ is negative definite, and $-\mathbf{V}'$ is negative semi-definite. Then, the criterion will be satisfied for $\beta_S - r$.

Therefore, if we introduce some sensing noise to an initially entangled system, there exists a critical amount of additional sensing noise for which an entangling-disentangling transition occurs, and the system becomes separable. Beyond this point, the system cannot get entangled again as we continue introducing more sensing noise. Conversely, before the entangling-disentangling transition (when the system is entangled), we cannot induce separability by removing sensing noise from the system. These statements prove our initial claim that the entangling-disentangling transition is unique with respect to the amount of sensing noise present.

Lastly, from Eqs. (3.18c), (3.26), (3.28), and (3.29), $M(t) \propto \Omega_q^2$, and $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \propto \Omega_q$. Therefore, the term $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ in Eq. (3.29) is independent of Ω_q . We also have that $\tilde{\mathbf{V}}^{bb}$ and $\mathbf{K}^b$ are already independent of Ω_q . Therefore, Ω_q factors out of Eq. (3.29), making the entangling-disentangling transition independent of the interaction strength between the mechanical oscillator, and the output light field.

Therefore, this transition is universal with respect to the interaction strength. To restate our definition of universality, we call a transition universal if it i) exists for a finite level of sensing noise, ii) is unique with respect to the amount of sensing noise, and iii) is independent of the coherent coupling Ω_q .

After the transition (i.e. for an entangled state), we saw numerically that the negativity increases with Ω_q , which supports our intuition that the optomechanical entanglement is strengthened with an increased interaction strength. However, most interestingly, the universality implies that, for a noise configuration before the transition (i.e. for a separable state), one cannot make the state entangled by increasing the interaction strength.

Example Case

To derive a simple analytical bound on the maximum allowable environmental noise in order to observe optomechanical entanglement, we assume white noise spectra parametrized with Ω_F and Ω_S for the force and the sensing noise, respectively; cf [47, 65]. More specifically, we assume that their spectra are in the form of

$$S_{\hat{n}_F}(\Omega) = \frac{2\Omega_F^2}{\omega_m}, \quad S_{\hat{n}_S}(\Omega) = \frac{2\omega_m}{\Omega_S^2}. \quad (3.60)$$

Consequently, $\chi_F(\Omega) = 2\Omega_F^2|\chi(\Omega)|^2$, and $\chi_S(\Omega) = 2/\Omega_S^2$.

Furthermore, we assume the *free-mass limit*, i.e. the mechanical resonance frequency ω_m of the oscillator is much smaller than the other frequencies in the system. In this limit, $\Omega_F, \Omega_S \gg \omega_m, \gamma_m$. Lastly, we assume the presence of passive losses occurring due to e.g. a photodetector with non-unity efficiency, whose effect on the dynamics are described in Appendix A.5. Even though these losses act effectively as a sensing noise on the system, they do not couple to the detection channel with a rate of Ω_q , different from the sensing noise $\hat{n}_S$ in Eqs. (3.1). Hence, in the presence of such losses, optomechanical entanglement is no longer universal with respect to the coherent coupling Ω_q .

For a white force and sensing noise, and a photodetector efficiency of $\eta < 1$, the spectrum of M is given by (Eq. (3.18c))

$$\frac{M(\Omega)}{2\eta\Omega_q^2} = \frac{1}{\Omega_S^2} + |\chi(\Omega)|^2 \left[\Omega_F^2 + \gamma_m\Omega + \frac{\Omega_q^2}{2}(1 - \eta) \right]. \quad (3.61)$$

In order to apply the Wiener-Hopf method, we need to write $M(\Omega)$ as a product of its causal and anti-causal poles and zeros, i.e. $M(\Omega) = M(\Omega)_+ M(\Omega)_-$, where

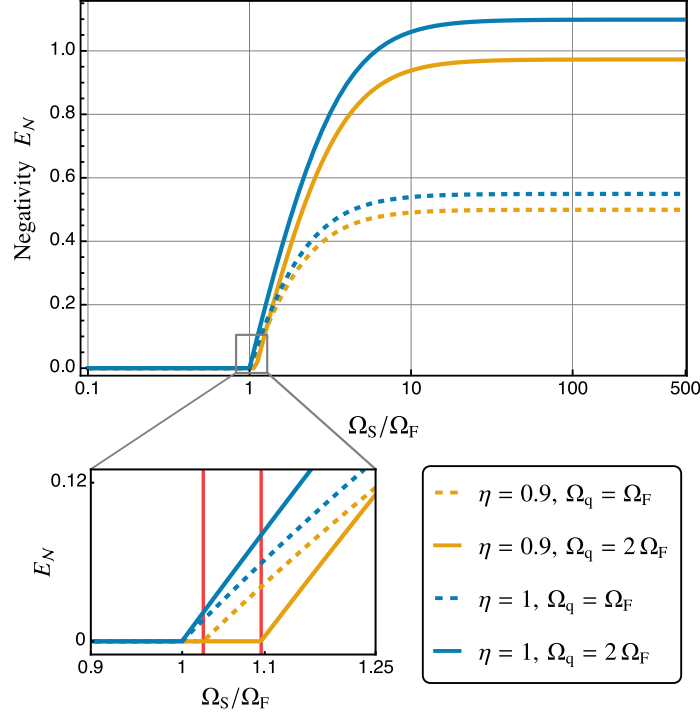


Figure 3.2: Logarithmic negativity for the case where the input light field is the vacuum state, for a photodetector efficiency of $\eta = 0.9$ (plotted in yellow), and $\eta = 1$ (plotted in blue), and white environmental noise sources. We fix the force noise level with $\Omega_F/(2\pi) = 10$ Hz, and vary the sensing noise level in the range of $\Omega_S/(2\pi) \in [1, 5000]$ Hz. The parameters of the mechanical oscillator are chosen as $\omega_m/(2\pi) = 1$ Hz and $\gamma_m/(2\pi) = 0.01$ Hz. We observe that for a unity photodetector efficiency ($\eta = 1$), the entangling transition is characterized by $\Omega_F \approx \Omega_S$, as observed numerically in Ref. [65]. For a non-unity photodetector efficiency ($\eta < 1$), the entangling transition is not universal with respect to the interaction strength Ω_q . For this case, the transition is characterized by $\sqrt{\Omega_F^2 + (1 - \eta)\Omega_q^2/2} \approx \Omega_S$, drawn by the red vertical lines in the inset.

$M(\Omega)_+ = M(\Omega)_-^\dagger$, and $M(\Omega)_+ (M(\Omega)_-)$ contains the causal (anti-causal) poles and zeros. In the free-mass limit, $M(\Omega)_+$ is given up to first order in ω_m, γ_m as

$$M(\Omega)_+ \approx \frac{\sqrt{2\eta}\Omega_q}{\Omega_S} \cdot \frac{(\Omega - \Omega_{z,+})(\Omega - \Omega_{z,-})}{\Omega^2 + i\gamma_m\Omega - \omega_m^2}, \quad (3.62)$$

where

$$\Omega_{z,\pm} = \frac{-i \pm 1}{\sqrt{2}} \sqrt{\Omega_S \Omega_N} \mp \frac{i\Omega_S \gamma_m}{4\Omega_N}. \quad (3.63)$$

with $\Omega_N = \sqrt{\Omega_F^2 + (1 - \eta)\Omega_q^2/2}$. To evaluate the entanglement indicator in Eq. (3.29), we need to obtain the expressions for $\tilde{\mathbf{V}}^{bb}$ and $\tilde{\mathbf{V}}_{\text{pt}}^{bv}$. From Eqs. (3.20) and

(3.26), we calculate

$$\tilde{\mathbf{V}}^{bb} = \frac{\Omega_N^2}{\gamma_m \omega_m} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{\mathbf{V}}_{\text{pt}}^{bv}(t) = \sqrt{\omega_m \eta} \Omega_q \begin{bmatrix} \chi(t) & 2\Omega_N^2 \Theta(t) \\ \frac{-1}{\omega_m} \dot{\chi}(t) & \frac{-1}{\omega_m} 2\Omega_N^2 \dot{\Theta}(t) \end{bmatrix} \quad (3.64)$$

where $\Theta(\Omega) = |\chi(\Omega)|^2$. Proceeding with the Wiener-Hopf method, we obtain up to the terms that contribute to constant order in ω_m, γ_m

$$\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T \approx \tilde{\mathbf{V}}^{bb} + \begin{bmatrix} 2^{3/2} \omega_m \sqrt{\frac{\Omega_N}{\Omega_S^3}} & -i + \frac{2\Omega_N}{\Omega_S} \\ i + \frac{2\Omega_N}{\Omega_S} & -\frac{2^{3/2}}{\omega_m} \sqrt{\frac{\Omega_N^3}{\Omega_S}} \end{bmatrix} \quad (3.65)$$

Therefore, Eq. (3.29) reduces to the following condition in the limit of $\gamma_m \rightarrow 0$:

$$\Omega_N < \Omega_S \iff \text{optomechanical entanglement.} \quad (3.66)$$

When $\eta = 1$, i.e. there are no passive losses, this relation gets modified as $\Omega_F < \Omega_S \iff \text{entanglement}$, which signifies that the presence of optomechanical entanglement depends on a simple trade-off between the relative strengths of the environmental noise sources. This bound has been observed numerically in [65]. For $\eta < 1$, $\Omega_N > \Omega_F$, therefore it becomes harder to achieve optomechanical entanglement for non-unity photodetector efficiency, and it is more advantageous to have a small coherent coupling Ω_q for this purpose.

We compute the logarithmic negativity numerically, assuming that the input light field is the vacuum state, and the environmental noise sources are white, characterized by Ω_F and Ω_S in Eq. (3.60). The results are given in Fig. 3.2. In order to work in the free-mass limit, we fix $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, $\Omega_F/(2\pi) = 10$ Hz, and vary Ω_S in the range of $\Omega_S/(2\pi) \in [1, 500]$ Hz, such that $\Omega_F, \Omega_S \gg \omega_m, \gamma_m$. First, we assume a perfect photodetector with unity efficiency ($\eta = 1$). This case is plotted with dashed and plain lines in blue, where the interaction strength is taken to be $\Omega_q/(2\pi) = 10$ Hz, and $\Omega_q/(2\pi) = 20$ Hz, respectively. The entangling threshold occurs at $\Omega_F \approx \Omega_S$, as observed in Fig. 3.2. Then, we assume a photodetector with non-unity efficiency ($\eta = 0.9$), which introduces passive losses to the system. We plot this case with dashed and plain lines in yellow, for $\Omega_q/(2\pi) = 10$ Hz, and $\Omega_q/(2\pi) = 20$ Hz, respectively. From Eq. (3.66), we compute the entangling threshold as $\Omega_S/\Omega_F = 1.025$ ($\Omega_S/\Omega_F = 1.095$) for $\Omega_q/(2\pi) = 10$ Hz ($\Omega_q/(2\pi) = 20$ Hz), which is drawn to the inset of Fig. 3.2 with a red, vertical curve. As observed analytically, passive losses impede achieving optomechanical entanglement. Furthermore, it is more advantageous to have a small interaction strength to obtain

entanglement (i.e. reach the entangling threshold). However, once the threshold is reached, a larger interaction strength provides more entanglement, quantified by a larger logarithmic negativity.

3.6 Generalization to arbitrary squeezing

Until this part of the chapter, we have assumed that the input light field consisted of vacuum fluctuations. Now, let us assume that arbitrary squeezing operations have been performed on the input field before interacting with the mechanical oscillator.

First, we show in Appendix A.2 that if the input field has been subjected to phase shifts or squeezing by a constant factor, the determinant condition in Eq. (3.29) is unchanged. Hence, such operations do not affect the entangling-disentangling transition. Therefore, we must apply a more general, potentially frequency-dependent transformation to the input light field in order to act on the entangling-disentangling transition.

A general, unitary transformation performed on the input field must be causal and symplectic. The causality condition is because we have defined the input field to be the temporal modes of the light during $t > 0$. Then, any non-causal transformation mixes the input and the output temporal modes, which conflicts with the causality of the dynamics.

The most general causal symplectic transformation can be written as (see Appendix A.3)

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow e^{i\phi(\Omega)} \begin{bmatrix} A(\Omega) & B(\Omega) \\ C(\Omega) & D(\Omega) \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \quad (3.67)$$

where $A(\Omega)$, $B(\Omega)$, $C(\Omega)$, and $D(\Omega)$ are real functions of Ω , and $A(\Omega)D(\Omega) - B(\Omega)C(\Omega) = 1 \forall \Omega$. The phase factor $e^{i\phi(\Omega)}$ ensures the causality of the transformation.

We proceed with writing down the covariance matrix of the system, given this transformation. First, we calculate the correlations between the quadratures of the output light field,

$$V_{11}^{vv}(\Omega) = T_1(\Omega), \quad (3.68a)$$

$$V_{12}^{vv}(\Omega) = T_{12}(\Omega) + \Omega_q^2 T_1(\Omega) \chi(-\Omega), \quad (3.68b)$$

$$V_{22}^{vv}(\Omega) = T_2(\Omega) + \Omega_q^4 T_1(\Omega) |\chi(\Omega)|^2 + \Omega_q^2 [\chi_{\text{EN}}(\Omega) + T_{12}(\Omega)(\chi(\Omega) + \chi(-\Omega))], \quad (3.68c)$$

where we have defined $T_1(\Omega) = A(\Omega)^2 + B(\Omega)^2$, $T_2(\Omega) = C(\Omega)^2 + D(\Omega)^2$, and $T_{12}(\Omega) = A(\Omega)C(\Omega) + B(\Omega)D(\Omega)$. Next, the covariances of the quadratures of the mechanical oscillator are given by

$$V_{11}^{bb}(\Omega) = \omega_m \chi_F(\Omega) + \omega_m \Omega_q^2 |\chi(\Omega)|^2 T_1(\Omega), \quad (3.69)$$

and $V_{12}^{bb}(\Omega) = i\Omega V_{11}^{bb}(\Omega)/\omega_m$, $V_{22}^{bb}(\Omega) = \Omega^2 V_{11}^{bb}(\Omega)/\omega_m$. We see that the variance associated with the force noise is unchanged compared to Eq. (3.8), and the variance due to the optomechanical interaction is scaled by the frequency-dependent factor of $T_1(\Omega)$ in the frequency domain. Finally, we compute the cross-spectra of the mechanical oscillator and the output light field:

$$V_{11}^{bv}(\Omega) = \Omega_q \sqrt{\omega_m} \chi(\Omega) T_1(\Omega), \quad (3.70a)$$

$$V_{12}^{bv}(\Omega) = \Omega_q \sqrt{\omega_m} T_{12}(\Omega) \chi(\Omega) + \Omega_q \sqrt{\omega_m} \chi_F(\Omega) + \Omega_q^3 \sqrt{\omega_m} |\chi(\Omega)|^2 T_1(\Omega) \quad (3.70b)$$

and $V_{2i}^{bv}(\Omega) = -i\Omega V_{1i}^{bv}(\Omega)/\omega_m$, $i = 1, 2$. To continue with the formalism of Sec. 3.5, let us recall the general form of the entanglement criterion. From Eq. (3.13) and (3.16a), we have:

$$\det \left[\mathbf{V}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot \left(\boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger + \boldsymbol{\Delta} \right) \cdot (\mathbf{V}_{\text{pt}}^{bv})^T \right] < 0 \iff \text{entanglement.} \quad (3.71)$$

where $\boldsymbol{\Delta}$, $\boldsymbol{\Sigma}$, and $\mathbf{D}_M$ are given as (see Eq. (3.16))

$$\boldsymbol{\Delta} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.72a)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} \cdot V_{12}^{vv}(\tau) + iV_{11}^{vv}(\tau)^{-1} & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (3.72b)$$

$$\mathbf{D}_M = \begin{bmatrix} M^{-1}(\tau) & 0 \\ 0 & M^{-1}(\tau) \end{bmatrix}. \quad (3.72c)$$

and $\mathbf{V}_{\text{pt}}^{bv}$ is defined in Eq. (3.11). To compute Eq. (3.71), we need to invert V_{11}^{vv} using the Wiener-Hopf method, therefore, we write it down in the Fourier domain as a product of its causal and anti-causal poles and zeros,

$$V_{11}^{vv}(\Omega) = T_1(\Omega) = T_1(\Omega)_+ T_1(\Omega)_-, \quad (3.73)$$

where $T_1(\Omega)_+^\dagger = T_1(\Omega)_-$, and $T_1(\Omega)_+$ ($T_1(\Omega)_-$) contains the causal (anti-causal) poles and zeros of $V_{11}^{vv}(\Omega)$. In the absence of any squeezing in the input light field, we had found a simplification in the entanglement criterion, which we referred to as

vacuum cancellation in Sec. 3.5. This simplification canceled out some of the terms in the entanglement criterion related to the mechanical oscillator; cf. Eq. (3.20). Specifically, we observed that the remaining terms were the variances of $\hat{b}_1$ and $\hat{b}_2$ generated due to the force noise only. Here, in the presence of frequency-dependent squeezing, we observe that the vacuum cancellation persists. To see this, we operate on $\mathbf{V}_{\text{pt}}^{bv}$ with Δ from the right,

$$(\mathbf{V}_{\text{pt}}^{bv} \cdot \Delta)(\Omega) = \begin{bmatrix} V_{11}^{bv} \cdot (V_{11}^{vv})^{-1} & 0 \\ -V_{21}^{bv} \cdot (V_{11}^{vv})^{-1} & 0 \end{bmatrix}. \quad (3.74)$$

Using Eq. (3.70) and Eq. (3.73), we can simplify this expression to obtain

$$\frac{\sqrt{\omega_m} \Omega_q}{T_1(\Omega)_+} \left[\frac{T_1(\Omega)_+ T_1(\Omega)_- \chi(\Omega)}{T_1(\Omega)_-} \right]_+ \begin{bmatrix} 1 & 0 \\ -\frac{i\Omega}{\omega_m} & 0 \end{bmatrix} = \sqrt{\omega_m} \Omega_q \chi(\Omega) \begin{bmatrix} 1 & 0 \\ -\frac{i\Omega}{\omega_m} & 0 \end{bmatrix} \quad (3.75)$$

Then, in the frequency domain, we have

$$\left(\mathbf{V}_{\text{pt}}^{bv} \cdot \Delta \cdot (\mathbf{V}_{\text{pt}}^{bv})^T \right) = \omega_m \Omega_q^2 |\chi(\Omega)|^2 T_1(\Omega) \begin{bmatrix} 1 & \frac{i\Omega}{\omega_m} \\ -\frac{i\Omega}{\omega_m} & \frac{\Omega^2}{\omega_m^2} \end{bmatrix} \quad (3.76)$$

This term is equal (in the frequency domain) to the variances of the quadratures of the mechanical oscillator arising from the optomechanical interaction, cf. the second term in Eq. (3.69). Therefore, we see that the vacuum cancellation is realized, and we can use Eq. (3.20) to test for entanglement, given that Σ and $\mathbf{D}_M$ are calculated accordingly.

Continuing with the calculation, we want to compute $\mathbf{V}_{\text{pt}}^{bv} \cdot \Sigma \cdot \mathbf{D}_M \cdot \Sigma^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$. For this purpose, we operate on $\mathbf{V}_{\text{pt}}^{bv}$ from the right with Σ . Using Eq. (3.11) and Eq. (3.72b), we obtain

$$\begin{bmatrix} V_{11}^{bv}(t) \cdot (V_{11}^{vv})^{-1} \cdot (i\delta(\tau) + V_{12}^{vv}(\tau)) - V_{12}^{bv}(t) & 0 \\ -V_{21}^{bv}(t) \cdot (V_{11}^{vv})^{-1} \cdot (i\delta(\tau) + V_{12}^{vv}(\tau)) + V_{22}^{bv}(t) & 0 \end{bmatrix}. \quad (3.77)$$

We can simplify this expression by plugging in what we have found for $V_{i1}^{bv}(t) \cdot (V_{11}^{vv})^{-1}$, $i = 1, 2$, from Eq. (3.75), which results in

$$\begin{bmatrix} \sqrt{\omega_m} \Omega_q \chi(t) \cdot (i\delta(\tau) + V_{12}^{vv}(\tau)) - V_{12}^{bv}(t) & 0 \\ -\frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}(t) \cdot (i\delta(\tau) + V_{12}^{vv}(\tau)) + V_{22}^{bv}(t) & 0 \end{bmatrix}. \quad (3.78)$$

To further simplify this expression, we want to compute $\chi(t) \cdot V_{12}^{vv}(\tau)$. Then, in the frequency domain, we have

$$\begin{aligned} \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_0^\infty dt' V_{12}^{vv}(t-t') \chi(t') \right\} &= \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_{-\infty}^\infty dt' V_{12}^{vv}(t-t') \chi(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} (T_{12}(\Omega) \chi(\Omega) + \Omega_q^2 |\chi(\Omega)|^2 T_1(\Omega)) \\ &= V_{12}^{bv}|_{\chi_F=0} \end{aligned} \quad (3.79)$$

where V_{12}^{bv} is given in Eq. (3.70b). Similarly,

$$\frac{\Omega_q}{\sqrt{\omega_m}} \mathcal{F} \left\{ \int_0^\infty dt' V_{12}^{vv}(t-t') \dot{\chi}(t') \right\} = V_{22}^{bv}|_{\chi_F=0}. \quad (3.80)$$

Then, similar to the case in Sec. 3.5, the terms related to the optomechanical interaction in V_{12}^{bv} and V_{22}^{bv} cancel out of Eq. (3.78).

We realize that thanks to these simplifications, the entanglement criterion in Eq. (3.71) is equivalent to the entanglement indicator defined in Eq. (3.29). However, while using Eq. (3.29), the operator M needs to be recomputed using Eq. (3.17). The expressions for the rest of the terms in Eq. (3.29), meaning $\tilde{\mathbf{V}}^{bb}$, $\tilde{\mathbf{V}}_{pt}^{bv}$, and $\mathbf{V}_M$ are unchanged compared to the case where the input field is the vacuum state.

Now, let us calculate $M(t, t')$, whose definition is given in Eq. (3.17). Different from Sec. 3.5, this operator has two time indices in the presence of arbitrary frequency-dependent squeezing. From the Wiener-Hopf method, we can first compute the columns of the matrix $V_{11}^{vv}(t, t')^{-1} \cdot [i\delta(t, t') + V_{12}^{vv}(t, t')]$. These columns will be functions of $t > 0$. Let us refer to them as $\{c_{t'}(t)\}$, where $t' > 0$ represents the column index, and t represents the row index, respectively. From Eqs. 3.68a and 3.68b, we have in the frequency domain

$$\begin{aligned} c_{t'}(\Omega) &= \frac{1}{T_1(\Omega)_+} \left[\mathcal{F} \{i\delta(t) + V_{12}^{vv}(-t)\} \frac{e^{i\Omega t'}}{T_1(\Omega)_-} \right]_+ \\ &= \frac{1}{T_1(\Omega)_+} \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} (i + T_{12}(\Omega)) \right]_+ \\ &\quad + \Omega_q^2 e^{i\Omega t'} \chi(\Omega). \end{aligned} \quad (3.81)$$

Then, the elements of the matrix

$$[-i\delta(t, t') + V_{21}^{vv}(t, t')] \cdot V_{11}^{vv}(t, t')^{-1} \cdot [i\delta(t, t') + V_{12}^{vv}(t, t')] \quad (3.82)$$

are given by

$$\int_0^\infty dx \left(-i\delta(t-x) + V_{21}^{vv}(x-t) \right) c_{t'}(x), \quad (3.83a)$$

$$= \int_{-\infty}^\infty dx \left(-i\delta(t-x) + V_{21}^{vv}(x-t) \right) c_{t'}(x), \quad (3.83b)$$

$$= \int_{-\infty}^\infty \frac{d\Omega}{2\pi} \mathcal{F}\{-i\delta(t-x) + V_{21}^{vv}(x-t)\} c_{t'}(\Omega). \quad (3.83c)$$

We were able to extend the lower integration limit to $-\infty$ in Eq. (3.83b) since the function $c_{t'}(t)$ is causal. Then, we used the Plancherel theorem to go from the time domain to the frequency domain in Eq. (3.83c). We can write Eq. (3.83c) explicitly as

$$\begin{aligned} & \int_{-\infty}^\infty \frac{d\Omega}{2\pi} \left\{ \frac{-i + T_{12}(\Omega)}{T_1(\Omega)_+} e^{-i\Omega t} \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} (i + T_{12}(\Omega)) \right]_+ \right. \\ & + \Omega_q^2 T_{12}(\Omega) e^{i\Omega(t'-t)} (\chi(\Omega) + \chi(-\Omega)) - i\Omega_q^2 e^{i\Omega(t'-t)} \\ & \left. (\chi(\Omega) - \chi(-\Omega)) + \Omega_q^4 e^{i\Omega(t'-t)} |\chi(\Omega)|^2 T_1(\Omega) \right\}. \end{aligned} \quad (3.84)$$

This expression describes the elements of a matrix indexed by $t, t' > 0$, where t and t' are the row and the column indices, respectively. We observe that this matrix is Hermitian, but not necessarily Toeplitz. The Hermitian property is seen from the fact that interchanging t and t' corresponds to the conjugation of this expression. However, this matrix can be written as a sum of a Hermitian matrix and a Hermitian Toeplitz matrix, where the Toeplitz part is the sum of the terms in Eq. (3.84) that are a function of $t - t'$ (i.e. the terms in 2nd and 3rd row).

Therefore, M will also be the sum of a Hermitian matrix and a Hermitian Toeplitz matrix. More specifically, using Eqs. (3.17), (3.68c), and (3.84), we find that M can be written as

$$M(t, t') = M_{\text{vac}}(t - t') + \Lambda(t, t'), \quad (3.85)$$

where $M_{\text{vac}}(t - t')$ is what we have found for M when the input light field is the vacuum state (given in Eq. (3.18c)), and $\Lambda(t, t')$ is some Hermitian operator independent of Ω_q , whose entries depend on the squeezing in the input light field. The full expression for $\Lambda(t, t')$ is written as

$$\Lambda(t, t') = T_2(t, t') - \int_{-\infty}^\infty \frac{d\Omega}{2\pi} \left(\frac{-i + T_{12}(\Omega)}{T_1(\Omega)_+} \right) e^{-i\Omega t} \cdot \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} (i + T_{12}(\Omega)) \right]_+. \quad (3.86)$$

We observe that when the input light field is the vacuum state, $T_2(t, t') = \delta(t - t')$, $T_{12}(\Omega) = 0$, and $T_1(\Omega)_\pm = 1$. Then, using Eq. (3.86), we find that $\Lambda = \mathbf{0}$ for this case, and $M(t, t') = M_{\text{vac}}(t - t')$.

We want to investigate whether optomechanical entanglement is achievable with arbitrary, frequency-dependent squeezing in the input light field. For this purpose, we first observe that when $\Omega_q \rightarrow \infty$, $M \rightarrow M_{\text{vac}}$, and the determinant condition in Eq. (3.29) is equivalent to its form when the input light field is the vacuum state for a given force and sensing noise. We refer to this configuration as the "vacuum case".

Now, we want to show that if the system is entangled for some Ω_q , it is bound to be entangled for $\Omega'_q > \Omega_q$. To show this, let us assume that the system is entangled for some Ω_q . We notice that the first two terms in Eq. (3.29) will be independent of Ω_q . The Ω_q dependency comes from the third term, where $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \propto \Omega_q$, and $M(t, t') = M_{\text{vac}}(t - t') + \Lambda(t, t')$, $M_{\text{vac}} \propto \Omega_q^2$. Then, the third term can be written in the form of

$$\begin{aligned} & \mathbf{C} \cdot \begin{bmatrix} \left(\frac{1}{\Omega_q^2} \Lambda + D \right)^{-1} & 0 \\ 0 & \left(\frac{1}{\Omega_q^2} \Lambda + D \right)^{-1} \end{bmatrix} \cdot \mathbf{C}^\dagger, \\ & \mathbf{C} = \frac{1}{\Omega_q} \tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \begin{bmatrix} -i\delta(\tau) & 0 \\ \delta(\tau) & 0 \end{bmatrix}, \quad D = \frac{1}{\Omega_q^2} M_{\text{vac}} \end{aligned} \quad (3.87)$$

where $\mathbf{C}$, D , and Λ are independent of Ω_q . Furthermore, notice that M is the Schur complement of the block $V_{11}^{\nu\nu}$ of the matrix $\mathbf{V}^{\nu\nu} + i\mathbf{K}^\nu$. Since $V_{11}^{\nu\nu}$ is positive definite, and $\mathbf{V}^{\nu\nu} + i\mathbf{K}^\nu$ is positive semi-definite (from the Heisenberg uncertainty principle), M is positive semi-definite for all Ω_q . Consequently, Λ and M_{vac} are also positive semi-definite, since $M = \Lambda$ when $\Omega_q = 0$, and $M \rightarrow M_{\text{vac}}$ in the limit of $\Omega_q \rightarrow \infty$. Also, recall that both D and Λ are Hermitian. As a result, we can write for $\Omega'_q > \Omega_q$

$$\begin{aligned} \frac{1}{\Omega_q'^2} \Lambda &\leq \frac{1}{\Omega_q^2} \Lambda, \\ \frac{1}{\Omega_q'^2} \Lambda + D &\leq \frac{1}{\Omega_q^2} \Lambda + D, \\ \left(\frac{1}{\Omega_q'^2} \Lambda + D \right)^{-1} &\geq \left(\frac{1}{\Omega_q^2} \Lambda + D \right)^{-1}. \end{aligned} \quad (3.88)$$

Therefore, if we refer to the matrix in the determinant condition in Eq. (3.29) as $\mathbf{V}_{\text{det}}(\Omega_q)$, we can write $\mathbf{V}_{\text{det}}(\Omega'_q) \leq \mathbf{V}_{\text{det}}(\Omega_q)$ for $\Omega'_q > \Omega_q$. Since the system

is entangled for Ω_q , the determinant of $\mathbf{V}_{\text{det}}(\Omega_q)$ is negative. Furthermore, since we are looking for the entanglement between one mode and the rest, $\mathbf{V}_{\text{det}}(\Omega_q)$ can have at most one negative eigenvalue (for this case, it has exactly one negative eigenvalue). Then, $\mathbf{V}_{\text{det}}(\Omega'_q)$ will also have a negative eigenvalue, and the system will be entangled. Hence, we proved that if optomechanical entanglement exists for Ω_q , it also exists for $\Omega'_q > \Omega_q$.

Consequently, we can make the following statements about the nature of optomechanical entanglement when the input light field is squeezed in a frequency-dependent fashion. First, we observe that universality with respect to the interaction strength, Ω_q , does not hold, since the entanglement indicator in Eq. (3.29) depends on Ω_q . Hence, increasing Ω_q may allow optomechanical entanglement to form. However, the transition is still unique with respect to the amount of environmental noise. Second, if there exists optomechanical entanglement for any frequency-dependent squeezing configuration, it will also exist when the input light field is in the vacuum state. Therefore, it is "easier" to achieve entanglement for a given force and sensing noise without frequency-dependently squeezing the input light field. In other words, the hyperspace of force and sensing noise spectra for which there exists optomechanical entanglement for the vacuum case encapsulates the space when the input light field is squeezed arbitrarily⁷. We observed numerically that after optomechanical entanglement is achieved, the amount of entanglement, quantified by the logarithmic negativity, can be greater or less than the case where the input light field is the vacuum state, depending on the frequency-dependent squeezing transformation.

In Fig. 3.3, we depict example configurations in which frequency-dependent squeezing suppresses the quantum noise spectra as

$$S(\Omega) \rightarrow e^{-2r} S(\Omega), \quad (3.89)$$

where $r > 0$ is the squeeze factor. This is achieved by first frequency-independently squeezing the light by e^r , and then, by filtering it with a detuned cavity [138] (see Appendix A.4). We again assume white force and sensing noises, whose spectrum at the phase quadrature of the output light field is plotted in Fig. 3.3a. Note that the force noise spectrum scales as Ω^{-2} in amplitude after being transduced by the mechanical oscillator. In Fig. 3.3b, we plot the boundary of the entangling-disentangling transition as a function of the force and sensing noise parameters Ω_F

⁷Note that the space is unchanged for frequency-independent squeezing.

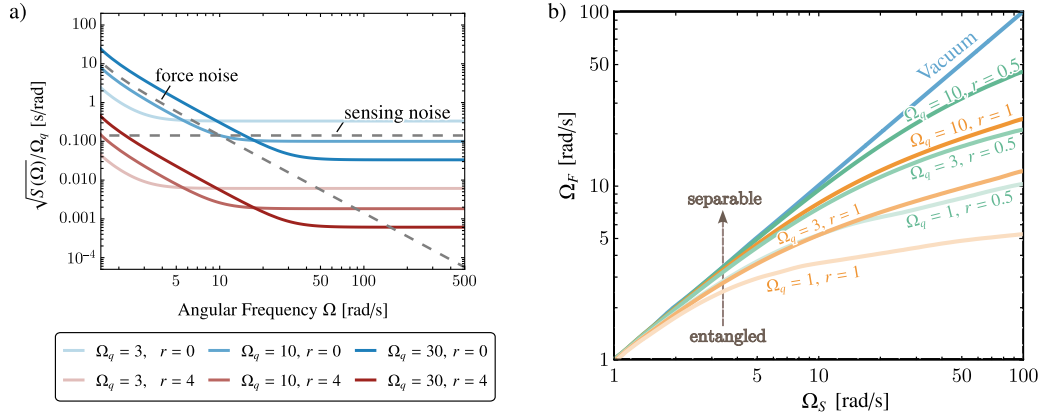


Figure 3.3: Impact of frequency-dependent squeezing on the entangling-disentangling transition. a) Amplitude spectra $\sqrt{S(\Omega)}$ normalized by the interaction strength Ω_q , as a function of different interaction strengths and squeezing levels. Spectra of the force and the sensing noises are shown in dashed lines. Although Markovian at the input, the force noise is transduced by the mechanical oscillator, causing it to decrease as Ω^2 in amplitude at the output. b) The entangling-disentangling transition with respect to the force and the sensing noises in the system, for different squeezing configurations. The optomechanical state is entangled for the noise configurations below their respective curves. When the input light is in the vacuum state (blue curve), the transition is independent of the interaction strength. We observe that as Ω_q increases, the transitions of the other configurations approach that of the vacuum case, in terms of the parameters for which they occur.

and Ω_S : as mentioned above, the transition occurs for $\Omega_F = \Omega_S$ when the input light field is the vacuum state. However, for frequency-dependent squeezing, the transition is harder to achieve, e.g., it requires a lower sensing noise level (larger Ω_S) for a given force noise level Ω_F .

3.7 Discussion for Frequency-Dependent Squeezing

In Sec. 3.6, it was proven that when the input light field is squeezed in a frequency-dependent fashion, it is harder to achieve optomechanical entanglement compared to the vacuum input case, i.e. there exists a region of the parameter space where optomechanical entanglement is observed only for the latter configuration. Intuitively, this result can be understood in the following way: when the ingoing light field is the vacuum state, the modes corresponding to different frequencies (or arrival times with respect to the surface of the oscillator) are not correlated. Conversely, when the ingoing light field is frequency-dependently squeezed, there is non-zero correlation between these modes. Then, when we consider the outgoing light field that has

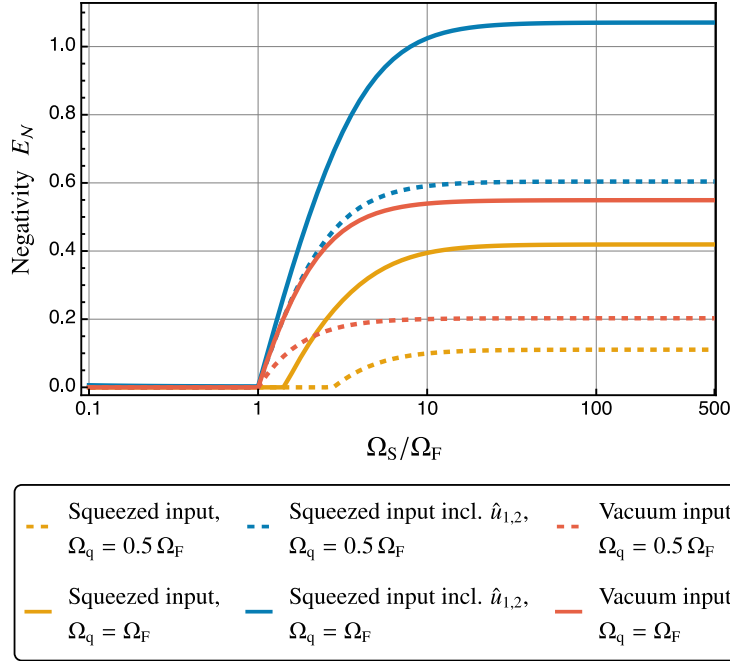


Figure 3.4: Logarithmic negativity for the case where the input light field is frequency-dependently squeezed with $r = 1$ (c.f. Eq. (3.89)), and for the vacuum input. We further consider the cases where we compute the entanglement between the mechanical oscillator at time $t = 0$ and the light field, where the field is considered to be i) the outgoing field that has left the oscillator during $t < 0$, ii) the joint field combining the outgoing field that has left the oscillator during $t < 0$, and the ingoing field that has yet to interact with the oscillator, but will be reflected from the oscillator during $t > 0$. These cases (for a squeezed input) are plotted in yellow and blue curves, respectively. For the vacuum input plotted in red, we only consider case i). We observe that for a squeezed input, entanglement is harder to achieve for case i), and the entangling transition is not universal with respect to the interaction strength Ω_q . However, for case ii), universality is restored.

reflected from the mechanical oscillator during time $t < 0$, it will be entangled with the ingoing light field that has yet to interact with the oscillator during this time (note that this field will be reflected from the oscillator during time $t > 0$). Hence, while computing the entanglement between this field and the mechanical oscillator at time $t = 0$, the correlation will induce an additional noise to the system when the ingoing light modes are traced out. Therefore, we can expect that it is harder to get entangled for this case, as it will appear to be more noisy than the vacuum input case.

This extra noise can be overcome by computing the entanglement between the mechanical oscillator at time $t = 0$, and the joint field combining the outgoing field

that has left the oscillator during $t < 0$, and the ingoing field that has yet to interact with the oscillator. Since the joint field contains information about the light field during time $t \in (-\infty, \infty)$, the frequency-dependent squeezing can be "reversed"⁸ by whitening the joint field. This would not be possible if we had only access to the outgoing light field, as the whitening transformation will be non-causal in the presence of frequency-dependent squeezing.

This situation is illustrated in Fig. 3.4. We fix the parameters of the mechanical oscillator as $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, and assume white environmental noise sources parametrized by $\Omega_F/(2\pi) = 10$ Hz, and $\Omega_S/(2\pi) \in [1, 5000]$ Hz. We first plot the logarithmic negativity for the vacuum input with red curves, for interaction strengths of $\Omega_q/(2\pi) = 5$ Hz and $\Omega_q/(2\pi) = 10$ Hz, with dashed and plain curves, respectively. We repeat this calculation for a frequency-dependently squeezed ingoing field (plotted in yellow). The squeezing transformation is again in the form of Eq. (3.89). For more details about the realization, see Appendix A.4. We fix $r = 1$, and observe that the entangling transition is not universal with respect to the interaction strength Ω_q in the presence of frequency-dependent squeezing, as was shown in Sec. 3.6. Finally, we repeat the calculation for the same squeezed input, but we compute the entanglement between the oscillator and the joint field combining the outgoing and the ingoing light field during $t < 0$ (plotted in blue). We observe that the universality is restored, and the entangling transition occurs at $\Omega_F \approx \Omega_S$, as found in Sec. 3.5. Note that the amount of entanglement is the largest for this case.

3.8 Conclusions

In this chapter, we proved the existence of an entangling-disentangling transition in an optomechanical system consisting of a mechanical oscillator interacting with a light field in a pure state. We assumed arbitrary, possibly non-Markovian, environmental noise sources. Assuming Gaussian dynamics, we were able to compute necessary and sufficient conditions for entanglement using the PPT criterion.

First, assuming that the input light field is the vacuum state, we showed that the entangling-disentangling transition is independent of the interaction strength between the oscillator and the light field, hence, it is universal. Therefore, if the environmental noises are such that the system is not in the regime where optomechanical entanglement can occur, one cannot achieve it by increasing the interaction

⁸The squeezing cannot be reversed perfectly due to the interaction with the mechanical oscillator.

strength between the oscillator and the light field.

In the second part of the chapter, we assumed arbitrary Gaussian and causal unitary transformations applied to the light field, including frequency-dependent squeezing. We demonstrated that the entangling-disentangling transition is not affected by frequency-independent squeezing and phase shifts – i.e. it is equivalent to the vacuum input case. However, when the light field is squeezed in a frequency-dependent manner, we showed that entanglement is harder to achieve. More specifically, we proved that if there is no optomechanical entanglement when the input light field is the vacuum state, it is impossible to generate it by squeezing the input light field arbitrarily.

With the techniques developed here, experimentalists could inquire whether the joint stationary optomechanical state of their device is entangled—in theory. For high frequency devices typically affected by Markovian noises, this topic has been discussed theoretically for more than ten years [81, 175], and attempts to demonstrate stationary optomechanical entanglement are ongoing [113, 191, 104]. Providing a quantitative framework to make predictions in the presence of non-Markovian noises is crucial for macroscopic devices operating at low frequencies, since they are typically affected by non-Markovian environments [99].

Finally, let us consider the fact that before applying *coarse graining*, the system that consists of the heat bath, the oscillator and the light field is in a pure state. With knowledge from the bath, are we able to "reinstate" optomechanical entanglement? This is possible in principle, at least in some cases. In our model, $\hat{n}_S$ can be a classical random process, while $\hat{n}_F$ can be the superposition of a classical random process and quantum noise that drives the zero-point fluctuations of the oscillator. In principle, one can know the particular realizations of both classical random processes, and hence reduce the environmental noises of the oscillator to its zero-point level—far below the entangling-disentangling transition.

MACROSCOPIC QUANTUM ENTANGLEMENT IN LIGO

4.1 Introduction

Entanglement is one of the hallmarks of the "quantumness" of physical systems¹. Ideally, it is possible for macroscopic objects, massive and/or containing a high number of degrees of freedom, to be entangled with each other. Yet in practice, such macroscopic entanglement can be very delicate in the presence of decoherence. It is an intriguing challenge to create and verify macroscopic entanglement, which is often viewed as expanding the limits of the quantum regime.

Optomechanical systems are promising candidates for experimental demonstration of macroscopic entanglement, partly due to their theoretical robustness against mechanical decoherence imposed by coupling to a possibly highly populated thermal bath [81]. They can also be used to engineer the quantum state of the mechanical system [114], where the entanglement is generated by the momentum exchange between the light reflecting from the mechanical oscillator—a phenomenon known as radiation pressure. It is theoretically well understood and broadly discussed in the literature [82, 176, 177, 175, 59, 115]; see for example, Refs. [81, 11] for a review.

Entanglement in optomechanical devices has been widely studied and there have been several successful experimental realizations: stationary entanglement between simultaneous light tones mediated by an optomechanical device [16, 46], generation of entanglement between spaced mechanical oscillators both in the micro and macro regime via radiation pressure [219, 196, 147, 251], and optomechanical entanglement between the light field and the mechanical oscillator in a pulsed scheme [199] are examples of such demonstrations. There also exist many proposals in the literature to further study macroscopic quantum phenomena in optomechanical systems [223, 183, 233, 263, 206] and entanglement between coupled oscillators in the presence of non-Markovian baths [163]. In this chapter, we again consider stationary optomechanical entanglement, where the system parameters (e.g. the

¹This chapter is adapted from: S. Direkci et al., "Macroscopic quantum entanglement between an optomechanical cavity and a continuous field in presence of non-Markovian noise," *Phys. Rev. Research* **6**, 013175 (2024), DOI: 10.1103/PhysRevResearch.6.013175. Published by the American Physical Society under a Creative Commons Attribution 4.0 International (CC BY 4.0) license.

driving), and statistical behavior thereof, are not changing over time. Schemes to verify stationary optomechanical entanglement were proposed in [175, 176, 103, 262, 201], whereas to the best of our knowledge an experimental demonstration has not been performed yet.

Our system consists of a single mechanical mode interacting with an optical cavity mode and the quadratures of the light field exiting the cavity. At any time t , we study the bipartite entanglement that is present in the joint quantum state of the mechanical mode, optical mode, and the light that has exited the system during $t' \leq t$. See Fig. 1 of Ref. [175], which includes a space-time diagram that illustrates the configuration. In the regime where the dynamics are linear, the state is Gaussian, and the noise processes are Markovian (white), the open-system optomechanical dynamics is solvable analytically and the state of the system can be known exactly [82]. The white-noise model describes well devices with high-frequency oscillators, where only thermal excitations are expected, and in the limit of large bath temperature where $k_B T \gg \hbar \omega_m$, T is the temperature of the bath and ω_m is the resonance frequency of the oscillator [85].

In this chapter, we extend the description to non-Markovian Gaussian noise processes where analytical results are, to our knowledge, not available, thus requiring numerical methods. This approach is applicable whenever non-white-noise processes, such as structural damping [227, 189, 71, 99], are relevant. We extend the methods developed in [175], incorporating a cavity, and, more importantly, non-Markovian noise processes. The technique consists in computing the minimal symplectic eigenvalue of the partially transposed covariance matrix of the system, constructed with numerical methods, which provides a measure of appropriate bipartite entanglement.

We first investigate entanglement in a generalized setting, considering a heavy suspended oscillator with a low mechanical resonance frequency. This corresponds to the free-mass limit, where the mechanical resonance frequency is much smaller than the other characteristic frequencies of the system. We work in this limit to study the general behavior of heavy suspended oscillators affected by environmental decoherence, however the algorithm does not require this assumption. Subsequently, we stop working in the free-mass limit and focus our attention on the Advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) [52], using it as a case study. It has been recently shown that by injecting squeezed vacuum, the detector's quantum noise can in principle surpass the free-mass standard quantum

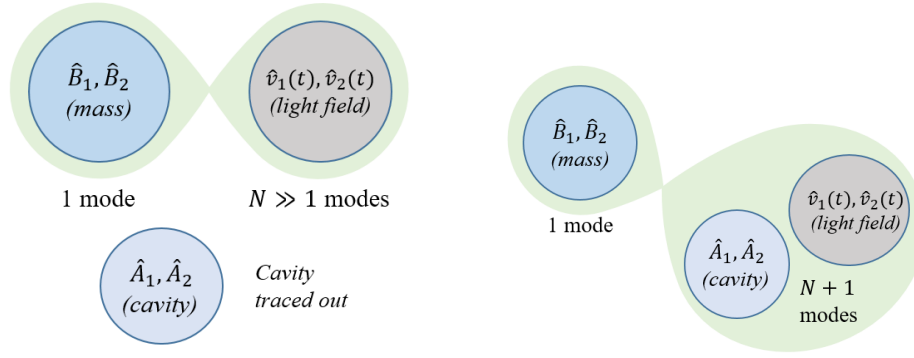


Figure 4.1: Figurative representation of the two different partitions that are used while testing for entanglement, which are defined by tracing over (top row) and *not* tracing over (bottom row) the cavity. Note that the system configuration is not changed, i.e. cavity is still present for both partitions.

limit (SQL) by 3 dB [278].

It is natural to ask whether this can already imply that aLIGO has built quantum entanglement between the mirrors and the light field. The answer to this question is non-trivial. First, from [278], we see that the level of classical noise is not yet below the SQL². Second, the strict definition of entanglement we use here requires integrating over all frequencies: it remains uncertain whether having noise below the SQL within a certain finite frequency band automatically leads to entanglement. Therefore, we parametrize aLIGO's noise curves to investigate regimes where entanglement, according to its strict definition, exists.

This chapter is organized as follows: In Sec. 4.2, we introduce the dynamics of the system and its equations of motion. In Sec. 4.3, we state our entanglement criterion and the covariance matrix of the system for two partitions of interest. To show the usefulness of our technique for systems with low mechanical resonance frequencies, we investigate entanglement in a generalized setting in Sec. 4.4. In Sec. 4.5, we give details about aLIGO's noise budget, and talk about how we model it in our calculations. Finally, in Sec. 4.6, we investigate whether there is entanglement between the mechanical oscillator and the light field at aLIGO for the partitions of interest, given different parametrizations of the classical noise curves.

²The quantum noise's beating of the SQL is only inferred by subtracting a classical noise floor that was obtained through calibration.

4.2 System Dynamics

Let us consider an optical cavity with a movable mirror, driven by a laser with frequency ω_0 close to one of the resonant frequencies of the cavity, $\omega_0 + \Delta$ [38]. The quantity Δ is often referred to as the detuning frequency of the cavity. For such a system, the linearized Hamiltonian in the interaction picture with the rotating-wave approximation is given by [47]

$$\begin{aligned} H/\hbar = & \omega_m \hat{B}^\dagger \hat{B} + \hbar \Delta \hat{A}^\dagger \hat{A} - \hbar G \hat{x} (\hat{A}^\dagger + \hat{A}) + \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \Omega \hat{c}^\dagger(\omega_0 + \Omega) \hat{c}(\omega_0 + \Omega) \\ & + i\sqrt{2\gamma} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} [\hat{A}^\dagger \hat{c}(\omega_0 + \Omega) - \hat{A} \hat{c}^\dagger(\omega_0 + \Omega)] , \end{aligned} \quad (4.1)$$

where $\hat{B}$ and $\hat{B}^\dagger$ are the annihilation and creation operators of the mechanical mode (center of mass motion of the mirror), $\hat{x}$ is the position of the center of mass of the mirror, ω_m is the mechanical resonance frequency, $\hat{A}$ and $\hat{A}^\dagger$ are the annihilation and creation operators of the cavity mode, $\hat{c}(\omega_0 + \Omega)$ and $\hat{c}^\dagger(\omega_0 + \Omega)$ are the annihilation and creation operators of the external vacuum light field at frequency $\omega_0 + \Omega$, G is the linear optomechanical coupling constant, and γ is the decay rate of the cavity mode. The position and momentum operators of the mirror are related to the creation and annihilation operators of the mechanical mode by

$$\hat{x} = \sqrt{\frac{\hbar}{M\omega_m}} \frac{(\hat{B} + \hat{B}^\dagger)}{\sqrt{2}} , \quad \hat{p} = \sqrt{\hbar M\omega_m} \frac{(\hat{B} - \hat{B}^\dagger)}{\sqrt{2}i} . \quad (4.2)$$

Note that for the sake of convenience we chose a displaced frame where all operators have zero mean. The mode operators satisfy the canonical commutation relations,

$$[\hat{A}, \hat{A}^\dagger] = [\hat{B}, \hat{B}^\dagger] = 1. \quad (4.3)$$

aLIGO detectors are power- and signal-recycled Fabry-Perot Michelson interferometers, which contain a high number of degrees of freedom. However, the core optomechanics can still be studied by the Hamiltonian given above; this reduction manifests itself in the "scaling-law" relations governing aLIGO's sensitivity as parameters of the signal-recycling cavity are modified [38]. From the scaling-law, the coupling constant G is related to the parameters of the interferometer by,

$$G = \sqrt{\frac{2\omega_0 P_c}{\hbar L c}} , \quad (4.4)$$

where L is the arm length of the interferometer (i.e. the cavity length), P_c is the power circulating inside the cavity, and c is the speed of light in vacuum.

We can transform the Hamiltonian such that the cavity mode $(\hat{A}, \hat{A}^\dagger)$ couples with the traveling wave at $z = 0$ (where the point-wise cavity interface is located). To do this, we notice that the symbol Ω in Eq. (4.1) is used to label a spatial mode which has a reduced wavenumber of Ω/c , and a free oscillation frequency of Ω . More specifically, $\omega_0 + \Omega$ is used to indicate a spatial mode whose wave number is $(\omega_0 + \Omega)/c$, where c is the speed of light. We can also write the same Hamiltonian in the spatial domain. Following [47] and setting $c = 1$, we define

$$\hat{c}(z) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \hat{c}(\omega_0 + \Omega) e^{i\Omega z}, \quad (4.5)$$

which represents a spatial mode of the light field with wave number Ω —or a temporal mode of the free light field with frequency Ω , since we assume no dispersion. The Hamiltonian can then be rewritten as

$$\begin{aligned} H = & \hbar\omega_m \hat{B}^\dagger \hat{B} + \hbar\Delta \hat{A}^\dagger \hat{A} - \hbar G \hat{x} (\hat{A}^\dagger + \hat{A}) - i\hbar \int_{-\infty}^{\infty} \hat{c}^\dagger(z) \partial_z \hat{c}(z) dz \\ & + i\hbar \sqrt{2\gamma} [\hat{A}^\dagger \hat{c}(z=0) - \hat{A} \hat{c}^\dagger(z=0)]. \end{aligned} \quad (4.6)$$

The only non-zero commutators for the creation and annihilation operators, as well as the spatial modes of the light field are

$$[\hat{c}(\omega_0 + \Omega), \hat{c}^\dagger(\omega_0 + \Omega')] = 2\pi\delta(\Omega - \Omega'), \quad (4.7a)$$

$$[\hat{c}(z), \hat{c}^\dagger(z')] = \delta(z - z'), \quad (4.7b)$$

where δ is the Dirac delta distribution. We then use $\hat{u} := \hat{c}(z = 0^-)$ and $\hat{v} := \hat{c}(z = 0^+)$ to label the field right before entering and right after exiting the cavity, respectively.

In this chapter, we restrict ourselves to $\Delta = 0$. In this resonant case, the system is unconditionally stable and it reaches a steady state, in which the Heisenberg equations can be solved using Fourier transformation³. To write down and solve the Heisenberg equations, instead of annihilation and creation operators we use the Caves-Schumaker quadrature operators [43, 236]:

$$\hat{u}_1(\Omega) = \frac{\hat{u}(\omega_0 + \Omega) + \hat{u}^\dagger(\omega_0 - \Omega)}{\sqrt{2}}, \quad (4.8a)$$

$$\hat{u}_2(\Omega) = \frac{\hat{u}(\omega_0 + \Omega) - \hat{u}^\dagger(\omega_0 - \Omega)}{\sqrt{2}i}, \quad (4.8b)$$

³We use the convention $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$.

where $\hat{u}_j^\dagger(\Omega) = \hat{u}_j(-\Omega)$. Quadratures $\hat{v}_1(\Omega)$ and $\hat{v}_2(\Omega)$ are defined from $\hat{v}(\omega_0 + \Omega)$ and $\hat{v}(\omega_0 - \Omega)$ in a similar fashion. Their commutation relations are [58]

$$[\hat{u}_1(\Omega), \hat{u}_2(\Omega')] = [\hat{v}_1(\Omega), \hat{v}_2(\Omega')] = i2\pi\delta(\Omega + \Omega'), \quad (4.9a)$$

$$[\hat{u}_j(\Omega), \hat{u}_j(\Omega')] = [\hat{v}_j(\Omega), \hat{v}_j(\Omega')] = 0, \quad (4.9b)$$

for $j = 1, 2$. Then, in the time domain, we have

$$\hat{u}_j(t) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \hat{u}_j(\Omega) e^{-i\Omega t}, \quad (4.10a)$$

$$[\hat{u}_1(t), \hat{u}_2(t')] = i\delta(t - t'), \quad (4.10b)$$

$$[\hat{u}_1(t), \hat{u}_1(t')] = [\hat{u}_2(t), \hat{u}_2(t')] = 0, \quad (4.10c)$$

for $j = 1, 2$. We similarly define quadrature operators $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$, in the *time domain*, with

$$\hat{A}_1(t) = \frac{\hat{A}(t) + \hat{A}^\dagger(t)}{\sqrt{2}}, \quad \hat{A}_2(t) = \frac{\hat{A}(t) - \hat{A}^\dagger(t)}{\sqrt{2}i}, \quad (4.11)$$

and similarly for $\hat{B}_{1,2}$. We also have

$$[\hat{A}_1(t), \hat{A}_1(t)] = [\hat{A}_2(t), \hat{A}_2(t)] = 0, \quad [\hat{A}_1(t), \hat{A}_2(t)] = i, \quad (4.12)$$

and the same for $\hat{B}_{1,2}$. Note here that the commutators are for same-time operators.

We include two classes of "classical" noises [58] in our system: a force noise $\hat{n}_F$ and a sensing noise $\hat{n}_X$, originally arising from a quantum treatment of the interaction of the system with its environment. We label them as "classical" noises because we assume that the details – and possible quantum limit – of a microscopic model of these noises are irrelevant (typically because they arise from thermal baths in the high temperature limit) unlike the noise arising from vacuum fluctuations, $\hat{u}_1(\Omega)$ and $\hat{u}_2(\Omega)$. Force noise affects the center-of-mass motion of the mechanical oscillator by introducing fluctuations in its momentum. We also introduce a velocity damping of the oscillator, with a damping rate γ_m . γ_m and n_F are associated with the heat bath(s) the mass is coupled to, with the value of γ_m and the spectrum of n_F related by the fluctuation-dissipation theorem [150]. Sensing noise affects how the position is measured by the light field. In our model below it arises from fluctuations of the reflecting surface that introduces noise in the cavity field.

In the Heisenberg picture, the dynamics are given by the Langevin equations of motion. In the Fourier domain they are written as

$$-i\Omega\hat{A}_1 = -\gamma\hat{A}_1 + \sqrt{2\gamma}\hat{u}_1, \quad (4.13a)$$

$$-i\Omega\hat{A}_2 = -\gamma\hat{A}_2 + \sqrt{2\gamma}\hat{u}_2 + \sqrt{2}G(\hat{x} + \hat{n}_X), \quad (4.13b)$$

$$-i\Omega\hat{x} = \hat{p}/M, \quad (4.13c)$$

$$-i\Omega\hat{p} = -\gamma_m\hat{p} - M\omega_m^2\hat{x} + \sqrt{2}\hbar G\hat{A}_1 + \hat{n}_F, \quad (4.13d)$$

$$\hat{v}_1 = \hat{u}_1 - \sqrt{2\gamma}\hat{A}_1, \quad (4.13e)$$

$$\hat{v}_2 = \hat{u}_2 - \sqrt{2\gamma}\hat{A}_2. \quad (4.13f)$$

We refer to Eqs. (4.13) as the Heisenberg equations for the rest of the chapter. It is straightforward to solve them to obtain $(\hat{x}, \hat{p}, \hat{A}_{1,2}, \hat{v}_{1,2})$ in terms of the input fields, $(\hat{u}_{1,2}, \hat{n}_X, \hat{n}_F)$, referred to as the input-output relations of the system. More specifically, quantum fluctuations in the in-going quadratures $\hat{u}_{1,2}(\Omega)$ drive the system's quantum noise [138]. From Eq. (4.13e) and (4.13f), reading the outgoing field quadratures are subject to noises in $\hat{u}_1$ and $\hat{u}_2$, giving rise to the *shot noise* (SN) for that readout strategy [138]. On the other hand, from Eq. (4.13a), we see that $\hat{u}_1$ drives $\hat{A}_1$, which in Eq. (4.13d) drives the momentum of the test mass, which then shows up in the position of the test mass via Eq. (4.13c), giving rise to *quantum radiation pressure noise* (QRPN), also known as back action noise in the literature. In general, the power spectrum of the SN is inversely proportional to circulating power in the cavity, while that of the QRPN is proportional to circulating power.

4.3 Entanglement Criteria and Partitions

Here, we summarize the entanglement criteria (PPT criterion) that we initially described in Sec. 3.4. The canonical commutation relations imply that $\mathbf{V} + \frac{1}{2}\mathbf{K}$ is positive semi-definite, where $\mathbf{V}$ is the covariance matrix with $\mathbf{V}_{ij} = \langle\{\hat{X}_i - \langle\hat{X}_i\rangle, \hat{X}_j - \langle\hat{X}_j\rangle\}\rangle/2$ and $\mathbf{K}_{ij} = [\hat{X}_i, \hat{X}_j]$ is the commutator matrix of the quadratures in the system. This relation can be stated as

$$\mathbf{V} + \frac{1}{2}\mathbf{K} \geq 0. \quad (4.14)$$

Here for an N -partite system containing N harmonic oscillators, the matrices $\mathbf{V}$ and $\mathbf{K}$ are $2N$ -dimensional.

To test for bipartite entanglement in a multimode system, we use the PPT criterion, which is necessary and sufficient to test for the separability of one of the modes from the rest for Gaussian systems [202, 4, 268].

To use the PPT criterion in this context, one obtains the partial-transposed covariance matrix $\mathbf{V}_{\text{pt}}$ by reverting the momentum of that one mode (which puts a minus sign on the column and the row that contains the momentum in question) [243]. The PPT criterion for separability is expressed as

$$\mathbf{V}_{\text{pt}} + \frac{1}{2}\mathbf{K} \geq 0 \Leftrightarrow \text{Separability} . \quad (4.15)$$

The amount of entanglement is quantified by the logarithmic negativity, $E_{\mathcal{N}}$ [261]. For a Gaussian state of N modes, it is defined as

$$E_{\mathcal{N}} = \sum_{j=1}^N \max\{0, -\ln(\tilde{\nu}_j)\} , \quad (4.16)$$

where $\tilde{\nu}_j$, $j = 1, \dots, N$ are the symplectic eigenvalues of the partially transposed covariance matrix, $\mathbf{V}_{\text{pt}}$, which are given by the absolute values of the eigenvalues of $\mathbf{K}^{-1}\mathbf{V}_{\text{pt}}$. For 1 vs. $N - 1$ mode partitions, only one of the symplectic eigenvalues of $\mathbf{V}_{\text{pt}}$ can have a magnitude smaller than 1 [238], therefore there can be at most one negative eigenvalue of $\mathbf{V}_{\text{pt}} + \frac{1}{2}\mathbf{K}$. We label the corresponding symplectic eigenvalue as $\tilde{\nu}_{\min}$.

Using the PPT criterion, we test for entanglement between the mechanical oscillator and the optical field in two ways: first, we construct the covariance matrix $\mathbf{V}$ with the mechanical mode and the modes of the light field, essentially tracing out the cavity mode. Here, we perform the partial transpose operation with respect to the mechanical oscillator. Second, we include the cavity mode in the covariance matrix while still taking the partial transpose with respect to the mechanical oscillator, which corresponds to measuring the entanglement between the oscillator and the joint system of the cavity plus external light field. The two ways of partitioning are depicted in Fig. 4.1. The elements of the covariance matrix $\mathbf{V}$ for both types of partitions, as well as the discretization of $\mathbf{V}$, can be found in Appendix B.1.

4.4 Entanglement in the Presence of Non-Markovian Noises

Due to the numerical nature of the algorithm, we can tackle any noise spectral density associated with $\hat{u}_1(\Omega)$, $\hat{u}_2(\Omega)$, $\hat{n}_F(\Omega)$ and $\hat{n}_X(\Omega)$ using the PPT criterion defined in Eq. (4.15) to determine whether entanglement is present for a given partition. Conversely, this problem is analytically solvable only for some simplified noise models to our knowledge, such as assuming all the noise sources to have a white spectrum [175, 103].

To show the usefulness of the method, we investigate entanglement in heavy suspended oscillators with relatively low mechanical resonance frequencies. Examples of such systems are aLIGO, KAGRA [247], and VIRGO [3], but also smaller devices such as those in [54, 146]. For such systems, the mechanical resonance frequency, ω_m , is much smaller than the other frequencies of the system, which is referred to as the *free-mass limit*. In this setting, ω_m essentially does not affect the dynamics. Furthermore, in this limit where $\Omega \gg \omega_m, \gamma_m$, the trade-off between shot noise and QRPN gives rise to the SQL [31], given by

$$S_{\text{SQL}}(\Omega) = \frac{2\hbar}{M\Omega^2}. \quad (4.17)$$

In the context of suspended oscillators, $\hat{n}_F(\Omega)$ is the force that gives rise to the suspension thermal noise, whereas $\hat{n}_X(\Omega)$ is an effective displacement that gives rise to coating thermal noise. When thermal noise is due to the internal friction of the suspension or the oscillator, the noise spectrum of $\hat{n}_F(\Omega)$ and $\hat{n}_X(\Omega)$ decreases as $1/\Omega$ above internal resonances, which is referred to as *structural damping* [227, 88, 189]. Evidently, structural damping gives rise to non-Markovian noises, and the position-referred noise spectral densities of $\hat{n}_F(\Omega)$ and $\hat{n}_X(\Omega)$ are given, in the free-mass limit, by

$$S_F(\Omega) = \frac{2\hbar}{M} \frac{\Omega_F^3}{|\Omega|^5}, \quad S_X(\Omega) = \frac{2\hbar}{M} \frac{1}{\Omega_X |\Omega|}, \quad (4.18a)$$

where Ω_F and Ω_X are the frequencies where the respective noise curves cross the SQL, given in Eq. (4.17). Accordingly, they encode the strength of the noise processes n_F and n_X , relative to the SQL level. For $\hat{n}_F(\Omega)$, the position-referred spectrum is related to the noise spectral density, labeled as $S_{n_F}(\Omega)$, with $S_F(\Omega) = S_{n_F}(\Omega)/M^2\Omega^4$, whereas for $\hat{n}_X(\Omega)$, the position-referred spectrum $S_X(\Omega)$ is also the noise spectral density $S_{n_X}(\Omega)$ ⁴. The incoming field quadratures $\hat{u}_j$ have uncorrelated white spectra given by Eq. (B.8), since we assume the incoming field to be at vacuum state.

In the limit of large cavity bandwidth, $\gamma \gg \Omega$ the cavity can be eliminated adiabatically

⁴To find spectra of the form of Eqs. (4.18), one assumes the free-mass and high- Q limits where the mechanical susceptibility behaves as $1/\Omega^2$; additionally, one must be in frequency ranges where S_{n_F} and S_{n_X} are constant beyond the polynomial roll-off – typical at frequencies above the resonance of suspended oscillators.

ically. Then, the equations of motion in (4.13) are modified as

$$\hat{v}_1(\Omega) = \hat{u}_1(\Omega), \quad (4.19a)$$

$$\hat{v}_2(\Omega) = \hat{u}_2(\Omega) + \alpha(\hat{x}(\Omega) + \hat{n}_X(\Omega)), \quad (4.19b)$$

$$-i\Omega\hat{p}(\Omega) = -\gamma_m\hat{p}(\Omega) - M\omega_m^2\hat{x}(\Omega) + \hbar\alpha\hat{u}_1(\Omega) + \hat{n}_F(\Omega), \quad (4.19c)$$

$$-i\Omega\hat{x}(\Omega) = \hat{p}(\Omega)/M, \quad (4.19d)$$

where $\alpha = \Omega_q\sqrt{M/\hbar}$ and $\Omega_q = 2G\sqrt{\hbar/M\gamma}$ is the characteristic interaction frequency. In the context of structural damping, there is no velocity damping, instead, the damping arises from a complex spring constant associated with the mechanical oscillator. Accordingly, Eq. (4.19b) is modified as

$$-i\Omega\hat{p}(\Omega) = -M\omega_m^2(1 + i\phi(\Omega))\hat{x}(\Omega) + \hbar\alpha\hat{u}_1(\Omega) + \hat{n}_F(\Omega), \quad (4.20)$$

where $\phi(\Omega)$ is referred to as the *loss angle*. When structural damping is present, $\phi(\Omega)$ is constant for a large band of frequencies and goes to zero as $\Omega \rightarrow 0$, however the dependence of $\phi(\Omega)$ to Ω depends on the properties of the material [227]. Numerically, we choose to model this with $\phi(\Omega) = \phi \cdot \Omega/(\Omega + \Omega_c)$, so that $\phi(\Omega) \approx \phi$ for $\Omega \gg \Omega_c$ and $\phi(0) = 0$ for some cutoff frequency Ω_c . Then, Ω_c determines the noise power of $\hat{n}_F$ and $\hat{n}_X$ at 0 Hz.

We fix $\omega_m/(2\pi) = 1$ Hz, $\Omega_F/(2\pi) = 100$ Hz, $\Omega_c/(2\pi) = 0.05$ Hz, $\phi = 0.05$, and vary Ω_X and Ω_q . Note that the aim of these choices is to ensure that $\Omega_F \gg \omega_m$, i.e. that the free-mass limit is justified. This choice also implies the large bath temperature limit where $k_B T \gg \hbar\omega_m$, T being the temperature of the bath. Furthermore, $\Omega_q \gg \omega_m$ ensures that the measurement of the system performed by light is faster than the dynamics of the system. Lastly, we perform the simulations by sampling the covariance matrix for $\tau = 0.1$ s, from $t = -0.1$ s to $t=0$ s, where t indicates the time (for the sampling of the covariance matrix, see Appendix B.1). Hence, we cannot resolve the very low frequency regime of Ω_c . Physically, this corresponds to the finite detection frequency resolution that renders the behavior of ϕ at the cutoff inaccessible. In this sense, one might expect Ω_c to be irrelevant. However, we see that it does affect the final characterisation of entanglement, due to its contribution to the total variance of the quadratures of the mechanical oscillator.

After specifying the low-frequency behavior of ϕ with Ω_c , the relevant parameters on which the presence of entanglement depends are Ω_q , Ω_F , and Ω_X . Then, working in the free-mass limit enables us to examine the general behavior of suspended oscillators with low mechanical resonance frequencies, classified by their coating

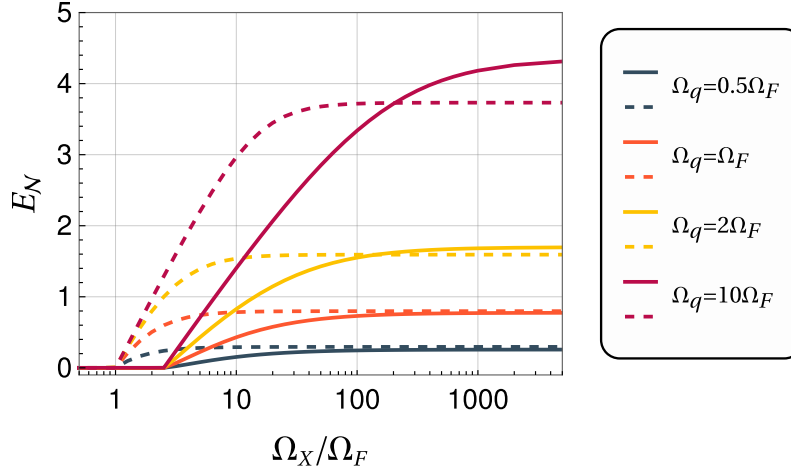


Figure 4.2: Logarithmic negativity between the mechanical oscillator and the outgoing light field in the free-mass limit as a function of Ω_X/Ω_F for various Ω_q/Ω_F . We plot the results for Markovian and non-Markovian force and sensing noise with dashed and plain lines, respectively. Note that entanglement does not exist for $\Omega_X/\Omega_F \lesssim 1$ with Markovian and for $\Omega_X/\Omega_F \lesssim 2.6$ with non-Markovian noise sources, for $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, $\phi = 0.05$, and a cutoff frequency of $\Omega_c/(2\pi) = 0.05$ Hz.

materials and suspension systems (i.e. the low-frequency behavior of ϕ). Therefore, we look for entanglement between the oscillator and the outgoing light field by varying the ratio Ω_X/Ω_F for various Ω_q in Fig. 4.2, and we plot the results with plain lines. We find that for $\Omega_c/(2\pi) = 0.05$ Hz, entanglement does not exist for $\Omega_X/\Omega_F \lesssim 2.6$ for any value of Ω_q . For $\Omega_X/\Omega_F \gtrsim 2.6$, the system is entangled for any (finite) Ω_q , and the entanglement increases monotonically with increasing Ω_X/Ω_F . This is in line with the universality of entanglement, as we discussed in Chapter 3. We also note that the threshold for Ω_X/Ω_F above which the system is entangled depends on the low-frequency cutoff Ω_c chosen in our model for the spectra of $\hat{n}_F$ and $\hat{n}_X$. We saw that the threshold is inversely proportional to the cutoff frequency.

In order to see the significance of non-Markovianity on the results, we repeat the same procedure with a white force and sensing noise. The noise spectra are given by $S_{n_F}(\Omega) = 2\hbar M \Omega_F^2$, $S_F(\Omega) = 2\hbar \Omega_F^2 / M \Omega^4$, and $S_{n_X}(\Omega) = S_X(\Omega) = 2\hbar / M \Omega_X^2$. The results can again be found in Fig. 4.2, plotted with dashed lines, where the system is not entangled for any Ω_q when $\Omega_X/\Omega_F \lesssim 1$, whereas for $\Omega_X/\Omega_F \gtrsim 1$, entanglement exists for all Ω_q and the amount of entanglement increases with increasing Ω_q .

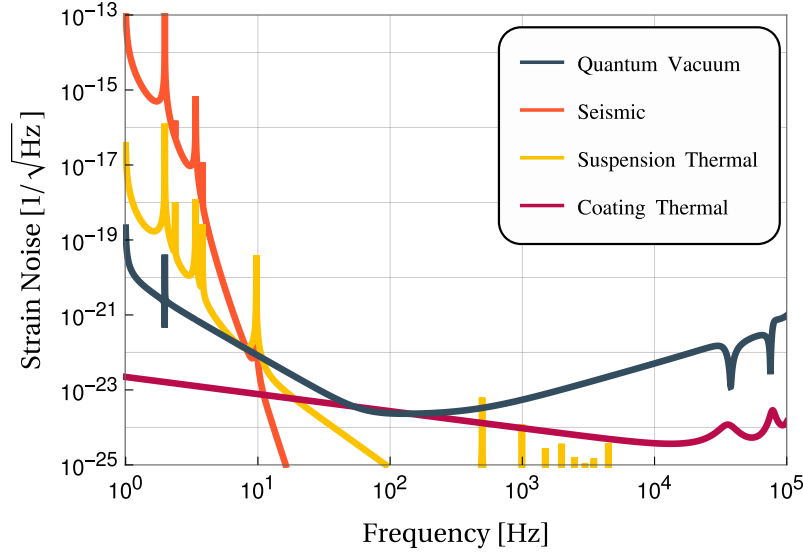


Figure 4.3: aLIGO noise budget obtained from pygwinc. Only the dominant classical noise sources are plotted, along with the quantum noise. The total force noise in the system is the sum of the seismic noise and the suspension thermal noise, which is effective at low frequencies. The coating Brownian thermal noise is taken as the main constituent of the sensing noise. As can be seen from the figure, quantum noise dominates over the sensing noise by a large margin at high frequencies.

4.5 Noise Model of aLIGO

The primary noise sources in aLIGO, other than the quantum noise, are the following [52]: seismic noise and suspension thermal noise are the main constituents of the force noise, and mirror coating thermal noise constitutes the sensing noise. The noise spectrum is dominated by seismic and thermal noise at low frequencies (up to 100 Hz), and quantum noise at high frequencies, cf. Fig. 4.3. The interferometer noise is stationary and Gaussian to very good approximation in the absence of glitches (i.e. transient noise artifacts) [1].

Seismic noise occurs because of the ground motion at the interferometer sites. This motion is $\sim 10^{-9} \text{ m}/\sqrt{\text{Hz}}$ at 10 Hz [171]. To provide isolation from this motion, the mirrors are suspended from quadruple pendulums [12]. The primary components of thermal noise are suspension thermal noise and coating Brownian noise. Suspension thermal noise occurs due to loss in the fused silica fibers used in the final suspension stage [52], whereas the coating Brownian noise (which is classified as a sensing noise) occurs due to the mechanical dissipation in the coatings [161]. Other types of sensing noise comprise many noise sources that are dominant at high frequencies, such as thermal fluctuations of the mirror's shape, optical losses, or photodetection

inefficiency [184]. The noise budget of aLIGO can be found in Figure 4.3.

Due to the classification above, we represent the sum of the seismic and the suspension thermal noise with $\hat{n}_F(\Omega)$, and the coating thermal noise with $\hat{n}_X(\Omega)$. We use the aLIGO noise budgets as given by the Python Gravitational Wave Interferometer Noise Calculator library (pygwinc) [222] and we model them with rational functions of Ω^2 . The noise spectra are modeled by

$$S_F^{LIGO}(\Omega) = \frac{\tau_F \alpha_{F_1}}{\left(\frac{\Omega}{\omega_F} \alpha_{F_2}\right)^{14} + 1}, \quad (4.21a)$$

$$S_X^{LIGO}(\Omega) = \tau_{X_1} \left(\frac{\Omega}{\omega_X}\right)^2 \alpha_{X_1} + \tau_{X_2} \alpha_{X_2}, \quad (4.21b)$$

where $S_F^{LIGO}(\Omega)$ is the spectrum of the force noise, and $S_X^{LIGO}(\Omega)$ is the spectrum of the sensing noise. We model $S_F^{LIGO}(\Omega)$ to decay as Ω^{-14} instead of Ω^{-16} (which is the expected behavior for quadruple suspension systems) since it performs better at approximating the global behavior. The power spectral densities are characterized by the time constants τ_F , τ_{X_1} , τ_{X_2} and cutoff frequencies ω_F , ω_X . The values of these parameters can be found in Appendix B.2, Table B.1. α_{F_1} , α_{F_2} , α_{X_1} , and α_{X_2} are dimensionless constants that will be used to change the noise curves in Section 4.6. Their effect on the noise curves can be seen in Fig. 4.4. If we set all of them to be unity, we get our model of aLIGO noise curves.

Even though the parameters of aLIGO are well-known, they need to be recomputed since we are reducing the antisymmetric mode of the interferometer to a single cavity. The relation between the parameters of the antisymmetric mode and the parameters of the reduced cavity has already been computed [38], however, we choose to work numerically and find the parameters of the reduced cavity G , γ , M , γ_m , ω_m , and L by fitting aLIGO's quantum noise spectrum found in pygwinc. The parameters for the modeled classical and quantum noise curves can be found in Appendix B.2, Table B.2.

4.6 Entanglement in aLIGO

In aLIGO, the spectrum is dominated by force noise for low frequencies and quantum noise for high frequencies, therefore we expect the sensing noise not to affect the entanglement significantly, which we observed with our numerics. Then, we focus on the effect of the force noise on the entanglement and use the parameters α_{F_1} and α_{F_2} to modify the force noise spectrum, and set $\alpha_{X_1} = \alpha_{X_2} = 1$ throughout all of the following subsections. We also introduce resonant modes to investigate the system

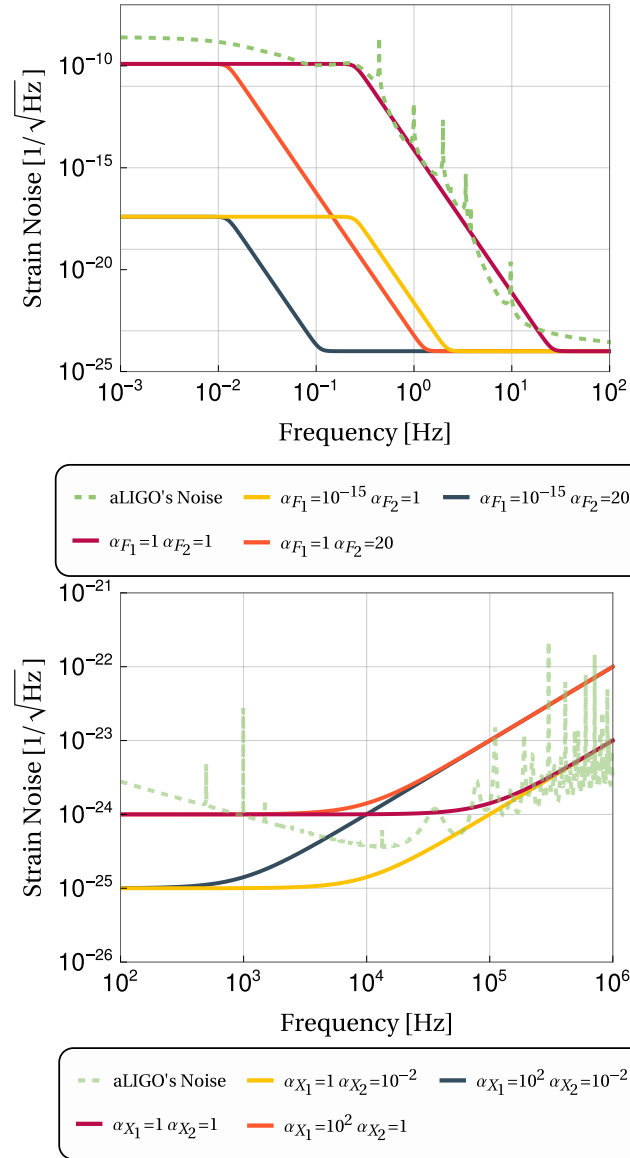


Figure 4.4: Force (top row) and sensing noise (bottom row) spectra parametrized by α_{F_1} , α_{F_2} , α_{X_1} , and α_{X_2} . The effect of α_{F_1} is to raise and lower the nominal noise strength below the cutoff frequency and α_{F_2} shifts the cutoff frequency. Similarly, α_{X_1} shifts the cutoff frequency where the sensing noise starts increasing as Ω^2 , whereas α_{X_2} shifts the nominal noise level.

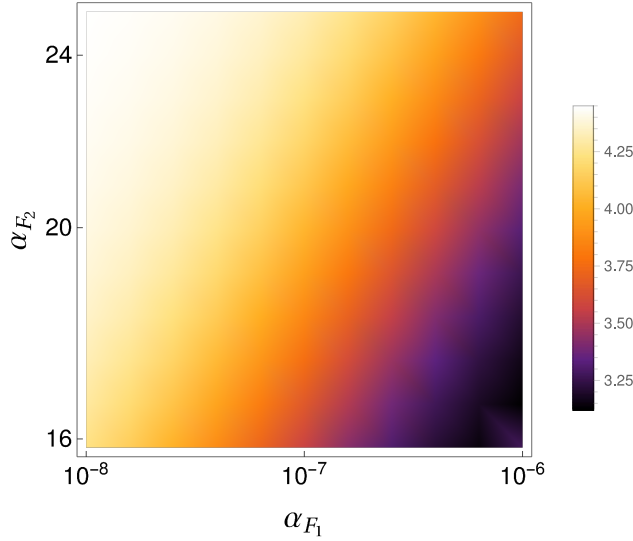


Figure 4.5: The effect of the force noise spectrum on the logarithmic negativity when we do not trace over the cavity. Note that the force noise levels increase toward the bottom-right of the figure and our fit of aLIGO’s operation regime is for $\alpha_{F_1} = \alpha_{F_2} = 1$.

as accurately as possible. Intuitively, we expect the entanglement to be destroyed in the presence of high classical noise levels.

Effect of Force Noise

First, we investigate the effect of the force noise spectrum on entanglement and we calculate the logarithmic negativity E_N as a function of α_{F_1} and α_{F_2} , for both of the partitions described in Sec. 4.3. For all pairs α_{F_1} and α_{F_2} here, we find larger logarithmic negativity values when we do not trace over the cavity. The results for the partition where we do not trace over the cavity can be found in Fig. 4.5. The amount of entanglement in the system diminishes when the force noise increases: that is, towards the bottom-right of the plot where α_{F_1} increases (proportional to the DC noise power) and α_{F_2} decreases (inversely proportional to the noise bandwidth), cf. Sec. 4.5 and Fig. 4.4. Our fit of aLIGO’s force noise level is for $\alpha_{F_1} = \alpha_{F_2} = 1$, hence this plot is for a comparatively low level of force noise. Further to the bottom-right, the numerics become unstable and do not converge due to the wide range of orders of magnitude entering the calculation; see Appendix B.2 for a discussion of the numerical implementation. Therefore, we cannot give a definite answer about optomechanical entanglement in aLIGO with our model yet.

Next, we look for entanglement in the absence of the seismic noise, since it is the

dominating contribution to the force noise of the system. Not being subjected to seismic noise is a realistic scenario for space-based gravitational-wave interferometers, such as the Laser Interferometer Space Antenna [265]. For aLIGO, in the absence of seismic noise, suspension thermal noise dominates the spectrum for low frequencies, which is modeled as

$$S_{ST}(\Omega) = \frac{\tau_{ST}}{(\Omega/\omega_{ST})^8 + 1}, \quad (4.22)$$

similar to how the total force noise was modeled in Sec. 4.5. The parameters τ_{ST} and ω_{ST} can be found in Table B.1. Without changing the other parameters in the system, we find that we can achieve negativities of 1.52 and 1.72 for the partitions where we do and do not trace over the cavity, respectively. This means that, in the absence of seismic noise, aLIGO has stationary optomechanical entanglement in its current operating regime.

Effect of Low-Frequency Resonances

Both the classical and the quantum noise curves contain many resonances, as can be seen from Fig. 4.3. The resonances in the seismic and suspension thermal noise arise from the displacement noises of the rigid-body resonant modes of the 4-stage suspension system [12]. These modes can be modeled with a sum of Lorentzians multiplying the force noise spectrum defined in Eq. (4.21). Then, the new formula defining the spectrum of the force noise is given as

$$S_F^R(\Omega) = \frac{\tau_F \alpha_{F_1}}{\left(\frac{\Omega}{\omega_F} \alpha_{F_2}\right)^{14} + 1} \left(1 \sum_i \frac{A_{v_i}^2}{(\Omega - \Omega_{v_i})^2 + (\frac{1}{2}\Gamma_{v_i})^2} \right) \quad \text{with } \Omega > 0, \quad (4.23a)$$

$$S_F^R(\Omega) = S_F^R(-\Omega), \quad (4.23b)$$

where the sum is over the resonant modes. The parameters Ω_{v_i} , Γ_{v_i} , and A_{v_i} are the mode frequencies, full widths at half maximum (FWHM), and the amplitudes of the Lorentzians, respectively, and they are listed in Appendix B.2, Table B.3 for the modes with the biggest relative amplitudes.

We investigate the effect of the resonant modes on the entanglement. In Fig. 4.6, with low force noise guaranteeing well-behaved numerics and setting $\alpha_{F_2} = 1$, we plot E_N for noise curves with and without these modes (orange and gray, resp.) and for both partitions (circles when the cavity is traced out). We see explicitly here that there is more entanglement in the partition where we do not trace over the cavity. For low noise (low α_{F_1}), the negativity remains unchanged, hence the

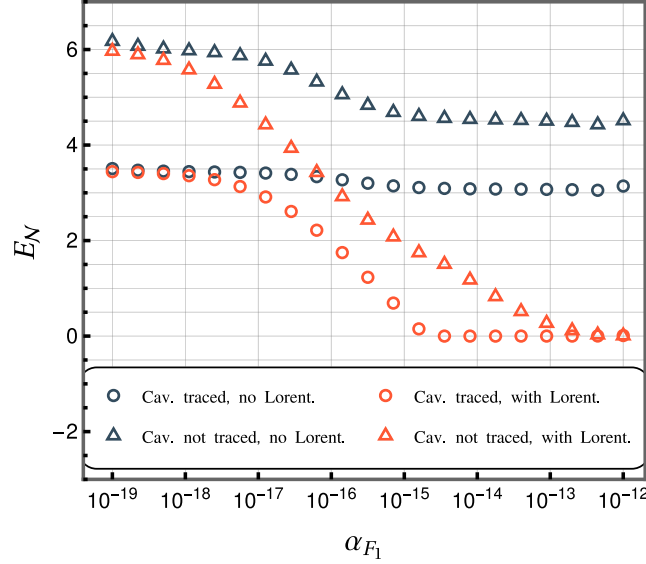


Figure 4.6: The effect of resonant modes on the logarithmic negativity, E_N , for both partitions. We see that the negativity reduces with increasing α_{F_1} or with the introduction of resonant modes. Note that $\alpha_{F_2} = 1$.

resonant modes do not affect the entanglement significantly in this regime. As the level of noise increases, resonant modes cause the logarithmic negativity to decrease faster than the negativities calculated without the resonant modes for both partitions. The system becomes separable when resonant modes are included for $\alpha_{F_1} \approx 10^{-15}$ and $\alpha_{F_1} \approx 10^{-12}$ when we do and do not trace over the cavity, respectively. It seems reasonable to expect that entanglement will not emerge when force noise becomes stronger. Hence, extrapolating $\alpha_{F_1} \rightarrow 1$, this is evidence (but not a rigorous proof) that aLIGO in its current operation regime (but without squeezed input) probably contains no optomechanical entanglement.

4.7 Discussion and Conclusions

In this chapter, we developed a framework to determine and quantify bipartite entanglement in an optomechanical system in the non-sideband-resolved regime, in the presence of non-Markovian Gaussian noises. The main novelty of our work is to enable the study of non-Markovian noise drives, which are common in devices with low mechanical frequencies, typically associated with large/macroscopic masses. Hence, we focused on macroscopic entanglement and used a free mass with structural damping and coating Brownian noise as an initial example, and then aLIGO as a more detailed case study. However, non-Markovian noise can also be found in optomechanical systems with higher frequencies [71, 99, 100, 173], and our

framework can be applied to them in the same way. Besides improving the quality and accuracy of predictions, modeling non-Markovian dynamics is rich in interesting physics and possibly useful phenomena: for example, in Ref. [173], they find that squashing/squeezing the mechanical state is less demanding in the presence of structural damping compared to Markovian viscous damping.

We tested for bipartite entanglement by looking at the separability between the mechanical oscillator and (i) the outgoing light field, and (ii) the joint system of the cavity and the outgoing light field. For low levels of classical noise, we saw that the latter partition is more entangled compared to the former. However, for high levels of classical noise, we did not see a significant advantage in using one partition over the other.

In the low mechanical frequency regime, where the free-mass limit approximation holds, we found that the presence of entanglement is independent of the coherent optomechanical interaction strength and depends only on the relative strength of force and sensing noise. This result is similar to that already found in [176] for white noise drive. Beyond the free-mass limit approximation, by parametrizing the noise curves of aLIGO, we were able to find a region of noise curves where entanglement exists, and we showed that there is a trade-off between the overall noise level and the cutoff frequency. Due to the high level of the current classical noise in aLIGO, we were not able to reach a definite conclusion in terms of the existence of entanglement in the system, even though it is unlikely to have a significant amount of entanglement based on our simulations. However, we saw that entanglement exists if we assume a system without seismic noise, even when the suspension thermal noise is still present. This is an important result since it shows that classical noises, even at very low frequencies, are able to demolish entanglement.

We also looked at how resonances in the noise curves of the system affect the amount of entanglement, and we saw that entanglement is more resistant (i.e. it disappears for higher levels of classical noise) for the partition where we test for the separability between the mechanical oscillator and the joint system of the cavity and the outgoing light field. For future work, we plan to develop better sampling strategies to overcome numerical instabilities.

BAYESIAN PHASE ESTIMATION WITH BLOCKS OF GHZ STATES

5.1 Introduction

Quantum metrology studies fundamental precision limits in physical measurements imposed by quantum physics¹. Recent advances in this field have led to novel protocols and improved precision for a variety of sensing devices: magnetometers [32, 18], atomic clocks [27, 224, 131, 169, 221, 68, 73, 41], nano-NMR [232, 28, 6, 231, 51], and atom interferometers [226, 195] to name a few. Many of these sensing applications can effectively be described as a single-shot phase estimation with an ensemble of N qubits, where an unknown phase ϕ , due to e.g. an electromagnetic field is imprinted on the qubits and is estimated after performing a measurement on the system.

A conventional phase estimation protocol is Ramsey interferometry, where the qubits are set to an initial state that is a superposition of the eigenstates of the computational basis, and are subjected to a projective measurement in the same basis after sensing ϕ . The precision limits of this scheme are well established: the estimated phase standard deviation (STD) scales as $1/\sqrt{N}$ when uncorrelated qubits are used as the initial state, also known as the standard quantum limit (SQL) [204]. Employing a suitable entangled state as the initial state, it is possible to achieve an improved estimation uncertainty that scales as $1/N$, referred to as the Heisenberg limit (HL) or Heisenberg scaling (HS) in the literature [84]. The initial state that achieves the minimum possible estimation uncertainty is referred to as the *optimal initial state*.

The two approaches to the phase estimation problem in the literature are (i) the frequentist approach, where the unknown phase ϕ is assumed to be a *deterministic* parameter and it can be measured many times. (ii) The Bayesian approach, where ϕ is assumed to be a *stochastic* variable and a single-shot phase estimation is performed [145]. In the Bayesian approach, the prior information about ϕ is encoded in a prior distribution $\mathcal{P}_{\delta\phi}(\phi)$, with a standard deviation of $\delta\phi$, also referred to as the prior

¹This chapter is adapted from: S. Direkci et al., "Heisenberg-limited Bayesian phase estimation with low-depth digital quantum circuits," npj Quantum Information **12**, 1 (2026), DOI: 10.1038/s41534-025-01177-9. Published by Springer Nature under a Creative Commons Attribution 4.0 International (CC BY 4.0) license.

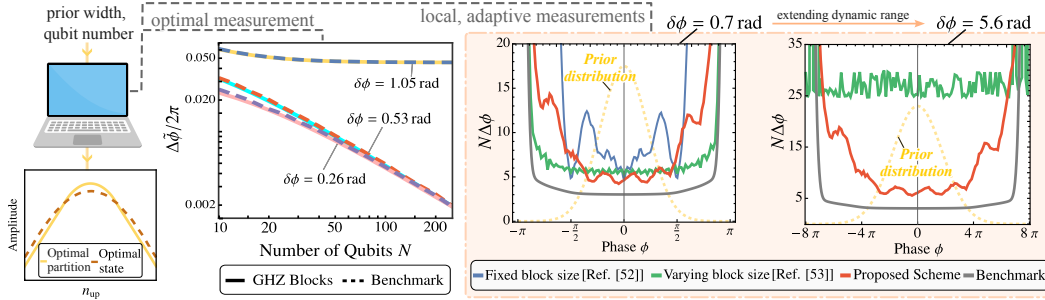


Figure 5.1: Summary of contributions. We propose a scheme that, given the prior width $\delta\phi$ and the total number of qubits N , computes the optimal partition over blocks of GHZ states, as well as optimal local, adaptive measurements. Partitions of blocks of GHZ states can be used to approximate the optimal initial states for phase estimation. If optimal measurements are available, blocks of GHZ states approximately reach the benchmark sensitivity for all prior widths. If local, adaptive measurements are used instead, we obtain Heisenberg scaling up to a constant overhead. Furthermore, optimal partitions can be rescaled to include slow atoms (atoms that accumulate phases of $\phi/2, \phi/4, \dots$), which enable an extended dynamic range beyond $\delta\phi \gg \pi$. We show that the proposed scheme surpasses other phase unwinding protocols in the literature that extend the dynamic range.

width. In this chapter, we consider the Bayesian approach, which is relevant to various applications, e.g. atomic clocks [131].

For the frequentist approach, it is well known that the optimal initial state is the Greenberger-Horne-Zeilinger (GHZ) state. Such states were analyzed in the literature extensively, in the context of Ramsey spectroscopy [25, 119, 174], quantum interferometry [120, 187], quantum lithography [57] and imaging [188, 122]. Conversely, for the Bayesian approach, the optimal initial state depends on the shape of the prior distribution: specifically, the prior width $\delta\phi$. For a small prior width ($\delta\phi < 1/N$, where N is the number of qubits), the GHZ state, combined with local readout, keeps on performing optimally [167]. However, for prior widths larger than the dynamical range of the GHZ states ($\delta\phi > 1/N$), they become sub-optimal. In this regime of wide prior phase distribution, the optimal initial state is the so-called *sine state*, a spin-squeezed state [141] that can be generated by the two-axis counter-twisting Hamiltonian [129]. Furthermore, the optimal measurement involves a Quantum Fourier Transform (QFT) followed by a projective measurement on the computational basis [37, 20, 249, 167]. Lastly, the ultimate estimation precision in this limit is given by the π -corrected HL, $\Delta\phi \approx \pi/N$ [125, 91].

There is an ongoing experimental effort to achieve below SQL precision levels [118]

and to reach the optimal limit in the wide prior regime [131]. A recent experiment achieved nearly optimal precision levels with trapped ions by optimizing over global squeezing operations [169]. However, such operations are not available (or scalable) in many quantum information platforms, e.g. atoms in tweezer arrays [73, 132], superconducting qubits [14, 207, 260], and more generally quantum computing devices that rely on circuits of few-qubit gates [192]. That is because generating a sine state and implementing the QFT measurement both require polynomial depth circuits [15, 48].

This raises the question: is it possible to saturate the fundamental precision limits, for any prior width, with constant to logarithmic depth digital quantum circuits? One particularly practical motivation behind this question is tweezer optical clocks [194, 168]. This technology emerges as a promising platform for quantum enhanced clocks, thanks to high fidelity quantum control and readout [73, 41]. However, since the quantum control in these clocks is limited to digital circuits, it is unknown whether the optimal precision can be obtained with state-of-the-art capabilities. Another motivation for this question is quantum phase estimation algorithms, which are performed using digital quantum devices. It is highly desirable to understand the fundamental precision limits of these algorithms when implemented with low-depth circuits [190, 158, 246].

One set of initial states that can be prepared efficiently by digital circuits is an ensemble of GHZ states with varying numbers of qubits. Such a state contains m_i copies of GHZ states with 2^{k_i} qubits, where $k_i, m_i \in \mathbb{N}$. We also refer to such states as *blocks of GHZ states*, and m_i as the block size. Several protocols using such initial states have been proposed in the literature [21, 128, 134, 110]. However, to our knowledge, the performance of these schemes in the Bayesian setting has not been investigated (i.e. different prior distributions or prior widths have not been considered).

Here, we analyze how such initial states perform and compare various measurement schemes with an increasing level of control. Specifically, we consider a scheme with a *fixed block size* adopted from Ref. [134] that employs non-adaptive, qubit-resolved (local) measurements, as well as a scheme with a *varying block size* [110] which further employs single-qubit rotations before measurement. We show that the scheme with a varying block size surpasses the scheme with a fixed block size and achieves HS up to a constant overhead, in the large prior width within the dynamic range regime ($\delta\phi = 0.7$ rad).

However, both schemes have several drawbacks. First, they offer exact solutions only for specific values of the number of qubits. For example, the scheme with a varying block size offers a solution only for qubit numbers $N = 2, 9, 26, 63, 140, \dots$. Extending these schemes to any number of qubits is highly desirable for near-term quantum devices that operate with a small number of qubits. Second, they consider a uniform prior phase distribution in the interval $[-\pi, \pi]$, which is not realistic for devices such as atomic clocks. This motivates extension to an arbitrary prior width, and Gaussian prior distributions, which are more relevant for atomic clocks.

We thus propose and analyze the following protocol: we numerically optimize over all possible partitions of a total number of N qubits into blocks of GHZ states with varying numbers of qubits, assuming a Gaussian prior distribution with an arbitrary prior width $\delta\phi$. As for the measurement strategy, we utilize local, adaptive measurements, instead of local, non-adaptive measurements used in Refs. [134, 110]. These measurements were introduced in [21], but the performance for Bayesian phase estimation was not studied. Given the initial state, we optimize over the local, adaptive measurements and show that this protocol achieves HS for any given prior width, and surpasses the previously suggested protocols.

To answer our initial question, we show that it is possible to approximately saturate fundamental precision limits for any prior width, with constant to logarithmic depth digital quantum circuits. We observe that the initial states used in the proposed scheme perform almost optimally and reach the fundamental limit of sensitivity for Bayesian phase estimation for all prior widths, if arbitrary level of control in measurement schemes is allowed.

Lastly, we discuss the application of these schemes in the context of atomic clocks, and propose a method to reach the ultimate limit of sensitivity. We do so by increasing the dynamic range of the proposed scheme, i.e. the interval for which the phase ϕ can be unambiguously estimated, by introducing *slow* atoms that accumulate fractional phases. We present a method to allocate the atoms into slow atoms and GHZ states efficiently, in order to extend the dynamic range with a negligible amount of additional atoms. We show that our method outperforms existing schemes that utilize slow atoms by comparing the number of atoms needed to obtain the same level of sensitivity.

5.2 Results

Formulation

We consider a Ramsey interferometry experiment where N qubits are initialized in the state $|\psi_i\rangle$, and undergo a unitary ϕ -encoding transformation $U(\phi)$ such that

$$U(\phi) = e^{-i\phi J_z}$$

$$J_z = \frac{1}{2} \sum_i \sigma_z^i, \quad i = 1, 2, \dots, N, \quad (5.1)$$

where σ_z^i is the Pauli Z operator for the i^{th} qubit, we set $\hbar = 1$, and $\phi \in \mathbb{R}$ is unknown. Let us denote the state after the encoding as $|\psi_f\rangle$, with $|\psi_f\rangle = U(\phi)|\psi_i\rangle$. Information about ϕ is obtained through performing positive operator valued measures (POVMs) on $|\psi_f\rangle$, i.e. a set of positive Hermitian operators $\{\Pi_x\}_x$ such that $\sum_x \Pi_x = \mathbf{1}$, where $\mathbf{1}$ is the identity operator. The probabilities of the measurement outcomes are given by the distribution $\{p(x|\phi)\}_x$, with $p(x|\phi) = \langle \psi_f | \Pi_x | \psi_f \rangle$. Given a measurement result, the phase is estimated by an estimator $\phi_{est}(x)$, and the accuracy of the phase estimation for any ϕ is defined by the mean-squared error (MSE),

$$(\Delta\phi)^2 = \sum_x p(x|\phi) (\phi - \phi_{est}(x))^2. \quad (5.2)$$

Given the MSE as a function of ϕ , we seek to minimize a cost function that quantifies the overall performance of the phase estimation, for all ϕ . For this purpose, we choose to utilize the Bayesian mean squared error (BMSE). Given a prior distribution of ϕ , $\mathcal{P}_{\delta\phi}(\phi)$, the BMSE is defined as

$$(\Delta\tilde{\phi})^2 = \int (\Delta\phi)^2 \mathcal{P}_{\delta\phi}(\phi) d\phi. \quad (5.3)$$

We refer to the square-root of the BMSE, $\Delta\tilde{\phi}$, as root Bayesian mean squared error (RBMSE), or the posterior width. We assume a Gaussian prior distribution with a standard deviation of $\delta\phi$, defined as

$$\mathcal{P}_{\delta\phi}(\phi) = \frac{1}{\sqrt{2\pi}(\delta\phi)^2} \exp \left[-\frac{\phi^2}{2(\delta\phi)^2} \right]. \quad (5.4)$$

$\delta\phi$ is also referred to as the prior width. Then, given a prior width, the problem of minimizing the BMSE reduces to optimizing over $|\psi_i\rangle$, $\{\Pi_x\}$, and $\{\phi_{est}(x)\}$. An efficient numerical algorithm for this task was given in [167]. This algorithm provides the minimal BMSE along with the optimal initial state, the POVM, and the optimal estimators that saturate it. We refer to this minimal BMSE as the optimal quantum interferometer (OQI). The OQI constitutes a benchmark for the maximum attainable sensitivity in Bayesian phase estimation.

5.3 Phase estimation for atomic clocks

An important application of phase estimation is in the context of optical atomic clocks [162], where a local oscillator (LO) with a frequency $\omega_L(t)$, specifically a clock laser, is locked to an atomic transition with a frequency ω_A . Due to the fluctuations in the laser frequency, $\omega_L(t)$ deviates from ω_A , and a phase $\phi := \int_0^T (\omega_A - \omega_L(t)) dt$ is accumulated. Here, T is the Ramsey time, and it will also be referred to as the interrogation time. Due to the fluctuating laser frequency, the accumulated phase ϕ is a stochastic variable, distributed according to a prior distribution with a prior width $\delta\phi$. For example, $\delta\phi$ grows with the Ramsey time as $\delta\phi \propto T$ for a laser frequency noise spectrum in the form of $S_\omega(f) \propto 1/f$ [154]. At the end of the interrogation, the accumulated phase ϕ is estimated from a population measurement of the atoms. This estimate is consequently used to correct the deviations in the laser frequency. After the measurement, the uncertainty on ϕ is reduced from $\delta\phi$ to $\Delta\tilde{\phi}$, the posterior width, given in Eq. (5.3). Finally, the atoms are reinitialized to perform a new interrogation. This process is described in Fig. 5.2a. During the detection and initialization time, the laser frequency is not interrogated, and this interval is referred to as dead time. Phase diffusion during the dead time of atomic clocks gives rise to the Dick effect [63]. This effect however can be circumvented by introducing an additional ensemble of atoms that enables implementation of a zero-dead time clock [22]. We ignore this effect in the following discussion. The clock stability is characterized by the Allan deviation of the frequency, $\sigma_y(\tau)$. For a clock operation with N_T interrogations, each having a Ramsey time T , the Allan deviation ideally follows [134]

$$\sigma_y(\tau) = \frac{\Delta\tilde{\phi}}{\omega_A \sqrt{\tau T}} \quad (5.5)$$

where ω_A is the frequency of the atomic transition as mentioned above, $\tau = N_T \cdot T$ is the total duration of the clock operation, or the total interrogation time, and $\Delta\tilde{\phi}$ is the posterior width. Note that here, we assume that $\Delta\tilde{\phi} \ll \delta\phi$. If this assumption fails, the effective uncertainty defined in Eq. (C.4) in Appendix C.1 should be used instead of the posterior width as the numerator of Eq. (5.5). Stable optical clocks are obtained through the minimization of $\sigma_y(\tau)$, therefore it is desirable to have long interrogation times T , and small posterior widths $\Delta\tilde{\phi}$. The posterior width $\Delta\tilde{\phi}$ depends on the initial state of the atoms, and the measurement strategy. For example, a coherent spin state (CSS), i.e. uncorrelated atoms, can be used as the initial state: an N -qubit CSS is defined as $|\psi_{\text{CSS}}\rangle = \left[(|0\rangle + |1\rangle)/\sqrt{2} \right]^{\otimes N}$. Phase estimation with a CSS results in the SQL, where the posterior width scales

as $\Delta\tilde{\phi} \propto 1/\sqrt{N}$. Using entanglement, it is possible to achieve a better scaling with N . For example, a GHZ state can be used as an initial state, where an N -qubit GHZ state reads $|\psi_{\text{GHZ}}\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. An N -qubit GHZ state achieves a posterior width of $\Delta\tilde{\phi} \propto 1/N$, for $\delta\phi \approx \pi/N$ or smaller.

As for the interrogation time T , it is typically limited by the LO noise, i.e. by γ_{LO} , the linewidth of the free-running laser. For T longer than γ_{LO}^{-1} , $\delta\phi \approx \pi$, leading to phase slip errors, i.e. phase estimation error arising from ϕ being outside of the interval $[-\pi, \pi]$. Since ϕ can be estimated up to modulo 2π , the phase estimation results in a $2\pi k$ error, $k \geq 1$, $k \in \mathbb{Z}$. These errors degrade the accuracy in phase estimation and consequently the Allan deviation. Therefore, for a naive clock operation using uncorrelated atoms, the maximum allowable Ramsey time is $T \approx \gamma_{\text{LO}}^{-1}$. Consequently, we see two scalings of the Allan deviation emerge as a function of τ : for $\tau < \gamma_{\text{LO}}^{-1}$, one can set $T = \tau$, such that $\sigma_y(\tau) \propto \tau^{-1}$. For $\tau > \gamma_{\text{LO}}^{-1}$, the Ramsey time is fixed to its maximum value, and $\sigma_y(\tau) \propto \tau^{-1/2}$ due to averaging over the many independent interrogations.

In a practical setting, optical atomic clocks are operated in the regime of $\tau > \gamma_{\text{LO}}^{-1}$. Since GHZ states perform optimally for $\delta\phi < \pi/N$, their maximum allowable Ramsey time is $T \approx (\gamma_{\text{LO}}N)^{-1}$, which is a factor of N shorter compared to the case of an N -qubit CSS. Then, the clock that uses an N -qubit GHZ state achieves a similar stability as the clock that uses an N -qubit CSS², providing no metrological gain. Therefore, optimal metrology seeks (i) protocols that extend the Ramsey time T beyond the LO noise limit, γ_{LO}^{-1} , (ii) protocols that show Heisenberg-scaling with respect to the number of atoms N , given a wide prior width, to achieve metrological advantage.

The first objective can be accomplished by introducing additional classical interrogations with uncorrelated atoms and performing phase unwinding [224, 26]. Phase unwinding is achieved by using ensembles of atoms that acquire smaller phases (which can be achieved by e.g. mid-circuit measurement and reset). These ensembles provide the ability to correct for phase slip errors, and thus, to extend the interrogation time such that $T = \tau$. We refer to the classical scheme that extends the Ramsey time in this way as the *best classical scheme*, and the clock that uses this scheme as the *best classical clock*.

Furthermore, the second objective can be achieved with the OQI. The OQI achieves the minimum possible posterior variance given any interrogation time. We refer to

²GHZ states achieve slightly smaller posterior variances, see Ref. [131] or Fig. 5.2b

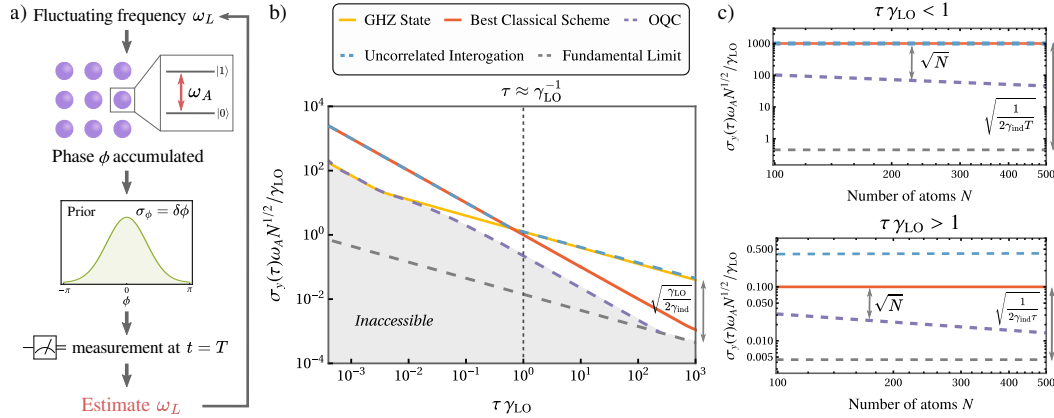


Figure 5.2: Operation of an optical atomic clock in the Bayesian context. (a) The fluctuating laser frequency $\omega_L(t)$ causes the atoms to accumulate a phase ϕ . The phase is measured at the end of the interrogation cycle, and is used to estimate the laser frequency. (b) Allan deviation $\sigma_y(\tau)$, normalized by $\gamma_{LO}/\omega_A N^{1/2}$, as a function of the total interrogation time τ , for various interrogation schemes, and $N = 200$ atoms. The vertical dashed line represents the coherence time limit due to local oscillator noise [235], which determines the maximum Ramsey time T . The Allan deviation decreases as $\tau^{1/2}$ after this point for the uncorrelated and GHZ interrogation. If phase slip errors are suppressed, it is possible for the stability to scale as τ^{-1} , as seen from the stabilities of the best classical scheme and the optimal quantum clock (OQC). The fundamental limit of sensitivity is set by the individual particle decoherence (Eq. (5.6)), which is plotted with a gray, dashed line. (c) Allan deviation $\sigma_y(\tau)$, normalized by $\gamma_{LO}/\omega_A N^{1/2}$, as a function of the total number of atoms N , for $\tau = 10^{-3}\gamma_{LO}^{-1}$ (top) and $\tau = 10\gamma_{LO}^{-1}$ (bottom). Note that the legend for the plots is shown in b. For $\tau\gamma_{LO} < 1$, the stabilities of all the clocks scale as $N^{1/2}$. Protocols that show HS can increase the clock stability by an additional factor of $N^{1/2}$. For $\tau\gamma_{LO} > 1$, T takes its largest value of $T \approx \gamma_{LO}^{-1}$ for the uncorrelated interrogation, causing the stability to scale as $\tau^{1/2}$. Extending the Ramsey time with the best classical scheme enables to obtain a stability that scales as τ , and using protocols that show HS, such as the OQC, can further boost the stability by a factor of $N^{1/2}$.

the clock that combines the OQI with phase unwinding techniques beyond the LO noise limit as the optimal quantum clock (OQC): the OQC achieves the maximum possible clock stability given N qubits, for any interrogation time.

Ultimately, the clock stability is limited by the individual particle decoherence arising from amplitude damping (comprised of scattering from trapping light and radiative decay), which restricts the linewidth of the atomic clock transition. The fundamental limit to phase sensitivity is given by [69, 61]

$$\sigma_{y,\min}(\tau) \approx \frac{1}{\omega_A} \sqrt{\frac{2\gamma_{\text{ind}}}{\tau N}} \quad (5.6)$$

where γ_{ind} is the rate of amplitude damping. Note that the fundamental limit scales as $N^{-1/2}$, where neither entanglement nor any classical protocol can retrieve the Heisenberg scaling [69, 61].

We illustrate the different regimes and the possible gain from quantum metrology in Fig. 5.2. For a fixed $\gamma_{\text{LO}}/\gamma_{\text{ind}} = 10^4$, we plot in Fig. 5.2b the Allan deviation $\sigma_y(\tau)$, with respect to the total interrogation time τ , for different protocols (see Appendix C.1 for more details). From the Figure, the stability of all of the protocols scales as τ^{-1} for small total interrogation times τ . However, the Allan deviation scales as $\tau^{-1/2}$ for $\tau > \gamma_{\text{LO}}^{-1}$, for uncorrelated atoms and the GHZ state. This leads to a factor of $\gamma_{\text{LO}}^{1/2}/(2\gamma_{\text{ind}})^{1/2}$ gap from the fundamental sensitivity limit given in Eq. (5.6), and plotted with gray dashed lines in Fig. 5.2b. The stability of the best classical clock, and the OQC, continue to scale as τ^{-1} for times shorter than the single-qubit decoherence limit ($\tau < \gamma_{\text{ind}}^{-1}$), as they assume the use of phase unwinding. Furthermore, the OQC surpasses the best classical clock since it shows HS in this time scale. The area below the OQC is marked as *inaccessible*, since it sets the benchmark for optimal sensitivity. Note that we assumed $\gamma_{\text{ind}} = 0$ when computing the performance of the OQC to simplify the calculations. As the total interrogation time τ approaches γ_{ind}^{-1} , the Allan deviation for all of the schemes start scaling as $\tau^{-1/2}$ due to the single-qubit noise.

In Fig. 5.2c, we plot the Allan deviation $\sigma_y(\tau)$, normalized by $\gamma_{\text{LO}}/\omega_A N^{1/2}$, with respect to the total number of atoms N , for $\tau = 10^{-3}\gamma_{\text{LO}}^{-1}$ (top) and $\tau = 10\gamma_{\text{LO}}^{-1}$ (bottom). For $\tau < \gamma_{\text{LO}}^{-1}$, the uncorrelated interrogation has the same stability as the best classical scheme. Since the OQC shows HS, the stability increases faster by a factor of $N^{1/2}$ compared to the other schemes and gets closer to the fundamental limit of sensitivity. For $\tau > \gamma_{\text{LO}}^{-1}$, the best classical clock performs better than the clock using uncorrelated interrogations by a factor of $(\tau\gamma_{\text{LO}})^{1/2}$ due to its extended

Ramsey time. On top of this, the OQC surpasses the best classical clock by a factor of $N^{1/2}$, due to its extended Ramsey time, and its HS. In summary, the maximum gain in sensitivity obtained by the OQC with respect to the sensitivity of the uncorrelated interrogation is $\sqrt{N}$ in the regime where $\tau\gamma_{\text{LO}} < 1$, and $\sqrt{N}\sqrt{\tau\gamma_{\text{LO}}}$ in the regime where $\tau\gamma_{\text{LO}} > 1$, respectively.

Blocks of GHZ States for Phase Estimation

It is desirable to use GHZ states as building blocks to construct initial states for phase estimation, as they are relatively easy to prepare. N -qubit GHZ states can be prepared using various short depth circuits: (i) a circuit depth of $O(\log N)$ assuming full-connectivity [182], (ii) constant depth if measurement and feedback strategies [250] or multi-qubit gates [41] are used, (iii) linear depth if only nearest-neighbor interactions are allowed [73]. GHZ states are optimal initial states for Bayesian phase estimation given a vanishing prior phase width and for the frequentist case. To show this, let us first define the J_x, J_y operators: $J_x = 1/2 \sum_{i=1}^N \sigma_x^i$, $J_y = 1/2 \sum_{i=1}^N \sigma_y^i$, where σ_x^i, σ_y^i are the X, Y Pauli operators on the i^{th} qubit, respectively. Consider a J_y measurement on the N -qubit GHZ state. The first and second moments of the parity operator, defined as $\Pi = (-1)^{J_y + N/2}$, are given by $\langle \Pi \rangle = \sin(N\phi)$, $\langle \Pi^2 \rangle = \cos(N\phi)^2$. Given M repetitions of the measurement, the MSE is

$$(\Delta\phi)^2 = \frac{(\Delta\Pi)^2}{M(\partial_\phi \langle \Pi \rangle)^2} \approx \frac{1}{MN^2} \quad \text{for } \phi \ll 1, \quad (5.7)$$

where $(\Delta\Pi)^2 = \langle \Pi^2 \rangle - \langle \Pi \rangle^2$. Thus, GHZ states attain the HL in the frequentist approach. However, this limit is unachievable in Bayesian quantum metrology, where only a single measurement is allowed ($M = 1$), and ϕ has a prior distribution with a non-zero width. Bayesian phase estimation resembles the frequentist phase estimation in the limit of small prior width, where $\delta\phi \ll \pi/N$, for which bimodal states close to GHZ states are the optimal initial states [167].

For $\delta\phi > \pi/N$, GHZ states cannot attain the HL due to their limited dynamic range. Measurement outcomes repeat themselves after a period of $2\pi/N$, causing phase wrap errors to appear outside of this period, and leading to a poor performance. In order to have both a high precision in estimation, and a large dynamic range, we look for ways to approximate the optimal initial states using a combination of GHZ states with varying numbers of qubits, i.e. blocks of GHZ states. Such a state reads

$$|\psi\rangle = \frac{1}{2^{\sum_i m_i/2}} \prod_i \left(|0\rangle^{\otimes 2^{k_i}} + |1\rangle^{\otimes 2^{k_i}} \right)^{\otimes m_i}, \quad (5.8)$$

where $|0\rangle, |1\rangle$ are the eigenstates of σ_z with a negative and positive eigenvalue, respectively, m_i is the size of the block that contains 2^{k_i} -qubit GHZ states, $k_i, m_i \in \mathbb{N}$, and the total number of qubits is given by $N = \sum_i m_i 2^{k_i}$. We want to find optimal partitions, i.e. $\{k_i\}, \{m_i\}$, for every N and optimal readout schemes and estimators. In Refs. [21, 128], it was shown that blocks of GHZ states can attain the HL up to a constant overhead given a flat prior in $[-\pi, \pi]$. This was achieved by setting $m_i \geq 3 \forall i$, and performing optimal measurements. Before outlining our protocol, let us first review existing schemes that utilized blocks of GHZ states.

Review of existing schemes

Protocols using blocks of GHZ states with different partitions (i.e. different sets of $\{k_i\}, \{m_i\}$ in Eq. (5.8)) have been studied in the literature [21, 110, 134]. We review and further analyze the performance of these schemes, focusing on the scheme in Ref. [134] to which we refer as the scheme with a fixed block size ($m_i = M$), and the scheme in Ref. [110] to which we refer as the scheme with a varying block size.

In the scheme with a fixed block size, all of the GHZ states (with $\{2^{k_i}\}_{k_i=0}^{k_{\max}}$ qubits) are repeated by a constant $m_i = M$. This constant is found from the implicit equation $M = \frac{16}{\pi^2} \log(N)$, where N is the total number of qubits. For example, for a state with 1, 2, and 4-qubit GHZ states, $M = 6$. For each block of 2^k -atoms GHZ states, $M/2$ of the GHZ states are measured in the local Pauli σ_x and the other $M/2$ in the σ_y basis, i.e. a dual quadrature readout is performed to increase the dynamic range. Ref. [134] assumes a uniform phase prior, which leads to an RBMSE of $\Delta\tilde{\phi} = \frac{8}{\pi} \sqrt{\log(N)}/N$, i.e. HS up to a logarithmic correction. We numerically calculate the RBMSE with two different estimation strategies: a bit-by-bit estimation strategy that was developed in [134] and the optimal Bayesian estimation. In the bit-by-bit estimation, we write $\phi \pmod{2\pi}$ as $\phi = 2\pi(0.Z_1 Z_2 Z_3 \dots)$, where $Z_j \in \{0, 1\}$ are the binary digits. By combining the results of the parity measurements of the GHZ states we estimate the binary digits Z_j of the phase ϕ . We find that while the Bayesian estimation achieves a sub-SQL BMSE, the bit-by-bit estimation does not achieve the expected precision.

In the scheme with a varying block size, the state consists of GHZ states with sizes of 2^{k_i} , where $k_i = 0, 1, \dots, k_{\max}$, and $m_i = m_{k_{\max}} + \mu(k_{\max} - k_i)$, $m_{k_{\max}}, \mu \in \mathbb{N}^+$. Then, the largest GHZ state has $2^{k_{\max}}$ qubits, and is repeated $m_{k_{\max}}$ times. Similarly, $2^{k_{\max}-1}$ -qubit GHZ states are repeated $m_{k_{\max}} + \mu$ times, etc. Before a measurement, the m_i copies of GHZ states are subjected to a set of single-qubit rotations θ_j , with

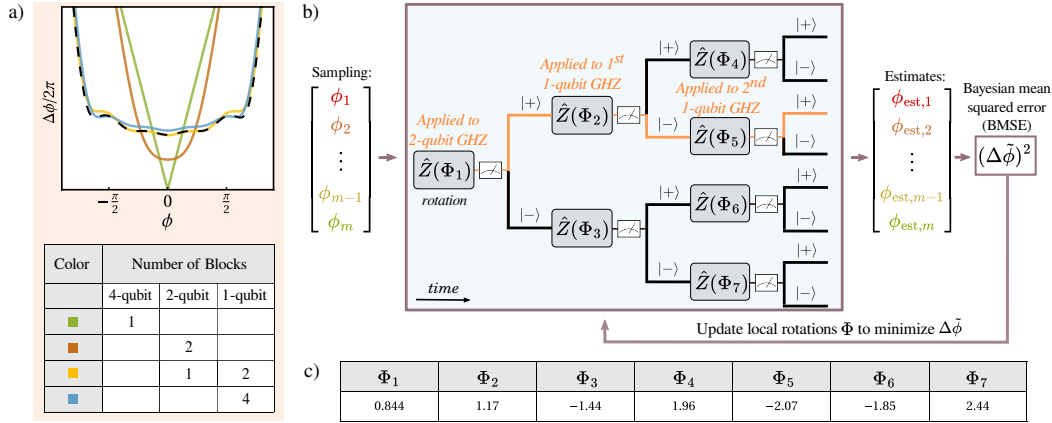


Figure 5.3: Summary of the introduced protocol for $N = 4$ qubits, and a prior width of $\delta\phi = \pi/3 \approx 1.05$ rad. (a) The possible partitions of the blocks of GHZ states for $N = 4$ qubits are given in the Table, and the MSE as a function of ϕ is plotted. For this prior width, the best-performing partition is plotted in yellow, highlighted in the Table. (b) Optimization over local, adaptive measurements. The measurement consists of successive $\hat{Z}(\Phi_i)$ operations and projective measurements. The $\hat{Z}(\Phi_i)$ operation corresponds to a rotation around the z -axis by Φ_i in the Bloch sphere picture, such that $\phi - \Phi_i$ is measured. Based on the measurement outcome, the next rotation angle is determined. During the optimization, m phases are sampled uniformly from the interval $[-6\delta\phi, 6\delta\phi]$, which are then used to compute the BMSE analytically. Gradient descent is used to tune the single-qubit rotations Φ_i in the adaptive measurement. This process is iterated until the BMSE converges to a minimum. (c) Numerical values (in units of radians) of the single-qubit rotations Φ_i at the end of the optimization.

$\theta_j = \pi \frac{j}{m_i}$, $j = 0, 1, \dots, m_i - 1$. Finally, they undergo parity measurements. Ref. [110] observed numerically that the maximum sensitivity is obtained for $m_{k_{\max}} = 2$, and $\mu = 3$. For example, to prepare an initial state with 2-qubit and 1-qubit GHZ states, this protocol employs two copies of 2-qubit GHZ states, and five copies of 1-qubit GHZ states. Furthermore, the 2-qubit (1-qubit) GHZ states are subjected to single-qubit rotations of $0, \pi/2$ ($0, \pi/5, \dots, 4\pi/5$). Ref. [110] showed that this scheme achieves HL up to a constant overhead of 2.03^3 . We implemented this scheme numerically with Gaussian prior distributions and observed that it achieves the sensitivity of the OQI up to a constant overhead that depends on the prior width. For example, for a prior width of $\delta\phi = 0.7$ rad, the overhead is calculated to be 1.66.

The schemes mentioned above raise several issues: First, they are not immediately

³Note that the cost function quantifying the performance of phase estimation in Ref. [110] is the Holevo variance, defined as $(\Delta\tilde{\phi})^2 = \int_0^{2\pi} 4 \sin^2\left(\frac{\phi - \tilde{\phi}}{2}\right) d\phi$, where $\tilde{\phi}$ is the estimator of ϕ .

applicable to atomic clocks since they assume a uniform prior distribution on the phase. To our knowledge, they have not been analyzed for Gaussian prior distributions, which are the relevant phase distributions for the regime where current atomic clocks operate.

Second, they correspond to very specific values of N , e.g. for the scheme with a varying block size, the considered states have a total number of $N = 2, 9, 26$ qubits for $k_{\max} = 0, 1, 2$. For intermediate numbers of qubits, it is necessary to modify these states by removing or adding GHZ states, which leads to sub-optimal states. Furthermore, the density of the qubit numbers suitable to these schemes decreases with increasing N , further complicating the use of them for large N . We therefore want to devise a scheme that will achieve optimal precision for every N .

These issues motivate us to find better schemes for Bayesian phase estimation that are applicable to all qubit numbers, N , and to Gaussian prior distributions. To this end, we use an adaptive protocol, inspired by Ref. [21]. We extend this protocol to Bayesian phase estimation and generalize it to all phase prior widths and all qubit numbers, N .

Proposed Adaptive Scheme

We now turn to outline our protocol. We design the protocol assuming a relatively small phase slip probability, i.e. $\delta\phi < \pi$, where $\delta\phi$ is the prior width. In the context of atomic clocks, this regime corresponds to a Ramsey time smaller than the local oscillator (LO) coherence time. In other words, $T\gamma_{\text{LO}} < 1$, where T is the Ramsey time and γ_{LO} is the linewidth of the laser (see Sec. 5.3). We extend the proposed scheme in the regime where $T\gamma_{\text{LO}} > 1$ through the introduction of phase unwinding.

Our goal is to get as close as possible to the performance of the OQI with low-depth circuits. The initial states and the POVMs of the OQI can be efficiently calculated, but they are difficult to implement: for a large prior width within the dynamic range ($\delta\phi \approx 1$ rad), the optimal state resembles a sine state [20], and the optimal measurements have high complexity [167] (for a derivation of the optimality of the sine state, see Appendix C.2 and Ref. [74]). Furthermore, optimizing over all possible digital circuits is quite formidable. Hence, we pursue a two-step approach. First, for a given atom number and prior width, we "mimic" the optimal initial state using simpler states, i.e. blocks of GHZ states. Then, given this approximated state, we optimize over the local measurements to minimize the BMSE.

A summary and illustration of the proposed adaptive scheme is given in Fig. 5.3

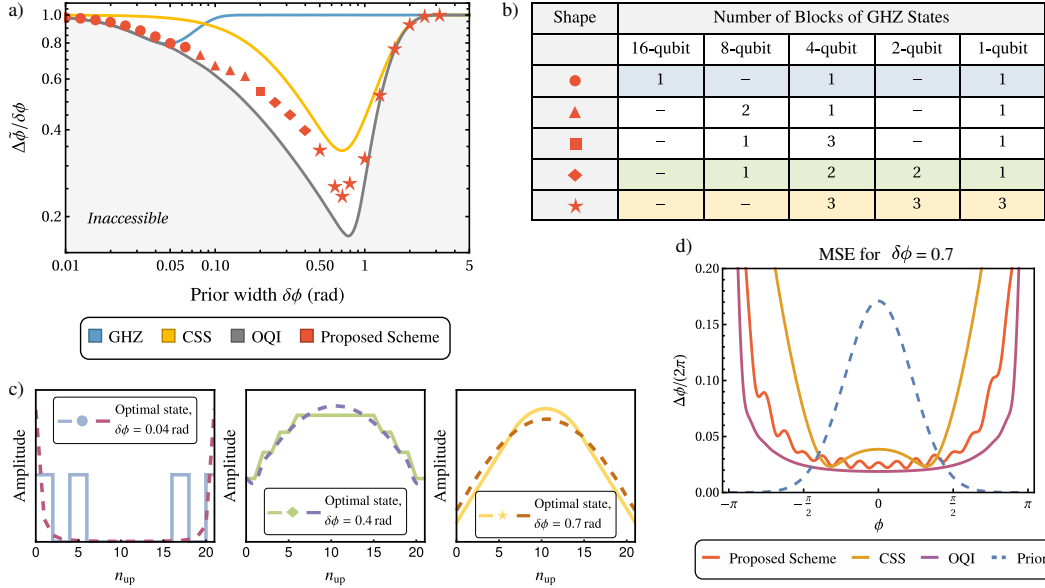


Figure 5.4: Performance of the proposed scheme for $N = 21$ qubits. (a) RBMSE $\Delta\tilde{\phi}$ as a function of the prior width, $\delta\phi$, for $N = 21$. The performance of the phase estimation with a 21-qubit GHZ state, 21-qubit CSS, the OQI, and the proposed scheme is plotted in blue, yellow, gray, and red, respectively. The different shapes of the red markers correspond to different ways of partitioning the qubits into blocks of GHZ states. (b) Optimal partitions of 21 qubits into blocks of GHZ states for different prior widths. For a small prior width the optimal strategy is to use the GHZ state with the maximum number of atoms possible. The remaining atoms are distributed to smaller GHZ states. In contrast, for a large prior width, it is more advantageous to use GHZ states with smaller numbers of atoms and distribute them equally. (c) The amplitudes of states with n_{up} excitations, $0 \leq n_{\text{up}} \leq 21$, for the optimal initial states of the OQI and the initial states obtained with blocks of GHZ states, for different prior widths. $n_{\text{up}} = 0$ ($n_{\text{up}} = 21$) corresponds to the state $|0\rangle^{\otimes N}(|1\rangle^{\otimes N})$. We see that for $\delta\phi \approx 1$ rad, the optimal initial state of the OQI resembles a sine state. (d) MSE of the OQI, CSS, and the proposed scheme for $N = 21$ and $\delta\phi = 0.7$ rad, plotted in purple, yellow, and red, respectively. The proposed scheme achieves a metrological gain of $g = 2.29$ (3.59 dB).

for $N = 4$ qubits, and a prior width $\delta\phi = \pi/3 \approx 1.05$ rad (such that phase slips are three-sigma events). We list all the possible partitions of $N = 4$ qubits into blocks of GHZ states, and plot the MSE as a function of ϕ of all such partitions in Fig. 5.3a. We remark that their MSE depends on the prior width through the measurement optimization that is based on it. The MSE of the OQI as a function of ϕ is given by the black, dashed line in Fig. 5.3a. The smallest BMSE with our scheme is obtained by the partition whose MSE is closest to the OQI, which corresponds here to the yellow line.

In Fig. 5.3b, local, adaptive measurements are depicted: a single-qubit rotation of Φ_i around the z -axis is applied before each measurement, such that $\phi - \Phi_i$ is measured. Depending on the measurement result, the single-qubit rotation that will be applied to the next GHZ state is chosen. This process is initiated with the largest GHZ state in the partition in terms of the number of qubits, and ends with the smallest GHZ state (see Fig. 5.3b, where an example of a measurement result is highlighted in orange). To find the optimal set of single-qubit rotations $\{\Phi\}$, m phases ϕ are uniformly sampled from the interval $[-6\delta\phi, 6\delta\phi]$, and the BMSE is calculated analytically, which is used to tune $\{\Phi\}$. This process is iterated until the BMSE converges to a minimum. In Fig. 5.3c, we list the numerical values of the single-qubit rotations obtained after performing the optimization for this case.

The proposed scheme can be realized experimentally with current quantum processors, where computing and sensing can be interleaved. State preparation and sequential feed-forward measurements can be implemented using e.g. Yb-171 nuclear spin qubits, where midcircuit measurements without entanglement were shown [193, 159]. We lay out an experimental procedure in the subsection "Experimental Realization".

Performance of the schemes

First, for a fixed qubit number, we analyze how the proposed scheme performs as we change the prior width, $\delta\phi$. Then, we assume a fixed prior width and observe the performance of all of the schemes as a function of the number of qubits, N . To quantify the increase in the sensitivity of phase estimation when the proposed scheme is used, we define the metrological gain $g = (\Delta\phi_{\text{CSS}})^2 / (\Delta\tilde{\phi})^2$ as our figure of merit, where $(\Delta\phi_{\text{CSS}})^2$ and $(\Delta\tilde{\phi})^2$ are the BMSE of the phase estimation performed using an N -qubit coherent spin state (CSS) and the proposed scheme, respectively. Note that the prior width and the number of qubits are the same for both schemes when calculating the gain, and we assume a single-quadrature readout for the CSS.

We study how the performance of our scheme, as well as the optimal initial state and partition depends on the prior width. The results for $N = 21$ qubits can be found in Fig. 5.4a. The performance of the OQI is indicated in gray, and the gray filling below this line is used to emphasize the fact that it is not possible to perform better than the OQI, i.e., this part of the parameter space is *inaccessible*. We also plot the performance of a GHZ state and a CSS for $N = 21$. We see that the GHZ state performs optimally for a small prior width, however, the performance quickly

deteriorates as the prior width grows. We plot the performance of the proposed scheme with red points, and see that the proposed scheme surpasses the performance of the CSS with 21 qubits for any prior width. The optimal performance, i.e. the minimum value of $\Delta\tilde{\phi}/\delta\phi$ for the proposed scheme occurs at $\delta\phi \approx 0.7$ rad. For each of the points of the proposed scheme, we (i) find the best partition for the given prior width, (ii) optimize over the local, adaptive measurements. The changing shapes of the red points indicate the different partitions of the 21 atoms into blocks of GHZ states: for example, the stars correspond to the partition with three 4-qubit, 2-qubit, and 1-qubit GHZ states, whereas the circles correspond to one 16-qubit, 4-qubit, and 1-qubit GHZ state. The full list of partitions corresponding to each shape can be found in Fig. 5.4b. We also plot the initial states of the proposed scheme and the initial states of the OQI for some of these shapes in Fig. 5.4c.

We then compare the MSE obtained with the proposed scheme, as a function of ϕ , to the performance of the OQI and the CSS for $\delta\phi = 0.7$ rad in Fig. 5.4d. As can be seen from the Figure, the OQI surpasses the CSS for all ϕ , performing optimally over almost the entire $[-\pi, \pi]$ range. We see that the proposed scheme also surpasses the CSS for almost all ϕ except a very small region around $\phi = \pm 3\pi/8$ rad, where the CSS performs slightly better. However, the proposed scheme has a much larger bandwidth and a better performance around $\phi = 0$ rad. We calculate the metrological gain at $\delta\phi = 0.7$ rad as $g = 2.29$ (3.59 dB).

To see how the performance of the different schemes scales with the qubit number N , we fix $\delta\phi = 0.7$ rad, from the observation that the proposed scheme works optimally around that point in Fig. 5.4⁴. In Fig. 5.5, we plot the performance of these schemes, as well as the sensitivities of the OQI and the CSS for $10 \leq N \leq 350$. We note that for our numerical simulations, we employed optimal Bayesian estimators. The blue, green, and red points in Fig. 5.5 correspond to the scheme with a fixed block size, the scheme with a varying block size, and the proposed scheme respectively. The implementation complexity of the proposed measurements also increases in this order: the scheme with a fixed block size requires a double-quadrature readout and local measurements only, whereas the other schemes require single-qubit rotations. Moreover, for the proposed scheme, the single-qubit rotations are determined adaptively, which thus requires mid-circuit measurements.

However, there is a significant difference in the discontinuous nature of the scheme

⁴Ref [131] observed that this point depends on the qubit number, N . However, $\delta\phi = 0.7$ rad is still close to the optimal point, hence relevant for qubit numbers that we work with.

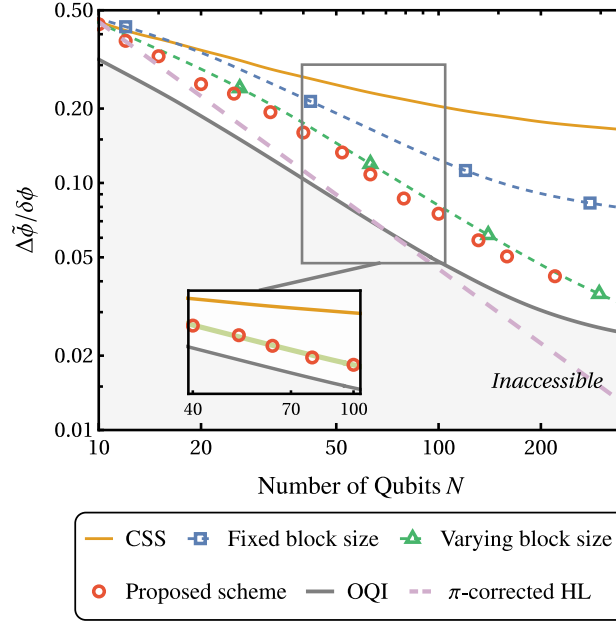


Figure 5.5: RBMSE as a function of number of qubits N , normalized by the prior width, for $\delta\phi = 0.7$ rad. We see that the CSS and the OQI constitute the SQL and HL, respectively. For larger qubit numbers, the BMSE of all of the schemes converges to a constant independent of N . We also plot the π -corrected HL in the absence of phase slip errors, which touches the OQI curve at $N \approx 70$. We observe that all of the schemes that use blocks of GHZ states show sub-SQL sensitivity. Furthermore, the scheme with a varying block size and the proposed scheme obtain HS. For all N , the proposed scheme obtains the maximal sensitivity, among existing schemes. *Inset.* We plot the RBMSE of the OQI, CSS, and the proposed scheme in the range $40 \leq N \leq 100$, with gray, yellow, and red, respectively. We see that the proposed scheme shows HS with an overhead of ≈ 1.56 , shown with the light green line.

with a fixed block size and the scheme with a varying block size, compared to the scheme of this chapter. For these schemes, the optimization over the blocks of GHZ states (Eq. (5.8)) is performed with respect to block sizes $\{m_i\}$, instead of N . Therefore, such schemes perform optimally only at certain N where $N = \sum_i m_i 2^{k_i}$ holds, which are the points that were used to show their performance in Fig. 5.5. For a general qubit number, such schemes must be modified by reducing or increasing the number of repetitions of certain blocks in order to fit the constraint of a fixed qubit number, which causes them to behave sub-optimally. In contrast, since we perform our optimization by fixing the number of qubits, the proposed scheme performs optimally for each N .

We observe from Fig. 5.5 that as the implementation complexity of the measurement

proposed by a protocol increases, so does its sensitivity. Accordingly, the proposed scheme and the scheme with a varying block size scale as the OQI, surpassing the scheme with a fixed block size for every N . These schemes are Heisenberg-limited, and their overheads are calculated to be 1.56 and 1.66, respectively. The smaller overhead belongs to the proposed scheme, and the fit is shown in the inset of Fig. 5.5. For example, the proposed scheme obtains a smaller posterior variance than that of the scheme with a varying block size by a factor of 1.22 (0.85 dB) for $N = 63$. We also compare these schemes for a smaller prior width ($\delta\phi = 0.2$ rad), and observe that the schemes continue to show HS. Furthermore, we observe that the proposed scheme surpasses the scheme with a varying block size by a larger margin. For example, for $N = 63$, the gain (in variance) obtained by the former over the latter is calculated to be 1.57 (1.97 dB).

Fundamental limits due to phase slip errors

Our analysis so far focused on achieving quantum-enhanced precision with low-depth circuits. However, another crucial limitation is phase slip errors, which arguably received less attention in the literature. Here, we first derive fundamental limits in precision due to phase slip errors. Then, we describe how to overcome such errors in the context of atomic clocks by proposing an efficient phase unwinding scheme.

Without phase slip errors, i.e. for any prior distribution with a negligible probability outside of the range of $[-\pi, \pi]$, it is well established that the minimal achievable RBMSE in the limit of large N is $\Delta\tilde{\phi} \approx \pi/N$ [125, 91], where N is the number of qubits. This is the relevant, π -corrected HL. Furthermore, the RBMSE achieved with N uncorrelated qubits is given by $\Delta\tilde{\phi} \approx 1/\sqrt{N}$, which is the SQL. In the limit of an infinite qubit number N , both the HL and the SQL approach zero.

However, any prior distribution with a support outside of the range $[-\pi, \pi]$ will give rise to an imperfect estimation, even in the limit of $N \rightarrow \infty$, since the phase can be estimated only up to modulo 2π . This means that if a phase ϕ is outside of this range, its estimation error is given by $2\pi k$ in the limit of a large qubit number $N \gg 1$, where $k = \lfloor (|\phi| + \pi)/2\pi \rfloor$. The effect of these errors is shown in Fig. 5.4a, where the RBMSE increases with increasing prior width $\delta\phi$ due to phase slip errors, in the regime where $\delta\phi > 1$ rad. For $\delta\phi \gg 1$, the RBMSE becomes approximately equal to $\delta\phi$, signifying that the phase estimation scheme does not provide any information about ϕ . Furthermore, in the presence of these errors, the RBMSE does not vanish

in the limit of $N \rightarrow \infty$. This is shown in Fig. 5.5: the RBMSE converges to a finite value in the limit of $N \rightarrow \infty$, which is determined by the statistics of these errors.

We now explicitly find this limit, i.e. we find the OQI BMSE and the CSS BMSE with a single quadrature readout in the limit of $N \rightarrow \infty$. These bounds are given in the following claim, in which we use the following expression of the BMSE (equivalent to Eq. (5.3)):

$$\begin{aligned} (\Delta\tilde{\phi})^2 &= (\delta\phi)^2 - \sum_x \frac{\left(\int \phi p(x|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi\right)^2}{\int p(x|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi} \\ &= (\delta\phi)^2 - \sum_x \phi_{\text{est}}(x)^2 p(x). \end{aligned} \quad (5.9)$$

Claim I. The OQI BMSE in the limit of $N \rightarrow \infty$ for any prior distribution $\mathcal{P}_{\delta\phi}(\phi)$ is

$$(\Delta\tilde{\phi})_{\text{OQI}}^2 = (\delta\phi)^2 - \int_{-\pi}^{\pi} \frac{\left(\sum_k \phi_k \mathcal{P}_{\delta\phi}(\phi_k)\right)^2}{\sum_k \mathcal{P}_{\delta\phi}(\phi_k)} d\phi, \quad (5.10)$$

where $\phi_k = \phi + 2\pi k$, $k \in \mathbb{Z}$.

Assuming a $J_\theta := \cos(\theta) J_x - \sin(\theta) J_y$ single quadrature readout, the CSS BMSE in the limit of $N \rightarrow \infty$ for any prior distribution $\mathcal{P}_{\delta\phi}(\phi)$ is

$$(\Delta\tilde{\phi})_{\text{SQL}}^2 = (\delta\phi)^2 - \int_{-\pi}^{\pi} d\phi \frac{\left(\sum_k (\phi_k - \theta) \mathcal{P}_{\delta\phi}(\phi_k - \theta) - (\phi_k + \theta) \mathcal{P}_{\delta\phi}(-\phi_k - \theta)\right)^2}{\sum_k \mathcal{P}_{\delta\phi}(\phi_k - \theta) + \mathcal{P}_{\delta\phi}(-\phi_k - \theta)} \quad (5.11)$$

We defer the detailed proof to Appendix C.5, and provide here a sketch of the proof. To demonstrate that Eq. (5.10) is the OQI BMSE in the limit of $N \rightarrow \infty$, we first show that it is a lower bound of the BMSE for any possible scheme. Then, we show that there exists a scheme that saturates it. The following minimization problem lower-bounds the OQI for any N :

$$\min_{\phi_{\text{est}}(\phi)} \int_{-\pi}^{\pi} \sum_k (\phi_{\text{est}}(\phi) - (\phi + 2\pi k))^2 p_{\delta\phi}(\phi + 2\pi k) d\phi, \quad (5.12)$$

where $\phi_{\text{est}}(\phi) : [-\pi, \pi] \rightarrow \mathbb{R}$. The BMSE given in Eq. (5.10) is a solution of this minimization problem. For attainability, we show in Appendix C.5 that a CSS interrogation with dual quadrature readout saturates this lower bound in the limit of $N \rightarrow \infty$.

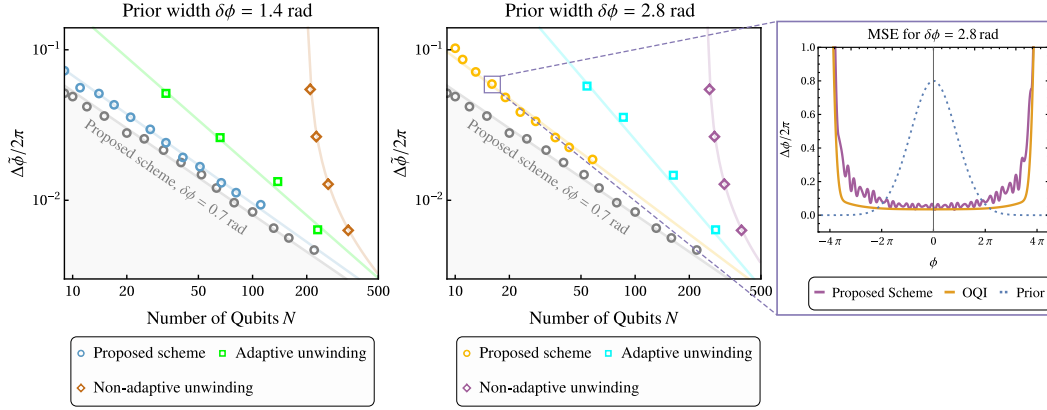


Figure 5.6: RBMSE $\Delta\tilde{\phi}$, normalized by 2π , as a function of the total qubit number N , for different prior widths. For small enough prior widths ($\delta\phi = 0.7$ rad), no slow atoms are needed. The RBMSE of the proposed scheme in this regime is plotted in gray. For larger prior widths, the number of atoms needed to extend the dynamic range depends on the *phase unwinding protocol*. We define non-adaptive and adaptive phase unwinding, also propose a more efficient phase unwinding protocol that extends the proposed scheme to large prior widths. Furthermore for the proposed scheme, in the limit of $N \gg 1$, the RBMSE for different prior widths converges to the RBMSE for $\delta\phi = 0.7$ rad. Therefore, in this limit, we can obtain a clock stability that scales as the total interrogation time τ even in the regime where $\tau\gamma_{\text{LO}} > 1$. *Inset.* MSE for the proposed phase unwinding scheme for $N = 16$ atoms and a prior width of $\delta\phi = 2.8$ rad. We see that the MSE is close to that of the OQI, and that the proposed scheme has a dynamic range of $\approx 8\pi$.

For the bound assuming a CSS J_θ quadrature readout, Eq. (5.11), we find the measurement probabilities $p(x_\theta|\phi)$, where x_θ is the outcome of $2J_\theta/N$ measurement, i.e. a measurement outcome of J_θ normalized with respect to $N/2$. In the limit of $N \rightarrow \infty$, $p(x_\theta|\phi) \rightarrow \delta(x_\theta - \cos(\phi + \theta))$. Inserting this into Eq. (5.9) leads to Eq. (5.11).

These bounds are the "plateaus" of the BMSE observed in Fig. 5.5: this is the remaining uncertainty that cannot be suppressed by increasing N . They increase as the prior width, $\delta\phi$, increases. We remark that the CSS bound shown in Fig. 5.5 corresponds to a J_y measurement. The difference between the bounds in Eqs. (5.10), (5.11) corresponds to the gap between the uncertainty of the OQI and the CSS in the limit of $N \gg 1$, as observed in Fig. 5.5.

Overcoming phase slip errors: phase unwinding schemes

For a wide phase prior distribution, phase slip errors are the main factor limiting the precision, preventing HS or even SQL for large N . In the context of atomic

clocks, this regime of a wide prior corresponds to a Ramsey time longer than the LO coherence time. In other words, $T\gamma_{\text{LO}} > 1$, where T is the Ramsey time, or the total interrogation time, and γ_{LO} is the linewidth of the laser (see Sec. 5.3). In this regime, phase slip errors dominate, and further Ramsey interrogations do not lead to any improvement in phase sensitivity. To suppress these phase slip errors, and to extend the Ramsey time beyond the LO noise limit, some techniques were devised in Refs. [224, 26]. These schemes introduce additional CSS interrogations with *slow* atoms, i.e. atoms that accumulate phases $\phi/2, \phi/4 \dots$ during the interrogation time, instead of accumulating a phase ϕ . We refer to such schemes as phase unwinding schemes.

These slow atoms that accumulate only a fractional phase can be obtained in several different ways, depending on the platform and the relevant phase noise. In the simplified case, where the laser frequency $\omega_L(t)$ is almost constant throughout the Ramsey time, they can be obtained through evolving the qubits for the relevant fraction of the Ramsey time T . In this case, a single qubit can accumulate a phase of $\phi/2^l$ for 2^l times. In other words, a single atom can be re-used as 2^l slow atoms with a phase of $\phi/2^l$. More generally, given an arbitrary stochastic process of the frequency, different methods should be employed and re-using of the qubits may not be possible. A reduced evolution rate in the presence of a general noise can be achieved using dynamical decoupling, as was shown in Refs. [224, 241]. The idea is to divide the Ramsey time into short time intervals of τ' , where τ' is shorter than the correlation time of the frequency noise. In each τ' interval we can apply a π -pulse after a period of $\tau' (1/2 + 1/2^{l+1})$, this would lead to an accumulated phase of $\phi/2^l$. Alternatively, midcircuit measurement and reset can be used [224]. With this method, each qubit corresponds to a single slow atom, and re-using of the atoms is not possible. An alternative method requires memory qubits, but allows re-using of sensing qubits: each memory qubit encodes a different slow atom, such that the sensing qubit will be entangled to different memory qubits at different periods of the τ' interval. This method requires multiple memory qubits and fast entangling gates between the sensors and memory qubits, but it allows more efficient use of the sensing qubits.

Let us first analyze and extend some existing phase unwinding methods that use slow atoms. The phase unwinding protocol in Ref. [224] defines $\phi = 2\pi P + \theta$, where $P \in \mathbb{Z}$, and $\theta = \phi \bmod 2\pi$. Assuming that m copies of slow atoms that accumulate phases of $\phi/2^l$, $l = 0, 1, \dots, l_{\text{max}}$ are available, it estimates P iteratively from the

following equation

$$P_{j-1} = 2P_j + \frac{2\beta_j - \beta_{j-1}}{2\pi} \quad (5.13)$$

where $j = 0, 1, \dots, l$, $P = P_0$, $P_{l_{\max}} = 0$, and β_j is the estimate of ϕ after performing dual quadrature readout on the j^{th} block of slow atoms. After estimating P , the posterior distribution on the phase resembles a uniform distribution in $[-\pi, \pi]$, in the limit of a large initial prior width $\delta\phi$. The protocol in Ref. [224] concludes by estimating θ from the slow atoms that accumulate a phase ϕ .

Alternatively, phase unwinding protocols can be combined with phase estimation using entangled atoms to obtain a higher precision. We can use the schemes in Refs. [134, 110], or the proposed scheme, for this purpose. We illustrate two possible ways, which we refer to as *adaptive* and *non-adaptive* phase unwinding. For non-adaptive phase unwinding, one can estimate P from the protocol in Ref. [224], and estimate θ with schemes that use GHZ blocks. Here, we assume that one can further constrain the posterior distribution on $\phi \bmod 2\pi$ after estimating P to be approximately a uniform distribution in the interval $[-\pi/2, \pi/2]$ instead of the interval $[-\pi, \pi]$, by including 2-qubit GHZ states in the protocol in Ref. [224]. We choose to do so since for a uniform prior distribution in $[-\pi, \pi]$, the BMSE is large due to the fact that small errors for phases close to $-\pi, \pi$ lead to large errors in estimating ϕ . In other words, the phase estimation error jumps to 2π for $|\phi|$ close to π , which is not suppressed when a uniform prior distribution in $[-\pi, \pi]$ is used.

In Fig. 5.6, we plot the RBMSE obtained by combining non-adaptive phase unwinding with the scheme with a varying block size (Ref. [110]). We analyze two prior widths, $\delta\phi = 1.4$ rad (plotted in brown), and $\delta\phi = 2.8$ rad (plotted in purple), where we plot the sensitivity as a function of the total number of qubits N . N includes both the atoms in the GHZ states, and the atoms used to extend the Ramsey time by the protocol in Ref. [224]. HS is not observed for the regime plotted in Fig. 5.6, due to the large overhead of additional atoms needed for this type of phase unwinding.

For adaptive phase unwinding, one can estimate both P and θ , thus estimate ϕ , written as ϕ_{est} . After the estimation, the posterior distribution of ϕ will approximately be a Gaussian distribution centered at ϕ_{est} , with a width of $\Delta\tilde{\phi}$, i.e. the RBMSE. Then, one can apply a global phase shift $-\phi_{\text{est}}$, and estimate $\phi - \phi_{\text{est}}$ with schemes that use GHZ blocks, assuming an effective prior width of $\Delta\tilde{\phi}$. We also plot the RBMSE obtained by adaptive phase unwinding in Fig. 5.6 for $\delta\phi = 1.4, 2.8$ rad in green and light blue, respectively. We observe that adaptive phase unwinding is more efficient

than non-adaptive phase unwinding, meaning that adaptive phase unwinding obtains the same sensitivity with a smaller number of atoms. We can explain this increase in performance by noticing that adaptive unwinding makes use of both P and θ before estimating ϕ with schemes that use GHZ blocks. Conversely, non-adaptive phase unwinding only computes P . Therefore, the posterior distribution on ϕ before doing the phase estimation with GHZ blocks is more localized in the former phase unwinding protocol, giving rise to a better sensitivity. In our simulations, we observed that when adaptive phase unwinding was optimized with respect to the number of slow atoms, the posterior width was in the vicinity of $\Delta\tilde{\phi} \approx 0.5$ rad.

We now outline a more efficient phase unwinding protocol that optimizes over all partitions of blocks of slow atoms and GHZ states. Assuming a large $\delta\phi$ such that phase slip errors dominate, we first choose an integer $l_{\max}$ such that $\delta\phi/2^{l_{\max}}$ is small enough. We now introduce slow atoms that accumulate phases $\phi/2^{l_{\max}}, \phi/2^{l_{\max}-1}, \dots, \phi/2$. For a given partition into blocks of slow atoms and GHZ states, we can write the relevant quantum state as:

$$|\psi\rangle \propto \prod_{l=1}^{l_{\max}} \left(|0\rangle + |1\rangle e^{i\phi 2^{-l}} \right)^{\otimes m_l} \otimes \prod_{k=0}^{k_{\max}} \left(|0\rangle^{\otimes 2^k} + |1\rangle^{\otimes 2^k} e^{i\phi 2^k} \right)^{\otimes m_k}. \quad (5.14)$$

The first term of the product is the slow or classical part, while the second term is the fast rotating part that contains GHZ states. The relation between N and the number of atoms in the different blocks requires special attention. Assuming that the same qubit can be used as a slow atom several times, e.g. a single qubit can accumulate a phase of $\phi/2^l$ for 2^l times, then the constraint on N is given by: $N = \lceil \sum_{l=-l_{\max}}^{k_{\max}} 2^l m_l \rceil$. If we cannot re-use qubits, then each slow atom corresponds to a single qubit, and the constraint on N is given by: $N = \sum_{l=-l_{\max}}^{-1} m_l + \sum_{k=0}^{k_{\max}} 2^k m_k$.

Given a total number of qubits N , we want to find an optimal partition of these N qubits into slow atoms and GHZ states. To do this we use a rescaling transformation that reverts the problem back to partitions of GHZ states only. We define $\phi' = \phi/2^{l_{\max}}$, and we use the following equality:

$$\begin{aligned} \sum_{\vec{n}} \int_{-\infty}^{\infty} (\phi - \phi_e)^2 p(\vec{n}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi = \\ 2^{2l_{\max}} \sum_{\vec{n}} \int_{-\infty}^{\infty} (\phi' - \phi'_e)^2 p(\vec{n}|\phi') \mathcal{P}_{\delta\phi/2^{l_{\max}}}(\phi') d\phi', \end{aligned} \quad (5.15)$$

which is obtained by changing the integration variable to $\phi' = \phi/2^{l_{\max}}$. This equality can be written as:

$$(\Delta\tilde{\phi})_{|\psi\rangle, \delta\phi}^2 = 2^{2l_{\max}} (\Delta\tilde{\phi}')_{|\psi'\rangle, \delta\phi/2^{l_{\max}}}^2, \quad (5.16)$$

where the l.h.s is our usual figure of merit: the BMSE in estimating ϕ with the original state, $|\psi\rangle$, and the original prior width $\delta\phi$. The r.h.s corresponds to the rescaled frame: it is $2^{2l_{\max}}$ times the BMSE in estimating $\phi' = \phi/2^{l_{\max}}$ with a prior width of $\delta\phi/2^{l_{\max}}$ and a rescaled state $|\psi'\rangle$. The rescaled state is:

$$|\psi'\rangle \propto \prod_{k=0}^{l_{\max}+k_{\max}} \left(|0\rangle^{\otimes 2^k} + |1\rangle^{\otimes 2^k} e^{i\phi' 2^k} \right)^{\otimes m_{k-l_{\max}}}. \quad (5.17)$$

Hence based on Eq. (5.15), we can optimize the BMSE of ϕ' in the rescaled frame using the techniques developed here, in which $|\psi'\rangle$ contains no slow atoms and the prior width is shrunk to $\delta\phi/2^{l_{\max}}$.

The rescaling transformation maps $\phi, \delta\phi, |\psi\rangle$ into $\phi', \delta\phi/2^{l_{\max}}, |\psi'\rangle$. Since the number of qubits in $|\psi'\rangle$ is not equal to the number of qubits in $|\psi\rangle$, N is also transformed into N' , given by $N' = 2^{l_{\max}} \sum_{l=-l_{\max}}^{k_{\max}} 2^l m_l$. The transformation of N depends on whether qubits can be re-used as several slow atoms or not. Assuming that re-using is possible, we see that $N' \approx 2^{l_{\max}} N$. However if re-using is not possible, $N' = 2^{l_{\max}} N - \sum_{l=0}^{l_{\max}} (2^{l_{\max}} - 2^l) m_{l-l_{\max}}$. Here, we similarly optimize the BMSE in the ϕ' frame for the relevant values of N' , and transform back to the BMSE in the ϕ frame using Eq. (5.16).

For example, we have calculated the optimal partition for $N' = 159, \delta\phi/2^{l_{\max}} = 0.7$ rad as $m_{k-l_{\max}} = 3, 2, 4, 3, 3, 2, k = 0, 1, \dots, 5$. We can rescale this partition by setting $l_{\max} = 3$, which corresponds to a prior width of $\delta\phi = 5.6$ rad in the rescaled frame. In this frame, the largest GHZ state is a 4-qubit GHZ state. Furthermore, assuming that re-using qubits is not possible, the rescaled frame contains $N = 12 + 2 \cdot 3 + 4 \cdot 2 = 26$ atoms. In summary, we were able to map the problem of finding the optimal partition of slow atoms and GHZ states into finding the optimal partition in presence of GHZ states only, which was already addressed and solved. In this way, we incorporated the slow atoms into the proposed scheme. The BMSE in presence of slow atoms can therefore be found using Eq. (5.16).

Finally, we plot the RBMSE obtained by the proposed phase unwinding scheme for $\delta\phi = 1.4, 2.8$ rad with blue and yellow, respectively, assuming that a qubit cannot be re-used as several slow atoms. We also plot the RBMSE of the proposed scheme for $\delta\phi = 0.7$ rad in gray as a reference since no slow atoms are used to extend the

dynamic range for this prior width. We observe that the proposed phase unwinding achieves the same precision as adaptive and non-adaptive phase unwinding with a much smaller number of slow atoms. Furthermore for large N , the RBMSE of the proposed scheme for $\delta\phi = 1.4, 2.8$ rad starts converging to the reference RBMSE ($\delta\phi = 0.7$), meaning that the number of slow atoms becomes negligible compared to the number of atoms of the GHZ blocks.

Then, in this limit, the interrogation time can be extended by the addition of a negligible number of atoms, resulting in the Allan deviation to scale as $\sigma_y(\tau) \propto \tau^{-1}$, where τ is the total interrogation time (see Sec. 5.3). Furthermore, the proposed phase unwinding protocol reduces the number of slow atoms needed to extend the interrogation time significantly compared to other phase unwinding protocols, as seen from Figure 5.6.

While the slow atoms methods presented above mitigate phase slip errors, they still do not recover the noiseless limit. Since the noiseless ultimate precision limit is the π -corrected HL⁵, the relevant question is whether there exists a method that achieves this limit for arbitrarily large prior widths and arbitrary N . The following claim provides an indication that the answer is positive: assuming ancillas and coherent control, we can construct slow replicas of optimal states that achieve HS.

Claim II. For every phase prior distribution $\mathcal{P}_{\delta\phi}(\phi)$, and for every number of probes, N , there exists a protocol that achieves a BMSE, $(\Delta\tilde{\phi})_{\mathcal{P}_{\delta\phi}(\phi), N}^2$, with

$$(\Delta\tilde{\phi})_{\mathcal{P}_{\delta\phi}(\phi), N}^2 = l^2 (\Delta\tilde{\phi})_{\text{OQI}, \mathcal{P}_{\delta\phi/l}(\phi), lN}^2, \quad (5.18)$$

where $(\Delta\tilde{\phi})_{\text{OQI}, \mathcal{P}_{\delta\phi/l}(\phi), lN}^2$ is the OQI BMSE with a rescaled prior distribution $\mathcal{P}_{\delta\phi/l}(\phi)$ and with lN probes, where $l \in \mathbb{N}_+$. This protocol assumes the use of ancillas and time dependent control.

Before proving the claim, let us first elaborate on its meaning. This protocol should allow us to achieve the π -corrected HL, π^2/N^2 , with arbitrarily large prior width. If we can find small enough $\delta\phi' = \delta\phi/l$ such that $(\Delta\tilde{\phi})_{\text{OQI}, \mathcal{P}_{\delta\phi/l}(\phi), lN}^2 = \pi^2/(lN)^2$, then this claim implies that the same scaling is achievable for $\delta\phi : (\Delta\tilde{\phi})_{\mathcal{P}_{\delta\phi}(\phi), N}^2 = \pi^2/N^2$.

Proof. We denote the OQI with $\mathcal{P}_{\delta\phi/l}(\phi)$ and lN probes as $(\Delta\tilde{\phi})_{\text{OQI}, \mathcal{P}_{\delta\phi/l}(\phi), lN}^2$. The optimal state that achieves it is denoted as $|\psi\rangle = \sum_{j=0}^{lN} c_j |j\rangle$. After the phase encoding, this state is given by $|\psi_\phi\rangle = \sum_{j=0}^{lN} c_j e^{-ij\phi} |j\rangle$. We have shown in Eq.

⁵The π -corrected HL is relevant when $\delta\phi N > 1$. If $\delta\phi N \ll 1$, the conventional HL, $1/N$, can be attained asymptotically [91].

(5.15) that the following rescaling transformation

$$\phi \mapsto \phi/l, \mathcal{P}_{\delta\phi/l}(\phi) \mapsto \mathcal{P}_{\delta\phi}(\phi), \quad (5.19a)$$

$$|\psi_\phi\rangle \mapsto |\psi_{\phi/l}\rangle = \sum_{j=0}^{Nl} c_j e^{-ij\phi/l} |j\rangle. \quad (5.19b)$$

transforms the BMSE, $(\Delta\tilde{\phi})_{\mathcal{P}_{\delta\phi}(\phi), N}^2$, as given in Eq. (5.18). This is the BMSE we want attain with N qubits, hence the only thing that is needed to show is that we can construct the following state (and phase encoding) $|\psi_{\phi/l}\rangle = \sum_{j=0}^{Nl} c_j e^{-ij\phi/l} |j\rangle$, and perform the optimal measurements using N qubits.

We provide a construction that uses ancillas and pulses that depend on the state of the ancillas. We first construct the state $|\psi\rangle = \sum_{j=0}^{Nl} c_j |j\rangle$ on the ancillas. In principle, $\log(Nl + 1)$ ancillas are sufficient to generate this state. It can also be generated by using a symmetric or a unary encoding, but this requires Nl ancillas. The N probes are initialized to the state $|N\rangle$, such that the initial state is $|N\rangle \left(\sum_{j=0}^{Nl} c_j |j\rangle \right)$. For every $|j\rangle$, we want to obtain the following phase accumulation: $|N\rangle|j\rangle \mapsto e^{-ij\phi/l} |N\rangle|j\rangle$. This can be achieved with $|j\rangle$ -dependent pulses that slow down the evolution, e.g. by adding a control Hamiltonian of $H_c = \sum_{j=0}^{Nl} f_j(t) J_x \otimes |j\rangle\langle j|$, with $f_j(t) = \pi\delta\left(t - T\frac{Nl+j}{2Nl}\right)$, where T is the Ramsey time ($\omega T = \phi$ ⁶). These ancilla-dependent pulses lead to the effective Hamiltonian of (after moving to the interaction picture with respect to H_c) $H = \sum_{j=0}^{Nl} \frac{j}{Nl} \omega J_z \otimes |j\rangle\langle j|$, and thus to the final state of $|N\rangle \left(\sum_{j=0}^{Nl} c_j e^{-ij\phi/l} |j\rangle \right)$. The state of the ancillas is now $|\psi_{\phi/l}\rangle$, hence by implementing the optimal measurement on the ancillas the desired BMSE is obtained.

Experimental Realization

Here, we provide a roadmap for the experimental implementation of the proposed scheme in existing quantum platforms. Specifically, neutral atom arrays have recently emerged as a promising platform for high fidelity control of qubits, where entanglement is mediated through Rydberg interactions [156, 157, 70, 159, 241, 73, 41].

⁶We assumed in the proof the simplest case of a fixed ω during the time evolution, i.e. $H = \omega J_z$. In most realistic cases, $\omega(t)$ fluctuates with time. In these cases, to obtain an effective Hamiltonian of $H = \sum_{j=0}^{Nl} \frac{j}{Nl} \omega(t) J_z \otimes |j\rangle\langle j|$, we need to apply π -pulses during time periods shorter than the correlation time of $\omega(t)$. Denoting such time intervals as τ , the ancilla-dependent control function $f_j(t)$ is then given by $f_j(t) = \sum_{k=0}^{T/\tau-1} \pi \left(\delta\left(t - \tau\left(k + \frac{Nl+j}{2Nl}\right)\right) + \delta(t - (k+1)\tau) \right)$

To implement the proposed scheme, the following experimental stages need to be carried out: (1) State preparation: high fidelity preparation of GHZ states of optical clock qubits. (2) Evolution period: LO phase is imprinted on the qubits. The evolution time should be shorter than the qubit coherence time (determined by amplitude damping or dephasing noise when slow atoms are used to extend the dynamic range). (3) Measurement: sequential local measurements should be performed, combined with adaptive single-qubit rotations. As the measurement is sequential and adaptive, it is crucial that when a qubit is measured, the remaining qubits to-be-measured are "frozen", i.e. decoupled from the LO phase and remain coherent during the measurement period, in order to achieve high sensitivity.

Due to the constraints above, we propose an implementation that would make use of interleaving sensing and computing with optical clock qubits and non-optical qubits with long coherence times, respectively. For the computing qubits, we can use nuclear spins in a metastable state (e.g. Yb-171). This system allows for long coherence times, high fidelity gates and protection from imaging light [126]. Or, we can use a hyperfine alkali qubit in a dual-species architecture. Let us explain what these implementations entail for state preparation and readout in the following.

In order to prepare the blocks of GHZ states in neutral atom array platforms, one can use high-fidelity entangling gates obtained through e.g. phase-modulated Rydberg pulses. Such gates have recently been implemented for optical [73, 41], as well as hyperfine and nuclear qubits [70, 164].

For fast and high-fidelity preparation of GHZ states, we can use nuclear qubits in a metastable state [165, 126]. For the clock operation, these qubits need to be mapped to optical clock qubits, which can be realized via a global π -pulse. This implementation allows for a faster state preparation that is robust to the clock laser noise. Another possible implementation is via a dual-species architecture, where state preparation is realized with e.g. the hyperfine states in rubidium [244]. In order to map these qubits to optical clock qubits, an interspecies SWAP gate is needed. Lastly, if state preparation is realized using optical clock qubits, a high-coherence laser (e.g. the clock laser systems of Refs. [29, 277]) can be used to achieve a high fidelity [73, 41].

Experimentally, the circuit to prepare an N -qubit GHZ state can be implemented through combining single-qubit rotations with controlled-Z (CZ) gates, which can be realized with two-qubit [73] or multi-qubit gates [157, 96, 41]. High-fidelity CZ gates allow for the construction of high-fidelity GHZ states. The state-of-the-

art fidelities are cited as 99.72(3)% for two-qubit CZ gates, and 99.963(2)% for single-qubit rotations, for systems that use Yb-171 nuclear spin qubits [185].

The proposed scheme requires the implementation of single-qubit rotations and sequential midcircuit measurements with feed-forward. As the initial states of the proposed scheme contain multiple copies of GHZ states with different sizes, the remaining states should be protected from the imaging light during the measurement of a GHZ state. This can be achieved by, for example, transporting the qubits to a separate readout zone before a measurement [24], or, by shelving the other qubits such that they are dark to the imaging light [159, 95, 164].

Let us describe how nuclear qubits can be used to achieve adaptive readout. After the accumulation of the LO phase, optical qubits can be mapped back to metastable nuclear qubits via another global π -pulse, so that they are decoupled from the LO and additional decoherence effects, as well as from the imaging light for the readout. Finally, a local π -pulse can be used to map one-by-one the GHZ states of nuclear qubits into GHZ states of optical or ground state qubits for sequential, local readout.

Finally, any experimental realization would suffer from decoherence, due to e.g. amplitude damping, state preparation errors and detection noise. We study the effect of these noises on the proposed scheme in Appendix C.6. Specifically, we analyze the effect of amplitude damping on blocks of GHZ states, and state preparation and measurement errors.

5.4 Methods

Approximating optimal initial states

First, we explain how the optimally performing partition is computed for a given qubit number N , and prior width $\delta\phi$. For this purpose, we fix N and write down all possible partitions into $\{k_i\}, \{m_i\}$, given the constraint $N = \sum_i m_i 2^{k_i}$. For all the partitions, we numerically calculate the minimum possible BMSE using the algorithm outlined in Ref. [167], assuming that optimal measurements (for example, the QFT operation followed by projective measurements) are available. The partition that achieves this minimum possible BMSE, for a given qubit number N and prior width $\delta\phi$, is referred to as the optimal partition. The optimal partition depends on the prior width: for a narrow prior distribution, it is desirable to use most of the qubits to construct a GHZ state with a high number of qubits, whereas for a wider prior distribution, we expect the optimal partition to include more blocks of GHZ states with smaller numbers of qubits.

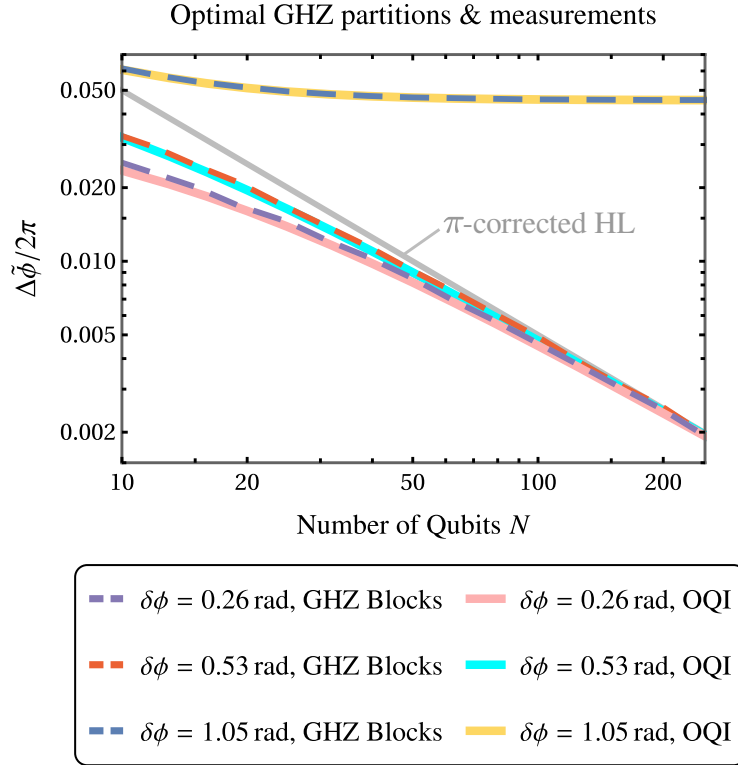


Figure 5.7: RBMSE normalized by 2π , as a function of the number of qubits N , for the optimal partition of the blocks of GHZ states, and the OQI. The OQI constitutes a benchmark for Bayesian phase estimation, and we assume that arbitrarily complex optimal measurements are available for the optimal partitions. The comparison to the OQI shows that the BMSE of the optimal partitions performs almost optimally for all prior widths. We also plot the π -corrected HL, defined as $\Delta\tilde{\phi} = \pi/N$ in the limit of $N \gg 1$ [125], with a gray line. The BMSE of the OQI (as well as the optimal partition) converges to the HL for all prior widths except $\delta\phi \approx 1.05$ rad, due to phase slip errors.

It is interesting to study how well these optimal partitions of blocks of GHZ states perform given that optimal measurements are applied, and how close their minimal BMSE is to the BMSE of the OQI. This analysis is shown in Fig. 5.7, where we compare the BMSE obtained by the optimal partitions with their optimal measurements to the BMSE of the OQI for different prior widths ($\delta\phi = \pi/12 \approx 0.26$ rad, $\pi/6 \approx 0.53$ rad, $\pi/3 \approx 1.05$ rad). We observe that optimal partitions achieve the performance of the OQI almost exactly for all qubit numbers. Therefore, given optimal measurements, approximating the optimal initial state with blocks of GHZ states causes almost no loss of sensitivity.

To get a better analytical understanding of how close the sensitivity obtained with

blocks of GHZ states can get to the OQI, we calculate the estimation error obtained by the state containing $M = 3$ copies of GHZ states with sizes $k = 0, 1, \dots, 2^{k_{\max}}$. Such a state can be written as

$$|\psi\rangle = \frac{1}{2^{3(k_{\max}+1)/2}} \sum_{n=0}^N \sqrt{f(n)} |n\rangle, \quad (5.20)$$

where $|n\rangle$ denotes the state with n excited qubits, i.e. n $|1\rangle$'s, that is symmetric under qubit-exchange operations. For this state, $f(n)$ is the coefficient multiplying x^n in the polynomial $\prod_{i=0}^{k_{\max}} (1 + x^{2^i})^3$. To simplify the analytical calculation we take a uniform prior distribution in $[-\pi, \pi]$ and a cost function of $4 \sin^2((\phi - \phi_{\text{est}})/2)$. This cost function is commonly used for this prior distribution, due to the $\phi = \phi + 2\pi$ symmetry, and it is equivalent to the standard $(\phi - \phi_{\text{est}})^2$ for small $\phi - \phi_{\text{est}}$. In Ref. [128], it was shown that the estimation error in this case can be written as

$$(\Delta\tilde{\phi})^2 = 2 \left[1 - \frac{1}{2^{3(k_{\max}+1)}} \sum_{n=0}^{N-1} \sqrt{f(n)f(n+1)} \right]. \quad (5.21)$$

Using this expression, we explicitly calculate $(\Delta\tilde{\phi})^2$ and find that it reduces to

$$(\Delta\tilde{\phi})^2 = \frac{9\sqrt{3} \ln(2 + \sqrt{3})}{2N^2} + O(N^{-3}) \approx \frac{10.26}{N^2}. \quad (5.22)$$

The detailed calculation can be found in Appendix C.3. Comparing this expression to the OQI with this prior and cost function, π^2/N^2 [20, 90], we get a very small overhead of ≈ 1.04 . Then, in accordance with the numerical results in Fig. 5.7, we observe that blocks of GHZ states can approach the optimal sensitivity limits significantly, if optimal measurements are available.

We note that the optimal partitions differ from the ones used by previous schemes. The latter are tailored for non-adaptive, local measurements, whereas the partitions of this work assume optimal measurements. We observed numerically that in general these optimal measurements cannot be replicated with non-adaptive, local measurement protocols.

Optimizing over local, adaptive measurements

Given the optimal partition, the optimal measurement strategy might be complex. For example, the optimal measurement includes performing a QFT in the limit of a large prior width within the dynamic range [74]. We therefore look for ways to perform a measurement that approximates the optimal measurement as much as possible. It has been shown that the QFT operation, followed by projective

measurements, can be approximated by performing adaptive measurements [20], where one starts measuring from the GHZ state with the highest k_i in Eq. (5.8), and performs single-qubit rotations to the rest of the states adaptively before their respective measurements.

Taking the optimal partitions of the previous section as our initial states, the adaptive measurement can be optimized with respect to the initial state and the prior width to minimize the BMSE in Eq. (5.3). First, defining $M = \sum_i m_i$, note that one needs to perform M measurements in total. For a parity measurement performed on a k -qubit GHZ state, preceded by a single-qubit rotation of Φ , the probabilities of obtaining an even or odd parity (denoted by $+$, $-$ respectively) are $P(\pm) = (1 \pm \cos(k(\phi - \Phi)))/2$. Furthermore, since there are M measurements, the number of possible outcomes is 2^M . Denoting the probability of each outcome by $p(\vec{n}, \vec{\Phi}|\phi)$, we write the total BMSE as

$$(\Delta\tilde{\phi})^2 = \int d\phi \mathcal{P}_{\delta\phi}(\phi) \sum_{\vec{n}} p(\vec{n}, \vec{\Phi}|\phi) (\phi - \phi_{est}(\vec{n}))^2 \quad (5.23)$$

where the sum is over all outcomes, or branches $\vec{n}$. Note that the single-qubit rotations before measurements, $\vec{\Phi}$, and the estimators, $\phi_{est}(\vec{n})$, are functions of the branches. We use the optimal Bayesian estimators, defined as

$$\phi_{est}(\vec{n}) = \frac{\int d\phi \phi p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi)}{\int d\phi p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi)} \quad (5.24)$$

We employ gradient descent techniques to find the optimal single-qubit rotations $\vec{\Phi}$ for a given initial state and prior width (see Appendix C.4 for more details about the optimization procedure).

5.5 Discussion

In this chapter, we analyzed phase estimation schemes that utilize state preparation with low-depth circuits in the Bayesian framework. Specifically, we focused on protocols that employ blocks of GHZ states with varying numbers of qubits, such as the protocols in Refs. [134, 110]. Inspired by these protocols, we proposed a scheme that combines such initial states with local, adaptive measurements, where we approximate optimal initial states for any prior width using such blocks of GHZ states. Constraining our optimization by fixing the number of qubits, N , we were able to achieve a good approximation for all N , which is one of the distinguishing features of the proposed scheme compared to other protocols that we have analyzed.

After obtaining the approximated optimal initial state, we optimized over local, adaptive measurements. These are implemented by applying single-qubit rotations (selected adaptively based on the previous measurement result) before a projective measurement. We analyzed the performance of the proposed scheme using Gaussian prior distributions. However, our optimization algorithm is completely general and can be applied to any prior distribution function.

Working with a large prior width within the dynamic range, we showed that it is possible to obtain Heisenberg scaling by using the protocol of Ref. [110] that combines non-adaptive measurements and single-qubit rotations with blocks of GHZ states. Moreover, we showed that the proposed protocol also exhibits Heisenberg scaling with a constant overhead of 1.56 for a prior width of $\delta\phi = 0.7$ rad, and outperforms the scheme of Ref. [110]. Since these schemes require logarithmic depth circuits, they may pave the way to a large-scale Heisenberg-limited atomic clock. A non-trivial challenge, however, is the adaptive local measurements, which require fast and non-destructive detection. This raises a question about how much we lose in sensitivity by restricting ourselves to non-adaptive measurements, for a general qubit number. We leave this question to future investigation.

Lastly, we proposed an efficient method to extend the dynamic range of the proposed scheme in order to obtain an enhanced precision. We showed that the optimal initial states of our scheme can be rescaled in order to obtain partitions that include slow atoms that accumulate fractional phases, as well as GHZ states. Compared to existing phase unwinding schemes, we showed that our method uses a smaller number of slow atoms.

Further future directions include a more detailed study of the effects of noise and decoherence. Our study of these effects can be improved by a closer examination of the optimal initial states and measurements given relevant noise processes. Such optimal measurements may involve error detection that mitigates the effect of noise, as in Ref. [135]. Lastly, inquiring whether modifying the partitioning into GHZ states adaptively can improve the performance is also an intriguing future direction.

EXTENDING THE DYNAMIC RANGE IN QUANTUM FREQUENCY ESTIMATION WITH SEQUENTIAL WEAK MEASUREMENTS

6.1 Introduction

In quantum frequency estimation, we seek to estimate an unknown frequency ω , given a dynamic range of $[\omega_0, \omega_0 + \delta\omega]$, with an optimal precision¹. Maintaining a high precision given an arbitrarily large frequency bandwidth, $\delta\omega$, is one of the fundamental challenges in quantum metrology, and it is typically referred to as the precision-bandwidth trade-off [224, 131, 169, 66, 160]. The standard estimation schemes rely on a Ramsey-type experiment: N qubits are initialized to an input state, rotate at the frequency ω for time T , and are subsequently measured. For an infinitesimally small bandwidth, longer evolution times enhance the precision, leading to estimation uncertainty that scales as $1/T$, known as Heisenberg scaling (HS) with respect to time [84]. However, a finite bandwidth restricts this improvement: an evolution time that is longer than $\pi/\delta\omega$ leads to phase slip errors that considerably degrade the precision. Consequently, the evolution time in systems such as optical atomic clocks [23, 162, 197, 194, 168, 221] is typically constrained by the inverse of the frequency bandwidth [154]. This phenomenon is prominent for the Greenberger-Horne-Zeilinger (GHZ) state, as it provides optimal sensitivity but suffers from a poor dynamic range, limiting its usefulness when $\delta\omega T$ is large [21, 167, 125, 131]. It is therefore highly desirable to develop schemes that suppress phase slip errors and achieve nearly optimal precision for a large bandwidth.

Several schemes successfully combine entanglement enhanced sensing with a relatively large dynamic range, i.e. $\delta\omega \approx \pi/T$ [21, 167, 131, 66, 134, 203]. However, in most practical applications it is important to extend the dynamic range even further: for example, in atomic clocks, larger dynamic range would improve the tolerance to laser noise and the clock stability. The fundamental question is, therefore, how to extend the dynamic range beyond this limit. Several methods have been

¹This chapter is adapted from: S. Direkci et al., "Extending the dynamic range in quantum frequency estimation with sequential weak measurements," *Phys. Rev. Research* **8**, L022046 (2026), DOI: 10.1103/2hyn-cqlq. Published by the American Physical Society under a Creative Commons Attribution 4.0 International (CC BY 4.0) license.

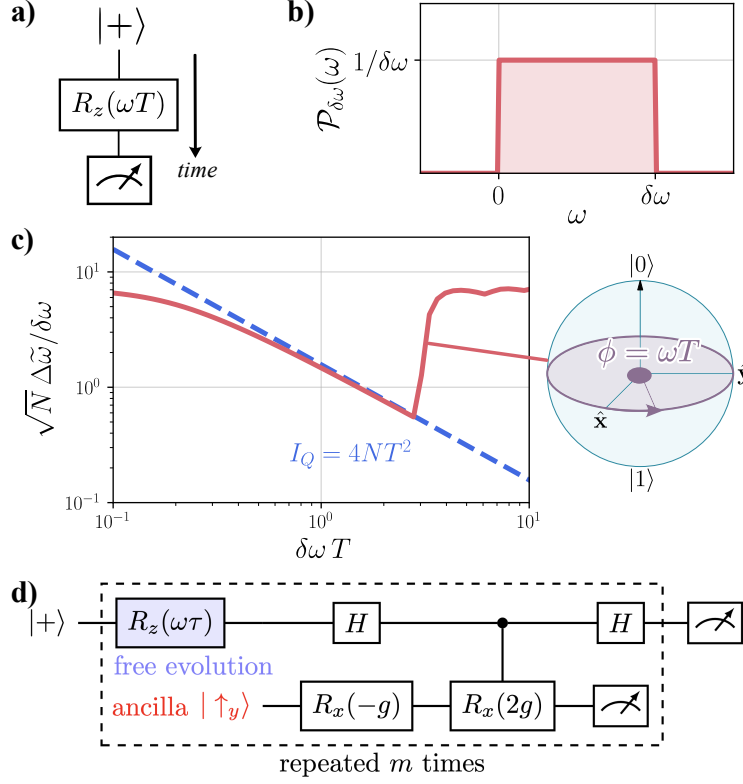


Figure 6.1: Dynamic range problem in frequency estimation: a) Circuit representation of the standard Ramsey experiment, where a strong measurement is applied after a phase accumulation time T . b) We consider a uniform prior distribution of the unknown frequency $\mathcal{P}_{\delta\omega}(\omega)$ in the interval $[0, \delta\omega]$. c) The square-root of the Bayesian mean squared error (BMSE) $\Delta\tilde{\omega}$, normalized by the prior width $\delta\omega$ and the number of qubits $N = 64$. The solid, red line corresponds to the measurement-optimized BMSE, which suffers from phase slips beyond $\delta\omega T = \pi$, as shown in the Bloch sphere picture. The dashed-blue line corresponds to the quantum Fisher information bound of $(\Delta\omega)^{-2} = I_Q = 4NT^2$. d) Circuit representation of the approach studied here: Ramsey experiment with sequential weak measurements.

proposed for this purpose, including cascaded protocols [224, 134, 66], quantum deamplification [160], and sequential weak measurements [242, 27]. However, it is unknown whether these schemes saturate the relevant precision bounds, and it is highly desirable to find schemes that are both optimal and easy to implement with state-of-the-art platforms.

We focus in this chapter on the weak measurement approach. While sequential weak measurements were proposed in the context of cavity-based atomic clocks [242, 27], there has been no systematic study of their precision limits in the context of Bayesian frequency estimation. In particular, it is unknown whether such schemes

achieve HS or saturate the precision bounds for a large bandwidth, especially for a limited number of atoms. Lastly, prior works assumed a large ensemble of atoms measured collectively [242, 27]; devising an optimal weak measurement protocol for circuit-based sensors [241, 73, 41] and for a small number of atoms remains an open challenge. We address these questions by designing a tunable sequential weak measurement protocol and analyze its performance for different configurations. We focus on the simplest case where the input state is a coherent spin state (CSS), and propose a protocol that asymptotically saturates the ultimate precision bounds for CSS for an arbitrarily large bandwidth. This protocol is thus an optimal "classical" protocol for frequency estimation.

6.2 Formulation

We consider a Ramsey type experiment: N qubits evolve according to the Hamiltonian $\mathcal{H} = \omega \sum_{j=1}^N \sigma_z^j$, where σ_z^j is the Pauli z operator acting on the j^{th} qubit, and ω is the unknown frequency to be estimated. We consider the Bayesian setting where ω has a uniform prior distribution $\mathcal{P}_{\delta\omega}(\omega) = \frac{1}{\delta\omega} \chi_{[0, \delta\omega]}(\omega)$ (χ is the indicator function). Note that any uniform prior distribution in the form of $[\omega_0, \omega_0 + \delta\omega]$ is equivalent to this distribution: we can retrieve the original prior by applying a unitary of $e^{i\omega_0 T \sum_j \sigma_z^j}$ before the measurement. The error in the estimation is quantified by the Bayesian mean squared error (BMSE), which is defined as the mean squared error (MSE) weighted by the prior distribution:

$$(\Delta\tilde{\omega})^2 = \int (\Delta\omega)^2 \mathcal{P}_{\delta\omega}(\omega) d\omega \quad (6.1)$$

where $(\Delta\omega)^2 := \sum_x p(x|\omega)(\omega - \omega_{\text{est}}(x))^2$ is the MSE, $(\Delta\tilde{\omega})^2$ is the BMSE, $\{p(x|\omega)\}_x$ are the probabilities of the measurement results, and $\{\omega_{\text{est}}(x)\}$ is the estimator function.

The standard Ramsey experiment is illustrated in Fig. 6.1a. The probes are initialized to a CSS, $|+\rangle^{\otimes N} = (|0\rangle + |1\rangle)^{\otimes N} / 2^{N/2}$, where $|+\rangle$ is an eigenstate of the Pauli x operator. They are subjected to a free evolution for a duration of T , and are finally measured. The BMSE of this protocol, when optimized over all possible measurements, is referred to as the optimal classical interferometer (OCI) [167] and is plotted in Fig. 6.1c. We observe that for interrogation times where $\delta\omega T > \pi$, the OCI BMSE increases rapidly, and converges to the prior width, $\Delta\tilde{\omega} \approx \delta\omega$, signifying that no information is gained from the estimation. The behavior is caused by phase slips: the unitary transformation describing the free evolution is periodic in ωT .

Therefore, the maximum interrogation time using this protocol is limited to $\pi/\delta\omega$, since longer times will give rise to phase slip errors.

To resolve this issue, we propose to perform sequential weak measurements during the free evolution. Weak measurements provide frequent monitoring of the state and allow us to track the oscillations, preventing phase slip errors [80, 55, 205, 50, 254]. They therefore can potentially extend the dynamic range beyond π/T .

Our scheme consists of applying a sequence of weak σ_x measurements in time intervals of τ . They are described by the Kraus operators

$$K_{\pm} = \frac{1}{\sqrt{2}} [\cos(g)I \pm \sin(g)\sigma_x] = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(g) & \pm \sin(g) \\ \pm \sin(g) & \cos(g) \end{bmatrix}, \quad (6.2)$$

where g is a tunable parameter that represents the measurement strength ($g = \pi/4$ corresponds to a projective measurement). They can be realized by weakly entangling each probe to an ancillary qubit and measuring the ancilla. Each step of the protocol involves the rotation $V_{\omega} = \cos(\omega\tau)I - i\sin(\omega\tau)\sigma_z$, followed by a weak measurement that transforms the state of the sensing qubit ρ as

$$\rho \rightarrow \frac{K_{\pm} \rho K_{\pm}^{\dagger}}{\text{Tr}[K_{\pm} \rho K_{\pm}^{\dagger}]}, \quad (6.3)$$

depending on the measurement outcome. Therefore, the sensing qubits evolve in stochastic quantum trajectories [121]. Given a total interrogation time T and a measurement period τ , this stochastic step is repeated for $m = \lfloor T/\tau \rfloor$ times². Note that, to prevent phase slips, the measurement period must satisfy $\tau < \frac{\pi}{2\delta\omega}$.

6.3 Ramsey Interferometry with Weak Measurements

To understand Ramsey interferometry with weak measurements evolution better, let us study the average dynamics described by the channel $\Lambda_{\omega}^{T/\tau}(\rho(0))$, where

$$\Lambda_{\omega}(\rho) = K_{+}^{(\omega)} \rho K_{+}^{(\omega)\dagger} + K_{-}^{(\omega)} \rho K_{-}^{(\omega)\dagger}, \quad (6.4)$$

and where $K_{\pm}^{(\omega)} = K_{\pm} V_{\omega}$. Similar to the previous Section, this channel represents averaging over all possible ancilla measurements during time $0 < t < T$. Expanding $\rho(0) = 1/2(I + \vec{r} \cdot \vec{\sigma})$, we obtain

$$\begin{pmatrix} r_x(\tau) \\ r_y(\tau) \\ r_z(\tau) \end{pmatrix} = \begin{pmatrix} \cos(2\omega\tau) & \sin(2\omega\tau) & 0 \\ -\cos(2g)\sin(2\omega\tau) & \cos(2g)\cos(2\omega\tau) & 0 \\ 0 & 0 & \cos(2g) \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}. \quad (6.5)$$

²Note that if we set $\tau = T$, $g = \pi/4$, this protocol reduces to the standard Ramsey experiment.

The dynamics of r_z is decoupled from that of r_x, r_y , and since $r_z(0) = 0$, it remains 0. Restricting ourselves to the block matrix of r_x, r_y , its eigenvalues are

$$\cos(2g)^{1/2} e^{\pm i\alpha} \quad \text{with } \alpha = \arccos\left(\frac{\cos(g)^2 \cos(2\omega\tau)}{\sqrt{\cos(2g)}}\right). \quad (6.6)$$

This implies that r_x, r_y decay with a rate of $\gamma = -\frac{1}{2\tau} \log(\cos(2g))$ and oscillate with a frequency of α/τ . Let us denote the probability of ancilla measurements at time $t = k\tau$, $k \in \mathbb{N}^+$ as $p(x_k|\omega)$, where $x_k = 0, 1$ correspond to the Kraus operators $K_+^{(\omega)}$ and $K_-^{(\omega)}$, respectively. The full solution of $r_x(k\tau)$ and $p(x_k = 0|\omega)$ is then given by

$$\begin{aligned} r_x(k\tau) &= \cos(2g)^{k/2} A \cos(\alpha k + \phi), \\ p(x_k = 0|\omega) &= \frac{1}{2} (1 + \sin(2g) r_x(k\tau)) \\ &= \frac{1}{2} \left[1 + \sin(2g) \cos(2g)^{k/2} A \cos(\alpha k + \phi) \right] \end{aligned}$$

where

$$\begin{aligned} A &= \sqrt{\frac{\cos(2g) \sin(2\omega\tau)^2}{\cos(2g) - \cos(g)^4 \cos(2\omega\tau)^2}}, \\ \phi &= \arcsin\left(\frac{-\sin(g)^2 \cos(2\omega\tau)}{\sqrt{\cos(2g) \sin(2\omega\tau)^2}}\right). \end{aligned} \quad (6.7)$$

Here, k denotes the number of the measurement, and α is given in Eq. (6.6). For $g^2 T/\tau \ll 1$, the parameters α, A, ϕ can be approximated as $\alpha = 2\omega\tau + O(g^4)$, $A = 1 + O(g^4)$, and $\phi = O(g^2)$. Then,

$$p(x_k = 0|\omega) \approx \frac{1}{2} [1 + \sin(2g) \cos(2g)^{k/2} \cos(2\omega k\tau)] \quad (6.8)$$

Let us now study the stochastic qubit trajectories defined by a string of measurement results. Contrary to the average dynamics, we keep track of the ancilla measurements, which indicate the trajectory of the sensing qubit. For this case, we can show that the state of the qubit and the measurement probabilities can be calculated iteratively. We first prove the following simple claim: The stochastic process of $K_{\pm}^{(\omega)}$ preserves the state in the σ_x - σ_y plane; i.e. if the initial state is in this plane, it will remain there at all time (ignoring decoherence effects, e.g. amplitude damping).

To show this we use the update rule of Eq. (6.3), which in this case reads $\rho \mapsto K_{\pm}^{(\omega)} \rho K_{\pm}^{(\omega)\dagger} / \text{Tr}(K_{\pm}^{(\omega)} \rho K_{\pm}^{(\omega)\dagger})$. Hence, it suffices to show that the (unnormalized)

operation $K_{\pm}^{(\omega)} \rho K_{\pm}^{(\omega)\dagger}$ does not take the state out of the σ_x - σ_y plane. Assuming the qubit to be in the state $\rho = 1/2 (I + \vec{r} \cdot \vec{\sigma})$, where $\vec{r} = (r_x, r_y, r_z)$, and $|\vec{r}| \leq 1$, this operation induces the following mapping of $\vec{r}$:

$$r_x \mapsto \cos(2\omega\tau) r_x + \sin(2\omega\tau) r_y \pm \sin(2g) \quad (6.9a)$$

$$r_y \mapsto -\cos(2g) \sin(2\omega\tau) r_x + \cos(2g) \cos(2\omega\tau) r_y \quad (6.9b)$$

$$r_z \mapsto \cos(2g) r_z \quad (6.9c)$$

since $r_z \mapsto r_z \cos(2g)$, a state initialized in the σ_x - σ_y plane ($r_z = 0$) will remain there. Hence, given an initial state of $|+\rangle$, the state of the qubit remains in the σ_x - σ_y plane for all possible measurement results, at all time. Since the state also remains pure, it is therefore fully characterized by its angle in the plane, θ , and it can be written as $\rho = \frac{1}{2} (I + \cos(\theta) \sigma_x + \sin(\theta) \sigma_y)$. For a given trajectory, let us denote as ρ_k and θ_k the state and the angle before the k -th measurement. The weak measurement probabilities at the k -th stage are then given by

$$p(x_k|\omega) := \text{Tr} \left(K_{(-1)^{x_k}} \rho_k K_{(-1)^{x_k}}^\dagger \right) = \frac{1}{2} [1 + (-1)^{x_k} \sin(2g) \cos(\theta_k)] \quad (6.10)$$

where $x_k = 0, 1$, $k \in \mathbb{N}^+$. Given that the k -th measurement outcome is x_k , the state is updated as follows:

$$\rho_{k+1} = \frac{1}{p(x_k|\omega)} V_\omega K_{(-1)^{x_k}} \rho_k K_{(-1)^{x_k}}^\dagger V_\omega^\dagger. \quad (6.11)$$

Inserting $\rho_k = \frac{1}{2} (I + \cos(\theta_k) \sigma_x + \sin(\theta_k) \sigma_y)$, we obtain the following recursive relation for θ_k :

$$\theta_{k+1} = \arg(\cos(\theta_k) + (-1)^{x_k} \sin(2g) + i \cos(2g) \sin(\theta_k)) - 2\omega\tau. \quad (6.12)$$

Note that in the limit of $g \ll 1$, this relation reduces to $\theta_{k+1} = \theta_k - (-1)^{x_k} 2g \sin(\theta_k) - 2\omega\tau + O(g^2)$, hence θ_k undergoes a scaled random walk, with a step size of $2g \sin(\theta_k)$. The angle and measurement probabilities can be thus calculated iteratively using Eq. (6.12).

We study two different protocols, illustrated in Fig. 6.1d: the *weak-only* protocol, in which only sequential weak measurements are performed, and the *weak-with-strong* protocol, in which the final weak measurement is replaced by a projective measurement on the sensing qubits.

6.4 Cramér-Rao bounds

We are now poised to study the precision limits of this scheme and compare them to the fundamental bounds. The ultimate precision limit is given by the Bayesian Cramér-Rao bound (or the van Trees inequality) [53, 215, 259]

$$(\Delta\tilde{\omega})^2 \geq [\bar{I}_C + I_\mathcal{P}]^{-1}, \quad (6.13)$$

where $\bar{I}_C := \int d\omega \mathcal{P}(\omega) I_C(\omega)$, $I_C = \int dx p(x|\omega) [\partial_\omega \log p(x|\omega)]^2$ is the classical Fisher information (CFI), and $I_\mathcal{P} = \int d\omega \mathcal{P}(\omega) [\partial_\omega \log \mathcal{P}(\omega)]^2$ denotes the information from the prior distribution. Since we are interested in the large bandwidth limit, i.e. $\delta\omega T \gg 1$, we have that $I_C \gg I_\mathcal{P}$, and Eq. (6.13) reduces to the Cramér-Rao bound (CRB): $(\Delta\tilde{\omega})^2 \gtrsim I_C^{-1}$. Furthermore, from the quantum Cramér-Rao bound [33], the CFI is upper bounded by the quantum Fisher information (QFI), i.e. $I_C \leq I_Q$. Then, the QFI is the ultimate limit of precision given an input state, and the benchmark for our parameter estimation.

For pure states, the QFI about the parameter ω encoded in $|\psi(\omega)\rangle$ reduces to $I_Q = 4 \left(\langle \partial_\omega \psi(\omega) | \partial_\omega \psi(\omega) \rangle - |\langle \psi(\omega) | \partial_\omega \psi(\omega) \rangle|^2 \right)$. In the special case of $|\psi(\omega)\rangle = \exp(-i\mathcal{H}(\omega)T) |\psi\rangle$, which corresponds to a qubit in a pure state evolving under some Hamiltonian $\mathcal{H}(\omega)$ parametrized by ω for time T , the QFI is further simplified to $4T^2 \text{var}_{|\psi\rangle}(\partial_\omega \mathcal{H})$. In our case, $\mathcal{H}$ is the free evolution Hamiltonian, i.e. $\mathcal{H} = \omega \sigma_z$. Hence, the optimal QFI is

$$I_Q = 4T^2 \max_{|\psi\rangle} \text{var}_{|\psi\rangle}(\sigma_z) = 4T^2. \quad (6.14)$$

It can be shown that this is the optimal QFI when optimized over all possible control strategies and initial states [200, 279], hence the fundamental precision limit for this problem is $\Delta\omega \geq \frac{1}{2T}$. For an N atom CSS, assuming separable control, the limit is given by $I_Q = 4NT^2$, hence $\Delta\omega \geq \frac{1}{2\sqrt{NT}}$. We refer to this bound as the QFI limit throughout the chapter. Furthermore, this scaling of the QFI is referred to as the Heisenberg limit (HL) with respect to the interrogation time T . In contrast, the CFI achieved by repeating a Ramsey protocol for $M = T/\tau$ times is found as $4N\tau^2 \cdot M = 4NT\tau$, which fails to achieve the HL.

We now turn to quantifying the performance of the proposed protocols. We first compute the CFI of this weak measurement scheme. Even though this is a local quantity, it provides a bound on the BMSE through the Bayesian Cramér-Rao bound (Eq. (6.13)) and can be calculated analytically. Therefore, we first show that the CFI can approach the QFI. We then demonstrate numerically the saturation of the CFI in the relevant Bayesian setting.

For this purpose, it is useful to define the parameter $\eta = g^2 T / \tau$, which quantifies the amount of back action on the sensing qubits due to the weak measurements: the back action decay time was found as $\gamma^{-1} \approx \tau / g^2$, therefore, $\eta = T / \gamma^{-1}$ describes how the total interrogation time compares to the decay time.

Expressions for the CFI

We first study the behavior of the CFI in the weak back action regime, i.e. $\eta \ll 1$. In this regime, the back action due to weak measurements is negligible. Therefore, measurement results for different times will approximately be independent of each other, and the total CFI can be written as a sum over the CFI obtained only from the k^{th} measurement, in the form of $I_C = \sum_k I_C[k]$. Furthermore, in this limit, $p(x_k|\omega)$ for a weak measurement can simply be written as $p(x_k|\omega) = [1 + (-1)^{x_k} \sin(2g) \cos(2k\omega\tau)] / 2$, $x_k = 0, 1$. Therefore, the CFI for the k^{th} weak measurement is calculated as

$$I_C[k] = \frac{4 \sin(2g)^2 k^2 \tau^2 \sin(2\omega k \tau)^2}{1 - \sin(2g)^2 \cos(2\omega k \tau)^2} \approx 4 \sin(2g)^2 k^2 \tau^2 \sin(2\omega k \tau)^2 \quad (6.15)$$

Thus, the total CFI of the weak-only protocol for a single qubit is given by

$$I_C^w = \sum_k I_C[k] \approx 2 \sin(2g)^2 T^3 / 3\tau \approx 8g^2 T^3 / 3\tau, \quad (6.16)$$

for $T/\tau \gg 1$. For N qubits, the total CFI is given by

$$\bar{I}_C^w \approx \frac{8g^2 N T^3}{3\tau} \quad (6.17)$$

This T^3 scaling of the CFI is typical for frequency estimation of classical signal [220, 248, 143, 232, 231, 77]. Here, even though the scaling with T is more favorable than HS, the absolute performance relative to the HL is poor since $\eta \ll 1$.

Now, let us find the CFI of the weak-with-strong protocol. In the regime where $\eta \ll 1$, the probability for the measurement outcomes of the projective measurement at the end of the interrogation is approximately given by

$$p_s(x_{T/\tau}|\omega) = \frac{1}{2} [1 + (-1)^{x_{T/\tau}} (\cos(2\omega T) + 2g \sin(2\omega T) s(\vec{x}, \omega\tau))] \quad (6.18)$$

where $s(\vec{x}, \omega\tau) := \sum_{k=1}^{T/\tau-1} (-1)^{x_k} \sin(2\omega k \tau)$, and $\vec{x} = (x_1, x_2, \dots, x_{T/\tau-1})$ are the measurement results of the weak measurements, and there is some back action on the qubit, to first order in g . Then, the CFI of this protocol for a single qubit can be

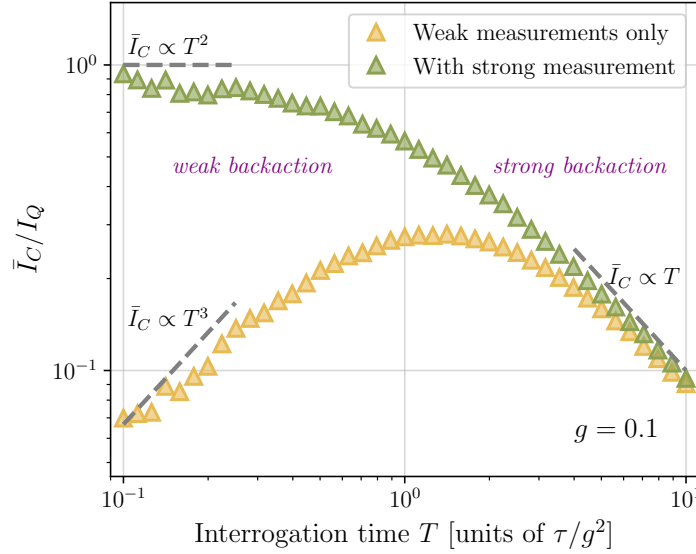


Figure 6.2: Average classical Fisher information (CFI) $\bar{I}_C$, normalized by the QFI, $I_Q = 4NT^2$, for the weak-only protocol (plotted in yellow), and weak-with-strong protocol (plotted in green). We fix the measurement strength as $g = 0.1$, measurement period as $\tau = 0.1$ s, and vary the interrogation time $T \in [1, 100]$ s. In the weak back action regime, i.e. $g^2T/\tau \ll 1$, the CFI for the weak-with-strong protocol saturates I_Q . The CFI for the weak-only protocols is much smaller, yet it increases rapidly as T^3 . Eventually, in the strong back action regime, $g^2T/\tau \gg 1$, the two protocols obtain the same CFI of $4NT\tau/g^2$.

written as

$$I_C^{\text{ws}} = \frac{4T^2 \sin^2(2\omega T)}{2T/\tau} \sum_{\vec{x}} \frac{\left[1 - 2g \left(\cot(2\omega T) s(\vec{x}, \omega\tau) + \frac{1}{2T} \frac{\partial s(\vec{x}, \omega\tau)}{\partial \omega} \right)\right]^2}{1 + (-1)^{xT/\tau} \sin(2\omega T) (\cot(2\omega T) + 2gs(\vec{x}, \omega\tau))}$$

$$\approx 4T^2 + O(g^2) \quad (6.19)$$

We note that this protocol saturates the ultimate sensitivity limit with respect to the total interrogation time T for a small measurement strength g , given by the QFI in Eq. (6.14). If we have N qubits, we will similarly have that $I_C^{\text{ws}} \approx 4NT^2 + O(g^2)$.

In the strong back action regime, $\eta \gg 1$, the CFI of both protocols scales as T . This linear scaling of the CFI is ubiquitous in sequential measurement schemes [75, 80, 275] and it is attributed to the state memory loss induced by the sequential measurements, i.e. the phase becomes uncorrelated with its initial value. We can interpret the overall behavior of the CFI as follows: in the weak back action regime not enough information is extracted from the state, while in the strong back action

regime the decoherence induced by the measurements degrades the CFI and prevents achieving HS with respect to time. The optimal g is thus at the crossover between the two regimes, where the trade-off between extracted information and dissipation is optimal, and HS can be achieved. For a given T , we find that the optimal g corresponds to $g^2 T / \tau \approx \sqrt{3/2}$. The resulting CFI is then $\bar{I}_C^w \approx 1.11 N T^2$. Hence, we can obtain HS with time using this weak-only scheme with an optimal g , where the CFI loses a factor of ≈ 3.60 compared to the QFI limit.

In the strong back action regime, we consider the following: a general upper bound to the CFI with sequential measurements of a probe was given in Ref. [75]. This is done by considering the map:

$$\Lambda(\omega_1, \omega_2)(\rho) := K_+^{(\omega_1)} \rho K_+^{(\omega_2)\dagger} + K_-^{(\omega_1)} \rho K_-^{(\omega_2)\dagger}. \quad (6.20)$$

Note that this is not a quantum channel, as it is not trace preserving (due to the different frequencies in $K_{\pm}^{(\omega_i)}$). The bound for a time T , i.e. after T/τ applications of Λ , is given by

$$I \leq 4 \partial_{\omega_1} \partial_{\omega_2} \log(\text{tr}(\rho(\omega_1, \omega_2)))|_{\omega_1=\omega_2=\omega}, \quad (6.21)$$

where $\rho(\omega_1, \omega_2) = \Lambda(\omega_1, \omega_2)^{T/\tau}(\rho)$. It can be shown that this bound is simplified to

$$I \leq 4T \partial_{\omega_1} \partial_{\omega_2} \eta|_{\omega_1=\omega_2=\omega} \quad (6.22)$$

where η is the largest eigenvalue of $\Lambda(\omega_1, \omega_2)$. Using Eqs. (6.21)-(6.22) for our case, we find $\partial_{\omega_1} \partial_{\omega_2} \eta|_{\omega_1=\omega_2=\omega} = \tau \cot(g)^2$. This upper bound thus reads

$$I \leq 4T \tau \cot(g)^2. \quad (6.23)$$

Note that in general, $\tau \cot(g)^2 \geq \gamma^{-1} = \left(-\frac{1}{2\tau} \log(\cos(2g))\right)^{-1}$, but in the limit of $g \ll 1$ we have that $\gamma^{-1} \approx \tau \cot(g)^2 \approx \frac{\tau}{g^2}$. Hence, in this limit,

$$I \leq 4T \gamma^{-1} \approx 4 \frac{\tau}{g^2} T. \quad (6.24)$$

While this bound holds for every T , it is tight only in the strong back action regime, i.e. $\eta \gg 1$. The intuition for this strong back action limit of the CFI is the following: For times shorter than $1/\gamma$, the frequency is sensed with Heisenberg scaling (HS), such that the CFI reaches $4/\gamma^2$. The measurements after $t = 1/\gamma$ are uncorrelated with the measurements during time $t < 1/\gamma$. Thus, the CFI will be equivalent to

the case where the measurement during $t < 1/\gamma$ is repeated $T\gamma$ times, which is calculated as $4T/\gamma$. Note that the analysis for this upper bound of the CFI is not modified when we perform a strong measurement at the end of the interrogation. This is because in this regime, the final projective measurement does not add any new information, since at the end of the interrogation the probes have already lost their phase coherence due to the weak measurements. Thus, we obtain the same linear CFI scaling $\bar{I}_C^{\text{ws}} \approx 4NT\tau/g^2$, as we had for $\bar{I}_C^{\text{w}}$. Weak measurements only reduce I_C^{ws} , but as we will show in the following, they are needed to saturate the CFI given a large dynamic range.

Numerically, we compute the CFI for a general protocol using a Monte Carlo sampling, i.e. by sampling K trajectories of measurement outcomes for N qubits. For large enough K , the CFI is approximately given by

$$I_C \approx \frac{1}{K} \sum_{\vec{x}} \left(\frac{\partial \ln p(\vec{x}|\omega)}{\partial \omega} \right)^2 \quad (6.25)$$

where the sum is over the sampled trajectories $\vec{x}$, and $p(\vec{x}|\omega)$ is the probability of obtaining a trajectory $\vec{x}$ given ω . We also numerically compute the derivative by calculating $p(\vec{x}|\omega + d\omega)$, $d\omega \ll 1$, for all $\vec{x}$.

6.5 Saturability of the CRB and the threshold effect

So far we have shown that the CFIs of both of the weak measurement protocols, $\bar{I}_C^{\text{w}}, \bar{I}_C^{\text{ws}}$, yield HS with appropriate values of g, τ, T . However, these quantities correspond to local estimation of ω , and they are not necessarily saturable in the relevant large bandwidth limit, $\delta\omega T \gg 1$. Hence, we study the minimal η, N for which $\bar{I}_C^{\text{w}}, \bar{I}_C^{\text{ws}}$ are saturated, given a uniform prior distribution in $[0, \delta\omega]$. In particular, we focus on whether they can be saturated for small enough η such that HS is achieved. Note that the prior distribution implicitly determines the measurement period for the weak measurement protocols, with $\tau = \pi/2\delta\omega$.

A similar problem was studied extensively in classical sensing, where a signal $s(t_n) = A \cos(2\omega t_n) + \nu_n$ is sampled at times $\{t_n = n\tau\}_{n=1}^M$ [220, 248]. Here, A is the amplitude of the signal and $\{\nu_n\}_{n=1}^M$ are i.i.d. Gaussian random variables, $\nu_n \sim N(0, \sigma^2)$. It was shown that this classical frequency estimation problem is plagued by a threshold effect with respect to the signal to noise ratio (SNR), defined as $\frac{A^2}{\sigma^2} \frac{T}{\tau}$. For low SNR, the posterior distribution is very close to the prior distribution, and the MSE is thus very close to the prior variance. Above a certain threshold of

the SNR, there is a sharp drop in the MSE, and it converges to the CRB. This is a well-known effect in signal estimation, studied in e.g. Refs. [220, 248, 143].

In our case we also sample a stochastic, time-dependent signal, through the weak measurements. The weak back action regime in our setting is equivalent to sampling a classical signal due to the negligible back action. We thus expect a similar threshold behavior, where the SNR in the limit of $g \ll 1$ can be defined as $Ng^2T/\tau = N\eta$. Let us derive this effect in the next subsection.

Threshold Derivation: Analytical Model for the Maximum Likelihood Estimator

Here, we derive the variance of the maximum likelihood estimator (MLE) in the regime where the number of qubits is large ($N \gg 1$), the measurement strength is small ($g^2 < \tau/T$), and the number of measurements is large $T/\tau \gg 1$. In this regime, the back action due to weak measurements is negligible; hence, subsequent measurements can be treated as approximately independent events. Let us denote the measurement results in the computational basis $\{|0\rangle, |1\rangle\}$ for a time $t = n\tau$ as $y_n = 0, 1$, and $x_+(n)$ stands for the number of the $y_n = 0$ outcomes normalized by the number of qubits, N . Denoting the probability of $y_n = 0$ as $p(y_n = 0|\omega)$, then in the limit of $N \gg 1$, $Np(y_n = 0|\omega), Np(y_n = 1|\omega) \gg 1$, we can use the central limit theorem and approximate the probability density function of $x_+(n)$ as

$$P(x_+(n)|\omega) \sim \frac{\sqrt{N} \exp\left(-\frac{N(x_+(n) - p(y_n=0|\omega))^2}{2p(y_n=0|\omega)(1-p(y_n=0|\omega))}\right)}{\sqrt{2\pi p(y_n=0|\omega)(1-p(y_n=0|\omega))}} \quad (6.26)$$

Here, we used the normal approximation to the binomial distribution, such that $x_+(n)$ is a Gaussian random variable with $E[x_+(n)] = p(y_n = 0|\omega)$, and $\text{var}(x_+(n)) = \frac{1}{N}p(y_n = 0|\omega)(1-p(y_n = 0|\omega))$. Note that in this weak back action regime $p(y_n|\omega)$ is given by $p(y_n|\omega) = \frac{1}{2} [1 + (-1)^{y_n} \sin(2g) \cos(2\omega n\tau)]$. Then, $E[x_+(n)] = \frac{1}{2} [1 + \sin(2g) \cos(2\omega n\tau)] \approx 1/2 + g \cos(2\omega n\tau)$, and $\text{var}(x_+(n)) \approx 1/4N$. In this limit the problem is thus equivalent to the problem of classical signal estimation [220]: we sample a classical signal $s(t_n) = \frac{1}{2} (1 + \sin(2g) \cos(2\omega t_n)) + \nu_n$ where $\{\nu_n\}_{n=1}^{T/\tau}$ are i.i.d Gaussian random variables $\nu_n \sim N\left(0, \frac{1}{4N}\right)$ and $t_n = n\tau$.

The MLE is therefore given by $\text{argmax}_{\omega} \left[\log \prod_{n=1}^{T/\tau} P(x_+(n)|\omega) \right]$. Using the probability distribution for $x_+(n)$ given in Eq. (6.26), we can write this expression

as

$$\begin{aligned}\omega_{\text{MLE}} &:= \underset{\omega}{\operatorname{argmin}} \sum_{n=1}^{T/\tau} (x_+(n) - p(y_n = 0|\omega))^2 \\ &= \underset{\omega}{\operatorname{argmin}} \sum_{n=1}^{T/\tau} \left(x_+(n) - \frac{1}{2} (1 + \sin(2g) \cos(2\omega n\tau)) \right)^2.\end{aligned}\quad (6.27)$$

For brevity, hereafter we will replace $x_+(n)$ with the more compact notation x_n . We expand the expression of Eq. (6.27) and discard the x_n^2 terms. In the limit that $g \ll 1$, the minimization above can be approximated as

$$\omega_{\text{MLE}} \approx \underset{\omega}{\operatorname{argmax}} \sum_{n=1}^{T/\tau} x_n \cos(2\omega n\tau) \quad (6.28)$$

The expression inside the maximization is the definition of the real part of a discrete time Fourier transform (DTFT), which is a continuous function of frequency. Since we work in the limit that $T \gg \tau$, the DTFT is closely linked to the discrete Fourier transform (DFT), i.e. we do not lose significant information by working with the discrete frequency bins instead of maximizing a continuous function. We can define the DFT operation as (matching the convention of the previous equations)

$$A_k = \frac{\tau}{T} \sum_{n=1}^{T/\tau} x_n \exp\left(i \frac{2\pi k n \tau}{T}\right), \quad k = 1, 2, \dots, T/\tau, \quad (6.29a)$$

$$x_n = \sum_{k=1}^{T/\tau} A_k \exp\left(-i \frac{2\pi k n \tau}{T}\right), \quad n = 1, 2, \dots, T/\tau \quad (6.29b)$$

Then, we see that $\omega_{\text{MLE}} \approx \pi/T \operatorname{argmax}_k \operatorname{Re}(A_k) = \pi/T \operatorname{argmax}_k B_k$, where we denoted the real part of A_k as B_k . Note that while the DFT frequencies are $\{\omega_k | \omega_k = \frac{\pi}{T} k\}_{k=1}^{T/\tau}$, in our case, because of the symmetry of the cosine function in Eq. (6.28), we cannot distinguish between $\omega_k = \frac{\pi}{T} k$ and $\omega_{m-k} = \frac{\pi}{T} (m - k)$. The likelihood function thus has this symmetry which means that there will be two identical global maxima: ω_k and ω_{m-k} . This is however not an issue for us since the prior frequency distribution corresponds to $[0, \frac{\pi m}{2T}]$ (where m is the total number of measurements $m = \lfloor T/\tau \rfloor$), therefore the maximization with respect to ω is only over the range $[0, \frac{\pi m}{2T}]$, which does not suffer from this symmetry. For simplicity, in what follows we will consider maximization over all of $\{\omega_k | \omega_k = \frac{\pi}{T} k\}_{k=1}^{T/\tau}$, where we understand that the maximum corresponds to a pair of frequencies from which the estimator is the frequency in the range of $[0, \frac{\pi m}{2T}]$.

We observe from numerics that the variance of the MLE has a strong dependence on the signal-to-noise-ratio (SNR). Here, as the amplitude of the signal is given by $\sin(2g) \approx O(g)$, $\text{SNR} \approx O(g^2 NT/\tau)$. For large enough SNR the MSE converges to the CRB, i.e. $(\Delta\omega_{\text{MLE}})^2 \approx I_C^{-1}$. Conversely, for small SNR, the likelihood is very close to a uniform distribution over $[0, \pi/2\tau]$. Therefore, the variance of the MLE will approximately be $\pi^2/48\tau^2$. This idea can be summarized in the form of

$$(\Delta\omega_{\text{MLE}})^2 \approx \text{Prob}(\text{no outlier}) \frac{1}{I_C} + \text{Prob}(\text{outlier}) \frac{\pi^2}{48\tau^2} \quad (6.30)$$

where the probability of having no outliers corresponds to the large SNR case mentioned above. Let us denote it as $1 - q$. We can write down this probability as

$$1 - q = \text{Prob}(B_k < B_{k^*} \ \forall k \neq k^*) \quad (6.31)$$

where k^* is the frequency bin that is associated with the true value of the frequency, and $B_k, k \neq k^*$ are the rest of the frequency bins. Therefore, an outlier occurs if the noise in one of these frequency bins exceeds the signal in the bin k^* . Without loss of generality, we can assume that $\omega^* = \pi/4\tau$ is the frequency that we are trying to estimate, which is approximately the mean value of all possible true frequencies. Then, the respective frequency bin is $k^* = \lfloor T/4\tau \rfloor$, or $k^* = \lceil 3T/4\tau \rceil$, where we define $\lfloor x \rfloor$ as the nearest integer to the real number x . For simplicity, let us assume that $\text{mod}(T, 4\tau) = 0$, so that $T/4\tau$ is an integer. The fact that there can be two correct frequency bins is due to the symmetry of the cosine in the definition of the MLE, also mentioned above. Then, x_n is a Gaussian random variable with $E[x_n] \approx 1/2 + g \cos(\pi n/2)$, and $\text{var}(x_n) \approx 1/4N$. From the definition of B_k ,

$$B_k = \frac{\tau}{T} \sum_{n=1}^{T/\tau} x_n \cos(2\pi kn\tau/T) \quad (6.32a)$$

$$= \frac{\tau}{T} \left(g \sum_{n=1}^{T/\tau} \cos(\pi n/2) \cos(2\pi kn\tau/T) + \sum_{n=1}^{T/\tau} v_n \cos(2\pi kn\tau/T) \right) \quad (6.32b)$$

where $\{v_n\}, n = 1, 2, \dots, T/\tau$ are independent zero-mean Gaussian random variables with a variance of $1/4N$. Since we assumed that $T/4\tau$ is an integer, the first term in this expression simplifies to

$$\sum_{n=1}^{T/\tau} \cos(\pi n/2) \cos(2\pi kn\tau/T) = \begin{cases} 0, & \text{if } k \in \{1, 2, \dots, T/\tau\} \setminus \{\frac{T}{4\tau}, \frac{3T}{4\tau}\} \\ \frac{T}{2\tau}, & \text{if } k \in \{\frac{T}{4\tau}, \frac{3T}{4\tau}\} \end{cases} \quad (6.33)$$

We then see that

$$B_k \sim \begin{cases} \mathcal{N}(0, \tau/8NT) & \text{if } k \neq \frac{T}{4\tau}, \frac{3T}{4\tau} \\ \mathcal{N}(g/2, \tau/8NT) & \text{otherwise.} \end{cases} \quad (6.34)$$

In order to calculate the outlier probability of Eq. (6.31), we also need to inquire whether there is a dependency between the variables B_k . Using the definition of B_k in Eq. (6.32), we can write down the normalized cross-correlation between B_{k_1} and B_{k_2} , $k_1, k_2 = 1, 2, \dots, T/\tau$,

$$\rho_{B_{k_1} B_{k_2}} = \frac{E[(B_{k_1} - E[B_{k_1}])(B_{k_2} - E[B_{k_2}])]}{\sigma_{B_{k_1}} \sigma_{B_{k_2}}} = \begin{cases} 0 & \text{if } k_1 \neq k_2, k_1 \neq \frac{T}{\tau} - k_2 \\ 1 & \text{otherwise.} \end{cases} \quad (6.35)$$

where $\sigma_{B_{k_1}} = \sqrt{\tau/8NT}$ is the standard deviation of B_{k_1} . Therefore, the total number of statistically uncorrelated frequency bins is $T/2\tau$. By using these independent frequency bins, we can redefine the probability of no outliers occurring in the MLE (in Eq. (6.31)) as

$$1 - q = \text{Prob}(B_k < B_{T/4\tau} \forall k \in \{1, 2, \dots, T/2\tau\} \setminus \{T/4\tau\}) \quad (6.36)$$

Since B_k for $k \neq T/4\tau$ are identically distributed, we can write this expression as

$$1 - q = \int dx \text{Prob}(B_1 < x)^{T/2\tau-1} \text{Prob}(B_{T/4\tau} = x) \quad (6.37a)$$

$$= \frac{1}{\sqrt{2^{\frac{T}{\tau}-1}} \pi \sigma^2} \int_0^\infty dx \left[\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) \right)^{\frac{T}{2\tau}-1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \left(1 - \text{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) \right)^{\frac{T}{2\tau}-1} e^{-\frac{(x+\mu)^2}{2\sigma^2}} \right] \quad (6.37b)$$

where $\mu = g/2$, and $\sigma^2 = \tau/8NT$, and $\text{erf}(x)$ is the error function defined with $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}$. Let us assume that $g^2 NT/\tau \gg 1$, i.e. $\mu/\sigma \gg 1$. In this limit, we expect the outlier probability to be small, as the SNR is large. Furthermore, the largest contribution to the integral will be around $x \approx \mu$, due to the exponential factor in the first term. The second term will peak at $x \approx -\mu$, which is not within the integration limits. Therefore, we expect this term to decrease monotonically as the integration variable x , increases, and we can ignore this term as it will be very

small compared to the first term. Therefore, we have

$$1 - q \approx \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty dx \left(1 - \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}\sigma}\right)\right)^{\frac{T}{2\tau}-1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (6.38a)$$

$$\approx \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty dx \left(1 - \frac{T}{4\tau} \operatorname{erfc}\left(\frac{x}{\sqrt{2}\sigma}\right)\right) e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (6.38b)$$

$$\approx 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \frac{T}{4\tau} \int_0^\infty dx \operatorname{erfc}\left(\frac{x}{\sqrt{2}\sigma}\right) e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (6.38c)$$

where $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ is the complementary error function. The approximation in Eq. (6.38b) holds when $\operatorname{erfc}(x/\sqrt{2}\sigma) T/\tau \ll 1$ for x near μ . Since we assumed that $\mu/\sigma \gg 1$, we can approximate the error function as $\operatorname{erfc}(x/\sqrt{2}\sigma) \approx e^{-x^2/2\sigma^2} \sqrt{2}\sigma/\sqrt{\pi}x(1 + O(x^{-2}))$. Then, Eq. (6.38b) holds when

$$\operatorname{erfc}\left(\frac{\mu}{\sqrt{2}\sigma}\right) \frac{T}{\tau} \approx e^{-g^2 NT/\tau} \sqrt{\frac{T}{g^2 \pi N \tau}} \ll 1, \quad (6.39)$$

or, when $g^2 NT/\tau \gg \log(T/\tau)$. Since $T/\tau \gg 1$, $T/\tau \gg \log(T/\tau)$, therefore this approximation holds when $g^2 \approx O(1/N)$ or larger. Furthermore, in Eq. (6.38c), we integrated over the constant term by extending the lower integration limit to $-\infty$, which is a good approximation when $\mu/\sigma \gg 1$. We can finally use the approximated complementary error function to compute $1 - q$:

$$1 - q \approx 1 - \frac{T}{4\pi\tau} e^{-\frac{\mu^2}{4\sigma^2}} \int_\epsilon^\infty dx \frac{1}{x} e^{-\frac{(x-\mu/2)^2}{\sigma^2}} \quad (6.40)$$

where ϵ is a small number close to zero to make sure that the integral converges. We employ Laplace's method to compute this expression for $\mu/\sigma \gg 1$. The integrand will peak near $x \approx \mu/2$, therefore we write

$$1 - q \approx 1 - \frac{T}{4\pi\tau} e^{-\frac{\mu^2}{4\sigma^2}} \frac{2}{\mu} \int_0^\infty dx e^{-\frac{(x-\mu/2)^2}{\sigma^2}} \approx 1 - \frac{T}{2\sqrt{\pi}\tau} \frac{\sigma}{\mu} e^{-\frac{\mu^2}{4\sigma^2}} \quad (6.41)$$

Plugging in μ and σ , we obtain for the probability of having no outliers

$$1 - q = 1 - \sqrt{\frac{T}{8\pi N \tau g^2}} e^{-g^2 NT/2\tau} \quad (6.42)$$

Therefore, given this probability, and the CFI of the weak-only protocol in Eq. (6.17), the variance of the MLE given in Eq. (6.30) can be computed as

$$(\Delta\omega_{\text{MLE}})^2 \approx \left(1 - \sqrt{\frac{T}{8\pi N \tau g^2}} e^{-g^2 NT/2\tau}\right) \frac{3\tau}{8g^2 NT^3} + \sqrt{\frac{T}{8\pi N \tau g^2}} e^{-g^2 NT/2\tau} \frac{\pi^2}{48\tau^2} \quad (6.43)$$

We want to find the region of the parameter space where the information obtained from the MLE is very close to the CFI. We can express this with the parameter ϵ , and write $I_{\text{MLE}} = (1 - \epsilon) I_C^w$, where $\epsilon \ll 1$, and $I_{\text{MLE}} = 1/\text{var}(\text{MLE})$. From Eq. (6.43), $1 - \epsilon$ is given by

$$1 - \epsilon = \left[1 - q + q \frac{\pi^2 g^2 N}{18} \left(\frac{T}{\tau} \right)^3 \right]^{-1} \approx 1 - q \frac{\pi^2 g^2 N}{18} \left(\frac{T}{\tau} \right)^3 \quad (6.44a)$$

$$= 1 - \frac{\pi^{3/2}}{36} \sqrt{\frac{g^2 N T}{2\tau}} \left(\frac{T}{\tau} \right)^3 e^{-g^2 N T / 2\tau} \quad (6.44b)$$

where $1 - q$ is the probability of having no outliers in Eq. (6.42). The approximation holds when $g^2 N (T/\tau)^3 \gg 1$. Since we already assumed that $g^2 N T / \tau \gg 1$ while deriving the probability of having no outliers, and since $T/\tau \gg 1$, this is a valid approximation. Rearranging this equation, we obtain

$$e^x \approx \frac{\sqrt{x} \pi^{3/2}}{\epsilon} \frac{1}{36} \left(\frac{T}{\tau} \right)^3 \quad (6.45)$$

where we defined $x = g^2 N T / 2\tau$. This is a transcendental equation, therefore we will find an approximate solution. Since $x \gg 1$, $x \gg \ln x$, and we have

$$x \approx \ln \left(\frac{\pi^{3/2}}{36\epsilon} \left(\frac{T}{\tau} \right)^3 \right) + \frac{1}{2} \ln(x) \approx \ln \left(\frac{\pi^{3/2}}{36\epsilon} \right) + 3 \ln \left(\frac{T}{\tau} \right) \quad (6.46)$$

To find the first order correction, let us write $x = x^* + \Delta x$, where x^* is given in Eq. (6.46), and assume that $\Delta x \ll x^*$. Plugging this into Eq. (6.45), and taking the logarithm of both sides, we find

$$\Delta x = \frac{1}{2} \ln(x^* + \Delta x) \approx \frac{1}{2} \ln(x^*) \quad \text{if } x^* \gg \Delta x \quad (6.47)$$

Therefore, the second order approximation to Eq. (6.45) is given by (we also plug in the definition of x):

$$\frac{g^2 N T}{2\tau} \approx \ln \left(\frac{\pi^{3/2}}{36\epsilon} \right) + 3 \ln \left(\frac{T}{\tau} \right) + \frac{1}{2} \ln \left[\ln \left(\frac{\pi^{3/2}}{36\epsilon} \right) + 3 \ln \left(\frac{T}{\tau} \right) \right] \quad (6.48)$$

Given the value for ϵ , and some of the system parameters, this solution defines the point at which the ratio between the information obtained from the MLE and the maximum obtainable information is $1 - \epsilon \approx 1$, as a function of a free system parameter. For example, we can fix the total interrogation time T , measurement period τ , and the number of qubits N , and find the minimum measurement strength

g necessary to surpass this threshold (and obtain a larger information from the MLE). Or, we can fix the parameters g , T , and τ to find the necessary number of qubits to reach this threshold. In order to operate in the favorable region where the CFI is maximized with respect to the measurement strength, g , we need to have $g^2 T/\tau \approx \sqrt{3/2} \sim O(1)$ for the weak-only protocol (see Fig. 6.2). Then, from Eq. (6.48), we have that the number of qubits to reach this threshold scales as $N \sim O(3 \ln(T/\tau))$.

Up to this point, we have only performed weak measurements on the qubits in order to estimate the frequency ω . Now, let us perform a strong projective measurement at the end of the interrogation, in addition to the weak measurements (i.e. apply the weak-with-strong protocol). For this case, the total CFI will be the sum of the CFI due to the weak and the strong measurement, which were computed to be $8Ng^2T^3/3\tau$, and $4NT^2$, respectively. Note that the total achievable CFI for this case is larger than the QFI, $4NT^2$, as we have neglected the back action due to the weak measurements.

Furthermore, we can still define the variance of the MLE with Eq. (6.30). Then, we need to find again the probability of having no outliers, $1 - q$. For this purpose, let us redefine the MLE:

$$\omega_{\text{MLE}} := \underset{\omega}{\operatorname{argmin}} \sum_{n=1}^{T/\tau-1} \left(x_n - \frac{1}{2} (1 + \sin(2g) \cos(2\omega n\tau)) \right)^2 + \left(x_{T/\tau} - \frac{1}{2} (1 + \cos(2\omega T)) \right)^2 \quad (6.49a)$$

$$\approx \underset{\omega}{\operatorname{argmax}} \sum_{n=1}^{T/\tau-1} (2g x_n \cos(2\omega n\tau)) + x_{T/\tau} \cos(2\omega T) - \cos(\omega T)^4 \quad (6.49b)$$

where we ignored the term proportional to $g^2 T/\tau$ as $g \ll 1$, and the remaining terms are $O(gT/\tau)$ and $O(1)$, respectively. We can again define

$$B'_k = \frac{\tau}{T} \left(\sum_{n=1}^{T/\tau-1} x_n \cos\left(2\pi k n \frac{\tau}{T}\right) + \frac{1}{2g} x_{T/\tau} \cos\left(2\pi k \frac{T}{\tau}\right) \right) \quad (6.50)$$

where we have defined B_k as the real part of A_k in Eq. (6.29), and $k = 1, 2, \dots, T/\tau$. Then, in the limit that $T/\tau \gg 1$, the optimization of the MLE reduces to the following: $\omega_{\text{MLE}} \approx \operatorname{argmax}_k 2gB'_k - \cos(\pi k T/\tau)^4 \tau/T$. As before, let us assume that the true value of the frequency ω is $\omega^* = \pi/4\tau$, approximately the mean value of possible (detectable) frequencies. Furthermore, let us assume that $\operatorname{mod}(T, 4\tau) = 0$,

such that the true value of the frequency corresponds to an integer value of k . Then, the MLE problem is modified as

$$\omega_{\text{MLE}} = \underset{k}{\operatorname{argmax}} 2gB'_k - \tau/T = \max_k \sum_{n=1}^{T/\tau-1} x_n \cos\left(2\pi kn \frac{\tau}{T}\right) \quad (6.51)$$

as $\cos(\pi k T/\tau) = \cos(2\pi k T/\tau) = 1$ for $k \in \mathbb{N}$. Then, we realize that the MLE is almost exactly the same as the case where we only had weak measurements, with the difference being that the summation in that case also included the T/τ^{th} sample, measured with strength g instead of 1. However, in the limit that $T/\tau \gg 1$, this difference is negligible. Therefore, we can employ the same expression for the probability of having no outliers as done for the case where we had weak measurements only (Eq. (6.42)).

Thus, the variance of the MLE can be written as

$$\operatorname{var}(\text{MLE}) \approx \frac{1 - \sqrt{\frac{T}{8\pi N\tau g^2}} e^{-g^2 NT/2\tau}}{\frac{8Ng^2 T^3}{3\tau} + 4NT^2} + \sqrt{\frac{T}{8\pi N\tau g^2}} e^{-g^2 NT/2\tau} \frac{\pi^2}{48\tau^2} \quad (6.52)$$

We can similarly define a parameter ϵ with $I_{\text{MLE}} = (1 - \epsilon) I_C$, $\epsilon \ll 1$, $I_C = 8g^2 NT^3/3\tau + 4NT^2$, and $I_{\text{MLE}} = 1/\operatorname{var}(\text{MLE})$. Then, ϵ is

$$\epsilon = 1 - \left[1 - q + q \frac{\pi^2 N}{12} \left(\frac{T}{\tau}\right)^2 \left(1 + \frac{2g^2 T}{3\tau}\right) \right]^{-1} \quad (6.53a)$$

$$\approx q \frac{\pi^2 N}{12} \left(\frac{T}{\tau}\right)^2 \left(1 + \frac{2g^2 T}{3\tau}\right) \quad (6.53b)$$

$$= \frac{\pi^{3/2}}{48} N \left(\frac{T}{\tau}\right)^3 \sqrt{\frac{2\tau}{g^2 NT}} \left(1 + \frac{2g^2 T}{3\tau}\right) e^{-g^2 NT/2\tau} \quad (6.53c)$$

where $1 - q$ is the probability of having no outliers in Eq. (6.42). Defining $x = g^2 NT/2\tau$, we obtain the following transcendental equation:

$$e^x \approx \frac{\pi^{3/2}}{48\epsilon} \frac{1}{\sqrt{x}} \left(\frac{T}{\tau}\right)^3 \left(N + \frac{4x}{3}\right) \quad (6.54)$$

Taking the logarithm on both sides, we obtain on the first order

$$x^* \approx \ln \left(\frac{\pi^{3/2}}{48\epsilon} N \left(\frac{T}{\tau}\right)^3 \right) + \frac{4x^*}{3N} \quad (6.55a)$$

$$x^* \approx \left[1 - \frac{4}{3N} \right]^{-1} \ln \left(\frac{\pi^{3/2}}{48\epsilon} N \left(\frac{T}{\tau}\right)^3 \right) \quad (6.55b)$$

Here, we assume that $4x^*/3N < 1$, i.e. $g^2T/\tau < 3/2$, in order to write $\ln(N + 4x/3) \approx \ln(N) + 4x/3N$. In order to find the second order approximation to the solution, let us write $x = x^* + \Delta x$. We obtain:

$$x^* + \Delta x \approx \ln\left(\frac{\pi^{3/2}}{48\epsilon}N\left(\frac{T}{\tau}\right)^3\right) - \frac{1}{2}\ln(x^* + \Delta x) + \frac{4}{3N}(x^* + \Delta x) \quad (6.56)$$

Plugging in the definition of x^* in Eq. (6.55b), we finally write that $\Delta x \approx -[1 - 4/3N]^{-1}\ln(x^*)/2$. Combining the first and second order terms, we find that the second order approximation to the solution to Eq. (6.54) is

$$\frac{g^2NT}{2\tau} \approx \left[1 - \frac{4}{3N}\right]^{-1} \left[\ln\left(\frac{\pi^{3/2}}{48\epsilon}N\left(\frac{T}{\tau}\right)^3\right) - \frac{1}{2}\ln\left(\ln\left(\frac{\pi^{3/2}}{48\epsilon}N\left(\frac{T}{\tau}\right)^3\right)\right) + \frac{1}{2}\ln\left(1 - \frac{4}{3N}\right) \right] \quad (6.57a)$$

$$\approx \ln\left(\frac{\pi^{3/2}}{48\epsilon}N\left(\frac{T}{\tau}\right)^3\right) - \frac{1}{2}\ln\left(\ln\left(\frac{\pi^{3/2}}{48\epsilon}N\left(\frac{T}{\tau}\right)^3\right)\right) \quad (6.57b)$$

where in the second line, $1 - 4/3N \approx 1$ as $N \gg 1$. Different from the previous case, in order to maximize the information gained from the measurement with respect to the measurement strength, g , we need to operate with the smallest possible g . Assuming that we fix g^2T/τ in Eq. (6.57), we see that the necessary number of qubits to surpass the threshold and get close to the CFI scales again as $N \sim O(3 \ln(T/\tau))$.

Numerical Simulation of the Performance

The threshold effect is illustrated in Fig. 6.3 and Fig. 6.4a. In Fig. 6.3, we plot the sensitivity obtained by the weak measurement protocols, quantified by the inverse of the BMSE $(\Delta\tilde{\omega})^2$, scaled by the QFI I_Q . We use the MLE to compute the BMSE. $(\Delta\tilde{\omega})^{-2}$ is compared to the CFI as a benchmark, plotted with gray lines. As the SNR is increased, e.g. by increasing g , the inverse of the BMSE starts to rapidly increase and saturates the CRB. Then, we can find where the sensitivity reaches the CFI in this regime when $g^2NT/\tau \gg 1$, for $N > 20$ using Eqs. (6.48) and (6.57). In Fig. 6.4a, we plot the BMSE obtained with the MLE as a function of T for varying numbers of qubits N . We again observe that the sensitivity approaches the CFI I_C^w , I_C^{ws} in the weak back action regime $g^2T/\tau < 1$ for $N > 20$.

In order to approach I_Q and achieve HS with time, the CRB needs to be saturated in the regime where $\tilde{I}_C^{ws} \approx I_Q$. Therefore, reaching this limit requires (i) weak measurement back action, $\eta < 1$, and (ii) large enough SNR, $N\eta > 1$. As shown in Fig. 6.4a, these two conditions are incompatible for small N : the saturability

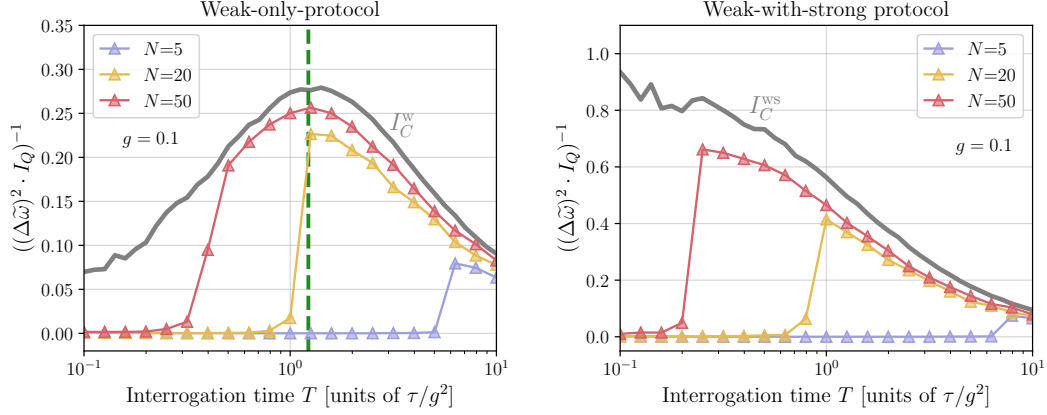


Figure 6.3: Performance of the weak-only protocol (left) and weak-with-strong protocol (right) with respect to the interrogation time T . We plot the inverse of the MSE scaled by the QFI I_Q , which quantifies the amount of information gained from the estimation. We fix the measurement strength as $g = 0.1$, and sample the unknown frequency from the uniform prior distribution in $[0, \delta\omega]$, where $\delta\omega = 5\pi$, such that the measurement period is chosen as $\tau = \pi/2\delta\omega = 0.1$ s. Finally, we plot the CFI of both protocols, I_C^w and I_C^{ws} , normalized by the QFI, as the benchmark. The CFIs are about $\omega = 5$ rad/s. The information gained from the estimation does not saturate the CFI for small SNR (i.e. small T) due to the threshold effect. As the SNR is proportional to the number of qubits N , we can saturate to the CFI faster (for smaller T) with a larger number of qubits. In order to obtain HS, this saturation needs to occur in the weak back action regime: the transition between the two regimes is marked by a green dashed line in the plot on the left.

occurs only in the strong back action regime. However, there exists a minimal N after which these two conditions are satisfied and HS can be achieved. In particular, we observe that as N increases, the saturability occurs at smaller η .

Let us analytically find the minimal N for which the CRB can be saturated. We aim to compute the parameters of the weak-with-strong protocol for which the BMSE is close to the CFI, i.e. $(\Delta\tilde{\omega})^{-2} = (1 - \epsilon) \bar{I}_C$, $\epsilon \ll 1$. In the limit of $\eta \ll 1$ and a large number of probes, $N \gg 1$, we can use Eq. (6.57) to find that $(\Delta\tilde{\omega})^{-2} = (1 - \epsilon) \bar{I}_C$ is attained for

$$N\eta \approx 2 \ln \left(\frac{\pi^{3/2}}{48\epsilon} \right) + 6 \ln \left(\frac{T}{\tau} \right). \quad (6.58)$$

For any system parameters g, T, τ , this expression provides the minimal number of probes N for which we are ϵ -close to the CFI. Since we can always tune g such that η remains constant, then the minimal N , for any given ϵ , grows as $6 \ln [(2\delta\omega T)/\pi]$. Hence it grows logarithmically with the number of phase wraps. While this loga-

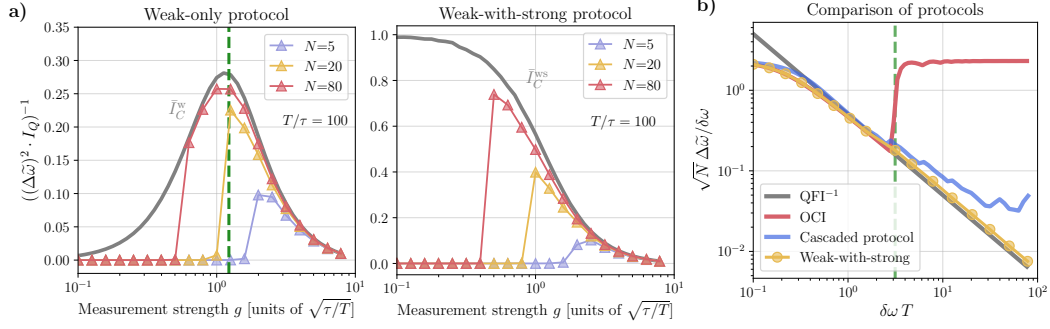


Figure 6.4: Performance of the proposed protocol with respect to the interrogation time T and the threshold effect. The frequency ω is sampled from a uniform distribution in $[0, \delta\omega]$, which determines the measurement period as $\tau = \pi/2\delta\omega$. a) We fix $\delta\omega = 5\pi$ (i.e. $\tau = 0.1$), $T = 10$, and plot the inverse of BMSE $(\Delta\tilde{\omega})^2$ scaled by the QFI I_Q for the two weak measurement protocols. $(\Delta\tilde{\omega})^2 \cdot I_Q = 1$ signifies saturating the QFI, and $\bar{I}_C/I_Q$ is plotted with gray, plain lines. We observe a threshold effect such that the attainability of the CFI $\bar{I}_C^w, \bar{I}_C^{ws}$ requires large enough SNR. The green, dashed, vertical line corresponds to $T = \sqrt{3/2}\tau/g^2$, which is a transition point between the weak and strong back action regimes. Achieving HS requires the threshold to occur before this line (which requires $N > 20$ qubits for this case). b) We fix $N = 64$, $\delta\omega = \pi$ ($\tau = 0.5$ s) and plot the square-root of the BMSE $\Delta\tilde{\omega}$ normalized by the prior width $\delta\omega$, and scaled by the square-root of the number of qubits N . The vertical green line corresponds to $\delta\omega T = \pi$: for larger interrogation times, phase slips degrade the sensitivity of standard protocols. We compare the performance of: the OCI [167], the proposed weak-with-strong protocol, and the cascaded protocol [224]. The latter two demonstrate an extended dynamic range, however the proposed protocol outperforms the cascaded protocol and achieves HS, with an overhead of 1.18 in $\Delta\tilde{\omega}$. The measurement strength g is optimized for each T for the proposed protocol.

ithmic dependence is in line with the scaling of N in previous schemes [134], we obtain a smaller overhead.

Finally, we want to explicitly show that this protocol achieves an almost optimal BMSE for long times, i.e. $\delta\omega T \gg 1$, and that it outperforms existing methods. To this end, we plot in Fig. 6.4b the square-root of the BMSE $\Delta\tilde{\omega}$, as a function of interrogation time for different sensing schemes for $N = 64$. We use the MLE in the extended bandwidth regime of $\delta\omega T \gg 1$ ³. Our weak-with-strong protocol is compared to the QFI bound, to the OCI, and to a more conventional dynamic range extension scheme: the cascaded protocol [224, 134, 241, 66]. The cascaded

³We use the optimal Bayesian estimator [131, 66] for $\delta\omega T < \pi$, since the prior information is significant in this regime.

protocol employs groups of atoms with shorter interrogation times $T/2^i$, $i > 0$, which are used to prevent phase slips.

We first observe that our weak-with-strong protocol performs very close to the QFI bound for the entire range of interrogation times, with a maximal overhead of 1.18 in $\Delta\tilde{\omega}$. We thus obtain an almost optimal HS that is immune to phase slip errors and persists in the limit of large $\delta\omega T$. This is in sharp contrast to the OCI that suffers from phase slip errors and becomes inefficient for $\delta\omega T > \pi$. We also observe that our protocol considerably outperforms the cascaded protocol. The intuitive reason behind this metrological gain is that, while the cascaded protocol allows optimization over only a finite set of partitions into blocks, the proposed weak measurement scheme offers greater tunability, as the optimization is over the continuous weak measurement strength g ⁴.

6.6 Conclusions and outlook

In this chapter, we developed and analyzed a weak measurement Ramsey protocol that achieves optimal precision over an extended dynamic range asymptotically. By optimizing over the weak measurement strength, we were able to achieve a nearly optimal sensitivity and HS in the large bandwidth limit, $\delta\omega T \gg 1$, given a large enough number of probes. We identified two separate conditions for obtaining a high sensitivity and a large bandwidth, and demonstrated that they can be satisfied simultaneously.

Some open questions and directions that can potentially improve the proposed protocol are as follows: first, we can reduce the number of ancillas by coupling only a subset of sensors to ancillas or coupling several sensors to the same ancilla. We can also improve the BMSE in the low SNR limit by adding more ancillas as memory qubits and performing a collective measurement on them [273, 10, 77]. This may allow achieving HS for smaller N . Finally, we can mitigate the effect of imperfect measurements by using error correction [153, 280, 42, 198]. Weak measurements could also be combined with entangled states [27] to achieve HS both with time, and the number of qubits, for an extended bandwidth.

⁴Note that if we plot $\Delta\tilde{\omega}$ for larger T , the optimal g will fall into the strong back action regime, causing the transition from HS to a T scaling in the obtained precision.

DISSIPATIVELY STABILIZED 0-N FOCK QUBITS FOR NOISE-BIASED QUANTUM COMPUTING

7.1 Introduction

Noise-biased physical qubits provide a promising route toward hardware-efficient fault-tolerant quantum computing [9, 257, 258, 101]¹. When one error channel is strongly suppressed relative to another, this asymmetry can be exploited to achieve improved thresholds and reduced overhead compared to unbiased architectures [212, 266]. Among noise-biased qubits, bosonic encodings have emerged as a platform in which such bias can be engineered at the hardware level [181, 178]. In particular, dissipatively stabilized cat qubits—logical states encoded in superpositions of coherent states [49] and protected by engineered dissipation—have demonstrated exponential suppression of bit-flip errors with increasing mean photon number, $|\alpha|^2$ [155, 214, 19, 217]. Related noise-biased bosonic encodings include Kerr-cat qubits [211, 98], squeezed-cat qubits [270, 228, 111], and pair-cat codes [8].

A central requirement in noise-biased architectures is that gates preserve the bias [212]. In particular, syndrome extraction in error correcting codes requires entangling gates between data and ancilla qubits, during which errors on the ancilla can propagate to the data qubit [9, 212]. If the ancilla is not itself noise-biased, these back-action errors can introduce bit flips on the data qubit and compromise the bias that the encoding was designed to provide. The two potential pathways to address this are i) using cat codes for ancillary qubits, then correcting propagating ancilla errors with an outer error correcting code, and ii) implementing hybrid protocols, where e.g. transmons are used as ancillary qubits [106]. For the former approach, although significant progress has been made for efficient implementations of bias-preserving gates between cat qubits [212, 271, 79, 78], the associated Hamiltonian engineering and coherence requirements remain nontrivial. In contrast, the latter approach benefits from less stringent experimental requirements. However, transmons can introduce additional decay and dephasing to the data qubits and thereby limit the achievable bias [64, 245]. These considerations motivate alternative noise-biased

¹This chapter is adapted from: S. Direkci et al., "Dissipatively Stabilized 0-n Fock Qubits for Noise-Biased Quantum Computing," arXiv preprint arXiv:2608.25959 (2026), DOI: 10.48550/arXiv.2608.25959.

qubit realizations whose protection mechanism is compatible with simpler couplings and modular device integration.

To this end, we propose a new type of noise-biased qubit, which we refer to as the *0-n Fock qubit*. Here, logical states are encoded in the Fock states $|0\rangle$ and $|n\rangle$ of an anharmonic oscillator, such as a single transmon operated at a suitably large E_J/E_C ratio, or a Kerr nonlinear oscillator. The key feature of this encoding is protection against single-photon loss: a photon-loss event takes $|n\rangle$ to $|n-1\rangle$, outside of the logical subspace, rather than directly inducing a logical bit flip. At least $\lceil n/2 \rceil$ such events must occur to produce a bit-flip $|1_L\rangle \leftrightarrow |0_L\rangle$, which can be suppressed with increasing number of levels in the qubit, analogous to the suppression of bit-flips with increasing $|\alpha|^2$ in cat codes [155]. By engineering dissipation that selectively removes population from unwanted intermediate Fock levels while leaving the states $\{|0\rangle, |n\rangle\}$ invariant, the system can be stabilized within the code manifold. This approach builds on established concepts in bosonic error correction and reservoir engineering, where tailored dissipation is used to confine dynamics to protected subspaces [209, 83, 270, 9, 152, 253].

Level-dependent dissipation is necessary to achieve the target qubit structure, which in turn requires nonlinearity to address each transition individually. Frequency-selective dissipation can be implemented by coupling the qubit capacitively to a linear chain of LC resonators that are themselves coupled to a bath mode. The effective spectral density of the bath viewed by the qubit can be shaped by engineering the inter-couplings of the resonators. Similar techniques have been used in the literature for various purposes, and are referred to as bath engineering [209, 186, 136]. Multi-mode on-chip filters and metamaterial resonators have recently been demonstrated as practical tools for bath engineering and leakage suppression in superconducting circuits [35, 180, 214, 252, 137]. In our setting, we observe that a small number (< 10) of filter modes are sufficient to achieve strong spectral selectivity while maintaining realistic device parameters.

To demonstrate the performance of the 0-n Fock qubit, we consider controlled-X (CX) gates with 0-n Fock qubits as control and stabilized cat qubits as target, enabling syndrome extraction for the repetition code with the Fock qubits as ancillas. These gates can be realized with only the simple, native dispersive coupling between the Fock qubit and the cat qubits (without additional Hamiltonian engineering). We find that bit-flip probabilities below 10^{-8} are achievable with realistic experimental considerations for $n = 9$ [264]. Furthermore, employing a repetition code with the

Fock ancilla, we find logical error rates reaching the megaquop regime ($p_L^* = 10^{-6}$) at a code distance of $d = 9$, in an experimentally realistic parameter regime. These results demonstrate that frequency-selectively stabilized Fock encodings provide a viable and compact route to realizing and scaling-up bias-preserving architectures, and suggest new directions for integrating bosonic qubits with engineered dissipation in scalable superconducting architectures.

This chapter is organized as follows. We first introduce the 0- n Fock qubit in Sec. 7.2, and show that it exhibits noise bias, with bit-flip times that scale exponentially with n . We then discuss the implementation of the 0- n qubit using the multimode bosonic filter in Sec. 7.3. Finally, in Sec. 7.4, we analyze its use as an ancilla for cat-qubit syndrome measurements for realistic experimental parameters.

7.2 Properties of the 0- n qubit

We define the 0- n Fock qubit using the lowest $N = n + 1$ Fock levels of an anharmonic oscillator. The logical states correspond to the lowest and the highest of these levels, with $|0_L\rangle = |0\rangle$, $|1_L\rangle = |n\rangle$. A schematic of the qubit can be found in Fig. 7.1a. For the stabilization of the qubit in the $\{|0\rangle, |n\rangle\}$ manifold, we assume engineered gain (decay) between levels m and $m + 1$ for $\lceil \frac{n}{2} \rceil \leq m \leq n - 1$ ($0 \leq m \leq \lfloor \frac{n}{2} \rfloor - 1$), $m \in \mathbb{N}$. Let us define the engineered dissipation rate as κ_s . If n is even, the Fock level $|n/2\rangle$ experiences both engineered gain and decay, which may hinder the stabilization. For this purpose, we assume that n is odd for the rest of the Sections.

In addition to the engineered dissipation κ_s in the designated levels mentioned above, the 0- n qubit is also subject to intrinsic single-photon events: decay ($\kappa_i^\downarrow$) and heating ($\kappa_i^\uparrow$). The corresponding Lindblad dissipators are $\kappa_i^\downarrow \mathcal{D}[\hat{a}]$ and $\kappa_i^\uparrow \mathcal{D}[\hat{a}^\dagger]$, where $\mathcal{D}[\hat{A}] := \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A}, \hat{\rho}\}$, and $\hat{a}$ is the annihilation operator associated with the Fock qubit. We consider the regime

$$\kappa_i^{\uparrow, \downarrow} \ll \kappa_s, \quad (7.1)$$

that is, the engineered dissipation rate greatly exceeds the intrinsic decay and heating rates.

To analyze the dynamics of this system, we consider an $N = n + 1$ level system with levels $m = 0, 1, \dots, n$, and dissipation between adjacent levels, we examine how the system evolves under two sets of Lindblad operators describing the decay and gain between the levels respectively, $\{(\kappa_j^d)^{1/2} |j\rangle \langle j+1|\}$ and $\{(\kappa_j^h)^{1/2} |j+1\rangle \langle j|\}$. The transition rates are then given by $\kappa_j^{d,h}$, and we assume that $\kappa_j^{d,h} \geq 0$, $\kappa_{-1}^{d,h} = \kappa_n^{d,h} = 0$.

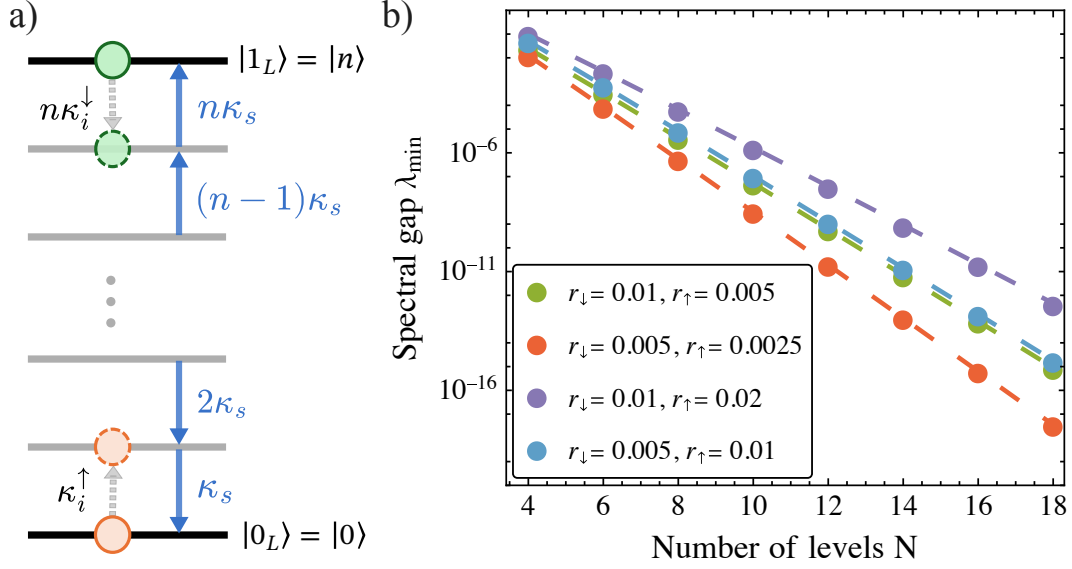


Figure 7.1: a) Schematic of the 0- n Fock qubit. We encode the logical states $\{|0_L\rangle, |1_L\rangle\}$ into the ground state $|0\rangle$ and the n -th excited state $|n\rangle$ of a Fock qubit. b) Smallest non-zero eigenvalue (in absolute magnitude) $\lambda_{\min}$ of the Lindbladian describing the evolution of the 0- n Fock qubit. We define $r_{\uparrow,\downarrow} := \kappa_i^{\uparrow,\downarrow}/\kappa_s \ll 1$ as the ratio of the rates between the intrinsic processes and the engineered dissipation. The numerical values are plotted with the markers, whereas the analytical fit given by Eq. (7.9) is plotted with the dashed lines.

The evolution of the density matrix $\hat{\rho}(t)$ can be described with [34]

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}(\hat{\rho}) = \sum_{j=0}^n |j\rangle\langle j| \left(\kappa_j^d \rho_{j+1,j+1} + \kappa_{j-1}^h \rho_{j-1,j-1} \right) - \frac{1}{2} \{ |j\rangle\langle j|, (\kappa_j^h + \kappa_{j-1}^d) \hat{\rho} \} \quad (7.2)$$

where $\rho_{j,j} := \langle j| \hat{\rho} |j\rangle$. Note that by construction, jump operators of the engineered dissipation annihilate both $|0\rangle$ and $|n\rangle$, so any logical state $c_0|0\rangle + c_1|n\rangle$ is a dark state of the engineered dissipation.

In Eq. (7.2), if the initial state is diagonal in the Fock eigenbasis, i.e. $\langle m|\hat{\rho}(0)|n\rangle = 0$ for all $m \neq n$, then the evolved state $\hat{\rho}(t)$ remains diagonal for all $t \geq 0$. Therefore, the evolution for this case reduces to a classical rate equation for the level populations, which can be described in the form of a continuous-time Markov chain (CTMC). In contrast, for an initial coherence $\hat{\rho}(0) = |m\rangle\langle n|$, $m \neq n$, the state decays with $\hat{\rho}(t) = \exp\left(-\frac{1}{2}(\kappa_m^t + \kappa_n^t)t\right)\hat{\rho}(0)$, $\kappa_n^t = \kappa_{n-1}^d + \kappa_n^h$ is the total rate of dissipation out of level n . Then, the Lindbladian yields the diagonalization $\mathcal{L} = \text{diag}[Q, \mathcal{L}_{\text{coh}}]$, where Q is a transition matrix describing how the diagonal elements evolve, and $\mathcal{L}_{\text{coh}}$ encodes the decay of the non-diagonal elements. $\mathcal{L}_{\text{coh}}$ does not admit steady-

state eigenoperators, whereas Q hosts a steady-state solution, characterized by a zero eigenvalue. We will elaborate more on the dynamics of the diagonal and the non-diagonal blocks in the following subsections.

For the 0-n Fock qubit, the transition rates of Eq. (7.2) are given by, for $0 \leq j < n$,

$$\frac{\kappa_j^d}{j+1} = \begin{cases} \kappa_s + \kappa_i^\downarrow & \text{if } j < \lfloor n/2 \rfloor \\ \kappa_i^\downarrow & \text{otherwise} \end{cases} \quad (7.3a)$$

$$\frac{\kappa_j^h}{j+1} = \begin{cases} \kappa_s + \kappa_i^\uparrow & \text{if } j \geq \lceil n/2 \rceil \\ \kappa_i^\uparrow & \text{otherwise} \end{cases} \quad (7.3b)$$

Note that the transition rates are multiplied by the bosonic enhancement factor for levels higher than the ground state.

In the following two subsections, we show that the 0-n Fock qubit exhibits a noise bias: bit-flip rates, governed by block Q of the Lindbladian, are suppressed exponentially with n ; while phase-flip rates, governed by block $\mathcal{L}_{\text{coh}}$, are enhanced linearly with n .

Evolution of diagonal elements and noise bias

Let us elaborate on the evolution of the diagonal elements of the density matrix. In the CTMC formalism, $\pi(t)$ denotes the population row vector over the $N = n + 1$ levels, where $\pi_j(t) = \rho_{j,j}(t)$ encodes the population at the j^{th} level. Its evolution is governed by

$$\pi(t) = \pi(0)e^{Qt}, \quad (7.4)$$

Since only the neighboring Fock levels are connected by single-photon processes, Q has a birth-death (tridiagonal) structure. The steady state of the full chain is found from $\pi_{\text{ss}}Q = \mathbf{0}$, where $\mathbf{0}$ is the null row vector. It can be written as

$$\begin{aligned} \frac{(\pi_{\text{ss}})_j}{(\pi_{\text{ss}})_{j+1}} &= \frac{\kappa_i^\downarrow}{\kappa_s + \kappa_i^\uparrow} \approx r_\downarrow, & \frac{n+1}{2} \leq j < n \\ \frac{(\pi_{\text{ss}})_{j+1}}{(\pi_{\text{ss}})_j} &= \frac{\kappa_i^\uparrow}{\kappa_s + \kappa_i^\downarrow} \approx r_\uparrow, & 0 \leq j < \frac{n-1}{2}, \\ \frac{(\pi_{\text{ss}})_{(n+1)/2}}{(\pi_{\text{ss}})_{(n-1)/2}} &= \frac{\kappa_i^\uparrow}{\kappa_i^\downarrow}. \end{aligned} \quad (7.5)$$

where we define $r_{\uparrow,\downarrow} := \kappa_i^{\uparrow,\downarrow}/\kappa_s \ll 1$. This state satisfies the detailed balance equations (i.e. the rate of the ingoing excitations is equal to the rate of the outgoing excitations for each level) [130], and the relative magnitudes of $\kappa_i^\downarrow$ and $\kappa_i^\uparrow$ determine whether most of the population is concentrated in the upper or lower levels of the qubit. Therefore, due to the intrinsic losses, the qubit cannot be stabilized perfectly to the code manifold. If it is initialized in the highest excited level, $|n\rangle\langle n|$, it will eventually decay to all of the lower levels, giving rise to a finite bit-flip probability. However, we can inquire how the timescale for the stabilization to the steady state scales as a function of n , in order to examine whether the increased number of levels provides a protection from bit-flips.

To analyze this, note that for odd n , the total number of levels N is even, and the Hilbert space separates into two halves connected only by weak intrinsic decay and heating processes. On short timescales, $t \ll 1/\kappa_i^{\uparrow,\downarrow}$, the system rapidly relaxes within each half-space to a metastable distribution governed by the ratio of the intrinsic decay or heating rate to the engineered dissipation rate. Initializing the state in $\hat{\rho} = |n\rangle\langle n|$, the corresponding metastable state (for upper levels) will be given by

$$\pi_{\text{ms}}^\uparrow \approx \frac{1}{\sum_{j=0}^{\frac{n-1}{2}} (r_\downarrow)^j} \left[0, \dots, 0, (r_\downarrow)^{\frac{n-1}{2}}, \dots, r_\downarrow, 1 \right], \quad (7.6)$$

obtained from detailed balance between neighboring levels. We can also define the metastable state associated with the lower levels as $\pi_{\text{ms}}^\downarrow$, where the ratio of populations between two levels is similarly given by $r_\uparrow$, in the lower half of the qubit.

We define a bit-flip as the transfer of population from the upper half ($\frac{n+1}{2} \leq m \leq n$) to the lower half ($0 \leq m \leq \frac{n-1}{2}$) of the qubit. The leakage rate out of this metastable state is, due to intrinsic decay, given by $\frac{n+1}{2}\kappa_i^\downarrow$, where $\frac{n+1}{2}$ is the bosonic enhancement factor. We can similarly write the metastable state corresponding to the lower half of the qubit, where the leakage rate to the upper half would be given by $\frac{n+1}{2}\kappa_i^\uparrow$. When population leaks to the opposite half-space, the system quickly relaxes to the corresponding metastable state in time $t \sim O(1/\kappa_s)$. Then, the population vector at some time t can be approximated as $\pi(t) \approx (1-x)\pi_{\text{ms}}^\uparrow + x\pi_{\text{ms}}^\downarrow$. The bit-flip probability at this time is therefore $p_{\text{bf}} = \sum_{j=0}^{(n-1)/2} \pi_j(t) \approx x$. The evolution of $\pi(t)$ is given by $\frac{d}{dt}\pi(t) = \pi(t)Q$. Using this expression and the explicit structure of Q ,

the rate-of-change of the bit-flip probability is found as

$$\begin{aligned} \frac{dp_{\text{bf}}}{dt} &\approx (1 - p_{\text{bf}}) \sum_{j=0}^{\frac{n-1}{2}} (\pi_{\text{ms}}^{\uparrow} \mathcal{Q})_j + p_{\text{bf}} \sum_{j=0}^{\frac{n-1}{2}} (\pi_{\text{ms}}^{\downarrow} \mathcal{Q})_j \\ &= \frac{(n+1)\kappa_s}{2} \left[(1 - p_{\text{bf}}) (r_{\downarrow})^{\frac{n+1}{2}} - p_{\text{bf}} (r_{\uparrow})^{\frac{n+1}{2}} \right]. \end{aligned} \quad (7.7)$$

Together with the initial condition $p_{\text{bf}}(0) = 0$, the bit-flip probability is therefore

$$p_{\text{bf}}(t) \approx \frac{1 - \exp\left(-\kappa_s t \frac{n+1}{2} \left[(r_{\downarrow})^{\frac{n+1}{2}} + (r_{\uparrow})^{\frac{n+1}{2}} \right]\right)}{1 + (r_{\uparrow}/r_{\downarrow})^{\frac{n+1}{2}}}. \quad (7.8)$$

In the limit of a small exponent, $p_{\text{bf}}(t) \approx \kappa_s t \frac{n+1}{2} (r_{\downarrow})^{\frac{n+1}{2}}$. Then, the effective bit-flip rate

$$\lambda_{\text{bf}} = \kappa_s \frac{n+1}{2} \left[(r_{\downarrow})^{\frac{n+1}{2}} + (r_{\uparrow})^{\frac{n+1}{2}} \right] \quad (7.9)$$

decreases exponentially with $\frac{n+1}{2}$, up to a linear factor in n . This establishes a hardware-level noise bias: logical bit flips require $O(\lceil n/2 \rceil)$ sequential intrinsic decay events and are exponentially suppressed with increasing n . The 0- n Fock qubit therefore exhibits a noise bias arising from the separation between fast engineered stabilization and weak intrinsic excitation or decay.

The effective bit-flip rate can also be found from the smallest (in magnitude) non-zero eigenvalue of the generator matrix $\mathcal{Q}$, i.e. the spectral gap. Physically, the spectral gap corresponds to the slowest relaxation mode of the CTMC. For the case outlined here, this is the transfer of population between the upper and lower halves of the levels, which corresponds to a bit-flip. All faster modes, with rates $\sim \kappa_s$, describe relaxation within each half-space to the respective metastable distributions.

We plot the spectral gap for different values of $r_{\downarrow, \uparrow}$, as a function of the number of levels N of the 0- n qubit, in Fig. 7.1b. The numerical samples are plotted with the data points, whereas the analytical rate λ_{bf} (Eq. (7.9)) is plotted with the dashed lines in the corresponding colors. From the Figure, we observe that the analytical prediction is a very good fit to the numerics.

Evolution of non-diagonal elements and phase coherence

Following a similar analysis to the previous subsection, we derive the phase-flip rate of the 0- n Fock qubit, given engineered dissipation κ_s , intrinsic heating $\kappa_i^{\uparrow}$, and intrinsic decay $\kappa_i^{\downarrow}$. For this purpose, we define $|\pm_{0n}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |n\rangle)$, and initialize

the qubit in $|+_{0n}\rangle$. Given the system state $\hat{\rho}(t)$ at some time, we assume an encoding that maps population in the upper and lower half-spaces to $|n\rangle$ and $|0\rangle$, respectively. Given the encoding, the phase-flip probability is given by $p_{\text{pf}}(t) := 1 - \langle +_{0n} | \hat{\rho} | +_{0n} \rangle$.

From Eq. (7.2), we observe that off-diagonal terms $|0\rangle\langle n|$ and $|n\rangle\langle 0|$ will decay with rate $\frac{1}{2}(n\kappa_i^\downarrow + \kappa_i^\uparrow)$. Thus, the phase flip probability is found as

$$p_{\text{pf}}(t) = \frac{1}{2} \left(1 - \exp \left(-\frac{t}{2} [n\kappa_i^\downarrow + \kappa_i^\uparrow] \right) \right). \quad (7.10)$$

$p_{\text{pf}}(t) \approx \frac{n\kappa_i^\downarrow t}{4}$ for $\frac{t}{2}[n\kappa_i^\downarrow + \kappa_i^\uparrow] \ll 1$, $\kappa_i^\uparrow \sim \kappa_i^\downarrow$, and $n \gg 1$. This probability grows linearly with n . Furthermore, since phase coherence is only affected by heating to levels above $|0\rangle$, or decay to levels below $|n\rangle$, it is not affected by engineered dissipation. This is also demonstrated by the independence of this expression from the engineered dissipation rate, κ_s .

In contrast with the exponential suppression of bit-flips, phase-flips are enhanced linearly with the number of levels in the 0- n Fock qubit. This scaling is directly analogous to stabilized cat qubits, which exhibit exponentially suppressed bit flips and linearly enhanced phase flips as the size of the bosonic encoding increases [181, 211, 155, 98]. However, different from cat qubits, where the suppression rate of bit-flips is independent of the engineered dissipation rate κ_2 or the intrinsic dephasing rate κ_ϕ , the suppression rate in the Fock qubit improves as the ratio between the engineered dissipation rate and the intrinsic error rate increases.

7.3 Proposed Experimental Implementation

In the previous Section, we have described how the 0- n Fock qubit is noise-biased, assuming that we can engineer the required dissipation structure. In this section, we describe how this dissipation can be realized. The 0- n Fock qubit requires that, in addition to the engineered decay on transitions in the lower half of the level ladder, and engineered gain on transitions in the upper half, each type of engineered dissipation must act only on its designated transitions: unwanted gain on lower levels or decay on upper levels would compromise the stabilization.

To enable this dissipation structure, an anharmonic oscillator such as a Kerr oscillator or a transmon is necessary. In these nonlinear oscillators, each transition $|i\rangle \leftrightarrow |i+1\rangle$ occurs at a distinct frequency, with neighboring transitions separated by the anharmonicity η . This converts the problem of level-selective dissipation into one of frequency-selective dissipation: if we can engineer a bath whose spectral density

has a sharp passband, we can address each transition independently based on its frequency.

To realize such a bath, we couple the qubit to a linear chain of LC resonators terminated by a lossy mode (see Fig. 7.2a). This chain acts as a bandpass filter, where excitations at frequencies in the passband propagate through the chain and are dissipated, while those outside the passband are reflected. As described in detail below, engineered decay and gain can be obtained from two different couplings to the first filter mode: a beam splitter interaction, and a two-mode squeezing interaction, respectively. By choosing the drive frequencies of these interactions, the transitions in the lower half of the qubit can be placed within the decay passband while the upper-half transitions lie outside it (and vice versa for gain). In this way, the single multimode filter simultaneously provides both the frequency-selective decay and the frequency-selective gain required to stabilize the $\{|0\rangle, |n\rangle\}$ manifold.

Multimode filter implementation

Let us first consider engineering decay or gain on a single harmonic oscillator mode $\hat{d}$, with frequency ω_d , before extending to the case of the multi-level anharmonic oscillator. This mode is connected to a linear chain of identical harmonic oscillators $\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M$, with frequency ω_f . The qubit is connected to the first filter mode with (i) a beam splitter interaction, and (ii) a two-mode squeezing interaction. The Hamiltonian in the interaction picture with respect to the bare Hamiltonians of the modes, within the rotating-wave approximation (RWA) can be written as

$$\hat{H} = g_{\text{bs}} \hat{d} \hat{f}_1^\dagger e^{i\Delta_{\text{bs}} t} + g_{\text{tm}} \hat{d}^\dagger \hat{f}_1^\dagger e^{i\Delta_{\text{tm}} t} + \sum_{i=1}^{M-1} J_{i,i+1} \hat{f}_i \hat{f}_{i+1}^\dagger + \text{h.c.} \quad (7.11)$$

where the first term denotes the beam splitter interaction, second term denotes the two-mode squeezing interaction, and the third term denotes the coupling between adjacent filter modes. Furthermore, we assume dissipation on the final filter mode, with a dissipator $\kappa_f \mathcal{D}[\hat{f}_M]$. The schematic of this system is plotted in Fig. 7.2a.

Such filters have been used extensively in the literature [252, 44, 213, 137] to shape the effective bath seen by the computational qubits, in order to prevent leakage and increase coherence. We describe the resulting dynamics in the following subsection.

Effective dissipation

In the weak-coupling limit $g_{\text{bs}}, g_{\text{tm}} \ll J_i, \kappa_f \forall i$, the eigenmodes of the coupled system can be obtained perturbatively. For the eigenmode associated with the *bare*

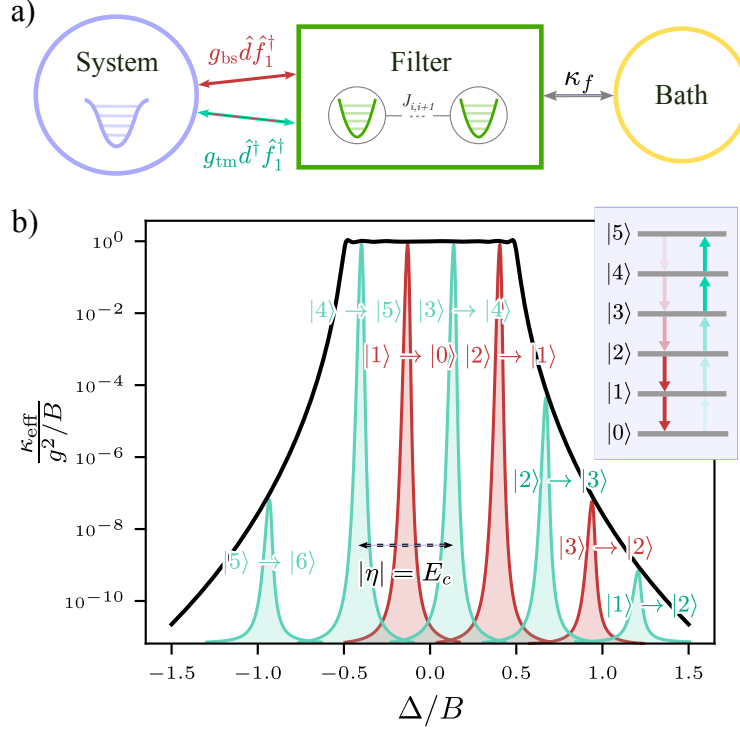


Figure 7.2: a) Schematic of the implementation for the 0-n Fock qubit. The qubit (system, drawn in purple) is connected to a multimode filter with coupling rates g_{bs} and g_{tm} for the beam splitter and the two-mode squeezing interaction, respectively. The excitations travel along the filter and are dissipated to a bath, with a rate κ_f . b) Effective rates κ_{eff} of engineered processes between adjacent levels of a 0-n Fock qubit, with $n = 5$. Decay and gain are engineered via the beam splitter and two-mode squeezing coupling with the filter. The rates are plotted as a function of the detuning Δ , normalized with the filter bandwidth B . We observe that $\frac{\kappa_{eff}}{g^2/B} \approx 2\pi$, where g is the coupling rate for the corresponding interaction. For an implementation of the Fock qubit with the Kerr Hamiltonian, the difference in detuning between adjacent transitions corresponds to the anharmonicity η .

system mode, i.e. $\hat{d}$, it can be shown that a two-mode squeezing interaction induces gain while a beam splitter interaction induces decay in the form of $\hat{d}(t) = \hat{d}(0)e^{-\kappa_{eff}t}$, where we refer to κ_{eff} as the effective gain or decay rate. Then, by coupling the system mode to a chain of oscillators, we can engineer dissipation that can either inject or extract excitations from the mode. Moreover, the dissipation profile can be shaped by optimizing over the parameters $\{J_i\}$ and κ_f .

Since the dissipation is observed in the eigenmodes of the joint system, i.e. the *dressed* modes that include the system mode and the filter modes, there are some subtleties with how the joint system evolves. We can use adiabatic ramping of the

interaction rate $g_{\text{bs, tm}}$ to achieve a smooth transition from the bare eigenmodes to the dressed eigenmodes. If the ramping is slow enough, it can be shown that the effective dynamics on a given mode $\hat{v}$ look like:

$$\hat{v}(t) = \sum_i c_i(0) \exp\left(\int_0^t ds \lambda_i(s)\right) \hat{u}_i(t) \quad (7.12)$$

where $\{c_i\}$ are the coefficients determining the decomposition over the eigenmodes $\{\hat{u}_i\}$ at time $t = 0$, $\{\lambda_i\}$ are the corresponding eigenvalues, and $\hat{v}(t)$ is the Heisenberg operator of $\hat{v}$. Ramping the interaction rate from zero at $t = 0$ up to $g_{\text{bs, tm}}$ and then down to zero at $t = T$, where T is the total simulation time, results in $c_i(0) = 1$ for $i = i^*$, $c_i(0) = 0$ otherwise, where $\hat{u}_{i^*} = \hat{u}_{i^*}(T) = \hat{d}$. Then, the effective dynamics look like $\hat{d}(t) = \exp(\int_0^t ds \lambda_{i^*}(s)) \hat{d}(0)$. This results in an effective gain or decay on the system mode $\hat{d}$, depending on the sign of the corresponding eigenvalue integrated in time.

Numerically, we observed that filter optimization results in a bandpass filter with a bandwidth B , where the dissipation rate is $\kappa_{\text{eff}} \approx 0$ for $|\Delta_{\text{bs, tm}}| > \frac{B}{2}$, and $\kappa_{\text{eff}} \approx 2\pi g_{\text{bs, tm}}^2/B$ otherwise. More details about the filter optimization can be found in Appendix E.1.

For the multi-level anharmonic oscillator case, we aim to simultaneously engineer gain or decay on multiple transitions with differing frequencies. For this purpose, we consider the analogous model of multiple Gaussian modes with frequencies $\omega_{d,1}, \omega_{d,2}, \dots$. We show in Appendix E.2 that if these modes are far detuned enough, i.e. $|\omega_{d,i} - \omega_{d,j}| \gg g_{\text{bs, tm}}^2/B$, the dissipation is independent for each mode. In other words, gain or decay is achieved for each mode independently using the same filter, with Lindblad dissipators $\kappa_{\text{eff},i} \mathcal{D}[\hat{d}_i]$ ($\kappa_{\text{eff},i} \mathcal{D}[\hat{d}_i^\dagger]$) for the decaying (gain-experiencing) modes, where $\kappa_{\text{eff},i}$ is the effective dissipation rate stated above. This independence is what allows a single filter to realize the level-resolved dissipation structure assumed in Sec. 7.2.

Implementation with the Kerr Hamiltonian

Building on the results from the previous subsection, showing how we can simultaneously engineer loss or gain on multiple modes with distinct frequencies, we now show how to implement the required dissipation for the 0-n Fock qubit. To implement frequency-selective dissipation, we consider a realization based on the Kerr nonlinearity. The qubit Hamiltonian is [144]

$$\hat{H} = \omega_d \hat{a}^\dagger \hat{a} - K (\hat{a}^\dagger)^2 \hat{a}^2 \quad (7.13)$$

where K is the self-Kerr nonlinearity. The energies of Fock levels are given by $E_n = \omega_d n - Kn(n-1)$ for $n \geq 0$. Defining the transition frequency between adjacent levels as $f_{i(i+1)} := E_{i+1} - E_i$, we find $f_{i(i+1)} = \omega_d - 2Ki$ for $i \geq 0$. The fundamental transition frequency is $f_{01} = \omega_d$, and the anharmonicity is $\eta := f_{12} - f_{01} = -2K$. To connect with transmon parameters in the regime $E_J/E_C \gg 1$, we identify $\omega_d = \sqrt{8E_J E_C} - E_C$ and $K = E_C/2$. With this parametrization, adjacent transitions are separated in frequency by $|f_{i(i+1)} - f_{(i+1)(i+2)}| = 2K = E_C$.

The key observation is that each transition $|i\rangle \leftrightarrow |i+1\rangle$ occurs at a distinct frequency due to the anharmonicity. This allows individual transitions to be addressed independently, provided the frequency separation exceeds the effective linewidth of the engineered dissipation. Specifically, if $E_C \gg g_{\text{bs, tm}}^2/B$, each transition can be treated as a separate mode coupled to the filter.

The detunings in Eq. (7.11) for the transition between levels $|i\rangle$ and $|i+1\rangle$ are:

$$\Delta_{\text{bs},i} = \omega_f + \omega_{\text{bs}} - \omega_d + 2Ki, \quad (7.14a)$$

$$\Delta_{\text{tm},i} = \omega_f + \omega_{\text{tm}} + \omega_d - 2Ki. \quad (7.14b)$$

The detuning for the beam splitter coupling increases with Fock level index, while it decreases for the two-mode squeezing coupling. By choosing the drive frequencies ω_{bs} and ω_{tm} appropriately, we can place the transitions in the lower half of the qubit inside the filter passband (enabling engineered decay) while keeping the upper-half transitions outside the passband (and vice versa for gain). The placement of transitions relative to the filter passband is illustrated in Fig. 7.2b, where engineered decay is shown in red and gain in blue.

Parameter scaling and constraints

We now analyze the constraints on system parameters and how they scale with the number of levels N .

Transmon well depth. Implementing the 0- n Fock qubit with a transmon requires a sufficiently deep cosine potential to confine N levels. The number of confined levels scales as $N_{\text{levels}} \sim \sqrt{E_J/(2E_C)}$ for $E_J/E_C \gg 1$ [264]. Fixing the fundamental transition frequency at $f_{01} = 5$ GHz together with the level-confinement requirement $E_J/(2E_C) \approx N^2$ uniquely determines E_J and E_C as functions of N .

Scaling with N . For $N \gg 1$, we have $f_{01} \approx 4NE_C$, which implies $E_C \propto 1/N$. Since the bandwidth scales as $B \propto E_C N$, it approaches a constant value for large

N . The independent dissipation assumption requires $E_C \gg g^2/B$, or $g^2 \ll E_C B \sim E_C^2 N$. Setting $g^2 \sim E_C^2 N$ (with appropriate prefactors ensuring $g^2/E_C^2 \ll N$), the coupling strength decreases as $g^2 \propto 1/N$. Consequently, the effective engineered dissipation rate scales as

$$\kappa_{\text{eff}} \sim \frac{g^2}{B} \propto \frac{1}{N}. \quad (7.15)$$

Therefore, the stabilization time, i.e. the timescale $\sim 1/\kappa_{\text{eff}}$ over which the engineered dissipation returns population from unwanted intermediate levels to the code manifold $\{|0\rangle, |n\rangle\}$, increases linearly with N . This timescale is relevant because it contributes to the duration of each syndrome-extraction cycle when the 0- n Fock qubit is used for error correction (see Sec. 7.4).

In the regime dominated by intrinsic loss (compared to heating), plugging $\kappa_i^\downarrow/\kappa_s = \beta N$ into Eq. (7.9) and minimizing over N , the minimal bit-flip rate is achieved for $N^* \approx 1/(e\beta)$. Beyond this value, increasing the number of levels N is not beneficial for the suppression of bit-flips. Since our regime of interest is $\beta N \ll 1$, we operate well below this critical value. We also note that this scaling follows from fixing f_{01} , which forces $E_C \propto 1/N$. If instead E_C is held fixed while E_J is increased with N (as in Sec. 7.4, where this suppresses charge-noise dephasing), the bandwidth B is fixed and g^2/B need not decrease, so κ_{eff} can be kept approximately constant.

Filter roll-off and out-of-band suppression. Outside the filter bandwidth, residual engineered dissipation persists due to the finite roll-off of the filter response, as visible for $|\Delta| > B/2$ in Fig. 7.2b. The roll-off can be steepened by increasing the number of filter modes M or by optimizing the couplings $\{J_i\}$ and bath dissipation κ_f (see Appendix E.1). For the parameters considered here, $M \sim 7$ modes suffice to suppress unwanted transitions below the intrinsic loss rates. We elaborate on this in Sec. 7.4.

Protection against excitation above $|n\rangle$. The 0- n Fock qubit is susceptible to intrinsic heating, which can excite population to levels above $|n\rangle$. While our design does not actively protect against such processes, intrinsic relaxation from these higher levels back to $|n\rangle$ provides natural suppression. However, it is crucial that engineered gain to levels above $|n\rangle$ remains below the intrinsic decay rates. This is demonstrated in Fig. 7.2b for the $|5\rangle \rightarrow |6\rangle$ transition. Achieving sufficient suppression requires $M \sim 7$ filter modes for the parameters that we consider.

7.4 0-n qubit as an ancilla

Having established that the 0-n Fock qubit exhibits noise bias and can be realized with current experimental tools, we now consider its use as an ancilla for cat qubit architectures in order to obtain noise-biased gates. In such architectures, the ancilla mediates syndrome extraction by coupling to the data qubits through entangling gates. Repetition codes are a natural choice for biased-noise qubits, as when bit-flip errors are exponentially suppressed, the dominant remaining errors are phase flips, which can be corrected using these codes [102, 94, 151]. In such implementations, logical errors are corrected by measuring X stabilizers, which require controlled- X (CX) gates applied to data qubits, controlled by the ancilla.

X errors on the ancilla during syndrome extraction using CX gates can propagate to the cat qubit, limiting the achievable noise bias [9, 212]. The 0-n Fock qubit, as it is inherently noise biased, can surpass this limitation compared to architectures where the ancilla is a standard transmon qubit (without any engineered dissipation) [106]. Moreover, the simple, native dispersive coupling between the Fock qubit and the cat data qubits suffices to implement the CX gate.

In light of this, we examine the performance of a CX gate conditioned on the 0-n Fock qubit, where we define a bit-flip in the following manner. Starting from the logical state $|0_L\rangle$ of two cat qubit resonators, and the $|+_{0n}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$ state of the ancilla, two CX gates are implemented to measure the XX stabilizer of the resonators in the repetition code. The cat qubits are then stabilized to the code space via a two-photon dissipation process. A bit-flip occurs if the resulting state of the cat (code) qubits has odd parity, i.e. if they are in the $|0_L\rangle \otimes |1_L\rangle$, or $|1_L\rangle \otimes |0_L\rangle$ state.

Physically, the bit-flip probability arises because intermediate Fock populations ($0 < k < n$) contribute partial phase-space rotations to the cat qubit that do not map $|0_L\rangle$ cleanly onto itself or $|1_L\rangle$, leaving residual overlap with the wrong logical state after its stabilization. The role of the engineered dissipation is therefore to suppress these intermediate populations, concentrating the ancilla in the $\{|0\rangle, |n\rangle\}$ manifold, where the CX gate acts as either the identity or a full bit-flip.

The use of the 0-n Fock qubit as an ancilla is demonstrated in Fig. 7.3a. First, the superposition state $|+_{0n}\rangle$ is prepared using X -pulses between adjacent levels. The qubit is then stabilized dissipatively, and coupled to stabilized cat qubits. CX gates conditioned on the ancilla state are implemented sequentially, after which the cat qubits are stabilized, and the ancilla is measured in the X basis.

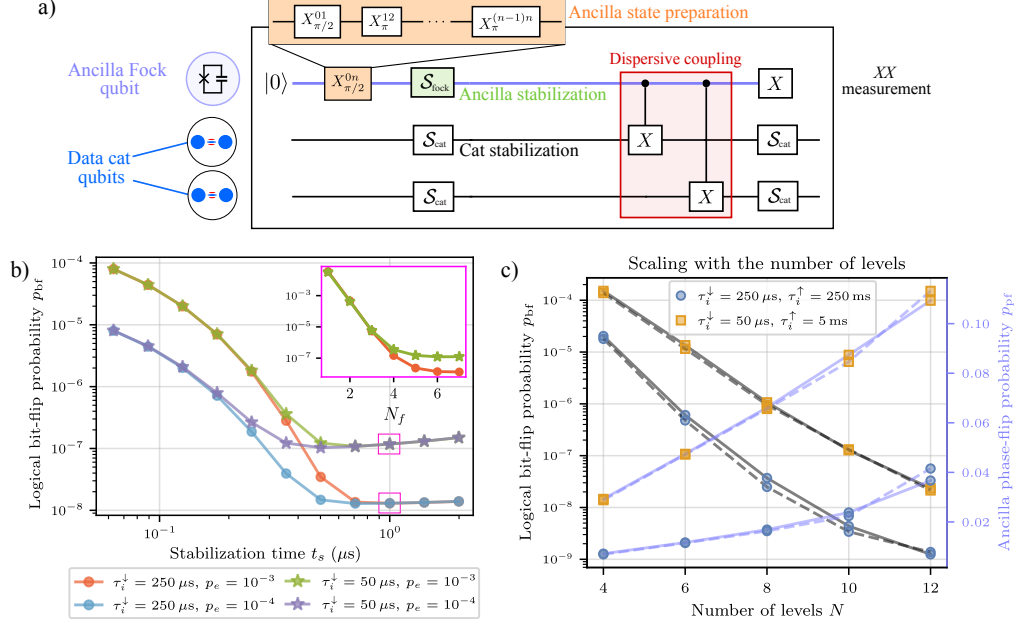


Figure 7.3: Use of the 0- n Fock qubit as an ancilla for noise-biased CX gates on stabilized cat qubits. a) Schematic of the protocol. The ancilla is prepared in the $|+_{0n}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |n\rangle)$ state using X pulses, then dissipatively stabilized (indicated by S_{fock}) using the implementation in Sec. 7.3. Two CX gates are applied sequentially to two stabilized cat qubits (the stabilization is indicated by S_{cat}) coupled dispersively to the ancilla, after which the cat qubits are restabilized and the ancilla is measured in the X basis. b) Logical bit-flip probability p_{bf} of the CX gate as a function of the stabilization time t_s , for $n = 7$, $|\alpha|^2 = 10$, and a filter with $N_f = 7$ filter modes. We assume perfect CX gates and consider two intrinsic noise configurations ($\tau_i^\downarrow = 50 \mu\text{s}$, $\tau_i^\uparrow = 5 \text{ms}$, and $\tau_i^\downarrow = 250 \mu\text{s}$, $\tau_i^\uparrow = 250 \text{ms}$) and pulse error probabilities ($p_e = 10^{-4}$, $p_e = 10^{-3}$). p_{bf} decays exponentially with stabilization time before converging to the bit-flip floor. *Inset*: dependence of p_{bf} on N_f , showing that ~ 7 modes suffice to suppress spurious engineered dissipation. c) p_{bf} and ancilla phase-flip probability p_{pf} as a function of the number of levels $N = n + 1$ in the Fock qubit. We consider the intrinsic single-photon transitions during the CX gates. The gate time is assumed as $t_g = 200 \text{ns}$, $p_e = 10^{-3}$, and $|\alpha|^2 = 10$. The dashed lines correspond to analytic predictions. Bit-flip probabilities below 10^{-8} can be obtained with $N \geq 10$ levels.

Note that one can also consider a fully Fock-qubit architecture where both data qubits and ancilla are implemented accordingly. However, this approach would require bias-preserving gates between Fock qubits, including the logical X operation connecting $|0\rangle$ to $|n\rangle$. Such an operation is not straightforward to implement experimentally, thus, we focus here on only a Fock qubit ancilla.

Bit-flip probability for the Fock-cat implementation

Let us assume capacitive coupling between the 0-n Fock qubit and the cat qubit. If the coupling strength is weak relative to their detuning, i.e. $g \ll \Delta$, they can exhibit dispersive coupling in the form of $\hat{H} = \chi_0(\hat{a}^\dagger \hat{a}) \otimes (\hat{d}^\dagger \hat{d})$ (we set $\hbar = 1$), where $\hat{a}$ and $\hat{d}$ are the annihilation operators of the Fock and cat qubit, respectively. The coupling strength is of order $\chi_0 \sim Kg^2/\Delta^2$ [144]. The evolution under this interaction will result in a rotation of the cat qubit in the phase space, proportional to the level number of the Fock qubit, in the form of:

$$e^{-i\chi_0 t(\hat{a}^\dagger \hat{a}) \otimes (\hat{d}^\dagger \hat{d})} |m\rangle \otimes |\alpha\rangle = |m\rangle \otimes |\alpha e^{-i\chi_0 m t}\rangle. \quad (7.16)$$

For $m = 0$, the cat qubit is unaffected, so the gate acts as the identity. In contrast, if $m\chi_0 t = \pi$, $|\alpha\rangle \leftrightarrow |-\alpha\rangle$, hence $|0_L\rangle \leftrightarrow |1_L\rangle$ and this operation implements the X operator. Conditioning the cat rotation on the Fock-qubit level in this way realizes a CX gate controlled by the Fock qubit.

We define the logical bit-flip probability $\tilde{p}_{\text{bf}}(\theta)$ as a function of the rotation angle $0 \leq \theta \leq \pi$:

$$\tilde{p}_{\text{bf}}(\theta) = \frac{1}{2} \left[1 - (1 - M_\alpha(\theta))^2 \right], \quad (7.17a)$$

$$M_\alpha(\theta) := \frac{2|\alpha|^2}{\sinh(2|\alpha|^2)} \int_0^\theta d\phi \sin(\phi) I_0\left(2|\alpha|^2 \sin(\phi)\right) \quad (7.17b)$$

For more details, see Appendix E.3. Furthermore, for a superposition over different Fock states of the control qubit, the total bit-flip probability is a linear function of the bit-flip probability for each Fock state, as coherence terms in the form of $|m\rangle \langle n|$, $m \neq n$, will not contribute when the control qubit is traced out. Since the rotation angle in phase space is proportional to m for the m^{th} Fock level, if we set the rotation angle to be $\theta = \pi$ for the highest level $|n\rangle$, the total bit-flip probability p_{bf} is found as

$$p_{\text{bf}} = \sum_{k=0}^n \tilde{p}_{\text{bf}}\left(\frac{\pi k}{n}\right) P(k) \quad (7.18)$$

where $P(k) := \langle k | \hat{\rho} | k \rangle$ is the population at the k^{th} level. We show in the following subsections that without stabilization, increasing n populates more intermediate levels through pulse errors, each of which contributes a partial rotation that degrades the cat qubit. Engineered dissipation, as described in Sec. 7.3, resolves this by pumping intermediate populations back to $|0\rangle$ and $|n\rangle$ before the CX gate is applied. Accordingly, we first examine the bit-flip probability of the 0-n Fock qubit in the

absence of engineered dissipation. Then, we present the numerical results with stabilization.

Bit-flip probability without stabilization

Our aim is to prepare the superposition state $(|0\rangle + |n\rangle)/\sqrt{2}$ with imperfect pulses. Let us define $|\pm_{ab}\rangle := (|a\rangle \pm |b\rangle)/\sqrt{2}$. We first prepare the superposition state $|\pm_{01}\rangle$ from the ground state $|0\rangle$ by performing an imperfect $\pi/2$ pulse. We assume the following Kraus operators during this operation:

$$\hat{K}_0 = \sqrt{1 - p_e} (\mathbb{1} - \mathbb{1}_{01} + |+\rangle_{01}\langle 0| + |-\rangle_{01}\langle 1|) \quad (7.19a)$$

$$\hat{K}_1 = \sqrt{p_e} \mathbb{1} \quad (7.19b)$$

where $\mathbb{1} := \sum_m |m\rangle\langle m|$ is the identity operator across all levels, and $\mathbb{1}_{01} = |0\rangle\langle 0| + |1\rangle\langle 1|$ is the identity operator for the $\{|0\rangle, |1\rangle\}$ subspace. Then, starting from the initial state $|0\rangle\langle 0|$, the resulting qubit state will be $(1 - p_e) |+\rangle_{01}\langle +_{01}| + p_e |0\rangle\langle 0|$. The remaining pulses will be π pulses performed between levels $|m\rangle$ and $|m + 1\rangle$, for $1 \leq m < n$. We assume the following Kraus operators:

$$\hat{K}_0^m = \sqrt{1 - p_e} (\mathbb{1} - 2 |-\rangle_{m(m+1)}\langle -_{m(m+1)}|), \quad (7.20a)$$

$$\hat{K}_1^m = \sqrt{p_e} \mathbb{1} \quad (7.20b)$$

After the π pulses, the state of the control qubit can be written as

$$\hat{\rho}_n = (1 - p_e)^n |+\rangle_{0n}\langle +_{0n}| + p_e |0\rangle\langle 0| + p_e \sum_{i=1}^{n-1} (1 - p_e)^i |+\rangle_{0i}\langle +_{0i}| \quad (7.21)$$

Using this state as the control, we can perform CX gates on two resonators encoded in the form of cat qubits. After this gate, the total bit-flip probability is found as

$$p_{\text{bf}} \approx \frac{p_e n}{\sqrt{2\pi^3} |\alpha|} \quad (7.22)$$

for $|\alpha|, n \gg 1$. Then, the bit-flip probability of the CX gate grows linearly with n and p_e up to first order in p_e : this scaling demonstrates the need for stabilization on the 0-n Fock qubit in order to obtain suppression of bit-flip probabilities with larger n .

Bit-flip probability with stabilization

Engineered dissipation ensures the stabilization of the 0-n Fock qubit to minimize the bit-flip probability. First, notice that since the bit-flip probability is independent

of the coherences of the 0- n Fock qubit, we can work with the CTMC description to compute the steady state of the qubit in the idealized case (cf. Sec. 7.2). For short timescales $t \sim 1/\kappa_s$, the spectral gap is negligible if $\kappa_i^\downarrow, \kappa_i^\uparrow \ll \kappa_s$, and the qubit converges to the metastable states for the upper and lower sections independently. Then, the bit-flip probability also converges to a fixed value determined by the population distribution across the qubit: we refer to this value as the *bit-flip floor*. In this analysis, we disregard the heating to levels above $|n\rangle$, which is negligible for sufficiently short stabilization times.

The bit-flip floor depends on the highest level number n , mean photon number of the cat qubits $|\alpha|^2$, and the relative strength of intrinsic loss processes $\kappa_i^\downarrow/\kappa_s, \kappa_i^\uparrow/\kappa_s$. We observe that this floor decays exponentially with n for relatively small n , whereas the scaling is modified to $1/n^2$ in the limit of $n \gg 1$. For a fixed n , the bit-flip probability decays exponentially with stabilization time t before converging to the bit-flip floor. For an extended discussion on the bit-flip floor, see Appendix E.4.

We analyze the convergence to the bit-flip floor numerically in Fig. 7.3b. We fix $n = 7$, $|\alpha|^2 = 10$, and examine the logical bit-flip probability p_{bf} as a function of the stabilization time, t_s , for two noise configurations. These configurations display distinct intrinsic decay and heating times $\tau_i^{\downarrow,\uparrow} = 1/\kappa_i^{\downarrow,\uparrow}$. For the first one, we assume $\tau_i^\downarrow = 50 \mu\text{s}$ and $\tau_i^\uparrow = 5 \text{ ms}$, whereas for the second one, $\tau_i^\downarrow = 250 \mu\text{s}$ and $\tau_i^\uparrow = 250 \text{ ms}$. Furthermore, we consider pulse error probabilities of $p_e = 10^{-3}$ and $p_e = 10^{-4}$. Lastly, we use a filter with $N_f = 7$ modes. For long stabilization times where the bit-flip probability approaches the bit-flip floor (e.g. $t_s = 1 \mu\text{s}$), we observe that this number is sufficient to suppress spurious engineered dissipation between levels, as can be seen in the inset of this Figure. For these times, bit-flip probabilities are independent of the pulse error probability p_e , in contrast to the case at the initialization (i.e. $t_s \ll 1 \mu\text{s}$). For longer stabilization times, the bit-flip probabilities start increasing due to spurious heating to levels above $|n\rangle$, which are not inhibited via the engineered dissipation. We note that this Figure assumes a perfect CX gate with no intrinsic decay or heating events.

In Fig. 7.3c, we plot the logical bit-flip probability p_{bf} , as well as the ancilla phase-flip probability p_{pf} , as a function of the number of levels N in the Fock qubit. For the latter, the cat qubits are initialized in state $|+_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$, and the Fock qubit is initialized as $|+_{0n}\rangle$. After subsequent CX gates conditioned on it, the Fock qubit is measured in the X basis, and the phase-flip probability is defined as $p_{\text{pf}} = 1 - \langle +_{0n} | \hat{\rho} | +_{0n} \rangle$, where $\hat{\rho}$ is the final state of the qubit. In Eq. (7.10), we have

derived that for the idealized case of single-photon transitions between adjacent levels due to loss or engineered dissipation, the phase-flip probability grows linearly with n for $\tilde{\kappa}t \ll 1$, where $\tilde{\kappa} = n\kappa_i^\downarrow + \kappa_i^\uparrow \approx n\kappa_i^\downarrow$ is the total depopulation rate of level $|n\rangle$ (assuming $\kappa_i^\uparrow \sim \kappa_i^\downarrow$ and $n \gg 1$), and t is the total time dedicated to stabilization and the CX gates. This property is in line with other noise biased qubits, where bit-flips are suppressed exponentially with n , whereas phase-flips are enhanced linearly with n .

We now quantify how the bit-flip and phase-flip probabilities scale with the number of levels N when intrinsic single-photon events during the CX gate are included (referred to as the imperfect CX gate), showing that bit-flips remain exponentially suppressed, approaching 10^{-8} for $N \geq 10$, while phase-flips grow only linearly. We set the per-pulse error as $p_e = 10^{-3}$, mean photon number of cat qubits as $|\alpha|^2 = 10$, the Fock qubit stabilization time as $t_s = 1 \mu\text{s}$, and consider the same noise configurations as above. The two configurations are indicated with yellow square markers and blue circular markers, for high and low noise, respectively. We assume a CX gate time of $t_g = 200 \text{ ns}$, which is experimentally feasible [212, 106, 213]. For both the logical bit-flips and the ancilla phase-flips, we plot the analytically calculated probabilities with dashed lines of the respective color. For this purpose, we use the approximate engineered dissipation rates obtained perturbatively (cf. Appendix E.1), and use the CTMC description of the evolution of the diagonal elements of the Fock qubit. We show in Appendix E.5 that the coherences of this qubit do not affect the bit-flip probability when the evolution can be described with the single-photon jump operators, as in Eq. (7.2). For this setting, we propose a solution for the evolution of the bit-flip probability which also takes into account single-photon events during the gate. We observe that the analytical prediction matches the numerical results for all cases. Thus, the assumptions of (i) independent dissipators for each transition within the Fock qubit, and (ii) adiabatic evolution of the bare system mode hold in this regime.

In summary, taking into consideration all intrinsic single-photon events during the ancilla stabilization and the CX gate, bit-flip probabilities below 10^{-8} can be achieved with a Fock qubit with $N \geq 10$ levels. The bit-flip probability is approximately exponentially suppressed with N , mirroring the scaling of cat qubits with mean photon number $|\alpha|^2$, achieved through a complementary mechanism: the separation between levels $|0\rangle$ and $|n\rangle$ in a multi-level, anharmonic oscillator, rather than the separation of coherent states in phase space. These results indicate that the

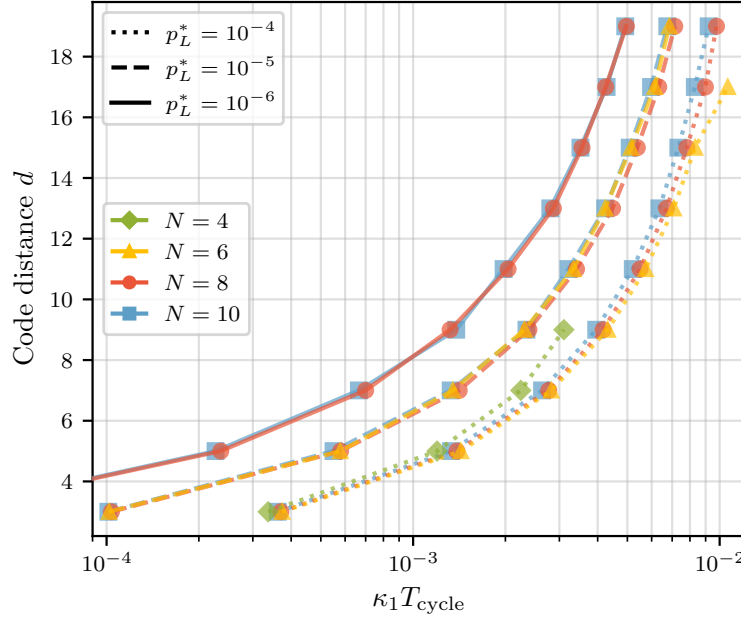


Figure 7.4: Average logical error per round p_L^* when the 0-n Fock qubit is used as an ancilla in the repetition code, paired with cat data qubits. The contours for $p_L^* \in \{10^{-4}, 10^{-5}, 10^{-6}\}$ are shown as a function of $\kappa_1 T_{\text{cycle}}$ and the code distance d , where κ_1 is the single-photon loss rate of the cat qubits, and T_{cycle} is the cycle time for a single round of error correction. We work with a phenomenological noise model, where Z errors are applied on the ancilla and the data qubits at the start of each error correction round. The ancilla Z errors occur only due to single-photon events ($\tau_i^\downarrow = 250 \mu\text{s}$, $\tau_i^\uparrow = 250 \text{ ms}$) during the stabilization ($t_s = 500 \text{ ns}$), or the CX gates ($t_g = 200 \text{ ns}$): we assume large enough E_J/E_C such that dephasing due to charge noise is sufficiently suppressed. Lastly, the cat qubits have a mean photon number of $|\alpha|^2 = 10$.

0-n Fock qubit ancilla can provide exponential bit-flip suppression comparable to cat-cat architectures, while only requiring engineered beam splitter and two-mode squeezing couplings for engineered dissipation. Moreover, simple, native dispersive couplings suffice to implement CX gates between Fock and cat qubits that preserve both qubits' exponential noise bias. This approach thus substantially simplifies the experimental requirements relative to bias-preserving cat-cat gates.

Logical error rates with the repetition code

We now examine the logical error rates achievable when the 0-n Fock ancilla is used for syndrome extraction in a repetition code. Here, as in the previous subsection, syndrome measurements are performed using two CX gates conditioned on the ancilla applied to the neighboring data qubits, followed by measurement of the

ancilla in the X basis.

We simulate the performance of a distance- d repetition code [44, 213] under a phenomenological noise model in which both ancilla and data qubits undergo Z errors at the start of each round, prior to the CX gates [62]. For the cat data qubits, the dominant mechanism is single-photon loss, which induces Z errors with probability $p_Z^{\text{data}} = (1 - e^{-2|\alpha|^2\kappa_1 T_{\text{cycle}}})/2 \approx |\alpha|^2\kappa_1 T_{\text{cycle}}$ in the regime $|\alpha|^2\kappa_1 T_{\text{cycle}} \ll 1$, where κ_1 is the loss rate, and T_{cycle} is the duration of one syndrome extraction round. Similarly, Z errors on ancilla qubits can occur due to single-photon loss and heating events, or dephasing. The latter can be mitigated by operating in the large E_J/E_C regime, where E_J is increased and f_{01} is adjusted so that E_C remains fixed, exponentially suppressing charge-noise-induced dephasing [264]. Therefore, we ignore dephasing effects and consider only intrinsic single-photon events. A more pessimistic noise budget that includes dephasing is presented in Appendix E.6.

In order to combat the effect of single-photon events as much as possible, we shorten the stabilization time to $t_s = 500$ ns, which is sufficient to approach the bit-flip floor (see Fig. 7.3b). The gate time is set to $t_g = 200$ ns, as in the previous subsection. With the stabilization and gate times fixed, the ancilla Z error probability per round is set entirely by t_s and t_g , and does not vary with T_{cycle} : only the data-qubit error probability p_Z^{data} scales with T_{cycle} . Furthermore, syndrome measurements are assumed to be subject to a readout error with probability $p_{\text{meas}} = 0.01$, and we consider the lower intrinsic noise configuration from the previous subsection, with $\tau_i^\downarrow = 250$ μs and $\tau_i^\uparrow = 250$ ms. Lastly, the mean photon number of a cat qubit is taken as $|\alpha|^2 = 10$.

Given these parameters, we simulate d rounds of a distance- d repetition code, and find the logical error rate per round occurring due to phase-flip events. We then compute the overall logical error rate p_L^* by adding the residual bit-flip contribution per round, which is found assuming imperfect CX gates with intrinsic single-photon events. The bit-flip contribution is found by taking the per-stabilizer bit-flip probability, as found in Fig. 7.3c in the previous subsection, and multiplying it by the number of stabilizer measurements per round, $d - 1$. In Figure 7.4, we plot the contours along which the overall logical error rate reaches 10^{-4} , 10^{-5} and 10^{-6} as a function of $\kappa_1 T_{\text{cycle}}$ and the code distance d , for $N \in \{4, 6, 8, 10\}$. The various logical error rate contours are shown with dotted, dashed, and plain lines, whereas the various N are plotted with different colors. From the Figure, we observe that for a given logical error rate, the contours for different N overlap significantly, which

signifies that data phase-flips are the dominant source of error in this regime, rather than ancilla phase-flips.

Thus, thanks to the exponential suppression of bit-flips achieved by the 0-n Fock ancilla, we can achieve error rates as small as 10^{-6} . Note that this target is achieved for $N \geq 8$ due to the bit-flip contribution to the logical error surpassing the target value for smaller N . In contrast, we are able to reach higher logical error rates with these N , for example, $p_L^* = 10^{-4}$. For comparison, achieving $p_L^* = 10^{-6}$ with a standard transmon ancilla, which lacks exponential noise bias, would require the ability to correct ancilla-transmon-induced bit flips with a more hardware-intensive code, like the surface code.

The $p_L^* = 10^{-6}$ logical error rate (i.e. the megaquop regime [210]) is achieved in the experimentally realistic regime of $\kappa_1 T_{\text{cycle}} \sim 10^{-3}$, corresponding to a storage-mode lifetime of $1/\kappa_1 \sim 2$ ms for the $T_{\text{cycle}} \sim 2 \mu\text{s}$ cycle times considered here. Single-photon lifetimes exceeding 10 ms have been demonstrated in bare superconducting cavities [216, 179], and current cat-qubit experiments operate within an order of magnitude of this target [213]. Furthermore, if one allows for a larger code distance of $d \sim 17$, this target can be achieved with $\kappa_1 T_{\text{cycle}} \sim 4 \cdot 10^{-3}$, corresponding to a lifetime of $1/\kappa_1 \sim 500 \mu\text{s}$.

The cycle time can in principle be reduced significantly further by pipelining the ancilla operations across consecutive rounds. While one ancilla is engaged in the CX gates and subsequent syndrome extraction, the next ancilla can be prepared and stabilized in parallel, so that ancilla preparation no longer contributes to the cycle time. Moreover, because each ancilla is decoupled from the filter once its stabilization is complete, the same multimode filter can be shared between consecutive ancillas rather than requiring a dedicated filter per ancilla. This pipelining could halve the effective cycle time to $\sim 1 \mu\text{s}$ without modifications to the underlying hardware, bringing the operating regime to $\kappa_1 T_{\text{cycle}} \sim 10^{-3}$ for $1/\kappa_1 \sim 1$ ms. These results demonstrate that the 0-n Fock ancilla preserves the noise bias of the cat data qubits well enough that ancilla-induced bit flips no longer dominate the logical error budget for moderate N . The hybrid cat-Fock architecture therefore provides a viable route to experimental realization of low logical error rates.

7.5 Conclusions

We have introduced the 0-n Fock qubit, a noise-biased bosonic encoding in which the logical states are defined by the Fock levels $|0\rangle, |n\rangle$. The protection mechanism relies

on the fact that single-photon loss events map the logical manifold to intermediate states rather than directly inducing bit flips. As a consequence, logical bit-flip errors require $\lceil n/2 \rceil$ sequential loss events and are therefore exponentially suppressed with increasing n , establishing a hardware-level noise bias analogous to the bit-flip suppression in cat qubits. However, distinct from cat qubits, the Fock qubit can be implemented with multi-level anharmonic oscillators, such as a transmon, or a Kerr nonlinear oscillator.

We have shown that this qubit can be stabilized using frequency-selective dissipation. By coupling the oscillator to a short chain of LC resonators, the effective bath spectral density can be shaped such that unwanted intermediate Fock states are selectively depopulated while the logical manifold remains invariant. Importantly, this stabilization can be realized with a small number (< 10) of filter modes and realistic circuit parameters, indicating that the required bath engineering is compatible with current superconducting circuit technology.

Beyond its intrinsic protection properties, the 0-n qubit is particularly well suited for hybrid architectures with cat qubits. In dissipatively stabilized cat qubits, operations associated with the logical X degree of freedom are naturally implemented through phase-space rotations, whereas the logical Z operator requires distinguishing $|\pm\alpha\rangle$, which is experimentally more demanding. In contrast, for the Fock qubit, the Z operator can be implemented straightforwardly via dispersive coupling to a resonator, while direct access to the logical X degree of freedom is more difficult. Thus, a hybrid architecture combining cat data qubits with 0-n ancilla qubits enables efficient and bias-preserving CX gates, which are central to syndrome extraction in fault-tolerant protocols.

Accordingly, we studied the use of the 0-n qubit in CX gates for syndrome measurements on cat qubits, and found that bit-flip probabilities below 10^{-8} are achievable with a Fock qubit with $N = 10$ levels under experimentally realistic conditions. This demonstrates that the 0-n Fock qubit can serve as a bias-preserving ancillary resource without requiring the same degree of Hamiltonian engineering as fully bosonic cat-cat interactions, while avoiding the bias limitations associated with transmon-based hybrid architectures.

Chapter 8

CONCLUSIONS

This thesis examined probing and harnessing entanglement, noise, and dissipation across various case studies in the fields of quantum metrology and computing. Here, we summarize the main findings.

In Chapters 3 and 4, we focused on a system where a mechanical oscillator is driven by a light field, and we investigated the entanglement between these two parties. Assuming Gaussian dynamics and arbitrary, possibly non-Markovian environmental noise, we derived necessary and sufficient conditions for entanglement through the PPT criterion. Chapter 3 established that for a vacuum input, the entangling-disentangling transition is independent of the optomechanical coupling strength, such that a system outside the entangling regime cannot be brought into it by strengthening the interaction between the mechanical oscillator and the light field. Furthermore, entanglement cannot be created by frequency-dependent squeezing of the input field if the vacuum-driven system is separable. Applying this framework to aLIGO with a numerical treatment of its non-Markovian noise in Chapter 4, we found that the present level of classical noise makes a definitive verdict difficult and renders significant entanglement unlikely. We observed that the main limiting factor is the large seismic noise at low frequencies: in the absence of this noise, stationary optomechanical entanglement is achievable in the current operating regime of the interferometer, which includes its suspension thermal noise. This chapter shows that classical noise, even at very low frequencies, suffices to destroy stationary macroscopic entanglement.

In Chapters 5 and 6, we tackled quantum phase and frequency estimation. In Chapter 5, we proposed a strategy that achieves Heisenberg scaling up to a constant overhead, using probe states implementable with logarithmic-depth circuits. These states are constructed using blocks of GHZ states, and the phase is estimated from a local, adaptive readout. The scheme can also be used in the extended dynamic range regime where the phase imprinted on the qubits extends beyond a 2π interval with the addition of slow atoms. In Chapter 6, we exclusively focused on the dynamic range problem using a weak-measurement Ramsey protocol that asymptotically attains the classical precision limit over an extended dynamic range. Optimizing

the measurement strength due to the trade-off between the frequency information gained from the weak measurements and the resulting measurement back action, and combining the protocol with a final projective measurement, we showed that high sensitivity and large bandwidth can be achieved simultaneously.

Finally, in Chapter 7, we introduced the $0-n$ Fock qubit, a noise-biased bosonic encoding whose logical states are the Fock levels $|0\rangle$ and $|n\rangle$. Because single-photon loss maps these states to intermediate Fock levels which are not in the logical code space, a logical bit flip requires $\lceil n/2 \rceil$ loss events and is exponentially suppressed in n , yielding a hardware-level noise bias comparable to that of cat qubits but realizable in multilevel anharmonic oscillators, such as a transmon or a Kerr oscillator. We showed that the qubit can be stabilized with frequency-selective dissipation using fewer than ten filter modes with realistic circuit parameters. We used this qubit as a bias-preserving ancilla for cat-qubit architectures and showed that it enables CX gates with bit-flip probabilities below 10^{-8} for $n = 9$, supporting logical error rates as low as 10^{-6} under experimentally realistic conditions.

These results span three different settings, but share common concepts: the role of entanglement, noise, and dissipation in quantum systems. In the optomechanical setting, noise is the factor that limits how much entanglement the system can harbor, in metrology, measurement back action sets the optimal measurement strength, and in the computing setting, the same dissipative dynamics that would otherwise destroy quantum information is instead engineered to protect it. This is the two-facet picture of quantum systems laid out in the introduction: the same object can be probed or harnessed, in order to achieve non-classical behavior or (metrological or computational) quantum advantage.

BIBLIOGRAPHY

- [1] B P Abbott et al. “A guide to LIGO–Virgo detector noise and extraction of transient gravitational-wave signals”. In: *Classical and Quantum Gravity* 37.5 (Feb. 2020), p. 055002. doi: [10.1088/1361-6382/ab685e](https://doi.org/10.1088/1361-6382/ab685e).
- [2] M. Abdi et al. “Effect of phase noise on the generation of stationary entanglement in cavity optomechanics”. In: *Phys. Rev. A* 84 (3 Sept. 2011), p. 032325. doi: [10.1103/PhysRevA.84.032325](https://doi.org/10.1103/PhysRevA.84.032325).
- [3] F Acernese et al. “Advanced Virgo: a second-generation interferometric gravitational wave detector”. In: *Classical and Quantum Gravity* 32.2 (Dec. 2014), p. 024001. doi: [10.1088/0264-9381/32/2/024001](https://doi.org/10.1088/0264-9381/32/2/024001).
- [4] Gerardo Adesso and Fabrizio Illuminati. “Entanglement in continuous variable systems: Recent advances and current perspectives”. In: *Journal of Physics A: Mathematical and Theoretical* 40.28 (July 2007). arXiv:quant-ph/0701221, pp. 7821–7880. issn: 1751-8113, 1751-8121. doi: [10.1088/1751-8113/40/28/S01](https://doi.org/10.1088/1751-8113/40/28/S01).
- [5] Gerardo Adesso, Sammy Ragy, and Antony R. Lee. “Continuous Variable Quantum Information: Gaussian States and Beyond”. In: *Open Systems & Information Dynamics* 21.01n02 (2014), p. 1440001. doi: [10.1142/S1230161214400010](https://doi.org/10.1142/S1230161214400010). eprint: <https://doi.org/10.1142/S1230161214400010>.
- [6] Nati Aharon et al. “NV center based nano-NMR enhanced by deep learning”. In: *Scientific Reports* 9.1 (Nov. 2019), p. 17802. issn: 2045-2322. doi: [10.1038/s41598-019-54119-9](https://doi.org/10.1038/s41598-019-54119-9).
- [7] Dorit Aharonov. “Quantum to classical phase transition in noisy quantum computers”. In: *Phys. Rev. A* 62 (6 Nov. 2000), p. 062311. doi: [10.1103/PhysRevA.62.062311](https://doi.org/10.1103/PhysRevA.62.062311).
- [8] Victor V. Albert et al. “Pair-cat codes: autonomous error-correction with low-order nonlinearity”. In: *Quantum Sci. Technol.* 4 (2019), p. 035007. doi: [10.1088/2058-9565/ab1e69](https://doi.org/10.1088/2058-9565/ab1e69). arXiv: [1801.05897](https://arxiv.org/abs/1801.05897).
- [9] P. Aliferis and J. Preskill. “Fault-tolerant quantum computation against biased noise”. In: *Phys. Rev. A* 78 (2008), p. 052331. doi: [10.1103/PhysRevA.78.052331](https://doi.org/10.1103/PhysRevA.78.052331).
- [10] Richard R Allen et al. “Quantum Computing Enhanced Sensing”. In: *arXiv preprint arXiv:2501.07625* (2025).
- [11] Markus Aspelmeyer, Tobias J. Kippenberg, and Florian Marquardt. “Cavity optomechanics”. In: *Reviews of Modern Physics* 86.4 (Dec. 2014), pp. 1391–1452. doi: [10.1103/revmodphys.86.1391](https://doi.org/10.1103/revmodphys.86.1391).

- [12] S M Aston et al. “Update on quadruple suspension design for Advanced LIGO”. In: *Classical and Quantum Gravity* 29.23 (Oct. 2012), p. 235004. DOI: [10.1088/0264-9381/29/23/235004](https://doi.org/10.1088/0264-9381/29/23/235004).
- [13] Gopalakrishnan Balasubramanian et al. “Ultralong spin coherence time in isotopically engineered diamond”. In: *Nature Materials* 8.5 (May 2009), pp. 383–387. ISSN: 1476-1122, 1476-4660. DOI: [10.1038/nmat2420](https://doi.org/10.1038/nmat2420).
- [14] Feng Bao et al. “Fluxonium: An Alternative Qubit Platform for High-Fidelity Operations”. In: *Phys. Rev. Lett.* 129 (1 June 2022), p. 010502. DOI: [10.1103/PhysRevLett.129.010502](https://doi.org/10.1103/PhysRevLett.129.010502).
- [15] Adriano Barenco et al. “Approximate quantum Fourier transform and decoherence”. In: *Phys. Rev. A* 54 (1 July 1996), pp. 139–146. DOI: [10.1103/PhysRevA.54.139](https://doi.org/10.1103/PhysRevA.54.139).
- [16] S. Barzanjeh et al. “Stationary entangled radiation from micromechanical motion”. In: *Nature* 570.7762 (June 2019), pp. 480–483. ISSN: 1476-4687. DOI: [10.1038/s41586-019-1320-2](https://doi.org/10.1038/s41586-019-1320-2).
- [17] Sh. Barzanjeh et al. “Reversible Optical-to-Microwave Quantum Interface”. In: *Phys. Rev. Lett.* 109 (13 Sept. 2012), p. 130503. DOI: [10.1103/PhysRevLett.109.130503](https://doi.org/10.1103/PhysRevLett.109.130503).
- [18] Tillmann Baumgratz and Animesh Datta. “Quantum Enhanced Estimation of a Multidimensional Field”. In: *Phys. Rev. Lett.* 116.3 (Jan. 2016), p. 030801. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.116.030801](https://doi.org/10.1103/PhysRevLett.116.030801).
- [19] C. Berdou et al. “One Hundred Second Bit-Flip Time in a Two-Photon Dissipative Oscillator”. In: *PRX Quantum* 4 (2023), p. 020350. DOI: [10.1103/PRXQuantum.4.020350](https://doi.org/10.1103/PRXQuantum.4.020350). arXiv: [2204.09128](https://arxiv.org/abs/2204.09128).
- [20] D. W. Berry and H. M. Wiseman. “Optimal States and Almost Optimal Adaptive Measurements for Quantum Interferometry”. In: *Phys. Rev. Lett.* 85.24 (Dec. 2000), pp. 5098–5101. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.85.5098](https://doi.org/10.1103/PhysRevLett.85.5098).
- [21] D. W. Berry et al. “How to perform the most accurate possible phase measurements”. In: *Phys. Rev. A* 80 (5 Nov. 2009), p. 052114. DOI: [10.1103/PhysRevA.80.052114](https://doi.org/10.1103/PhysRevA.80.052114).
- [22] G. W. Biedermann et al. “Zero-Dead-Time Operation of Interleaved Atomic Clocks”. In: *Phys. Rev. Lett.* 111.17 (Oct. 2013), p. 170802. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.111.170802](https://doi.org/10.1103/PhysRevLett.111.170802).
- [23] BJ Bloom et al. “An optical lattice clock with accuracy and stability at the 10⁻¹⁸ level”. In: *Nature* 506.7486 (2014), pp. 71–75.
- [24] Dolev Bluvstein et al. “Logical quantum processor based on reconfigurable atom arrays”. In: *Nature* 626.7997 (Feb. 2024), pp. 58–65. ISSN: 1476-4687. DOI: [10.1038/s41586-023-06927-3](https://doi.org/10.1038/s41586-023-06927-3).

- [25] J. J. Bollinger et al. “Optimal frequency measurements with maximally correlated states”. In: *Phys. Rev. A* 54.6 (Dec. 1996), R4649–R4652. ISSN: 1050-2947, 1094-1622. DOI: [10.1103/PhysRevA.54.R4649](https://doi.org/10.1103/PhysRevA.54.R4649).
- [26] J. Borregaard and A. S. Sørensen. “Efficient Atomic Clocks Operated with Several Atomic Ensembles”. In: *Phys. Rev. Lett.* 111 (9 Aug. 2013), p. 090802. DOI: [10.1103/PhysRevLett.111.090802](https://doi.org/10.1103/PhysRevLett.111.090802).
- [27] J. Borregaard and A. S. Sørensen. “Near-Heisenberg-Limited Atomic Clocks in the Presence of Decoherence”. In: *Phys. Rev. Lett.* 111.9 (Aug. 2013), p. 090801. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.111.090801](https://doi.org/10.1103/PhysRevLett.111.090801).
- [28] J. M. Boss et al. “Quantum sensing with arbitrary frequency resolution”. In: *Science* 356.6340 (May 2017), pp. 837–840. ISSN: 0036-8075, 1095-9203. DOI: [10.1126/science.aam7009](https://doi.org/10.1126/science.aam7009).
- [29] Tobias Bothwell et al. “Resolving the gravitational redshift across a millimetre-scale atomic sample”. In: *Nature* 602.7897 (Feb. 2022), pp. 420–424. ISSN: 1476-4687. DOI: [10.1038/s41586-021-04349-7](https://doi.org/10.1038/s41586-021-04349-7).
- [30] William Bowden et al. “Improving the Q Factor of an Optical Atomic Clock Using Quantum Nondemolition Measurement”. In: *Physical Review X* 10.4 (Dec. 2020), p. 041052. ISSN: 2160-3308. DOI: [10.1103/PhysRevX.10.041052](https://doi.org/10.1103/PhysRevX.10.041052).
- [31] Vladimir B. Braginsky, Farid Ya Khalili, and Kip S. Thorne. *Quantum Measurement*. Cambridge University Press, 1992. DOI: [10.1017/CBO9780511622748](https://doi.org/10.1017/CBO9780511622748).
- [32] J. B. Brask, R. Chaves, and J. Kołodyński. “Improved Quantum Magnetometry beyond the Standard Quantum Limit”. In: *Physical Review X* 5.3 (July 2015), p. 031010. ISSN: 2160-3308. DOI: [10.1103/PhysRevX.5.031010](https://doi.org/10.1103/PhysRevX.5.031010).
- [33] Samuel L. Braunstein and Carlton M. Caves. “Statistical distance and the geometry of quantum states”. In: *Phys. Rev. Lett.* 72 (22 May 1994), pp. 3439–3443. DOI: [10.1103/PhysRevLett.72.3439](https://doi.org/10.1103/PhysRevLett.72.3439).
- [34] Heinz-Peter Breuer and Francesco Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, Jan. 2006. ISBN: 9780199213900. DOI: [10.1093/acprof:oso/9780199213900.001.0001](https://doi.org/10.1093/acprof:oso/9780199213900.001.0001).
- [35] Nicholas T. Bronn et al. “Broadband filters for abatement of spontaneous emission in circuit quantum electrodynamics”. In: *Appl. Phys. Lett.* 107 (2015), p. 172601. DOI: [10.1063/1.4934867](https://doi.org/10.1063/1.4934867). arXiv: [1508.01743](https://arxiv.org/abs/1508.01743).
- [36] B. M. Brubaker et al. “Optomechanical Ground-State Cooling in a Continuous and Efficient Electro-Optic Transducer”. In: *Phys. Rev. X* 12 (2 June 2022), p. 021062. DOI: [10.1103/PhysRevX.12.021062](https://doi.org/10.1103/PhysRevX.12.021062).
- [37] V. Bužek, R. Derka, and S. Massar. “Optimal Quantum Clocks”. In: *Phys. Rev. Lett.* 82 (10 Mar. 1999), pp. 2207–2210. DOI: [10.1103/PhysRevLett.82.2207](https://doi.org/10.1103/PhysRevLett.82.2207).

- [38] Alessandra Buonanno and Yanbei Chen. “Scaling law in signal recycled laser-interferometer gravitational-wave detectors”. In: *Physical Review D* 67.6 (Mar. 2003). DOI: [10.1103/physrevd.67.062002](https://doi.org/10.1103/physrevd.67.062002).
- [39] Jonathan J. Burnett et al. “Decoherence benchmarking of superconducting qubits”. In: *npj Quantum Information* 5.1 (June 2019), p. 54. ISSN: 2056-6387. DOI: [10.1038/s41534-019-0168-5](https://doi.org/10.1038/s41534-019-0168-5).
- [40] Herbert B. Callen and Theodore A. Welton. “Irreversibility and Generalized Noise”. In: *Phys. Rev.* 83 (1 July 1951), pp. 34–40. DOI: [10.1103/PhysRev.83.34](https://doi.org/10.1103/PhysRev.83.34).
- [41] Alec Cao et al. “Multi-qubit gates and Schrödinger cat states in an optical clock”. In: *Nature* 634.8033 (2024), pp. 315–320.
- [42] Nadav Carmel and Nadav Katz. “Hybrid Logical-Physical Qubit Interaction as a Post Selection Oracle”. In: *arXiv preprint arXiv:2306.05027* (2023).
- [43] Carlton M. Caves and Bonny L. Schumaker. “New formalism for two-photon quantum optics. I. Quadrature phases and squeezed states”. In: *Physical Review A* 31.5 (May 1985), pp. 3068–3092. ISSN: 0556-2791. DOI: [10.1103/PhysRevA.31.3068](https://doi.org/10.1103/PhysRevA.31.3068).
- [44] Christopher Chamberland et al. “Building a Fault-Tolerant Quantum Computer Using Concatenated Cat Codes”. In: *PRX Quantum* 3.1 (Feb. 2022), p. 010329. ISSN: 2691-3399. DOI: [10.1103/PRXQuantum.3.010329](https://doi.org/10.1103/PRXQuantum.3.010329).
- [45] Jasper Chan et al. “Laser cooling of a nanomechanical oscillator into its quantum ground state”. In: *Nature* 478.7367 (Oct. 2011), pp. 89–92. ISSN: 1476-4687. DOI: [10.1038/nature10461](https://doi.org/10.1038/nature10461).
- [46] Junxin Chen et al. “Entanglement of propagating optical modes via a mechanical interface”. In: *Nature Communications* 11.1 (Feb. 2020), p. 943. ISSN: 2041-1723. DOI: [10.1038/s41467-020-14768-1](https://doi.org/10.1038/s41467-020-14768-1).
- [47] Yanbei Chen. “Macroscopic quantum mechanics: theory and experimental concepts of optomechanics”. In: *Journal of Physics B: Atomic, Molecular and Optical Physics* 46.10 (May 2013). Publisher: IOP Publishing, p. 104001. ISSN: 0953-4075. DOI: [10.1088/0953-4075/46/10/104001](https://doi.org/10.1088/0953-4075/46/10/104001).
- [48] R. Cleve and J. Watrous. “Fast parallel circuits for the quantum Fourier transform”. In: *Proceedings 41st Annual Symposium on Foundations of Computer Science*. 2000, pp. 526–536. DOI: [10.1109/SFCS.2000.892140](https://doi.org/10.1109/SFCS.2000.892140).
- [49] P. T. Cochrane, G. J. Milburn, and W. J. Munro. “Macroscopically distinct quantum-superposition states as a bosonic code for amplitude damping”. In: *Phys. Rev. A* 59 (1999), p. 2631. DOI: [10.1103/PhysRevA.59.2631](https://doi.org/10.1103/PhysRevA.59.2631).
- [50] D Cohen et al. “Achieving the ultimate precision limit with a weakly interacting quantum probe”. In: *npj Quantum Information* 6.1 (2020), p. 83.

- [51] D. Cohen et al. “Confined Nano-NMR Spectroscopy Using NV Centers”. In: *Advanced Quantum Technologies* 3.11 (Nov. 2020), p. 2000019. ISSN: 2511-9044, 2511-9044. DOI: [10.1002/qute.202000019](https://doi.org/10.1002/qute.202000019).
- [52] The LIGO Scientific Collaboration. “Advanced LIGO”. In: *Classical and Quantum Gravity* 32.7 (Mar. 2015), p. 074001. DOI: [10.1088/0264-9381/32/7/074001](https://doi.org/10.1088/0264-9381/32/7/074001).
- [53] Harald Cramer. *Mathematical methods of statistics (PMS-9), volume 9*. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, Sept. 1946.
- [54] Jonathan Cripe et al. “Measurement of quantum back action in the audio band at room temperature”. In: *Nature* 568.7752 (Mar. 2019), pp. 364–367. DOI: [10.1038/s41586-019-1051-4](https://doi.org/10.1038/s41586-019-1051-4).
- [55] K. S. Cujia et al. “Tracking the precession of single nuclear spins by weak measurements”. In: *Nature* 571.7764 (July 2019), pp. 230–233. ISSN: 0028-0836, 1476-4687. DOI: [10.1038/s41586-019-1334-9](https://doi.org/10.1038/s41586-019-1334-9).
- [56] A. V. Cumming et al. “Large-scale Monolithic Fused-Silica Mirror Suspension for Third-Generation Gravitational-Wave Detectors”. In: *Phys. Rev. Appl.* 17 (2 Feb. 2022), p. 024044. DOI: [10.1103/PhysRevApplied.17.024044](https://doi.org/10.1103/PhysRevApplied.17.024044).
- [57] Milena D’Angelo, Maria V. Chekhova, and Yanhua Shih. “Two-Photon Diffraction and Quantum Lithography”. In: *Phys. Rev. Lett.* 87.1 (June 2001), p. 013602. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.87.013602](https://doi.org/10.1103/PhysRevLett.87.013602).
- [58] Stefan L. Danilishin and Farid Ya. Khalili. “Quantum Measurement Theory in Gravitational-Wave Detectors”. In: *Living Reviews in Relativity* 15.1 (Apr. 2012), p. 5. ISSN: 1433-8351. DOI: [10.12942/lrr-2012-5](https://doi.org/10.12942/lrr-2012-5).
- [59] Stefan L. Danilishin, Farid Ya. Khalili, and Haixing Miao. “Advanced quantum techniques for future gravitational-wave detectors”. In: *Living Reviews in Relativity* 22.1 (Apr. 2019). DOI: [10.1007/s41114-019-0018-y](https://doi.org/10.1007/s41114-019-0018-y).
- [60] Rafał Demkowicz-Dobrzański, Jan Czajkowski, and Pavel Sekatski. “Adaptive Quantum Metrology under General Markovian Noise”. In: *Phys. Rev. X* 7 (4 2017), p. 041009. DOI: [10.1103/PhysRevX.7.041009](https://doi.org/10.1103/PhysRevX.7.041009).
- [61] Rafal Demkowicz-Dobrzański and Lorenzo Maccone. “Using Entanglement Against Noise in Quantum Metrology”. In: *Phys. Rev. Lett.* 113 (25 Dec. 2014), p. 250801. DOI: [10.1103/PhysRevLett.113.250801](https://doi.org/10.1103/PhysRevLett.113.250801).
- [62] Eric Dennis et al. “Topological quantum memory”. In: *J. Math. Phys.* 43 (2002), p. 4452. DOI: [10.1063/1.1499754](https://doi.org/10.1063/1.1499754).
- [63] G. Dick. *Proceedings of the Nineteenth Annual Precise Time and Time Interval (PTTI) Applications and Planning Meeting, Redondo Beach, CA, 1987*. Naval Observatory, Washington, DC, 1987, p. 133.

- [64] S. Ding et al. “Quantum control of an oscillator with a Kerr-cat qubit”. In: *Nat. Commun.* 16 (2025), p. 5143. DOI: [10.1038/s41467-025-60352-w](https://doi.org/10.1038/s41467-025-60352-w).
- [65] S. Direkci et al. “Macroscopic quantum entanglement between an optomechanical cavity and a continuous field in presence of non-Markovian noise”. In: *Phys. Rev. Res.* 6 (1 Feb. 2024), p. 013175. DOI: [10.1103/PhysRevResearch.6.013175](https://doi.org/10.1103/PhysRevResearch.6.013175).
- [66] Su Direkci et al. “Heisenberg-limited Bayesian phase estimation with low-depth digital quantum circuits”. In: *npj Quantum Information* 12.1 (Jan. 2026). ISSN: 2056-6387. DOI: [10.1038/s41534-025-01177-9](https://doi.org/10.1038/s41534-025-01177-9).
- [67] Lu-Ming Duan et al. “Inseparability Criterion for Continuous Variable Systems”. In: *Physical Review Letters* 84.12 (2000).
- [68] William J. Eckner et al. “Realizing spin squeezing with Rydberg interactions in an optical clock”. In: *Nature* 621.7980 (Sept. 2023), pp. 734–739. ISSN: 1476-4687. DOI: [10.1038/s41586-023-06360-6](https://doi.org/10.1038/s41586-023-06360-6).
- [69] B. M. Escher, R. L. de Matos Filho, and L. Davidovich. “General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology”. In: *Nature Physics* 7.5 (May 2011), pp. 406–411. ISSN: 1745-2481. DOI: [10.1038/nphys1958](https://doi.org/10.1038/nphys1958).
- [70] Simon J. Evered et al. “High-fidelity parallel entangling gates on a neutral-atom quantum computer”. In: *Nature* 622.7982 (Oct. 2023), pp. 268–272. ISSN: 1476-4687. DOI: [10.1038/s41586-023-06481-y](https://doi.org/10.1038/s41586-023-06481-y).
- [71] S.A. Fedorov et al. “Evidence for structural damping in a high-stress silicon nitride nanobeam and its implications for quantum optomechanics”. In: *Physics Letters A* 382.33 (2018). Special Issue in memory of Professor V.B. Braginsky, pp. 2251–2255. ISSN: 0375-9601. DOI: <https://doi.org/10.1016/j.physleta.2017.05.046>.
- [72] Alessandro Ferraro, Stefano Olivares, and Matteo G. A. Paris. “Gaussian states in continuous variable quantum information”. In: (2005). DOI: [10.48550/ARXIV.QUANT-PH/0503237](https://arxiv.org/abs/10.48550/ARXIV.QUANT-PH/0503237).
- [73] Ran Finkelstein et al. “Universal quantum operations and ancilla-based read-out for tweezer clocks”. In: *Nature* 634.8033 (Oct. 2024), pp. 321–327. ISSN: 1476-4687. DOI: [10.1038/s41586-024-08005-8](https://doi.org/10.1038/s41586-024-08005-8).
- [74] Nicolai Friis et al. “Flexible resources for quantum metrology”. In: *New Journal of Physics* 19.6 (June 2017), p. 063044. ISSN: 1367-2630. DOI: [10.1088/1367-2630/aa7144](https://doi.org/10.1088/1367-2630/aa7144).
- [75] Søren Gammelmark and Klaus Mølmer. “Fisher Information and the Quantum Cramér-Rao Sensitivity Limit of Continuous Measurements”. In: *Physical Review Letters* 112.17 (Apr. 2014), p. 170401. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.112.170401](https://doi.org/10.1103/PhysRevLett.112.170401).

- [76] C. W. Gardiner and P. Zoller. *Quantum noise : a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics*. 3rd ed. Springer complexity. Berlin: Springer, 2004. ISBN: 9783540223016.
- [77] James W. Gardner et al. *Bayesian frequency estimation at the fundamental quantum limit*. 2025. arXiv: [2507.02811](https://arxiv.org/abs/2507.02811) [quant-ph].
- [78] Ronan Gautier, Mazyar Mirrahimi, and Alain Sarlette. “Designing High-Fidelity Zeno Gates for Dissipative Cat Qubits”. In: *PRX Quantum* 4 (2023), p. 040316. DOI: [10.1103/PRXQuantum.4.040316](https://doi.org/10.1103/PRXQuantum.4.040316).
- [79] Ronan Gautier, Alain Sarlette, and Mazyar Mirrahimi. “Combined Dissipative and Hamiltonian Confinement of Cat Qubits”. In: *PRX Quantum* 3 (2022), p. 020339. DOI: [10.1103/PRXQuantum.3.020339](https://doi.org/10.1103/PRXQuantum.3.020339). arXiv: [2112.05545](https://arxiv.org/abs/2112.05545).
- [80] Tuvia Gefen et al. “Quantum spectroscopy of single spins assisted by a classical clock”. In: *Physical Review A* 98.1 (July 2018), p. 013844. ISSN: 2469-9926, 2469-9934. DOI: [10.1103/PhysRevA.98.013844](https://doi.org/10.1103/PhysRevA.98.013844).
- [81] C. Genes et al. “Chapter 2 Quantum Effects in Optomechanical Systems”. In: *Advances in Atomic Molecular and Optical Physics*. Vol. 57. Advances In Atomic, Molecular, and Optical Physics. San Diego: Academic Press, 2009, pp. 33–86. DOI: [https://doi.org/10.1016/S1049-250X\(09\)57002-4](https://doi.org/10.1016/S1049-250X(09)57002-4).
- [82] C. Genes et al. “Robust entanglement of a micromechanical resonator with output optical fields”. In: *Phys. Rev. A* 78 (3 Sept. 2008), p. 032316. DOI: [10.1103/PhysRevA.78.032316](https://doi.org/10.1103/PhysRevA.78.032316).
- [83] Jeffrey M. Gertler et al. “Protecting a Bosonic Qubit with Autonomous Quantum Error Correction”. In: *Nature* 590.7845 (Feb. 2021). arXiv:2004.09322 [quant-ph], pp. 243–248. ISSN: 0028-0836, 1476-4687. DOI: [10.1038/s41586-021-03257-0](https://doi.org/10.1038/s41586-021-03257-0).
- [84] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. “Advances in quantum metrology”. In: *Nature Photonics* 5.4 (Apr. 2011), pp. 222–229. ISSN: 1749-4885, 1749-4893. DOI: [10.1038/nphoton.2011.35](https://doi.org/10.1038/nphoton.2011.35).
- [85] Vittorio Giovannetti and David Vitali. “Phase-noise measurement in a cavity with a movable mirror undergoing quantum Brownian motion”. In: *Phys. Rev. A* 63 (2 Jan. 2001), p. 023812. DOI: [10.1103/PhysRevA.63.023812](https://doi.org/10.1103/PhysRevA.63.023812).
- [86] D. Andrew Golter et al. “Optomechanical Quantum Control of a Nitrogen-Vacancy Center in Diamond”. In: *Phys. Rev. Lett.* 116 (14 Apr. 2016), p. 143602. DOI: [10.1103/PhysRevLett.116.143602](https://doi.org/10.1103/PhysRevLett.116.143602).
- [87] Gabriela González. “Suspensions thermal noise in the LIGO gravitational wave detector”. In: *Classical and Quantum Gravity* 17.21 (Nov. 2000), p. 4409. DOI: [10.1088/0264-9381/17/21/305](https://doi.org/10.1088/0264-9381/17/21/305).

- [88] Gabriela I. González and Peter R. Saulson. “Brownian motion of a torsion pendulum with internal friction”. In: *Physics Letters A* 201.1 (1995), pp. 12–18. ISSN: 0375-9601. DOI: [https://doi.org/10.1016/0375-9601\(95\)00194-8](https://doi.org/10.1016/0375-9601(95)00194-8).
- [89] C. Gonzalez-Ballester et al. “Levitodynamics: Levitation and control of microscopic objects in vacuum”. In: *Science* 374.6564 (2021), eabg3027. DOI: [10.1126/science.abg3027](https://doi.org/10.1126/science.abg3027). eprint: <https://www.science.org/doi/pdf/10.1126/science.abg3027>.
- [90] Wojciech Górecki. “Heisenberg limit beyond quantum Fisher information”. In: *arXiv preprint arXiv:2304.14370* (2023).
- [91] Wojciech Górecki et al. “ π -Corrected Heisenberg Limit”. In: *Phys. Rev. Lett.* 124.3 (Jan. 2020), p. 030501. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.124.030501](https://doi.org/10.1103/PhysRevLett.124.030501).
- [92] J. M. H. Gosling et al. “Sensing directional noise baths in levitated optomechanics”. In: *Phys. Rev. Res.* 6 (1 Jan. 2024), p. 013129. DOI: [10.1103/PhysRevResearch.6.013129](https://doi.org/10.1103/PhysRevResearch.6.013129).
- [93] Daniel Gottesman, Alexei Kitaev, and John Preskill. “Encoding a qubit in an oscillator”. In: *Phys. Rev. A* 64 (1 June 2001), p. 012310. DOI: [10.1103/PhysRevA.64.012310](https://doi.org/10.1103/PhysRevA.64.012310).
- [94] Élie Gouzien et al. “Performance Analysis of a Repetition Cat Code Architecture: Computing 256-bit Elliptic Curve Logarithm in 9 Hours with 126133 Cat Qubits”. In: *Phys. Rev. Lett.* 131 (2023), p. 040602. DOI: [10.1103/PhysRevLett.131.040602](https://doi.org/10.1103/PhysRevLett.131.040602). arXiv: [2302.06639](https://arxiv.org/abs/2302.06639).
- [95] T. M. Graham et al. “Midcircuit Measurements on a Single-Species Neutral Alkali Atom Quantum Processor”. In: *Phys. Rev. X* 13 (4 Dec. 2023), p. 041051. DOI: [10.1103/PhysRevX.13.041051](https://doi.org/10.1103/PhysRevX.13.041051).
- [96] T. M. Graham et al. “Multi-qubit entanglement and algorithms on a neutral-atom quantum computer”. In: *Nature* 604.7906 (Apr. 2022), pp. 457–462. ISSN: 1476-4687. DOI: [10.1038/s41586-022-04603-6](https://doi.org/10.1038/s41586-022-04603-6).
- [97] S. Gras and M. Evans. “Direct measurement of coating thermal noise in optical resonators”. In: *Phys. Rev. D* 98 (12 Dec. 2018), p. 122001. DOI: [10.1103/PhysRevD.98.122001](https://doi.org/10.1103/PhysRevD.98.122001).
- [98] A. Grimm et al. “Stabilization and operation of a Kerr-cat qubit”. In: *Nature* 584.7820 (Aug. 2020), pp. 205–209. ISSN: 0028-0836, 1476-4687. DOI: [10.1038/s41586-020-2587-z](https://doi.org/10.1038/s41586-020-2587-z).
- [99] S. Gröblacher et al. “Observation of non-Markovian micromechanical Brownian motion”. In: *Nature Communications* 6.1 (July 2015), p. 7606. ISSN: 2041-1723. DOI: [10.1038/ncomms8606](https://doi.org/10.1038/ncomms8606).

- [100] Simon Gröblacher et al. “Demonstration of an ultracold micro-optomechanical oscillator in a cryogenic cavity”. In: *Nature Physics* 5.7 (July 2009), pp. 485–488. ISSN: 1745-2481. DOI: [10.1038/nphys1301](https://doi.org/10.1038/nphys1301).
- [101] Jérémie Guillaud, Joachim Cohen, and Mazyar Mirrahimi. “Quantum computation with cat qubits”. In: *SciPost Phys. Lect. Notes* (2023), p. 72. DOI: [10.21468/SciPostPhysLectNotes.72](https://doi.org/10.21468/SciPostPhysLectNotes.72). arXiv: [2203.03222](https://arxiv.org/abs/2203.03222).
- [102] Jérémie Guillaud and Mazyar Mirrahimi. “Repetition Cat Qubits for Fault-Tolerant Quantum Computation”. In: *Phys. Rev. X* 9 (2019), p. 041053. DOI: [10.1103/PhysRevX.9.041053](https://doi.org/10.1103/PhysRevX.9.041053). arXiv: [1904.09474](https://arxiv.org/abs/1904.09474).
- [103] C. Gut et al. “Stationary optomechanical entanglement between a mechanical oscillator and its measurement apparatus”. In: *Physical Review Research* 2.3 (Aug. 2020), p. 033244. ISSN: 2643-1564. DOI: [10.1103/PhysRevResearch.2.033244](https://doi.org/10.1103/PhysRevResearch.2.033244).
- [104] Corentin Gut. “Stationary optomechanical entanglement detection”. PhD thesis. Vienna, Austria: University of Vienna, 2024. DOI: [10.25365/thesis.77115](https://doi.org/10.25365/thesis.77115).
- [105] G D Hammond et al. “Reducing the suspension thermal noise of advanced gravitational wave detectors”. In: *Classical and Quantum Gravity* 29.12 (June 2012), p. 124009. ISSN: 1361-6382. DOI: [10.1088/0264-9381/29/12/124009](https://doi.org/10.1088/0264-9381/29/12/124009).
- [106] Connor T. Hann et al. “Hybrid Cat-Transmon Architecture for Scalable, Hardware-Efficient Quantum Error Correction”. In: *PRX Quantum* 6.3 (July 2025), p. 030305. ISSN: 2691-3399. DOI: [10.1103/75x7-5ysv](https://doi.org/10.1103/75x7-5ysv).
- [107] Jan Harms and Conor M Mow-Lowry. “Suspension-thermal noise in spring–antispring systems for future gravitational-wave detectors”. In: *Classical and Quantum Gravity* 35.2 (Dec. 2017), p. 025008. ISSN: 1361-6382. DOI: [10.1088/1361-6382/aa9e28](https://doi.org/10.1088/1361-6382/aa9e28).
- [108] Gregory M. Harry et al. “Thermal noise from optical coatings in gravitational wave detectors”. In: *Appl. Opt.* 45.7 (Mar. 2006), pp. 1569–1574. DOI: [10.1364/AO.45.001569](https://doi.org/10.1364/AO.45.001569).
- [109] E. D. Herbschleb et al. “Ultra-long coherence times amongst room-temperature solid-state spins”. In: *Nature Communications* 10.1 (Aug. 2019), p. 3766. ISSN: 2041-1723. DOI: [10.1038/s41467-019-11776-8](https://doi.org/10.1038/s41467-019-11776-8).
- [110] B. L. Higgins et al. “Demonstrating Heisenberg-limited unambiguous phase estimation without adaptive measurements”. In: *New Journal of Physics* 11.7 (July 2009), p. 073023. ISSN: 1367-2630. DOI: [10.1088/1367-2630/11/7/073023](https://doi.org/10.1088/1367-2630/11/7/073023).
- [111] Timo Hillmann and Fernando Quijandría. “Quantum error correction with dissipatively stabilized squeezed-cat qubits”. In: *Phys. Rev. A* 107 (2023), p. 032423. DOI: [10.1103/PhysRevA.107.032423](https://doi.org/10.1103/PhysRevA.107.032423). arXiv: [2210.13359](https://arxiv.org/abs/2210.13359).

- [112] Melvyn Ho et al. “Witnessing Optomechanical Entanglement with Photon Counting”. In: *Phys. Rev. Lett.* 121 (2 July 2018), p. 023602. DOI: [10.1103/PhysRevLett.121.023602](https://doi.org/10.1103/PhysRevLett.121.023602).
- [113] Jason Hoelscher. “Generation and Detection of Quantum Entanglement in Optomechanical Systems”. PhD thesis. University of Vienna, 2017.
- [114] Sebastian G. Hofer and Klemens Hammerer. “Entanglement-enhanced time-continuous quantum control in optomechanics”. In: *Physical Review A* 91.3 (Mar. 2015), p. 033822. ISSN: 1050-2947, 1094-1622. DOI: [10.1103/PhysRevA.91.033822](https://doi.org/10.1103/PhysRevA.91.033822).
- [115] Sebastian G. Hofer et al. “Quantum entanglement and teleportation in pulsed cavity optomechanics”. In: *Phys. Rev. A* 84 (5 Nov. 2011), p. 052327. DOI: [10.1103/PhysRevA.84.052327](https://doi.org/10.1103/PhysRevA.84.052327).
- [116] A. S. Holevo. “Covariant measurements and imprimitivity systems”. In: *Quantum Probability and Applications to the Quantum Theory of Irreversible Processes*. Ed. by Luigi Accardi, Alberto Frigerio, and Vittorio Gorini. Berlin, Heidelberg: Springer Berlin Heidelberg, 1984, pp. 153–172. ISBN: 978-3-540-38798-5.
- [117] Ting Hong et al. “Brownian thermal noise in multilayer coated mirrors”. In: *Phys. Rev. D* 87 (8 Apr. 2013), p. 082001. DOI: [10.1103/PhysRevD.87.082001](https://doi.org/10.1103/PhysRevD.87.082001).
- [118] Onur Hosten et al. “Measurement noise 100 times lower than the quantum-projection limit using entangled atoms”. In: *Nature* 529.7587 (Jan. 2016), pp. 505–508. ISSN: 1476-4687. DOI: [10.1038/nature16176](https://doi.org/10.1038/nature16176).
- [119] S. F. Huelga et al. “Improvement of Frequency Standards with Quantum Entanglement”. In: *Phys. Rev. Lett.* 79.20 (Nov. 1997), pp. 3865–3868. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.79.3865](https://doi.org/10.1103/PhysRevLett.79.3865).
- [120] Pieter Kok Hwang Lee and Jonathan P. Dowling. “A quantum Rosetta stone for interferometry”. In: *Journal of Modern Optics* 49.14-15 (2002), pp. 2325–2338. DOI: [10.1080/0950034021000011536](https://doi.org/10.1080/0950034021000011536).
- [121] Theodoros Ilias. “Biasing quantum trajectories for enhanced sensing”. In: *Physical Review A* 111.4 (Apr. 2025), p. 042432. ISSN: 2469-9926, 2469-9934. DOI: [10.1103/PhysRevA.111.042432](https://doi.org/10.1103/PhysRevA.111.042432).
- [122] Yonatan Israel, Shamir Rosen, and Yaron Silberberg. “Supersensitive Polarization Microscopy Using NOON States of Light”. In: *Phys. Rev. Lett.* 112.10 (Mar. 2014), p. 103604. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.112.103604](https://doi.org/10.1103/PhysRevLett.112.103604).
- [123] Vijay Jain et al. “Direct Measurement of Photon Recoil from a Levitated Nanoparticle”. In: *Phys. Rev. Lett.* 116 (24 June 2016), p. 243601. DOI: [10.1103/PhysRevLett.116.243601](https://doi.org/10.1103/PhysRevLett.116.243601).

- [124] Kamiel Janssens et al. “Correlated 0.01–40 Hz seismic and Newtonian noise and its impact on future gravitational-wave detectors”. In: *Phys. Rev. D* 109 (10 May 2024), p. 102002. doi: [10.1103/PhysRevD.109.102002](https://doi.org/10.1103/PhysRevD.109.102002).
- [125] Marcin Jarzyna and Rafał Demkowicz-Dobrzański. “True precision limits in quantum metrology”. In: *New Journal of Physics* 17.1 (Jan. 2015), p. 013010. issn: 1367-2630. doi: [10.1088/1367-2630/17/1/013010](https://doi.org/10.1088/1367-2630/17/1/013010).
- [126] Alec Jenkins et al. “Ytterbium Nuclear-Spin Qubits in an Optical Tweezer Array”. In: *Phys. Rev. X* 12 (2 May 2022), p. 021027. doi: [10.1103/PhysRevX.12.021027](https://doi.org/10.1103/PhysRevX.12.021027).
- [127] E. Joos and H. D. Zeh. “The emergence of classical properties through interaction with the environment”. In: *Zeitschrift für Physik B Condensed Matter* 59.2 (June 1985), pp. 223–243. issn: 1431-584X. doi: [10.1007/BF01725541](https://doi.org/10.1007/BF01725541).
- [128] Tomasz Kaftal and Rafał Demkowicz-Dobrzański. “Usefulness of an enhanced Kitaev phase-estimation algorithm in quantum metrology and computation”. In: *Phys. Rev. A* 90 (6 Dec. 2014), p. 062313. doi: [10.1103/PhysRevA.90.062313](https://doi.org/10.1103/PhysRevA.90.062313).
- [129] Dariusz Kajtoch and Emilia Witkowska. “Quantum dynamics generated by the two-axis countertwisting Hamiltonian”. In: *Phys. Rev. A* 92 (1 July 2015), p. 013623. doi: [10.1103/PhysRevA.92.013623](https://doi.org/10.1103/PhysRevA.92.013623).
- [130] N. G. van Kampen. *Stochastic Processes in Physics and Chemistry*. 3rd ed. Amsterdam: North-Holland, 2007.
- [131] Raphael Kaubruegger et al. “Quantum Variational Optimization of Ramsey Interferometry and Atomic Clocks”. In: *Physical Review X* 11.4 (Dec. 2021), p. 041045. issn: 2160-3308. doi: [10.1103/PhysRevX.11.041045](https://doi.org/10.1103/PhysRevX.11.041045).
- [132] Adam M. Kaufman and Kang-Kuen Ni. “Quantum science with optical tweezer arrays of ultracold atoms and molecules”. In: *Nature Physics* 17.12 (Dec. 2021), pp. 1324–1333. issn: 1745-2481. doi: [10.1038/s41567-021-01357-2](https://doi.org/10.1038/s41567-021-01357-2).
- [133] Steven M Kay. *Fundamentals of statistical signal processing: estimation theory*. Prentice-Hall, Inc., 1993.
- [134] E. M. Kessler et al. “Heisenberg-Limited Atom Clocks Based on Entangled Qubits”. In: *Phys. Rev. Lett.* 112 (19 May 2014), p. 190403. doi: [10.1103/PhysRevLett.112.190403](https://doi.org/10.1103/PhysRevLett.112.190403).
- [135] Timm Kielinski, Piet O Schmidt, and Klemens Hammerer. “GHZ protocols enhance frequency metrology despite spontaneous decay”. In: *Science Advances* 10.43 (2024), eadr1439.
- [136] D. Kienzler et al. “Quantum harmonic oscillator state synthesis by reservoir engineering”. In: *Science* 347 (2015), pp. 53–56. doi: [10.1126/science.1261033](https://doi.org/10.1126/science.1261033).

- [137] Gihwan Kim et al. *Fast Unconditional Reset and Leakage Reduction of a Tunable Superconducting Qubit via an Engineered Dissipative Bath*. arXiv:2411.02950 [quant-ph]. Nov. 2024. doi: [10.48550/arXiv.2411.02950](https://doi.org/10.48550/arXiv.2411.02950).
- [138] H. J. Kimble et al. “Conversion of conventional gravitational-wave interferometers into quantum nondemolition interferometers by modifying their input and/or output optics”. In: *Phys. Rev. D* 65 (2 Dec. 2001), p. 022002. doi: [10.1103/PhysRevD.65.022002](https://doi.org/10.1103/PhysRevD.65.022002).
- [139] Diederik P. Kingma and Jimmy Ba. “Adam: A Method for Stochastic Optimization”. In: *3rd International Conference on Learning Representations, ICLR 2015, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings*. Ed. by Yoshua Bengio and Yann LeCun. 2015.
- [140] Anastasia V. Kisil et al. “The Wiener–Hopf technique, its generalizations and applications: constructive and approximate methods”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 477.2254 (Oct. 2021). ISSN: 1471-2946. doi: [10.1098/rspa.2021.0533](https://doi.org/10.1098/rspa.2021.0533).
- [141] Masahiro Kitagawa and Masahito Ueda. “Squeezed spin states”. In: *Phys. Rev. A* 47 (6 June 1993), pp. 5138–5143. doi: [10.1103/PhysRevA.47.5138](https://doi.org/10.1103/PhysRevA.47.5138).
- [142] Emanuel Knill, Raymond Laflamme, and Lorenza Viola. “Theory of Quantum Error Correction for General Noise”. In: *Phys. Rev. Lett.* 84 (11 Mar. 2000), pp. 2525–2528. doi: [10.1103/PhysRevLett.84.2525](https://doi.org/10.1103/PhysRevLett.84.2525).
- [143] L. Knockaert. “The Barankin bound and threshold behavior in frequency estimation”. In: *IEEE Transactions on Signal Processing* 45.9 (Sept. 1997), pp. 2398–2401. ISSN: 1053587X. doi: [10.1109/78.622965](https://doi.org/10.1109/78.622965).
- [144] Jens Koch et al. “Charge-insensitive qubit design derived from the Cooper pair box”. In: *Phys. Rev. A* 76 (2007), p. 042319. doi: [10.1103/PhysRevA.76.042319](https://doi.org/10.1103/PhysRevA.76.042319). arXiv: [cond-mat/0703002](https://arxiv.org/abs/cond-mat/0703002).
- [145] Jan Kolodynski. *Precision bounds in noisy quantum metrology*. 2015. arXiv: [1409.0535 \[quant-ph\]](https://arxiv.org/abs/1409.0535).
- [146] Kentaro Komori et al. “Attonewton-meter torque sensing with a macroscopic optomechanical torsion pendulum”. In: *Phys. Rev. A* 101 (1 Jan. 2020), p. 011802. doi: [10.1103/PhysRevA.101.011802](https://doi.org/10.1103/PhysRevA.101.011802).
- [147] Shlomi Kotler et al. “Direct observation of deterministic macroscopic entanglement”. In: *Science* 372.6542 (May 2021), pp. 622–625. doi: [10.1126/science.abf2998](https://doi.org/10.1126/science.abf2998).
- [148] S. Kroker et al. “Brownian thermal noise in functional optical surfaces”. In: *Physical Review D* 96.2 (July 2017). ISSN: 2470-0029. doi: [10.1103/physrevd.96.022002](https://doi.org/10.1103/physrevd.96.022002).

- [149] Aleksander Kubica et al. “Erasure Qubits: Overcoming the T_1 Limit in Superconducting Circuits”. In: *Phys. Rev. X* 13 (4 Nov. 2023), p. 041022. DOI: [10.1103/PhysRevX.13.041022](https://doi.org/10.1103/PhysRevX.13.041022).
- [150] R Kubo. “The fluctuation-dissipation theorem”. In: *Reports on Progress in Physics* 29.1 (Jan. 1966), p. 255. DOI: [10.1088/0034-4885/29/1/306](https://doi.org/10.1088/0034-4885/29/1/306).
- [151] François-Marie Le Régent et al. “High-performance repetition cat code using fast noisy operations”. In: *Quantum* 7 (2023), p. 1198. DOI: [10.22331/q-2023-12-06-1198](https://doi.org/10.22331/q-2023-12-06-1198).
- [152] Z. Leghtas et al. “Confining the state of light to a quantum manifold by engineered two-photon loss”. In: *Science* 347.6224 (2015), p. 853. DOI: [10.1126/science.aaa2085](https://doi.org/10.1126/science.aaa2085).
- [153] Yink Loong Len et al. “Quantum metrology with imperfect measurements”. In: *Nature Communications* 13.1 (Nov. 2022), p. 6971. ISSN: 2041-1723. DOI: [10.1038/s41467-022-33563-8](https://doi.org/10.1038/s41467-022-33563-8).
- [154] Ian D Leroux et al. “On-line estimation of local oscillator noise and optimisation of servo parameters in atomic clocks”. In: *Metrologia* 54.3 (Mar. 2017), p. 307. DOI: [10.1088/1681-7575/aa66e9](https://doi.org/10.1088/1681-7575/aa66e9).
- [155] Raphaël Lescanne et al. “Exponential suppression of bit-flips in a qubit encoded in an oscillator”. In: *Nature Physics* 16.5 (May 2020), pp. 509–513. ISSN: 1745-2473, 1745-2481. DOI: [10.1038/s41567-020-0824-x](https://doi.org/10.1038/s41567-020-0824-x).
- [156] Harry Levine et al. “High-Fidelity Control and Entanglement of Rydberg-Atom Qubits”. In: *Phys. Rev. Lett.* 121.12 (Sept. 2018), p. 123603. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.121.123603](https://doi.org/10.1103/PhysRevLett.121.123603).
- [157] Harry Levine et al. “Parallel Implementation of High-Fidelity Multiqubit Gates with Neutral Atoms”. In: *Phys. Rev. Lett.* 123 (17 Oct. 2019), p. 170503. DOI: [10.1103/PhysRevLett.123.170503](https://doi.org/10.1103/PhysRevLett.123.170503).
- [158] Haoya Li, Hongkang Ni, and Lexing Ying. “Adaptive low-depth quantum algorithms for robust multiple-phase estimation”. In: *Physical Review A* 108.6 (2023), p. 062408.
- [159] Joanna W. Lis et al. “Midcircuit Operations Using the omg Architecture in Neutral Atom Arrays”. In: *Phys. Rev. X* 13 (4 Nov. 2023), p. 041035. DOI: [10.1103/PhysRevX.13.041035](https://doi.org/10.1103/PhysRevX.13.041035).
- [160] Qi Liu et al. “Enhancing Dynamic Range of Sub-Standard-Quantum-Limit Measurements via Quantum Deamplification”. In: *Phys. Rev. Lett.* 135 (4 July 2025), p. 040801. DOI: [10.1103/25ds-9724](https://doi.org/10.1103/25ds-9724).
- [161] Yuk Tung Liu and Kip S. Thorne. “Thermoelastic noise and homogeneous thermal noise in finite sized gravitational-wave test masses”. In: *Phys. Rev. D* 62 (12 Nov. 2000), p. 122002. DOI: [10.1103/PhysRevD.62.122002](https://doi.org/10.1103/PhysRevD.62.122002).

- [162] Andrew D. Ludlow et al. “Optical atomic clocks”. In: *Rev. Mod. Phys.* 87 (2 June 2015), pp. 637–701. doi: [10.1103/RevModPhys.87.637](https://doi.org/10.1103/RevModPhys.87.637).
- [163] Max Ludwig, K. Hammerer, and Florian Marquardt. “Entanglement of mechanical oscillators coupled to a nonequilibrium environment”. In: *Phys. Rev. A* 82 (1 July 2010), p. 012333. doi: [10.1103/PhysRevA.82.012333](https://doi.org/10.1103/PhysRevA.82.012333).
- [164] Shuo Ma et al. “High-fidelity gates and mid-circuit erasure conversion in an atomic qubit”. In: *Nature* 622.7982 (Oct. 2023), pp. 279–284. issn: 1476-4687. doi: [10.1038/s41586-023-06438-1](https://doi.org/10.1038/s41586-023-06438-1).
- [165] Shuo Ma et al. “Universal Gate Operations on Nuclear Spin Qubits in an Optical Tweezer Array of ^{171}Yb Atoms”. In: *Phys. Rev. X* 12 (2 May 2022), p. 021028. doi: [10.1103/PhysRevX.12.021028](https://doi.org/10.1103/PhysRevX.12.021028).
- [166] Yiqiu Ma et al. “Proposal for gravitational-wave detection beyond the standard quantum limit through EPR entanglement”. In: *Nature Physics* 13.8 (Aug. 2017), pp. 776–780. issn: 1745-2473, 1745-2481. doi: [10.1038/nphys4118](https://doi.org/10.1038/nphys4118).
- [167] Katarzyna Macieszczak, Martin Fraas, and Rafał Demkowicz-Dobrzański. “Bayesian quantum frequency estimation in presence of collective dephasing”. In: *New Journal of Physics* 16.11 (Oct. 2014), p. 113002. issn: 1367-2630. doi: [10.1088/1367-2630/16/11/113002](https://doi.org/10.1088/1367-2630/16/11/113002).
- [168] Ivaylo S. Madjarov et al. “An Atomic-Array Optical Clock with Single-Atom Readout”. In: *Phys. Rev. X* 9 (4 Dec. 2019), p. 041052. doi: [10.1103/PhysRevX.9.041052](https://doi.org/10.1103/PhysRevX.9.041052).
- [169] Christian D Marciniak et al. “Optimal metrology with programmable quantum sensors”. In: *Nature* 603.7902 (2022), pp. 604–609. doi: [10.1038/s41586-022-04435-4](https://doi.org/10.1038/s41586-022-04435-4).
- [170] Paulina Marian and Tudor A. Marian. “Entanglement of Formation for an Arbitrary Two-Mode Gaussian State”. In: *Phys. Rev. Lett.* 101 (22 Nov. 2008), p. 220403. doi: [10.1103/PhysRevLett.101.220403](https://doi.org/10.1103/PhysRevLett.101.220403).
- [171] D. V. Martynov et al. “Sensitivity of the Advanced LIGO detectors at the beginning of gravitational wave astronomy”. In: *Phys. Rev. D* 93 (11 June 2016), p. 112004. doi: [10.1103/PhysRevD.93.112004](https://doi.org/10.1103/PhysRevD.93.112004).
- [172] L. McCuller et al. “Frequency-Dependent Squeezing for Advanced LIGO”. In: *Phys. Rev. Lett.* 124 (17 Apr. 2020), p. 171102. doi: [10.1103/PhysRevLett.124.171102](https://doi.org/10.1103/PhysRevLett.124.171102).
- [173] Chao Meng et al. “Measurement-based preparation of multimode mechanical states”. In: *Science Advances* 8.21 (2022), eabm7585. doi: [10.1126/sciadv.abm7585](https://doi.org/10.1126/sciadv.abm7585). eprint: <https://www.science.org/doi/pdf/10.1126/sciadv.abm7585>.

- [174] V. Meyer et al. “Experimental Demonstration of Entanglement-Enhanced Rotation Angle Estimation Using Trapped Ions”. In: *Phys. Rev. Lett.* 86.26 (June 2001), pp. 5870–5873. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.86.5870](https://doi.org/10.1103/PhysRevLett.86.5870).
- [175] Haixing Miao, Stefan Danilishin, and Yanbei Chen. “Universal quantum entanglement between an oscillator and continuous fields”. In: *Physical Review A* 81.5 (May 2010). Publisher: American Physical Society, p. 052307. DOI: [10.1103/PhysRevA.81.052307](https://doi.org/10.1103/PhysRevA.81.052307).
- [176] Haixing Miao et al. “Achieving ground state and enhancing optomechanical entanglement by recovering information”. In: *New Journal of Physics* 12.8 (Aug. 2010), p. 083032. DOI: [10.1088/1367-2630/12/8/083032](https://doi.org/10.1088/1367-2630/12/8/083032).
- [177] Haixing Miao et al. “Probing macroscopic quantum states with a sub-Heisenberg accuracy”. In: *Phys. Rev. A* 81 (1 Jan. 2010), p. 012114. DOI: [10.1103/PhysRevA.81.012114](https://doi.org/10.1103/PhysRevA.81.012114).
- [178] Marios H. Michael et al. “New Class of Quantum Error-Correcting Codes for a Bosonic Mode”. In: *Phys. Rev. X* 6 (2016), p. 031006. DOI: [10.1103/PhysRevX.6.031006](https://doi.org/10.1103/PhysRevX.6.031006).
- [179] Ofir Milul et al. “Superconducting Cavity Qubit with Tens of Milliseconds Single-Photon Coherence Time”. In: *PRX Quantum* 4.3 (2023). ISSN: 2691-3399. DOI: [10.1103/prxquantum.4.030336](https://doi.org/10.1103/prxquantum.4.030336).
- [180] Mohammad Mirhosseini et al. “Superconducting metamaterials for waveguide quantum electrodynamics”. In: *Nat. Commun.* 9 (2018), p. 3706. DOI: [10.1038/s41467-018-06142-z](https://doi.org/10.1038/s41467-018-06142-z). arXiv: [1802.01708](https://arxiv.org/abs/1802.01708).
- [181] Mazyar Mirrahimi et al. “Dynamically protected cat-qubits: a new paradigm for universal quantum computation”. In: *New Journal of Physics* 16.4 (Apr. 2014). arXiv:1312.2017 [quant-ph], p. 045014. ISSN: 1367-2630. DOI: [10.1088/1367-2630/16/4/045014](https://doi.org/10.1088/1367-2630/16/4/045014).
- [182] Gary J Mooney et al. “Generation and verification of 27-qubit Greenberger-Horne-Zeilinger states in a superconducting quantum computer”. In: *Journal of Physics Communications* 5.9 (Sept. 2021), p. 095004. DOI: [10.1088/2399-6528/ac1df7](https://doi.org/10.1088/2399-6528/ac1df7).
- [183] Helge Müller-Ebhardt et al. “Entanglement of Macroscopic Test Masses and the Standard Quantum Limit in Laser Interferometry”. In: *Phys. Rev. Lett.* 100 (1 Jan. 2008), p. 013601. DOI: [10.1103/PhysRevLett.100.013601](https://doi.org/10.1103/PhysRevLett.100.013601).
- [184] Helge Müller-Ebhardt et al. “Quantum-state preparation and macroscopic entanglement in gravitational-wave detectors”. In: *Physical Review A* 80.4 (Oct. 2009). DOI: [10.1103/physreva.80.043802](https://doi.org/10.1103/physreva.80.043802).
- [185] J. A. Muniz et al. *High-fidelity universal gates in the ^{171}Yb ground state nuclear spin qubit*. 2024. arXiv: [2411.11708](https://arxiv.org/abs/2411.11708) [quant-ph].

- [186] K. W. Murch et al. “Cavity-Assisted Quantum Bath Engineering”. In: *Phys. Rev. Lett.* 109 (2012), p. 183602. DOI: [10.1103/PhysRevLett.109.183602](https://doi.org/10.1103/PhysRevLett.109.183602). arXiv: [1207.0053](https://arxiv.org/abs/1207.0053).
- [187] Tomohisa Nagata et al. “Beating the Standard Quantum Limit with Four-Entangled Photons”. In: *Science* 316.5825 (May 2007), pp. 726–729. ISSN: 0036-8075, 1095-9203. DOI: [10.1126/science.1138007](https://doi.org/10.1126/science.1138007).
- [188] Magued B. Nasr et al. “Demonstration of Dispersion-Canceled Quantum-Optical Coherence Tomography”. In: *Phys. Rev. Lett.* 91.8 (Aug. 2003), p. 083601. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.91.083601](https://doi.org/10.1103/PhysRevLett.91.083601).
- [189] Abraham R Neben et al. “Structural thermal noise in gram-scale mirror oscillators”. In: *New Journal of Physics* 14.11 (Nov. 2012), p. 115008. DOI: [10.1088/1367-2630/14/11/115008](https://doi.org/10.1088/1367-2630/14/11/115008).
- [190] Hongkang Ni, Haoya Li, and Lexing Ying. “On low-depth algorithms for quantum phase estimation”. In: *Quantum* 7 (2023), p. 1165.
- [191] Ramon Moghadas Nia. “Multimode optomechanics in the strong cooperativity regime : towards optomechanical entanglement with micromechanical membranes”. PhD thesis. University of Vienna, 2018.
- [192] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Vol. 2. Cambridge university press Cambridge, 2001.
- [193] M. A. Norcia et al. “Midcircuit Qubit Measurement and Rearrangement in a ^{171}Yb Atomic Array”. In: *Phys. Rev. X* 13 (4 Nov. 2023), p. 041034. DOI: [10.1103/PhysRevX.13.041034](https://doi.org/10.1103/PhysRevX.13.041034).
- [194] Matthew A. Norcia et al. “Seconds-scale coherence on an optical clock transition in a tweezer array”. In: *Science* 366.6461 (2019), pp. 93–97. DOI: [10.1126/science.aay0644](https://doi.org/10.1126/science.aay0644). eprint: <https://www.science.org/doi/pdf/10.1126/science.aay0644>.
- [195] Caspar F. Ockeloen et al. “Quantum Metrology with a Scanning Probe Atom Interferometer”. In: *Phys. Rev. Lett.* 111.14 (Oct. 2013), p. 143001. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.111.143001](https://doi.org/10.1103/PhysRevLett.111.143001).
- [196] C. F. Ockeloen-Korppi et al. “Stabilized entanglement of massive mechanical oscillators”. In: *Nature* 556.7702 (Apr. 2018), pp. 478–482. ISSN: 1476-4687. DOI: [10.1038/s41586-018-0038-x](https://doi.org/10.1038/s41586-018-0038-x).
- [197] Eric Oelker et al. “Demonstration of 4.8×10^{-17} stability at 1 s for two independent optical clocks”. In: *Nature Photonics* 13.10 (2019), pp. 714–719.
- [198] Yingkai Ouyang. “Robust projective measurements through measuring code-inspired observables”. In: *npj Quantum Information* 10.1 (2024), p. 104.

- [199] T. A. Palomaki et al. “Entangling Mechanical Motion with Microwave Fields”. In: *Science* 342.6159 (2013), pp. 710–713. doi: [10.1126/science.1244563](https://doi.org/10.1126/science.1244563). eprint: <https://www.science.org/doi/pdf/10.1126/science.1244563>.
- [200] Shengshi Pang and Andrew N. Jordan. “Optimal adaptive control for quantum metrology with time-dependent Hamiltonians”. In: *Nature Communications* 8.1 (Mar. 2017). Publisher: Springer Science and Business Media LLC. ISSN: 2041-1723. doi: [10.1038/ncomms14695](https://doi.org/10.1038/ncomms14695).
- [201] M. Paternostro et al. “Creating and Probing Multipartite Macroscopic Entanglement with Light”. In: *Phys. Rev. Lett.* 99 (25 Dec. 2007), p. 250401. doi: [10.1103/PhysRevLett.99.250401](https://doi.org/10.1103/PhysRevLett.99.250401).
- [202] Asher Peres. “Separability Criterion for Density Matrices”. In: *Physical Review Letters* 77.8 (Aug. 1996), pp. 1413–1415. ISSN: 0031-9007, 1079-7114. doi: [10.1103/PhysRevLett.77.1413](https://doi.org/10.1103/PhysRevLett.77.1413).
- [203] Luca Pezzè and Augusto Smerzi. “Heisenberg-Limited Noisy Atomic Clock Using a Hybrid Coherent and Squeezed State Protocol”. In: *Phys. Rev. Lett.* 125 (21 Nov. 2020), p. 210503. doi: [10.1103/PhysRevLett.125.210503](https://doi.org/10.1103/PhysRevLett.125.210503).
- [204] Luca Pezzè et al. “Quantum metrology with nonclassical states of atomic ensembles”. In: *Reviews of Modern Physics* 90.3 (Sept. 2018), p. 035005. ISSN: 0034-6861, 1539-0756. doi: [10.1103/RevModPhys.90.035005](https://doi.org/10.1103/RevModPhys.90.035005).
- [205] Matthias Pfender et al. “High-resolution spectroscopy of single nuclear spins via sequential weak measurements”. In: *Nature Communications* 10.1 (Feb. 2019). Publisher: Springer Science and Business Media LLC. ISSN: 2041-1723. doi: [10.1038/s41467-019-08544-z](https://doi.org/10.1038/s41467-019-08544-z).
- [206] Stefano Pirandola et al. “Macroscopic Entanglement by Entanglement Swapping”. In: *Phys. Rev. Lett.* 97 (15 Oct. 2006), p. 150403. doi: [10.1103/PhysRevLett.97.150403](https://doi.org/10.1103/PhysRevLett.97.150403).
- [207] Alexander P. M. Place et al. “New material platform for superconducting transmon qubits with coherence times exceeding 0.3 milliseconds”. In: *Nature Communications* 12.1 (Mar. 2021), p. 1779. ISSN: 2041-1723. doi: [10.1038/s41467-021-22030-5](https://doi.org/10.1038/s41467-021-22030-5).
- [208] M. B. Plenio. “Logarithmic Negativity: A Full Entanglement Monotone That is not Convex”. In: *Phys. Rev. Lett.* 95 (9 Aug. 2005), p. 090503. doi: [10.1103/PhysRevLett.95.090503](https://doi.org/10.1103/PhysRevLett.95.090503).
- [209] J. F. Poyatos, J. I. Cirac, and P. Zoller. “Quantum Reservoir Engineering with Laser Cooled Trapped Ions”. In: *Phys. Rev. Lett.* 77 (1996), pp. 4728–4731. doi: [10.1103/PhysRevLett.77.4728](https://doi.org/10.1103/PhysRevLett.77.4728).
- [210] John Preskill. “Beyond NISQ: The Megaquop Machine”. In: *ACM Transactions on Quantum Computing* 6.3 (Apr. 2025), pp. 1–7. ISSN: 2643-6817. doi: [10.1145/3723153](https://doi.org/10.1145/3723153).

- [211] Shruti Puri, Samuel Boutin, and Alexandre Blais. “Engineering the quantum states of light in a Kerr-nonlinear resonator by two-photon driving”. In: *npj Quantum Information* 3.1 (Apr. 2017), p. 18. ISSN: 2056-6387. DOI: [10.1038/s41534-017-0019-1](https://doi.org/10.1038/s41534-017-0019-1).
- [212] Shruti Puri et al. “Bias-preserving gates with stabilized cat qubits”. In: *Science Advances* 6.34 (Aug. 2020), eaay5901. ISSN: 2375-2548. DOI: [10.1126/sciadv.aay5901](https://doi.org/10.1126/sciadv.aay5901).
- [213] Harald Putterman, Kyungjoo Noh, Connor T. Hann, et al. “Hardware-efficient quantum error correction via concatenated bosonic qubits”. In: *Nature* 638.8052 (Feb. 2025), pp. 927–934. ISSN: 0028-0836, 1476-4687. DOI: [10.1038/s41586-025-08642-7](https://doi.org/10.1038/s41586-025-08642-7).
- [214] Harald Putterman et al. “Preserving Phase Coherence and Linearity in Cat Qubits with Exponential Bit-Flip Suppression”. In: *Physical Review X* 15.1 (Mar. 2025), p. 011070. ISSN: 2160-3308. DOI: [10.1103/PhysRevX.15.011070](https://doi.org/10.1103/PhysRevX.15.011070).
- [215] C Radhakrishna Rao. *Selected papers of C. R. Rao*. Ed. by S Gupta et al. Selected Papers of C. R. Rao. Nashville, TN: John Wiley & Sons, Mar. 1995.
- [216] Matthew Reagor et al. “Reaching 10 ms single photon lifetimes for superconducting aluminum cavities”. In: *Applied Physics Letters* 102.19 (May 2013). ISSN: 1077-3118. DOI: [10.1063/1.4807015](https://doi.org/10.1063/1.4807015).
- [217] U. Réglade et al. “Quantum control of a cat qubit with bit-flip times exceeding ten seconds”. In: *Nature* 629 (2024), pp. 778–783. DOI: [10.1038/s41586-024-07294-3](https://doi.org/10.1038/s41586-024-07294-3).
- [218] Florentin Reiter and Anders S. Sørensen. “Effective operator formalism for open quantum systems”. In: *Physical Review A* 85.3 (Mar. 2012). arXiv:1112.2806 [quant-ph], p. 032111. ISSN: 1050-2947, 1094-1622. DOI: [10.1103/PhysRevA.85.032111](https://doi.org/10.1103/PhysRevA.85.032111).
- [219] Ralf Riedinger et al. “Remote quantum entanglement between two micromechanical oscillators”. In: *Nature* 556.7702 (Apr. 2018), pp. 473–477. ISSN: 1476-4687. DOI: [10.1038/s41586-018-0036-z](https://doi.org/10.1038/s41586-018-0036-z).
- [220] D. Rife and R. Boorstyn. “Single tone parameter estimation from discrete-time observations”. In: *IEEE Transactions on Information Theory* 20.5 (Sept. 1974), pp. 591–598. ISSN: 0018-9448. DOI: [10.1109/TIT.1974.1055282](https://doi.org/10.1109/TIT.1974.1055282).
- [221] John M. Robinson et al. “Direct comparison of two spin-squeezed optical clock ensembles at the 10^{-17} level”. In: *Nature Physics* 20.2 (Feb. 2024), pp. 208–213. ISSN: 1745-2481. DOI: [10.1038/s41567-023-02310-1](https://doi.org/10.1038/s41567-023-02310-1).
- [222] Jameson Graef Rollins et al. *pygwinc: Gravitational Wave Interferometer Noise Calculator*. Astrophysics Source Code Library, record ascl:2007.020. July 2020. ascl: [2007.020](https://ascl.net/2007.020).

- [223] Oriol Romero-Isart. “Quantum superposition of massive objects and collapse models”. In: *Phys. Rev. A* 84 (5 Nov. 2011), p. 052121. DOI: [10.1103/PhysRevA.84.052121](https://doi.org/10.1103/PhysRevA.84.052121).
- [224] T. Rosenband and D. R. Leibbrandt. *Exponential scaling of clock stability with atom number*. 2013. arXiv: [1303.6357 \[quant-ph\]](https://arxiv.org/abs/1303.6357).
- [225] Matteo A. C. Rossi et al. “Noisy Quantum Metrology Enhanced by Continuous Nondemolition Measurement”. In: *Physical Review Letters* 125.20 (Nov. 2020). Publisher: American Physical Society (APS). ISSN: 0031-9007, 1079-7114. DOI: [10.1103/physrevlett.125.200505](https://doi.org/10.1103/physrevlett.125.200505).
- [226] Leonardo Salvi et al. “Squeezing on Momentum States for Atom Interferometry”. In: *Phys. Rev. Lett.* 120.3 (Jan. 2018), p. 033601. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.120.033601](https://doi.org/10.1103/PhysRevLett.120.033601).
- [227] Peter R. Saulson. “Thermal noise in mechanical experiments”. In: *Phys. Rev. D* 42 (8 Oct. 1990), pp. 2437–2445. DOI: [10.1103/PhysRevD.42.2437](https://doi.org/10.1103/PhysRevD.42.2437).
- [228] David S. Schlegel, Fabrizio Minganti, and Vincenzo Savona. “Quantum error correction using squeezed Schrödinger cat states”. In: *Phys. Rev. A* 106 (2022), p. 022431. DOI: [10.1103/PhysRevA.106.022431](https://doi.org/10.1103/PhysRevA.106.022431). arXiv: [2201.02570](https://arxiv.org/abs/2201.02570).
- [229] Maximilian Schlosshauer. “Quantum decoherence”. In: *Physics Reports* 831 (2019). Quantum decoherence, pp. 1–57. ISSN: 0370-1573. DOI: <https://doi.org/10.1016/j.physrep.2019.10.001>.
- [230] Maximilian Schlosshauer-Selbach. “Decoherence and the quantum-to-classical transition”. In: 2008.
- [231] Simon Schmitt et al. “Optimal frequency measurements with quantum probes”. In: *npj Quantum Information* 7.1 (Apr. 2021), p. 55. ISSN: 2056-6387. DOI: [10.1038/s41534-021-00391-5](https://doi.org/10.1038/s41534-021-00391-5).
- [232] Simon Schmitt et al. “Submillihertz magnetic spectroscopy performed with a nanoscale quantum sensor”. In: *Science* 356.6340 (May 2017), pp. 832–837. ISSN: 0036-8075, 1095-9203. DOI: [10.1126/science.aam5532](https://doi.org/10.1126/science.aam5532).
- [233] Roman Schnabel. “Einstein-Podolsky-Rosen–entangled motion of two massive objects”. In: *Phys. Rev. A* 92 (1 July 2015), p. 012126. DOI: [10.1103/PhysRevA.92.012126](https://doi.org/10.1103/PhysRevA.92.012126).
- [234] Pascal Scholl et al. “Erasure conversion in a high-fidelity Rydberg quantum simulator”. In: *Nature* 622.7982 (Oct. 2023), pp. 273–278. ISSN: 1476-4687. DOI: [10.1038/s41586-023-06516-4](https://doi.org/10.1038/s41586-023-06516-4).
- [235] Marius Schulte et al. “Prospects and challenges for squeezing-enhanced optical atomic clocks”. In: *Nature Communications* 11.1 (Nov. 2020). ISSN: 2041-1723. DOI: [10.1038/s41467-020-19403-7](https://doi.org/10.1038/s41467-020-19403-7).

- [236] Bonny L. Schumaker and Carlton M. Caves. “New formalism for two-photon quantum optics. II. Mathematical foundation and compact notation”. In: *Phys. Rev. A* 31 (5 May 1985), pp. 3093–3111. doi: [10.1103/PhysRevA.31.3093](https://doi.org/10.1103/PhysRevA.31.3093).
- [237] Pavel Sekatski et al. “Quantum metrology with full and fast quantum control”. In: *Quantum* 1 (2017), p. 27. issn: 2521-327X. doi: [10.22331/q-2017-09-06-27](https://doi.org/10.22331/q-2017-09-06-27).
- [238] Alessio Serafini. *Quantum Continuous Variables*. CRC Press, July 2017. doi: [10.1201/9781315118727](https://doi.org/10.1201/9781315118727).
- [239] Alessio Serafini. *Quantum continuous variables: a primer of theoretical methods*. Boca Raton: CRC Press, Taylor & Francis Group, CRC Press is an imprint of the Taylor & Francis Group, an informa business, 2017. isbn: 978-1-4822-4634-6.
- [240] Prasoon K. Shandilya et al. “Optomechanical interface between telecom photons and spin quantum memory”. In: *Nature Physics* 17.12 (Dec. 2021), pp. 1420–1425. issn: 1745-2481. doi: [10.1038/s41567-021-01364-3](https://doi.org/10.1038/s41567-021-01364-3).
- [241] Adam L. Shaw et al. “Multi-ensemble metrology by programming local rotations with atom movements”. In: *Nature Physics* 20.2 (Feb. 2024), pp. 195–201. issn: 1745-2481. doi: [10.1038/s41567-023-02323-w](https://doi.org/10.1038/s41567-023-02323-w).
- [242] N Shiga and M Takeuchi. “Locking the local oscillator phase to the atomic phase via weak measurement”. In: *New Journal of Physics* 14.2 (Feb. 2012), p. 023034. issn: 1367-2630. doi: [10.1088/1367-2630/14/2/023034](https://doi.org/10.1088/1367-2630/14/2/023034).
- [243] R. Simon. “Peres-Horodecki Separability Criterion for Continuous Variable Systems”. In: *Physical Review Letters* 84.12 (Mar. 2000), pp. 2726–2729. issn: 0031-9007, 1079-7114. doi: [10.1103/PhysRevLett.84.2726](https://doi.org/10.1103/PhysRevLett.84.2726).
- [244] Kevin Singh et al. “Dual-Element, Two-Dimensional Atom Array with Continuous-Mode Operation”. In: *Phys. Rev. X* 12 (1 Mar. 2022), p. 011040. doi: [10.1103/PhysRevX.12.011040](https://doi.org/10.1103/PhysRevX.12.011040).
- [245] V. V. Sivak et al. “Recovering Quantum Coherence of a Cavity Qubit Coupled to a Noisy Ancilla through Real-Time Feedback”. In: *Phys. Rev. X* 14 (2024), p. 041056. doi: [10.1103/PhysRevX.14.041056](https://doi.org/10.1103/PhysRevX.14.041056).
- [246] Joseph G Smith, Crispin HW Barnes, and David RM Arvidsson-Shukur. “Adaptive Bayesian quantum algorithm for phase estimation”. In: *Physical Review A* 109.4 (2024), p. 042412.
- [247] Kentaro Somiya et al. “Detector configuration of KAGRA—the Japanese cryogenic gravitational-wave detector”. In: *Classical and Quantum Gravity* 29.12 (June 2012), p. 124007. doi: [10.1088/0264-9381/29/12/124007](https://doi.org/10.1088/0264-9381/29/12/124007).

- [248] A. Steinhardt and C. Bretherton. “Thresholds in frequency estimation”. In: *ICASSP '85. IEEE International Conference on Acoustics, Speech, and Signal Processing*. Vol. 10. Tampa, FL, USA: Institute of Electrical and Electronics Engineers, 1985, pp. 1273–1276. doi: [10.1109/ICASSP.1985.1168170](https://doi.org/10.1109/ICASSP.1985.1168170).
- [249] G.S. Summy and D.T. Pegg. “Phase optimized quantum states of light”. In: *Optics Communications* 77.1 (June 1990), pp. 75–79. issn: 00304018. doi: [10.1016/0030-4018\(90\)90464-5](https://doi.org/10.1016/0030-4018(90)90464-5).
- [250] Nathanan Tantivasadakarn et al. *Long-range entanglement from measuring symmetry-protected topological phases*. Jan. 2022.
- [251] Rodrigo A. Thomas et al. “Entanglement between distant macroscopic mechanical and spin systems”. In: *Nature Physics* 17.2 (Feb. 2021), pp. 228–233. issn: 1745-2481. doi: [10.1038/s41567-020-1031-5](https://doi.org/10.1038/s41567-020-1031-5).
- [252] Ted Thorbeck et al. *High-fidelity gates in a transmon using bath engineering for passive leakage reset*. arXiv:2411.04101 [quant-ph]. Nov. 2024. doi: [10.48550/arXiv.2411.04101](https://doi.org/10.48550/arXiv.2411.04101).
- [253] S. Touzard et al. “Coherent Oscillations inside a Quantum Manifold Stabilized by Dissipation”. In: *Phys. Rev. X* 8 (2018), p. 021005. doi: [10.1103/PhysRevX.8.021005](https://doi.org/10.1103/PhysRevX.8.021005). arXiv: [1705.02401](https://arxiv.org/abs/1705.02401).
- [254] Benedikt Tratzmiller et al. “Limited-control metrology approaching the heisenberg limit without entanglement preparation”. In: *Physical Review A* 101.3 (2020), p. 032347.
- [255] Lucia Trozzo and Francesca Badaracco. “Seismic and Newtonian Noise in the GW Detectors”. In: *Galaxies* 10.1 (2022). issn: 2075-4434. doi: [10.3390/galaxies10010020](https://doi.org/10.3390/galaxies10010020).
- [256] Mankei Tsang and Carlton M. Caves. “Evading Quantum Mechanics: Engineering a Classical Subsystem within a Quantum Environment”. In: *Phys. Rev. X* 2 (3 2012), p. 031016. doi: [10.1103/PhysRevX.2.031016](https://doi.org/10.1103/PhysRevX.2.031016).
- [257] David K. Tuckett, Stephen D. Bartlett, and Steven T. Flammia. “Ultrahigh Error Threshold for Surface Codes with Biased Noise”. In: *Phys. Rev. Lett.* 120 (2018), p. 050505. doi: [10.1103/PhysRevLett.120.050505](https://doi.org/10.1103/PhysRevLett.120.050505).
- [258] David K. Tuckett et al. “Fault-tolerant thresholds for the surface code in excess of 5% under biased noise”. In: *Phys. Rev. Lett.* 124 (2020), p. 130501. doi: [10.1103/PhysRevLett.124.130501](https://doi.org/10.1103/PhysRevLett.124.130501).
- [259] Harry L Van Trees. *Detection, estimation, and modulation theory, part I*. New York: Wiley-Interscience, Apr. 2004.
- [260] J. Verjauw et al. “Path toward manufacturable superconducting qubits with relaxation times exceeding 0.1 ms”. In: *npj Quantum Information* 8.1 (Aug. 2022), p. 93. issn: 2056-6387. doi: [10.1038/s41534-022-00600-9](https://doi.org/10.1038/s41534-022-00600-9).

- [261] G. Vidal and R. F. Werner. “Computable measure of entanglement”. In: *Physical Review A* 65.3 (Feb. 2002), p. 032314. ISSN: 1050-2947, 1094-1622. DOI: [10.1103/PhysRevA.65.032314](https://doi.org/10.1103/PhysRevA.65.032314).
- [262] D. Vitali et al. “Optomechanical Entanglement between a Movable Mirror and a Cavity Field”. In: *Phys. Rev. Lett.* 98 (3 Jan. 2007), p. 030405. DOI: [10.1103/PhysRevLett.98.030405](https://doi.org/10.1103/PhysRevLett.98.030405).
- [263] Mei Wang et al. “Macroscopic quantum entanglement in modulated optomechanics”. In: *Phys. Rev. A* 94 (5 Nov. 2016), p. 053807. DOI: [10.1103/PhysRevA.94.053807](https://doi.org/10.1103/PhysRevA.94.053807).
- [264] Zihao Wang et al. “High- E_J/E_C transmon qubits with up to 12 levels”. In: *Physical Review Applied* 23.3 (Mar. 2025), p. 034046. ISSN: 2331-7019. DOI: [10.1103/PhysRevApplied.23.034046](https://doi.org/10.1103/PhysRevApplied.23.034046).
- [265] Gudrun Wanner. “Space-based gravitational wave detection and how LISA Pathfinder successfully paved the way”. In: *Nature Physics* 15.3 (Mar. 2019), pp. 200–202. ISSN: 1745-2481. DOI: [10.1038/s41567-019-0462-3](https://doi.org/10.1038/s41567-019-0462-3).
- [266] P. Webster, S. D. Bartlett, and D. Poulin. “Reducing the overhead for quantum computation when noise is biased”. In: *Phys. Rev. A* 92 (2015), p. 062309. DOI: [10.1103/PhysRevA.92.062309](https://doi.org/10.1103/PhysRevA.92.062309).
- [267] Ulrich Weiss. *Quantum Dissipative Systems*. 4th. WORLD SCIENTIFIC, 2012. DOI: [10.1142/8334](https://doi.org/10.1142/8334). eprint: <https://www.worldscientific.com/doi/pdf/10.1142/8334>.
- [268] R. F. Werner and M. M. Wolf. “Bound Entangled Gaussian States”. In: *Physical Review Letters* 86.16 (Apr. 2001), pp. 3658–3661. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.86.3658](https://doi.org/10.1103/PhysRevLett.86.3658).
- [269] Howard M. Wiseman and Gerard J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2009.
- [270] Qian Xu et al. “Autonomous quantum error correction and fault-tolerant quantum computation with squeezed cat qubits”. In: *npj Quantum Inf.* 9 (2023), p. 78. DOI: [10.1038/s41534-023-00746-0](https://doi.org/10.1038/s41534-023-00746-0).
- [271] Qian Xu et al. “Engineering fast bias-preserving gates on stabilized cat qubits”. In: *Physical Review Research* 4.1 (Feb. 2022), p. 013082. ISSN: 2643-1564. DOI: [10.1103/PhysRevResearch.4.013082](https://doi.org/10.1103/PhysRevResearch.4.013082).
- [272] Fei Yan et al. “Distinguishing Coherent and Thermal Photon Noise in a Circuit Quantum Electrodynamical System”. In: *Phys. Rev. Lett.* 120.26 (June 2018), p. 260504. ISSN: 0031-9007, 1079-7114. DOI: [10.1103/PhysRevLett.120.260504](https://doi.org/10.1103/PhysRevLett.120.260504).
- [273] Dayou Yang, Susana F Huelga, and Martin B Plenio. “Efficient information retrieval for sensing via continuous measurement”. In: *Physical Review X* 13.3 (2023), p. 031012.

- [274] Huan Yang et al. “Macroscopic Quantum Mechanics in a Classical Space-time”. In: *Phys. Rev. Lett.* 110 (17 Apr. 2013), p. 170401. DOI: [10.1103/PhysRevLett.110.170401](https://doi.org/10.1103/PhysRevLett.110.170401).
- [275] Yaoling Yang, Victor Montenegro, and Abolfazl Bayat. “Extractable information capacity in sequential measurements metrology”. In: *Physical Review Research* 5.4 (2023), p. 043273.
- [276] Xiong Yao et al. “Long-Range Optomechanical Interactions in SiN Membrane Arrays”. In: *Phys. Rev. X* 15 (1 Jan. 2025), p. 011014. DOI: [10.1103/PhysRevX.15.011014](https://doi.org/10.1103/PhysRevX.15.011014).
- [277] Aaron W. Young et al. “Half-minute-scale atomic coherence and high relative stability in a tweezer clock”. In: *Nature* 588.7838 (Dec. 2020), pp. 408–413. ISSN: 1476-4687. DOI: [10.1038/s41586-020-3009-y](https://doi.org/10.1038/s41586-020-3009-y).
- [278] Haocun Yu et al. “Quantum correlations between light and the kilogram-mass mirrors of LIGO”. In: *Nature* 583.7814 (2020), pp. 43–47.
- [279] Sisi Zhou and Liang Jiang. “Asymptotic Theory of Quantum Channel Estimation”. In: *PRX Quantum* 2.1 (Mar. 2021). Publisher: American Physical Society (APS). ISSN: 2691-3399. DOI: [10.1103/prxquantum.2.010343](https://doi.org/10.1103/prxquantum.2.010343).
- [280] Sisi Zhou, Spyridon Michalakis, and Tuvia Gefen. “Optimal Protocols for Quantum Metrology with Noisy Measurements”. In: *PRX Quantum* 4 (4 Oct. 2023), p. 040305. DOI: [10.1103/PRXQuantum.4.040305](https://doi.org/10.1103/PRXQuantum.4.040305).
- [281] Sisi Zhou et al. “Achieving the Heisenberg limit in quantum metrology using quantum error correction”. In: *Nature Communications* 9.1 (Jan. 2018), p. 78. ISSN: 2041-1723. DOI: [10.1038/s41467-017-02510-3](https://doi.org/10.1038/s41467-017-02510-3).
- [282] Wojciech Hubert Zurek. “Decoherence, einselection, and the quantum origins of the classical”. In: *Rev. Mod. Phys.* 75 (3 May 2003), pp. 715–775. DOI: [10.1103/RevModPhys.75.715](https://doi.org/10.1103/RevModPhys.75.715).

Appendix A

SUPPLEMENTARY INFORMATION FOR CHAPTER III

A.1 The Wiener-Hopf method

Here, we first re-state the conventions used in dot products between operators and functions. Assume real functions of t , $f(t)$, $g(t)$, and a real operator $A(t - t')$, for $t, t' > 0$. First, let us write down the inner product between the functions f, g : assuming that the functions are column vectors, we have

$$\begin{aligned} f^T \cdot g &:= \int_0^\infty f(t) g(t) dt \\ &= \int_{-\infty}^\infty f(t) g(t) dt \quad \text{if } f(t) \text{ or } g(t) \text{ is causal.} \\ &= \int_{-\infty}^\infty f(\Omega)^\dagger g(\Omega) \frac{d\Omega}{2\pi} \end{aligned} \tag{A.1}$$

where T denotes the transpose operation such that f^T is a row vector, † denotes the complex conjugate operation, and $f(\Omega)$ is the Fourier transform of $f(t)$. Note that we used the Plancherel theorem in the last line. We use the following Fourier transform convention:

$$\begin{aligned} f(\Omega) &= \mathcal{F}\{f(t)\} = \int_{-\infty}^\infty dt f(t) e^{i\Omega t} \\ f(t) &= \mathcal{F}^{-1}\{f(\Omega)\} = \int_{-\infty}^\infty \frac{d\Omega}{2\pi} f(\Omega) e^{-i\Omega t} \end{aligned} \tag{A.2}$$

such that, for example,

$$f(\Omega) = \frac{1}{b + i(a - \Omega)}, \quad a > 0, b > 0 \implies f(t) = e^{-(ia+b)t} H(t), \tag{A.3}$$

where $H(t)$ is the Heaviside step function. Then, we see that the spectrum of a causal (anti-causal) function has poles located on the lower (upper)-half of the complex plane.

The dot product between the operator A and the function f is given as

$$\begin{aligned} A \cdot f &= h(t) := \int_0^\infty A(t' - t) f(t') dt', \quad t > 0 \\ f^T \cdot A &= (A^T \cdot f)^T = l(t) := \int_0^\infty A(t - t') f(t') dt', \quad t > 0 \end{aligned} \tag{A.4}$$

where $h(t)$ and $l(t)$ are column and row vectors, respectively. Then, we see that the transpose operation corresponds to a time inversion for operators. Furthermore, if $f(t)$ is a causal function of t (i.e. $f(t) = 0$ for $t < 0$), we can change the integration to be between $-\infty, \infty$, and can write $h(\Omega) = A(-\Omega)f(\Omega) = A(\Omega)^\dagger f(\Omega)$, and $l(\Omega) = A(\Omega)f(\Omega)$. Note that $A(-\Omega) = A(\Omega)^\dagger$ since A is a real operator.

Now, assume an operator M , and functions h, g , which are related to each other with

$$\int_0^\infty M(t-t')h(t')dt' = g(t), \quad t > 0 \quad (\text{A.5})$$

Eq. (A.5) implies that $h = M^{-1}g$. In Chapter 3, we want to find h given M and g , in order evaluate the characteristic polynomial (Eq. (3.13)). For this purpose, we use the Wiener-Hopf method (See Ref. [140] for a review). First, let us define $M(\Omega)$, $g(\Omega)$ as the spectrum of M and g , respectively, and assume that they are rational functions of Ω . Notice that Eq. (A.5) can be rewritten as

$$\int_{-\infty}^\infty dt' M(t-t')h(t')_+ dt' = g(t), \quad t > 0, \quad (\text{A.6})$$

where

$$f(t)_+ = [f(t)]_+ = \begin{cases} f(t) & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.7})$$

which implies that

$$\begin{aligned} \left[\int_{-\infty}^\infty dt' [M(t-t')h(t')_+ - g(t)] \right]_+ &= [s(t)]_+ = 0, \\ s(t) &:= \int_{-\infty}^\infty dt' [M(t-t')h(t')_+ - g(t)] \end{aligned} \quad (\text{A.8})$$

i.e. $s(t)$ is an anti-causal function of time. In the frequency domain, we have that

$$s(\Omega) = \sum_k \frac{r(\Omega_k, \sigma_k)}{(\Omega - \Omega_k)^{\sigma_k}} = s(\Omega)_+ + s(\Omega)_-, \quad (\text{A.9})$$

for any rational function $s(\Omega)$, where the sum is the partial fraction decomposition of the spectrum, and it is over the poles Ω_k of $s(\Omega)$, with multiplicity σ_k . $r(\Omega_k, \sigma_k)$ is the residue at $\Omega = \Omega_k$. Furthermore, $s(\Omega)_+$ is given by

$$s(\Omega)_+ = \mathcal{F}\{s(t)_+\} = \sum_k \frac{r(\Omega_k, \sigma_k)}{(\Omega - \Omega_k)^{\sigma_k}} \quad \text{with } \text{Im}(\Omega_k) < 0 \quad (\text{A.10})$$

where $\text{Im}(\Omega_k)$ is the imaginary part of Ω_k . From Eq. (A.8), we know that

$$\begin{aligned} s(\Omega) &= M(\Omega)h(\Omega)_+ - g(\Omega) \\ \implies s(\Omega)_+ &= [M(\Omega)h(\Omega)_+ - g(\Omega)]_+ = 0. \end{aligned} \quad (\text{A.11})$$

Since $M(\Omega)$ is a rational function, we can write it as $M(\Omega) = M(\Omega)_+ M(\Omega)_-$, where $M(\Omega)_+$ ($M(\Omega)_-$) contains the causal (anti-causal) zeros and poles of $M(\Omega)$. Furthermore, $M(\Omega)_- = M(\Omega)_+^\dagger$ since $M(-t) = M(t)^\dagger$ in the Chapter 3. We have

$$\begin{aligned} & [M(\Omega)_+ M(\Omega)_- h(\Omega)_+ - g(\Omega)]_+ \\ &= \left[M(\Omega)_+ h(\Omega)_+ - \frac{g(\Omega)}{M(\Omega)_-} \right]_+ = 0 \end{aligned} \quad (\text{A.12})$$

where the second equality comes from the fact that multiplying the spectrum of an anti-causal function with a rational function with anti-causal roots and poles does not change the causality of the function. Therefore, we find

$$\begin{aligned} h(\Omega)_+ &= \frac{1}{M(\Omega)_+} \left[\frac{g(\Omega)}{M(\Omega)_-} \right]_+, \\ h(t) &= \mathcal{F}^{-1} \left\{ \frac{1}{M(\Omega)_+} \left[\frac{g(\Omega)}{M(\Omega)_-} \right]_+ \right\}, \quad t > 0. \end{aligned} \quad (\text{A.13})$$

A.2 Universal entanglement in the presence of frequency-independent squeezing

We consider two classes of symplectic transformations on the input light field, (i) phase shifts, and (ii) squeezing the amplitude and phase quadratures with a constant squeezing factor. We show that both of these transformations result in the same determinant condition for optomechanical entanglement computed for the case when the input light field consists of vacuum fluctuations, given in Eq. (3.29). First, let us consider phase shift operations. Such a transformation can be represented by

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow e^{i\phi(\Omega)} \begin{bmatrix} \cos \theta(\Omega) & -\sin \theta(\Omega) \\ \sin \theta(\Omega) & \cos \theta(\Omega) \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \quad (\text{A.14})$$

where $\theta(\Omega)$ is a real function of Ω , and the phase factor $e^{i\phi(\Omega)}$ ensures the causality of the transformation. We compute the light field sector of the covariance matrix in the frequency domain as

$$\begin{aligned} V_{11}^{vv}(\Omega) &= \cos^2 \theta(\Omega) + \sin^2 \theta(\Omega) = 1, \\ V_{12}^{vv}(\Omega) &= \cos^2 \theta(\Omega) \Omega_q^2 \chi(-\Omega) - \cos \theta(\Omega) \sin \theta(\Omega) \\ &\quad + \cos \theta(\Omega) \sin \theta(\Omega) + \sin^2 \theta(\Omega) \Omega_q^2 \chi(-\Omega) \\ &= \Omega_q^2 \chi(-\Omega), \\ V_{22}^{vv}(\Omega) &= |\cos(\theta(\Omega)) \Omega_q^2 \chi(\Omega) - \sin(\theta(\Omega))|^2 \\ &\quad + |\sin(\theta(\Omega)) \Omega_q^2 \chi(\Omega) + \cos(\theta(\Omega))|^2 + \Omega_q^2 \chi_{\text{EN}}(\Omega) \\ &= 1 + \Omega_q^2 \chi_{\text{EN}}(\Omega) + \Omega_q^4 |\chi(\Omega)|^2 \end{aligned} \quad (\text{A.15})$$

Then, we see that the light-field sector of the covariance matrix is unchanged with respect to the case where the input light field is the vacuum state. Similarly, we find that all of the elements of the covariance matrix are unchanged. Hence, the condition for the force and the sensing noise spectra for which the entangling-disentangling transition occurs is unaltered.

Now, we consider squeezing the quadratures of the light field with a constant squeezing factor. Such a transformation looks like the following

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \quad (\text{A.16})$$

where r is a real number. In order to utilize the formalism of Chapter 3, we can perform a symplectic transformation that simply rescales $\hat{v}_1(t)$ and $\hat{v}_2(t)$ as $\hat{v}_1(t) \rightarrow e^{-r} \hat{v}_1(t)$, $\hat{v}_2(t) \rightarrow e^r \hat{v}_2(t)$. Then, the light field sector of the covariance matrix is computed after the symplectic transformation as

$$\begin{aligned} V_{11}^{vv}(\Omega) &= 1, & V_{12}^{vv}(\Omega) &= e^{2r} \Omega_q^2 \chi(-\Omega), \\ V_{22}^{vv}(\Omega) &= 1 + e^{2r} \Omega_q^2 \chi_{\text{EN}}(\Omega) + e^{4r} \Omega_q^4 |\chi(\Omega)|^2 \end{aligned} \quad (\text{A.17})$$

The covariances of the quadratures of the mechanical oscillator are

$$\begin{aligned} V_{11}^{bb}(\Omega) &= \omega_m \chi_F(\Omega) + \omega_m \Omega_q^2 |\chi(\Omega)|^2 e^r, \\ V_{22}^{bb}(\Omega) &= \frac{\Omega^2}{\omega_m^2} V_{11}^{bb}(\Omega), \end{aligned} \quad (\text{A.18})$$

and $V_{12}^{bb}(0) = V_{21}^{bb}(0) = 0$ due to symmetry. Lastly, the correlations between the quadratures of the mechanical oscillator and the light field are computed as

$$\begin{aligned} V_{11}^{bv}(\Omega) &= \sqrt{\omega_m} e^r \Omega_q \chi(\Omega), \\ V_{12}^{bv}(\Omega) &= \sqrt{\omega_m} e^r \Omega_q \chi_F(\Omega) + \sqrt{\omega_m} e^{3r} \Omega_q^3 |\chi(\Omega)|^2, \end{aligned} \quad (\text{A.19})$$

and $V_{21}^{bv}(\Omega) = -\frac{i\Omega}{\omega_m} V_{11}^{bv}(\Omega)$, $V_{22}^{bv}(\Omega) = -\frac{i\Omega}{\omega_m} V_{12}^{bv}(\Omega)$. From Eqs. (A.17-A.19), we notice that the effect of the frequency independent squeezing is such that the interaction strength Ω_q is rescaled to $e^r \Omega_q$. Since we have proved the universality of the entangling-disentangling transition with respect to the interaction strength (see Section 3.5), rescaling it will not affect the transition. Therefore, we conclude that the condition for the force and the sensing noise spectra for which the entangling-disentangling transition occurs is unaltered if the input light field is squeezed by a constant squeezing factor.

A.3 General symplectic transformations on the light field

We want to write down an expression for a general, causal symplectic transformation applied to the light field. Such a transformation can be written down as a 2-by-2 matrix of operators $\mathbf{S}$, with

$$\begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} \rightarrow \mathbf{S} \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_{11}(\tau) & S_{12}(\tau) \\ S_{21}(\tau) & S_{22}(\tau) \end{bmatrix} \quad (\text{A.20})$$

where $S_{ij}(t)$ are causal functions of time, $i, j = 1, 2$. In order to be a valid symplectic transformation, $\mathbf{S}$ must satisfy [5]

$$\mathbf{S}^T \mathbf{K} \mathbf{S} = \mathbf{K}, \quad \mathbf{K} = \begin{bmatrix} 0 & \delta(\tau) \\ -\delta(\tau) & 0 \end{bmatrix} \quad (\text{A.21})$$

where $\mathbf{K}$ is the commutator matrix. Writing $\mathbf{S}^T \mathbf{K} \mathbf{S} = \mathbf{K}$ explicitly,

$$\begin{aligned} & \int_0^\infty dt' \begin{bmatrix} S_{11}(\tau_1)S_{21}(\tau_2) - S_{21}(\tau_1)S_{11}(\tau_2) & S_{11}(\tau_1)S_{22}(\tau_2) - S_{21}(\tau_1)S_{12}(\tau_2) \\ S_{12}(\tau_1)S_{21}(\tau_2) - S_{22}(\tau_1)S_{11}(\tau_2) & S_{12}(\tau_1)S_{22}(\tau_2) - S_{22}(\tau_1)S_{12}(\tau_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \delta(t - t'') \\ -\delta(t - t'') & 0 \end{bmatrix}, \end{aligned} \quad (\text{A.22})$$

with $\tau_1 := t' - t$, $\tau_2 := t'' - t'$. We can extend the lower limit of this integral to $-\infty$ due to the causality of $S_{ij}(t)$, $i, j = 1, 2$. Thus, we can express these constraints in the frequency domain, by defining $S_{ij}(\Omega) = \int_{-\infty}^\infty d\tau S_{ij}(\tau) e^{i\Omega\tau}$, $i, j = 1, 2$. We have:

$$S_{11}^\dagger(\Omega)S_{21}(\Omega), S_{12}^\dagger(\Omega)S_{22}(\Omega) \in \mathbb{R} \quad \forall \Omega \in \mathbb{R} \quad (\text{A.23a})$$

$$S_{11}^\dagger(\Omega)S_{22}(\Omega) - S_{12}^\dagger(\Omega)S_{21}(\Omega) = 1 \quad \forall \Omega \in \mathbb{R} \quad (\text{A.23b})$$

We realize that we can write down a symplectic transformation satisfying these constraints in the following manner,

$$\begin{aligned} \mathbf{S}(\Omega) &= e^{i\phi(\Omega)} \begin{bmatrix} A(\Omega) & B(\Omega) \\ C(\Omega) & D(\Omega) \end{bmatrix}, \\ A(\Omega), B(\Omega), C(\Omega), D(\Omega), \phi(\Omega) &\in \mathbb{R} \quad \forall \Omega, \\ A(\Omega)D(\Omega) - B(\Omega)C(\Omega) &= 1 \quad \forall \Omega \end{aligned} \quad (\text{A.24})$$

where the "phase" $\phi(\Omega)$ ensures causality by canceling out the anti-causal poles of $A(\Omega)$, $B(\Omega)$, $C(\Omega)$ and $D(\Omega)$. Furthermore, we can replace $A(\Omega)$, $B(\Omega)$, $C(\Omega)$ and

$D(\Omega)$ with other real functions of Ω such that

$$A(\Omega) = \cos \theta(\Omega) \cosh r(\Omega) + \cos \rho(\Omega) \sinh r(\Omega) \quad (\text{A.25a})$$

$$B(\Omega) = -\sin \theta(\Omega) \cosh r(\Omega) + \sin \rho(\Omega) \sinh r(\Omega) \quad (\text{A.25b})$$

$$C(\Omega) = \sin \theta(\Omega) \cosh r(\Omega) + \sin \rho(\Omega) \sinh r(\Omega) \quad (\text{A.25c})$$

$$D(\Omega) = \cos \theta(\Omega) \cosh r(\Omega) - \cos \rho(\Omega) \sinh r(\Omega) \quad (\text{A.25d})$$

where $r(\Omega), \theta(\Omega), \rho(\Omega) \in \mathbb{R} \forall \Omega$. It is straightforward to check that this form satisfies the constraints in Eq. (A.23).

A.4 Realization of frequency-dependent squeezing

We aim to find the parameters of a filter that squeezes the input light field such that the spectrum of the quantum noise in the output is squeezed by the same amount at every frequency, i.e. the shot noise and the quantum radiation pressure noise (QRPN) is squeezed simultaneously [138, 172]. First, we write down the input/output relations for the light field only,

$$\begin{bmatrix} \hat{v}_1(\Omega) \\ \hat{v}_2(\Omega) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Omega_q^2 \chi(\Omega) & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \quad (\text{A.26})$$

Assuming that the input light field is in the vacuum state, the spectrum of the quantum noise $S(\Omega)$ in the phase quadrature $\hat{v}_2$ is given by

$$S(\Omega) = 1 + \Omega_q^4 |\chi(\Omega)|^2 \quad (\text{A.27})$$

We observe that the first term contributes as a shot noise, whereas the second term is the QRPN since it is transduced by the mechanical oscillator. In order to realize frequency-dependent squeezing, we first squeeze the input light field in a frequency-independent manner (for example, with propagation through a bulk crystal), and filter it through a detuned Fabry-Perot cavity. Such an operation will have the following transfer function in the frequency domain

$$\frac{1}{\tilde{C}} \begin{bmatrix} \Omega^2 + \gamma^2 - \Delta^2 & -2\gamma\Delta \\ 2\gamma\Delta & \Omega^2 + \gamma^2 - \Delta^2 \end{bmatrix} \cdot \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix} \quad (\text{A.28})$$

where $\tilde{C} = \Delta^2 - (\Omega + i\gamma)^2$. γ and Δ are the cavity linewidth and detuning, respectively, and r is the squeezing factor, with $r, \gamma, \Delta \in \mathbb{R}$, $r, \gamma > 0$. Plugging this in Eq. (A.26),

assuming a high-Q oscillator (i.e. $\omega_m \gg \gamma_m$), $\hat{v}_2$ will be in the form of

$$\begin{aligned} \hat{v}_2 = & \frac{e^r}{\tilde{C}} \left(-2\gamma\Delta + \frac{\Omega_q^2 (\gamma^2 + \Omega^2 - \Delta^2)}{\Omega^2 - \omega_m^2} \right) \hat{u}_1 \\ & + \frac{e^{-r}}{\tilde{C}} \left(\Delta^2 - \gamma^2 - \Omega^2 - \frac{\Omega_q^2 2\gamma\Delta}{\Omega^2 - \omega_m^2} \right) \hat{u}_2 \end{aligned} \quad (\text{A.29})$$

where $\hat{u}_1, \hat{u}_2$ are the vacuum fluctuations. We notice that the coefficient multiplying the amplitude quadrature in Eq. (A.29) can be set to zero with an appropriate choice of γ, Δ . More specifically, if we set

$$\gamma = \sqrt{\frac{-\omega_m^2 + \sqrt{\omega_m^4 + \Omega_q^4}}{2}}, \quad \Delta = \frac{\Omega_q^2}{2\gamma}, \quad (\text{A.30})$$

we obtain that the spectrum of the quantum noise is modified as

$$S(\Omega) \rightarrow e^{-2r} S(\Omega) \quad (\text{A.31})$$

Therefore, the spectrum is squeezed with a fixed factor of e^{-2r} .

A.5 Passive losses

Here, we consider passive (detection) losses, affecting the readout of the output light field. Such losses occur due to processes such as the finite quantum efficiency of the photodetector. They can be modeled as [58]

$$\hat{v}'_i(t) = \sqrt{\eta} \hat{v}_i(t) + \sqrt{1-\eta} \hat{e}_i(t), \quad i = 1, 2, \quad (\text{A.32})$$

where $\hat{v}'_i(t)$ is the measured quadrature of the output light field, $\hat{e}_i(t)$ is an additional quantum noise that is assumed to be in the vacuum state, and $0 < \eta < 1$ is the efficiency of the photodetector. Then, $S_{e_i e_i} = 1, i = 1, 2$, and $S_{e_1 e_2} = 0$. We can rewrite the equations of motion in Eqs. (3.1) in terms of $\hat{v}'_{1,2}(t)$ as

$$\hat{v}'_1(t) = \sqrt{\eta} \hat{u}_1(t) + \sqrt{1-\eta} \hat{e}_1(t), \quad (\text{A.33a})$$

$$\begin{aligned} \hat{v}'_2(t) = & \sqrt{\eta} \left[\hat{u}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}} (\hat{b}_1(t) + \hat{n}_s(t)) \right] \\ & + \sqrt{1-\eta} \hat{e}_2(t), \end{aligned} \quad (\text{A.33b})$$

$$\dot{\hat{b}}_2(t) = -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q}{\sqrt{\omega_m}} \hat{u}_1(t) + \hat{n}_F(t), \quad (\text{A.33c})$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t), \quad (\text{A.33d})$$

If the ingoing light field is not frequency-dependently squeezed, such that $\hat{u}_{1,2}(t)$ denote vacuum fluctuations, we can combine the operators $\hat{u}_{1,2}(t)$ and $\hat{e}_{1,2}(t)$ in the form a white noise. In this way, Eqs. (A.33) will resemble the initial equations of motion in the absence of passive losses (Eqs. (3.1)). Defining $\hat{u}'_{1,2}(t) = \sqrt{\eta} \hat{u}_{1,2}(t) + \sqrt{1-\eta} \hat{e}_{1,2}(t)$, we observe that $\hat{u}'_{1,2}$ is a white noise with $S_{u'_i u'_i} = 1$, $i = 1, 2$, and $S_{u'_1 u'_2} = 0$, since $\hat{u}_{1,2}$ and $\hat{e}_{1,2}$ are independent noise sources. Making sure to preserve the dynamics of the oscillator in terms of the amount of driving noise, we will obtain

$$\hat{v}'_1(t) = \hat{u}'_1(t), \quad (\text{A.34a})$$

$$\hat{v}'_2(t) = \hat{u}'_2(t) + \frac{\Omega_q \sqrt{\eta}}{\sqrt{\omega_m}} (\hat{b}_1(t) + \hat{n}_S(t)), \quad (\text{A.34b})$$

$$\begin{aligned} \dot{\hat{b}}_2(t) = & -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q \sqrt{\eta}}{\sqrt{\omega_m}} \hat{u}'_1(t) \\ & + \hat{n}_F(t) + \frac{\Omega_q \sqrt{1-\eta}}{\sqrt{\omega_m}} \hat{\xi}(t), \end{aligned} \quad (\text{A.34c})$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t), \quad (\text{A.34d})$$

where $\hat{\xi}$ is an additional noise in the form of vacuum fluctuations. We notice that Ω_q is scaled by $\sqrt{\eta}$, and there appears to be an additional white force noise on the mechanical oscillator with a constant spectrum of $\Omega_q^2(1-\eta)/\omega_m$, due to the rescaling of Ω_q by η . Note that the total force noise on the oscillator is, as expected, unchanged. Due to the dependency of this additional force noise to the coherent coupling Ω_q , the entangling-disentangling transition is no longer universal with respect to Ω_q .

Then, we encounter two distinct limits: if $S_{\hat{n}_F}(\Omega) \gg \Omega_q^2(1-\eta)/\omega_m$, the entangling-disentangling transition is approximately unaffected. However, we expect the amount of entanglement in the system—if there exists any—to reduce because of the rescaling of Ω_q by η . On the other hand, if $S_{\hat{n}_F}(\Omega) \ll \Omega_q^2(1-\eta)/\omega_m$, it is more advantageous to have a small coupling Ω_q in order to achieve entanglement, as the additional force noise in the system is approximately proportional to Ω_q .

Appendix B

SUPPLEMENTARY INFORMATION FOR CHAPTER IV

B.1 Structure of the Covariance Matrix

The mirror and the cavity, at $t = 0$, constitute two modes, whereas there are an infinite number of modes in the outgoing light field given by the quadratures $\hat{v}_1(t), \hat{v}_2(t)$, $t \in (-\infty, 0)$. In continuum coordinates, we can first write down the commutator matrix,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^B & & \\ & \mathbf{K}^A & \\ & & \mathbf{K}^v \end{bmatrix}, \quad (\text{B.1})$$

with

$$\mathbf{K}^A = \mathbf{K}^B = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \mathbf{K}^v = \begin{bmatrix} 0 & i\delta(t-t') \\ -i\delta(t-t') & 0 \end{bmatrix}. \quad (\text{B.2})$$

Note that $\mathbf{K}^v$ is a 2×2 block matrix, but each block is infinite dimensional, with columns and rows indexed by t and t' , respectively. The indices t and t' each run through all negative real numbers, $(-\infty, 0]$. In order to represent these covariance matrices in a less ambiguous and more operational way, we shall adopt an index notation, in which j, k, l, m are discrete and run through 1 and 2, while t and t' are continuous and run through negative real numbers. We can then write

$$K_{jk}^A = i\epsilon_{jk}, \quad K_{lm}^B = i\epsilon_{lm}, \quad K_{lt,mt'}^v = i\epsilon_{lm}\delta(t-t'). \quad (\text{B.3})$$

Note that for $\mathbf{K}^v$ we have a two-dimensional row index (l, t) , as well as a two-dimensional column index (m, t') , to label quadrature and arrival time.

The covariance matrix $\mathbf{V}$ of the system contains the steady state correlations between the mechanical mode at $t = 0$, the cavity mode at $t = 0$, and the outgoing light field modes for $t < 0$. It can be written as

$$\mathbf{V} = \left[\begin{array}{cc|c} V^{BB} & V^{BA} & V^{Bv} \\ V^{AB} & V^{AA} & V^{Av} \\ \hline V^{vB} & V^{vA} & V^{vv} \end{array} \right] \equiv \left[\begin{array}{cc} V^{QQ} & V^{Qv} \\ V^{vQ} & V^{vv} \end{array} \right]. \quad (\text{B.4})$$

Here each of A and B represent two dimensions, while v represents an infinite number of dimensions. For computational purposes we can group A and B together

as Q , or

$$\hat{Q}_J = \hat{Q}_{(1,2,3,4)} = (\hat{B}_1, \hat{B}_2, \hat{A}_1, \hat{A}_2). \quad (\text{B.5})$$

Here we shall use upper-case Latin indices to run from 1 to 4, to label quadratures in the mechanical and cavity modes. We can write

$$V_{JK}^{QQ} = \frac{1}{2} \langle \hat{Q}_J \hat{Q}_K + \hat{Q}_K \hat{Q}_J \rangle = \int_0^\infty \frac{d\Omega}{2\pi} S_{Q_J Q_K}(\Omega), \quad (\text{B.6})$$

where the (one-sided) cross spectrum is defined via

$$S_{XY}(\Omega) \delta(\Omega - \Omega') = 2\pi \langle \hat{X}(\Omega) \hat{Y}^\dagger(\Omega') + \hat{Y}^\dagger(\Omega') \hat{X}(\Omega) \rangle, \quad (\text{B.7})$$

and they can be obtained from solutions to the Heisenberg Equations, as well as the spectra

$$S_{u_i u_j}(\Omega) = \delta_{ij}, \quad (\text{B.8})$$

and the prescriptions we use for the spectra of $S_{n_X}(\Omega)$ and $S_{n_F}(\Omega)$. The uncorrelated white spectra between $\hat{u}_1$ and $\hat{u}_2$ in Eq. (B.8) result from the ingoing light field being in its vacuum state. We shall discuss the magnitude and frequency dependence of $S_{n_X}(\Omega)$ and $S_{n_F}(\Omega)$ in depth in the next appendix.

For elements that involve v , we shall still use lt for column indices and mt' for row indices. We then have

$$V_{J,mt'}^{Qv} = \frac{1}{2} \langle \hat{Q}_J \hat{v}_m(t') + \hat{v}_m(t') \hat{Q}_J \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_{Q_J v_m}(\Omega) e^{i\Omega t'}, \quad (\text{B.9})$$

and

$$V_{lt,mt'}^{vv} = \frac{1}{2} \langle \hat{v}_l(t) \hat{v}_m(t') + \hat{v}_m(t) \hat{v}_l(t') \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_{v_l v_m}(\Omega) e^{-i\Omega(t-t')}. \quad (\text{B.10})$$

Note that $V_{J,mt'}^{Qv} = V_{mt',J}^{vQ}$.

In numerical computations, we will have to convert the continuum of $t, t' \leq 0$ into a finite grid. This means we will sample a finite duration T with a step size of Δt . We shall still use lower-case Latin indices to run from 1 to 2, and upper case Latin indices to run from 1 to 4, while we use Greek indices, for example, $\alpha = 0, 1, 2, \dots, T/\Delta t \equiv \mathcal{N} - 1$, to replace t . We shall write

$$K_{lt,mt'}^v \rightarrow K_{l\alpha,m\alpha'}^v = i\epsilon_{lm} \delta_{\alpha\alpha'}. \quad (\text{B.11})$$

Note that a Kronecker delta now replaces the Dirac delta. For the covariance matrix, we replace

$$V_{J,mt'}^{Qv} \rightarrow V_{J,m\alpha'}^{Qv} = \frac{\sqrt{\Delta t}}{2} \langle \hat{Q}_J \hat{v}_m(t_{\alpha'}) + \hat{v}_m(t_{\alpha'}) \hat{Q}_J \rangle, \quad (\text{B.12})$$

which are $4 \times 2\mathcal{N}$ and $2\mathcal{N} \times 4$ dimensional matrices (with $V_{J,mt'}^{vQ}$), and

$$V_{lt,mt'}^{vv} \rightarrow V_{l\alpha,m\alpha'}^{vv} = \frac{\Delta t}{2} \langle \hat{v}_l(t_\alpha) \hat{v}_m(t_{\alpha'}) + \hat{v}_m(t_\alpha) \hat{v}_l(t_{\alpha'}) \rangle, \quad (\text{B.13})$$

which is a $2\mathcal{N} \times 2\mathcal{N}$ -dimensional matrix. For the discrete sampling times we have defined $t_\alpha = -\left(\alpha + \frac{1}{2}\right)\Delta t$, where the additional $\frac{1}{2}\Delta t$ provides a faster convergence in numerics. The entire covariance matrix is then $(2\mathcal{N}+4) \times (2\mathcal{N}+4)$ -dimensional.

Our particular convention of inserting Δt at various places of the matrix is associated with our convention of discretizing vectors. For a generic variable, in the continuum form, we can always express it as

$$X = \alpha^j \hat{A}_j + \beta^j \hat{B}_j + \int_{-\infty}^0 \xi^j(t) \hat{v}_j(t) dt = \gamma^J \hat{Q}_J + \int_{-\infty}^0 \xi^J(t) \hat{v}_J(t) dt, \quad (\text{B.14})$$

where we have used upper indices for vector components, and we have grouped α^j and β^j into γ^J . The variance of X , which is formally written as $\mathbf{X}^\dagger \mathbf{V} \mathbf{X}$, will then be

$$\begin{aligned} \mathbf{X}^\dagger \mathbf{V} \mathbf{X} &= \frac{1}{2} \gamma^J \langle \hat{Q}_J \hat{Q}_K + \hat{Q}_K \hat{Q}_J \rangle \gamma^K + \int_{-\infty}^0 \gamma^J \langle \hat{Q}_J \hat{v}_m(t') + \hat{v}_m(t') \hat{Q}_J \rangle \xi^m(t') dt' \\ &+ \frac{1}{2} \iint_{-\infty}^0 \xi^l(t) \langle \hat{v}_l(t) \hat{v}_m(t') + \hat{v}_m(t) \hat{v}_l(t') \rangle \xi^m(t') dt dt'. \end{aligned} \quad (\text{B.15})$$

As we convert the integrals in Eq. (B.15) into summations, $\int$ will become Σ , while dt will become Δt . We shall take $\xi^{m\alpha} = \xi^m(t_\alpha) \sqrt{\Delta t}$. Together with Eq. (B.12)–(B.13), the fully discretized version of Eq. (B.15) will then be

$$\mathbf{X}^\dagger \mathbf{V} \mathbf{X} = \gamma^J V_{JK}^{QQ} \gamma^K + \gamma^J V_{J,m\alpha}^{Qv} \xi^{m\alpha} + \xi^{l\beta} V_{l\beta,K}^{vQ} \gamma^K + \xi^{l\alpha} V_{l\alpha,m\beta}^{vv} \xi^{m\beta}. \quad (\text{B.16})$$

In this convention, the usual vector norm for the discretized version of a function of time coincides with the L^2 -norm of that function. It can also be checked that discretized matrices in Eqs. (B.11)–(B.13), when contracted with vectors in this convention, lead to the appropriate integrals. Note that if a $\delta(t_\alpha - t'_\alpha)$ shows up in Eq. (B.13), we will take $\Delta t \delta(t_\alpha - t'_\alpha) \rightarrow \delta_{\alpha\alpha'}$, as in Eq. (B.11).

Table B.1: Classical Noise Model Parameters

Parameter	Symbol	Value	Units
Force Noise Time Constant	τ_F	$1.6 \cdot 10^{-20}$	s
Force Noise Cutoff Frequency	ω_F	$2\pi \cdot 0.25$	rad/s
Sensing Noise Time Constant 1	τ_{X_1}	10^{-50}	s
Sensing Noise Time Constant 2	τ_{X_2}	10^{-48}	s
Sensing Noise Cutoff Frequency	ω_X	$2\pi \cdot 10^4$	rad/s
Suspension Thermal Noise Time Constant	τ_{ST}	$3.1 \cdot 10^{-35}$	s
Suspension Thermal Noise Cutoff Frequency	ω_{ST}	$2\pi \cdot 1.9 \cdot 10^3$	rad/s

Table B.2: aLIGO Parameters

Parameter	Symbol	Value	Units
Mechanical Resonant Frequency	ω_m	$2\pi \cdot 0.9991$	rad/s
Mirror Mass	M	9.446	kg
Cavity Decay Rate	γ	$2\pi \cdot 424.6$	rad/s
Arm Length	L	3.995	km
Circulating Power	P_c	322.7	kW
Laser Wavelength	λ	1064	nm
Mechanical Damping	γ_m	$2\pi \cdot 10^{-3}$	rad/s

Corresponding to the discussion at the end of Sec. 4.3 (also shown in Fig. 4.1), here we consider entanglement between: (i) mass at $t = 0$ and the outgoing light field that had emerged during $t \leq 0$ and (ii) mass and the joint system of the cavity mode as well as light that had emerged during $t \leq 0$. In case (i), we simply throw away elements involving A in both $\mathbf{K}$ and $\mathbf{V}$, while in case (ii) we consider the full matrices. In both cases, $\mathbf{V}_{\text{pt}}$ is obtained by adding a minus sign to the column involving $\hat{B}_2$ and the row involving $\hat{B}_2$ —but not the diagonal element at which they intersect.

B.2 Numerical Implementation for aLIGO's Noise

The parameters used in modeling the classical and quantum noise curves, including the resonant modes, were calculated by minimizing the mean-squared error between the actual noise curves of aLIGO taken from pygwinc and the theoretical models, characterized by the parameters of interest, sampled logarithmically in frequency. Table B.1 contains the parameters for the classical force and sensing noise in aLIGO defined in Eqs. (4.21), Table B.2 contains the parameters of aLIGO introduced in the optomechanical Hamiltonian of Sec. 4.2, and lastly, Table B.3 contains the parameters of the resonant modes present in aLIGO's noise curves, modeled with

Table B.3: Resonant Mode Parameters

Parameter	Symbol	Value	Units
Mode Frequency	Ω_{v_1}	$2\pi \cdot 0.441$	rad/s
	Ω_{v_2}	$2\pi \cdot 0.995$	rad/s
	Ω_{v_3}	$2\pi \cdot 1.98$	rad/s
	Ω_{v_4}	$2\pi \cdot 2.37$	rad/s
	Ω_{v_5}	$2\pi \cdot 3.38$	rad/s
	Ω_{v_6}	$2\pi \cdot 3.81$	rad/s
	Ω_{v_7}	$2\pi \cdot 9.73$	rad/s
Full Width at Half Maximum	Γ_{v_1}	$2\pi \cdot 1.92 \cdot 10^{-3}$	rad/s
	Γ_{v_2}	$2\pi \cdot 5.63 \cdot 10^{-5}$	rad/s
	Γ_{v_3}	$2\pi \cdot 2.11 \cdot 10^{-5}$	rad/s
	Γ_{v_4}	$2\pi \cdot 1.44 \cdot 10^{-1}$	rad/s
	Γ_{v_5}	$2\pi \cdot 1.45 \cdot 10^{-4}$	rad/s
	Γ_{v_6}	$2\pi \cdot 1.65 \cdot 10^{-3}$	rad/s
	Γ_{v_7}	$2\pi \cdot 1.03 \cdot 10^{-3}$	rad/s
Amplitude	A_{v_1}	159	rad/s
	A_{v_2}	93.8	rad/s
	A_{v_3}	538	rad/s
	A_{v_4}	235	rad/s
	A_{v_5}	353	rad/s
	A_{v_6}	27.4	rad/s
	A_{v_7}	78.0	rad/s

Eq. (4.23).

In our simulations, we use $dt = 0.25$ ms and $T = 0.1$ s, which corresponds to sampling the light field at 4000 Hz and working with the outgoing field emitted from the cavity between $t = -0.1$ s and $t = 0$ s. We achieve numerical convergence with these parameters. To quantify the amount of entanglement in the system, we use the logarithmic negativity defined in Eq. (4.16). However, this is possible only for low levels of classical noise. For high levels of classical noise, classical correlations dominate over quantum correlations, which causes the cross-correlation values in the system to cover a wide-range of orders of magnitude, mostly due to the 14th power of Ω in our force noise model Eq. (4.21). For aLIGO parameters, the entries of the covariance matrix span about 20 orders of magnitudes, while we attempt to find a symplectic eigenvalue of order 1—this is numerically an extremely challenging problem.

Numerical errors also arise because of time-binning with an insufficient resolution.

Thus, we lose precision on the numerically determined covariance matrices. This affects the smallest symplectic eigenvalue $\tilde{\nu}_{min}$ to the point that it cannot be used to measure entanglement with the logarithmic negativity. Numerical imprecision can lead to covariance matrices that do not satisfy Heisenberg uncertainty bound Eq. (4.14), they thus do not correspond to a bona fide state and we call them *non-physical*. One way to get around this loss of precision is to use the PPT criterion as a yes/no test only, renouncing the magnitude information of $\tilde{\nu}_{min}$. The sampling frequency during time binning should be higher than the Nyquist rate of the system (i.e. twice the largest frequency in the system), since the entries of the covariance matrix contain correlations from all frequencies. In our system, the largest frequency is the cavity decay rate $\gamma = 424$ Hz. Therefore, we choose $dt < 1/(2 \cdot 424) \approx 1.2$ ms. However, as we decrease dt , we are limited by computational resources, such as the RAM size, or time. The parameter dt is limited to an optimal range determined by this trade-off. We thus develop the following strategy: we first quantify the amount of numerical errors in the system by computing the most negative eigenvalue (if it exists) of $\mathbf{V} + \frac{1}{2}\mathbf{K}$ before and after the partial transpose operation, denoted as $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$, respectively. Then, we decide that entanglement exists if $\lambda_{\mathcal{B}} > 0$ and $\lambda_{\mathcal{N}} < 0$, or if $\lambda_{\mathcal{B}} < 0$ and $|\lambda_{\mathcal{N}}| \gg |\lambda_{\mathcal{B}}|$. Furthermore, we decide that the system is separable if $\lambda_{\mathcal{B}} > 0$ and $\lambda_{\mathcal{N}} > 0$.

Two case studies about this strategy can be found in Fig. B.1 where we fix $T = 0.1$ s, change dt , and examine how $\lambda_{\mathcal{N}}$ and $\lambda_{\mathcal{B}}$ change by plotting $\lambda_{\mathcal{N}}$ and $\lambda_{\mathcal{B}}$. We decide on entanglement if $\lambda_{\mathcal{B}} \geq 0$ and $\lambda_{\mathcal{N}} < 0$, or $\lambda_{\mathcal{B}} < 0$, $\lambda_{\mathcal{N}} < 0$, and $|\lambda_{\mathcal{N}}| \geq 100|\lambda_{\mathcal{B}}|$. Our criteria for convergence is a relative change smaller than 5% for both $\lambda_{\mathcal{N}}$ and $\lambda_{\mathcal{B}}$ as we change dt . In Fig. B.1a, we work with a low level of classical force noise and set $\alpha_{F_1} = 10^{-15}$, $\alpha_{F_2} = 15$ in Eq. (4.21). We see that $\lambda_{\mathcal{N}}$ and $\lambda_{\mathcal{B}}$ converge with $\lambda_{\mathcal{N}}$ changing by 0.053%, 0.068%, and $\lambda_{\mathcal{B}}$ changing by 4.9%, 0.77% before and after tracing over the cavity, respectively, for $dt = 0.1$ ms. The system is entangled for both partitions since $\lambda_{\mathcal{B}} \geq 0$ and $\lambda_{\mathcal{N}} < 0$. Furthermore, $\lambda_{\mathcal{B}}$ becomes positive and converges after $dt \sim 1$ ms, or a sampling frequency of 1000 Hz; consistent with the discussion above relating physicality to Nyquist rate of ~ 850 Hz. We also see that $\lambda_{\mathcal{N}}$ converges for similar values of dt from Fig. B.1a.

In Fig. B.2, we plot the light-field section of the eigenvector associated with the (converged) minimal eigenvalue of $\mathbf{V}_{pt} + \frac{1}{2}\mathbf{K}$, for the partition where we do not trace over the cavity and with $\alpha_{F_1} = 10^{-15}$, $\alpha_{F_2} = 15$. It corresponds to a temporal mode of the free electromagnetic field outside the cavity. It is that particular mode

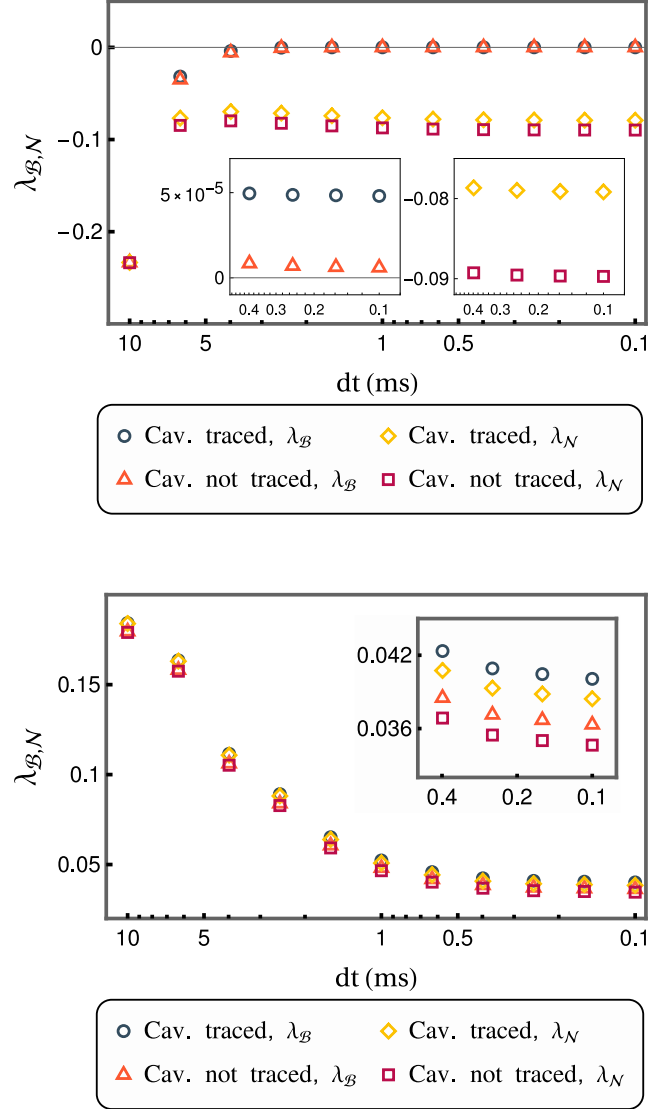


Figure B.1: $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ for $\alpha_{F_1} = 10^{-15}$ and $\alpha_{F_2} = 15$ [top row, (a)]; $\alpha_{F_1} = 10^{-8}$, $\alpha_{F_2} = 10$, and $\alpha_{X_2} = 10^3$ [bottom row, (b)]. We set $T = 0.1$ s, and $dt \in [0.1, 10]$ ms. Note that dt decreases to the right of the plots. In Figure a, entanglement exists for both partitions, and $\lambda_{\mathcal{B}}(\lambda_{\mathcal{N}})$ is re-plotted in the inset on the left(right) for $dt \in [0.1, 0.5]$ ms. In Figure b, the state is separable for both partitions, and $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ are re-plotted in the inset for $dt \in [0.1, 0.5]$ ms.

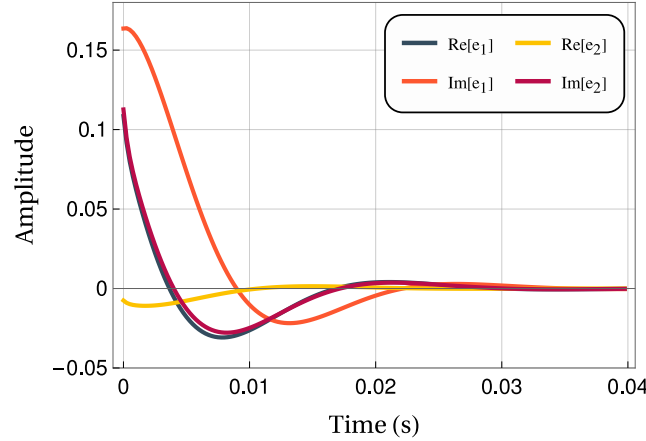


Figure B.2: The real and imaginary parts of the eigenvector for the negative eigenvalue of $\mathbf{V}_{\text{pt}} + \frac{1}{2}\mathbf{K}$ for $\alpha_{F_1} = 10^{-15}$, $\alpha_{F_2} = 15$, denoted by e_1 and e_2 , which correspond to the first and second halves of the entire eigenvector, respectively. The reason behind this slicing is the block-matrix structure of the light-field sector of the covariance matrix. Furthermore, since the light field modes are continuous in time, e_1 and e_2 are also functions of time.

associated with the (sole) negative eigenvalue that is entangled with the joint system mechanics plus cavity. The four curves correspond to the real and imaginary parts of $\hat{v}_1(t)$ and $\hat{v}_2(t)$. They have the form of smooth decaying oscillations with the same frequency and decay rate, but differing by a phase. This form of the mode functions was predicted for a white force noise in [175]; which gives us confidence in the correctness of our study. Also, exponentially decaying demodulation pulses were used to demonstrate optomechanical entanglement [199, 115] and proposed for a demonstration in the stationary regime [103]. We fit functions in the form of $e^{-\gamma_* t} \sin(\omega_* t + \theta)$ to each curve, which results in $\omega_*/(2\pi) \approx 40$ Hz, and $\gamma_*/(2\pi) \approx 25$ Hz. In the frequency domain, exponentially decaying harmonic oscillations are Lorentzians, centered at $\pm\omega_*$ and with a bandwidth (FWHM) $2\gamma_*$. In aLIGO's noise budget (Fig. 4.3), these Lorentzians are on the low frequency side of the low-noise band and their halfwidth at half-maximum to the left crosses the quantum noise, where it is not yet dominated by suspension thermal and seismic noises—although we saw in Sec. 4.6 that the latter is probably the main mechanism preventing optomechanical entanglement. We add that Lorentzians are heavy-tail distributions, being a possible reason why even lower frequency components matter.

In Fig. B.1b, we set $\alpha_{F_1} = 10^{-8}$, $\alpha_{F_2} = 10$, and $\alpha_{X_2} = 10^3$, causing the sensing noise to dominate over quantum noise for frequencies in the 30 – 2000 Hz band. We again see that λ_N and λ_B converge with λ_N changing by 1.3%, 1.2%, and λ_B changing

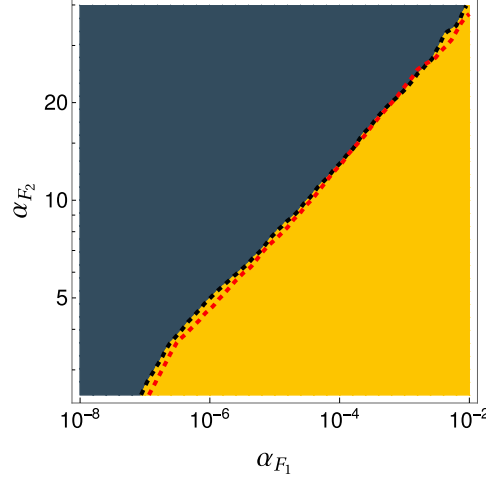


Figure B.3: Contour plot depicting the effect of the force noise spectrum on our numerical precision for both partitions. The force noise levels increase toward the bottom-right of the figure. The black and the red dashed lines separate the regions where numerics converge from the regions where numerics fail for the partition where we do and do not trace over the cavity, respectively. The region where numerics converge for both partitions is marked in gray, whereas numerics fail for both partitions toward the bottom right of the figure, past the red dashed line, in the yellow region.

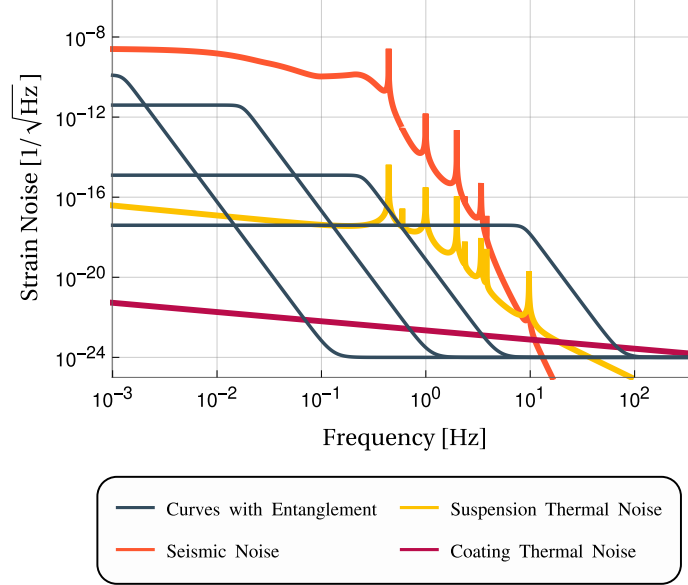


Figure B.4: Examples of force noise spectra along the boundary where we lose numerical precision (defined with $|\lambda_N| = 100|\lambda_B|$) for the partition where we do not trace over the cavity, plotted in black. aLIGO noise curves are also plotted for comparison. The black curves are parametrized with (from left to right) $\alpha_{F_1} = 1$ and $\alpha_{F_2} = 200$, $\alpha_{F_1} = 10^{-3}$ and $\alpha_{F_2} = 14$, $\alpha_{F_1} = 10^{-10}$ and $\alpha_{F_2} = 1$, and $\alpha_{F_1} = 10^{-15}$ and $\alpha_{F_2} = 0.029$.

by 1.2%, 1.1% before and after tracing over the cavity, respectively, for $dt = 0.1$ ms. Since $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ are positive for both partitions, we conclude that there is no entanglement in the system for either partition. When we increase the classical noise level we see that convergence is much harder to achieve. Furthermore, $\lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{N}}$ are negative, and $\lambda_{\mathcal{B}} \sim \lambda_{\mathcal{N}}$ for every value of dt regardless of when we do or do not trace over the cavity. Therefore, we cannot conclude that there is entanglement for either of the partitions. These case studies show that we can use $\lambda_{\mathcal{B}}$ as an indicator of the "non-physicality" of the covariance matrix $\mathbf{V}$ introduced by finite-sampling and high levels of classical noise in the system, and that the negativity of $\lambda_{\mathcal{N}}$ is not enough to decide on entanglement when we consider the numerics.

For our model of aLIGO's non-Markovian noises, Eqs. (4.21), we study the numerical stability of broad noise regimes, parametrized by the pair α_{F_j} , $j = 1, 2$. We set $\alpha_{X_j} = 1$, $j = 1, 2$, since we saw that force noise had a greater impact on numerical stability than sensing noise in our simulations for aLIGO's noise. In Fig. B.3, we plot the boundary between noise regimes where the numerics converge and the computed covariance matrices are physical (in gray) and those regimes where the numerics fail (either at converging or at generating physical covariance matrices or both) with the available computing resources (in yellow). As a matter of fact, in all the operation regimes in the gray region where the numerics work, the system is entangled. This means that our model predicts optomechanical entanglement in aLIGO, if its classical noise is in this gray region. Recall that the current status of aLIGO corresponds to $\alpha_{F_j} = \alpha_{X_j} = 1$, $j = 1, 2$, far to the bottom-right in the undecidable yellow region.

If we follow the red dashed line in Fig. B.3, we continuously sample the force noise spectrum over the boundary where our numerics converge, and the system is entangled, for the partition where we do not trace over the cavity. This boundary corresponds to a lower limit of the maximum amount of classical noise allowed in the system in order to have entanglement. We plot some of these noise curves in Fig. B.4. Note that if we set α_{F_1} or α_{F_2} to be unity, the corresponding pairs of α_{F_1} , α_{F_2} situated on this boundary would be $\alpha_{F_1} = 1$ and $\alpha_{F_2} \geq 200$, or for $\alpha_{F_2} = 1$ and $\alpha_{F_1} \leq 10^{-10}$. The corresponding noise curves are also plotted in Fig. B.4.

Appendix C

SUPPLEMENTARY INFORMATION FOR CHAPTER V

C.1 Allan deviation for various clocks

To calculate the clock stability for a given protocol, we need to find the optimal interrogation time T to operate with. This optimal interrogation time is a function of the total interrogation time τ . For an uncorrelated interrogation, $T = \tau$ for $\tau < \gamma_{\text{LO}}^{-1}$, and $T \approx \gamma_{\text{LO}}^{-1}$ for $\tau > \gamma_{\text{LO}}^{-1}$. Here, we calculate the exact value of T numerically.

The unknown phase ϕ of the LO is a stochastic variable, sampled from a prior distribution with a variance of $(\delta\phi)^2$. The variance grows with the interrogation time T , in the form of $\delta\phi = \gamma_{\text{LO}} T$, for a $1/f$ laser noise frequency spectrum. Therefore, the probability of a phase slip occurring increases with T . However, the clock stability $\sigma_y(\tau)$ also increases with T , which results in an optimal T to balance this trade-off.

We assume a posterior variance $(\Delta\tilde{\phi})^2$ for a clock protocol, in absence of phase slips. The broadening of the posterior variance due to phase slips can be represented by

$$\begin{aligned} (\Delta\phi)_{\text{slip}}^2 &\approx 2 \sum_{i=1}^{\infty} (2k\pi)^2 \int_{(2k-1)\pi}^{(2k+1)\pi} d\phi \frac{1}{\sqrt{2\pi}(\delta\phi)^2} e^{-\frac{\phi^2}{2(\delta\phi)^2}} \\ &= \sum_{k=1}^{\infty} (2k\pi)^2 \left[\text{erf}\left(\frac{(2k+1)\pi}{\sqrt{2}\delta\phi}\right) - \text{erf}\left(\frac{(2k-1)\pi}{\sqrt{2}\delta\phi}\right) \right] \end{aligned} \quad (\text{C.1})$$

for $(\Delta\tilde{\phi})^2 \ll 1$, or $N \gg 1$. Then, assuming that this effect is independent for each clock cycle, $\sigma_y(\tau)$ is calculated as

$$\sigma_y(\tau) = \frac{1}{\omega_A \sqrt{\tau T}} \left[(\Delta\tilde{\phi})^2 + \frac{\tau}{T} (\Delta\phi)_{\text{slip}}^2 \right]^{1/2} \quad (\text{C.2})$$

Then, the optimal interrogation time T is found from minimizing Eq. (C.2), resulting in the minimum Allan deviation

$$\sigma_y^*(\tau) = \min_T \sigma_y(\tau) \quad (\text{C.3})$$

To obtain the Allan deviation curve in Fig. 5.2b and Fig. 5.2c for uncorrelated atoms, we plot Eq. (C.3) for $N = 200$. We find numerically that the Allan deviation, normalized by $N^{1/2}$ is approximately independent of N .

For a clock interrogation using an N -atom GHZ state, we take a different approach. First, we redefine Eq. (5.5) in Sec. 5.3 for short interrogation times, where the posterior variance is close to the prior width, i.e., $\Delta\tilde{\phi} \approx \delta\phi$. In this limit, the assumption that we made in Sec. 5.3 ($\Delta\tilde{\phi} \ll \delta\phi$) breaks down, and the uncertainty of a stable clock is given by $\Delta\tilde{\phi}_{\text{eff}}$, with

$$(\Delta\tilde{\phi}_{\text{eff}})^2 = [(\Delta\tilde{\phi})^{-2} - (\delta\phi)^{-2}]^{-1} \quad (\text{C.4})$$

which we refer to as the *effective uncertainty*. The numerator in Eq. (5.5) in Sec. 5.3 is therefore given by this $\Delta\tilde{\phi}_{\text{eff}}$. Now, assume a parity measurement on the N -atom GHZ state: the measurement has two outcomes, even or odd parity. The probabilities of obtaining these results are given by $P(\pm) = (1 \pm \cos(N\phi))/2$. Then, using optimal estimators, the BMSE is calculated as

$$(\Delta\tilde{\phi})^2 = (\delta\phi)^2 - 2 \frac{\left(\frac{1}{2} \int_{-\infty}^{\infty} \phi \sin(N\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \right)^2}{1/2} = (\delta\phi)^2 - N^2(\delta\phi)^4 e^{-N^2(\delta\phi)^2} \quad (\text{C.5})$$

Plugging in $\delta\phi = \gamma_{\text{LO}}T$, the effective uncertainty is given by

$$\Delta\tilde{\phi}_{\text{eff}} = \frac{\left(e^{N^2(\gamma_{\text{LO}}T)^2} - N^2(\gamma_{\text{LO}}T)^2 \right)^{1/2}}{N} \quad (\text{C.6})$$

Then, the optimal Allan deviation for a GHZ interrogation can be obtained from minimizing $\sigma_y(\tau)$ with respect to the interrogation time T , given the effective uncertainty in Eq. (C.6). In Fig. 5.2b, we plot the Allan deviation, normalized up to some prefactors, for an $N = 200$ atom GHZ state.

When slow atoms are introduced to a given clock protocol, we can extend the interrogation time T beyond the LO noise limit, i.e. $T = \tau$ even in the regime where $\tau > \gamma_{\text{LO}}^{-1}$. Here, we assume that enough slow atoms are introduced such that the phase slip probability is small, and that the number of slow atoms is negligible compared to the initial number of atoms N . Therefore, in this regime, the Allan deviations of the best classical clock and the OQC are given by $1/\omega_A \sqrt{N} \tau$, and by $\pi/\omega_A N \tau$, respectively. The stabilities of these clocks eventually reach the fundamental limit of sensitivity (up to a possible constant factor of order one, see Ref. [69]), given in Eq. (5.6) in Sec. 5.3.

C.2 Optimality of the sine state

Here, we show that the phase estimation protocol where the initial state is an N -qubit sine state, and the measurement is a QFT followed by a projective measurement on

the basis states, attains the HL in the limit of large qubit number, $N \gg 1$. This regime is equivalent to having a large prior width within the dynamic range [167]. The N -qubit sine state is given by

$$|\psi_{\text{sine}}\rangle = \sqrt{\frac{2}{N+2}} \sum_{m=0}^N \sin\left(\frac{\pi(m+1)}{N+2}\right) |m\rangle \quad (\text{C.7})$$

where the states $\{|m\rangle\}$ are the symmetric-under-permutations eigenvectors of the angular momentum operator in the z -direction J_z , with $J_z = \frac{1}{2} \sum_i \sigma_z^i$, $i = 1, 2, \dots, N$, and σ_z^i is the Pauli Z operator for the i^{th} qubit. After undergoing a unitary evolution $U(\phi) = e^{-i\phi J_z}$, the sine state transforms into

$$|\psi\rangle = \sqrt{\frac{2}{N+2}} \sum_{m=0}^N e^{-im\phi} \sin\left(\frac{\pi(m+1)}{N+2}\right) |m\rangle \quad (\text{C.8})$$

up to a phase factor. Performing a QFT, followed by a projective measurement on the basis $\{|m\rangle\}$ can be represented with a POVM $E_k = |e_k\rangle \langle e_k|$, $k = 0, 1, \dots, N$, where

$$|e_k\rangle = \sqrt{\frac{1}{N+1}} \sum_{m=0}^N e^{-im\frac{2\pi k}{N+1}} |m\rangle \quad (\text{C.9})$$

And the probability of obtaining an outcome k after the measurement is given by $p(k) = |\langle \psi | E_k | \psi \rangle|^2$,

$$\begin{aligned} p(k) &= \frac{2}{(N+1)(N+2)} \left| \sum_{m=0}^N e^{im(\frac{2\pi k}{N+1} - \phi)} \sin\left(\frac{\pi(m+1)}{N+2}\right) \right|^2 \\ &\approx 2 \int_0^1 \int_0^1 dx dy e^{i(x-y)(2\pi k - N\phi)} \sin(\pi x) \sin(\pi y) \\ &= \frac{8\pi^2 \cos^2(N\phi/2)}{[\pi^2 - (2\pi k - N\phi)^2]^2} \end{aligned} \quad (\text{C.10})$$

for $N \gg 1$ and $k = 0, 1, \dots, N$. Note that we assume no phase slips, $\phi \in [0, 2\pi]$, such that $p(k)$ is non-zero only in this interval. Given the probabilities of measurement outcomes $p(k)$, we can derive the statistics of the measurement results.

Let us denote the random variable that corresponds to the measurement outcomes as O_{QFT} . O_{QFT} therefore takes the values $\{k\}_{-N/2}^{N/2}$ with probabilities $p(k)$. The first

and second moments of O_{QFT} are given by

$$\begin{aligned}\langle O_{\text{QFT}} \rangle &= \sum_{m=-N/2}^{N/2} \frac{8\pi^2 \cos^2(N\phi/2) m}{[\pi^2 - (2\pi m - N\phi)^2]^2} \\ \langle O_{\text{QFT}}^2 \rangle &= \sum_{m=-N/2}^{N/2} \frac{8\pi^2 \cos^2(N\phi/2) m^2}{[\pi^2 - (2\pi m - N\phi)^2]^2}\end{aligned}\quad (\text{C.11})$$

where we define $\bar{k} = \lfloor N\phi/(2\pi) \rfloor \approx N\phi/(2\pi)$. In the limit of $N \gg 1$, these expressions are approximately

$$\begin{aligned}\langle O_{\text{QFT}} \rangle &\approx \sum_{m=-\infty}^{\infty} \frac{8\pi^2 \cos^2(N\phi/2) m}{[\pi^2 - (2\pi m - N\phi)^2]^2} = \frac{N\phi}{2\pi} \\ \langle O_{\text{QFT}}^2 \rangle &\approx \sum_{m=-\infty}^{\infty} \frac{8\pi^2 \cos^2(N\phi/2) m^2}{[\pi^2 - (2\pi m - N\phi)^2]^2} = \left(\frac{N\phi}{2\pi}\right)^2 + \frac{1}{4} \\ (\Delta O_{\text{QFT}})^2 &= \langle O_{\text{QFT}}^2 \rangle - \langle O_{\text{QFT}} \rangle^2 = \frac{1}{4}\end{aligned}\quad (\text{C.12})$$

where $(\Delta O_{\text{QFT}})^2$ is the variance of O_{QFT} . We can thus take the estimator of ϕ in the limit of $N \gg 1$ to be $\frac{2\pi}{N} O_{\text{QFT}}$. The MSE of this estimator is then

$$(\Delta\phi)^2 = \frac{(2\pi)^2}{N^2} (\Delta O_{\text{QFT}})^2 = \frac{\pi^2}{N^2}.\quad (\text{C.13})$$

We thus attain the π -corrected HL for the entire interval $\phi \in [0, 2\pi]$, for $N \gg 1$. Note that this result is independent of the basis states $\{|m\rangle\}$.

C.3 Overhead for three blocks with optimal measurements

Here, we calculate the phase estimation error for the scheme where the initial state consists of GHZ states with sizes of 2^{k_i} , where $k_i = 0, 1, \dots, k_{\max}$, and each state has $M = 3$ copies. For the readout, we assume optimal measurements are available. The total number of qubits is given by $N = 3(2^{k_{\max}+1} - 1)$. As mentioned in Chapter 5, such a state can be written as

$$|\psi\rangle = \frac{1}{(N/3 + 1)^{3/2}} \sum_{n=0}^N \sqrt{f(n)} |n\rangle.\quad (\text{C.14})$$

We can explicitly calculate the coefficients $f(n)$, with

$$f(n) = \begin{cases} (n+1) \left(\frac{n}{2} + 1\right) & \frac{1}{3} > \frac{n}{N} \geq 0 \\ \frac{(N+3)^2}{12} + \left(n - \frac{N-1}{2}\right) \left(n - \frac{N-3}{2}\right) & \frac{2}{3} \geq \frac{n}{N} \geq \frac{1}{3} \\ (N-n+1) \left(\frac{N-n}{2} + 1\right) & 1 \geq \frac{n}{N} > \frac{2}{3} \end{cases}.\quad (\text{C.15})$$

We assume a uniform prior in $[-\pi, \pi]$, and choose the commonly used cost function of $4 \sin^2((\phi - \phi_{\text{est}})/2) \approx (\phi - \phi_{\text{est}})^2$. It was shown in Ref. [128], that in this case, given the state in Eq. (C.14) and the optimal measurements, $(\Delta\tilde{\phi})^2$ is given by

$$(\Delta\tilde{\phi})^2 = 2 \left[1 - \frac{S_f}{(N/3 + 1)^3} \right], \quad S_f = \sum_{n=0}^{N-1} \sqrt{f(n)f(n+1)}. \quad (\text{C.16})$$

Let us use the analytical expression for $f(n)$ in Eq. (C.15) to explicitly calculate the $(\Delta\tilde{\phi})^2$ for large N . First, we can write S_f as

$$S_f = \sum_{n=0}^{\frac{N}{3}-1} (n+2) \sqrt{(n+1)(n+3)} - \frac{(N+3)^2}{12} + \sum_{n=0}^{\frac{N-3}{6}} \sqrt{\left(\frac{(N+3)^2}{12} - n^2 \right)^2 - n^2} \quad (\text{C.17})$$

The sum S_f can be expanded using the Euler-Maclaurin formula,

$$\sum_m^n g(i) = \int_m^n g(x) dx + \sum_{k=1}^p \frac{B_k}{k!} \left(g^{(k-1)}(n) - g^{(k-1)}(m) \right) + R_p, \quad (\text{C.18})$$

where B_k are the Bernoulli numbers, R_p is an error term that depends on the order of the expansion, and $g^{(k)}(x)$ denotes the k th derivative of the function g evaluated at x . The BMSE can be written as $\alpha/N + \beta/N^2 + O(1/N^3)$, where we want to calculate α and β . Since the sum S_f is multiplied by a factor $\approx N^{-3}$, the terms that will contribute to α and β in S_f will scale as N^2 and N , respectively. Then, we do not need to find the constant terms in S_f , which reduces the number of terms that we need to calculate in the expansion significantly. Accordingly, we find

$$\sum_{n=0}^{\frac{N}{3}-1} (n+2) \sqrt{(n+1)(n+3)} = \frac{N^3}{81} + \frac{N^2}{6} + \frac{5N}{9} + O(1) \quad (\text{C.19})$$

and

$$\sum_{n=0}^{\frac{N-3}{6}} \sqrt{\left(\frac{(N+3)^2}{12} - n^2 \right)^2 - n^2} = \frac{2N^3}{81} + \frac{N^2}{4} + \frac{(17 - \frac{3\sqrt{3}}{2} \ln(2 + \sqrt{3}))N}{18} + O(1) \quad (\text{C.20})$$

Plugging these sums into S_f , we find that

$$(\Delta\tilde{\phi})^2 = \frac{9\sqrt{3} \ln(2 + \sqrt{3})}{2} \frac{1}{N^2} + O(N^{-3}). \quad (\text{C.21})$$

C.4 Optimization of the adaptive measurement

Here, we show the optimization over the single-qubit rotations in the adaptive measurement. The BMSE can be rewritten as

$$(\Delta\tilde{\phi})^2 = \int \sum_{\vec{n}} p(\vec{n}, \vec{\Phi}|\phi) (\phi - \phi_{est}(\vec{n}))^2 \mathcal{P}_{\delta\phi}(\phi) d\phi \quad (\text{C.22})$$

where the optimal Bayesian estimator for each branch, $\phi_{est}^*(\vec{n})$, is given by

$$\phi_{est}^*(\vec{n}) = \frac{\int \phi p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi}{\int p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi} \quad (\text{C.23})$$

which can be thought of as the expected value of ϕ given that the branch $\vec{n}$ has been sampled. Eq. (C.22) and (C.23) result in the minimum possible BMSE, given by

$$(\Delta\tilde{\phi}^*)^2 = (\delta\phi)^2 - \sum_{\vec{n}} \frac{\left(\int \phi p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \right)^2}{\int p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi} \quad (\text{C.24})$$

$p(\vec{n}, \vec{\Phi}|\phi)$ contains the rotations angles (system phases) $\vec{\Phi}$ that will be performed after each measurement along the branch $\vec{n}$. For example, if the initial state contains one block of a 2-atom GHZ state and two blocks of 1-atom GHZ states, there are $2^3 = 8$ possible branches depending on measurement results, occurring with probabilities

$$p([i, j, k], \vec{\Phi}|\phi) = \theta_i \left(\frac{2(\phi - \Phi_1)}{2} \right) \theta_j \left(\frac{\phi - \Phi_{2+i}}{2} \right) \theta_k \left(\frac{\phi - \Phi_{4+2i+j}}{2} \right),$$

$$\theta_i(x) = \begin{cases} \cos^2(x), & \text{if } i = 0 \\ \sin^2(x), & \text{if } i = 1 \end{cases} \quad (\text{C.25})$$

for $i, j, k \in \{0, 1\}$. The branch vector $[i, j, k]$ indicates the measurement result of a parity measurement on a GHZ block with m atoms: 0 (1) represents even (odd) parity.

We now want to minimize $\Delta\tilde{\phi}^*$ with respect to the rotation angles $\vec{\Phi}$. For this purpose, one needs to obtain $\partial\Delta\tilde{\phi}^*/\partial\Phi_i = 0$ for every i . $\partial\Delta\tilde{\phi}^*/\partial\Phi_i$ is found as

$$\frac{\partial\Delta\tilde{\phi}^*}{\partial\Phi_i} = \sum_{\vec{n}} \phi_{est}^*(\vec{n})^2 \int \partial_{\Phi_i} p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi$$

$$- 2\phi_{est}^*(\vec{n}) \int \phi \partial_{\Phi_i} p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \quad (\text{C.26})$$

Note that $\partial_{\Phi_i} p(\vec{n}, \vec{\Phi}|\phi) = 0$ if the branch $\vec{n}$ does not contain Φ_i . Assuming that Φ_i is contained in the branch $\vec{n}$, and that it is the l^{th} measurement, performed on a GHZ state with m atoms, from Eq. (C.25), we see that $\partial_{\Phi_i} p(\vec{n}, \vec{\Phi}|\phi)$ has two possible forms:

$$\partial_{\Phi_i} p(\vec{n}, \vec{\Phi}|\phi) = \begin{cases} m p(\vec{n}, \vec{\Phi}|\phi) \tan\left(\frac{m(\phi - \Phi_i)}{2}\right), & \text{if } n_l = 0 \\ -m p(\vec{n}, \vec{\Phi}|\phi) \cot\left(\frac{m(\phi - \Phi_i)}{2}\right), & \text{if } n_l = 1 \end{cases} \quad (\text{C.27})$$

Plugging the expression for the derivative in Eq. (C.27) into Eq. (C.26), we obtain

$$\begin{aligned} \frac{\partial \Delta \tilde{\phi}^*}{\partial \Phi_i} &= \sum_{\vec{n}, \Phi_i \text{ in } \vec{n}} \phi_{est}^*(\vec{n})^2 \int f(n_l, \Phi_i|\phi) p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \\ &\quad - 2\phi_{est}^*(\vec{n}) \int \phi f(n_l, \Phi_i|\phi) p(\vec{n}, \vec{\Phi}|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \end{aligned} \quad (\text{C.28})$$

with

$$f(n_l, \Phi_i|\phi) = \begin{cases} m \tan\left(\frac{m(\phi - \Phi_i)}{2}\right), & \text{if } n_l = 0 \\ -m \cot\left(\frac{m(\phi - \Phi_i)}{2}\right), & \text{if } n_l = 1 \end{cases} \quad (\text{C.29})$$

and m is the number of atoms contained in the GHZ block such that the feedback Φ_i is applied after its measurement.

To find the system phases that minimize the BMSE, we perform a gradient descent, where the system phases are updated according to $\Phi_i \rightarrow \Phi_i - \alpha \frac{\partial \Delta \tilde{\phi}^*}{\partial \Phi_i}$, where α is the step size and the derivative is given in Eq. (C.28). We implement the Adam optimizer [139] to perform the gradient descent.

C.5 Proof of claim I (OQI and CSS Plateaus)

We provide here a detailed proof for the BMSE expressions in the limit of $N \rightarrow \infty$ given in Chapter 5. For completeness, the statement of the claim is:

Claim. The OQI BMSE in the limit of $N \rightarrow \infty$ for any prior distribution $\mathcal{P}_{\delta\phi}(\phi)$ is

$$(\Delta \tilde{\phi})_{\text{OQI}}^2 = (\delta\phi)^2 - \int_{-\pi}^{\pi} \frac{\left(\sum_k \phi_k \mathcal{P}_{\delta\phi}(\phi_k) \right)^2}{\sum_k \mathcal{P}_{\delta\phi}(\phi_k)} d\phi, \quad (\text{C.30})$$

where $\phi_k = \phi + 2\pi k$, $k \in \mathbb{Z}$.

Assuming a $J_\theta := \cos(\theta) J_x - \sin(\theta) J_y$ single quadrature readout, the CSS BMSE

in the limit of $N \rightarrow \infty$ for any prior distribution $\mathcal{P}_{\delta\phi}(\phi)$ is

$$(\Delta\tilde{\phi})_{\text{SQL}}^2 = (\delta\phi)^2 - \int_{-\pi}^{\pi} d\phi \frac{\left(\sum_k (\phi_k - \theta) \mathcal{P}_{\delta\phi}(\phi_k - \theta) - (\phi_k + \theta) \mathcal{P}_{\delta\phi}(-\phi_k - \theta) \right)^2}{\sum_k \mathcal{P}_{\delta\phi}(\phi_k - \theta) + \mathcal{P}_{\delta\phi}(-\phi_k - \theta)} \quad (\text{C.31})$$

Proof. We first prove the OQI BMSE, given in Eq. (C.30). To this end, we first prove that it is a lower bound to the BMSE for any possible scheme. Then, we show that there exists a scheme that saturates it. Observe that a solution to the following minimization problem provides a lower bound for any scheme:

$$\begin{aligned} & \min_{\phi_{\text{est}}(x)} \min_{p(x|\phi)} \sum_x \int_{-\infty}^{\infty} (\phi_{\text{est}}(x) - \phi)^2 p(x|\phi) \mathcal{P}_{\delta\phi}(\phi) d\phi \\ & \text{subject to: } \sum_x p(x|\phi) = 1, \quad p(x|\phi) = p(x|\phi + 2\pi). \end{aligned} \quad (\text{C.32})$$

The first step is to show that we can map this into an equivalent minimization problem that involves ϕ_{est} only:

$$\min_{\phi_{\text{est}}(\phi)} \int_{-\pi}^{\pi} \sum_k (\phi_{\text{est}}(\phi) - (\phi + 2\pi k))^2 \mathcal{P}_{\delta\phi}(\phi + 2\pi k) d\phi. \quad (\text{C.33})$$

To see this, observe first that due to the constraint $p(x|\phi) = p(x|\phi + 2\pi)$, we can write the BMSE as:

$$\int_{-\pi}^{\pi} \sum_k \sum_x (\phi_{\text{est}}(x) - \phi_k)^2 p(x|\phi) \mathcal{P}_{\delta\phi}(\phi_k) d\phi \quad (\text{C.34})$$

where $\phi_k = \phi + 2\pi k$, $k \in \mathbb{Z}$. For each ϕ , there exists an $x_{\min,\phi}$ for which $\sum_k (\phi_{\text{est}}(x_{\min,\phi}) - \phi_k)^2 \mathcal{P}_{\delta\phi}(\phi_k)$ is minimal. Therefore, the optimal $p(x|\phi)$ is $p(x|\phi) = \delta(x - x_{\min,\phi})$. Denoting $\phi_{\text{est}}(x_{\min,\phi})$ as $\phi_{\text{est}}(\phi)$, we thus want to minimize $\sum_k (\phi_{\text{est}}(\phi) - \phi_k)^2 \mathcal{P}_{\delta\phi}(\phi_k)$ over $\phi_{\text{est}}(\phi)$ for each ϕ . This is a convex minimization problem with respect to $\phi_{\text{est}}(\phi)$, and the optimal $\phi_{\text{est}}(\phi)$ is given by:

$$\phi_{\text{est}}(\phi) = \frac{\sum_k \phi_k \mathcal{P}_{\delta\phi}(\phi_k)}{\sum_j \mathcal{P}_{\delta\phi}(\phi_j)}. \quad (\text{C.35})$$

To get the expression of Eq. (C.30), we use BMSE expression of Eq. (5.9). For this lower bound, the set of outcomes is the continuous range $[-\pi, \pi]$, hence:

$$(\Delta\phi)^2 = (\delta\phi)^2 - \int_{-\pi}^{\pi} \phi_{\text{est}}(\phi)^2 p(\phi) d\phi = (\delta\phi)^2 - \int_{-\pi}^{\pi} \frac{\left(\sum_k \phi_k \mathcal{P}_{\delta\phi}(\phi_k) \right)^2}{\sum_k \mathcal{P}_{\delta\phi}(\phi_k)} d\phi, \quad (\text{C.36})$$

which is the desired lower bound.

For tightness, we show that a CSS interrogation with a dual quadrature readout attains this bound in the limit of $N \rightarrow \infty$. Let us first define a dual quadrature measurement: given a $2N$ -qubits CSS, we split it into a product of two identical N -qubits CSS: $|\psi_{\phi,2N}\rangle = |\psi_{\phi,N}\rangle|\psi_{\phi,N}\rangle$. We measure the $J_x = \frac{1}{2} \sum_{i=1}^N \sigma_x^i$ operator of the first $|\psi_{\phi,N}\rangle$ and the $J_y = \frac{1}{2} \sum_{i=N+1}^{2N} \sigma_y^i$ operator of the second $|\psi_{\phi,N}\rangle$. Let us denote $x_x = \frac{2N_x}{N}$, $x_y = \frac{2N_y}{N}$, where $N_\theta \in \{-N/2, \dots, N/2 - 1, N/2\}$ is the outcome of J_θ measurement. x_x, x_y have independent shifted binomial distributions. In the limit of large N , the joint distribution (x_x, x_y) converges to the multivariate Gaussian distribution $p(x_x, x_y|\phi) \rightarrow N([\cos(\phi), \sin(\phi)]^T, \Sigma = \text{diag}(s_x^2, s_y^2))$, where $s_x^2 = \sin(\phi)^2/N$, $s_y^2 = \cos(\phi)^2/N$. Therefore, in the limit of $N \rightarrow \infty$ we obtain

$$\begin{aligned} p(x_x, x_y|\phi) &\rightarrow \delta(x_x - \cos(\phi)) \delta(x_y - \sin(\phi)) \\ &\Rightarrow p(r, \varphi|\phi) = \delta(r - 1) \left(\sum_{k \in \mathbb{Z}} \delta(\varphi - \phi - 2\pi k) \right), \end{aligned} \quad (\text{C.37})$$

where $r = \sqrt{x_x^2 + x_y^2}$, $\varphi = \arg(x_x + ix_y)$. This $p(\varphi|\phi)$, yields the optimal ϕ_{est} of Eq. (C.35), hence the bound is saturated. This proves the first part of the claim, the OQI BMSE.

Let us now compute the bound for a CSS interrogation with a J_θ single quadrature readout. Similar to above, we denote $x_\theta = \frac{2N_\theta}{N}$. x_θ has a shifted binomial distribution with an average of $\cos(\phi + \theta)$ and a variance of $\sin(\phi + \theta)^2/N$. Therefore, in the limit of $N \rightarrow \infty$, $p(x_\theta|\phi)$ converges to

$$\delta(x_\theta - \cos(\phi + \theta)) = \sum_{k \in \mathbb{Z}} \delta(\phi_k - \arccos(x_\theta) + \theta) + \delta(\phi_k + \arccos(x_\theta) + \theta) \quad (\text{C.38})$$

Inserting this into Eq. (5.9) results in Eq. (C.31).

C.6 Impact of noise on the proposed scheme

In atomic clocks, beyond local oscillator noise, we consider the impact of amplitude damping [272, 39, 156, 109, 13], imperfect state preparation due to finite gate fidelities, and measurement errors. To estimate the effect of such noise processes on the phase estimation, we again define the metrological gain $g_D = (\Delta\phi_{\text{CSS}})^2/(\Delta\tilde{\phi})^2$ as our figure of merit, where $(\Delta\phi_{\text{CSS}})^2$ and $(\Delta\tilde{\phi})^2$ are the BMSE of the phase estimation performed using a noiseless CSS, and the BMSE of the proposed scheme in the presence of the given noise process, respectively.

Amplitude damping

Let us analyze the effect of amplitude damping noise on GHZ states and on the resulting probabilities. A k -qubit GHZ state, $|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes k} + e^{ik\phi}|1\rangle^{\otimes k})$, undergoes a k -qubit amplitude damping channel:

$$\rho \rightarrow \varepsilon^{\otimes k}(\rho), \quad \varepsilon(\rho) = p_a \sigma_- \rho \sigma_+ + \sqrt{1 - p_a} \sigma_+ \sigma_- \rho \sqrt{1 - p_a} \sigma_+ \sigma_-,$$

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{C.39})$$

where $p_a > 0$ is the amplitude damping probability. The GHZ state is thus transformed to an ensemble $\sum_{\vec{v}} c(\vec{v}) \rho_{\vec{v}}$, where $\vec{v} \in \{0, 1\}^N$, such that $v_i = 0$ (1) indicates loss (no loss) in the i^{th} qubit, and $\rho_{\vec{v}} = |\vec{v}\rangle\langle\vec{v}|$. $c(\vec{v})$ are given by

$$c(\vec{0}) = \frac{1}{2} \left(1 + (1 - p_a)^k \right),$$

$$c(\vec{v}) = \frac{1}{2} (1 - p_a)^{\sum_i v_i} p_a^{1 - \sum_i v_i}, \quad \text{for } \vec{v} \neq \vec{0}. \quad (\text{C.40})$$

The phase information is now encoded only in $|\vec{0}\rangle$ which equals to

$$|\vec{0}\rangle = \frac{1}{\sqrt{2c(\vec{0})}} \left(|0\rangle^{\otimes k} + (1 - p_a)^{k/2} e^{ik\phi} |1\rangle^{\otimes k} \right). \quad (\text{C.41})$$

Let us derive the probabilities obtained from performing local Pauli X measurements on the state. We denote the ± 1 eigenstates of Pauli X as $|\pm\rangle$ respectively and the probability of obtaining m times $|-\rangle$ as $p(m)$. The probability for an even parity is the probability of obtaining an even m and is therefore equal to: $p(\text{even}) = \sum_{m \text{ even}} p(m)$. Note that $p(\text{even}) = \sum_{\vec{v}} c(\vec{v}) p(\text{even}|\vec{v})$. For any $\vec{v} \neq \vec{0}$, $p(\text{odd}|\vec{v}) = p(\text{even}|\vec{v}) = \frac{1}{2}$. Let us denote the contribution from $\sum_{\vec{v} \neq \vec{0}} \rho_{\vec{v}}$ as $p_{\vec{v} \neq \vec{0}}(\text{even})$, $p_{\vec{v} \neq \vec{0}}(\text{odd})$. We have

$$\begin{aligned} \sum_{\vec{v} \neq \vec{0}} p_{\vec{v} \neq \vec{0}}(\text{even}) &= \sum_{\vec{v} \neq \vec{0}} p_{\vec{v} \neq \vec{0}}(\text{odd}) = \sum_{\vec{v} \neq \vec{0}} c(\vec{v}) p(\text{even}|\vec{v}) \\ &= \sum_{\vec{v} \neq \vec{0}} (1 - p_a)^{\sum_i v_i} p_a^{1 - \sum_i v_i} \frac{1}{4} = \frac{1}{4} \left(1 - (1 - p_a)^k \right). \end{aligned} \quad (\text{C.42})$$

Now, let us calculate the contribution from $\vec{v} = \vec{0}$, denoted as $p_{\vec{0}}(\text{even})$, $p_{\vec{0}}(\text{odd})$. For any state $\alpha|0\rangle^{\otimes k} + \beta|1\rangle^{\otimes k}$, we have:

$$\alpha|0\rangle^{\otimes k} + \beta|1\rangle^{\otimes k} = \left(\frac{1}{\sqrt{2}} \right)^k \left(\alpha (|+\rangle + |-\rangle)^k + \beta (|+\rangle - |-\rangle)^k \right). \quad (\text{C.43})$$

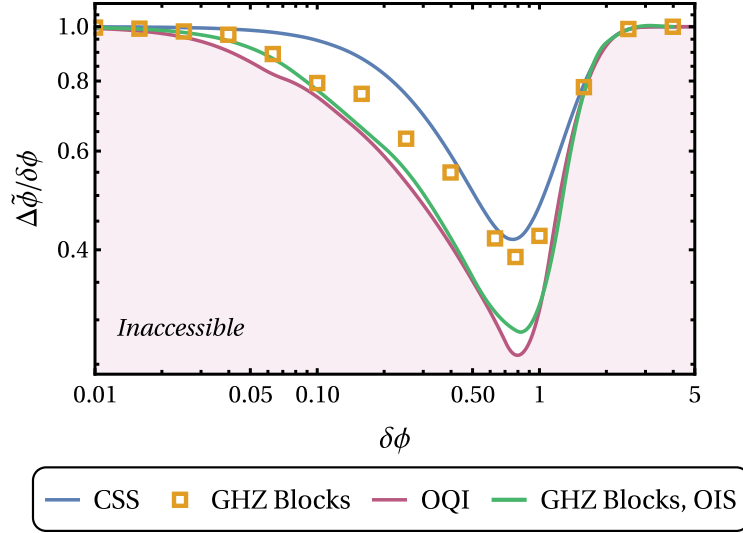


Figure C.1: RBMSE $\Delta\tilde{\phi}$ as a function of the prior width $\delta\phi$, normalized by $\delta\phi$, for $N = 12$ and in the presence of amplitude damping with a probability of $p_a = 10^{-2}$. The benchmark is the noiseless OQI, plotted in purple. The 12-qubit CSS for $p_a = 10^{-2}$ is also shown in the plot in blue. The minimum RBMSE obtained from optimal partitioning into blocks of GHZ states—i.e., the optimal initial state (OIS) for blocks of GHZ states—for $p_a = 10^{-2}$, assuming that optimal measurements are available, is plotted in green. We notice that this curve is very close to the noiseless OQI. Finally, the RBMSE given optimal partitioning into blocks of GHZ states and local, adaptive measurements is plotted with yellow rectangles. We see that in the presence of amplitude damping, even though the optimal partitions perform very close to the OQI when optimal measurements are available, there is a large gap between the minimum achievable sensitivity and the sensitivity achieved when local, adaptive measurements are used.

Therefore, the probability of measuring m times $|-\rangle$ given this state is $p(m) = \left(\frac{1}{2}\right)^k \binom{k}{m} |\alpha + (-1)^m \beta|^2$, and thus the probability for odd/even m is given by: $p_{\pm} = \frac{1}{2} |\alpha \pm \beta|^2$. Applying this to our case, i.e. $|\vec{0}\rangle$, then $\alpha = \frac{1}{\sqrt{2c(\vec{0})}}$, $\beta = \frac{(1-p)^{k/2} e^{ik\phi}}{\sqrt{2c(\vec{0})}}$, and thus

$$\begin{aligned} p_{\vec{0}}(\text{even}) &= \frac{1}{4} \left(1 + (1-p_a)^k + 2(1-p_a)^{k/2} \cos(k\phi) \right), \\ p_{\vec{0}}(\text{odd}) &= \frac{1}{4} \left(1 + (1-p_a)^k - 2(1-p_a)^{k/2} \cos(k\phi) \right). \end{aligned} \quad (\text{C.44})$$

Summing up the contributions from all $\vec{v}$ we obtain that $p(\text{even})$, $p(\text{odd})$ are:

$$p(\text{even}) = \frac{1}{2} \left(1 + (1-p_a)^{k/2} \cos(k\phi) \right), \quad (\text{C.45})$$

$$p(\text{odd}) = \frac{1}{2} \left(1 - (1-p_a)^{k/2} \cos(k\phi) \right). \quad (\text{C.46})$$

We considered Pauli X measurement, however any other Pauli operator in the $X - Y$ plane would give equivalent results: For a measurement of $\cos(\phi_0) X + \sin(\phi_0) Y$ the probabilities will take the same form and ϕ will be changed to: $\phi \rightarrow \phi - \phi_0$.

Note that given an initial GHZ state, amplitude damping noise commutes with the unitary phase encoding, $U(\phi)$. We can therefore assume that amplitude damping acts first, followed by the phase encoding. This can be easily verified: given $\rho = |\psi\rangle\langle\psi|$ of a k -qubit GHZ state, we have

$$\epsilon^{\otimes k} \left(U(\phi) \rho U(\phi)^\dagger \right) = U(\phi) \left(\epsilon^{\otimes k}(\rho) \right) U(\phi)^\dagger = |\psi'_\phi\rangle\langle\psi'_\phi| + \sum_{\vec{v} \neq \vec{0}} \rho_{\vec{v}}, \quad (\text{C.47})$$

where $|\psi'_\phi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes k} + (1 - p_a)^{k/2} e^{-ik\phi} |1\rangle^{\otimes k} \right)$.

Now, let us examine how amplitude damping affects the proposed scheme with and without optimal measurements. A case study is shown in Fig. C.1. Here, we work with 12 qubits and assume an amplitude damping probability of $p_a = 10^{-2}$. The purple line represents the RBMSE of the noiseless OQI, i.e. without any amplitude damping effects. The green line is the RBMSE obtainable by using optimal partitions of GHZ states if optimal measurements are available. We see that the RBMSE of the optimal partitions, even in the presence of amplitude damping, can get very close to the RBMSE of the OQI. However, local, adaptive measurements are not enough to achieve this precision: we show the performance of the proposed scheme when such measurements are used with yellow rectangles, and see that there is a gap between that and the RBMSE achievable with optimal measurements. Intuitively, this gap can be explained by the following reasoning: since for the adaptive measurement, the single-qubit rotation applied to a given qubit relies so much on the previous measurement results, any error during the previous measurements changes the trajectory of the future measurements, resulting in a large error in phase estimation. We conjecture that this issue can be mitigated by using error detection scheme that checks whether the state remained in the GHZ code space. Note that for every GHZ state with $k \geq 2$ atoms, decay errors can be detected by measuring if the GHZ state remains in the code space of $|0\rangle^{\otimes k}, |1\rangle^{\otimes k}$, which can be achieved by e.g., measuring the stabilizers of this repetition code space. We therefore detect a decay by measuring if the state is outside of this code space. Given a decay detection, the state has no phase information, and therefore we can ignore its measurement result. Otherwise, we can proceed with the same, adaptive, local measurements. This scheme converts amplitude damping noise into an erasure noise [149, 234]

and involves only error detection and not error correction. This idea was recently proposed and analyzed also in Ref. [135].

Let us illustrate the advantage in using such error detection scheme in the frequentist case. Given a k -qubit GHZ state, the probabilities of even/odd parities are $\frac{1}{2} \left(1 \pm (1 - p_a)^{k/2} \cos(k\phi) \right)$, while with error detection the relevant probabilities are given by $p_{\bar{0}}$ (even), $p_{\bar{0}}$ (odd). By direct calculation of the Cramer-Rao Bound (CRB), it can be observed that in the limit of $(1 - p_a)^k \ll 1$, the MSE given by the latter case is smaller by a factor of two than the former case. Hence, error detection leads to a factor of two improvement.

For simplicity, we choose to work with channels with small decay probabilities p_a , such that the optimal partition is unchanged (i.e. the BMSE obtained by this partition is relatively close to the BMSE obtainable by the optimal initial state of the noiseless case). In the context of atomic clocks, the decay probability is given by $p_a = 1 - \exp(-2\gamma_{\text{ind}}\tau)$ [192], where γ_{ind} is the individual, uncorrelated, decay rate out of the clock state, due to radiative decay or scattering from trapping light. τ is the total interrogation time. Then, working with small decay probabilities corresponds to operating with a short total interrogation time, such that $p_a \approx 2\gamma_{\text{ind}}\tau$, $\tau \ll \gamma_{\text{ind}}^{-1}$. This is the relevant regime where protocols using entanglement can provide considerably better atomic clock stability [119].

We simulate the performance of the proposed scheme for the amplitude damping probabilities $p_a = 10^{-3}$ and $p_a = 10^{-2}$ with $N = 21$ qubits in Fig. C.2. We compare the RBMSE obtained with the proposed scheme to the RBMSE obtained with a 21-qubit CSS and the noiseless OQI as a function of the prior width, $\delta\phi$. We see that for small p_a , the proposed scheme continues surpassing the performance of the 21-qubit CSS, and achieves a metrological gain of $g_D = 1.85$ (2.67 dB) for $p_a = 10^{-3}$, and $g_D = 1.51$ (1.78 dB) for $p_a = 10^{-2}$, at $\delta\phi = 0.7$ rad.

Regarding the adaptive protocol, amplitude damping renders the optimization over the single-qubit rotations used during the adaptive measurement scheme challenging since it flattens the hypersurface over which the gradient descent is being performed. Furthermore, due to the sequential nature of the measurement, errors performed during the beginning of the measurement sequence can alter the resulting estimate of the phase significantly, resulting in errors increasing with the probability of decay, p_a . This can be seen from Fig. C.2, where the performance of the scheme degrades when p_a is increased from 10^{-3} to 10^{-2} . As mentioned above, this issue can be mitigated by using error detection measurements, instead of local measurements,

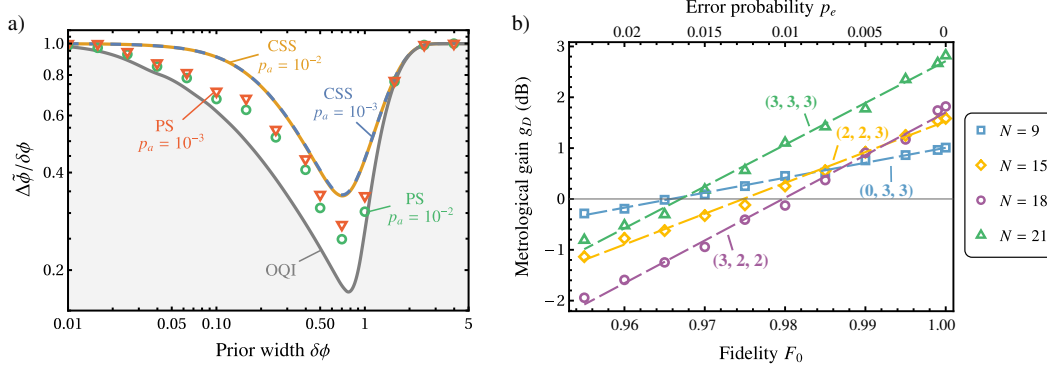


Figure C.2: Impact of noise on the proposed scheme. (a) RBMSE $\Delta\tilde{\phi}$, as a function of the prior width, $\delta\phi$, for $N = 21$ and various probabilities of decay p_a . As before, the RBMSE is normalized by the prior width. We plot the RBMSE of the 21-qubit CSS and the proposed scheme (PS on the Figure) for $p_a = 10^{-3}$ and $p_a = 10^{-2}$. Our benchmark is the noiseless OQI, plotted in gray. Increase in the RBMSE due to amplitude damping is most prominent around the prior width of $\delta\phi = 0.7$ rad. (b) The metrological gain as a function of the effective fidelity per qubit F_0 , or the bit-flip measurement error probability p_e . We plot the performances of the optimal partitions for qubit numbers of $N = 9, 15, 18, 21$. The partitions are shown in the plot in the form of (x, y, z) , where x, y , and z are the number of 4-qubit, 2-qubit, and 1-qubit GHZ states, respectively. We fit lines with different slopes to the performances of the scheme for different N , signifying an exponential decay in the metrological gain with decreasing F_0 , or increasing p_e . We observe a correlation between the decay rate and the number of copies of the 4-qubit GHZ state: the partitions with $N = 21$ and $N = 18$ both have three 4-qubit GHZ states, and they have similar decay rates.

for each GHZ state.

GHZ state preparation and single-qubit measurement errors

State preparation and measurement errors result in a parity signal with a limited contrast, $C < 1$, where the outcomes of a parity measurement performed on an N -qubit GHZ state with contrast $C(N)$ are written as $P(\pm) = [1 \pm C(N) \cos(N\phi)]/2$, where $\{+, -\}$ denote even, odd parity. Let us show this for measurement errors: assuming the noisy measurement is given by a symmetric bit-flip channel that precedes a perfect projective measurement in the $|+\rangle, |-\rangle$ basis, the measurement operators of each qubit are given by $M_{\pm} = (1 - p_e)|\pm\rangle\langle\pm| + p_e|\mp\rangle\langle\mp|$ with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and p_e being the bit-flip probability. The probabilities of even/odd parities are then: $P(\pm) = P(\pm)p_{\text{even}} + P(\mp)p_{\text{odd}}$, where p_{even} is the probability of an even number of bit-flip errors occurring, given by $p_{\text{even}} = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{i} p_e^{2i} (1 -$

$p_e)^{N-2i}$, and $p_{\text{odd}} = 1 - p_{\text{even}}$. These errors therefore result in a contrast of $C(N) = p_{\text{even}} - p_{\text{odd}} = (1 - 2p_e)^N$.

A bit-flip measurement error is equivalent to a phase-flip error just before the readout pulse. Since phase-flip errors commute with the ϕ -encoding unitary $U(\phi)$, this is equivalent to a phase flip error occurring during state preparation. Hence, such preparation errors are completely equivalent to bit-flip measurement errors, and would lead to the same reduced parity. For imperfect state preparation, we therefore assume a model where the contrast scales with the qubit number as $C(N) = F_0^N$, where F_0 denotes the effective fidelity per qubit.

The metrological gain g_D in dB for various qubit numbers N , as a function of the effective fidelity per qubit F_0 , or the bit-flip probability p_e can be found in Figure C.2. Note that a metrological gain above 0 dB signifies that the proposed scheme surpasses the noiseless CSS interrogation for the respective qubit number. The partitions used for the proposed scheme are plotted in the Figure in the form of (x, y, z) , where x, y , and z denote the number of repetitions of the 4-qubit, 2-qubit, and 1-qubit GHZ state. Since we aim to understand how the gain scales with the fidelity or the error probability as a function of the particular partitions, we choose to examine the case of (i) $N = 9$, which does not contain any 4-qubit GHZ states, (ii) $N = 15$, which contains two 4-qubit GHZ states, (iii) $N = 18$ and $N = 21$, which both contain three 4-qubit GHZ states. We fit functions in the form of $g_D = A \cdot \exp(-B(1 - F_0)) = A \cdot \exp(-2Bp_e)$ to the metrological gains of different N , plotted with dashed lines. The decay rate B for $N = 9, 15, 18$ and 21 is calculated to be 6.78, 13.9, 19.3, and 18.9, respectively. Interestingly, the gain appears to be mainly a function of the number of repetitions of the largest GHZ state, the larger this number is, the quicker the gain decays with decreasing fidelity, or increasing bit-flip error probability. It is approximately independent of the number of repetitions of the smaller blocks of GHZ states (the 2-qubit and 1-qubit GHZ states in this case).

More sophisticated measurement schemes might be employed to overcome the decay in the metrological gain due to measurement errors. It has been shown that in the presence of noisy measurements (assuming no other noise source), attaining HS depends on the available control operations. For example, HS is attainable in this regime when global control operations and post-processing methods are used, however it is elusive when local control operations are available only [153, 280]. We defer studying the effect of using more sophisticated measurement schemes on the proposed scheme to future work.

Appendix D

SUPPLEMENTARY INFORMATION FOR CHAPTER VI

D.1 Imperfect Weak Measurements

Let us consider the following noisy ancilla measurement, in which the ancilla is measured through the following positive operator-valued measure (POVM):

$$M_0 = (1 - p_e) |0\rangle\langle 0| + p_e |1\rangle\langle 1|, \quad M_1 = (1 - p_e) |1\rangle\langle 1| + p_e |0\rangle\langle 0|, \quad (\text{D.1})$$

instead of the perfect projective measurement POVM of $\{\Pi_0 = |0\rangle\langle 0|, \Pi_1 = |1\rangle\langle 1|\}$. This noisy measurement can be due to a symmetric bit-flip noise inflicted on the ancilla followed by a perfect projective measurement, or due to an error in the detection process itself. The probability of a bit-flip is parametrized through p_e . The sensor and ancilla density matrix following this bit-flip noise is given by:

$$\begin{aligned} \Lambda(\rho_s \otimes \rho_a) = & \left((1 - p_e) K_+ \rho K_+^\dagger + p_e K_- \rho K_-^\dagger \right) \otimes |1\rangle\langle 1| \\ & + \left((1 - p_e) K_- \rho K_-^\dagger + p_e K_+ \rho K_+^\dagger \right) \otimes |0\rangle\langle 0| \end{aligned} \quad (\text{D.2})$$

where $K_\pm$ are the weak measurement Kraus operators given in Eq. (6.2). Hence, the state of the sensor after this measurement is updated as follows:

$$\rho \rightarrow \frac{1}{p(x|\omega)} \left[(1 - p_e) K_{(-1)^x} \rho K_{(-1)^x}^\dagger + p_e K_{(-1)^{x+1}} \rho K_{(-1)^{x+1}}^\dagger \right]. \quad (\text{D.3})$$

$p(x|\omega)$ correspond to the probabilities of the measurement results, and $x = 0, 1$. In order to further understand the effect of such a channel on the sensing qubit, let us denote $\rho = 1/2 (I + \vec{r} \cdot \vec{\sigma})$, where $\vec{r} = (r_x, r_y, r_z)$. Furthermore, let us define $\phi = \tan^{-1}(r_y/r_x)$. The measurement probabilities after the ancilla readout will be given by

$$\begin{aligned} p(x|\omega) = & \frac{1}{2} \left[1 + (-1)^x (1 - 2p_e) \sin(2g) (r_x \cos(2\omega\tau) + r_y \sin(2\omega\tau)) \right] \\ = & \frac{1}{2} \left[1 + (-1)^x (1 - 2p_e) \sin(2g) r \cos(2\omega\tau - \phi) \right], \end{aligned} \quad (\text{D.4})$$

where we defined $r = |\vec{r}|$. The update on the parameters r, ϕ after the measurement are calculated as

$$r^2 \rightarrow \frac{r^2 (1 - \sin(2g)^2 \sin(\phi')^2) + (-1)^x 2p_c \sin(2g) r'_x + p_c^2 \sin(2g)^2}{(1 + (-1)^x \sin(2g) p_c r'_x)^2}, \quad (\text{D.5a})$$

$$\phi \rightarrow \arg(r'_x + (-1)^x p_c \sin(2g) + i r \cos(2g) \sin(\phi')), \quad (\text{D.5b})$$

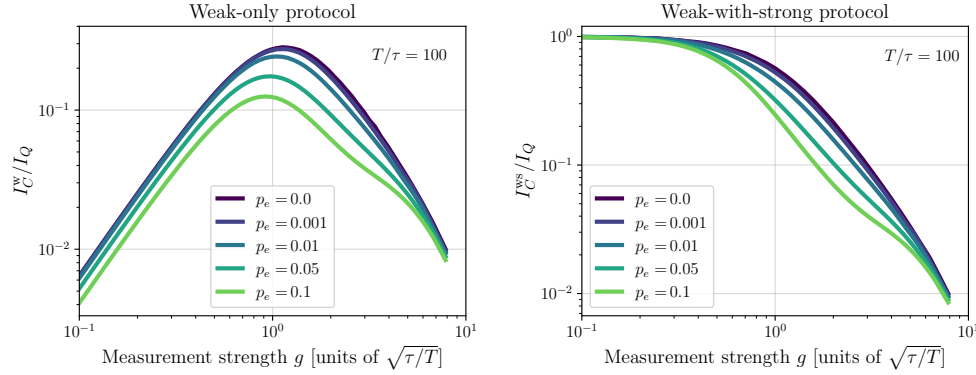


Figure D.1: Effect of imperfect weak measurements on the CFI, with respect to the measurement strength g . We plot the CFI I_C normalized by the QFI I_Q as a function of g and for different error probabilities p_e . The left plot corresponds to the weak-only protocol, and the right one to the weak-with-strong protocol. We fix the interrogation time, the measurement period, and the frequency-to-be-estimated as $T = 10$ s, $\tau = 0.1$ s, and $\omega = 5$ rad, respectively. Note that in the weak-with-strong protocol (plotted on the right) we assume that the strong projective measurement is noiseless.

where $\phi' = \phi - 2\omega\tau$, $p_c = 1 - 2p_e$, $r'_x = r \cos(\phi')$, and ϕ are initialized at the beginning of interrogation to $r = 1$, $\phi = 0$, as the sensing qubit is in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. For a perfect measurement, $p_e = 0$, r is unchanged throughout the interrogation, as the sensing qubit remains in a pure state. However, for $p_e \neq 0$, the sensing qubit becomes a mixed state due to imperfect measurements, and r undergoes a stochastic process (in addition to ϕ).

In what follows we derive the noisy CFI in the weak back action regime. An analytical derivation of the CFI in the strong back action regime is left for future work. In the weak back action limit, Eqs. (D.5) can be approximated as

$$r \rightarrow r + (-1)^x 2g (1 - 2p_e) r (1 - r^2) \cos(\phi - 2\omega\tau), \quad (\text{D.6a})$$

$$\phi \rightarrow \phi - 2\omega\tau - (-1)^x 2g (1 - 2p_e) \sin(\phi - 2\omega\tau). \quad (\text{D.6b})$$

Then, we observe that up to first order in g , an initial $r = 1$ (pure state) will remain equal to 1 throughout the interrogation. Combined with the expression for the measurement probabilities in Eq. (D.4), it can be seen that in this limit, the imperfect measurement effectively rescales the measurement strength g as $g \rightarrow g(1 - 2p_e)$. Therefore, the CFI of the weak-only protocol in this limit is $8Ng^2(1 - 2p_e)^2T^3/3\tau$. Regarding the CFI with the weak-with-strong protocol in this regime, it can be seen that CFI remains $4NT^2 - O(g^2)$ and the sensitivity gap compared to the noiseless

case is negligible. In the strong back action regime however the effect of p_e is different: the gap from the noiseless CFI becomes larger and we observe that the scaling with time is no longer T but modified to T^α with $\alpha < 1$. In the strong back action regime, the CFI of both weak measurement protocols converge.

We plot the effect of imperfect measurements on the CFI in Figs. D.1 and D.2. In Fig. D.1, we plot the CFI as a function of g for different measurement error values, p_e , for both protocols. As we found analytically, in the weak back action regime, the CFI with imperfect measurements is $I_C^w \approx 8Ng^2(1-2p_e)^2T^3/3\tau$ for the weak-only protocol. Hence, the gap from the noiseless CFI in this limit is $(1-2p_e)^2$, and it is observed on Fig. D.1a. Furthermore, we observe that the CFI of the weak-with strong protocol is $I_C^{ws} \approx 4NT^2$ in the limit of small measurement strength $g \ll 1$. In Fig. D.2, we plot the CFI as a function of the interrogation time T and different values of p_e . We notice that the CFI of both protocols converge to the same value in the strong back action regime, where the scaling with T is modified due to imperfect measurements.

We can compute the threshold for which the information obtained from the MLE gets close to the CFI. For this purpose, we again work in the weak back action regime $g^2T/\tau \ll 1$, and assume a large number of qubits and measurements ($N, T/\tau \gg 1$). From the discussion above, we can compute the thresholds given in Eq. (6.48) and Eq. (6.57) in this limit by rescaling $g \rightarrow g(1-2p_e)$.

To correct a finite number of bit-flip errors, we can prepare the ancilla qubits in an n -qubit GHZ state, i.e. in the state $|\psi_{\text{GHZ}}\rangle (|1^n\rangle + i|0^n\rangle)/\sqrt{2}$. Furthermore, one can engineer the following unitary interaction between the sensing qubit and the ancilla state: $U = \exp(-ig\sigma_x \otimes \sum_i \sigma_x^i)$, where the summation of the Pauli x operators is over the ancilla qubits. After the free evolution and this unitary, the joint state of the sensing qubit and the ancillae $\rho_s \otimes \rho_a$ will be the following:

$$\begin{aligned} \Lambda(\rho_s \otimes \rho_a) = & K_+\rho_s K_+^\dagger \otimes |1\rangle^{\otimes n} \langle 1|^{\otimes n} + iK_-\rho_s K_+^\dagger \otimes |0\rangle^{\otimes n} \langle 1|^{\otimes n} \\ & - iK_+\rho_s K_-^\dagger \otimes |1\rangle^{\otimes n} \langle 0|^{\otimes n} + K_-\rho_s K_-^\dagger \otimes |0\rangle^{\otimes n} \langle 0|^{\otimes n}, \end{aligned} \quad (\text{D.7})$$

where $K_\pm$ are the weak measurement Kraus operators given in Eq. (6.2), and ρ_s is the state of the sensing qubit. A noisy measurement on the ancilla qubits will be described by the POVM in Eq. (D.1). Let us act on this state with the POVM element $M_0^m M_1^{n-m}$. The probability of obtaining this element will be given by $\text{Tr}[\Lambda(\rho_s \otimes \rho_a) I \otimes M_0^m M_1^{n-m}] = p_e^m (1-p_e)^{n-m} \text{Tr}[K_+\rho_s K_+^\dagger] + (1-p_e)^m p_e^{n-m} \text{Tr}[K_-\rho_s K_-^\dagger]$. Furthermore, after tracing out the ancilla qubits, the sensing qubit will be in the

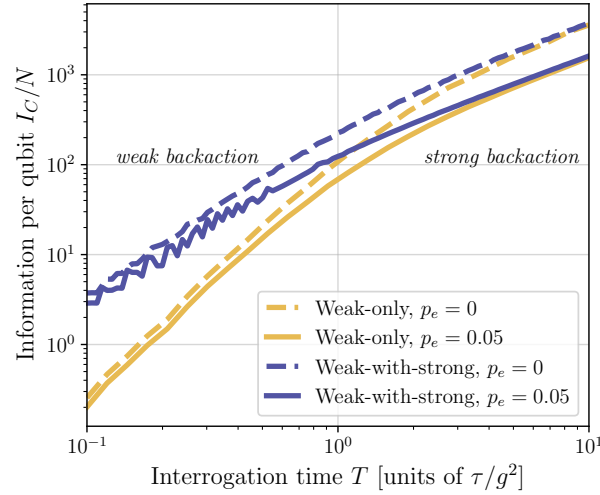


Figure D.2: Effect of imperfect weak measurements on the CFI, with respect to the interrogation time T . We plot the CFI I_C normalized by the number of qubits N for the weak-only protocol and the weak-with-strong protocol with yellow and purple lines, respectively. We fix the measurement strength and period as $g = 0.1$ and $\tau = 0.1$ s. Furthermore, we fix the frequency to be estimated as $\omega = 5$ rad/s. We observe that in the weak back action limit where $g^2 T / \tau \ll 1$, the imperfect measurement effectively scales the measurement strength as $g \rightarrow g(1 - 2p_e)$, but the scaling with respect to T is approximately unaffected such that the yellow dashed and plain curves are parallel to each other. In the strong back action limit, the CFI for the protocols with imperfect measurements (plain lines) converge to the classical T scaling faster compared to the protocols with perfect measurements (dashed lines). We observed numerically that for larger p_e , the T scaling gets modified to T^α , $\alpha < 1$.

state of

$$\rho_s \rightarrow \frac{p_e^m (1 - p_e)^{n-m} K_+ \rho_s K_+^\dagger + (1 - p_e)^m p_e^{n-m} K_- \rho_s K_-^\dagger}{p_e^m (1 - p_e)^{n-m} \text{Tr}[K_+ \rho_s K_+^\dagger] + (1 - p_e)^m p_e^{n-m} \text{Tr}[K_- \rho_s K_-^\dagger]} \quad (\text{D.8})$$

For a very weak measurement, $\text{Tr}[K_- \rho_s K_-^\dagger] \approx \text{Tr}[K_+ \rho_s K_+^\dagger]$. Let us also assume without loss of generality that $m \ll n$, $n \gg 1$. Then, the probability of the state of the sensing qubit being $K_- \rho_s K_-^\dagger$ after obtaining the POVM element $M_0^m M_1^{n-m}$ is on the order of

$$\frac{p_e^{n-2m}}{p_e^{n-2m} + (1 - p_e)^{n-2m}} \approx \left(\frac{p_e}{1 - p_e} \right)^{n-2m} \quad (\text{D.9})$$

Therefore, the conditional error probability is suppressed with $(p_e / (1 - p_e))^n$.

D.2 Weak Measurements with Light

Here, we consider a related but separate system, where the atoms are stored in a cavity. In this system, weak measurements can be performed by coupling a cavity mode dispersively to the atomic spin [30, 27]. Such a coupling results in the interaction Hamiltonian $\mathcal{H}_{\text{int}} = \chi \hat{J}_x \hat{X}$, where $\hat{X}$ is the position operator of the light field, $\hat{J}_x$ is the x component of the total angular momentum operator $\hat{\mathbf{J}}$ of the atoms, and χ is some interaction strength. χ is related to the measurement strength g in the proposed protocol. After the interaction, homodyne measurement is performed on the momentum operator $\hat{P}$ of light field to probe the atomic spin. Let us assume that the atoms are initialized in a coherent state $|+\rangle = \left[(|0\rangle + |1\rangle) / \sqrt{2} \right]^{\otimes N}$, where N is the number of atoms. Therefore, we initially have: $\langle \hat{J}_y \rangle = \langle \hat{J}_z \rangle = 0$, $\langle \hat{J}_x \rangle = N/2$, $\langle \hat{J}_y^2 \rangle = \langle \hat{J}_z^2 \rangle = N/4$, and $\langle \hat{J}_x^2 \rangle = N^2/4$, such that $\Delta \hat{J}_y = \Delta \hat{J}_z = \sqrt{N/4}$, and $\Delta \hat{J}_x = 0$.

The free evolution can be described by a rotation of the total angular momentum vector on the collective Bloch sphere, quantified by a rotation matrix $\mathbf{U}(\omega\tau)$, where ω is the frequency to be estimated and τ is the free evolution time. A weak measurement is performed immediately after the free evolution, where the outgoing momentum operator of the light field is given by

$$\hat{P}_{\text{out}} = \hat{P}_{\text{in}} - \chi t' \hat{J}_x. \quad (\text{D.10})$$

where t' is the weak measurement time. We assume that the probing light has vacuum statistics at any time, such that $\langle \hat{X}_{\text{in}} \rangle = \langle \hat{P}_{\text{in}} \rangle = \langle \hat{X}_{\text{in}} \hat{P}_{\text{in}} \rangle = 0$, and $\langle \hat{X}_{\text{in}}^2 \rangle = \langle \hat{P}_{\text{in}}^2 \rangle = 1/2$. The weak measurement will cause a back action on the collective angular momentum of the atoms, which also can be described by a rotation on the Bloch sphere, quantified by a rotation matrix $\mathbf{M}(\hat{\Pi})$, with $\hat{\Pi} = \chi t' \hat{X}$. We describe the collective dynamics of the atoms using the Heisenberg picture of the angular momentum operators $\hat{\mathbf{J}}(t)$, $t > 0$. The effect of one cycle of free evolution and weak measurement can be written as

$$\hat{\mathbf{J}}(t + \tau) = \mathbf{M}(\hat{\Pi}) \mathbf{U}(\omega\tau) \hat{\mathbf{J}}(\tau), \quad (\text{D.11a})$$

$$\mathbf{U}(\omega\tau) = \begin{bmatrix} \cos(2\omega\tau) & -\sin(2\omega\tau) & 0 \\ \sin(2\omega\tau) & \cos(2\omega\tau) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{D.11b})$$

$$\mathbf{M}(\hat{\Pi}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\hat{\Pi}) & -\sin(\hat{\Pi}) \\ 0 & \sin(\hat{\Pi}) & \cos(\hat{\Pi}) \end{bmatrix} \quad (\text{D.11c})$$

In order to find the sensitivity with respect to ω at any given time t , we can calculate

$$(\Delta\omega)^2 \approx \frac{(\Delta\hat{J}_x(t))^2}{(\partial\langle\hat{J}_x(t)\rangle/\partial\omega)^2} \quad (\text{D.12})$$

Therefore, we want to find $J_x(T)$ at the end of the interrogation, where T is the interrogation time. In order to compare a protocol that uses the light field to perform weak measurements to our protocol, we assume that weak measurements with light are performed periodically (with a period of τ), and a projective measurement on the atomic ensemble is performed at the end of the interrogation. Furthermore, we aim to take the limit of infinitesimally weak measurements $\chi t' \rightarrow 0$, in order to observe if this protocol also reaches the QFI limit of $4NT^2$.

First, to calculate the first order moment of $\hat{J}_x$, we can write from Eq. (D.11)

$$\langle\hat{\mathbf{J}}(n\tau)\rangle = \prod_{i=1}^n \langle\mathbf{M}_i(\hat{\Pi})\rangle \mathbf{U}(\omega\tau) \langle\hat{\mathbf{J}}(0)\rangle \quad (\text{D.13})$$

where $\mathbf{M}_i(\hat{\Pi})$ is the back action matrix from the i^{th} weak measurement. In order to find $\langle\mathbf{M}_i(\hat{\Pi})\rangle$, we observe that for linear operators on Gaussian states, we have: $\langle e^{\hat{A}} \rangle = e^{\langle\hat{A}\rangle} e^{(\Delta\hat{A})^2/2}$. Then, $\langle\cos(\hat{\Pi})\rangle = \cos((\chi t'/2)^2)$, $\langle\sin(\hat{\Pi})\rangle = \sin((\chi t'/2)^2)$. To make the analysis simpler, we can add another rotation matrix after each weak measurement to cancel this average rotation. Then, we will have at the end of the last measurement: $\langle\hat{J}_x(T)\rangle = \cos(2\omega T)\langle\hat{J}_x(0)\rangle - \sin(2\omega T)\langle\hat{J}_y(0)\rangle = \cos(2\omega T)N/2$. To find the second moments, we can do a similar analysis where we evolve the Heisenberg operators of $\{\hat{J}_{x,y,z}, \hat{J}_{x,y,z}\}$. We find in the limit that $T/\tau \gg 1$, $N \gg 1$

$$(\Delta\hat{J}_x(T))^2 = \frac{N}{4} \sin(2\omega T)^2 - \frac{(\chi t')^2}{64} N \frac{T}{\tau} \cot(\omega\tau) \sin(4\omega T) \quad (\text{D.14})$$

Therefore, from Eq. (D.13), we find that

$$\frac{(\Delta\hat{J}_x(T))^2}{(d\langle\hat{J}_x(T)\rangle/d\omega)^2} \approx \frac{1}{4NT^2} \quad (\text{D.15})$$

in the limit that $\chi t' \rightarrow 0$. Therefore, we observe that $(\Delta\omega)^2 \approx I_Q$ with this protocol. Then, the protocol that employs light to perform periodic weak measurements on the total angular momentum of an atomic ensemble, with a projective measurement at the end of the interrogation, reaches the ultimate sensitivity limit for coherent states in the limit of infinitesimally weak measurements. Note that the QFI, I_Q , is saturated asymptotically, i.e. in the limit of $N \rightarrow \infty$.

For non-zero measurement strengths $\chi t'$, the analysis gets significantly more complex, since weak measurements can squeeze the total angular momentum of the atomic ensemble, giving rise to non-classical scaling with respect to the number of atoms N [225]. This behavior can be recreated in our protocol by coupling a single ancilla to multiple qubits (compare with the current protocol where a single qubit interacts with a single ancilla). We leave the details of such a protocol to future work.

Appendix E

SUPPLEMENTARY INFORMATION FOR CHAPTER VII

E.1 Filter optimization

From the Gaussian approximation, the effective decay or gain rate on the dressed mode that corresponds to the bare system mode is found perturbatively as

$$\kappa_{\text{eff}}(\Delta) \approx 2g^2 \left| \text{Im} \left(\Delta - \frac{J_{1,2}^2}{\Delta - \frac{J_{2,3}^2}{\ddots \frac{\Delta - i\frac{\kappa_f}{2}}{\ddots}}} \right)^{-1} \right| \quad (\text{E.1})$$

given that $g \ll J_{i,i+1} \forall i$. Here, g is the interaction rate with the filter ($g_{\text{bs,tm}}$ in Chapter 7), and Δ is the detuning ($\Delta_{\text{bs,tm}}$ in Chapter 7). For multiple modes, we can also show that the rates can be found this way using the respective interaction rates and detunings if the detuning of two modes Δ_i and Δ_j satisfy $|\Delta_i - \Delta_j| \gg \frac{g^2}{B}$. To optimize over the effective rate given $\{J_{i,i+1}\}$ and κ_f , we define the following filter cost function:

$$C(r) = \int_{-1/2}^{1/2} d\left(\frac{\Delta}{B}\right) \left(\frac{\kappa_{\text{eff}}(\Delta/B)}{\kappa_{\text{eff}}(0)} - 1 \right)^2 + 2r \int_{1/2}^{\infty} d\left(\frac{\Delta}{B}\right) \left(\frac{\kappa_{\text{eff}}(\Delta/B)}{\kappa_{\text{eff}}(0)} \right)^2. \quad (\text{E.2})$$

This cost function is parametrized by r , which represents the ratio of importance of out-of-passband errors to in-passband errors. The roll-off speed of the filter increases with larger r , however, we observe larger-amplitude oscillations in κ_{eff} in the passband. Then, r can be tuned to change the "flatness" of the filter. In Chapter 7, we set $r \in [10^2, 10^3]$ while optimizing over the filter, for which we observed a satisfactory roll-off speed.

E.2 Independence of dissipators

In this Appendix, we show that the engineered dissipation induced by the multi-mode filter can be decomposed into independent Lindblad dissipators acting on each transition $|k\rangle \leftrightarrow |k-1\rangle$ of the system mode, provided that the transition frequencies are sufficiently separated. We present two complementary derivations. In Appendix E.2, we treat the adjacent transitions as independent Gaussian modes

coupled to the filter, and derive the independence condition $|\Delta_i - \Delta_j| \gg g^2/B$ via Heisenberg-picture perturbation theory. In Appendix E.2, we work directly with the anharmonic system Hamiltonian and adiabatically eliminate the filter modes using the effective operator formalism of Refs. [218, 44]. Both approaches yield the same independence condition and the same leading-order form for the effective dissipation rates.

With the Gaussian approximation

In order to simplify the dynamics analytically, we can approximate the transitions within the multilevel qubit as independent Gaussian modes with annihilation operators $\hat{d}_i$, $i = 1, 2, \dots$. We refer to these modes as the system modes. Similar to Eq. (7.11), the interaction Hamiltonian with respect to the bare Hamiltonian of the system modes, within the RWA, can be written as

$$\hat{H} = g \hat{f}_1^\dagger \left(\sum_{j \in s_d} \hat{d}_j e^{i\Delta_j t} + \sum_{j \in s_g} \hat{d}_j^\dagger e^{i\Delta_j t} \right) + \sum_{i=1}^{M-1} J_{i,i+1} \hat{f}_i \hat{f}_{i+1}^\dagger + \text{h.c.} \quad (\text{E.3})$$

where we assumed the same interaction rate g for both the beam splitter and the two-mode squeezing interaction, $\{\hat{f}_i\}$ are the filter annihilation operators, and s_d, s_g denote the sets for which the respective modes experience engineered decay or gain, respectively. We also engineer dissipation on the final filter mode, with a dissipator $\kappa_f \mathcal{D}[\hat{f}_M]$. The evolution of the annihilation or creation operators under this Hamiltonian can be described within the Heisenberg picture:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{f}}(t) \end{bmatrix} &= \left(\mathbf{M}^{(0)} + g \mathbf{M}^{(1)} \right) \cdot \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{f}}(t) \end{bmatrix} \\ &= B \left(\begin{bmatrix} \mathbf{S} & \\ & \mathbf{F} \end{bmatrix} + \tilde{g} \begin{bmatrix} \mathbf{0} & \mathbf{C}_1 \\ \mathbf{C}_2 & \mathbf{0} \end{bmatrix} \right) \cdot \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{f}}(t) \end{bmatrix}. \end{aligned} \quad (\text{E.4})$$

Here, $\hat{\mathbf{f}} := (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_{M-1})^T$ is the vector containing the annihilation operators of the filter modes, whereas $\hat{\mathbf{s}}$ is the vector containing the annihilation (creation) operators of the system modes in s_d (s_g). With this convention, we can represent both dynamics in a single equation. Lastly, $B > 0$ is the bandwidth of the filter, with $g \ll B$, $g \ll J_i \forall i$, and $g \ll \kappa_f$. We use the bandwidth to define $\tilde{g} := g/B$ as a small, dimensionless perturbation.

We can choose a rotating frame such that $\mathbf{S}$ is a diagonal matrix containing the elements of $\{\Delta_j\}$. $\mathbf{C}_1$ and $\mathbf{C}_2$ are the (S, M) dimensional coupling matrices with $S = |s_d| + |s_g|$. For example, if all of the modes are coupled with the beam splitter

(two-mode squeezing) interaction, $\mathbf{C}_1 = \mathbf{C}_2^T$ ($\mathbf{C}_1 = -\mathbf{C}_2^T$). Since the modes only couple to the first filter mode, only the first column of $\mathbf{C}_1$ and row of $\mathbf{C}_2$ is nonzero. The type of interaction determines the sign of the elements of the first column of $\mathbf{C}_1$, whereas the elements in the first row of $\mathbf{C}_2$ are always given by $-\tilde{g}$. Lastly, due to the symmetry of $\mathbf{S}$ and $\mathbf{F}$, $\mathbf{M}$ is symmetric. However, $\mathbf{F}$ is a complex matrix if $\kappa_f > 0$: thus, the eigenvalues of $\mathbf{F}$ are complex in general.

We aim to quantify the effect of the coupling with the filter on the system modes. Notice that if all of the detunings Δ_j are distinct, the eigenvector matrix of $\mathbf{S}$ is the identity matrix, and the respective eigenvalues are Δ_j/B . Let us write the perturbed, diagonal eigenvalue matrix as $\mathbf{\Lambda}^{(0)} + \tilde{g}\mathbf{\Lambda}^{(1)} + \tilde{g}^2\mathbf{\Lambda}^{(2)}$, and the corresponding perturbed eigenvector matrix as $\mathbf{V}^{(0)} + \tilde{g}\mathbf{V}^{(1)} + \tilde{g}^2\mathbf{V}^{(2)}$. We assume that eigenvectors are the column vectors. We also define submatrices in the form of

$$\mathbf{\Lambda}^{(i)} := \begin{bmatrix} \mathbf{\Lambda}_S^{(i)} & \\ & \mathbf{\Lambda}_F^{(i)} \end{bmatrix}, \quad \mathbf{V}^{(i)} := \begin{bmatrix} \mathbf{V}_{SS}^{(i)} & \mathbf{V}_{SF}^{(i)} \\ \mathbf{V}_{FS}^{(i)} & \mathbf{V}_{FF}^{(i)} \end{bmatrix}. \quad (\text{E.5})$$

Note that $\mathbf{V}_{SF}^{(0)} = \mathbf{V}_{FS}^{(0)} = \mathbf{0}$. From grouping the terms on the same order in the eigenvalue equation, we first write

$$\mathbf{M}^{(0)}\mathbf{V}^{(1)} + \mathbf{M}^{(1)}\mathbf{V}^{(0)} = \mathbf{V}^{(0)}\mathbf{\Lambda}^{(1)} + \mathbf{V}^{(1)}\mathbf{\Lambda}^{(0)} \quad (\text{E.6})$$

Multiplying from the left by $(\mathbf{V}^{(0)})^T$, we obtain

$$\mathbf{\Lambda}^{(1)} = \text{diag}((\mathbf{V}^{(0)})^T \mathbf{M}^{(1)} \mathbf{V}^{(0)}) = \mathbf{0}, \quad (\text{E.7})$$

and

$$\mathbf{M}^{(0)}\mathbf{V}^{(1)} - \mathbf{V}^{(1)}\mathbf{\Lambda}^{(0)} = -\mathbf{M}^{(1)}\mathbf{V}^{(0)}. \quad (\text{E.8})$$

Then, we notice that on the first order, the eigenvalues are not modified due to the connection with the filter modes. For the first order perturbation in the eigenvectors, we find

$$\mathbf{S}\mathbf{V}_{SS}^{(1)} - \mathbf{V}_{SS}^{(1)}\mathbf{\Lambda}_S^{(0)} = \mathbf{0} \quad (\text{E.9a})$$

$$\mathbf{F}\mathbf{V}_{FF}^{(1)} - \mathbf{V}_{FF}^{(1)}\mathbf{\Lambda}_F^{(0)} = \mathbf{0} \quad (\text{E.9b})$$

$$\mathbf{F}\mathbf{V}_{FS}^{(1)} - \mathbf{V}_{FS}^{(1)}\mathbf{\Lambda}_S^{(0)} = -\mathbf{C}_2\mathbf{V}_{SS}^{(0)} \quad (\text{E.9c})$$

$$\mathbf{S}\mathbf{V}_{SF}^{(1)} - \mathbf{V}_{SF}^{(1)}\mathbf{\Lambda}_F^{(0)} = -\mathbf{C}_1\mathbf{V}_{FF}^{(0)} \quad (\text{E.9d})$$

From the first two subequations, we observe that the diagonal blocks of the first order perturbation is parallel to the unperturbed eigenvectors themselves. This perturbation can then be eliminated (or ignored) via a renormalization of the perturbed

eigenvalue matrix. In contrast, the non-diagonal blocks require solving a Sylvester equation. In order to simplify the next order perturbation, we choose to work in the gauge where $\mathbf{V}_{SS}^{(1)} = \mathbf{V}_{FF}^{(1)} = 0$. For the second order perturbation of $\mathbf{\Lambda}^{(2)}$, we find:

$$\mathbf{M}^{(0)}\mathbf{V}^{(2)} + \mathbf{M}^{(1)}\mathbf{V}^{(1)} = \mathbf{V}^{(0)}\mathbf{\Lambda}^{(2)} + \mathbf{V}^{(1)}\mathbf{\Lambda}^{(1)} + \mathbf{V}^{(2)}\mathbf{\Lambda}^{(0)} \quad (\text{E.10})$$

Multiplying with $(\mathbf{V}^{(0)})^T$ from the left and substituting $\mathbf{\Lambda}^{(1)} = 0$, we obtain

$$\begin{aligned} \mathbf{\Lambda}^{(2)} &= \text{diag} \left((\mathbf{V}^{(0)})^T \mathbf{M}^{(1)} \mathbf{V}^{(1)} \right) \\ &= \text{diag} \left(\begin{bmatrix} (\mathbf{V}_{SS}^{(0)})^T \mathbf{C}_1 \mathbf{V}_{FS}^{(1)} & (\mathbf{V}_{FF}^{(0)})^T \mathbf{C}_2 \mathbf{V}_{SF}^{(1)} \end{bmatrix} \right) \end{aligned} \quad (\text{E.11})$$

Finally, the second order eigenvector perturbation results in

$$\mathbf{S}\mathbf{V}_{SS}^{(2)} - \mathbf{V}_{SS}^{(2)}\mathbf{\Lambda}_S^{(0)} = \mathbf{V}_{SS}^{(0)}\mathbf{\Lambda}_S^{(2)} - \mathbf{C}_1\mathbf{V}_{FS}^{(1)} \quad (\text{E.12a})$$

$$\mathbf{F}\mathbf{V}_{FF}^{(2)} - \mathbf{V}_{FF}^{(2)}\mathbf{\Lambda}_F^{(0)} = \mathbf{V}_{FF}^{(0)}\mathbf{\Lambda}_F^{(2)} - \mathbf{C}_2\mathbf{V}_{SF}^{(1)} \quad (\text{E.12b})$$

$$\mathbf{F}\mathbf{V}_{FS}^{(2)} - \mathbf{V}_{FS}^{(2)}\mathbf{\Lambda}_S^{(0)} = \mathbf{0} \quad (\text{E.12c})$$

$$\mathbf{S}\mathbf{V}_{SF}^{(2)} - \mathbf{V}_{SF}^{(2)}\mathbf{\Lambda}_F^{(0)} = \mathbf{0} \quad (\text{E.12d})$$

Here, since the spectra of $\mathbf{S}$ and $\mathbf{F}$ are guaranteed to be different (the spectra of $\mathbf{F}$ is complex when $\kappa_f > 0$ whereas the spectra of $\mathbf{S}$ is real), the unique solution of the last two subequations are $\mathbf{V}_{SF}^{(2)} = \mathbf{V}_{FS}^{(2)} = \mathbf{0}$. The magnitude of the second order perturbations in the system eigenvalues are thus given by the first row of $\mathbf{V}_{FS}^{(1)}$, up to a shift in the sign depending on the type of interaction with the filter. From Eq. (E.9), we find $\mathbf{V}_{FS}^{(1)}$ as (substituting $\mathbf{V}_{SS}^{(0)} = \mathbf{I}$):

$$(\mathbf{V}_{FS}^{(1)})_{ki} = \sum_j (\mathbf{V}_{FF}^{(0)})_{kj} \frac{((\mathbf{V}_{FF}^{(0)})^T \mathbf{C}_2)_{ji}}{s_i - f_j} \quad (\text{E.13})$$

where s_i, f_i are the eigenvalues of $\mathbf{S}, \mathbf{F}$, respectively. $\mathbf{V}_{SF}^{(1)}$ can also be written similarly,

$$(\mathbf{V}_{SF}^{(1)})_{ki} = \frac{(\mathbf{C}_1 \mathbf{V}_{FF}^{(0)})_{ki}}{f_i - s_k} \quad (\text{E.14})$$

Then, for the eigenvector perturbation to hold, we need to have

$$|s_i - f_j| \gg \tilde{g} \quad \forall \quad i, j. \quad (\text{E.15})$$

Because the filter is designed for a flat response of bandwidth B , its M modes have comparable linewidths, with no anomalously narrow mode. Hence, $\min_j |\text{Im}[f_j]| \sim$

$O(1/M^2)$, and $|s_i - f_j| \geq |\text{Im}[f_j]| \gtrsim 1/M^2$, and we need to have $\tilde{g} \ll 1/M^2$. Furthermore, for the second order perturbation analysis in the eigenvalues to hold, the perturbations need to be smaller than the spacing between the unperturbed eigenvalues, i.e.

$$|\Delta_i - \Delta_j| \gg g^2/B, \quad |f_i - f_j| \gg g^2/B, \forall i, j. \quad (\text{E.16})$$

If this condition is satisfied, the perturbative approach holds. The nonzero $\mathbf{V}_{ss}^{(2)}$ signifies that the different Gaussian modes hybridize when connected to the filter, which gives rise to a dressed eigenbasis. However, this is a second order effect. Each mode in the dressed eigenbasis experiences an effective dissipation, where the dissipation rate is $O(g^2/B)$, as seen from the second order correction in the eigenvalues. In contrast, if there is a subspace of system modes with identical detunings, one needs to apply degenerate perturbation theory, where the correct eigenvector matrix corresponds to linear superpositions of the corresponding degenerate system modes.

Without the Gaussian approximation

Here, let us forego the Gaussian approximation on the adjacent transitions in the system mode, and consider the anharmonic oscillator with the Hamiltonian $\hat{H}_0 = -K\hat{d}^\dagger\hat{d}^\dagger\hat{d}\hat{d} + \omega_d\hat{d}^\dagger\hat{d}$. In order to approximate the effective dynamics, we will adiabatically eliminate the filter modes. For this purpose, we follow the formalism of Refs. [218, 44]. First, we define a *ground subspace* with Hamiltonian $\hat{H}_g$, and an *excited subspace* with Hamiltonian $\hat{H}_e$. The ground subspace describes the stable, non-decaying modes, whereas the excited subspace describes the excitations decaying back to the ground subspace.

For our filter system, given the small coupling rate $g \ll J_{i,i+1}$, $g \ll \kappa_f$, excitations in the filter modes will decay rapidly to the bath. Then, we can identify the ground subspace as the subspace where the filter is in its vacuum state $|0_f\rangle$. Similarly, the excited subspace can be identified as the subspace where the filter has an excitation. In both subspaces, the system mode can be in any state. To simplify the dynamics, let us go into an interaction frame with respect to the $\hat{H}_0$ of the system mode, and the bare Hamiltonian of the filter modes, $\sum_{i=1}^{M-1} \omega_f \hat{f}_i^\dagger \hat{f}_i$. In this frame, we rewrite the Hamiltonian with the projection operators $\hat{P}_g := |0_f\rangle\langle 0_f|$ and $\hat{P}_e := I_f - |0_f\rangle\langle 0_f| \approx$

$\sum_{k=1}^M |e_k\rangle \langle e_k|$ with $|e_k\rangle := \hat{f}_k^\dagger |0_f\rangle$ as

$$\begin{aligned} \hat{H} &= \begin{bmatrix} \hat{P}_g & \hat{P}_e \end{bmatrix} \begin{bmatrix} \hat{H}_g & \hat{V}_+ \\ \hat{V}_- & \hat{H}_e \end{bmatrix} \begin{bmatrix} \hat{P}_g \\ \hat{P}_e \end{bmatrix} \\ &= g \hat{f}_1^\dagger \left(\hat{d}(t) e^{i(\omega_f + \omega_{bs})t} + \hat{d}^\dagger(t) e^{i(\omega_f + \omega_{tm})t} \right) + \sum_{i=1}^{M-1} J_{i,i+1} \hat{f}_i \hat{f}_{i+1}^\dagger + \text{h.c.}, \end{aligned} \quad (\text{E.17})$$

where I_f is the identity operator on the filter modes, and

$$\hat{d}(t) = \sum_{k \geq 1} \sqrt{k} |k-1\rangle \langle k| e^{-i(\omega_d - 2K(k-1))t} \quad (\text{E.18})$$

is the system mode in the interaction frame. Then, we can define the level-dependent detunings as in Eq. (7.14). Doing so, we find

$$\hat{H}_g = 0 \quad (\text{E.19a})$$

$$\hat{H}_e = I_s \otimes \left(\sum_{i=1}^{M-1} J_{i,i+1} |e_i\rangle \langle e_{i+1}| + \text{h.c.} \right) \quad (\text{E.19b})$$

$$\hat{V}_+ = g \sum_{k \geq 1} \sqrt{k} \left(|k-1\rangle \langle k| e^{i\Delta_{bs,k}t} + |k\rangle \langle k-1| e^{i\Delta_{tm,k}t} \right) \otimes |e_1\rangle \langle 0_f| \quad (\text{E.19c})$$

$\hat{V}_- = \hat{V}_+^\dagger$, and $|e_1\rangle = \hat{f}_1^\dagger |0_f\rangle$ is a single excitation on the first filter mode. The operator $\hat{V}_+$ describes the transition from the ground subspace to the excited subspace. Let us denote the separate time-dependent terms as

$$\hat{V}_+ = \sum_{x \in T} \sum_{k \geq 1} \hat{V}_{+,k}^x e^{i\Delta_{x,k}t} \quad (\text{E.20})$$

with $T = \{\text{bs}, \text{tm}\}$, $\hat{V}_{+,k}^{\text{bs}} = g\sqrt{k} |k-1\rangle \langle k| \otimes |e_1\rangle \langle 0_f|$, $\hat{V}_{+,k}^{\text{tm}} = g\sqrt{k} |k\rangle \langle k-1| \otimes |e_1\rangle \langle 0_f|$, and $\hat{V}_{-,k}^x = (\hat{V}_{+,k}^x)^\dagger$. We similarly write the Lindblad operator $\mathcal{D}[\sqrt{\kappa_f} \hat{f}_M]$ as $\hat{L} = \sqrt{\kappa_f} |0_f\rangle \langle e_M|$. Then, the unitary evolution in the excited subspace described by $\hat{H}_e$ is combined with the Lindblad operator to form a non-Hermitian operator that describes the full dynamics in this subspace,

$$\begin{aligned} \hat{H}_{\text{NH}} &= \hat{H}_e - \frac{i}{2} \hat{L}^\dagger \hat{L} \\ &= I_s \otimes \left(\sum_{i=1}^{M-1} J_{i,i+1} |e_i\rangle \langle e_{i+1}| + \text{h.c.} \right) - I_s \otimes \frac{i\kappa_f}{2} |e_M\rangle \langle e_M| \end{aligned} \quad (\text{E.21})$$

Given this operator, the effective Hamiltonian in the ground subspace is found as

$$\begin{aligned} \hat{H}_{\text{eff}} &= -\frac{1}{2} \sum_{x,y \in T} \sum_{k,n \geq 1} \hat{V}_{-,k}^x [\hat{F}_{x,k} + \hat{F}_{y,n}^\dagger] \\ &\quad \hat{V}_{+,n}^y e^{i(\Delta_{y,n} - \Delta_{x,k})t} \end{aligned} \quad (\text{E.22})$$

with $\hat{F}_{x,k} := (\hat{H}_{\text{NH}} - \Delta_{x,k})^{-1}$. The sum over the bs-bs and tm-tm terms in $\hat{V}_+$ and $\hat{V}_-$ will give rise to time-independent contributions, whereas the bs-tm cross terms will oscillate with frequencies $|\Delta_{\text{bs},k} - \Delta_{\text{tm},k-1}|$, $k \geq 2$. The time-dependent terms can be dropped with the RWA since $|\Delta_{\text{bs},i} - \Delta_{\text{tm},k}| \sim 2\omega_d \gg g^2/\kappa_f$. This approximation will result in

$$\begin{aligned} \hat{H}_{\text{eff}} \approx & -\frac{g^2}{2} \sum_{k \geq 1} k \left[\langle e_1 | \hat{F}_{\text{bs},k} + \hat{F}_{\text{bs},k}^\dagger | e_1 \rangle |k\rangle \langle k| + \right. \\ & \left. \langle e_1 | \hat{F}_{\text{tm},k} + \hat{F}_{\text{tm},k}^\dagger | e_1 \rangle |k-1\rangle \langle k-1| \right] \otimes |0_f\rangle \langle 0_f| \end{aligned} \quad (\text{E.23})$$

Thus, the interaction with the excited subspace results in an $O(g^2)$ perturbation in the energy levels of the system in the ground subspace, and leaves the eigenbasis unchanged as cross terms within the same channel (bs-bs or tm-tm with $k \neq l$) vanish identically by the orthogonality of the projectors $|k\rangle \langle k-1|$ for different k . Similarly, we can write the effective Lindblad operator in this space as

$$\begin{aligned} \hat{L}_{\text{eff}}(t) &= \hat{L} \sum_{x \in T} \sum_{k \geq 1} \hat{F}_{x,k} \hat{V}_{+,k}^x e^{i\Delta_{x,k}t} \\ &= \sum_{k \geq 1} \left(r_k^{\text{bs}} \hat{L}_k e^{i\Delta_{\text{bs},k}t} + r_k^{\text{tm}} \hat{L}_k^\dagger e^{i\Delta_{\text{tm},k}t} \right), \end{aligned} \quad (\text{E.24a})$$

$$\hat{L}_k := g\sqrt{\kappa_f} \sqrt{k} |k-1\rangle \langle k|, \quad (\text{E.24b})$$

where $r_k^x := \langle e_M | \hat{F}_{x,k} | e_1 \rangle$. Then, we observe that in the ground subspace, interacting with the filter with the beam splitter (two-mode squeezing) interaction will cause an effective Lindblad operator $\propto \hat{d}$ ($\propto \hat{d}^\dagger$), resulting in engineered decay (gain). The engineered dissipation rates are proportional to g^2 , the filter interaction rate to the second power. Finally, let us show the approximate independence of the engineered dissipation on each transition $|k\rangle \leftrightarrow |k+1\rangle$. The effective Lindblad master equation in the ground subspace can be written as

$$\dot{\hat{\rho}} = -i[\hat{H}_{\text{eff}}, \hat{\rho}] + \hat{L}_{\text{eff}} \hat{\rho} \hat{L}_{\text{eff}}^\dagger - \frac{1}{2} \{ \hat{L}_{\text{eff}}^\dagger \hat{L}_{\text{eff}}, \hat{\rho} \} \quad (\text{E.25})$$

Here, the terms oscillating at different frequencies in $\hat{L}_{\text{eff}}(t)$ will give rise again to oscillating terms. However, now, there will be bs-bs and tm-tm cross terms as well. The RWA can be applied if, for all k, n ,

$$|\Delta_{x,k} - \Delta_{x,n}| \gg g^2 \kappa_f \sqrt{kn} |r_k^x r_n^x|, \quad (\text{E.26})$$

for $x \in T$. From the tridiagonal structure of $\hat{H}_{\text{NH}}$, $|r_n^x|$ is found exactly as

$$|r_n^x| = \frac{\prod_{i=1}^M J_i}{|\det(\hat{H}_{\text{NH}} - \Delta_{x,n})|} \quad (\text{E.27})$$

assuming that $J_i > 0 \forall i$. For an upper bound on the required separation of the detunings, we want to estimate an upper bound on the r.h.s. of this inequality, i.e. an upper bound on the obtainable effective decay rate obtained with the filter. For this purpose, assuming that $\{J_i\}, \kappa_f$ is such that the filter response is almost flat, we can set $\Delta_{x,k} = 0$ (i.e. the transition is in the middle of the filter). Then, $|r_n^x|$ is given by

$$|r_n^x| = \begin{cases} \frac{\prod_{i=1}^M J_i}{\prod_{i=0}^{(M-2)/2} J_{2i+1}^2} & \text{if } M \text{ is even} \\ \frac{2 \prod_{i=1}^M J_i}{\kappa_f \prod_{i=0}^{(M-3)/2} J_{2i+1}^2} & \text{if } M \text{ is odd} \end{cases} \quad (\text{E.28})$$

For a bandpass filter, we will have that $J_i \sim \kappa_f \sim O(B)$: thus, $|r_n^x| \sim O(1/B)$, and the condition in Eq. (E.26) reduces to

$$|\Delta_{x,k} - \Delta_{x,n}| \gg \frac{g^2 \sqrt{kn}}{B} \quad (\text{E.29})$$

If this condition is satisfied for $x \in T$, we can rewrite Eq. (E.25) as

$$\dot{\hat{\rho}} = -i[\hat{H}_{\text{eff}}, \hat{\rho}] + \sum_{k \geq 1} \left(|r_k^{\text{bs}}|^2 \mathcal{D}[\hat{L}_k] \hat{\rho} + |r_k^{\text{tm}}|^2 \mathcal{D}[\hat{L}_k^\dagger] \hat{\rho} \right) \quad (\text{E.30})$$

The coherent off-diagonal terms dropped by the RWA in this equation correspond to the non-zero $\mathbf{V}_{SS}^{(2)}$ perturbation retained in the Gaussian approximation of Appendix E.2. Both are controlled by the same small parameter $g^2/(B|\Delta_i - \Delta_j|)$, and do not affect the leading-order engineered dissipation rates. Thus, the effective dynamics can be approximated with independent jump operators $g^2 \kappa_f k |r_k^{\text{bs}}|^2 \mathcal{D}[|k-1\rangle\langle k|]$ describing engineered decay, and $g^2 \kappa_f k |r_k^{\text{tm}}|^2 \mathcal{D}[|k\rangle\langle k-1|]$ describing engineered gain, $k \geq 1$. Lastly, the engineered dissipation rates can be suppressed with a far-detuned interaction (i.e. large $\Delta_{x,k}$), since $|r_k^x| \ll 1$ if $\Delta_{x,k} \gg J_{i,i+1}$, $\Delta_{x,k} \gg \kappa_f$.

E.3 CX bit-flip probability

Here, the logical states of the data qubits are defined from $|\pm\rangle = \mathcal{N}_\pm(|\alpha\rangle \pm |-\alpha\rangle)$, where $\mathcal{N}_\pm = (2 \pm 2e^{-2|\alpha|^2})^{-1/2}$. These states are perfectly orthogonal to each other, $\langle + | - \rangle = 0$. The logical states are

$$|0_L\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1_L\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \quad (\text{E.31})$$

Writing the logical states in the coherent state basis, we obtain

$$|k_L\rangle = |\alpha\rangle \frac{(\mathcal{N}_+ + (-1)^k \mathcal{N}_-)}{\sqrt{2}} + |-\alpha\rangle \frac{(\mathcal{N}_+ - (-1)^k \mathcal{N}_-)}{\sqrt{2}} \quad (\text{E.32})$$

with $k = 0, 1$. For $\alpha \gg 1$, $|0_L\rangle \approx |\alpha\rangle$, and $|1_L\rangle \approx |-\alpha\rangle$.

Let us assume a control qubit is prepared in $|\psi\rangle = |m\rangle$ in the Fock basis. We apply the dispersive coupling Hamiltonian $H = \chi_0(\hat{a}^\dagger \hat{a}) \otimes (\hat{d}^\dagger \hat{d})$, where $\hat{a}$ ($\hat{d}$) is the annihilation operator of the control (data) qubit. Initializing the data qubit in $|0_L\rangle$, we apply this Hamiltonian on the state $|m\rangle \otimes |0_L\rangle$ for a time t . Given that we have

$$e^{-i\chi_0 t(\hat{a}^\dagger \hat{a}) \otimes (\hat{d}^\dagger \hat{d})} |m\rangle \otimes |\alpha\rangle = |m\rangle \otimes |\alpha e^{-i\chi_0 m t}\rangle, \quad (\text{E.33})$$

this interaction will result in the following state:

$$|m\rangle \otimes \left(|\tilde{\alpha}\rangle \frac{(N_+ + N_-)}{\sqrt{2}} + |-\tilde{\alpha}\rangle \frac{(N_+ - N_-)}{\sqrt{2}} \right) \quad (\text{E.34})$$

Here, $\tilde{\alpha} = \alpha e^{-i\theta}$, and $\theta := \chi_0 m t$ is the rotation angle. We then trace the control qubit out and dissipatively stabilize the data qubit such that it is contained in the logical manifold $\{|0_L\rangle, |1_L\rangle\}$. The stabilization is performed with a two-photon process. The initial state can be written as

$$\hat{\rho}_i(\theta) = a_+^2 |\tilde{\alpha}\rangle \langle \tilde{\alpha}| + a_-^2 |-\tilde{\alpha}\rangle \langle -\tilde{\alpha}| + a_+ a_- (|\tilde{\alpha}\rangle \langle -\tilde{\alpha}| + |-\tilde{\alpha}\rangle \langle \tilde{\alpha}|) \quad (\text{E.35})$$

where $a_\pm = \frac{(N_\pm \pm N_-)}{\sqrt{2}}$. The evolution of the dissipatively stabilized data qubit will obey

$$\hat{\rho}(t) = \kappa_2 \mathcal{D}[\hat{d}^2 - \alpha^2] \hat{\rho}(t) \quad (\text{E.36})$$

where κ_2 is the engineered dissipation rate, and $\mathcal{D}[\hat{A}] := \hat{A} \hat{\rho} \hat{A}^\dagger - \frac{1}{2} \{\hat{A}^\dagger \hat{A}, \hat{\rho}\}$. The asymptotic steady state at the end of the stabilization is in the form of

$$\begin{aligned} \hat{\rho}_\infty = & c_{++} |+\rangle\langle +| + (1 - c_{++}) |-\rangle\langle -| \\ & + c_{+-} |+\rangle\langle -| + c_{+-}^* |-\rangle\langle +|. \end{aligned} \quad (\text{E.37})$$

We can write [181]

$$c_{++} = 2\mathcal{N}_+^2 \frac{1 + e^{-2|\alpha|^2}}{2} = \frac{1}{2}, \quad (\text{E.38})$$

$$c_{+-} = \frac{2i\mathcal{N}_+ \mathcal{N}_- |\alpha|^2}{\sqrt{e^{4|\alpha|^2} - 1}} F_\alpha(\theta) = \frac{i|\alpha|^2 e^{-2|\alpha|^2}}{1 - e^{-4|\alpha|^2}} F_\alpha(\theta) \quad (\text{E.39})$$

where we have defined

$$F_\alpha(\theta) := \int_0^\pi d\phi e^{-i(\phi-\theta)} I_0(2|\alpha|^2 \sin(\phi-\theta)) \quad (\text{E.40})$$

Here, I_0 is the modified Bessel function of the first kind. c_{++}, c_{+-} characterize the final state exactly. Let us now derive the bit-flip probability if a CX gate is applied to two data qubits, conditioned on the Fock state $|m\rangle$. We refer to this probability as $p_{\text{bf}}(\theta)$. First, defining the joint state of the control and the data qubit after a rotation, given in Eq. (E.34), as $|m\rangle \otimes |\xi\rangle$, we will now have a state $|m\rangle \otimes |\xi\rangle \otimes |\xi\rangle$. Tracing out the control qubit, the data qubits will be in the form of $\hat{\rho}_\infty \otimes \hat{\rho}_\infty$, where $\hat{\rho}_\infty$ is characterized in Eqs. (E.37), (E.38). The bit-flip probability can then be written as

$$\begin{aligned}\tilde{p}_{\text{bf}}(\theta) &= 1 - (\langle 0_L | \hat{\rho}_\infty | 0_L \rangle)^2 - (\langle 1_L | \hat{\rho}_\infty | 1_L \rangle)^2 \\ &= \frac{1}{2} - 2\text{Re}[c_{+-}]^2 \\ &= \frac{1}{2} \left[1 - \left(\frac{|\alpha|^2}{\sinh(2|\alpha|^2)} \text{Re}[F_\alpha(\theta)] \right)^2 \right]\end{aligned}\quad (\text{E.41})$$

From the definition of the Bessel function, $\frac{|\alpha|^2}{\sinh(2|\alpha|^2)} \text{Re}[F_\alpha(\theta)]$ can be rewritten as the following for $0 \leq \theta \leq \pi$:

$$\begin{aligned}& \frac{|\alpha|^2}{\sinh(2|\alpha|^2)} \sum_{m=0}^{\infty} \frac{|\alpha|^{4m}}{(m!)^2} \int_0^\pi d\phi \sin^{2m+1}(\phi - \theta) \\ &= 1 - \frac{2|\alpha|^2}{\sinh(2|\alpha|^2)} \sum_{m=0}^{\infty} \frac{|\alpha|^{4m}}{(m!)^2} \int_0^\theta d\phi \sin^{2m+1}(\phi) \\ &= 1 - \frac{2|\alpha|^2}{\sinh(2|\alpha|^2)} \int_0^\theta d\phi \sin(\phi) I_0(2|\alpha|^2 \sin(\phi)) \\ &:= 1 - M_\alpha(\theta)\end{aligned}\quad (\text{E.42})$$

which gives the expression in Eq. (7.17) in Chapter 7.

Given a control qubit in the state of Eq. (7.21) in Chapter 7, let us compute the total bit-flip probability from the CX gate on two data qubits. We again assume dispersive coupling between the control and the data qubit. Notice that the terms that are non-diagonal in the Fock basis in Eq. (7.21) will lead to zero contribution in the final state of the data qubits, when the control qubit is traced out. Then, the initial state of the data qubits before the stabilization is written as

$$\frac{1+p}{2} \hat{\rho}_i(0) \otimes \hat{\rho}_i(0) + \frac{(1-p)^n}{2} \hat{\rho}_i(n\theta) \otimes \hat{\rho}_i(n\theta) + \frac{p}{2} \sum_{j=1}^{n-1} (1-p)^j \hat{\rho}_i(j\theta) \otimes \hat{\rho}_i(j\theta) \quad (\text{E.43})$$

where θ is the rotation angle. We set $n\theta = \pi$, such that the data qubits undergo a π -pulse $|0_L\rangle \rightarrow |1_L\rangle$ if the control qubit is in state $|n\rangle$. The first two terms do not

contribute to the bit-flip probability after the stabilization (since $p_{\text{bf}}(0) = p_{\text{bf}}(\pi) = 0$). Then, Eq. (7.18) in Chapter 7 is found to be

$$p_{\text{bf}} = \frac{p}{2} \sum_{k=1}^{n-1} (1-p)^k \tilde{p}_{\text{bf}} \left(\theta = \frac{\pi k}{n} \right) \quad (\text{E.44})$$

Linearizing for $p \ll 1$, we find that $p_{\text{bf}} \approx \frac{p}{2} \sum_{k=0}^n \tilde{p}_{\text{bf}} \left(\theta = \frac{\pi k}{n} \right)$. Here, note that we extended the summation limits to include $k = 0$ and $k = n$, as their contribution to the sum is zero. Using the definition for $p_{\text{bf}}(\theta)$ in Eq. (7.17) of Chapter 7, we expand this expression as

$$\begin{aligned} p_{\text{bf}} &\approx \frac{p}{2} \sum_{k=0}^n \left[M_{\alpha} \left(\frac{\pi k}{n} \right) - \frac{1}{2} M_{\alpha}^2 \left(\frac{\pi k}{n} \right) \right] \\ &\approx \frac{pn}{2} \int_0^1 dz \left[M_{\alpha}(\pi z) - \frac{1}{2} M_{\alpha}^2(\pi z) \right] \end{aligned} \quad (\text{E.45})$$

where we converted the summation over the Fock levels to an integral in the limit that $n \gg 1$. From the symmetry of $M(\pi z)$, we have that $M_{\alpha}(\pi z) = 2 - M_{\alpha}(\pi(1-z))$ for $z \in [\frac{1}{2}, 1]$. Then, we find:

$$\int_0^1 dz M_{\alpha}(\pi z) = \int_0^{\frac{1}{2}} dz M_{\alpha}(\pi z) + \int_{\frac{1}{2}}^1 dz [2 - M_{\alpha}(\pi(1-z))] = 1 \quad (\text{E.46})$$

We can similarly rewrite the second term in Eq. (E.45) as

$$\begin{aligned} \int_0^1 dz M_{\alpha}^2(\pi z) &= \int_0^{\frac{1}{2}} dz M_{\alpha}^2(\pi z) + \int_{\frac{1}{2}}^1 dz [2 - M_{\alpha}(\pi(1-z))]^2 \\ &= 2 - 4 \int_0^{\frac{1}{2}} dz M_{\alpha}(\pi z) + 2 \int_0^{\frac{1}{2}} dz M_{\alpha}^2(\pi z) \end{aligned} \quad (\text{E.47})$$

In order to approximate this expression, we notice that the integrand in the definition of $M_{\alpha}(\theta)$ in Eq. (E.42) is sharply peaked around $\pi/2$. In the vicinity of $\pi/2$, we can approximate $I_0(2|\alpha|^2 \sin \phi) \approx \exp(2|\alpha|^2 \sin \phi) / \sqrt{4\pi|\alpha|^2 \sin \phi}$, and $\sin(\phi) \approx 1 - \frac{(\frac{\pi}{2} - \phi)^2}{2}$. Then, using the Laplace method, we rewrite $M_{\alpha}(\theta)$ as

$$\begin{aligned} M_{\alpha}(\theta) &\approx \frac{|\alpha|/\sqrt{\pi}}{\sinh(2|\alpha|^2)} \int_{-\pi/2}^{\theta-\pi/2} d\phi e^{2|\alpha|^2(1-\frac{u^2}{2})} \\ &\approx \frac{|\alpha|/\sqrt{\pi}}{\sinh(2|\alpha|^2)} \int_{-\infty}^{\theta-\pi/2} d\phi e^{2|\alpha|^2(1-\frac{u^2}{2})} \end{aligned} \quad (\text{E.48})$$

For $0 \leq \theta \leq \pi/2$, we can further write

$$M_{\alpha}(\theta) \approx 1 - \text{erf} \left(|\alpha| \left(\frac{\pi}{2} - \theta \right) \right), \quad (\text{E.49})$$

where $\text{erf}(z)$ is the erf function. Finally, plugging this into Eq. (E.47), we obtain

$$\int_0^1 dz M_\alpha^2(\pi z) \approx 1 + 2 \int_0^{\frac{1}{2}} dz \text{erf}^2(|\alpha|\pi z) \approx 2 - \left(\frac{2}{\pi}\right)^{3/2} \frac{1}{|\alpha|} \quad (\text{E.50})$$

for $|\alpha| \gg 1$. Thus, Eq. (E.45) can finally be approximated as $p_{\text{bf}} \approx \frac{pn}{\sqrt{2\pi^3}|\alpha|}$ for $|\alpha|, n \gg 1$.

E.4 Bit-flip floor

We now derive the bit-flip floor analytically. We assume the ideal setting of the 0-n Fock qubit where we engineer dissipation with rate κ_s between adjacent levels in the lower and upper half spaces. The transition rates are summarized in Eq. (7.3), and we also assume that intrinsic loss processes are much weaker than the engineered dissipation $\kappa_i^{\uparrow, \downarrow} \ll \kappa_s$.

In this setting, we observed that the system stabilizes to a metastable state in short time-scales $t \sim 1/\kappa_s$. As the populations in each level converge to their steady state values, the CX bit-flip probability when the qubit is used as the control qubit also converges to a fixed value. We refer to this probability as the *bit-flip floor*. Let us analyze how the bit-flip floor varies as a function of the number of levels in the qubit N , and the mean photon number in the cat qubit α . Furthermore, let us examine the speed of convergence to the bit-flip floor.

Fixing $|\alpha|^2 = 10$, $\kappa_i^\downarrow = 10^{-3}\kappa_s$, and $\kappa_i^\uparrow = 10^{-5}\kappa_s$, we plot the total bit-flip probability as a function of the stabilization time t for various n in Fig E.1a. From the Figure, we observe that the bit-flip probability decays exponentially with time before converging to the bit-flip floor.

Then, we plot the scaling of the bit-flip floor with respect to the number of levels N for various $|\alpha|$ in Fig E.1b. We fix $\kappa_i^\uparrow = 10^{-2}\kappa_i^\downarrow$, and define $r_\downarrow = \kappa_i^\downarrow/\kappa_s \ll 1$. We plot the case where $r_\downarrow = 1^{-3}$ with the circles, and $r_\downarrow = 1^{-4}$ with the stars. For relatively small N , the bit-flip floor decays exponentially with N , where we observe that the decay rate is proportional to $|\alpha|^2$. As $N \gg 1$, the scaling is modified into a power law. Let us derive this scaling analytically, assuming that the limiting loss process for the bit-flip floor is intrinsic decay, i.e. $\kappa_i^\downarrow \gg \kappa_i^\uparrow$.

First, notice that given the initial state before the stabilization in Eq. (7.21), we have

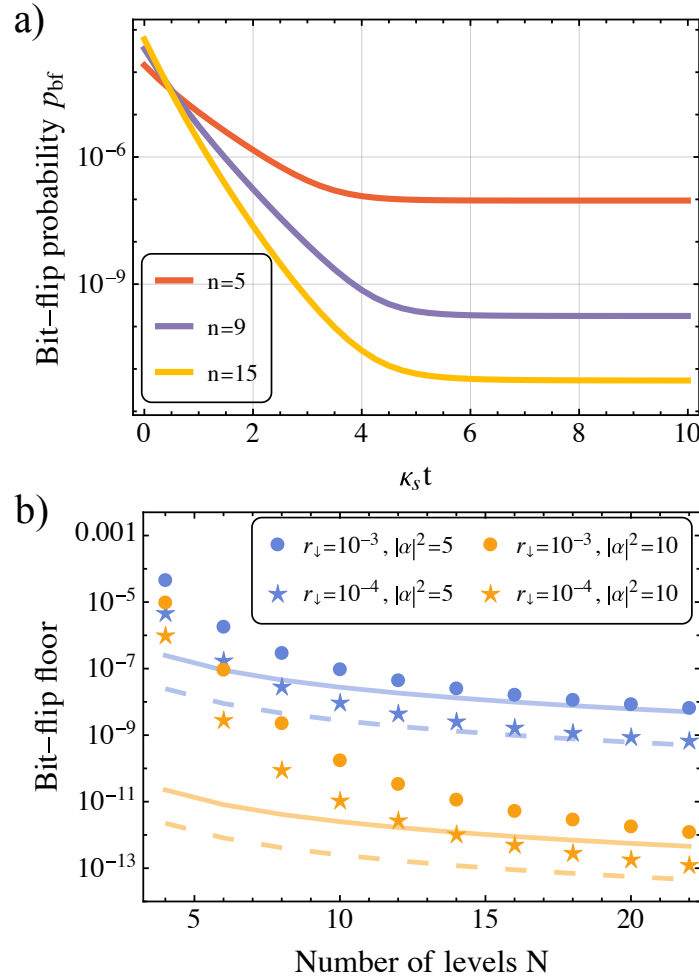


Figure E.1: Bit-flip probability p_{bf} of the CX gate on cat qubits with mean photon number $|\alpha|^2$, when the 0- n Fock qubit is used as the control qubit. a) On relatively short timescales $t \sim 1/\kappa_s$, the 0- n Fock qubit stabilizes and p_{bf} converges to a fixed value referred to as the bit-flip floor, inversely proportional to n . b) Bit-flip floor as a function of the number of levels $N = n + 1$ of the Fock qubit. The floor decays exponentially with n for small n , however the scaling is modified to $1/n^2$ for $n \gg 1$ (see Eq. (E.56)). We set $r_{\downarrow} = \frac{\kappa_{\downarrow}}{\kappa_s}$ to 10^{-3} (plotted with circles) and 10^{-4} (plotted with stars), for $|\alpha|^2 = 5$ (plotted in blue) and $|\alpha|^2 = 10$ (plotted in yellow). The analytical fit for large n is plotted with the respective colors, with plain and dashed lines for $r_{\downarrow} = 10^{-3}$ and $r_{\downarrow} = 10^{-4}$, respectively.

that

$$P(0) = \frac{1}{2} (p + (1-p)^n), \quad (\text{E.51a})$$

$$P(j) = \frac{p}{2} (1-p)^j, \quad 1 \leq j \leq n-1, \quad (\text{E.51b})$$

$$P(n) = \frac{1}{2} (1-p)^n. \quad (\text{E.51c})$$

Since $\kappa_i^\downarrow \gg \kappa_i^\uparrow$, the bit-flip errors will arise from intrinsic loss to lower levels in the upper half of the 0-n Fock qubit. The total population in these levels is initially $\sum_{j=(n+1)/2}^n P(j) = \frac{1}{2}(1-p)^{(n+1)/2}$. On the timescale $t \sim 1/\kappa_s$, the populations in the upper half will converge to the metastable state given in Eq. (7.6) of Chapter 7. Then, the populations in this section can be written as

$$P(j) \approx \frac{(1-p)^{\frac{n+1}{2}}}{2} \left(\frac{\kappa_i^\downarrow}{\kappa_s + \kappa_i^\uparrow} \right)^{n-j} \approx \frac{(1-p)^{\frac{n+1}{2}}}{2} \left(\frac{\kappa_i^\downarrow}{\kappa_s} \right)^{n-j} \quad (\text{E.52})$$

for $\frac{n+1}{2} \leq j \leq n$. The level populations decay exponentially as we go to lower levels from n . In contrast, the bit-flip probability of a level increases. However, for large enough n , the decay in populations will be faster than the increase in the bit flip probability. Then, the highest contribution to the total bit-flip probability will be due to level $n-1$. The bit-flip probability can therefore be approximated as

$$p_{\text{bf}} \approx \frac{(1-p)^{\frac{n+1}{2}}}{2} \left(\frac{\kappa_i^\downarrow}{\kappa_s} \right) \tilde{p}_{\text{bf}} \left(\pi - \frac{\pi}{n} \right) \quad (\text{E.53})$$

where

$$\tilde{p}_{\text{bf}} \left(\pi - \frac{\pi}{n} \right) = \frac{1}{2} \left[1 - \left(1 - M_\alpha \left(\pi - \frac{\pi}{n} \right) \right)^2 \right] = M_\alpha \left(\frac{\pi}{n} \right) - \frac{1}{2} M_\alpha^2 \left(\frac{\pi}{n} \right) \quad (\text{E.54})$$

as $M(\pi - \frac{\pi}{n}) = 2 - M(\frac{\pi}{n})$. Let us approximate $M(\frac{\pi}{n})$ for $n \gg 1$. Defining $x := \pi/n$, $x \ll 1$, we have:

$$\begin{aligned} M(x) &= \frac{2|\alpha|^2}{\sinh(2|\alpha|^2)} \int_0^x d\phi \sin(\phi) I_0(2|\alpha|^2 \sin(\phi)) \\ &\approx \frac{|\alpha|^2 x^2}{\sinh(2|\alpha|^2)}, \quad \text{for } x \ll 1 \end{aligned} \quad (\text{E.55})$$

Then, for large enough n such that $M(\frac{\pi}{n}) \ll 1$, we find

$$p_{\text{bf}} \approx \frac{\pi^2 |\alpha|^2 (1-p)^{\frac{n+1}{2}}}{2n^2 \sinh(2|\alpha|^2)} \left(\frac{\kappa_i^\downarrow}{\kappa_s} \right). \quad (\text{E.56})$$

We thus observe that the bit-flip floor decays as $1/n^2$ in the limit of $n \rightarrow \infty$. This analytical expression is plotted with plain lines for $r_{\downarrow} = 10^{-3}$, and dashed lines for $r_{\downarrow} = 10^{-4}$ in Fig. E.1b. For both cases, blue curves represent $|\alpha|^2 = 5$, whereas yellow curves represent $|\alpha|^2 = 10$. We observe that this function is a good fit in the limit of $N \gg 1$.

E.5 Analytical solution to bit-flip probability after CX gates with decay

For a large number of filter modes, performing the full simulation of the stabilization of the 0-n Fock qubit, as well as the noisy CX gates conditioned on it can be computationally challenging. Because of this, it is valuable to approximately quantify the bit-flip probabilities with an analytical method. We show that such a method can be obtained here.

First, we have shown in Section 7.2 that the evolution of the 0-n Fock qubit during an ideal stabilization can be separated into two components: the evolution of the diagonal elements, $|i\rangle\langle i|$, and the coherences, $|i\rangle\langle j|$, $i \neq j$. The former can be computed with a CTMC, whereas the latter only shrinks in amplitude, where the rate can be computed given the decay and heating rates κ_i^d , κ_i^h between levels $|j\rangle$ and $|j+1\rangle$. These rates are obtained analytically in a perturbative fashion (cf. Eq. (E.1)).

After the stabilization, the 0-n Fock qubit is used to perform the CX gates on the cat qubits. The interaction Hamiltonian between the Fock qubit and a cat qubit is in the form of $H = \chi \hat{n}_a \hat{n}_d$, where $\hat{n}_x = \hat{x}^\dagger \hat{x}$, and $\hat{a}$ ($\hat{d}$) is the annihilation operator on the Fock (cat) qubit. Furthermore, the CX gate is subject to intrinsic single-photon events. For the CX gate on M cat qubits, their final state can be written as

$$\hat{\Psi}_f = \Phi^M(\text{Tr}_F(\Lambda^M(\hat{\rho} \otimes \hat{\Psi}_i))) \quad (\text{E.57})$$

where $\hat{\Psi}_i := |0_L\rangle\langle 0_L|^{\otimes M}$, $\Lambda^M(\cdot) := \Lambda_M \circ \dots \circ \Lambda_1$, $\Phi^M(\cdot) := \Phi_M \circ \dots \circ \Phi_1$, $\Phi_i(\cdot) := I^{\otimes M-i} \otimes \Phi(\cdot) \otimes I^{i-1}$, and the trace is over the Fock qubit. Furthermore, we define Λ_i to be the channel Λ applied to the control qubit and the i^{th} qubit, with the identity acting on all remaining subsystems. Here, Λ is the CX gate, and Φ is the stabilization of the cat qubit via two-photon dissipation. Given this final state, the bit-flip probability is

$$1 - (\langle 0_L |^{\otimes M} \hat{\Psi}_f | 0_L \rangle^{\otimes M} + \langle 1_L |^{\otimes M} \hat{\Psi}_f | 1_L \rangle^{\otimes M}). \quad (\text{E.58})$$

We observe that the CX interaction satisfies

$$\Lambda(|i\rangle\langle j| \otimes \hat{\rho}) \propto |i\rangle\langle j| \otimes \hat{\sigma} \text{ for } i \neq j, \quad (\text{E.59})$$

i.e. the coherence terms of the Fock qubit are unaffected up to an amplitude rescaling due to decoherence, and the cat qubit is transformed from $\hat{\rho}$ into $\hat{\sigma}$. Due to the tracing out of the Fock qubit, these terms will not contribute to $\hat{\Psi}_f$. Furthermore, there is no cross-talk between them and the diagonal terms $|i\rangle\langle i|$ of the Fock qubit. Then, we can ignore the coherences of the initial state of the Fock qubit while calculating the bit-flip probability. Because of this, we work with the initial Fock qubit state in the form of $\hat{\rho} = \sum_j |j\rangle\langle j|$.

We can then compute the effect of Λ using the Lindbladian in Eq. (7.2) with the CX Hamiltonian $H = \chi \hat{n}_a \hat{n}_d$. For coherent states $|\alpha\rangle, |\beta\rangle$, the Hamiltonian satisfies

$$[H, |j\rangle\langle j| \otimes |\alpha\rangle\langle\beta|] = \chi j |j\rangle\langle j| \otimes [\hat{n}_d, |\alpha\rangle\langle\beta|] \quad (\text{E.60})$$

This term acts on the initial coherent state superposition to rotate them in the phase space over time, and the Fock qubit is unaffected. Conversely, the Lindbladian $\mathcal{L}(\hat{\rho})$ in Eq. (7.2) causes single photon decay and heating events, causing transitions between adjacent Fock levels, and leaves the coherent states unchanged. Therefore, we propose the following ansatz for the evolution of the joint state of the Fock qubit $\sum_j p_j |j\rangle\langle j|$ and the dyad $|\alpha\rangle\langle\beta|$:

$$\hat{\sigma} = \sum_j \int d\theta P_j(t, \theta) |j\rangle\langle j| \otimes |\alpha e^{-i\theta}\rangle\langle\beta e^{-i\theta}| \quad (\text{E.61})$$

where $P_j(0, \theta) = p_j \delta(\theta) \forall j$, $\delta(x)$ is the Dirac delta function, and $\sum_j p_j = 1$. For easier notation, let us define $\alpha e^{-i\theta} = \alpha_\theta$, $\beta e^{-i\theta} = \beta_\theta$. Then,

$$\frac{d\hat{\sigma}}{dt} = \sum_j \int d\theta \partial_t P_j(t, \theta) |j\rangle\langle j| \otimes |\alpha_\theta\rangle\langle\beta_\theta| \quad (\text{E.62})$$

Also, notice that

$$\begin{aligned} \frac{d|\alpha_\theta\rangle}{d\theta} &= e^{-|\alpha|^2/2} \sum_n (-in) \frac{(\alpha e^{-i\theta})^n}{\sqrt{n!}} |n\rangle \\ &= -i\hat{n} |\alpha e^{-i\theta}\rangle \end{aligned} \quad (\text{E.63})$$

Then, $[\hat{n}, |\alpha_\theta\rangle\langle\beta_\theta|] = i \frac{d}{d\theta} (|\alpha_\theta\rangle\langle\beta_\theta|)$. Using this identity, $-i[H, \hat{\sigma}]$ can be written as (up to the constant factor χ):

$$\begin{aligned} &\sum_j \int d\theta j P_j(t, \theta) |j\rangle\langle j| \otimes \frac{d}{d\theta} (|\alpha_\theta\rangle\langle\beta_\theta|) \\ &= - \sum_j \int d\theta \partial_\theta j P_j(t, \theta) |j\rangle\langle j| \otimes |\alpha_\theta\rangle\langle\beta_\theta| \end{aligned} \quad (\text{E.64})$$

Let us also denote $\mathcal{L}(\hat{\sigma})$ with

$$\sum_{j,l} \int d\theta M_j(l) P_j(t, \theta) |j\rangle \langle j| \otimes |\alpha_\theta\rangle \langle \beta_\theta| \quad (\text{E.65})$$

The evolution of $\hat{\sigma}$ is found from

$$\frac{d\hat{\sigma}}{dt} = -i[H, \hat{\sigma}] + \mathcal{L}(\hat{\sigma}). \quad (\text{E.66})$$

From Eqs. (E.64) and (E.65), we observe this equation can be rewritten in terms of $P_j(t, \theta)$ only. Doing so, we obtain the following partial differential equation

$$\partial_t P_j(t, \theta) = -\chi j \partial_\theta P_j(t, \theta) + \sum_l M_j(l) P_j(t, \theta). \quad (\text{E.67})$$

This equation can be solved by going into the Fourier domain, where $\tilde{P}_j(t, k) = \int d\theta e^{-ik\theta} P_j(t, \theta)$.

$$\begin{aligned} \partial_t \tilde{P}_j(t, k) &= -i\chi j k \tilde{P}_j(t, k) + \sum_l M_j(l) \tilde{P}_j(t, k) \\ &= \sum_l M'_j(l) \tilde{P}_j(t, k) \end{aligned} \quad (\text{E.68})$$

Solving this for each j results in obtaining the evolution of $\hat{\sigma}$. Let us refer to the solution at the end of the CX interaction as $P_j(T, \theta)$. We notice that this solution is independent of the coherent state amplitudes α, β . Using this property, and given the initial state of the cat qubit before the interaction is in the form of

$$|0_L\rangle \langle 0_L| = a_{++} |\alpha\rangle \langle \alpha| + a_{--} |-\alpha\rangle \langle -\alpha| + a_{+-} (|\alpha\rangle \langle -\alpha| + |-\alpha\rangle \langle \alpha|), \quad (\text{E.69})$$

The final state will therefore result in

$$\begin{aligned} \hat{\sigma}_f &= \sum_j \int d\theta P_j(T, \theta) |j\rangle \langle j| \otimes (a_{--} |-\alpha_\theta\rangle \langle -\alpha_\theta| \\ &\quad + a_{++} |\alpha_\theta\rangle \langle \alpha_\theta| + a_{+-} (|\alpha_\theta\rangle \langle -\alpha_\theta| + |-\alpha_\theta\rangle \langle \alpha_\theta|)) \end{aligned} \quad (\text{E.70})$$

For the two-resonator cost function considered in Chapter 7, we need to apply the same CX interaction to a secondary cat qubit resonator initialized in $|0_L\rangle \langle 0_L|$. For an easier computation, we can commute the stabilization of the first qubit in Eq. (E.57), as well as the projection into the logical subspace $\{|0_L\rangle, |1_L\rangle\}$, with this interaction. Thus, we obtain two possible initial Fock qubit states depending on the result of this projection. For both of these states, one can resolve the evolution for $P_j(t, \theta)$ for the second resonator, which is then stabilized and projected into the logical subspace. Finally, the bit-probability can be computed from Eq. (E.58) with $M = 2$.

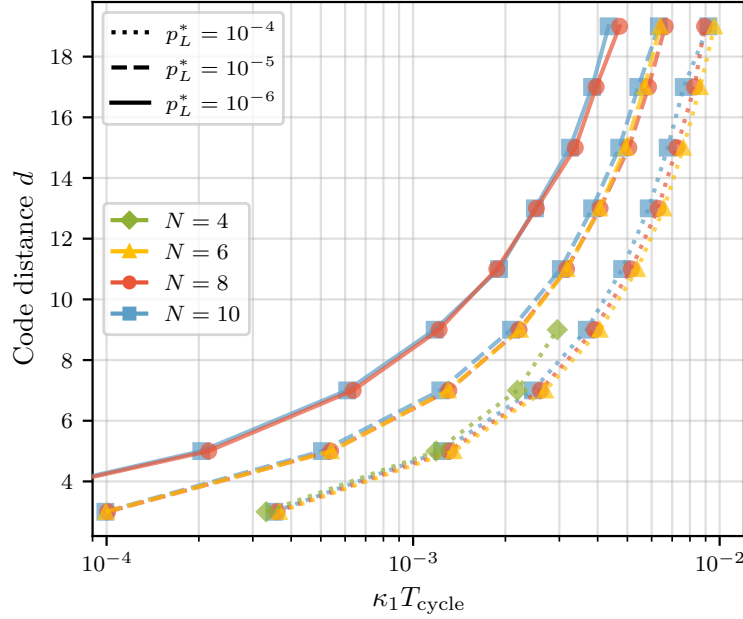


Figure E.2: Average logical error per round p_L^* when the 0-n Fock qubit is used as an ancilla in the repetition code, paired with cat data qubits, with a dephasing budget on the Fock qubit. The contours for p_L^* are shown as a function of $\kappa_1 T_{\text{cycle}}$ and the code distance d , where κ_1 is the single-photon loss rate of the cat qubits, and T_{cycle} is the cycle time for a single round of error correction. Z errors are applied on the ancilla and the data qubits at the start of each error correction round. We assume single-photon events on the ancilla with $\tau_i^\downarrow = 250 \mu\text{s}$, $\tau_i^\uparrow = 250 \text{ ms}$, and dephasing such that the overall Z error probability p_Z^{ancilla} is doubled. Lastly, we assume a stabilization time $t_s = 500 \text{ ns}$, CX gate time $t_g = 200 \text{ ns}$, and cat mean photon number $|\alpha|^2 = 10$.

E.6 Logical error rates with a dephasing budget

Here, we recompute the average logical error per round, given the 0-n Fock qubit as the ancilla and cat data qubits, for the repetition code. In Sec. 7.4, we have ignored the dephasing effects as we assumed that E_J can be sufficiently increased such that dephasing due to charge noise is suppressed exponentially in E_J/E_C . Here, we relax this assumption and work in a regime where dephasing results in a loss of coherence equal to that of the intrinsic single-photon events. In Sec. 7.2, we have computed that single-photon events result in a decay of coherence terms as $\hat{\rho}(t) = e^{-\frac{1}{2}\kappa_c t} \hat{\rho}(0)$, where $\hat{\rho}(0) = |m\rangle\langle n|$, $\kappa_c = \kappa_m^t + \kappa_n^t$, and $\kappa_n^t = \kappa_{n-1}^d + \kappa_n^h$ is the total rate of the outgoing excitations from level n . Then, the coherence $|0\rangle\langle n|$ decays with a total rate $\kappa_c = n\kappa_i^\downarrow + (n+2)\kappa_i^\uparrow \approx n\kappa_i^\downarrow$. Similarly, dephasing with a Lindblad operator $\hat{L}_\phi = \sqrt{\kappa_\phi} \hat{a}^\dagger \hat{a}$ results in a loss of coherence with rate $\kappa_c = \kappa_\phi(m-n)^2$. Thus, we set $\kappa_\phi = \kappa_i^\downarrow/n$ for the 0-n Fock qubit, so that the decay due to dephasing is equal to

that due to the single-photon loss.

The recomputed average logical error rates (including the bit-flip contribution) can be found in Fig. E.2. We plot the contours $p_L^* \in \{10^{-4}, 10^{-5}, 10^{-6}\}$ with dotted, dashed, and plain lines, respectively. We work with $N \in \{4, 6, 8, 10\}$, which are represented with the different colors. We set the Fock qubit stabilization time as $t_s = 500$ ns, and the CX gate time as $t_g = 200$ ns: thus, the total Z and X errors on the ancilla are fixed and are independent of the cycle time T_{cycle} . T_{cycle} determines the amount of dephasing on the data qubits, with $p_Z^{\text{data}} \approx |\alpha|^2 \kappa_1 T_{\text{cycle}}$, $|\alpha|^2 \kappa_1 T_{\text{cycle}} \ll 1$, as in Sec. 7.4. Here, κ_1 is the single-photon loss rate of the cat qubits. Finally, we set the mean photon number as $|\alpha|^2 = 10$.

Comparing Fig. E.2 with Fig. 7.4 in Sec. 7.4, we observe that a given logical error rate for a code distance d is achieved for smaller $\kappa_1 T_{\text{cycle}}$, i.e. for a smaller dephasing rate of the cat qubit. This is expected, as the additional dephasing budget causes an increased ancilla phase-flip probability, which deteriorates the performance of the repetition code. However, this performance change does not require an order-of-magnitude change in $\kappa_1 T_{\text{cycle}}$.