APPLICATIONS OF THE POTENTIAL ANALOGY IN NETWORK ANALYSIS

Thesis by

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ABSTRACT

The analogy existing between linear, lumped parameter network functions and the complex potential function of line charges is applied to various problems in network analysis and synthesis. A means of determining the stream function directly is described. It is shown how the analogy may be used for both steady state and transient analysis when the zeros and poles of the network function are known. In addition, methods of determining the poles of special networks by means of the analogy are described.

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I INTRODUCTION

The potential analogy useful in network analysis relates the potential and stream functions of a two-dimensional field to the amplitude and phase functions, respectively, of a corresponding network. How this analogy comes about can be seen by considering the general rational algebraic fraction F(s) which arises in the solution of linear network problems by operational means (ref. 1).

$$F(s) = \frac{P(s)}{Q(s)} = K\,\frac{s^p + \alpha_{p-1} s^{p-1} + \cdots \alpha_1 s + \alpha_0}{s^q + \beta_{q-1} s^{q-1} + \cdots \beta_1 s + \beta_0} \qquad (1)$$

If the roots of the equation $P(s) = 0$ are $-a_i$, and those of $Q(s) = 0$ are $-b_i$, then the fraction may be written

$$F(s) = K\,\frac{\prod_{i=1}^{p}(s+a_i)}{\prod_{i=1}^{q}(s+b_i)} \qquad (2)$$

It follows that

$$\log\left[F(s)\right] = \log K + \sum_{i=1}^{p} \log(s+a_i) - \sum_{i=1}^{q} \log(s+b_i) \qquad (3)$$

While F(s) may represent any linear network function, let it here be the dimensionless transfer ratio $\frac{E_o}{E_i}$. If A is the logarithmic amplitude and B the phase of the transfer ratio (ref. 2), then

$$\log\left[F(s)\right] = \log \frac{E_o}{E_i}(s) = \log e^{A(s) + jB(s)} = A(s) + jB(s) \qquad (4)$$

To see how these functions are related to the field of a two-dimensional potential distribution, consider the complex potential function $W(s) = U(s) + jV(s)$ of a line charge (ref.3) perpendicular to the s-plane and piercing it at $s = -\gamma_i$

$$W(s) = -2q \log (s + \gamma_i) = U(s) + jV(s) \qquad (5)$$

Here $U(s)$ is the potential and $V(s)$ the stream function of $W(s)$. Comparing (5) with (3) and (4), it is seen that $W(s)$ can be made identical to $\log F(s)$, by making it represent the field of $p + q$ charges of strength $\frac{1}{2}$; one of negative sign being placed at each of the zeros of $F(s)$, $-a_i$, and one of positive sign being placed at each of the poles, $-b_i$. Doing so makes the potential function U correspond to the amplitude A and the stream function V correspond to the phase B of $F(s)$. In addition, the poles and zeros of the network transfer ratio, which become the negative and positive logarithmic singularities of $\log F(s)$, correspond to the sources and sinks of the equivalent field. These relationships constitute the potential analogy.

Practical application of the analogy to the solution of network problems dates from 1945, when Hansen and Lundstrom (ref. 4) experimentally measured an impedance function by using an electrolytic tank to obtain the desired potential distribution. In a later paper, Huggins (ref. 5) pointed out the value of certain transformations in simplifying the experimental problems posed by the limitation to a tank of finite size. Both of these papers were limited in scope to steady-state analysis. More recently, the potential analogy has been applied to problems in transient analysis (ref. 6, 7). Results similar to some given in reference 7 and in this paper have been independently obtained by Evans (ref. 8), who used an approach not involving the potential analogy.

II EXPERIMENTAL APPARATUS

Although many useful results can be obtained from the analogy from purely theoretical considerations which make use of the mathematical techniques of potential theory, there are important applications which require experimental means for producing the analogous fields. Since the fields required are those of line charges, the potential function may be produced in an electrolytic tank (4, 5, 6), or on recently available conducting paper (9), by introducing and removing current at points corresponding to the poles and zeros of F(s). Consideration must be given to the problems posed by the limitation to a tank of finite size, since in general the boundary, by its distortion of the field, introduces non-negligible errors in the readings. Hansen and Lundstrom (4) used a circular tank, and applied a first-order correction to the readings to compensate for the influence of the boundary. Huggins (5), making use of the symmetry of the field about the real axis, transformed the upper half of the s-plane into a strip, using the logarithmic transformation $s' = \log s$. The logarithmic coordinates, in which an increase in length of the strip of an amount $\frac{\log 10}{\pi}$, or 0.733, times its width is equivalent to an increase in radius of the circular tank by a factor of ten, make possible a tank of convenient dimensions in which the boundary distortion is entirely negligible. A clever method of avoiding boundary distortion in a circular tank, involving a double current sheet, is described in reference 6.

Once having established the potential distribution, the logarithmic amplitude of F(s) for any value of s is then determined by measuring the potential at the corresponding point in the s-plane. The phase of F(s), which is analogous to the stream function, can be determined from the potential distribution by making use of the relations between the conjugate functions U and V. Letting $s = \alpha + j\omega$, these relations are

$$\frac{\partial U}{\partial \omega} = -\frac{\partial V}{\partial \alpha} \qquad \frac{\partial U}{\partial \alpha} = \frac{\partial V}{\partial \omega} \qquad \text{(3, p. 72)}$$

$$V_1 - V_0 = \int_0^1 dV = \int_0^1 \left(\frac{\partial V}{\partial \omega} d\omega + \frac{\partial V}{\partial \alpha} d\alpha\right) = \int_0^1 \left(\frac{\partial U}{\partial \alpha} d\omega - \frac{\partial U}{\partial \omega} d\alpha\right)$$

Since the value of U is known by inspection along the entire real axis, as will be demonstrated below, the path of integration can always be taken parallel to the imaginary axis, making the second term of the integral zero. The derivative $\frac{\partial U}{\partial \alpha}$, which is approximated by $\frac{\Delta U}{\Delta \alpha}$, can be determined by using closely spaced pick-up probes separated a distance $\Delta\alpha$. Two procedures have been used to evaluate the phase in this manner. In references 4, 5, and 6, readings of $\frac{\Delta U}{\Delta \alpha}$ are taken at a series of stations and integrated numerically to give the phase. In reference 9, a set of permanently mounted pick-up probes are scanned mechanically and the integration performed electronically to give the phase. This method, though rapid, is limited to measurements made along the frequency axis, restricting the usefulness of the analogy to steady-state analysis only.

Still a third method may be used to determine the phase, one which is less time consuming than the first and more versatile than the second. Since the value of the phase at any point is the algebraic sum of the phase contributions of each of the factors of F(s), measurements of these individual contributions can be used as the basis of a phase function analyzer. For a typical numerator factor $(s+a_i)$, the phase at a point $\alpha+jw$ is $\tan^{-1}\frac{w}{\alpha+a_i}$. If a_i is complex, then, letting $a_i=c+jd$, the phase at s is $\tan^{-1}\frac{w+d}{\alpha+c}$. In either case, it is the angle between the line correcting the point $-a_i$ to s, and the positive real axis. If the sum of angles of the factors of the denominator is subtracted from that of the numerator, the result is the phase of F(s).

A phase function analyzer, which simultaneously measures and adds these angles, has been constructed. As shown in Figure 1, it consists of a frame, a number of potentiometers which act as goniometers, a power supply, a control box, and a voltmeter. A schematic diagram of the electrical circuits employed is given in Figure 2. In operation, the frame is placed over a sheet of graph paper which represents the s-plane. A potentiometer, fitted with a slotted arm, is positioned on the frame at each zero and pole of F(s). The slotted arms are constrained by the pointer pin to pass through the point at which the phase is to be measured. On-off and reversing switches in the control box supply voltage of proper polarity to the potentiometer, those representing poles requiring opposite polarity to those representing zeros. The coils of the potentiometers used are linear with shaft rotation to within an accuracy of 0.25%, which allows an angular measurement accuracy of about ± one degree. The angular gap in each coil is only 4 degrees, which enables measurements to be made almost everywhere in the plane. The voltage of each potentiometer arm is then proportional to the shaft angle, which is the phase angle of a factor of F(s). These voltages are summed by a resistive network in the control box, and the sum read by the voltmeter which indicates the phase.

The method of determining the amplitude function which has been found most convenient is the use of conducting paper cut in a logarithmic strip. The experimental apparatus is shown in Figure 3. Current sources and sinks at the zeros and poles of F(s) establish a potential distribution in the strip which is proportional to the logarithmic amplitude of F(s). The currents are adjusted to a convenient value of about 5 milliamperes by rheostats in series with the power supply. The potential measured by the voltmeter at any point is then proportional to the logarithmic amplitude of F(s) at that point, and can be calibrated in decibels. Sources or sinks must be included at zero and infinity, represented by the left and right edges of the strip, when log [F(s)] is singular at these points.

III STEADY STATE AND TRANSIENT ANALYSIS

In steady state analysis, the driving function is sinusoidal and only the steady state component of the response is calculated. In this case, the amplitude and phase of the transfer ratio F(s) is given by

$$A = \log |F(jw_i)|$$
$$B = \text{Arg } F(jw_i)$$

where a_i is the frequency of the driving function (1, p. 176). The frequency response curves A(jw) and B(jw) can therefore be determined from the field of log [F(s)] by measuring the amplitude and phase along the frequency axis.

The fields of familiar network functions have been plotted in Figures 4 and 6 to illustrate their use in determining the frequency response curves. Figure 4 shows the equipotentials for two decibel increments and streamlines for 10° increments of two line charges of the same sign. They were obtained by superimposing graphically the fields of the individual charges, before the experimental apparatus described above had been constructed. By properly positioning the s-plane on the field, the steady state frequency response curves for a damped oscillator of any damping ratio may be determined, simply by reading the values of A and B along the frequency axis. The normalized network transfer ratio and the frequency response curves for two values of the damping ratio ζ are shown in Figure 5.

The field of two opposite charges, which consists entirely of circles (ref. 3, p. 75), is plotted in Fig. 6. Again, the equipotentials are two decibels apart and the streamlines 10° apart. The transfer ratio of an analogous phase lead network and the frequency response curves for two values of maximum phase lead are shown in Fig. 7.

Because the poles and zeros of any network function either are real or occur in conjugate pairs, the corresponding fields are symmetric about the real axis so that no flux crosses it. The real axis can therefore be a non-conducting boundary without disturbing the field. When the s-plane is transformed into a strip by the logarithmic transformation $s' = \log s$, only the upper half of the s-plane need be transformed, and the positive and negative real axes of the s-plane become the non-conducting lower and upper edges, respectively, of the strip. To illustrate this transformation, the field of Figure 6 has been mapped in the log s-plane in Figure 8. The origin of the s-plane was chosen so that the field represents the transfer ratio $\frac{s+.5}{s+2}$. Let $s = \alpha + jw = re^{j\theta}$, then $\log s = \log r + j\theta$. The abscissa indicated in Figure 8 is r and the ordinate is θ, in degrees.

As mentioned previously, the logarithmic transformation makes convenient the physical dimensions necessary to produce a sufficiently low boundary distortion. The transformation is also valuable from a purely analytical standpoint - it makes evident, in many important cases, a symmetry in the field which is not obvious from the position of the singularities in the s-plane. In the example plotted in Figure 8, it is obvious by inspection that the vertical line midway between the zero and pole must be an equipotential, and that this equipotential is therefore a circle in the s-plane. By shifting the origin before transforming, it can be seen that every equipotential is a circle in the s-plane, a fact which requires some algebraic manipulation to deduce by standard means. A knowledge of logarithmic symmetry, when it exists, reduces the region in the s-plane which need be covered by the phase function analyzer to determine the phase function throughout the plane. In addition, it makes the shape in the s-plane of critical flux lines evident without measurement or calculation. As will be shown later, this simplifies the labor in finding the roots of equations when logarithmic symmetry exists.

The potential analogy can be used to determine the coefficients and phase angles of the various terms of the transient response as well as those of the steady state terms. The analytical method of transient analysis by means of the Laplace Transformation results in a function of s which must be inverted to yield the desired function of time. This requires that the residues of the function at the various poles be evaluated. These residues, which for complex terms are in general complex, appear as the coefficients and phase angles in the various terms of the time function. In the field corresponding to the function, the residue at any pole can be determined by removing the pole and measuring the amplitude and phase due to the remainder of the function at that point. The difficulty of setting up the field experimentally compared to that of straightforward analytical solution makes it impractical at present to use the analogy for this purpose for an ordinary problem. If it were desired to determine the response of a system for a number of initial conditions, the use of the analogy could prove practical, since only the zeros of the function are shifted with changing initial conditions.

In concluding the discussion of steady state and transient analysis by means of the analogy, it might be pointed out that obtaining the steady state response by means of the analogy is merely a special case of the method used for transient analysis, in which the forcing function is sinusoidal and the residue at the poles of the forcing function only are obtained.

IV SOLUTION OF NETWORK EQUATIONS

In the applications of the potential analogy treated so far, a knowledge of the zeros and poles of the network function F(s) has been assumed. However, the problem of finding the roots of P(s) and especially of Q(s) is usually the most difficult step in the treatment of network problems. It is therefore gratifying that the analogy itself can be used in determining the roots of polynomials. By this property it affords the user valuable insight regarding the effects of loading, coupling, and feedback on the natural modes of networks.

The method of using the potential analogy in determining the roots of polynomials to be described is dependent upon a knowledge of the field of a given distribution of logarithmic singularities. Let the polynomial to be factored be

$$f(s) = a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_o$$

Let a minor $M_i(s)$ of this polynomial be defined as any term or group of terms contained in f(s), and let its complement $N_i(s)$ be $f(s) - M_i(s)$. Then

$$f(s) = M_i(s) + N_i(s)$$

The equation $f(s) = 0$ can then be written as

$$\frac{M_i(s)}{N_i(s)} = -1 \quad \text{or} \quad \log \frac{M_i(s)}{N_i(s)} = 0 + j\pi$$

The roots of f(s) are those values of s which satisfy the above equation. If the roots of $M_i(s)$ and $N_i(s)$ are known, the equation can be solved by a single set of field measurements.

If $f(s)$ contains no pair of complementary minors whose roots are known, a preliminary step must be taken to find those of an arbitrarily chosen pair. The roots of $M_i(s)$ and $N_i(s)$ are used in generating the field $\log \frac{M_i(s)}{N_i(s)}$. The n points in the field at which the potential is zero and the phase π radians or 180 degrees are then the n roots of $f(s)$.

Before demonstrating the application of this method to the solution of particular network problems, several examples of its use in finding the roots of polynomials will be given. Consider first the quadratic $s^2 + as + b = 0$

Let $M = s^2 + as = s(s+a)$ and let $N = b$.

Then $\frac{s(s+a)}{b} = -1$ or $s(s+a) = -b$

The field required for the solution is that of two charges of the same sign, one at the origin and the other at $-a$. The complete field is shown in Figure 4. Because only the 180° phase lines are needed for the solution, however, these lines have been redrawn separately in Figure 9. Several features of the solution of equations by this method are demonstrated by this diagram. The real axis must be a flux line because the charges are symmetric about the real axis when the coefficients are real. More specifically, a line segment on the real axis having an even number of singularities to its right is a zero degree or $2n\pi$ radians flux line, while one having an odd number of singularities to its right is a 180 degree or $(2n+1)\pi$ radians flux line. By symmetry, the 180 degree flux lines between the two charges meet at a neutral point on the real axis and become the vertical flux line whose constant abscissa is $-\frac{a}{2}$. Roots of the equation are constrained to lie on the 180 degree flux lines, their position being determined by the potential requirement that $A = \log |s(s+a)| = \log b$. Values of A along the flux lines can be obtained from the equipotentials of Figure 4, the zero of potential being determined by its known value of $2 \log(\frac{a}{2})$ at the neutral point.

If the roots are desired as a function of the coefficient a, then M should be chosen to include the terms not involving a. Thus, $M = s^2 + b$ and $N = as$, giving $\frac{s^2+b}{s} = -a$ or $\frac{(s+j\sqrt{b})(s-j\sqrt{b})}{s} = -a$

The critical flux lines for this charge configuration is shown in Figure 10. The existence of the neutral point on the real axis and the circular path of the flux line from the neutral point to the imaginary axis is obvious from the charge distribution in the logarithmic strip. The location of the root along the flux line may be obtained from the potential requirement that $A = \log a$. Alternately, if the roots are complex, they are given by the intersection of the flux lines of Figure 9 with those of Figure 10.

The general solution of a cubic equation requires a family of curves, since either of two coefficients can be varied relative to a third. Consider the cubic $s^3 + as^2 + bs + bc = 0$. Let the minors be chosen as $M = s^3 + as^2$ and $N = bs + bc$. The equation may then be written

$$\frac{s^2(s+a)}{s+c} = -b$$

The analogous field has a double charge at the origin, corresponding to the double zero, a single charge of the same sign at $-a$, and one of opposite sign at $-c$, as shown in Figure 11. If $a = c$, the two charges of single magnitude and opposite polarity coincide, their external field dropping to zero. Since all values of the stream and potential function are present at the coincident point, this point is a root. The other two roots, for this special case, lie on the imaginary axis, since it is a 180 degree flux line because of the double charge at the origin. The 180 degree flux lines for $a = 3c$ are shown in Figure 11, and the family of critical flux lines obtained for various values of the ratio a/c is shown in Figure 12. The phase function analyzer previously described was used to determine these flux lines. For all values of a/c, the portion of the real axis between a and c is a 180 degree flux line. For $a/c = 9$, a neutral point exists at $-3c$; for $a/c > 9$, a portion of the flux lines terminating at infinity is detached from the lines originating at the origin, and only the detached portion is shown in Figure 12. The position of the roots along the flux lines is determined by the value of b, and can be obtained from the potential requirement $A = \log b$.

A similar family of critical flux lines, which arises in the solution of the particular group of quartic equations $s^4 + 2s^3 + s^2 + as + ab = 0$, is shown in Figure 13. They were obtained by separating the quartic into $M = s^4 + 2s^2 + s^2$ and $N = as + ab$, giving

$$\frac{s^2(s+1)^2}{s+b} = -a$$

A double charge at the origin and at -1 and one of opposite sign at -b were simulated by the phase function analyzer for the values of b indicated to obtain the 180 degree flux lines of Figure 13.

The effect of coupling on the modes of two inductively coupled circuits can be determined in a similar application of the analogy. Consider the tuned circuits shown in Figure 14a. The modes of the coupled network are the zeros of the main determinant; these can be found for any value of coupling coefficient k when the uncoupled modes are known, as follows. Expanding the determinant gives $(L_1C_1s^2 + R_1C_1s + 1)(L_2C_2s^2 + R_2C_2s + 1) - M^2C_1C_2s^4$. This can be written in terms of the uncoupled modes and the coefficient of coupling as

$$\frac{[(s+\alpha_1)^2+\omega_1^2][(s+\alpha_2)^2+\omega_2^2] - k^2 s^4}{\omega_1^2\,\omega_2^2}$$

where w_i is the undamped resonant frequency, and α_i the damping coefficient, of the ith circuit when no coupling exists, and k, the coefficient of coupling, is $\frac{M}{\sqrt{L_1L_2}}$.

The equation for the coupled modes becomes

$$\frac{[(s+\alpha_1)^2+\omega_1^2][(s+\alpha_2)^2+\omega_2^2]}{s^4} = k^2$$

The solution by potential analogy requires a charge at each of the uncoupled modes and a fourth order charge of opposite sign at the origin. The flux lines along which the roots must lie are the zero degree lines, since the right hand side of the equation is positive. These lines, for a particular distribution of uncoupled modes, are shown in Figure 14b. The arrows on the lines show the direction of travel of the coupled roots with increasing k; for

$k = o$ they coincide with the uncoupled modes. For the case of small damping of the tuned circuits and low values of k, the variation of the field in the vicinity of the two roots can be considered to be due to these two roots alone, the rest of the singularities contributing a nearly constant field in this small region. When this holds, the field of Figure 4 can be used to determine the steady state frequency response curves. The ratio of the separation of the roots to their distance from the frequency axis determines the position of the frequency axis in Figure 4. As the coupled roots become separated more and more with increasing k, the frequency axis gets closer and closer to the singularities in Figure 4, giving rise to the familiar family of frequency response curves shown in Figure 14c.

The effect of loading on the modes of a ladder network can be demonstrated in connection with the network of Figure 15a. The nodal equation for the n-1 node is $-E_n + E_{n-1}\,(s+2) - E_{n-2} = 0$. The voltage transfer ratio for any number of meshes in cascade can be determined by solving the above difference equation by means of the potential analogy. If the voltage at the output node E_o is 1, then that at the first node, E_1, becomes $s+1$, since $\frac{E_o}{E_1} = \frac{1}{s+1}$. These two boundary conditions, inserted in the difference equation, give for E_2

$$E_2 = E_1\,(s+2) - E_o = (s+1)\,(s+2) - 1$$

The zeros of this expression, which are the poles of the transfer ratio $\frac{E_o}{E_2}$, occur at the points where $(s+1)\,(s+2) = 1$. In general, for the n'th node, the poles of $\frac{E_o}{E_n}$ are the points at which the ratio $\frac{E_{n-1}(s+2)}{E_{n-2}} = 1$. For this network, the poles of the transfer ratio for all nodes are constrained to the negative real axis because of the absence of inductors and active elements. This makes the logarithmic strip well suited for an analogous solution, since the insertion of currents representing the singularities of the n-1 and n-2 nodes and the search for poles of the n'th node take place along the upper edge of the strip only. The solution for the poles of the n'th node by standard means requires first that the n'th degree polynomial be found by solution of the network equations, and secondly that the polynomial be factored to determine its roots. Both of these steps are avoided in the solution by analogy. The location of the zero degree flux lines and the zeros of potential which determine the poles are shown for the first few nodes in Figure 15b. If the individual meshes of the network were completely isolated from one another by means of vacuum tubes, the poles would all fall at -1. The spreading of the poles outward from -1 is the result of the finite load presented each mesh.

If a network whose poles and zeros are known is used as the only frequency function in a feedback amplifier, the potential analogy can be used to determine the variation with feedback of the modes of the amplifier. Let the amplifier have a forward gain of μ and a feedback factor B. Let $F(s)=\frac{P(s)}{Q(s)}$ be the transfer ratio of the network used in the amplifier. If it is used in the forward loop, the overall transfer ratio of the amplifier becomes

$$\frac{E_o}{E_i} = \frac{\frac{P(s)}{Q(s)}\mu}{1+\frac{P(s)}{Q(s)}\mu B} = \frac{\mu P(s)}{Q(s)+P(s)\mu B}$$

When used in the feedback loop,

$$\frac{E_o}{E_i} = \frac{\mu}{1+\frac{P(s)}{Q(s)}\mu B} = \frac{\mu\, Q(s)}{Q(s)+P(s)\mu B}$$

The two cases differ only in the numerator, whose zeros in the first case coincide with those of P(s) and in the second with Q(s). For both configurations, the poles are the zeros of $Q(s)+P(s)\mu B$. Proceeding as before, the denominator is written

$$\frac{Q(s)}{P(s)} = -\mu B$$

Only the 180 degree flux lines and the values of potential along them in the field analogous to the left hand side are needed to solve this equation. For the network described in Figure 16, however, the complete analogous field has been determined, and is shown in Figure 17, the values of the equipotentials being indicated in decibels. The intersections of the 180 degree phase lines with the equipotential whose value is $20 \log_{10} \mu B$, locates the poles of the feedback network.

The inverse of the values of the field along the frequency axis give the steady state response of $\frac{P(s)}{Q(s)}$, which is plotted in Figure 18. Conversely, if one starts with the steady state response, the entire field can be reconstructed, either by analytic continuation or its graphical equivalent, flux plotting. Both processes are simplified when the field is desired only in regions neighboring the frequency axis. Flux plotting is especially convenient when some of the singularities of the field are known; in the present example of feedback circuit analysis this is generally the case. When sufficient portions of the field are thus determined, the locations of the closed loop poles are readily obtained for various values of loop gain by the process described above.

These examples have demonstrated only a few of the many possible applications of the potential analogy in network analysis. The practical range of application would be greatly increased if more convenient means were available for generating the amplitude and phase functions. It is hoped that future workers will uncover new applications and better tools for exploiting them.

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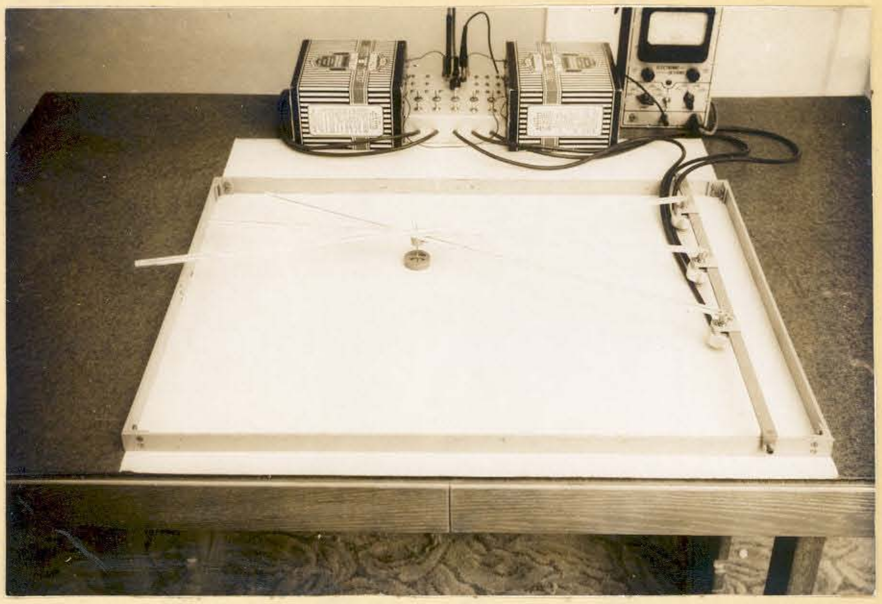
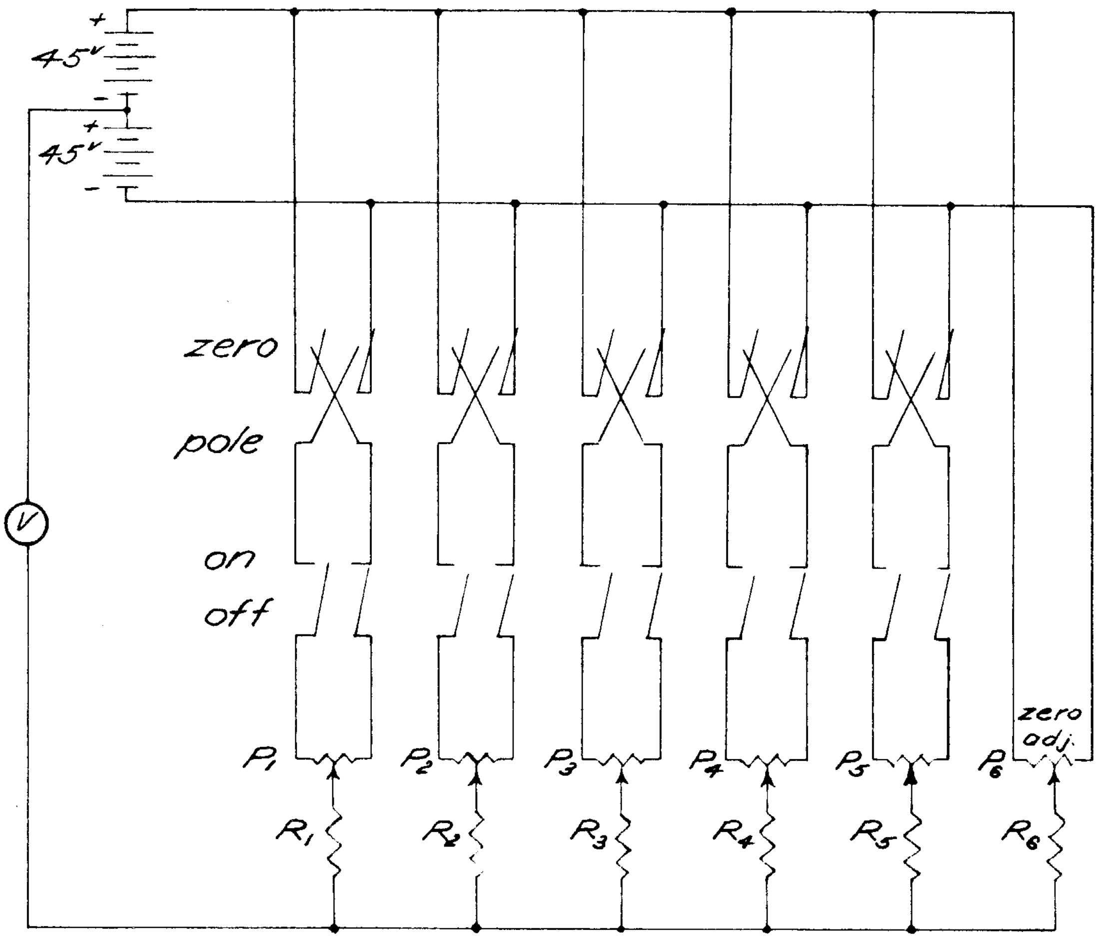


PHASE FUNCTION ANALYZER

Figure 1

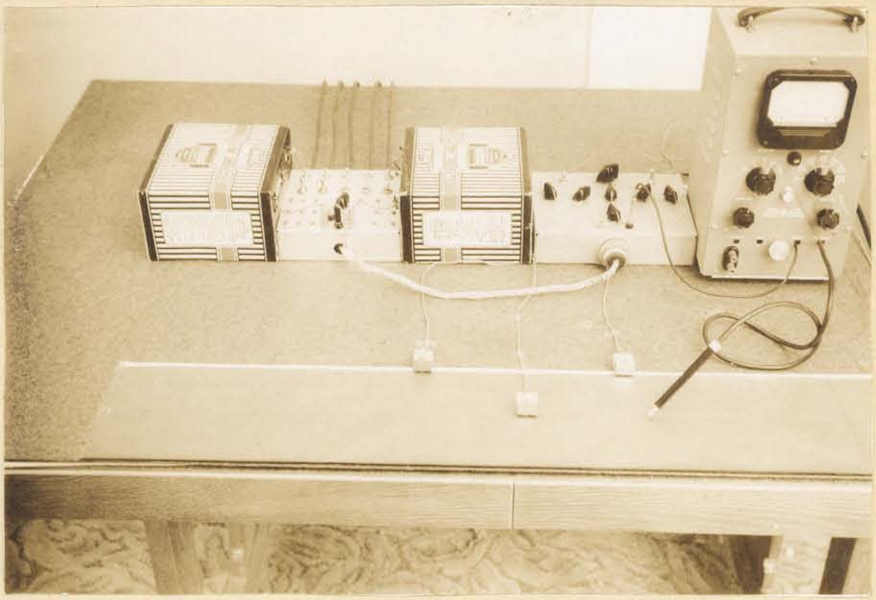


$P_{1 \dots 5}$: 20,000 Ω, 0.25 % linearity, wire-wound potentiometers

$R_{1 \dots 5}$: 2,000,000 Ω wire-wound resistors matched to within 0.1 %

PHASE FUNCTION ANALYZER

FIGURE 2



SET-UP FOR MEASURING POTENTIAL DISTRIBUTION

Figure 3

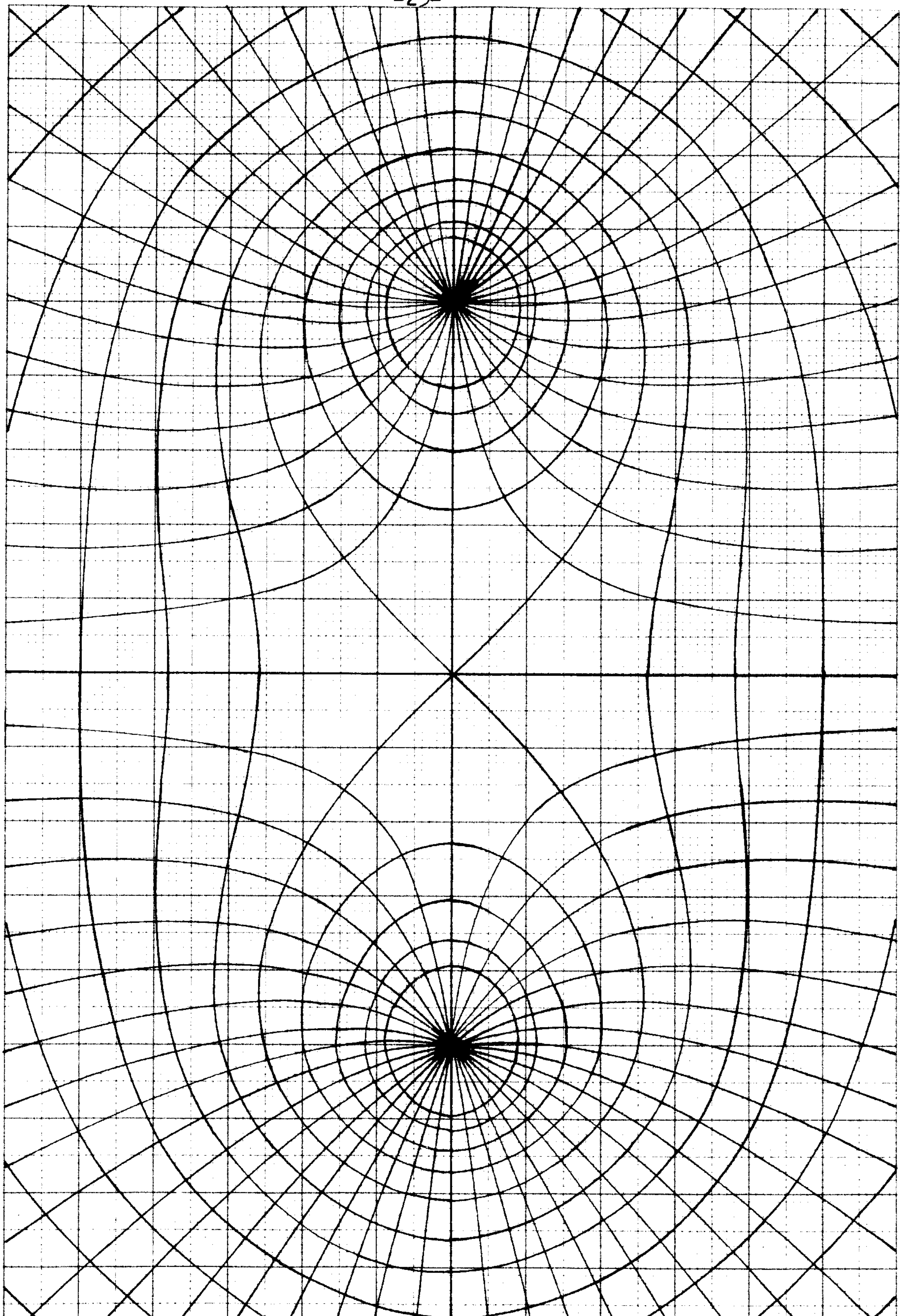


FIGURE 4

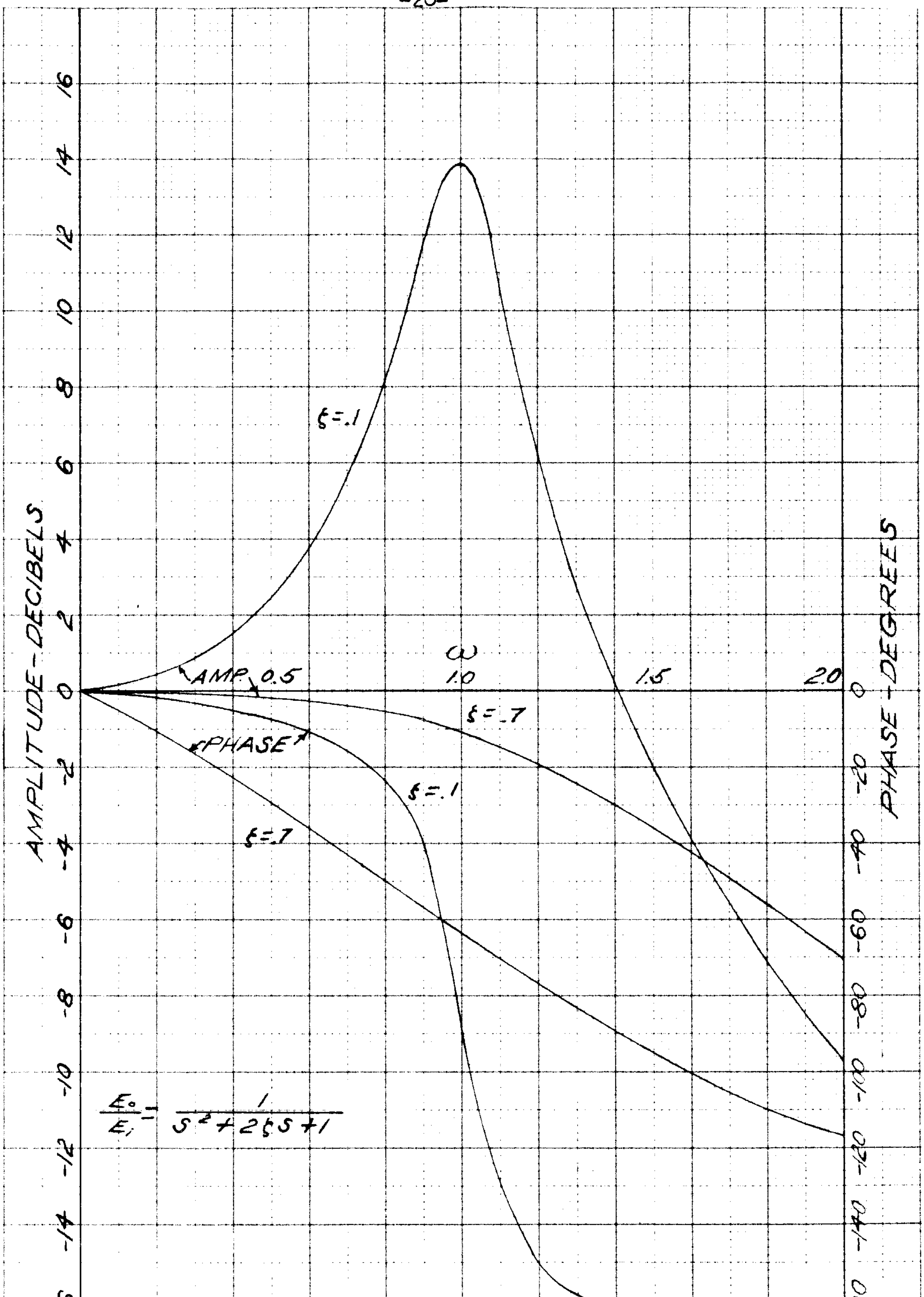

AMPLITUDE - DECIBELS
PHASE - DEGREES
16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
0
-20
-40
-60
-80
-100
-120
-140
-160
ω
0.5
1.0
1.5
2.0
AMP.
PHASE
ξ=.1
ξ=.7
ξ=.1
ξ=.7
$\frac{E_o}{E_i} = \frac{1}{s^2+2\xi s+1}$

FIGURE 5

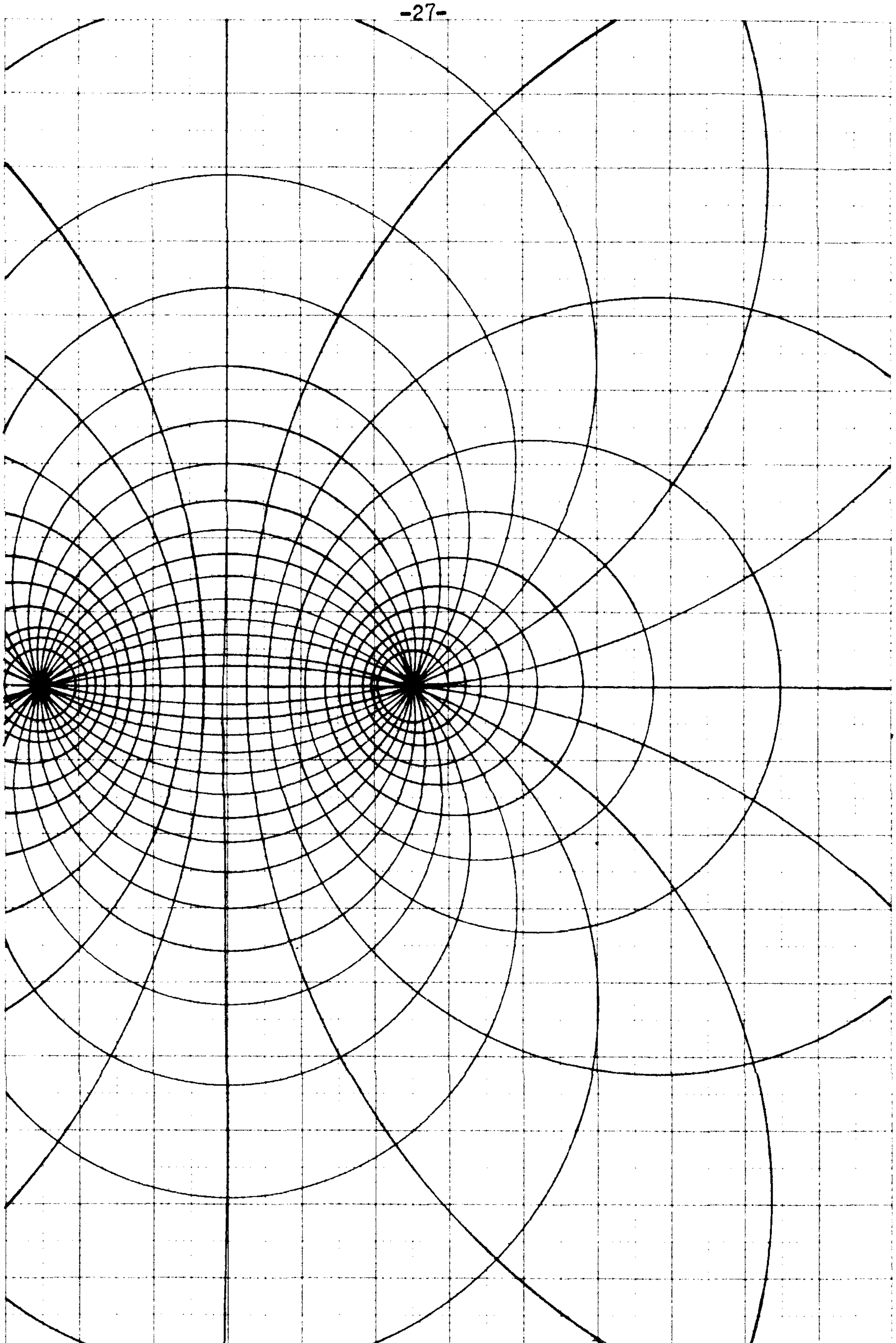


FIGURE 6

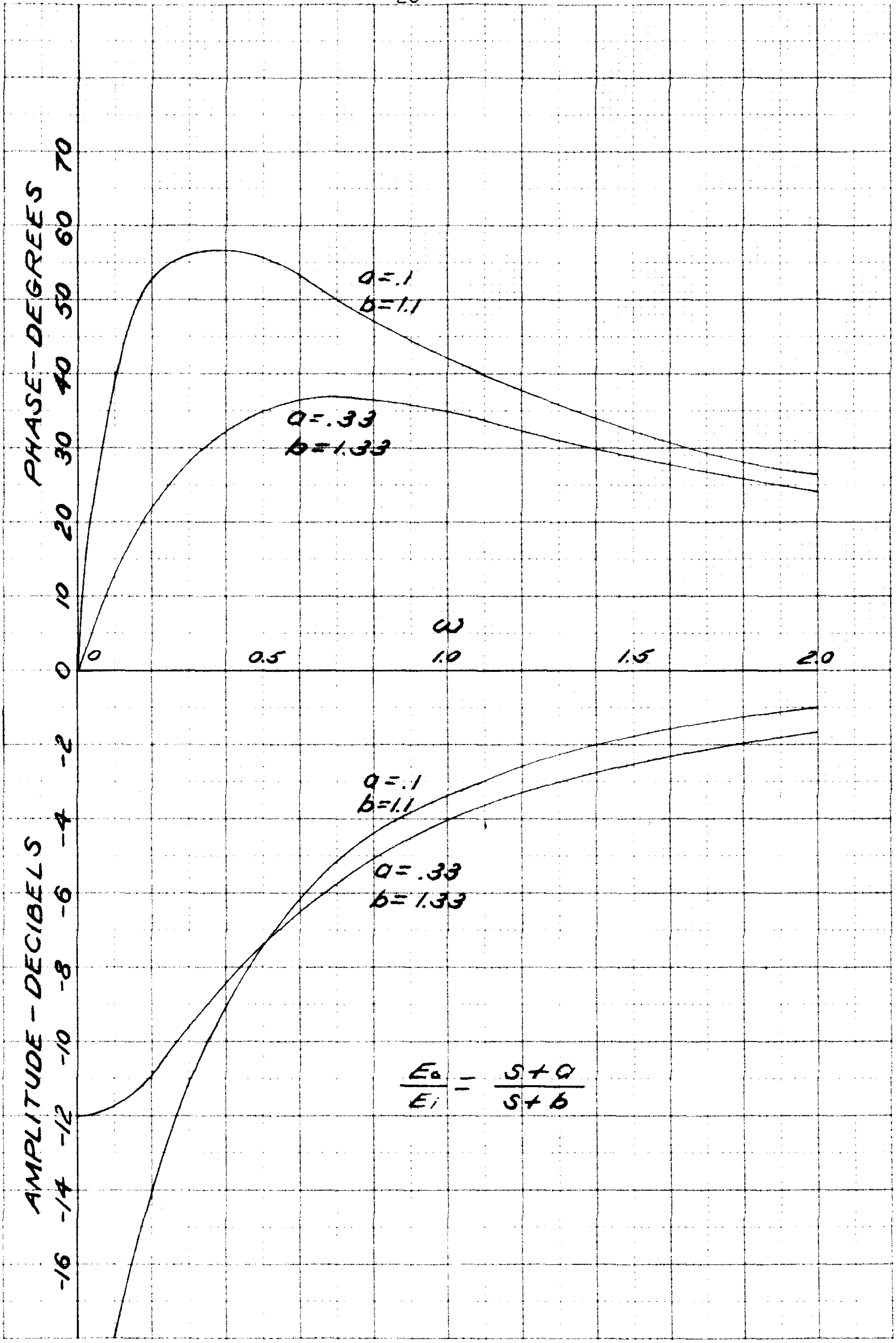
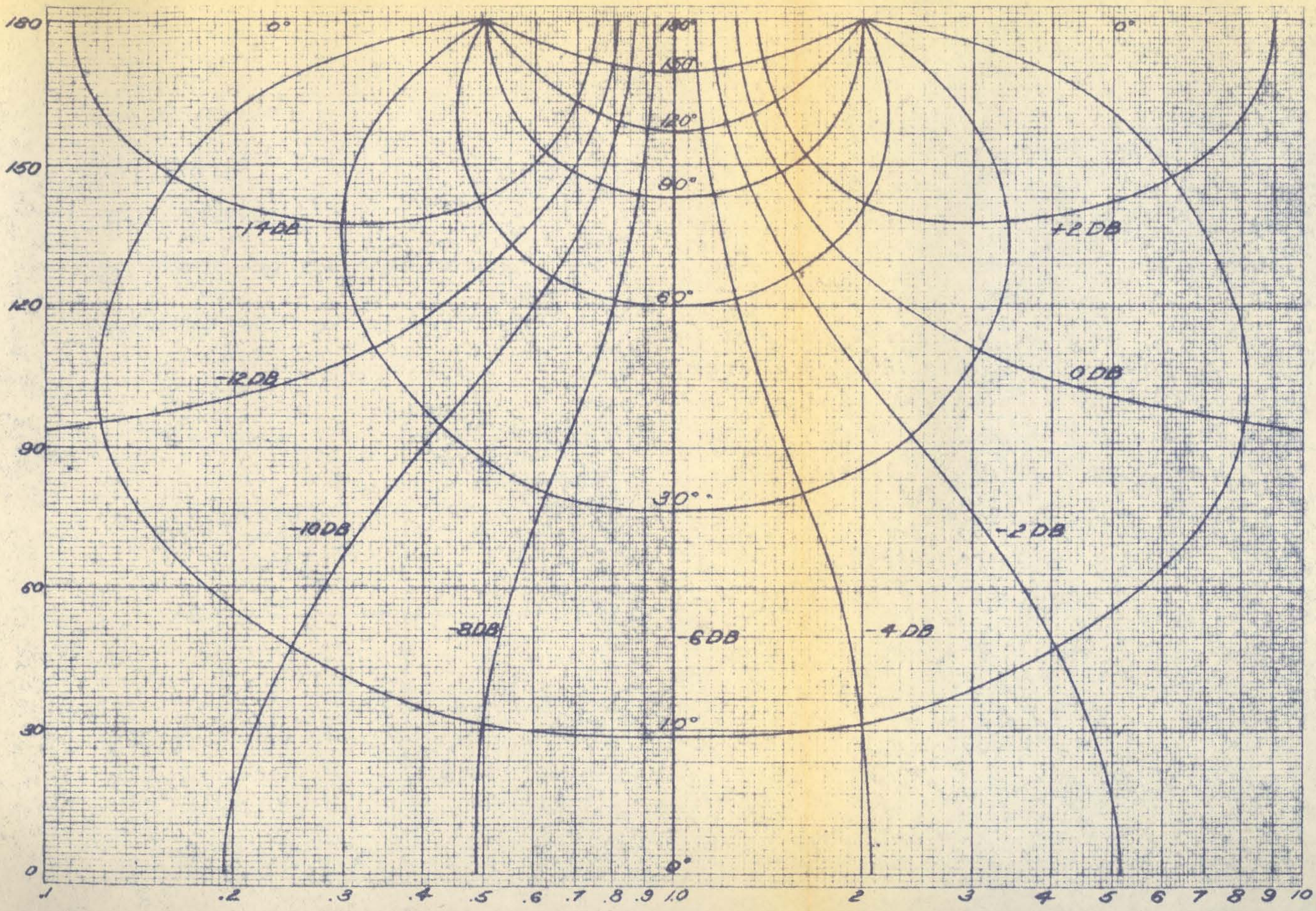

PHASE - DEGREES
70
60
50
40
30
20
10
0
a=.1
b=1.1
a=.33
b=1.33
ω
0
0.5
1.0
1.5
2.0
AMPLITUDE - DECIBELS
-2
-4
-6
-8
-10
-12
-14
-16
a=.1
b=1.1
a=.33
b=1.33
$\frac{E_o}{E_i} = \frac{S+a}{S+b}$

FIGURE 7



180
150
120
90
60
30
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
2
3
4
5
6
7
8
9
10
0°
180°
150°
120°
90°
60°
30°
10°
0°
+2 DB
0 DB
-2 DB
-4 DB
-6 DB
-8 DB
-10 DB
-12 DB
-14 DB

FIGURE 8

$$s^2 + as + b = 0$$

$$s(s+a) = -b$$

$$F(s) = s(s+a)$$

$$\log F(s) = \log s + \log (s+a)$$

(singularities at 0 and $-a$)

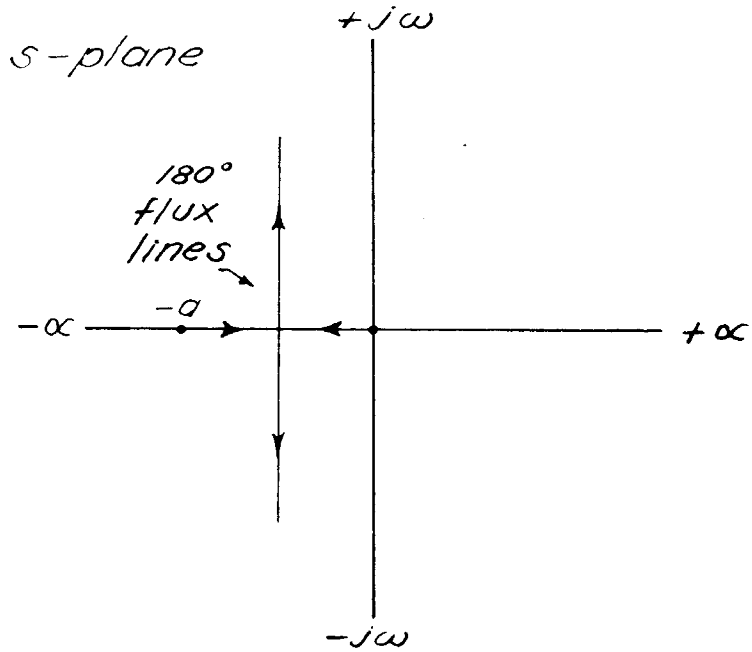


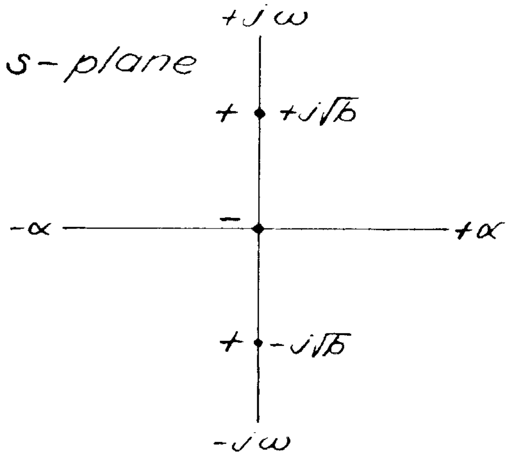
FIGURE 9

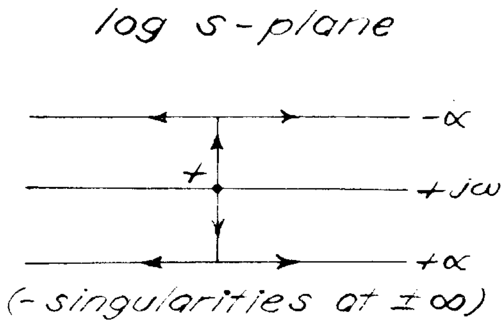
$$s^2 + as + b = 0$$

$$\frac{s^2+b}{s} = -a$$

$$F(s) = \frac{s^2+b}{s} = \frac{(s+j\sqrt{b})(s-j\sqrt{b})}{s}$$

$$\log F(s) = \log(s+j\sqrt{b}) + \log(s-j\sqrt{b}) - \log s$$





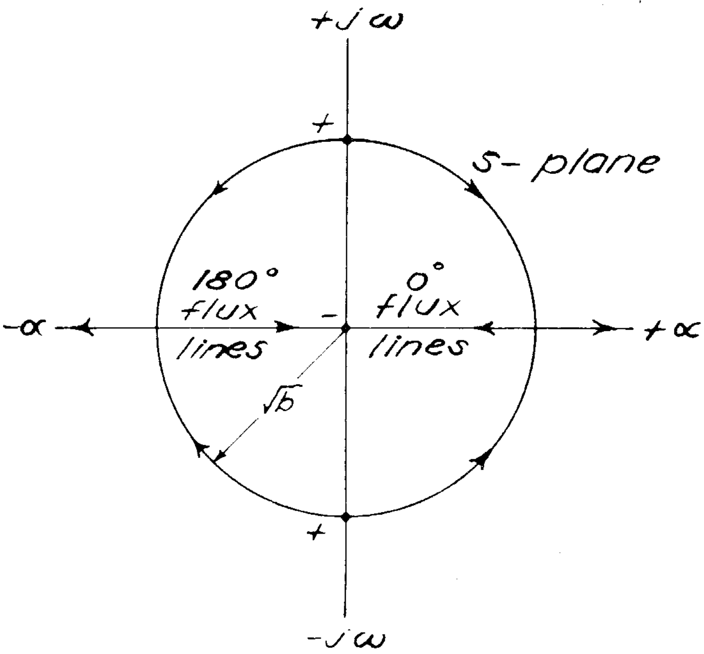


FIGURE 10

$$s^3 + as^2 + bs + bc = 0$$

$$\frac{s^2(s+a)}{s+c} = -b$$

$$F(s) = \frac{s^2(s+a)}{s+c}$$

$$\log F(s) = 2\log s + \log(s+a) - \log(s+c)$$

singularities: $+2$ at 0, $+1$ at $-a$, -1 at $-c$

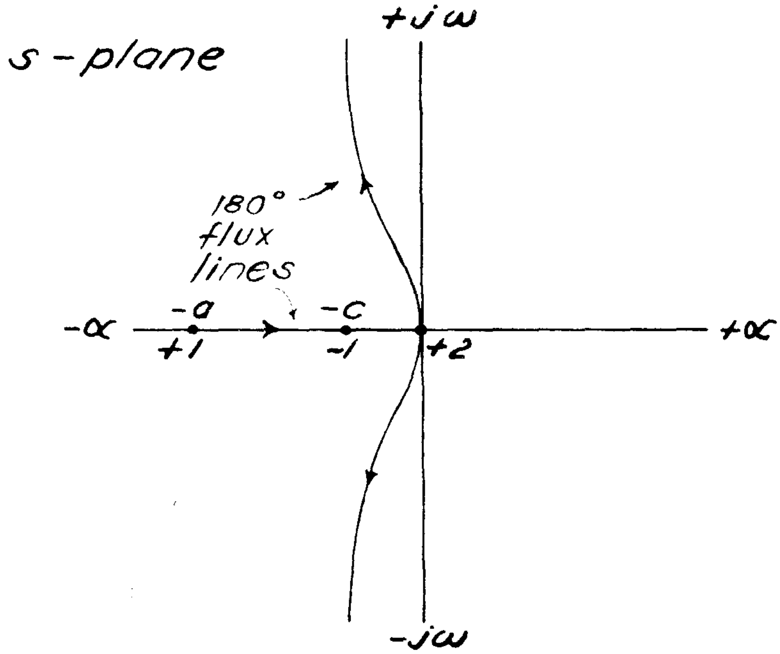
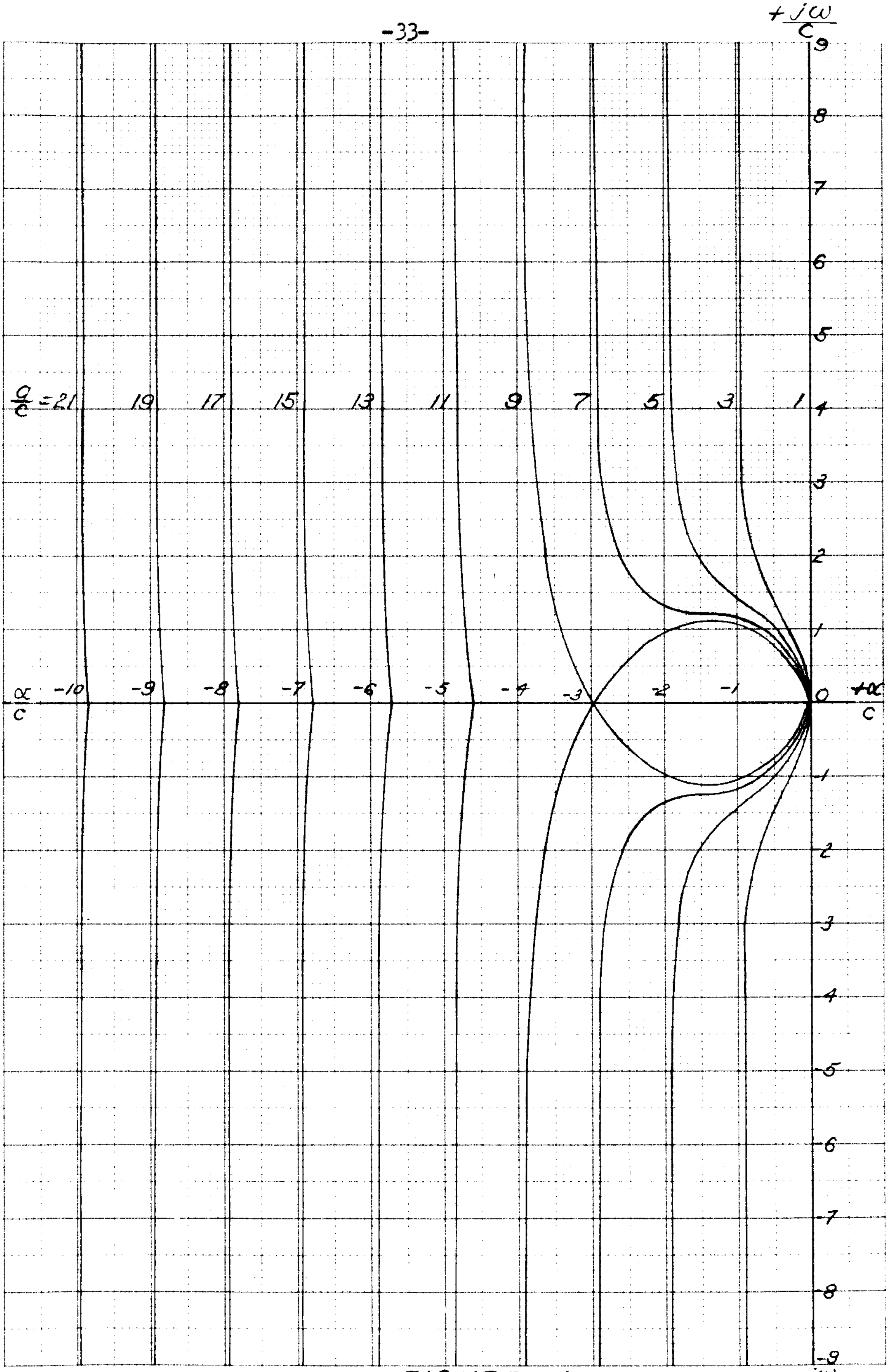


FIGURE 11



$+\frac{j\omega}{c}$
$\frac{\alpha}{c}=21$
19
17
15
13
11
9
7
5
3
1
$\frac{\alpha}{c}$
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
$+\frac{\alpha}{c}$
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
9
$-\frac{j\omega}{c}$

FIGURE 12

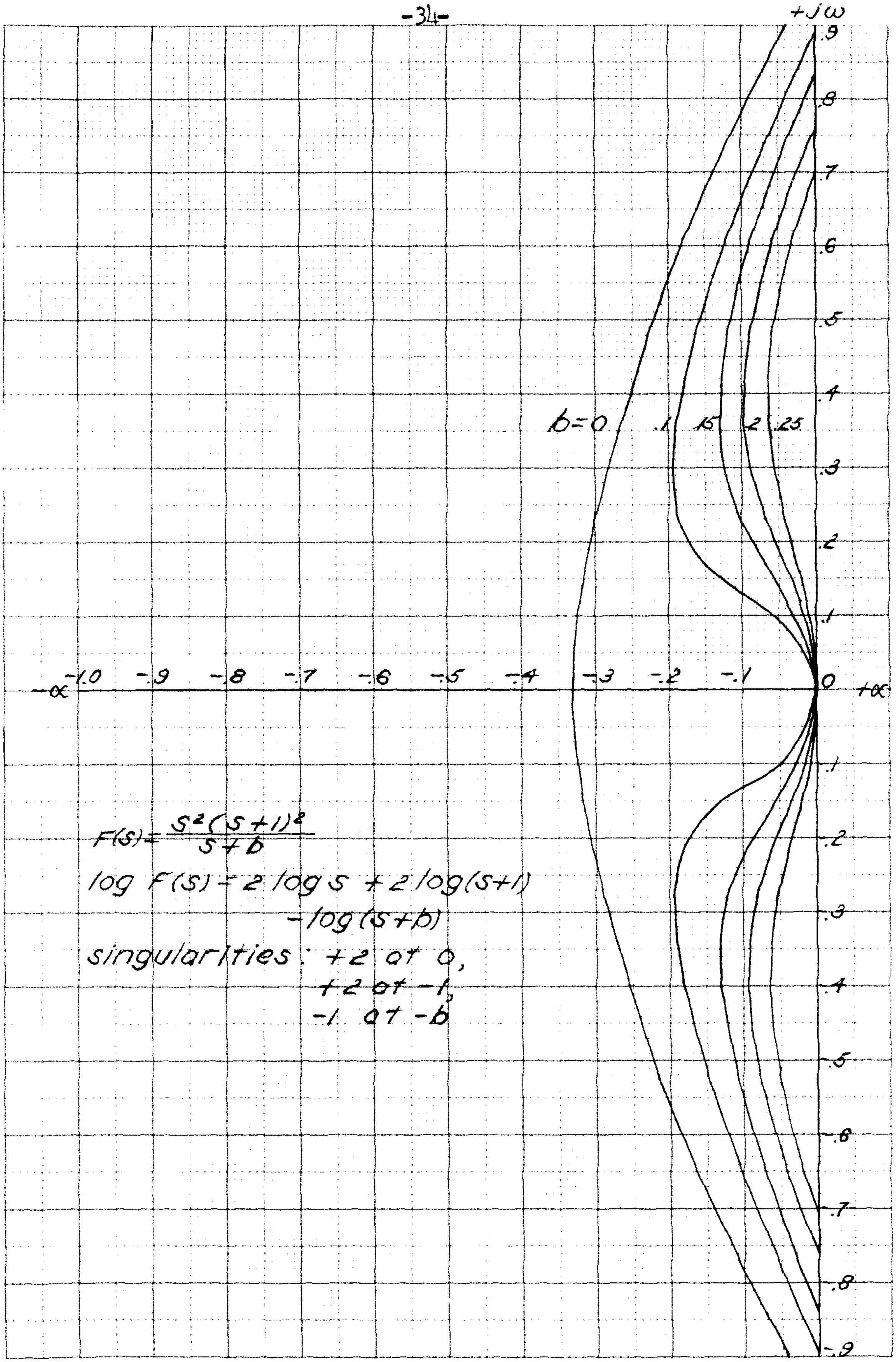
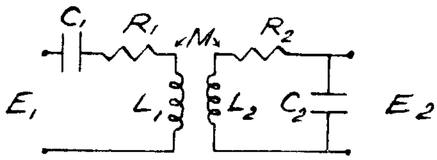

+jω
-jω
-α
+α
-1.0 -.9 -.8 -.7 -.6 -.5 -.4 -.3 -.2 -.1 0
.9 .8 .7 .6 .5 .4 .3 .2 .1
b=0 .1 .15 .2 .25
F(s)= s²(s+1)² / s+b
log F(s) = 2 log s + 2 log(s+1)
- log(s+b)
singularities: +2 at 0,
+2 at -1,
-1 at -b

FIGURE 13



a.

$$\frac{E_2}{E_1} = \frac{MC_1 S}{(L_1C_1S^2 + R_1C_1S + 1)(L_2C_2S^2 + R_2C_2S + 1) - M^2C_1C_2S^2}$$

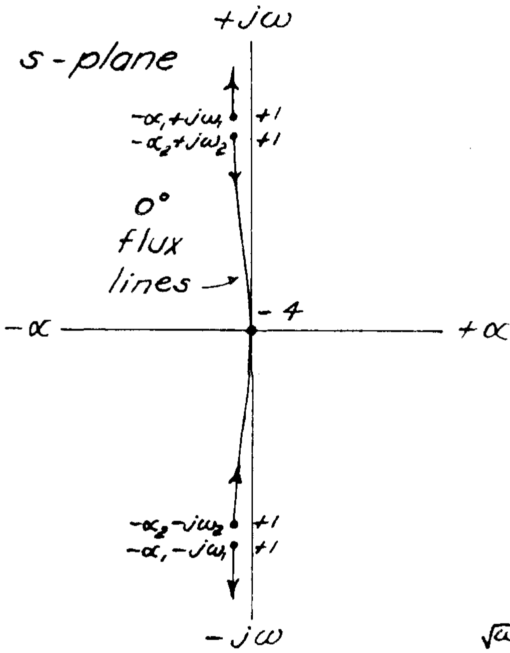
Denominator:

$$L_1L_2C_1C_2\left\{[(S+\alpha_1)^2 + \omega_1^2][(S+\alpha_2)^2 + \omega_2^2] - k^2S^4\right\}$$

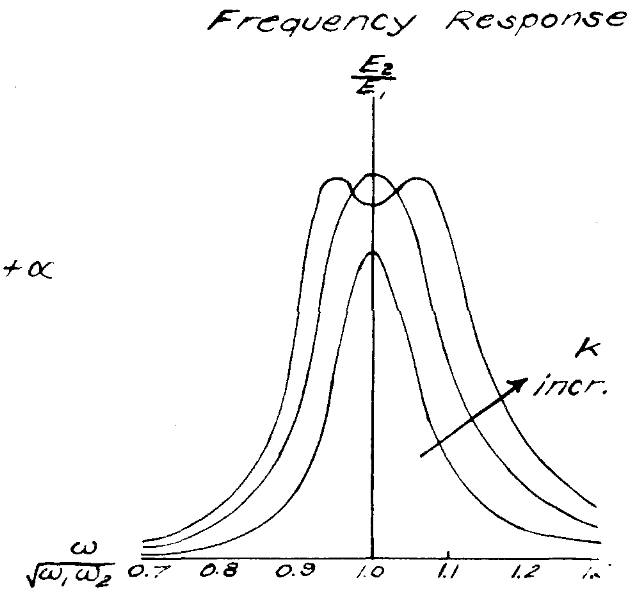
where: $\alpha_{1,2} = \dfrac{R_{1,2}}{2L_{1,2}}$ $\qquad \omega_{1,2}^2 = \dfrac{1}{L_{1,2}C_{1,2}} - \dfrac{R_{1,2}^2}{4L_{1,2}^2}$

$$k^2 = \frac{M^2}{L_1L_2}$$

$$F(s) = \frac{[(S+\alpha_1)^2 + \omega_1^2][(S+\alpha_2)^2 + \omega_2^2]}{S^4} = k^2 \text{ at roots}$$

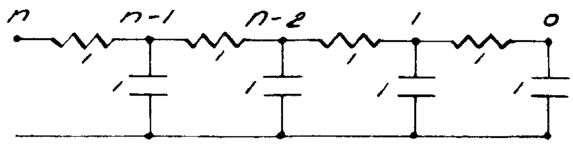


b.



c.

FIGURE 14



a.

$$-E_n + E_{n-1}(s+2) - E_{n-2} = 0$$

$$E_n = E_{n-1}(s+2) - E_{n-2}$$

$$E_0 = 1$$

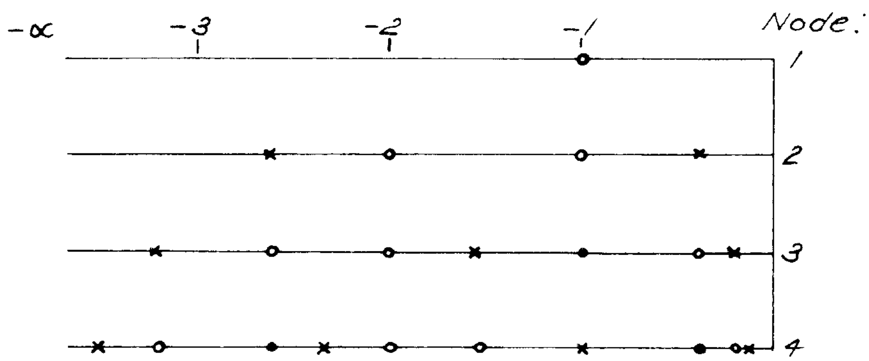
$$E_1 = s+1$$

$$E_2 = (s+1)(s+2) - 1$$

zeros of E_2 : $(s+1)(s+2) = 1$

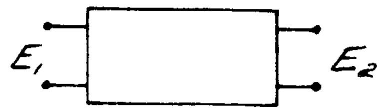
zeros of E_n : $\dfrac{E_{n-1}(s+2)}{E_{n-2}} = 1$

KEY:
zero - o pole - • solution - x

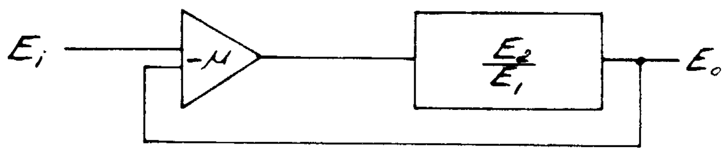


b.

FIGURE 15



$$\frac{E_2}{E_1} = \frac{130}{[(S+1)^2+5^2](S+5)}$$

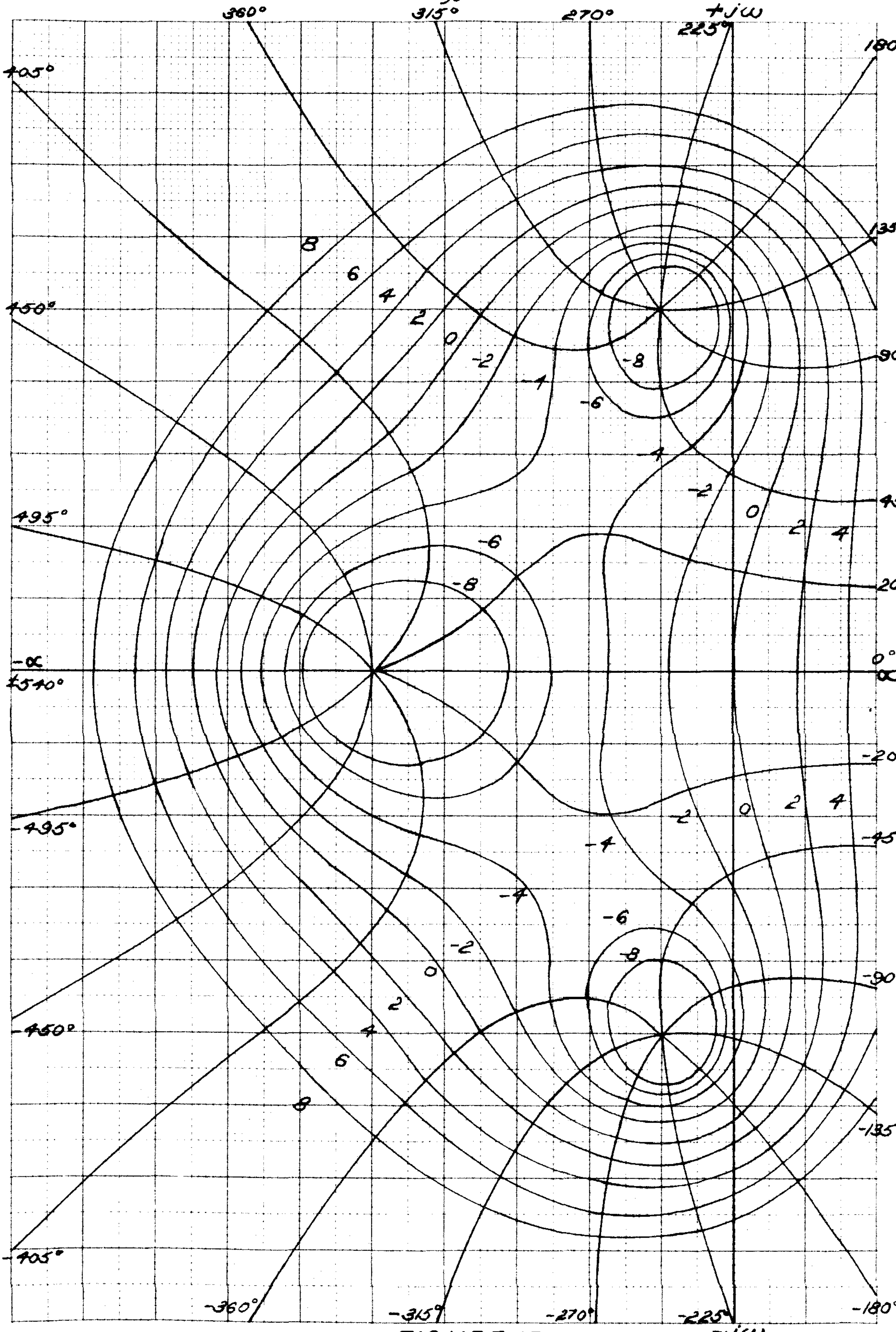


$$\frac{E_o}{E_i} = \frac{-130\mu}{[(S+1)^2+5^2](S+5)+130\mu}$$

Closed Loop Poles:

$$\frac{[(S+1)^2+5^2](S+5)}{130} = -\mu$$

FIGURE 16



360°
315°
270°
+jω
225°
180°
405°
135°
450°
90°
45°
495°
20°
-α
-540°
0°
α
-20°
-495°
-45°
-90°
-450°
-135°
-405°
-360°
-315°
-270°
-225°
-jω
-180°
8
6
4
2
0
-2
-4
-6
-8

FIGURE 17

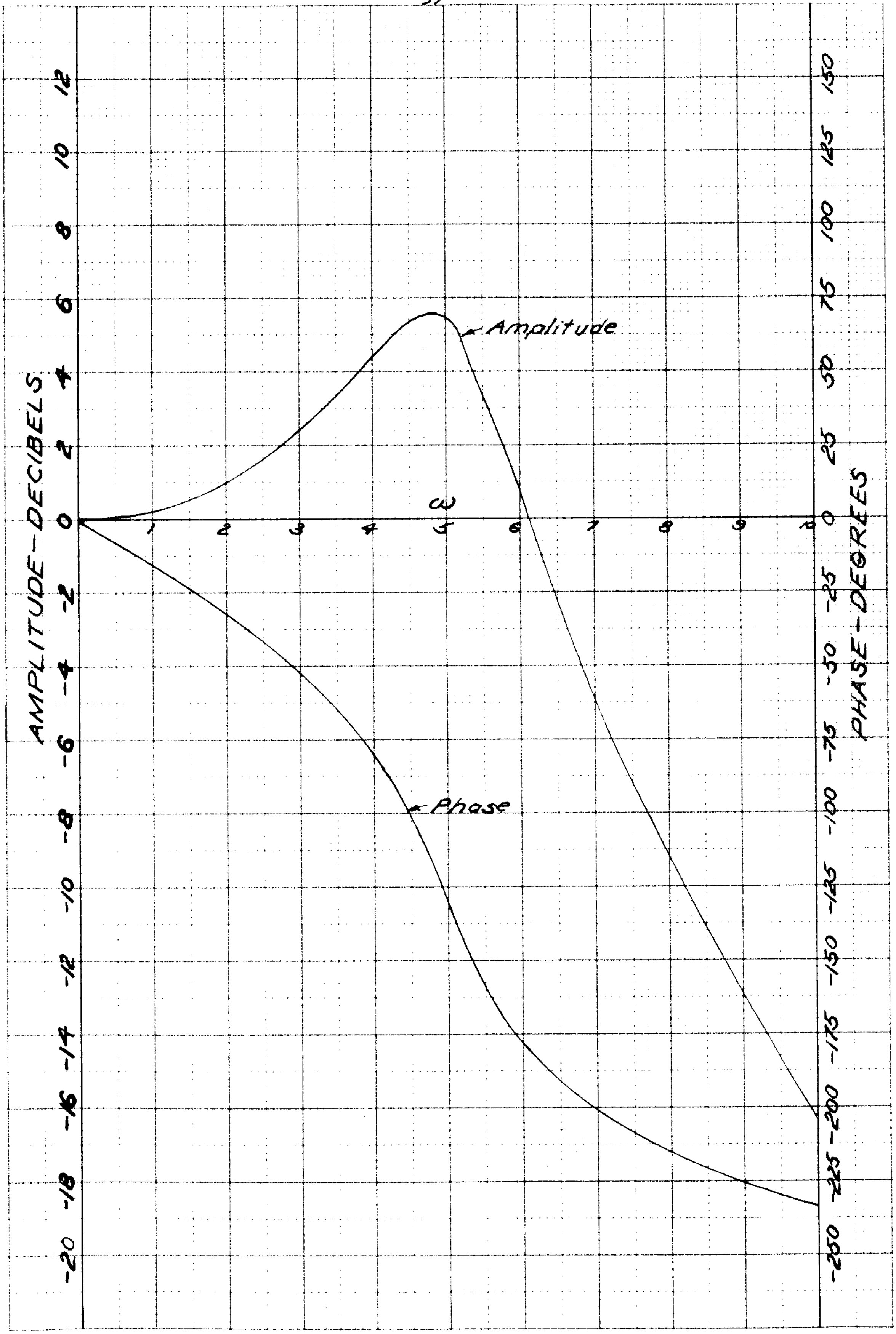

AMPLITUDE-DECIBELS
PHASE-DEGREES
Amplitude
Phase
ω
12 10 8 6 4 2 0 -2 -4 -6 -8 -10 -12 -14 -16 -18 -20
150 125 100 75 50 25 0 -25 -50 -75 -100 -125 -150 -175 -200 -225 -250
1 2 3 4 5 6 7 8 9 10

FIGURE 18