A STUDY OF CROSS POLARIZATION EFFECTS

IN PARABOLOIDAL ANTENNAS

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Vassilios Kerdemelidis

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ABSTRACT

In this report the induced surface current method is used to investigate the spatial structure of the radiated electric field for a number of paraboloidal antennas. The paraboloids are excited by three different types of feeds, namely, a small electric dipole, an elemental plane wave source, and a rectangular horn.

For the case of electric dipole excitation, formulas are derived that show the following characteristics:

(i) For a reflector of constant ratio of focal length to the aperture diameter the magnitude of the cross-polarized lobe nearest to the antenna axis (paraboloid axis) remains constant relative to the maximum of the main lobe of the principally-polarized wave and is independent of the aperture size.

(ii) For a given aperture size the magnitude of the cross-polarized component relative to its own principally-polarized maximum decreases with the focal length.

(iii) The position of the maximum of the cross-polarized lobe depends only on the aperture size and is independent of the focal length.

The problem of cross-polarization is also solved by using a simple model which gives results that are in surprisingly close agreement with those obtained by the more complete expressions. In addition this crude model explains the angular variation of the

amplitude of cross-polarization component at angles not necessarily small from the paraboloid axis.

For a paraboloid excited by an elemental Huyghens source the cross-polarization in the forward direction is reduced but the component in the laterally-directed radiation is increased relative to that of an electric dipole.

In the case of the horn-excited paraboloid we obtain a formula that explains the experimentally-observed large cross-polarization.

Finally, we show that the problems of the paraboloids excited by a small electric dipole and a plane (Huyghens) source are merely particular cases of the horn excitation problem.

Partial List of Symbols

α, β, ϕ	the paraboloidal coordinates
$\beta = \beta_o$	equation of the paraboloidal reflector surface
$\alpha = \alpha_o$	size of the paraboloid
$f = \beta_o^2/2$	paraboloid focal length
D	aperture of the paraboloid
q	α_o/β_o
Z	$k\, \alpha_o \beta_o \sin \theta$
w	$k\, \alpha_o^2 (1 - \cos \theta)$
β	$w/Z = q \tan \frac{\theta}{2}$
θ_{cr}	$2 \tan^{-1}(\frac{1}{q})$, the critical angle
$Z_o = \eta$	the free space impedance
A_o, A_2, A_4	the coefficients in the expansion

$$(1 + q^2 t^2)^{-2} = A_o + t^2 A_2 + t^4 A_4$$

$$A = \frac{1}{2}(A_o + \frac{1}{2} A_2 + \frac{1}{3} A_4)$$

For electric dipole:

$$N_x = N_{ox} \mathcal{N}_x$$

$$= N_{ox} \left\{ \beta_o^2 I_o^{(1)}(Z) + \alpha_o^2 \cos 2\phi \, I_2^{(3)}(Z) \right\}$$

$$N_y = N_{ox} \sin 2\phi \, \mathcal{N}_y$$

$$= N_{ox} \sin 2\phi \left\{ \alpha_o^2 I_2^{(3)}(Z) \right\}$$

$$N_z = i N_{ox} \frac{\alpha_o}{\beta_o} \left\{ \alpha_o^2 I_1^{(4)}(Z) - \beta_o^2 I_1^{(2)}(Z) \right\} \cos \phi$$

$$\mathcal{N}_z = \frac{i\alpha_o}{\beta_o} \left\{ \alpha_o^2 I_1^{(4)}(Z) - \beta_o^2 I_1^{(2)} \right\}$$

η	antenna efficiency, or alternately,
$\eta = q^2/Z^2$	
G	the antenna gain
$E_{\tilde{\theta}}$	electric field polarized in $\tilde{\theta}$ direction (see Fig.5)
$E_{\tilde{\phi}}$	electric field polarized in the $\tilde{\phi}$ direction, called the cross-polarized component
$E_{\tilde{\theta}n}, E_{\tilde{\phi}n}$	the above field components normalized to $E_{\tilde{\theta}}$ maximum

1. INTRODUCTION

1.1 General

The transmission of signals through space requires wave-launching and receiving devices (antennas). Since for various reasons the maximum transmitter power may be limited, the efficiency of launching and reception of the electromagnetic energy is of great importance.

A large number of various types of antennas has been developed. The type of antenna used depends on its function; antennas have been developed for high-gain narrow-band operation, for broad-band low-gain operation, nearly isotropic, with sharp spatial characteristics, broad radiation characteristics, etc.

Classification of Antennas. From the above one can see that there are many ways of classifying antennas. One possible means of antenna differentiation can be the frequency dependence of the various antenna parameters. This type of classification is of importance, for example, in multichannel communication systems or any systems utilizing broad-band signals.

Antenna Fundamentals (1)

It seems appropriate, at this point, to define a number of terms associated with antennas in general.

Antenna Pattern. The graphic plot of the magnitude of a field component at every point in space is called the absolute field (component) pattern. The field intensity may be expressed in units relative to its value at some reference direction or relative to the field of some reference antenna.

Power Pattern. The plot of the time-average power flow per unit solid angle at each point is the power pattern. This plot may be the total power pattern or the power due to a particular field component.

Antenna Gain. Antenna gain is defined as the ratio of the power per unit solid angle in the direction of maximum radiation of antenna to the power per unit solid angle in the direction of maximum radiation of a reference antenna for the same input power. This ratio is usually denoted by the symbol G, and the usual reference antenna is either the isotropic or the half-wave antenna. An isotropic antenna is a fictitious source that radiates equally in all directions.

Feed. The feed is the primary antenna radiator or exciter.

Reflector. The reflector is a metallic body used for focussing the primary radiation into a sharp or other required pattern. A common reflector is a section of a paraboloid of revolution.

Antenna. Antenna is the composite feed-reflector system.

Polarization. The polarization of electromagnetic radiation is defined as the direction of the vibration of the electric vector. If an antenna is designed to operate with a given (principal) polarization, then any energy radiated with a polarization at right angles to the principal represents loss. The knowledge of this spurious or cross-polarized component, as it is usually called, may be of considerable importance in some antenna applications.

1.2 Paraboloidal Reflector Antenna (2)

A paraboloidal antenna (see Fig. 1) is made up of a primary source, called the exciter or the feed, and a section, usually circular, of a paraboloid of revolution, called the reflector, The exciters may take many forms: the most common types are the electric dipole and horns.

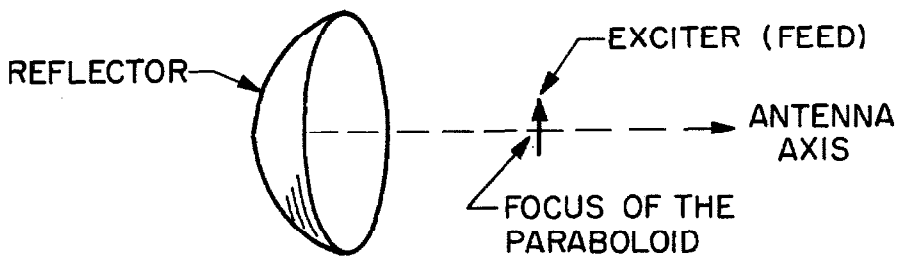

Fig. 1. Configuration of a dipole-excited paraboloidal antenna

Its broad-bandedness, high gain and desirable mechanical properties combine to make the paraboloidal reflector antenna one of the most popular in the microwave range. Some of the desirable mechanical properties of the antenna are:

(i) Smaller size of radomes for a given large gain compared with yagis or horn-paraboloidal reflector antennas.

(ii) Simplicity of structure and absence of highly resonant lengths.

The use of this type of antenna in radio astronomy, radar, and radio-telephone trunk lines, requires a good knowledge of its gain properties, its over-all radiation, and its polarization characteristics.

One of the earlier works on the paraboloidal antenna appears to be that of R. Darbord (3) who used a geometrical optics approach to deduce the reflected field in the aperture. This is the so-called "aperture" method. Darbord, however, did not compute the radiation characteristics of the antenna. Morita (4) incorrectly used the field at the surface of the reflector as the aperture field to compute the radiation. Aperture field is the field at the opening of the reflector.

Wwedensky (5), employing the aperture method, computed the far-zone field. However, he neglected the cross-polarized component.

E. U. Condon (6) took into account the cross-polarized aperture component and, using the same approach and approximations, computed the cross-polarized far-zone field. For dipole excitation he found that there were four symmetrically-placed sidelobes very close to the main lobe and at 45° to the principal planes. Planes of symmetry or the principal planes for the paraboloidal reflector antenna are the xz- and yz-planes (see Fig. 2).

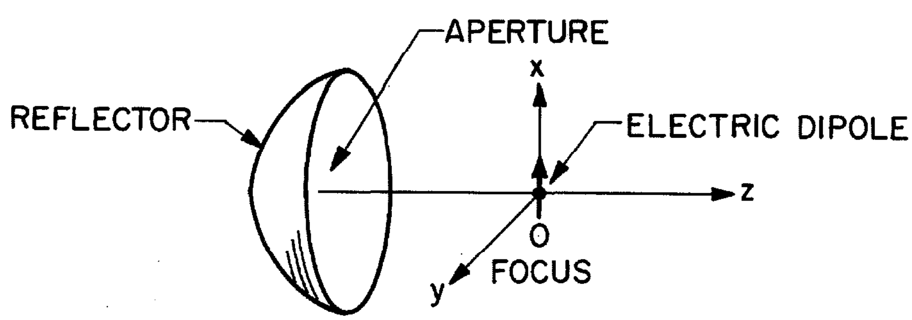

Fig. 2.

The cross-polarization sidelobes have magnitudes considerably greater than the first sidelobes of the principal polarization and, consequently, are troublesome in radar applications. The above sidelobes are sometimes called "Condon lobes".

Using the aperture* method, E.M.T. Jones (7) computed the principal polarization in the principal planes and the cross-polarization fields of paraboloids excited by dipoles and combinations of dipoles. Jones found that a reflector fed by a certain combination of an electric and a magnetic dipole, gave zero cross-polarized component at the aperture and therefore in the far-zone field. This certain combination of electric and magnetic dipoles gives a ratio of electric and magnetic fields that is the same as the free-space impedance. It was found experimentally (7,8) that paraboloids excited by plane wave sources such as horns and waveguide radiators gave much larger cross-polarized components than predicted theoretically. This latter discrepancy had been attributed by Kinber (8) to the fact that the field inside a waveguide is not a single plane wave but a combination of two plane waves at some angle to each other; and that the combination of electric and magnetic dipoles does not give y-directed components in the zero order approximation, only.

Cutler's (9) investigations showed that the effect of the phase variation at the feed due to its physical size has only a small effect on the gain patterns of the antenna.

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*The terms "aperture" and "current" methods are explained more fully in Section 2.1.

D. Carter (10) formulated the far-zone field of a paraboloid in the form of an integral. He assumed an axisymmetric illumination and using a digital computer, evaluated the fields in principal planes up to an azimuthal angle of 90°.

The radiation characteristics of antennas beyond this angle ($\theta = 90^{\circ}$) have received little attention. H. N. Kritikos (11) used an extended aperture method to compute the field on the axis at the back of the paraboloid. His simple method gives results that are in agreement with those of Tartakovskii (12). In the Extended Aperture method Kritikos considered the spherical wavefront of the exciter as the source of the far-zone field and assumed that the only effect of the reflector was to block part of this wavefront. However, by virtue of the approximations, this method is applicable only near the axis at the back of the paraboloidal antenna.

The only approach similar to the present treatment was that due to Kinber, but he did not evaluate the integrals and investigated only the radiation near the axis of the paraboloid and then only in the principal planes and planes at 45° to the principal. In his analysis Kinber assumed an electric dipole feed.

In all of the above-mentioned papers approximation methods were used to deduce the radiation field. No account was taken of the exciter near-zone field, the effect of the paraboloid's curvature on the reflector current, or the effect of the finite size of the reflector. No exact solution to the problem is known at present. However, corrections to the assumed approximate field distributions have been derived by a number of investigators.

J. B. Keller (13) introduced the notion of diffracted rays to obtain a correction to the geometrical optics theory. Keller's method gives good results in directions close to the edges of the reflector.

Other corrections to the geometrical optics approximations were made by L. B. Tartakovskii and V. L. Tandit (14,15). However, their results, especially the edge correction, are of little practical use due to the complicated form of the resultant expressions.

The so-called shadow correction (15) in improving the estimate of the surface current magnitude on the reflector, takes account of the fact that the field behind an infinite reflector is zero.

The case of the infinite paraboloid excited by an electric dipole was investigated by E. Pinney (16) and later by I. P. Skal'skaya (17). Pinney used Laguerre functions in his solution of the problem and the results were given in the form of double series. Skal'skaya's results are in the form of contour integrals. In both cases the results are usable only in the limit of small wavelengths (geometrical optics approximation).

The above brief summary of the technical literature provides an idea of the state of the art as far as the theoretical analysis of the paraboloidal reflector antennas in the microwave range is concerned. The two papers on the wide-angle radiation from the antennas are those of D. Carter (10) and of L. B. Tartakovskii (12). Carter's results are not amenable to physical interpretation and Tartakovskii's results, while quite interesting, have a number of serious shortcomings. These are:

(i) The $\frac{1}{r}$ dependence of the field from the exciter to the reflector is not taken into account.

(ii) The ϕ-dependence is approximated unnecessarily.

(iii) The arbitrary form of the illumination assumed, although mathematically tractable, is not of a form easily realizable in practice. Also, his results are given in a form where the principal and the cross-polarized radiation are not easily distinguishable.

The cross-polarization component of paraboloidal antennas excited by electric dipoles had been investigated by both Jones and Kinber in a rather limited way near the antenna axis.

Thus in no one single work is the problem of principal and cross-polarization radiation solved in such a way that the results are applicable to all points in space. Also, no work considers the combinations of electric and magnetic dipoles in combinations other than those giving plane waves.

In our paper we evaluate the fundamental electric dipole integrals. These expressions are then combined to obtain the required components, whether principal or cross-polarized, of the far-zone radiation field for the cases of the electric dipole alone and combinations of electric and magnetic dipoles.

1.3 Objectives of this Study

The objective of this paper is the study of the radiation characteristics of a paraboloidal antenna for a number of feeds. We wish to gain a physical insight into the polarization properties of these antennas. We also wish to obtain solutions of this problem without resorting to the approximation introduced by the above-mentioned workers, so that the results will be applicable throughout all space. In particular, we want to investigate more fully the polarization near the antenna axis and, if possible, deduce simple formulas that describe the polarization structure for the feeds considered.

The effects of variation of the ratio

$$\frac{\text{Focal Length } (f)}{\text{Aperture Diameter } (D)}$$

on the cross-polarized and principal polarizations for the following two cases will be investigated:

(i) Varying f, keeping D constant

(ii) Varying D, keeping ratio f/D constant.

In his paper, Jones shows how a plane wave is constructed by a certain combination of electric and magnetic dipoles. Other combinations of these dipoles have not been investigated until recently (18). We shall investigate the effect on the components of the paraboloid surface current of the variation of the relative magnitudes of the two dipoles. We shall also look into the possibilities of synthesis of the required combinations of the dipoles by the use of waveguide feeds.

Lastly, we will explain from our theoretical results the unexpectedly large cross-polarization observed by Jones in his experimental investigations of a paraboloid excited by a horn.

2. THEORY

2.1 Introduction

In this paper the general microwave antenna problem is first formulated. The basic formulas obtained are then approximated into forms that are practically tractable. From these approximate formulas the two most commonly-used approaches, namely the "aperture" and the induced surface "current" methods, are discussed. Assumptions common to both methods are:

(i) The reflector is considered to be in the far-zone field of the feed antenna.

(ii) The pattern and hence the current of the feed antenna is not affected by the presence of the reflector.

(iii) Plane-wave boundary conditions are assumed to hold, i.e., the radius of curvature of the reflector is large compared with the wavelength of operation and the induced current on the reflector is given by $\underline{J}_s = 2(\underline{n} \times \underline{H}^{inc})$. Here $\underline{H}^{inc}$ is the incident magnetic field of the feed.

In all cases the current on the shadow side of the reflector is assumed to be zero.

In the induced current method, the currents flowing on the conductors are found by the use of the above assumptions and the radiated fields are computed by taking these currents as the new sources.

In the aperture method, one further assumption is made that the energy travels between the reflector and the aperture as predicted by the geometrical optics. This means that the currents or fields at the reflector surface may be projected onto the aperture with the appropriate phase factor. It is then obvious that one cannot expect the diffracted field computed from the aperture distribution to give the correct result anywhere, except near the axis of the paraboloid. The correct surface of integration is the reflector surface. For shallow reflectors the aperture approach should give comparable results for angles not necessarily small.

Thus the superiority of the surface current method over the aperture method lies in the following facts:

(i) Correct region of integration assures, within the approximations made, the validity of the results everywhere.

(ii) Methods may be devised to compensate for the effects of the reflector edge, curvature, and the near field of the exciter.

(iii) All the cross-polarized components of the far field may be determined at all points in space.

2.2 Formulation of the Microwave Antenna Problem

The rigorous solution of antenna problems presents great mathematical difficulties. Only a very small number of cases has been solved exactly up to the present time. The usual method of solving the microwave antenna problem is to divide it into two--the internal and the external. The internal problem consists of finding the field at some open part of the antenna structure (the aperture). The external problem reduces to the determination of the radiation field at large distances from the antenna for a given field distribution at the aperture. Before considering the two problems in detail we describe a principle that will be of use in the later analysis.

Principle of Equivalence (19). As a rule, in antenna theory the electromagnetic field is computed from either a given distribution of currents and charges or from the field given at a defined surface (aperture). The principle of equivalence may be stated as: "For electromagnetic fields established by given sources, the surface currents and charges and the tangential and normal components of the field vectors $\underline{E}$ and $\underline{H}$ at the surface are equivalent". Stated mathematically,

$$\begin{aligned} \underline{n} \times \underline{E} &= \underline{J}_s^m \\ \underline{n} \times \underline{H} &= - J_s^e \\ \underline{n} \cdot \underline{E} &= - \frac{1}{\epsilon} \rho_s^e \\ \underline{n} \cdot \underline{H} &= - \frac{1}{\mu} \rho_s^m \end{aligned} \qquad (1)$$

On the surface of an ideal conductor these reduce to $\underline{n} \times \underline{E} = 0$ and $\underline{n} \cdot \underline{H} = 0$. In general, however, when the defined surface passes through a dielectric medium, the equivalent magnetic surface currents and charges must be taken into consideration.

Formulation of the Problem in Two Parts

Consider the system depicted below (Fig. 3).

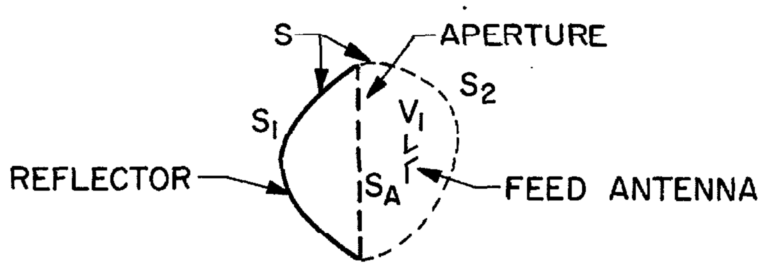


Fig. 3

The whole space is divided into two regions. Let the antenna region be V_i and the outside V_a . The two regions are connected through surface S_2 . The internal region is enclosed by the surface $S = S_1 + S_2$. A known current distribution on the feed antenna is assumed. We now divide the problem into two parts, the internal and the external. The internal problem entails the determination of the fields inside volume V_i and the external the finding of the fields in the volume V_a . These fields are mutually coupled and this coupling is expressed by equations 1. The vectors $\underline{E}$ and $\underline{H}$ satisfy Maxwell's equations in the entire unbounded space and the tangential component of $\underline{E}$ vanishes on S_1 . Also, the tangential components of $\underline{E}$ and $\underline{H}$ remain continuous across the dielectric-air interface. The fields, of course, satisfy the edge conditions at the sharp edges of the conductors and the radiation condition insures uniqueness of the results. The basis of the method is the solution of the internal problem neglecting coupling to the external problem through the surface S_2 . Field values on S_2 are then used to solve the external problem. In the solution of the internal problem the conditions are usually idealized. Since the surface S encloses all the sources, no foreign currents flow in space V_a , and the external fields are due to phenomena in region V_i or their resultant on the surface S . We are now free to choose the aperture of the antenna. The fundamental difference between the direct (current) and the indirect (aperture) methods is that in the former one considers the fields or currents at the conducting surface S_1 , while in the latter a further approximation is involved in estimating the aperture field. This further approximation consists in assuming that the energy travels in straight lines between the reflector surface and aperture plane, and consequently limits the accuracy of the results to certain directions. For any arbitrary distribution of currents and charges in space, the electric and magnetic fields $\underline{E}$, $\underline{H}$ are given by the following expressions (20), assuming $e^{-i\omega t}$ time dependence.

$$\underline{E} = - \nabla \phi + i\omega \underline{A} - \frac{1}{\epsilon} \nabla \times \underline{A}^e$$

$$\underline{H} = \nabla \times \underline{A} - \mu \nabla \phi_m + i\omega\mu \, \underline{A}^e \qquad (2)$$

and where the functions ϕ, $\underline{A}$, ϕ_m and $\underline{A}^e$ satisfy the equations

$$\nabla^2 \phi + k^2 \phi = - \rho/\epsilon$$

$$\nabla^2 \underline{A} + k^2 \underline{A} = - \mu \underline{J}$$

$$\nabla^2 \phi_m + k^2 \phi_m = - \rho_m/\mu$$

$$\nabla^2 \underline{A}^e + k^2 \underline{A}^e = - \epsilon \underline{J}^m \qquad (3)$$

$$\underline{B} = \nabla \times \underline{A} \qquad \nabla \cdot \underline{A} = i\omega\mu\epsilon \, \phi$$

$$\underline{D} = -\nabla \times \underline{A}^e \qquad \nabla \cdot \underline{A}^e = i\omega \mu \epsilon \phi_m$$

where ρ_m is defined as a magnetic charge, i.e., $\nabla \cdot \underline{B} = \rho_m$ and $\underline{J}^m$ is defined as a magnetic current. These quantities satisfy Maxwell's equations

$$\nabla \times \underline{H} = \underline{J} - i\omega \underline{D} \qquad \nabla \cdot \underline{D} = \rho$$

$$\nabla \times \underline{E} = i\,\omega \underline{B} \qquad \nabla \cdot \underline{B} = 0$$

and

$$\nabla \times \underline{H} = - i\omega \underline{D} \qquad \nabla \cdot \underline{D} = 0$$

$$\nabla \times \underline{E} = - \underline{J}^{m} + i\omega \underline{B} \qquad \nabla \cdot \underline{B} = \rho_m$$

The solutions of the inhomogeneous wave equations are of the forms:

$$\underline{A} = \mu \int_V \underline{J}(\underline{r}') \; G(|\underline{r} - \underline{r}'|) \; dV'$$

$$\underline{A}^e = \epsilon \int_V \underline{J}^m(\underline{r}') \; G(|\underline{r} - \underline{r}'|) \; dV' \tag{4}$$

where

$$G(|\underline{r} - \underline{r}'|) = \frac{e^{ik|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|}$$

From equation 3

$$- \nabla \phi = \frac{i}{\omega \epsilon \mu} \nabla(\nabla \cdot \underline{A})$$

and

$$- \nabla \phi_m = \frac{i}{\omega \epsilon \mu} \nabla(\nabla \cdot \underline{A}^e) \quad .$$

Substituting this into equations 2, we obtain

$$\underline{E} = i\omega \left[\underline{A} + \frac{1}{k^2} \nabla(\nabla \cdot \underline{A}) \right] - \frac{1}{\epsilon} \left[\nabla \times \underline{A}^e \right] \tag{5}$$

$$\underline{H} = \frac{1}{\mu} \left[\nabla \times \underline{A} \right] + i\omega \left[\underline{A}^e + \frac{1}{k^2} \nabla(\nabla \cdot \underline{A}^e) \right] \tag{6}$$

Cast these into operators forms (Ref. 20, p.24)

$$\underline{E} = i\omega \left(\underline{\underline{u}} + \frac{1}{k^2} \nabla\nabla \right) \cdot \underline{A} - \frac{1}{\epsilon} \nabla \times \underline{A}^e$$

This may now be rewritten as

$$\underline{E} = i\omega\mu \int_V \left[(\underline{\underline{u}} + \frac{1}{k^2} \nabla\nabla) \frac{e^{ik|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|} \right] \cdot \underline{J}(\underline{r}')\, dV'$$

$$- \int_V (\nabla \frac{e^{ik|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|}) \times \underline{J}^m(\underline{r}')\, dV' \qquad (7)$$

since

$$\nabla \frac{e^{ik|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|} = - \nabla' \frac{e^{ik|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|}$$

Then we may rewrite

$$(\underline{\underline{u}} + \frac{1}{k^2} \nabla\nabla)\; G(\underline{r},\underline{r}') = (\underline{\underline{u}} + \frac{1}{k^2} \nabla'\nabla')\; G(\underline{r},\underline{r}')$$

with the double gradient now operating with respect to the primed coordinates only.

In the far-zone defined by

$$r \gg r' \qquad \text{and} \qquad kr \gg 1$$

where $r = \sqrt{\underline{r} \cdot \underline{r}}$ and $r' = \sqrt{\underline{r}' \cdot \underline{r}'}$, the following approximation is valid

$$|\underline{r} - \underline{r}'| = \sqrt{r^2 + r'^2 - 2\underline{r} \cdot \underline{r}'}$$

$$\approx r - \underline{e}_r \cdot \underline{r}'$$

where $\underline{e}_r = \underline{r}/r$.

Thus, replacing $e^{ik(|\underline{r} - \underline{r}'|)}$ by $e^{ik(r - \underline{e}_r \cdot \underline{r}')}$ and $\frac{1}{|\underline{r} - \underline{r}'|}$ by $1/r$, we obtain

$$(\underline{\underline{u}} + \frac{1}{k^2}\nabla\nabla)\, G(\underline{r},\underline{r}') = \frac{e^{ikr}}{4\pi r}\,(\underline{\underline{u}} + \frac{1}{k^2}\nabla'\nabla')\, e^{-ik\underline{e}_r \cdot r'}$$

$$= (\underline{\underline{u}} - \underline{e}_r\, \underline{e}_r)\, \frac{e^{ikr}}{4\pi r}\, e^{-ik\underline{e}_r \cdot \underline{r}'}$$

Since $(\underline{\underline{u}} - \underline{e}_r\, \underline{e}_r) \cdot \underline{J} = \underline{J} - \underline{e}_r(\underline{e}_r \cdot \underline{J})$

$$= -\underline{e}_r \times (\underline{e}_r \times \underline{J})$$

we finally get the expression for the far-zone electric field:

$$\underline{E}(\underline{r}) = -\frac{i\omega\mu}{4\pi r}\, e^{ikr}\, \underline{e}_r \times \left[\underline{e}_r \times \int_V e^{-ik\underline{e}_r \cdot \underline{r}'}\, \underline{J}(\underline{r}')dV'\right] + $$

$$+ \frac{ik\, e^{ikr}}{4\pi r}\left[\underline{e}_r \times \int_V \underline{J}^m(\underline{r}')\, e^{ik\underline{e}_r \cdot \underline{r}'}\, dV'\right] \qquad (8)$$

or

$$\underline{E}(\underline{r}) = \left(-\frac{i\omega\mu}{4\pi r}\, e^{ikr}\right)\, \underline{e}_r \times \int_V \left\{\underline{e}_r \times \underline{J}(\underline{r}') + \right.$$

$$\left. + \frac{1}{Z_o}\, J^m(\underline{r}')\right\}\, e^{-ik\underline{e}_r \cdot \underline{r}'}\, dV' \qquad (9)$$

where $c\mu = \sqrt{\mu/\epsilon} = Z_o$.

In problems where $\underline{J}$'s are surface currents, we substitute $\underline{J} \rightarrow \underline{J}_s$ and $\underline{J}^m \rightarrow \underline{J}_s^m$ and the volume of integration becomes a surface of integration. The above derived formula for the far-zone field holds for any regular surface, i.e., surface having a defined tangent plane at all points.

If the surface of integration is a conductor, then $\underline{J}_s^m = 0$ and we have the current formula:

$$\underline{E}(\underline{r}) = -\frac{i\omega\mu\, e^{ikr}}{4\pi r}\,\underline{e}_r \times \left(\underline{e}_r \times \int_S J_s(\underline{r}')\, e^{-ik\underline{e}_r \cdot \underline{r}'} dV'\right) \qquad (9a)$$

If, however, the surface of integration is S_A , then $\underline{J}_s^m$ must be taken into account. Using the principle of equivalence, we substitute into equation 9 for $\underline{J}_s = -(\underline{n} \times \underline{H}^{inc})$ and $\underline{J}_s^m = (\underline{n} \times \underline{E})$ and obtain the expressions for the far-zone field:

$$\underline{E}(\underline{r}) = \frac{i\omega\mu\, e^{ikr}}{4\pi r}\,\underline{e}_r \times \int_{S_{aperture}} \left[\underline{e}_r \times (\underline{n} \times \underline{H}^{inc}) - \frac{1}{Z_o}(\underline{n} \times \underline{E})\right] e^{-ik\underline{e}_r \cdot \underline{r}'} dS' \qquad (10)$$

In our analysis the incident fields will be due to electric dipole alone and combinations of electric and magnetic dipoles. The exciters will be assumed to be at the focus of the paraboloid of revolution.

2.3 Approximate Solution

Consider the paraboloid shown in Fig. 4 with the origin of the rectangular coordinates at the focus. The exciter is assumed to be small relative to the wavelength of operation, and located at the focus.

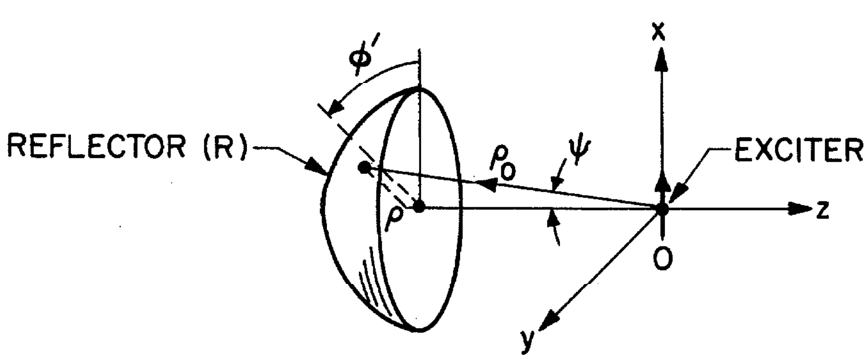


Fig. 4. Geometry of Paraboloid

We wish to compute the current on the surface R of the reflector by assuming plane-wave boundary conditions

$$\underline{J} = 2(\underline{n} \times \underline{H}) \tag{11}$$

i.e., employing the usual current method.

Let $\underline{\rho}_o$ be a unit radius-vector from origin to a point on the reflector surface,

$$\therefore \quad \underline{\rho}_o = \sin\psi \cos\phi' \, \underline{e}_x + \sin\psi \sin\phi' \, \underline{e}_y - \cos \quad \underline{e}_z \tag{12}$$

Also, let $\underline{n}$ be a unit vector, normal to the paraboloidal surface at the same point as for $\underline{\rho}_o$,

$$\therefore \quad \underline{n} = -\sin\frac{\psi}{2}\cos\phi' \, \underline{e}_x - \sin\frac{\psi}{2}\sin\phi' \, \underline{e}_y + \cos\frac{\psi}{2}\, \underline{e}_z \tag{13}$$

Expressions 12 and 13 are derived in the appendix A1.

The relation between the $\underline{E}$ and $\underline{H}$ fields in the radiation zone is

$$\underline{H}^{inc} = \frac{1}{\eta}(\underline{\rho}_o \times \underline{E}^{inc}) \tag{14}$$

where η is the free-space impedance. Then substituting equation 14 into equation 11, we obtain

$$\underline{J} = \frac{2}{\eta}\left[n \times (\underline{\rho}_o \times \underline{E}^{inc})\right].$$

By vector expansion formula this may be put into the form

$$\underline{J} = \frac{2}{\eta}\left[(\underline{n} \cdot \underline{E}^{inc})\, \underline{\rho}_o - (\underline{\rho}_o \cdot \underline{n})\, \underline{E}^{inc}\right]$$

but

$$(\underline{n} \cdot \underline{\rho}_o) = - \cos \frac{\psi}{2}$$

$$\underline{J}_s = \frac{2}{\eta} (\underline{n} \cdot \underline{E}^{inc}) \underline{\rho}_o + \cos \frac{\psi}{2} \underline{E}^{inc} \tag{15}$$

specific types of feeds will now be considered

3. ELECTRIC DIPOLE FEED

3.1 Formulation of the Dipole Integrals

The far-zone electric field due to a small electric dipole at the origin of coordinates and directed along the x-axis is given by

$$\underline{E} = \frac{E_o e^{ikr}}{r} \Big[(\cos^2\psi \cos^2\phi' + \sin^2\phi') \underline{e}_x - \frac{\sin 2\phi' \sin^2\psi}{2} \underline{e}_y + \frac{\sin 2\psi \cos \phi}{2} \underline{e}_z \Big] \tag{16}$$

The paraboloidal system of coordinates is given by (21)

$$x = \alpha\beta \cos \phi$$

$$y = \alpha\beta \sin \phi$$

$$z = \frac{1}{2} (\alpha^2 - \beta^2)$$

$$r = \sqrt{x^2 + y^2 + z^2} = \frac{1}{2}(\alpha^2 + \beta^2) = \sqrt{\rho^2 + z^2} \tag{17}$$

The equation of the reflector in paraboloidal coordinates is

$$\beta = \beta_o$$

$$0 \leq \alpha \leq \alpha_o$$

For points on the surface of the paraboloid, since $\beta = \beta_o$, we have

$$\cos \psi = - \frac{\alpha^2 - \beta_o^2}{\alpha^2 + \beta_o^2}$$

$$\sin \psi = \frac{2\alpha\beta_o}{\alpha^2 + \beta_o^2}$$

$$\sin \frac{\psi}{2} = \frac{\alpha}{\sqrt{\alpha^2 + \beta_o^2}}$$

$$\cos \frac{\psi}{2} = \frac{\beta_o}{\sqrt{\alpha^2 + \beta_o^2}} \tag{18}$$

The current on the surface $\beta = \beta_o$ in paraboloidal coordinates is then

$$\underline{J}_s = \frac{4E_o}{\eta} e^{\frac{ik\beta_o^2}{2}} \Big\{ \beta_o(\beta_o^2 - \alpha^2 \cos 2\phi') \, \underline{e}_x - \beta_o \alpha^2 \sin 2\phi' \underline{e}_y - \alpha(\alpha^2 - \beta_o^2) \cos \phi' \underline{e}_z \Big\} \frac{e^{\frac{ik\alpha_o^2}{2}}}{(\alpha^2 + \beta_o^2)^{5/2}} \tag{19}$$

where we have used equations 18, 16 and 13 in equation 15. The far-zone field due to these currents may now be found by the equation 9a, where $\underline{J}_s$ is the surface current density confined to surface $\beta = \beta_o$.

The radiation vector $\underline{N}$ is defined as (22)

$$\underline{N} = \int_S e^{-ik \, \underline{e}_r \cdot \underline{r}'} \, \underline{J}(\underline{r}') \, ds' \tag{20}$$

i.e., $$\underline{E} = -\frac{i\omega\mu}{4\pi r} e^{ikr} \underline{e}_r \times (\underline{e}_r \times \underline{N}) \tag{21}$$

From Appendix A1 we have

$$-\underline{e}_r \cdot \underline{r}' = \left[-\frac{\alpha^2 - \beta_o^2}{2} \cos\theta - \alpha\beta_o \sin\theta \cos(\phi - \phi') \right]$$

and element of area dS' is

$$dS' = \alpha\beta_o \sqrt{\alpha^2 + \beta_o^2}\, d\alpha\, d\phi' \quad .$$

Angles θ, ϕ, ϕ' are defined in Fig. 15 , Appendix 1. The radiation vector, equation 20, becomes on substitution of these values of $\underline{e}_r \cdot \underline{r}'$ and dS' :

$$\underline{N} = \frac{4\beta_o}{\eta} E_o e^{\frac{ik\beta_o^2}{2}(1+\cos\theta)} \int_{\phi=0}^{2\pi} \int_{\alpha=0}^{\alpha_o} \left[\beta_o(\beta_o^2 - \alpha^2 \cos 2\phi')\, \underline{e}_x - \beta_o\, \alpha^2 \sin 2\phi'\, \underline{e}_y - \alpha(\alpha^2 - \beta_o^2) \cos\phi'\, \underline{e}_z \right] \times$$

$$\times \frac{e^{\frac{ik\alpha_o^2}{2}(1-\cos\theta) - ik\alpha\beta_o \sin\theta \cos(\phi-\phi')}\, \alpha\, d\alpha\, d\phi'}{(\alpha^2 + \beta_o^2)^2}$$

Let $N_{ox} = \frac{4}{\eta} E_o e^{\frac{ik\beta_o^2}{2}(1+\cos\theta)} \frac{2\pi\alpha_o^2}{\beta_o^2}$. Then the radiation vector may be written as

$$\underline{N} = \frac{N_{ox}}{2\pi\alpha_o^2/\beta_o^4} \int_0^{2\pi}\int_0^{\alpha_o} \Big[(\beta_o^2 - \alpha^2 \cos 2\phi') \underline{e}_x - \alpha^2 \sin 2\phi' \underline{e}_y -$$

$$- \frac{\alpha}{\beta_o}(\alpha^2 - \beta_o^2)\cos\phi' \, e_y \Big] e^{\frac{ik\alpha^2}{2}(1 - \cos\theta) - ik\alpha\beta_o \sin\theta\cos(\phi - \phi')}$$

$$\times \frac{\alpha \, d\alpha \, d\phi'}{(\alpha^2 + \beta_o^2)^2} \qquad (22)$$

We now write the radiation vector in terms of its rectangular components

$$\underline{N} = N_x \underline{e}_x + N_y \underline{e}_y + N_z \underline{e}_z \qquad (23)$$

We will consider each component separately. The component N_x will be considered in some detail; the other components are derived in a similar way.

The integration with respect to ϕ' will be carried out first. We have (23)

$$e^{-i\lambda\cos(\phi - \phi')} = J_o(\lambda) + 2\sum_{n=1}^{\infty} (-i)^n J_n(\lambda) \cos n(\phi - \phi')$$

Substituting $\lambda = k\alpha\beta_o \sin\theta$ and performing the integration with respect to ϕ', we get for N_x

$$N_x = \frac{N_{ox}}{2\pi\alpha_o^2/\beta_o^4} 2\pi \int_0^{\alpha_o} \beta_o^2 J_o(Z'\alpha) + \alpha^2 \cos\phi J_2(Z'\alpha) \frac{e^{\frac{ik\alpha^2}{2}(1 - \cos\theta)}}{(\alpha^2 + \beta_o^2)^2} \alpha \, d\alpha$$

where $Z' = k\beta_o \sin\theta$.

Now let $\alpha = \alpha_o t$, then equation 22 may be written as

$$N_x = N_{ox} \beta_o^4 \int_0^1 \left[\beta_o^2 J_o(Zt) + \alpha_o^2 t^2 \cos 2\phi \, J_2(Zt) \right] \frac{e^{\frac{iwt^2}{2}} t \, dt}{(\alpha_o^2 t^2 + \beta_o^2)^2}$$

where $Z = k \alpha_o \beta_o \sin \Theta$

$w = k \alpha_o^2 (1 - \cos \Theta)$.

Setting $\alpha_o / \beta_o = q$, we obtain

$$N_x = N_{ox} \int_0^1 \left[\beta_o^2 J_o(Zt) + \alpha_o^2 \cos 2\phi \, J_2(Zt) \right] \frac{e^{\frac{iwt^2}{2}} t \, dt}{(1 + q^2 t^2)^2} \tag{24}$$

These integrals will be evaluated in the following pages in exact and approximate forms.

3.2 Exact Evaluation of the Dipole Integrals

Consider the integral

$$I_o^{(1)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt) \, t \, dt}{(1 + q^2 t^2)^2} \tag{25}$$

If we let $Zt = V$, $\eta^2 = q^2/Z^2$, $i\mu = \frac{iw}{Z^2}$, then equation 25 becomes

$$I_o^{(1)}(Z) = \frac{1}{Z^2} \int_0^Z \frac{e^{\frac{i\mu V^2}{2}} J_o(V) \, V \, dV}{(1 + \eta^2 V^2)^2}$$

Integrating by parts we obtain the double series

$$I_o^{(1)}(Z) = \frac{e^{\frac{iw}{2}}}{(1+q^2)^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(m+n)!(-iw)^m J_{m+n+1}(Z)}{n!\ m!\ Z^{m+n+1}} \tag{26}$$

Now

$$I_1^{(2)}(Z) = -\frac{\partial I_o^{(1)}}{\partial Z} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^2 dt}{(1+q^2t^2)^2}$$

$$= \frac{-e^{\frac{iw}{2}}}{(1+q^2)^2} \sum_{n,m=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(m+n)!(-iw)^m}{n!\ m!}$$

$$\times \left\{ \frac{J'_{m+n+1}}{Z^{m+n+1}} - \frac{m+n+1}{Z^{m+n+2}} J_{m+n+1} \right\} .$$

But $\quad J'_{m+n+1}(Z) = -J_{m+n+2}(Z) + \frac{m+n+1}{Z} J_{m+n+1}(Z)$

Thus,

$$I_1^{(2)}(Z) = \frac{e^{\frac{iw}{2}}}{(1+q^2)^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2q^2)^n (n+1)!(m+n)!(-iw)^m J_{m+n+2}(Z)}{(1+q^2)^n\ n!\ m!\ Z^{m+n+1}} \tag{27}$$

Similarly, all the integrals of the form

$$I_p^{(s)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_p(Zt)\ t^s dt}{(1+q^2t^2)^2}$$

can be derived from $I_o^{(1)}(Z)$, equation 25.

We are now in a position to evaluate the radiation vector components (see Appendix A2).

From equation 24 we then have

$$N_x = N_{ox} \left[\beta_o^2 I_o^{(1)} + \alpha_o^2 \cos 2\phi \, I_2^{(3)}(Z) \right] .$$

Similarly,

$$N_y = N_{ox} \alpha_o^2 \sin 2\phi \, I_2^{(3)}(Z)$$

and

$$N_z = \frac{i \alpha_o}{\beta_o} \cos \phi \, N_{ox} \left[\alpha_o^2 I_1^{(4)}(Z) - \beta_o^2 I_1^{(2)} \right]$$

Also

$$N_x = N_{ox} \mathcal{N}_x$$

$$N_y = N_{ox} \sin 2\phi \, \mathcal{N}_y$$

$$N_z = N_{ox} \cos \phi \, \mathcal{N}_z \quad .$$

Here $I_o^{(1)}$, $I_1^{(2)}$, etc., are expressed in the form of a series and these series will converge as long as factor $\frac{w}{Z} \leq 1$, i.e., $q \tan \frac{\theta}{2} \leq 1$, or $\theta \leq \theta_{cr}$, where

$$\theta_{cr} = 2 \tan^{-1}(\frac{1}{q}) = 2 \tan^{-1} (\frac{\beta_o}{\alpha_o})$$

In the region $\theta > \theta_{cr}$, the shadow region, the integrals will have to be evaluated in inverse powers of w/Z .

Thus in the range of θ where $\pi \geq \theta > \theta_{cr}$, we start with the original integral, equation 25:

$$I_o^{(1)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt) \, t \, dt}{(1 + q^2 t^2)^2}$$

Substitute, as before $Zt = V$, $i\mu = \frac{iw}{Z^2} = \lambda$, $\eta^2 = q^2/Z^2$.

Then

$$I_o^{(1)}(Z) = \frac{1}{Z^2}\int_0^Z \frac{e^{\frac{i\mu V^2}{2}} J_o(Zt)\ V\ dV}{(1+\eta^2 V^2)^2}$$

Integrating by parts, we obtain the series

$$I_o^{(1)*} = \frac{1}{Z^2\ (1+\eta^2 V^2)^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2q^2)^n (m+n)!(n+1)!\ J_m(V)\ e^{\frac{i\mu V^2}{2}}}{(1+\eta^2 V^2)^n\ Z^{2n}\ m!\ n!\ \lambda^n (\lambda V)^m}\Bigg|_0^Z$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2q^2)^n}{(1+q^2)^{n+2}}\ \frac{(n+1)!(m+n)!}{n!\ m!}\ \frac{e^{\frac{iw}{2}}}{(iw)^{m+n+1}}\ Z^m J_m(Z) -$$

$$- \frac{1}{(1+q^2)^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2q^2)^n}{2^m}\ \frac{(m+n)!(n+1)!}{n!\ (m!)^2}\ \frac{Z^{2m}}{(iw)^{m+n+1}} \tag{28}$$

For $\theta > \theta_{cr}$, $|iw| \gg 1$ so that it suffices to take $n = 0$.

$$I_o^{(1)*} = \frac{e^{\frac{iw}{2}}}{(1+q^2)^2}\ \frac{1}{iw}\sum_{m=0}^{\infty} \frac{J_m(Z)}{(iw/Z)^m} - \frac{e^{Z^2/2iw}}{iw(1+q^2)^2} \tag{29}$$

which is a rapidly converging form. Here the second term is obtained from the summation of the corresponding series.

The other integrals for the range of θ , $\theta_{cr} < \theta \leq \pi$, can be evaluated from $I_o^{(1)*}(Z)$ by differentiation as before. Thus in the range $\pi \geq \theta > \theta_{cr}$ we obtain for the radiation vector components:

$$N_x^* = N_{ox}\left\{\beta_o^2\ I_o^{(1)*} + \alpha_o^2 \cos 2\phi\ I_2^{(3)*}\right\}$$

$$N_y^* = N_{ox} \, \alpha_o^2 \sin 2\phi \, I_2^{(3)*}$$

$$N_z^* = \frac{iN_{ox} \, \alpha_o}{\beta_o} \left\{ \alpha_o^2 \, I_1^{(4)*} - \beta_o^2 \, I_1^{(2)*} \right\} \cos \phi$$

and

$$\mathcal{N}_z^{\rho} = \frac{i\alpha_o}{\beta_o} \cdot \left\{ \alpha_o^2 \, I_1^{(4)*} - \beta_o^2 \, I_1^{(2)*} \right\}$$

etc.

Also

$$\underline{N} = \underline{e}_x \, N_x^* + \underline{e}_y \, N_y^* + \underline{e}_z \, N_z^* \, . \qquad (30)$$

Values of $I_i^{(j)*}$ and their derivations are given in Appendix A3. Some of the series are expressed in an alternative form in A4.

3.3 Approximate Evaluation of Dipole Integrals.

For ease of computation the integral forms of $I_0^{(1)}$, $I_1^{(2)}$, etc. can be simplified further (7). In the case of interest α varies in the range $0 \leq \alpha \leq \alpha_o \leq \beta_o$ or $t \leq 1$, so that the expression $\frac{1}{(1+q^2 \, t^2)^2}$ will vary slowly with t. Then this expression may be approximated by a polynomial to a high degree of accuracy. Using the least squares method (24) of approximation, we express the function

$$\frac{1}{(1 + \alpha_o^2/\beta_o^2 \, t^2)^2}$$

as a polynomial $A_o + A_2 t^2 + A_4 t^4$. The constants A_o, A_2, A_4 are functions of α_o, β_o and will have to be computed for each pair of values (α_o, β_o).

Consider the integral, equation 25, again

$$I_o^{(1)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt)\ t\ dt}{(1+q^2t^2)^2} .$$

Substitute the polynomial expansion for the denominator and integrate in series, to obtain

$$I_o^{(1)}(Z) = \Big\{(A_1 + A_2 + A_4)\frac{1}{Z}\sum_{m=0}^{\infty}\left(-\frac{iw}{Z}\right)^m J_{m+1}(Z) -$$

$$- \frac{2}{Z^2}(A_2 + 2A_4)\sum_{m=0}^{\infty}(m+1)\left(-\frac{iw}{Z}\right)^m J_{m+2}(Z) +$$

$$+ \frac{4A_4}{Z^3}\sum_{m=0}^{\infty}(m+1)(m+2)\left(-\frac{iw}{Z}\right)^m J_{m+3}(Z)\Big\}\ e^{\frac{iw}{2}} . \quad (31)$$

Similarly for $I_2^{(3)}(Z)$,

$$I_2^{(3)}(Z) = \int_0^1 \frac{J_2(Zt)\ e^{\frac{iwt^2}{2}}\ t^3 dt}{(1+q^2t^2)^2}$$

$$= \Big\{(A_o + A_2 + A_4)\frac{1}{Z}\sum_{m=0}^{\infty}\left(-\frac{iw}{Z}\right)^m J_{m+3}(Z) -$$

$$- \frac{2}{Z^2}(A_2 + 2A_4)\sum_{m=0}^{\infty}(m+1)\left(-\frac{iw}{Z}\right)^m J_{m+4}(Z) +$$

$$+ \frac{4A_4}{Z^3}\sum_{m=0}^{\infty}(m+1)(m+2)\left(-\frac{iw}{Z}\right)^m J_{m+5}(Z)\Big\}\ e^{\frac{iw}{2}} . \quad (32)$$

We may now compute radiation vector N_x , given as before by

$$N_x = N_{ox} \left\{ I_o^{(1)} \beta_o^2 + \alpha_o^2 \cos 2\phi \, I_2^{(3)} \right\} = N_{ox} \mathcal{N}_x .$$

The remaining radiation vector components are found in the same way.

The argument of $J_m(Z)$ varies over a wide range of values. For large values of Z, $J_m(Z)$ converges very slowly so that if $w/Z \geq 1$ questions of convergence arise, i.e., for the range where $\theta > \theta_{cr}$. In this case we evaluate the integrals in inverse powers of w/Z and we get the expressions

$$I_o^{(1)} \simeq I_o^{(1)*} = \int_0^1 e^{\frac{iwt^2}{2}} J_o(Zt)\, t\, (A_o + A_2 t^2 + A_4 t^4)\, dt$$

$$= \left\{ \frac{e^{\frac{iw}{2}}}{iw} \left[(A_o + A_2 + A_4) \sum_{m=0}^{\infty} \left(\frac{-i}{w/Z}\right)^m J_m(Z) \; - \right.\right.$$

$$- \frac{2}{iw} (A_2 + 2A_4) \sum_{m=0}^{\infty} (m+1) \left(\frac{-i}{w/Z}\right)^m J_m(Z) \; +$$

$$\left. + \frac{4A_4}{(iw)^2} \sum_{m=0}^{\infty} (m+1)(m+2) \left(\frac{-i}{w/Z}\right)^m J_m(Z) \right]$$

$$- \frac{1}{iw} \left[A_o \sum_{m=0}^{\infty} \frac{Z^m}{2^m m!} \left(\frac{-i}{w/Z}\right)^m \; - \right.$$

$$- \frac{2A_2}{iw} \sum_{m=0}^{\infty} \frac{(m+1)\, Z^m}{2^m\, m!} \left(\frac{-i}{w/Z}\right)^m \; +$$

$$\left.\left. + \frac{4A_4}{(iw)^2} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)\, Z^m}{2^m\, m!} \left(\frac{-i}{w/Z}\right)^m \right] \right\} \qquad (33)$$

The series of the form

$$S_n^{(\ell)} = \sum_{m=0}^{\infty} \frac{(m+1)(m+2)\cdots(m+n)}{2^m\ (m+\ell)!} \frac{Z^m}{(iw/Z)^m}$$

may be put into the following closed form

$$S_n^{(\ell)} = \frac{1}{Z^\ell} \frac{\partial^{n-\ell}}{\partial Z^{n-\ell}} (Z^n\ e^{Z/2i\beta}) \qquad (34)$$

for $n \geq \ell$ and $\beta \equiv w/Z$. For the derivation of formula 34, see Appendix A7.

The radiation vector components are given in Appendix A5 for $0 \leq \theta \leq \theta_{cr}$ and in Appendix A6 for $\pi \geq \theta > \theta_{cr}$.

3.4 Computation of Results

The feed antenna is usually linearly polarized in some direction and the polarization of the composite antenna is referred to this initial polarization. Field component polarized in the same direction as the feed is called principal polarization, and the component at right angles to it is called the cross-polarized field.

In our case the electric dipole is directed along the x-axis so the polarization components will be related to this axis.

Introduce a new system of coordinates $(r,\tilde{\theta},\tilde{\phi})$ as shown in Fig. 5 with the polar axis along the dipole (x-) axis. We wish to express the new radiation vector components $N_{\tilde{\theta}}$ and $N_{\tilde{\phi}}$ in terms of N_x, N_y and N_z components. We have

$$N_{\tilde{\theta}} = - N_x \sin\tilde{\theta} + \cos\tilde{\theta}(N_y \cos\tilde{\phi} + N_z \sin\tilde{\phi})$$

$$N_{\tilde{\phi}} = - N_y \sin\tilde{\phi} + N_z \cos\tilde{\phi} \quad .$$

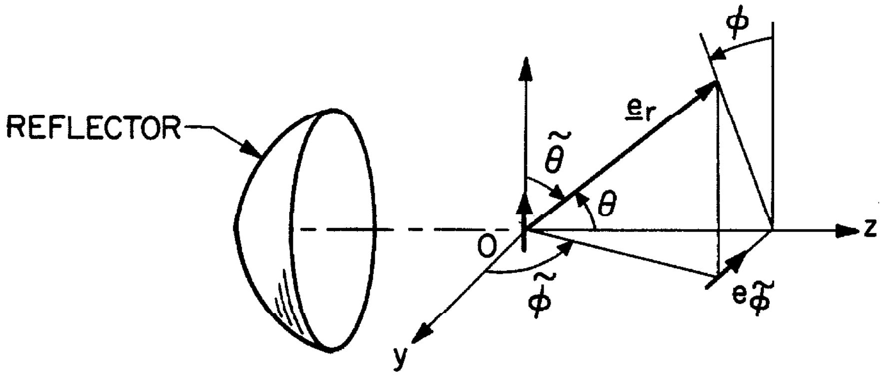


Fig. 5. Dipole Coordinate System

The relations between the (θ,ϕ) and $(\tilde{\theta},\tilde{\phi})$ systems (see Fig. 5) are -

$$\cos\tilde{\theta} = \sin\theta\cos\phi \qquad \cos\tilde{\phi} = \frac{\sin\theta\sin\phi}{\sqrt{1-\sin^2\theta\cos^2\phi}}$$

$$\sin\tilde{\theta} = \sqrt{1-\sin^2\theta\cos^2\phi} \qquad \sin\tilde{\phi} = \frac{\cos\theta}{\sqrt{1-\sin^2\theta\cos^2\phi}}$$

From the expressions for the field given by equation 21, we have

$$\underline{E} \sim (\underline{e}_r \times (\underline{e}_r \times \underline{N})) \tag{35}$$

If we write

$$\underline{N} = \underline{e}_r N_r + \underline{e}_{\tilde{\theta}} N_{\tilde{\theta}} + \underline{e}_{\tilde{\phi}} N_{\tilde{\phi}}$$

then substitute into equation 35, we get

$$\underline{E} \sim -(\underline{e}_{\tilde{\theta}} N_{\tilde{\theta}} + \underline{e}_{\tilde{\phi}} N_{\tilde{\phi}}) \tag{36}$$

so that components of the field $E_{\tilde{\theta}}$ and $E_{\tilde{\phi}}$ are proportional to $N_{\tilde{\theta}}$ and $N_{\tilde{\phi}}$, respectively, i.e.,

$$E_{\tilde{\theta}} \sim - N_{\tilde{\theta}} \quad , \qquad E_{\tilde{\phi}} \sim - N_{\tilde{\phi}}$$

or

$$E_{\tilde{\theta}}^{(s)} = \frac{i\omega\mu \, e^{ikr}}{4\pi r} \left\{ - N_x \sin \tilde{\theta} + \cos \tilde{\theta}(N_y \cos \tilde{\phi} + N_y \sin \tilde{\phi}) \right\} \quad (37)$$

$$E_{\tilde{\phi}} = \frac{i\omega\mu \, e^{ikr}}{4\pi r} \left\{ - N_y \sin \tilde{\phi} + N_z \cos \tilde{\phi} \right\} \quad (38)$$

Here $E_{\tilde{\phi}}$ is a new component of the electric field that did not exist in the original feed polarization.

Now $E_{\tilde{\phi}}$ may be rewritten as

$$E_{\tilde{\phi}} = \frac{i\omega\mu \, e^{ikr}}{4\pi r} \left\{ - \mathcal{N}_y \cos \theta + \mathcal{N}_z \frac{\sin \theta}{2} \right\} \sin 2\phi \quad (39)$$

where $\mathcal{N}_i$'s are defined earlier.

To avoid cumbersome calculations, the electric field components are normalized with respect to the maximum value of the principally polarized radiation, i.e., the maximum of $E_{\tilde{\theta}}$ which is given by

$$E_{\tilde{\theta}}\Big|_{\theta=0} \sim \frac{i\omega\mu \, e^{ikr}}{4\pi r} \left[- N_x\Big|_{\theta=0} \right] \quad (37a)$$

$$\left| E_{\tilde{\theta}}\Big|_{\theta=0} \right| \sim \frac{\omega\mu}{4\pi r} \left(\frac{4}{\eta_o} E_o \quad 2\pi \alpha_o^2\right) A \quad (37b)$$

where

$$A = \frac{1}{2}\left(\frac{1}{1+q^2}\right)$$ in the exact solution of dipole integrals, and,

$$A = \frac{1}{2}\left(A_o + \frac{A_2}{2} + \frac{A_4}{3}\right)$$ in the approximate case.

Then for any point in space, the normalized expressions for $E_{\tilde{\theta}}$ and $E_{\tilde{\phi}}$ are given by

$$E_{\tilde{\theta}n} = \frac{e^{i\Delta}}{A\beta_o^2\sqrt{1-\sin^2\theta\cos^2\phi}}\left\{-(1-\sin^2\theta\cos^2\phi)\mathcal{N}_x^{\rho} + \right.$$

$$\left. + \frac{\sin^2\theta\sin^2 2\phi}{2}\mathcal{N}_y^{\rho} + \frac{\sin 2\theta\cos^2\phi}{2}\mathcal{N}_z^{\rho}\right\} + i\,\frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{2k\,\alpha_o^2\,A} \qquad (40)$$

$$E_{\tilde{\phi}n} = \frac{e^{i\Delta}}{1-\sin^2\theta\cos^2\phi}\,\frac{1}{A\beta_o^2}\left\{\mathcal{N}_y^{\rho}\cos\theta - \mathcal{N}_z^{\rho}\frac{\sin\theta}{2}\right\}\sin 2\phi \qquad (41)$$

where $\Delta = \frac{k\beta_o^2}{2}(1+\cos\theta)$ and

$$i\,\frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{2k\,\alpha_o^2\,A} = \frac{i\sin\tilde{\theta}}{2k\,\alpha_o^2\,A}$$

is the contribution of the dipole field to the principal field. In equations 37a and 37b we have neglected the effect of the dipole field on the maximum lobe, since it is relatively small.

In the forward direction $\theta \leq \theta_{cr}$ the principal field without the dipole field contribution will be called the scattered field. It is due to the reflector currents alone and is given by

$$E_{\tilde{\theta}n}^{(s)} = \frac{e^{i\Delta}}{A\beta_o^2 \sqrt{1 - \sin^2\theta \cos^2\phi}} \left\{ -(1 - \sin^2\theta \cos^2\phi)\mathcal{N}_x + \right.$$

$$\left. + \frac{\sin^2\theta \sin^2 2\phi}{2} 2\phi \mathcal{N}_y + \frac{\sin 2\phi \cos^2\phi}{2} \mathcal{N}_z \right\} \qquad (42)$$

The radiated fields are computed from formulas 40, 41, or 42 by using the appropriate values of $\mathcal{N}$'s . Care must be taken to use the correct expression for the ranges $0 \le \theta \le \theta_{cr}$ and $\theta_{cr} < \theta \le \pi$.

3.5 Theoretical Results Obtained for the Case of a Reflector Excited by a Small Electric Dipole

3.5 (i) Antenna Gain in Principal Polarization

Let us start with equation for $E_{\tilde{\theta}n}$. The maximum of $E_{\tilde{\theta}n}$ occurs along the axis of the paraboloid, i.e., at $\theta = 0$

$$E_{\tilde{\theta}n}\Big|_{\theta \to 0} \sim \frac{e^{ik\beta_o^2}}{A\beta_o^2} \left\{ -\beta_o^2 I_o^{(1)}(Z) \right\}_{\theta \to 0} - \frac{i}{2k\alpha_o^2 A}$$

From equation 26 we have

$$I_o^{(1)}(Z)\Big|_{\substack{\theta = 0 \\ m = 0}} = \frac{1}{(1 + q^2)^2} \sum_{n=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!}{(n+1)!\ 2^{n+1}}$$

$$= \frac{1}{2(1 + q^2)}$$

Hence

$$\frac{E_{\tilde{\theta}}}{E_{dipole}} = \left[\frac{i\ e^{ik\beta_o^2}\ k\alpha_o^2}{(1 + q^2)} - 1 \right]$$

Then the gain G with respect to a small electric dipole is

$$G = 10 \log_{10} \eta \left(\frac{E_{\tilde{\theta}}}{E_{dip}}\right)^2$$

or

$$G = 10 \log_{10} \eta \left\{ \left(\frac{k \alpha_o^2}{1+q^2}\right)^2 + 1 - \frac{k \alpha_o^2}{1+q^2} \sin k \beta_o^2 \right\} \tag{43}$$

where η is now the efficiency of the antenna defined as the ratio of power incident on the reflector to the total power radiated by the dipole. It is given by equation 7.26 of reference (19),

$$\eta = \frac{1}{2} + \frac{3}{8} \cos \theta_{cr} + \frac{1}{8} \cos^3 \theta_{cr}$$

The sine term in equation 43 is due to phase angle between the paraboloid radiation and direct radiation from the dipole. For large paraboloids

$$k \alpha_o^2 \gg 1 \quad .$$

Then the gain is

$$G = 10 \log_{10} \eta \left(\frac{k \alpha_o^2}{1+q^2}\right)^2 \tag{44}$$

Usually we have $q \leq 1$. For $q < 1$, we obtain approximately

$$G = 20 \log_{10}(k \alpha_o^2) - 8.6 q^2 + 10 \log_{10} \eta$$

Fig. 10 shows graphs of principal polarization patterns for various apertures.

3.5 (ii) Front-to-Back Ratio

This is defined as the ratio of power radiated per unit solid angle in the forward direction ($\theta = 0$) to the power radiated per unit solid angle in the back direction ($\theta = \pi$).

From the exact evaluation of the integral as given by equation 26, we find, using equation 40

$$E_{\tilde{\theta}}\Big|_{\theta=0} = \left(\frac{ik\,\alpha_o^2\, e^{ik\beta_o^2}}{1+q^2} - 1 \right) \times E_{dipole}\Big|_{\theta=0} \tag{46}$$

Similarly for $E_{\tilde{\theta}}$ at $\theta = \pi$, from equation 40

$$E_{\tilde{\theta}}\Big|_{\theta=\pi} = \left[\frac{2k\,\alpha_o^2}{(1+q^2} \sum_{n=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!\, e^{ik\alpha_o^2 + \frac{i\pi}{2}}}{(2ik\,\alpha_o^2)^{n+1}} - 1 \right] \times E_{dip}\Big|_{\theta=0} \tag{47}$$

If $k\,\alpha_o^2$ is large, one may take the first term of the series in equation 47. Actually, the series may be summed in a closed form. To this end let us consider (25)

$$S(y) = \sum_{n=0}^{\infty} \left(\frac{q^2}{1+q^2}\right)^n \frac{(n+1)}{ik\,\alpha_o^2)^n}\, y^n \tag{48}$$

where $S(1)$ is equal to the infinite series of equation 47.

Integrating equation 48 term by term, we get

$$\int S(y)\, dy = \left[\sum_{n=0}^{\infty} \left(\frac{q^2}{1+q^2}\right)^2 \frac{y^n}{(ik\,\alpha_o^2)^n} \right] y \tag{49}$$

This sum converges as long as

$$\left| \frac{q^2}{1+q^2} \, \frac{y}{ik\alpha_o^2} \right| < 1 \quad .$$

Now

$$\frac{q^2}{1+q^2} < 1$$

and

$$\left| ik\,\alpha_o^2 \right| \quad \text{is a large number.}$$

If we assume $y < k\,\alpha_o^2$ then we may sum the right hand side of equation 49 as an infinite geometric series, i.e.

$$\int S(y)\,dy = \frac{y}{1 - \dfrac{q^2 y}{ik\,\alpha_o^2(1+q^2)}} \quad .$$

From this

$$S(y) = \frac{\partial}{\partial y}\left(\frac{y}{1-Ay}\right)$$

$$= \frac{1}{(1-Ay)^2} \quad .$$

Hence

$$S(1) = \frac{1}{(1-A)^2}$$

where

$$A = \frac{q^2}{ik\,\alpha_o^2(1+q^2)} = -\,iB$$

Then from equations 43 and 44 we obtain for the front-to-back ratio F/B , the following expression:

$$\frac{F}{B} = \frac{\left|E_{\tilde{\theta}n}\Big|_{\theta=0}\right|^2}{\left|E_{\tilde{\theta}n}\Big|_{\theta=\pi}\right|^2} = \frac{(k\alpha_o^2/(1+q^2))^2 - 1 - (k\alpha_o^2/(1+q^2))\ \sin k\beta_o^2}{(1+\frac{1}{(1+q^2)^4}) + \frac{2}{(1+q^2)^2\ (1+B^2)}\left[(1-B^2)\cos k\alpha_o^2 + 2B\ \sin k\alpha_o^2\right]} \tag{50}$$

$$\approx \left(\frac{k\alpha_o^2}{1+q^2}\right)\left\{\frac{\frac{k\alpha_o^2}{1+q^2} - \sin k\beta_o^2}{1 + \frac{2}{(1+q^2)^2}\cos k\alpha_o^2}\right\} \tag{51}$$

From this approximate form, equation 51, for the F/B ratio, we see that the effect of $k\alpha_o^2$ on the ratio is more important than that of $k\beta_o^2$. For small q and large $k\alpha_o^2$ the maximum F/B is

$$\max F/B \simeq G \quad ,$$

where G is now the gain in the principal polarization, but defined as

$$G = \eta\left(\frac{k\alpha_o^2}{1+q^2}\right)^2$$

3.5 (iii) Antenna Gain Versus Focal Length for a Constant Aperture D .

If we neglect the contribution of the direct radiation from the electric dipole onto the maximum lobe of the paraboloidal antenna, then the gain is given by equation 44.

If G_o , the gain for $f/D = 0.25$, is taken as reference-- f_o being the focal length of this reference antenna and D the aperture diameter--then for constant aperture D and focal length $f^{(n)} = n\ f_o$ we have

$$\alpha_o^{(n)} = \alpha_o/\sqrt{n}$$

and

$$\beta_o^{(n)} = \alpha_o\sqrt{n} = \beta_o\sqrt{n}$$

where

$$\alpha_o^2 = \beta_o^2 = \frac{D}{2} = 2f_o$$

Then the ratio of power gains for the two antennas of same aperture and the ratio of their focal lengths n, is

$$R_n = \frac{G_n}{G_o}$$

$$\approx 4\left(\frac{n}{1+n^2}\right)^2 .$$

This ratio R_n has a maximum value at $n = 1$. However, a plot of R_n against n shows that it is not a symmetrical function of n about its maximum at $n = 1$. This is shown analytically as follows

$$\frac{dR_n}{dn} = \frac{2(1-n^2)}{(1+n^2)^3}$$

Let $n = 1 \pm \delta$, $\delta > 0$. We obtain

$$\delta\left\{\left|\frac{dR_n}{dn}\right|_{1+\delta}\right| - \left|\frac{dR_n}{dn}\right|_{1-\delta}\left|\right\} = 6\ \delta^3 > 0$$

Here we see that R_n decreases slower on the side $n > 1$. This is due to the fact that the illumination of the reflector of focal length greater than f_o is more uniform in the E-plane than that of the reflector of focal length smaller than f_o. This uniformity of illumination also explains some of the experimental results of Jones where the half-power angle in the E-plane tends to a constant value with the increase in focal length.

3.5 (iv) Cross-Polarization

Because of the directions of the current components on the reflector, one would expect that the cross-polarization component near the antenna axis will be mainly due to the y-directed currents. On the basis of this observation we shall investigate the position and magnitude of the cross-polarization maximum near the main lobe.

(a) The position of the cross-polarization maximum near the axis will now be found. Using the results of the approximate analysis and assuming Z very small, i.e., near the antenna axis, we have approximately from equations 41 and 31, 32, etc.

$$E_{\phi n}^{\sim}\Big|_{\theta\to 0} \sim \left\{ C_1 \frac{J_3(Z)}{Z} - C_2 \frac{J_4(Z)}{Z^2} + C_3 \frac{J_5(Z)}{Z^3} \right\} \tag{52}$$

where

$$C_1 = A_o + A_2 + A_4$$

$$C_2 = 2(A_2 + 2A_4)$$

$$C_3 = 8A_4$$

To determine the position of $E_{\phi n}^{\sim}$ maximum, we differentiate it with respect to θ and equate to zero. Accordingly,

$$\frac{\partial E_{\tilde{\phi}n}}{\partial\theta} \sim \left\{ 2\left[C_1 \frac{J_3}{Z^2} - C_2 \frac{J_4}{Z^3} + C_3 \frac{J_5}{Z^4} \right] - \left[C_1 \frac{J_4}{Z} - C_2 \frac{J_5}{Z^2} + C_3 \frac{J_6}{Z^3} \right] \right\} = 0 \qquad (53)$$

As $Z \to 0$ $(\theta \to 0)$, $J_n(Z) \sim (\frac{Z}{2})^n \frac{1}{n!}$, and $(Z \sim k\alpha_o\beta_o\theta)$.

Substituting these approximate values into equation (53), we obtain

$$\frac{\partial E_{\tilde{\phi}_n}}{\partial\theta} \sim \left\{ \gamma_1 Z - \gamma_2 Z^3 \right\} = 0 \qquad (54)$$

where

$$\gamma_1 = 2\left[\frac{C_1}{48} - \frac{C_2}{4!16} + \frac{C_3}{5!32} \right] \quad \text{and}$$

$$\gamma_2 = \frac{1}{4!16}\left[C_1 - \frac{C_2}{10} + \frac{C_3}{120} \right] .$$

From equation 54 we find

$$Z = \sqrt{\frac{\gamma_1}{\gamma_2}}$$

Since $Z \sim k\alpha_o\beta_o \sin\theta$, we have for the angular position of the maximum the expression

$$\sin\theta \approx \theta_{rad} \approx \frac{1}{k\alpha_o\beta_o} \sqrt{\frac{\gamma_1}{\gamma_2}} .$$

For a small change in the range of α_o , β_o we expect that constants A_o, A_2 and A_4 , and consequently constants γ_1, γ_2 vary only slightly. For $D \sim 30\lambda$, we thus have

$$\theta_{rad} \approx \frac{4}{k\alpha_o\beta_o} \quad . \tag{55}$$

For a given constant aperture, if the ratio of focal lengths of two paraboloids is n , where

$$\alpha_o^{(n)} = \frac{\alpha_o}{\sqrt{n}} \quad , \qquad \beta_o^{(n)} = \beta_o \sqrt{n} \quad ,$$

and aperture $\quad D = \alpha_o\beta_o = \text{const.},$

then if

$$\theta_{rad}^{(o)} = \frac{4}{k\,\alpha_o\beta_o}$$

$$\theta_{rad}^{(n)} = \frac{4}{k\alpha_o^{(n)}\beta_o^{(n)}} = \frac{4}{\dfrac{k\alpha_o\beta_o}{\sqrt{n}}\sqrt{n}}$$

$$= \frac{4}{k\alpha_o\beta_o} = \theta_{rad}^{(o)} \quad , \quad \text{a constant.}$$

Here $\theta^{(o)}$ and $\theta^{(n)}$ are the positions of maxima for two different focal lengths f_o and f_n . Hence the angle at which the maximum of cross-polarization occurs is a function of the aperture only, and is independent of the focal length. It tends toward the paraboloidal axis with the increase of the aperture size. As a check we compute the angles of the cross-polarization maxima for two apertures

$D = 25.8\lambda$ (for $\lambda = 3$ cm, $\alpha_o = \beta_o = 6.2$ cm$^{1/2}$) $\theta_{max}^{o} \approx 2.7^{o}$

$D = 37.2\lambda$ (and $\lambda = 3$ cm, $\alpha_o = \beta_o = 7.4$ cm$^{1/2}$) $\theta_{max}^{o} \approx 1.9^{o}$

These values agree very well with those computed from the complete expressions and plotted in Fig. 6.

(b) We now investigate the variation of the maximum of the cross-polarization with focal length for a given constant aperture. The approximate formula for the relative magnitudes of the maxima near the antenna axis (i.e., θ is small) is given by

$$E_{\tilde{\phi}n}\Big|_{\theta\sim 0} \sim \frac{e^{i\Delta}\alpha_o^2 \cos\theta}{A\,\beta_o^2}\left[\frac{C_1}{48} - \frac{C_2}{4!16} + \frac{C_3}{5!32}\right] Z^2 \sin 2\phi$$

$$\left|E_{\tilde{\phi}n}\right| = \frac{\sin 2\phi}{A}\, q^2\, \gamma_o\, (k\alpha_o\beta_o)^2\, \theta^2_{max}$$

But

$$\theta_{max} = \frac{4}{k\alpha_o\beta_o}$$

therefore,

$$\left|E_{\tilde{\phi}n}\right|_{\theta\sim 0} \sim \left(\frac{16\sin 2\phi\,\gamma_o}{A}\right) q^2$$

where

$$\gamma_o = \left(\frac{C_1}{48} - \frac{C_2}{4!16} + \frac{C_3}{5!32}\right) \quad \text{and} \quad q = \alpha_o/\beta_o\ .$$

When $\alpha_o = \beta_o$, i.e., $f/D = 0.25$, we obtain the largest maximum of the cross-polarization component. Its magnitude is given by

$$\left|E_{\tilde{\phi}n}\right| \sim \frac{16\sin 2\phi}{A}\gamma_o \qquad (q^2 = 1)$$

and to the first order is independent of the aperture size. This magnitude is, of course, given relative to the maximum of the principal polarization so that it increases in absolute value with the size

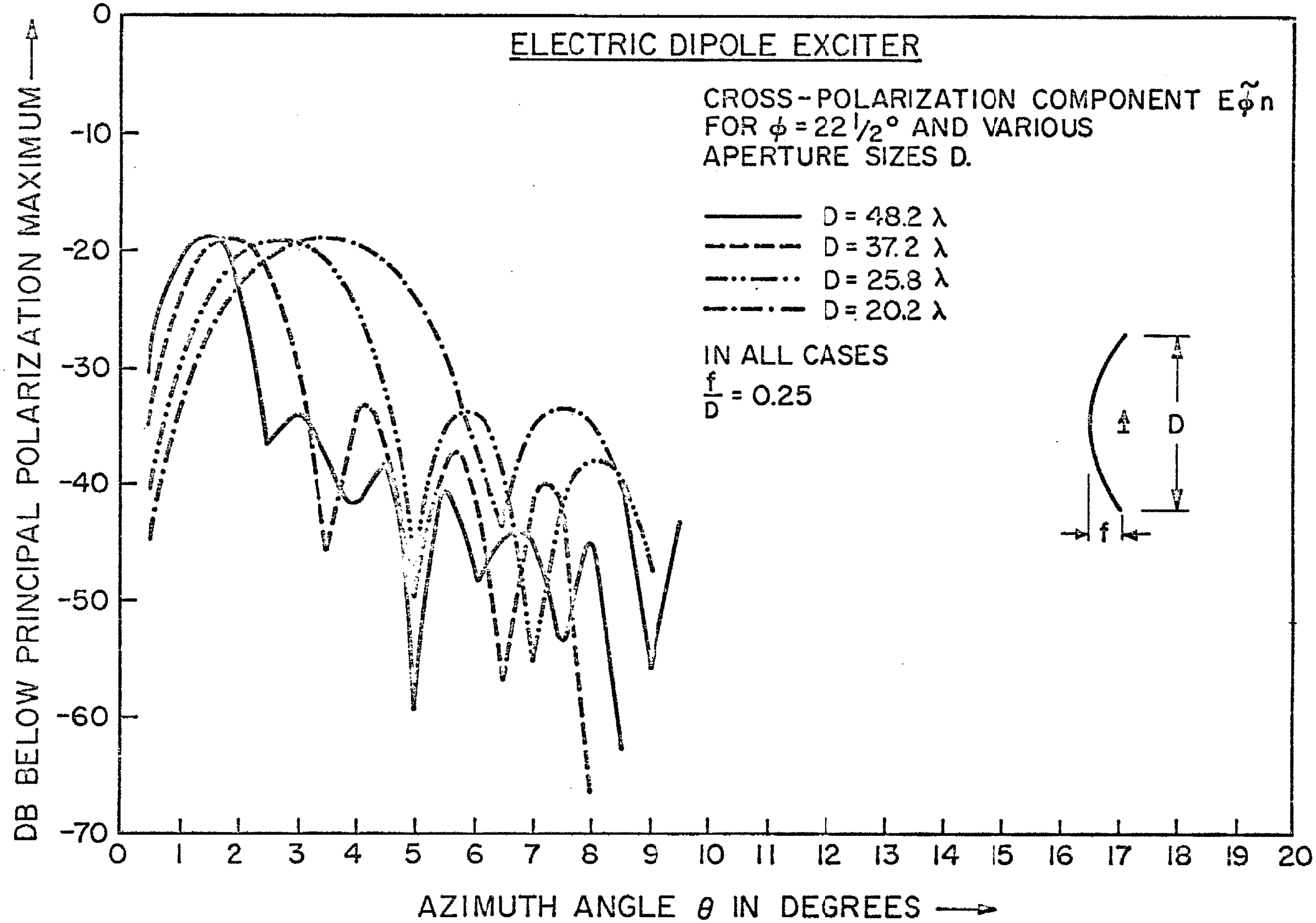

ELECTRIC DIPOLE EXCITER
CROSS-POLARIZATION COMPONENT $E_{\tilde{\phi}n}$
FOR $\phi = 22\frac{1}{2}°$ AND VARIOUS
APERTURE SIZES D.
D = 48.2 λ
D = 37.2 λ
D = 25.8 λ
D = 20.2 λ
IN ALL CASES
$\frac{f}{D} = 0.25$
D
f
DB BELOW PRINCIPAL POLARIZATION MAXIMUM
0
-10
-20
-30
-40
-50
-60
-70
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
AZIMUTH ANGLE θ IN DEGREES

FIG. 6

of the aperture. However, for a given constant aperture $D = 2\alpha^2 = 4f_o$ and a ratio of focal lengths n, the ratio of magnitudes between the two maxima is given by

$$R_n = \frac{q_n^2}{q_o^2} = \frac{q_o^2}{n^2}\frac{1}{q_o^2},$$

or

$$R_n = 1/n^2 \qquad (56)$$

For example, when $f_n/f_o = 2$ we get

$$20 \log_{10} R_n = -12 \text{ db} .$$

This agrees with the results in Fig. 7.

3.6 Interpretation of the Results on the Basis of a Simple Model

Now we shall consider a simple model to explain the observed experimental results, namely, (i) the maximum of cross-polarization components approaches the paraboloidal axis with the increase in aperture size, and (ii) the magnitude of this maximum decreases with the focal length for a given constant aperture. The purpose of the present approach is to give a physical insight into the mechanism by which the cross-polarization pattern is generated.

From the considerations of the surface current on the reflector we find that its y-component increases monotonically from the apex of the paraboloid to a maximum at $\alpha_{max}^2 = 2\beta_o^2 = 4f$. This places the maximum outside the aperture planes of the antennas under consideration and consequently the largest value of the y-directed current component will occur at the rim of the reflector. The y-directed component of current that flows on the reflector surface is due to its

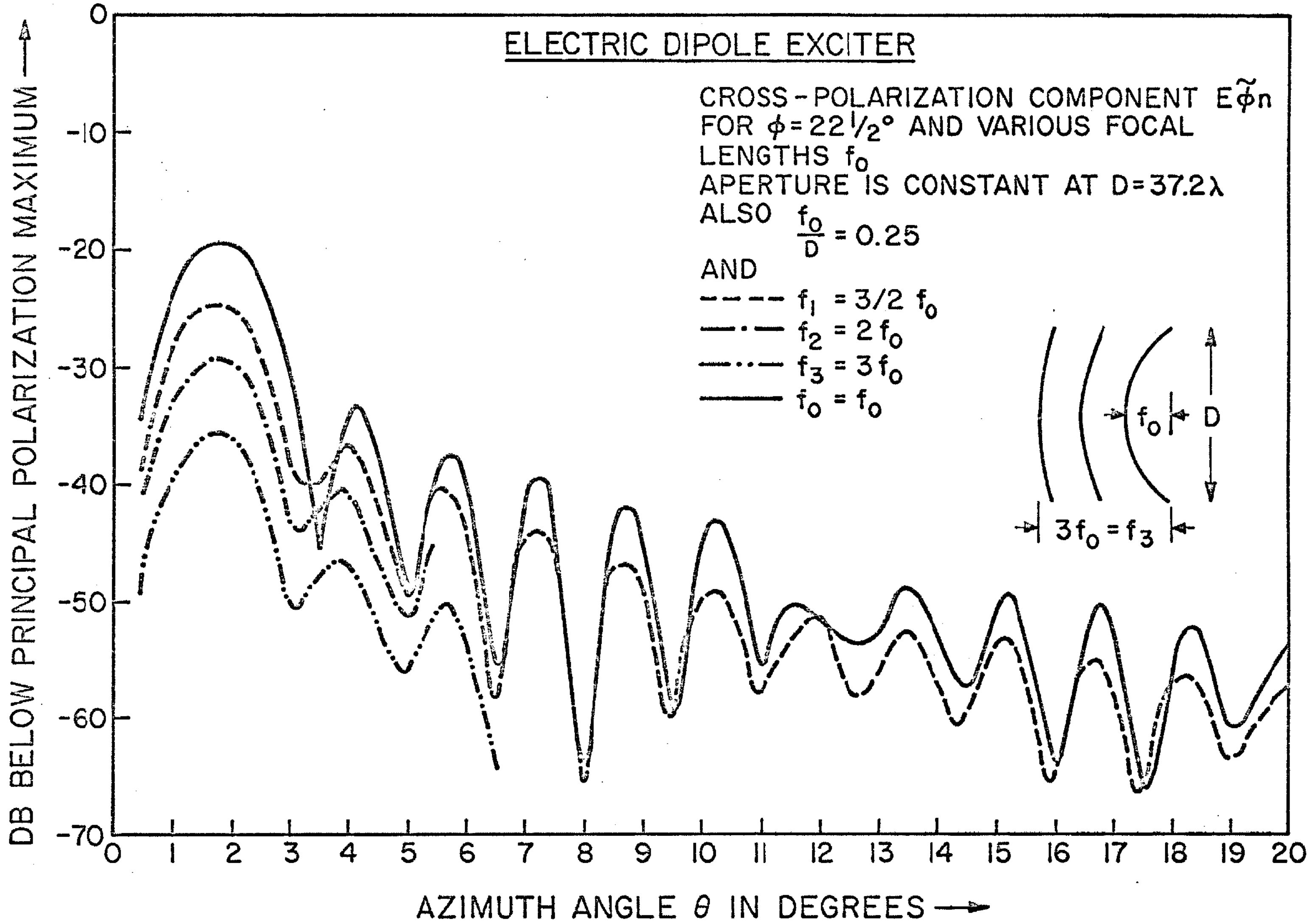

ELECTRIC DIPOLE EXCITER
CROSS-POLARIZATION COMPONENT $E_{\tilde{\phi}n}$
FOR $\phi = 22\frac{1}{2}°$ AND VARIOUS FOCAL
LENGTHS f_o
APERTURE IS CONSTANT AT $D = 37.2\lambda$
ALSO $\frac{f_o}{D} = 0.25$
AND
$f_1 = 3/2\ f_o$
$f_2 = 2\ f_o$
$f_3 = 3\ f_o$
$f_o = f_o$
f_o
D
$3f_o = f_3$
DB BELOW PRINCIPAL POLARIZATION MAXIMUM
0
-10
-20
-30
-40
-50
-60
-70
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
AZIMUTH ANGLE θ IN DEGREES

FIG. 7

curvature.

Because of the symmetry of the problem, one would reasonably expect that we can get a rough approximation to the currents on the paraboloid by postulating that the y-component of polarization is generated by four current filaments equally spaced at the aperture plane and at angles $\phi = 90^\circ$ to each other. These four current filaments may be considered as four dipoles placed at $\phi = 45^\circ, 135^\circ, 225^\circ, 315^\circ$ and directed as shown in Fig. 8

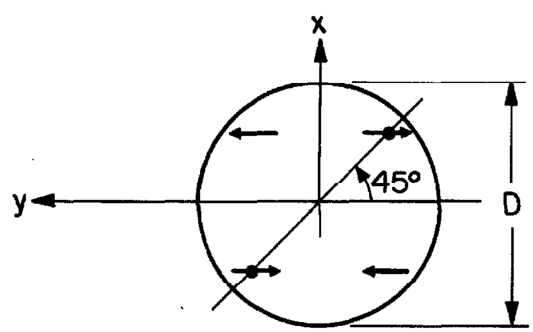


Fig. 8. Model for y-directed Polarization in the Forward Direction

The resultant field due to these four dipoles is given by the following expression

$$E_y \sim E_{\tilde{\phi}} \quad \frac{A_o e^{ikr}}{r} \quad \sqrt{1 - \sin^2\theta \sin^2\phi} \; \left\{ \cos\left[\frac{kD}{2} \sin\theta \cos(\phi - \frac{\pi}{4}) \right] - \right.$$
$$\left. - \cos\left[\frac{kD}{2} \sin\theta \sin(\phi - \frac{\pi}{4}) \right] \right\} \qquad (57)$$

Here $\sqrt{1 - \sin^2\theta \sin^2\phi}$ is the field of each individual dipole, and the curly bracket is the form factor of the four equally-spaced point sources with their different phases taken into account.

To find the principal maximum of the cross-polarized radiation from the expression 57, we assume that the dipole field is uniform for small values of θ , i.e.

$$\sqrt{1 - \sin^2\theta \sin^2\phi} \sim 1 .$$

The stationary values of equation 57 w.r.t. θ are given by the expression

$$\frac{\partial E_\phi^\gamma}{\partial\theta} \sim \frac{kD \sin\theta}{2} \left\{ \cos(\phi - \frac{\pi}{4}) \sin\left[\frac{kD}{2} \sin\theta \cos(\phi - \frac{\pi}{4})\right] - \right.$$

$$\left. - \sin(\phi - \frac{\pi}{4}) \sin\left[\frac{kD}{2} \sin\theta \sin(\phi - \frac{\pi}{4})\right]\right\} = 0 \tag{58}$$

By symmetry, the maxima will occur in the planes $\phi = \pi/4$ and $\phi = 3\pi/4$ and hence we consider the particular plane $\phi = \pi/4$: In this case equation 58 reduces to

$$\frac{\partial E_\phi^\gamma}{\partial\theta} \sim \frac{kD}{2} \sin(\frac{kD}{2} \sin\theta) \cos\theta = 0$$

Near the axis $\theta = 0$, $\cos\theta \sim 1$, thus $\frac{kD}{2} \sin(\frac{kD}{2} \sin\theta) = 0$. This means that $\frac{kD}{2} \sin\theta = n\pi$, but $k = 2\pi/\lambda$. Hence,

$$\frac{2\pi}{\lambda} \frac{D}{2} \sin\theta = n\pi .$$

Solving for θ , we get

$$\theta = \sin^{-1} (\frac{n}{D/\lambda}) . \tag{59}$$

Near the antenna axis $\sin\theta \sim \theta$ and we obtain the approximate formula for the position of the cross-polarization principal maximum

$$\theta_{max} \sim (\frac{n}{D/\lambda}) . \tag{60}$$

Since the distance of the effective current filaments from the axis of the parabola will be somewhat less than $D/2$, one can see how well this rough model explains the positions of the observed maxima. Formula 59 shows that maxima will occur at regular intervals. For $D = 37.2\lambda$, for instance, this interval is $\Delta\theta^{o} \approx 2^{o}$, thus, in good agreement with the graphs of the complete expressions, Figs. 6, 7 and 9.

For a given aperture the distances between the assumed effective current filaments are, to a first approximation, independent of the focal length. Hence the position of the first maximum will remain constant.

However, if the focal length is increased while the aperture is kept constant, the largest current will still occur at the rim, but its magnitude will decrease. This decrease in the magnitude of the y-directed current filament explains the reduction in the y-directed radiation component with the increase in the focal length. One would also expect this reduction in magnitude to occur on the grounds that the reflector curvature decreases with focal length.

Incidentally, the complete expression 57 shows that the cross-polarization in the planes $\phi = 0$, $\pi/2$ is zero, as would be expected from the symmetry of the problem.

Fig. 9 shows graphs of principal and cross-polarization patterns for $\phi = 22\frac{1}{2}^{o}$, $f/0 = 0.25$ and aperture $D = 25.8\lambda$. For this aperture $\theta_{max} = \Delta\theta \approx 2.2^{o}$ which is in good agreement with the results of the graphs.

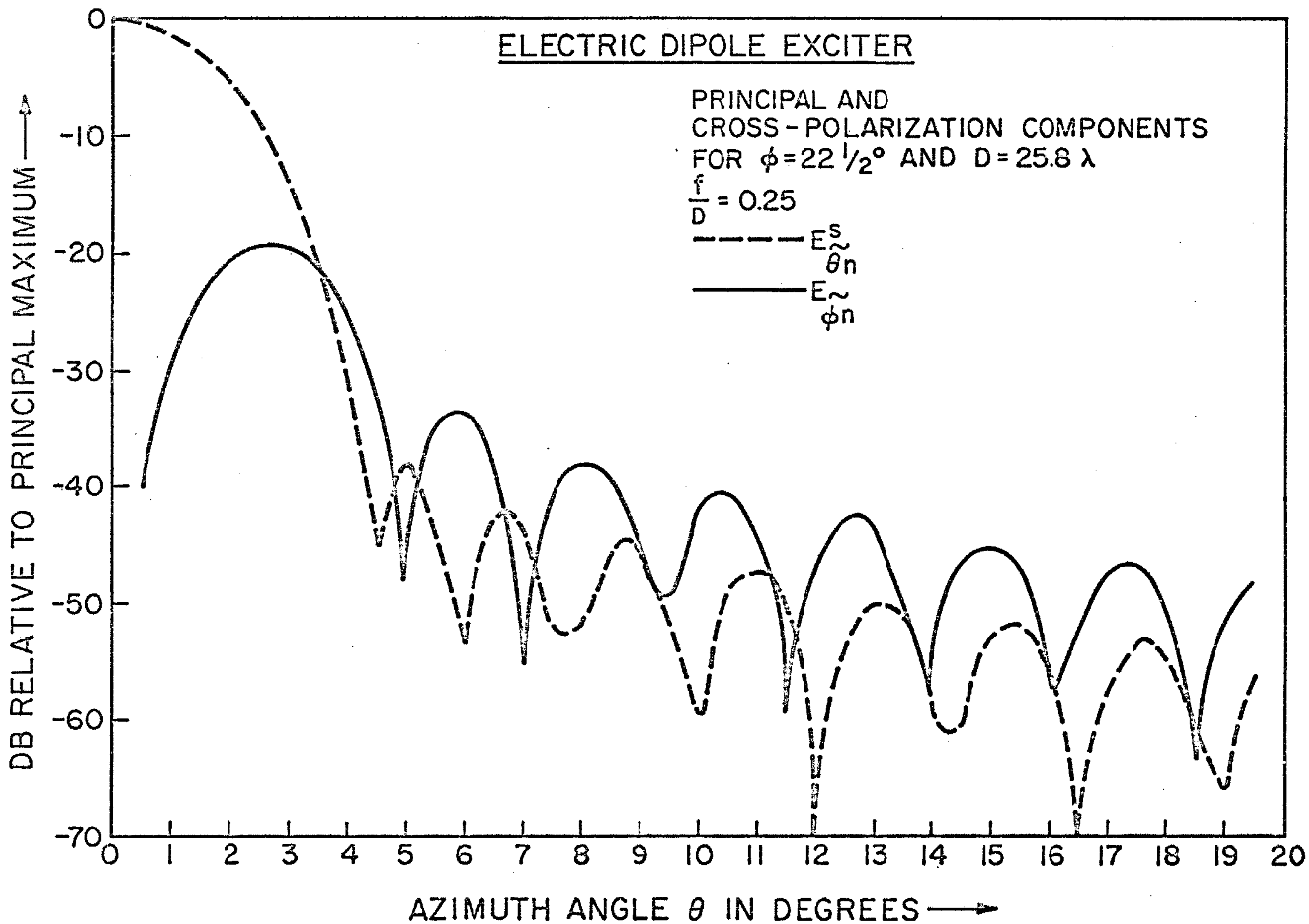

ELECTRIC DIPOLE EXCITER
PRINCIPAL AND
CROSS-POLARIZATION COMPONENTS
FOR φ=22 1/2° AND D=25.8 λ
f/D = 0.25
DB RELATIVE TO PRINCIPAL MAXIMUM
AZIMUTH ANGLE θ IN DEGREES
0
-10
-20
-30
-40
-50
-60
-70
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

FIG. 9

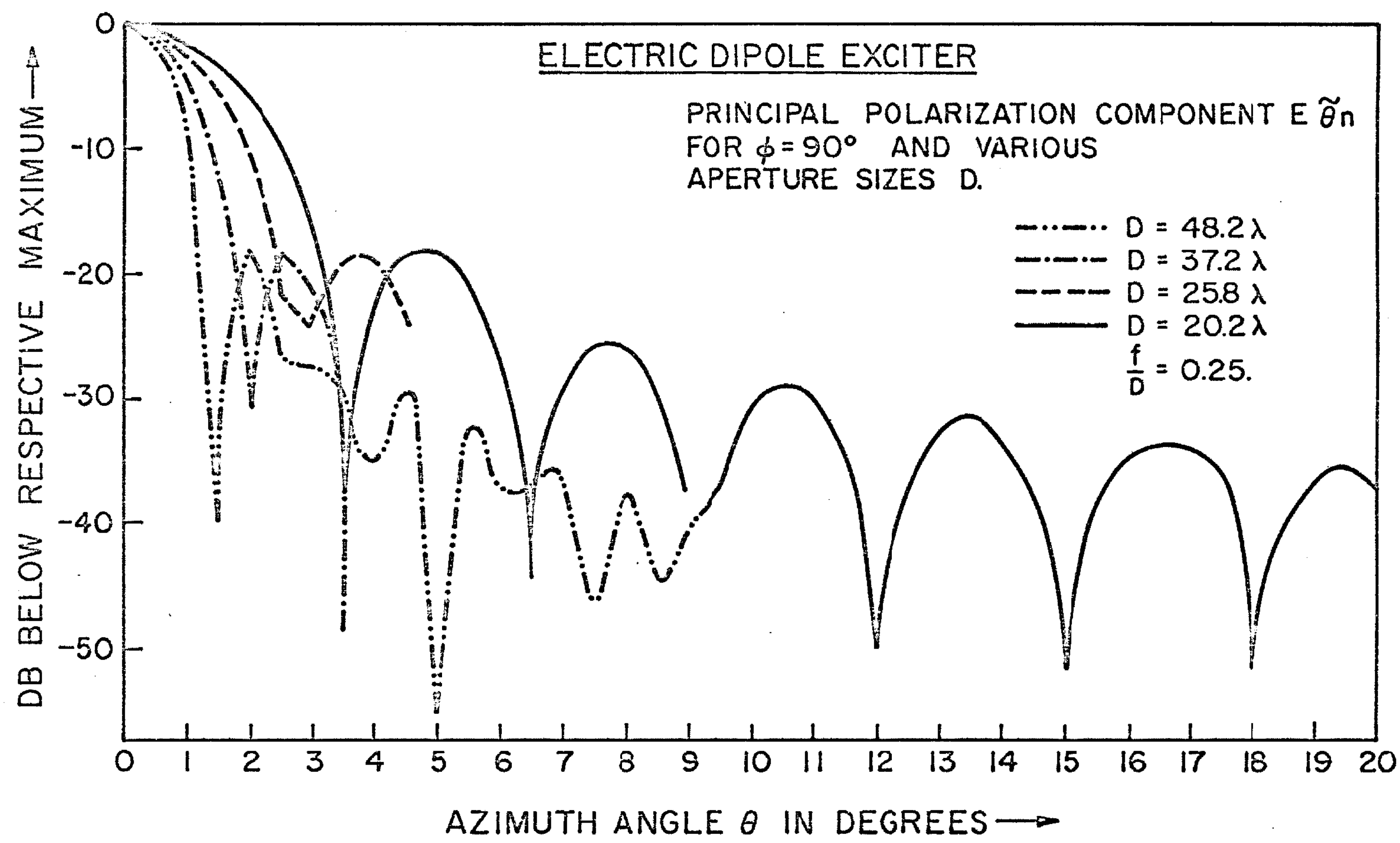

ELECTRIC DIPOLE EXCITER
PRINCIPAL POLARIZATION COMPONENT $E_{\tilde{\theta}n}$
FOR $\phi = 90°$ AND VARIOUS
APERTURE SIZES D.
D = 48.2 λ
D = 37.2 λ
D = 25.8 λ
D = 20.2 λ
$\frac{f}{D} = 0.25.$
DB BELOW RESPECTIVE MAXIMUM
0
-10
-20
-30
-40
-50
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
AZIMUTH ANGLE θ IN DEGREES

FIG. 10

4. PLANE WAVE SOURCES

4.1 Introduction

In his paper Jones showed that a combination of electric and magnetic dipoles oriented at right angles to each other and having a certain ratio of electric to magnetic currents gives a plane wave source. In our coordinate system, Fig. 5, the electric dipole is directed along the x-axis and the magnetic dipole along the y-axis.

Jones also showed that in the zeroth order approximation a plane wave source induces no y-directed current on the reflector. Thus cross-polarization is only due to the z-directed current. Because of its orientation, the z-directed current component will contribute little to the cross-polarization near the axis of the paraboloid. However, Jones found experimentally that for horn radiators the magnitude of the cross-polarization component is comparable to that of an electric dipole alone. His experiments also showed that the cross-polarization decreases as the horn is flared out in the H-plane.

Kinber explained the phenomenon by the fact that the waves in the horn exciter cannot be considered as one plane wave, but as a combination of at least two plane waves propagating at an angle to each other. He also claimed that, in the zeroth order approximation only, the combination of electric and magnetic dipoles gives no y-directed component.

In the following we show how to obtain the plane wave source by the combination of electric and magnetic dipoles and also show that the higher order contribution to the y-polarized current is due to the radially-directed induction field of the dipole and is $1/k\alpha_o^2$ times smaller than that given by the electric dipole alone. Thus for large apertures (i.e., for large $k\alpha_o^2$) which Jones used in the experiment, it is obvious that the first order correction to the y-directed component of current would be of the order of 20 db below that of the electric dipole alone, while the experimentally-determined cross-polarization for horns was only slightly below that due to the small electric dipole by itself.

Finally, we consider the excitation of the paraboloidal reflector antennas by horn radiators.

4.2 Huyghens Source

The far-zone field for a magnetic dipole located at the origin and oriented along the y-axis of coordinates is given by the expression

$$\underline{E}_m^{inc} = \frac{M\, e^{ikr}}{r} \left\{ \underline{e}_x \cos\psi + \underline{e}_z \sin\psi \cos\phi' \right\} \qquad (61)$$

where

M is a constant for a given magnetic dipole.

Other symbols have the same meaning as for the electric dipole.

The surface current on the paraboloid due to this magnetic dipole is given by

$$\underline{K}_m = \frac{2}{\eta} \left\{ (\underline{n} \cdot \underline{E}_m^{inc})\, \underline{\rho}_o - (\underline{\rho}_o \cdot \underline{n})\, \underline{E}_m^{inc} \right\} \qquad (62)$$

Substituting equations 12, 13 and 61 into equation 62 we get, after some manipulation

$$\underline{K}_m = \frac{2M}{\eta} \left\{ \cos\frac{\psi}{2} \left[(1 - \cos\psi)\cos^2\phi' + \cos\psi \right] \underline{e}_x \right. +$$

$$+ \sin\frac{\psi}{2} \sin\psi \frac{\sin 2\phi'}{2} \underline{e}_y +$$

$$\left. + \left[-\cos\phi'(\sin\frac{\psi}{2}\cos\psi) + \cos\phi'(\sin\psi\cos\frac{\psi}{2}) \right] \underline{e}_z \right\} \quad (63)$$

The current on the reflector due to a small electric dipole at the origin (focus) and directed along the x-axis is given by equation 16. Then the total current on the surface of the paraboloid is the sum of the currents given by the expressions 16 and 63.

$$\underline{K}_T = \underline{K}_E + \underline{K}_m \quad (64)$$

If now we adjust the dipole currents so that $M = E_o$ or ratio of moments given by

$$\frac{dy\ I_m}{dv\ I_e} = \eta \quad \text{(the free space impedance)},$$

where I_m is the equivalent magnetic current

I_e is the electric dipole current,

then equation 64 becomes

$$\underline{K}_T = \frac{2E_o}{\eta} \left\{ \cos\frac{\psi}{2}(1 + \cos\psi)\, \underline{e}_x + \cos\phi' \sin\psi \cos\frac{\psi}{2}\, \underline{e}_z \right\} \quad (65)$$

or in the paraboloidal coordinates

$$\underline{K}_T = \frac{2E_o}{\eta} \left\{ \frac{2\beta_o^3}{(\alpha^2 + \beta_o^2)^{3/2}} \underline{e}_x + \cos\phi' \frac{2\alpha\beta_o^2}{(\alpha^2 + \beta_o^2)^{3/2}} \underline{e}_z \right\} \quad (66)$$

The radiation vector for the surface currents given by equation 66 is then

$$\underline{N} = \frac{4\beta_o E_o}{\eta} e^{\frac{ik\beta_o^2}{2}(1 + \cos\theta)} \int_{\emptyset=0}^{2\pi} \int_{\alpha=0}^{\alpha_o} \left[2\beta_o^3 \underline{e}_x + \right.$$

$$\left. + 2\beta_o^2 \alpha \cos\emptyset' \underline{e}_z \right] e^{\frac{ik\alpha^2}{2}(1 - \cos\theta) - ik\alpha\beta_o \sin\theta \cos(\emptyset - \emptyset')} \frac{\alpha \, d\alpha \, d\emptyset'}{(\alpha^2 + \beta_o^2)^2} \quad (67)$$

or

$$\underline{N} = N_x \underline{e}_x + N_z \underline{e}_z \quad (68)$$

where

$$N_x = 2\beta_o^2 N_{ox} \int_0^1 \frac{J_o(Zt)\, e^{\frac{iwt^2}{2}} t \, dt}{(1 + q^2 t^2)^2} \quad (69)$$

and

$$N_z = - 2i\alpha_o \beta_o \cos\emptyset \, N_{ox} \int_0^1 \frac{J_1(Zt) e^{\frac{iwt^2}{2}} t^2 dt}{(1 + q^2 t^2)^2} \quad (70)$$

We may rewrite equations 69 and 70 in terms of

$I_o^{(1)}$, $I_1^{(2)}$, etc., as follows

$$N_x = 2\beta_o N_{ox} I_o^{(1)}$$

and

$$N_z = -2i\alpha_o\beta_o \cos\emptyset \, N_{ox} I_1^{(2)} \quad (71)$$

If the aperture illumination is tapered, then we may use a factor $(1 - \gamma t^2)^n$ to take this into account. Here γ is the required tapering.

Let us now consider the case of tapered illumination with $n = 1$, then we may rewrite equations 69 and 70 as

$$N_x = 2\beta_o^2 N_{ox} \int_0^1 (1 - \gamma t^2) \frac{J_o(Zt)\, e^{\frac{iwt^2}{2}}\, t\, dt}{(1 + q^2 t^2)^2}$$

$$= 2\beta_o^2 N_{ox} \left\{ I_o^{(1)} - \gamma I_o^{(3)} \right\} \quad . \qquad (72)$$

Similarly for N_z

$$N_z = -2i\, \alpha_o \beta_o \cos \phi\, N_{ox} \left\{ I_1^{(2)} - \gamma I_1^{(4)} \right\} \qquad (73)$$

where the values of the integrals are determined as for the case of the dipole excitation. These values are given in Appendix A2.

For the range $0 \leq \theta \leq \theta_{cr}$ the normalized far-zone fields are given by the following modified expressions

$$E_{\tilde{\phi} n} = \frac{e^{i\Delta} \sin 2\phi}{N_o \sqrt{1 - \sin^2\theta \cos^2\phi}} \left(-\mathcal{N}_z'\right) \sin \theta \qquad (74)$$

and

$$E_{\tilde{\theta} n} = \frac{e^{i\Delta}}{N_o \sqrt{1 - \sin^2\theta \cos^2\phi}} \left\{ -(1 - \sin^2\theta \cos^2\phi) \mathcal{N}_x' + \frac{\sin 2\phi \cos^2\phi \mathcal{N}_z'}{2} \right\} \qquad (75)$$

with the notation

$$\mathcal{N}_x' = 2\beta_o^2 \left\{ I_o^{(1)} - \gamma I_o^{(3)} \right\}$$

and

$$\mathcal{N}'_z = -2\, i \alpha_o \beta_o \left\{ I_1^{(2)} - \gamma I_1^{(4)} \right\}$$

and the normalization constant for the illumination taper γ is given by

$$N_o = \frac{2\beta_o^2}{2q^2} \left\{ \frac{q^2 + \gamma}{1 + q^2} - \gamma \log_e (1 + q^2) \right\} .$$

We now investigate the contributions of the first order dipole fields to cross-polarization in y-direction.

Let us consider the expressions for the fields of small electric and magnetic dipoles. From Kraus (1), p. 133, we write for the electric dipole fields

$$E_\Theta = \epsilon_\Theta \left\{ \frac{1}{r} + \frac{i}{kr^2} \right\} \sin \tilde{\Theta}$$

$$E_r = 2\epsilon_\Theta \left\{ \frac{i}{kr^2} \right\} \cos \tilde{\Theta}$$

and for the magnetic dipole field, p. 158 of reference (1),

$$E_{\phi m} = \epsilon_\Theta \left\{ \frac{1}{r} + \frac{i}{kr^2} \right\} \sqrt{1 - \sin^2\Theta \, \sin^2\phi}$$

where

ϵ_Θ is a constant

$\tilde{\Theta}$ is the angle defined in Fig. 5, i.e.,

$\cos \tilde{\Theta} = \sin \Theta \cos \phi$

$\sin \tilde{\Theta} = \sqrt{1 - \sin^2\Theta \cos^2\phi}$

When we combine the fields due to E_Θ and $E_{\phi m}$ we see that the first order correction term to the y-directed current is zero as in the case of the zeroth order. The only contribution is that due to the component E_r of the electric dipole. This component is proportional to

$$N_y \sim \frac{8}{k\beta_o^2} N_o \int_0^1 \frac{J_1(Zt)\, e^{\frac{iwt^2}{2}}\, t^2 dt}{(1+q^2t^2)^2} \sin \phi \qquad (76)$$

For $k\beta_o^2 \sim 80$, which corresponds to the case where $D = 37.2\lambda$, the contribution is 20 db below that due to the dipole zeroth order component. Thus the experimentally-observed large cross-polarization cannot be explained by this argument.

4.4 Horn Exciters

For a transverse electric (TE) wave propagating in a waveguide of height b and width a , we have the following field components (26) (See Fig. 10)

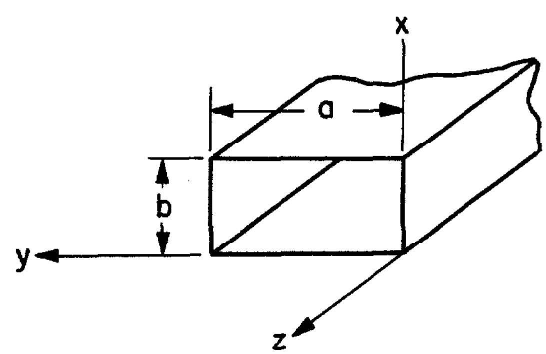


Fig. 10

$$E_y = -Z_{TE} H_x = \frac{i\omega\mu}{k_z^2} B\, e^{-\gamma Z}\, \frac{\pi m}{b} \cos \frac{\pi \ell y}{a} \sin \frac{\pi m x}{b}$$

$$E_x = Z_{TE} H_y = \frac{-i\omega\mu}{k_z^2} B\, e^{-\gamma Z}\, \frac{\pi \ell}{a} \sin \frac{\pi \ell y}{a} \cos \frac{\pi m x}{b} \qquad (77a)$$

where $Z_{TE} = -\frac{i\omega\mu}{\gamma}$ is the waveguide impedance for the $TE_{\ell m}$ mode. Also,

$$\gamma^2 = k_z^2 - \frac{\omega^2}{c^2}$$

and

$$k_z^2 = (\frac{\pi\ell}{a})^2 + (\frac{\pi m}{b})^2 \tag{77b}$$

The lowest (fundamental) mode for this type of guide is the TE_{10} mode, given by $\ell = 1, m = 0$. There is no variation of field strength in the x-direction in this mode.

For TE_{10} mode we get from equations 74

$$E_y = 0$$

$$E_x = Z_{TE} H_y = \frac{-i\omega\mu}{k_z^2} Be^{-\gamma Z} \frac{\pi}{a} \sin \frac{\pi y}{a} \tag{78}$$

$$Z_{TE} = \frac{-i\omega\mu}{\gamma}$$

If, keeping the height constant, we flare out the waveguide in the H-plane gradually, the TE_{10} mode will still carry most of the wave energy. Since the transition is assumed to be smooth, we may neglect the disturbance (higher order harmonics) caused by it. The width a of the guide is now a variable. If we let $a = n(\lambda/2)$ then substituting for a in the expressions for k_z and γ we get

$$k_z^2 = (\frac{\pi}{n\frac{\lambda}{2}})^2 \quad \text{and} \quad \gamma^2 = (\frac{\pi}{n\frac{\lambda}{2}})^2$$

or

$$\gamma^2 = -(\frac{2\pi}{\lambda})^2 (1 - \frac{1}{n^2}) \tag{79}$$

Solving equation 79 for γ we obtain

$$\gamma = \pm\, ik\left(1 - \frac{1}{n^2}\right)^{1/2} \tag{80}$$

Here we take the negative square root for propagation in the positive z-direction.

If we set $\gamma = -ik\,\nu'$, then the expressions 78 for the field become

$$E_x = Z_{TE}\, H_y = \frac{-i\omega\mu}{-ik\nu}\, H_y$$

or

$$E_x = \frac{Z_o}{\nu'}\, H_y \tag{81}$$

where

Z_o is the free space impedance

Thus for TE_{10} wave

$$Z_{TE} = \frac{Z_o}{\sqrt{1 - \frac{1}{n^2}}}$$

and

$$\nu' = \sqrt{1 - \frac{1}{n^2}} \tag{82}$$

For plane waves in free space we have

$$\frac{E_x}{H_y} = Z_o$$

so that the quantity ν' is a measure of deviation in the dipole magnitude ratio from that of the plane wave dipole ratio.

If $\nu' = 1$, we have plane wave conditions. This requires that from equation 82

$$\nu' = \sqrt{1 - \frac{1}{n^2}} = 1$$

or

$$n \to \infty$$

Thus plane wave conditions will occur only for an infinitely flared guide.

For the case of $\nu' = 0$, equation 82 gives

$$\sqrt{1 - \frac{1}{n^2}} = 0$$

or

$$n \to 1$$

then

$$\frac{E_x}{H_y} \sim \infty$$

i.e.,

$$E_x \gg H_y$$

and so electric dipole conditions result. If the waveguide is cut to form a horn radiator we have, neglecting any discontinuity effects at the aperture, the ratio between the equivalent electric and magnetic dipoles relative to the free-space ratio

$$\frac{1}{Z_o}\left(\frac{E_x}{H_y}\right) = \frac{1}{\sqrt{1 - \frac{1}{n^2}}} = \frac{1}{\nu'}$$

where $0 \leq \nu' \leq 1$.

In general the aperture impedance is different from that of free space so that we will have reflected waves in the horn. If we define the reflection coefficient Γ as

$$\Gamma = \frac{E_x^{refl}}{E_x^{inc}} = \frac{Z_o - Z_{TE}}{Z_o + Z_{TE}}$$

this gives

$$E_x^{Tot} = E_x^{inc} + E_x^{refl} = (1 + \Gamma) E_x^{inc}$$

and

$$H_y^{Tot} = (1 - \Gamma)\ H_y^{inc} = (1 - \Gamma)\ \frac{E_x^{inc}}{Z_{TE}} \tag{83}$$

From equations 81 and 83 we obtain

$$\frac{1}{Z_o}\ \frac{E_x^T}{H_y^T} = \frac{1 + \Gamma}{1 - \Gamma}\ \frac{1}{\nu'} = \frac{1}{\nu} \quad .$$

Here ν is the actual value of the ratio of the two effective fundamental dipoles. We then have

$$1 \leq \frac{1}{\nu} \leq \frac{1}{\nu'}$$

or

$$\nu' \leq \nu \leq 1 \tag{84}$$

Thus the actual value of the ratio ν will be somewhat greater than that computed by equation 82.

We are now ready to compute the radiation components from a paraboloid excited by a combination of electric and magnetic dipoles whose ratio of magnitudes relative to the free-space ratio is denoted by ν . Current on the surface of the paraboloid is given by

$$\underline{K}_E + \nu \underline{K}_m = \frac{2E_o}{\eta} \left\{ \cos\frac{\psi}{2} \left[(1+\nu) \left(\frac{\cos\psi + 1}{2}\right) - (1-\cos\psi)(1-\nu)\frac{\cos 2\phi}{2} \right] \underline{e}_x - (1-\nu)(1-\cos\psi)\cos\frac{\psi}{2}\frac{\sin 2\phi}{2}\underline{e}_y + \cos\phi \left[\sin\frac{\psi}{2}\cos\psi(1-\nu) + \nu\sin\psi\cos\frac{\psi}{2} \right] \underline{e}_z \right\} \tag{85}$$

Introducing the paraboloidal coordinates for the trigonometric functions ψ and integrating to find the far-zone fields, we get for the radiation vector components

$$N_x = N_{ox} \left[\beta_o^2 (1+\nu) I_o^{(1)} + \alpha_o^2 (1-\nu) \cos 2\phi \, I_2^{(3)} \right]$$

$$N_y = N_{ox} \sin 2\phi \, \alpha_o^2 (1-\nu) I_2^{(3)}$$

and

$$N_z = N_{ox} \left[\alpha_o^2 (1-\nu) I_1^{(4)} + \beta_o^2 (1+\nu) I_1^{(2)} \right] \cos\phi \left(\frac{i\alpha_o}{\beta_o}\right)$$

where the integrals $I_o^{(1)}$, $I_1^{(2)}$, etc. have already been evaluated in the electric dipole case, i.e., they can be evaluated exactly or approximately with different series for $0 \le \theta \le \theta_{cr}$ and for $\theta_{cr} < \theta \le \pi$.

The only parameter that requires further computation is ν and this depends on the reflection coefficient Γ (equation 81). Thus the values of Γ must be known in each particular case.

In Fig. 12 we plot the magnitude of the y-directed component of radiation vector relative to that of an electric dipole.

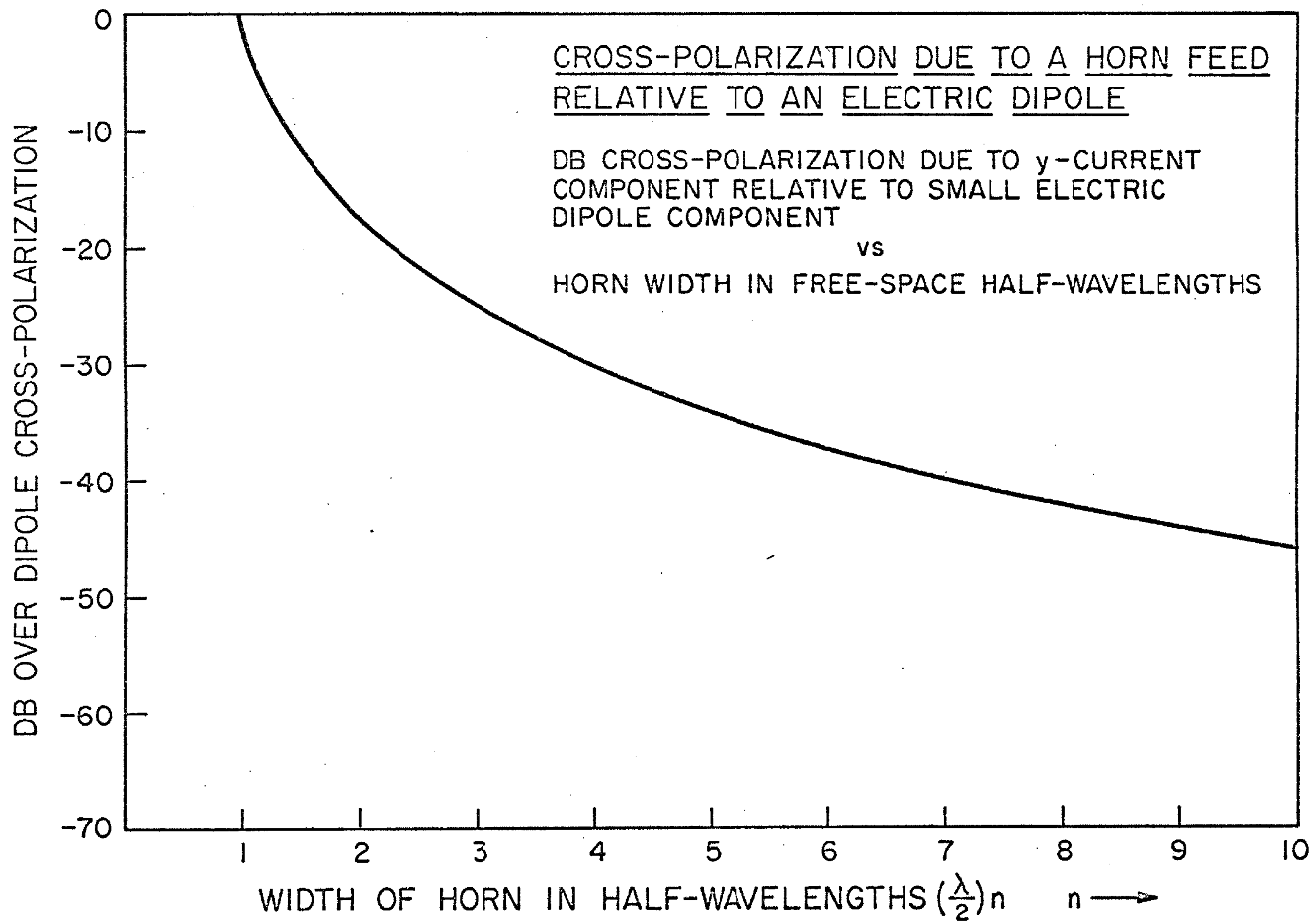

CROSS-POLARIZATION DUE TO A HORN FEED RELATIVE TO AN ELECTRIC DIPOLE
DB CROSS-POLARIZATION DUE TO y-CURRENT COMPONENT RELATIVE TO SMALL ELECTRIC DIPOLE COMPONENT
vs
HORN WIDTH IN FREE-SPACE HALF-WAVELENGTHS
DB OVER DIPOLE CROSS-POLARIZATION
0
-10
-20
-30
-40
-50
-60
-70
1
2
3
4
5
6
7
8
9
10
WIDTH OF HORN IN HALF-WAVELENGTHS $(\frac{\lambda}{2})n$
n →

FIG. 12

5. CONCLUSIONS

In this theoretical work we have derived a number of functions of fundamental importance in the paraboloidal antenna theory. Using these functions we investigate the space structure of the far-zone fields for a number of sources. The functions can also be used to find any one of the polarization components.

The electric dipole illumination and the integrals arising from the dipole-induced currents on the reflector surface are of basic importance, since the integrals arising from any exciting source can be easily deduced from those for the electric dipole.

With the aid of the dipole integrals, we investigate the radiation characteristics of the paraboloidal reflector antennas excited by combinations of electric and magnetic dipoles. In our case these dipoles are oriented at 90° to each other. By varying the relative magnitudes of the dipole currents we can simulate a variety of sources such as horn exciter, plane wave sources and loop antennas.

Using a waveguide as the exciter and varying the H-plane flare we find that we can realize a whole series of various combinations of electric and magnetic dipoles. Thus the electric dipole and plane wave exciters are but special cases of the horn feed problem. Simple approximate formulas are derived that show how the required ratio of the dipole magnitudes can be obtained by varying the H-plane width of the waveguide.

For the electric dipole we derive formulas that give the approximate angular variation cross-polarization maxima near the main lobe of the antenna. Formulas are also derived that relate the magnitude of this cross-polarization maxima to the main lobe magnitude.

These formulas can easily be modified to the case of waveguide excitation. The positions of the maxima of cross-polarization, for same reflector aperture, will not change as can be seen from the physical reasoning. However, the magnitudes of these maxima will be modified by a factor related to the ratio of the horn aperture to the waveguide width, which is the same thing as the ratio of the electric to magnetic dipole magnitudes. The results obtained explain in a quantitative way the large cross-polarization measured by Jones for a paraboloidal reflector excited by a small horn.

The effect of the discontinuity at the open end of the horn may be taken into account by the experimentally-found reflection coefficients. This reflection causes a change in the ratio of magnitudes of the equivalent dipoles.

The cross-polarization at wide angles (i.e., near 90°) is found to be due to the combination of the y-directed and the z-(axially)-directed currents. The analysis shows that the latter component of current increases in magnitude with the flaring-out of the horn radiator. In fact, the difference between this component excited by an electric dipole and that by an infinitely-flared guide is more than 6db, i.e., twice that due to the electric dipole alone.

The rapid angular variation of the magnitude of the cross-polarized component may be explained in the case of the electric dipole by the rough model of four dipoles.

Plots of the principal (fundamental) and cross-polarized components show very small variations of magnitude with the azimuthal angle θ , indicating that a good portion of the energy lost in the side lobes is concentrated near $\theta = 90^{\circ}$ to the parabola axis. This constitutes a large part of the literally-radiated cross-polarized energy component, even though larger maxima occur in the forward and backward directions close to the paraboloid axis.

In conclusion, although the rectangular horn exciters are simple in construction, they do not provide the lowest cross-polarization for the case of paraboloidal reflectors. Consequently, rectangular horns are not recommended as sources if minimum cross-polarization is a requirement.

Appendix A1

(a) To find the unit vectors $\underline{n}$ and $\underline{\rho}_o$ (Fig. 13)

(i) $\underline{n}$ is the unit normal to the paraboloidal surface.

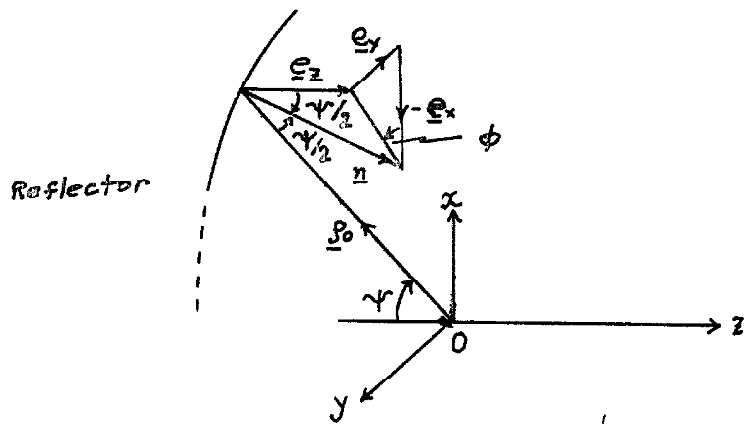


Fig. 13

Components $n_z = |\underline{n}| \cos\frac{\psi}{2}$, $n_x = -|\underline{n}| \sin\frac{\psi}{2} \cos\phi$,

$$n_y = -|\underline{n}| \sin\frac{\psi}{2} \sin\phi$$

$$\text{but} \quad |\underline{n}| = 1$$

so
$$\underline{n} = -\sin\frac{\psi}{2}\cos\phi\,\underline{e}_x - \sin\frac{\psi}{2}\sin\phi\,\underline{e}_y + \cos\frac{\psi}{2}\,\underline{e}_z$$

(ii) Radius vector from origin to point (ψ,ϕ) , as before

$$\rho_x = -|\underline{\rho}_o| \cos\psi \quad , \quad \rho_y = |\underline{\rho}_o| \sin\psi\cos\phi \ ,$$

$$\rho_x = |\underline{\rho}_o| \sin\psi\sin\phi$$

or
$$\underline{\rho}_o = \sin\psi\sin\phi\,\underline{e}_x + \sin\psi\cos\phi\,\underline{e}_y - \cos\psi\,\underline{e}_z$$

(b) Element of surface area in paraboloidal coordinates (Fig. 14)

(α, β, ϕ)

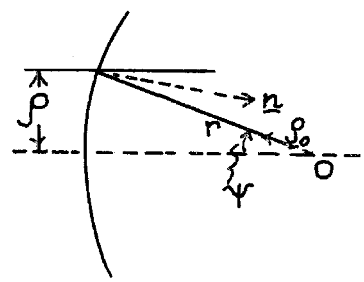


Fig. 14

$$dS = (\rho d\phi)\left(\frac{r d\psi}{\cos\frac{\psi}{2}}\right)$$

Now $\rho = \alpha\beta$

$$d\psi = \frac{\sqrt{(dx)^2 + (dy)^2}}{r}\cos\frac{\psi}{2}$$

For $\beta = \beta_o$, $d\psi = \dfrac{\sqrt{\alpha^2 + \beta_o^2}}{r}\cos\dfrac{\psi}{2}\, d\alpha$

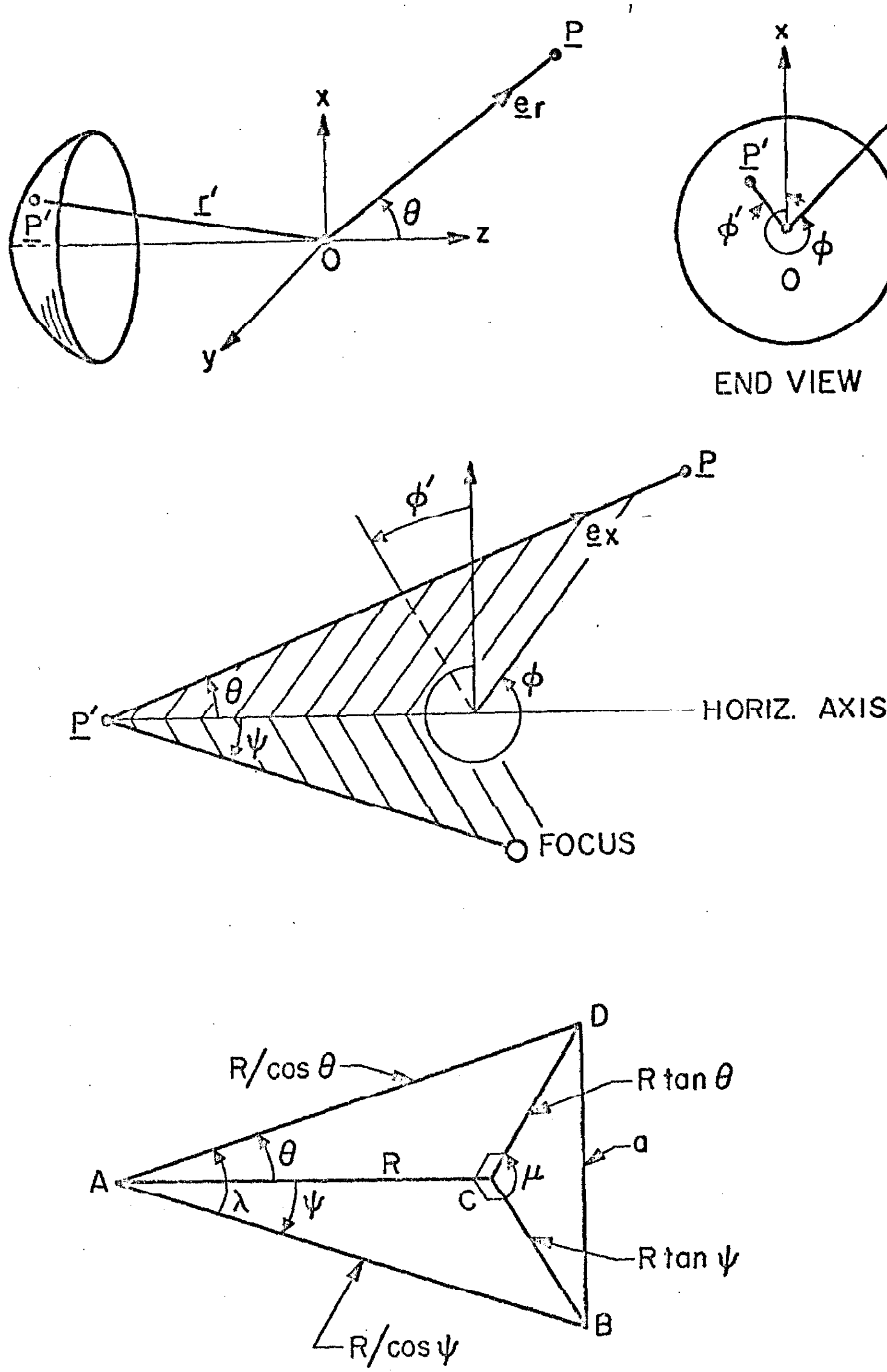
Therefore,

$$dS = \alpha\beta_o\sqrt{\alpha^2 + \beta_o^2}\; d\phi'\; d\alpha$$

(c) To determine the value of $(\underline{e}_r \cdot \underline{r}')$ in paraboloid coordinates, (see Fig. 15)

In triangle BCD we use the cosine rule to find an expression for a

$$a^2 = R^2\left\{\tan^2\theta + \tan^2\psi - 2\tan\theta\tan\chi\cos\mu\right\}$$


P
x
er
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P′
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y
x
P
P′
φ′
φ
O
END VIEW
P
φ′
ex
θ
φ
P′
ψ
HORIZ. AXIS
FOCUS
D
R/cos θ
R tan θ
θ
a
A
R
μ
λ
ψ
C
R tan ψ
B
R/cos ψ

FIG. 15

where

$$\cos\mu = \cos(\phi - \phi')$$

Consider triangle ABD, formed by

$$AB = R/\cos\psi \;, \quad AD = R/\cos\theta \;, \quad \text{and} \quad BD = a$$

Using the cosine rule in triangle ABD we get for a^2

$$\frac{R^2}{\cos^2\psi} + \frac{R^2}{\cos^2\theta} - \frac{2R^2\cos\lambda}{\cos\theta\cos\psi} = a^2 =$$

$$= R^2\,[\tan^2\theta + \tan^2\psi + 2\tan\psi\tan\theta\cos(\phi - \phi')]$$

Solve for $\cos\lambda$

$$\cos\lambda = \cos\theta\cos\psi - \sin\theta\sin\psi\cos(\phi - \phi')$$

We may now compute $\underline{e}_r \cdot \underline{r}' = |\underline{r}'|\cos(\pi - \lambda)$

$$\therefore \quad \underline{e}_r \cdot r' = -|\underline{r}'|\,[\cos\psi\cos\theta - \sin\psi\sin\theta\cos(\phi - \phi')]$$

Substitute for $\sin\psi$, $\cos\psi$ their equivalent in terms of α,β and obtain

$$\underline{e}_r \cdot \underline{r}' = -\left[-\frac{\alpha^2 - \beta_o^2}{2}\cos\theta - \alpha\beta\sin\theta\cos(\phi - \phi')\right]$$

Appendix A2

Exact Solutions of Radiation Integrals for Electric Dipole

For $0 \leq \theta \leq \theta_{cr}$

where $\theta_{cr} = 2 \tan^{-1}(\frac{1}{q})$ and $q = \alpha_o/\beta_o$

$$I_o^{(1)} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt)\ t\ dt}{(1 + q^2t^2)^2}$$

$$= \frac{e^{\frac{iw}{2}}}{(1+q^2)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2q^2)^n (n+1)!(m+n)!(-iw)^m J_{m+n+1}(Z)}{(1+q^2)^n\ n!\ m!\ Z^{m+n+1}}$$

$$I_o^{(2)} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^2 dt}{(1 + q^2)^2}$$

$$= \frac{e^{\frac{iw}{2}}}{(1 + q^2)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2q^2)^n (n+1)!(m+n)!(-iw)^m J_{m+n+2}(Z)}{(1+q^2)^n\ n!\ m!\ Z^{m+n+1}}$$

$$I_1^{(4)} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^4 dt}{(1+q^2t^2)^2}$$

$$= \frac{e^{\frac{iw}{2}}}{(1+ q^2)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2q^2)^n (n+1)!(-iw)^m J_{(m,n)}}{(1+q^2)^n\ n!\ m!\ Z^{m+n+1}}$$

where $J(m,n) = [Z\, J_{m+n+2}(Z) - 2(m+n+1)\, J_{m+n+3}(Z)]$

$$I_2^{(3)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_2(Zt)\, t^3 dt}{(1 + q^2t^2)^2}$$

$$= \frac{e^{\frac{iw}{2}}}{(1+q^2)^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(m+n)!(-iw)^m J_{m+n+3}(Z)}{n!\; m!\; Z^{m+n+1}}$$

$$I_o^{(3)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt)\, t^3 dt}{(1+q^2t^2)^2}$$

$$= \frac{2}{i} \frac{\partial}{\partial w} \left\{ I_o^{(1)}(Z) \right\} \quad .$$

Appendix A3

Exact Evaluation of Radiation Integrals for Electric Dipole Exciter

Case: $\theta_{cr} < \theta \leq \pi$

$$I_0^{(1)*} = \frac{e^{\frac{iw}{2}}}{(iw)(1+q^2)^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(m+n)!\ Z^m J_m(Z)}{n!\ m!\ (iw)^{m+n}}$$

$$- \frac{1}{(iw)(1+q^2)^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2q^2)^n}{2^m} \frac{(n+1)!(m+n)!\ Z^{2m}}{n!\ (m!)^2\ (iw)^{m+n}}$$

$$I_1^{(2)*} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^2 dt}{(1+q^2t^2)^2}$$

$$= - \frac{\partial I_1^{(1)*}}{\partial Z}$$

$$= - \frac{e^{\frac{iw}{2}} Z}{(iw)^2(1+q^2)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(n+m+1)\ Z^m J_m(Z)}{n!(m+1)!\ (iw)^{m+n}}$$

$$+ \frac{Z}{(iw)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2q^2)^n (n+1)!\ (m+n+1)!\ Z^{2m}}{2^m\ n!\ m!(m+1)!(iw)^{m+n}}$$

$$I_2^{(3)*} = \frac{1}{Z} I_1^{(2)*} - \frac{\partial}{\partial Z} I_1^{(2)*} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_2(Zt)\ t^3 dt}{(1+q^2t^2)^2}$$

$$= \frac{e^{\frac{iw}{2}} Z}{(iw)^3(1+q^2)^2} \sum_{n,m=0}^{\infty} \left(\frac{2q^2}{1+q^2}\right)^n \frac{(n+1)!(m+n+2)!\ Z^m J_m(Z)}{n!\ (m+2)!\ (iw)^{m+n}}$$

$$- \sum_{n,m=0}^{\infty} (2q^2)^n \frac{(n+1)!(m+n+2)!\ Z^{2m+2}}{(iw)^{m+n+3}\ 2^m (m!)\ (m+2)}$$

$$I_1^{(4)*} = \frac{2}{i} \frac{\partial}{\partial w} I_1^{(2)*}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^4 dt}{(1+q^2t^2)^2}$$

$$= \sum_{n,m=0}^{\infty} \frac{(2q^2)^n (n+1)!(m+n+1)!}{(1+q^2)^{2+n}\ n!(m+1)!} \left[\frac{(m+n+2) - \frac{iw}{2}}{(iw)^{m+n+3}} e^{\frac{iw}{2}}\right] Z^{m+1} J_m(Z)$$

$$- 2 \sum_{n,m=0}^{\infty} \frac{(2q^2)^n (n+1)!(m+n+2)!}{2^m\ m!\ (m+1)!} \cdot \frac{Z^{2m+1}}{(iw)^{m+n+3}}$$

Appendix A4

Summation of Series

$$S_1^{(2)*} = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{n+2}} \sum \frac{(m+2)\ \cdots\ (m+n+1)}{2^m\ m!\ (i\beta)^m} Z^{m+1}$$

$$\int S_1^{(2)*}\ dZ^n = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{n+2}} \sum_{m=0}^{\infty} Z^{n+1} \frac{Z^m}{m!\ (2i\beta)^m}$$

$$= \sum \frac{(2q^2)^n\ (n+1)}{(iw)^{n+2}} (Z^{n+1}\ e^{Z/2i\beta})$$

$$S_1^{(2)*} = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{n+2}} \frac{\partial^n}{\partial Z^n} (Z^{n+1}\ e^{Z/2i\beta})$$

$$S_1^{(4)*} = \sum_{n=0} \frac{(2q^2)^n (n+1)}{(iw)^{n+3}} \sum_{m=0}^{\infty} \frac{(m+2)\ \cdots\ (m+n+2)}{(2i\beta)^m} \frac{Z^{m+1}}{m!}$$

$$S_1^{(4)*}\ dZ^{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)(2q^2)^n}{(iw)^{n+3}} \sum_{m=0}^{\infty} Z^{n+2} \frac{Z^m}{(2i\beta)^m} \frac{1}{m!}$$

$$\int S_1^{(4)*} = \sum_{n=0}^{\infty} (n+1) \frac{(2q^2)^n}{(iw)^{n+3}} \frac{\partial^{n+1}}{\partial Z^{n+1}} (Z^{n+2}\ e^{Z/2i\beta})$$

$$S_2^{(3)*} = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{n+3}} \sum_{m=0}^{\infty} \frac{(m+3)\ \cdots\ (m+n+2)\quad Z^{m+2}}{m!\ (2i\beta)^m}$$

$$\int S_2^{(3)*}\ dZ^n = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{3+n}} \left(Z^{n+2}\ e^{Z/2i\beta}\right)$$

$$S_2^{(3)*} = \sum_{n=0}^{\infty} \frac{(2q^2)^n\ (n+1)}{(iw)^{3+n)}} \frac{\partial^n}{\partial Z^n} \left(Z^{n+2}\ e^{Z/2i\beta}\right)$$

Appendix A5

Approximate Evaluation of Radiation Integrals for the Case of the Electric Dipole

Case: $0 \leq \theta \leq \theta_{cr}$

$$I_o^{(1)}(Z) = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_o(Zt)\ t\ dt}{(1 + q^2t^2)^2}$$

Assume $\frac{1}{(1+q^2t^2)^2} = A_o + t^2 A_2 + t^4 A_4$, then

$$I_o^{(1)}(Z) = e^{\frac{iw}{2}} \Big\{ (A_o + A_2 + A_4) \frac{1}{Z} \sum_{m=0}^{\infty} (- \frac{iw}{Z})^m J_{m+1}(Z) -$$

$$- \frac{2}{Z^2}(A_2 + 2A_4) \sum_{m=0}^{\infty} (m+1)(- \frac{iw}{Z})^m J_{m+2}(Z) +$$

$$+ \frac{4\ A_4}{Z^3} \sum_{m=0}^{\infty} (m+1)(m+2)(- \frac{iw}{Z})^m J_{m+3}(Z) \Big\}$$

$$I_1^{(2)} = \int_0^1 \frac{e^{\frac{iwt^2}{2}} J_1(Zt)\ t^2 dt}{(1 + q^2t^2)^2}$$

$$= e^{\frac{iw}{2}} \Big\{ (A_1 + A_2 + A_4) \frac{1}{Z} \sum_{m=0}^{\infty} (- \frac{iw}{Z})^m J_{m+2}(Z)$$

$$= \frac{2}{Z^2} (A_2 + 2A_4) \sum_{m=0}^{\infty} (m+1)(- \frac{iw}{Z})^m J_{m+3}(Z)$$

$$+ \frac{4A_4}{Z^3} \sum_{m=0}^{\infty} (m+1)(m+2)(- \frac{iw}{Z})^m J_{m+4}(Z) \Big\}$$

$$I_2^{(3)}(Z) = e^{\frac{iw}{2}} \left\{ (A_o + A_2 + A_4) \frac{1}{Z} \sum_{m=0}^{\infty} \left(-\frac{iw}{Z}\right)^m J_{m+3}(Z) \right.$$

$$- \frac{2}{Z^2} (A_2 + 2A_4) \sum_{m=0}^{\infty} \left(-\frac{iw}{Z}\right)^m J_{m+4}(Z) \, (m+1) +$$

$$\left. + \frac{4A_4}{Z^3} \sum_{m=0}^{\infty} (m+1)(m+2) \left(-\frac{iw}{Z}\right)^m J_{m+5}(Z) \right\}$$

$$I_1^{(4)} = e^{\frac{iw}{2}} \left\{ (A_o + A_2 + A_4) \frac{1}{Z} \sum_{m=0}^{\infty} \left(-\frac{iw}{Z}\right)^m J_{m+2}(Z) \right. -$$

$$- \frac{2}{Z^2} (A_o + 2A_2 + 3A_4) \sum_{m=0}^{\infty} (m+1) \left(-\frac{iw}{Z}\right)^m J_{m+3}(Z) +$$

$$+ \frac{4}{Z^3} (A_2 + 3A_4) \sum_{m=0}^{\infty} (m+1)(m+2) \left(-\frac{iw}{Z}\right)^m J_{m+4}(Z) -$$

$$\left. - \frac{8A_4}{Z^4} \sum_{m=0}^{\infty} (m+1)(m+2)(m+3) \left(-\frac{iw}{Z}\right)^m J_{m+5}(Z) \right\}$$

Appendix A6

Approximate Evaluation of Electric Dipole Integrals

For region: $\theta_{cr} < \theta \leq \pi$

$$I_o^{(1)*} = \frac{e^{\frac{iw}{2}}}{iw} (A_o + A_2 + A_4) \sum_{m=0}^{\infty} \frac{J_m(Z)}{(iw/Z)^m} -$$

$$- \frac{2}{(iw)} (A_2 + 2A_4) \sum_{m=0}^{\infty} (m+1) \frac{J_m(Z)}{(iw/Z)^m}$$

$$+ \frac{4A_4}{(iw)^2} \sum_{m=0}^{\infty} (m+1)(m+2) \frac{J_m(Z)}{(iw/Z)^m}$$

$$- \frac{1}{(iw)} A_o \sum_{m=0}^{\infty} \frac{Z^m}{2^m \; m!(iw/Z)^m}$$

$$- \frac{2A_2}{(iw)} \sum_{m=0}^{\infty} \frac{(m+1) \; Z^m}{2^m \; m! \; (iw/Z)^m}$$

$$+ \frac{4A_4}{(iw)^2} \sum_{m=0}^{\infty} \frac{(m+1)(m+2) \; Z^m}{2^m \; m! \; (iw/Z)^m}$$

Set $w/Z = \gamma$

$$I_1^{(4)*}(Z) = \frac{e^{\frac{iw}{2}}}{iw} \left\{ \left[(A_o + A_2 + A_4) \sum_{m=0}^{\infty} \frac{J_{1+m}(Z)}{(i\gamma)^m} - \right. \right.$$

$$- \frac{2}{iw} (2A_o + 3A_2 + 4A_4) \sum_{m=0}^{\infty} \frac{J_{1+m}(Z)}{(i\gamma)^m} +$$

$$+ \frac{4}{(iw)^2} (A_o + 3A_2 + 6A_4) \sum \frac{(m+2)}{m!} \frac{J_{1+m}(Z)}{(i\gamma)^m} .$$

$$- \frac{8}{(iw)^3} (A_2 + 4A_4) \sum \frac{(m+3)!}{m!} \frac{J_{1+m}(Z)}{(i\gamma)^m} + \frac{16A_4}{(iw)^4} \sum \frac{(m+4)!}{m!} \frac{J_{1+m}(Z)}{(i\gamma)^m}$$

$$- \frac{2Z}{(iw)^3} \left[A_o \sum \frac{(m+2)!}{m!} \frac{Z^m}{(m+1)!\ (2i\gamma)^m} - \right.$$

$$\left. - \frac{2A_2}{(iw)} \sum \frac{(m+3)!\ Z^m}{m!\ 2^m (m+1)!(i\gamma)^m} + \frac{4A_4}{(iw)^2} \sum \frac{(m+4)!}{m!(m+1)!} \frac{Z^m}{(2i\gamma)^m} \right.$$

$$I_1^{(2)*}(Z) = \left\{ \frac{e^{\frac{iw}{2}}}{iw} \left[(A_o + A_2 + A_4) \sum_{m=0}^{\infty} \frac{J_{m+1}(Z)}{(i\gamma)^m} - \right. \right.$$

$$- \frac{2}{iw} (A_o + 2A_2 + 3A_4) \sum_{m=0}^{\infty} (m+1) \frac{J_{m+1}(Z)}{(i\gamma)^m}$$

$$+ \frac{4}{(iw)^2} (A_2 + 3A_4) \sum_{m=0}^{\infty} (m+1)(m+2) \frac{J_{m+1}(Z)}{(i\gamma)^m} -$$

$$\left. - \frac{8A_4}{(iw)^3} \sum_{m=0}^{\infty} \frac{(m+3)!}{m!} \frac{J_{m+1}(Z)}{(i\gamma)^m} \right]$$

$$+ \frac{Z}{(iw)^2} \left[A_o \sum \frac{(m+1)\ Z^m}{(m+1)!(2i\gamma)^m} - \frac{2A_2}{iw} \sum \frac{(m+2)!}{m!(m+1)!} \frac{Z^m}{(2i\gamma)^m} \right.$$

$$\left. \left. + \frac{4A_4}{(iw)^2} \sum \frac{(m+3)!\ Z^m}{m!(m+1)!(2i\gamma)^m} \right] \right\}$$

$$I_2^{(3)*}(Z) = \left\{ \frac{e^{\frac{iw}{2}}}{iw} \left[(A_o + A_2 + A_4) \sum_{m=0}^{\infty} \frac{J_{m+2}(Z)}{(i\gamma)^m} - \right.\right.$$

$$- \frac{2}{(iw)} (2A_o + 3A_2 + 4A_4) \sum_{m=0}^{\infty} (m+1) \frac{J_{m+2}(Z)}{(i\gamma)^m} +$$

$$+ \frac{4}{(iw)^2} (A_o + 3A_2 + 6A_4) \sum_{m=0}^{\infty} \frac{(m+2)!}{m!} \frac{J_{m+2}(Z)}{(i\gamma)^m} -$$

$$- \frac{8}{(iw)^3} (A_2 + 4A_4) \sum_{m=0}^{\infty} \frac{(m+3)!}{m!} \frac{J_{m+2}(Z)}{(i\gamma)^m} +$$

$$\left. + \frac{16A_4}{(iw)^4} \sum_{m=0}^{\infty} \frac{(m+4)!}{m!} \frac{J_{m+2}(Z)}{(i\gamma)^m} \right] +$$

$$+ \frac{Z^2}{(iw)^3} \left[- A_o \sum_{m=0}^{\infty} \frac{Z^m}{m!\,(2i\gamma)^m} + \right.$$

$$+ \frac{2A_2}{(iw)} \sum_{m=0}^{\infty} \frac{(m+3)!}{m!\,(m+2)!} \frac{Z^m}{(2i\gamma)^m} -$$

$$\left.\left. - \frac{4A_4}{(iw)^2} \sum \frac{(m+4)!}{m!\,(m+2)!} \frac{Z^m}{(2i\gamma)^m} \right] \right\}$$

Appendix A7

Closed Form Sum for Series of Type

$$\sum_{m=0}^{\infty} \frac{(m+1)(m+2)\cdots(m+n)\, Z^m}{(m+\ell)!\,(2i\gamma)^m} = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!\,(m+\ell)!} \frac{Z^\ell}{(2i\gamma)^m}$$

The series converges for large enough values of m. This is shown in the following. Since the series is really the sum of two series, real and imaginary, we consider convergence of each one separately.

m^{th} term:

$$u_m = \frac{(m+1)\cdots(m+n)\, Z^m}{(m+\ell)!\,(2i\gamma)^m}$$

$(m+2)^{nd}$ term:

$$u_{m+2} = \frac{(m+3)\cdots(m+n+2)\, Z^{m+2}}{(m+\ell+2)!\,(2i\gamma)^{m+2}}$$

Ratio

$$\frac{u_{m+2}}{u_m} = \frac{(m+n+1)(m+n+2)\, Z^2}{(m+1)(m+2)(m+\ell+1)(m+\ell+2)} \quad \frac{1}{(2i\gamma)^2}$$

For large m $(m > n, \ell)$ Ratio $\rightarrow \frac{Z^2}{m!} \frac{1}{(2\beta)^2}$

Now for $\Theta > \Theta_{cr}$, $\beta > 1$ always , so that as long as $m \doteq Z/2$, $u_{m+2}/u_m < 1$, and the series will converge for sufficiently large m .

In series

$$S_{n\ell} = \sum_{n=0}^{\infty} \frac{(m+1)(m+2)\cdots(m+n)\,Z^m}{2^m (i\gamma)^m\,(m+\ell)!}$$

$n \geq \ell$ always, so that we may write

$$S_{n\ell} = \sum_{m=0}^{\infty} \frac{(m+\ell+1)\cdots(m+n)\,Z^n}{2^m (i\gamma)^m\,m!}$$

$$Z^\ell S_{n\ell} = \sum_{m=0}^{\infty} \frac{(m+\ell+1)\cdots(m+n)\,Z^{n+\ell}}{2^m (i\gamma)^m\,m!}$$

and

$$\int\cdots\int Z^\ell S_{n\ell}(Z)\,dZ^{n-\ell} = \sum_{m=0}^{\infty} \frac{Z^{m+n-\ell+\ell}}{(2i\gamma)^m\,m!}$$

$$= Z^n e^{Z/2i\gamma}$$

so

$$S_{n\ell}(Z) = \frac{1}{Z^\ell}\frac{\partial^{n-\ell}}{\partial Z^{n-\ell}}\left(Z^n e^{Z/2i\gamma}\right)$$

i.e.

$$\sum_{m=0}^{\infty} \frac{(m+1)(m+2)\cdots(m+n)\,Z^m}{(m+\ell)!\,(2i\gamma)^m} = \frac{1}{Z^\ell}\left[\frac{\partial^{n-\ell}}{\partial Z^{n-\ell}}\left(Z^n e^{Z/2i\gamma}\right)\right]$$

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