CALCULATION OF πNN AND πNN* COUPLING SHIFTS, USING THE DASHEN-FRAUTSCHI METHOD

Thesis by

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ABSTRACT

In this thesis the Dashen-Frautschi method is applied
to find shifts from SU(2) symmetry of πNN and πNN^*
coupling constants. Three new features of this type of
calculation are discovered. First, we observe that the
same A matrix used to connect s channel and u channel
residue shifts also relates u channel contributions
such as those due to γN exchange to analogous processes
in the s channel. Secondly, under the assumption that all
photon masses are the same (for fixed ΔI), we note that
for linear D functions the A matrix has one eigenvalue
exactly equal to one (for each ΔI), but the corresponding
component of the driving term <u>vanishes</u>. This result
depends essentially only on Clebsch-Gordan and crossing
coefficients. The third discovery is that when viewed
in a space of larger dimension, eigenvalues near one and
"no enhancement" are easy to understand. We also note
that when including infrared corrections, residues are <u>not</u>
in general proportional to products of coupling constants.

Since the driving terms are approximately orthogonal
to the eigenvectors with eigenvalues near one, there is
no simple dominant pattern, and thus our numerical results
are sensitive to details which are poorly known. We do
believe that our calculation gives correct order of

magnitude estimates; the observable parts of the $\delta R/R_o$'s are found to be less than one percent. The experimental data are presently unclear but at least do not decisively indicate larger residue shifts than this.

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I. Introduction

Recently a self-consistent S-Matrix method for calculating perturbations on strong interactions has been proposed by Dashen and Frautschi.[1] This method has been applied with considerable success in determining mass and coupling shifts and has provided a dynamical enhancement mechanism with a wide variety of applications.[2-23] The basic ingredient in this method is the "bootstrap" hypothesis, in which particles are thought of as being bound states of each other. Mathematically one then expects particles to appear as poles in scattering amplitudes. In the absence of symmetry-breaking effects certain masses are degenerate, and coupling constants satisfy certain symmetry relations. In the presence of symmetry breaking, however, one has mass splittings and residue shifts. We then expect equations such as [6]

$$\vec{\delta R} = A^{RR} \vec{\delta R} + A^{RM} \frac{\vec{\delta M}}{M} + \vec{D} , \qquad (1)$$

where $\vec{\delta M}/M$ is a vector containing mass shifts, A^{RR} and A^{RM} are matrices, $\vec{D}$ is a "driving term" arising from processes which were not present in the original unper-

turbed case, and $\overrightarrow{\delta R}$ is a vector containing residue

shifts. The Dashen-Frautschi method gives us equations

of the type (1) and presents us with a way of calculating

A^{RR} , A^{RM} , and $\overrightarrow{D}$.

In this thesis we apply the Dashen-Frautschi method

to calculate deviations from SU(2) symmetry of πNN and

πNN* coupling constants. Our calculation includes the

"bootstrap" terms in which residue shifts act on each

other, as well as all driving terms found to contribute

in the energy region under consideration. Three new

features of this type of calculation are discovered. The

first discovery (Sec. XII) is that the matrix A^{RR} which

connects residue shifts in the u channel to residue

shifts in the s channel $\underline{also}$ serves to relate u channel

processes such as ɣN exchange to the analogous processes

in the s channel. More explicitly, if $\overrightarrow{D}_{1}$ is a vector

arising due to the existence of intermediate states in

the s channel, then

$$\overrightarrow{\delta R} \cong A^{RR} \overrightarrow{\delta R} + (I-A^{RR}) \overrightarrow{D}_{1} + \quad \text{(mass shift and t channel terms).}$$

I.e.,

$$(I-A^{RR})(\overrightarrow{\delta R} - \overrightarrow{D}_{1}) \cong \quad \text{(mass shift and t channel terms).} \quad (2)$$

Although $\vec{D}_1$ thus cannot cause enhancement, it is found that some of the components of $\vec{D}_1$ are not small (namely, those with I=0 and I=2) and hence should not be neglected in calculations of this type.

The second discovery (Sec. XIII) is that even though each of our "self-consistent" matrices has exactly one eigenvalue equal to one (or near one), the corresponding eigenvector is not enhanced when considering ohly the lowest mass processes! This result depends essentially only on crossing and Clebsch-Gordan coefficients and is at first quite surprising, especially when we consider the good agreement with experiment that has been obtained in previous calculations under the assumption that the eigenvector with eigenvalue near one (if there is only one such eigenvector) is enhanced.[3,4,7,16,17,19] We do find, however, that mass shifts are enhanced in our model. Most of the previous successful calculations involved enhanced mass shifts, or mass shifts driving coupling shifts. In our present problem the mass shifts happen to have a small effect.

The third discovery (Sec. XIV) is that the occurrence of eigenvalues near one and the no enhancement conclusion can be understood qualitatively by looking at a larger space which is physically less appropriate but

mathematically simpler; in this larger space these properties are easily seen to hold. This also makes clearer the occurrence of eigenvalues near one in previous residue shift studies. An additional technical point we note is that when infrared terms are included in the scattering amplitude, it is no longer correct to assume the residue of a pole in this amplitude is proportional to a product of two coupling constants (see Sec. XV).

Since the driving terms are approximately orthogonal to the eigenvectors with eigenvalues near one (i.e., "no enhancement"), there is no single dominant pattern, and thus our numerical results are sensitive to details which are not well known. Thus the numerical results of our calculation are not reliable, but we expect that the theory does at least produce reliable order of magnitude estimates. Our numerical results are small (partly as a consequence of no enhancement); we predict π-N P-wave residue shifts of less than one percent of the unperturbed values. Experimentally there are no measurements precise enough to compare with predictions as small as ours, though at least there is no definite evidence for much larger residue shifts.

The plan of this thesis is as follows: Sections II

and IV give the Dashen-Frautschi dispersion relations
used, as well as definitions and a discussion of D
functions. These two sections are included for complete-
ness and do not represent new material. Sec. III contains
a discussion of the infrared divergence problem arising
in this type of calculation. Sections V through XII
describe the explicit evaluation of the dispersion
integrals, using as input both "bootstrap" and driving
terms. The relation of intermediate states such as YN
exchange in the u channel to contributions due to the
existence of direct channel intermediate states is
established in Sec. XII. In Sec. XIII we find an
analysis of the results and the conclusion of no enhance-
ment. The numerical results of the computation are found
in Tables II - VIII. Sec. XIV contains a discussion of
residue shifts in an expanded space,-which, as mentioned
above, is helpful in understanding some of our results.
Sec. XV examines the relation between residue shifts and
coupling constants. In Sec. XVI we have a discussion
of experimental data. In Sec. XVII we compare our
calculation with related analyses.

II. Basic Dispersion Formulas

In this section we review the derivation of the DF dispersion relations[1,6] used in our calculation of πNN and πNN^* residue shifts. We temporarily neglect consideration of the infrared divergence in the amplitude associated with virtual infrared photons; this problem will be dealt with in the next section. We consider the $J = \frac{1}{2}^+$ and $J = \frac{3}{2}^+$ partial wave projections of the pion-nucleon scattering amplitude and note that near the poles due to the "direct channel" processes in Fig. 1 we have

$$A(I_i I_3 \rightarrow I_f I_3 ; J = \tfrac{1}{2}^+) \cong \frac{R(I_i I_3 \rightarrow I_f I_3 ; J = \tfrac{1}{2}^+)}{W - M} \quad \text{(near W=M)};$$

$$\tag{3}$$

$$A(I_i I_3 \rightarrow I_f I_3 ; J = \tfrac{3}{2}^+) \cong \frac{R(I_i I_3 \rightarrow I_f I_3 ; J = \tfrac{3}{2}^+)}{W - M^*} \quad \text{(near W=M}^*\text{)}.$$

Here I_i and I_f represent the initial and final total isospins of the πN system; I_3 is the third component of isospin, W is the center of mass energy, M is the nucleon mass, M^* is the $N^*(1238)$ mass, the R's are (constant) dimensionless residues, and the A's are π-N scattering amplitudes with the normalization

$$A = \frac{e^{2i\eta} - 1}{2i\rho} \quad ; \quad \rho = \frac{q\left[(W-M)^2 - \mu^2\right]}{W^2}$$

(q is the center of mass momentum, and μ is the pion mass).
In the absence of symmetry breaking (i.e., in the "unper-
turbed case") the scattering amplitudes would not depend
on I_3 and total isospin would be conserved. Since we are
assuming symmetry breaking due to electromagnetism, how-
ever, the amplitudes depend on I_i, I_f, and I_3 (I_3 is still
conserved, of course).

We now write the analogues of (3) for the unperturbed
π-N amplitudes and subtract the two cases, obtaining (to
first order in small quantities)

$$\delta A(I_i I_3 \rightarrow I_f I_3; J=\tfrac{1}{2}+) \cong \frac{\delta R(I_i I_3 \rightarrow I_f I_3; J=\tfrac{1}{2}+)}{W-M_o} + \frac{R_o(I_i \rightarrow I_f; J=\tfrac{1}{2}+)\,\delta M}{(W-M_o)^2} \qquad (4)$$

(near $W=M_o$; an analogous result holds for the $J=\tfrac{3}{2}+$ case).
(Here δA, for example, means the perturbed minus the
unperturbed amplitude, and zero subscripts indicate
unperturbed values.) Now let $D_{11}(W)$ be any function of W
which has a zero at $W=M_o$. Then

$$D_{11}^2(W)\,\delta A(I_i I_3 \rightarrow I_f I_3; J=\tfrac{1}{2}+; W) \cong R_o(I_i \rightarrow I_i; J=\tfrac{1}{2}+)\,\delta M$$

and

$$\frac{d}{dW}\left[D_{11}^2(W)\,\delta A(I_i I_3 \rightarrow I_i I_3; J=\tfrac{1}{2}+; W)\right] \cong \begin{array}{l} R_o(I_i \rightarrow I_i; J=\tfrac{1}{2}+)\,\delta M\,D_{11}''(M) \qquad (5) \\[2mm] + \delta R(I_i I_3 \rightarrow I_i I_3; J=\tfrac{1}{2}+) \end{array}$$

(near $W=M_o$; an analogous result holds for the $J=\tfrac{3}{2}+$ case).

(We have assumed for convenience that $\left.\frac{D_{11}(W)}{W-M}\right|_{W=M} = 1$;
the normalization of $D(W)$ would cancel in the final
formulas in any case.) Next we write integrals for each
of the left hand sides of (5) as a function of W:

$$D_{11}^2(W)\,\delta A(W) = \frac{1}{2\pi i}\int_C \frac{D_{11}^2(W')\,\delta A(W')}{W'-W}\,dW' ;$$

$$\frac{d}{dW}\left[D_{11}^2(W)\,\delta A(W)\right] = \frac{1}{2\pi i}\int_C \frac{D_{11}^2(W')\,\delta A(W')}{(W'-W)^2}\,dW'.$$

(C is a contour in the counter-clockwise direction around
W.) Evaluating each of these integrals at $W=M$ and com-
bining this result with (5), we obtain

$$R_0(\tfrac{1}{2}\to\tfrac{1}{2};J=\tfrac{1}{2}+)\,\delta M = \frac{1}{2\pi i}\int_C \frac{D_{11}^2(W')\delta A(\tfrac{1}{2}I_3\to\tfrac{1}{2}I_3;J=\tfrac{1}{2}+;W')}{W'-M}\,dW';$$

$$\delta R(\tfrac{1}{2}I_3\to\tfrac{1}{2}I_3;J=\tfrac{1}{2}+) = \frac{1}{2\pi i}\int \frac{D_{11}^2(W')\delta A(\tfrac{1}{2}I_3\to\tfrac{1}{2}I_3;J=\tfrac{1}{2}+;W')}{(W'-M)^2}\left[1-D_{11}''(M)(W'-M)\right].$$

(6)

(Analogous relations hold for $I_i=I_f=\tfrac{3}{2}$; $J=\tfrac{3}{2}+)$).

When $I_i\neq I_f$ we use a simpler form (note that $R_0=0$ for
this case). Here

$$D_{31}(W)D_{11}(W)\,\delta A(I_i I_3\to I_f I_3;J=\tfrac{1}{2}+) \cong \frac{\delta R(I_i I_3\to I_f I_3;J=\tfrac{1}{2}+)D_{31}(W)D_{11}(W)}{W-M}\ \text{(near $W=M$)}.$$

Hence (writing a Cauchy integral for $D_{31}(W)D_{11}(W)\delta A(W)$)

$$\delta R(I_i I_3 \rightarrow I_f I_3; J=\tfrac{1}{2}+) = \frac{1}{2\pi i}\int_C \frac{D_{31}(W')\,D_{11}(W')\delta A(I_i I_3 \rightarrow I_f I_3; J=\tfrac{1}{2}+; W')}{W'-M}dW'. \quad (7)$$

(Analogous relations hold for the $J=\tfrac{3}{2}+$ case.) Here $D_{31}(W)$ is any function of W which has no poles or zeroes at $W=M$; we are setting $D_{31}(M)=1$.

The next step is to expand each contour in the above integrals around all the singularities of the integrand except the initial direct channel pole at $W=M$ (there will also be a contour at infinity). The specific D functions used are discussed in Section IV and will prove convenient when integrating over the right hand cuts of the integrand. Before delving further into the dispersion integral formalism, however, we need to discuss the infrared divergences arising in a calculation of residue shifts. This problem is dealt with in the following section.

III. Infrared Divergences

The scattering amplitude for $\pi N \to \pi N$ will contain an infrared multiplicative correction arising from virtual infrared photons.[24,25] We shall first discuss the relevance of infrared factors to residue shifts in scattering amplitudes and then to cross sections. Our conclusions are summarized at the end of this section.

For purposes of orientation it will be useful first to consider a non-relativistic example. We look at the asymptotic form of the wave function for a charged, spinless particle scattering in a potential

$$V = V_{short-range} + V_{Coulomb} \quad .$$

Then the scattered wave is (see for example, reference (26))

$$\psi_{scattered} \xrightarrow{(r \to \infty)} \frac{1}{r} e^{ikr} e^{-i\gamma \ln 2kr} \left[f_{Coul}(\theta) + f'(\theta) \right] ,$$

where

$$f_{Coul}(\theta) = \frac{-\gamma}{2k \sin^2 \frac{\theta}{2}} e^{-i\gamma \ln \sin^2 \frac{\theta}{2} + 2i\sigma_0}$$

$$f'(\theta) = \frac{1}{2ik} \sum_\ell (2\ell+1) e^{2i\sigma_\ell} \left(e^{2i\eta_\ell} - 1 \right) P_\ell(\cos \theta)$$

θ = scattering angle $\hbar k$ = momentum

$\sigma_\ell = \arg \Gamma(\ell + 1 + i\gamma)$ υ = velocity .

$$\gamma = e^2 / \hbar \upsilon$$

We next note that if the P wave has a resonance at $W = M^*$, then

$$\frac{e^{2i\eta_1} - 1}{2ik} \cong \frac{(W-M)^2 - \mu^2}{W^2} \cdot \frac{R}{W - M^*} \qquad (W \cong M^*).$$

We set $R = R_o + \Delta R$ (since R and R_o are finite, ΔR is finite). Thus, near $W = M^*$,

$$\psi_{scattered} \xrightarrow{(r \to \infty)} \frac{e^{ikr}}{r} e^{-i\gamma \ln 2kr} \left[f_{Coul}(\theta) + e^{2i\sigma_1} 3P_1 \left(\frac{(W-M)^2 - \mu^2}{W^2} \right) \frac{R_o + \Delta R}{W - M^*} \right].$$

Expanding to order α, the coefficient of $\frac{1}{W - M^*}$ is hence

$$\frac{e^{ikr}}{r} 3P_1 e^{2i\sigma_1} \frac{(W-M)^2 - \mu^2}{W^2} \left(1 - i\gamma \ln 2kr \right) \left(R_o + \Delta R \right). \qquad (8)$$

The effective residue at the pole is thus seen to contain a finite piece (ΔR), as well as an infinite piece $(-R_o i\gamma \ln 2kr)$ arising from the infinite Coulomb phase of the scattered wave.

Using the above example as a guide, we now examine the relativistic case. The total $\pi N \to \pi N$ scattering amplitude A is a product of a non-infrared piece $(A_o + \Delta A)$ and an infrared piece arising from infrared virtual

photons.[24,25] To first order in the fine structure
constant α ,

$$A(I_i I_3 \rightarrow I_s I_3) = A_o + \Delta A + \sum_i A_o^i \alpha B^i ,$$

where B^i includes the contributions of virtual infrared
photons.[24,25] (The sum is a linear combination of terms
arising from $\pi_1 N_1 \rightarrow \pi_2 N_2$ amplitudes.) We can take the P
wave projections of this amplitude according to a standard
formula for such projections[27,28] :

$$\frac{P}{q} A(I_i I_3 \rightarrow I_s I_3; J) = \frac{1}{2} \int_{-1}^{+1} \left[F_1(x) P_1(x) + F_2(x) P_2(x) \right] dx ,$$

where F_1 and F_2 are standard expressions involving
Legendre polynomials, and x is the cosine of the scattering
angle. We easily obtain

$$A(I_i I_3 \rightarrow I_s I_3; J) = A_o(I_i \rightarrow I_s; J) + \Delta A(I_i I_3 \rightarrow I_s I_3; J)$$

$$+ \frac{\alpha}{2} \frac{q}{P} \int_{-1}^{+1} \sum_i \left[f_1^{oi} B^i P_1(x) + f_2^{oi} B^i P_2(x) \right] dx.$$

Assuming a resonance in A at $W = M^*$, for example, we write

$$A(I_i I_3 \to I_f I_3; J) \cong \frac{R_0}{W-M_0^*} + \Delta\left(\frac{R}{W-M^*}\right) + \frac{\alpha}{2} \frac{R_0}{W-M_0^*} \int_{-1}^{+1} dx\, B\left[\underline{P}_2' \underline{P}_1 - \underline{P}_1 \underline{P}_2'\right]^{-} \tag{9}$$

$$= \frac{R_0 + \Delta R + \frac{\alpha}{4} R_0 \int_{-1}^{+1} dx (3x^2+1) B\Big|_{W=M_0^*}}{W-M_0^*} + \frac{R_0\, \delta M^*}{(W-M_0^*)^2} \quad (W \cong M^*).$$

The infrared divergence is contained in the integral involving B. ΔR is finite. (The reader will recall that the analogous situation held in the non-relativistic example above.)

We next note that the separation of the total scattering amplitude into a non-infrared term and an infrared term (involving B) was not unique. If we write $B(s,t) = a(s,t) \ell n\, \lambda/b(s,t)$ (λ is a fictitious photon mass), then the function $b(s,t)$ is not uniquely determined ($a(s,t)$ is uniquely defined, however, and can be found by explicitly computing the divergent part of the relevant Feynman diagrams). For convenience we choose $b(s,t) =$ constant $= m_\rho$. With this choice of B, equation (9) takes the form

$$A(I_i I_3 \to I_f I_3; J) \cong \frac{R_0 + \Delta R + \frac{\alpha}{4} R_0 \ell n \frac{\lambda}{m_\rho} \int_{-1}^{+1} dx (3x^2+1)\, a\Big|_{W=M_0^*}}{W-M_0^*}$$

$$+ \frac{R_0\, \delta M^*}{(W-M_0^{*})^2} \qquad (W \cong M^*).$$

Thus the total residue of the simple pole of A is given by

$$R_{total} = R_0 + \Delta R + d(M^*)\ln \lambda/m_\rho \; ,$$

where

$$d(W) = \frac{\alpha}{4} R_0 \int_{-1}^{+1} dx \begin{pmatrix} 3x^2+1 \, ; \\ 1 \end{pmatrix} a \Big|_W \qquad \left(J = \begin{matrix} \frac{3}{2}+ \\ \frac{1}{2}+ \end{matrix}\right).$$

We now wish to use Dashen-Frautschi dispersion integrals to calculate residue shifts. We will show how to calculate ΔR this way; the $d(M^*)\ln \lambda/m_\rho$ piece (if desired explicitly) could be obtained directly from the Feynman diagrams. The reader will recall that in deriving the dispersion integrals (6) and (7), no properties of δA other than its pole were actually used. Thus, if we write

$$\delta A = A - A_0$$

$$\delta \hat{A} = A - A_0 - F(W) ,$$

where A, A_0, and F(W) refer to $(I_i I_3 \to I_f I_3 ; J)$, and where F(W) is some function of W, then one could equally well use the dispersion relations (6) and (7) (written with $\delta\hat{A}$ amplitudes) to obtain the residues corresponding to $\delta\hat{A}.$[32] In expanding the contours of the dispersion integrals, we would then need to integrate around any cuts of $D_i D_f F$ in addition to those involved in the δA integrals.

One could in principle choose $\delta \hat{A}$ amplitudes free of infrared divergences or infinite phase shifts and use this type of amplitude in the dispersion integrals (6) and (7). We shall instead use amplitudes which do not have any infrared divergent terms in the residue at the pole, but which still could give spurious infrared divergences in an approximate evaluation of the dispersion integrals. The amplitudes we shall use in writing Dashen-Frautschi dispersion integrals are

$$\delta q(W) = (A - A_0) - \grave{A}_0' \frac{d\binom{M}{or}}{\grave{R}_0'} \ln \lambda/m_\rho \qquad (11)$$

$$\left(I_i I_3 \to I_f I_3, J = \tfrac{\frac{1}{2}+}{\frac{3}{2}+} \right); \grave{A}_0', \grave{R}_0' : \text{Set } I_i' = I_f' = J.$$

Near the pole in A our amplitude equals (in the case $J = \frac{3}{2}+$, for example):

$$\delta q(W) \cong \frac{\left(\Delta R + d(M) \ln \frac{\lambda}{m_\rho} \right) - \grave{R}_0' \left[\frac{d(M)}{\grave{R}_0'} \ln \frac{\lambda}{m_\rho} \right]}{W - M_0^*} \quad + \quad \frac{R_0 \delta M^*}{(W - M_0^*)^2}$$

$$= \frac{\Delta R}{W - M_0^*} \quad + \frac{R_0 \delta M^*}{(W - M_0^*)^2} \quad . \qquad (12)$$

Writing a dispersion integral for ΔR, we have (for $I_i = I_f = \frac{3}{2}$, for example):

$$\Delta R = \frac{1}{2\pi i} \int_C \frac{D_{33}^2(W) \, \delta q(W)}{(W - M^*)^2} \left[1 - D_{33}''(M^*)(W - M^*) \right] dW ;$$

$$\therefore \Delta R = \frac{1}{2\pi i} \int_C \frac{D_{33}^2(w)\,(A-A_o)}{(w-M_o^*)^2} \left[1 - D_{33}''(M^*)(w-M^*)\right] dw$$

$$- \frac{d(M^*)}{R_o}\left(\ln\frac{\lambda}{m_\rho}\right)\frac{1}{2\pi i}\int_C \frac{D_{33}^2(w)\,A_o(w)}{(w-M_o^*)^2}\left[1 - D_{33}''(M^*)(w-M^*)\right] dw.$$

The first integral on the right hand side (and its analogues for other spin and isospin processes) will be studied in detail in the following sections. This type of integral gives us $(R_{total} - R_o)$, which we will denote by δR, and it will be evaluated by expanding the contour C. The second integral on the right hand side will not be evaluated by expanding the contour, but instead is trivially seen (by direct evaluation about the pole of A_o enclosed by C) to give $-d(M^*)\ln\lambda/m_\rho$. Thus

$$\Delta R = \delta R - d\left(\begin{smallmatrix}M\\ or\\ M^*\end{smallmatrix}\right)\ln\lambda/m_\rho \quad \left(J = \begin{smallmatrix}\frac{1}{2}+\\ or\\ \frac{3}{2}+\end{smallmatrix}\right). \tag{13}$$

Now since ΔR is finite, an exact evaluation of the dispersion relation for ΔR should give a finite result. In evaluating the integrals approximately, however, a spurious infrared divergence, similar to the spurious infrared divergence of mass shift calculations[2], may arise. We shall use a prescription to eliminate such spurious infrared divergences. There has been some

discussion[2,29] in the literature about possible prescriptions. The method we use is simply to set $\ln \lambda = \ln m_\rho$ [29] in each of the divergent parts of our residue shifts ΔR. This prescription can at least be expected to give correct order of magnitude results, although the detailed values of the residue shifts - even their signs - are questionable. Thus, if our approximate calculation gives $\delta R = a_1 + b_1 \ln \lambda / c_1$ then we set

$$\Delta R = \left(a_1 + b_1 \ln \frac{m_\rho}{c_1}\right) - d \binom{M}{\text{or} \atop M^*} \ln \frac{m_\rho}{m_\rho} = a_1 + b_1 \ln \frac{m_\rho}{c_1} \tag{14}$$

$$= \delta R \Big|_{\ln \lambda = \ln m_\rho}.$$

Hence the total residue of the pole in the partial wave amplitude A is:

$$R_{total} = R_0 + \left(a_1 + b_1 \ln \frac{m_\rho}{c_1}\right) + d \binom{M}{\text{or} \atop M^*} \ln \lambda / m_\rho.$$

In general, the total scattering amplitude will be given by

$$A(\pi_1 N_1 \to \pi_2 N_2) = \left\{ \frac{\left[R_0 + \left(a_1 + b_1 \ln \frac{m_\rho}{c_1}\right)\right]}{W - \binom{M \text{ or}}{M^*}} \left[\begin{array}{c} \text{function} \\ \text{of } \theta \end{array}\right]_{\pi_1 N_1 \to \pi_2 N_2} + \cdots \right\} e^{\alpha B(s,t)},$$

where we must remember that ΔR depended on the choice of

B, so that we must use $B = a(s,t) \ln \lambda / m_\rho$ in the expression involving B.

In our method of solving for $\Delta R = a_1 + b_1 \ln(m_\rho / c_1)$, ΔR is not a function of energy. $B(s,t)$ $\underline{is}$ a function of W. Thus the effective residue of the amplitude does depend on the energy. In our calculation the portion of the effective residue depending on energy depends on the convention chosen for B. A different choice of B would give different values for the residue of the pole in the amplitude as a function of W, whereas the W dependence of the residue is in principle uniquely defined. This difficulty arises because in our method of calculation we do not derive the energy dependence of ΔR, and away from the value of W at the pole it represents an uncertainty in the final answer obtained for ΔR.

Having thus looked at residue shifts in amplitudes, we now examine a typical cross section. The $\pi N \rightarrow \pi N$ scattering cross section will contain corrections due to real as well as virtual infrared photons, and the resonant piece of the differential cross section for $\pi_1 N_1 \rightarrow \pi_2 N_2$ is

$$\frac{d\sigma}{d\Omega} \cong (1 + 3x^2) \left| \frac{\rho (R_o + \Delta R)}{q (W - M^*)} \right|^2 \left(1 + 2\alpha \left[\tilde{B} + Re\, B \right] \right)$$

$$(W \cong M^*)$$

(Here $\tilde{B}$ is the correction from real infrared photons; its infrared divergence cancels that of Re B.) Hence the resonant part of the total cross-section is

$$\sigma \cong \frac{8\pi}{q^2} \left| \frac{\rho(R_0 + \Delta R)}{W - M^*} \right|^2 \left[1 + \frac{\alpha}{2} \int_{-1}^{+1} dx (3x^2 + 1)(\tilde{B} + \text{Re }B) \right]. \qquad (16)$$

When expressing $\Delta R(\pi, N_1 \rightarrow \pi_2 N_2; J = \frac{3}{2}+)$ in terms of $\Delta R(I_i I_z \rightarrow I_f I_z; J)$'s, we must include pieces with $I_i \neq I_f$.

We now summarize the conclusions reached in this section. We first examined a non-relativistic example and noted that the residue of the pole in a scattering amplitude contained an infinite piece. We then noted that the residue in the relativistic case will also be the sum of a finite and an infinite piece, and that the separation into these two pieces is not unique. We chose a certain convention for separating the two pieces and then wrote a Dashen-Frautschi dispersion relation for the finite piece ΔR, using as input an amplitude having ΔR as the residue of its direct pole. We then chose a prescription to eliminate the spurious infrared divergence arising in an approximate evaluation of the dispersion integral for ΔR. This prescription yielded

$$\Delta R = \delta R \Big|_{\ell n \lambda = \ell n \, m_\rho} \qquad \left(\text{where } \delta R = R_{total} - R_0\right), \qquad (14)$$

and we saw that the total scattering amplitude is given by

$$A(\pi_1 N_1 \to \pi_2 N_2) = \left\{ \frac{\left[R_0 + \Delta R\right]}{W - \left(\begin{smallmatrix} M \\ or \\ M^* \end{smallmatrix}\right)} \left|\begin{bmatrix} function \\ of \quad \theta \end{bmatrix}\right|_{\pi_1 N_1 \to \pi_2 N_2} + \cdots \right\} \left(1 + \alpha \, B(s,t)\right), \qquad (15)$$

where we need to remember the convention chosen for the finite part of B. Finally, we noted that

$$\sigma_{\pi_1 N_1 \to \pi_2 N_2} \cong \frac{8\pi}{q^2} \left| \frac{\rho \left(R_0 + \Delta R\right)_{\pi_1 N_1 \to \pi_2 N_2}}{W - M^*} \right|^2 \left[1 + \frac{\alpha}{2} \int_{-1}^{+1} dx \, (3x^2 + 1)(\breve{B} + Re\,B) \right] \qquad (16)$$

$$\left(W \cong M^*\right).$$

In the following sections we shall evaluate the residue shifts δR corresponding to the amplitudes $\delta A = A - A_0$. These residue shifts are given by integrals of the form (6) and (7). After finding the δR's, we take the piece which does not vanish as $\lambda \to 0$ and evaluate it at $\lambda = m_\rho$ to obtain the ΔR's $(\Delta R = \delta R \big|_{\ell n \lambda = \ell n \, m_\rho})$. Before doing the dispersion integrals for R, however, we shall first discuss various definitions that will prove

convenient in dealing with our residue shifts.

IV. Definitions and D Functions

Since δA is the difference between the perturbed and unperturbed π-N scattering amplitudes, it receives contributions from changes in the parameters of the original symmetric theory as well as from additional processes which occur only in the presence of electromagnetism. Thus δA contains pieces proportional to shifts in residues as well as pieces arising from such quantities as mass shifts, one-photon exchange, and inelastic intermediate states. In general, then, we may write[6]

$$
\begin{pmatrix} \dfrac{\overrightarrow{\delta m}}{m_0} \\[2mm] \overrightarrow{\delta R} \end{pmatrix} = \begin{pmatrix} A^{mm} & A^{mR} \\[2mm] A^{RM} & A^{RR} \end{pmatrix} \begin{pmatrix} \dfrac{\overrightarrow{\delta m}}{m_0} \\[2mm] \overrightarrow{\delta R} \end{pmatrix} + \begin{pmatrix} \overrightarrow{D_m} \\[2mm] \overrightarrow{D_R} \end{pmatrix} , \qquad (17)
$$

where $\overrightarrow{\delta R}$ is a vector with components $\delta R(I_i I_z \to I_f I_z ; J)$, $\overrightarrow{\delta m}/m_0$ has components $\delta m(I, I_z, J)/m_0$, the A's are matrices, and the $\overrightarrow{D}$'s are vectors (called "driving vectors") containing the contributions from quantities other than residue or mass shifts.

As has been shown previously,[6] there is a basis in which A^{RR} is block-diagonal; since this greatly simplifies the arithmetic, it is the basis we shall use.[6,7] We define

$$\delta R_I(I_i \to I_f; J) = \sum_{I_3} \langle I_i I_3; I0 | I_f I_3 \rangle \sqrt{\frac{2I+1}{2I_f+1}} \; \delta R(I_i I_3 \to I_f I_3; J). \quad (18)$$

(Actually, a minus sign is inserted in front of the summation for the case of $\delta R_I(\frac{3}{2} \to \frac{1}{2}; J=\frac{1}{2}+)$.) Here $\langle I_i I_z; I0 | I_f I_z \rangle$ is a Clebsch-Gordan coefficient for coupling the initial isospin state with the isospin state $| I0 \rangle$ to form a state of final isospin $| I_f I_z \rangle$. We use the Condon-Shortley phase conventions for the Clebsch-Gordan coefficients as, for example, employed on the wallet cards of A. Rosenfeld et al.[30] δm_I and $\delta A_I(I_i \to I_f; J; W)$ are defined in exact analogy to δR_I. The δR_I's are listed explicitly in Table I.

The matrices defined by

$$\mathfrak{m}^{I,0}_{I_2 I_1} = \langle I_1 I_3; I0 | I_2 I_3 \rangle \sqrt{\frac{2I+1}{2I_2+1}}$$

are orthogonal in the sense that $\mathrm{Tr}[\mathfrak{m}^{I0}_{I_2 I_1}] \, \mathrm{transpose}[\mathfrak{m}^{I'0}_{I_2 I_1}] = \delta_{II'}$, and hence it is not surprising that A^{RR} is block-diagonal in the new basis. (One could verify explicitly that mass shifts and residue shifts only contribute to residue shifts with the same subscript I; a proof of this is found in reference (6).) Thus

$$\begin{pmatrix} \dfrac{\overrightarrow{\delta m}_I}{m_0} \\[4mm] \overrightarrow{\delta R_I} \end{pmatrix} = \begin{pmatrix} A_I^{MM} & A_I^{MR} \\[4mm] A_I^{RM} & A_I^{RR} \end{pmatrix} \begin{pmatrix} \dfrac{\overrightarrow{\delta m}_I}{m_0} \\[4mm] \overrightarrow{\delta R_I} \end{pmatrix} + \begin{pmatrix} \overrightarrow{D_{m,I}} \\[4mm] \overrightarrow{D_{R,I}} \end{pmatrix}. \tag{19}$$

Since we have already found dispersion relations for the quantities $\delta R(I_i I_z \rightarrow I_f I_z; J)$, we can take suitable linear combinations (using the definition of the δR_I's) to find dispersion relations for the residue shifts $\delta R_I(I_i \rightarrow I_f; J)$. We easily obtain for the residue shifts under consideration

$$\delta R_I(\tfrac{1}{2} \rightarrow \tfrac{1}{2}; J=\tfrac{1}{2}+) = \frac{1}{2\pi i}\int \frac{D_{11}^2(w')\,\delta A_I(\tfrac{1}{2}\rightarrow\tfrac{1}{2};J=\tfrac{1}{2}+;w')}{(w'-M)^2}\left[1-D_{11}''(M)(w'-M)\right]dw'$$
$$(I=0,1,2)$$

$$\delta R(\tfrac{3}{2}\rightarrow\tfrac{1}{2}; J=\tfrac{1}{2}+) = \frac{1}{2\pi i}\int \frac{D_{31}(w')\,D_{11}(w')\,\delta A_I(\tfrac{3}{2}\rightarrow\tfrac{1}{2};J=\tfrac{1}{2}+;w')}{w'-M}dw'$$
$$(I=1,2)$$

$$\delta R(\tfrac{3}{2}\rightarrow\tfrac{3}{2}; J=\tfrac{3}{2}+) = \frac{1}{2\pi i}\int \frac{D_{33}^2(w')\,\delta A_I(\tfrac{3}{2}\rightarrow\tfrac{3}{2};J=\tfrac{3}{2}+;w')}{(w'-M^*)^2}\left[1-D_{33}''(M^*)(w'-M^*)\right]dw' \tag{20}$$
$$(I=0,1,2)$$

$$\delta R(\tfrac{3}{2}\rightarrow\tfrac{1}{2}; J=\tfrac{3}{2}+) = \frac{1}{2\pi i}\int \frac{D_{13}(w')\,D_{33}(w')\,\delta A_I(\tfrac{3}{2}\rightarrow\tfrac{1}{2};J=\tfrac{3}{2}+;w')}{w'-M^*}dw'$$
$$(I=1,2).$$

(Here the δA_I's are defined in analogy with (18).) In doing these integrals, we will find that often contributions in one integral are simply proportional (via Clebsch-Gordan coefficients) to similar contributions in other integrals with the same I_i, I_f, and $J^{(31,32,6)}$ thus much of the work will be simplified.

The next step is to evaluate the above integrals. Each δA_I is (by definition) a linear combination of partial wave amplitudes. The reader is referred to references (27) and (28) for projection formulae for obtaining the partial wave contributions and the location of cuts in these amplitudes. There will be additional cuts arising from new processes such as one-photon exchange and the γN intermediate state. In doing the integrals we shall investigate the "near-by" cuts; since we are unable to evaluate most of the "far-away" cuts, we shall neglect all distant singularities. The main reasons [7] that this may not be such a bad approximation are the belief that the dispersion integrals converge rapidly (assuming the nucleon is not elementary) and the success of past "bootstrap" calculations including only near-by contributions.

Before evaluating the actual contributions to the dispersion integrals (20), we discuss the D functions we

shall use. There has been some discussion regarding the
explicit form that such D's should take.[34,19] The
derivation of our dispersion integrals did not depend on
the form of the D functions (except that they have certain
zeroes), and the choice of the explicit form of the D
functions is a matter of convenience. A discussion of
this question is found in reference (19), and the essential
points are as follows: We wish our D functions to
deemphasize those pieces of the left hand cut corresponding
to (less well-known) exchanges of higher mass particles.
It would also be convenient if in the low energy region
above threshold the D functions had approximately the
negative of the phase of the scattering amplitude (i.e.,
phase of $D = -\mathrm{Re}\,\eta_i^o$, where η_i^o is the unperturbed phase shift
of the π-N partial wave scattering amplitude). It is not
easy to satisfy both these criteria at once; the Balazs
D function[2] represents a fairly good compromise.

On the other hand, for certain theoretical discussions
linear D functions are most convenient. Thus we shall do
the dispersion integrals twice - once with Balazs curved
D functions and once with linear D functions. Explicitly,
we use

$$D_{13}(w) = 1 \qquad\qquad D_{31}(w) = 1$$

$$D_{11}(W) = W - M \qquad\qquad D_{33}(W) = W - M^*$$

("linear D functions")

and

$$D_{13}(W) = 1 \qquad\qquad D_{31}(W) = 1$$

$$D_{11}(W) = (W-M)\left(\frac{M-W_0}{W-W_0}\right) \qquad D_{33}(W) = (W-M^*)\left(\frac{M^*-W_0}{W-W_0}\right)$$

$$\left(W_0 = \frac{7M}{3}\right) \qquad\qquad \text{("curved D functions").}$$

We now proceed with an evaluation of the dispersion integrals (20).

V. The A^{RR} Matrix

In this section we obtain the A^{RR} matrix relating residue shifts in the crossed channel to residue shifts in the direct channel. The method for this has been explained in references (6) and (7) and is included here for completeness. We consider the contributions to the integrals (20) from the diagrams in Fig. 2; to evaluate the contribution from the nearby short cuts (treated as poles) of these diagrams, we shall use crossing symmetry. (Consistent with using the Condon-Shortley phase conventions for Clebsch-Gordan coefficients, we let $\pi^{\pm} \rightarrow |1 \pm 1\rangle$; $\pi^{\circ} \rightarrow |1\ 0\rangle$; $\overline{\pi^{\pm}} \rightarrow -|1 \pm 1\rangle; \overline{\pi^{\circ}} \rightarrow |1\ 0\rangle$.) We expand each $\delta A\ (I_i I_z \rightarrow I_f I_z; J\ )$ in terms of π-N amplitudes, cross the pions to obtain amplitudes that are a function of the energy in the crossed (u) channel, recombine the resulting π-N states to states of total I and I_z , and finally take the appropriate linear combinations of the $\delta A(I_i I_z \rightarrow I_f I_z; J)$'s to obtain linear relations between the δA_I's in the direct (s) channel and the δA_I's in the u channel. Taking the low-energy approximation $W^u \cong 2M - W^s$ (where W^u , for example, is the energy in the u channel), we then express the poles in the u channel amplitudes in terms of W^s and perform the integrations around the poles in (20), keeping the terms proportional to residue shifts. The results of

this analysis are presented in Table II. The A^{RR} matrix

for linear D is determined completely by crossing and

Clebsch-Gordon coefficients; the A^{RR} matrix for curved D

is quite similar to the one for linear D. The A^{RR} matrix

for linear D has three eigenvalues exactly equal to one

(one each for I=0, 1, and 2), and the A^{RR} matrix for

curved D has three corresponding eigenvalues near one

(see Table III). Thus, from Equation (17) one might

expect the corresponding eigenvectors to be enhanced, but,

as will be seen in Sec. XIII, the relevant components of

the driving vector $\vec{D}$ are negligible, and no such enhance-

ment occurs. The occurrence of eigenvalues near one and

the phenomenon of no enhancement from low-mass states will

be made more understandable in Sec. XIV.

VI.   N, N*, and π Mass Shifts

We must, of course, also consider the effect of mass shifts on our residue shifts. There are two types of mass shifts, namely exchanged mass shifts (arising from internal particles) and external mass shifts (arising from external pions and nucleons).[2,6] The effect of exchanged mass shifts δm^{exch} is easy to find, using the analysis of the preceding section. We again express the s channel δA_I's in terms of u channel δA_I's. A pole in the u channel at $W^u = m^{exch}$ gives rise to a pole in the s channel at $W^s = 2M - m^{exch}$. Thus a u channel term such as

$$\delta\left(\frac{R}{W^u - m^{exch}}\right) \quad \text{equals}$$

$$\delta\left(\frac{-R}{W^s - (2M - m^{exch})}\right) = \frac{-\delta R}{W^s - (2M_0 - m_0^{exch})} + \frac{R_0\, \delta m^{exch}}{\left(W^s - (2M_0 - m_0^{exch})\right)^2}.$$

We again perform the integrations around the poles in the integrals (20), this time keeping the terms proportional to δm^{exch} . The results of this analysis are presented in Table IV and are seen to depend strongly on the choice of D function.

To obtain the contributions from external mass shifts to our residue shifts, a slightly more elaborate method is necessary. We shall use a relation obtained from mass invariance arguments[6,31,32] to obtain the I=0 effects.

Some of the I=1 and I=2 effects will then be related to these via Clebsch-Gordan coefficients[6,31,32] the remaining $I = 1$ and $I = 2$ effects will be calculated by considering the appropriate integrals directly.

First we look at the I=0 contributions from external masses to our residue shifts. We shall use two relations derived from mass invariance (for an explicit derivation, see reference (6)). We assume negligible contributions from mass shifts other than those of the pion, nucleon, or N^* resonance (see Sec. VIII for a discussion of ρ, ω, and ϕ effects). Then mass invariance implies

$$0 = A_{I=0}^{R(\frac{1}{2}\to\frac{1}{2}; J=\frac{1}{2}+), M}\sqrt{2} + A_{I=0}^{R(\frac{1}{2}\to\frac{1}{2}; J=\frac{1}{2}+), \mu}\sqrt{3} + A_{I=0}^{R(\frac{1}{2}\to\frac{1}{2}; J=\frac{1}{2}+), M^{*exch}}\sqrt{4}$$

and (21)

$$0 = A_{I=0}^{R(\frac{3}{2}\to\frac{3}{2}; J=\frac{3}{2}+), M}\sqrt{2} + A_{I=0}^{R(\frac{3}{2}\to\frac{3}{2}; J=\frac{3}{2}+), \mu}\sqrt{3} + A_{I=0}^{R(\frac{3}{2}\to\frac{3}{2}; J=\frac{3}{2}+), M^{*exch}}\sqrt{4} .$$

(The $A_{I=0}$'s are defined by (19); they are simply quantities giving the contribution of mass shifts to our residue shifts.) Since the nucleon mass shifts can occur either with exchanged nucleons or with external nucleons, we see that each contribution from nucleon mass shifts actually is a sum of two terms:

$$A_I^{R(I_i \to I_f),M} = A_I^{R(I_i \to I_f),M^{exch}} + A_I^{R(I_i \to I_f),M^{ext}} \quad , \quad (22)$$

We next make the assumption that compared with the nucleon terms $A_{I=0}^{R,M^{ext}}$, the pion contributions $A_{I=0}^{R,\mu}$ can be neglected. This is justified as follows: The pion terms $A_{I=0}^{R,\mu}$ can arise from the left hand short cuts (which we have been treating as poles) and from the right hand cut. (We are neglecting external mass contributions to "t-channel" processes such as ρ, ω, or ϕ exchange (see Sec. VIII). The one-photon diagram is already of first order in α (the fine structure constant), so any mass perturbations on this diagram would be of second order in small quantities and are hence also neglected.) We shall neglect pion contributions from the left hand short cuts (see reference (7) for a discussion of this approximation). On the right hand cut the external mass contributions come from $\delta\rho$ (see Sec. IX), and the relative magnitude from this source of the nucleon external mass pieces $A_{I=0}^{R,M^{ext}}$ compared with the pion contributions $A_{I=0}^{R,\mu}$ should be about 5:1. Thus we see that as a lowest order approximation we can neglect the pion term $A_{I=0}^{R,\mu}$ compared with the external nucleon mass term. Using this approximation, together with (21) and (22) (setting I=0), and

employing the results for exchanged mass shifts (Table III)
we solve for the external nucleon term $A_{I=0}^{R, M^{ext}}$ (see Table V).
(The analysis is done twice; once for linear D functions
and once for curved D functions.)

Having thus obtained the I=0 contributions from
external mass shifts, we now turn to the I=1 contributions
(from external mass shifts) to our residue shifts. There
is no pion contribution, since $\delta\mu_{I=1} = \sqrt{\frac{1}{2}}(m_{\pi^+} - m_{\pi^-}) = 0$ (π^+
and π^- have the same inertial mass). We shall express
the nucleon terms $A_{I=1}^{R(\frac{1}{2} \to \frac{1}{2}; J=\frac{1}{2}+)M^{ext}}$ and $A_{I=1}^{R(\frac{3}{2} \to \frac{3}{2}; J=\frac{3}{2}+), M^{ext}}$
in terms of the corresponding $A_{I=0}$'s by means of Clebsch-
Gordan coefficients.[6] There are two equivalent ways of
doing this. The first is to note that there is a direct
analogy to the action of external mass shifts on <u>mass</u>
shifts; we take as an example the external nucleon mass
shifts. Then[33]

$$\delta m_{I=\begin{smallmatrix}0\\1\end{smallmatrix}} = \sqrt{\frac{1}{2}}(\delta m_p \pm \delta m_n)$$

$$= (\text{constant})\sqrt{\frac{1}{2}}\left[\left(\frac{2}{3}\delta m_n^{ext} + \frac{1}{3}\delta m_p^{ext}\right) \pm \left(\frac{1}{3}\delta m_n^{ext} + \frac{2}{3}\delta m_p^{ext}\right)\right]$$

$$= (\text{constant})\begin{pmatrix}\delta m_{I=0}\\-\frac{1}{3}\delta m_{I=1}\end{pmatrix} \quad ;$$

hence $A_{I=1}^{M,M^{ext}} \div A_{I=0}^{M,M^{ext}} = -1/3$. Exactly analogous steps

can be taken for residue shifts having $I_i = I_f$.

The second (equivalent) way to find $A_{I=1}^{R(\frac{1}{2}\to\frac{1}{2}; J=\frac{1}{2}+),M^{ext}}$

and $A_{I=1}^{R(\frac{1}{2}\to\frac{3}{2}; J=\frac{3}{2}+),M^{ext}}$ is to look at the integrals (20)

directly. We expand each $\delta A_{I=0}$ amplitude in terms of

$\delta A(I_i I_z \to I_f I_z; J)$'s and expand these latter amplitudes

in terms of initial and final π-N particle amplitudes

$\delta A(\pi_1 N_1 \to \pi_2 N_2; J)$. (The contribution from $\delta A(\pi^+ p \to \pi^+ p; J=\frac{3}{2}+)$

due to external nucleon mass shifts, for example, will be

$\frac{\partial A_0}{\partial M^{ext}}(\pi^+ p \to \pi^+ p; J=\frac{3}{2}+)\cdot \delta m_p^{ext}$.) The next step is to reex-

press the $\frac{\partial A_0}{\partial M^{ext}}(\pi_1 N_1 \to \pi_2 N_2 \quad ; \quad J)$'s in terms of total

isospin amplitudes $\frac{\partial A_0}{\partial M^{ext}}(I_i \to I_i; J)$. The result for

$\delta R_I(\frac{1}{2}\to\frac{1}{2} \quad ; J=\frac{1}{2}+)$, for example, is

$$\delta R_{I=0 \atop 1}(\tfrac{1}{2}\to\tfrac{1}{2}; J=\tfrac{1}{2}+) \leftrightarrow \frac{1}{2\pi i}\int dW' \frac{D_{11}^2(W')}{(W'-M)^2}\left[1 - D_{11}''(M)(W'-M)\right]\frac{\partial A_0(\tfrac{1}{2}\to\tfrac{1}{2}; J=\tfrac{1}{2}+)}{\partial M^{ext}}\cdot M\cdot$$

$$\cdot \begin{pmatrix} \delta M_{I=0}/M \\ -\tfrac{1}{3}\,\delta M_{I=1}/M \end{pmatrix} ,$$

(23)

so that the same conclusions are obtained as by method one

(described in the previous paragraph). The numerical

results are to be found in Table V.

To find the contributions of external nucleon mass

shifts to residue shifts with $I=1$, but with $I_i \neq I_f$, we no

longer have the mass analogue (or the corresponding

$A_{I=0}^{R,M^{ext}}$'s) of the $I_i = I_f$ case described above. Thus, to find the nucleon terms $A_{I=1}^{R(\frac{3}{2} \to \frac{1}{2}; J=\frac{1}{2}+), M^{ext}}$ and $A_{I=1}^{R(\frac{3}{2} \to \frac{1}{2}; J=\frac{3}{2}+), M^{ext}}$, we must do the corresponding integrals in (20) directly. In evaluating the integrals, we note that there is no contribution from the right hand cut (see Sec. X). To find the contribution from the left hand cut, we expand as in the previous paragraph to obtain the analogue of (23) for the $I_i \neq I_f$ residue shifts. Next we use crossing to relate the A_o's to A_o's of the crossed channel; we approximate the crossed channel cuts by means of poles, reexpress the poles in terms of the integration variable, and do the integrals around the poles. The results are again to be found in Table V.

Finally, we turn to the I=2 external mass shifts (these are just pion mass shifts). When $I_i = I_f$ $\left(=\frac{3}{2}+\right)$, we again use Clebsch-Gordan coefficients to find the ratio of the pion terms $A_{I=2}^{R(\frac{3}{2} \to \frac{3}{2}; J=\frac{3}{2}+), \mu}$ and $A_{I=0}^{R(\frac{1}{2} \to \frac{1}{2}; J=\frac{3}{2}+), \mu}$. Since the latter term is assumed small, the former one is also small, and we shall neglect it even though $\delta\mu_{I=2}$ is not small; hopefully this is not a bad approximation. When $I_i \neq I_f$ there is no integral over the right hand cut (see Sec. X), and, in analogy with a case for I=0 mentioned above, we neglect any contributions from the integral over the short left hand cuts.

In conclusion, then, we have evaluated the contributions to our residue shifts from exchanged and external mass shifts. The results, which depend strongly on the D functions used, are listed in Tables IV and V.

The remaining contributions to our residue shifts will come from ˙t-channel processes, modifications in intermediate s channel states, and from u channel processes such as γN exchange. We now proceed to evaluate these pieces. First we look at the ˙t-channel˙ process of one-photon exchange.

VII. One-Photon Exchange

The diagram whose contribution we wish to consider is shown in Fig. 3. This diagram will contribute a $\ln \lambda$ term (λ is a fictitious photon mass) to our residue shifts, and we shall use a prescription (discussed in Sec. III) to eliminate this spurious divergence. The one-photon exchange diagram also gave a spurious divergence to the neutron-proton mass difference,[2] and there the numerical contribution (and even the sign) of the P wave projection of the diagram depended strongly on the prescription used to eliminate the spurious divergence. Since we do not have a unique way of choosing the prescription to eliminate the $\ln \lambda$ divergence in our calculation, we expect that the numerical contributions (as well as the signs involved) of the one-photon exchange diagram to our residue shifts are particularly unreliable, although the order of magnitude of the contributions should be correct.

We now discuss the input used in evaluating the one-photon exchange diagram. In writing the amplitude, we use approximate low-energy one-pole form factors, derived from the (two-pole) results of Hand, Miller, and Wilson.[35] We neglect the (small) scalar anomalous magnetic moment of the nucleon.[2] More explicitly, we set

$$F_1^p + F_1^n = \frac{m_1^2}{m_1^2 - t}$$

$$\kappa^p F_2^p + \kappa_n F_2^n = 0$$

$$\left(F_1^p + \kappa^p F_2^p\right) - \left(F_1^n + \kappa^n F_2^n\right) = 4.70 \frac{m_3^2}{m_3^2 - t}$$

$$\kappa^p F_2^p - \kappa_n F_2^n = 3.70 \frac{m_4^2}{m_4^2 - t} \quad .$$

(We also use $F_\pi = \dfrac{m_\rho^2}{m_\rho^2 - t}$).

Here $t = -2q^2(1 - \cos\Theta_{qq'})$, where $\Theta_{qq'}$ is the center of mass scattering angle; κ^p and κ^n are the nucleon anomalous magnetic moments ($\kappa^p = 1.79$ and $\kappa^n = -1.91$ Bohr magnetons); F_1 and F_2 are the (Dirac and Pauli) nucleon form factors; and F_π is the pion form factor. In doing the analysis, we give the photon a mass λ , as discussed in Sec. III. We set $m_1^2 = m_2^2 = m_3^2 = 20\ m_\pi^2$ [2] and $m_\rho = 763$ Mev.

We shall use the projection formula[27,28]

$$\delta A(\pi_1 N_1 \to \pi_2 N_2 ; J = \tfrac{1}{2}^+, \tfrac{3}{2}^+) = \frac{1}{32\pi}\left[\frac{(W+M)^2 - \mu^2}{(W-M)^2 - \mu^2} \left(\mathcal{Q}_1 + (W-M)\,\mathcal{B}_1\right) + \left(-\mathcal{Q}_{\frac{0}{2}} + (W+M)\,\mathcal{B}_2\right)\right] \quad . \tag{25}$$

Here $\alpha_\ell = \int_{-1}^{+1} dx\, P_\ell(x)\, \alpha$, $\beta_\ell = \int_{-1}^{+1} dx\, P_\ell(x)\, \beta$ ($x = \cos \Theta_{\vec{q}_i, \vec{q}_f}$),

where α and β are obtained from

$$S_{fi} = \delta_{fi} + i\,(2\pi)^4\, N_f\, N_i\, \delta^4(\overrightarrow{P_f - P_i})\, T_{fi} ; \qquad (26)$$

$$T_{fi} = \bar{u}_{p_f}\left[\alpha + \frac{\beta}{2}\,(q_i + q_f)_\mu\, \gamma^\mu\right] u_{p_i} , \qquad (27)$$

where S_{fi} is an S matrix element, the N's are normalization factors, and we use the metric $a_\mu b^\mu = a_0 b - \vec{a}\cdot\vec{b}$ (p_f, p_i, q_f, and q_i are the final and initial nucleon and pion four-vectors, respectively).

In doing the integrals (20) with one-photon exchange as input, we use the static approximation ($q^2 = (W-M)^2 - \mu^2$) for α_2 (this eliminates the far-away left hand cut (in the region $W \cong -M$) of α_2), since we do not wish to include the far-away left hand cuts of the integrand (the factors which multiply $\alpha_1, \beta_1, \beta_0,$ and β_2 tend to deemphasize these left hand cuts anyway, so no static approximation is made here). (The basic method used in evaluating the one-photon exchange integrals is due to Dashen.[2]) The next step is simply to expand the contour around the right hand cut to infinity (this makes evalua-

tion easier). For linear D functions we then evaluate
the contribution along the path at infinity, while for
curved D there is an additional contribution coming from
the pole (of the D function) at $W = W_o$, encountered in
expanding the contour. The results from the above analysis
are presented in Table VI. The results do depend on the
form of D. We note that for linear D and fixed I, the
ratios of the one-photon contributions depend only on
Clebsch-Gordan coefficients, while modifications occur
for curved D functions. The values of the one-photon
contributions depend on the (fictitious) photon mass λ ;
the effective value of λ is discussed in Sec. III.
We note (Table VI) that for fixed I and linear D the ratios
of the components of the one-photon exchange contributions
are independent of the values of the nucleon anomalous
magnetic moments, the positions of the simple poles in
the form factors, and the value of the photon mass
(as long as λ is assumed the same for each residue
shift with the same I value).

VIII. ρ, ω and ϕ Meson Contributions

In this section we consider the contributions to our residue shifts from ρ, ω, and ϕ meson exchange. $\delta m_{\rho I=1}$ vanishes and hence does not give any contribution. Since the ω and ϕ do not couple to two pions in the unperturbed case, the only ω and ϕ contributions could come from changes in the $\omega\pi\pi$ or $\phi\pi\pi$ couplings; hence any contributions from ρ, ω, or ϕ exchange would come only from $\delta m_{\rho I \neq 1}$ and from shifts in ρNN, $\rho\pi\pi$, $\omega\pi\pi$, and $\phi\pi\pi$ couplings. These mass and coupling shifts, however, are not known; thus we are unable to determine the contributions to our residue shifts due to ρ, ω, or ϕ exchange.

Our procedure will be to neglect any such states (except, of course, that we use the ρ in form factors for one-photon exchange). We do this in the hope that higher mass states such as the ρ, ω, and ϕ, when done in a complete analysis (including form factors at all vertices), do not contribute as much as lower mass states such as the photon. Some further comments about ρ, ω, and ϕ exchange (with respect to the "enhancement mechanism") are to be found in Sec. XIV.

IX. $\gamma\pi$ Cut

As the last low mass t-channel process, we look at the $\gamma\pi$ exchange state [36] (see Fig. 4). The possible importance of this state will be studied by examining the $\gamma\pi$ inelastic state in the processes $\rho\to\omega$ (Fig. 5) and $\rho\to\rho$. (These processes seem natural when one uses form factors at the $\pi\pi\text{-}\gamma\pi$ and $\gamma\pi\text{-}\bar{N}N$ vertices.) We shall compare the discontinuity in the $\rho\to\omega$ (and $\rho\to\rho$) amplitude arising from the $\gamma\pi$ intermediate state with the discontinuity in the same amplitude due to the γ intermediate state. The discontinuity of the total $\rho\to\omega$ scattering amplitude is given by unitarity of the S matrix ((16), (17)):

$$\Im m\, T_{\omega\leftarrow\rho} = \frac{(2\pi)^4}{2}\left[\frac{1}{2\omega_\gamma}\sum_{\epsilon's}\int\delta^4(P_\gamma - P_\rho)T^*_{\gamma\leftarrow\omega}T_{\gamma\leftarrow\rho}\frac{d^3\vec{k}_\gamma}{(2\pi)^3}\right.$$

$$\left. +\frac{1}{2\omega_\pi}\frac{1}{2\omega_\gamma}\sum_{\epsilon's}\iint\delta^4(P_{\pi\gamma}-P_\rho)T^*_{\pi\gamma\leftarrow\omega}T_{\pi\gamma\leftarrow\rho}\frac{d^3\vec{k}_\gamma\,d^3\vec{k}_\pi}{(2\pi)^6}+\cdots\right]. \tag{28}$$

We set $T_{\gamma\leftarrow\rho}=\gamma_{\rho\gamma}$ and $T_{\pi\gamma\leftarrow\rho}=f_{\rho\pi\gamma}\epsilon_{\mu\nu\sigma\tau}k_\gamma^\mu k_\rho^\nu \epsilon_\gamma^\sigma \epsilon_\rho^\tau$. Here $\gamma_{\rho\gamma}$ and $f_{\rho\pi\gamma}$ are constants; the k's are four-momenta (see reference (37)); we use analogous expressions for $T_{\gamma\leftarrow\omega}$ and $T_{\gamma\pi\leftarrow\omega}$; the analogue of (28) is written for Im $T_{\rho\leftarrow\rho}$.

Performing the summations and integrations in (28), we have:

$$\text{Im } T_{\omega \leftarrow \rho} = \pi \gamma_{\rho\gamma} \gamma_{\omega\gamma} \frac{\delta(x)}{\mu^2} + \frac{(x-1)^3}{24X} \mu^4 f_{\rho\pi\gamma} f_{\omega\pi\gamma} + \cdots \qquad (x = \frac{t}{\mu^2}).$$

But $f_{\rho\pi\gamma} / \gamma_{\rho\gamma}$ and $f_{\omega\pi\gamma} / \gamma_{\omega\gamma}$ can be found from reference (34), and thus

$$\text{Im } T_{\omega \leftarrow \rho} = \pi \gamma_{\rho\gamma} \gamma_{\omega\gamma} \frac{\delta(x)}{\mu^2} + (3.5 \cdot 10^{-5}) \gamma_{\rho\gamma} \gamma_{\omega\gamma} \frac{(x-1)^3}{\mu^2 X} + \cdots .$$

The first term in this sum is from the γ intermediate state, while the second term arises from the $\gamma\pi$ intermediate state. Since the above $\gamma\pi$ contribution to Im $T_{\omega \leftarrow \rho}$ is so small compared to the contribution of the γ intermediate state, we are motivated not to keep the $\gamma\pi$ exchange state in $\pi N \to \pi N$ scattering when computing $\Delta I = 1$ residue shifts. An exactly analogous study can be made for the $\rho \to \rho$ t channel amplitude, and thus we also neglect any $\gamma\pi$ exchange contributions to residue shifts with $\Delta I = 0$ or $\Delta I = 2$.

Having now considered the t-channel processes in the low energy region, we turn to the right hand cut and find its contributions to our residue shifts.

X. The Right Hand Cut; πN and γN Intermediate States

In this section we shall examine the contribution to
the dispersion integrals (20) arising from integration
around the right hand cut. A typical integrand in (20)
is proportional to $D_k D_j \delta A_I$, and when integrating around
the right hand cut we remember that the D functions as
well as the δA's have right hand discontinuities. The
plan is as follows:[38] First we shall compute Im δA
by use of unitarity, including (in principle) all possible
intermediate states. (In actuality we only include the
πN and γN intermediate states; other possible
contributions to Im δA are discussed in Sec. XI.) This
will give us a relation between Im δA and Re δA.

We shall then first investigate the general case
where the D functions have phases along the right hand
cut. We note that Im $D_i D_j \delta A$ = Im $D_i D_j$ Re δA + Re $D_i D_j$ Im δA.
Combining this equation with the above relation between
Im δA and Re δA, we then obtain an expression for Im $D_i D_j \delta A$
which involves (if one were to assume certain phases for
the D functions) a possible $\delta \rho$ term and a term containing
photoproduction multipoles. The $\delta \rho$ term was already
implicitly incorporated in Sec. V, and thus we would only
need to integrate over the term containing photoproduction
multipoles.

The plan of this section is different from that of previous sections in that we consider all the contributions to $\text{Im}\,\delta A$ __simultaneously__ and __then__ integrate over $\text{Im}\,D_i D_f \delta A$ (times appropriate functions of W; see (20)). δA does indeed involve a sum of terms, but we do not calculate the contributions of each term separately to $\text{Im}\,D_i D_f \delta A$ (although this could be done in principle); instead it seems easier to consider all of $\text{Im}\,\delta A$ at once.

We shall now proceed with the above program. First we obtain the discontinuity of $\delta A(I_i I_3 \to I_f I_3; J)$ across the right hand cut. The π-N partial wave scattering amplitudes have a right hand cut due to direct channel intermediate states, and the discontinuity of the amplitudes across this cut is given by employing unitarity of the S matrix.[38] More explicitly, we have

$$\text{Im} \left\langle f \left| f(\theta) + i\,\vec{\sigma} \cdot \frac{\left(\vec{q}_f \times \vec{q}_i\right)}{\left|\vec{q}_f \times \vec{q}_i\right|} \, g(\theta) \right| i \right\rangle$$

$$= \left(\frac{M}{4\pi W}\right) \frac{(2\pi)^4}{2} \sum_n N_n^2 \, \delta^4(P_n - P_i)\, T_{nf}^* \, T_{ni},$$

(30)

where $|i\rangle$ and $|f\rangle$ are two-component spinors, $\vec{\sigma}$ denotes the 2x2 Pauli matrices, $\vec{q}_i$ and $\vec{q}_f$ are the initial

and final pion momenta, $f(\theta)$ and $g(\theta)$ are standard linear combinations of partial wave amplitudes and Legendre polynomials, and the T_{jk}'s are T matrix elements (defined by (26)). Equation (30) does not involve any approximations.

The contribution of the πN intermediate state to $\text{Im } A(I_i I_z \to I_f I_z; J)$ is (with our normalization for the amplitudes A)

$$\rho \left| A(I_i I_3 \to I_i I_3; J) \right|^2 . \tag{31}$$

When $I_i \neq I_f$, the corresponding contribution (obtained in a straightforward way from (30)) to $\text{Im} A(I_i I_z \to I_f I_z; J)$ is

$$\rho \left[A^*(I_f I_3 \to I_f I_3; J) A(I_i I_3 \to I_f I_3; J) \right.$$
$$\left. + A^*(I_i I_3 \to I_f I_3; J) A(I_i I_3 \to I_i I_3; J) \right]. \tag{32}$$

Another state which can contribute to the discontinuity across the right hand cut is the γN intermediate state. To obtain this contribution, we insert the general form[39]

$$i\,\vec{\sigma}\cdot\vec{\epsilon}\,\mathcal{F}_1(\theta) + \vec{\sigma}\cdot\vec{q}_f\,\vec{\sigma}\cdot\vec{k}\times\vec{\epsilon}\,\frac{\mathcal{F}_2(\theta)}{kq} + i\,\vec{\sigma}\cdot\vec{k}\,\vec{q}_f\cdot\vec{\epsilon}\,\frac{\mathcal{F}_3(\theta)}{kq} + i\,\vec{\sigma}\cdot\vec{q}_f\,\vec{q}_f\cdot\vec{\epsilon}\,\frac{\mathcal{F}_4(\theta)}{q^2}$$

of the photoproduction amplitude into the right hand side of (30). (k_γ is the photon momentum.) (The above form for photoproduction can be obtained by assuming Lorentz invariance and gauge invariance.[40]) Summing over photon spin states and taking various linear combinations of the nucleon spin states $|i\rangle$ and $|f\rangle$, we can separate the pieces contributing to $f(\theta)$ from those contributing to $g(\theta)$, and from this we can find the contribution to the imaginary part of our partial wave amplitudes $A(I_i I_z \rightarrow I_f I_z; J)$. The contribution to $\mathrm{Im}A(I_i I_z \rightarrow I_f I_z; J)$ for the case $I_i = I_f$ is[41]

$$\frac{2 k_\gamma q}{\rho}\left[\,\left|M_{1\pm}^{I_i I_3}\right|^2 + 3\left|E_{1\pm}^{I_i I_3}\right|^2\,\right] \qquad . \qquad (33)$$

(Here $M_{1\pm}$ and $E_{1\pm}$ are photoproduction multipoles; "$1\pm$" means $J = 1 \pm \frac{1}{2}$. Naturally, $E_{1-}=0$.)

When $I_i \neq I_f$, the result at first sight appears more involved, but we can simplify if we note that the T matrix element for $\pi N \rightarrow \pi N$ is symmetric (this assumption was already used in writing (30); thus

$$\operatorname{Im} T_{fi} = \operatorname{Im} \frac{T_{fi} + T_{if}}{2} = \frac{(2\pi)^4}{2} \sum_n N_n^2 \, \delta^4(P_n - P_i) \left[\frac{T_{nf}^* T_{ni} + T_{ni}^* T_{mf}}{2} \right] \quad (34)$$

Using (34), we find that the contribution to $\operatorname{Im} A(I_1 I_{\bar{z}} \rightarrow I_f I_{\bar{z}}')$ from photoproduction is

$$\frac{2 k_\gamma q}{\rho} \operatorname{Re} \left[M_{1\pm}^{I_f I_3 *} M_{1\pm}^{I_i I_3} + 3 E_{1\pm}^{I_f I_3 *} E_{1\pm}^{I_i I_3} \right], \quad (35)$$

while the contribution from the πN state is (this replaces (22))

$$\rho \operatorname{Re} \left[A^*(I_f I_3 \rightarrow I_f I_3; J) A(I_f I_3 \rightarrow I_i I_3; J) \right.$$
$$\left. + A^*(I_i I_3 \rightarrow I_f I_3; J) A(I_i I_3 \rightarrow I_i I_3; J) \right] \quad (36)$$
$$(I_i \neq I_f).$$

Contributions to $\operatorname{Im} A$ from states other than πN and γN will be discussed in a later section.

Since in the integrands of (20) we actually have functions of the form $D_i D_j \delta A$, we now investigate the discontinuity across the right hand cut of such quantities. We first investigate the general case where the D functions have phases along the right hand cut.

We set

$$A_0\left(\tfrac{1}{2}\to\tfrac{1}{2}; J=\tfrac{1}{2}+\right) = \frac{e^{2i\eta_{11}^0}-1}{2i\rho_0} \tag{37}$$

$$A_0\left(\tfrac{3}{2}\to\tfrac{3}{2}; J=\tfrac{3}{2}+\right) = \frac{e^{2i\eta_{33}^0}-1}{2i\rho_0}$$

$$D_i = |D_i|\,e^{-i\delta_i},$$

where η_{11}^0 and η_{33}^0 are (unperturbed) phase shifts, and the δ_i are real. First we consider the case $I_i = I_f$. Then

$$\mathrm{Im}\,\delta A(I_i I_3 \to I_i I_3; J) = \delta\,\mathrm{Im}\,A(I_i I_3 \to I_i I_3; J)$$

$$= \delta\left(\rho\,|A(I_i I_3 \to I_i I_3; J)|^2\right) + \frac{2k_\gamma q}{\rho}\left[\left|M_{1\pm}^{I_i I_3}\right|^2 + 3\left|E_{1\pm}^{I_i I_3}\right|^2\right]. \tag{38}$$

But

$$\delta\left(\rho\,|A|^2\right) = \delta\rho\,|A_0|^2 + 2\rho_0\left[\mathrm{Re}A_0\,\mathrm{Re}\,\delta A + \mathrm{Im}A_0\,\mathrm{Im}\,\delta A\right]. \tag{39}$$

Now $\mathrm{Im}\,D^2\,\delta A = \mathrm{Im}D^2\cdot\mathrm{Re}\,\delta A + \mathrm{Re}D^2\,\mathrm{Im}\,\delta A$; solving (38) and (39) for $\mathrm{Re}\,\delta A$ and using (37), we find that

$$\mathcal{I}m \, D^2(I_i \to I_i \,; J) \, \delta A(I_i I_3 \to I_i I_3 \,; J) \quad =$$

$$\frac{|D(I_i \to I_i \,; J)|^2}{\sin(2 Re\,\eta_i^\circ)} \left\{ \mathcal{I}m \, \delta A(I_i I_3 \to I_i I_3 \,; J) \sin 2\left([Re\,\eta_i^\circ] - \delta_i\right) \right.$$

$$(40)$$

$$\left. + e^{2\mathcal{I}m\,\eta_i^\circ} \sin 2\delta_i \left[\delta\rho(I_i I_3 \to I_i I_3)|A_0(I_i \to I_i \,; J)|^2 + \frac{2k_\gamma q}{\rho}\left(|M_{1t}^{I_i I_3}|^2 + 3|E_{1t}^{I_i I_3}|^2\right) \right] \right\}$$

$$(I_i = I_f).$$

When $I_i \neq I_f$, a modified result is obtained. Here

$$\mathcal{I}m \, \delta A(I_i I_3 \to I_f I_3 \,; J) = \delta\left(\rho \, Re\left[A^*(I_f I_3 \to I_f I_3 \,; J) A(I_f I_3 \to I_i I_3 \,; J) \right. \right.$$

$$\left. + A^*(I_i I_3 \to I_f I_3 \,; J) A(I_i I_3 \to I_i I_3 \,; J)\right])$$

$$+ \frac{2k_\gamma q}{\rho} \, Re\left[M_{1t}^{I_f I_3 *} M_{1t}^{I_i I_3} + 3\, E_{1t}^{I_f I_3 *} E_{1t}^{I_i I_3}\right]$$

$$(41)$$

$$= \rho_0 \left(Re\left[\delta A(I_i I_3 \to I_f I_3 \,; J)\right]\left[Re\left(A_0(I_f \to I_f \,; J) + A_0(I_i \to I_i \,; J)\right)\right] \right.$$

$$\left. + \mathcal{I}m\left[\delta A(I_i I_3 \to I_f I_3 \,; J)\right]\left[\mathcal{I}m\left(A_0(I_f \to I_f \,; J) + A_0(I_i \to I_i \,; J)\right)\right] \right)$$

$$+ \frac{2k_\gamma q}{\rho} \, Re\left[M_{1t}^{I_f I_3 *} M_{1t}^{I_i I_3} + 3\, E_{1t}^{I_f I_3 *} E_{1t}^{I_i I_3}\right].$$

Using $\text{Im} D_i D_f \delta A = \text{Im} D_i D_f \text{Re} \delta A + \text{Re} D_i D_f \text{Im} \delta A$ together
with (37) and (41), we obtain

$$\text{Im}\, D(I_i \to I_i; J) D(I_f \to I_f; J) \delta A(I_i I_3 \to I_f I_3; J) \quad =$$

$$\left\{ \frac{\left| D(I_i \to I_i; J) D(I_f \to I_f; J) \right|}{\sin 2 \text{Re}\,\eta_f^0\, e^{-2 \text{Im}\,\eta_f^0} + \sin 2 \text{Re}\,\eta_i^0\, e^{-2 \text{Im}\,\eta_i^0}} \right\} \text{Im}\, \delta A(I_i I_3 \to I_f I_3; J). \tag{42}$$

$$\cdot \left[\sin\left(2[\text{Re}\,\eta_f^0] - \delta_i - \delta_f\right) e^{-2 \text{Im}\,\eta_f^0} + \sin\left(2[\text{Re}\,\eta_i^0] - \delta_i - \delta_f\right) e^{-2 \text{Im}\,\eta_i^0} \right]$$

$$+ 2 \sin(\delta_i + \delta_f) \frac{2 k_\gamma q}{\rho} \text{Re}\left[M_{1\pm}^{I_f I_3}{}^* M_{1\pm}^{I_i I_3} + 3 E_{1\pm}^{I_f I_3}{}^* E_{1\pm}^{I_i I_3} \right] \Biggr\}$$

$$(I_i \neq I_f).$$

(Indices i and f denote initial and final isospins,
respectively.) We see that (except for a $\delta\rho$ term) this
expression would reduce to (40) when $I_i = I_f$.

Now we see the simplification that would arise on the
right hand cut if we had $\delta_f = \text{Re}\,\eta_f^0$; we then would have

$$\text{Im}\, D^2(I_i \to I_i; J)\, \delta A(I_i I_3 \to I_i I_3; J) \quad =$$

$$|D(I_i \to I_i; J)|^2 e^{2\operatorname{Im}\eta_i^0} \left\{ \delta\rho(I_i I_3 \to I_i I_3)|A_0(I_i \to I_i; J)|^2 \right. \tag{43}$$

$$\left. + \frac{2 k_\gamma q}{\rho}\left[|M_{1\pm}^{I_i I_3}|^2 + 3|E_{1\pm}^{I_i I_3}|^2 \right] \right\} \qquad (I_i = I_f),$$

and

$$\operatorname{Im} D(I_i \to I_i; J) D(I_f \to I_f; J) \delta A(I_i I_3 \to I_i I_3; J) = \tag{44}$$

$$\frac{|D(I_i \to I_i; J) D(I_f \to I_f; J)| \, 2 \sin \operatorname{Re}(\eta_i^0 + \eta_f^0)}{\sin 2\operatorname{Re}\eta_f^0 \, e^{-2\operatorname{Im}\eta_f^0} + \sin 2\operatorname{Re}\eta_i^0 \, e^{-2\operatorname{Im}\eta_i^0}} \cdot \frac{2 k_\gamma q}{\rho} \cdot$$

$$\cdot \operatorname{Re}\left[M_{1\pm}^{I_f I_3 *} M_{1\pm}^{I_i I_3} + 3 E_{1\pm}^{I_f I_3 *} E_{1\pm}^{I_i I_3} \right] \qquad (I_i \neq I_f) .$$

(If δ_j were to deviate slightly from $\operatorname{Re}\eta_j^0$, this would only produce a second order effect in (40) and (42).)

Now the explicit D functions we have been using (linear D and curved D) have no phases along the right hand cut. Thus we would set $\operatorname{Im} D_i D_j \delta A = D_i D_j \operatorname{Im} \delta A$ along this cut. But unitarity only gives us $\operatorname{Im} \delta A$ in terms of $\operatorname{Re} \delta A$ (Equations (38), (39), and (41)), and we have no model with which to make an independent estimate of $\operatorname{Re} \delta A$. Thus we do not have any way to find numerical values for $\operatorname{Im} \delta A$. This situation is described in terms of δ(phase shift) in an appendix of reference (19). What we shall do

here to estimate $\mathrm{Im} D_i D_j \delta A$ is to set $\rightarrow \delta_i$ (the phase of
the D function)$= -\mathrm{Re}\, \eta_i^0$ into (40) and (42) (we also let
$W_0 \rightarrow \infty$ in the curved D functions) and thus use Equations
(43) and (44) when actually doing the numerical integra-
tion along the right hand cut. This is admittedly only an
approximation, and can only give us an idea of the order
of magnitude of the terms involved. The approximation is
not so bad for amplitudes with small phase shifts, but is
doubtful whenever a (33) phase shift is involved.

The next question is clearly what to use for the
photoproduction multipoles entering in (43) and (44). In
principle we could use either experimentally or theoreti-
cally determined multipoles. (Of course, for $M \le W \le M + \mu$
it is not possible to obtain direct measurements.) We
simply use the largest (N^* resonance and Born part of $M_{1\pm}$
from the e amplitudes) parts of the CGLN[39] multipoles.[42]
These multipoles are expected to be valid up to the N^*
(1238) resonance region, and the dominant parts thereof
(N^* resonance and pole term) are well established
experimentally.[40,43,44] (Recent work on photoproduction
(off protons) by Höhler and Schmidt[43] indicates that for
$E_\gamma \le 500 \mathrm{Mev}$, keeping only the N^* resonance and the pole
term is indeed a good approximation in $M_{1+}^{I=\frac{3}{2}}$, while
keeping only the pole term in M_{1-} , for example, may

introduce an error of about 30% at $E_\gamma = 300$ Mev. Thus

we expect that integrals involving only $M_{1+}^{I=\frac{3}{2}}$ will

be more accurate than the (numerically smaller) integrals

involving additional multipoles. In doing the integrals

over the right hand cut using (43) and (44), we shall set

all absorption parameters (due to complex unperturbed

phase shifts) equal to one, which is a good approximation

for $T_\pi \leq 300$ Mev.$^{(45)}$ $\pi-N$ phase shifts are obtained from

the analysis of Roper et al.[45] In using the dominant

parts of the CGLN multipoles, we remember to insert

appropriate minus signs when applicable, since their

phase conventions differ from ours.[42]

Since the residue shift integrals (20) actually

involve δA_I's, we next take appropriate linear combina-

tions of the quantities in (41) or (42) to obtain the

input due to the πN and γN intermediate states. We

do the integrals over the pieces containing multipoles

numerically (integrating to $W \cong 1.37M$); the results are

presented in Table VII.

The term in (40) involving $\delta\rho$ contains contributions

from external mass shifts. This type of contribution has

already been included in Sec. V.

The photoproduction results can be explained

qualitatively by noting that for residue shifts with $I=1$,

one isoscalar vertex and one isovector vertex are needed,[2] whereas for I=0 or 2 no isoscalar vertex is required. The available isoscalar vertex (nucleon spin flip) is small, and hence the I=1 contributions from the γN intermediate state are expected to be small.[2] For I=0 and I=2 larger contributions are expected, especially for $I_i = I_f = \frac{3}{2}+$, since the photoproduction amplitude here is large due to N^{*} resonance effects.

XI. The Right Hand Cut; Additional Contributions

In the previous section we analyzed the contribution
(to the residue shifts) of the right hand cut, keeping
only the πN and γN intermediate states in the unitarity
condition for Im δA. An additional intermediate state to
consider is the $\pi N \gamma$ direct channel intermediate state.
To obtain an idea of the relative sizes involved, we
compare the cross section for $\pi N \rightarrow \pi N \gamma$ with the cross
section for $\pi N \rightarrow \gamma N$ at a typical energy (the ratio of these
cross sections is just equal to the ratio of the respective
contributions to the imaginary part of the forward
$\pi N \rightarrow \pi N$ scattering amplitude).

We could use either theory or experiment to obtain
the $\pi N \rightarrow \pi N \gamma$ cross section. The theory developed by
Cutkosky[46] was used by Carruthers,[47] but the numerical
results do not agree with experiment.[48,49] Hence it is
preferable to use experimental data, even though these
are meager. Using data at 224 Mev[48] we see that[48]

$$\frac{\sigma_{\pi^- p \rightarrow \pi^- p \gamma}}{\sigma_{\pi^- p \rightarrow \gamma n}} \cong \frac{.04 \; mb}{.6 \; mb} = \frac{1}{15} \; .$$

(The cross section for $\pi^- p \rightarrow \pi^- p \gamma$ is measured directly
(with $E_\gamma \geq 50$ Mev); $\sigma_{\pi^- p \rightarrow \gamma n}$ is found by using detailed
balance arguments.) Since our calculation is not

expected to be accurate to one part in fifteen, we simply neglect the $\pi N \gamma$ intermediate state.

We also neglect the $N^* \gamma$ intermediate state. This state is expected to give only a small contribution to $\operatorname{Im} \delta A$, both because the $\pi N \gamma$ contribution is small, and because the available phase space for $\pi N \rightarrow N^* \gamma$ is quite small at the energies we are considering.

The remaining contribution to the unitarity condition for $\operatorname{Im} \delta A$ (in the low-energy range we are considering) will come from the $\pi \pi N$ inelastic intermediate state. Similarly to the analysis of the $\pi N \gamma$ state, we compare total cross sections to obtain an idea of the magnitude involved. Experimental data [50] give $\sigma_{\pi^- p \rightarrow \pi^- \pi^+ n} \cong 1.4\text{mb}$ and $\sigma_{\pi^- p \rightarrow \pi^- \pi^\circ p} \cong .4\text{mb}$ at $T_{\pi_{incident}} = 400$ Mev (laboratory energy). Since these are small compared with $\pi N \rightarrow \pi N$ cross sections, we neglect the $\pi \pi N$ intermediate state.

Having thus completed the survey of contributions to the residue shifts from the right hand cut, we turn to the remaining contributions, namely those parts of the u channel cut not coming from N or N* exchange.

XII. The Left Hand Cut - Relation to the Right Hand

Cut - γN and πN Exchange

Up to now we have analyzed contributions to our

residue shifts coming from N and N^* exchange, one-

photon exchange and related "t-channel" processes, and

from the right hand cut. The remaining contributions to

the residue shifts will come from processes such as

πN, γN, $\gamma \pi N$, and γN^* exchange, and $\pi \pi N$ exchange (i.e.,

from the analogues of the intermediate states contributing

to the right hand cut). These remaining contributions

(which contribute to the left hand cut) will now be

simply related to quantities on the right hand cut.

We shall see that if $\vec{D}_1$ denotes the contribution

to our residue shifts due to the presence of intermediate

states in the direct channel, then exchange of the same

states gives the approximate term $-A^{RR}\vec{D}_1$, where A^{RR} is

the A matrix connecting u channel residue shifts to s

channel residue shifts. I.e.,

$$\vec{\delta R} \cong A^{RR} \vec{\delta R} + \left(I - A^{RR} \right) \vec{D}_1 + \text{[mass shift and}$$
$$\text{t channel terms]}$$

Hence

$$\left(\mathbb{I}-A^{RR}\right)\left(\overrightarrow{\delta R}-\overrightarrow{D_1}\right) \cong \quad \text{(mass shift and t channel terms).} \quad (2)$$

This result for exchanged states such as πN, γN $\gamma\pi N$, γN^*, and $\pi\pi N$ is approximate but holds for curved as well as linear D functions. We shall assume the static crossing relations; this approximation will be sufficient to derive (2) in the linear D case. For curved D functions we make the additional approximation of evaluating the coefficient of $\text{Im}\delta A$ (in the integrand of the dispersion integral) at $W=M$ (when $J=\frac{1}{2}+$) or at $W=M^*$ (when $J=\frac{3}{2}+$). Thus we will derive (2) for curved D functions as well.

We shall now proceed with the explicit proof of Equation (2). The method for the sample case $\delta R_I(\frac{3}{2}\to\frac{1}{2};J=\frac{1}{2}+)$ is illustrated as follows:

$$\delta R_I\left(\tfrac{3}{2}\to\tfrac{1}{2};J=\tfrac{1}{2}+\right) = \frac{1}{2\pi i}\int\left(\frac{M-W_0}{W'-W_0}\right)\delta A_\alpha(W')\,dW'$$

(α represents I, I_i, I_f, and J)

$$= \frac{1}{2\pi i}\int_0\left(\frac{M-W_0}{\omega+M-W_0}\right)\left[\delta B_\alpha(\omega+i\epsilon)-\delta B_\alpha(\omega-i\epsilon)\right]d\omega$$

($\omega=W-M$; $B(\omega\pm i\epsilon)=A(W+i\epsilon)$)

$$= \frac{1}{2\pi i}\int_0\left(\frac{M-W_0}{-x+M-W_0}\right)\left[\delta B_\alpha(-x+i\epsilon)-\delta B_\alpha(-x-i\epsilon)\right]dx$$

$$(x = -\omega)$$

$$\cong C_{\alpha\beta} \frac{1}{2\pi i} \int_0^0 \left(\frac{M-W_0}{-x+M-W_0}\right)\left[\delta B_\beta (x-i\epsilon) - \delta B_\beta (x+i\epsilon)\right](-dx)$$

(amplitudes in the s channel are related to amplitudes in the u channel via a matrix denoted by C)

$$= -C_{\alpha\beta} \frac{1}{\pi} \int_0 \left(\frac{M-W_0}{-x+M-W_0}\right) \mathcal{Im}\, \delta B_\beta (x+i\epsilon)\, dx$$

$$= -\dot{C}_{\alpha\beta} \frac{1}{\pi} \int_M \left(\frac{M-W_0}{2M-W-W_0}\right) \mathcal{Im}\, \delta A_\beta (W+i\epsilon)\, dW \qquad (45)$$

$$(W = x + M).$$

In relating s and u channel amplitudes (all with $\ell=1$), we are using static crossing relations. This approximation is valid, since we intend to keep only contributions from pieces of the left hand cut fairly near the nucleon pole.

First we look at the linear D case. Then (45) reduces to

$$\frac{1}{2\pi i} \int_{\text{L.H. cut, } u \text{ channel}\atop\text{piece}} \delta A_\alpha (W')\, dW' \cong -C_{\alpha\beta} \frac{1}{\pi} \int_M^{(\infty)} \mathcal{Im}\, \delta A_\beta (W+i\epsilon)\, dW. \qquad (46)$$

If β represents $(I_i = I_f = J)$ or $(I_i \neq I_f; J)$, then the factor multiplying $-C_{\alpha\beta}$ is just the contribution of the right hand cut to our integrals (this contribution was estimated in a previous section). When $(I_i = I_f \neq J)$, on the other hand, we did not previously calculate the corresponding integral over the right hand cut, since the residues with this type of β would be of second order in symmetry breaking. Thus for these β's we now look at the right hand side of (46) directly. Numerically these terms are not expected to be large. The multipoles due to the γN intermediate state are not large, and the $I \neq J$ phase shifts are small. In addition, these terms numerically have no component along eigenvectors of A^{RR} with eigenvalue equal to one. Thus

$$\delta R_\alpha \leftrightarrow -C_{\alpha\beta} \left[\vec{D_i}\right]_\beta , \tag{47}$$

where $\left[\vec{D_i}\right]_\beta$ is the contribution to the direct channel residue shifts due to direct channel intermediate states.

But now we observe that the crossing coefficients $C_{\alpha\beta}$ have already been used previously. We shall now show explicitly that $\left(A^{RR}\right)_{\alpha\beta} = \left(C\right)_{\alpha\beta}$ when linear D functions are used. To obtain this relation, we turn back to Sec. V and show in more detail how A^{RR} was derived. For linear D

functions the dispersion integrals (20) are very simple:

$$\delta R_\alpha = \frac{1}{2\pi i} \int \delta A_\alpha (W^s) \, dW^s \qquad . \tag{48}$$

In Sec. V we then reexpressed (via $\pi N \to \pi N$ crossing and Clebsch-Gordan coefficients) the s channel amplitude $\delta A_\alpha(W^s)$ in terms of u channel amplitudes. I.e. (in the static limit),

$$\delta A_\alpha (W^s) = C_{\alpha\beta} \, \delta A_\beta (W^u) .$$

The $C_{\alpha\beta}$ in this expression are just the $C_{\alpha\beta}$ encountered previously in this section. Next we noted that the amplitudes in the u channel, $\delta A_\beta(W^u)$, had poles as a function of W^u (the relevant s channel diagrams are shown in Fig. 2). I.e.,

$$\delta A_\beta (W^u) \cong \delta \left(\frac{R_\beta}{W^u - (M \text{ or } M^*)} \right) .$$

Since in our dispersion integrals we actually integrate over W^s, we reexpressed the poles of $\delta A_\beta(W^u)$ in terms of the variable of integration W^s. I.e. (in the static limit),

$$\delta A_\beta(W^u) \cong \delta\left(\frac{R_\beta}{-W^s + (2M - \frac{M_{or}}{M^*})}\right).$$

Thus for the linear D case we had

$$\delta R_\alpha \longleftrightarrow \frac{1}{2\pi i}\int_{\mathcal{P}} C_{\alpha\beta}\, \delta A_\beta(W^u)\, dW^s$$

$$= C_{\alpha\beta}\frac{1}{2\pi i}\int_{\mathcal{P}}\frac{\delta R_\beta}{-W^s + (2M - \frac{M_{or}}{M^*})}\, dW^s \quad + \text{ mass terms}$$

$$= \qquad C_{\alpha\beta}\; \delta R_\beta \qquad\qquad + \text{ mass terms.}$$

By definition of A^{RR}, we hence see that $(A^{RR})_{\alpha\beta} = (C)_{\alpha\beta}$ when using linear D functions. Hence we now see that

$$\delta R_\alpha \longleftrightarrow -\left(A^{RR}\right)_{\alpha\beta}\left[\vec{D}_1\right]_\beta \tag{49}$$

when using linear D functions. ($\vec{D}_1$ is the contribution due to the existence of s channel intermediate states.)

Next we must consider the situation for curved D functions. Looking at (45) and its analogues, and evaluating the quantity in curved brackets at W=M when $J=\frac{1}{2}+$ and at W=M* when $J = \frac{3}{2}+$, we find

$$\delta R_{\alpha} \overset{\sim}{\longleftrightarrow} -\left(A^{RR}\right)_{\alpha\beta} \frac{1}{\pi} \int_M \mathcal{Im}\, \delta A_{\beta}(W'+i\epsilon)\, dW'. \qquad (50)$$

Here $\left(A^{RR}\right)_{\alpha\beta}$ is the (α,β) element of the A^{RR} matrix for curved D obtained in Sec. V and tabulated in Table I. (50) is valid for any α. In writing (50) we are neglecting those β's for which $I_i = I_f \neq J$, as in the previous case of linear D functions.

Now $\overrightarrow{D_1}$ (for curved D functions) is approximately equal to

$$\frac{1}{\pi} \int_M \mathcal{Im}\, \delta A_{\beta}(W'+i\epsilon)\, dW',$$

and thus (50) implies

$$\delta R_{\alpha} \overset{\sim}{\longleftrightarrow} -\left(A^{RR}\right)_{\alpha\beta} \left[\overrightarrow{D_1}\right]_{\beta} \qquad\qquad \text{(curved D).}$$

Thus, for both linear and curved D functions,

$$\delta R_{\alpha} \overset{\sim}{\longleftrightarrow} -\left(A^{RR}\right)_{\alpha\beta} \left[D_1\right]_{\beta}$$

is the contribution from πN, γN, $\gamma\pi N$, γN^*, and $\pi\pi N$ exchange. Since in finding $\overrightarrow{D_1}$ (Sections X and XI) we neglected the $\gamma\pi N$, γN^*, and $\pi\pi N$ states, these states are negligible.

XIII. Analysis of Contributions - No Enhancement

In the preceding sections we have investigated all the near-by contributions to our residue shifts and have found numerical values for all the significant states, using first linear and then curved D functions. The complete numerical results of this study have been presented in Tables II - VII.

A discovery we shall make is that even though the matrices A_I^{RR} each have one eigenvalue equal to one (or near one), the corresponding eigenvectors are not enhanced! For the linear D case there is one eigenvalue exactly equal to one (for each I), but the relevant driving terms have exactly zero component along the enhanced eigenvectors. This result depends only on crossing and Clebsch-Gordan coefficients and is independent of the values of the nucleon magnetic moments, the positions of the simple poles in the nucleon form factors, and the value of the vanishing photon mass λ (as long as λ is assumed the same for each residue shift with the same I value). These results are derived below and will be made qualitatively more plausible in the following section.

We now investigate whether the "enhancement

mechanism", which has been so successfully employed in other studies,[3,4,7,16,17,19] is also relevant to our case. By "enhancement mechanism" we mean that if the matrix A^{RR} has one eigenvalue near one, then it is likely that the corresponding eigenvector is enhanced compared to the other eigenvectors of this matrix. The analysis of the preceding sections has shown us that

$$\vec{\delta R_I} \cong A_I^{RR}\,\vec{\delta R_I} + A^{RM}\,\frac{\vec{\delta m_I}}{m_o} + \vec{D}_{1-\gamma} + (I - A^{RR})\vec{D}_1 \,, \qquad (52)$$

where A^{RR} and A^{RM} are defined by (17), $\vec{D}_{1-\gamma}$ is the contribution to our residue shifts from one-photon exchange, and $\vec{D}_1$ is the contribution due to the existence of s channel intermediate states. Thus

$$(I - A_I^{RR})(\vec{\delta R_I} - \vec{D}_1) \cong A_I^{RM}\,\frac{\vec{\delta m_I}}{m_o} + \vec{D}_{1-\gamma} \,. \qquad (53)$$

Denoting the eigenvalues and eigenvectors of A^{RR} by λ_j and $\vec{v}_j$, we can find vectors $\vec{w}_i$ with the property $\vec{w}_i \cdot \vec{v}_j = \delta_{ij}$. We then define a matrix F by

$$F_{ij} = (\vec{v}_j)_i \,. \qquad \text{(component i of vector j);}$$

then $(F^{-1})_{ij} = (\vec{w_i})_j$, and (53) implies

$$(I - \Lambda) F^{-1} (\vec{\delta R_I} - \vec{D_1}) \cong F^{-1} (A^{RM} \frac{\vec{\delta m}}{m_0} + \vec{D_{1-\gamma}}) , \qquad (54)$$

where Λ is a diagonal matrix: $\Lambda_{ij} = \lambda_i \delta_{ij}$ (no summation convention).

   If A_I^{RR} has exactly one eigenvalue near one, then $(\vec{\delta R_I} - \vec{D_1})$ will lie mostly along the corresponding eigenvector, unless the component of $(A_I^{RM} \vec{\delta m}/m_0 + \vec{D_{1-\gamma}})$ along that eigenvector is very small. Numerical results for the eigenvalues, eigenvectors, and "reciprocal eigenvectors" $\vec{w_i}$ of A_I^{RR} (these are listed in Table III) show that when D is linear, there is one eigenvalue of A_I^{RR} exactly equal to one (for each I). But the right hand side of (54) has no component corresponding to this eigenvector, as can be seen by explicitly inserting the linear D results for F , $\vec{D_{1-\gamma}}$, and A_I^{RM} into (54) (using the assumption that for fixed I the numerical value of the "photon mass" λ is the same). Thus we are led not to expect any enhancement mechanism for linear D! This result depends only on crossing and Clebsch-Gordan coefficients, since A_I^{RR} and the ratios of the components of $\vec{D_{1-\gamma}}$ (for fixed I) depend only on these coefficients (A_I^{RM} equals zero for the linear D case).

Turning to the curved D case, we see from Table III that there is one eigenvalue of A_I^{RR} near one (for each I). The right hand side of (52) is modified from the linear D case, but a precise estimate is difficult to obtain, since not all of the N* masses are known. (See references (51) and (52) for available N* mass differences.) We shall take the hint of "no enhancement" given us by the linear D case and shall simply use equations (53) directly to find the residue shifts δR_I (see below).

This conclusion of "no enhancement" for linear or curved D functions is at first sight quite surprising, when we remember that previous studies[3,4,7,16,17,19] (which did not calculate as many driving terms, however) where enhancement was assumed have led to very good agreement with experiment. Actually, our result is not in direct contradiction with the previous successful applications of the enhancement mechanism. Most of these calculations involved mass shifts, or mass shifts driving coupling shifts. The only successful case not driven by masses was the case of the weak parity-violating coupling shifts, where the driving term is not known. Now in our model the mass shift contributions for curved D are non-zero and will indeed be enhanced. This effect does not determine the direction of symmetry breaking in our case,

however, since our mass shifts are either small (nucleon

masses) or are assumed small (N^* masses).

Before solving equations (53) directly to find our

residue shifts, we need to investigate whether these

equations change much under projecting out states with the

wrong charge conjugation property.$^{(53)}$ I.e., we require

$\delta g(\pi^- p \to n) = - g(\pi^+ n \to p)$ (i.e.,$C=+1$), and we wish to project

out states satisfying $\delta g(\pi^- p \to n) = + \delta g(\pi^+ n \to p)$ ($C=-1$). These

conditions can be translated into conditions on residue

shifts (since residues are in general proportional to

products of coupling constants):

$$\delta R_{I=1}\left(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+\right) = \sqrt{2}\, \delta R_{I=1}\left(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+\right) \quad (C=+1) \tag{55}$$

$$\delta R_{I=1}\left(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+\right) = -2\sqrt{2}\, \delta R_{I=1}\left(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+\right) \quad (C=-1). \tag{56}$$

We want to satisfy (55) and wish to project out states

satisfying (56). Forming the matrix P which satisfies

this requirement, we then find that numerically $P\, A^{RR}_{I=1}\, P$

is quite different from $A^{RR}_{I=1}$ (P does not affect states

with $I=0$ or 2). I.e., the matrix $A^{RR}_{I=1}$ is not at all charge

conjugation invariant. Thus we expect that the numerical

results for $I=1$ will be especially unreliable. Keeping

this in mind, we now solve (53) for our δR's.

For the curved D case we set

$$\vec{\delta R}_I = \vec{D}_1 + (I - A^{RR})^{-1}\left[A^{RM}\frac{\vec{\delta m}}{m_o} + \vec{D}_{1-r}\right] \qquad (57)$$

When using linear D functions, we note that A_I^{RR} has an eigenvalue equal to one; thus $(I - A_I^{RR})^{-1}$ does not exist, and we see that

$$\vec{\delta R}_I = \vec{D}_1 + a_I \vec{x}_h + \vec{x}_p \,. \qquad (58)$$

Here a_I is an undetermined constant, $\vec{x}_h$ satisfies $(I - A_I^{RR})\vec{x}_h = 0$ (i.e., $\vec{x}_h$ is the eigenvector with eigenvalue equal to one), and $\vec{x}_p$ is a particular solution of $(I - A_I^{RR})\vec{x} = \vec{D}_{1-r}$.

The results for δR_I, found from (57) and (58), are shown in Table VIII. $\delta R_{I=0}$ is undetermined for the curved D case, since the I=0 mass shifts are of course undetermined ($\delta R_{I=0}$ for the linear D case does not depend on mass shifts). We have evaluated the $\ell n\lambda$ pieces of the δR's at $\lambda = m_\rho$, as discussed in Sec. III. The constants "a_I" for the linear D case (I=1,2) were fixed by setting the components of δR_I for this case equal to the components of δR_I for the curved D case and taking the average values of the a_I's thus obtained. We have set all N* mass shifts equal to zero (see references (51)

and (52) for available experimental N^* mass shifts).

We see that the values of the ΔR_I's for linear and curved D functions agree fairly well. The absolute numerical values obtained are, however, very strongly dependent on the value of the photon mass λ chosen. Another source of uncertainty is of course that we have simply set all N^* mass shifts equal to zero. We note that there is no enhancement of the eigenvectors with eigenvalues equal to (or near) one, and the residue shifts are all approximately of the order of the fine structure constant α. The conclusion of no enhancement and the order of magnitude of the final numerical results are both probably qualitatively correct, in spite of the uncertainties of the calculation.

XIV. A Larger Space

In this section we shall consider a larger set of residue shifts than in the previous sections. This larger set is physically less appropriate, but mathematically simpler. We shall find that in this larger space some of the empirical properties of the previous calculation become more transparent. In particular, we shall gain an understanding of why one might expect eigenvalues of A^{RR} near one and why the corresponding components of some of the driving terms vanish. This understanding applies to SU(3) calculations as well.

We first note that in the previous sections we calculated only ten of the twenty residues shifts in pion-nucleon P wave amplitudes. (There is a total of twenty such amplitudes, since there are two P wave projections, corresponding to $J=\frac{1}{2}+$ and $J=\frac{3}{2}+$, for each of the ten pion-nucleon scattering amplitudes.) We assumed (in the previous sections) that some of the residue shifts (namely, $\delta R(\frac{1}{2} \to \frac{1}{2} ; J=\frac{3}{2}+)$ and $\delta R(\frac{3}{2} \to \frac{3}{2} ; J=\frac{1}{2}+)$) were small. These residues correspond to symmetry breaking at both vertices simultaneously; they would arise in a first order symmetry-breaking calculation such as ours if there were physical particles with $(I=\frac{1}{2}+, J=\frac{3}{2}+)$ and $(I=\frac{3}{2} , J=\frac{1}{2}+)$. The assumption of time-reversal invariance in the above

sections further reduced the number of residue shifts to
be calculated. If we had calculated all twenty residue
shifts, we would have obtained an equation of the form

$$\begin{pmatrix} \delta R_a - D_{1,a} \\ \delta R_b - D_{1,b} \end{pmatrix} = \begin{pmatrix} A_{aa}^{RR} & A_{ab}^{RR} \\ A_{ba}^{RR} & A_{bb}^{RR} \end{pmatrix} \begin{pmatrix} \delta R_a - D_{1,a} \\ \delta R_b - D_{1,b} \end{pmatrix} + \begin{pmatrix} mass\ shift\ and \\ t\ channel\ terms \end{pmatrix}$$

(59)

where δR_a is the set of residue shifts we calculated in
the previous sections, δR_b is the set of ten additional
residue shifts, and the A^{RR}'s are matrices. A_{aa}^{RR} is
related to the matrix we called A^{RR} in the previous
sections.

We note from the way in which A^{RR} is calculated that
for linear D functions A^{RR} is just the crossing matrix
relating scattering amplitudes in the s channel to
scattering amplitudes in the u channel. We now choose a
basis in our twenty-dimensional space in which the
crossing matrix (denoted by C) takes on a particularly
simple form. We choose as our basis the twenty linear
combinations

$$
\begin{bmatrix}
\delta R\left(\pi^+ p \to \pi^+ p;\ J=\tfrac{1}{2}+\right) \pm\ \delta R\left(\pi^- p \to \pi^- p;\ J=\tfrac{1}{2}+\right) \\[4pt]
\delta R\left(\pi^+ n \to \pi^+ n;\ J=\tfrac{1}{2}+\right) \pm\ \delta R\left(\pi^- n \to \pi^- n;\ J=\tfrac{1}{2}+\right) \\[4pt]
\delta R\left(\pi^+ n \to \pi^0 p;\ J=\tfrac{1}{2}+\right) \pm\ \delta R\left(\pi^0 n \to \pi^- p;\ J=\tfrac{1}{2}+\right) \\[4pt]
\delta R\left(\pi^- p \to \pi^0 n;\ J=\tfrac{1}{2}+\right) \pm\ \delta R\left(\pi^0 p \to \pi^+ n;\ J=\tfrac{1}{2}+\right) \\[4pt]
\delta R\left(\pi^0 p \to \pi^0 p;\ J=\tfrac{1}{2}+\right) \\[4pt]
\delta R\left(\pi^0 n \to \pi^0 n;\ J=\tfrac{1}{2}+\right) \\[16pt]
\delta R\left(\pi^+ p \to \pi^+ p;\ J=\tfrac{3}{2}+\right) \pm\ \delta R\left(\pi^- p \to \pi^- p;\ J=\tfrac{3}{2}+\right) \\[4pt]
\delta R\left(\pi^+ n \to \pi^+ n;\ J=\tfrac{3}{2}+\right) \pm\ \delta R\left(\pi^- n \to \pi^- n;\ J=\tfrac{3}{2}+\right) \\[8pt]
\text{etc.}
\end{bmatrix}
\tag{60}
$$

Then the crossing matrix C is

$$
C = \begin{pmatrix} -\tfrac{1}{3} C_{\shortparallel} & \tfrac{4}{3} C_{\shortparallel} \\[6pt] \tfrac{2}{3} C_{\shortparallel} & \tfrac{1}{3} C_{\shortparallel} \end{pmatrix}, \text{ where } \quad C_{\shortparallel} = \begin{pmatrix} 1 & -1 & & & & O \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \\ O & & & & & 1 \end{pmatrix}.
\tag{61}
$$

(The numbers $-\tfrac{1}{3}$, $\tfrac{4}{3}$, $\tfrac{2}{3}$, and $\tfrac{1}{3}$ in C are just the static crossing coefficients relating the spin states $J=\tfrac{1}{2}+$ and $J=\tfrac{3}{2}+$ of the s channel to those of the u channel.) This matrix has eigenvalues $\lambda_i = \pm 1$. The eigenvalues

$\lambda = \pm 1$ are actually to be expected, since $\lambda_t = \pm 1$ is a
general property of crossing matrices relating complete
sets of states. The complete set of states in our speci-
fic case is the set of all P wave π-N scattering amplitudes
in the static limit.

The eigenvalues encountered in the previous sections
were those of $A^{RR}_{10 \times 10}$, namely, the eigenvalues of a
"truncated" matrix. By "truncated" we mean that the space
of residue shifts is only ten-dimensional; residue shifts
with symmetry breaking at both vertices simultaneously are
not included, and the assumption of time-reversal
invariance further reduces the number of residue shifts.
Since the untruncated A^{RR} matrix for the linear D case
has many eigenvalues equal to one, it is thus perhaps not
so surprising that the truncated matrix for linear D also
has (some) eigenvalues equal to one. Thus it is now not
so surprising when in calculations in a truncated space,
one obtains eigenvalues equal to one (or near one, when
D is curved).

The next step is to examine enhancement of eigen-
vectors with eigenvalues equal to (or near) one. We
again look at the linear D case in our twenty-dimensional
space. We recall that in the truncated (ten-dimensional)
space the driving term due to one-photon exchange had no

component along the eigenvector with eigenvalue equal to
one. We now see whether the analogous conclusion is true
in the larger space. In this larger space $\vec{D}_{1-\gamma}$ has a very
simple form. The components of $\vec{D}_{1-\gamma}$ are given (in our
formalism) by a contour integral using the one-photon
exchange amplitude, $\delta A_{1-\gamma}$, as input. $\delta A_{1-\gamma}$ has opposite
signs for π^+ and π^- scattering off the same nucleon and is
zero whenever at least one of the external pions is neutral.
One finds by explicit calculation that $\delta A_{1-\gamma}$ does not
depend on whether $J = \frac{1}{2}+$ or $J = \frac{3}{2}+$. Thus we see that in our
basis system

$$\vec{D}_{1-\gamma} = \begin{pmatrix} \overrightarrow{D_{1,1-\gamma}} \\ \overrightarrow{D_{1,1-\gamma}} \end{pmatrix} , \text{ where } \overrightarrow{D_{1,1-\gamma}} = \begin{pmatrix} 0 \\ 2a \\ 0 \\ 2b \\ 0 \end{pmatrix} , \tag{62}$$

and a and b are numbers.

But now we note from (61) and (62) that $C\, D_{1-\gamma} = -D_{1-\gamma}$, and
since $C = A^{RR}$, we see that $\vec{D}_{1-\gamma}$ is an eigenvector of A^{RR}
with eigenvalue minus one. Hence $\vec{D}_{1-\gamma}$ does not have any
component along eigenvectors of A^{RR} that have eigenvalues
equal to plus one. I.e., the eigenvectors of A^{RR} with
eigenvalues equal to plus one are <u>not enhanced</u> by the one-
photon exchange contribution. We thus are no longer
surprised by the fact that one-photon exchange in the
truncated ten-dimensional space also did <u>not cause</u>

<u>enhancement</u> of eigenvectors with eigenvalues equal to one
(or near one, when using curved D functions).

One might ask how general the <u>no</u> <u>enhancement</u> conclu-
sion is. I.e., would the driving terms for ρ exchange
or ω or ϕ exchange also not contain any components along
eigenvectors of A^{RR} with eigenvalues equal to one?
Writing the analogues of (62) for ρ and ω or ϕ exchange,
we note that

$$\vec{D}_{\substack{\rho_\omega \\ \phi}} = \begin{pmatrix} \vec{D}_{1,\,\rho_\omega} \\ \phi \\ \vec{D}_{1,\,\rho_\phi} \end{pmatrix} , \tag{63}$$

since this form depends only on spin properties (not on
mass or isospin properties) and is hence the same as for
one-photon exchange. For ρ exchange we find

$$\vec{D}_{1,\rho} = \begin{pmatrix} 0 \\ 2a' \\ 0 \\ 2b' \\ 2c' \\ 0 \\ 2d' \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{64}$$

(a', b', c', and d' are
numbers),

and thus $C\,D_\rho = -D_\rho$, in exact analogy with the one-
photon exchange case. For ω exchange we obtain a

similar expression:

$$\vec{D}_{1,\omega} = \begin{pmatrix} 0 \\ ja'' \\ 0 \\ jb'' \\ 0 \end{pmatrix} \tag{65}$$

(a″ and b″ are numbers),

and thus $C\ D_\omega = -D_\omega$. Similarly, $C\ D_\phi = -D_\phi$. (We note that in the absence of any symmetry breaking there would be no ω or ϕ exchange, since ω and ϕ do not couple to two pions in this case.) The way the above vectors ($\vec{D}_{1,\rho}$, $\vec{D}_{1,\omega}$ and $\vec{D}_{1,\phi}$) were deduced is as follows: First we note that C invariance implies that the ρ, ω, and ϕ do not couple to $\pi°\pi°$. The Pauli principle and angular momentum conservation imply that the two pions at the vector meson-$\pi\pi$ vertex are in a relative I=1 state. This state is anti-symmetric in charge space, and we are able to deduce the relative signs of appropriate vector meson-$\pi\pi$ couplings. In this manner we arrive at (64) and (65).

For $\gamma\pi$ exchange we note that C invariance again implies no $\pi°\pi°$ coupling to the exchanged state. The incoming and outgoing pions would still be in a relative I=1 state and thus the conclusion of "no enhancement" is valid for $\gamma\pi$ exchange. The reason that I must equal one is that if one assumes G parity conservation <u>of that part of the</u>

photon <u>which</u> <u>couples</u>, then G parity is not conserved at the $\left(\gamma_{isovector}^{\pi}\right)\pi\pi$ vertex: $G(\pi\ \gamma_{isovector}) = -1$; $G(\pi\pi) = +1$. We note that exchange of a C=+1 meson (such as the f^{o} and A2) could cause enhancement, since the two external pions would be in an I=0 or I=2 state, and these states have the opposite symmetry to the I=1 states encountered above.

Thus we see that while ρ, ω, ϕ, and γ_{π} exchange do not cause enhancement of eigenvectors of A^{RR} with eigenvalues equal to one (or near one), exchange of C=+1 mesons could indeed cause enhancement.

The arguments of the preceding two paragraphs were made in the framework of the twenty-dimensional space. It would not be surprising if the conclusions of no enhancement from ρ, ω, or ϕ were to hold approximately true in the truncated, ten-dimensional space as well. Explicit expressions for exchange of the ρ, ω, and ϕ confirm this expectation.

Viewed in a larger, extended space, the eigenvalues near one encountered in previous SU(3) residue shift calculations[17,19] also appear less surprising. Since the A^{RR} matrix for linear D functions is again just the crossing matrix, we might indeed expect eigenvalues near one. Of course, in SU(3) calculations one has an

additional parameter, the F/D ratio of BBπ coupling, so things are not quite so simple. But qualitatively we can understand the occurrence of eigenvalues near one in the SU(3) residue shift calculations.

XV.   Residues and Coupling Constants

We have now obtained our predictions for residue shifts in the $\pi N \to \pi N$ P wave scattering amplitudes, and we have also gained a qualitative understanding of the occurrence of eigenvalues near one and "no enhancement" from one-photon (or vector meson) exchange.   In the next section we shall wish to compare our results with experiment.   Before doing this, however, we first examine the relation of our residue shifts to coupling constants.   We shall discover that coupling constants are not applicable, and that when including infrared corrections to scattering amplitudes, it is not correct to say the residue is proportional to a product of two coupling constants.   We now examine this in more detail.

Consider the three processes

a) $\pi^0 p \to \pi^0 p$

b) $\pi^0 p \to \pi^+ n$

c) $\pi^+ n \to \pi^+ n$.

Then the $J = \frac{1}{2} +$ partial wave amplitude contains a pole term (Equation (9) modified for $J = \frac{1}{2} +$):

$$A\left(\pi_1 N_1 \to \pi_2 N_2 ; J = \tfrac{1}{2} +\right) = \frac{\left[R_0 + \Delta R + \frac{\alpha}{4} R_0 \int_{-1}^{+1} dx \, B\right]}{W - M} \quad + \cdots$$

We take the limit of low center of mass energies, so that the nucleon is almost stationary. Then the B function in the integrand will contain a possible infinite phase (which we do not consider in the following) and (aside from finite terms) a $\ln \lambda$ term arising from virtual photons connecting to external pion lines. We now consider this $\ln \lambda$ term. In process a) there is no such term, since the π^0 is uncharged. Let us now make the assumption that residues are proportional to products of coupling constants. Then $g(\pi^0 p \to p)$ has no $\ln \lambda$ term. Now we look at process b); since $g(\pi^0 p \to p)$ has no $\ln \lambda$ term, the $\ln \lambda$ term in b) contributes only to $g(\pi^+ n \to p)$. Thus we have found the $\ln \lambda$ part of $g(\pi^+ n \to p)$. But if we use c) to obtain this $\ln \lambda$ part, we get a different answer. The reason is easy to see: the $\ln \lambda$ terms can come from a virtual photon connected to one external pion line or also from a virtual photon connecting (in case c)) the initial and final pion lines. It is this latter type of $\ln \lambda$ that causes the discrepancy between the $\ln \lambda$ pieces of $g(\pi^+ n \to p)$. What is happening is that a branch point (of a box diagram) overlaps the pole term in the scattering amplitude as $\lambda \to 0$. Thus we must conclude that when infrared contributions are included, it is not advisable to speak of coupling constants; residues do not factor in

this case.

We note that if residues of scattering amplitudes do not factor, then it is not clear whether one can equate the non-infrared pieces of residues in NN→NN amplitudes (or any other amplitudes) to non-infrared pieces of similar residues in $\pi N \to \pi N$ scattering. Equating such quantities would be straightforward only in the limit where corrections from infrared photons are negligible.

Armed with the above knowledge, we now look at experimental results to see if any precise comparison with our predictions can be made.

XVI. <u>Comparison</u> <u>With</u> <u>Experimental</u> <u>Data</u>

Precise experimental information on π-N residue shifts is scarce. The experimental error bars on direct determinations of residues are large, and no definite conclusions can be reached. Other (theoretical) work on charge independence indicates that the residue shifts should be small.

We shall now proceed with a more detailed examination of the experimental and theoretical work that has been done under the assumption that infrared corrections can be neglected. Under this assumption it is permissible to deal with coupling constants; first we look at the data concerning $g^2_{\pi NN}$.

There are several ways which in principle could be used to obtain g^2 from experiment. The most accurate method has been to use dispersion relations together with pion-nucleon scattering results;[54,55]   this yields a value of $f^2 = .081 \pm .0018$.

Another way of determining g^2 is through the use of N-N scattering data; by varying g^2 a best fit for pp phase shifts is obtained. The value of g^2 determined in this way from pp and np data agrees with the value found by using the pp data alone (the np data by itself would not be complete).[56]   The value $g^2 = 13.8 \pm 1.9$ is

obtained.[56]   In an article by Seamon et al it is shown (using a model potential) that inclusion of electrostatic effects in the analysis could raise this value of g^2 by about 5%.[57]

One can also determine g^2 by multiplying $\frac{d\sigma}{d\Omega}$ by the pole denominator and obtaining a best fit to N-N scattering data. Such analyses yield a value of $g^2 = 14.7 \pm .9$.[58]

One could use photoproduction data to obtain f^2 (by extrapolating to the pole), but Höhler and Schmidt[43] point out that the errors are so large that it is better to treat the pole as a measured point.[43,59]

In summary, then, there are several ways to obtain g^2 from experiment, but the error bars are fairly large. One cannot, thus, yet determine any δg's from experiment. Our results are consistent with the experimental situation in that they predict only small violations of SU(2) symmetry; very large effects would contradict the experimental evidence.

We may next ask about the relevance of a prediction of π-N residue shifts to other work on charge independence in nuclear physics. Evidence for charge independence of the nucleon-nucleon interaction can be obtained in several ways. Wilkinson[60] has examined properties of mirror

nuclei to conclude that charge symmetry of the N-N inter-
action holds quite well; the same author has also studied
energy levels in light nuclei to conclude [60,61] that
complete charge independence probably holds to within
one percent.

More direct evidence on charge independence comes
from the $^1S_0$ nucleon-nucleon scattering lengths.
Experimentally these scattering lengths are: [62-65]

$$a_{nn} = -16.4 + 1.9 \text{ fm}$$
$$a_{np} = -23.678 + 0.028 \text{ fm}$$
$$a_{pp} = -7.815 + 0.008 \text{ fm} \quad (\cong -17 \text{ fm for the}$$
$$\text{nuclear part).}$$

The nuclear part of these quantities would be equal if
charge independence were exactly true, and various
calculations have been performed to explain the observed
discrepancy between a_{np} and a_{nn}. Heller et al[66] con-
clude that this discrepancy can probably be explained in
several ways, one of which is to assume $g_{\pm}^2 / g_0^2 = 1.035$
and to use this in the corresponding one-pion exchange
potential ($g_{\pm}$ is the coupling to charged pions, and g_0
is the coupling to neutral pions; these quantities would
be equal in magnitude under exact charge independence).
Moravcsik[67] points out that the scattering lengths are

very dependent on the potential. Schneider and Thaler[68] include form factors and pion mass shifts; they conclude that charge independence is not contradicted. Noyes[69] uses pion mass shifts and finds that $g^2_{\pm p} / g^2_{o\ p}$ 14.636/14.002 would explain a_{np}, but he notes that other effects (in addition to δg^2) could be of equal importance, and thus one cannot reach any definite conclusions concerning δg^2. Henley and Morrison[65] include pion mass splittings and find they can explain a_{np} in several ways, one of which is to assume $|g_+/g_o| \cong .98$. Goldberg[13] uses the Dashen-Frautschi method to calculate $a_{nn} - a_{np}$ and finds that nucleon and pion mass shifts, together with πNN coupling shifts, give a prediction of $a_{nn} - a_{np}$ in qualitative agreement with experiment.

A third way to obtain information on charge independence comes from β decay. Blin-Stoyle et al[70] conclude that if one assumes a conserved vector current, then the experimental results are consistent with a few percent of charge independence violation.

Thus we see that the observed deviations from charge independence are not larger than several percent.[71] Coupling constant shifts of this order of magnitude could, according to some of the calculations, explain the difference between a_{nn} and a_{np}, but other effects could also account

for the difference. Thus we are unable to reach any
definite conclusions about the splitting of N-N residues
needed to explain deviations from charge independence in
nuclear physics. The only thing one can say is that the
effects are small, and this is indeed consistent with our
prediction of small residue shifts. Thus, with respect
to residue shifts for $J = \frac{1}{2}+$, we must conclude that while
our residue shifts are consistent with experimental results,
there is no measurement available for precise numerical
comparison with our predictions.

Having thus examined the situation when $J = \frac{1}{2}+$, we
now turn to experimental evidence on π-N residue shifts
with $J = \frac{3}{2}+$. The corresponding residue shifts are
directly related to total cross sections in the manner
described at the end of Sec. III. As a complication we
note that the N* resonance experimentally does not have
a pure Breit-Wigner shape. Thus even when neglecting
any infrared corrections (which one should include, however,
since the theoretical residue shifts ΔR are of the same
order of magnitude), the usual resonance formula

$$\sigma \cong \frac{8\pi}{q^2} \left| \frac{-\Gamma/2}{W - \mathrm{Re}\, M^* + i\Gamma/2} \right|^2 \qquad (66)$$

is not adequate when considering effects of SU(2)
symmetry breaking. This can easily be investigated
mathematically for the sample case of $\pi^+ p \to \pi^+ p$ (where $I_i =$
$I_f = \frac{3}{2} +$). We write the $J = \frac{3}{2}^+$ partial wave amplitude near $W = M^*$

$$\frac{e^{2i\eta} - 1}{2i\rho} \simeq \frac{R\left(\frac{3}{2} \frac{3}{2} \to \frac{3}{2} \frac{3}{2}; J = \frac{3}{2} +\right)}{W - \text{Re} M^* + i\Gamma/2} + \delta B, \qquad (W \simeq M^*)$$

where η is the phase shift, and δB is a small correction
term. Expanding about the unperturbed SU(2) values at $W =$
ReM* , we have

$$\frac{-\left(e^{-2 \mathscr{I}m \eta} e^{2i\left(\eta - \frac{\pi}{2}\right)} + 1\right)}{2i\left(\rho_0 + \delta\rho\right)} = \frac{R_0 + \delta R}{\frac{1}{2}\left(\Gamma_0 + \delta\Gamma\right)} + \delta B.$$

Collecting zero order and first order terms,

$$-\Gamma_0/2 = \rho_0 R_0$$

$$-\delta\Gamma/2 = \delta\left(\rho R\right) + \frac{\rho_0 \Gamma_0 i \delta B}{2} + \left[-\mathscr{I}m\eta + i\left(\eta - \frac{\pi}{2}\right)\right]\frac{\Gamma_0}{2} . \qquad (67)$$

Since $(\Gamma_0/2)\text{Im}\,\eta$ is of the same order as $\rho\delta R$,[72] for example,
we cannot conclude that $\frac{-\delta\Gamma}{2} = \delta(\rho R)$, and hence it is not
correct to set $-\Gamma/2 = \rho R$ in the numerator of Equation (66).

Analysis of $\pi^\pm p$ data [51] using the effective
assumption that residue shifts are large compared to terms
like $\text{Im}\,\eta$ and neglecting infrared corrections, has

yielded a value of

$$\Gamma(\tfrac{3}{2}\ \tfrac{3}{2}) \ - \ \Gamma(\tfrac{3}{2}\ -\tfrac{1}{2}) \ = \ .4 \pm 3.1 \text{ Mev}$$

$$m(\tfrac{3}{2}\ \tfrac{3}{2}) \ - \ m(\tfrac{3}{2}\ -\tfrac{1}{2}) \ = -.45 \pm .85 \text{ Mev}.$$

Study of the inelastic processes $nn \rightarrow np\pi^-$ and $pp \rightarrow np\pi^+$ has given[52]

$$\Gamma(\tfrac{3}{2}\ -\tfrac{3}{2}) \ - \ \Gamma(\tfrac{3}{2}\ \tfrac{3}{2}) \ = 25 \pm 23 \text{ Mev}$$

$$m(\tfrac{3}{2}-\tfrac{3}{2}) \ - \ m(\tfrac{3}{2}\ \tfrac{3}{2}) \ = \ 7.9 \pm 6.8 \text{ Mev}.$$

Both of these results are consistent with small residue shifts, but the error bars are too large to permit numerical comparison with the theory.

We thus see that our predictions of small residue shifts for $J = \tfrac{3}{2} +$ are consistent with the experimental situation. For any further comparison between theory and experiment, however, available data would have to be analyzed taking into account very small correction terms, and the error bars would have to be smaller. Additional experiments which could give information on N^* parameters are[51,73] π° photoproduction (yielding information on the N^{*+}) and pion scattering on neutrons.

XVII. Comparison With Related Analyses

In this section we compare our calculation with related work by Dashen et al.[17,19] The calculations by these authors have also included predictions for πNN and πNN^* coupling shifts. These predictions were obtained[17,19] by calculating an SU(3) symmetric A^{RR} matrix and also evaluating mass shift contributions to residue shifts. A double enhancement of the octet eigenvector with eigenvalue nearest one was found, and this eigenvector was then used (together with experimental baryon mass splittings) to predict the coupling shifts. This calculation differs from ours in that it predicts only the I=1 part of the coupling shifts and also does not calculate driving terms other than mass shift contributions. Dashen et al obtain I=1 πNN $\delta g/g$'s which are very small (of the order of $.1\alpha$ or less) and the estimate $\delta\Gamma(N^{*-} - N^{*++}) \cong 1$ Mev. Our $\delta R/R_o$'s are roughly of order α ; this result agrees with the $\delta\Gamma$ estimate but is larger than the $\delta g/g\big|_{\pi NN}$ results of Dashen et al.

Table I.   Linear combinations of residue shifts

used in the analysis.

$$\delta R_{I=0 \atop 1}(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+) = \sqrt{\tfrac{1}{2}}\left[\, \delta R(\tfrac{1}{2}\tfrac{1}{2} \to \tfrac{1}{2}\tfrac{1}{2}; J=\tfrac{1}{2}+) \pm \right.$$
$$\left. \delta R(\tfrac{1}{2}\,\tfrac{-1}{2} \to \tfrac{1}{2}\,\tfrac{-1}{2}; J=\tfrac{1}{2}+)\right]$$

$$\delta R_{I=0 \atop 2}(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+) = \sqrt{\tfrac{1}{4}}\left[\, \delta R(\tfrac{3}{2}\tfrac{3}{2} \to \tfrac{3}{2}\tfrac{3}{2}; J=\tfrac{3}{2}+) \pm \right.$$
$$\delta R(\tfrac{3}{2}\,\tfrac{1}{2} \to \tfrac{3}{2}\,\tfrac{1}{2}; J=\tfrac{3}{2}+) \pm \delta R(\tfrac{3}{2}\,\tfrac{-1}{2} \to \tfrac{3}{2}\,\tfrac{-1}{2}; J=\tfrac{3}{2}+)$$
$$\left. + \delta R(\tfrac{3}{2}\,\tfrac{-3}{2} \to \tfrac{3}{2}\,\tfrac{-3}{2}; J=\tfrac{3}{2}+)\right]$$

$$\delta R_{I=1}(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+) = \sqrt{\tfrac{1}{20}}\left[3\delta R(\tfrac{3}{2}\tfrac{3}{2} \to \tfrac{3}{2}\tfrac{3}{2}; J=\tfrac{3}{2}+) + \right.$$
$$\delta R(\tfrac{3}{2}\tfrac{1}{2} \to \tfrac{3}{2}\,\tfrac{1}{2}; J=\tfrac{3}{2}+) - \delta R(\tfrac{3}{2}\,\tfrac{-1}{2} \to \tfrac{3}{2}\,\tfrac{-1}{2}; J=\tfrac{3}{2}+)$$
$$\left. -3\delta R(\tfrac{3}{2}\,\tfrac{-3}{2} \to \tfrac{3}{2}\,\tfrac{-3}{2}; J=\tfrac{3}{2}+)\right]$$

$$\delta R_{I=\tfrac{1}{2}}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+) = -\sqrt{\tfrac{1}{2}}\left[\, \delta R(\tfrac{3}{2}\tfrac{1}{2} \to \tfrac{1}{2}\tfrac{1}{2}; J=\tfrac{1}{2}+) \pm \right.$$
$$\left. \delta R(\tfrac{3}{2}\,\tfrac{-1}{2} \to \tfrac{1}{2}\,\tfrac{-1}{2}; J=\tfrac{1}{2}+)\right]$$

$$\delta R_{I=\tfrac{1}{2}}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{3}{2}+) = \sqrt{\tfrac{1}{2}}\left[\, \delta R(\tfrac{3}{2}\tfrac{1}{2} \to \tfrac{1}{2}\tfrac{1}{2}; J=\tfrac{3}{2}+) + \right.$$
$$\left. \delta R(\tfrac{3}{2}\,\tfrac{-1}{2} \to \tfrac{1}{2}\,\tfrac{-1}{2}; J=\tfrac{3}{2}+)\right]$$

Table II.   The matrix A^{RR}.

a) Linear D functions

b) Curved D functions

a)

Column headers (left to right):

$$\begin{pmatrix} I=0 \\ \tfrac12\to\tfrac12 \\ \tfrac12+ \end{pmatrix} \begin{pmatrix} I=0 \\ \tfrac32\to\tfrac32 \\ \tfrac32+ \end{pmatrix} \begin{pmatrix} I=1 \\ \tfrac12\to\tfrac12 \\ \tfrac12+ \end{pmatrix} \begin{pmatrix} I=1 \\ \tfrac32\to\tfrac12 \\ \tfrac12+ \end{pmatrix} \begin{pmatrix} I=1 \\ \tfrac32\to\tfrac12 \\ \tfrac12+ \end{pmatrix} \begin{pmatrix} I=1 \\ \tfrac32\to\tfrac32 \\ \tfrac32+ \end{pmatrix} \begin{pmatrix} I=2 \\ \tfrac32\to\tfrac12 \\ \tfrac12+ \end{pmatrix} \begin{pmatrix} I=2 \\ \tfrac32\to\tfrac12 \\ \tfrac32+ \end{pmatrix} \begin{pmatrix} I=2 \\ \tfrac32\to\tfrac32 \\ \tfrac32+ \end{pmatrix}$$

Matrix a) (circles denote zero blocks):

- Column 1: $\tfrac19$, $\tfrac{4\sqrt2}{9}$
- Column 2: $\tfrac{8\sqrt2}{9}$, $\tfrac19$
- Central 4×4 block (columns 3–6):

$$\begin{pmatrix} -\tfrac{5}{27} & -\tfrac{4\sqrt2}{27} & -\tfrac{16\sqrt2}{27} & -\tfrac{4\sqrt{40}}{27} \\ -\tfrac{2\sqrt2}{27} & -\tfrac{5}{27} & -\tfrac{20}{27} & \tfrac{4\sqrt{20}}{27} \\ -\tfrac{4\sqrt2}{27} & -\tfrac{10}{27} & \tfrac{5}{27} & -\tfrac{\sqrt{20}}{27} \\ -\tfrac{2\sqrt{40}}{27} & \tfrac{4\sqrt{20}}{27} & -\tfrac{2\sqrt{20}}{27} & -\tfrac{1}{27} \end{pmatrix}$$

- Bottom-right 3×3 block (columns 7–9):

$$\begin{pmatrix} -\tfrac49 & -\tfrac49 & -\tfrac89 \\ -\tfrac29 & \tfrac19 & \tfrac29 \\ -\tfrac89 & \tfrac49 & -\tfrac19 \end{pmatrix}$$

b)

Matrix b) (circles denote zero blocks):

- Column 1: $\tfrac19$, $\tfrac{5\sqrt2}{12}$
- Column 2: $\tfrac{64\sqrt2}{75}$, $\tfrac{7}{75}$
- Central 4×4 block (columns 3–6):

$$\begin{pmatrix} -\tfrac{5}{27} & -\tfrac{4\sqrt2}{27} & -\tfrac{128\sqrt2}{225} & -\tfrac{32\sqrt{40}}{225} \\ -\tfrac{2\sqrt2}{27} & -\tfrac{5}{27} & -\tfrac{16}{27} & \tfrac{16\sqrt{20}}{135} \\ -\tfrac{\sqrt2}{9} & -\tfrac{5}{18} & \pm\tfrac19 & -\tfrac{\sqrt{20}}{45} \\ -\tfrac{5\sqrt{40}}{72} & \tfrac{5\sqrt2}{36} & -\tfrac{14\sqrt{20}}{225} & -\tfrac{7}{225} \end{pmatrix}$$

- Bottom-right 3×3 block (columns 7–9):

$$\begin{pmatrix} -\tfrac14 & -\tfrac{16}{45} & -\tfrac{32}{45} \\ -\tfrac16 & \tfrac{1}{15} & \tfrac{2}{15} \\ -\tfrac56 & \tfrac{28}{75} & -\tfrac{7}{75} \end{pmatrix}$$

<u>Table</u> <u>III.</u> Eigenvalues λ_i, eigenvectors $\vec{v}_i$, and reciprocal eigenvectors (defined in the text) $\vec{w}_i$ of A^{RR}. ($\vec{v}_i \cdot \vec{v}_i = 1.$)

a) Linear D functions

b) Curved D functions

a)

$I=0$ $\lambda_1 = 1$ $v_1 = \sqrt{\tfrac{1}{3}}\,(\sqrt{2},1)$ $w_1 = \tfrac{\sqrt{3}}{2}(\sqrt{\tfrac{1}{2}},1)$

   $\lambda_2 = -\tfrac{7}{9}$ $v_2 = \sqrt{\tfrac{1}{3}}\,(\sqrt{2},-1)$ $w_2 = \tfrac{\sqrt{3}}{2}(\sqrt{\tfrac{1}{2}},-1)$

$I=1$ $\lambda_1 = 1$ $v_1 = \sqrt{\tfrac{1}{3}}(\sqrt{\tfrac{1}{2}},-1,\tfrac{1}{2},\tfrac{-\sqrt{5}}{2})$ $w_1 = \sqrt{\tfrac{1}{3}}\,(\tfrac{1}{2}\sqrt{\tfrac{1}{2}},-1,1,\tfrac{-\sqrt{5}}{2})$

   $\lambda_2 = \tfrac{5}{7}$ $v_2 = \tfrac{1}{6\sqrt{2}}(5\sqrt{2},1,-4,-\sqrt{5})$ $w_2 = \tfrac{\sqrt{2}}{21}(5\sqrt{2},2,-16,-2\sqrt{5})$

   $\lambda_3 = -\tfrac{7}{7}$ $v_3 = \tfrac{\sqrt{5}}{6}(\sqrt{2},2,1,-\sqrt{\tfrac{1}{5}})$ $w_3 = \tfrac{\sqrt{5}}{12}(\sqrt{2},4,4,\tfrac{-2}{\sqrt{5}})$

   $\lambda_4 = -1$ $v_4 = \sqrt{\tfrac{1}{57}}(4\sqrt{2},-2,1,2\sqrt{5})$ $w_4 = \tfrac{\sqrt{19}}{7\sqrt{3}}(\sqrt{2},-1,1,\sqrt{5})$

$I=2$ $\lambda_1 = 1$. $v_1 = \tfrac{1}{3}(2,-1,-2)$ $w_1 = \tfrac{3}{5}(1,-1,-1)$

   $\lambda_2 = \tfrac{-1}{9}$ $v_2 = \sqrt{\tfrac{1}{6}}(1,2,-1)$ $w_2 = \tfrac{\sqrt{6}}{10}(1,4,-1)$

   $\lambda_3 = -1$ $v_3 = \sqrt{\tfrac{1}{2}}(1,0,1)$ $w_3 = \sqrt{\tfrac{1}{2}}\,(1,0,1)$

b)

$I=0$. $\lambda_1 = .95$ $v_1 = (.82,.57)$ $w_1 = (.61,.87)$

   $\lambda_2 = -.74$ $v_2 = (.82,-.58)$ $w_2 = (.61,-.88)$

$I=1$ $\lambda_1 = .85$ $v_1 = (-.55,.48,-.15,.66)$ $w_1 = (-.39,.61,-.43,.72)$

   $\lambda_2 = .45$ $v_2 = (.81,.22,-.52,-.16)$ $w_2 = (.37,.26,-1.20,-.16)$

   $\lambda_3 = -.67$ $v_3 = (.55,.73,.35,-.18)$ $w_3 = (.23,.78,.76,-.19)$

   $\lambda_4 = -.91$ $v_4 = (.78,-.21,.12,.57)$ $w_4 = (.52,-.39,.41,.81)$

$I=2$ $\lambda_1 = .81$ $v_1 = (-.65,.27,.71)$ $w_1 = (-.64,.60,.59)$

   $\lambda_2 = -.08$ $v_2 = (.38,.82,-.43)$ $w_2 = (.20,1.02,-.21)$

   $\lambda_3 = -.87$ $v_3 = (.69,.02,.73)$ $w_3 = (.74,.01,.68)$

Table IV.   Contributions of exchanged masses to
residue shifts.

a) Linear D functions

b) Curved D functions

a) All exchanged mass contributions equal zero.

b)	A_I^{R,M^*exch}	$A_I^{R,M^{exch}}$
$\delta R_{I=0}(\frac{1}{2} \to \frac{1}{2} ; J=\frac{1}{2}+)$	1.79α	0.00α
$\delta R_{I=0}(\frac{3}{2} \to \frac{3}{2} ; J=\frac{3}{2}+)$	$.24\alpha$	1.96α
$\delta R_{I=1}(\frac{1}{2} \to \frac{1}{2} ; J=\frac{1}{2}+)$	-1.33α	0.00α
$\delta R_{I=1}(\frac{3}{2} \to \frac{1}{2} ; J=\frac{1}{2}+)$	2.36α	$-.87\alpha$
$\delta R_{I=1}(\frac{3}{2} \to \frac{1}{2} ; J=\frac{3}{2}+)$	$-.44\alpha$	-1.31α
$\delta R_{I=1}(\frac{3}{2} \to \frac{3}{2} ; J=\frac{3}{2}+)$	$-.08\alpha$	-1.46α
$\delta R_{I=2}(\frac{1}{2} \to \frac{1}{2} ; J=\frac{1}{2}+)$	-3.16α	$---$
$\delta R_{I=2}(\frac{3}{2} \to \frac{1}{2} ; J=\frac{3}{2}+)$	$.59\alpha$	$---$
$\delta R_{I=2}(\frac{3}{2} \to \frac{3}{2} ; J=\frac{3}{2}+)$	$-.24\alpha$	$---$

Table V.   Contributions of external mass shifts to

residue shifts.

a) Linear D functions

b) Curved D functions

a) All external mass contributions equal zero.

b)

	$_A R, M^{ext}_I$	$_A R, \mu^{ext}_I$
$\delta R_{I=0} (\tfrac{1}{2} \to \tfrac{1}{2} ; J=\tfrac{1}{2}+)$	$-2.53\,\alpha$	$0\ \alpha$
$\delta R_{I=0} (\tfrac{3}{2} \to \tfrac{3}{2} ; J=\tfrac{3}{2}+)$	$-2.30\ \alpha$	$0\ \alpha$
$\delta R_{I=1} (\tfrac{1}{2} \to \tfrac{1}{2} ; J=\tfrac{1}{2}+)$	$.84\ \alpha$	$---$
$\delta R_{I=1} (\tfrac{3}{2} \to \tfrac{1}{2} ; J=\tfrac{1}{2}+)$	$-4.70\ \alpha$	$---$
$\delta R_{I=1} (\tfrac{3}{2} \to \tfrac{1}{2} ; J=\tfrac{3}{2}+)$	$2.36\ \alpha$	$---$
$\delta R_{I=1} (\tfrac{3}{2} \to \tfrac{3}{2} ; J=\tfrac{3}{2}+)$	$-1.71\ \alpha$	$---$
$\delta R_{I=2} (\tfrac{3}{2} \to \tfrac{1}{2} ; J=\tfrac{1}{2}+)$	$---$	$0\ \alpha$
$\delta R_{I=2} (\tfrac{3}{2} \to \tfrac{1}{2} ; J=\tfrac{3}{2}+)$	$---$	$0\ \alpha$
$\delta R_{I=2} (\tfrac{3}{2} \to \tfrac{3}{2} ; J=\tfrac{3}{2}+)$	$---$	$0\ \alpha$

Table VI.   Contributions from one-photon exchange.

a) Linear D functions

b) Curved D functions

	a)	b)
$\delta R_{I=0} \left(\frac{1}{2} \to \frac{1}{2}; J=\frac{1}{2}+\right)$	$\frac{2\sqrt{2}}{3} \alpha \left(S_0 - \frac{7}{8} Q_0\right)$	$\alpha\left(-.93 \ln\frac{M}{\lambda} + .76\right)$
$\delta R_{I=0} \left(\frac{3}{2} \to \frac{3}{2}; J=\frac{3}{2}+\right)$	$-\frac{2}{3}\alpha \left(\;"\;\right)$	$-\alpha\left(-.97 \ln\frac{M}{\lambda} + .74\right)$
$\delta R_{I=1}\left(\frac{1}{2} \to \frac{1}{2}; J=\frac{1}{2}+\right)$	$-\frac{2\sqrt{2}}{3}\alpha \left(P_0\right)$	$-\alpha\left(-.64 \ln\frac{M}{\lambda}+.28\right)$
$\delta R_{I=1}\left(\frac{3}{2} \to \frac{1}{2}; J=\frac{1}{2}+\right)$	$\frac{2}{3}\alpha \left(\;"\;\right)$	$\alpha\left(-1.43 \ln\frac{M}{\lambda}+1.31\right)$
$\delta R_{I=1}\left(\frac{3}{2} \to \frac{1}{2}; J=\frac{3}{2}+\right)$	$-\frac{2}{3}\alpha \left(\;"\;\right)$	$-\alpha\left(-1.07 \ln\frac{M}{\lambda}+1.08\right)$
$\delta R_{I=1}\left(\frac{3}{2} \to \frac{3}{2}; J=\frac{3}{2}+\right)$	$-\frac{2\sqrt{2}}{3}\alpha\left(\;"\;\right)$	$-\alpha\left(-1.77 \ln\frac{M}{\lambda}+1.39\right)$
$\delta R_{I=2}\left(\frac{3}{2} \to \frac{1}{2}; J=\frac{1}{2}+\right)$	$-\frac{2}{3}\alpha \left(S_0 - \frac{7}{8} Q_0\right)$	$-\alpha\left(-1.58 \ln\frac{M}{\lambda}+1.09\right)$
$\delta R_{I=2}\left(\frac{3}{2} \to \frac{1}{2}; J=\frac{3}{2}+\right)$	$\frac{2}{3}\alpha \left(\;"\;\right)$	$\alpha\left(-1.18 \ln\frac{M}{\lambda}+1.31\right)$
$\delta R_{I=2}\left(\frac{3}{2} \to \frac{3}{2}; J=\frac{3}{2}+\right)$	$-\frac{4}{3}\alpha \left(\;"\;\right)$	$-\alpha\left(-1.93 \ln\frac{M}{\lambda}+1.48\right)$

$$S_0 = 4.70\, P_0, \quad Q_0 = 3.70\, P_0, \quad P_0 = m_\rho^2 m_1^2 \left[A\ln\lambda + B\ln m_\rho + C\ln m_1\right]$$

$$\left(A = \left(m_\rho^2 m_1^2\right)^{-1}, \; B = \left(m_\rho^2 (m_\rho^2 - m_1^2)\right)^{-1}, \; C = \left(m_1^2(m_1^2 - m_\rho^2)\right)^{-1}\right),$$

$$\text{thus}\quad P_0 = \left[-\ln\frac{M}{\lambda} + .82\right].$$

Table VII.   Contributions due to the existence of the γN direct-channel intermediate state.

$\delta R_{I=0}\left(\frac{1}{2}\to\frac{1}{2};J=\frac{1}{2}\right)$ $.72\,\alpha$

$\delta R_{I=0}\left(\frac{3}{2}\to\frac{3}{2};J=\frac{3}{2}\right)$ $3.35\,\alpha$

$\delta R_{I=1}\left(\frac{1}{2}\to\frac{1}{2};J=\frac{1}{2}\right)$ $-.35\,\alpha$

$\delta R_{I=1}\left(\frac{3}{2}\to\frac{1}{2};J=\frac{1}{2}\right)$ $.13\,\alpha$

$\delta R_{I=1}\left(\frac{3}{2}\to\frac{1}{2};J=\frac{3}{2}\right)$ $-.22\,\alpha$

$\delta R_{I=1}\left(\frac{3}{2}\to\frac{3}{2};J=\frac{3}{2}\right)$ $0.00\,\alpha$

$\delta R_{I=2}\left(\frac{3}{2}\to\frac{1}{2};J=\frac{1}{2}\right)$ $-.51\,\alpha$

$\delta R_{I=2}\left(\frac{3}{2}\to\frac{1}{2};J=\frac{3}{2}\right)$ $.66\,\alpha$

$\delta R_{I=2}\left(\frac{3}{2}\to\frac{3}{2};J=\frac{3}{2}\right)$ $-3.35\,\alpha$

<u>Table</u> <u>VIII.</u>   Residue shifts ΔR (neglecting N^* mass differences), obtained by evaluation of the N, N^*, γN, πN, and γ exchange diagrams and the πN and γN direct channel intermediate states.

a) Linear D functions

b) Curved D functions

	a)	b)
$\delta R_{I=0}(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+)$	$\sqrt{2}\, a_{I=0} + 2.2\alpha$	undetermined
$\delta R_{I=0}(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+)$	$a_{I=0} - .89\alpha$	"
$\delta R_{I=1}(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+)$	$-2.9\,\alpha$	-2.7α
$\delta R_{I=1}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+)$	$4.6\,\alpha$	$4.9\,\alpha$
$\delta R_{I=1}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{3}{2}+)$	$-2.9\,\alpha$	-2.9α
$\delta R_{I=1}(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+)$	4.0α	$4.1\,\alpha$
$\delta R_{I=2}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+)$	-2.5α	$-2.3\,\alpha$
$\delta R_{I=2}(\tfrac{3}{2} \to \tfrac{1}{2}; J=\tfrac{3}{2}+)$	$2.1\,\alpha$	$2.2\,\alpha$
$\delta R_{I=2}(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+)$	$-2.3\,\alpha$	-2.4α

$a_{I=0}$ = undetermined constant

$R_0(\tfrac{1}{2} \to \tfrac{1}{2}; J=\tfrac{1}{2}+) \cong -11.1$

$R_0(\tfrac{3}{2} \to \tfrac{3}{2}; J=\tfrac{3}{2}+) \cong -5.6$
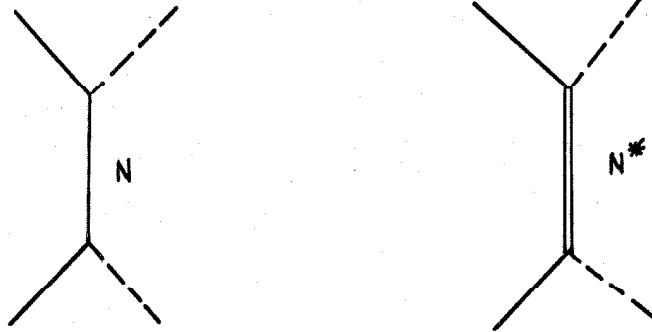
Figure 1   Pole terms in the direct channel
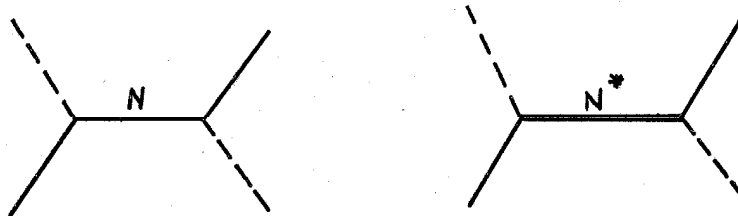

Figure 2   Diagrams with N and N* exchange
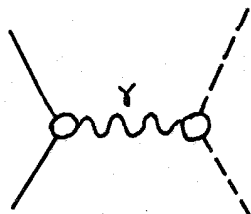
Figure 3   One-photon exchange
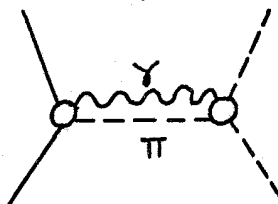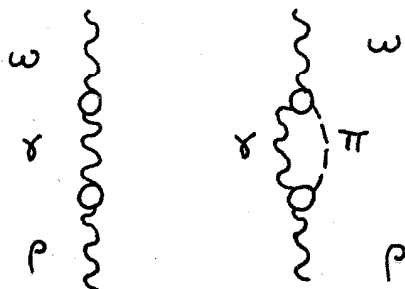

Figure 4   γπ exchange

Figure 5   Intermediate states in the t channel

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