EXPANSION PROCEDURES AND SIMILARITY LAWS FOR TRANSONIC FLOW

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ABSTRACT

The transonic flow past slender bodies and thin wings is investigated with the use of a general theory of expansion procedures. It is assumed that the solutions for the velocity components possess asymptotic expansions of a very general form, and the differential equations and boundary conditions for the first and higher approximations are obtained by applying appropriate limiting procedures to the full equations. The following cases are treated: 1) bodies of revolution at zero incidence; 2) bodies of nearly circular cross-section, at zero incidence; 3) bodies of revolution at an angle of attack; and 4) thin wings at zero incidence. Certain first-order similarity laws are derived for these problems, and the order of magnitude of the error is stated in each case.

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1. INTRODUCTION

Linearized theory has found wide practical application in predicting the flow fields around thin wings and bodies, over a wide range of subsonic and supersonic flight velocities. But the transonic speed range must be excluded. Typically the linear solution for the streamwise velocity perturbation becomes infinite as the free-stream Mach number M approaches one. For example, this quantity is proportional to $|1 - M^2|^{-1/2}$ for flow past a two-dimensional airfoil, and proportional to $\log|1 - M^2|$ at the surface of a slender body of revolution. For a given wing or body, these solutions violate the assumption of small disturbances, and are therefore not correct, in some range of Mach numbers near one.

A solution for any of the velocity components obtained by linearized theory is actually the first term in an asymptotic expansion presumed to be valid as one or more geometric parameters (thickness ratio, angle of attack, etc.) approaches zero. The error in the linear approximation at a fixed point in the flow can be made arbitrarily small by taking a sufficiently small value of the parameter, but the large streamwise velocities obtained for M near one indicate a nonuniformity in the expansion. Uniform validity of these solutions with respect to Mach number is achieved only if M is bounded away from one. The nonuniformity should not be surprising, because a flow near Mach number one must contain both subsonic and supersonic velocities. The differential equation describing a purely subsonic or purely supersonic flow is, respectively, everywhere elliptic or everywhere hyperbolic in nature, and solutions to the two types of equations show entirely different dependence on the boundary conditions. For a mixed flow we would therefore expect the governing equation to be one which has the possibility of changing in type. Since the sonic line, along which the change occurs, is not fixed in advance, the equation is further expected to be nonlinear. These ideas have been used by several authors as the basis for deriving first-order differential equations and boundary conditions for transonic flow over bodies of revolution and thin wings (see, for example, Refs. 1, 2, and 3).

A more complete understanding of the transonic approximations can be achieved by regarding the approximate transonic equations as part of a systematic expansion procedure. The velocity components are assumed to possess asymptotic expansions of a very general form, and the differential equations and boundary conditions for the first and higher approximations are obtained by applying appropriate limiting procedures to the full equations. In principle it becomes possible to compute higher-order terms, or at least to estimate errors.

The procedures are found to bear close resemblance to the methods used in Refs. 4 and 5 in deriving expansions for viscous flow over bodies for low Reynolds numbers. In the viscous problem the limiting process requires that the Reynolds number approach zero; for the transonic problem some geometric parameter must go to zero and at the same time the Mach number must approach one. In both cases, terms in an "outer" expansion which satisfies the boundary condition at infinity are of a different order of magnitude near the body than far away. An additional "inner" expansion is convenient, and in some cases necessary, for calculating various flow quantities near the body surface. The inner expansion is not uniformly valid at infinity, and a matching with the outer expansion is therefore necessary. In the transonic case the approximate outer expansion is found to be the nonlinear equation predicted above, while if the configuration is slender the first-order inner equation is Laplace's equation. These equations correspond to the Oseen equations and the Stokes equations, respectively, for flow at low Reynolds numbers. The formal difference between the inner and outer expansions lies in the choice of space coordinates to be held fixed as the small parameter goes to zero. Determination of the correct forms of coordinate distortion, i.e. the proper length scales, represents part of the derivation of the first approximation.

The need of a coordinate distortion for the transonic outer expansion can be demonstrated either from the requirement of a nonlinear differential equation or from the following heuristic argument. In the case of a slightly supersonic flow one expects that the Mach cone from the foremost point on the body should appear in the solution for the first approximation. Consequently the proper length scale is given by the tangent of the Mach angle, $\frac{1}{\sqrt{|1 - M^2|}}$. Since this quantity increases toward infinity as M approaches one, it is not possible to use only the original physical coordinates. If the form of the differential equation is also considered, one can show that a completely general transonic theory is obtained only if the disturbance potential in the outer limit is of order $|1 - M^2|$. In other words the streamwise perturbation velocity should be of the same order as the difference between the free-stream flow velocity and speed of sound. This much information can be obtained without specifying anything about the body, except that the body dimensions should change, as M approaches one, in such a manner that the flow disturbances disappear in the limit.

For the inner expansion the length scale should be of the same order of magnitude as some characteristic dimension of the body. In the case of a body of revolution the relevant quantity is the thickness, while for a wing it will be shown that the span should be used.

The final step in determining the form of expansion is to relate the Mach number to the body dimensions, that is to determine the order of magnitude of $|1 - M^2|$ in terms of certain geometric parameters. Using concepts from a general theory of expansion procedures (5), a principle for matching the inner and outer expansions can be derived. For slender configurations the result obtained states essentially that the outer solution for small values of the (outer) radial coordinate should be compared with the inner solution for large values of the (inner) radial coordinate.

In the course of the derivations it is shown that the order-of-magnitude relation chosen between $|1 - M^2|$ and the body dimensions is the only one which permits discussion of all possible transonic flow patterns. The relation thus established may be expressed by the statement that some appropriate function of M and the body dimensions must remain constant in the limit. Such a function defines a similarity parameter. Certain similarity laws may be derived which allow comparison of the velocities, pressures, and forces for a family of similar flows characterized by a fixed numerical value of the similarity parameter. Similarity rules are given, for example, in Refs. 1, 2, and 3. The present procedures are believed to clarify the earlier derivations; furthermore, the order of magnitude of the error is given for each of the similarity laws stated, and the possibility of deriving higher approximations is indicated in each case.

The appropriate form of expansion is derived first for a body of revolution at zero incidence. In order to illustrate the considerations which dictate the choice of a particular transonic expansion procedure, the discussion is introduced by an investigation of the non-uniformities which appear for a subsonic expansion. It is shown that four major considerations must be taken into account in deriving a transonic expansion: 1) the Mach number must approach one as the body thickness approaches zero; 2) a shrinking of the radial coordinate is necessary in order to obtain an approximate "outer" differential equation which admits solutions capable of satisfying the boundary condition at infinity; 3) an "inner" expansion, satisfying the boundary condition at the body and matching with the outer expansion, is desirable in order to obtain a description of the flow which is uniform in a strict sense of the word; 4) the shock relations must be considered in order to show that the vorticity is of a very small order of magnitude.

Consideration of the outer limit shows that there are only three possibilities for a first-order equation if the boundary condition at infinity is to be satisfied: 1) a linear equation describing a purely subsonic or purely supersonic flow; 2) a nonlinear equation which does not contain the Mach number and which can only describe a flow for M exactly equal to one; and 3) a nonlinear equation which does include a dependence on M. The third of these possibilities is clearly the most general. The form of the desired equation is therefore known, and the mathematical reasoning turns out to verify the heuristic argument given previously. Next it is pointed out that the vorticity is negligible, at least in the first approximation, and potential functions may therefore be introduced to describe the first-order outer and inner solutions. It is shown that the leading term of the inner solution must obey Laplace's equation, and an outline of the justification for matching inner and outer solutions is given. Since the approximate inner representation satisfies a relatively simple equation, the form of the solution satisfying the boundary condition can be stated. The solution contains a term representing a line source along the body axis plus an unspecified function of the streamwise coordinate which could be written explicitly only if it were possible to solve the nonlinear outer equation. The form of solution for the second approximation is then derived in a similar manner. First-order similarity laws are given (equations 3.81 and 3.84) for the pressure at the body surface and for the drag, including in each case a statement of the order of magnitude of the error. These rules can be reduced to the results of Ref. 3, which were derived in a quite different manner. In Ref. 6 the procedures outlined above are applied to the special case of a slender cone at zero incidence in a slightly supersonic flow. The conical property of the flow is used to reduce the approximate nonlinear partial differential equation to an ordinary differential equation and the complete solution for the first approximation is obtained by numerical integration.

As a second example, the expansions are extended to include the effects of small perturbations in cross-section shape. The ratio of maximum change in radius to the original thickness ratio is required to approach zero in the limit. It is pointed out that the use of an inner expansion to supplement the outer expansion is in general essential for this problem, while for the case of a circular cross-section it is a considerable convenience rather than an absolute necessity. If the axial distribution of cross-section area is held fixed, it is shown that small changes in shape can produce only a very small change in drag, if the shape of the body at the base satisfies certain requirements (equations 4.22 and 4.23).

The expansions for a body of revolution at incidence are also derived. For this case a second small parameter, the angle of attack, must be considered. For greatest generality the angle of attack is taken to be of the same order of magnitude as the thickness ratio. Except for the presence of a second parameter, the procedures are fundamentally the same as for the case of zero incidence. It is shown that to first order the lift coefficient, referred to the base area, has the same value as predicted by the linearized theory of subsonic or supersonic flow; the order of magnitude of the error is also stated (equation 5.20).

The problem of transonic flow over a thin wing is especially interesting, and can be treated by procedures which are quite similar to those used for slender bodies. The essential new feature lies in the presence of a parameter representing the aspect ratio. It is assumed that the wing has symmetrical profile sections, and only the case of zero incidence is considered. A mathematical argument is again given to justify the heuristic arguments stated previously, and it is shown that the length scale and disturbance potential for the outer limit must be of the same order of magnitude relative to $|1 - M^2|$ as for slender bodies. Given these relations, one must distinguish between two possible cases, which differ in the behavior of the aspect ratio in the limit.

In the first case, the inner and outer expansions coincide exactly, and the wing span is distorted (as $M \to 1$) in the same manner as the radial coordinate. Therefore the wing span must vary in a prescribed manner as the thickness tends to zero and the Mach number approaches one. Specifically, the reduced aspect ratio, proportional to the product of maximum span times $\sqrt{|1 - M^2|}$, must be kept fixed in the limit. This fact is expressed by defining a second similarity parameter (see also Ref. 2). A wing subject to these restrictions will be called "nonslender".

If on the other hand it is assumed that the inner and outer expansions may be different, an essential change in the nature of the problem occurs. The restriction on the order of magnitude of the wing span is no longer a specific one; it is only required that the reduced aspect ratio vanish in the limit, and that the ratio of thickness to span tend to zero so that the wing may be considered "thin". For this case the wing is called "slender". We should note that the term "slender" for the transonic regime is less restrictive than for the subsonic or supersonic regimes because of the importance of Mach number in the definition; the actual wing span may even increase slowly toward infinity in the limit. Since the span need not depend on the small parameters in any more definite fashion, there is only a single similarity parameter, as for the case of a body of revolution. It turns out that this parameter depends only on the Mach number and the maximum cross-section area of the wing, and may be written in exactly the same form as for the body of revolution; this is also true of the corresponding parameter for nonslender wings. For the slender wing the first-order outer solution again obeys a nonlinear partial differential equation, and it can be shown that this solution possesses rotational symmetry. Perhaps the most significant difference from the previous case is that near the wing the approximate solution now obeys a relatively simple equation, the Laplace equation; that is, the leading term of the inner expansion is described by this equation. The first-order inner solution may be expressed by a source distribution, but as for slender bodies an unknown function of the streamwise coordinate must be included. Knowledge of the form of solution is, however, sufficient for determining a similarity law, of the form given in Ref. 7, for the pressure at the wing surface (equation 6.31). The order of magnitude of the second approximation will depend on the specific relation between span and thickness, but a completely general error estimate for the similarity rule can be obtained by considering all possible relations.

2. BASIC EQUATIONS

The basic differential equations are statements concerning the conservation of mass, momentum, and energy for a continuous fluid. The equation of state is taken to be that for a perfect gas, and it is assumed that the viscosity and heat conduction are zero. Since various flow quantities will change discontinuously across shock waves, it is also necessary to include equations expressing the conservation laws in the form of jump conditions for shock waves. Boundary conditions are imposed requiring that the flow be tangent to the body surface, and that the velocity approach the free-stream value far away from the body.

The continuity equation and the momentum equation are as follows:

$$\text{div}\ \rho\ \vec{q} = 0 \tag{2.1}$$

$$\vec{q} \cdot \nabla \vec{q} + \frac{1}{\rho} \nabla p = 0 \tag{2.2}$$

where ρ is the density, $\vec{q}$ the velocity vector, and p the pressure; the space coordinates used are non-dimensional, with the reference length taken to be the length of the body. The energy equation expresses the fact that stagnation enthalpy is conserved along streamlines. Assuming the perfect gas law,

$$\vec{q} \cdot \nabla \left(\frac{q^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \right) = 0 \tag{2.3}$$

where γ is the ratio of specific heats. These three differential equations, together with the shock relations, comprise the basic system of equations. An expansion procedure applied to this system would require an assumed form of expansion for the pressure and density as well as for the velocity components. For convenience, therefore, the equations are modified so that it will be necessary to assume expansions only for the velocities.

The momentum relation may be expressed in a form involving the vorticity $\vec{\omega}$:

$$\nabla \left(\frac{q^2}{2} \right) + \vec{\omega} \times \vec{q} + \frac{1}{\rho} \nabla p = 0 \tag{2.4a}$$

Taking the scalar product with $\vec{q}$,

$$\vec{q} \cdot \nabla \left(\frac{q^2}{2} \right) + \frac{1}{\rho} \vec{q} \cdot \nabla p = 0 \tag{2.4b}$$

Combining equations 2.3 and 2.4b, it is found that

$$\vec{q} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0 \tag{2.5}$$

It can be shown from the definition of the entropy s that

$$\frac{p}{\rho^\gamma} = \frac{p_\infty}{\rho_\infty^\gamma} \, e^{\frac{s-s_\infty}{c_v}} \tag{2.6}$$

where c_v is the specific heat at constant volume, and p_∞, ρ_∞, s_∞ indicate free-stream conditions. Therefore equation 2.5 is equivalent to a statement that in a flow without shock waves the entropy remains constant along a streamline.

The entropy must increase across a shock wave, but it can be shown that the stagnation enthalpy is conserved. Since the flow is uniform far upstream, the energy equation can be rewritten in an integrated form valid throughout the flow field:

$$\frac{q^2}{2} + \frac{a^2}{\gamma - 1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma - 1} = \frac{1}{2}\,\frac{\gamma + 1}{\gamma - 1}\, a^{*2} \tag{2.7}$$

where U is the free-stream velocity, $a = \sqrt{\frac{\gamma p}{\rho}}$ is the local speed of sound, and a* is the critical speed of sound. Equations 2.1, 2.4b, and 2.5 may now be combined to give another relation between $\vec{q}$ and a:

$$a^2 \operatorname{div} \vec{q} = \vec{q} \cdot \nabla \left(\frac{q^2}{2} \right) \tag{2.8}$$

Elimination of a^2 from the last two equations leads to an equation containing only the velocity components.

Taking the gradient of equation 2.7 and replacing $\nabla \left(\frac{q^2}{2} \right)$ by the value given in equation 2.4a, a vorticity law can be derived:

$$\vec{\omega} \times \vec{q} = \frac{\rho^{\gamma - 1}}{\gamma - 1} \nabla \left(\frac{p}{\rho^\gamma} \right) \tag{2.9}$$

This result is equivalent to Crocco's theorem, which relates the vorticity and the entropy gradient. Equation 2.5 may be regarded as a consequence of equation 2.9, as is seen by taking the scalar product with $\vec{q}$. Far upstream from the body, the flow is isentropic and $\vec{\omega} = 0$, while directly behind a curved shock the entropy will vary from one streamline to another. Equation 2.5 states that $\frac{p}{\rho^\gamma}$ is constant along streamlines; hence the order of magnitude of the vorticity downstream of a shock wave can be determined from equation 2.9 by finding the order of magnitude of $\nabla\left(\frac{p}{p_\infty}\frac{\rho_\infty^\gamma}{\rho^\gamma}\right)$ immediately behind the shock. The shock relations will provide the necessary expressions for the changes in pressure and density in terms of the jump in velocity. It will be shown later that $\nabla\left(\frac{p}{p_\infty}\frac{\rho_\infty^\gamma}{\rho^\gamma}\right)$ is of a very small order of magnitude compared to the body thickness, and the flow may therefore be considered irrotational in the first few approximations.

The conservation laws applied to shock waves are stated, for example, in Ref. 8, and are modified to give the shock polar equation plus expressions for the pressure, density, and wave angle in terms of the velocities. The resulting equations are written here for an axially symmetric case in which the flow upstream of the shock is uniform:

$$\frac{q_r^2}{U^2} = \left(1 - \frac{q_x}{U}\right)^2 \frac{\frac{q_x}{U} - \frac{1}{\gamma+1}\left(\gamma - 1 + \frac{2}{M^2}\right)}{\frac{2}{\gamma+1} - \frac{q_x}{U} + \frac{1}{\gamma+1}\left(\gamma - 1 + \frac{2}{M^2}\right)} \qquad (2.10)$$

$$\frac{p}{p_\infty} = 1 + \gamma M^2\left(1 - \frac{q_x}{U}\right) \qquad (2.11)$$

$$\frac{\rho_\infty}{\rho} = \frac{q_x}{U} - \frac{\frac{q_r^2}{U^2}}{1 - \frac{q_x}{U}} \qquad (2.12)$$

$$\tan\theta_w = \frac{1 - \frac{q_x}{U}}{\frac{q_r}{U}} \qquad (2.13)$$

where q_x and q_r are the axial and radial velocity components immediately behind the shock, M is the free-stream Mach number, and θ_w is the shock wave angle. The equations are easily generalized to cases where the flow ahead of the shock is not uniform, and to flows not having axial symmetry.

Equations 2.7 through 2.13 will constitute the modified basic system of equations. In the following sections, expansion procedures will be derived for various specific cases by finding suitable expansion forms for the velocity components, substituting into the preceding equations, and performing appropriate limiting procedures. The boundary conditions necessary will be discussed for each case.

3. BODY OF REVOLUTION AT ZERO INCIDENCE

General considerations

The determination of an appropriate form for a transonic expansion is based on the following general considerations:

1) The expansions are assumed asymptotic, valid as the body thickness δ approaches zero. In some sense they should also be asymptotic as the Mach number M approaches one, and it will be shown that the quantities δ and M-1 should not go to zero independently. Other geometric parameters in a particular problem may also be required to approach limits in a specified manner.

2) The differential equation for the first approximation must admit solutions which satisfy the boundary conditions at the body and at infinity. To achieve this possibility, it will become necessary to shrink the radial coordinate, in a manner to be prescribed. The resulting equation will have the non-linear properties desired for the description of a mixed subsonic-supersonic flow.

3) For slender configurations, spatial derivatives in the transverse plane become large near the body. It is convenient, although not always essential, to use an "outer" expansion which satisfies the boundary condition at infinity, and an "inner" expansion which satisfies the boundary condition at the body; a condition for matching the two expansions can be derived.

4) Since shock waves will be present, the flow can only be approximately isentropic, and it is necessary that the shock

relations be satisfied. It will be shown that the shock polar equation requires the velocity components to be of different orders of magnitude in the limit.

<u>Exact equations and boundary conditions</u>

The derivation of a transonic expansion based on these considerations will be discussed in detail for the flow over a slender body of revolution at zero incidence. Cylindrical coordinates x, r, θ are used, and the body surface is described by

$$S(x,r;\delta) = r - \delta F(x) = 0 \qquad (3.1)$$

where δ is the ratio of maximum radius to length, or one-half the thickness ratio. The function $F(x)$ will be assumed to have several continuous derivatives; however, if it is desired only that the pressure at the body be accurate to order δ^2, it will be sufficient to assume that $F'(x)$ and $F''(x)$ are continuous. Nondimensional perturbation velocities are defined by

$$\begin{aligned} \frac{q_x}{U}(x,r;\delta,M) &= 1 + u(x,r;\delta,M) \\ \frac{q_r}{U}(x,r;\delta,M) &= v(x,r;\delta,M) \end{aligned} \qquad (3.2)$$

The dependence upon γ is not mentioned explicitly, since it is assumed that γ will have the same value for all problems to be considered; also, the dependence upon the other parameters, and upon the independent variables, will not always be written out in full.

The boundary condition at the body surface is given by

$$\vec{q} \cdot \nabla S = 0 \qquad \text{at} \qquad r = \delta F(x) \tag{3.3}$$

Substituting 3.1 and 3.2,

$$v(x,\delta F) = \left\{1 + u(x,\delta F)\right\} \delta F' \tag{3.4}$$

It is also required that u and v approach zero far away from the body.

To obtain the differential equation for axisymmetric flow, equation 2.7 is first rewritten as follows:

$$\frac{a^2}{U^2} + (\gamma - 1)\frac{(1+u)^2 + v^2}{2} = \frac{1}{M^2} + \frac{\gamma-1}{2} = \frac{\gamma+1}{2}\,\frac{a^{*2}}{U^2} \tag{3.5}$$

In combining with 2.8 this gives the result

$$\begin{aligned}(1 - M^2)u_x + v_r + \frac{1}{r}v = M^2\Big\{&(\gamma + 1)uu_x + v(u_r + v_x) \\ &+ (\gamma - 1)\, u\, (v_r + \frac{1}{r}v) + v^2v_r + uv(u_r + v_x) \\ &+ \frac{\gamma - 1}{2}v^2u_x + \frac{\gamma + 1}{2}u^2u_x \\ &+ \frac{\gamma - 1}{2}(u^2 + v^2)(v_r + \frac{1}{r}v)\Big\}\end{aligned} \tag{3.6}$$

In nondimensional form the two components of the vorticity equation 2.9 are

$$(u_r - v_x)(1 + u) = -\frac{1}{\gamma(\gamma-1)M^2} \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \frac{\partial}{\partial r} \left(\frac{p}{p_\infty} \frac{\rho_\infty^\gamma}{\rho^\gamma}\right)$$

$$(u_r - v_x)v = \frac{1}{\gamma(\gamma-1)M^2} \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \frac{\partial}{\partial x} \left(\frac{p}{p_\infty} \frac{\rho_\infty^\gamma}{\rho^\gamma}\right) \tag{3.7}$$

Nonuniformities in expansions for subsonic flow

The familiar first-order transonic equation is obtained by taking a certain limit of equation 3.6. In order to clarify the reasoning which dictates the choice of particular limiting procedures, it is convenient first to consider limits which lead to the linearized and slender-body equations. For simplicity the subsonic case is treated rather than the supersonic; no shock waves are present, and the flow is isentropic to any order of magnitude.

The Prandtl-Glauert equation is often derived simply by neglecting all nonlinear terms in the full differential equation. Stated in more precise terms, the derivation may be interpreted as the first step of an expansion procedure. It is assumed that the exact solution for the perturbation potential possesses an expansion of the form

$$\Phi(x,r;\delta,M) \sim \sum_{i=1} \varepsilon_i(\delta)\varphi_i(x,r;M)$$

where (3.8)

$$\lim_{\delta \to 0} \varepsilon_1(\delta) = 0$$

$$\lim_{\delta \to 0} \frac{\varepsilon_{n+1}(\delta)}{\varepsilon_n(\delta)} = 0$$

(Questions of uniformity will be discussed in the following paragraphs.) The expansion is substituted into the differential equations and boundary conditions, and δ is allowed to approach zero while x, r, and M are held fixed. For the first approximation, one finds that

$$\varepsilon_1 = \delta^2$$

$$\beta^2 \varphi_{1_{xx}} + \varphi_{1_{rr}} + \frac{1}{r}\varphi_{1_r} = 0$$

$$\lim_{\delta \to 0} \left[r\varphi_{1_r}(x,r) \right]_{r=\delta F(x)} = F(x)F'(x) \qquad (3.9)$$

$$\varphi_1(x,r) \to 0 \qquad \text{as} \qquad x^2 + r^2 \to \infty$$

where $\beta^2 = 1 - M^2$. The solution for $\varphi_1(x,r)$ is found by distributing sources along the body axis:

$$\varphi_1(x,r) = -\frac{1}{2}\int_0^1 \frac{F(\xi)F'(\xi)\, d\xi}{\sqrt{(x-\xi)^2 + \beta^2 r^2}} \qquad (3.10)$$

Similar considerations may be used in obtaining higher-order solutions.

A useful approximation to $\varphi_1(x,r)$ is obtained by expanding equation 3.10 for small values of r:

$$\varphi_1(x,r) \sim F(x)F'(x) \log r + g(x;M) \qquad (3.11)$$

where the dependence of $g(x;M)$ upon M is contained in a term proportional to log β (see, e.g., Ref. 9). Because of this behavior, the solution, as expressed by either 3.11 or 3.10, can be made uniformly valid with respect to M only if M is bounded away from one. This means that given an integer n, a Mach number $M_0 < 1$, and a positive number ε, one may find a number δ_0 such that

$$\frac{\Phi - \sum_{i=1}^{n} \varepsilon_i \varphi_i}{\varepsilon_n} < \varepsilon \qquad (3.12)$$

if $\delta < \delta_0$ and if $0 \leq M \leq M_0 < 1$. (Uniformity with respect to the space variables will be considered later.) It is therefore impossible to obtain a transonic solution for axisymmetric flow over a slender body by first allowing δ to approach zero with x, r, and M fixed, and then taking the limit as M approaches one. Instead, limiting procedures must be considered for which $\delta \to 0$ and $M \to 1$ simultaneously. The parameter to be held constant will then be a function of both δ and M.

For the special case of flow at M = 1, a possible procedure is to take the limit of the differential equation as M approaches one, and then to let δ approach zero. Since it can be shown that the flow is nearly isentropic, the first approximation to the solution may be expressed in terms of a potential. If x and r are held fixed during the limit procedure, the resulting equation is Laplace's equation in the transverse plane. The equation does not have the desired nonlinear properties, and the general solution can not satisfy the boundary condition at infinity. A revision of the limit process described above is therefore necessary: some sort of coordinate distortion must be used so that one or more of the nonlinear terms may be of the same order of magnitude as the Laplacian. This conclusion is equivalent to a statement that the velocity components must be of different orders of magnitude in the limit.

Even for a fixed value of M different from unity, certain nonuniformities with respect to the space coordinates will occur. The subsonic approximation for $u(x,r)$ is in general nonuniform with respect to x because of singularities in $g'(x)$ at the front and rear of the body; for example, $g'(x)$ behaves logarithmically at the ends of a pointed body. Methods have been suggested for removing this sort of nonuniformity, but will not be discussed here. A nonuniformity in r also appears which is somewhat less obvious, but important in the present context. The quantity $v^2 v_r$ in equation 3.6 contributes a term to the potential which is $O(\delta^6 r^{-2})$. In order to obtain all terms which are $O(\delta^4)$ at the body surface, it is therefore necessary to carry out the expansion to $O(\delta^6)$. If a definition of uniformity analogous to equation 3.12 is used, the assumed expansion 3.8 cannot be uniformly valid all the way to the body surface.

Since the expansion 3.8 satisfies the boundary condition at infinity, it is reasonable to assume that it is uniformly valid as $\delta \to 0$ except in some finite region including the body; such an expansion will be called an "outer" expansion. It is then convenient to find an interpretation of slender-body theory which treats the slender-body solution as the beginning of an "inner" expansion valid near the body surface. Calculation, for example, of the δ^4 term in the pressure at the body can then be achieved in a more simple, yet still systematic, fashion.

Qualitatively speaking, slender-body theory is based on the idea that derivatives with respect to x are small compared to derivatives with respect to r, if r is sufficiently small. In a neighborhood of the body, the equation for $\varphi_1(x,r)$ reduces approximately to Laplace's equation, and the solution satisfying the boundary condition at the body is given by equation 3.11. The function $g(x;M)$ is not determined by slender-body considerations, but must be obtained by expanding a solution of the complete Prandtl-Glauert equation.

In order to formulate an inner expansion which is uniformly valid in a neighborhood of the body, a new radial coordinate is introduced:

$$r^* = \frac{r}{\delta} \tag{3.13}$$

Now δ is allowed to approach zero while x, r^*, and M are held fixed, and the equation of the body surface $r^* = F(x)$ is independent of δ. The differential equation for the first few approximations is Laplace's, and the initial terms are exactly the slender-body solution. Since this solution becomes infinite as $r^* \to \infty$, the region of uniform validity is bounded. It will be shown that the regions of uniformity of the inner and outer expansions can be made to overlap, and consequently that a matching procedure can be devised. Since the boundary condition at the body determines only a term proportional to $\log r^*$, the matching is essential for the determination of the complete inner solution.

<u>Expanded form of basic equations</u>

For a transonic problem, it may be expected that the use of inner and outer solutions will again be desirable. The inner expansion is written in terms of r^*, and the outer expansion in terms of another radial coordinate $\tilde{r}$, where

$$\tilde{r} = r\ f(\delta) \tag{3.14}$$

and $f(\delta)$ is to be determined. Since it is necessary to consider limiting procedures for which $\delta \to 0$ and $M \to 1$ simultaneously, the parameter M is replaced by a parameter K, which is a function of both M and δ; the definition of K depends upon the limiting process.

The following expansions are assumed:

$$\begin{aligned}
u(x,r;\delta,M) &\sim \sum_{i=1} \varepsilon_i(\delta) u_i(x,\tilde{r};K) \\
&\sim \sum_{i=1} \lambda_i(\delta) u_i^*(x,r^*;K) \\
v(x,r;\delta,M) &\sim \sum_{i=1} \nu_i(\delta) v_i(x,\tilde{r};K) \\
&\sim \sum_{i=1} \kappa_i(\delta) v_i^*(x,r^*;K)
\end{aligned} \tag{3.15}$$

Since these representations are assumed asymptotic,

$$\lim_{\delta \to 0} \varepsilon_1(\delta) = 0$$

$$\lim_{\delta \to 0} \frac{\varepsilon_{n+1}(\delta)}{\varepsilon_n(\delta)} = 0$$

$$u_1(x,\tilde{r};K) = \lim_{\substack{\delta \to 0 \\ x,\tilde{r},K \text{ fixed}}} \frac{u(x,r;\delta,M)}{\varepsilon_1(\delta)} \tag{3.16}$$

$$u_n(x,\tilde{r};K) = \lim_{\substack{\delta \to 0 \\ x,\tilde{r},K \text{ fixed}}} \frac{u(x,r;\delta,M) - \sum_{i=1}^{n-1} \varepsilon_i(\delta) u_i(x,\tilde{r};K)}{\varepsilon_n(\delta)}$$

Similar statements hold for the other three expansions.

Substituting the outer expansions into the differential equation 3.6,

$$(1-M^2)(\varepsilon_1 u_{1_x} + \varepsilon_2 u_{2_x} + \varepsilon_3 u_{3_x} + \ldots) + f(\nu_1 v_{1_{\tilde{r}}} + \nu_2 v_{2_{\tilde{r}}} + \nu_3 v_{3_{\tilde{r}}} + \ldots)$$
$$+ \frac{f}{\tilde{r}} (\nu_1 v_1 + \nu_2 v_2 + \nu_3 v_3 + \ldots)$$
$$= M^2 \Big\{ (\gamma + 1)(\varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3 + \ldots)(\varepsilon_1 u_{1_x} + \varepsilon_2 u_{2_x} + \varepsilon_3 u_{3_x} + \ldots)$$
$$+ (\nu_1 v_1 + \ldots)(f\varepsilon_1 u_{1_{\tilde{r}}} + \ldots + \nu_1 v_{1_x} + \ldots)$$
$$+ (\gamma - 1)(\varepsilon_1 u_1 + \ldots) f(\nu_1 v_{1_{\tilde{r}}} + \ldots + \frac{1}{\tilde{r}} \nu_1 v_1 + \ldots)$$
$$+ (\nu_1 v_1 + \ldots)^2 f(\nu_1 v_{1_{\tilde{r}}} \ldots) + (\varepsilon_1 u_1 + \ldots)(\nu_1 v_1 + \ldots)(f\varepsilon_1 u_{1_{\tilde{r}}} \ldots + \nu_1 v_{1_x} + \ldots)$$
$$+ \frac{\gamma - 1}{2} (\nu_1 v_1 + \ldots)^2 (\varepsilon_1 u_{1_x} + \ldots) + \frac{\gamma + 1}{2} (\varepsilon_1 u_1 + \ldots)^2 (\varepsilon_1 u_{1_x} + \ldots)$$
$$+ \frac{\gamma - 1}{2} (\varepsilon_1^2 u_1^2 + \ldots + \nu_1^2 v_1^2 + \ldots) f(\nu_1 v_{1_{\tilde{r}}} + \ldots + \frac{1}{\tilde{r}} \nu_1 v_1 + \ldots) \Big\} \tag{3.17}$$

In terms of the inner solution, the equation becomes

$$(1 - M^2)(\lambda_1 u_{1_x}^* + \lambda_2 u_{2_x}^* + \lambda_3 u_{3_x}^* + \dots) + \frac{1}{\delta}(\kappa_1 v_{1_{r*}}^* + \kappa_2 v_{2_{r*}}^* + \kappa_3 v_{3_{r*}}^* + \dots)$$
$$+ \frac{1}{\delta r^*}(\kappa_1 v_1^* + \kappa_2 v_2^* + \kappa_3 v_3^* + \dots)$$
$$= M^2\Big\{(\gamma + 1)(\lambda_1 u_1^* + \dots)(\lambda_1 u_{1_x}^* + \dots)$$
$$+ (\kappa_1 v_1^* + \dots)(\frac{1}{\delta}\lambda_1 u_{1_{r*}}^* + \dots + \kappa_1 v_{1_x}^* + \dots)$$
$$+ (\gamma - 1)(\lambda_1 u_1^* + \dots)\frac{1}{\delta}(\kappa_1 v_{1_{r*}}^* + \dots + \frac{1}{r^*}\kappa_1 v_1^* + \dots)$$
$$+ (\kappa_1 v_1^* + \dots)^2 \frac{1}{\delta}(\kappa_1 v_{1_{r*}}^* + \dots)$$
$$+ (\lambda_1 u_1^* + \dots)(\kappa_1 v_1^* + \dots)(\frac{1}{\delta}\lambda_1 u_{1_{r*}}^* + \dots + \kappa_1 v_{1_x}^* + \dots)$$
$$+ \frac{\gamma - 1}{2}(\kappa_1 v_1^* + \dots)^2(\lambda_1 u_{1_x}^* + \dots) + \frac{\gamma + 1}{2}(\lambda_1 u_1^* + \dots)^2(\lambda_1 u_{1_x}^* + \dots)$$
$$+ \frac{\gamma - 1}{2}(\lambda^2 u_1^{*2} + \dots + \kappa_1^2 v_1^{*2} + \dots)\frac{1}{\delta}(\kappa_1 v_{1_{r*}}^* + \dots + \frac{1}{r^*}\kappa_1 v_1^* + \dots)\Big\} \quad (3.18)$$

The boundary conditions are

$$u_n(x,\tilde{r};K)\ ,\ v_n(x,\tilde{r};K) \to 0 \quad \text{as} \quad \tilde{r}^2 + x^2 \to \infty \quad (3.19)$$

and

$$\kappa_1 v_1^*(x,F) + \kappa_2 v_2^*(x,F) + \kappa_3 v_3^*(x,F) + \dots$$
$$= \Big\{1 + \lambda_1 u_1^*(x,F) + \lambda_2 u_2^*(x,F) + \dots\Big\}\delta F' \quad (3.20)$$

The outer expansions are also substituted into the shock relations.

Retaining only terms which might be necessary for a first approximation,

$$(v_1 v_1 + \dots)^2$$
$$= (\varepsilon_1 u_1 + \dots)^2 \frac{(1 + \varepsilon_1 u_1 + \dots) - \left\{1 + \frac{2}{\gamma+1}(1 - M^2) + \dots\right\}}{\frac{2}{\gamma+1} - (1 + \varepsilon_1 u_1 + \dots) + \left\{1 + \frac{2}{\gamma+1}(1 - M^2) + \dots\right\}} \qquad (3.21)$$

$$\frac{p}{p_\infty} = 1 - \gamma M^2(\varepsilon_1 u_1 + \dots) \qquad (3.22)$$

$$\frac{\rho_\infty}{\rho} = 1 + \varepsilon_1 u_1 + \dots + \frac{v_1{}^2 v_1{}^2 + \dots}{\varepsilon_1 u_1 + \dots} \qquad (3.23)$$

$$\tan \theta_w = - \frac{\varepsilon_1 u_1 + \dots}{v_1 v_1 + \dots} \qquad (3.24)$$

These expansions describe shock waves in a uniform stream; similar equations could be derived for the more general case. The beginning of the vorticity expansion may be obtained from equation 3.7:

$$(f\varepsilon_1 u_{1_{\tilde{r}}} - v_1 v_{1_x} + \dots)(1 + \varepsilon_1 u_1 + \dots)$$
$$= - \frac{1}{\gamma(\gamma - 1)M^2}(1 + \dots) f \frac{\partial}{\partial \tilde{r}} \left(\frac{p}{p_\infty} \frac{\rho_\infty^\gamma}{\rho^\gamma} \right) \qquad (3.25)$$

$$(f\varepsilon_1 u_{1_{\tilde{r}}} - v_1 v_{1_x} + \dots)(v_1 v_1 + \dots) = \frac{1}{\gamma(\gamma - 1)M^2}(1 + \dots) \frac{\partial}{\partial x} \left(\frac{p}{p_\infty} \frac{\rho_\infty^\gamma}{\rho^\gamma} \right)$$

First-order equations for outer expansion

The preceding equations allow a number of possible order-of-magnitude relations among the various functions of δ. These relations can not be expressed uniquely as equalities — for example, any assumed result could be altered by an arbitrary multiplicative constant. However, for simplicity the order relations will be written as equalities or inequalities.

Equation 3.21 leads to one of three possible order relations:

$$\nu_1 = \varepsilon_1\sqrt{|1 - M^2|} > \varepsilon_1^{3/2} \tag{3.26a}$$

$$\nu_1 = \varepsilon_1^{3/2} > \varepsilon_1\sqrt{|1 - M^2|} \tag{3.26b}$$

$$\nu_1 = \varepsilon_1^{3/2} = \varepsilon_1\sqrt{|1 - M^2|} \tag{3.26c}$$

With the use of 3.26, the largest terms in equation 3.17 are seen to give

$$(1 - M^2)\varepsilon_1 u_{1_x} + \nu_1 f(v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1) = (\gamma + 1)\varepsilon_1^2 u_1 u_{1_x} + \cdots \tag{3.27}$$

In any limiting process for which the functions $\varepsilon_i(\delta)$ and $\nu_i(\delta)$ have the properties prescribed in 3.16, and for which M is bounded, the neglected terms will disappear in the first approximation. Since equation 3.27 is to describe an outer solution, the approximate equations must admit solutions which are capable of satisfying the boundary condition at infinity. Again three possibilities arise:

$$(1 - M^2)\varepsilon_1 u_{1_x} + v_1 f(v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1) = 0 \tag{3.28a}$$

$$v_1 f(v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1) = (\gamma + 1)\varepsilon_1{}^2 u_1 u_{1_x} \tag{3.28b}$$

$$(1 - M^2)\varepsilon_1 u_{1_x} + v_1 f(v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1) = (\gamma + 1)\varepsilon_1{}^2 u_1 u_{1_x} \tag{3.28c}$$

Equations 3.26a, b, c and 3.28a, b, c correspond to limiting processes for which, respectively,

$$\lim_{\delta \to 0} \frac{1 - M^2}{\varepsilon_1} = \infty \tag{3.29a}$$

$$\lim_{\delta \to 0} \frac{1 - M^2}{\varepsilon_1} = 0 \tag{3.29b}$$

$$\lim_{\delta \to 0} \frac{1 - M^2}{\varepsilon_1} = K = \text{const.} \tag{3.29c}$$

Consider first the limit 3.29a. If M is held fixed as $\delta \to 0$, the shock polar relation gives $v_1 = \varepsilon_1$ for supersonic flow, and the condition of irrotationality gives the same result for subsonic flow. Then $f(\delta) = 1$, i.e. $\tilde{r} = r$, and the usual linearized equation is obtained. We have seen that this approach cannot lead to a transonic approximation. Now suppose that $M \to 1$ sufficiently slowly that $|1 - M^2| \gg \varepsilon_1$. The governing equation is 3.28a, with $v_1 = \varepsilon_1 \sqrt{|1 - M^2|}$ and $f = \sqrt{|1 - M^2|}$. Since $u(x,r) = O(\varepsilon_1) = o(1 - M^2)$, the flow will be entirely subsonic or entirely supersonic, if δ is sufficiently small. The solution cannot be a uniform approximation to any mixed flow, because there is an error $O(1 - M^2)$, which by assumption is of larger order than ε_1.

If instead we first allow M to approach one, and then take the limit as $\delta \to 0$, the shock polar and the differential equation give $v_1 = \varepsilon_1^{3/2}$ and $f = \varepsilon_1^{1/2}$. This procedure gives the first term of a double expansion in δ and $(1 - M^2)$. Essentially the same result is obtained by considering the more general limit 3.29b and allowing M to approach one sufficiently rapidly that $|1 - M^2| \ll \varepsilon_1$. Again the first approximation 3.28b describes a flow at M = 1, and it can be argued that the result is not a uniform representation for flow at any other Mach number (see immediately below).

The expected qualitative features of the relevant mixed-flow patterns are described, for example, in Ref. 10. A flow at M = 1 is seen to have certain properties which are different from flows at any other value of M. Approaching from upstream infinity, the flow decelerates, and then accelerates to supersonic velocities as it passes over the body. The sonic line begins at some point on the surface, and extends to infinity. A shock may appear at the rear of the body, or else compression waves may coalesce to form a shock at some distance away; immediately behind the shock the velocity is still supersonic. If now the Mach number is slightly less than one, the sonic line does not extend to infinity. The shock at the rear remains, and the supersonic region is terminated by a shock further downstream; both shocks end at a finite distance from the axis. For somewhat smaller values of M the supersonic region is terminated by a shock originating at some point on the body. If M is slightly greater than one, there is a detached shock in front, followed by a finite subsonic region; the sonic line originates on the body, and again there is a shock at the rear of the body. It can therefore be seen that for Mach numbers very close to one, the flow near the body may approximate the flow at $M = 1$, but the conditions far away must be different. Near the limits of the transonic regime the flow patterns are different everywhere.

The limit 3.29c is therefore the only one which has the possibility of yielding uniform approximations to the various types of transonic flows. Hence we should choose the order relation 3.26c and the differential equation 3.28c. It is expected that as $\delta \to 0$ with K fixed the essential nature of the flow pattern will be preserved, although the shape of shocks and streamlines in the physical plane will be distorted in a manner which can be defined. Thus a particular value of K characterizes a family of similar flows, and K may be called a similarity parameter. It is important to notice that a physically reasonable differential equation is obtained if $\delta \to 0$ with K fixed, and K subsequently approaches zero. That is, the expansion is expected to be uniformly valid in some range of values of K which includes zero, and flow at $M = 1$ is simply the special case obtained by setting $K = 0$.

From the expressions for pressure and density behind a shock,

$$\frac{p}{\rho^{\gamma}} = \frac{p_{\infty}}{\rho_{\infty}^{\gamma}} \left\{ 1 + o(\varepsilon_1) \right\} \tag{3.30}$$

Since this quantity is conserved along streamlines, the relation holds everywhere downstream of a shock wave, and the vorticity law 3.25 shows that the flow is irrotational in the first approximation. The same type of argument is valid for a flow with more than one shock.

We have therefore established

$$Ku_{1_x} + v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1 = (\gamma + 1) u_1 u_{1_x}$$

$$u_{1_{\tilde{r}}} - v_{1_x} = 0$$

$$\tilde{r} = \varepsilon_1^{\ 1/2}\, r \tag{3.31}$$

$$K = \frac{1 - M^2}{\varepsilon_1}$$

$$\nu = \varepsilon_1^{\ 3/2}$$

Matching with inner expansion

In order to find $\varepsilon_1(\delta)$, a matching with the inner solution will be derived. From the boundary condition 3.20 and the differential equation 3.18,

$$\kappa_1 = \delta$$

$$v_1{}^*(x,F) = F' \tag{3.32}$$

$$v_{1_{r*}}^* + \frac{1}{r*} v_1{}^* = 0$$

The solution for $v_1{}^*$ which satisfies the boundary condition is

$$v_1{}^*(x,r^*;K) = \frac{FF'}{r^*} \tag{3.33}$$

The argument that irrotationality is preserved in the first approximation as δ approaches zero with r^* fixed is more complicated than before. In order to apply the shock relations, the limit must be performed in such a manner that the point of observation remains at the shock. It is therefore necessary to let x vary slightly. Consider the case of an attached shock at the nose of a pointed body. The expansion 3.24 for the shock wave angle shows that at the shock

$$r^* = \frac{\tilde{r}}{\varepsilon_1^{1/2}\,\delta} = \frac{R_1(x)}{\varepsilon_1^{1/2}\,\delta} + \ldots \tag{3.34}$$

In order to stay at the shock and keep x fixed, we must also keep $\tilde{r}$ fixed. Assuming $R_1(x)$ to be linear near $x = 0$, we may also stay at the shock by maintaining r^* constant and allowing x to approach zero as $\varepsilon_1^{1/2}\,\delta$. That is, a new coordinate $x^* = x + O(\varepsilon_1^{1/2}\,\delta)$ should be used. Since $F(x)$ is also linear near $x = 0$, $v = O(\varepsilon_1^{1/2}\delta^2)$ as $\delta \to 0$ with r^* and x^* fixed. From the shock polar, it is seen that $v \ll u$, and the remaining part of the argument for irrotationality proceeds as before.

It follows that

$$u(x,r;\delta,M) \sim A(x;\delta,K) + \delta^2(F'^2 + FF'') \log r^* \tag{3.35}$$

If the coordinate x is retained, the expressions for u and v may be nonuniform at the ends of the body and at shock waves. It will be assumed that the nonuniformities could be removed by suitable procedures, and the approximations will be considered uniformly valid for

$$F(x) \leq r^* \leq r_0^* < \infty \qquad (3.36)$$

where r_0^* is independent of δ.

In this particular case, the approximate outer equation contains all the terms of the corresponding inner equation, and it is reasonable to expect that the first-order representation of the outer solution will be uniformly valid everywhere. This conclusion can be proved with the use of some general concepts relating to expansion procedures. The regions of uniform validity of the inner and outer representation will be shown to overlap, and a matching of the approximate solutions is therefore possible. In Ref. 5 these ideas are applied to problems of viscous flow over bodies at low Reynolds numbers. There the first-order equation for the outer expansion is the Oseen equation, while the first term of the inner solution is described by the Stokes equation. An outline of the matching procedure is given here in order to illustrate another application of the general theory. The complete proofs will not be given, and the ideas will be discussed in detail only for the relatively simple first-order axisymmetric flow.

Consider a class of functions $\psi(\delta)$ such that

$$\lim_{\delta \to 0} \psi(\delta) = 0$$
$$\lim_{\delta \to 0} \frac{\varepsilon_1^{1/2}\,\delta}{\psi(\delta)} < \infty \qquad (3.37)$$

That is, $\psi \to 0$ as $\delta \to 0$, but ψ should not approach zero any faster than $\varepsilon_1^{1/2}\,\delta$. A new radial coordinate $r^{(\psi)}$ is defined by

$$r^{(\psi)} = \frac{\tilde{r}}{\psi} = \frac{\varepsilon_1^{1/2}\delta}{\psi} r^* \qquad (3.38)$$

For the special case $\psi = \varepsilon_1^{1/2}\delta$, $r^{(\psi)}$ will be equal to r^*. The limiting process associated with $r^{(\psi)}$ requires that $\delta \to 0$ with x, $r^{(\psi)}$, and K fixed; this limit will be denoted by $\lim_\psi$.

The first term in a representation of $v(x,r;\delta,M)$ obtained by applying $\lim_\psi$ will be expressed by

$$v(x,r;\delta,M) \sim \pi_1(\delta)\; v_1^{(\psi)}(x,r^{(\psi)};K) \qquad (3.39)$$

Taking the limit of the full differential equation 3.6 and boundary condition 3.4,

$$\begin{aligned} & v_{1_{r^{(\psi)}}}^{(\psi)} + \frac{1}{r^{(\psi)}} v_1^{(\psi)} = 0 \\ & \pi_1 v_1^{(\psi)} \left(x, \frac{\varepsilon_1^{1/2}\delta}{\psi} F\right) = \delta F' \end{aligned} \qquad (3.40)$$

The approximate differential equation thus has the same form for the entire class of functions $\psi(\delta)$ as for the special case $r^{(\psi)} = r^*$. The solution to 3.40 gives

$$v(x,r;\delta,M) \sim \frac{\varepsilon_1^{1/2}\delta^2}{\psi} \frac{FF'}{r^{(\psi)}} \qquad (3.41)$$

Comparison with the results for κ_1 and v_1^* shows that the result 3.41 is identical to the first term of the inner expansion; hence we have extended the region of uniform validity of the inner solution .

Since the same representation is obtained for any choice of $\psi(\delta)$, subject to the restrictions 3.37, uniform validity has been achieved for

$$F(x) \leq r^* \leq r_0^* \frac{\psi}{\varepsilon_1^{1/2}\delta}$$

or (3.42)

$$\varepsilon_1^{1/2}\delta F(x) \leq \tilde{r} \leq r_0^* \psi$$

where ψ may approach zero arbitrarily slowly, and r_0^* is independent of δ.

The outer solution for $v(x,r)$ is approximated by

$$v(x,r;\delta,M) \sim \varepsilon_1^{3/2} v_1(x,\tilde{r};K) \tag{3.43}$$

which is expected to be uniformly valid for $\tilde{r}_0 \leq \tilde{r} < \infty$, where $\tilde{r}_0$ is independent of δ. Since the representation is assumed asymptotic, the error, measured by $\varepsilon_1^{-3/2}(v - \varepsilon_1^{3/2} v_1)$, must approach zero if $\delta \to 0$ with x, $\tilde{r}$, and K fixed. The function expressing the error may have a singularity at $\tilde{r} = 0$, but it is always possible to let $\tilde{r} \to 0$ sufficiently slowly as $\delta \to 0$ that the dependence upon δ still dominates. The domain of uniformity may therefore be extended, and there exist functions $\psi_0(\delta)$ which belong to the set of functions $\psi(\delta)$ and which tend to zero sufficiently slowly that the error indicated tends to zero if $\tilde{r}_0\psi_0(\delta) \leq \tilde{r} < \infty$. According to equation 3.42, the inner solution is valid for $\varepsilon_1^{1/2}\delta F(x) \leq \tilde{r} \leq r_0^* \psi_0(\delta)$, where we may choose $r_0^* > \tilde{r}_0$. Hence there is an overlap domain $\tilde{r}_0 \psi_0 (\delta) \leq \tilde{r} \leq r_0{}^* \psi_0 (\delta)$ where the inner and outer solutions are both valid. The two representations may therefore be matched in this domain.

To find the form of v_1 in the overlap domain, we note that $\lim_\psi$ applied to the equation for v_1 yields the same equation as the inner equation. Hence in the overlap domain

$$v_1(x,\tilde{r};K) \sim \frac{S_1(x)}{\tilde{r}} = \frac{S_1(x)}{\psi\, r^{(\psi)}} \tag{3.44}$$

and

$$v(x,r;\delta,M) \sim \frac{\varepsilon_1{}^{3/2}}{\psi} \frac{S_1(x)}{r^{(\psi)}} \tag{3.45}$$

The fact that the representations 3.41 and 3.45 agree in this domain may be expressed by the following limit:

$$\lim_{\psi} \frac{\dfrac{\varepsilon_1{}^{3/2}}{\psi} \dfrac{S_1(x)}{r^{(\psi)}} - \dfrac{\varepsilon_1{}^{1/2}\,\delta^2}{\psi} \dfrac{FF'}{r^{(\psi)}}}{\dfrac{\varepsilon_1{}^{3/2}}{\psi}} = 0 \tag{3.46}$$

Therefore

$$\begin{aligned} \varepsilon_1 &= \delta^2 \\ v_1(x,\tilde{r};K) &= \frac{FF'}{\tilde{r}} + \ldots \end{aligned} \tag{3.47}$$

The definitions for $\tilde{r}$, K, and v_1 are obtained by substituting the value of ε_1 into equation 3.31.

With the use of equation 3.35, a matching condition analogous to 3.46 is obtained for u_1:

$$\lim_{\psi} \frac{\varepsilon_1 u_1(x,\tilde{r};K) - A(x;\delta,K) - \delta^2(F'^2 + FF'')\log\left(\frac{\psi}{\varepsilon_1^{1/2}\,\delta}\, r^{(\psi)}\right)}{\varepsilon_1} = 0 \quad (3.48)$$

Therefore

$$\begin{aligned} A(x;\delta,K) &= 2\,\delta^2 \log \delta\,(F'^2 + FF'') + \delta^2 g_1'(x;K) \\ u_1(x,\tilde{r};K) &= (F'^2 + FF'')\log \tilde{r} + g_1'(x;K) + \dots \end{aligned} \quad (3.49)$$

where $g_1'(x;K)$ is undetermined, as in the slender-body solution 3.11. Substituting $A(x;\delta,K)$ into the inner representation 3.35 for $u(x,r)$, it is seen that

$$\begin{aligned} \lambda_1 &= \delta^2 \log \delta \\ \lambda_2 &= \delta^2 \\ u_1^*(x,r^*;K) &= 2(F'^2 + FF'') \\ u_2^*(x,r^*;K) &= (F'^2 + FF'')\log r^* + g_1'(x;K) \end{aligned} \quad (3.50)$$

Summary of first-order results

Since it has been shown that the vorticity is of smaller order than δ^2, the exact solution to the full equations may be represented approximately by means of a potential $\Phi(x,r;\delta,M)$. The error thus incurred can be determined, in principle, by continuing the expansion procedure in terms of the velocities until the shock relations show a change in entropy. In order to condense the results, it will be assumed that this potential function possesses inner and outer asymptotic expansions, with the understanding that the expansions must be terminated before the vorticity becomes an essential part of the problem.

The assumed expansions for Φ are

$$\Phi(x,r;\delta,M) \sim \sum_{i=1}^{n} \varepsilon_i(\delta)\varphi_i(x,\tilde{r};K)$$

$$\sim \sum_{i=1}^{n} \lambda_i(\delta)\ \chi_i(x,r^*;K)$$

$$\tilde{r} = \delta\ r \tag{3.51}$$

$$r^* = \frac{r}{\delta}$$

$$K = \frac{1 - M^2}{\delta^2}$$

It has been shown that a family of similar mixed flows can be obtained only if the Mach number is allowed to approach one in such a manner that K remains constant; this quantity is essentially the same as the similarity parameter introduced by von Kármán in Ref. 1. Formulas can be derived which relate the velocities at corresponding points in two similar flows, i.e. at points where x and $\tilde{r}$ have the same values.

The first approximation to the outer solution is given by

$$\varepsilon_1 = \delta^2$$

$$K\varphi_{1_{xx}} + \varphi_{1_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{1_{\tilde{r}}} = (\gamma + 1)\varphi_{1_x}\varphi_{1_{xx}}$$

$$\begin{aligned}\varphi_1(x,\tilde{r};K) &= S_1(x)\ \log\tilde{r} + g_1(x;K) + \tilde{r}^2\ \log^2\tilde{r}\ \left(\frac{\gamma+1}{4}\ S_1'S_1''\right)\\ &+ \tilde{r}^2\ \log\tilde{r}[(\gamma+1)(S_1'g_1'' + S_1''g_1' - 2S_1'S_1'') - KS_1''] \\ &+ \frac{1}{4}\tilde{r}^2[(\gamma+1)(\tfrac{3}{2}S_1'S_1'' - S_1'g_1'' - S_1''g_1' + g_1'g_1'') \\ &+ K(S_1'' - g_1'')] + O(\tilde{r}^4\ \log^3\tilde{r})\end{aligned} \tag{3.52}$$

$$S_1(x) = FF'$$

The additional terms in φ_1 are obtained from the differential equation by an iteration procedure, as suggested in the derivation of equation 3.44.

It is also of interest to determine the behavior of the solution at the body surface. As $\delta \to 0$, the body shrinks, and a point on the surface obviously does not follow the rule $\tilde{r}$ = constant. The inner expansion provides the formulas needed to relate the velocities in two similar flows for a given point on the body; a similarity law for the pressure at the body is derived at the end of this section. The initial terms of the inner expansion are

$$\lambda_1 = \delta^2 \log \delta$$

$$\lambda_2 = \delta^2$$

$$X_{1_{r^*r^*}} + \frac{1}{r^*} X_{1_{r^*}} = 0 \tag{3.53}$$

$$X_{2_{r^*r^*}} + \frac{1}{r^*} X_{2_{r^*}} = 0$$

$$X_1(x, r^*; K) = 2\, FF'$$

$$X_2(x, r^*; K) = FF' \log r^* + g_1(x; K)$$

Since the quantity log δ is not very large for physically reasonable values of the thickness ratio, the terms of order $\delta^2 \log \delta$ and δ^2 may be of very nearly the same size, and will be considered together as a "first-order" solution.

The results given in equation 3.53 are similar to the results of slender-body theory, in that the dominant terms represent the potential due to a source line of strength proportional to the local rate of change of body cross-section area. The term X_1 is, however, twice as large as the corresponding term in subsonic or supersonic flow. The essential transonic part of the problem is the nonlinear interaction between sources, represented by the function $g_1(x;K)$, which can be determined only by solving the nonlinear equation for φ_1 .

Derivation of higher approximations

In principle the methods outlined may be used to obtain any desired order of accuracy. The procedure will be continued here to give two more terms in each expansion.

After subtracting the first-order terms, the differential equation 3.17 for the outer expansion becomes

$$
\begin{aligned}
K\delta^2(\varepsilon_2 u_{2_x} + \varepsilon_3 u_{3_x} + \dots) + \delta(v_2 v_{2_{\tilde{r}}} + v_3 v_{3_{\tilde{r}}} + \dots) + \frac{\delta}{\tilde{r}}(v_2 v_2 + v_3 v_3 + \dots) \\
= (\gamma + 1)[\delta^2 \varepsilon_2 (u_1 u_{2_x} + u_2 u_{1_x}) + \delta^2 \varepsilon_3 (u_1 u_{3_x} + u_3 u_{1_x}) + \dots] \\
-(\gamma + 1)K\delta^6 u_1 u_{1_x} + \delta^6 v_1 (u_{1_{\tilde{r}}} + v_{1_x}) + (\gamma - 1)\delta^6 u_1 (v_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} v_1) \\
+ \frac{\gamma + 1}{2} \delta^6 u_1^2 u_{1_x} + \dots
\end{aligned}
\qquad (3.54)
$$

In order to satisfy the boundary condition that u_i and v_i approach zero at infinity, the homogeneous part of each approximate equation, starting with the second, must be the same. Specifically, the equations for all higher approximations are linear, with variable coefficients, and may have known forcing terms on the right-hand side:

$$
K u_{i_x} + v_{i_{\tilde{r}}} + \frac{1}{\tilde{r}} v_i - (\gamma + 1)(u_1 u_{i_x} + u_i u_{1_x}) = \dots \qquad (3.55)
$$

Therefore

$$
v_i = \varepsilon_i \delta \qquad (3.56)
$$

for every value of i.

Next the inner solution is considered. From the boundary condition 3.20 it follows that

$$\kappa_2 = \delta^3 \log \delta$$
$$\kappa_3 = \delta^3 \tag{3.57}$$

and

$$v_2^*(x,F) = 2S_1'F'$$
$$v_3^*(x,F) = S_1'F' \log F + g_1'F' \tag{3.58}$$

The differential equation 3.18 gives

$$v_{2_{r*}}^* + \frac{1}{r*} v_2^* = 0$$

$$v_{3_{r*}}^* + \frac{1}{r*} v_3^* = v_1^*(u_{2_{r*}}^* + v_{1_x}^*) + v_1^{*2} v_{1_{r*}}^* \tag{3.59}$$

$$= 2 \frac{S_1 S_1'}{r*^2} - \frac{S_1^3}{r*^4}$$

Solutions to these equations which satisfy the boundary conditions are

$$v_2^* = \frac{2S_1S_1'}{r^*}$$

$$v_3^* = \frac{1}{r^*}(-S_1S_1'\log F + g_1'S_1 - \frac{1}{2}S_1F'^2) + 2S_1S_1'\frac{\log r^*}{r^*} + \frac{1}{2}\frac{S_1^3}{r^{*3}} \tag{3.60}$$

It is again necessary to relate the terms of the inner and outer expansions by means of a matching procedure. The justification of the matching is fundamentally the same as for the first approximation, and the existence of overlapping regions of uniform validity can be demonstrated. It can then be shown that, for the outer expansion,

$$v_2 = \varepsilon_2\delta = \delta^5 \log \delta$$

$$v_3 = \varepsilon_3\delta = \delta^5$$

$$Ku_{2_x} + v_{2_{\tilde{r}}} + \frac{1}{\tilde{r}}v_2 - (\gamma+1)(u_1u_{2_x} + u_2u_{1_x}) = 0 \tag{3.61}$$

$$Ku_{3_x} + v_{3_{\tilde{r}}} + \frac{1}{\tilde{r}}v_3 - (\gamma+1)(u_1u_{3_x} + u_3u_{1_x})$$
$$= v_1(u_{1_{\tilde{r}}} + v_{1_x}) - 2\gamma Ku_1u_{1_x} + \frac{(2\gamma-1)(\gamma+1)}{2}u_1^2u_{1_x}$$

The last equation has been simplified by using the equation for the first approximation. As in the derivation of equation 3.44, expansions of v_2 and v_3 for small $\tilde{r}$ are obtained by an iteration procedure. Retaining only the largest terms in equation 3.61 and solving the resulting equation, the initial terms are found to be

$$v_2 = \frac{S_2(x)}{\tilde{r}} + \ldots$$

$$v_3 = 2S_1(x)S_1'(x)\,\frac{\log \tilde{r}}{\tilde{r}} + \frac{S_3(x)}{\tilde{r}} + \ldots \qquad (3.62)$$

Matching with the inner solution then gives

$$S_2 = -\,2\,S_1S_1'$$

$$S_3 = -\,S_1S_1'\,\log F + g_1'S_1 - \frac{1}{2}\,S_1F'^2$$

It is seen that S_3 depends on the parameter K.

The additional relations needed are found by using the shock conditions and the vorticity equation. While the argument will again be given only for an attached shock at the nose of the body, the methods could also be used for the other possible shock configurations.

According to equation 3.34, the first approximation to the shock shape is $\tilde{r} \sim R_1(x)$. Since the position of the shock cannot be fixed in advance, the determination of $R_1(x)$ is part of the solution for the first approximation. To obtain higher-order terms in the shock relations, the velocities will be expanded in series about $\tilde{r} = R_1(x)$. The largest terms in the shock polar equation 3.21 give a relation between u_1 and v_1 immediately behind the shock:

$$[v_1(x,R_1)]^2 = [u_1(x,R_1)]^2[\,\frac{\gamma+1}{2}\,u_1(x,R_1) - K] \qquad (3.63)$$

If the error $\tilde{r} - R_1(x)$ is assumed to be larger than $\delta^2 \log \delta$ in order of magnitude, the shock polar leads to an additional relation between u_1 and v_1, which is not justifiable. The expression 3.24 for the shock wave angle can therefore be rewritten with an estimate of the error:

$$\tan \theta_w = -\frac{1}{\delta} \frac{u_1(x,R_1)}{v_1(x,R_1)} + O(\delta \log \delta) \tag{3.64}$$

It follows that the second term in the asymptotic representation of the shock shape must be of order $\delta^2 \log \delta$; similarly, the third term is $O(\delta^2)$. The shock position is then described by

$$\tilde{r} = R_1(x) + \delta^2 \log \delta \, R_2(x) + \delta^2 R_3(x) + \dots \tag{3.65}$$

and the velocities immediately behind the shock may be expressed by

$$\begin{aligned} u &= \delta^2 u_1(x,R_1) + \delta^4 \log \delta \, [u_2(x,R_1) + R_2 u_{1\tilde{r}}(x,R_1)] \\ &\quad + \delta^4 [u_3(x,R_1) + R_3 u_{1\tilde{r}}(x,R_1)] + \dots \\ v &= \delta^3 v_1(x,R_1) + \delta^5 \log \delta [v_2(x,R_1) + R_2 v_{1\tilde{r}}(x,R_1)] \\ &\quad + \delta^5 [v_3(x,R_1) + R_3 v_{1\tilde{r}}(x,R_1)] + \dots \end{aligned} \tag{3.66}$$

Rewriting the shock polar equation 3.21 to include the largest second-order terms,

$$[\delta^3 v_1 + \delta^5 \log \delta (v_2 + R_2 v_{1_{\tilde{r}}}) + \dots]^2$$

$$= [\delta^2 u_1 + \delta^4 \log \delta (u_2 + R_2 u_{1_{\tilde{r}}}) + \dots]^2 \tag{3.67}$$

$$\times \frac{\delta^2 u_1 + \delta^4 \log \delta \, (u_2 + R_2 u_{1_{\tilde{r}}}) + \dots - \frac{2}{\gamma + 1} K\delta^2 + \dots}{\frac{2}{\gamma + 1} - \delta^2 u_1 + \dots + \frac{2}{\gamma + 1} K\delta^2 + \dots}$$

where all functions depending on $\tilde{r}$ are to be evaluated at $\tilde{r} = R_1(x)$. The terms of order $\delta^8 \log \delta$ now give a relation between u_2 and v_2 rather than a second relation between u_1 and v_1. The equations for pressure and density become

$$\frac{p}{p_\infty} = 1 - \gamma(1 - K\delta^2)[\delta^2 u_1 + \delta^4 \log \delta (u_2 + R_2 u_{1_{\tilde{r}}}) + \delta^4 (u_3 + R_3 u_{1_{\tilde{r}}}) + \dots] \tag{3.68}$$

$$\frac{\rho_\infty}{\rho} = 1 + \delta^2 u_1 + \delta^4 \log \delta (u_2 + R_2 u_{1_{\tilde{r}}}) + \delta^4 (u_3 + R_3 u_{1_{\tilde{r}}} + \frac{v_1^2}{u_1}) + \dots \tag{3.69}$$

Making use of equation 3.63,

$$\frac{p}{\rho^\gamma} = \frac{p_\infty}{\rho_\infty^\gamma} [1 + o(\delta^4)] \tag{3.70}$$

Since this quantity remains constant along streamlines, the relation is valid throughout the flow field. The first of equations 3.25 for the vorticity can be rewritten to include more terms:

$$[\delta^3(u_{1_{\tilde{r}}} - v_{1_x}) + \delta^5 \log \delta (u_{2_{\tilde{r}}} - v_{2_x}) + \delta^5 (u_{3_{\tilde{r}}} - v_{3_x}) + \ldots][1 + \delta^2 u_1 + \ldots]$$

$$= - \frac{1}{\gamma(\gamma - 1)(1 - K\delta^2)} (1 + \ldots)\, \delta \frac{\partial}{\partial \tilde{r}} \left(\frac{p}{p_\infty} \frac{\rho_\infty^\gamma}{\rho^\gamma} \right) \qquad (3.71)$$

It follows that the flow is irrotational in the second and third approximations:

$$u_{2_{\tilde{r}}} - v_{2_x} = 0$$

$$u_{3_{\tilde{r}}} - v_{3_x} = 0 \qquad (3.72)$$

The argument could be continued to show that irrotationality of the inner solution is also preserved.

Results for higher approximations

The description of the flow by a potential Φ therefore introduces an error due to the entropy variations which is of smaller order than δ^4, and the expansions 3.51 may be continued to include terms of order δ^4. In the outer expansion the results for the higher approximations are:

$$\varepsilon_2 = \delta^4 \log \delta$$

$$\varepsilon_3 = \delta^4$$

$$K\varphi_{2_{xx}} + \varphi_{2_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{2_{\tilde{r}}} - (\gamma+1)(\varphi_{1_x}\varphi_{2_{xx}} + \varphi_{2_x}\varphi_{1_{xx}}) = 0$$

$$K\varphi_{3_{xx}} + \varphi_{3_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{3_{\tilde{r}}} - (\gamma+1)(\varphi_{1_x}\varphi_{3_{xx}} + \varphi_{3_x}\varphi_{1_{xx}})$$

$$= 2\varphi_{1_{\tilde{r}}}\varphi_{1_{x\tilde{r}}} - 2\gamma K\varphi_{1_x}\varphi_{1_{xx}} + \frac{1}{2}(\gamma+1)(2\gamma-1)\varphi_{1_x}^{\ 2}\varphi_{1_{xx}} \qquad (3.73)$$

$$\varphi_2(x,\tilde{r};K) = S_2(x)\log\tilde{r} + g_2(x;K) + O(\tilde{r}^2\log^2\tilde{r})$$

$$\varphi_3(x,\tilde{r};K) = S_1(x)S_1'(x)\log^2\tilde{r} + S_3(x;K)\log\tilde{r} + g_3(x,K) + O(\tilde{r}^2\log^2\tilde{r})$$

$$S_2(x) = -2S_1(x)S_1'(x)$$

$$S_3(x;K) = -S_1(x)S_1'(x)\log F(x) + g_1'(x;K)S_1(x) - \frac{1}{2}S_1(x)F'^2(x)$$

The functions $g_2(x;K)$ and $g_3(x;K)$ can be determined only by solving the full equations for φ_2 and φ_3. These equations are linear, but the coefficients involve $g_1(x;K)$, which is part of the solution of the nonlinear equation for φ_1.

Results for the inner expansion are:

$$\lambda_3 = \delta^4 \log \delta$$

$$\lambda_4 = \delta^4$$

$$\chi_{3_{r^*r^*}} + \frac{1}{r^*}\chi_{3_{r^*}} = 0$$

$$\chi_{4_{r^*r^*}} + \frac{1}{r^*}\chi_{4_{r^*}} = 2\chi_{2_{r^*}}\chi_{2_{xr^*}} + \chi^2_{2_{r^*}}\chi_{2_{r^*r^*}} \qquad (3.74)$$

$$\chi_3(x,r^*;K) = 2S_1(x)S_1'(x) \log r^* + g_2(x;K)$$

$$\chi_4(x,r^*;K) = S_1(x)S_1'(x) \log^2 r^* - \frac{1}{4}\frac{S_1^{\,3}(x)}{r^{*2}}$$

$$+ S_3(x;K) \log r^* + g_3(x;K)$$

<u>Similarity laws</u>

The pressure at the body surface may now be determined from the inner expansions of the velocities. Rewriting the irrotational condition 3.70,

$$\frac{p}{p_\infty} = \left(\frac{a^2}{U^2} M^2\right)^{\gamma/(\gamma-1)} [1 + o(\delta^4)] \qquad (3.75)$$

Substituting equation 3.5,

$$\frac{p}{p_\infty} = [1 - \frac{\gamma-1}{2} M^2(2u + u^2 + v^2)]^{\gamma/(\gamma-1)}[1 + o(\delta^4)] \qquad (3.76)$$

The pressure coefficient is defined by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} = \frac{2}{\gamma M^2}\left(\frac{p}{p_\infty} - 1\right) \qquad (3.77)$$

In terms of the inner expansion,

$$C_p = -2\delta^2 \log \delta\, X_{1_x} - 2\delta^2 X_{2_x} - \delta^2 X^2_{2_{r*}} + O(\delta^4 \log \delta) \quad (3.78)$$

The last equation is of special interest because it shows quite clearly that a quadratic term must be included in the expression for the pressure at the body surface. The same equation is obtained for subsonic and supersonic flows, with different definitions of the functions X_1 and X_2. While the result is now well known, the theoretical justification is not entirely clear unless appropriate limiting procedures are considered. For the linearized cases, an approximate solution is usually expressed by means of a potential which describes the flow at a fixed point in space and which satisfies the Prandtl-Glauert equation in the limit as δ approaches zero. A point on the body surface, however, is not a fixed point, but a point which moves toward the axis as $\delta \to 0$. Obviously the velocities need not have the same order of magnitude under the two limiting processes. To perform calculations at a point on the body, it is necessary to work with equations written in coordinates which remain fixed in the limit, so that the dependence upon δ appears explicitly. The proper expression for the pressure is then obtained directly by the procedure of the preceding paragraph.

For the transonic case, the pressure coefficient at the body surface is

$$C_p = -\delta^2 \log \delta (4S_1') - \delta^2(2S_1' \log F + 2g_1' + F'^2)$$

$$+ \quad O(\delta^4 \log \delta) \tag{3.79}$$

A first-order similarity law for the pressure at the body may now be formulated as follows:

$$\frac{C_p}{\delta^2} + 2S_1' \log \delta^2 F + F'^2 = fn(x;K) + O(\delta^2 \log \delta) \tag{3.80}$$

In this form the rule is a statement that for bodies of similar shapes there is a transonic similarity for the nonlinear interaction effects described by $g_1(x;K)$. Another form of the similarity law is

$$\frac{C_p}{\delta^2} + 4S_1' \log \delta = fn(x;K) + O(\delta^2 \log \delta) \tag{3.81}$$

The result was given in Ref. 3, in essentially the same form but without an estimate of the error.

The drag coefficient, based on the maximum cross-sectional area $\pi \delta^2$, is defined by an integration of the pressure over the body surface:

$$C_D = 2 \int_0^1 C_p \, FF' \, dx \tag{3.82}$$

If the body is pointed at both ends, the drag reduces to

$$C_D = -4 \delta^2 \int_0^1 FF' \, g_1' \, dx + O(\delta^4 \log \delta) \tag{3.83}$$

The similarity law for the drag of similar bodies is

$$\frac{C_D}{\delta^2} = \text{fn}(K) + O(\delta^2 \log \delta) \tag{3.84}$$

4. BODY OF NEARLY CIRCULAR CROSS-SECTION

Form of expansion

The derivation of a transonic expansion procedure can be carried out in much the same manner for bodies of noncircular cross-section. The body may be described by

$$S(x,r,\theta;\delta,\tau) = 0 = r - \delta F(x)\sqrt{1 + \tau G(x,\theta)}$$

where (4.1)

$$\int_0^{2\pi} G(x,\theta)\, d\theta = 0$$

With this representation, the distribution of cross-sectional area is the same as for the body of revolution defined by $r = \delta F(x)$. The quantity τ may be a function of δ, and indicates the relative order of magnitude of the deviations from a circular section. A limiting process will be used which allows δ to approach zero, and the role of τ in this procedure must be determined. If τ decreases as δ goes to zero, the first approximation will be the same as for a body of revolution. On the other hand, if τ increases, the cross-section shape will approach that of a wing, or of some other configuration with thickness small compared to one of the other dimensions. In order to study a general class of noncircular sections, the logical choice would be to keep τ constant in the limit, i.e. to regard τ as a second similarity parameter.

For the present analysis, however, the deviation from a circular section will be assumed small, in order to allow an expansion for small

τ. One possibility is to let δ approach zero with τ fixed, and then to take the limit as τ goes to zero. Alternatively, τ might be allowed to decrease with δ, since for this problem it is expected that the first approximation will be the axisymmetric result. The latter conception of the procedure has an advantage because, having specified the dependence of τ upon δ, it is possible to discuss the relative orders of the two quantities in the solution.

As before, an inner expansion will be used to describe the flow close to the body, and an outer expansion will represent the solution farther away. For the body of revolution it can be seen that continuing the outer expansion to include terms up to the order of magnitude of δ^{10} would give a representation which is uniformly valid everywhere to order δ^4. Presumably one could obtain a representation uniformly valid everywhere to any desired order simply by taking a sufficient number of terms in the outer expansion. The inner solution therefore is not essential in principle, but serves as a considerable aid in computation. In the present case, however, it will become clear that the use of two expansions is in general essential.

Since the transonic flow over a slender body of revolution is approximately irrotational, it is expected that the initial terms in the velocity expansions for nearly circular cross-sections may also be represented by means of potential functions. The procedures used before will lead to the same coordinate distortions and similarity parameter as for the axisymmetric problem. Since it is assumed that the solution is close to that for a body of revolution, the inner expansion is assumed to be of the form

$$\Phi(x,r,\theta;\delta,M,\tau) \sim \delta^2 \log \delta\, X_1(x,r^*;K) + \delta^2 X_2(x,r^*;K) + \delta^4 \log \delta\, X_3(x,r^*;K) + \ldots + \sum_{i=1}^{n} \mu_i(\delta,\tau)\varphi_i^*(x,r^*,\theta;K) \qquad (4.2)$$

where the functions $X_i(x,r^*,\theta;K)$ are the results obtained for axial symmetry, and $\mu_i(\delta,\tau)$ must approach zero if $\tau \to 0$ with δ held fixed. The latter quantities form a decreasing sequence in terms of δ, but the ordering must of course depend on the order of magnitude of τ with respect to δ. An analogous expansion is assumed for the outer solution.

Approximate solution

The full differential equation in terms of velocities differs from equation 3.6 only in the presence of a third velocity component $w(x,r,\theta)$. If a potential Φ is assumed, and the expansion 4.2 is substituted for Φ, an equation analogous to 3.18 is obtained. Subtracting the terms which are independent of τ, and keeping only the largest remaining terms,

$$\frac{1}{\delta^2}(\mu_1\varphi_{1_{r^*r^*}}^* + \mu_2\varphi_{2_{r^*r^*}}^* + \ldots) + \frac{1}{\delta^2}\frac{1}{r^*}(\mu_1\varphi_{1_{r^*}}^* + \mu_2\varphi_{2_{r^*}}^* + \ldots) + \frac{1}{\delta^2}\frac{1}{r^{*2}}(\mu_1\varphi_{1_{\theta\theta}}^* + \mu_2\varphi_{2_{\theta\theta}}^* + \ldots)$$

$$= \frac{1}{\delta^2}\left\{\mu_1\delta^2(X_{2_{r^*}}\varphi_{1_{xr^*}}^* + \varphi_{1_{r^*}}^* X_{2_{xr^*}}) + \ldots\right\} + \frac{1}{\delta^4}\left\{\mu_1\delta^4(X_{2_{r^*}}^2\,\varphi_{1_{r^*r^*}}^* + 2\varphi_{1_{r^*}}^*\,X_{2_{r^*}}X_{2_{r^*r^*}}) + \ldots\right\} + \ldots \qquad (4.3)$$

The boundary condition is again given by the requirement of tangent flow. Generalizing equation 3.4 to the present case

$$v(x,\delta F\sqrt{1+\tau G},\theta) = \left\{1 + u(x,\delta F\sqrt{1+\tau G},\theta)\right\} \frac{\partial}{\partial x}(\delta F\sqrt{1+\tau G})$$

$$+ w(x,\delta F\sqrt{1+\tau G},\theta)\frac{1}{\delta F\sqrt{1+\tau G}}\frac{\partial}{\partial\theta}(\delta F\sqrt{1+\tau G}) \qquad (4.4)$$

Introducing Φ and its expansion,

$$\frac{1}{\delta}\left\{\delta^2\log\delta\, X_{1_{r^*}}(x,F\sqrt{1+\tau G}) + \delta^2 X_{2_{r^*}}(x,F\sqrt{1+\tau G}) + \ldots\right\}$$

$$+\frac{1}{\delta}\left\{\mu_1\varphi_{1_{r^*}}^*(x,F\sqrt{1+\tau G},\theta) + \mu_2\varphi_{2_{r^*}}^*(x,F\sqrt{1+\tau G},\theta) + \ldots\right\}$$

$$= \left\{1 + \delta^2\log\delta\, X_{1_x}(x,F\sqrt{1+\tau G}) + \ldots + \mu_1\varphi_{1_x}^*(x,F\sqrt{1+\tau G},\theta) + \ldots\right\}$$

$$\times\,\delta\,\frac{FF' + \frac{1}{2}\tau(F^2G)_x}{F\sqrt{1+\tau G}} + \frac{\tau G_\theta}{2\delta F(1+\tau G)^{3/2}}\left\{\mu_1\varphi_{1_\theta}^*(x,F\sqrt{1+\tau G},\theta) + \ldots\right\}$$

$$(4.5)$$

Substituting for the functions $X_i(x,r^*)$ and performing Taylor expansions about $r^* = F$,

$$\delta\,\frac{S_1}{F}\left(1 - \frac{1}{2}\tau G + \frac{3}{8}\tau^2G^2 + \ldots\right) + \delta^3\log\delta\,\frac{2S_1S_1'}{F}\left(1 - \frac{1}{2}\tau G + \ldots\right) + \ldots$$

$$+\frac{\mu_1}{\delta}\left\{\varphi_{1_{r^*}}^*(x,F,\theta) + \frac{\tau FG}{2}\varphi_{1_{r^*r^*}}^*(x,F,\theta) + \ldots\right\} + \frac{\mu_2}{\delta}\left\{\varphi_{2_{r^*}}^*(x,F,\theta) + \ldots\right\}$$

$$= \frac{\delta}{F}\left\{FF' + \frac{1}{2}\tau(F^2G)_x\right\}\left\{1 - \frac{1}{2}\tau G + \frac{3}{8}\tau^2G^2 + \ldots\right\}\left\{1 + \delta^2\log\delta\cdot 2S_1' + \ldots\right\}$$

$$+ \frac{\tau}{2\delta}\,\frac{G_\theta}{F}(1+\ldots)\left\{\mu_1\varphi_{1_\theta}^*(x,F,\theta) + \ldots\right\} \qquad (4.6)$$

The boundary condition is now applied at the mean surface $r^* = F(x)$.

Terms in 4.6 which are independent of τ will cancel because of the definition of the functions $\chi_i(x,r^*)$. The largest remaining terms determine μ_1 and the boundary condition for $\varphi_1^*(x,r^*,\theta)$:

$$\mu_1 = \tau\,\delta^2$$
$$\varphi_{1\,r^*}^*(x,F,\theta) = \frac{1}{2F}(F^2G)_x \tag{4.7}$$

Depending on the magnitude of τ, μ_2 will be either $\tau\,\delta^4\log\delta$ or $\tau^2\delta^2$. In the discussion of equation 4.17 it will be pointed out that terms linear in τ cannot contribute to the drag, so we will take

$$\mu_2 = \tau^2\delta^2$$
$$\varphi_{2\,r^*}^*(x,F,\theta) = -\frac{G}{4F}(F^2G)_x - \frac{FG}{2}\varphi_{1\,r^*r^*}^*(x,F,\theta) + \frac{G_\theta}{2F}\varphi_{1\,\theta}^*(x,F,\theta) \tag{4.8}$$

without necessarily requiring that $\tau\,\delta^4\log\delta \ll \tau^2\delta^2$. From 4.3 it is seen that the differential equation is Laplace's for both φ_1^* and φ_2^*:

$$\varphi_{1\,r^*r^*}^* + \frac{1}{r^*}\varphi_{1\,r^*}^* + \frac{1}{r^{*2}}\varphi_{1\,\theta\theta}^* = 0$$
$$\varphi_{2\,r^*r^*}^* + \frac{1}{r^*}\varphi_{2\,r^*}^* + \frac{1}{r^{*2}}\varphi_{2\,\theta\theta}^* = 0 \tag{4.9}$$

The form of solution can be determined if it is possible to expand $G(x,\theta)$ in a Fourier series. We therefore take

$$G(x,\theta) = \sum_{n=-\infty}^{\infty} G_n(x)\, e^{in\theta} \qquad (4.10)$$

where $G_o(x)$ is xero and $G_{-n}(x)$ is the complex conjugate of $G_n(x)$. Substituting in the boundary condition for $\varphi_1{}^*$,

$$\varphi_{1\,r^*}^*(x,r^*,\theta) = \frac{1}{2F} \sum_{n=-\infty}^{\infty} (F^2 G_n)'\, e^{in\theta} \qquad (4.11)$$

A general solution to Laplace's equation contains terms of the form $r^{*m} e^{\pm im\theta}$, where m may be any integer. If positive values of m were admitted, the matching condition would give terms in the outer solution which were too large, i.e. of order τ if m = 1, $\frac{\tau}{\delta^2}$ if m = 2, etc. Since τ can be of any order of magnitude smaller than one, these terms can become infinite as $\delta \to 0$, or at least larger than the source term from the axisymmetric solution. Therefore the result for $\varphi_1{}^*$ is

$$\varphi_1{}^*(x,r^*,\theta) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{F^{|n|}}{|n|\, r^{*|n|}} (F^2 G_n)'\, e^{in\theta} \qquad (4.12)$$

In the outer solution there will appear terms of order δ^2, $\delta^4 \log \delta$, and δ^4 which are the same as for the body of revolution. Using a matching condition as before, the first term depending on θ is of order $\tau\delta^4$, and the potential $\varphi_1{}^*(x,r^*,\theta)$ will also lead to terms of order $\tau\delta^6$, $\tau\delta^8$, etc. Since $\varphi_1{}^*$ is in general an infinite series, one could take arbitrarily many terms in the outer solution and still not be able to find the complete expression for $\varphi_1{}^*$. In other words, it is not possible to obtain an expansion which is uniformly valid everywhere to order $\tau\delta^2$ simply by taking a sufficiently large number of terms in the outer solution. In this respect the present expansions differ from the ones obtained for axial symmetry.

The boundary condition for φ_2^* becomes

$$\varphi_{2_{r^*}}^*(x,F,\theta) = \frac{1}{4F}\left\{\sum_{n=-\infty}^{\infty} G_n e^{in\theta}\right\}\left\{\sum_{n=-\infty}^{\infty} |n|(F^2G_n)' e^{in\theta}\right\}$$

$$+ \frac{1}{4F}\left\{\sum_{n=-\infty}^{\infty} nG_n e^{in\theta}\right\}\left\{\sum_{n=-\infty}^{\infty} \frac{n}{|n|}(F^2G_n)' e^{in\theta}\right\} \qquad (4.13)$$

and the solution for $\varphi_2^*(x,r^*,\theta)$ is

$$\varphi_2^*(x,r^*,\theta) = -\frac{1}{4}\sum_{n=-\infty}^{\infty} c_n(x)\frac{1}{|n|}\frac{F^{|n|}}{r^{*|n|}} e^{in\theta} \qquad (4.14)$$

The coefficient $c_m(x)$ is defined by

$$\sum_{m=-\infty}^{\infty} c_m(x)e^{im\theta} = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} G_j(F^2G_k)'(|k| + j\frac{k}{|k|})e^{i(j+k)\theta} \qquad (4.15)$$

and the value $c_o(x) = 0$ is included in the definition.

Surface pressures and drag

An expression for the pressure coefficient is obtained in the same manner as for the body of revolution, and the expanded result is analogous to equation 3.78. At the body surface,

$$C_p = -2\Big\{\delta^2 \log \delta\, \chi_{1_x}(x, F\sqrt{1+\tau G}) + \delta^2 \chi_{2_x}(x, F\sqrt{1+\tau G}) + \ldots$$

$$+ \tau\delta^2 \varphi_{1_x}^*(x, F\sqrt{1+\tau G}, \theta) + \tau^2\delta^2 \varphi_{2_x}^*(x, F\sqrt{1+\tau G}, \theta) + \ldots\Big\}$$

$$- \frac{1}{\delta^2}\Big\{\delta^2 \chi_{2_{r*}}(x, F\sqrt{1+\tau G}) + \ldots + \tau\delta^2 \varphi_{1_{r*}}^*(x, F\sqrt{1+\tau G}, \theta)$$

$$+ \tau^2\delta^2 \varphi_{2_{r*}}^*(x, F\sqrt{1+\tau G}, \theta) + \ldots\Big\}^2$$

$$- \frac{1}{\delta^2 F^2(1+\tau G)}\Big\{\tau\delta^2 \varphi_{1_\theta}^*(x, F\sqrt{1+\tau G}, \theta) + \ldots\Big\}^2 + \ldots \qquad (4.16)$$

Expressions for χ_{1_x} and χ_{2_x} are substituted from equation 3.53, and the terms representing the radial velocity component are replaced by the right-hand side of the boundary condition 4.5. Expanding all terms in Taylor series about $r^* = F$,

$$C_p = -2\Big\{\delta^2 \log \delta \cdot 2S_1' + \delta^2 S_1'[\log F + \tfrac{1}{2}\tau G - \tfrac{1}{4}\tau^2 G^2 + \ldots]$$

$$+ \delta^2 g_1' + \ldots + \tau\delta^2[\varphi_{1_x}^*(x,F,\theta) + \tfrac{1}{2}\tau F G \varphi_{1_{xr*}}^*(x,F,\theta) + \ldots]$$

$$+ \tau^2\delta^2[\varphi_{2_x}^*(x,F,\theta) + \ldots] + \ldots\Big\}$$

$$-\Big\{\frac{\delta}{F}[FF' + \tfrac{1}{2}\tau(F^2G)_x][1 - \tfrac{1}{2}\tau G + \tfrac{3}{8}\tau^2 G^2 + \ldots][1 + \delta^2 \log \delta \cdot 2S_1' + \ldots\Big\}$$

$$+ \frac{\tau}{\delta}\,\frac{G_\theta}{2F}[1 + \ldots][\tau\delta^2 \varphi_{1_\theta}^*(x,F,\theta) + \ldots]\Big\}^2$$

$$- \frac{1}{\delta^2 F^2}[1 + \ldots]\Big\{\tau\delta^2 \varphi_{1_\theta}^*(x,F,\theta) + \ldots\Big\}^2 \qquad (4.17)$$

Considerable simplification occurs in the expression for the drag. In terms of the surface pressure, the drag coefficient is defined by

$$C_D = \frac{1}{\pi\delta^2} \int_0^1 dx \int_0^{2\pi} C_p \delta F \sqrt{1+\tau G}\, \frac{\partial}{\partial x} (\delta F \sqrt{1+\tau G})\, d\theta$$

$$= \frac{1}{\pi} \int_0^1 dx \int_0^{2\pi} C_p \left\{ FF' + \frac{1}{2}\tau (F^2 G)_x \right\} d\theta \qquad (4.18)$$

Substituting for C_p and combining terms where possible,

$$C_D = C_{D_o} + \frac{1}{2\pi}\tau \int_0^1 dx \int_0^{2\pi} C_{p_o} (F^2G)_x \, d\theta$$

$$- \frac{1}{\pi}\tau\delta^2 \int_0^1 dx \int_0^{2\pi} \left\{ F'(F^2G)_x + F^2F'F''G + 2FF'\varphi_{1_x}^* (x,F,\theta) \right\} d\theta$$

$$+ \frac{1}{\pi}\tau^2\delta^2 \int_0^1 dx \int_0^{2\pi} \Big\{ -\frac{1}{2} FF'(3F'^2 + FF'')G^2 - \frac{1}{2} F^2(4F'^2 + FF'')GG_x$$

$$- \frac{3}{4} F^3F'G_x{}^2 - (F^2G)_x \varphi_{1_x}^*(x,F,\theta) - F^2F'G\varphi_{1_{xr^*}}^*(x,F,\theta)$$

$$- 2FF'\varphi_{2_x}^* (x,F,\theta) - F'^2 G_\theta \varphi_{1_\theta}^* (x,F,\theta) - \frac{F'}{F} \varphi_{1_\theta}^{*2} (x,F,\theta) \Big\} d\theta \qquad (4.19)$$

where C_{p_o} and C_{D_o} are the results obtained for the body of revolution $r = \delta F(x)$. Throughout the preceding development the Fourier expansions of terms linear in τ never contain a term which is independent of θ. This will also be true for the terms of order $\tau\, \delta^4 \log \delta$ and smaller which contain τ only to the first power. All such terms will drop out upon integration over θ. There will be no term in C_D which is linear in τ, and for purposes of determining drag no generality has been lost by taking μ_2 equal to $\tau^2\delta^2$.

The last integral in equation 4.19 can be simplified by using relations of the form

$$\int_0^{2\pi} G^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} G_n G_{-n} \tag{4.20}$$

Each integral of a double sum may therefore be replaced by a single summation to be integrated over x. After rearranging terms, the remaining integration is carried out explicitly to give

$$C_D = C_{D_o} + \frac{1}{2}\tau^2\delta^2 \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{|n|}(F^2 G_n)'(F^2 G_{-n})' - F^2 F' G_n (3F' G_{-n} + 2F G_{-n}') \right\} \Bigg|_{x=1} + o(\tau^2\delta^2) \tag{4.21}$$

where it has been assumed that the body has a pointed nose. The summation reduces to zero in three cases. For a body with a pointed base,

$$F(1) = 0 \tag{4.22a}$$

and consequently all terms in the sum disappear. If instead the base is blunt, but the rate of change of shape with x is zero, then

$$F'(1) = G_x(1,\theta) = 0 \tag{4.22b}$$

and the sum again vanishes. Lastly, the sum is zero if

$$G(1,\theta) = G_x(1,\theta) = 0 \tag{4.22c}$$

i.e. if the base is blunt and circular and the derivative of the shape perturbation $G(x,\theta)$ is zero at the base.

For the three types of bodies described, therefore, the change in drag due to small deviations from a circular cross-section is of smaller order than $\tau^2\delta^2$:

$$C_D = C_{D_o} + o(\tau^2\delta^2) \tag{4.23}$$

where δ represents the order of magnitude of the thickness ratio and τ the order of the perturbations in cross-section shape.

5. BODY OF REVOLUTION AT ANGLE OF INCIDENCE

Expansion procedure

The expansion procedure for circular bodies at small angles of attack is basically the same as for zero incidence, but there is a considerable increase in the complexity of the equations. Cylindrical coordinates x,r,θ are used, such that the x-axis coincides with the body axis. The angle of incidence is denoted by α, and the undisturbed velocity has a small component in the r,θ plane which is of the order of α. If q_x, q_r, q_θ are the total velocity components, and u,v,w the nondimensional perturbations to the free-stream velocity,

$$\frac{q_x}{U}(x,r,\theta;\delta,M,\alpha) = \cos\alpha + u(x,r,\theta;\delta,M,\alpha)$$

$$\frac{q_r}{U}(x,r,\theta;\delta,M,\alpha) = \sin\alpha\sin\theta + v(x,r,\theta;\delta,M,\alpha) \qquad (5.1)$$

$$\frac{q_\theta}{U}(x,r,\theta;\delta,M,\alpha) = \sin\alpha\cos\theta + w(x,r,\theta;\delta,M,\alpha)$$

The boundary condition at infinity therefore requires that u,v,w approach zero as the distance from the body increases. At the body surface $r = \delta F(x)$, the condition of tangent flow gives

$$v(x,\delta F,\theta) = \left\{\cos\alpha + u(x,\delta F,\theta)\right\}\delta F' - \sin\alpha\sin\theta \qquad (5.2)$$

The square of the velocity is given by

$$\frac{q^2}{U^2} = 1 + 2u \cos \alpha + 2 \sin \alpha(v \sin \theta + w \cos \theta)$$

$$+ u^2 + v^2 + w^2 \qquad (5.3)$$

Substituting 5.1 and 5.3 in 2.7 and 2.8, and eliminating a^2,

$$\left\{1 - \frac{\gamma-1}{2} M^2[2u \cos \alpha + 2 \sin \alpha (v \sin \theta + w \cos \theta) + u^2 + v^2 + w^2]\right\}$$

$$\times \left\{u_x + v_r + \frac{1}{r} v + \frac{1}{r} w_\theta\right\}$$

$$= M^2\left\{[\cos \alpha + u][u_x \cos \alpha + \sin \alpha(v_x \sin \theta + w_x \cos \theta)\right.$$

$$+ uu_x + vv_x + ww_x] + [\sin \alpha \sin \theta + v][u_r \cos \alpha$$

$$+ \sin \alpha(v_r \sin \theta + w_r \cos \theta) + uu_r + vv_r + ww_r]$$

$$+ \frac{1}{r} [\sin \alpha \cos \theta + w][u_\theta \cos \alpha + \sin \alpha(v_\theta \sin \theta + w_\theta \cos \theta$$

$$\left.+ v \cos \theta - w \sin \theta) + uu_\theta + vv_\theta + ww_\theta]\right\} \qquad (5.4)$$

After rearranging terms, the equation is stated in a form corresponding to the relation 3.6 for $\alpha = 0$:

$$(1 - M^2)u_x + v_r + \frac{1}{r} v + \frac{1}{r} w_\theta = M^2\Big\{\sin\alpha\cos\alpha\,[(u_r + v_x)\sin\theta$$

$$+ (\frac{1}{r} u_\theta + w_x)\cos\theta] + \sin^2\alpha[-u_x + (v_r \sin\theta + \frac{1}{r} v_\theta\cos\theta)\sin\theta$$

$$+ (w_r\sin\theta + \frac{1}{r} w_\theta\cos\theta)\cos\theta + \frac{1}{r}(v\cos\theta - w\sin\theta)\cos\theta]$$

$$+ \cos\alpha[(\gamma + 1)uu_x + v(u_r + v_x) + w(\frac{1}{r} u_\theta + w_x)$$

$$+ (\gamma - 1)u(v_r + \frac{1}{r} v + \frac{1}{r} w_\theta)] + \sin\alpha[(\gamma - 1)u_x(v\sin\theta$$

$$+ w\cos\theta) + (\gamma - 1)(v\sin\theta + w\cos\theta)(v_r + \frac{1}{r} v + \frac{1}{r} w_\theta)$$

$$+ u(u_r + v_x)\sin\theta + u(\frac{1}{r} u_\theta + w_x)\cos\theta$$

$$+ (\frac{1}{r} v_\theta + w_r)(v\cos\theta + w\sin\theta) + 2vv_r\sin\theta + \frac{2}{r} ww_\theta\cos\theta$$

$$+ \frac{1}{r} w(v\cos\theta - w\sin\theta)] + [v^2 v_r + \frac{1}{r} w^2 w_\theta + uv(u_r + v_x)$$

$$+ uw(\frac{1}{r} u_\theta + w_x) + vw(\frac{1}{r} v_\theta + w_r) + \frac{\gamma + 1}{2} u^2 u_x + \frac{\gamma - 1}{2}(v^2 + w^2)u_x$$

$$+ \frac{\gamma - 1}{2}(u^2 + v^2 + w^2)(v_r + \frac{1}{r} v + \frac{1}{r} w_\theta)]\Big\} \qquad (5.5)$$

The limiting procedure requires that δ and α approach zero simultaneously, and the choice of an appropriate relationship between the two parameters will depend on the generality desired of the expansion. If α should decrease more rapidly than δ, the first approximation would be the same as for zero incidence. On the other hand, if α should go to zero rather slowly, the first-order solution would contain no source term and would correspond to flow around a thin needle-like body inclined to the free stream. Since neither of these results is considered sufficiently general, α will be taken proportional to δ:

$$\alpha = A\delta \tag{5.6}$$

where A is to be held constant as δ approaches zero. This choice will include the body at zero incidence as the special case $A = 0$, while large values of A will correspond to the case of an extremely thin body at an angle of attack.

The use of inner and outer expansions is again helpful, and would in fact be necessary from a practical point of view for determining the form of the δ^4 terms in the potential at the body surface. As for the case of zero incidence, the inner solution should be expressed in terms of a radial coordinate which is of order one at the body surface; the first approximation to the outer solution must satisfy a nonlinear equation; and the expansion procedure should be capable of describing all the expected types of transonic flow patterns. Thus the procedures used before apply without any change, and the definitions of $\tilde{r}$, r^*, and K are the same as in equation 3.51. It can again be shown that the entropy changes due to the presence of shock waves are of small order, and for the first few orders of magnitude the flow may be represented by a potential. The assumed forms of expansion are therefore

$$\begin{aligned} \Phi(x,r,\theta;\delta,M,\alpha) &\sim \sum_{i=1}^{n} \varepsilon_i(\delta)\varphi_i(x,\tilde{r},\theta;K,A) \\ &\sim \sum_{i=1}^{n} \mu_i(\delta)\, X_i(x,r^*,\theta;K,A) \end{aligned} \tag{5.7}$$

Form of approximate solution

The assumed expansions are now substituted into the full differential equation 5.5. In terms of an outer expansion, the equation becomes

$$K\delta^2(\varepsilon_1\varphi_{1_{xx}} + \varepsilon_2\varphi_{2_{xx}} + \varepsilon_3\varphi_{3_{xx}} + \dots) + \delta^2(\varepsilon_1\varphi_{1_{\tilde{r}\tilde{r}}} + \varepsilon_2\varphi_{2_{\tilde{r}\tilde{r}}} + \varepsilon_3\varphi_{3_{\tilde{r}\tilde{r}}} + \dots)$$

$$+ \delta^2\frac{1}{\tilde{r}}(\varepsilon_1\varphi_{1_{\tilde{r}}} + \varepsilon_2\varphi_{2_{\tilde{r}}} + \varepsilon_3\varphi_{3_{\tilde{r}}} + \dots) + \delta^2\frac{1}{\tilde{r}^2}(\varepsilon_1\varphi_{1_{\theta\theta}} + \varepsilon_2\varphi_{2_{\theta\theta}} + \varepsilon_3\varphi_{3_{\theta\theta}} + \dots)$$

$$= (1 - K\delta^2)\Big\{2\delta(A\delta - \frac{1}{6}A^3\delta^3 + \dots)(1 - \frac{1}{2}A^2\delta^2 + \dots)[(\varepsilon_1\varphi_{1_{x\tilde{r}}} + \varepsilon_2\varphi_{2_{x\tilde{r}}}$$

$$+ \varepsilon_3\varphi_{3_{x\tilde{r}}} + \dots)\sin\theta + \frac{1}{\tilde{r}}(\varepsilon_1\varphi_{1_{x\theta}} + \varepsilon_2\varphi_{2_{x\theta}} + \varepsilon_3\varphi_{3_{x\theta}} + \dots)\cos\theta]$$

$$+ (A\delta - \frac{1}{6}A^3\delta^3 + \dots)^2[-(\varepsilon_1\varphi_{1_{xx}} + \varepsilon_2\varphi_{2_{xx}} + \varepsilon_3\varphi_{3_{xx}} + \dots)$$

$$+ \delta^2(\varepsilon_1\varphi_{1_{\tilde{r}\tilde{r}}} + \dots)\sin^2\theta + 2\delta^2\frac{1}{\tilde{r}}(\varepsilon_1\varphi_{1_{\tilde{r}\theta}} + \dots)\cos\theta\sin\theta$$

$$+ \delta^2\frac{1}{\tilde{r}^2}(\varepsilon_1\varphi_{1_{\theta\theta}} + \dots)\cos^2\theta + \delta^2\frac{1}{\tilde{r}}(\varepsilon_1\varphi_{1_{\tilde{r}}} + \dots)\cos^2\theta$$

$$- 2\delta^2\frac{1}{\tilde{r}^2}(\varepsilon_1\varphi_{1_{\theta}} + \dots)\cos\theta\sin\theta] + (1 - \frac{1}{2}A^2\delta^2 + \dots)[(\gamma + 1)(\varepsilon_1\varphi_{1_x}$$

$$+ \varepsilon_2\varphi_{2_x} + \varepsilon_3\varphi_{3_x} + \dots)(\varepsilon_1\varphi_{1_{xx}} + \varepsilon_2\varphi_{2_{xx}} + \varepsilon_3\varphi_{3_{xx}} + \dots)$$

$$+ 2\delta^2(\varepsilon_1\varphi_{1_{\tilde{r}}} + \dots)(\varepsilon_1\varphi_{1_{x\tilde{r}}} + \dots) + \frac{2\delta^2}{\tilde{r}^2}(\varepsilon_1\varphi_{1_{\theta}} + \dots)(\varepsilon_1\varphi_{1_{x\theta}} + \dots)$$

$$+ \delta^2(\gamma - 1)(\varepsilon_1\varphi_{1_x} + \dots)(\varepsilon_1\varphi_{1_{\tilde{r}\tilde{r}}} + \dots + \frac{1}{\tilde{r}}\varepsilon_1\varphi_{1_{\tilde{r}}} + \dots + \frac{1}{\tilde{r}^2}\varepsilon_1\varphi_{1_{\theta\theta}} + \dots)]$$

$$+ (A\delta + \dots)[(\gamma - 1)(\varepsilon_1\varphi_{1_{xx}} + \dots)\delta(\varepsilon_1\varphi_{1_{\tilde{r}}}\sin\theta + \dots + \frac{1}{\tilde{r}}\varepsilon_1\varphi_{1_{\theta}}\cos\theta + \dots)$$

$$+ 2\delta(\varepsilon_1\varphi_{1_x} + \dots)(\varepsilon_1\varphi_{1_{x\tilde{r}}}\sin\theta + \dots + \frac{1}{\tilde{r}}\varepsilon_1\varphi_{1_{x\theta}}\cos\theta + \dots) + \dots]$$

$$+[\frac{\gamma + 1}{2}(\varepsilon_1\varphi_{1_x} + \dots)^2(\varepsilon_1\varphi_{1_{xx}} + \dots) + \dots]\Big\} \qquad (5.8)$$

The inner expansion gives

$$\frac{1}{\delta^2}(\mu_1\chi_{1_{r*r*}}+\mu_2\chi_{2_{r*r*}}+\mu_3\chi_{3_{r*r*}}+\cdots)+\frac{1}{\delta^2 r*}(\mu_1\chi_{1_{r*}}+\mu_2\chi_{2_{r*}}+\mu_3\chi_{3_{r*}}+\cdots)$$

$$+\frac{1}{\delta^2 r*^2}(\mu_1\chi_{1_{\theta\theta}}+\mu_2\chi_{2_{\theta\theta}}+\mu_3\chi_{3_{\theta\theta}}+\cdots)$$

$$=(1+\cdots)\Big\{(A\delta+\cdots)(1+\cdots)\frac{2}{\delta}[(\mu_1\chi_{1_{xr*}}+\cdots)\sin\theta$$

$$+\frac{1}{r*}(\mu_1\chi_{1_{x\theta}}+\cdots)\cos\theta+\cdots]+(A\delta+\cdots)^2\frac{1}{\delta^2}[(\mu_1\chi_{1_{r*r*}}+\cdots)\sin^2\theta$$

$$+\frac{2}{r*}(\mu_1\chi_{1_{r*\theta}}+\cdots)\cos\theta\sin\theta+\frac{1}{r*^2}(\mu_1\chi_{1_{\theta\theta}}+\cdots)\cos^2\theta$$

$$+\frac{1}{r*}(\mu_1\chi_{1_{r*}}+\cdots)\cos^2\theta-\frac{2}{r*^2}(\mu_1\chi_{1_\theta}+\cdots)\cos\theta\sin\theta+\cdots]$$

$$+(1+\cdots)\frac{2}{\delta^2}[(\mu_1\chi_{1_{r*}}+\cdots)(\mu_1\chi_{1_{xr*}}+\cdots)+\frac{1}{r*^2}(\mu_1\chi_{1_\theta}+\cdots)(\mu_1\chi_{1_{x\theta}}+\cdots)$$

$$+\cdots]+(A\delta+\cdots)\frac{1}{\delta^3}[\frac{2}{r*}(\mu_1\chi_{1_{r*}}+\cdots)(\mu_1\chi_{1_{r*\theta}}+\cdots)\cos\theta$$

$$+\frac{2}{r*^2}(\mu_1\chi_{1_\theta}+\cdots)(\mu_1\chi_{1_{r*\theta}}+\cdots)\sin\theta$$

$$+2(\mu_1\chi_{1_{r*}}+\cdots)(\mu_1\chi_{1_{r*r*}}+\cdots)\sin\theta+\frac{2}{r*^3}(\mu_1\chi_{1_\theta}+\cdots)(\mu_1\chi_{1_{\theta\theta}}+\cdots)\cos\theta$$

$$-\frac{2}{r*^3}(\mu_1\chi_{1_\theta}+\cdots)^2\sin\theta]+\frac{1}{\delta^4}[(\mu_1\chi_{1_{r*}}+\cdots)^2(\mu_1\chi_{1_{r*r*}}+\cdots)$$

$$+\frac{1}{r*^4}(\mu_1\chi_{1_\theta}+\cdots)^2(\mu_1\chi_{1_{\theta\theta}}+\cdots)$$

$$+\frac{1}{r*^2}(\mu_1\chi_{1_{r*}}+\cdots)(\mu_1\chi_{1_\theta}+\cdots)(2\mu_1\chi_{1_{r*\theta}}+\cdots-\frac{1}{r*}\mu_1\chi_{1_\theta}+\cdots)+\cdots]\Big\} \quad (5.9)$$

The tangency condition at the body surface becomes

$$\frac{1}{\delta}\left\{\mu_1 X_{1_{r*}}(x,F,\theta) + \mu_2 X_{2_{r*}}(x,F,\theta) + \mu_3 X_{3_{r*}}(x,F,\theta) + \dots\right\}$$

$$= \left\{1 - \frac{1}{2}A^2\delta^2 + \dots + \mu_1 X_{1_x}(x,F,\theta) + \dots\right\}\delta F'$$

$$- (A\delta - \frac{1}{6}A^3\delta^3 + \dots)\sin\theta \qquad (5.10)$$

A matching condition between the inner and outer expansions provides the necessary additional relations, and the procedure for finding the form of the solution is exactly the same as in the case of zero incidence. For the inner expansion the initial terms must be solutions to Laplace's equation. To satisfy the boundary condition, a source term plus a term proportional to $\sin\theta$ are needed, both of which must be of order δ^2. Solutions showing the desired dependence upon θ are $\frac{1}{r*}\sin\theta$ and $r*\sin\theta$, but the latter possibility is excluded, since otherwise the matching condition would require a term of order one in the outer expansion. Additional functions of x, of unspecified order of magnitude, may also appear, and the potential is of the form

$$\Phi(x,r,\theta;M,\delta,\alpha) \sim B(x;\delta,K,A) + \delta^2(FF'\log r* + AF^2\frac{\sin\theta}{r*}) \qquad (5.11)$$

In the outer solution the first term is again of order δ^2, but the differential equation is more complicated:

$$\varepsilon_1 = \delta^2$$

$$K\varphi_{1_{xx}} + \varphi_{1_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{1_{\tilde{r}}} + \frac{1}{\tilde{r}^2}\varphi_{1_{\theta\theta}} \qquad (5.12)$$

$$= 2A\sin\theta\,\varphi_{1_{x\tilde{r}}} + 2A\frac{\cos\theta}{\tilde{r}}\varphi_{1_{x\theta}} - A^2\varphi_{1_{xx}} + (\gamma+1)\varphi_{1_x}\varphi_{1_{xx}}$$

If now $\tilde{r}$ is small, the largest terms are the three terms of the Laplacian. The first terms in an expansion of φ_1 for small $\tilde{r}$ must satisfy Laplace's equation, and the expansion may be continued by means of an iteration procedure:

$$\Phi(x,\tilde{r},\theta;K,A) = S_1(x) \log \tilde{r} + g_1(x;K) + AS_1'(x)\, \tilde{r} \log \tilde{r} \sin \theta$$

$$+ h_1(x;K,A)\, \tilde{r} \sin \theta + \frac{1}{4} (\gamma + 1) S_1'(x)\, S_1''(x)\, \tilde{r}^2 \log^2 \tilde{r}$$

$$+ \; O(\tilde{r}^2 \log \tilde{r}) \qquad (5.13)$$

The matching condition is used to show that $S_1(x)$ equals the source strength $F(x)F'(x)$, and that other singular terms such as $\frac{1}{\tilde{r}} \sin \theta$ should not appear in φ_1.

A function $h_1(x)$ appears which is analogous to $g_1(x)$ in the sense that it can be determined only by finding a complete solution to the nonlinear equation 5.12 which satisfies the boundary condition at infinity. In the subsonic or supersonic case the corresponding term is the first term of an expansion for small r of a solution to the Prandtl-Glauert equation which becomes infinite far away from the body. One might then expect, or hope, that $h_1(x)$ is zero for transonic flow. However, the term $\tilde{r} \sin \theta$ does represent a solution to the homogeneous equation, and the present procedure does not provide any justification for omitting it. A term proportional to $\tilde{r} \cos \theta$ would also be a solution, but is excluded because the flow must be symmetrical about a vertical plane through the body axis.

Determination of the form of the approximate inner solution can now be completed by matching with the outer expansion:

$$
\begin{aligned}
&\mu_1 = \delta^2 \log \delta \\
&\mu_2 = \delta^2 \\
&X_{1_{r*r*}} + \frac{1}{r*} X_{1_{r*}} + \frac{1}{r*^2} X_{1_{\theta\theta}} = 0 \\
&X_{2_{r*r*}} + \frac{1}{r*} X_{2_{r*}} + \frac{1}{r*^2} X_{2_{\theta\theta}} = 0 \\
&X_{1_{r*}}(x,F,\theta) = 0 \\
&X_{2_{r*}}(x,F,\theta) = F'(x) - A \sin\theta \\
&X_1(x,r*,\theta) = 2F(x)F'(x) \\
&X_2(x,r*,\theta) = F(x)F'(x) \log r* + g_1(x;K) + AF^2(x) \frac{\sin\theta}{r*}
\end{aligned} \tag{5.14}
$$

The real need for using an inner expansion would become apparent in deriving higher-order approximations. Equations 5.8 through 5.10 are written out in sufficient detail that all terms necessary for carrying out the expansion to order δ^4 can be obtained without returning to the full differential equation and boundary condition. It is obvious that the equations become quite lengthy for terms of this order. Continuation of the procedure would be tedious, but would probably not require an unreasonably large amount of work. However, certain of the terms in the potential which are of order δ^4 at the body surface are of order δ^{14} in the outer limit, and it would therefore be virtually impossible to obtain uniform validity to order δ^4 by using only an outer expansion.

Pressure and lift

Since the vorticity is again of very small order, the pressure is given approximately by the isentropic relation

$$\frac{p}{p_\infty} \sim \left\{1 + \frac{\gamma - 1}{2} M^2\left(1 - \frac{q^2}{U^2}\right)\right\}^{\gamma/(\gamma-1)} \tag{5.15}$$

In terms of the inner expansion, the pressure coefficient is

$$C_p = -2\delta^2 \log \delta X_{1_x} - \delta^2\Big(2X_{2_x} + 2A \sin\theta X_{2_{r*}} + 2A \frac{\cos\theta}{r*} X_{2_\theta}$$

$$+ X_{2_{r*}}^{\;2} + \frac{1}{r*^2} X_{2_\theta}^{\;2}\Big) + o(\delta^4 \log^2 \delta) \tag{5.16}$$

Substituting for X_1 and X_2, and evaluating the result at the body surface,

$$C_p = -\delta^2 \log \delta[4S_1'] - \delta^2[2S_1' \log F + 2g_1' + F'^2$$

$$+ 4AF' \sin\theta + A^2 + 2A^2 \cos 2\theta] + O(\delta^4 \log^2 \delta) \tag{5.17}$$

A similarity law analogous to equation 3.81 could be formulated.

In the first approximation, the lift force perpendicular to the free stream equals the force normal to the body axis. The lift coefficient referred to the base area is then given by

$$C_L \sim -\frac{1}{\pi\,\delta^2 F^2(1)} \int_0^1 \int_0^{2\pi} C_p \sin\theta\, \delta F(x)\, d\theta\, dx \tag{5.18}$$

Integration over θ drops out all terms in equation 5.17 except the one containing a factor of $\sin\theta$; the relative error in C_L will be of the same order as in the pressure.

$$C_L = \frac{4A\delta}{F^2(1)} \int_0^1 F(x)F'(x)\, dx \left\{1 + O(\delta^2 \log^2 \delta)\right\} \tag{5.19}$$

The first-order lift coefficient in terms of α is determined simply by completing the integration and substituting α for $A\delta$. The error estimate in terms of α and δ is obtained by showing that the error in C_p is at most $O(\delta^4 \log^2\delta, \alpha^4 \log^2\delta)$, and then repeating the derivation of C_L. In this manner it is found that

$$C_L = 2\alpha + O\left(\alpha\delta^2 \log^2\delta\, , \frac{\alpha^4}{\delta} \log^2\delta\right) \tag{5.20}$$

In the first approximation, therefore, the lift coefficient is the same as for subsonic and supersonic velocities. Since the coefficient is defined in terms of the base area, the first-order lift force is zero if the rear of the body is pointed. The two terms given in the error estimate are of the same order of magnitude in a mathematical sense, but for very large or very small A, the actual values could differ considerably. Furthermore this representation of the error shows that the expansion is not uniformly valid for large A; uniformity is achieved for $0 \le |A| \le A_0 < \infty$.

The procedure could be continued to obtain "second-order" terms in the potential which are of order $\delta^4 \log^2 \delta$, $\delta^4 \log \delta$, and δ^4. The unspecified function $h_1(x)$ would then appear in the inner expansion, and would probably contribute to the lift. One would therefore expect to obtain only a second-order similarity law for the lift, instead of a second-order formula expressed entirely in terms of known functions.

It should be pointed out that the expansion for circular bodies at incidence can not be considered as a special case of the expansion derived in Section 4 for bodies of nearly circular cross-section. Suppose the coordinate system for the body at an angle of attack were chosen so that the x-axis passes through the nose of the body, but is parallel to the free stream instead of coincident with the body axis. The boundary condition still requires that the component of the free-stream velocity normal to the body surface be cancelled at every point on the surface. This condition could also be interpreted as the boundary condition for a body of noncircular section at zero incidence; however, the deviations from a circular section would be of order α, i.e. of the same order as δ, while for the problem discussed in Section 4 the deviations are of smaller order than δ. The limit procedure of this section is therefore not equivalent to that of Section 4.

6. THIN WINGS AT ZERO INCIDENCE

Introduction

In this section we shall consider wings at zero incidence, in the limit of very small thickness. The thickness ratio is measured by a parameter δ, which will tend to zero, and the maximum semispan is denoted by b. The unit of length is taken equal to the maximum chord; hence b is essentially an aspect ratio.

We first discuss some general properties of the outer expansion which are valid for an arbitrary body shape; it will only be assumed that in the limit some body dimension tends to zero in such a manner that the flow disturbances due to the body are small. To fix the ideas, consider a slightly supersonic flow and let the Mach number decrease toward one in the limit. At the station x = 1 the radius of a Mach cone from a point with x = 0 is $\frac{1}{\sqrt{|1 - M^2|}}$; the radius will increase as M approaches one. It is expected that the outer limit should be chosen so that a point on the Mach cone remains fixed in the distorted coordinates. That is, one should define $\tilde{r}$ to be proportional to $r\sqrt{|1 - M^2|}$. For any other choice the Mach cone would disappear from the problem in the first approximation; either one would approach the free-stream conditions in the limit, or else a point $\tilde{r}$ = constant would remain too close to the body and the first-order solution would not satisfy the boundary condition at infinity. Furthermore, if the first approximation to the potential is of the form $\varepsilon_1\varphi_1(x,\tilde{r})$, it can be shown either from the shock polar or the equation of motion that a completely general transonic theory can be obtained only if ε_1 is proportional to $M^2 - 1$. The suggested selections of $\tilde{r}$ and ε_1 are in agreement with the results derived for a body of revolution, and it will be verified later that similar agreement is obtained for the wing problem. Thus it appears that the length scale for the outer expansion and the order of magnitude of the flow disturbances can be expressed in terms of M^2-1 independently of the body shape. On the other hand, it must be possible to express the magnitudes of the flow disturbances by means of geometric parameters defined in terms of the body dimensions. This is accomplished with the use of the inner expansion. By comparing the inner and outer representations one then can relate M^2-1, and hence ε_1 and the scale factor for the outer limit, to the body parameters.

In the inner expansion the proper unit of length should be equal to some characteristic dimension of the body. Returning now to the wing problem, one sees that there are two characteristic lengths, b and δ. It will be argued later that b rather than δ should be taken as the relevant dimension. If b increases as $\frac{1}{\sqrt{|1 - M^2|}}$, the length scale of the inner expansion is the same as that for the outer expansion. This is the case of "nonslender" wings. The wing span remains of the same order as the distance to the Mach cone, and the reduced aspect ratio, proportional to b $\sqrt{|1 - M^2|}$, is kept constant in the limit. (This is also true for the Prandtl-Glauert transformation of linear theory.) It is interesting to note that the significance of the reduced aspect ratio can be inferred directly from this purely physical argument, as well as indirectly by the mathematical reasoning given later. If on the other hand the wing span becomes negligible relative to the width of the Mach cone, the scales of the inner and outer expansions are different. In this case the reduced aspect ratio tends to zero and the wing is called slender. It should be noted that if the actual aspect ratio, proportional to b, remains fixed, or even increases sufficiently slowly, the wing will still be considered slender.

The preceding considerations suggest that the order of magnitude chosen for b will have a significant effect on the mathematical nature of the problem. A similar situation arises with respect to the order of magnitude of $|1 - M^2|$, and it will be useful to summarize the three possible cases which arise. It can be shown that for a body of revolution, of radius proportional to δ, the first-order equation is fundamentally different for the cases $|1 - M^2| \gg \delta^2$ and $|1 - M^2| \ll \delta^2$. The former case leads to a linear equation and the latter gives the flow for M exactly equal to one (see the discussion of equations 3.28 and 3.29). The change in the equation occurs at the critical case for which $1 - M^2$ is proportional to δ^2; for this case the first-order equation contains all the terms which appear in the other two cases. In Section 3 the relation $\frac{1 - M^2}{\delta^2}$ = constant was selected as the most general one, and an argument was given to justify the choice. The same type of reasoning will also be applied for the flow over a wing.

In a similar manner we will investigate the mathematical changes in the wing problem which depend on the relation between b and δ. First the behavior of $\frac{\delta}{b}$ in the limit should be examined. The case where this quantity is held constant does not really correspond to a wing, but instead should be considered as describing a slender body of noncircular cross-section. Such a body belongs to the class of shapes mentioned in the first paragraph of Section 4, where the parameter τ was chosen to be a constant. This case will not be of interest in the present discussion because the restriction to a "thin" wing implies that $\frac{\delta}{b}$ should tend to zero as δ goes to zero. That is, b is not allowed to approach zero as rapidly as δ. Second, we require that the first approximation include effects of aspect ratio, and not reduce to a two-dimensional problem. This requirement leads to a second limitation on the variation of b with δ. It will be shown that one may allow b to approach infinity as δ goes to zero, but that b should not increase more rapidly than $\delta^{-1/3}$. The cases $b\delta^{1/3}$= constant and $\frac{\delta}{b}$ = constant turn out to be critical cases at which there is an essential change in the mathematical nature of the problem. The cases of interest will be 1) the critical case $b\delta^{1/3}$ = constant and 2) the intermediate case for which $b\delta^{1/3} \to 0$ and $\frac{\delta}{b} \to 0$; the first case corresponds to a constant reduced aspect ratio, while for the second case the reduced aspect ratio will tend to zero.

In the first of these cases the fundamental mathematical assumption is that the inner and outer expansions coincide. The first term of the expansion satisfies the nonlinear transonic equation, and it is not possible to describe the flow near the wing by reducing this equation to the Laplace equation. The wing might therefore be called "nonslender". Since a single expansion is to be valid everywhere, it is expected that the wing semispan b should be distorted in the same manner as the y-coordinate. It is this consideration which will lead to the relation $b\delta^{1/3}$ = constant. The aspect ratio, proportional to b, will increase toward infinity as $\delta^{-1/3}$. Using the relation which will be obtained to relate M and δ, it will be shown that the reduced aspect ratio, proportional to $b\sqrt{|1 - M^2|}$, will be kept fixed in the limit; the mathematical development therefore leads to the result predicted previously by a heuristic argument.

In the second case of interest, b is allowed to vary with δ in such a manner that $b\delta^{1/3}$ and $\frac{\delta}{b}$ both tend to zero. This is the case of "slender" wings. The exact nature of the dependence of b on δ need not be specified; the relation between the two parameters will, however, affect the order of magnitude of terms in the second and higher approximations. The assumption that $b\delta^{1/3}$ approaches zero will be exactly equivalent to the assumption that the length scales of the inner and outer expansions are different. The reduced aspect ratio will also tend to zero in the limit; this fact affords still another possible definition of the problem.

The expansion procedure for this case will again be derived by studying the differential equation and boundary conditions. In determining the approximate form of expansion, however, it is convenient to make use of the idea that one expects the expansion for nonslender wings to remain valid if $b\delta^{1/3}$ is allowed to approach zero. That is, one assumes that each term in this expansion itself possesses an asymptotic expansion for small values of $b\delta^{1/3}$. Actually two such expansions are necessary -- an inner and an outer. Each of the two representations of the potential for a slender wing may therefore be stated as a double series. If the exact relation between b and δ were specified, the terms could be ordered so that each expansion would be given by a single series, with the decreasing asymptotic sequence expressed in terms of δ and $b\delta^{1/3}$ instead of just δ. For convenience it will be assumed that such an ordering is possible, and the inner and outer expansions will each be represented by a single series. In general it is expected that these series will not be expressed in terms of the variables used in the original expansion (for nonslender wings), but that the proper coordinates will involve $b\delta^{1/3}$ as well as δ. As indicated previously, the unit of length for the inner expansion is b. The scale for the outer expansion will be a function of δ and $b\delta^{1/3}$ which must be determined. It will be shown that this function should be proportional to $\frac{1}{\sqrt{|1 - M^2|}}$, as expected from physical considerations.

The leading term of the inner expansion obeys Laplace's equation, and the form of the first-order solution near the wing is given by the most general solution of this equation which satisfies the boundary condition at the wing surface. (For nonslender wings Laplace's equation is not valid, and it is not possible to determine anything about the first-order solution without solving the full nonlinear transonic equation.) The outer expansion will be of the same nature as in previous cases, and a matching of the two expansions will again be necessary. In the matching procedure, the outer expansion for small values of the (outer) radial coordinate should be compared with the inner expansion for large values of the (inner) radial coordinate. For the first approximation, the matching depends only on the distribution of cross-sectional area, and not on the details of cross-section shape. Hence the first term in the outer expansion is influenced only by the area distribution. Furthermore the coordinate distortion (i.e. the length scale) for the outer expansion, the order of magnitude of disturbances far away from the wing, and the similarity parameter relating M to b and δ all depend only on the maximum cross-sectional area. These three results may therefore be expressed in exactly the same form as for the body of revolution. By inserting appropriate constant factors, the same conclusion can be shown to be true for nonslender wings.

Thus one finds a close resemblance between the two wing problems and the case of a body of revolution. This result might not be wholly unexpected. Physical reasoning suggests that the length scale for the outer expansion can be expressed in terms of M, independently of the body shape. Starting from this assumption it can be shown, as indicated previously, that the most general transonic equation is obtained only if the disturbance potential far away from the body is of order $|1 - M^2|$. Detailed knowledge of the body shape would therefore be necessary only to establish the appropriate dependence, in the limit, of the Mach number on the characteristic body dimensions, and one might hope to find a unique order-of-magnitude relationship. For the cases treated the relation turns out to involve only the order of the body cross-sectional area. The same dependence might be expected for other cases, but further investigation would be necessary before complete generality could be assumed.

Nonslender wings

In rectangular coordinates, the differential equation obtained from 2.7 and 2.8 is

$$
\begin{aligned}
(1 - M^2)u_x + v_y + w_z = M^2\Big\{&(\gamma + 1)uu_x + v(u_y + v_x) + w(u_z + w_x) \\
&+ (\gamma - 1)u(v_y + w_z) + v^2 v_y + w^2 w_z + uv(u_y + v_x) + uw(u_z + w_x) \\
&+ vw(v_z + w_y) + \frac{\gamma + 1}{2} u^2 u_x + \frac{\gamma - 1}{2}(v^2 + w^2)u_x \\
&+ \frac{\gamma - 1}{2}(u^2 + v^2 + w^2)(v_y + w_z)\Big\} \qquad (6.1)
\end{aligned}
$$

The coordinates x,y,z and the perturbation velocities u,v,w are all non-dimensional, with the maximum chord and the free-stream velocity taken as reference quantities. From the three-dimensional form of the shock polar equation, it may be seen that v and w can not be of larger order than u, and the first-order condition of irrotationality follows directly. The flow may therefore be described approximately by a potential Φ, and equation 6.1 becomes

$$(1 - M^2)\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = (\gamma + 1)\Phi_x\Phi_{xx} + \dots \qquad (6.2)$$

The wing plan form is described by

$$s_1(x) < \frac{y}{b} < s_2(x) \qquad 0 < x < 1 \qquad (6.3)$$

where b is the ratio of maximum semispan to chord, and the functions $s_1(x)$ and $s_2(x)$ are of order one. For simplicity of notation, $s_1(x)$ and $s_2(x)$ are also assumed single-valued, but the methods to be used do not exclude other wing shapes. The wing surface is given by

$$S(x,y,z;\delta,b) = z \mp \delta h(x, \frac{y}{b}) = 0 \qquad (6.4)$$

where δ is one-half the thickness ratio, and the upper and lower signs refer to the upper and lower surfaces respectively. The boundary condition at the surface is determined by the requirement of tangent flow:

$$w(x,y,\delta h) = \left\{1 + u(x,y,\delta h)\right\}\delta \frac{\partial h}{\partial x} + v(x,y,\delta h)\,\frac{\delta}{b}\,\frac{\partial h}{\partial(\frac{y}{b})} \qquad (6.5)$$

where $h(x, \frac{y}{b})$ and its derivatives are assumed to be of order one. In 6.5 and in subsequent expressions the condition is written only for the upper surface; due to symmetry of the differential equation and boundary condition, the flow field must be symmetrical about the plane $z = 0$.

We will now investigate the behavior of the potential Φ in the limit as δ approaches zero. For the moment we will consider only the special mathematical case for which the inner and outer expansions coincide. For this case the wing will be called "nonslender". It will be shown that the quantity $b\delta^{1/3}$ is held fixed as δ goes to zero. As expected from physical considerations, the reduced aspect ratio also remains constant.

In writing an expansion for Φ, the same coordinate distortion will be used for both y and z, so that the terms Φ_{yy} and Φ_{zz} in equation 6.2 will both contribute to the differential equation for the first approximation. (If instead we were to choose Φ_{yy} of smaller order than Φ_{zz}, the first-order solution would describe the two-dimensional flow over an infinite wing; if Φ_{zz} were too small, the solution agreeing with the boundary conditions would turn out to be trivial.) Since it is expected that M and b in general will not be constant as δ goes to zero, these quantities are replaced by certain unspecified parameters which are expressed as combinations of the physical quantities. The new parameters will be denoted by $K(M,\delta)$ and $\sigma(b,\delta)$. The parameter K is determined in the same manner as for the body of revolution. The assumption that the inner and outer expansions coincide will lead to a relation between the orders of magnitude of b and δ, and the equation σ = constant then allows one to choose a suitable function for σ.

We therefore express Φ by the following asymptotic expansion:

$$\Phi(x,y,z;\delta,M,b) \sim \sum_{i=1}^{n} \varepsilon_i(\delta)\varphi_i(x,\tilde{y},\tilde{z};K,\sigma)$$

$$\tilde{y} = yf(\delta)$$

$$\tilde{z} = zf(\delta) \qquad (6.6)$$

$$K = K(M,\delta)$$

$$\sigma = \sigma(b,\delta)$$

where $\varepsilon_1(\delta) << 1$ and $\varepsilon_{i+1}(\delta) << \varepsilon_i(\delta)$.

Substituting 6.6 into the differential equation 6.2,

$$(1 - M^2)\varepsilon_1\varphi_{1_{xx}} + \varepsilon_1 f^2\varphi_{1_{\tilde{y}\tilde{y}}} + \varepsilon_1 f^2\varphi_{1_{\tilde{z}\tilde{z}}} = (\gamma + 1)\varepsilon_1^{\,2}\varphi_{1_x}\varphi_{1_{xx}} + \cdots \qquad (6.7a)$$

Using the arguments given previously it can be shown that all terms omitted in this expression are necessarily negligible in the first approximation, and that the four remaining terms should all be of the same order of magnitude. We may therefore choose the following relations among the parameters:

$$\varepsilon_1 = f^2$$

$$\frac{1 - M^2}{\varepsilon_1} = K = \text{constant} \qquad (6.7b)$$

These expressions are in agreement with the previous statements that f^2 and ε_1 should be proportional to $|1 - M^2|$. These parameters will now be related to the body parameters by considering the boundary conditions.

By assumption the outer expansion is valid near the wing, and it should therefore satisfy the boundary condition at the wing surface. At the wing surface, $\tilde{z}$ is of order $\delta f(\delta)$, and hence must tend to zero as δ approaches zero. We shall assume that $\varphi_{1_{\tilde{z}}}$ may be expanded in a Taylor series about $\tilde{z} = 0$. From equations 6.4 and 6.6 it can be shown that

$$\varepsilon_1 f \varphi_{1_{\tilde{z}}}(x,y,0;K,\sigma) + \ldots = \delta h_x(x, \frac{\tilde{y}}{bf}) + \ldots \tag{6.8a}$$

for

$$s_1(x) < \frac{\tilde{y}}{bf} < s_2(x)$$

All omitted terms are necessarily of higher order than the terms retained. By definition K and σ remain constant in the limit, and $\varepsilon_1 f(\delta)$ must therefore be of the same order of magnitude as δ. We may take the two quantities to be equal:

$$\varepsilon_1 f = \delta \tag{6.8b}$$

To obtain a proper description of the flow near the wing, a coordinate $\bar{y} = \frac{y}{b} = \frac{\tilde{y}}{bf}$ should be held fixed as δ goes to zero (see also the discussion immediately preceding equation 6.15). That is, for some neighborhood of the wing the y-coordinate should be stretched in the same manner as the wing semispan b. In terms of $\bar{y}$, the second argument of $\varphi_{1_{\tilde{z}}}$ in the boundary condition 6.8a is $\bar{y}bf$. If the outer expansion is to be valid at the wing surface, then the outer limit may be applied to the boundary condition. In order that $\varphi_{1\tilde{z}}(x,\bar{y}bf,0)$ depend on $\bar{y}$ in the limit, it is necessary that $bf(\delta)$ be held constant, i.e. that $\bar{y}$ and $\tilde{y}$ differ only by a constant factor. In other words, a single stretching of the y-coordinate is correct for the entire flow field, and must agree with the stretching of the wing; this is consistent with the original assumption that the inner and outer expansions coincide. The parameter σ is required to be a function of b and δ which remains constant in the limit; a convenient definition satisfying this requirement is

$$\sigma = bf = \text{constant} \tag{6.8c}$$

The results for the coordinate distortion and similarity parameters, obtained by combining equations 6.7b, 6.8b, and 6.8c, may be summarized as follows:

$$\begin{aligned} \tilde{y} &= \delta^{1/3} y \\ \tilde{z} &= \delta^{1/3} z \\ K &= \frac{1 - M^2}{\delta^{2/3}} \\ \sigma &= b\delta^{1/3} \end{aligned} \tag{6.9}$$

The similarity parameters K and σ are equivalent to those given in Ref. 2. Combining the definitions found for K and σ, it can be shown that for the present case the quantity $b\sqrt{|1 - M^2|}$ is held fixed as

δ approaches zero. Thus, as anticipated from a heuristic argument in the introductory paragraphs of this section, the mathematical assumption that the inner and outer expansions coincide has the physical significance that the reduced aspect ratio remains constant in the limit.

The equations describing the first approximation are obtained from 6.7 and 6.8:

$$\varepsilon_1 = \delta^{2/3}$$

$$K\varphi_{1_{xx}} + \varphi_{1_{\tilde{y}\tilde{y}}} + \varphi_{1_{\tilde{z}\tilde{z}}} = (\gamma + 1)\varphi_{1_x}\varphi_{1_{xx}} \tag{6.10}$$

$$\varphi_{1_{\tilde{z}}}(x,\tilde{y},0) = h_x(x, \frac{\tilde{y}}{\sigma})$$

In determining the second approximation one should first examine higher-order terms in the shock relations to show that vorticity may still be neglected. This can be done in a manner analogous to the discussion for a body of revolution. Then it can be shown that the second term in the expansion of Φ is described by the following:

$$\varepsilon_2 = \delta^{4/3}$$

$$\begin{aligned} K\varphi_{2_{xx}} + \varphi_{2_{\tilde{y}\tilde{y}}} + \varphi_{2_{\tilde{z}\tilde{z}}} = {} & (\gamma + 1)(\varphi_{1_x}\varphi_{2_{xx}} + \varphi_{2_x}\varphi_{1_{xx}}) + 2\varphi_{1_{\tilde{y}}}\varphi_{1_{x\tilde{y}}} \\ & + 2\varphi_{1_{\tilde{z}}}\varphi_{1_{x\tilde{z}}} - 2\gamma K\varphi_{1_x}\varphi_{1_{xx}} + \frac{1}{2}(\gamma + 1)(2\gamma - 1)\varphi_{1_x}^{\,2}\varphi_{1_{xx}} \end{aligned} \tag{6.11}$$

$$\varphi_{2_{\tilde{z}}}(x,\tilde{y},0) = \varphi_{1_x}(x,\tilde{y},0)h_x(x, \frac{\tilde{y}}{\sigma})$$

If values are given for K and σ, a change in δ distorts the flow in a relatively simple manner which can be specified (to a certain order of magnitude). Knowledge of one solution would therefore imply knowledge of an entire family of similar flows. One of the useful similarity laws is the rule for the pressure at the wing surface. Following the procedure of equations 3.75 through 3.77, the pressure coefficient is found to be

$$C_p = -2\delta^{2/3}\varphi_{1_x} + O(\delta^{4/3}) \tag{6.12}$$

Consequently the similarity rule is given by

$$\frac{C_p}{\delta^{2/3}} = fn(x,\tilde{y};K,\sigma) + O(\delta^{2/3}) \tag{6.13}$$

This expression may be used to relate the pressures acting on two affinely related wings having the same values of K and σ.

Expansions for slender wings

The second problem of interest is that of "slender" wings. There are actually three possible ways of defining the problem, all of which can be shown to be equivalent. From a mathematical point of view the fundamental assumption is that the inner expansion is essentially different from the outer expansion. Using a physical picture one could instead take the reduced aspect ratio as a basic parameter and begin the discussion with the requirement that $b\sqrt{|1 - M^2|}$ tend to zero. Or one could start from the case of nonslender wings by specifying that the quantity $\sigma = b\delta^{1/3}$ should approach zero in the limit. The last choice seems to be the most convenient; we will specify that σ must tend to zero with δ, subject only to the restriction that $\frac{\delta}{b}$ must also approach zero in order that the wing remain "thin". The exact nature of the relation between σ and δ will be important only in determining the order of terms in the second and higher approximations. In the present discussion we shall be interested primarily in the first approximation. It will be shown that for the given restrictions on σ there are only two possibilities for the order of magnitude of the largest neglected term, and a completely general error estimate can be written quite simply. It will therefore be sufficient to require

$$\lim_{\delta \to 0} \sigma = 0$$
$$\lim_{\delta \to 0} \frac{\sigma}{\delta^{4/3}} = 0 \qquad (6.14)$$

where σ is defined by $\sigma = b\delta^{1/3}$.

In writing the form of expansion we assume that the individual terms of the expansion for nonslender wings will possess (inner and outer) expansions for small σ. In principle the slender-wing solution could be obtained by actually performing such expansions, and each of the two representations of the potential would be expressed by a double series. The terms in these series could presumably be ordered if further information were given about the dependence of σ upon δ. Each of the expansions for Φ could then be written as a single series in terms of a decreasing sequence of functions which depend on both δ and σ. This will be the assumed form of expansion.

First the flow near the wing will be considered. In determining the proper coordinates for an inner expansion, the general rule is that the scale should be of the order of magnitude of the body dimension. For thin wings the characteristic dimension of the wing in the y-direction (the semispan b) differs in order of magnitude from the characteristic dimension in the z-direction (the thickness δ). It may appear at first glance that the coordinates to be used are $\frac{y}{b}$ and $\frac{z}{\delta}$. However, (as can be seen from later calculations) if these variables were used the first term of the potential would obey the trivial equation $\Phi_{zz} = 0$. If one instead uses $\frac{y}{b}$ and $\frac{z}{b}$, Laplace's equation $\Phi_{yy} + \Phi_{zz} = 0$ is obtained. Formally one could consider the first equation as an inner equation relative to the second equation. Solutions to the two could then be matched in the usual way. This leads, however, to trivialities which can easily be bypassed. (Note that the second equation contains the first, in the sense that if one introduces $\frac{y}{b}$ and $\frac{z}{\delta}$ into Laplace's equation one obtains $\Phi_{zz} = 0$ in the limit.) We therefore assume the inner expansion to be of the form

$$\Phi(x,y,z;\delta,M,b) \sim \sum_{i=1}^{n} \mu_i(\delta,\sigma)\bar{\varphi}_i(x,\bar{y},\bar{z};K)$$

$$\bar{y} = \frac{y}{b} \qquad (6.15)$$

$$\bar{z} = \frac{z}{b}$$

In the case of nonslender wings σ is constant and $\bar{y}$ is proportional to $\tilde{y}$. For the present case $\sigma \ll 1$, and it is to be expected that the coordinates $\bar{y}$ and $\tilde{y}$, and hence the corresponding expansions, will be fundamentally different. The inner expansion will therefore be matched with an outer expansion which satisfies the boundary condition at infinity. It is also expected that the coordinate distortion for the outer expansion may differ from that obtained in the case of nonslender wings, and may depend on σ as well as on δ. In cylindrical coordinates the outer expansion will be as follows:

$$\Phi(x,y,z;\delta,M,b) \sim \sum_{i=1}^{n} \varepsilon_i(\delta,\sigma)\varphi_i(x,\tilde{r},\theta;K)$$
$$\tilde{r} = rf(\delta,\sigma) = \bar{r}bf(\delta,\sigma) = \sqrt{\bar{y}^2 + \bar{z}^2}\; bf(\delta,\sigma) \tag{6.16}$$

In order to relate the present problem to the case of a body of revolution at zero incidence, it will be convenient to introduce the idea of an "equivalent" body of revolution. This body will be defined as the body of circular cross-section which has the same longitudinal distribution of cross-sectional area as the wing. The equivalent body will be defined by $r = \delta_e F(x)$, where δ_e is the maximum radius and the maximum area is

$$\pi\delta_e^{\,2} = \pi kb\delta = \pi k\sigma\delta^{2/3} \tag{6.17}$$

The constant of proportionality k depends only on cross-sectional geometry; e.g., for a delta wing of elliptic cross-section, k is equal to one. For the nonslender wing, the maximum cross-sectional area is proportional to $\delta^{2/3}$. According to 6.9, the coordinate distortion and the similarity parameter K can therefore be made the same as for the body revolution simply by inserting appropriate constant factors and rewriting the results in terms of δ_e. In the present case the same will be true, and the quantities K, f, and ε_1 will be expressed in terms of δ_e.

Form of slender-wing solution

Substituting the outer expansion into the full differential equation,

$$(1 - M^2)(\varepsilon_1\varphi_{1_{xx}} + \varepsilon_2\varphi_{2_{xx}} + \varepsilon_3\varphi_{3_{xx}} + \dots) + f^2(\varepsilon_1\varphi_{1_{\tilde{r}\tilde{r}}} + \varepsilon_2\varphi_{2_{\tilde{r}\tilde{r}}} + \varepsilon_3\varphi_{3_{\tilde{r}\tilde{r}}} + \dots)$$

$$+ f^2 \frac{1}{\tilde{r}} (\varepsilon_1\varphi_{1_{\tilde{r}}} + \varepsilon_2\varphi_{2_{\tilde{r}}} + \varepsilon_3\varphi_{3_{\tilde{r}}} + \dots) + f^2 \frac{1}{\tilde{r}^2}(\varepsilon_1\varphi_{1_{\theta\theta}} + \varepsilon_2\varphi_{2_{\theta\theta}} + \varepsilon_3\varphi_{3_{\theta\theta}} + \dots)$$

$$= M^2\Big\{(\gamma + 1)(\varepsilon_1\varphi_{1_x} + \varepsilon_2\varphi_{2_x} + \varepsilon_3\varphi_{3_x} + \dots)(\varepsilon_1\varphi_{1_{xx}} + \varepsilon_2\varphi_{2_{xx}} + \varepsilon_3\varphi_{3_{xx}} + \dots)$$

$$+ 2f^2(\varepsilon_1^2\varphi_{1_{\tilde{r}}}\varphi_{1_{x\tilde{r}}} + \dots + \frac{1}{\tilde{r}^2}\varepsilon_1^2\varphi_{1_\theta}\varphi_{1_{x\theta}} + \dots) + \frac{\gamma + 1}{2}(\varepsilon_1^3\varphi_{1_x}^2\varphi_{1_{xx}} + \dots)$$

$$+ (\gamma - 1)f^2(\varepsilon_1\varphi_{1_x} + \dots)(\varepsilon_1\varphi_{1_{\tilde{r}\tilde{r}}} + \varepsilon_1\frac{1}{\tilde{r}}\varphi_{1_{\tilde{r}}} + \varepsilon_1\frac{1}{\tilde{r}^2}\varphi_{1_{\theta\theta}} + \dots) + \dots\Big\} \quad (6.18)$$

Again a sufficiently general first-order transonic equation is obtained only if $(1 - M^2)$, ε_1, and f^2 are of the same order of magnitude as $\delta \to 0$. The approximate equation can be determined, and for small $\tilde{r}$ the largest terms in φ_1 will represent the velocity field due to fluid sources distributed along the x-axis:

$$K\varphi_{1_{xx}} + \varphi_{1_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{1_{\tilde{r}}} + \frac{1}{\tilde{r}^2}\varphi_{1_{\theta\theta}} = (\gamma + 1)\varphi_{1_x}\varphi_{1_{xx}}$$

$$\varphi_1 = S_1(x) \log \tilde{r} + g_1(x;K) + O(\tilde{r}^2 \log^2 \tilde{r}) \quad (6.19)$$

where $S_1(x)$ is the source strength and $g_1(x)$ describes the nonlinear interaction between sources. The source strength can be found by matching with the inner expansion, but $g_1(x)$ must remain unspecified because we do not know the full solution to the nonlinear equation.

Applying the inner limit to the differential equation,

$$
\begin{aligned}
&(1 - M^2)(\mu_1\bar{\varphi}_{1_{xx}} + \ldots) + \frac{\delta^{2/3}}{\sigma^2}(\mu_1\bar{\varphi}_{1_{\bar{y}\bar{y}}} + \mu_2\bar{\varphi}_{2_{\bar{y}\bar{y}}} + \ldots) \\
&\quad + \frac{\delta^{2/3}}{\sigma^2}(\mu_1\bar{\varphi}_{1_{\bar{z}\bar{z}}} + \mu_2\bar{\varphi}_{2_{\bar{z}\bar{z}}} + \mu_3\bar{\varphi}_{3_{\bar{z}\bar{z}}} + \ldots) \\
&= M^2\Big\{(\gamma + 1)(\mu_1{}^2\,\bar{\varphi}_{1_x}\bar{\varphi}_{1_{xx}} + \ldots) + 2\frac{\delta^{2/3}}{\sigma^2}(\mu_1{}^2\,\bar{\varphi}_{1_{\bar{y}}}\bar{\varphi}_{1_{x\bar{y}}} + \ldots \\
&\quad + \mu_1{}^2\,\bar{\varphi}_{1_{\bar{z}}}\bar{\varphi}_{1_{x\bar{z}}} + \ldots) + \frac{\delta^{4/3}}{\sigma^4}(\mu_1{}^3\,\bar{\varphi}_{1_{\bar{y}}}^{\,2}\,\bar{\varphi}_{1_{\bar{y}\bar{y}}} + \ldots + \mu_1{}^3\bar{\varphi}_{1_{\bar{z}}}^{\,2}\,\bar{\varphi}_{1_{\bar{z}\bar{z}}} \\
&\quad + \ldots + 2\mu_1{}^3\,\bar{\varphi}_{1_{\bar{y}}}\bar{\varphi}_{1_{\bar{z}}}\bar{\varphi}_{1_{\bar{y}\bar{z}}} + \ldots) + \ldots\Big\} \qquad (6.20)
\end{aligned}
$$

The expanded boundary condition is

$$
\begin{aligned}
&\frac{\delta^{1/3}}{\sigma}\left\{\mu_1\bar{\varphi}_{1_{\bar{z}}}(x,y,0) + \mu_2\bar{\varphi}_{2_{\bar{z}}}(x,y,0) + \mu_3\bar{\varphi}_{3_{\bar{z}}}(x,y,0) + \ldots\right\} \\
&\quad = \left\{1 + \mu_1\bar{\varphi}_{1_x}(x,\bar{y},0) + \ldots\right\}\delta h_x(x,\bar{y}) \\
&\quad + \frac{\delta^{1/3}}{\sigma}\left\{\mu_1\bar{\varphi}_{1_{\bar{y}}}(x,\bar{y},0) + \ldots\right\}\delta h_{\bar{y}}(x,\bar{y}) + \ldots \\
&\quad - \frac{\delta^{2/3}}{\sigma^2}\left\{\mu_1\bar{\varphi}_{1_{\bar{z}\bar{z}}}(x,\bar{y},0) + \ldots\right\}\delta h(x,\bar{y}) + \ldots \qquad (6.21)
\end{aligned}
$$

By analogy with previous cases it is expected that the leading term of the expansion should satisfy Laplace's equation. That this is true can be seen by comparing the orders of magnitude of the terms appearing in equation 6.20. Since the perturbation velocities must vanish in the limit, the quantities μ_1 and $\frac{1}{\sigma}\delta^{1/3}\mu_1$ must both tend to zero with δ. It follows that terms such as $\bar{\varphi}_{1_{\bar{y}}}\bar{\varphi}_{1_{x\bar{y}}}$ and $\bar{\varphi}^2_{1_{\bar{y}}}\bar{\varphi}_{1_{\bar{y}\bar{y}}}$ can not appear in the first-order equation. The assumption that inner and outer expansions are different can be shown to imply that the terms containing $\bar{\varphi}_{1_{xx}}$ will also be negligible in the first approximation. As anticipated we therefore find that the largest term in Φ must satisfy the Laplace equation.

The largest term in the right side of the boundary condition 6.21 is equal to $\delta h_x(x,\bar{y})$. Multiplying both sides of the equation by $\sigma\delta^{-1/3}$, it is seen that one of the terms in the inner expansion of Φ should be of order $\sigma\delta^{2/3}$; the order might also be stated as $b\delta$ or $\delta_e^{\,2}$. This term in the solution may be expressed by a distribution of two-dimensional sources at $z = 0$, of local strength proportional to $h_x(x,\bar{y})$. However, it is possible that the source term might not be the largest term in the expansion. Any function which depends on x, but not on $\bar{y}$ and $\bar{z}$, is also a solution to Laplace's equation, and its derivative with respect to $\bar{z}$ is identically zero. We must allow for the possibility that such a function may appear as the first term of the expansion. Thus we should include in the solution an unspecified function which depends on the variable x and on the parameters δ, σ, and K. It is assumed that the order of magnitude of the function may be greater than or equal to the order of $\sigma\delta^{2/3}$; otherwise the function would not be of interest for a first approximation. It will turn out from a matching with the outer expansion that two orders of magnitude will appear, namely $\sigma\delta^{2/3}\log\sigma$ and $\sigma\delta^{2/3}$. For the moment we express the beginning of the inner expansion in the following form:

$$\Phi(x,y,z;\delta,M,b) \sim A(x;\delta,K,\sigma)$$

$$+ \delta_e^{\,2} \frac{1}{\pi k} \int_{s_1(x)}^{s_2(x)} h_x(x,\bar{\eta}) \log\sqrt{(\bar{y}-\bar{\eta})^2 + \bar{z}^2}\, d\bar{\eta} \qquad (6.22)$$

Strictly speaking one should postpone discussion of the source term until the order of magnitude of $A(x;\delta,K,\sigma)$ has been determined. If this function should be of a rather large order relative to $\delta_e^{\,2}$, it is possible that Laplace's equation might be correct only to describe $A(x;\delta,K,\sigma)$, and that additional terms should be retained for the equation which leads to terms of order $\delta_e^{\,2}$. However, in the present case this difficulty will not arise, and it is convenient to consider the source term and the unspecified function simultaneously.

Following Kaplun's ideas (14), a principle for matching the inner and outer expansions may be derived by introducing a class of functions $\psi(\delta)$ and a coordinate $r^{(\psi)}$ such that

$$r^{(\psi)} = \frac{\tilde{r}}{\psi(\delta)} = \frac{bf(\delta,\sigma)}{\psi(\delta)}\,\bar{r}$$

$$\lim_{\delta\to 0} \psi(\delta) = 0 \qquad (6.23)$$

$$\lim_{\delta\to 0} \frac{bf(\delta,\sigma)}{\psi(\delta)} = 0$$

If δ is allowed to approach zero with $r^{(\psi)}$ fixed, then $\tilde{r} \to 0$ and $\bar{r} \to \infty$. The approximate differential equation obtained by taking this limit is again Laplace's. One therefore expects that there is a solution of the Laplace equation which satisfies the boundary condition at the wing and which is uniformly valid for $0 \le \tilde{r} \le B\psi(\delta)$, for any $\psi(\delta)$ which satisfies the conditions 6.23. According to the general concepts of matching, the first term of the outer expansion should be valid in a region defined by $C\psi_o(\delta) \le \tilde{r} < \infty$, for functions $\psi_o(\delta)$ which tend to zero sufficiently slowly. Taking $B > C$, it is seen that there is an overlap domain $C\psi_o(\delta) \le \tilde{r} \le B\psi_o(\delta)$ in which both representations are uniformly valid, and therefore a matching between the two is possible. In this domain $\tilde{r}$ is small and $\bar{r}$ is large. Hence a matching in the overlap domain means that the inner solution for large values of $\bar{r}$ should be compared with the outer solution for small values of $\tilde{r}$.

The general solution of Laplace's equation satisfying the boundary condition at the wing is given by equation 6.22. Expanding this expression for large values of $\bar{r} = \sqrt{\bar{y}^2 + \bar{z}^2}$,

$$\Phi \sim A(x;\delta,K,\sigma) + \log \bar{r} \int_{bs_1(x)}^{bs_2(x)} \delta h_x(x, \frac{\eta}{b})\, d\eta \qquad (6.24)$$

Since the integral is one-half the rate of change of cross-sectional area,

$$\Phi \sim A(x;\delta,K,\sigma) + \delta_e^{\,2} F(x)F'(x) \log \bar{r} \qquad (6.25a)$$

where, according to equation 6.17, the equation $r = \delta_e F(x)$ is used to define the body of revolution having the same area distribution as the wing. Similarly, using equation 6.19, one may write the first term of the outer expansion for small values of $\tilde{r}$:

$$\begin{aligned}\Phi &\sim \varepsilon_1\left\{S_1(x)\ \log \tilde{r} + g_1(x;K)\right\} \\ &= \varepsilon_1\left\{S_1(x)\ \log \bar{r} + S_1(x)\ \log bf + g_1(x;K)\right\}\end{aligned} \tag{6.25b}$$

The matching principle states essentially that these two expressions must be identical. Equating the two, and using the result $\varepsilon_1 = f^2$ obtained from considering the outer expansion, we may write the following:

$$\begin{aligned}\varepsilon_1 &= \delta_e^{\ 2} \\ f(\delta,\sigma) &= \delta_e \\ S_1(x) &= F(x)F'(x) \\ A(x;\delta,K,\sigma) &= \delta_e^{\ 2}F(x)F'(x)\ \log\ (b\delta_e) + \delta_e^{\ 2}g_1(x;K)\end{aligned} \tag{6.25c}$$

Thus the function $A(x;\delta,K,\sigma)$ contributes to the first two terms of the inner expansion; it can be verified that Laplace's equation is correct for both the first and second terms.

The results obtained thus far will be summarized immediately below.

First-order results and error estimates

For the outer expansion it is found that

$$\varepsilon_1 = \delta_e^{\,2}$$

$$K\varphi_{1_{xx}} + \varphi_{1_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{1_{\tilde{r}}} + \frac{1}{\tilde{r}^2}\varphi_{1_{\theta\theta}} = (\gamma + 1)\varphi_{1_x}\varphi_{1_{xx}}$$

$$\varphi_1(x,\tilde{r},\theta) = F(x)F'(x)\,\log\tilde{r} + g_1(x;K) + \ldots \qquad (6.26)$$

$$\tilde{r} = \delta_e r$$

$$K = \frac{1 - M^2}{\delta_e^{\,2}}$$

The initial terms of the inner expansion are given by

$$\mu_1 = \delta_e^{\,2}\,\log\,(b\delta_e) = O\,(\sigma\delta^{2/3}\log\,\sigma) = O\,(b\delta\,\log\,b\delta^{1/3})$$

$$\mu_2 = \delta_e^{\,2} = O\,(\sigma\delta^{2/3}) = O\,(b\delta)$$

$$\bar{\varphi}_{1_{\bar{y}\bar{y}}} + \bar{\varphi}_{1_{\bar{z}\bar{z}}} = 0$$

$$\bar{\varphi}_{2_{\bar{y}\bar{y}}} + \bar{\varphi}_{2_{\bar{z}\bar{z}}} = 0$$

$$\bar{\varphi}_{1_{\bar{z}}}(x,\bar{y},0) = 0 \qquad (6.27)$$

$$\bar{\varphi}_{2_{\bar{z}}}(x,\bar{y},0) = h_x(x,\bar{y})$$

$$\bar{\varphi}_1(x,\bar{y},\bar{z}) = F(x)F'(x)$$

$$\bar{\varphi}_2(x,\bar{y},\bar{z}) = \frac{1}{\pi k}\int_{s_1(x)}^{s_2(x)} h_x(x,\bar{\eta})\,\log\sqrt{(\bar{y} - \bar{\eta})^2 + \bar{z}^2}\,d\bar{\eta} + g_1(x;K)$$

The quantities μ_1 and μ_2 are written in terms of δ_e and b only for conciseness.

The resemblance between the wing problem and the case of a body of revolution can now be seen clearly. In each case the coordinate distortion is proportional to the square root of the maximum cross-section area; the order of magnitude of the perturbation potential far away from the body is the same as the order of the cross-sectional area; and the relation between Mach number and body dimensions, found by keeping the similarity parameter K constant in the limit, requires that $1 - M^2$ be proportional to the maximum cross-section area. It follows that the length scale of the outer expansion is of order $\frac{1}{\sqrt{|1 - M^2|}}$, in agreement with the value predicted from physical considerations. The results also illustrate the three properties of slender wings: the inner and outer length scales are different, the parameter $b\delta^{1/3}$ tends to zero with δ, and the reduced aspect ratio vanishes in the limit.

Since an estimate of the error at the wing surface is of interest, the derivation will be continued to determine the order of magnitude of μ_3. In this determination, one must consider the inner expansion of the differential equation, the boundary condition at the wing surface, and the matching with the outer expansion. If σ tends to zero relatively slowly, the nonlinear term appearing in the first equation of the outer expansion is expected to be relatively large in the inner limit, and will give the largest contribution to the error. Accordingly, for $\sigma \gg \delta^{1/3}$, the largest neglected term in the differential equation determines μ_3, and all neglected terms in the boundary condition are of smaller order.

$$\mu_3 = b^2\delta_e{}^4 \log^2(b\delta_e) = O(\sigma^4\delta^{2/3}\log^2\sigma) = O(b^4\delta^2\log^2 b\delta^{1/3})$$

$$\bar{\varphi}_{3_{\bar{y}\bar{y}}} + \bar{\varphi}_{3_{\bar{z}\bar{z}}} = (\gamma + 1)\bar{\varphi}_{1_x}\bar{\varphi}_{1_{xx}} \qquad (6.28)$$

$$\bar{\varphi}_{3_{\bar{z}}}(x,\bar{y},0) = 0$$

The same order of magnitude is obtained by a matching with the expansion 6.19 of φ_1 for small $\tilde{r}$.

On the other hand, if σ approaches zero more rapidly than $\delta^{1/3}$ the span approaches zero almost as rapidly as the thickness, and the largest part of the error arises from the Taylor expansion of the boundary condition about $z = 0$. If $\sigma \ll \delta^{1/3}$, the next approximate differential equation is Laplace's, and the solution consists of a distribution of sources. As in the first approximation, the corresponding term in the outer solution behaves as $\log \tilde{r}$. The proper matching then shows that μ_3 must be larger by a factor of $\log \sigma$ than the value required by the boundary condition:

$$\mu_3 = \frac{\delta_e{}^4}{b^2} \log(b\delta_e) = O(\delta^2 \log \sigma) = O(\delta^2 \log b\delta^{1/3})$$

$$\bar{\varphi}_{3_{\bar{y}\bar{y}}} + \bar{\varphi}_{3_{\bar{z}\bar{z}}} = 0 \qquad (6.29)$$

$$\bar{\varphi}_{3_{\bar{z}}}(x,y,0) = -h(x,\bar{y})\bar{\varphi}_{2_{\bar{z}\bar{z}}}(x,\bar{y},0)$$

Similarity law for C_p

A formula for the pressure coefficient is obtained in the same manner as before. Omitting all terms smaller than those neglected in Φ,

$$C_p = -2\delta_e^2 \log (b\delta_e)\bar{\varphi}_{1x} - 2\delta_e^2\bar{\varphi}_{2x}$$

$$+ O\left\{b^2\delta_e^4 \log^2(b\delta_e) \quad \frac{\delta_e^4}{b^2} \log (b\delta_e)\right\} \qquad (6.30)$$

With the use of equation 6.17 relating δ, b, and δ_e, a similarity law may be formulated for the pressure at the wing surface:

$$\frac{C_p}{\delta_e^2} + \left\{F(x)F''(x) + F'^2(x)\right\} \log (b^3\delta)$$

$$= fn(x,\frac{y}{b};K) + O\left\{b^3\delta \log^2(b^3\delta), \frac{\delta}{b} \log (b^3\delta)\right\} \quad (6.31)$$

A similar expression was derived in Ref. 9. The result may be used to compare surface pressures in any two problems for which the similarity parameter K and the wing shape functions $s_1(x)$, $s_2(x)$, and $h(x, \frac{y}{b})$ are the same. The result requires only that $b\delta^{1/3}$ and $\frac{\delta}{b}$ be small, and says nothing further about the relation between b and δ; in particular, $b\delta^{1/3}$ need not be constant.

If $b\delta^{1/3}$ is not small, the rule breaks down because the inner and outer expansions are no longer distinct, and the flow near the wing can not be described by the Laplace equation. The unspecified function in equation 6.31 must then depend on this quantity as well as on the ones indicated. In this case the similarity law 6.13 may be used. The present result is also incorrect if $\frac{\delta}{b}$ is not small, because the pressure at the wing surface can not be calculated at the plane $z = 0$, and the unknown function has to depend on $\frac{\delta}{b}$. If b is of order δ we should properly speak of a slender body rather than a thin wing. In Section 4 results were derived for the special case of slender bodies for which the deviations from a circular cross-section are small. A more general slender-body theory could be derived with the aid of the methods discussed in the present section.

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