THE INTERSTELLAR DUST AND GAS STRUCTURES

Thesis by

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Abstract

An investigation of the statistical properties of the mechanics of the dust and gas structures making up the interstellar medium is carried out. The aim is to get a comprehensive picture of the general nature of the motions and structure of the interstellar material that would follow from certain simple assumptions. Comparison with observation then shows to what extent the behavior of the interstellar material can be accounted for on such a basis and to what extent further factors, such as for example magnetic fields, will have to be studied in the hope of finding more satisfactory agreement.

In I the interstellar gas is considered. It is shown that the observed radial motions of the gas clouds, if unordered, give too high a kinetic temperature for one to account for the observed clumping of the gas into clouds. The stability of a clump of gas was investigated using a modification of the ideas commonly employed in the virial theorem. It is further shown that given discrete gas clouds, the unordered component of their proper motions must be even lower to prevent them from collapsing into extremely dense structures, possibly clusters of stars, after repeated low velocity collisions.

In II the observed dust structures are considered and their general evolution under the influence of the velocity field of the gas and the radiation field is worked out. It is shown that for a continuous distribution of dust sizes, certain types of finely striated structures cannot be formed.

From the inadequacies of this non-magnetic theory, one concludes that magnetic fields must play a very significant role

in the dynamics of the interstellar medium.

Finally, in III the collapse of a nonturbulent nonrotating dust cloud under its own gravitational field is very briefly considered yielding remarkably short collapse times of the order of only $3 \cdot 10^6$ years.

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Part I

Interstellar Gas Clouds

Most of the interstellar matter is in the gaseous state, only about one percent being dust. While the shape of a gas cloud cannot be observed because the gas in an H_I region can be observed only by the absorption lines it produces, the radial motions can be measured directly by the Doppler shift in the absorption lines. Thus, while for dust structures it is of interest to study theoretically the significance of the observed shapes, for gas clouds we shall be more interested in the motions since they are all that can be observed [1].

To begin, we shall consider several characteristics of the interstellar gas which will be of use in our later developments primarily by giving us an intuitive insight into the physical nature of gas clouds. We shall employ cgs units except that distances may sometimes be expressed in light years ($0.947 \cdot 10^{18}$cm) or parsecs ($3.086 \cdot 10^{18}$ cm) and masses in units of the mass of the sun ($1.97 \cdot 10^{33}$ gm).

Observations show that the gas is not distributed uniformly between the stars but is in clumps[1,2]. This is, of course, just what one would expect theoretically since Jeans has shown that if Θ is the temperature of a gas, $\bar{\rho}$ its mean density, w the molecular mass in grams, and G the gravitational constant, then for small fluctuations in density from the mean, the fluctuations will tend to increase or decrease in magnitude according as to whether the wavelength of the fluctuation is greater or less than λ, where

$$\lambda^2 = \frac{\pi \alpha k \theta}{G \bar{\rho} w} . \tag{1}$$

α is a number and lies in the range $(1,\gamma)$, where γ is the ratio of specific heats for the gas. α is the exponent in the relation between gas pressure and density.

$$\frac{P}{\bar{P}} = \left(\frac{\rho}{\bar{\rho}}\right)^{\alpha} . \tag{2}$$

Thus, if there is no heat lost by transfer to dust etc.[3] $\alpha = \gamma$. And if the compression is isothermal, $\alpha = 1$.

λ is given in Table I for temperatures of 100° K and $10{,}000^\circ$ K. For 100° K we assume that the gas is mainly H_2, altho at low densities this is probably not too good an assumption. At $10{,}000^\circ$ K the gas is, of course, H_1 (and is probably ionized altho this is not taken into account in the calculations). We take $\alpha = 1$ for Table I.

The mass in a cube of side λ is $\lambda^3 \bar{\rho}$. Thus

$$\lambda^3 \bar{\rho} = \left(\frac{\pi \alpha k \theta}{G w}\right)^{\frac{3}{2}} \frac{1}{\bar{\rho}^{\frac{1}{2}}} \tag{3}$$

We see that the mass in an unstable cell increases as $\bar{\rho}$ decreases. The masses are plotted in Table II for various densities.

It is of interest to note that $\lambda^3 \bar{\rho}$ for $\rho \sim 10^{-29}$ gm/cm^3 and $\theta = 10^4$ °K is comparable to galactic masses.

The characteristics of the individual clumps of gas or clouds are of particular interest. From observations a rough working model of the clouds is:

cloud gas density: 10 - 100 H atoms/cm^3

$2 \cdot 10^{-23}$ - $2 \cdot 10^{-22}$ gm/cm^3

Mean Density $\bar{\rho}$	Critical Length λ $\Theta = 100^\circ$ K	$\Theta = 10{,}000^\circ$ K
10^{-29} gm/cm^3	$1.40\cdot10^{23}$ cm $1.48\cdot10^{5}$ lt yr	$1.98\cdot10^{24}$ cm $2.09\cdot10^{6}$ lt yr
10^{-28}	$4.43\cdot10^{22}$ $4.68\cdot10^{4}$	$0.627\cdot10^{24}$ $0.663\cdot10^{6}$
10^{-27}	$1.40\cdot10^{22}$ $1.48\cdot10^{4}$	$1.98\cdot10^{23}$ $2.09\cdot10^{5}$
10^{-26}	$4.43\cdot10^{21}$ $4.68\cdot10^{3}$	$0.627\cdot10^{23}$ $0.663\cdot10^{5}$
10^{-25}	$1.40\cdot10^{21}$ $1.48\cdot10^{3}$	$1.98\cdot10^{22}$ $2.09\cdot10^{4}$
10^{-24}	$4.43\cdot10^{20}$ $4.68\cdot10^{2}$	$0.627\cdot10^{22}$ $0.663\cdot10^{4}$
10^{-23}	$1.40\cdot10^{20}$ $1.48\cdot10^{2}$	$1.98\cdot10^{21}$ $2.09\cdot10^{3}$
10^{-22}	$4.43\cdot10^{19}$ $4.68\cdot10^{1}$	$0.627\cdot10^{21}$ $0.663\cdot10^{3}$
10^{-21}	$1.40\cdot10^{19}$ $1.48\cdot10^{1}$	$1.98\cdot10^{20}$ $2.09\cdot10^{2}$
10^{-20}	$4.43\cdot10^{18}$ $4.68\cdot10^{0}$	$0.627\cdot10^{20}$ $0.663\cdot10^{2}$

Table I Lengths for Gravitational Instability

Mean Density $\bar{\rho}$	Critical Mass $\lambda^3\bar{\rho}$	
	$\Theta = 100^oK$	$\Theta = 10{,}000^oK$
10^{-29} gm/cm^3	$2.74\cdot10^{40}$ gm $1.39\cdot10^{7}$ $M_\odot$	$0.777\cdot10^{44}$ gm $3.94\cdot10^{10}$ $M_\odot$
10^{-28}	$0.867\cdot10^{40}$ $4.40\cdot10^{6}$	$2.45\cdot10^{43}$ $1.246\cdot10^{10}$
10^{-27}	$2.74\cdot10^{39}$ $1.39\cdot10^{6}$	$0.777\cdot10^{43}$ $3.94\cdot10^{9}$
10^{-26}	$0.867\cdot10^{39}$ $4.40\cdot10^{5}$	$2.45\cdot10^{42}$ $1.246\cdot10^{9}$
10^{-25}	$2.74\cdot10^{38}$ $1.39\cdot10^{5}$	$0.777\cdot10^{42}$ $3.94\cdot10^{8}$
10^{-24}	$0.867\cdot10^{38}$ $4.40\cdot10^{4}$	$2.45\cdot10^{41}$ $1.246\cdot10^{8}$
10^{-23}	$2.74\cdot10^{37}$ $1.39\cdot10^{4}$	$0.777\cdot10^{41}$ $3.94\cdot10^{7}$
10^{-22}	$0.867\cdot10^{37}$ $4.40\cdot10^{3}$	$2.45\cdot10^{40}$ $1.246\cdot10^{7}$
10^{-21}	$2.74\cdot10^{36}$ $1.39\cdot10^{3}$	$0.777\cdot10^{40}$ $3.94\cdot10^{6}$
10^{-20}	$0.867\cdot10^{36}$ $4.40\cdot10^{2}$	$2.45\cdot10^{39}$ $1.246\cdot10^{6}$

Table II   Masses for Gravitational Instability cloud diameter: 10 psc. or $3 \cdot 10^{19}$cm

cloud spacing: 10^{-4}clouds/psc^3. and along any straight line, 7 clouds/kilopsc.

For a density of 10^{-22}gm/cm^3 and a radius of 5 psc. we have a cloud mass of $1.535 \cdot 10^{36}$gm or 779 M☉, altho calculations on stability show that the observed densities are low by a factor of ten. Increasing the density of the cloud by a factor of ten gives a smeared out average of 10^{-23}gm/cm^3 which we see from Table II gives cells of just about the increased cloud mass.

If we assume that the clouds have random proper motions, then the classical mean free path between collisions is

$$L = \frac{1}{4\sqrt{2}\pi a^2 N} \sim 30\text{psc} \sim 100 \text{ lt yr}$$

where a is the cloud radius and N is the number of clouds per unit volume.

A relative velocity of 5 km/sec gives $2/3 \cdot 10^7$ years between collisions.

First, it is of interest to see how long it takes an object to fall into a cloud. We write that its acceleration due to gravity alone is

$$\ddot{r} = -\frac{GM}{r^2}$$

where M is the cloud mass and r is the distance of the object from the center of the cloud. Integrating this we obtain

$$t = \frac{GM}{\sqrt{-2\mathcal{E}^3}}\left\{\cos^{-1}\sqrt{-\frac{r_0\mathcal{E}}{GM}} - \cos^{-1}\sqrt{-\frac{r\mathcal{E}}{GM}} + \sqrt{-\frac{r_0\mathcal{E}}{GM}\left(1+\frac{r_0\mathcal{E}}{GM}\right)} - \sqrt{-\frac{r\mathcal{E}}{GM}\left(1+\frac{r\mathcal{E}}{GM}\right)}\right\}$$

where $\mathcal{E}$ is the total energy per unit mass. The constant of integration has been chosen so that $r = r_0$ when $t = 0$. If the particle began falling freely from infinity, then $\mathcal{E} \to 0$ and the time to fall from r_0 to r is

$$t = \frac{2}{3\sqrt{2GM}}(r_o^{3/2} - r^{3/2}) \qquad (5)$$

In Tables III, IV the fall times are given for $M = 779M_\odot$.

We shall ultimately be interested in the internal mechanics of a gas cloud so that we may obtain a relation between the cloud mass, radius, and temperature, and also to establish criteria for stability against radial collapse or expansion. Thus, if V is the gravitational potential of the cloud, and p is the gas pressure, we have, from the Navier-Stokes equations

$$\nabla p + \rho \nabla V = 0 \qquad (6)$$

for equilibrium. Or, if Θ is the temperature, and w the molecular mass in grams,

$$p = \frac{\rho}{w} k\Theta \qquad (7)$$

$$\nabla\left(\frac{\rho}{w}k\Theta\right) + \rho\nabla V = 0$$

But

$$-\nabla V = \frac{4\pi G}{r^2}\int_0^r \rho(\eta)\,\eta^2\,d\eta$$

And we have

$$\frac{k}{\rho}\nabla\left(\frac{\rho\Theta}{w}\right) = \frac{4\pi G}{r^2}\int_0^r \rho(\eta)\,\eta^2\,d\eta$$

or

$$\frac{r^2}{\rho}\frac{d}{dr}\left(\frac{\rho\Theta}{w}\right) = \frac{4\pi G}{k}\int_0^r \rho(\eta)\,\eta^2\,d\eta \qquad (9)$$

Differentiating, we obtain merely

$$\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d}{dr}\left(\frac{\rho\Theta}{w}\right)\right] = \frac{4\pi G}{k} r^2\rho^2 \qquad (10)$$

The solutions for this nonlinear differential equation have been worked out for many cases in the literature [4].

If a cloud is bumped by another cloud, the period of oscillation will be of the order of the time required for a sound wave to travel the diameter of the cloud. The speed of propagation of sound is given classically by

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma \overline{u^2}}{3}} = \sqrt{\frac{\gamma k\Theta}{w}} \qquad (11)$$

Initial Position r_o	Total Energy per gm. $\mathcal{E} = -\frac{GM}{r}$	Time to fall to r = 50psc	Time to fall to r =20psc	Time to fall to r =10psc	Time to fall to r = 5psc
100 psc $3.084 \cdot 10^{20}$ cm	$-3.31 \cdot 10^{8}$ ergs	$1.542 \cdot 10^{16}$ sec $4.89 \cdot 10^{9}$ yrs	$1.810 \cdot 10^{16}$ sec $5.73 \cdot 10^{8}$ yrs	$18.59 \cdot 10^{15}$ sec $5.90 \cdot 10^{8}$ yrs	$18.77 \cdot 10^{15}$ sec $59.5 \cdot 10^{7}$ yrs
50 psc $1.542 \cdot 10^{20}$ cm	$-0.662 \cdot 10^{9}$		0.583 1.85	6.38 2.02	6.57 20.8
20 psc $6.168 \cdot 10^{19}$ cm	$-1.656 \cdot 10^{9}$			1.376 0.436	1.583 5.02
10 psc $3.084 \cdot 10^{19}$ cm	$-3.31 \cdot 10^{9}$				0.500 1.584

Table III Time to fall to r starting from rest at r_o

Initial Positions r_o	Time to fall from r_o where r is 70 psc	50 psc	30 psc	20 psc	15 psc	10 psc
100 psc	$3.31 \cdot 10^{15}$ sec $105 \cdot 10^{6}$ yrs	$5.17 \cdot 10^{15}$ sec $163.8 \cdot 10^{6}$ yrs	$6168 \cdot 10^{15}$ sec $211 \cdot 10^{6}$ yrs	$7.28 \cdot 10^{15}$ sec $231 \cdot 10^{6}$ yrs	$7.53 \cdot 10^{15}$ sec $238 \cdot 10^{6}$ yrs	$7.73 \cdot 10^{15}$ sec $245 \cdot 10^{6}$ yrs
70 psc		1.855 58.8	3.36 106.4	3.96 125.4	4.22 133.6	4.42 140.0
50 psc			1.511 47.8	2.11 66.9	2.35 74.5	2.56 81.2
30 psc				0.595 18.85	0.844 26.7	1.056 33.5
20 psc					0.249 7.89	0.461 14.60
10 psc						0.222 7.04

Table IV Time to fall from r to r_o having started from infinity at rest.

where $\overline{u^2}$ is the mean square thermal velocity. The speed of sound is given in Table V along with the time required to propagate 10 parsecs. We choose $\gamma = 5/3$ for H_1 and $7/5$ for H_2.

The period of oscillation about the stable equilibrium is readily obtained from mechanics if we consider an idealized model of the cloud in which we assume the cloud to be a homogeneous sphere of radius a where we introduce the constraints that the angular velocity, $\dot{\varphi}$, is constant throughout the cloud and the radial motion is of the form

$$\dot{r} = \dot{a}\left(\frac{r}{a}\right). \qquad (12)$$

$\dot{a}$ is the radial velocity of the outer surface.

Thus, the spin energy is $\frac{1}{5}Ma^2\dot{\varphi}^2$, the radial motion is $\frac{3}{10}M\dot{a}^2$, and the gravitational potential energy is $-\frac{3GM^2}{a}$. We must also take into account the energy exchange with the general stellar radiation field. The energy from the radiation field, of course, produces the temperature changes in the cloud and also does the work in expansion of the cloud against its internal pressure.

To compute this energy exchange we remark that in our idealized homogeneous cloud,

$$\frac{\partial\theta}{\partial r} = \frac{\partial\rho}{\partial r} = 0 \qquad (13)$$

and we have implicitly assumed that all gravitational forces are exerted at the periphery. There is no gravitational interaction between interior molecules. Thus, as far as the computation of the energy exchange with the radiation field is concerned, we have a sort of balloon filled with nongravitating matter. The balloon exerts the necessary forces to simulate the gravitational field. Thus, for an expansion of the cloud, the work done by the gas is (w is the molecular weight)

Gas	Temperature °K	Speed of Sound cm/sec	Time to Travel 10 parsecs
H_1	10	$3.72\cdot10^4$	$8.29\cdot10^{14}$ sec $2.63\cdot10^7$ yrs
H_1	50	$8.31\cdot10^4$	$3.71\cdot10^{14}$ $1.175\cdot10^7$
H_1	100	$1.174\cdot10^5$	$2.63\cdot10^{14}$ $0.833\cdot10^7$
H_1	500	$2.63\cdot10^5$	$1.172\cdot10^{14}$ $3.71\cdot10^6$
H_1	1000	$3.715\cdot10^5$	$0.829\cdot10^{14}$ $2.63\cdot10^6$
H_1	5000	$8.31\cdot10^5$	$3.71\cdot10^{13}$ $1.175\cdot10^6$
H_1	10,000	$1.174\cdot10^6$	$2.63\cdot10^{13}$ $0.833\cdot10^6$
H_2	10	$2.41\cdot10^4$	$1.28\cdot10^{15}$ sec $4.06\cdot10^7$ yrs
H_2	50	$5.40\cdot10^4$	$5.71\cdot10^{14}$ $1.807\cdot10^7$
H_2	100	$7.63\cdot10^4$	$4.03\cdot10^{14}$ $1.279\cdot10^7$
H_2	500	$1.708\cdot10^5$	$1.805\cdot10^{14}$ $5.72\cdot10^6$
H_2	1000	$2.41\cdot10^5$	$1.28\cdot10^{14}$ $4.06\cdot10^6$
H_2	5000	$5.40\cdot10^5$	$5.71\cdot10^{13}$ $1.807\cdot10^6$
H_2	10,000	$7.63\cdot10^5$	$4.03\cdot10^{13}$ $1.279\cdot10^6$

Table V The speed of sound in H_1 and H_2.

$$\begin{aligned} dW &= p\,dV \\ &= p_0\left(\frac{\rho}{\rho_0}\right)^{\alpha} dV \\ &= p_0\left(\frac{a_0}{a}\right)^{3\alpha} 4\pi a^2\,da \\ &= p_0\, 4\pi a_0^{3\alpha} a^{2-3\alpha}\,da \end{aligned}$$

Thus

$$\begin{aligned} W &= \frac{4\pi a_0^{3\alpha} p_0}{3(1-\alpha)} \left(a^{3-3\alpha} - a_0^{3-3\alpha}\right) \\ &= \frac{MR\theta a_0^{3\alpha-3}}{w(1-\alpha)} \left(a^{3(1-\alpha)} - a_0^{3(1-\alpha)}\right) \end{aligned} \tag{14}$$

where W represents the work done in expanding the cloud from a_0 to a. The energy in the galactic radiation field is of the nature of a potential energy. This energy changes by -W if the cloud is expanded. Therefore, the Lagrangian is

$$L = \frac{Ma^2\dot{\varphi}^2}{5} + \frac{3M\dot{a}^2}{10} + \frac{3GM^2}{5a} + \frac{MR\theta a_0^{-3(1-\alpha)}}{w(1-\alpha)}\left(a^{3(1-\alpha)} - a_0^{3(1-\alpha)}\right) \tag{15}$$

The equations of metion are

$$\frac{d}{dt}\left(\frac{2Ma^2\dot{\varphi}}{5}\right) = 0$$

$$\frac{3M\ddot{a}}{5} - \frac{2Ma\dot{\varphi}^2}{5} + \frac{3GM^2}{5a^2} - \frac{3MR\theta a_0^{3(\alpha-1)}}{w} a^{2-3\alpha} = 0 \tag{16}$$

Hence,

$$\frac{2Ma^2\dot{\varphi}}{5} = p_s = \text{constant}$$

so that

$$\ddot{a} - \frac{25p_s^2}{6M^2a^3} + \frac{GM}{a^2} - \frac{5R\theta a_0^{3(\alpha-1)}}{w} a^{2-3\alpha} = 0 \tag{17}$$

Now, to integrate the equation of motion, we may multiply thru by $\dot{a}$ as an integrating factor so that

$$\frac{1}{2}\frac{d}{dt}(\dot{a}^2) + \frac{25p_s^2}{12M^2}\frac{d}{dt}\left(\frac{1}{a^2}\right) - GM\frac{d}{dt}\left(\frac{1}{a}\right) - \frac{5R\theta a_0^{3(\alpha-1)}}{3w(1-\alpha)}\frac{d}{dt}a^{3(1-\alpha)} = 0$$

$$\dot{a}^2 + \frac{25p_s^2}{6M^2a^2} - \frac{2GM}{a} - \frac{10R\theta a_0^{3(\alpha-1)}}{3w(1-\alpha)} a^{3(1-\alpha)} = C = \text{constant} \tag{18}$$

$$t - t_\circ = \int_{a_\circ}^{a} \frac{da}{\sqrt{\frac{10R\theta_\circ}{3w(1-\alpha)}\left(\frac{a}{a_\circ}\right)^{w} + \frac{2GM}{a} - \frac{25p_s^2}{6M^2a^2} + C}} \tag{19}$$

For our purposes however, only an approximate value of the period of oscillation is required. Thus, rather than to attempt to evaluate the above integral, let us consider oscillations of small amplitude so that

$$a = a_\circ + \xi$$

$$\dot{a} = \dot{\xi}$$

$$\ddot{a} = \ddot{\xi}$$

where $a_\circ$ is the stable equilibrium radius.

Thus, the equation of motion becomes

$$\ddot{\xi} + \xi\left[\frac{25p_s^2}{2M^2a_\circ^4} - \frac{2GM}{a_\circ^3} - \frac{5R\theta_\circ}{w}(2 - 3\alpha)\right] = \frac{5R\theta_\circ}{wa_\circ} + \frac{25p_s^2}{6M^2a_\circ^3} - \frac{GM}{a_\circ^2} \tag{20}$$

But, if $a_\circ$ is an equilibrium radius, then the right hand member must vanish. We obtain just the equation for the equilibrium radius a_1 when the temperature at equilibrium is $\theta_\circ$.

$$\frac{5R}{w}a_1^2 - GMa_1 + \frac{25p_s^2}{6M^2} = 0 \tag{21}$$

Thus, the period of oscillation is

$$P = \frac{2\pi}{\sqrt{\frac{25p_s^2}{2M^2a_1^4}\left(\frac{5}{3} - \alpha\right) - \frac{3\alpha GM}{a_1}}} = \frac{2\pi}{\sqrt{\frac{GM}{a_1^3} - \frac{5R\theta}{wa_1^2}(5 - 3\alpha)}} \tag{22}$$

We see that for small p_s and therefore small a_1, the period becomes fairly short. P is given in Table VI for a cloud of 1000M⊙ and $\theta_\circ = 100^\circ$ K, $\alpha = 2/3$, $w = 2$ for various equilibrium radii.

We see that if $\alpha = 5/3$ as it would in an adiabatic H_{II} region, there exist equilibrium radii from 0 to ∞ and P increases indefinitely with a_1. It is of interest to note that the time to approach the equilibrium temperature [3] for $\rho = 10^{-22}$gm/cm^3 in an H_I region is much less than the period

Equilibrium Radius a	Period of Oscillation P $M = 10^2 M_\odot$	$M = 10^3 M_\odot$	$M = 10^4 M_\odot$	$M = 10^5 M_\odot$	$M = 10^6 M_\odot$
10^{15} cm	$1.74 \cdot 10^9$ sec. $0.552 \cdot 10^2$ yrs.	$0.548 \cdot 10^9$ sec. $1.74 \cdot 10$ yrs.			
10^{16} cm	$0.562 \cdot 10^{11}$ $1.78 \cdot 10^3$	$1.74 \cdot 10^{10}$ $0.55 \cdot 10^3$	$0.548 \cdot 10^{10}$ sec $1.74 \cdot 10^2$ yrs		
10^{17} cm	$2.39 \cdot 10^{12}$ $0.759 \cdot 10^5$	$0.56 \cdot 10^{12}$ $1.782 \cdot 10^4$	$1.74 \cdot 10^{11}$ $0.552 \cdot 10^4$	$0.548 \cdot 10^{11}$ sec $1.74 \cdot 10^3$ yrs	
$2.11 \cdot 10^{17}$ cm	∞				
10^{18} cm		$2.39 \cdot 10^{13}$ $0.759 \cdot 10^6$	$0.56 \cdot 10^{13}$ $1.782 \cdot 10^5$	$1.74 \cdot 10^{12}$ $0.552 \cdot 10^5$	$0.548 \cdot 10^{12}$ sec $1.74 \cdot 10^4$ yrs
$2.11 \cdot 10^{18}$ cm		∞			
10^{19} cm			$2.39 \cdot 10^{14}$ $0.759 \cdot 10^7$	$0.562 \cdot 10^{13}$ $1.782 \cdot 10^6$	$1.74 \cdot 10^{13}$ sec $0.552 \cdot 10^6$ yrs
$2.11 \cdot 10^{19}$ cm			∞		
10^{20} cm				$2.39 \cdot 10^{15}$ $0.759 \cdot 10^8$	$0.562 \cdot 10^{14}$ sec $1.782 \cdot 10^7$ yrs
$2.11 \cdot 10^{20}$ cm				∞	
10^{21} cm					$2.39 \cdot 10^{16}$ sec $0.759 \cdot 10^9$ yrs
$2.11 \cdot 10^{21}$ cm					∞

Table VI Period of Radial Oscillation of a Gas Cloud About Its Stable Equilibrium Radius ($\theta_\circ = 100^\circ$K) of oscillation of a cloud.

While we are on the subject of motions produced by collisions etc., it is of interest to consider the turbulent internal motions of a cloud.

The kinematic viscosity, ν, is exceedingly high because of the low density, but the large size of the clouds gives large Reynolds numbers. For H_2 the viscosity is approximately given by

$$\eta = 1/2 \cdot 10^{-5}\sqrt{\theta} \text{ poises}$$

For a density of 10^{-22} gm/cm^3 and 100°K we obtain the kinematic viscosity as $\frac{1}{2}\cdot 10^{18}$. For a region 10 parsecs across with velocity differences of 2 km/sec, the Reynolds number is $\sim 10^7$.

From experiment, and as expected from theory[5], we have that

$$\frac{100\nu L}{u\lambda^2} \cong 1 \qquad (23)$$

where L is the size of the largest eddies present in the turbulence, u is the velocity of the turbulence, and λ is the characteristic length of the smallest eddies. Thus, the Reynold's number of the smallest eddies is

$$R_\lambda = \frac{u\lambda}{\nu} = \sqrt{\frac{100Lu}{\nu}} \qquad (24)$$

For clouds with radii of the order of 10^{19}cm we see that $L \sim 10^{18}$cm, i.e. one order of magnitude smaller, the reason being that eddies 10^{19}cm will simply be a rigid rotation of the cloud which is preserved by conservation of angular momentum of the system. If we take $L = 10^{18}$cm, $u = 2\cdot 10^5$cm/sec, and $\nu = \frac{1}{2}\cdot 10^{18}$ cgs, then $R_\lambda = 6.3\cdot 10^3$, the expected order of magnitude.

We are primarily interested in the energy dissipation and the decay time for the turbulent motion. The energy dissipation ϵ, in ergs per sec per gm is

$$\epsilon = 15 \frac{\nu u^2}{\lambda^2} \tag{25}$$

so that in terms of the largest eddies we have

$$\epsilon = \frac{3u^3}{20L}$$

But, in terms of the turbulent motion, we have the kinematic result that

$$\epsilon = -\frac{d}{dt}\left(\frac{u^2}{2}\right) = -u\frac{du}{dt}$$

$$\frac{du}{dt} = -\frac{3u^2}{20L}$$

And

$$\frac{1}{u} - \frac{1}{u_\circ} = \frac{3}{20L}\, t \tag{26}$$

where $u_\circ$ is the initial velocity. Thus the decay time is of the order of

$$\frac{20L}{3u_\circ} \tag{27}$$

If $u_\circ = 2\cdot 10^5$, we have $\sim 10^6$ years for the relaxation time. We note that this is much less than the 10^7 years between collisions computed on the basis of random translational velocities of the clouds, although it is of the same order of magnitude or larger than the period for radial oscillations of a cloud.

Now let us turn our attention to an investigation of the stability of a gas cloud against radial collapse or expansion.

We find it convenient to use the virial in a rather unconventional manner. Thus, if m_i, $\bar{r}_i$, and $\bar{F}_i$ are the mass, position, and force on the i th particle in the cloud, then

$$\bar{F}_i = m_i \ddot{\bar{r}}_i$$

$$\bar{r}_i \cdot \bar{F}_i = m_i \bar{r}_i \cdot \ddot{\bar{r}}_i$$

Summing over all the particles in the cloud, and introducing a Legendre transformation,

$$\sum_i \bar{r}_i \cdot \bar{F}_i = \frac{d}{dt}\sum_i m_i \bar{r}_i \cdot \dot{\bar{r}}_i - \sum_i m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i$$

This is customarily written in the form

$$\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i = \sum_i \bar{r}_i \cdot \bar{F}_i + 2T \qquad (28)$$

where $\bar{p}_i$ is the momentum of the i th particle and T is the kinetic energy of the internal motions.

For gravitational interaction

$$\bar{F}_i = \sum_j \frac{Gm_i m_j}{r_{ij}^3} \bar{r}_{ij}$$

Thus

$$\sum_i \bar{r}_i \cdot \bar{F}_i = G \sum_{i,j}{}' \frac{m_i m_j}{r_{ij}^3} \bar{r}_i \cdot \bar{r}_{ij}$$

But $\bar{r}_i + \bar{r}_{ij} = \bar{r}_j$. Thus

$$\sum_i \bar{r}_i \cdot \bar{F}_i = \frac{G}{2} \sum_{i,j}{}' \frac{m_i m_j}{r_{ij}^3} (\bar{r}_i \cdot \bar{r}_{ij} + \bar{r}_j \cdot \bar{r}_{ij} - r_{ij}^2)$$

Now $\bar{r}_{ij} = -\bar{r}_{ji}$. Thus

$$\sum_i \bar{r}_i \cdot \bar{F}_i = \frac{G}{2} \sum_{i,j}{}' \frac{m_i m_j}{r_{ij}^3} (\bar{r}_i \cdot \bar{r}_{ij} - \bar{r}_j \cdot \bar{r}_{ji} - r_{ij}^2)$$

Since i,j are indices which will be summed out, and since we have a finite sum, we may interchange them in the second term on the right hand side. Thus the expression reduces to

$$\sum_i \bar{r}_i \cdot \bar{F}_i = -\frac{G}{2} \sum_{i,j}{}' \frac{m_i m_j}{r_{ij}}$$

$$\sum_i \bar{r}_i \cdot \bar{F}_i = V \qquad (29)$$

where V is the gravitational potential energy of the system. Putting (29) into (28) we have

$$\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i = V + 2T \qquad (30)$$

For equilibrium the left hand member of (30) vanishes and we have

$$T = -\frac{1}{2}V \qquad (31)$$

We are interested in stability as well as equilibrium. Therefore we wish to consider the variation in $\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ rather than just its zeros.

For convenience we divide T into two parts so that

$$T = T_o + T_e \qquad (32)$$

T_e is that part of the kinetic energy in motions unordered over dimensions of the order of the mean free path of the gas molecules. And T_o is the remainder. Thus T_o is the energy of all motions ordered over dimensions of the order of the mean free path. T_θ is therefore the energy of the purely thermal motions. T_o is due to turbulence, general rotation, etc.

Since the internal motions of the gas molecules are not included in (30),

$$T_\theta = \frac{3R}{2w}\int_{Volume} \rho_H \theta d\tau \qquad (33)$$

where w is the molecular weight, ρ_H the gas density, and θ the absolute temperature. Thus $\frac{3R}{2w}$ is the specific heat for the three translational degrees of freedom. θ must, of course, be measured by an observer moving with whatever ordered motions are present in the gas.

(31) and (32) give

$$T_o + T_\theta = -\frac{1}{2}V \qquad (34)$$

for equilibrium.

If this equilibrium is to be stable we see that for the cloud slightly expanded from the equilibrium size, $\frac{d}{dt}\sum_i \overline{p}_i\cdot\overline{r}_i$ must be negative and for the cloud slightly contracted it must be positive. Thus if $T_\theta + T_o$ does not decrease sufficiently rapidly on expansion or increase sufficiently rapidly on contraction, the cloud will be unstable.

First, let us consider a cloud in which $T_o = 0$. Then (30) reduces to

$$\frac{d}{dt}\sum_i \overline{p}_i\cdot\overline{r}_i = 2T_\theta + V$$

Let us consider a cloud in which the temperature, pressure, and density are related by

$$\theta = \theta_o\left(\frac{\rho}{\rho_o}\right)^{\alpha-1}$$
$$\text{or}\quad p = p_o\left(\frac{\rho}{\rho_o}\right)^{\alpha} \qquad (35)$$

Now, if a is the cloud radius, we see that $\theta \propto a^{3(1-\alpha)}$ and $V \propto a^{-1}$. Thus, for stability

$$3(1-\alpha) < -1$$

$$\alpha > \frac{4}{3}$$

The physical explanation is, of course, that the larger molecules have too many internal degrees of freedom so that energy transfered from the gravitational field is wasted in exciting the internal degrees of freedom and an insufficient fraction goes to increasing T_0. For an adiabatic change

$$\gamma = 1 + \frac{2}{f} \qquad (37)$$

so that $\gamma > \frac{4}{3}$ is equivalent to $f < 6$ where f is the number of degrees of freedom of the molecule.

Therefore, a cloud of monatomic or diatomic molecules would be stable if we consider adiabatic changes only. But triatomic and larger molecules could not form a stable cloud.

Now, with the conditions prevailing in the interstellar gas clouds [3], α is at most equal to unity in H_I regions and usually somewhat less, ranging from 1/2 to 1 depending upon the dust grain density etc. In H_{II} regions α is very nearly equal to unity lying in the range $1 \leq \alpha \leq 1.04$. We see then that (20) is never satisfied by the interstellar clouds and hence they can never be stable with $T_0 = 0$.

Another way of looking at this is to consider the coupling between the molecules and the rest of the galaxy by the galactic radiation field. The degree of coupling varies, of course. One may write

$$\alpha = 1 + \frac{2}{F}$$

and interpret F as the effective number of degrees of freedom of the molecule. If $\alpha = 1$, $F = \infty$ and we say that the coupling to the rest of the galaxy is complete. Thus the internal motions of the molecules comprise the entire galaxy which is effectively infinite. If $\alpha = 1.04$, $F = 50$, i.e. the coupling is not complete. If, on the other hand, $\alpha < 1$, then $F < 0$ and we can only interpret it is more than complete coupling, so that an addition of energy due to compression reduces the temperature.

Because we are not interested in transient effects such as shock waves, etc., the exchange of energy with the internal degrees of freedom may be considered effectively instantaneous for those degrees coupled by electromagnetic radiation as well as by chemical bonds. And we see that physically as well as mathematically, the interpretation of the energy stored at distant points in the radiation field as being equivalent to energy stored in the internal motions of the gas molecule is valid.

If we introduce turbulence so that $T_0 \neq 0$, then because it is ordered over distances of the order of the mean free path of the gas molecules, it will not tend to excite the internal degrees of freedom to the extent that unordered motions will. This reduces the effective value of f and if a cloud were sufficiently turbulent, could produce stability even for $\alpha < 1$. If we let Θ be the temperature of the unordered and ordered motions, together, then we write for small fluctuations in ρ,

$$\Theta = \Theta_0 \left(\frac{\rho}{\rho_0}\right)^{A-1} \qquad (38)$$

and

$$\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i = 12\pi R \int_0^\infty \frac{\rho(r)}{W} \Theta(r)\, r^2\, dr + V \qquad (39)$$

Actually, of course, if H is the temperature of the turbulence,

$$\Theta = \theta + H \qquad (40)$$

$$\Theta = \theta_0 \left(\frac{\rho}{\rho_0}\right)^{\alpha-1} + H_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{2}{3}}$$

For small oscillations about ρ_0, let

$$\rho = \rho_0 + \epsilon$$

Then

$$\Theta = \theta_0 + H_0 + \frac{\epsilon}{\rho_0}\left[\theta_0(\alpha - 1) + \frac{2}{3}H_0\right] + O(\epsilon^2)$$

$$\Theta = (\theta_\circ + H_\circ)\left[1 + \frac{\epsilon}{\rho_\circ}\,\frac{\theta_\circ(\alpha - 1) + 2/3 H_\circ}{\theta_\circ + H_\circ} + O(\epsilon^2)\right.$$

And Θ varies as

$$\Theta = \Theta_\circ\left(\frac{\rho}{\rho_\circ}\right)^{A-1}$$

where $\Theta_\circ = \theta_\circ + H_\circ$ and $A = \frac{\theta_\circ\alpha + 5/3\ H_\circ}{\theta_\circ + H_\circ}$. We have stability if

$$A > \frac{4}{3}$$

i.e.

$$\frac{\theta_\circ\alpha + 5/3\ H_\circ}{\theta_\circ + H_\circ} > \frac{4}{3}$$

or

$$H_\circ > (4 - 3\alpha)\theta_\circ \tag{41}$$

The limiting values of $\frac{H_\circ}{\theta_\circ}$ are shown in Fig. 1. For $\alpha > 4/3$, there exist stable equilibriums for all $\frac{H_\circ}{\theta_\circ}$.

It has been shown, however, that turbulence damps out very rapidly. Thus whatever stability there may be due to turbulence, it will be short lived compared to the other processes in which we shall be interested, and therefore of no importance. At the most it would introduce a brief delay in the unstable mechanical processes. Therefore, in the following discussion we shall assume that there is no turbulence and we shall consider the effect of the cloud having nonzero angular momentum or spin, p_s.

Because all clouds have at least some angular momentum, this will be the case of general interest. Therefore, we shall be interested in investigating it quantitatively in detail. In order to facilitate such investigation we shall introduce the idealization that the cloud is spherical in spite of p_s and homogeneous in spite of the gravitational field. The approximation to the actual clouds existing in space is admittedly rough. The most serious approximation is probably in assuming that the cloud is spherical when actually it is more like an oblate spheroid. However, to take this into account introduces a tremendous amount of labor because of the transcendental equations involved. Therefore, we shall use the spherically symmetric model for all of the following development. Then, when we have a clearer picture of what occurs in the cloud, we shall investigate the characteristics of an oblate sphheroidal cloud which might make a significant difference in our result. We will find, happily, that the only difference will be in the numerical factors occurring in some of the coefficients.

We assume that the cloud spins as a rigid body with angular velocity $\dot{\varphi}$. Thus, if M is the cloud mass and a the cloud radius,

$$p_s = \frac{2}{5} Ma^2\dot{\varphi} \tag{42}$$

The spin energy is

$$T_o = \frac{5p_s^2}{4Ma^2} \tag{43}$$

The gravitational potential energy is

$$V = -\frac{3GM^2}{5a} \tag{44}$$

We assume a uniform temperature distribution. Thus where w is the molecular weight,

$$T_\theta = \frac{3MR\theta}{2w} \tag{45}$$

This is valid for all molecules since we are interested only in the translational degrees of freedom. (30) becomes

$$\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_1 \cdot \overline{r}_1 = \frac{3MR\theta}{2w} + \frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a} \tag{46}$$

(35) may be written as

$$\theta = \theta_o\left(\frac{a_o}{a}\right)^{3(\alpha-1)} \tag{47}$$

so that

$$\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_1 \cdot \overline{r}_1 = \frac{3MR\theta_o}{2w}\left(\frac{a_o}{a}\right)^{3(\alpha-1)} + \frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a} \tag{48}$$

The equilibrium radii are given by

$$\frac{3MR\theta_o}{2w}\left(\frac{a}{a_o}\right)^{3(1-\alpha)} + \frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a} = 0 \tag{49}$$

For small p_s this has two positive real roots (excluding $a = \infty$ if $\alpha < 1$.). If p_s is too large there are no roots. For stability,

$$\frac{d}{da}\left(\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i\right) < 0$$

Thus, one root, the larger, will be an unstable equilibrium and the smaller root will be stable.

Because of the difficulty for a general α in determining the maximum p_s allowing positive real roots of (30), we shall carry out our calculations for $\alpha = 1$, typical of H_{II} regions, and for $\alpha = 2/3$ typical of H_I regions.

Therefore, if we denote equilibrium radii by a_1, we have

$$\alpha = 1 : \quad a_1 = \frac{wGM}{10\ R\Theta_o}\left[1 \pm \sqrt{1 - \frac{250\ R\Theta_o p_s^2}{3wG^2M^4}}\right]$$

$$\alpha = \frac{2}{3} : \quad \frac{3MR\Theta_o}{2w}\frac{a_1^3}{a_o} - \frac{3GM^2}{10}a_1 + \frac{5p_s^2}{4M} = 0 \qquad (50)$$

The cubic for $\alpha = 2/3$ is most readily solved by plotting. Of interest is the maximum value of p_s giving real positive roots. For $\alpha = 1$, the maximum value is obviously

$$\alpha = 1 : \quad p_s \leq \frac{GM^2}{5}\sqrt{\frac{3w}{10R\Theta_o}} \qquad (51)$$

The maximum p_s for $\alpha = 2/3$ is obtained by writing the cubic as

$$Aa^3 - Ba + C = 0 \qquad (52)$$

As is readily seen, the maximum p_s gives a multiple root so that the cubic is of the form

$$(a - a_1)^2(a - a_2) = 0$$

Thus

$$a^3 - a^2(2a_1 + a_2) + a(a_1(a_1 + 2a_2)) - a_1^2 a_2 = 0$$

Comparing coefficients with (52) we have

$$2a_1 + a_2 = 0$$
$$a_1(a_1 + 2a_2) = -\frac{B}{A} \qquad (53)$$
$$-a_1^2 a_2 = \frac{C}{A}$$

Thus, the maximum p_s is

$$\alpha = 2/3 : \quad p_s \leq \frac{2M^2}{5}\left(\frac{G^3 w a_0}{15 M R \theta_0}\right)^{\frac{1}{4}} \tag{55}$$

The physical explanation for the maximum value of p_s is readily obtained by comparing the spin energy and the thermal energy,

$$\frac{5p_s^2}{4Ma^2} \qquad\qquad 3\frac{MR\theta_0}{2w}\left(\frac{a}{a_0}\right)^{3(1-\alpha)}.$$

The spin energy varies as a^{-2} and if $\alpha \leq 4/3$, p_s cannot be too large, or else by the time a is sufficiently large to make the spin energy small, the thermal energy is proportionately too large, and $\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ will not vanish as it must for an equilibrium. If p_s is below the maximum value, then, provided the cloud is not expanded beyond its larger and unstable equilibrium position, $\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ is negative and the cloud tends to contract until the spin energy can dominate by virtue of a^{-2}, thus giving the smaller equilibrium configuration.

To show how the equilibrium size of a cloud varies with p_s, $\frac{1}{2}\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ is given in Fig. 2 for various p_s. The calculations were carried out for $\alpha = 1$, $2/3$. We see that a decrease in p_s by a factor of ten from its maximum value will cause a decrease in the equilibrium radius by a factor $\sim 10^2$. We note that for radii much larger than a_0 (in the calculations a_0 was chosen as 5 parsecs) where $\alpha < 1$, the main contribution to $\frac{1}{2}\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ is from the temperature which increases with a, whereas for $a << a_0$ it is the angular momentum term. Thus $\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i$ is quite insensitive to p_s for $a >> a_0$ and to θ_0 for $a << a_0$. And for $a >> a_0$ it is quite sensitive to θ_0 and for $a << a_0$ it is quite sensitive to p_s. Further, we notice that as p_s decreases, the lack of dependence on p_s extends to smaller and smaller radii, even to $a << a_0$.

For small p_s, near the stable equilibrium the thermal energy is negligible for $\alpha \leq 1$ so that

$$\frac{1}{2}\frac{d}{dt}\sum_i \bar{p}_i \cdot \bar{r}_i = \frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a}$$

And the stable equilibrium, the smaller root of

$$\frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a} = 0$$

is

$$a = \frac{25p_s^2}{6GM^3} \tag{56}$$

Thus, the equilibrium cloud radius rapidly approaches zero as p_s decreases.

For small p_s we see, then, that a, because it is insensitive to temperature, is nearly independent of α . By inspection of Fig. 3 one sees that the main difference between clouds in H_I regions, $\alpha \sim 2/3$ and in H_{II} regions, $\alpha \sim 1$, is the rate at which the cloud would expand to infinity from its larger equilibrium position. An H_{II} cloud would expand more rapidly because of its higher temperature, but the temperature in an H_I region rises rapidly as the density falls and ultimately becomes nearly as large as the temperature in an H_{II} region. Thus, at large radii the difference between the expansion rates is not great.

If we were to consider the gravitational collapse of an initially infinite homogeneous gas atmosphere, we see that for no turbulence, the collapsing cells would either reach a stable equilibrium if $\alpha > 4/3$, or if, as is typical of the interstellar material, $\alpha \leq 1$, would collapse completely. Actually, of course, one would expect some turbulence. For cells of the size considered, 100 lt yrs, (Table I), such things as galactic rotation and the mass motion of stars due to extension of the spiral arms etc. would be the most important.

Using Tables I, II we see that if we had a medium of hydrogen at 100° K and 10^{-24} gm/cm^3, then we would expect cells with masses of the order of $4 \cdot 10^4$ $M_\odot$ to form. So that the cloud would not collapse to unduly small radii we make $\frac{p_s}{M^2} = 10^{-13}$ gm^{-1}cm^{2}sec^{-1}. Over distances of the order of 200 lt yrs this amount of angular momentum requires velocity differences of the order of 1 km/sec in the uncollapsed homogeneous medium. Interestingly enough, this is just about one half the shearing in the vicinity of the sun due to galactic rotation.

Let us consider the lower limit of the density and masses of clouds of various sizes. The minimum mass for a given size or the maximum size for a given mass occurs for the limiting value of p_s given by (51) and (55). The radius of the equilibrium sige is

$$a = \begin{cases} \frac{wGM}{10R\theta_i} & \text{for } \alpha = 1 \\ \frac{wGM}{15R\theta_i} & \text{for } \alpha = \frac{2}{3} \end{cases} \tag{57}$$

And

$$M = \begin{cases} \frac{10aR\theta_i}{wG} & \text{for } \alpha = 1 \\ \frac{15aR\theta_i}{wG} & \text{for } \alpha = \frac{2}{3} \end{cases} \tag{58}$$

$$\rho = \begin{cases} \frac{15R\theta_i}{2\pi wGa^2} & \text{for } \alpha = 1 \\ \frac{45R\theta_i}{4\pi wGa^2} & \text{for } \alpha = \frac{2}{3} \end{cases} \tag{59}$$

The results are given in Table VII. In H_I regions, $\alpha = \frac{2}{3}$, $w = 2$, $\theta = 100^\circ$K. In H_{II} regions $\alpha = 1$, $w = 1$, $\theta = 5000^\circ$ K.

We see that in H_I regions, when it is recalled that the listed values of ρ are the minimum values, if we consider clouds of 10 parsec diameter, the densities will be of the order of 10^{-21}gm/cm^3 which is one order of magnitude denser than the working model initially posed.

We see that in H_{II}regions because of the larger thermal velocities, very large densities are required for equilibrium

H_I Region, $w = 2$, $\alpha = 2/3$, $\theta = 100°K$

Cloud Radius	Minimum Density	Minimum Mass
a	ρ	M
10^{17}cm	$2.23 \cdot 10^{-17}$gm/cm^3	$0.623 \cdot 10^{35}$gm $3.16 \cdot 10$ $M_\odot$
10^{18}cm	$2.23 \cdot 10^{-19}$	$0.623 \cdot 10^{36}$ $3.16 \cdot 10^{2}$
$3.16 \cdot 10^{18}$cm	$2.23 \cdot 10^{-20}$	$1.97 \cdot 10^{36}$ $1.00 \cdot 10^{3}$
10^{19}cm	$2.23 \cdot 10^{-21}$	$0.623 \cdot 10^{37}$ $3.16 \cdot 10^{3}$
$3.16 \cdot 10^{19}$cm	$2.23 \cdot 10^{-22}$	$1.97 \cdot 10^{37}$ $1.00 \cdot 10^{4}$
10^{20}cm	$2.23 \cdot 10^{-23}$	$0.623 \cdot 10^{38}$ $3.16 \cdot 10^{4}$
$3.16 \cdot 10^{20}$cm	$2.23 \cdot 10^{-24}$	$1.97 \cdot 10^{38}$ $1.00 \cdot 10^{5}$
10^{21}cm	$2.23 \cdot 10^{-25}$	$0.623 \cdot 10^{39}$ $3.16 \cdot 10^{5}$

H_{II} Region, $w = 1$, $\alpha = 1$, $\theta = 5000°K$

Cloud Radius	Minimum Density	Minimum Mass
10^{17}cm	$1.49 \cdot 10^{-15}$	$0.935 \cdot 10^{37}$ $0.474 \cdot 10^{4}$
10^{18}cm	$1.49 \cdot 10^{-17}$	$0.935 \cdot 10^{38}$ $0.474 \cdot 10^{5}$
$3.16 \cdot 10^{18}$cm	$1.49 \cdot 10^{-18}$	$2.95 \cdot 10^{38}$ $1.50 \cdot 10^{5}$
10^{19}cm	$1.49 \cdot 10^{-19}$	$0.935 \cdot 10^{39}$ $0.474 \cdot 10^{6}$
$3.16 \cdot 10^{19}$cm	$1.49 \cdot 10^{-20}$	$2.95 \cdot 10^{39}$ $1.50 \cdot 10^{6}$
10^{20}cm	$1.49 \cdot 10^{-21}$	$0.935 \cdot 10^{40}$ $0.474 \cdot 10^{7}$
$3.16 \cdot 10^{20}$cm	$1.49 \cdot 10^{-22}$	$2.95 \cdot 10^{40}$ $1.50 \cdot 10^{7}$
10^{21}cm	$1.49 \cdot 10^{-23}$	$0.935 \cdot 10^{41}$ $0.474 \cdot 10^{8}$

Table VII Minimum Mass and Density of Gas Clouds and that these densities are at least two orders of magnitude greater than the working model.

It is of some interest to consider how violent a collision a cloud can withstand without being expanded beyond its larger equilibrium radius and doomed to infinity. In general, a collision between two clouds will produce both ordered and unordered internal motions. The ordered motions will be in the form of angular momentum. We consider turbulence and thermal motions separately. First, if a cloud suddenly finds its internal motions greatly increased, it will expand with a velocity not greater than that of the internal motions. At 1 km/sec, 10^6 years are required to travel 1 psc. At 10 km/sec, 10^5 years.

The relaxation times for the temperature at the high densities one finds in a gas cloud are much less than this. Thus, transient thermal velocities produced by the collision will not be sufficiently long lived to be effective in producing expansion because of the time required to order themselves in an expansion and the inelastic collisions which would result during the ordering process.

The turbulent motions are more persistent, lasting of the order of $5 \cdot 10^6$ years. And, of course, the angular momentum of the cloud is preserved until the next collision of the order of 10^7 years later. Therefore, we shall lump together all motions with correl ations varying as slowly or slower than turbulence and refer to them as ordered motions. By this definition, we see that the unordered motions are of no interest.

The energy required to lift the cloud against its gravitational field from the smaller, a_1, to the larger equilibrium radius, a_2, is

$$GM^2\left(\frac{1}{a_1} - \frac{1}{a_2}\right) \qquad (60)$$

Some of this energy will be supplied by the thermal motions maintained by the radiation field, but of course, so long as $a < a_2$, it will never be enough to carry on the expansion by itself.

If we assume that a collision takes place when the cloud has a radius a_1, angular momentum $p_{s\infty}$, and temperature θ_1, and that after collision the angular momentum is p_s and the temperature of the thermal motions plus the turbulent motions gives an effective temperature θ, then the energy imparted by the collision is

$$\frac{5}{4Ma_1^2}(p_s^2 - p_{s\infty}^2) + \frac{3MR}{2w}(\theta - \theta_1) \qquad (61)$$

where we assume that the collision is sufficiently short lived that the cloud radius does not change appreciably during the exchange of angular momentum. If θ_T is the effective temperature of the turbulent motions, we see that due to the short relaxation time of the true temperature immediately after collision, $\theta = \theta_1 + \theta_T$.

We inquire now as to the conditions on p_s and θ necessary to expand the cloud to infinity. From (48) we have that

$$\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i = \frac{3MR\theta_1}{2w}\left(\frac{a}{a_1}\right)^{3(1-\alpha)} + \frac{5p_{s\infty}^2}{4Ma^2} - \frac{3GM^2}{10a} \qquad (62)$$

for the cloud with temperature θ_1 when $a = a_1$. Now, after the collision the angular momentum is p and the temperature is $\theta = \theta_1 + \theta_T$ where θ_T is the temperature of the turbulence. Since $\theta_T(a)$ does not excite the internal motions of the molecules, it varies as $\theta_T\left(\frac{a_1}{a}\right)^2$, i.e. as for an adiabatic expansion of a gas possessing three degrees of freedom. Therefore, (62) becomes

$$\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i = \frac{3MR}{2w}\left[\theta_1\left(\frac{a}{a_1}\right)^{3(1-\alpha)} + \theta_T\left(\frac{a_1}{a}\right)^2\right. + \frac{5p_s^2}{4Ma^2} - \frac{3GM^2}{10a} \qquad (63)$$

Now, a_1 is assumed to be the equilibrium radius so that (62) gives

$$\frac{3MR\theta_1}{2w} + \frac{5p_{so}^2}{4Ma_1^2} - \frac{3GM^2}{10a_1} = 0$$

The condition that (63) have positive real roots is that θ_T and p_s be as small or smaller than the values which satisfy the following relations. (See (54) for $\alpha = 2/3$)

$$\alpha = 1: \qquad \frac{5wp_s^2}{6M^2R\theta_1} + \frac{\theta_T}{\theta_1} a_1^2 = \frac{w^2G^2M}{100R^2\theta_1^2}$$

$$\alpha = \frac{2}{3}: \qquad \frac{5wp_s^2}{6M^2R\theta_1} + \frac{\theta_T}{\theta_1} a_1^2 = \frac{2}{15\sqrt{3}} \left(\frac{wGMa_1^{1/3}}{5R\theta_1}\right)^{\frac{3}{2}} \qquad (64)$$

If p_s and θ_T are less than given in (64), then $\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i$ before and after collision is as shown in Fig. 4.

Immediately after the collision and until $a = a'$, $\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i > 0$ and the expansion is accelerating. For $a' < a < a_2$ the expansion is decelerating and making use of the radial momentum (which we have not included in (62)) gained for $a_1 < a < a'$. This momentum carries the cloud up to (and beyond if there is any excess) a_2 where the expansion will continue by itself drawing the necessary energy from the radiation field.

If p_s and θ_T are larger than given in (64), then, after collision, $\frac{1}{2}\frac{d}{dt}\sum_i \overline{p}_i \cdot \overline{r}_i$ vanishes for no value of a as shown in Fig. 5. Thus, the expansion is increasing in weighted momentum for all values of a after collision.

We are interested in determining p_s and θ_T so that the cloud is just lifted to a_2. This will be the minimum value required to expand the cloud to infinity.

The rate of expansion is given by (18) if $\theta_T = 0$. In general, however, we must consider θ_T. Thus, we write in place of (18), the more general expression

$$\dot{a}^2 + \frac{25p_s^2}{6M^2a^2} - \frac{2GM}{a} - \frac{10R}{3w}\left[\frac{1}{1-\alpha}\,\theta_i\left(\frac{a}{a_1}\right)^{3(1-\alpha)} + \frac{1}{1-\gamma}\left(\frac{a}{a_1}\right)^{3(1-\gamma)}\right] = C \quad (65)$$

for $\alpha \neq 1$. And

$$\dot{a}^2 + \frac{25p_s^2}{6M^2a^2} - \frac{2GM}{a} - \frac{10R}{3w}\left[3\theta_i \ln\left(\frac{a}{a_1}\right) + \frac{1}{1-\gamma}\theta_r\left(\frac{a}{a_1}\right)^{3(1-\gamma)}\right] = C' \quad (65)$$

for $\alpha = 1$. Of course, $\gamma = 5/3$ for reasons already given.

Now, we assume that $\dot{a} = 0$ when $a = a_1$. This allows us to evaluate the constants of integration.

$$\alpha \neq 1: \quad \dot{a}^2 + \frac{25p_s^2}{6M^2}\left(\frac{1}{a^2} - \frac{1}{a_1^2}\right) - 2GM\left(\frac{1}{a} - \frac{1}{a_1}\right)$$
$$- \frac{10R}{3w}\left[\frac{1}{1-\alpha}\theta_i\left(\left(\frac{a}{a_1}\right)^{3(1-\alpha)} - 1\right) + \frac{1}{1-\gamma}\theta_r\left(\left(\frac{a}{a_1}\right)^{3(1-\gamma)} - 1\right)\right] = 0 \quad (66)$$

$$\alpha = 1: \quad \dot{a}^2 + \frac{25p_s^2}{6M^2}\left(\frac{1}{a^2} - \frac{1}{a_1^2}\right) - 2GM\left(\frac{1}{a} - \frac{1}{a_1}\right)$$
$$- \frac{10R}{3w}\left[3\theta_i \ln\left(\frac{a}{a_1}\right) + \frac{1}{1-\gamma}\theta_r\left(\left(\frac{a}{a_1}\right)^{3(1-\ )} - 1\right)\right] = 0 \quad (67)$$

Since we are interested in the p_s and θ that will just expand the cloud to its larger and unstable equilibrium radius, a_2, we see that $\dot{a}$ must vanish at a_2. Further, since a_2 is an equilibrium radius, $\ddot{a}$ will vanish as well. This, then, gives us two conditions which allow us to determine a_2 and the relation between p_s, θ_r, and θ_1.

We consider the two examples $\alpha = 2/3$, $\alpha = 1$. For $\alpha = 2/3$, we have from (56) that $\dot{a}$ vanishes when

$$\frac{25p_s^2}{6M^2}\left(\frac{1}{a^2} - \frac{1}{a_1^2}\right) - 2GM\left(\frac{1}{a} - \frac{1}{a_1}\right)$$
$$- \frac{10R}{3w}\left[3\theta_i\left(\frac{a}{a_1} - 1\right) + \left(-\frac{3}{2}\right)\theta_r\left(\left(\frac{a_1}{a}\right)^2 - 1\right)\right] = 0$$

But this contains the factor $\left(\frac{1}{a} - \frac{1}{a_1}\right)$ giving the real root $a = a_1$. Dividing this out leaves

$$\frac{25p_s^2}{6M^2}\left(\frac{1}{a}+\frac{1}{a_1}\right)-2GM-\frac{10R}{3w}\left[-3\theta_1 a-\frac{3}{2}a_1{}^2\theta_1\left(\frac{1}{a}+\frac{1}{a_1}\right)\right]=0$$

which may be rewritten as

$$\frac{10R\theta_1 a^2}{w}+\left(\frac{25p_s^2}{6M^2a_1}-2GM+\frac{5R\theta_1 a_1}{w}\right)a+\left(\frac{25p_s^2}{6M^2}+\frac{5R\theta_T a_1{}^2}{w}\right)=0 \qquad (58)$$

If we observe that $\dot{a}=0$ at $a=a_2$ gives a double root in (58), then we have immediately that

$$a_2=-\frac{\left(\frac{25p_s^2}{6M^2a_1}-2GM+\frac{5R\theta_T a_1}{w}\right)}{\frac{20R\theta_1}{w}} \qquad (59)$$

And

$$\left(\frac{25p_s^2}{6M^2a_1}-2GM+\frac{5R\theta_T a_1}{w}\right)^2=\frac{40R\theta_1}{w}\left(\frac{25p_s^2}{6M^2}+\frac{5R\theta_T a_1{}^2}{w}\right) \qquad (60)$$

Now, the kinetic energy of the cloud immediately after collision is

$$T=\frac{5p_s^2}{4Ma_1{}^2}+\frac{3R\theta_T M}{2w} \qquad (60')$$

excluding the thermal energy, $\frac{3R\theta_1 M}{2w}$. Thus (60) gives

$$\left(T-\frac{3GM^2}{5a_1}\right)^2=\frac{12R\theta_1 MT}{w} \qquad (61)$$

$$T=\frac{1}{2}\left(\frac{6}{5}\frac{GM^2}{a_1}+\frac{12R\theta_1 M}{w}\right)\left[1\pm\sqrt{1-\frac{\frac{36G^2M^4}{5a_1{}^2}}{\frac{6GM^2}{5a_1}+\frac{12R\theta_1 M}{w}}}\;\right] \qquad (62)$$

This, then, is the kinetic energy which a cloud ($\alpha = 2/3$) will have immediately after a collision if it is to just expand to its larger equilibrium radius a_2.

Now, for $\alpha = 1$, (57) gives

$$\frac{25p_s^2}{6M^2}\left(\frac{1}{a^2}-\frac{1}{a_1{}^2}\right)-2GM\left(\frac{1}{a}-\frac{1}{a_1}\right)-\frac{10R\theta_1}{w}\ln\frac{a}{a_1}+\frac{5R\theta_T a_1{}^2}{w}\left(\frac{1}{a^2}-\frac{1}{a_1{}^2}\right)$$

$$=0$$

And (59) gives

$$\frac{5k\theta_1 a^2}{w} - GMa + \frac{10a_1{}^2T}{3M} = 0$$

where T is as defined in (60). The simultaneous solution of these two equations obviously gives a transcendental equation. Therefore we do not attempt an exact solution for a particular collision.

By examining (62) we see that T is of the same order of magnitude as $\frac{3GM^2}{5a_1}$. Thus, approximately,

$$T > \frac{1}{2}\ \frac{3GM^2}{5a_1}$$

in order that the cloud expand from a_1. Therefore, as an estimate, we write

$$T \sim \frac{1}{2}\ \frac{3GM^2}{5a_1}$$

for a limiting collision. For $M = 10^4 M_\odot$, $a_1 = 5$ psc, T corresponds to a velocity of 2.26 km/sec. How much of this is supplied by the collision depends, of course, on the angular momentum before the collision. We see, then, that collisions at speeds corresponding to the observed radial motions would in general disrupt the clouds and cause them to expand to infinity. However, if the velocities of collision are much less than the above figure, as will be discussed later, we can expect very few collisions to result in expansion to the continuum of the participating clouds.

Let us now turn our attention from the mechanics of an isolated gas cloud to the problem of considering the effects of momentum exchanges between clouds.

There is some evidence[2] that there is equipartition of the energy of translation of the gas clouds. If this is true, then it implies that the losses of energy by collision are not as great as the sources of translational velocity (e.g. galactic rotation etc.) This seems a fairly reasonable result since really head on collisions between clouds where considerable energy is lost should be relatively rare, i.e. $\sim 10^8$ years.

Thus, if T is the effective temperature of translation, so that

$$\frac{1}{2}M\overline{v^2} = \frac{3}{2}kT \tag{64}$$

and if the number of clouds with masses in the interval (M, M + dM) is n(M)dM, then the number of clouds with velocity components in $(v^1, v^1 + dv^1)$ and mass in (M, M + dM) is

$$n(M)\left(\frac{M}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{Mv_i v^i}{2kT}\right) \, dM \, dv^1 dv^2 dv^3 \tag{65}$$

where we are neglecting the gravitational interactions of the clouds. Thus the observed distribution of radial velocities if there is no ordered motion present would be

$$n(M)\left(\frac{M}{2\pi kT}\right)^{\frac{1}{2}} \exp\left(-\frac{M\, v_r^2}{2kT}\right) \tag{66}$$

Because of the nature of the conclusions which will be reached it is desirable to consider the problem in a very qualitative manner using only physical arguments before going on to establishing the behavior of the clouds on a more rigorous mathematical basis. Thus, to begin with, we write down the Hamiltonian of the cloud as

$$H = \frac{1}{2M} p_i p^i + \frac{5p_s^2}{4Ma^2} \tag{67}$$

where $p^i = Mv^i$. This neglects temperature effects and so is actually valid only for $\alpha = 1$. But as will be seen, the characteristics in which we shall be interested are not too drastically affected. Probably the worst error is in omitting the gravitational term giving the self energy of the cloud,

$$- \frac{3GM}{5a} .$$

This was omitted because it was desired to obtain this effect by physical arguments on the actual collision phenomenon. Since p_i and p_s appear quadratically, we expect equipartition between the translational and rotational degrees of freedom.

Now, for a gas cloud, we have from (51) and (55) that

$$p_s \leq \begin{cases} \frac{2}{5} M^2 \left(\frac{G^3 w a_o}{15 M R \theta_o} \right)^{1/4} & \text{for } \alpha = 2/3 \\ \frac{1}{5} GM^2 \left(\frac{3w}{10 R \theta_o} \right)^{1/2} & \text{for } \alpha = 1 \end{cases}$$

But, from equipartition of energy, since rotation comprises three degrees of freedom, we have

$$\frac{5p_s^2}{4Ma^2} = \frac{1}{2} Mv^2$$

where v is the average velocity of translation of the cloud. Therefore, if this equipartition is not to disperse the clouds by violating (51) and (55), we have

$$v \leq \frac{1}{Ma} \sqrt{\frac{5}{2}} \begin{cases} \frac{2}{5} M^2 \left(\frac{G^3 w a_o}{15 M R \theta_o} \right)^{1/4} & \text{for } \alpha = 2/3 \\ \frac{1}{5} GM^2 \left(\frac{3w}{10 R \theta_o} \right)^{1/2} & \text{for } \alpha = 1 \end{cases}$$

Then, if $M = 10^4 M_\odot$, $\theta_o = 100^\circ$ K, $a_o = 10^{19}$cm, $a = 10^{19}$cm, $w = 2$, we have for $\alpha = 2/3$ that $v \leq 2.69$ km/sec. But this is the average v. This means only that the average spin will not disperse the cloud. To have some assurance that a fair percentage of the clouds will survive several collisions we should probably require that the average relative velocity of the clouds be 1 km/sec or less which we see is a very low value.

To return to (75) we see that the distribution in p_s is given by

$$C\ f(p_1)\ \exp\left(-\frac{5p_s^2}{4Ma^2kT}\right) \qquad (76)$$

where C is a constant and $f(p_1)$ is the distribution in momentum given in (73).

Now, a and p_s are related by (49). These give, for no turbulence,

$$p_s^2 = \begin{cases} \frac{6M^2R\theta_1}{5wa_1}\left(\frac{GMwa_1}{5R\theta_1}\,a - a^3\right) & \text{for } \alpha = 2/3 \\ \frac{6M^2R\theta_1}{5w}\left(\frac{GMw}{5R\theta_1}\,a - a^2\right) & \text{for } \alpha = 1 \end{cases}$$

or

$$p_s = F(\alpha, a)$$

Those clouds in the interval $(p_s\ ,\ p_s + dp_s)$ will also lie in the interval $(a, a + da)$ in configuration space where

$$dp = F'(\alpha, a)da$$

Thus, (76) gives for the distribution over a,

$$\frac{1}{2}\,Cf(p_1)\begin{cases} \dfrac{\sqrt{\frac{6M^2R\theta_1}{5wa_1}}\left(\frac{GMwa_1}{5R\theta_1} - 3a^2\right)}{\sqrt{\frac{GMw}{5R\theta_1}\,a_1a - a^3}}\ \exp\left[-\frac{3MR\theta_1}{2wa_1kT}\left(\frac{GMwa_1}{5R\theta_1}\,\frac{1}{a} - a\right)\right] \\ \dfrac{\sqrt{\frac{6M^2R\theta_1}{5w}}\left(\frac{GMw}{5R\theta_1} - 2a\right)}{\sqrt{\frac{GMw}{5R\theta_1}\,a - a^2}}\ \exp\left[-\frac{3MR\theta_1}{2wkT}\left(\frac{GMw}{5R\theta_1}\,\frac{1}{a} - 1\right)\right] \end{cases}$$

for $\alpha = 2/3$ and $\alpha = 1$ respectively. Or, if $\lambda \equiv \frac{GMw}{5R}$, $\beta \equiv \sqrt{\frac{6M^2R\theta_1}{5wa_1}}$, then we have

$$\frac{1}{2}\,C\beta f(p_1)\begin{cases} \dfrac{\lambda a_1 - 3a^2}{\sqrt{\lambda a_1 a - a^3}}\ \exp\left[-\frac{3MR\theta_1}{2wa_1kT}\left(\frac{\lambda a_1}{a} - a\right)\right] & \text{for } \alpha = 2/3 \\ \dfrac{\sqrt{a_1}(\lambda - 2a)}{a - a^2}\ \exp\left[-\frac{3MR\theta_1}{2wkT}\left(\frac{\lambda}{a} - 1\right)\right] & \text{for } \alpha = 1 \end{cases}$$

We see that both expressions vanish as a approaches zero because of the infinite spin **energy, i.e.** $p_s^2/a^2 \longrightarrow \infty$ as $p_s \longrightarrow 0$. However, from (18), we see that the contribution of the gravitational potential energy to the Hamiltonian would more than offset this effect if we had included it.

Let us disregard this vanishing at a = 0 and consider the effect of repeated collisions and the resulting trend toward equilibrium.

Suppose that initially all clouds have a radius a and angular momentum p_s. After the clouds have begun to undergo collisions, the individual angular momenta, of course, scatter from the initial value, p_s. If p_s was the value expected for equipartition of energy, then approximately as many would have p_s increased as decreased. Now, let us consider those for which p_s, and, of course a, were decreased.

First, their collision cross section has been decreased. Thus, for this reason, the clouds with small angular momentum will tend to remain in that region longer because their collision rate is lower.

Secondly, while the gravitational momentum exchanges will be essentially unaltered as far as the components of linear momentum are concerned, the exchanges of angular momentum by this means will be greatly reduced and will on the average show an increasing tendency to decrease p_s rather than increase it because for small p_s, the angular velocity becomes large.

Now, the gravitational exchange of angular momentum depends mainly on the tidal effects produced by the inhomogeneity of the field of one cloud in the vicinity of the other. The tidal forces are thus proportional to the radius of the cloud in which they are produced and their moment is thus proportional to a^2. Thus, the magnitude of the angular momentum exchanges decrease with a as a^2. Or dimensionally, we write

$$[\Delta p_s] = ML^2T^{-1}$$

Thus, we write

$$\Delta p_s \propto \frac{Ma^2}{\tau}$$

for the passage of clouds at a given distance of closest approach, which, of course, does not vary with a. M is the cloud mass, a its radius, and τ the time of interaction. But τ and M are independent of p_s. Thus, for a given group of distinct clouds, $\Delta p_s \propto a^2$.

Further, from (70) $p_s \propto a^{1/2}$ for small a. But $p_s \propto a^2\omega$. Thus

$$a^{1/2} \propto a^2\omega$$

and the peripheral velocity is

$$a\omega \propto \left(\frac{1}{a}\right)^{1/2}$$

which we see approaches infinity as a approaches zero. Thus those momentum exchanges in which the clouds approach sufficiently close to exceed ωa become less probably as a decreases, and a preponderant fraction of the exchanges only serves to further decrease p_s and increase ω.

Therefore, we conclude that clouds finding themselves with small p_s will remain in this region for longer periods than in regions of large p_s because of the decreased momentum exhange rate. And further, that having reached a region of small p_s, it becomes increasingly probable that p_s will durther decrease. This, then, implies instability at small radii in addition to that at large radii already discussed.

Conversely, clouds finding themselves with an excess p_s will tend to go toward larger p_s rather than return to the norm. Thus, even with collision velocities of the order of 1 km/sec, there would be a trend toward the contimuum.

We may develop the above assertions more rigorously on the basis of statistical mechanics. We use (43) and describe the rotation more generally in terms of Euler's angles.

Now, the energy of a gas cloud is altered partly by exchange with other clouds during a collision and partly by exchange with the radiation field. Therefore, if we are to consider the entire system, we must include the radiation field. Therefore, we assume that associated with each cloud is a space filled with electromagnetic radiation. Each space is sufficiently large that it remains at constant temperature in spite of the exchanges with its associated cloud. When the radius of the cloud is a_0 and the temperature of the cloud θ_0, we denote the energy in the associated radiation space plus the thermal energy of the cloud by U_0. Then, from (44), for $a \neq a_0$, we have that the energy in radiation space plus the thermal energy in the cloud is U where

$$U_0 = U + \frac{MR\theta_0}{w(1-\alpha)}\left(\left(\frac{a}{a_0}\right)^{3(1-\alpha)} - 1\right)$$

The Lagrangian for the cloud from (15) is, then,

$$L = \frac{1}{5} Ma^2(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) + \frac{3}{10} M\dot{a}^2 + \frac{3}{5}\frac{GM^2}{a} + \frac{1}{2}M\overline{u^2} - U$$

where $\frac{1}{2}M\overline{u^2}$ is the contribution of the turbulent motions. However, because the turbulence damps out rapidly and is assumed to be replenished by an energy input through the translational degrees of freedom, (See the discussion immediately above (72)), we neglect its effects and henceforth will omit the term from the Lagrangian and Hamiltonian. It should be pointed out that we are still considering a homogeneous spherical cloud of radius a.

The conjugate momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{2}{5} Ma^2\dot{\theta}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = \frac{2}{5} Ma^2(\dot{\varphi} + \dot{\psi}\cos\theta)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = \frac{2}{5} Ma^2(\dot\psi + \dot\varphi \cos\Theta)$$

$$p_a = \frac{\partial L}{\partial \dot a} = \frac{3}{5} M\dot a$$

Thus, the Hamiltonian is

$$H = \frac{1}{4Ma^2}\left[p_\Theta^2 + \frac{1}{\sin^2\Theta}\left(p_\psi^2 + p_\varphi^2 - 2p_\psi p_\varphi \cos\Theta\right)\right] + \frac{5p_a^2}{6M}$$
$$- \frac{3GM^2}{5a} + U_0 - \frac{MR\Theta_0}{w(1-\alpha)}\left[\left(\frac{a}{a_0}\right)^{3(1-\alpha)} - 1\right]$$

Therefore, it follows that the distribution in μ-space[6] is

$$C \exp\left(\frac{-H}{kT}\right)$$

where T is the effective temperature of the system and C is a constant.

Now, we are primarily interested in the distribution over a, which we shall denote by A(a). Then

$$A(a) = C \int_0^{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-\frac{H}{kT}\right) dp_a\, dp_\Theta\, d\varphi\, d\psi\, dp_\psi\, dp_\varphi\, d\Theta$$

The indicated order of integration is found to be as convenient as any. The integration then consists of repeated applications of the relation

$$\int_{-\infty}^{+\infty} \exp\left(-\alpha\mu^2 - \beta\mu\right) d\mu = \sqrt{\frac{\pi}{\alpha}}\, \exp \frac{\beta^2}{4\alpha} \qquad \mathrm{Re}\,\alpha > 0$$

We obtain

$$A(a) = \frac{64\pi^4\sqrt{6}}{25}\, CM^2k^2T^2 \exp\left(-\frac{u_0}{kT} + \frac{MR\Theta_0}{kTw(1-\alpha)}\right)$$
$$a^3 \exp\left[\frac{1}{kT}\left(\frac{3GM^2}{5a} - \frac{MR\Theta_0}{w(1-\alpha)}\left(\frac{a}{a_0}\right)^{3(1-\alpha)}\right)\right] \tag{78}$$

We see that A(a) and $\int_0^{+\infty} A(a)\,da$ diverge for $\alpha < 4/3$. A(a) is infinite at the origin and at infinity. Further $\int_0^\beta A(a)da$ and $\int_\beta^\infty A(a)da$ are both infinite for $\beta > 0$. Therefore, we conclude that in the equilibrium configuration, all clouds will have collapsed to solids or expanded into the continuum. This is, of course, the conclusion which we reached on the basis of physical arguments.

At this point let us investigate the oblate spheroidal model to see what, if any, significant differences will be introduced. As it turns out, our main interest lies with the divergence at a = 0, ∞. Therefore, we investigate these two extremes. For intermediate values of a, the difference, although certainly not zero, is of little interest.

We shall use the results of Lamb, Hydrodynamics 6th ed Sections 373, 374. We describe the oblate spheroid by its semi-major axis a and its semi-minor axis c. Then, the effective radius R is

$$R = a^{2/3} c^{1/3}$$

Further, the parameter ζ which is a measure of the eccentricity of the ellipsoid is defined by

$$a = \frac{(\zeta^2 + 1)^{1/2}}{\zeta} c$$

Then, from 373 (12), the gravitational potential energy of a homogeneous oblate spheroid is

$$V = -\frac{3GM^2}{5R}\left(\frac{\zeta^2 + 1}{\zeta^2}\right)^{1/3} \zeta \cot^{-1}\zeta$$

And its angular momentum is

$$H^2 = \frac{6GM^3R}{25}\left(\frac{\zeta^2 + 1}{\zeta^2}\right)^{2/3}\left[(3\zeta^2 + 1)\cot^{-1}\zeta - 3\zeta^2\right]$$

From 374 (6) and the remarks following 374 (9) we see that ζ does not approach zero as R approaches infinity. Thus, c does not vanish and the potential energy behaves similar to that of a sphere as R approaches infinity. Therefore, the oblate spheroid introduces no essential changes at infinity.

Let us now turn our attention to the somewhat more obscure situation when R goes to zero. To determine ζ we need one further relation and that is

$$T = -\frac{1}{2} V$$

But

$$T = \frac{5H^2}{4Ma^2}$$

$$= \frac{5H^2}{4MR^2} \left(\frac{\zeta^2}{\zeta^2 + 1}\right)^{1/3}$$

Thus

$$T = \frac{3M^2}{10R} \left(\frac{\zeta^2 + 1}{\zeta^2}\right)^{1/3} \left[(3\zeta^2 + 1)\zeta \cot^{-1}\zeta - 3\zeta^2\right]$$

And hence the equilibrium relation between the kinetic and potential energy gives

$$\zeta^3 \cot^{-1}\zeta = \zeta^2$$

We have a double root at $\zeta = 0$ and the rest at

$$\zeta = \cot \frac{1}{\zeta}$$

giving an infinite number of roots in all.

The interpretation of this, for a gas cloud, however, is not so simple as for a liquid drop. The reason is that the volume of the gas cloud may vary, say, as the temperature changes. Therefore, we shall solve the problem somewhat indirectly by considering the most extreme divergence possible of the oblate spheroidal model from the spherical model.

This occurs when small values of a are reached and the temperature has fallen so low as to be unable to support the cloud in the direction parallel to the axis of rotation. Then the cloud becomes a disk, i.e.

$$c \longrightarrow 0$$

$$\zeta \longrightarrow 0$$

Putting

$$R = (a^3)^{1/3}\left(\frac{\zeta^2}{\zeta^2 + 1}\right)^{1/8}$$

we have

$$\lim_{\zeta \to 0} V = -\frac{3}{5}\frac{GM^2}{a}\frac{(\zeta^2 + 1)^{\frac{1}{6}}}{\zeta^{1/3}}\left(\frac{\zeta^2 + 1}{\zeta^2}\right)^{1/3}\zeta \cot^{-1}\zeta$$

$$= -\frac{3GM}{5a}\ \frac{\pi}{2}$$

And we see that V still diverges as 1/a as $a \longrightarrow 0$. Thus, there is no essential difference introduced at a = 0 by the oblate model and we return to our result based on calculations from the simpler spherical model.

We interpret the result as follows: First, from observation, gas clouds are known to exist as individual units. Therefore, they cannot be in statistical equilibrium. That is, an individual cloud is a stable unit, but by collision with other clouds it may lose its angular momentum and collapse, or gain angular momentum and expand to infinity. The observed clouds, therefore, have not undergone a sufficient number of collisions to do this. But we have shown that with the observed cloud density and velocity a cloud would be expected to undergo of the order of 60 collisions in $4 \cdot 10^8$ years. This should be ample collisions to pretty well depopulate space of gas clouds. Since there are so many clouds, we conclude, then, that the collision rate is much lower than this. This requires, since the spacing of the clouds and the magnitude of their radial velocities is fixed by observation, that their velocities be not random as assumed when calculating the above collision rate, but rather ordered, so that only a very small portion of the observed velocity is random. We must require that the

60 collisions in $4 \cdot 10^8$ years be cut to say 5 or 10. Thus, since 5 km/sec random velocity was used to calculate the collision rate, giving 60 collisions in $4 \cdot 10^8$ years, we see that the random velocity must be of the order of 0.5 km/sec or less. This makes T correspondingly low so that only a very small fraction of the clouds lost by instability will be lost to $a = \infty$. It also implies that any tendency toward equilibrium will be toward very low p_s corresponding to the low T.

We conclude that the rather large observed radial velocities are due to such ordered processes as galactic rotation, extension of the spiral arms, and possibly orbits in the gravitational field of other clouds. The idea of large complexes of clouds satisfies the above restrictions and follows very nicely from Table I.

It should be pointed out that the difficulty involved in forming protostars from gas and dust clouds caused by the spin angular momentum of the clouds is actually nonexistent unless one considers clouds sufficiently isolated or having their velocities sufficiently ordered that their collision rate is less than, say, one collision every 10^8 years. Thus, the collisions transfer the angular momentum of a cloud into angular momentum of the colliding clouds relative to each other allowing the clouds to collapse individually. It would therefore be of interest to consider the dynamics of the collapse of a gas cloud in some detail. Extensive use of Spitzer's [3] work on temperatures should be made. One would probably find, instead of forming single stars or star systems, that because of the large masses of the gas clouds, clusters of stars would tend to be formed simultaneously on the basis of Tables I, II extended to higher densities and lower temperatures.

Now, we have two essential criticisms of the above gas cloud model based on gravitational forces alone. The first is that we require gas densities of the order of 10^{-21}gm/cm^3 for stable clouds of the size observed and very low proper motion to keep even these clouds from being dispersed on collision. 10^{-21} is ten or a hundred times denser than observations indicate. The second objection is that the Hamiltonian for gravitational forces alone gives a divergence in the equilibrium state at zero and infinite cloud radius. This can be gotten around at least in part by restrictions on the translational velocity of the clouds. These two difficulties, however, serve to point up the inadequacy in considering a purely gravitational model. This same difficulty with gravitational forces alone will be met in Part II. Therefore, it is felt that the conclusions reached above are fairly strong evidence for the existence of other forces, probably magnetic since we cannot offhand think of what others would be significant.

II

Interstellar Dust Structures

The interstellar dust is observable from its extinction properties and reddening effect. It constitutes of the order of 1% of the matter in the interstellar medium, and yet it is the most readily observable component. For this reason the shapes of the dust structures can be observed in some considerable detail making it of interest to study theoretically the significance of detailed shapes rather than just the general dimension-energy characteristics as was done for the gas clouds which cannot so readily be observed.

From the very irregular shapes of the observed absorbing structures it is obvious that we can at best hope to construct a statistical theory and speak only of general trends and averages. In this manner we shall consider below some of the observable characteristics of the dust structures.

One may observe the color excess of a star and thereby determine the smeared out average dust density in the intervening space. Or, one may observe the actual geometrical shape of a given collection of dust. It is this latter type of observation which we shall attempt to interpret theoretically.

At present, it seems that the most satisfactory origin of the dust is from a process of condensation of some of the heavier components of the interstellar gas.[7] Thus we envisage dust forming in more or less regular globs or clouds which are internally homogeneous. They are probably limited in size by the gas distribution. Such dust structures are readily observable in space. But, there is also an immense amount of striated and filamentary structure which cannot be explained simply on the basis of condensation. Such structures imply an evolution from their initially less eccentric homogeneous form.

To investigate the possible nature of this evolution toward filamentary structures, one first remarks that the dust tends to move to a large extent with the gas. Due to galactic rotation etc. one expects to find shearing gradients in the gas and hence in the dust. It is first shown below that a globule of dust placed in a shearing field will be drawn out into a filament,(7). Filaments produced in this manner are compared to the observed filaments and striations and it will be seen that the comparison is favorable to an interpretation of the evolution toward filaments and striations as being due to shearing fields caused by the passage of early type stars, galactic rotation etc.

Now, it is possible to extend these results by considering dust grains of a given size only and employing the equations of motion for the dust to show that any sort of shearing fields due to random turbulence or ordered velocity fields will produce in the dust structures an irreversible trend toward striated and filamentary forms. That is, a dust cloud, which we assume has condensed from the heavier atoms in the gas into a more or less regularly shaped form, will be pulled out into filaments by whatever velocity fields are present, and there exists no reverse process, i.e. to take the filaments back into more evenly smeared out structures.

This may be very nicely demonstrated by a drop of ink placed in a fairly quiescent bowl of water. A white bowl is preferable rendering greater contrast and making the fine structure visible. If the water is not too turbulent, i.e. if it has sat for a few minutes since the bowl was filled, the structures produced are quite striking in their resemblance to the actual observed structures in interstellar space.

It must be pointed out, however, that it is evidently a fortuitious similarity because the ink moves exactly with the water whereas the dust lags behind the gas by an amount depending on the grain size. Further, it is shown that there is some evidence of magnetic effects in the actual motion of the dust in interstellar space whereas there is certainly none in the water.

A fairly important topic is the preservation of filaments. That is, if a filament lies, say, very near to a large cloud, there is no mechanism for merging filament and cloud if we consider grains of only one size. Winds, gravitational fields, and radiation pressure as well as thermal diffusion cannot merge two distinct objects. This is demonstrated formally by showing the the visibility, W, defined in (11), does not decrease with time.

Finally, the effects of a continuous distribution of grain sizes are considered. Since the mean free path for the collision of grains with grains is long, $\sim$ light years, we may obtain the resultant structure by superimposing the independent structures for each grain size. It is this consideration which leads to the conclusion that the long fine curved striations, such as are observed in the Pleiades, cannot be accounted for without assuming either a very limited range of grain sizes or else some mechanism such as magnetic fields to keep the dust confined to such narrow ribbons while being drawn around a curved path.

We turn now to a more mathematical discussion and development of the above assertions.

In the following development the c.g.s. system will be used exclusively except that lengths may be expressed in astronomical units, light years, or parsecs when indicated. There will be many idealizations introduced and rigor will in general be sacrificed for the purpose of obtaining results.

We first assume that the dust density, ρ_D, is everywhere less than the gas or hydrogen density, ρ_H, except possibly in a dust globule which will be given special consideration when required. In most regions, if indeed not all,

$$\rho_D \leq 0.01 \rho_H$$

Hence the gravitational field of the dust may be neglected. We assume that the only significant momentum transfer from the dust to the gas is by the relative motion of the dust thru the gas. On the other hand, the dust may receive momentum from the gravitational field of the gas and stars, from the radiation field, and of course directly from the gas by relative motion.

We shall describe the motion of the gas by the three velocity components, w^i, and that of the dust by v^i. And, of course, we shall consider the Lagrangian viewpoint. We represent the time rate of momentum transfer from the radiation field to the dust per unit mass of dust by $k_1 F_1{}^i(x^j,t)$. (See Appendix) If we represent the momentum transfer from the gravitational field per unit mass of dust by $k_2 F_2{}^i(x^j,t)$, then, of course,

$$k_2 F_{2i} = -\psi_{,i}$$

where ψ is the scalar potential describing the gravitational field. The equations of motion for a dust grain are

$$\frac{Dv^i}{Dt} = \gamma(w^i - v^i) + k_1F_1{}^i(x^j,t) + k_2F_2{}^i(x^j,t) \tag{1}$$

or

$$\frac{Dv^i}{Dt} = \gamma(w^i - v^i) + F^i(x^j,t) \tag{2}$$

where

$$F^i \equiv k_1F_1{}^i + k_2F_2{}^i \tag{3}$$

and

$$\gamma \equiv \frac{\pi\sigma^2}{M}\rho_H\bar{u} . \tag{4}$$

σ and M are the dust grain radius and mass respectively and $\bar{u}$ is the mean thermal velocity of the hydrogen gas.

Now, we see that $J(\theta,\varphi,x^j,t)$, (see Appendix), is a continuous function, and hence $F_1{}^i(x^j,t)$ and $F_2{}^i(x^j,t)$, so long as $\rho_H + \rho_D$ is everywhere finite, are continuous functions of position and time. Hence, $F^i(x^j,t)$ is a continuous function with a derivative everywhere.

If we consider now the motion of the dust on the basis of (2), we are assuming that the dust moves as a fluid implying that the thermal motions of the dust do not produce significant diffusion. That such diffusion is indeed negligible follows from a consideration of the random thermal velocities of the dust grains. Assuming equipartition with the gas, then for $\sigma = 10^{-5}$cm, $M = 4.2 \cdot 10^{-15}$gm, the latter corresponding to an internal density of 1 gm/cm^3, we have

$$v = \sqrt{\frac{3kT}{M}}$$

which for 100° K gives 3.1 cm/sec. This would be the velocity of expansion into a vacuum. The presence of the gas reduces such motion to diffusion which would progress at a speed much less than v. Therefore, since in 10^7 years, 1 cm/sec gives

20 a.u., a negligible distance, we henceforth neglect the effects of thermal diffusion and adopt (2) forthwith.

Now, the interstellar gas is in turbulent motion because of the shearing velocity fields introduced by the passage of type O and B stars, galactic rotation, the extension of the spiral arms, etc., and because of the collision of gas clouds. Thus, there will be shearing gradients at most places in the gas. Since the dust tends to follow the motion of the gas due to the preponderance of the term $\gamma(w^i - v^i)$ in (2), it follows that v^i will be a sort of turbulent field. This immediately suggests a means of producing filaments and striations. For let us consider dust originally in the form of a cube as shown in Fig.6. We consider v^i to be of the form

$$v_x = v_o\left(\frac{y}{y_o}\right), \quad v_y = 0, \; v_z = 0$$

The cube initially has sides of length a. At subsequent times we shall be interested in the width, w, and length, L, as shown in Fig.6.

Obviously

$$w = a \sin\alpha$$

$$L = \frac{a}{\sin\alpha}$$

Further

$$\cot\alpha = \left(\frac{a}{y_o} v_o t\right)\frac{1}{a} = \frac{v_o t}{y_o}$$

Therefore

$$\sin\alpha = \frac{1}{\left[1 + \left(\frac{v_o t}{y_o}\right)^2\right]^{1/2}}$$

so that

$$w = \frac{a}{\left[1 + \left(\frac{v_o t}{y_o}\right)^2\right]^{1/2}}$$

$$L = a\left[1 + \left(\frac{v_o t}{y_o}\right)^2\right]^{1/2}$$

$$\frac{L}{w} = 1 + \left(\frac{v_o t}{y_o}\right)^2 \tag{5}$$

We see that L/w increases with time indicating that the cube is drawn into a filament.

To apply (5), let us compute very roughly what order of L/w we might expect as a result of the passage of, say, an O star through a region. The radiation field of such an early type star will exert forces on the surrounding gas and, in general, will exchange considerable momentum with the region.

If the star has a relative velocity, V, and this is distributed over a distance, L, then we expect shearing gradients of the order of V/L. The time which this gradient exists, without taking into account the lingering inertial turbulence effects is of the order of 2L/V. Because of the continuance of shearing gradients in the turbulent field by inertial forces, we write nL/V and remark that certainly $n > 2$. Thus, the total effect should give

$$\frac{v_o}{y_o} t = \frac{V}{L}\left(n \frac{L}{V}\right) = n \tag{6}$$

Thus (5) yields

$$\frac{L}{w} = 1 + n^2 \tag{7}$$

At this point it would be well to compare the theory with observations. It must first be pointed out that one cannot measure directly the velocity gradients in interstellar space. And further, one cannot determine L and w for a structure unless the distance to the structure is known because only the angular dimensions are available. Therefore, while one may readily determine L/w for the observed projection of any visible structure, it is only for those structures whose distance is known that the actual magnitudes of the necessary shearing gradients may be estimated.

We consider the dust structures in the Pleiades. The distance is known and is about 140 parsecs. The striations in this region are remarkable in appearance and will serve as clear cut examples for some later conclusions.

The lengths of the striations or filaments are up to $3 \cdot 10^{18}$ cm and the widths vary from $4 \cdot 10^{17}$ for the long straight filaments to the limit of resolution, say, $1 \cdot 10^{16}$ for the fine less regular striations.

L/w has values from, say 5 to 25 which, from (7), is covered by n in the interval (2,5). Thus, L/w seems to be satisfactorily accounted for.

To compare the observed and computed values of L and w we must estimate a. The desired estimate of a depends, of course, on the value of n used. We simply remark that if $n = 4$, then to obtain the maximum observed L, $3 \cdot 10^{18}$ cm, we require only that $a = 0.7 \cdot 10^{18}$ cm which seems to be a reasonable result.

It might be well to point out that if we write (7) as

$$n(t) = \sqrt{\frac{L}{w} - 1} \qquad (9)$$

and recall that n(t) is also the integral over the shearing gradient,

$$n(t) = \int_0^t \frac{\partial v_i}{\partial x^j}\, dt \qquad i \neq j \qquad (10)$$

then the observed value of L/w gives us the history of the shearing field in terms of the parameter n(t). Actually, probably little practical use can be made of this because even if we were to consider a case where there was no turbulence, the velocities would be sufficiently high that the linearizing of the Navier-Stokes equation, as in Stokes' law for a sphere moving thru a viscous medium, would not be valid.

The only feature of the dust structures in the Pleiades which seems a little extraordinary in light of the above discussion is the remarkable straightness of the observed projections of some of the filaments. $n > 2$, as is required to give filaments of such L/w, implies turbulence. But the observed straight projections imply no eddies in their vicinity in the plane perpendicular to the line of sight. Now, due to the fact that the larger eddies required to produce such long filaments are only a little smaller than the region considered, the statistics are naturally very poor so that the restriction on the orientation of the eddies is not entirely out of the question. However, it does appear to be something to be held in mind for later use.

The question of diffusion by the microturbulences, which is included in the more general treatment to follow, can best be disposed of at present by remarking that it is of significance only in that small eddies may produce filaments beyond the limit of resolution of the observer and hence could give an apparent diffusion effect [8].

We see then that turbulence and shearing can be expected to produce filaments and striations from initially globular structures. Let us proceed now to consider briefly a few points to facilitate our understanding of the dynamics of the dust grains before going on to the general development of the evolution of the dust structures.

First, since γ is proportional to σ^4, and F depends upon σ, (2) indicates that the trajectories of the grains of different size can be expected to be markedly different. To investigate this further we point out that with a dust density of $2 \cdot 10^{-24}$ gm/cm^3 and $\sigma = 10^{-5}$cm, the mean free path for collision of a dust grain with another dust grain is

$$L = \frac{1}{4\sqrt{2}\pi N \sigma^2} = \frac{10^{19}}{2\sqrt{2\pi}} \sim 1 \text{ lt yr}$$

Thus the dust grains of different sizes exchange very little momentum and to a fair degree of approximation may be assumed to move independently, as assumed in (2). Therefore, one would expect considerable dispersion of the different sized dust grains. This effect will be taken up at greater length below. And for the present, due to the large value of L, we shall consider only those grains with radii in $(\sigma; \sigma + d\sigma)$.

Let us now consider the general evolutionary trends of a dust structure. We shall begin with a more or less homogeneous cloud which presumably has condensed from the gas in a quiescent region. The reason for assuming a quiescent region is to eliminate the unnecessary complication of simultaneously considering both the initial condensing process involving the creation of grains in once clear regions and the evolution of the dust.

We remark that the inhomogeneities, e.g. edges, internal density fluctuations, etc., are the observable characteristics, and so we shall confine our attention to them. Of great interest will be the quantity analogous to the visibility defined in optical diffraction and interference phenomena.

Since we can only observe the projection of a three dimensional cloud on the two dimensional subspace perpendicular to the line of sight, we might well define the visibility as

$$V \equiv \frac{\tau(x_2{}^i) - \tau(x_1{}^i)}{\tau(x_2{}^i) + \tau(x_1{}^i)} \qquad i = 1,2$$

where x^i ($i = 1,2$) are the coordinates in the subspace and $x_1{}^i$ and $x_2{}^i$ are respectively the coordinates of two adjacent minimum and maximum points of the optical depth, τ.

However, because it is ρ_D and not τ which is directly obtainable from the equations of motion, we shall go one step further and define

$$W \equiv \frac{\rho_D(x_2{}^i) - \rho_D(x_1{}^i)}{\rho_D(x_2{}^i) + \rho_D(x_1{}^i)} \qquad i = 1,2,3 \quad (11)$$

where the subscripts have the same significance as before and we are now considering three dimensional space.

Now, while W is not simply proportional to V, at least it is physically obvious that their functional relationship is invariant to similarity transformations. Further, for thick clouds, $V \longrightarrow 0$ although W may not. For clouds of small extent in the line of sight $V \cong W$. We see then that while V and W are not simply related analytically, there exists a close physical analogy.

Now, we shall laterbe interested in the time dependence of W. Therefore, we shall look into the matter here so that we need not break the chain of thought when the dependence is needed in the development of the evolutionary trend of the dust structures.

Thus, to determine W we use (11). Then we relate $\rho_D(x^i,t)$ to the velocity field, v^i, by the Lagrangian equation of continuity,

$$\frac{D\rho_D}{Dt} + \rho_D v^i{}_{,i} + \rho_D H(x^i,t) = 0 \qquad (12)$$

and then determine $v^i{}_{,i}$ from (3). We shall use the summation convention for italic indices but not for Greek. $H(x^i,t)$ gives losses due to sublimation and is a function of the gas density, ρ_H , and the radiation density,

$$\frac{1}{c}\int_0^{2\pi}\int_0^{\pi} J(\Theta,\varphi,x^i,t)\sin\Theta\, d\Theta\, d\varphi.$$

If we denote the initial position of a particle now at x^i by x_o^i, then we may write for the Lagrangian coordinates x^i,

$$x^i = x^i(x_o^j,t) \tag{13}$$

Since $F^i(x^j,t)$ is a continuous function of x^j, then so long as we neglect the discontinuous effects not taken into account in (2), (e.g. thermal motions of the dust grains, their finite size and separation, and the discontinuities introduced by the passage of somewhat larger solid bodies), x^i, which is determined by (2), will be everywhere a continuous function of x_o^i for all t.

Integrating (12) for a Lagrangian observer, we obtain,

$$\rho_D[x^i(x_0^j,t),t] = \rho_D[x_0^i,0]\ \exp\left[-\left(\int_0^t v^i{}_{,i}\ dt + \int_0^t H\ dt\right)\right] \tag{14}$$

We shall now attempt to estimate the nature of $v^i{}_{,i}$ from (2).
From (2) we write,

$$\frac{Dv^i{}_{,i}}{Dt} = \gamma w^i{}_{,i} + F^i{}_{,i} + (\gamma_{,i}w^i - \gamma_{,i}v^i) - \gamma v^i{}_{,i} \tag{15}$$

It will be found convenient to introduce the following substitutions. If $\frac{D}{Dt}$ denotes differentiation for an observer moving with the dust and $\frac{\mathcal{D}}{\mathcal{D}t}$ denotes differentiation for an observer moving with the gas but evaluated at the dust grain under consideration, then for purposes of physical interpretation we write

$$\frac{D\gamma}{Dt} = \frac{\partial\gamma}{\partial t} + \gamma_{,i}v^i$$

$$\frac{\mathcal{D}\gamma}{\mathcal{D}t} = \frac{\partial\gamma}{\partial t} + \gamma_{,i}w^i \tag{16}$$

And (15) becomes

$$\frac{Dv^i{}_{,i}}{Dt} = \gamma w^i{}_{,i} + F^i{}_{,i} + \left(\frac{\mathcal{D}\gamma}{\mathcal{D}t} - \frac{D\gamma}{Dt}\right) - \gamma v^i{}_{,i} \tag{17}$$

Further, since

$$\gamma w^i{}_{,i} = \frac{\pi\sigma^2\,\bar{u}}{M}\rho_H\, w^i{}_{,i}$$

and the equation of continuity for ρ_H gives

$$\frac{\mathcal{D}\rho_H}{\mathcal{D}t} + \rho_H w^i{}_{,i} - \rho_D H(x^j,t) = 0\ ,$$

we write

$$\rho_H w^i{}_{,i} \cong -\frac{\mathcal{D}\rho_H}{\mathcal{D}t}$$

because $\rho_D << \rho_H$. Thus

$$\gamma w^i{}_{,i} = -\frac{\pi\sigma^2\bar{u}}{M}\frac{\mathcal{D}\rho_H}{\mathcal{D}t} \tag{18}$$

We also note that

$$\frac{\mathcal{D}\gamma}{\mathcal{D}t} = \frac{\pi\sigma^2}{M}\left(\bar{u}\frac{\mathcal{D}\rho_H}{\mathcal{D}t} + \rho_H\frac{\mathcal{D}\bar{u}}{\mathcal{D}t}\right) \tag{19}$$

neglecting the variation in σ due to H for the present development. Thus, combining (18) and (19), we obtain (17) as

$$\frac{Dv^i{}_{,i}}{Dt} = \left(F^i{}_{,i} - \frac{D\gamma}{Dt} + \frac{\pi\sigma^2}{M}\rho_H\frac{\mathcal{D}\bar{u}}{\mathcal{D}t}\right) - \gamma v^i{}_{,i} \tag{20}$$

The solution of (20) is given in Dw 891.1 as

$$v^i{}_{,i} = \exp\left(-\int_0^t\gamma dt\right)\left[\int_0^t\left(F^i{}_{,i} - \frac{D\gamma}{Dt} + \frac{\pi\sigma^2}{M}\rho_H\frac{\mathcal{D}\bar{u}}{\mathcal{D}t}\right)\exp\left(\int_0^\eta\gamma d\mu\right)d\eta + C\right] \tag{21}$$

Because $\gamma \sim 10^{-15}$, the term, C, becomes negligible in, say, 10^6

years, so we neglect it and write merely

$$v^i{}_{,i} = \int_0^t \left(F^i{}_{,i} - \frac{D\gamma}{Dt} + \frac{\pi\delta^2}{M}\rho_H \frac{\mathfrak{D}\bar{u}}{\mathfrak{D}t} \right) \exp\left(-\int_\eta^t \gamma \, d\mu\right) d\eta \qquad (22)$$

Now, except for the dependence of temperature on ρ_D, γ is independent of ρ_D. Actually, $\bar{u}$ varies only very slowly with ρ_D.[3] Thus, we make a slight approximation in $v^i{}_{,i}$ if we consider

$$\frac{\partial \bar{u}}{\partial \rho_D} = 0$$

If we were to include the temperature dependence, we would obtain the same result. The only change would be a more complex physical picture with nothing to be gained.

Therefore, γ, $\frac{D\gamma}{Dt}$, and $\rho_H \frac{\mathfrak{D}\bar{u}}{\mathfrak{D}t}$ are taken as independent of ρ_D.

It would probably be well to point out that while there is indeed some correlation between ρ_H and ρ_D, they are to be considered as entirely independent since any such correlation was established during the initial condensation process. Thus, ρ_H and ρ_D are independent, again not considering H, as far as cause-effect processes at the time $t > 0$ are concerned.

To consider the remaining term in (22), $F^i{}_{,i}$, we write from (3) that

$$F^i{}_{,i} = k_1 F_1{}^i{}_{,i} + k_2 F_2{}^i{}_{,i}$$

The gravitational contribution, $F_2{}^i{}_{,i}$, is independent of ρ_D if $\rho_D \ll \rho_H$. i.e. so long as we stay away from dust globules, if indeed such structures do exist. Thus, if we write

$$k_2 F_2{}^i{}_{,i} = g^i{}_{,i} + G^i{}_{,i} \qquad (23)$$

where $g^i{}_{,i}$ is that part contributed by ρ_D and $G^i{}_{,i}$ by ρ_H, then usually, and possibly always,

$$g^i{}_{,i} \ll G^i{}_{,i}$$

Similarly, the momentum transfer from the radiation field is (see Appendix)

$$k_1 F_1^{\,i} = \frac{\sigma^2}{Mc} \int_0^{4\pi} J(\theta,\varphi,x^j,t)\cos\theta \, d\Omega$$

where $d\Omega$ is the differential of solid angle and θ is the angle between $d\Omega$ and the x^i direction.

Part of the anisotropy in the radiation field will be due to anisotropy of the location of the radiation sources. This part is independent of ρ_D. However, the part due to the anisotropic location of absorbing matter is not independent of the distribution of ρ_D because it is the dust which does the bulk of the absorbing of radiation. Thus, let us write

$$k_1 F_1^{\,i}{}_{,i} = h^i{}_{,i} + H^i{}_{,i} \tag{24}$$

where $h^i{}_{,i}$ is that part depending on ρ_D and $H^i{}_{,i}$ not on ρ_D.

Therefore, we write (22) as

$$v^i{}_{,i} = \int_0^t \left[G^i{}_{,i} + H^i{}_{,i} - \frac{D\gamma}{Dt} + \frac{\pi\sigma^2}{M}\rho_H \frac{\mathfrak{D}\bar{u}}{\mathfrak{D}t} \right] \exp\left(-\int_\eta^t \gamma \, d\mu\right) d\eta$$

$$+ \int_0^t (g^i{}_{,i} + h^i{}_{,i}) \exp\left(-\int_\eta^t \gamma \, d\mu\right) d\eta \tag{25}$$

If

$$u^i{}_{,i} \equiv \int_0^t \left[G^i{}_{,i} + H^i{}_{,i} - \frac{D\gamma}{Dt} + \frac{\pi\sigma^2}{M}\rho_H \frac{\mathfrak{D}\bar{u}}{\mathfrak{D}t} \right] \exp\left(-\int_\eta^t \gamma \, d\mu\right) d\eta$$

$$\nu^i{}_{,i} \equiv \int_0^t (g^i{}_{,i} + h^i{}_{,i}) \exp\left(-\int_\eta^t \gamma \, d\mu\right) d\eta\,, \tag{26}$$

Then

$$v^i{}_{,i} = u^i{}_{,i} + \nu^i{}_{,i} \tag{27}$$

where $u^i{}_{,i}$ is independent of ρ_D.

If we average by operating with $\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} dt$ where T is long compared to the short period fluctuations due to oscillation of gas clouds, shock waves, etc., although short compared to the gross processes such as any general trend for the gas to go from irregular structures to the continuum etc., then $\overline{u^i{}_{,i}}$ is a slowly varying function of position, and, incidentally, of time, where the bar denotes the operation. The same operation on $\upsilon^i{}_{,i}$ giving $\overline{\upsilon^i{}_{,i}}$, because of the sometimes rapid irreversible processes involved in the dependence on ρ_D, namely a trend toward increasing ρ_D, will be a slowly varying function of time and position for the radiative processes but not for the gravitational part. We write, then,

$$\overline{v^i{}_{,i}} = \overline{u^i{}_{,i}} + \overline{\upsilon^i{}_{,i}} \tag{28}$$

Now we must investigate H. This, of course, involves a discussion of the sublimation process. To accomplish this we shall simply consider a very idealized process because of the mathematical difficulties and remark that it is certainly qualitatively similar to the actual process.

Therefore, let us consider dust with a density,

$$\rho_D = \rho_D(x^i, t)$$

Since this dust will consist of grains of all radii, σ , in the interval, say, $(0, 10^{-4})$, we must for the purpose of detailed analysis consider $P(x^i, t, \sigma)d\sigma$, this being the density of the dust with radii in the interval $(\sigma, \sigma + d\sigma)$.

Obviously

$$\rho_D(x^j, t) = \int_0^\infty P(x^j, t, \sigma)\, d\sigma \tag{29}$$

Let there be incident on the dust from $x = +\infty$, plane parallel monochromatic radiation. Then, if $I_\infty(y,z)$ is the radiation intensity at $x = +\infty$, it follows that for absorption only,

$$I(x^1,t) = I_\infty(y,z)\ \exp(-\int_x^\infty K(\eta,t)d\eta) \qquad (30)$$

where K is the absorption coefficient. If

$$\frac{\partial\rho}{\partial y} = \frac{\partial\rho}{\partial z} = \frac{\partial I_\infty}{\partial y} = \frac{\partial I_\infty}{\partial z} = 0 ,$$

then the absorption and extinction coefficients become indentical. And (30) is valid for extinction. If $f(\sigma)$ is the absorption per gm/cm^3 for grains in $(\sigma, \sigma + d\sigma)$, then

$$K(x,t) = \int_0^\infty P(x,t,\sigma)f(\sigma)d\sigma \qquad (31)$$

where we have omitted writing y,z since they are constant.

It is obvious, then, that if C is the heat of sublimation, and if we neglect re-emission of absorbed energy, sublimation takes place, decreasing ρ according to

$$C\frac{\partial\rho}{\partial t} = -I\int_0^\infty P(x,t,\sigma)f(\sigma)d\sigma = -IK$$

Thus

$$\frac{\partial}{\partial t}\int_0^\infty P(x,t,\sigma)d\sigma = -\frac{I_\infty}{C}\ \exp(-\int_x^\infty K(\eta,t)\ d\eta)\ K(x,t) \qquad (32)$$

(32) may be formally integrated. However, the results are just what one would expect. Therefore, we shall merely discuss two special cases. The first is when the absorption by the dust is so heavy that only the outermost striation is exposed to any appreciable radiation. In this case the exposed striations may be sublimed away, but all others will remain intact. The second case is when the absorption is quite low

and the incident radiation is absorbed over distances somewhat larger than the separation of adjacent maxima. Then maxima and minima will be suppressed at the same fractional rate, i.e.

$$H(x_2^i,t) \cong H(x_1^i,t) , \qquad (33)$$

and W will be unaffected. Therefore, we state that except for the actual disappearance of a filament by resublimation, H will have no significant effect on W.

We are now in a position to consider W. Since (28) is of more interest than (27), we write (14) as

$$\rho_D\left[x^i(x_0^j,t),t\right] = \rho_D\left[(x_0^i,0)\right] \exp\left(-\int_0^t \overline{v^i}_{,i}dt - \int_0^t H\,dt\right) \qquad (34)$$

Thus, (28) and (34) make (11) into

$$W = \frac{\rho_D(x_{20}^i,\ 0)\exp\left[-\left(\int_0^t u^i_{,i}(2)dt + \int_0^t \upsilon^i_{,i}(2)dt + \int_0^t H(2)dt\right)\right]}{\rho_D(x_{20}^i,0)\ \exp\left[-\left(\int_0^t u^i_{,i}(2)dt + \int_0^t \upsilon^i_{,i}(2)dt + \int_0^t H(2)dt\right)\right]} \qquad (35)$$
$$\frac{-\ \rho_D(x_{10}^i,0)\exp\left[-\left(\int_0^t u^i_{,i}(1)dt + \int_0^t \upsilon^i_{,i}(1)dt + \int_0^t H(1)dt\right)\right]}{+\ \rho_D(x_{10}^i,0)\exp\left[-\left(\int_0^t u^i_{,i}(1)dt + \int_0^t \upsilon^i_{,i}(1)dt + \int_0^t H(1)dt\right)\right]}$$

The internal scripts 1,2 refer to the points $x_1{}^i$ and $x_2{}^i$ respectively. Now, from (33)

$$H(2) = H(1)$$

Further, since $x_1{}^i$ and $x_2{}^i$ are separated by only half the width of a striation and $\overline{u^i_{,i}}$ is independent of ρ_D , we write

$$\overline{u^i_{,i}}(2) = \overline{u^i_{,i}}(1)$$

Thus, W reduces to

$$W = \frac{\rho_D(x^i_{20},0)\exp(-\int_0^t v^i{}_{,i}(2)dt) - \rho_D(x^i_{10},0)\exp(-\int_0^t v^i{}_{,i}(1)dt)}{\rho_D(x^i_{20},0)\exp(-\int_0^t v^i{}_{,i}(2)dt) + \rho_D(x^i_{10},0)\exp(-\int_0^t v^i{}_{,i}(1)dt)}$$

Taking the Lagrangian derivative for observers moving with the dust at $x_1{}^i$ and $x_2{}^i$, we have

$$\frac{DW}{Dt} = \frac{2\rho_D(x^i_{20},0)\rho_D(x^i_{10},0)\exp(-\int_0^t(v^i{}_{,i}(2) + v^i{}_{,i}(1))dt(v^i{}_{,i}(1) - v^i{}_{,i}(2))}{\left(\rho_D(x^i_{20},0)\exp(-\int_0^t v^i{}_{,i}(2)dt) + \rho_D(x^i_{10},0)\exp(-\int_0^t v^i{}_{,i}(1)dt)\right)^2}$$

Thus, $\frac{DW}{Dt}$ is greater or less than zero according as $v^i{}_{,i}(1)$ is greater or less than $v^i{}_{,i}(2)$. Further, since radiation pressure and the gravitational field of the dust tend to increase condensations of dust, we see that in general

$$h^i{}_{,i}(2) \leq h^i{}_{,i}(1)$$
$$g^i{}_{,i}(2) \leq g^i{}_{,i}(1)$$

so that

$$v^i{}_{,i}(2) \leq v^i{}_{,i}(1) \qquad (53)$$

And we have that

$$\frac{DW}{Dt} \geq 0 \qquad (54)$$

Thus, that part of $v^i{}_{,i}$ which is independent of ρ_D does not affect W, on the average, and the part dependent on ρ_D can only increase W.

We have now discussed enough phenomena to begin considering the general evolutionary trend of the dust structures.

From (13) and the fact that $x^i(x_0{}^i,t)$ is a continuous function of $x_0{}^i$ and, of course, of t, we may demonstrate what might be termed the <u>preservation of neighbors</u>: A grain initially in the first order neighborhood of another grain will, for all finite values of t, remain in a first order neighborhood of the other grain. And also a grain outside a first order neighborhood of another will, for all finite values of t, remain outside a first order neighborhood of the other grain.

Now, it will be recalled that thermal diffusion of the dust is negligible. Thus, we may consider a space in which the gravitational and radiation forces give $v^i{}_{,i} \equiv 0$, e.g. where they are negligible. In this space filled with gas we assume that there is initially one or more homogeneous clouds of dust which are each much larger than the smallest eddies present. While we require that $v^i{}_{,i} = 0$, we do not restrict the off diagonal terms of $v^i{}_{,j}$. We assume that the field w^i is subsonic turbulence so that in general, $v^i{}_{,j} \neq 0$. It follows immediately from (12) that if we neglect H, then $\frac{D\rho_D}{Dt} = 0$

Now, the turbulence will tend to mix the dust through the space, i.e. if at a time t the simply connected surface S(t) just encloses the finite dusty region in the space, and if V(t) is the volume contained in S(t), then because of the turbulence[8],

$$\frac{D}{Dt} V(t) > 0 \qquad (55)$$

This, together with the above statement that

$$\frac{D\rho_D}{Dt} = 0 \qquad (56)$$

implies that the original dust, in order to fill the increasing V(t), must form an inhomogeneous mixture with clear gas. But preservation of neighbors mentioned above implies that what was initially a simply connected cloud must remain a simply connected structure. This implies a filamentary structure for the dust as the only alternative. Since $\frac{DW}{Dt} \geq 0$, there is no dispersion or trend back toward homogeneity. And we conclude that clouds undergo an irreversible evolution from more or less homogeneous cumulus structures toward filamentary and striated structures.

Now, let us consider the generalization of this theorem to include the actual existing physical conditions found in space, i.e. $v^i{}_{,i} \neq 0$. This can probably be most easily accomplished by introducing a time dependent transformation of the space coordinates which will put us in a space in which the divergence of the velocity is identically zero, i.e. we shall determine a transformation giving a space in which the dust density is constant. Then, if we can establish that the transformation has no effect on the statistical distribution of shapes, we need only repeat the arguments given above for $v^i{}_{,i} \equiv 0$.

Thus, we wish to introduce the transformation

$$\bar{x}^i = \bar{x}^i(x^j,t) \tag{57}$$

such that the density, $\bar{\rho}_D$, in $\bar{x}^i$ space, defined by

$$\bar{\rho}_D \, \delta\bar{\tau} = \rho_D \, \delta\tau$$

is independent of time, i.e.

$$\frac{D\bar{\rho}_D}{Dt} = 0 \tag{58}$$

In x^i space we choose rectangular cartesian coordinates so that

$$\delta\tau = \prod_i \delta x^i \qquad i = 1,2,3$$

We define $\delta\bar{\tau}$ by (59)

$$\delta\bar{\tau} \equiv \prod_i \delta\bar{x}^i \qquad i = 1,2,3$$

which is valid if we remain sufficiently close to the origin in $\bar{x}^i$ where the coordinates are chosen so as to be rectangular cartesian coordinates at least in a first order neighborhood of the origin.

Thus, we write by way of definition of density in $\overline{x}^i$ space,

$$\overline{\rho}_0 = \rho_0 \frac{\delta\tau}{\delta\overline{\tau}} \tag{60}$$

Combining this with (58) and taking $\delta\tau$ as constant we have

$$\frac{D}{Dt}\left(\frac{\rho_0}{\delta\overline{\tau}}\right) = 0$$

or

$$\frac{1}{\rho_0}\frac{D\rho_0}{Dt} = \frac{1}{\delta\overline{\tau}}\frac{D}{Dt}(\delta\overline{\tau}) \tag{61}$$

(12) reduces this to

$$\frac{1}{\delta\overline{\tau}}\frac{D}{Dt}(\delta\overline{\tau}) + v^i{}_{,i} + H = 0 \tag{62}$$

And (59) gives

$$\frac{1}{\overline{x}^i}\frac{D\delta\overline{x}^i}{Dt} + v^i{}_{,i} + H = 0 \tag{63}$$

Now, it is not necessary, but it is certainly sufficient, if we satisfy (63) by requiring that

$$\frac{1}{\delta\overline{x}^\alpha}\frac{D\delta\overline{x}^\alpha}{Dt} + v^\alpha{}_{,\alpha} + \frac{1}{3}H = 0 \tag{64}$$

where we recall that when Greek indices are used, the summation convention is not observed. Thus, the eccentricity of a structure in $\overline{x}^i$ space is not affected by an anisotropic compression.

Thus, integrating (64) we have

$$\delta\overline{x}^\alpha = C^\alpha \exp\left(-\int_0^t (v^\alpha{}_{,\alpha} + \frac{1}{3}H)\,dt\right) \tag{65}$$

where the integration is for a Lagrangian observer and C^α is a constant of integration. If we let $\delta\overline{x}^\alpha = \delta x^\alpha$ when $t = 0$, then

$$\delta\overline{x}^\alpha = \delta x^\alpha \exp\left(-\int_0^t (v^\alpha{}_{,\alpha} + \frac{1}{3}H)\,dt\right) \tag{66}$$

From preservation of neighbors it follows that such a Lagrangian integration can be carried out and, therefore, the transformation (57) does in fact exist. To find (57) explicitly we write

$$\frac{\partial \bar{x}}{\partial x} \equiv \frac{\delta \bar{x}}{\delta x} = \exp(-\int_0^t (v^\alpha{}_{,\alpha} + 1/3\ H)dt)$$

Thus

$$\bar{x}^\alpha = \int_0^x \exp(-\int_0^t (v^\alpha{}_{,\alpha} + \tfrac{1}{3} H)dt)dx^\alpha \qquad (67)$$

where we have arbitrarily put all constants of integration equal to zero. We see that the origins in $\bar{x}^i$ and x^i space coincide.

To investigate the properties of (57) or (67) and demonstrate that the transformation does not affect the statistical distribution of eccentricities of the structures, we expand $\bar{x}^i(x^j,t)$ in a power series about the point $x^i(x_0{}^j,t)$.

$$\bar{x}^j(x^i,t) = \bar{x}^j(x^i(x_0{}^k,t),t) + \bar{x}^j{}_{,l}(x^i(x_0{}^k,t),t)(x^l - x^l(x_0{}^k,t)) + O^2(x^l - x^l(x_0{}^k,t)) \qquad (68)$$

which is a valid representation of $\bar{x}^j(x^i,t)$ for x^l sufficiently close to $x^l(x_0{}^k,t)$ since in general $\bar{x}^j{}_{,l} \neq 0$. We see that the transformation takes spheres into ellipsoids. And that in general it will not affect the statistical distribution of eccentricities of dust structures since, while a cloud with its long axis parallel to the $\bar{x}^\beta$ axis in a region where $\left|\frac{\partial \bar{x}^\beta}{\partial x^\beta}\right| < \left|\frac{\partial \bar{x}^\alpha}{\partial x^\alpha}\right|, \left|\frac{\partial \bar{x}^\gamma}{\partial x^\gamma}\right|$ will have its eccentricity reduced by the transformation, clouds with their axes parallel to the $\bar{x}^\alpha$ or $\bar{x}^\gamma$ axes will have their eccentricities increased.

This depends, of course, on the fact that the anisotropies in the transformation are independent of the direction of orientation of the dust structures. That this is indeed the case follows directly from the fact that the transformation (57) as given by (67) depends upon the diagonal terms, $v^\alpha{}_{,\alpha}$, of $v^i{}_{,j}$ whereas the orientation of the filaments and eccentricities depends upon the off diagonal terms. From G.I. Taylor, Proc. Roy. Soc. 151 433 we have that there is no correlation between the diagonal and off diagonal terms in $w^i{}_{,j}$, i.e. $a_2 = a_5 = a_7 = 0$. Thus, from physical arguments we expect no correlation between the diagonal and off diagonal terms in $v^i{}_{,j}$. Hence the anisotropy of the transformation and the eccentricity of the clouds are independent. And we expect no net effect on the statistical distribution of eccentricities by the transformation.

In the immediate neighborhood of a dust globule the upper limit on $x^i - x^i(x_o{}^k,t)$ required to keep (68) a valid approximation could conceivably be less than the width of a striation, and so, globules would not preserve the ellipsoidal form of an initially ellipsoidal cloud of the usual dimension observed. But, we see that on the basis of (54) filaments are not destroyed in such a region, and so, our physical arguments apply anyway.

We are now ready to consider the general evolution of dust structures in velocity fields where $v^i{}_{,i} \neq 0$.

First, we point out that since the transformation, (57), is a single valued continuous function of x^j,t, preservation of neighbors in x^j-space implies preservation of neighbors in $\overline{x}^j$-space. Further, the pseudo-density in $\overline{x}^j$-space satisfies

$$\frac{D\bar{\rho}}{Dt} = 0$$

And secondly, the velocity field in $\overline{x}^j$-space, although not a true turbulent velocity field as in x^i-space, will nonetheless be quite similar to one. Thus, as before, we expect a mixing of the dust into initially clear parts of space which together with (58) implies an inhomogeneous mixture of dust and gas.

And, preservation of neighbors implies that the inhomogeneities be filamentary and not globular.

Therefore, because of the nature of (57), filaments in $\bar{x}^j$-space imply filaments and striations in x^i-space. And we have that dust structures undergo an irreversible evolution from initial relatively homogeneous clouds with low eccentricity toward filamentary structures with ever increasing eccentricity.[9] This implies that the eccentricity of a structure is a measure of the amount of turbulence in the region since the structure initially condensed.

It might be well to point out the consequences of some of the implicit assertions in the above development. The most striking is probably that in spite of the trend toward filaments,

$$\frac{D\rho_o}{Dt} \cong 0$$

in regions where there is not an excessive amount of contraction of the dust structures due to radiation etc. To consider a numerical example, let us observe the consequences of this condition for a dark filament observed to lie across the projection of a large homogeneous obscuring cloud. Let us assume that the optical depth of the filament is 1/20 the optical depth of the large cloud. i.e.

$$\tau_{fil} = \frac{1}{20} \tau_{cld}$$

Further, let us assume that the thickness of the filament is 0.01 the thickness of the cloud. i.e.

$$l_{fil} = \frac{1}{100} l_{cld}$$

Thus

$$\frac{\rho_{fil}}{\rho_{cld}} = \frac{\tau_{fil}}{\tau_{cld}} \frac{l_{cld}}{l_{fil}} = \frac{1}{20} 100 = 5$$

which is a perfectly reasonable result. Thus, the dust cloud from which the filament was originally drawn had a density of five times the large obscuring cloud. Actually, of course, it need not have been this dense originally since radiation pressure makes $\frac{D\rho_D}{Dt} > 0$.

We note in passing that many filamentary structures will be unobservable because their optical depth is too small even though the initial thicker homogeneous clouds from which they were drawn may have had $\tau = 1$ or greater. And it is on the drawing out of relatively dense, although not necessarily large clouds, that will produce the observable filaments of narrow width as are sometimes seen.

Now we must recall that the above development was carried out for particles of a given radius, σ, We must consider the result if the dust consists of a distribution of sizes. To get an idea of what might be expected we consider a single dust grain with a velocity of 1 km/sec parallel to the x axis relative to a stationary gaseous medium in a field free region of space. Then (2) reduces to

$$\ddot{x} = -\gamma\dot{x} \qquad \gamma = \text{constant}$$

$$\dot{x} = \dot{x}_0 \exp(-\gamma t)$$

where $\dot{x}_0$ is the velocity (1 km/sec) at $t = 0$. Thus

$$x = \frac{\dot{x}_0}{\gamma}(1 - \exp(-\gamma t))$$

so that $x = 0$ when $t = 0$. Then $t = \infty$,

$$x = x_\infty = \frac{\dot{x}_0}{\gamma} .$$

If the dust grain density is δ , then from (4) it follows that

$$x_\infty = \dot{x}_0 \frac{4\sigma\delta}{3\rho_H \bar{u}}$$

With $\dot{x}_o = 10^5$ cm/sec, $\overline{u} = 10^5$ cm/sec which corresponds to a gas temperature of 50° K, $\rho_H = 10^{-23}$gm/cm^3, $\delta = 1$ gm/cm^3, and $\sigma = 10^{-5}$ cm, then $x_\infty = \frac{4}{3} \cdot 10^{18}$cm $= \frac{4}{3}$ light year.

Since x_∞ is proportional to σ, it follows that a grain of twice this size will coast twice as far. Therefore, for even a very small spread in observable grain sizes, we would readily expect to disperse the grains over distances of the order of light years. A globule suddenly struck by a wind would become a long streamer with the largest grains to windward. Structures are observed which are very suggestive of such a phenomenon. It is possible, of course, that the structure be due to other causes such as a vortex ring. We note finally that being drawn out in the shearing field of a turbulent eddy a globule would become a sheet, the heavier grains spewing out away from the center of the eddy.

To consider the amount by which the trajectories of two particles of different size will diverge from each other in an eddy, let us consider the motion of a particle in a gas cloud rotating as a rigid body with angular velocity Ω . We consider the case where $F^1 = 0$. That is, we assume that the gas in the region considered is held together primarily by a pressure gradient and not entirely by, say, a gravitational field which would act on the dust as well.

Thus (2) becomes

$$\begin{aligned} \ddot{x} &= -\gamma(\Omega y + \dot{x}) \\ \ddot{y} &= -\gamma(-\Omega x + \dot{y}) \end{aligned} \qquad (64)$$

where the gas is rotating about the z axis. Solving simultaneously, we obtain, with $D = \frac{d}{dt}$

$$(D^4 + 2\gamma D^3 + \gamma^2 D^2 + \gamma^2\Omega^2)\begin{pmatrix}x\\y\end{pmatrix} = 0 \qquad (65)$$

We consider, then, the characteristic equation

$$X^4 + 2\gamma X^3 + \gamma^2 X^2 + \gamma^2\Omega^2 = 0 \qquad (66)$$

The roots are

$$X = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 \pm 2i\Omega\left(\frac{\gamma}{2}\right)}$$

where the two $\pm$ are independent. For convenience we write

$$R^2 = \left(\frac{\gamma}{2}\right)^4 + 4\Omega^2\left(\frac{\gamma}{2}\right)^2$$

$$\tan\varphi = \frac{2\Omega}{\frac{\gamma}{2}} = \frac{4\Omega}{\gamma} \qquad (67)$$

The roots of (66) become, then,

$$X_i = \left(-\frac{\gamma}{2} \pm R^{1/2}\cos\frac{\varphi}{2}\right) \pm iR^{1/2}\sin\frac{\varphi}{2} \qquad (68)$$

$$i = 1,2,3,4$$

The solutions of (64) are of the form

$$x = \sum_{i=1}^{4} A_i \exp(X_i t) \qquad y = \sum_{i=1}^{4} B_i \exp(X_i t) \qquad (69)$$

Putting (69) into (64) gives

$$\begin{aligned} A_i(X_i^2 + \gamma X_i) + \gamma\Omega B_i &= 0 \\ -\gamma\Omega A_i + (X_i^2 + \gamma X_i)B_i &= 0 \end{aligned} \qquad (70)$$

(68) guarantees that the determinant of the coefficients will vanish. From (70)

$$B_i = -A_i \frac{X_i(X_i + \gamma)}{\lambda\Omega} \qquad (71)$$

From (68)

$$X_i(X_i + \gamma) = \pm i\Omega\gamma \text{ (two + \& two -)} \qquad (72)$$

If we define

$$\tau_1 = \frac{1}{R^{1/2}\cos\frac{\varphi}{2} - \frac{\gamma}{2}} \;,\; \tau_2 = \frac{1}{R^{1/2}\cos\frac{\varphi}{2} + \frac{\gamma}{2}} \;,\; \omega = R^{1/2}\sin\frac{\varphi}{2} \quad (73)$$

then we have

$$\begin{aligned} x &= C_1\exp(\tfrac{t}{\tau_1})\cos\omega(t + t_1) + C_2\exp(-\tfrac{t}{\tau_2})\cos\omega(t + t_2) \\ y &= C_1\exp(\tfrac{t}{\tau_1})\sin\omega(t + t_1) - C_2\exp(-\tfrac{t}{\tau_2})\sin\omega(t + t_2) \end{aligned} \quad (74)$$

If we put in the initial conditions that when $t = 0$,

$$x = a,\ y = 0;\ \dot{x} = 0,\ \dot{y} = 0,$$

then we find that

$$\tan\omega t_1 = \frac{\omega\tau_2(\tau_2 - \tau_1)}{\tau_1 + \tau_2 + 2\tau_1\tau_2\omega^2} = -\frac{\frac{\gamma}{2}\sin\frac{\varphi}{2}}{R^{1/2} + \frac{\gamma}{2}\cos\frac{\varphi}{2}}$$

$$\tan\omega t_2 = \frac{\omega\tau_1(\tau_2 - \tau_1)}{\tau_1 + \tau_2 + 2\tau_1\tau_2\omega^2} = -\frac{\frac{\gamma}{2}\sin\frac{\varphi}{2}}{R^{1/2} + \frac{\gamma}{2}\cos\frac{\varphi}{2}} \quad (75)$$

$$C_1 = \frac{a}{2R^{1/2}}\sqrt{R + R^{1/2}\gamma\cos\frac{\varphi}{2} + \frac{\gamma^2}{4}}$$

$$C_2 = \frac{a}{2R^{1/2}}\sqrt{R - R^{1/2}\gamma\cos\frac{\varphi}{2} + \frac{\gamma^2}{4}}$$

The trajectories computed from (74) and (75) for two dust grains with radii 10^{-5}cm and $2\cdot10^{-5}$cm starting from rest at a distance of one parsec from the center of an eddy with an angular velocity of $3.24\cdot10^{-14}\text{sec}^{-1}$ so that the linear speed at one parsec is 1 km/sec, are shown in Fig. 7. The eddy is presumed to be in an H_I region and is computed for H_2 with $\rho_H = 10^{-23}$ gm/cm^3, $\Theta = 100^\circ$ K. The dust grains are assumed to have a density of one gm/cm^3.

We see that the trajectories of the two particles with radii 10^{-5}cm and $2\cdot10^{-5}$cm diverge considerably.

If we consider the case where the grains have not started from rest but have been moving with the cloud for some time, the separation is much more striking. For, if $x = a$, $y = 0$ when $t = 0$, then

$$x = a \exp\left(\frac{t}{\tau}\right) \cos\omega t$$
$$y = a \exp\left(\frac{t}{\tau_1}\right) \sin\omega t \qquad (76)$$

Fig. 8 is plotted from (76) using the same numerical values as for Fig. 7. We see that for the steady state motion, the separation is more rapid than when starting from rest since the particles have their full speeds for the entire period rather than having to be accelerated from rest during the time considered.

Therefore, if we wish to generalize our statements concerning the evolution of the geometry of the dust structures to include dust structures in which the dust grains have a distribution of radii, we take advantage of the relatively long mean free path for the collision of dust with dust and superimpose the independently moving distributions of the different sized grains.

Let the dust density at the point x^i for grains with radii in $(\sigma, \sigma + d\sigma)$ be $P(x^i, \sigma)d\sigma$. Then

$$\rho_0(x^i) = \int_0^\infty P(x^i, \sigma)\, d\sigma \qquad (77)$$

We have shown that the dust in $(\sigma, \sigma + d\sigma)$ undergoes an irreversible evolution toward filamentary and striated structures. Then, neglecting momentum exchange by collision of dust grains in $(\sigma, \sigma + d\sigma)$ with all other grains, we have that all structures of grains of a given size undergo such evolution and the resultant is given by (77), the superposition of the filaments from each interval $(\sigma, \sigma + d\sigma)$. Actually, since it is the extinction that is observed, it would be more to the point to consider $\chi(x^i)$ where

$$\varkappa(x^1) = \int_0^{\infty}\int_{\lambda_1}^{\lambda_2} Q(\sigma,\eta)\ P(x^1,\sigma)\ d\eta\ d\sigma \qquad (78)$$

$Q(\sigma,\lambda)$ is the extinction per gram of dust in $(\sigma,\sigma + d\sigma)$ of the radiation in $(\lambda, \lambda + d\lambda)$.

Now, in the case that the shearing field producing the filaments is essentially constant in direction over the region considered, $\rho_D(x^1)$ will have very much the same form in space as $P(x^1,\sigma)$. That is, the filaments will be straight and there will be some separation of grains sizes, the larger grains not reacting quite so readily to the shearing field, but the overall appearance of the filament will not be significantly changed. Of course, it may be possible under certain circumstances to observe the dispersion of different sized grains by the effect on the reddening of transmitted light.

If we consider a region where the shearing gradients are due primarily to turbulence, so that the trajectories of the dust grains are not straight lines, then $\rho_D(x^1)$ and $P(x^1,\sigma)$ will not be very similar. Fig. 9,10 show the separation of grains for $\sigma = 10^{-5}$cm and $2\cdot 10^{-5}$cm starting as a single cloud from rest and steady state motion respectively in a gas cloud rotating as a rigid body. The separation were computed for the same conditions as Fig. 7,8. After $8\cdot 10^{13}$ sec or $2.53\cdot 10^{6}$ years the grains are separated as outlined by the broken line.

Let us summarize our results: The dust in $(\sigma,\sigma + d\sigma)$ undergoes an irreversible evolution toward filamentary structures. The net result is a superposition of the filaments of dust grains of various sizes. Due to the inertia of the dust grains the superimposed state will not be strikingly striated or filamentary unless the filaments happen to be straight or separated by distances greater than the dispersion of each filament. In the case where the filaments in each $(\sigma, \sigma + d\sigma)$ are not straight, the resultant will not have much fine structure, i.e. no striations $<$ light year in width because of the separation of the different sized dust particles. In any case, whether the resultant be filamentary or not, there will be a marked sorting of particles, as shown in the figures, due to the dependence on σ of the trajectories computed from (3).

Now, it will be pointed out that there are observed filaments and striations, e.g. Pleiades, which are not straight, altho, of course, some are straight, and which have widths of the order of 10^{16} cm or 0.01 lt yrs. On the basis of the above theory, we must either assume a remarkably narrow range in size of the visible dust grains, or we must assume that the picture is not so simple as was assumed and that there are other forces at work. The best guess is magnetic forces due to the relative motion of net charge in the clouds. There is little that can be done to rigorously apply magnetohydrodynamics in such cases because, first, the ion densities at any particular point in a cloud are not known and the relative velocities of the various components of the cloud cannot be observed. Further, the mathematical complexities involved in the solution of the equations of magnetohydrodynamics are immense [10]. It would seem that for the present, the most fruitful approach would be to apply magnetohydrodynamic theory to the observed structures for the purpose of calculating the necessary ion densities, turbulent velocities, etc. Unfortunately, the results obtained would not be unique.

The Condensation of Dust Globules

It may be of some interest to consider the collapse of the special dust structures usually referred to as globules. The collapse under radiation pressure and under gravity is discussed in the literature [11]. The following discussion is concerned with the general nature of the collapse under gravity.

The radial force on a particle in a spherical cloud due to radiation pressure varies as the cross section of the cloud or as R^2, where R is the cloud radius, whereas the force of gravity varies as $1/R^2$ during contraction. Obviously, then, even though radiation pressure greatly exceeds gravity at large radii, there will come a time after sufficient contraction when gravity and radiation forces will be comparable, and thereafter gravity will exceed the radiation pressure. For an opaque cloud such as we are considering above we obtain, by equating the total radiation force over the surface of the cloud to the integral of the gravitational forces over the interior, the result

$$\frac{3GM^2}{a^6}\int_0^a r^3\,dr = \frac{1}{6}\,\delta\,4\pi a^2$$

$$a^4 = \frac{9GM^2}{8\pi\,\delta} \qquad (83)$$

where a is the cloud radius at which gravity and radiation fordes are equal, M is the cloud mass, δ the radiation density, ergs/cm^3, in the radiation field in which the cloud is assumed to be immersed, and G is the gravitational constant.

For $M = 1\ M_\odot$ and $\delta = 5.2 \cdot 10^{-13}$ ergs/cm^3 we obtain $a = 0.654 \cdot 10^{18}$ cm = 0.692 lt yrs. Thus, for cloud radii less than a, gravitational forces greatly predominate.

For a quiescent cloud, neglecting radiation pressure, (2) of Part II reduces to

$$\dot{v} = \gamma(w - v) - \frac{GM(r)}{r^2} \qquad (84)$$

where M(r) is the mass in the sphere of radius r, r being the Lagrangian radial coordinate of the dust grain under consideration, v the radial velocity of the dust grain so that $v = \dot{r}$, and w is the radial velocity of the gaseous substratum. We may rewrite (84) as

$$\ddot{r} + \gamma\dot{r} + \frac{GM}{r^2} = \gamma w \qquad (85)$$

Because of the complexities of the physics and the non-linearity of the equation it is not possible at present to give a rigorous solution to (85). Therefore, we shall resort to more devious methods to obtain simply an estimate of the collapse time.

First, we conclude that $w \neq 0$ since with a gas density of 10 atoms / cm^3 at 100° K, the gas and radiation pressures are the same. Hence for radii less than given in (83), the gas pressure would be less than the momentum exchange term $\gamma\dot{r}$ because the dust grains, except in the final stages of collapse, will be traveling at very nearly their terminal velocities in the gravitational field. We expect then that $w < 0$. But, if $w < 0$, then probably $\dot{\gamma} > 0$ since γ contains ρ_H, the gas density, as a factor. Thus, it is difficult to estimate the effect of a nonstationary gas medium.

Therefore, we solve

$$\ddot{r} + \gamma \dot{r} + \frac{GM}{r^2} = 0 \qquad (86)$$

i.e. we expediently assume that w = 0, and shall later attempt to estimate the effect of w.

Now (86) is nonlinear and there is no immediately obvious integrating factor. Thus, we must take advantage of the fact that at large radii the grains travel at their terminal velocities, i.e. the inertial term $\ddot{r}$ is negligible, and

$$\dot{r} + \frac{GM}{\gamma} \frac{1}{r^2} = 0 . \qquad (87)$$

And for small cloud radii, i.e. after collapse has proceeded for a time, the inertial term is all important yielding

$$\ddot{r} + \frac{GM}{r^2} = 0 \qquad (88)$$

Thus, (87) and (88) give respectively

$$\dot{r} = - \frac{GM}{\gamma} \frac{1}{r^2}$$
$$\dot{r} = -\sqrt{\frac{2GM}{r} + 2E} \qquad (89)$$

for the radial velocities where E is the total energy per unit mass of the dust grain. Integrating, we obtain respectively

$$(r_o^3 - r^3) = \frac{3GM}{\gamma} t \qquad (90)$$

$$\frac{-GM}{\sqrt{-2E^3}} \left\{ \cos^{-1}\sqrt{-\frac{rE}{GM}} - \cos^{-1}\sqrt{-\frac{r_oE}{GM}} + \sqrt{-\frac{rE}{GM}\left(1 + \frac{rE}{GM}\right)} - \sqrt{-\frac{r_oE}{GM}\left(1 + \frac{r_oE}{GM}\right)} \right\} = t - t_o \qquad (91)$$

In (90), $r = r_o$ when $t = 0$. In (91) we also have $r = r_o$ when $t = t_o$. Thus E is the total energy per gram, $1/2\ v_o^2 + \frac{GM}{r_o}$.

To determine the radius at which $\ddot{r}$ and $\gamma\dot{r}$ are comparable, i.e. at what cloud radius neither approximation is very good and we must change from (90) to (91), we remark that for large r, and therefore early in the collapse, (87) is valid and

$$\dot{r} = -\frac{GM}{\gamma}\frac{1}{r^2}$$

Thus

$$\ddot{r} = \frac{2GM}{\gamma}\frac{\dot{r}}{r^3} = -\frac{2G^2M^2}{\gamma^2}\frac{1}{r^5}$$

This, then, is the inertial force while the grain is traveling at its terminal velocity. It becomes equal to the momentum loss to the gas when

$$\ddot{r} = \gamma\dot{r}$$

or

$$r^3 = \frac{2GM}{\gamma^2}$$

Now

$$\gamma = \frac{\pi\sigma^2 \bar{u}\rho_H}{M_d}$$

where M_d is the mass of a dust grain. Thus, if we take $\sigma = 10^{-5}$cm, $\bar{u} = 1.775\cdot10^5$ cm/sec which corresponds to H_2 at 300° K, $\rho_H = 10^{-23}$gm/cm^3, $M_d = 4.02\cdot10^{-15}$ gm, then we find that $\gamma = 1.39\cdot10^{-13}$ sec^{-1}. Thus for M = 1 $M_\odot$ we have

$$r_0 = 2.38\cdot10^{17}\text{cm} = 0.252 \text{ lt yrs}$$

Hence, for r less than this value we must use (88). Therefore, let us compute the time required for a particle to fall from a to r_0 and r_0 to r = 0 if it has just the terminal velocity, (87), until it passes r_0 but thereafter falls according to (88). The time of fall from a to r_0 at the terminal velocity is $3\cdot10^6$ yrs. The terminal velocity at r_0 is $\frac{GM}{\gamma}\frac{1}{r_0^2}$ or $1.67\cdot10^4$ cm/sec. Hence

$$E = \frac{1}{2}\dot{r}_o^{\,2} - \frac{GM}{r_o} = -4.13\cdot 10^8 \text{ ergs/gm}$$

Then $\frac{GM}{E} = -3.18\cdot 10^{17}$cm and (91) gives the time of fall as

$$t - t_o = -\frac{GM}{\sqrt{-2E^3}}\left\{1 - \cos^{-1}\sqrt{-\frac{r_oE}{GM}} - \sqrt{-\frac{r_oE}{GM}\left(1 + \frac{r_oE}{GM}\right)}\right\}$$

$$= 4.87\cdot 10^{11} \text{ sec.} = 1.54\cdot 10^4 \text{ yrs.}$$

This is an extremely short period of time. Hence, we conclude that only short periods of momentum transfer from the gravitational field are required to supply the collapse momentum for $r < r_o$. And that if much time is to be consumed it must be because of momentum losses to the gas, i.e. $\gamma(w^i - v^i)$. But for the collapse considered, $\gamma(w^i - v^i)$ was less than $\dot{v}^i$, and this means that either the above value is a good approximation or else γ must be greatly increased by the increasing gas temperature and density.

Thus it would be well to estimate the time which might be required to dissipate the collapse energy, thereby furnishing a lower limit to the collapse time since, as will be seen, it will be less than the $3.0\cdot 10^6$ yrs required by the mechanics of the collapse.

While it is very difficult if not impossible to write down exactly how the collapse energy will be dissipated, one pictures simply a heating of the hydrogen by compression and by the passage of the dust. Since the time of equipartition between dust and gas for their ordered as well as their unordered velocity components is of the order of thousands of years, this energy is passed on quickly to the dust grains and then radiated. Now, the problem of radiation transfer in the interior of such a cloud is as yet unsolved. Hence, we shall rely on a rather arbitrary assumption to obtain a result. We shall assume that the outer surface of the cloud is maintained at 10° K.

For a homogeneous cloud, the gravitational potential energy is $-\frac{3GM^2}{5a}$ where a is the radius of the cloud. Hence, if σT^4 represents the radiation per cm^2 on the surface of the cloud,

$$\frac{d}{dt}\left(\frac{3GM^2}{5a}\right) = 4\pi\sigma T^4 a^2$$

Thus, if $a = a_o$ when $t = 0$,

$$t = \left(\frac{1}{a^3} - \frac{1}{a_o{}^3}\right)\frac{GM^2}{20\pi\sigma T^4}$$

If we choose $a = 100$ a.u. and $a_o \sim$ light year,

$$t = 0.68 \cdot 10^5 \text{ years.}$$

We conclude, therefore, that the collapse from r_o should probably take place in $3 \cdot 10^5$ years with no difficulty with the assumptions of quiescence, no angular momentum, etc. under which the development was carried out.

References

1. Walter S. Adams, Ap.J. 109 354
2. Fred L. Whipple, Harvard Observatory Monographs No. 7, Centenial Symposia
3. L. Spitzer Jr., Malcolm P. Savedoff, Ap. J. 111 593
4. Emden, Gaskugeln
5. G. K. Batchelor, Proc. Camb. Phil. Soc. 43 533 (1947)
6. R. C. Tolman, Principles of Statistical Mechanics 33.1
7. H. C. van de Hulst, Harvard Observatory Monographs No. 7, Centenial Symposia
8. G. I. Taylor, Proc. Roy. Soc. 151 465
9. R. C. Tolman, Principles of Statistical Mechanics, pg 178
10. W. E. Elsasser, Phys. Rev. 79 183 (1950)
11. F. L. Whipple, Ap. J. 104 1(1946); Bart J. Bok, Harvard Observatory Monographs No. 7; Lyman Spitzer Jr., Harvard Observatory Monographs No. 7; Fred L. Whipple, Harvard Observatory Monographs No. 7.

Appendix

Momentum Exchange Rate Between Radiation and Dust

Because radiation pressure has significant effects on the motion of the dust in interstellar space, it is of interest to express quantitatively the momentum transfer rate to a dust particle by the radiation field.

We consider a dust grain with an effective cross section of $\pi\left[\sigma(\theta,\varphi)\right]^2$ for radiation incident from the θ,φ direction. $\pi\sigma^2$ is the effective cross section for the transfer of momentum by radiation of all wavelengths taking into account scattering effects etc. If $\frac{\partial I}{\partial\omega}d\omega$ is the intensity of the radiation lying in the solid angle $d\omega$ (i.e. ergs/sec/cm^2/steradian), then the energy density of the radiation in $d\omega$ is $\frac{1}{c}\frac{\partial I}{\partial\omega}d\omega$, and the force on the dust particle is

$$\frac{\pi}{c}\,\sigma^2(\theta,\varphi)\,\frac{\partial I}{\partial\omega}\,d\omega$$

Now, in an isotropic black body radiation field it is customary to use the quantity J which is the energy streaming across unit area per second. It can be shown that

$$\frac{\partial I}{\partial\omega} = \frac{J}{\pi}$$

But in our case $\frac{\partial I}{\partial\omega}$ depends upon θ,φ. Thus we write $\frac{J}{\pi}$ for $\frac{\partial I}{\partial\omega}$ and remark that J is the energy streaming across unit area per second if the field were isotropic with $\frac{\partial I}{\partial\omega}$ everywhere the value which it has at θ,φ. Thus, the force is

$$\frac{1}{c}\,\sigma^2(\theta,\varphi)\;J(\theta,\varphi,x^j,t)\;d\omega$$

where

$$J(\theta,\varphi,x^j,t) \;=\; \frac{\partial I}{\partial\omega}(\theta,\varphi)$$

The x^j and t are included to take into account the fact that J may vary from place to place and with time. The total force in the z direction is, then,

$$F_z(x^j,t) \;=\; -\frac{1}{c}\int_{\Omega}\sigma^2(\theta,\varphi)\;J(\theta,\varphi,x^j,t)\;\cos\theta\;d\omega$$

$$F_z(x^j,t) \;=\; -\frac{1}{c}\int_0^{2\pi}\!\!\int_0^{\pi}\;\sigma^2(\theta,\varphi)\;J(\theta,\varphi,x^j,t)\cos\theta\sin\theta\;d\theta d\varphi$$

If we assume that $\sigma(\theta,\varphi) \;=\; \sigma(\pi-\theta,\varphi+\pi)$, then

$$F_z(x^j,t) \;=\; \frac{1}{c}\int_0^{2\pi}\!\!\int_0^{\frac{\pi}{2}}\sigma^2(\theta,\varphi)\Big[J(\pi-\theta,\varphi+\pi,x^j,t) - J(\theta,\varphi,x^j,t)\Big]\;\cos\theta\;\sin\theta\;d\theta\;d\varphi$$

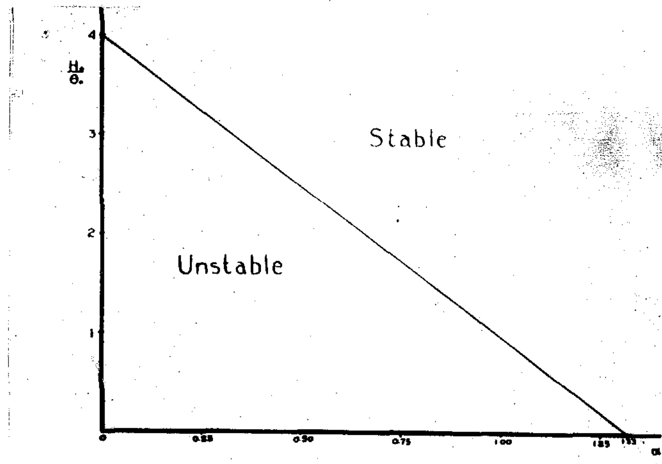

Fig. 1 Minimum Ratio of ordered to Unordered Temperature for Stability

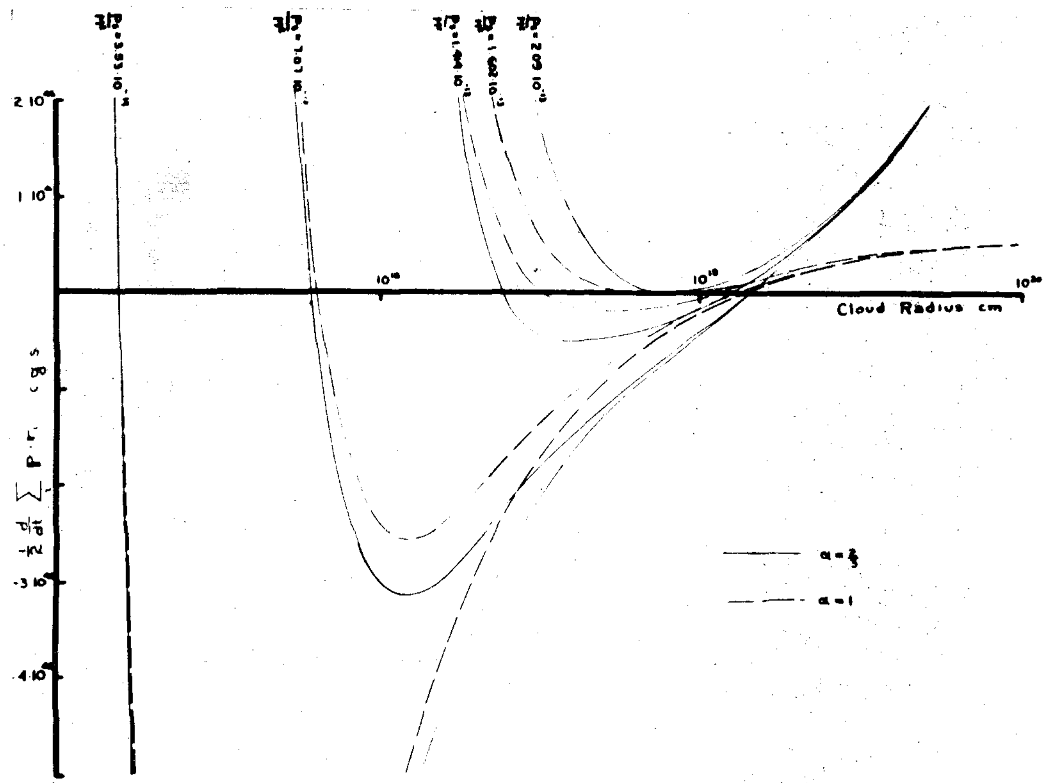

Fig 2 Expansion and Contraction Acceleration for Gas Clouds with Mass 1000 Mo

Angular Momentum p_0 cgs, Temperature = 50°K for a = 1.542·10^{19}cm = 5 psc

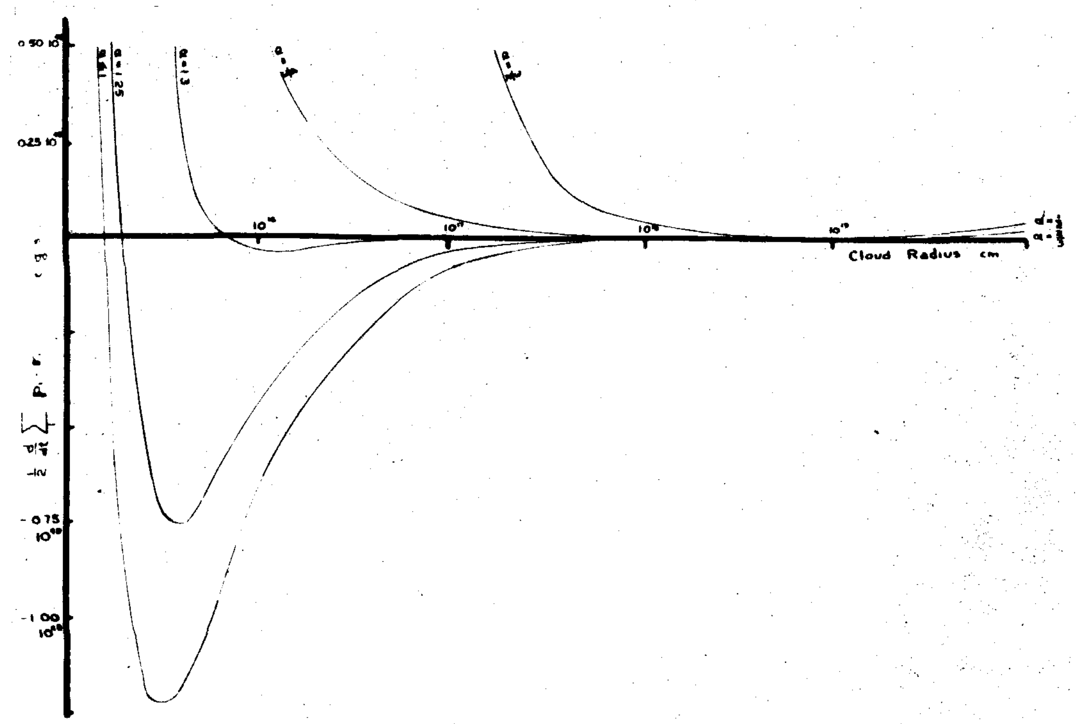

Fig. 3 Expansion and Contraction Acceleration for Gas Clouds with Mass 1000 M⊙

Angular Momentum = 3.11 10^{58} gm cm²/sec , Temperature = 100°K for a = 10^{19} cm

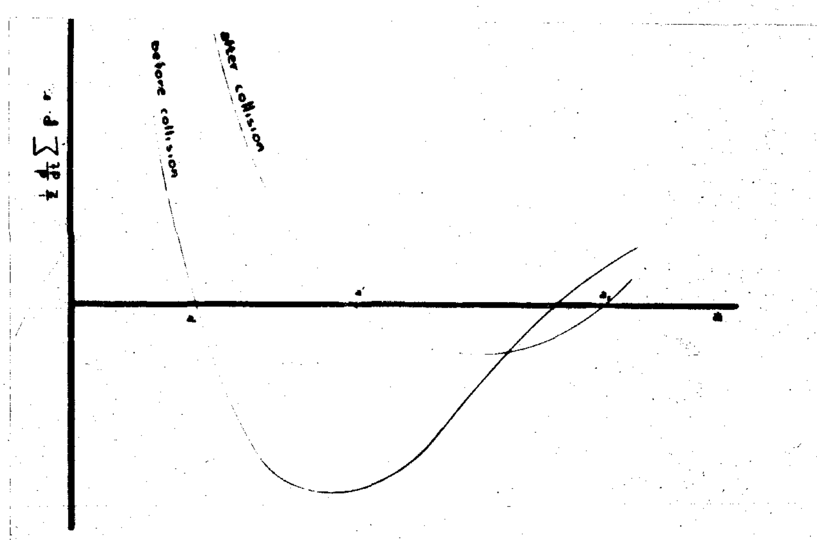

Fig. 4

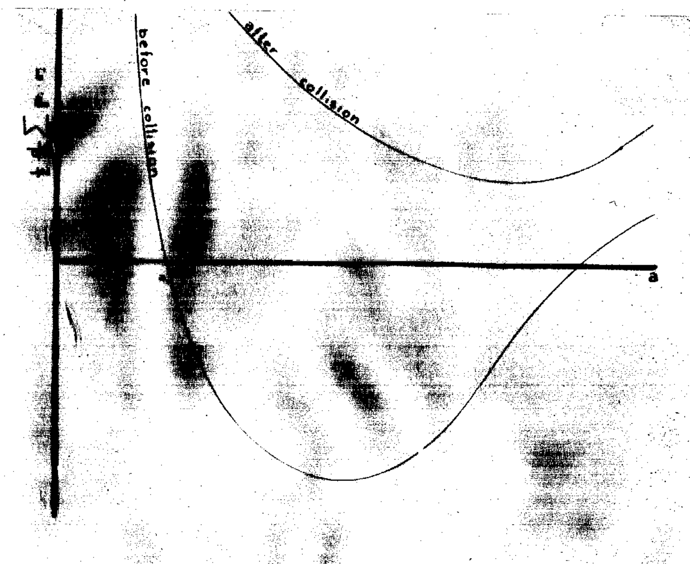

Fig 5

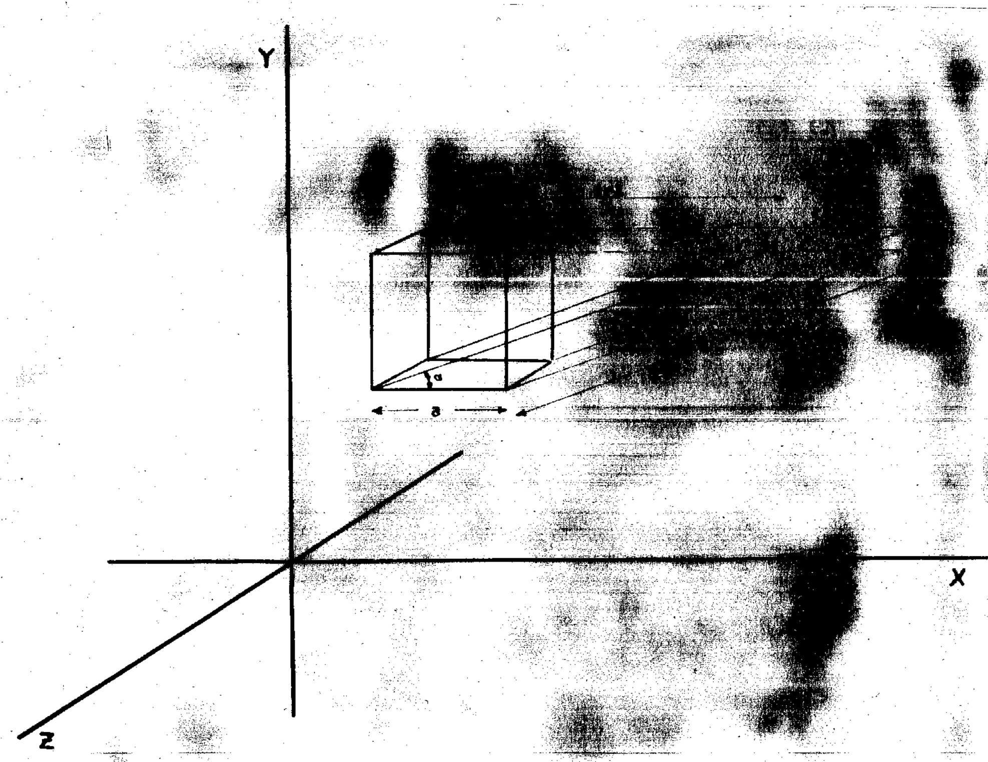

Fig. 6

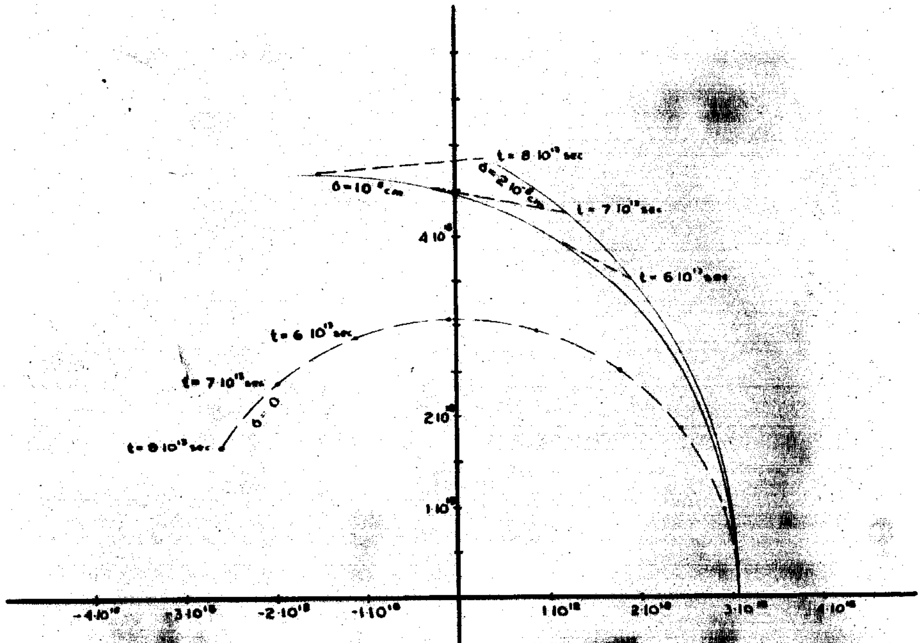

Fig 7 Trajectories of Dust Grains with Radii 10^{-5} cm and $2 \cdot 10^{-5}$ cm Starting from Rest in a Rotating Cloud Gas Cloud

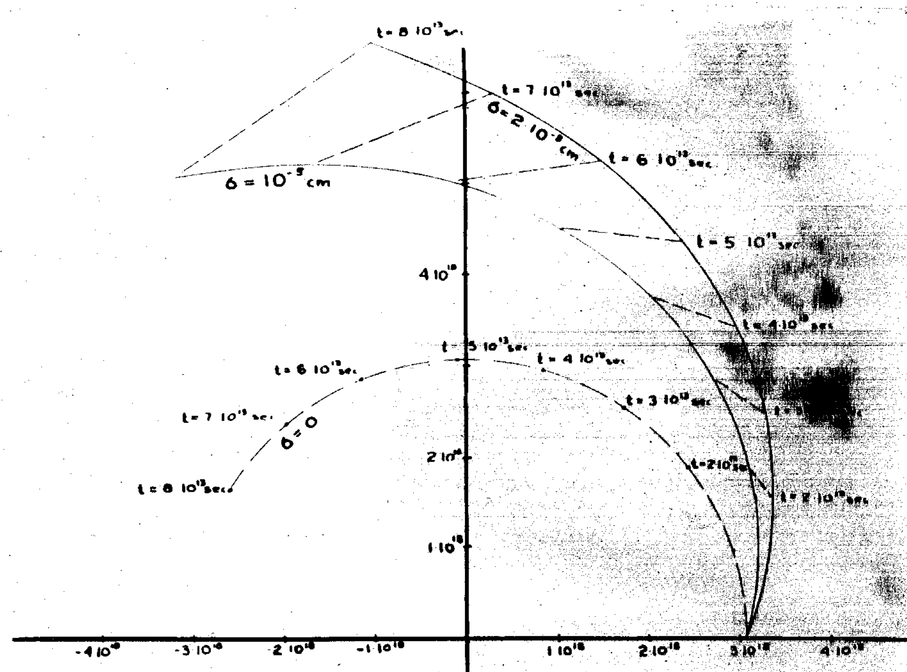

Fig 8 Trajectories of Dust Grains with Radii 10^{-5} cm and $2 \cdot 10^{-5}$ cm for Steady State Motion in a Rotating Gas Cloud

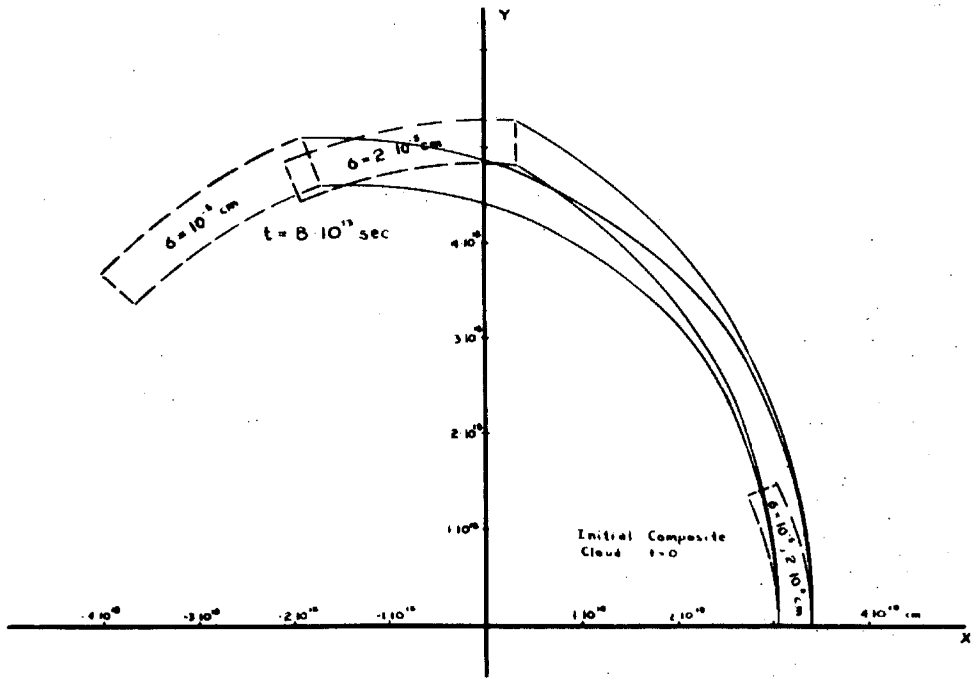

Fig. 9 The Separation of Dust Grains 10^{-5} cm, $2\ 10^{-5}$ cm Starting From Rest

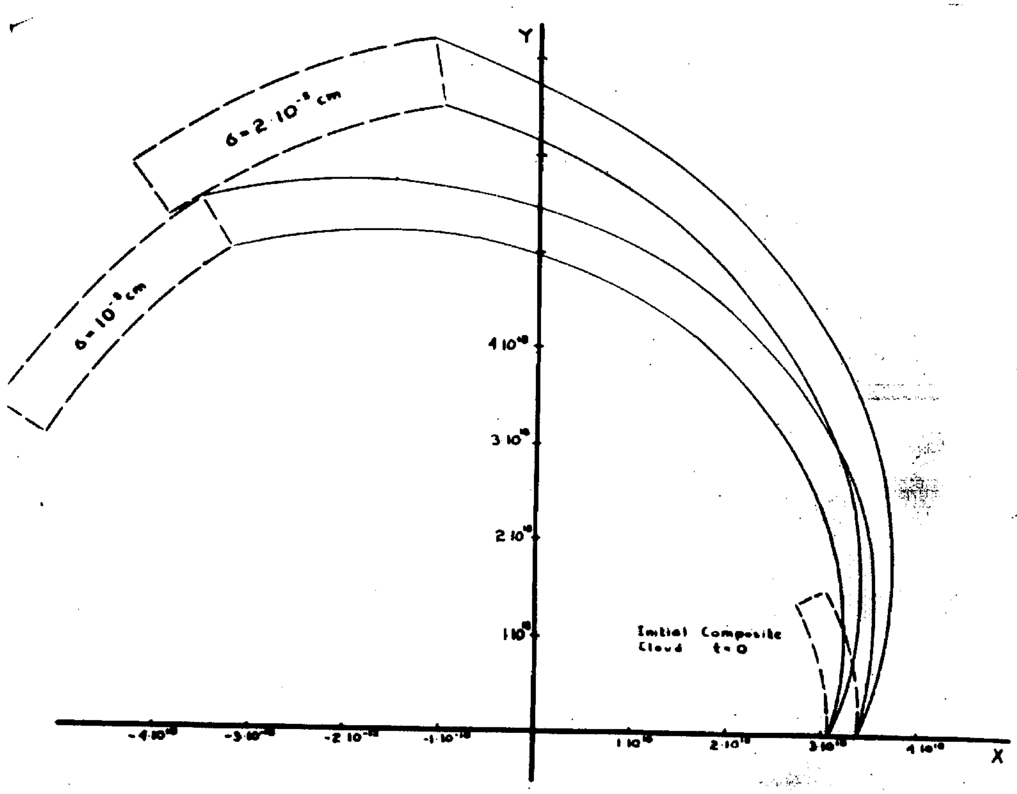

Fig 10 The Separation of Dust Grains 10^{-5} cm, $2\ 10^{-5}$ cm Starting From Steady State At t = 0