ON THE DYNAMICAL DETERMINATION OF HIGH ENERGY SCATTERING CROSS SECTIONS

Thesis by

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DEDICATION

This thesis is dedicated, with esteem and affection, to my wife, Margot.

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ABSTRACT

This thesis deals with the problem of obtaining a quantitative understanding of high energy cross sections and angular distributions and their connection with low energy resonances exchanged in crossed channels. The work presented here takes as its starting point the conjecture that scattering amplitudes may be expressed as a sum of Regge poles.

In a general introduction we summarize the basic ideas and current status of the Regge conjecture. Here we also review briefly the main results of this work and try to cast them into perspective before plunging into details.

In Part II of the thesis a detailed analysis of NN and $N\bar{N}$ scattering on the basis of the Regge hypothesis is carried out. The Regge expansions of a set of ten invariant amplitudes describing NN-scattering are presented, with residues expressed in factorized form. Expressions involving both the full Legendre functions and their asymptotic forms are given. Spin sums are carried out to obtain simple and convenient expressions for the contributions of the P, ρ, ω, and P' trajectories to the differential cross sections. The optical theorem has been applied to find the contribution of the P, P', ρ, and ω trajectories to the spin-averaged total cross sections. Finally, the available data on the total and differential cross sections for NN-scattering has been analysed to extract information about the Regge pole parameters. The possible effect of the spin structure of the amplitudes and the variation with energy of the Legendre functions has been taken into account.

In Part III, we show that the analytic properties of the Regge parameters plus the unitarity condition satisfied by the partial wave amplitude lead to a set of coupled non-linear integral equations for the Regge pole parameters.

We then show that these equations can be written in a very simple form which makes many of their mathematical properties transparent and permits their numerical solution by iteration.

These equations have been solved numerically in several interesting cases. In the potential theory case, where our results could be compared with those obtained from the Schrödinger equation, the agreement was good in most cases.

In the relativistic case, we calculated the position of the Pomeranchuk trajectory, the ρ-meson trajectory and the second vacuum trajectory P'. Inelastic contributions were neglected. One notable result of this set of calculations is that the function $\mathrm{Re}\,\alpha(t)$ for the Pomeranchuk trajectory as determined by our equations agrees well with the results obtained by Foley et al. from an analysis of the $\pi^- p$ angular distributions in the range $-0.8\ (\mathrm{BeV/c})^2 < t < -0.2\ (\mathrm{BeV/c})^2$. No spin 2 resonance is found to lie on this trajectory. As for the ρ-trajectory, we find that $\alpha_\rho(t)$, $-0.8\ (\mathrm{BeV/c})^2 < t \leq 0$, is larger than 0.9 for a wide range of input parameters. The width of the ρ resonance, as determined by our equations, is several times larger than the experimental width. This probably means that inelastic contributions must be included to obtain a correct value for the width.

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Part I

General Introduction

This thesis has two parts. In the first part we express the cross sections for high energy nucleon-nucleon scattering in terms of Regge pole parameters (1). This is carried out using the prescription of Frautschi, Gell-Mann and Zachariasen (2, 3). In the second part a derivation and numerical solution of a set of equations which provide an approximate dynamical determination of the Regge pole parameters is given. Taken together, the techniques employed in the two parts of this thesis provide a means for the dynamical determination of elementary particle cross sections at high energies.

In order to cast the results to be presented here into perspective, we shall summarize the basic ideas and current status of the Regge conjecture in this Introduction.

In the past two years experiments have been made which establish the existence of a number of new states, which are unstable and in most cases have spin ≥ 1. These are frequently interpreted as resonant configurations of other particles. At such a time, it is especially important to have a method with which to discuss the properties of these composite states.

The basis for such a method was laid by Regge (1), who studied the asymptotic behavior of the scattering amplitude,as determined by the Schroedinger equation, when dynamical resonances and bound states were present. He was able to find the asymptotic form of these amplitudes for large momentum transfer, and to relate this behavior to the properties of composite states. This basic idea was soon applied to give a prescription (2, 3) for the large energy $(s \to \infty)$, low momentum transfer $(t \lesssim 0)$ behavior of two-body scattering amplitudes in field theory.

We shall formulate this prescription for the two-body process of figure 1.

One starts from the partial wave expansion in the t-channel of the scattering amplitude $A(\cos\theta_t, t)$;

$$A(\cos\theta_t, t) = \sum_{\ell=0}^{\infty} (2\ell + 1) A(\ell, t) P_\ell(\cos\theta_t), \tag{1}$$

where $t = (p_3 + p_1)^2$ = c.m. energy squared, θ_t = c.m. scattering angle in the t-channel and $\cos\theta_t = -1 + \frac{2s}{4m^2 - t}$. We wish to study the asymptotic form of $A(\cos\theta_t, t)$ for t fixed, $|\cos\theta_t|$ large. For this purpose we transform equation (1), which fails to converge for large $\cos\theta_t$, into the contour integral

$$A(\cos\theta_t, t) = \frac{1}{2\pi i} \oint d\ell (2\ell + 1) A(\ell, t) P_\ell(-\cos\theta_t) \frac{\pi}{\sin\pi\ell} . \tag{2}$$

ℓ is now complex, and $A(\ell, t)$ is the analytic continuation to complex ℓ of the usual partial wave amplitude. The contour surrounds the positive real axis.

Extrapolating to the relativistic case results proven by Regge in potential theory (for potentials expressible as a superposition of Yukawa potentials), we assume a) that we may distort the contour in equation (2) to the line from $-1/2 - i\infty$ to $-1/2 + i\infty$ and b) that in so doing we encounter, for $t >$ threshold $= t_0$, only simple poles

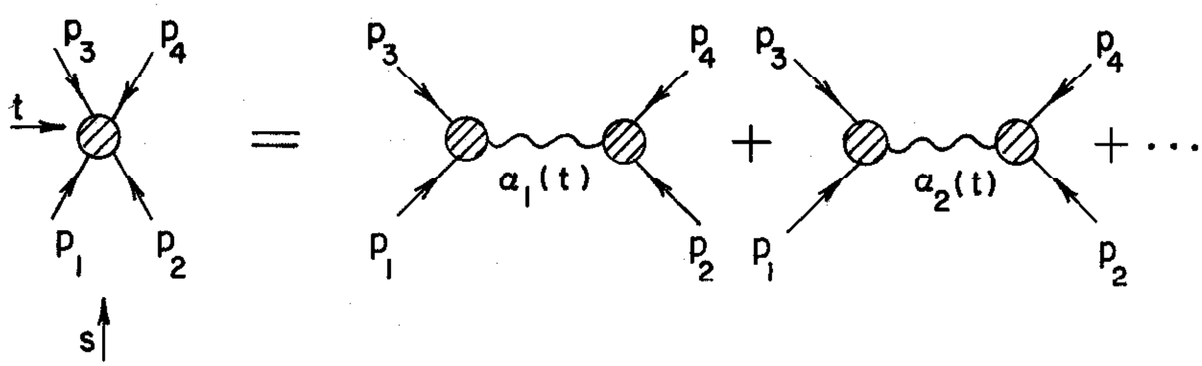

Figure 1. The scattering process $p_1 + p_2 \rightarrow p_3 + p_4$ (s-reaction) related to exchange of Regge particles in the process $p_1 + \overline{p}_3 \rightarrow \overline{p}_2 + p_4$ (t-reaction).

of $A(\ell, t)$ which correspond to dynamical resonances having the quantum numbers (other than J) of the t-channel. (This assumption will be discussed shortly.) We then obtain

$$A(\cos\theta_t, t) = \sum_n \beta_n(t)\, P_{\alpha_n(t)}(-\cos\theta_t)\, (\sin\pi\, \alpha_n(t))^{-1}$$

$$- \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} d\ell (2\ell + 1)\, A(\ell, t)\, P_\ell(-\cos\theta_t)\, \pi(\sin\pi\ell)^{-1} \quad (3)$$

$$\pm (-\cos\theta_t \rightarrow \cos\theta_t).$$

For $t <$ threshold, the poles of $A(\ell, t)$ are on the real ℓ-axis and correspond to bound states.

The Regge poles move in the ℓ-plane, their position is given by $\alpha(t)$ and their residue is related to $\beta(t)$. The additional terms in $P_\alpha(+\cos\theta_t)$ arise because in the relativistic case forces in the t-channel are generated by exchanges in both the s and u-channels. Each of these contributions can be treated separately and identically (2, 3).

We now consider t fixed and $> t_o$, and let $|\cos\theta_t| \rightarrow \infty$. In this asymptotic limit, only the pole terms in equation (3) remain because the line integral vanishes as $|\cos\theta_t|^{-1/2}$. Thus we have, asymptotically,

$$A(\cos\theta_t, t) \xrightarrow[|\cos\theta_t| \rightarrow \infty]{} \sum_n \beta_n(t)\, (\sin\pi\, \alpha_n)^{-1}\, \frac{1}{2} \left[P_{\alpha_n}(-\cos\theta_t) \pm P_{\alpha_n}(\cos\theta_t)\right]$$

$$\longrightarrow \sum_n \beta_n(t) \left[\frac{1 \pm e^{-i\pi\, \alpha_n(t)}}{2 \sin\pi\, \alpha_n(t)}\right] \left(\frac{2s}{4m^2 - t}\right)^{\alpha_n(t)} \quad (4)$$

The region $t > t_0$, $\cos\theta_t$ large is an unphysical one in the t-channel. However, we may analytically continue this representation without change (4) to the region $t \leq 0$, s large, which region corresponds to physical scattering in the s-channel.

The above prescription thus relates the high energy, low momentum transfer scattering in the s-channel to the exchange of composite objects (dynamical resonances and bound states) in the t-channel.

The Regge conjecture as just outlined was originally believed to have the following interesting consequences.

1) One can construct the contribution of an elementary particle of spin ℓ to a scattering amplitude $A(s, t)$. One finds near the pole

$$A(s, t) = [c\, P_\ell(\cos\theta_t) / t - t_R] + \cdots \quad . \tag{5}$$

In the unphysical region this gives an asymptotic behavior s^ℓ, ℓ a fixed integer. One can show in certain cases (2, 5) that this large s behavior results for all t, so that we can suppose that an elementary particle of spin ℓ exchanged in the t-channel will contribute a term $\sim s^\ell$ to the high energy, low momentum transfer scattering in the s-channel.

We have seen that a Regge pole contributes a term $\sim s^{\alpha(t)}$ to the asymptotic form of the amplitude. If this behavior is associated with the exchange of a composite object, as is usually supposed, and if the behavior s^ℓ is characteristic of the exchange of an elementary particle, then we may decide whether a state with given mass and quantum numbers is composite or elementary by comparing the form of the cross sections predicted in the two cases with experiment.

2) In the 10-25 Gev energy range, the πN and NN total cross sections appear to approach (different) constant values as the energy s increases. In addition, the πN and NN angular distributions show diffraction peaks in this energy range.

From the optical theorem, we know

$$\sigma_{tot}(s) \underset{s\to\infty}{\longrightarrow} (16\pi / s)\, \mathrm{Im}\, A(s, o). \qquad (6)$$

Combined with the Regge asymptotic form of the amplitude, this means (6) that $\alpha_i(o) \leq 1$, for each Regge trajectory i. This suggests (2, 7) the existence of a Regge trajectory with the quantum numbers of the vacuum and $\alpha(o) = 1$, the Pomeranchuk trajectory, which would result in a <u>constant</u> total cross section in the high energy limit. Moreover, exchange of any Regge particle leads, in the simplest cases, to angular distributions with the characteristic diffraction form

$$d\sigma / dt \underset{s\to\infty}{\longrightarrow} F(t)\, s^{2\alpha(t)-2} . \qquad (7)$$

The <u>amount</u> of diffraction shrinking depends critically on the slope of the trajectory for small, negative t.

3) It is a consequence of the Regge formalism that a set of resonances or bound states, all having the same quantum numbers including J-parity, but having different values of J and occuring at different energies, will all lie along the same Regge trajectory $\alpha(t)$. This leads to an interesting new principle for classifying the many newly observed resonances, and for correlating some of their properties (8, 9a).

In a more speculative vein, we might also add the following points.

4) The supposition that all the strongly interacting particles appear in the dispersion relations with the Regge asymptotic behavior seems to supply a criterion for the composite nature of these particles in terms of the computability of their masses and effective coupling strengths by means of the "bootstrap" principle (8, 9). Such calculations have always faced the difficulty that unitarity plus analyticity in themselves offered no clear prescription for the high energy behavior of the amplitudes. If one looked to perturbation theory for guidance, one almost always found a high energy behavior so divergent that subtractions had to be made in the dispersion relations, and consequently new undetermined parameters had to be introduced. If, however, the amplitudes have the Regge asymptotic behavior and if, for some t, Re $\alpha_m(t) < 0$ ($\alpha_m(t)$ being the position of the Regge pole which lies furthest to the right in the angular momentum plane), then the amplitude will converge as $s \to \infty$ at a rate which, according to Froissant (6), precludes any arbitrary subtractions in s. By analytic continuation in t, one finds that all subtraction terms are determined. In this sense, the assumed Regge asymptotic behavior for the amplitudes provides a set of boundary conditions which completes the S-matrix description of a two-body scattering process.

5) The usual perturbation theory applied to the exchange of particles of spin > 1 (or $= 1$ in the case of charged particles) is divergent in each order in such a way that it cannot be renormalized. This behavior results from a singular high energy behavior of the amplitudes in which the high spin particle is exchanged. If the

particle exchanged does not have the high energy behavior indicated by lowest order perturbation theory, but rather the Regge behavior, this divergence might be avoided, and the renormalization effects become finite corrections to be made in each order. It would then seem possible to apply perturbation theory to discuss the properties of vector mesons and particles of higher spin. Whether this technique would prove feasible for practical calculations depends on the size of the coupling constants involved and the accuracy with which one can use a knowledge of the Regge parameters to compute the renormalization effects. In any case, it should improve somewhat the logic behind the present perturbation theory, and possibly also be useful for discussing symmetries shared by the new resonances. Furthermore, one could apply these methods to particles with spin 0 or 1/2. Here one can renormalize by subtraction, but it would still be interesting to see, for example, if the electron self-mass becomes finite if the photon is a Regge particle.

When an interesting new idea like the Regge hypothesis appears on the scene, one important task is to work out its experimental consequences and test them as thoroughly as possible. Until very recently, available experimental data could be suitably compared to the predictions of the Regge theory only in the case of high energy nucleon-nucleon scattering. It seems, moreover, that more precise and varied experiments than are presently available can most feasibly be carried out on the nucleon-nucleon system. For these reasons a detailed Regge pole analysis of nucleon-nucleon scattering, including the full spin structure of the amplitudes, has been carried out in the first part of this thesis*. We have tried to explain the observed total cross sections and angular distributions by including the contributions of the Pomeranchuk trajectory and a few of the prominant resonances (ω, ρ, η, π).

Now I would like to comment on the current status of various aspects of the Regge theory.

Analysis of the data on total cross sections in pp, $p\bar{p}$ and np scattering at energies in the range 10-30 Gev yields the following main conclusions.

1) The existence of a Regge trajectory $\alpha_p(t)$ resulting in constant total cross sections is consistent with the data only if other trajectories make large contributions (ω-trajectory) to $p\bar{p}$ scattering and if there exists a singularity P' in the J-plane introduced explicitly to cancel the contribution of the ω-trajectory to the total pp cross sections. This means that in the energy range explored so far the Pomeranchuk trajectory dominates the cross sections only when the other contributions cancel out (pp cross section) and not because all the other contributions are small.

2) The data indicates that $\alpha_\omega(0) \approx 0.3$ and $\alpha_\rho(0) \approx 0.4$. This shows that the ρ and ω resonances do not contribute to the cross sections with the fixed spin behavior s^1 expected from lowest order perturbation theory, but in fact contribute like Regge particles.

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*Here only the general conclusions and results will be summarized. More detailed summaries, as well as comparisons with other treatments of the problem, will be found in the introductions to each separate part of the thesis.

3) An analysis of the data on the pp elastic angular distributions in the energy range 10-25 Gev shows that the observed diffraction peaking can be understood as a consequence of the existence of a dominant P-trajectory. It appears very difficult to get unambiguous information about the t-dependence of the parameters associated with the lower lying Regge trajectories on the basis of such analyses, because too many independent parameters are involved.

Were we to take such an analysis at face value, we might say that the main features of high energy nucleon-nucleon scattering can be consistently and usefully accounted for by the Regge theory, and that we have obtained an experimental verification of the "composite" nature of the ρ and ω resonances.

However, serious doubt has been cast on such conclusions by several recent results.

First, there is some reason to believe (10) that when multiparticle states are included in the analysis of relativistic scattering processes, the analyticity properties of the S-matrix in the J-plane will be complicated by the presence of cuts in addition to simple poles. If these cuts exist, and are important at low t, it is hard to see how any useful analysis of the data for the purpose of determining the Regge parameters could be carried out. It would thus be nearly impossible to get any reasonably clear cut experimental tests of the Regge predictions about total cross sections and diffraction peaks, or to ascertain from experiment whether the function $\alpha(t)$ associated with a given pole has the Regge behavior. Needless to say, the data analysis carried out in Part I of this thesis and in similar works would be almost totally invalidated, should such cuts be present.

Secondly, Gell-Mann and co-workers (11, 12a), investigating the high energy behavior of various renormalizable field theories, have found a most interesting result.

They consider elastic scattering of a neutral vector meson i) by a spin 1/2 nucleon and ii) by a spin 0 nucleon. The scattering problem is formulated using ordinary renormalized perturbation theory; both the vector meson and the nucleons are "elementary" and are represented in the Lagrangian by their own elementary fields. In lowest order, the nucleons (spin 1/2 or spin 0) appear as fixed singularities in the complex angular momentum plane. <u>However,</u> one finds when one computes to all orders the radiative corrections to this process due to the exchange of the vector meson that in case i) the nucleon pole acquires the Regge behavior in that the amplitude has the form:

$$b(\sqrt{u})\ s^{\alpha(\sqrt{u})} \ ; \quad \alpha(m_N) = 0 \tag{8}$$

while no such result obtains in case ii).

This result shows that the Regge asymptotic form is not <u>necessarily</u> indicative of the exchange of a particle of composite structure. In this case, the experimental test for the composite nature of an exchanged particle mentioned above loses its rationale, and we are again back to distinguishing a composite from an elementary object on the basis of some computability criterion, or perhaps on the basis of scattering phase shifts (Levinson's theorem).

In addition, this result seems to demolish what little understanding we had of the "physical" meaning of the Regge asymptotic behavior, i.e. the less singular behavior of the amplitudes is due to a natural cut-off being introduced because the exchanged particle, being composite, is extended in space.

Thirdly, recent data on πN angular distributions show little or no diffraction shrinkage. Some people have argued that an analysis of the pp angular distributions, properly restricted to the asymptotic regime, also indicates no diffraction shrinking. The situation appears too fluid to permit any clear evaluation at present. However, it should be noted that neither of the above results is <u>inconsistent</u> with the idea of a Pomeranchuk trajectory. Strictly speaking, the mere fact that an amplitude has an asymptotic form dominated by a sum of Regge poles makes <u>no</u> prediction whatever regarding the angular distributions, without some dynamical determination of the parameters and a knowledge of the properties of cuts in the J-plane.

These results, <u>if true</u>, do further support our previous conclusion that the analysis of the cross sections must be complicated by the presence of several trajectories contributing in an important way. Moreover, they strip every bit of positive experimental confirmation from the notion of a dominating Pomeranchuk trajectory, with the exception that the observed cross sections do appear to approach constants as predicted. Even this evidence is marred by the circumstance that the $p\bar{p}$ cross sections do not satisfy the Pomeranchuk theorems, at least at presently attainable energies.

Accepting the existence of the Pomeranchuk trajectory, it is possible that a spin 2 resonance, C, having the quantum numbers of the vacuum and mass $m_c^2 \sim 1$ (Gev)^2 may lie on this trajectory. Such a resonance should show up as a peak in $I = 0$ $\pi\pi$ scattering. Preliminary evidence for such a spin 2 resonance has been found. <u>If</u> it can be established that this resonance really lies on the Pomeranchuk trajectory, this would be a very important confirmation of the Regge theory for two reasons; i) the existence of this resonance would be a really qualitatively new effect following from the existence of the Pomeranchuk trajectory, ii) the prediction is a rather clean one in that the possible existence of Regge cuts should not lead to any ambiguities in experimentally establishing the existence of this spin 2 resonance.

For this reason the possibility of grouping the new resonances in Regge families, and of using this information to correlate the resonance parameters with observed total cross sections and angular distributions remains as an interesting application of the Regge theory. To make good use of this possibility, however, it seems essential to have a method to determine the Regge pole parameters dynamically. This circumstance, plus the realization that experimental cross sections probably cannot be usefully analysed to get information about the Regge parameters, provides the primary motivation for the work in the second part of this thesis. Here we use the principles of analyticity and unitarity to derive a set of singular, non-linear integral equations satisfied by the Regge parameters. These equations have been solved numerically* and applied to determine the Regge

*In solving these equations numerically, we have used an "on-line" computing center as developed by Drs. G. J. Culler and B. D. Fried of the Thompson Ramo Wooldridge Corporation. This system has proved to be of particular value in our problem in instances where it was extremely difficult to devise a convergent iteration scheme. For a detailed description of their computing facility, see reference (13).

parameters in some potential theory and relativistic problems.

In the case of scattering in a simple Yukawa potential, where it was possible to compare our results for the Regge parameters with those previously obtained by solving the Schröedinger equation, reasonably good agreement was found for a wide range of potential strengths.

In the relativistic case, solutions have been obtained for the positions $\alpha(t)$ of the Pomeranchuk and ρ-meson trajectories. Our result $\alpha_p'(t)$ for the Pomeranchuk trajectory agrees quite well with that recently obtained by Foley et al. (35) from an analysis of $\pi^- p$ angular distributions, but no spin 2 resonance is found to lie on this trajectory. We find a value of $\alpha_\rho(t)$, $-0.8\ (\text{BeV}/c)^2 < t \leq 0$ which seems to be consistent with experiment (52). How these results should be interpreted is somewhat unclear, and will no doubt remain so until we see if an understanding of high energy NN-scattering can also be obtained on the basis of these equations.

The potential applications of Regge ideas to bootstrap calculations and to ordinary field theory have not so far been exploited to any significant extent, with the exception of the work cited in reference (9b), (11) and (12). Therefore, no further comments on these ideas will be made here.

Part II

REGGE POLE ANALYSIS OF NUCLEON-NUCLEON SCATTERING

1. INTRODUCTION

In part II of this thesis we shall discuss nucleon-nucleon and nucleon-anti-nucleon scattering at high energies $(s \to \infty)$ and low momentum transfer $-s \ll t < 0$. It is in this regime of momentum and energy that the Regge pole hypothesis, in terms of which we shall treat NN and $N\bar{N}$ scattering, finds its most immediate application.

The general features of the nucleon-nucleon problem have already been discussed in terms of Regge poles (2). Simple expressions have been obtained for various differential cross sections on the basis of an analysis which ignored the spin structure of the amplitudes. Perhaps the most characteristic result of such a simple Regge pole analysis, which should also come out of any more detailed Regge analysis, is the prediction of a diffraction cross section which, as energies become arbitrarily large, and momentum transfers remain small, has the functional form

$$\left(\frac{d\sigma}{dt}\right)/\left(\frac{d\sigma}{dt}\right)_{t=0} = F(t)\ (s/s_0)^{2[\alpha(t)-1]} \quad . \qquad (1.1)$$

Recent data (14) on pp-scattering in the range $15 < s/2m_N^{\ 2} < 25$, $0 < -t/2m_N^{\ 2} < 3$ have been analyzed (15) in terms of Equation (1.1), with the important result that at least the most general features of the Regge hypothesis (as applied to nucleon scattering) seem to be consistent with experiment.*

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*For qualifications of this statement, see the General Introduction.

The nucleon-nucleon system is of intrinsic importance in elementary particle and nuclear physics. The complicated spin structure of the amplitudes means that there will be many independent physical quantities in the NN and $N\bar{N}$ system which can be expressed in terms of Regge poles. With these, more detailed and precise experimental consequences of the Regge hypothesis can be deduced, and their investigation will lead to correspondingly more stringent tests of the Regge hypothesis. In terms of experimental feasibility, the nucleon-nucleon system appears to be the most suitable for further detailed experimental verification of the Regge pole conjecture. For all these reasons, we feel that the nucleon-nucleon system merits a thorough treatment based on the Regge pole hypothesis, which is given in the following.

Consequently, we present in Section 2 the leading terms in the Regge expansions of a set of ten invariant amplitudes, which are free of kinematic singularities, describing NN and $N\bar{N}$ scattering. We discuss the possible transitions in NN and $N\bar{N}$ scattering between states of given parity, spin and isospin. These are conveniently summarized in terms of $\tau P[=(\text{signature})(\text{parity})]$ and $(-)^I GP$. The selection rules which result reduce the number of independent amplitudes describing the scattering which arises from a given Regge pole. Regge expansions for the helicity amplitudes are also obtained in this section.

The expansions we derive in this section are of interest regardless of whether the set of important singularities in the angular momentum plane consists of poles only or contains also cuts. However, the usefulness of the Regge asymptotic expansion for data analysis will be seriously impaired if cuts play a very significant role.

The functions $b_i(u)$ occurring in the Regge expansions are related to certain coupling strengths. In Section 3 we establish the precise relationships in a number of particular cases by comparing the Regge amplitude to the corresponding Feynman amplitude at the pole.

We should like to mention at this point that other discussions of the Regge expansions of the NN and $N\bar{N}$ amplitudes have also been carried out (3, 16-18), and some of the results of Section 2 of this thesis are contained in these papers. In particular, Gell-Mann (3) has presented his expressions for the amplitude in "factorized" form, as shall also be done in this paper. In addition, he has analyzed in a most interesting way the question of the presence of "ghosts" in these amplitudes. Muzinich (18), in his discussion of the Regge expansions of NN and $N\bar{N}$ amplitudes, considers a problem not discussed here; namely, he shows (on the basis of the Mandelstam representation) that the Froissart (6) analytic continuation of the partial-wave helicity amplitudes can be carried out for the NN problem, where the particles are spinors.

In Section 4 we discuss in detail the cross sections for NN and $N\bar{N}$ scattering. The contributions of the P, ρ, ω, and P' trajectories are all discussed. All spin sums are carried out explicitly.

In Section 5 we turn to an analysis of existing data on NN and $N\bar{N}$ scattering in terms of the Regge pole hypothesis. Our analysis is based on the data of Diddens et al (14, 15) and of Lindenbaum et al (19). We find that an analysis which includes the full variation of the Legendre functions with energy, as well as the spin structure of the amplitudes, does not change the basic conclusions (20, 21) of the Regge analysis of total cross sections. A second vacuum trajectory, introduced by K. Igi (22) is consistent with the data. However, because the $\sigma_{p\bar{p}}$ data (19) are so far from satisfying the Pomeranchuk theorem, and because the σ_{np} data, containing the Glauber correction, are so unreliable, the conclusions of such an analysis must be regarded as highly tentative.

The angular distributions have been expressed in terms of the Regge pole parameters. If only the Pomeranchuk trajectory is included, the differential cross sections can be expressed in terms of essentially one function, a result which becomes clear when the differential cross sections are expressed in terms of helicity amplitudes (23, 24). The available data have been used to determine this function; we find it has a linear behavior for $0 < -t \leq 0.40$ $(\text{GeV})^2$, and is a constant in this region if $s_0 = 1$ $(\text{GeV})^2$.

Throughout this section of the thesis, our emphasis has been on exploring the detailed experimental consequences of the Regge hypothesis as applied to the nucleon-nucleon system. It is hoped that this effort will instigate more elaborate experimental investigations, designed to test critically the predictions made here. We wish to check as thoroughly as possible by experiment whether this approach to elementary particle physics has a firm basis in the facts of nature.

2. REGGE EXPANSIONS FOR NUCLEON-NUCLEON SCATTERING AMPLITUDES

The Regge pole contributions to the amplitude may be deduced from the partial-wave expansion of the amplitude in the cross channel according to the prescription of Frautschi, Gell-Mann, and Zachariasen (2, 3). To obtain them, we may employ the matrices of Goldberger, Grisaru, MacDowell, and Wong (25) who have discussed the application of the Mandelstam representation to the NN problem. GGMW, and Amati, Leader, and Vitale (26) have shown that only if the NN scattering amplitude is expressed in terms of Fermi invariants are the associated invariant functions free of kinematic singularities.

In order to facilitate comparison with previous work, we shall adopt the notation introduced by GGMW. The nucleon-nucleon scattering amplitude is written as

$$T = \sum_I \beta^I \Big[F_S{}^I(s, u, t)\, S + F_T{}^I(s, u, t)\, T + F_A{}^I(s, u, t)\, A + F_V{}^I(s, u, t)\, V + F_P{}^I(s, u, t)\, P\Big] ,$$

where

$$S = \bar{u}(p_1')\, 1\, u(p_1)\, \bar{u}(p_2')\, 1\, u(p_2)$$

$$T = \tfrac{1}{2}\, \bar{u}(p_1')\, \sigma_{\mu\nu}\, u(p_1)\, \bar{u}(p_2')\, \sigma_{\mu\nu}\, u(p_2)$$

$$A = \bar{u}(p_1')\, i\, \gamma_5\, \gamma_\mu\, u(p_1)\, \bar{u}(p_2')\, i\, \gamma_5\, \gamma_\mu\, u(p_2)$$

$$V = \bar{u}(p_1')\, \gamma_\mu\, u(p_1)\, \bar{u}(p_2')\, \gamma_\mu\, u(p_2)$$

$$P = \bar{u}(p_1')\, \gamma_5\, u(p_1)\, \bar{u}(p_2')\, \gamma_5\, u(p_2) \quad , \tag{2.0a}$$

$$\beta^1 = \tfrac{1}{4}\Big[(\bar{s}_1, \vec{\tau}\, s_1)\cdot(\bar{s}_2, \vec{\tau}\, s_2) + 3(\bar{s}_1, s_1)(\bar{s}_2, s_2)\Big] , \tag{2.0b}$$

$$\beta^0 = \tfrac{1}{4}\Big[- (\bar{s}_1, \vec{\tau}\, s_1)\cdot(\bar{s}_2, \vec{\tau}\, s_2) + (\bar{s}_1, s_1)(\bar{s}_2, s_2)\Big] , \tag{2.0c}$$

and

$$s = -(p_1 + p_2)^2$$

$$u = -(p_2' - p_1)^2$$

$$t = -(p_1' - p_1)^2 \quad . \tag{2.0d}$$

In the above, the s_i represent isopinors.

The inclusion of the isotopic spin factors, which we usually drop for the sake of simplicity, is accomplished quite readily by making use of the matrix

$$(\Lambda_{II'}) = \frac{1}{2} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \quad , \tag{2.0e}$$

which relates the invariant functions F^I, considered as a two-component vector in the isotopic spin index, to the relevant functions in the t-channel with isospin I'. The first row and column refer to $I = 0$, and the second to $I = 1$.

Throughout this part of the thesis, we shall suppose that the Regge pole is in the t-channel. However, we may briefly indicate here how to pass from the t-channel to the u-channel, or vice versa $(t \rightleftharpoons u)$. This corresponds to interchanging p_1' and p_2'. The following changes are thereby produced: (i) the full amplitude changes sign; (ii) the spinors of the final particles are interchanged, $u(p_1') \rightleftharpoons u(p_2')$; (iii) in the C.M. system, the scattering angle changes from θ to $\pi - \theta$; (iv) finally, the isospin projection operator β^0 changes sign while β^1 does not. The matrix (2.0e) then becomes

$$(\Lambda_{II'}) = -\frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} . \qquad (2.0e')$$

It should be noticed that Equation (2.0e') already includes the sign change mentioned in (i) above.

The first part of the problem is to ascertain the contributions to the five invariant functions, F_S, F_T, F_A, F_V, and F_P, resulting from a Regge pole characterized by definite values of G, I, P and signature (τ). This may be expedited by employing some of the formulas of GGMW. (In the following kinematic considerations, we shall omit the isospin factor.) With the aid of Equation (2.6) of GGMW, we see that

$$\begin{pmatrix} \tilde{S} - S \\ \tilde{T} + T \\ \tilde{A} - A \\ \tilde{V} + V \\ \tilde{P} - P \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 & 1 & 1 & 1 & 1 \\ 6 & 2 & 0 & 0 & 6 \\ 4 & 0 & -6 & 2 & -4 \\ 4 & 0 & 2 & 2 & -4 \\ 1 & 1 & -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} S \\ T \\ A \\ V \\ P \end{pmatrix} , \qquad (2.1)$$

and, consequently,

$$\begin{pmatrix} F_S \\ F_T \\ F_A \\ F_V \\ F_P \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 & 6 & 4 & 4 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & -6 & 2 & -1 \\ 1 & 0 & 2 & 2 & -1 \\ 1 & 6 & -4 & -4 & -3 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} , \qquad (2.2)$$

where

$$T = F_1(\tilde{S} - S) + F_2(\tilde{T} + T) + F_3(\tilde{A} - A) + F_4(\tilde{V} + V) + F_5(\tilde{P} - P) \quad .$$

The set of invariant functions $\{F_1, F_2, F_3, F_4, F_5\}$ have nice symmetries under the interchange $u \longleftrightarrow t$ due to the generalized Pauli principle, but in the Regge pole considerations, it is much more convenient to work with our unsymmetrized functions. Inverting Equation (4.24) of GGMW, we have

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \frac{\pi}{2} \begin{pmatrix} 1 & 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 3 & 0 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} , \qquad (2.3)$$

and using Equations (4.27) and (4.28) of GGMW to relate G_i and $\overline{G}_i$, we obtain:

$$\begin{pmatrix} F_S \\ F_T \\ F_A \\ F_V \\ F_P \end{pmatrix} = \frac{\pi}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \overline{G}_3 \\ \overline{G}_4 \\ \overline{G}_5 \end{pmatrix} . \qquad (2.4)$$

Thus the $\overline{G}_i$ of GGMW are exactly the same as the choice of invariant functions convenient for our analysis.

The partial-wave decomposition of the G's may be obtained by using Equation (4.33) of GGMW, which in our notation reads

$$\overline{G}(\overline{f}) = \begin{pmatrix} 1/E^2 & 0 & m^2/E^2p^2 & -z/E^2 & -z/m^2 \\ 0 & 0 & 0 & -1/p^2 & -E^2/m^2p^2 \\ 0 & 0 & -1/p^2 & 0 & 0 \\ 0 & 0 & 0 & 1/p^2 & 1/p^2 \\ 0 & -1/p^2 & 0 & -z/p^2 & -z(E^2+m^2)/m^2p^2 \end{pmatrix} \qquad (2.5)$$

where $4p^2 = t - 4m^2$, $4E^2 = t$, and $z = -(1 + \frac{2s}{t-4m^2})$, together with equations which relate the f_i to their partial-wave forms.

The angular functions employed for this purpose were evaluated from the reduction formulas of Jacob and Wick (27) with the following results:

$$\overline{f}_1 = \frac{E}{p}\,(2J + 1)\; P_J(z)\; \overline{f}_o^{\,J} \qquad (2.6a)$$

$$\overline{f}_2 = \frac{E}{p}\,(2J + 1)\; P_J(z)\; \overline{f}_{11}^{\,J} \qquad (2.6b)$$

$$\overline{f}_5 = -\frac{m}{p}\,(2J + 1)\; P_J'(z)\; \frac{\overline{f}_{12}^{\,J}}{\sqrt{J(J+1)}} \qquad (2.6c)$$

$$\overline{f}_3 = \frac{E}{p}\,\frac{(2J+1)}{J(J+1)} \left\{ \left[P_J' + \frac{z(2P_{J-1}' - J(J-1)\,P_J)}{1 - z^2} \right] \overline{f}_1^{\,J} - \left[\frac{2P_{J-1}' - J(J-1)\,P_J}{1 - z^2} \right] \overline{f}_{22}^{\,J} \right\} \qquad (2.6d)$$

$$\overline{f}_4 = \overline{f}_3\; (\overline{f}_1^{\,J} \longleftrightarrow \overline{f}_{22}^{\,J}) \quad . \qquad (2.6e)$$

We can now easily obtain the Regge amplitudes corresponding to a given trajectory. Use of Equations (2.4) and (2.5) can be made to obtain the partial-wave expansions of the amplitudes $F_S, \ldots, F_P$. These are summarized in Table I for states of the $N\bar{N}$ system classified by the quantum numbers τP, J, and $(-)^I GP$. A more detailed classification of the states of the $N\bar{N}$ system is presented in Table II.

Mandelstam (28) has shown that it is likely that the true asymptotic expansion of the amplitudes in the sense of Regge involves Legendre functions of the second kind rather than those of the first kind. The transition from the Regge expansion to the modified expansion amounts to the replacement (3) of $P_\alpha(x)$ by $\mathcal{P}_\alpha(x)$, where $\mathcal{P}_\alpha(x) = -[\tan \pi\alpha\, Q_{-\alpha-1}(x)]/\pi$. In this paper we shall write the expansion formally in terms of the P_α for typographical reasons only; in practice, it makes no difference in the data analysis whether one uses P_α or the more correct $\mathcal{P}_\alpha$.

The next step is to factor out the threshold behavior in the functions f. We may do so by introducing the functions b_i according to the definitions:

$$\bar{f}_{11}^{\alpha}(t) = \left[-\frac{\alpha!\ \sqrt{\pi}}{2^{\alpha+1}(\alpha+1/2)!}\ \left(\frac{p^3}{2\pi E}\right)\left(\frac{2s_o}{4m^2}\right)\left(\frac{t-4m^2}{2s_o}\right)^{\alpha}\right]$$

$$\times\ \left[\frac{1+\sigma\, e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\right]\ b_{11}(t) \qquad (2.7a)$$

TABLE I. Partial-wave expansions of the invariant NN scattering amplitudes associated with the exchange of an object in the t-channel with quantum numbers τP, J, and $(-)^I GP$.

Quantum Numbers of Object Exchanged			Partial-Wave Expansion of Amplitudes F_S, F_T, F_A, F_V, F_P
τP	J	$(-)^I GP$	
+	α	+	$F_S = -\frac{2\pi E(2\alpha+1)}{p^3}\left\{\bar{f}_{11}^{\alpha} P_\alpha(z) + z\left[P'_\alpha + \frac{z(2P'_{\alpha-1} - \alpha(\alpha-1)P_\alpha)}{1-z^2}\right]\frac{\bar{f}_{22}^{\alpha}}{\alpha(\alpha+1)}\right.$
			$\left. - z(E^2/m^2+1)\frac{m}{E}\frac{P'_\alpha}{\sqrt{\alpha(\alpha+1)}}\bar{f}_{12}^{\alpha}\right\}$
			$F_T = -\frac{2\pi E(2\alpha+1)}{p^3}\left\{\left[P'_\alpha + \frac{z(2P'_{\alpha-1} - \alpha(\alpha-1)P_\alpha)}{1-z^2}\right]\frac{\bar{f}_{22}^{\alpha}}{\alpha(\alpha+1)} - \frac{E}{m}\frac{P'_\alpha}{\sqrt{\alpha(\alpha+1)}}\bar{f}_{12}^{\alpha}\right\}$
			$F_A = \frac{2\pi E(2\alpha+1)}{p^3}\frac{\left[2P'_{\alpha-1} - \alpha(\alpha-1)P_\alpha\right]}{1-z^2}\frac{\bar{f}_{22}^{\alpha}}{\alpha(\alpha+1)}$
			$F_V = \frac{2\pi E(2\alpha+1)}{p^3}\left\{\left[P'_\alpha + \frac{z(2P'_{\alpha-1} - \alpha(\alpha-1)P_\alpha)}{1-z^2}\right]\frac{\bar{f}_{22}^{\alpha}}{\alpha(\alpha+1)} - \frac{m}{E}\frac{P'_\alpha}{\sqrt{\alpha(\alpha+1)}}\bar{f}_{12}^{\alpha}\right\}$
			$F_P = -\frac{2\pi(2\alpha+1)}{pE}\left\{\left[zP'_\alpha + \left(\frac{m^2}{p^2}+z^2\right)\frac{(2P'_{\alpha-1} - \alpha(\alpha-1)P_\alpha)}{1-z^2}\right]\frac{\bar{f}_{22}^{\alpha}}{\alpha(\alpha+1)} - z\frac{E}{m}P'_\alpha\frac{\bar{f}_{12}^{\alpha}}{\sqrt{\alpha(\alpha+1)}}\right\}$

TABLE I (continued)

Quantum Numbers of Object Exchanged: τP	J	$(-)^{I_{GP}}$	Partial-Wave Expansion of Amplitudes F_S, F_T, F_A, F_V, F_P
$-$	α	$-$	$F_S = F_T = F_A = F_V = 0$ $F_P = \frac{2\pi(2\alpha+1)}{pE} \quad P_\alpha(z)\, \bar{f}_o^{\,\alpha}(t)$
$-$	α	$+$	$F_S = \frac{2\pi E(2\alpha+1)}{p^3} \quad \frac{z}{1-z^2} \left[2P'_{\alpha-1} - \alpha(\alpha-1)\, P_\alpha\right] \frac{\bar{f}_1^{\,\alpha}}{\alpha(\alpha+1)}$ $F_T = F_S/z$ $F_A = -\frac{2\pi E(2\alpha+1)}{p^3} \quad \left[P'_\alpha + \frac{z(2P'_{\alpha-1} - \alpha(\alpha-1)\, P_\alpha)}{1-z^2}\right] \frac{\bar{f}_1^{\,\alpha}}{\alpha(\alpha+1)}$ $F_V = -F_T$ $F_P = -\frac{m^2}{E^2} F_A + \frac{p^2}{E^2} F_S$

TABLE II. Number and type of scattering states of the $N\bar{N}$-system. Of the quantum numbers shown, an independent set consists of the combinations σP and $(-)^I GP$.

$\sigma = (-)^J$	$P = -(-)^L$	S	I	C	G	J^P	States	observed resonances with same quantum numbers	numbers of independent amplitudes for given I*
+	−	0	0	+	+	$0^-, 2^-, \ldots$	$^1S_0, {}^1D_2, \ldots$	χ	1
+			1	+	−			π	
+	−	1	0	−	−	$2^-, 4^-, \ldots$	$^3D_2, \ldots$	—	1
+			1	−	+				
+	+	1	0	+	+	$0^+, 2^+, \ldots$	$^3P_0, {}^3P_2 - {}^3F_2, \ldots$	P, P', ABC	3
+			1	+	−				
−	+	0	0	−	−	$1^+, 3^+, \ldots$	$^1P_1, \ldots$	—	1
−			1	−	+				
−	+	1	0	+	+	$1^+, 3^+, \ldots$	$^3P_1, \ldots$	—	1
−			1	+	−				
−	−	1	0	−	−	$1^-, 3^-, \ldots$	$^3S_1 - {}^3D_1, \ldots,$	ω	3
−			1	−	+			ρ	

*An exception arises for $J = 0$, when there are only two terms altogether.

$$\frac{\bar{f}_{12}^{\alpha}(t)}{\sqrt{\alpha(\alpha+1)}} = \frac{E}{m}\left[-\frac{\alpha!\,\sqrt{\pi}}{2^{\alpha+1}(\alpha+1/2)!}\left(\frac{p^3}{2\pi E}\right)\left(\frac{2s_o}{4m^2}\right)\left(\frac{t-4m^2}{2s_o}\right)^{\alpha}\right] \times \left[\frac{1+\sigma\, e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\right] b_{12}(t) \qquad (2.7b)$$

$$\frac{\bar{f}_{22}^{\alpha}(t)}{\alpha(\alpha+1)} = \frac{E^2}{m^2}\left[-\frac{\alpha!\,\sqrt{\pi}}{2^{\alpha+1}(\alpha+1/2)!}\left(\frac{p^3}{2\pi E}\right)\left(\frac{2s_o}{4m^2}\right)\left(\frac{t-4m^2}{2s_o}\right)^{\alpha}\right] \times \left[\frac{1+\sigma\, e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\right] b_{22}(t) \qquad (2.7c)$$

$$\bar{f}_{o}^{\alpha}(t) = \frac{E^2}{p^2}\left[-\frac{\alpha!\,\sqrt{\pi}}{2^{\alpha+1}(\alpha+1/2)!}\left(\frac{p^3}{2\pi E}\right)\left(\frac{2s_o}{4m^2}\right)\left(\frac{t-4m^2}{2s_o}\right)^{\alpha}\right] \times \left[\frac{1+\sigma\, e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\right] b_{o}(t) \qquad (2.7d)$$

$$\frac{\bar{f}_{1}^{\alpha}(t)}{\alpha(\alpha+1)} = \left(\frac{4m^2}{t-4m^2}\right)\frac{t}{4m^2}\left[-\frac{\alpha!\,\sqrt{\pi}}{2^{\alpha+1}(\alpha+1/2)!}\left(\frac{p^3}{2\pi E}\right)\left(\frac{2s_o}{4m^2}\right)\left(\frac{t-4m^2}{2s_o}\right)^{\alpha}\right] \times \left[\frac{1+\sigma\, e^{-i\pi\alpha(t)}}{2\sin\pi\alpha(t)}\right] b_{1}(t) \quad . \qquad (2.7e)$$

The new expressions for the amplitudes in Table I may be written conveniently in terms of the b_i and (3)

$$Z_{\alpha}^{\tau}(s,t) = [Z_{\alpha}(s,t) + \tau\, Z_{\alpha}(u,t)]/(1+\tau\, e^{-i\pi\alpha(t)}) \quad ,$$

where

$$Z_\alpha(s,\,t) = e^{-i\pi\alpha}\ \frac{\alpha!\ \sqrt{\pi}}{2^\alpha(\alpha-1/2)!}\ \left(\frac{t-4m^2}{2s_o}\right)^\alpha P_\alpha\left[-(1+\frac{2s}{t-4m^2})\right] . \qquad (2.8)$$

The asymptotic behavior of these functions is independent of the signature:

$$Z_\alpha(s,\,t) = \left(\frac{2s+t-4m^2}{2s_o}\right)^\alpha - \frac{\alpha(\alpha-1)(t-4m^2)^2}{2(2\alpha-1)\ 4s_o^{\ 2}}\left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-2} + \ldots$$

$$= \left(\frac{2s+t-4m^2}{2s_o}\right)^\alpha \left[1 - \frac{\alpha(\alpha-1)}{2(2\alpha-1)}\,x^2 + \ldots\right] , \qquad (2.9)$$

where $x = (t-4m^2)/(2s+t-4m^2)$, and

$$Z'_\alpha(s,\,t) = dZ_\alpha/d(s/s_o) = \alpha\left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-1} \left[1 - \ldots\right] . \qquad (2.10)$$

Upon substituting the Z_α into the amplitudes of Table I, we obtain formulas for the Regge pole terms in the NN-scattering amplitude.

Thus far in our analysis, we have not incorporated the hypothesis that the Regge pole terms are factorizable (29). The effect of this property is to reduce the number of independent invariant functions, $b_i(t)$, from three to two in the case where the Regge trajectory has the quantum number $\tau P = +$. It results in no change for those contributions to the invariant functions arising from trajectories with $\tau P = -$, since there is only one invariant function, $b_o(t)$ or $b_1(t)$, associated with such poles.

The relations

$$t\,F_P + 4m^2\,F_A + (2s+t-4m^2)\,F_T = 0 \quad ,$$

and

$$(t\,F_V + 4m^2\,F_T)\quad\left[2\,P'_{\alpha-1}(-1 - \frac{2s}{t-4m^2}) - \alpha(\alpha-1)\,P_\alpha\right]$$

$$+ F_A(4m^2 - t)\left[(1 - z^2)\,P'_\alpha + z(2P'_{\alpha-1} - \alpha(\alpha-1)\,P_\alpha)\right] = 0\ ,$$

are valid for the contributions from Regge poles with $\tau P = +$, irrespective of whether the coupling to the pole may be factorized. The additional relation imposed by the factorizability of the pole contribution, however, may not be expressed in the simple form of a linear relation between invariant functions. Rather, it leads to expressions for all the invariant functions as a bilinear form in two functions instead of as a linear form in three.

The functions $\bar{f}^J_{11}(t)$, $\bar{f}^J_{12}(t)$, and $\bar{f}^J_{22}(t)$ of GGMW, which appear first in our Equation (2.6),are the elements of a 2×2 symmetric reaction matrix. The assumption that it may be factorized is equivalent to choosing the representation:

$\bar{f}^J_{11}(t) = [\bar{f}^J_{1+}(t)]^2$, $\bar{f}^J_{12}(t) = \bar{f}^J_{1+}(t)\ \bar{f}^J_{2+}(t)$, $\bar{f}^J_{22}(t) = [\bar{f}^J_{2+}(t)]^2$. It is natural, therefore, to introduce the functions $b_{1+}(t)$ and $b_{2+}(t)$ so that

$$b_{11}(t) = [b_{1+}(t)]^2\ ,$$

$$b_{12}(t) = [b_{1+}(t)]\ [b_{2+}(t)]\ ,$$

$$b_{22}(t) = [b_{2+}(t)]^2\ ,$$

which when inserted into Equation (2.7) yields the final form of the Regge amplitudes for NN scattering. These are given in Tables III, IV, V in their exact form. The leading terms in the series, valid for

$s \gg 4m^2 - t$, are to be found in Tables VI, VII and VIII.

It has recently been shown (23) that the Regge analysis of scattering problems involving spin may be decisively simplified if helicity amplitudes are introduced. Although we shall not use this method in our thesis, we shall for the sake of completeness express the helicity amplitudes in terms of our factored residues b_{1+} and b_{2+}. This is most easily accomplished by using first Equations (4.17a-e) of GGMW to relate the helicity amplitudes $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$ to the $\{F_1, \dots, F_5\}$, and then the inverse of our Equation (2.2) to relate them to F_S, F_T, F_A, F_V, F_P. Use of Table III then yields the desired results, which are the following: (see page 36)

($T = 8\pi\, s^{1/2}\, \phi$, and we use m_N as the energy unit)

TABLE III. Regge amplitudes for a pole in the t-channel with quantum numbers $\tau P = +$, $(-)^I GP = +$. In the tables, $\zeta = (1 + \tau\ e^{-i\pi\alpha})/2 \sin \pi\alpha$.

$$F_S = \zeta \frac{2s_o}{(4m^2)^2} \left\{ 4m^2\, Z_\alpha(s,t) \left[b_{1+}(t)\right]^2 + t \left[\frac{(s + \frac{t-4m^2}{2})}{s_o} Z'_\alpha \right.\right.$$

$$\left. - \frac{\alpha(\frac{t-4m^2}{2} + s)^2}{s(s+t-4m^2)} \left(\frac{2(\frac{t-4m^2}{2s_o})^2}{2\alpha-1} Z'_{\alpha-1} - (\alpha-1)\, Z_\alpha \right) \right] \left[b_{2+}(t)\right]^2$$

$$\left. - (t + 4m^2) \left(\frac{s + \frac{t-4m^2}{2}}{s_o} \right) Z'_\alpha \left[b_{1+}(t)\, b_{2+}(t) \right] \right\}$$

$$F_T = -\zeta \frac{2s_o}{(4m^2)^2} \left\{ t \left(\frac{t-4m^2}{2}\right) \left[\frac{Z'_\alpha}{s_o} - \frac{\alpha(s + \frac{t-4m^2}{2})}{s(s+t-4m^2)} \left(\frac{2}{2\alpha-1} \left(\frac{t-4m^2}{2s_o}\right)^2 Z'_{\alpha-1} - (\alpha-1)\, Z_\alpha\right) \right] \right.$$

$$\left. \times \left[b_{2+}(t)\right]^2 - t\left(\frac{t-4m^2}{2s_o}\right) Z'_\alpha\, b_{1+}(t)\, b_{2+}(t) \right\}$$

$$F_A = \zeta \frac{2s_o}{(4m^2)^2} \frac{\alpha t (\frac{t-4m^2}{2})^2}{s(s+t-4m^2)} \left[\frac{2}{2\alpha-1} \left(\frac{t-4m^2}{2s_o}\right)^2 Z'_{\alpha-1} - (\alpha-1)\, Z_\alpha \right] \left[b_{2+}(t)\right]^2$$

$$F_V = -\zeta \frac{2s_o}{(4m^2)^2} \left\{ \left[-\frac{Z'_\alpha}{s_o} + \frac{\alpha(s + \frac{t-4m^2}{2})}{s(s+t-4m^2)} \left(\frac{2}{2\alpha-1} \left(\frac{t-4m^2}{2s_o}\right)^2 Z'_{\alpha-1} - (\alpha-1)\, Z_\alpha\right) \right] \times \right.$$

$$\left. t\left(\frac{t-4m^2}{2}\right) \left[b_{2+}(t)\right]^2 + 4m^2 \left(\frac{t-4m^2}{2s_o}\right) Z'_\alpha\, b_{1+}(t)\, b_{2+}(t) \right\}$$

$$F_P = \zeta \frac{2s_o}{(4m^2)^2} \left\{ (t-4m^2) \left[\frac{(s + \frac{t-4m^2}{2})}{s_o} Z'_\alpha - \frac{\alpha \left[m^2(t-4m^2) + (s + \frac{t-4m^2}{2})^2 \right]}{s(s+t-4m^2)} \times \right.\right.$$

$$\left. \left[\frac{2}{2\alpha-1} \left(\frac{t-4m^2}{2s_o}\right)^2 Z'_{\alpha-1} - (\alpha-1)\, Z_\alpha \right] \right] \left[b_{2+}(t)\right]^2$$

$$\left. - \frac{(t-4m^2)}{2s_o} (2s + t - 4m^2)\, Z'_\alpha\, b_{1+}(t)\, b_{2+}(t) \right\}$$

TABLE IV. Regge amplitudes for a pole in the t-channel with quantum numbers $\tau P = -$, $(-)^I GP = -$.

$$F_S = F_T = F_A = F_V = 0$$

$$F_P = -\zeta \, \frac{2s_o}{4m^2} \, Z_\alpha \, b_o(t)$$

TABLE V. Regge amplitudes for a pole in the t-channel with quantum numbers $\tau P = -$, $(-)^I GP = +$.

$$F_S = -\zeta \, \frac{t}{4m^2} \frac{s_o}{s} \, \frac{\alpha(s + \frac{t-4m^2}{2})}{(s+t-4m^2)} \left[\frac{2}{2\alpha-1} \left(\frac{t-4m^2}{2s_o}\right)^2 Z'_{\alpha-1} - (\alpha-1) \, Z_\alpha \right] b_1(t)$$

$$F_T = \frac{-(t-4m^2)}{2s + t - 4m^2} \, F_S$$

$$F_A = -\zeta \, \frac{t}{4m^2} \, Z'_\alpha \, b_1(t) - F_S$$

$$F_V = -F_T$$

$$F_P = \zeta \, Z'_\alpha \, b_1(t) + F_S$$

TABLE VI. Leading terms in the expansion of the Regge amplitude for a pole in the t-channel with quantum numbers $\tau P = +$, $(-)^I GP = +$.

$$F_S \to \zeta \frac{2s_o}{(4m^2)^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} \left\{ 4m^2 \left[b_{1+}(t)\right]^2 \left[1 - \frac{\alpha(\alpha-1)}{2(2\alpha-1)} x^2\right] - \alpha(t+4m^2)\right.$$

$$\times \left[b_{1+}(t)\, b_{2+}(t)\right] \left[1 - \frac{(\alpha-1)(\alpha-2)}{2(2\alpha-1)} x^2\right] + \alpha t \left[b_{2+}(t)\right]^2$$

$$\left. \times \left[\alpha - \frac{(\alpha-1)(\alpha-2)^2 x^2}{2(2\alpha-1)}\right]\right\}$$

$$F_T \to \zeta \frac{2s_o}{(4m^2)^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} t\, \alpha\, x \left[b_{1+}(t) - \alpha\, b_{2+}(t)\right] b_{2+}(t)$$

$$F_A \to -\zeta \frac{2s_o}{(4m^2)^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} t\, \alpha(\alpha-1)\, x^2 \left[b_{2+}(t)\right]^2$$

$$F_V \to \zeta \frac{2s_o}{(4m^2)^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} \alpha\, x \left[\alpha\, t\, b_{2+}(t) - 4m^2\, b_{1+}(t)\right] b_{2+}(t)$$

$$F_P \to \zeta \frac{2s_o}{(4m^2)^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} \alpha(t-4m^2)\, b_{2+}(t) \left\{- b_{1+}(t) \left[1 - \frac{(\alpha-1)(\alpha-2)}{2(2\alpha-1)} x^2\right]\right.$$

$$\left. + b_{2+}(t) \left[\alpha - (\alpha-1)\, x^2 \left(\frac{(\alpha-2)^2}{2(2\alpha-1} + \frac{4m^2}{4m^2-t}\right)\right]\right\}$$

TABLE VII. Leading terms in the expansion of the Regge amplitudes for a pole in the t-channel with quantum numbers $\tau P = -$, $(-)^I GP = -$.

$$F_S = F_T = F_A = F_V = 0$$

$$F_P \to -\zeta \; \frac{2s_o}{4m^2} \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha} b_o(t) \left[1 - \frac{\alpha(\alpha-1)\, x^2}{2(2\alpha-1)}\right]$$

TABLE VIII. Leading terms in the expansion of the Regge amplitudes for a pole in the t-channel with quantum numbers $\tau P = -$, $(-)^I GP = +$.

$$F_S \to \zeta \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-1} \alpha(\alpha-1) \frac{t}{4m^2} \; b_1(t)$$

$$F_T \to -\zeta \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-1} \alpha(\alpha-1)\, x^2 \frac{t}{4m^2} \; b_1(t)$$

$$F_A \to -\zeta \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-1} \alpha^2 \frac{t}{4m^2} \; b_1(t)$$

$$F_V = -F_T$$

$$F_P \to -\zeta \left(\frac{2s+t-4m^2}{2s_o}\right)^{\alpha-1} \alpha \left[(\alpha-1) \frac{t}{4m^2} + 1\right] b_1(t)$$

$$\begin{pmatrix} 4(s+t-4) & -2ts(s+t-4) & \dot{Z}_\alpha \frac{t}{4}\left\{s(st+2t-8)+2(t-4)^2\right\} - \frac{st^2}{4}\; Y_\alpha \\ st & -2ts(s+t-4) & \dot{Z}_\alpha \frac{t}{2}\,(2s+t-4)(s+t-4)-t\; Y_\alpha(s+t-4) \\ 4(s+t-4) & -2ts(s+t-4) & \dot{Z}_\alpha \frac{t}{4}\,(st-2t+8)(s+t-4) - \frac{st^2}{4}\; Y_\alpha \\ -st & 2ts(s+t-4) & -\dot{Z}_\alpha \frac{t}{2}\,(2s+t-4)(s+t-4)+t(s+t-4)\; Y_\alpha \\ 1 & -\frac{1}{4}\,[st+4s+4t-16] & \dot{Z}_\alpha\,(2s+t-4)\,\frac{t}{8} - \frac{t}{4}\; Y_\alpha \end{pmatrix}$$

$$\times \begin{pmatrix} b_{1+}^2\, Z_\alpha \\ b_{1+}\, b_{2+}\, \dot{Z}_\alpha \\ b_{2+}^2 \end{pmatrix} \left[\frac{2s_o\, \zeta}{(s-4)\;4}\right] = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ \hline 2\sqrt{-st}\;(s+t-4) \end{pmatrix} \qquad (2.11)$$

We have introduced the following abbreviations:

$$\dot{Z} = dZ/ds = Z'/s_o \quad ,$$

$$\zeta = (1 + \tau\, e^{-i\pi\alpha})/2 \sin \pi\alpha \quad ,$$

$$Y_\alpha = \alpha\left[\frac{2}{2\alpha-1}\left(\frac{t-4}{2s_o}\right)^2 Z'_{\alpha-1} - (a-1)\; Z_\alpha\right] \quad .$$

In the asymptotic limit $s \to \infty$, $t \lesssim 0$ the relation between the helicity amplitudes and b_{1+}, b_{2+} simplifies to

In the asymptotic limit $s \to \infty$, $t \leq 0$ the relation between the helicity amplitudes and b_{1+}, b_{2+} simplifies to

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \frac{s_o \zeta}{2} \left(\frac{s}{s_o}\right)^{\alpha} \begin{pmatrix} 4 & -t & t \\ t & -t & 4 \\ 4 & -t & t \\ -t & t & -4 \\ -2\sqrt{-t} & \sqrt{-t}\,(1+\frac{t}{4}) & -2\sqrt{-t} \end{pmatrix} \begin{pmatrix} b_{1+}^2 \\ 2\alpha\, b_{1+}\, b_{2+} \\ \frac{t}{4}\alpha^2\, b_{2+} \end{pmatrix} \tag{2.12}$$

and a simple relation between the helicity amplitudes is revealed:

$$T_1 \to T_3$$

$$T_2 = - T_4 \quad . \tag{2.13}$$

3. ASYMPTOTIC AMPLITUDES DUE TO THE EXCHANGE OF P, ω, ρ, and P' MESONS

In this section we construct the contributions to the invariant amplitudes describing NN scattering arising from the exchange of the P, ω, ρ, and π mesons. We can then compare the asymptotic forms of these expressions to those given by the Regge theory applied to the corresponding trajectories, and identify the residues b_i with appropriate coupling constants by comparing the amplitudes (2) at $t = m_\rho^{\,2}$, $m_\pi^{\,2}$, ...

We shall first consider the Pomeranchuk trajectory, having the quantum numbers of the vacuum and $\alpha_P(0) = 1$. It is possible that there is a spin 2^+ resonance occurring on this trajectory (30) at $t \sim 1\ (\mathrm{GeV})^2$. We may identify the Pomeranchuk pole residues with the coupling constants of this spin 2 resonance to the nucleon. To do this, we must first construct the contribution of a spin 2 meson, C, to the invariant amplitudes describing NN-scattering.

The propagator for a spin 2 meson must be a tensor of rank four. Its most general form is therefore

$$D_{\mu\nu\lambda\sigma}(q^2) = a\ \{\delta_{\mu\lambda}\,\delta_{\nu\sigma} + \delta_{\mu\sigma}\,\delta_{\nu\lambda} + B\,\delta_{\mu\nu}\,\delta_{\lambda\sigma}$$
$$+ C(q_\lambda\, q_\sigma\, \delta_{\mu\nu} + q_\mu\, q_\nu\, \delta_{\lambda\sigma})/m^2$$
$$+ D(q_\nu\, q_\sigma\, \delta_{\mu\lambda} + q_\mu\, q_\lambda\, \delta_{\nu\sigma} + q_\nu\, q_\lambda\, \delta_{\mu\sigma} + q_\mu\, q_\sigma\, \delta_{\nu\lambda})/m^2$$
$$+ E(q_\mu\, q_\nu\, q_\lambda\, q_\sigma)/m^4\}\quad (q^2 + m^2)^{-1} \quad , \tag{3.1}$$

where we have taken into account the symmetries $D_{\mu\nu\lambda\sigma} = D_{\nu\mu\lambda\sigma}$ and $D_{\mu\nu\lambda\sigma} = D_{\lambda\sigma\mu\nu}$. At the pole $q^2 = -m^2$, the propagator is divergenceless, $q_\mu\ D_{\mu\nu\lambda\sigma} = 0$, and traceless,* $D_{\mu\mu\lambda\sigma} = 0$. From these two conditions we find $B = C = -2/3$, $D = 1$, and $E = 4/3$. The factor a is determined to have the value $1/2$ so that if a polarization tensor $\epsilon_{\mu\nu}$ of the meson is normalized to 1, then $\epsilon_{\mu\nu}\ D_{\mu\nu\lambda\sigma}\ \epsilon_{\lambda\sigma} = 1$.

In the Born approximation, the coupling of the C meson to two nucleons takes the form

$$\frac{1}{4m_N^{\;2}}\,(\chi_{CNN} - \xi_{CNN})\ \Sigma_\mu\ \Sigma_\nu + \frac{i}{4m_N}\ \xi_{CNN}\ [\Sigma_\mu\ \gamma_\nu + \Sigma_\nu\ \gamma_\mu] \quad ,$$

where $\Sigma_\mu = (p + p')_\mu$.

From these results we readily see that the C-meson pole term in the amplitude has the form

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*The author wishes to thank Dr. W. G. Wagner for clarifying the meaning of this condition to him.

$$T = \frac{1}{t-m_C^2} \; \bar{u}_2' \left[\frac{(\chi_{CNN} - \xi_{CNN})}{4m_N^2} \Sigma'_\mu \Sigma'_\nu + \frac{i\,\xi_{CNN}}{4m_N} (\Sigma'_\mu \gamma_\nu + \Sigma'_\nu \gamma_\mu) \right] u_2$$

$$\times \bar{u}_1' \left[\frac{(\chi_{CNN} - \xi_{CNN})}{4m_N^2} \Sigma_\mu \Sigma_\nu + \frac{i\,\xi_{CNN}}{4m_N} (\Sigma_\mu \gamma_\nu + \Sigma_\nu \gamma_\mu) \right] u_1$$

$$- \frac{1}{3} \frac{1}{t-m_C^2} \bar{u}_2' u_2 \bar{u}_1' u_1 \left[\chi_{CNN} \left(\frac{t}{4m_N^2} - 1\right) - \xi_{CNN} \frac{t}{4m_N^2} \right]^2 , \quad (3.2)$$

where $\Sigma'_\mu = (p_2 + p_2')_\mu$, $\Sigma_\mu = (p_1 + p_1')_\mu$.

Using Equation (2.21) of ALV, which states that

$$2i\, m_N [\bar{u}_1' \Sigma'_\mu \gamma_\mu u_1 \bar{u}_2' u_2 + \bar{u}_1' u_1 \bar{u}_2' \Sigma_\mu \gamma_\mu u_2]$$

$$= (4m_N^2 - t - 2s)(S + P) - 4m_N^2 V + t\, T \quad (3.3a)$$

and

$$\bar{u}_1' \Sigma'_\mu \gamma_\mu u_1 \bar{u}_2' \Sigma_\nu \gamma_\nu u_2$$

$$= -(2s + t - 4m_N^2) V + t\, A - 4m_N^2 P , \quad (3.3b)$$

we find that the C pole terms in the invariant amplitudes are:

$$(4m_N^2)^2 (t-m_C^2) F_S = \chi_{CNN}(\chi_{CNN} - \xi_{CNN})(4m_N^2 - t - 2s)^2$$

$$- \frac{1}{3} \left[\chi_{CNN}(t - 4m_N^2) - t\,\xi_{CNN}\right]^2 , \quad (3.4a)$$

$$(4m_N^2)^2 (t-m_C^2) F_T = (\chi_{CNN} - \xi_{CNN})\, \xi_{CNN}\, t(4m_N^2 - t - 2s) , \quad (3.4b)$$

$$(4m_N{}^2)^2\ (t-m_C{}^2)\ F_A = -2\ \xi_{CNN}^2\ m_N{}^2\ t, \tag{3.4c}$$

$$(4m_N{}^2)^2\ (t-m_C{}^2)\ F_V = -4\ \chi_{CNN}\ \xi_{CNN}\ m_N{}^2\ (4m_N{}^2 - t - 2s)\quad , \tag{3.4d}$$

$$(4m_N{}^2)^2\ (t-m_C{}^2)\ F_P = 8\ \xi_{CNN}^2\ m_N{}^4 + (\chi_{CNN} - \xi_{CNN}) \times \xi_{CNN}(4m_N{}^2 - t - 2s)^2 \quad . \tag{3.4e}$$

These expressions may be compared to those in Table VII. In particular, we can, at $t = m_C{}^2$, identify the Pomeranchuk Regge pole parameters α, b_{1+}, and b_{2+} with various properties of the C meson. At the position of the resonance, $t = m_C{}^2$, we must have $\mathrm{Re}\ \alpha_P(m_C{}^2) = 2$. Also $\mathrm{Im}\ \alpha_P(m_C{}^2) = I_P$ is related to the width (2)

$$m_C\ \Gamma_C = I_P/\epsilon_C \quad , \tag{3.5}$$

where $\epsilon_C = \mathrm{Re}\ \left[d\alpha_P(t)/dt\ \Big|_{t=m_C{}^2}\right]$, and we find

$$\frac{b_{1+}^P\ (m_C{}^2)}{\sqrt{\pi}\ \epsilon_C\ s_o} = \frac{(\chi_{CNN} - \xi_{CNN}) + \dfrac{m_C{}^2}{4m_N{}^2 - m_C{}^2}\ \xi_{CNN}}{m_N}$$

$$\frac{b_{2+}^P\ (m_C{}^2)}{\sqrt{\pi}\ \epsilon_C\ s_o} = \frac{2m_N\ \xi_{CNN}}{(4m_N{}^2 - m_C{}^2)} \tag{3.6}$$

In a similar way, we may compare the Feynman amplitude corresponding to ρ exchange with the associated Regge pole contribution, to identify the residues $b_i^\rho(t)$ at $t = m_\rho^2$. For the ρ-pole term in the amplitude, we have

$$T(s, t) = - \gamma_{NN\rho}^2 \bar{u}_1' \left[\gamma_\mu - \frac{\mu_{NN\rho}}{2m_N} \sigma_{\mu\nu}(p_1' - p_1)_\nu\right] \tau_\alpha u_1$$

$$\times \frac{\bar{u}_2' \quad \gamma_\mu - \frac{\mu_{NN\rho}}{2m_N} \sigma_{\mu\nu}(p_2' - p_2)_\nu \quad \tau_\alpha u_2}{t - m_\rho^2} \qquad (3.7)$$

which can be reduced to the form

$$T(s, t) = \frac{1}{2} \binom{-3}{1} \left(\frac{-2 \gamma_{NN\rho}^2}{t - m_\rho^2}\right) \left\{S[4m_N^2 - 2s - t] \frac{\mu_{\rho NN}}{4m_N^2}\right.$$

$$+ P \left[1 - \frac{2s+t}{4m_N^2}\right] \mu_{\rho NN}(1 + \mu_{\rho NN}) + \frac{\mu_{\rho NN}(1 + \mu_{\rho NN})}{4m_N^2} t T$$

$$\left. + (1 + \mu_{\rho NN}) V\right\} \quad , \qquad (3.8)$$

using the relations

$$-u_{p'} \sigma_{\mu\nu}(p' - p)_\nu u_p = u_{p'}[2m_N \gamma_\mu + i(p' + p)_\mu] u_p \quad ,$$

$$(p_1 + p_1')_\mu (p_2 + p_2')_\mu = u - s = 4m_N^2 - t - 2s \quad , \qquad (3.9)$$

and Equation (3.3b). In the column vector, the first row refers to $I = 0$ in the s-channel, and the second to $I = 1$. Near the ρ-meson pole, therefore, we have (dropping isospin factors):

$$F_S = \left(\frac{-2\,\gamma^2_{NN\rho}}{t - m_\rho^2}\right) \frac{(4m_N^2 - t - 2s)}{4m_N^2}\ \mu_{\rho NN} \quad , \tag{3.10a}$$

$$F_T = \left(\frac{-2\,\gamma^2_{NN\rho}}{t - m_\rho^2}\right) \frac{t}{4m_N^2}\ \mu_{\rho NN}(1 + \mu_{\rho NN}) \quad , \tag{3.10b}$$

F_A is not singular , (3.10c)

$$F_V = \left(\frac{-2\,\gamma^2_{NN\rho}}{t - m_\rho^2}\right) (1 + \mu_{\rho NN}) \quad , \tag{3.10d}$$

$$F_P = \left(\frac{-2\,\gamma^2_{NN\rho}}{t - m_\rho^2}\right) \frac{(4m_N^2 - t - 2s)}{4m_N^2}\ \mu_{\rho NN}(1 + \mu_{\rho NN}) \quad . \tag{3.10e}$$

We compare these results to those arising from the ρ trajectory. At $t = m_\rho^2$, Re $\alpha_\rho(m_\rho^2) = 1$, and as before, we have

$$I_\rho = m_\rho\ \Gamma_\rho\, \epsilon_\rho \quad ,$$

where $\alpha_\rho(m_\rho^2) = 1 + i\, I_\rho$ and $\epsilon_\rho = \mathrm{Re}\left[d\alpha_\rho(t)/dt\ \Big|_{t=m_\rho^2}\right]$. We then find that the $b_i^\rho(t)$ are related, at $t = m_\rho^2$, to the coupling constants $\gamma_{\rho NN}$ and $\mu_{\rho NN}$ as follows:

$$\pm \begin{cases} (1 - \frac{m_\rho^2}{4m_N^2})\, b_{1+}^{\rho}\,(m_\rho^2)/\sqrt{\pi\,\epsilon_\rho/2} & = 2\,\gamma_{\rho NN}[1 + \mu_{\rho NN}\, m_\rho^2/4m_N^2] \quad (3.11a) \\ (1 - \frac{m_\rho^2}{4m_N^2})\, b_{2+}^{\rho}\,(m_\rho^2)/\sqrt{\pi\,\epsilon_\rho/2} & = 2\,\gamma_{\rho NN}(1 + \mu_{\rho NN}) \quad . \quad (3.11b) \end{cases}$$

The corresponding formulas for the ω Regge pole are exactly analogous to those of the ρ, since the only difference is that of isospin, which we take care of with the matrix Λ, (see Equation 2.0e).

Of those Regge poles associated with meson systems having zero spin, the most prominent contributor to the NN-scattering amplitude is likely to be that corresponding to the pion, since $\text{Re}\ \alpha_\pi(t)$ is zero for the lowest t, $\alpha_\pi(m_\pi^2) = 0$. This trajectory has $I = 1$, $C = +$, $\tau = +$, $\tau P = -$. Near $t = m_\pi^2$,

$$T(s, t) = - g_{NN\pi}^2 \frac{P}{t - m_\pi^2} \binom{-3}{1}$$

and therefore

$$\frac{b_0^\pi(m_\pi^2)}{\pi\,\epsilon_\pi} = \frac{2\, g_{NN\pi}^2}{\sqrt{2s_0}} \quad .$$

In considering various trajectories which may contribute to NN scattering, we should like to mention briefly some recent speculations on the existence of another trajectory with $C = +$, for which $\alpha(0)$ lies in the region 0 to 1. Igi (22) has shown that the data on

π^+p and π^-p scattering require some singularity in the J-plane which lies in the region 0 to 1 for forward scattering. Let this singularity, which has $C = +$, and $\tau = +$ since it is coupled to the two-pion system, be labelled P'. As we shall see in Section 5, and as suggested on the basis of a spinless treatment of NN scattering by Hadjioannou et al. (21), such a singularity is also needed to cancel the contribution of the ω Regge pole in NN scattering. It must therefore have $I = 0$, and is a companion to the Pomeranchuk trajectory in that they both have the quantum numbers of the vacuum. Igi has suggested (22) that the P' be associated with the ABC anomaly (31 - 33), but this seems inappropriate because the trajectory associated with the ABC anomaly must have $\alpha \approx 0$ near $t = 0$. If the P' singularity is a pole, rather than a branch cut, there exists the possibility of associating P' with a resonance with $J = 2$ in the region $t > 4m_\pi^2$. We would look for such a resonance in the 1 to 1.5 GeV region, which still remains virtually unexplored. However, the P' trajectory may not reach the line $\text{Re}\,\alpha = 2$, or even if it does, $\text{Im}\,\alpha$ may be large, so that a resonance would not occur.

S. Mandlestam (10) has recently investigated the contribution of a class of multiparticle intermediate states to the partial-wave amplitude. He concludes that they give rise to cuts in the angular-momentum plane which are in general present up to $t = 0$. If this conclusion is correct, it appears to us much more plausible to regard the P' singularity as the cut associated with the P pole, rather than as a second vacuum trajectory. For further comments, see Section 5.

4. CROSS SECTIONS FOR NUCLEON-NUCLEON SCATTERING

A. Elastic Differential Cross Sections

The nucleon-nucleon elastic differential cross section may be written as a bilinear form in the amplitudes $F_I (I = S, T, A, V, P)$:

$$d\sigma/dt = \frac{1}{16\pi s(s - 4m_N^2)} \sum_{I,I'} C_{II'}(s, t) F_{I'}^* F_I \quad . \tag{4.1}$$

The coefficients $C_{II'}$ were obtained by carrying out the spin sums:

$$4C_{II'} = \mathrm{Tr}[(m - i\not{p}_1) \bar{O}_{I'}(m - i\not{p}_{1'}) O_I] \times$$
$$\mathrm{Tr}[(m - i\not{p}_2) O_{I'}(m - i\not{p}_{2'}) O_I], \tag{4.2}$$

where $O_S = 1$, $O_T = \frac{1}{\sqrt{2}} \sigma_{\mu\nu}$, $O_A = i\gamma_5 \gamma_\mu$, $O_V = \gamma_\mu$, and $O_P = \gamma_5$. The results are summarized in Table IX.

With these coefficients we may construct the contribution to the differential cross sections for NN and $N\bar{N}$ scattering from the P, P', ρ and ω trajectories.

TABLE IX. Spin sums for NN scattering.

Coefficient	Form in terms of s and t.
C_{SS}	$(4m_N^2 - t)^2$
$C_{ST} = C_{TS}$	$t(2s + t - 4m_N^2)$
$C_{SA} = C_{AS}$	0
$C_{SV} = C_{VS}$	$4m_N^2[2s + t - 4m_N^2]$
$C_{SP} = C_{PS}$	0
C_{TT}	$2[2s + t - 4m_N^2]^2 + 8m_N^2(t - 4m_N^2) + 48m_N^4$
$C_{TA} = C_{AT}$	$12m_N^2 [2s + t - 4m_N^2]$
$C_{TV} = C_{VT}$	$12m_N^2 t$
$C_{TP} = C_{PT}$	$t[2s + t - 4m_N^2]$
C_{AA}	$[2s + t - 4m_N^2]^2 + (t - 4m_N^2)^2 + 16m_N^4$
$C_{AV} = C_{VA}$	$2t[2s + t - 4m_N^2]$
$C_{AP} = C_{PA}$	$4m_N^2 t$
C_{VV}	$[2s + t - 4m_N^2]^2 + t(t + 8m_N^2)$
$C_{VP} = C_{PV}$	0
C_{PP}	t^2

For pp scattering we find

$$16\pi\, s(s - 4m_N^2)\,\frac{d\sigma_{pp}}{dt} = D_{PP}\left[\frac{2s+t-4m_N^2}{2s_o^P}\right]^{2\alpha(t)}$$

$$+ 2\,D_{P\omega}\left[\frac{2s+t-4m_N^2}{2s_o^P}\right]^{\alpha_P(t)}\left[\frac{2s+t-4m_N^2}{2s_o^\omega}\right]^{\alpha_\omega(t)}$$

$$+ 2\,D_{PP'}\left[\frac{2s+t-4m_N^2}{2s_o^P}\right]^{\alpha_P(t)}\left[\frac{2s+t-4m_N^2}{2s_o^{P'}}\right]^{\alpha_{P'}(t)} + D_{\omega\omega}\left[\frac{2s+t-4m_N^2}{2s_o^\omega}\right]^{2\alpha_\omega(t)}$$

$$+ 2\,D_{\omega P'}\left[\frac{2s+t-4m_N^2}{2s_o^\omega}\right]^{\alpha_\omega(t)}\left[\frac{2s+t-4m_N^2}{2s_o^{P'}}\right]^{\alpha_{P'}(t)} + D_{P'P'}\left[\frac{2s+t-4m_N^2}{2s_o^{P'}}\right]^{2\alpha_{P'}(t)} .$$

(4.3)

The result for $p\bar{p}$ scattering is the same except for a minus sign on the terms $D_{P\omega}$ and $D_{P'\omega}$. Since no $I = 1$ pole in included, $d\sigma_{np}/dt \approx d\sigma_{pp}/dt$.

Making use of the expressions in Tables VI and IX, we find for the coefficients in Equation (4.3) the remarkably simple expressions,

$$\sin^2\frac{\pi\alpha_P(t)}{2}\,D_{PP}(t) = \left[(b_{1+}^P)^2 - \alpha_P^{\,2}\,\frac{t(b_{2+}^P)^2}{4m_N^2}\right]^2 (s_o^P)^2\left(1 - \frac{t}{4m_N^2}\right)^2 , \quad (4.4)$$

and

$$\frac{\sin \pi\alpha_P \sin \pi\alpha_\omega D_{P\omega}}{[1 + \tau_P \cos \pi\alpha_P + \tau_\omega \cos \pi\alpha_\omega + \tau_P \tau_\omega \cos \pi(\alpha_P - \alpha_\omega)]}$$

$$= \left[b_{1+}^{P} b_{1+}^{\omega} - \frac{t}{4m_N^2} \alpha_P \alpha_\omega b_{2+}^{P} b_{2+}^{\omega} \right]^2 s_o^{P} s_o^{\omega} (1 - \frac{t}{4m_N^2})^2 \quad , \qquad (4.5)$$

where τ_P, τ_ω indicate the signature of the P, ω trajectory. All the other D functions can be obtained simply by changing the indices.

The circumstance that the coefficients D_{ij} are perfect squares is a result of the facts that the amplitudes can be factored and that all particles are nucleons.

These same results can be obtained in a very simple way by expressing the cross section in terms of helicity amplitudes, and using the results of Section 2 to relate the helicity amplitudes to the factored residues b_{1+} and b_{2+} (23, 24). This same method also allows a simple derivation of the expressions for the polarized cross sections (23, 24).

B. Total Cross Sections

By the optical theorem, the total cross section is related to the imaginary part of the forward scattering amplitude,

$$\sigma_{tot}(s) = \frac{-1}{\sqrt{s(s-4m_N^2)}} \quad \text{Im } T(s, t = 0) \quad . \qquad (4.6)$$

To apply this formula, we need to evaluate the spin average of each of the Fermi invariants in the forward direction. We shall do this by computing the two helicity amplitudes T(++, ++) and T(+-, -+) for

$t = 0$, where the $\pm$ signs denote the helicities of particles (1', 2', 2,1). We find

	(++, ++)	(+-, -+)
S	$4m_N^2$	$4m_N^2$
T	$-4m_N^2$	$4m_N^2$
A	$-(2s-4m_N^2)$	$(2s-4m_N^2)$
V	$(2s-4m_N^2)$	$(2s-4m_N^2)$
P	0	0

(4.7)

Since a trajectory with quantum numbers $\tau P = -$, $(-)^I GP = -$ gives a contribution only to F_P, it makes no contribution to the total cross section. In particular, there will be no terms in formulas for the total cross sections arising from the π-meson and η-meson trajectories. The contributions to the spin-averaged, total cross sections from the P, P', ω and ρ trajectories are

$$\sigma_{pp} = \{B^P - B^\omega R_\omega(v) + B^{P'} R_{P'}(v) - B^\rho R_\rho(v)\} (1 - 1/v^2)^{-1/2} \tag{4.8a}$$

$$\sigma_{np} = \{B^P - B^\omega R_\omega(v) + B^{P'} R_{P'}(v) + B^\rho R_\rho(v)\} (1 - 1/v^2)^{-1/2} \tag{4.8b}$$

$$\sigma_{p\bar{p}} = \{B^P + B^\omega R_\omega(v) + B^{P'} R_{P'}(v) + B^\rho R_\rho(v)\} (1 - 1/v^2)^{-1/2} \tag{4.8c}$$

where

$$B = (2m_N^2/s_o)^{(\alpha(0)-1)}[b_{1+}(0)]^2$$

$$v = s/2m_N^2 - 1$$

$$R(v) = \frac{\sqrt{\pi}\ \ [\alpha(0)]!}{2^{\alpha(0)}[\alpha(0) - 1/2]!\ v} \quad P_{\alpha(0)}(v) \quad .$$

5. AN ANALYSIS OF RECENT DATA ON NN AND N$\bar{N}$ SCATTERING

We have analyzed the data reported by Diddens et al. (14) on the total cross sections for pp and np scattering and that of Lindenbaum et al. (19) on the $\bar{p}p$ cross sections. We find that the presently available data indicate:

$$B^{P} \approx 38 \text{ mb} \quad , \quad B^{P'} \approx 53 \text{ mb} \quad , \quad B^{\omega} \approx 48 \text{ mb} \quad , \quad B^{\rho} \approx -9 \text{ mb} \quad ,$$

$$\alpha_{P'} \approx 0.3 \quad , \quad \alpha_{\omega} \approx 0.3 \quad , \quad \alpha_{\rho} \approx 0.4 \quad . \qquad (5.1)$$

We should like to make several comments on our analysis and its results:

i) The inclusion of the nucleon's spin does not give any appreciable modification of the structure of the Regge analysis of the total cross sections.

ii) A study of the Legendre functions $P_{\alpha(0)}(v)$ indicates that for $\alpha < 2$, P_{α} is represented by its leading term to better than 10% for $v > 2$. Since $v = (E_{lab})/m$, it is certainly sufficient, for incident energies above 2 GeV, to keep only the leading term in any practical analysis of data. Moreover, the replacement of the Legendre functions of the first kind, $P_{\alpha}(v)$, by Legendre functions of the second kind, $Q_{-\alpha-1}(v)$, does not alter the fact that only the first term, v^{α}, in the expansion of these functions need be kept in the analysis, even though the $Q_{-\alpha-1}(v)$ are singular at $v = +1$. This simplifies the analysis, but eliminates the hope that perhaps the introduction of a second vacuum trajectory with α in the range 0 to 1 could be avoided provided that one included the full contribution from the

Regge poles on the Pomeranchuk, omega, rho and "ABC" trajectories.

iii) Our analysis requires that the location of a possible second vacuum pole, $\alpha_{P'}(0)$, be significantly larger than zero, so that it is unlikely that the trajectory could be associated with the ABC anomaly.

iv) Our results are somewhat different from those of Hadjioannou <u>et al</u>. (21), who arbitrarily assumed $\alpha_{\omega}(0) = \alpha_{P'}(0) = 0.5$ and neglected the ρ trajectory.

v) The sign of the ρ term is opposite to that of the ω term. If a pole analysis is to be taken at all seriously, this is puzzling since it should be positive. This discrepancy may well arise from present inaccuracies in the np data. Alternatively, this may mean that the cut associated with the ρ trajectory is not small near $t = 0$, and indeed overrides the pole part of the contribution.

vi) We can interpret our results for the P and P' trajectories as follows. The analysis of the data indicates the presence of an additional singularity besides the P, ω and ρ poles. This we attribute to a cut associated with the P trajectory. If the cut is approximated, near $t = 0$, by a pole, then this pole is described by the parameters we have associated with the P', and whose numerical values are as given above. In so doing, we have ignored possible cuts associated with the ρ and ω.

vii) This analysis suggests a possible explanation for the apparent lack of shrinkage (34, 35) in the πp diffraction peaks. Note that the pp cross sections receive contributions from the P, P', and ω trajectories. (We suppose the ρ contribution to be small.)

Each of these contributions is individually large, but the contribution of the P' is cancelled out by that of the ω, leaving just the P as the dominant contributor. In πp scattering, on the other hand, the ω can not contribute at all, which leaves the P' as a competitor of the P. These two contributions could well combine to give a resultant shrinkage which is much less rapid, over a given range of s, than that observed in pp scattering. Note that this explanation does not depend in any essential way on the supposition that the P' is a pole, rather than a cut associated with the P trajectory.

Finally, we have analyzed the data of Diddens et al. (15) on the pp elastic differential cross sections. These data lie in thr range

$12 < \frac{s}{2m_N^2} - 1 = E_L/m_N < 28$ and $0 \leq -t < 0.60$ GeV^2.

Only the Pomeranchuk contribution was included. The cross section is then given by Equations (4.3) and (4.4),

$$\left[\frac{d\sigma}{dt} \Big/ \frac{d\sigma}{dt}\Big|_{t=0}\right] = \left[\left(b_{1+}^2(t) - \alpha^2(t)\, b_{2+}^2(t)\, \frac{t}{4m_N^2}\right) \left(1 - \frac{t}{4m_N^2}\right) \frac{1}{b_1^2(0)} \right]^2$$

$$\times \left[\frac{2s + t - 4m_N^2}{2s_0} \right]^{2\alpha_P(t)-2} \quad . \qquad (5.2)$$

We note that in this one-pole approximation, the differential cross section involves only one unknown function, namely,

$$F(t) = b_{1+}^2(t) - \alpha^2(t)\, b_{2+}^2(t)\, \frac{t}{4m_N^2} \quad . \qquad (5.3)$$

We assume for $\alpha(t)$ the linear behavior

$$\alpha(t) = 1 + t \tag{5.4}$$

in accord with existing data.

According to Gell-Mann's ghost suppression mechanism (3) the residue $F(t)$ must contain a factor $\alpha(t)$ in order to eliminate the possibility of a ghost at $\alpha = 0 (t \sim -1\ (\mathrm{GeV})^2)$. The resulting quantity, $F(t)/\alpha(t)$, we expect to be nearly constant for small negative t.

The arbitrary parameter s_o is to be chosen so that $F(t)/\alpha(t)$ varies as slowly as possible with t. We try the values $s_o = 1, 2, 3\ (\mathrm{GeV})^2$. Results are summarized in Figure 2. We see from the figure that the function $F(t)/\alpha(t)$ has a linear behavior for $t \lesssim -0.40\ (\mathrm{GeV})^2$. Beyond this point, $F(t)/\alpha(t)$ shows a marked increase reflecting a corresponding increase in the experimental value of $d\sigma/dt$. The graphs show quite clearly that the function $F(t)/\alpha(t)$ is most nearly constant for $s_o = 1\ (\mathrm{GeV})^2$.

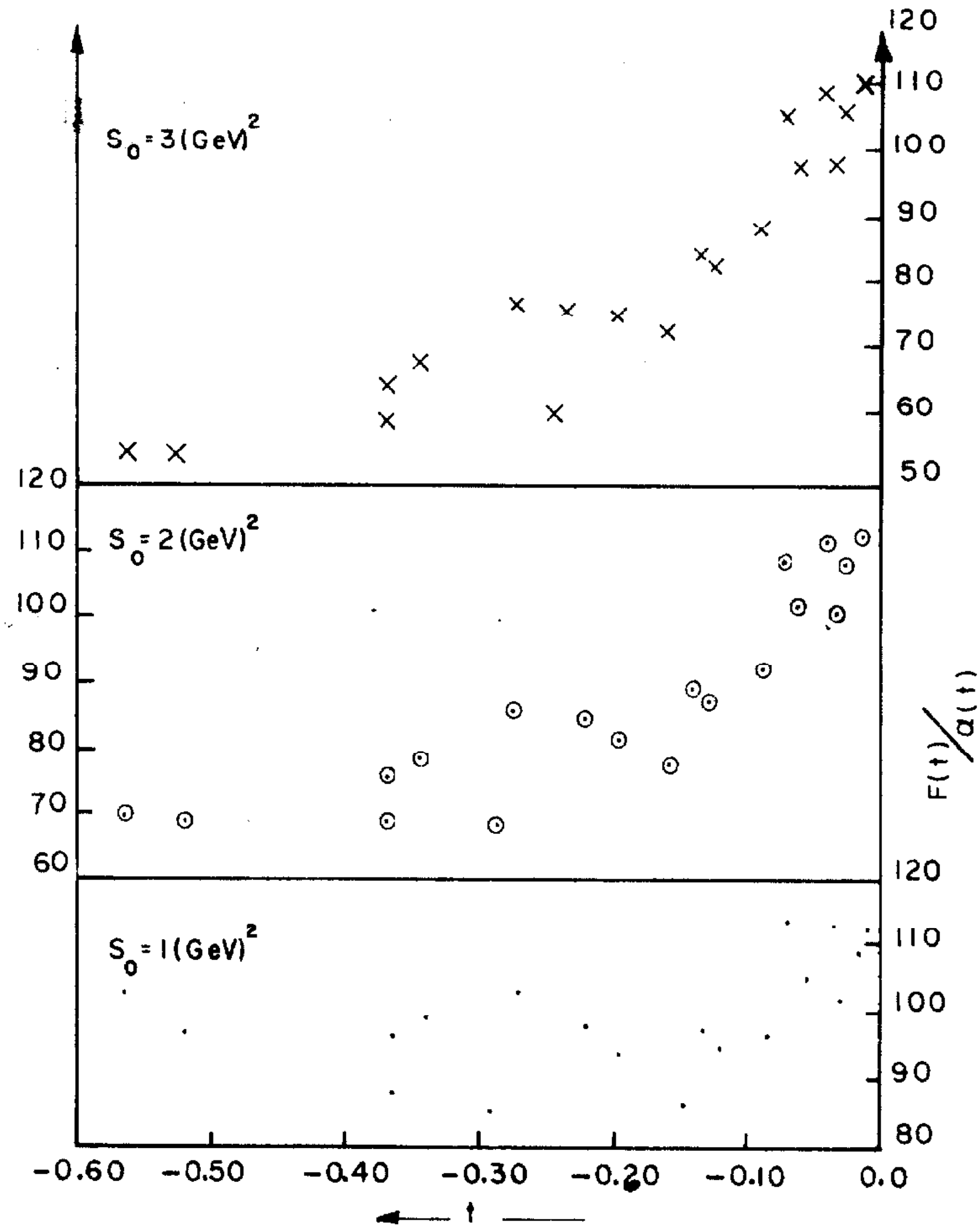


FIGURE 2: $F(t)/\alpha(t)$ vs. t for $s_o = 1,2,3$ $(GeV)^2$. The experimental uncertainty in each point is typically about 15%

Part III

ON THE DYNAMICAL DETERMINATION OF THE REGGE POLE PARAMETERS

1. INTRODUCTION

If Regge poles are to play an important role in understanding the properties of high energy scattering cross sections and of the many newly observed resonances, it appears essential to have a method for the dynamical determination of the Regge pole parameters. This belief is based on the following considerations.

(1) Recent measurements of the angular distributions in πp and pp scattering (15, 34, 35) at high energies $(15 < s/2m_N^2 < 25)$ have been analyzed on the basis of a Regge pole model. The constancy of the total cross sections in the two systems at these energies at first suggested that one can assume that the dominant contribution to the cross sections comes from the Pomeranchuk trajectory. That this assumption cannot be correct in both cases, at least as far as the differential cross sections are concerned, is shown by the facts that almost no diffraction shrinking is observed in the πp system while considerable shrinking is observed in the pp system. If the hypothesis that Regge poles dominate the high energy scattering is still valid, it must mean that in the present energy range the analysis of the cross sections is complicated by the presence of several trajectories contributing in an important way. If this is the case it would seem that reasonably clear cut experimental tests of the Regge predictions about total cross sections and diffraction peaks would be possible only if the Regge pole parameters involved were known functions.

(2) There is some reason to believe (10) that when multiparticle states are included in the analysis of relativistic scattering processes, the analyticity properties of the S-matrix in the J-plane will be complicated by the presence of cuts in addition to simple poles. This circumstance would result in further ambiguities in the interpretation of experimental data, which would be somewhat alleviated if the pole parameters were known.

(3) It is a consequence of the Regge formalism that a set of resonances or bound states, all having the same quantum numbers including J-parity, but having different values of J and occurring at different energies, will all lie along the same Regge trajectory (8, 9a) $\alpha(t)$. The existence of Regge cuts should not lead to any ambiguities in experimentally establishing the existence and properties of any such resonances. For this reason the possibility of grouping the new resonances in Regge families, and of correlating a set of resonance parameters with each other and with the observed total cross sections and angular distributions remains as an interesting application of the Regge theory. To make good use of this possibility, however, it again seems essential to have a method with which to determine the Regge pole parameters.

It is our purpose in this part of the thesis to make use of the analytic properties of the Regge pole parameters $\alpha(t)$ and $r(t)$ plus the unitarity condition satisfied by the partial wave amplitude to derive a coupled set of integral equations which determine the location $\alpha(t)$ and the residue $r(t)$ of a Regge pole as functions of t. We shall discuss a few of the formal properties of these equations and obtain numerical solutions of them in several interesting cases.

In Section 2 we discuss the analytically continued partial wave amplitude and the unitarity condition relating the Regge parameters. Among the more important assumptions that we shall make here are:

i) Validity of the Mandelstam representation with real singularities only.

ii) That the partial wave amplitudes may be analytically continued to complex ℓ without encountering natural boundaries. The possibility of essential singularities, in particular (36) those at $\ell = -1, -2, \ldots,$ is not excluded here or in the discussion of the analyticity of $\alpha(t)$ and $r(t)$.

iii) Applicability of a "chopped off" unitarity condition in which only two-particle intermediate states are kept. We do not thereby limit ourselves to elastic scattering.

iv) Finally, the unitarity condition is employed in a form which is valid only when $\text{Im}\,\alpha(t)$ is small. This latter condition implies that the influence of the coupling of one Regge pole to another is neglected. We feel that this approximation can be improved upon once a way is found to express the partial wave amplitude entirely in terms of Regge parameters without a background term.

In Section 3 we investigate the analyticity of $\alpha(t)$ and $r(t)/(q_i q_j)^{\alpha(t)}$ and show that these functions are real analytic with only right hand cuts in t providing the trajectories do not cross. The case in which the trajectories cross has been discussed by H. Cheng (37). We also discuss in this section a possible essential singularity in the partial wave amplitude at negative, integral values of ℓ and show that it is consistent with unitarity and that its presence does not alter the fact that $\alpha(t)$ and $r(t)$ have only right hand cuts in t. Finally we note in this section how these results are modified if the Regge pole being considered is a Fermion, in which case it is more convenient to discuss the Regge parameters as functions of the $\sqrt{t}$, rather than t (38, 39).

In the fourth section we express the real analyticity of $\alpha(t)$ via a dispersion relation. In many cases one can write a convenient dispersion relation for the residue $r(t)$ as well. However, we shall need for our applications only the fact that $r(t)/(q_i\ q_j)^{\alpha(t)}$ is real analytic. In this section we also consider possible subtractions in the dispersion relations and the threshold behavior of $\alpha(t)$ and $r(t)$.

Taken together, the dispersion relation for $\alpha(t)$, the real analyticity of $r(t)/(q_i\ q_j)^{\alpha(t)}$ and the approximate unitarity condition derived in Section 2 form a coupled set of integral equations which determine the Regge parameters $\alpha(t)$ and $r(t)$.

In Section 5 we show how to transform this set of equations so as to obtain an integral equation involving the single unknown function $\mathrm{Im}\,\alpha(t)$. Once $\mathrm{Im}\,\alpha(t)$ is obtained by solving this equation, we obtain $\mathrm{Re}\,\alpha(t)$ and the residue $r(t)$ by performing simple integral transforms.

Because the equations we use are approximate, it is very desirable to compare our results for the Regge parameters with those obtained in some rigorous way. This is possible only in potential theory. Consequently, in Section 6 we specialize the equations derived in Section 5 to their non-relativistic form. We also make in Section 6 a number of comments on the more formal mathematical properties of these equations, especially those related to the uniqueness question.

In Section 7 we present our calculations of the Regge parameters in the case of scattering in a single Yukawa potential of unit range. A wide variety of potential strengths are considered. These results are critically compared to those obtained by Ahmadzadeh, Durke and Tate (40) and by Lovelace and Masson (41).

In Section 8 we solve the equations for the case of relativistic $\pi\pi$ scattering. In the case of $\pi\pi$ scattering, we have obtained the positions and residues of the poles describing the Pomeranchuk trajectory, the ρ-meson trajectory and the second vacuum trajectory introduced by K. Igi (22). The properties of the P-trajectory as computed from our equations agree well with those ascertained by Foley et al. (35) from the analysis of the π^-p angular distributions. We use our results on the ρ-meson trajectory to obtain $\alpha_\rho(t)$, $t \leq 0$, which governs the energy dependence of $\sigma_{\pi^+p} - \sigma_{\pi^-p}$ and of the corresponding angular distributions.

Finally, in Section 9, we summarize the conclusions reached in this paper and outline a number of interesting problems which remain to be investigated.

2. THE ANALYTICALLY CONTINUED PARTIAL WAVE AMPLITUDE AND THE UNITARITY CONDITION

It is well known that the conventional partial wave amplitude (ℓ a non-negative integer) for the scattering process $a + b \rightarrow c + d$ in which the particles have masses m_a, m_b, m_c, m_d, can be expressed as

$$A_{ij}(\ell,t) = \frac{1}{\pi}\int_{s_o}^{\infty} \frac{ds}{2q_iq_j}\, Q_\ell\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}}{2q_iq_j}\right) D^s_{ij}(t,s)$$

$$+ \frac{(-1)^\ell}{\pi}\int_{u_o}^{\infty} \frac{du}{2q_iq_j}\, Q_\ell\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)}}{2q_iq_j}\right) D^u_{ij}(u,t), \tag{2.1}$$

if ℓ is large enough so that the integrals in (2.1) converge.

In the equation, q_j and q_i are the C.M. momenta of the incoming and outgoing particles in the t-channel, respectively, j is the state $a + b$ and i is the state $c + d$, and $D^s_{ij}(t,s)$, $D^u_{ij}(t,u)$ are the absorptive parts of the scattering amplitude $A_{ij}(s,t,u)$ in the s and u channels. Since $Q_\ell(z)$ is a meromorphic function of ℓ with poles at the negative integers $\ell = -1, -2, \ldots$, Equation (2.1) provides an analytic continuation of $A(\ell,t)$ if the integrals on the right hand side of Equation (21.) converge (6). For large z, $Q_\ell(z) \propto 1/z^{\ell+1}$, hence the integrals in Equation (2.1) converge uniformly in the region $\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha$ if $D^s_{ij}(t,s)$ and $D^u_{ij}(t,u)$ diverge no faster than s^α and u^α, respectively, for large s and u. The factor $(-1)^\ell$ for ℓ complex can be defined in various ways. For example, we can define it to be either $e^{i\pi\ell}$ or $e^{-i\pi\ell}$; however, we observe that $(-1)^\ell$ for ℓ an integer takes the value ± 1 according as ℓ is even or odd. We can therefore choose the two independent amplitudes

$$A^{\pm}_{ij}(\ell,t) = \frac{1}{\pi}\int_{s_o}^{\infty} \frac{ds}{2q_iq_j} Q_\ell \left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)}}{2q_iq_j} \right) D^s_{ij}(t,s)$$

$$\pm \frac{1}{\pi}\int_{u_o}^{\infty} \frac{du}{2q_iq_j} Q_\ell \left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)}}{2q_iq_j} \right) D^u_{ij}(t,u) \quad , \tag{2.2}$$

which correspond to amplitudes with plus or minus signature, (2).

Making use of the formula

$$Q_\ell(z) = \frac{\Gamma^2(\ell+1)}{2\Gamma(2\ell+2)} \left(\frac{z-1}{2}\right)^{-\ell-1} F(\ell+1,\ \ell+1;\ 2\ell+2;\ \frac{2}{1-z}) \quad ,$$

we obtain

$$B^{\pm}_{ij}(\ell,t) \equiv A^{\pm}_{ij}(\ell,t)/(q_iq_j)^\ell$$

$$= \frac{\Gamma^2(\ell+1)}{4\Gamma(2\ell+2)} \frac{1}{\pi} \left[\int_{s_o}^{\infty} \left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)} - 2q_iq_j}{4} \right)^{-\ell-1} D^s_{ij}(t,s) \right.$$

$$\times F\left(\ell+1,\ \ell+1;\ 2\ell+2,\ -\frac{4q_iq_j}{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)} - 2q_iq_j}\right) ds$$

$$\pm \int_{u_o}^{\infty} du \left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)} - 2q_iq_j}{4} \right)^{-\ell-1} D^u_{ij}(t,u)$$

$$\left. \times F\left(\ell+1,\ \ell+1;\ 2\ell+2;\ -\frac{4q_iq_j}{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2 + q_j^2)(m_d^2 + q_i^2)} - 2q_iq_j}\right) \right] \tag{2.3}$$

valid in the region $\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha$. We can observe at this point that the function $B^{\pm}_{ij}(\ell,t)$ defined in (2.3) is a real analytic function satisfying

$$B^{\pm}_{ij}(\ell,t) = (B^{\pm}_{ij}(\ell^*, t^*))^* \quad . \tag{2.4}$$

We shall now show that each of the amplitudes $A^{\pm}_{ij}(\ell,t)$ satisfy the unitarity condition. We first define $(A^{\pm}_{ij}(\ell,t))^+$ as

$$(A^{\pm}_{ij}(\ell,t))^+ = \frac{1}{\pi}\int_{s_o}^{\infty} \frac{ds}{2q_iq_j}\, Q_\ell\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2+q_j^2)(m_c^2+q_i^2)}}{2q_iq_j}\right) D^s_{ij}(t^-,s)$$

$$\pm \frac{1}{\pi}\int_{u_o}^{\infty} \frac{du}{2q_iq_j}\, Q_\ell\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2+q_j^2)(m_d^2+q_i^2)}}{2q_iq_j}\right) D^u_{ij}(t^-,u) \quad .$$

In the above, $t^- = t - i\epsilon$. For $\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha$, we have

$$\left[A^{\pm}_{ij}(\ell,t) - (A^{\pm}_{ij}(\ell,t))^+\right]/2i =$$

$$\frac{1}{\pi}\int_{s_o}^{\infty} \frac{ds}{2q_iq_j}\, Q_\ell\left(\frac{s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2+q_j^2)(m_c^2+q_i^2)}}{2q_iq_j}\right)\rho_{ij}(t,s)$$

$$\pm \frac{1}{\pi}\int_{u_o}^{\infty} \frac{du}{2q_iq_j}\, Q_\ell\left(\frac{u - m_a^2 - m_d^2 + 2\sqrt{(m_a^2+q_j^2)(m_d^2+q_i^2)}}{2q_iq_j}\right)\rho_{ij}(t,u) \ . \tag{2.5}$$

Now, the unitarity condition for the scattering amplitued $A(s,t,u)$ reads (43)

$$\rho_{ij}(t,s) = \sum_k \frac{\theta(t-t_k)}{4\omega\, q_k\, q_i\, q_j\, \pi} \left[\iint \frac{(D^s_{ki}(t,s'))^* \, D^s_{kj}(t,s'')\, ds'\, ds''}{\sqrt{x^2 + x_k'^2 + x_k''^2 - 2x\, x_k'\, x_k'' - 1}} \right.$$

$$\left. + \iint \frac{(D^u_{ki}(t,u'))^* \, D^u_{kj}(t,u'')\, du'\, du''}{\sqrt{x^2 + y_k'^2 + y_k''^2 - 2x\, y_k'\, y_k'' - 1}} \right] \quad t > t_o \quad . \qquad (2.6)$$

In Equation (2.6), $x = \left[s - m_a^2 - m_c^2 + 2\sqrt{(m_a^2 + q_j^2)(m_c^2 + q_i^2)} \right] / 2q_i q_j$,

$$x_k' = \left[s' - m_c^2 - m_{k1}^2 + 2\sqrt{(m_c^2 + q_i^2)(m_{k1}^2 + q_k^2)} \right] / 2q_i q_k \quad ,$$

$$x_k'' = \left[s'' - m_a^2 - m_{k1}^2 + 2\sqrt{(m_a^2 + q_j^2)(m_{k1}^2 + q_k^2)} \right] / 2q_j q_k \quad ,$$

$$y_k' = \left[u' - m_c^2 - m_{k2}^2 + 2\sqrt{(m_c^2 + q_i^2)(m_{k2}^2 + q_k^2)} \right] / 2q_i q_k \quad ,$$

$$y_k'' = \left[u'' - m_a^2 - m_{k2}^2 + 2\sqrt{(m_a^2 + q_j^2)(m_{k2}^2 + q_k^2)} \right] / 2q_j q_k$$

, where q_k is the C.M. momentum in the intermediate state k which contains two particles whose masses are m_{k1} and m_{k2}, and ω is the C.M. energy of the system. The integration above is over the region where

$x > x_k' x_k'' + \sqrt{(1 - x_k'^2)(1 - x_k''^2)}$ for the first term in (2.6), and a similar region for the second term. Also, $\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$, t_k is the threshold for the intermediate state k, and t_o the mass squared of the multi-particle state with the lowest energy and the same quantum numbers as state i. Likewise,

$$\rho_{ij}(t,u) = \sum_k \frac{1}{4\omega\, q_k q_i q_j\, \pi} \left[\iint \frac{(D^s_{ki}(t,s'))^* D^u_{kj}(t,u'')\, ds'\, du''}{\sqrt{x^2 + x'^2_k + y''^2_k + 2x\, x'_k\, y''_k - 1}} \right.$$

$$\left. + \iint \frac{(D^u_{ki}(t,u'))^* D^s_{kj}(t,s'')\, du'\, ds''}{\sqrt{x^2 + y'^2_k + x''^2_k + 2x\, x''_k\, y'_k - 1}} \right] \theta(t-t_k),\ t > t_o \quad , \qquad (2.7)$$

with a similar range of integration. From the formula (44)

$$\int_0^\infty \frac{Q\ (x)\ \theta(x - x_1 - x_2 - \sqrt{(1-x_1^2)(1-x_2^2)})}{\sqrt{x^2 + x'^2 + x''^2 - 2x\, x'\, x'' - 1}}\ dx = Q_\ell(x_1)\, Q_\ell(x_2),$$

we obtain by substituting (2.6) and (2.7) into (2.4) and some algebraic manipulation

$$\left[(A^{\pm}_{ij}(\ell,t) - (A^{\pm}_{ij}(\ell,t))^+ \right] /2i = \sum_k \frac{q_k}{\omega} (A_{ki}(\ell,t))^+ A_{kj}(\ell,t)\ \theta(t-t_k),\ t > t_o, \qquad (2.8)$$

valid in the region $\mathrm{Re}\ \ell > \mathrm{Re}\ \alpha$. Now both sides of Equation (2.8) are analytic functions of ℓ, and by analytic continuation (2.8) holds in the whole region of ℓ where $A(\ell,t)$ is analytic.

For the case of equal mass, elastic scattering, the argument of Q_ℓ in $A^{\pm}(\ell,t)$ of (2.8) is always greater than 1. As $Q_\ell(z)$ is continuous on the real axis of z if $z > 1$, we have in the elastic scattering case

$$(A^{\pm}(\ell,t))^+ \quad = \quad (A^{\pm}(\ell^*,\ t))^* \quad ,$$

and (2.8) takes the familiar form

$$\left[A^{\pm}(\ell,t) - (A^{\pm}(\ell^*,t))^*\right]/2i = \frac{q}{\omega} (A(\ell^*, t))^* A(\ell,t) . \qquad (2.9)$$

Writing $A^{\pm}(\ell,t) = \frac{\omega}{q} \frac{S^{\pm}(\ell,t) - 1}{2i}$, we also have from (2.9)

$$S^{\pm}(\ell,t)(S^{\pm}(\ell^*,t))^* = 1. \qquad (2.10)$$

We assume that $A(\ell,t)$ has a pole at $\ell \approx \alpha(t)$. Then if we compare the residue of both sides of (2.8) at $\ell = \alpha$, we obtain (omitting signature)

$$\frac{r_{ij}}{2i} = \sum_k \frac{q_k}{\omega} (A_{ki}(\alpha,t))^+ \, r_{kj}(t) \, \theta(t-t_k) \quad . \qquad (2.11)$$

Equating the real part and the imaginary part of (2.11) would give us two algebraic equations relating α and r_{ij} along the positive real axis (for $t > t_o$).

Several interesting consequences follow from (2.11). First, as was pointed out by Gribov and Pomeranchuk (29b), Equation (2.11) requires that $r_{ij}(t)$ is factorized and we have relations like (29a)

$$r_{ii}(t) = r_{ij}^2(t)/r_{jj}(t) \quad . \qquad (2.12)$$

Secondly, from (2.12) we see that if any two of $r_{ij}(t)$, $r_{ii}(t)$, and $r_{jj}(t)$ are non-zero, then the third would be non-zero. Hence, if a Regge pole occurs in any two of the amplitudes $A_{ii}(\ell,t)$, $A_{ij}(\ell,t)$, $A_{jj}(\ell,t)$, it automatically occurs in the third. The unitarity condition thus implies that the <u>same</u> Regge poles occur in all channels (2, 8). Also, since in the unitarity condition (2.9) we include only those intermediate states k which have the same conserved quantum numbers as states i and j, a Regge pole trajectory is characterized by a set of conserved quantum numbers. Since the $A^{\pm}(\ell,t)$ satisfy (2.11) separately, a Regge pole trajectory has a definite signature (2).

Equation (2.11) also leads directly to the form of the unitarity condition as it will be used in this work. We suppose that $A(\ell,t)$ has a pole at $\ell \approx \alpha(t)$ and that $\mathrm{Re}\,\alpha(t_1) > -1/2$ at the threshold energy t_1. Then we may write,

$$A^{\pm}_{ij}(\ell,t) = A^{*}_{ij}(\ell^{*},t) \approx \frac{-r^{*}_{ij}(t)}{\alpha^{*}(t) - \alpha(t)} \quad . \tag{2.13}$$

Substitution of equation (2.13) into equation (2.11) then gives the result we seek

$$r_{ij} = \frac{1}{\mathrm{Im}\,\alpha(t)} \sum_{k} \frac{q_k}{\omega} r^{*}_{ki}\, r_{kj}\, \theta(t - t_k) \quad . \tag{2.14}$$

We wish to emphasize at this point that the unitarity condition as expressed in (2.14) is approximate in two important respects.

First, intermediate states of more than two particles have not been included. The extent to which a scattering process can be described by two-body intermediate states is not clear, here or in any such application of the unitarity condition. We note, however, that this short-coming can probably be removed, at least in principle, when techniques for handling multi-particle intermediate states are developed.

Second, we have approximated $A_{ij}(\ell,t)$ by $r_{ij}/(\ell - \alpha(t))$, which is valid only for $\ell \approx \alpha(t)$ and hence only for $\mathrm{Im}\,\alpha(t)$ small. Attempts to improve this are being made. At any rate we know that for some range of t extending upwards from the elastic threshold $\mathrm{Im}\,\alpha(t)$ is indeed small. By putting enough subtractions in the dispersion relations we can hope to make the contribution from large t unimportant in the low energy region. We might therefore expect the functions $\alpha(t)$ and $r(t)$ which we obtain with these equations to be accurately given for t in the low energy range.

3. THE ANALYTICITY OF $\alpha(t)$ AND $r(t)$

In this section we shall present arguments to make plausible the hypothesis that the function $\alpha(t)$ and $r(t)$, for a boson trajectory, are analytic functions of t.

We shall start from the assumption that $A(\ell,t)$ can be analytically continued to the whole ℓ plane. In the proof, the possibility of essential singularities, in particular (36) those at $\ell = -1, -2, \ldots$, is indicated as well as the possibility of the crossing of two Regge trajectories.

We first evaluate explicitly the discontinuity across the left-hand cut of $A(\ell,t)$. We shall, for simplicity, take $m_a = m_b = m_c = m_d = 1$ and $s_o = u_o = 4$. We then obtain

$$h^{\pm}(\ell,t) = \frac{A^{\pm}(\ell,\ t+i\epsilon) - A^{\pm}(\ell,\ t-i\epsilon)}{2i}$$

$$= - \frac{1}{\pi} \int_4^{-t} \frac{2ds}{t-4}\ Q_\ell(1 + \frac{2s}{t-4-i\epsilon})\ \rho_{su}(s,\ 4-t-s)$$

$$\mp \frac{1}{\pi} \int_4^{-t} \frac{2du}{t-4}\ Q_\ell(1 + \frac{2u}{t-4+i\epsilon})\ \rho_{su}(4-u-t,\ u) + \frac{1}{2} \int_4^{-(t-4)} \frac{2ds}{t-4}\ e^{-i\ell\pi}$$

$$\times P_\ell(-1 - \frac{2s}{t-4})\ D^s(t+i\epsilon,\ s) \pm \frac{1}{2} \int_4^{-(t-4)} e^{-i\pi\ell}\ \frac{2du}{t-4}\ P_\ell(-1 - \frac{2u}{t-4})\ D^u(t-i\epsilon,\ u) \quad , \tag{3.1}$$

for $t < 0$.

Since the range of integration in Equation (3.1) is finite, this equation gives $h^{\pm}(\ell,t)$ for all ℓ. We thus see from Equation (3.1) that $h^{\pm}(\ell,t)$ is a meromorphic function of ℓ, with simple poles at the negative integers $\ell = -1, -2, -3, \ldots$. It has been shown by Gribov and Pomeranchuk (36) that an essential singularity is required to exist in $A(\ell,t)$ at these points.

If we write

$$A^{\pm}(\ell,t) = \frac{t^n}{\pi} \int_{-\infty}^{0} dt\ \frac{h^{\pm}(\ell,t')}{(t'-t)t'^n} + A_1^{\pm}(\ell,t) \quad , \tag{3.2}$$

where n is large enough for the integral to converge, then $A_1^{\pm}(\ell,t)$ is an analytic function of t with a right-hand cut only. Since no Regge pole can come from the term

$$\frac{t^n}{\pi} \int_{-\infty}^{0} dt' \; \frac{h^{\pm}(\ell,t')}{(t'-t)t'^n} \quad ,$$

all Regge trajectories are determined by the equation

$$F^{\pm}(\alpha,t) \equiv \frac{1}{A_1^{\pm}(\alpha,t)} = 0 \quad ;$$

or
$$\frac{d\alpha(t)}{dt} = - \frac{\partial F(\alpha,t)}{\partial t} \Big/ \frac{\partial F(\alpha,t)}{\partial \alpha} \quad . \qquad (3.3)$$

Here the signature has been omitted.

We shall first investigate the right-hand side of (3.3) at the points $\alpha = -1, -2, -3, \ldots$. Let us assume that the most singular term of $F(\ell,t)$ at $\ell = -n$, is of the form (45) $\exp \sum_i \frac{a_i(t)}{(\ell+n)^i}$, where i is a number (not necessarily an integer). Then we have

$$\frac{d\alpha}{dt} = - \Sigma \; \frac{a_i(t)}{(\alpha+n)^i} \Big/ - \Sigma \; \frac{i a_i(t)}{(\alpha+n)^{i+1}} = 0 \quad . \qquad (3.4)$$

Thus the right-hand side of (3.3) is regular at $\alpha = -n$, for this type of essential singularity. We have no proof that the essential singularity in the partial wave amplitude actually has the (rather general) form we have chosen.

The right-hand side of (3.3) is thus seen to be an analytic function of t with only a right-hand cut, and entire in α, provided that $\partial F(\alpha,t)/\partial\alpha \neq 0$ for all (α,t). Solving (3.3) thus gives α as an analytic function of t with a right-hand cut only. Then $r(t)$, which is equal to $\mathrm{Res}\,(A_1(\ell,t))_{\ell=\alpha(t)}$, is also an analytic function of t with only a right-hand cut.

If at a given point (α_o, t_b) we have $\partial F(\alpha,t)/\partial\alpha = 0$ as well as $F(\alpha,t) = 0$, which is the condition that two or more Regge poles cross at the point, then $\alpha(t)$ is not analytic at t_b since $d\alpha(t)/dt$ equals infinity there. Assume $\alpha(t_b)$ is not infinite, then t_b is a branch point for $\alpha(t)$ and a branch cut will arise.

Since $A_{ij}(\ell,t)/(q_i q_j)^\ell$ is a real analytic function of t and ℓ, $F(\ell,t)/(q_i q_j)^\ell$ is a real analytic function of t and ℓ. Thus for t negative and real, $F(\alpha,t) = 0$ implies $F(\alpha^*,t) = 0$. Therefore, either there are two Regge pole trajectories $\alpha_1(t)$ and $\alpha_2(t)$ which are complex conjugate to each other for negative t, and hence satisfy

$$\alpha_1(t) = \alpha_2^*(t^*) \tag{3.5a}$$

for all t (as $\alpha_1(t)$ and $\alpha_2(t)$ are analytic functions of t), or

$$\alpha(t) = \alpha^*(t) \tag{3.5b}$$

for t negative and $\alpha(t)$ is a real analytic function of t. In the latter case, $r_{ij}(t)/(q_i q_j)^\alpha$ is also a real analytic function of t. In the discussion that follows, we shall assume that this in fact is the case.

The location of the right-hand cut of $\alpha(t)$ and $r(t)$ is seen to coincide with that of the right-hand cut in t of the scattering amplitude $A(s,t)$, hence it starts from the mass squared of the lowest energy multi-particle state which has the same conserved quantum numbers (except for the angular momentum) as $\alpha(t)$. For example, if we consider the Regge trajectory $\alpha_\omega(t)$ which gives the 3π resonance $I = 0$, $J = 1$ at 787 MeV, the branch cut starts at $9m_\pi^2$.

The analytic properties of the functions $\alpha(t)$ and $r(t) / (q_i q_j)^{\alpha(t)}$ are unchanged if scattering by particles of unequal mass is considered. However in this case the function $r(t)$, without the kinematic factor $(q_i q_j)^{\alpha(t)}$, will acquire an additional right hand cut extending from 0 to $(m_a - m_b)^2$.

For a fermion Regge trajectory, it has been found (38, 39) that the Regge parameters are best discussed as functions of $\sqrt{t}$, in order to avoid kinematic singularities. Consider the scattering of a fermion by a spinless boson. There are two partial-wave amplitudes, $A(j^+, \sqrt{t})$ and $A(j^-, \sqrt{t})$, corresponding to the two states with the total angular momentum j and parity $(-1)^{j \pm 1/2}$. Both of these are analytic functions of $\sqrt{t}$ with branch cuts: (i) from $\sqrt{t_o}$ to ∞, where t_o is the energy squared of the lowest mass state having the appropriate quantum numbers; (ii) from $-\sqrt{t_o}$ to $-\infty$; (iii) from $-i\infty$ to $i\infty$. The third kind of branch cut corresponds to a left-hand cut in t in the boson case. A generalization of the arguments presented in this section then shows that $\alpha(\sqrt{t})$ and $r_{ij}(\sqrt{t})$ are analytic functions of $\sqrt{t}$ with a right-hand cut from $\sqrt{t_o}$ to ∞ and a left-hand cut from $-\sqrt{t_o}$ to $-\infty$, in addition to those arising from the crossing of two Regge trajectories.

4. DISPERSION RELATIONS FOR THE REGGE POLE PARAMETERS

In the preceding section we have shown, assuming that $A(\ell,t)$ is an analytic function of ℓ (possibly with essential singularities) and the validity of the Mandelstam representation, that $\alpha(t)$ and $r(t)$ are analytic functions of t with branch cuts only along the positive real axis.

The function $\alpha(t)$ is assumed to have a behavior at infinity which permits us to express its real analyticity by means of a dispersion relation of the simple form

$$\alpha(t) = \alpha_o + \frac{t - t_o}{\pi} \int_{T_o}^{\infty} \frac{\text{Im}\, \alpha(t')\, dt'}{(t'-t)(t'-t_o)} \quad , \tag{4.1}$$

where $T_o = (m_a + m_b)^2$. For the case of equal mass scattering, we can also write a simple dispersion relation for the real analytic function

$c(t) = r(t)\, e^{-i\pi\alpha(t)}$;

$$c(t) = c(t_o) + \frac{t - t_o}{\pi} \int_{T_o}^{\infty} \frac{\text{Im}\, c(t')\, dt'}{(t'-t)(t'-t_o)} \tag{4.2}$$

If the masses of the scattering particles are not all equal it is not so easy to write a convenient dispersion relation involving the residue function $r(t)$. The difficulty is that because $\alpha(t)$ presumably approaches a negative quantity as $t \to \pm\infty$, it is not clear that we can write a dispersion relation for $r(t)/q^{2\alpha(t)}$ in the once-subtracted form of Equation (4.1). We can avoid this difficulty in the case of equal mass scattering by dealing with the function $r(t)\, e^{-i\pi\alpha(t)}$. But if we consider the scattering of particles of unequal mass, a dispersion relation for $r(t)$ would be complicated by the presence of kinematic cuts coming from the factor $q^{2\alpha(t)}$.

We will see in the following section that for the purpose of obtaining equations for the Regge pole parameters from the principles of analyticity and unitarity it is wholly adequate simply to know that $r(t)/q^{2\alpha(t)}$ is real analytic, and no occasion will arise where it is necessary to have a dispersion relation for $r(t)/q^{2\alpha(t)}$. Therefore, we can avoid the complications mentioned above.

For a fermion Regge trajectory, the dispersion relation for $\alpha(w)$, $w = \sqrt{t}$, becomes (46)

$$\alpha_+(w) = \alpha_+(w_o) + \frac{w-w_o}{\pi} \int_{w_T}^{\infty} \frac{\mathrm{Im}\, \alpha_+(w')\, dw'}{(w' - w)(w' - w_o}$$

$$- \frac{w-w_o}{\pi} \int_{w_T}^{\infty} \frac{\mathrm{Im}\, \alpha_-(w')\, dw'}{(w' + w)(w' + w_o)} \quad , \qquad (4.3)$$

where $w_T = (m_a + m_b)$ is the total C.M. threshold energy of the system and w_o is the energy at the point of subtraction. $\alpha_+(w)$ and $\alpha_-(w)$ are Regge poles in the amplitudes $A(j^+, w)$ and $A(j^-, w)$, $j^{\pm} = \ell \pm 1/2$, respectively. These two amplitudes satisfy the unitarity conditions separately and, as before, it relates $\alpha_{\pm}(w)$ to $r_{\pm}(w)$ on the right hand cut.

In writing the dispersion relation (4.3), and throughout this work, we shall disregard the possibility that two Regge trajectories may cross. This problem is treated in a paper by Cheng (37).

The two integral transforms (4.1) and (4.2), plus the two algebraic equations (2.14) which, expressing the unitarity condition, relate $\alpha(t)$ and $r(t)$ along the cut on the positive real axis, form a set of coupled, singular, non-linear integral equations which must be satisfied by the four unknown functions Re $\alpha(t)$, Im $\alpha(t)$, Re $c(t)$ and Im $c(t)$ describing a given Regge trajectory. In the following sections we shall show that these equations can be transformed into a considerably simpler form which makes many of their mathematical properties transparent and permits their solution by iteration. In the remainder of this section we shall discuss various questions that bear either on the question of subtractions, or on the threshold behavior of the Regge parameters.

The singularity of $\alpha(t)$ and $r(t)$ at infinity is not known in the relativistic case. In the case of potential scattering in the potential

$$v(r) = \int_{m^2}^{\infty} \sigma(\mu^2) \frac{e^{-\mu r}}{r} d\mu^2 ,$$

it has been found that (8, 37, 41)

$$\alpha_n(q^2) \xrightarrow[|q^2| \to \infty]{} -n - i \frac{\int_{m^2}^{\infty} \sigma(\mu^2)\, d\mu^2}{2q} \tag{4.4}$$

$$\beta_n(q^2) \xrightarrow[|q^2| \to \infty]{} - \frac{-\pi(2n-1) \int_{m^2}^{\infty} d\mu^2\, \sigma(\mu^2)}{2q^2} , \quad n=1,2,3,\ldots, \tag{4.5}$$

where $\alpha_n(q^2)$ is the n^{th} Regge trajectory and q^2 is the energy. We see from this that no subtractions are necessary, although it may be convenient for practical purposes to make some. In the relativistic case, therefore, it may be reasonable to conjecture that $\alpha(t)$ and $r(t)$ have no singularity at infinity. This is the most appealing conjecture from the theoretical point of view. However, to be on the safe side, we shall usually prefer to make subtractions. How many subtractions are to be made really depends on the specific problem and on what one is willing to supply from the outside as subtraction constants as compared to what one wishes to predict.

We next wish to investigate the behavior of $\alpha(t)$ and $r(t)$ near a threshold. This is of interest for two reasons: (i) The behavior of $\alpha(t)$ and $r(t)$ near a threshold can be rigorously established. It is very important to make use of this information on the functional form of $\alpha(t)$ and $r(t)$ in obtaining approximate solutions to our integral equations. (ii) If subtractions in $r(t)$ are made for those values of t corresponding to thresholds, the subtraction constants can be shown to vanish, hence no additional parameters are introduced.

The behavior of $\alpha(t)$ near threshold in the relativistic many-channel problem has been shown by Barut (47) to be the same as in the elastic scattering case, if only two-particle intermediate states are considered. We shall apply his arguments to obtain the threshold behavior for $\beta(t) = -\pi[2\alpha(t) + 1]\, r(t)$.

The unitarity condition for $B(\ell,t)$ takes the matrix form

$$\frac{B(\ell,t+i\epsilon) - B^*(\ell^*, t+i\epsilon)}{2i} = B^*(\ell^*,t+i\epsilon)\, \rho(t)\, B(\ell,t+i\epsilon) \tag{4.6}$$

where $\rho(t)$ is the matrix with elements

$$\rho_{ij}(t) = \delta_{ij}\ \theta(t-t_i)\ (q_i q_j)^{\ell + \frac{1}{2}}/\omega \quad . \tag{4.7}$$

Making use of the fact that $B(\ell,t) = B^*(\ell^*, t^*)$, we have

$$\frac{B^{-1}(\ell, t+i\epsilon) - B^{-1}(\ell, t-i\epsilon)}{2i} = - \rho(t) \quad , \tag{4.8}$$

which gives the discontinuity of $B(\ell,t)$ across the right-hand cut. Write

$$B^{-1}(\ell,t) = (Y(\ell,t) + R)/\cos \pi\ell \tag{4.9}$$

where

$$R_{ij}(\ell,t) = q_i^{2\ell+1} e^{-i\pi(\ell + \frac{1}{2})} \delta_{ij}\ \theta(t-t_i)/\omega \quad .$$

Then $Y(\ell,t)$ is analytic in t with only a left-hand cut, since R has the same discontinuity across the right-hand cut as $B^{-1}(\ell,t)$. The matrix $Y(\ell,t)$ is the analogue of the Y function previously introduced by Barut and Zwangiger (42) and by Cheng and Gell-Mann (48). From (4.9) we have

$$B(\ell,t) = \frac{\mathrm{adj}\ (Y + R)}{\det\ (Y + R)} \quad \cos \pi\ell \quad , \tag{4.10}$$

and the Regge poles are given by the zeroes of $\det(Y + R)$. Now suppose t is near the threshold t_i of state i so that q_i is small, then we have $\det(Y(\alpha,t) + R(\alpha,t))$

$$= \det(Y(\alpha,t) + R'(\alpha,t)) + R_{ii}(\alpha,t)\ [Y(\alpha,t) + R(\alpha,t)]_{ii} = 0 \quad , \tag{4.11}$$

where $R'(\alpha,t)$ is the matrix obtained from R by deleting the element $R_{ii}(\alpha,t)$, and $[Y(\alpha,t) + R(\alpha,t)]_{ii}$ is the cofactor of the element ii of $(Y + R)$. If we write $F(\alpha,t) = \det(Y(\alpha,t) + R'(\alpha,t))/[Y(\alpha,t) + R(\alpha,t)]_{ii}$, Equation (4.11) becomes

$$F(\alpha,t) = - q_i^{2\alpha+1} e^{-i\pi(\alpha + \frac{1}{2})}/\omega \quad , \qquad (4.12)$$

where $F(\alpha,t)$ is an analytic function of t in the neighborhood of the threshold t_i.

If $\mathrm{Re}\, \alpha(t_i) > -\frac{1}{2}$, then $F(\alpha(t_i), t_i) = 0$. Expanding $F(\alpha,t)$ in a Taylor series and writing $\alpha_i = \alpha(t_i)$, we find

$$\left.\frac{\partial F(\alpha,t)}{\partial \alpha}\right|_{\substack{\alpha=\alpha_i \\ t=t_i}} (\alpha-\alpha_i) + \left.\frac{\partial F(\alpha,t)}{\partial t}\right|_{\substack{\alpha=\alpha_i \\ t=t_i}} (t-t_i) = - q_i^{2\alpha_i+1} e^{-i\pi(\alpha_i + \frac{1}{2})}/\omega \ ,$$

or

$$\alpha(t) \approx \alpha_i + a(t-t_i) + b\, q_i^{2\alpha_i+1} e^{-i\pi(\alpha_i + \frac{1}{2})} \quad . \qquad (4.13)$$

Also

$$\beta_{ij}(t) = - (q_i q_j)^{\alpha(t)} \pi(2\alpha(t) + 1)\ \mathrm{Res}\ (B(\ell,t))_{\ell=\alpha(t)}$$

$$\approx c(q_i q_j)^{\alpha_i} \quad .$$

If $\mathrm{Re}\, \alpha_i < \frac{1}{2}$, then (4.13) shows $F(\alpha_i, t_i) = \infty$. Writing (4.13) in the form

$$\frac{1}{F(\alpha,t)} = -\omega\, q_i^{-2\alpha(t)-1} \quad e^{i\pi(\alpha + \frac{1}{2})}$$

and expanding $1/F(\alpha,t)$ in a Taylor series now gives us

$$\alpha(t) \approx \alpha_i + e(t-t_i) + f\, q_i^{-2\alpha_i-1} \quad e^{i\pi(\alpha_i + \frac{1}{2})} , \tag{4.14}$$

and

$$\beta(t) \approx g\, q_i^{-3\alpha_i-2} \quad q_j^{\alpha_i} , \tag{4.15}$$

where a, b, c, d, e, f, g are constants. At the elastic threshold these constants are all real and are related (42, 48). We see that for inelastic two-body scattering $\beta_{ij}(t)$ will vanish at <u>two</u> points, i.e., will vanish if $q_i \to 0$ <u>or</u> if $q_j \to 0$, provided that either $\text{Re}\,\alpha(t_i) > 0$ or $\text{Re}\,\alpha(t_i) < -\frac{2}{3}$. The function $\beta_{ii}(t)$, however, vanishes only at one point. We also see that subtractions made at any threshold of $\beta(t)$ do not introduce new parameters because $\beta_{ij}(t) \to 0$ as $t \to t_i$.

5. FORMULATION OF A SET OF INTEGRAL EQUATIONS FOR THE REGGE POLE PARAMETERS: RELATIVISTIC CASE

In the preceding three sections we have developed the idea that the dispersion relation for $\alpha(t)$

$$\alpha(t) = \alpha_o + \frac{t - t_o}{\pi} \int_{T_o}^{\infty} \frac{\text{Im}\,\alpha(t')\,dt'}{(t'-t)(t'-t_o)} , \tag{5.1}$$

the real analyticity of $r(t)/(q_i q_j)^{\alpha(t)}$, and the unitarity condition in the approximate form

$$r(t) = \text{Im}\, \alpha(t)(\omega/q) \quad , \quad t > T_o \tag{5.2}$$

may provide an appropriate set of equations for the dynamical determination of the Regge pole parameters.

It will be recalled that the kinematic variables introduced in the above equations are the following:

$t = 4\omega^2$ = total C.M. energy squared in the t-channel

$$= m_a^2 + m_b^2 + 2\left[\sqrt{(m_a^2 + q^2)(m_b^2 + q^2)} + q^2\right] \tag{5.3a}$$

and $$q^2 = \frac{\left[t - (m_a + m_b)^2\right]\left[t - (m_a - m_b)^2\right]}{4t} \quad , \tag{5.3b}$$

where q = C.M. momentum of an incoming or outgoing particle.

F. Zachariasen (49) has pointed out that we can use equations (5.1), (5.2) and the real analyticity of $r(t)/(q_i q_j)^{\alpha(t)}$ to derive a very simple integral equation for $\text{Im}\,\alpha(t)$. We shall show in this section how this can be done.

Since we know that the function $r(t)/q^{2\alpha(t)}$ is real analytic, we have

$$r^*(t) = r(t^*)\, e^{-2i\pi\alpha(t^*)} \; , \; t > T_o \quad . \tag{5.4a}$$

According to Equation (5.2), $r(t^+)$ is real. Therefore,

$$r(t^+) = r(t^-)\, e^{-2i\pi\alpha(t^-)} \quad , \quad t > T_o \tag{5.4b}$$

where $t^{\pm} = t \pm i\epsilon$. Let us write

$$\frac{r(t)}{q^{2\alpha(t)}} = F(t)\, e^{\psi(t)} \quad , \tag{5.5}$$

where $F(t)$ is a rational function of t, and $\psi(t)$ is an analytic function of t cut from T_o to ∞. The discontinuity of $\psi(t)$ across the branch cut can be obtained from (5.4) and (5.5);

$$\psi(t^+) - \psi(t^-) = -2i \, \text{Im}\, \alpha(t) \, \ln q^2 \quad . \tag{5.6}$$

Consequently we can apply Cauchy's theorem to the analytic function $\psi(t)$ to find

$$\psi(t) = -\frac{(t-t_o)}{\pi} \int_{T_o}^{\infty} \frac{\ln (q'^2)\, \text{Im}\, \alpha(t')}{(t'-t)(t'-t_o)}\, dt' \, , \tag{5.7}$$

where we have normalized $\psi(t)$ so that $\psi(t_o) = 0$. Equations (5.1), (5.5) and (5.8) give

$$r(t) = F(t)\, q^{2\alpha_o} \exp \left\{ -\frac{(t-t_o)}{\pi} \int_{T_o}^{\infty} \frac{\ln \left(\frac{q'^2}{q^2}\right) \text{Im}\, \alpha(t')}{(t'-t)(t'-t_o)}\, dt' \right\}, \tag{5.8}$$

where the dispersion relation for $\alpha(t)$, Equation (5.1), has been used to replace $\alpha(t)$ in (5.5) by the right side of (5.1). Equations (5.2) and (5.8) then give

$$\text{Im}\, \alpha(t) = \frac{q}{\omega}\, F(t)\, q^{2\alpha_o} \exp \left\{ -\frac{t-t_o}{\pi} \int_{T_o}^{\infty} \frac{\ln \frac{q'^2}{q^2}}{t'-t} \frac{\text{Im}\, \alpha(t')}{t'-t_o}\, dt' \right\}, \; t > T_o. \tag{5.9}$$

One point may be worth noting. We know that $\mathrm{Im}\,\alpha(t)$ is always real, but α_o may be complex and at first sight the right side of (5.9) may appear to be complex. However, we can easily see that we can replace α_o by $\mathrm{Re}\,\alpha_o$ and the integral by its Cauchy principal value, and then the right side of (5.9) is actually real.

Now let us determine the function $F(t)$. We shall assume that $r(t)$ has no poles, in which case $F(t)$ is entire in t. We obtain from Equation (5.9) that

$$\mathrm{Im}\,\alpha(t) \xrightarrow[t \to \infty]{} \lambda F(t)\, q^{2\alpha_o} \exp\left\{ - \frac{\ln q^2}{\pi} \int_{T_o}^{\infty} \frac{\mathrm{Im}\,\alpha(t')}{t' - t_o}\, dt' \right\}$$

$$= \lambda F(t)\, q^{2\alpha(\infty)} \quad , \qquad (5.10)$$

where λ is a constant. If we now require that $\mathrm{Im}\,\alpha(t)$ vanishes as $t \to \infty$, then $F(t)$ being entire is a polynomial of order n satisfying the inequality

$$n < -\alpha(\infty) \quad . \qquad (5.11)$$

Moreover, from (5.8) we find

$$q_o^{2\alpha_o} \; F(t_o) = r(t_o) \quad . \qquad (5.12)$$

Thus we can infer that the general form of $F(t)$ is

$$F(t) = \frac{r(t_o)}{q_o^{2\alpha_o}} \prod_{i=1}^{n} \left(\frac{t - t_i}{t_o - t_i}\right) \quad , \qquad (5.13)$$

where the t_i specify the location of the zeroes of $r(t)$. We shall go into the question of zeroes, and the connection between the number of zeros and the asymptotic behavior of $\mathrm{Im}\,\alpha(t)$ more fully in Section 6 which treats the potential theory case. If we suppose that the trajectory of interest has 0 or 1 zero, for example, then the resultant equations take the form:

$$\operatorname{Im}\alpha(t) = \frac{q}{\omega}\, r(t_o)\; 1\; \left(\frac{q}{q_o}\right)^{2\alpha_o} \exp\left\{-\frac{t-t_o}{\pi}\int_{T_o}^{\infty} \frac{\ell n\, \frac{q^2}{q^2}}{t'-t}\, \frac{\operatorname{Im}\alpha(t')}{t'-t_o}\, dt'\right\} \quad (5.14)$$

or

$$= \frac{q}{\omega}\, r(t_o)\, \left(\frac{t-t_1}{t_o-t_1}\right) \left(\frac{q}{q_o}\right)^{2\alpha_o} \exp\left\{-\frac{t-t_o}{\pi}\int_{T_o}^{\infty} \frac{\ell n\, \frac{q'^2}{q^2}}{t'-t}\, \frac{\operatorname{Im}\alpha(t')}{t'-t_o}\, dt'\right\},\; t > T_o. \quad (5.15)$$

Equations (5.14) and (5.15) are the desired results. What we have achieved is a decoupling of Equations (5.1) and (5.2) so as to obtain an integral equation involving the single unknown function $\operatorname{Im}\alpha(t)$. Once we have solved for $\operatorname{Im}\alpha(t)$, we can obtain $\operatorname{Re}\alpha(t)$ by performing a simple Hilbert transform. For $t > T_o$, $r(t)$ is obtained algebraically from the unitarity condition (5.2) and for other values of t it can be obtained from the dispersion relation for $r(t)\, e^{-i\pi\alpha(t)}$ if $m_a = m_b$, and from Equation (5.8) in the general case.

Equation (5.14) has many attractive features. It incorporates the known threshold behavior of $\operatorname{Im}\alpha(t)$, it exhibits the possible zeros of $r(t)$ explicitly, it has a reasonable asymptotic behavior and it is in a form which suggests the possibility of a solution by some iteration procedure. If this is the case it is plausible that the solution is unique if $\operatorname{Re}\alpha_o$, $r(t_o)$ and the location of any possible zeros in $r(t)$ is given. These and other properties of the integral equation (5.14) for $\operatorname{Im}\alpha(t)$ are discussed in Section 6.

A similar set of equations can also be derived in case a fermion Regge pole is exchanged in the t-channel (46).

6. FORMULATION AND DISCUSSION OF A SET OF INTEGRAL EQUATIONS FOR THE REGGE PARAMETERS: POTENTIAL THEORY CASE

In this section we shall turn to the formulation of a set of integral equations for the Regge parameters in the case when the scattering may be described by a superposition of Yukawa potentials. This topic is of interest because the most clear cut check on the validity of our approximate form of the unitarity condition comes from a comparison of our results for the Regge parameters, computed for a single Yukawa potential, with the existing results found by numerically solving the Schrödinger equation (40). Secondly, we can establish in this case several rather precise theorems regarding the properties of the integral equation which we shall derive for $\mathrm{Im}\,\alpha(\nu)$. Here $\nu = k^2$ is the energy.

We recall that $\alpha(\nu)$ and $r(\nu)/\nu^{\alpha(\nu)}$ are both real analytic functions of ν cut from 0 to ∞, when crossing of trajectories is neglected. The approximate form for the unitarity condition reads

$$r(\nu) = \frac{\mathrm{Im}\,\alpha(\nu)}{\sqrt{\nu}} \quad . \tag{6.1}$$

Writing

$$\frac{r(\nu)}{\nu^{\alpha}} = F(\nu) \quad e^{\psi(\nu)}$$

and applying the same procedure as in the relativistic case, we obtain

$$\operatorname{Im}\alpha(\nu) = r(\nu_o)\,\sqrt{\nu}\,\left(\frac{\nu}{\nu_o}\right)^{\alpha_o} \prod_{i=1}^{n} \left(\frac{\nu-\nu_i}{\nu_o-\nu_i}\right) \exp\left[-\frac{(\nu-\nu_o)}{\pi} \int_0^\infty \frac{\ln\frac{\nu'}{\nu}}{(\nu'-\nu)(\nu'-\nu_o)} \operatorname{Im}\alpha(\nu')d\nu\right.$$

$$\nu > 0 \quad , \qquad (6.2)$$

and

$$r(\nu) = r(\nu_o)\left(\frac{\nu}{\nu_o}\right)^{\alpha_o} \prod_{i=1}^{n} \left(\frac{\nu-\nu_i}{\nu_o-\nu_i}\right) \exp\left[-\frac{(\nu-\nu_o)}{\pi} \int_0^\infty \frac{\ln\frac{\nu'}{\nu}}{(\nu'-\nu)(\nu'-\nu_o)} \operatorname{Im}\alpha(\nu')\, d\nu' ,\right]$$

$$(6.3)$$

where ν_o denotes the point of subtraction and ν_i, $i = 1, \ldots, n$ gives the location of the zeros of $r(\nu)$.

We would now like to show the relationship between the number of zeroes of $r(\nu)$ and the asymptotic behavior of the Regge parameters.

From Equation (6.2) we have

$$\operatorname{Im}\alpha(\nu) \to \nu^{\alpha(\infty) + n + \frac{1}{2}} \quad , \nu \to \infty \qquad (6.4)$$

and

$$\operatorname{Im}\alpha(\nu) \to \nu^{\alpha(0) + \frac{1}{2}} \quad , \nu \to 0 \ . \qquad (6.5)$$

Equation (6.4) shows that the number of zeroes is bounded by

$$n < -\alpha(\infty) - \frac{1}{2} \ , \qquad (6.6)$$

and Equation (6.5) gives the familiar threshold behavior of the Regge poles in the right-hand plane

$$\mathrm{Re}\ \alpha(0) > -\frac{1}{2} \quad . \tag{6.7}$$

Equation (6.6) is actually independent of our approximation. To prove this statement we recall that $r(\nu)/\nu^{\alpha(\nu)}$ is a real analytic function of ν with a cut from 0 to ∞, neglecting those coming from the crossing of trajectories. We can therefore write

$$r(\nu) = \nu^{\alpha(\nu)}\, F(\nu)\, e^{U(\nu)} \quad , \tag{6.8}$$

where $U(\nu)$ is analytic in ν cut from 0 to ∞. The function $U(\nu)$ can be written in the form

$$U(\nu) = \frac{\nu-\nu_0}{\pi} \int_0^\infty \frac{1}{(\nu'-\nu)} \frac{\Delta U(\nu')}{(\nu'-\nu_0)}\, d\nu' \tag{6.9}$$

where $\Delta U(\nu)$ is the discontinuity of $U(\nu)$ across the cut. Now (8, 37, 41)

$$\mathrm{Im}\ \alpha(\nu) \rightarrow -g^2/2\sqrt{\nu} \quad ,$$

$$r(\nu) \rightarrow -g^2/2\nu \quad , \quad \text{as} \quad \nu \rightarrow \infty \tag{6.10}$$

and $r(\nu)$ is real for ν real. Thus

$$\frac{r(\nu_+)\nu_+^{-\alpha(\nu_+)}}{r(\nu_-)\nu_-^{-\alpha(\nu_-)}} \longrightarrow \exp\ \{-2i\ \mathrm{Im}\ \alpha(\nu)\,\ell n\ \nu\} \quad ,$$

and therefore

$$\Delta\, U(\nu) \to -\, 2i\ \mathrm{Im}\ \alpha(\nu)\, \ln\ \nu, \text{ as } \quad \nu \to \infty\ , \qquad (6.11)$$
$$\to 0.$$

Thus we have,

$$U(\nu) \to -\frac{1}{\pi} \int_0^\infty \frac{\Delta\, U(\nu')\, d\nu'}{\nu' - \nu_o} \quad , \nu \to \infty\ , \qquad (6.12)$$

which is a finite number. Equations (6.8) and (6.12) together give

$$r(\nu) \to \nu^{\alpha(\infty)+n} \quad , \quad \nu \to \infty \quad , \qquad (6.13)$$

where n is the number of zeroes of $r(\nu)$. Comparing (6.10) and (6.13), we conclude that

$$n = -\, \alpha(\infty) - 1 \quad . \qquad (6.14)$$

The first Regge trajectory, having $\alpha(\infty) = -1$, therefore has no zero, while the second, the third, ... trajectories have one, two, ... zeroes respectively. It is easily shown (46) that all of the above statements remain true when the cuts arising from the crossing of trajectories are included.

Now let us discuss Equation (6.2) for the leading trajectory, which has no zero. Then we may write

$$U(\nu) = -\, \frac{r(\nu_o)}{\nu_o^{\alpha_o}} \quad \frac{(\nu-\nu_o)}{\pi} \int_0^\infty \frac{\ln \frac{\nu'}{\nu}}{(\nu'-\nu)(\nu'-\nu_o)}\, \nu'^{\,\alpha_o + 1/2}\, e^{U(\nu')}\, d\nu'$$

and

$$\text{Im}\,\alpha(\nu) = \frac{r(\nu_o)}{\nu_o^{\alpha_o}} \quad \nu^{\alpha_o + 1/2}\, e^{U(\nu)} \quad , \nu > 0 \quad . \tag{6.15}$$

If we take the subtraction point at $\nu = 0$, then we obtain

$$U(\nu) = -\lambda \frac{\nu}{\pi} \int_0^\infty \frac{\ln \frac{\nu'}{\nu}}{(\nu'-\nu)} \quad \nu'^{\alpha(0) - 1/2} \quad e^{U(\nu')} \, d\nu' \tag{6.16a}$$

and

$$\text{Im}\,\alpha(\nu) = \lambda\, \nu^{\alpha(0) + 1/2}\, e^{U(\nu)} \quad , \nu > 0 \quad ,$$

where

$$\lambda = \lim_{\nu \to 0} \frac{r(\nu)}{\nu^{\alpha(\nu)}} \quad .$$

And, if we take the subtraction point at $\nu = \infty$, then we have

$$U(\nu) = -\frac{\lambda}{\pi} \int_0^\infty \frac{\ln \frac{\nu'}{\nu}}{(\nu'-\nu)} \quad \nu'^{\alpha(\infty) + 1/2}\, e^{U(\nu')} \, d\nu' \tag{6.16b}$$

and $$\text{Im}\,\alpha(\nu) = \lambda\, \nu^{\alpha(\infty) + 1/2}\, e^{U(\nu)}, \quad \nu > 0$$

where $$\lambda = \lim_{\nu \to \infty} \frac{r(\nu)}{\nu^{\alpha(\nu)}} \quad .$$

We should like to point out several interesting consequences of Equations (6.16a) and (6.16b). We require $\text{Im}\,\alpha(0) = 0$ and $\text{Im}\,\alpha(\infty) = 0$. Thus the following inequalities have to be satisfied

$$\alpha(0) > -\frac{1}{2} \tag{6.17a}$$

$$\alpha(\infty) < -\frac{1}{2} \quad . \tag{6.17b}$$

If we take $\alpha(\infty) = -1$, which is correct for the leading trajectory, then (6.16b) shows that $\mathrm{Im}\,\alpha(\nu)$ has the correct asymptotic form as $\nu \to \infty$, providing $\lambda = g^2/2$. The solution of (6.16b), which is the equation having the subtraction point at $\nu = \infty$, should thus be expected to give a good approximation to $\alpha(\nu)$ and $r(\nu)$ at large ν. For the same reason, the solution of (6.16a), which gives the correct threshold behavior, should approximate $\alpha(\nu)$ and $r(\nu)$ accurately at small ν. It has been pointed out to the authors (50) that (6.16b) is dependent on the coupling constant g^2 only and is independent of the range μ of the potential. But (6.16b) is good only for ν large, and when the energy is large the mass can usually be neglected. In fact, the asymptotic forms for $\alpha(\nu)$ and $\beta(\nu)$ have been shown to be independent of μ. It is therefore natural that the range of the potential does not enter in (6.16b). On the other hand, if we make a subtraction at $\nu = 0$, or at some point ν_0 near zero, then the solution will be accurate at low energy if the subtraction constants $\alpha(\nu_0)$ and $r(\nu_0)$ are both supplied. It should be noticed that if we make a subtraction at some finite point ν_0, then the solution of (6.16) would not automatically give $\alpha(\infty) = -1$, in disagreement with the known behavior of the trajectory. However, in this case, we expect the solution to be accurate only at <u>low</u> energy, and its behavior at $\nu = \infty$ cannot in general be expected to be given in a precisely correct way using our approximate equations.

Suppose we have two functions $U^{(1)}(\nu)$ and $U^{(2)}(\nu)$ satisfying (6.16a) with different subtraction constants λ_1 and λ_2 but the same subtraction constant $\alpha(0)$. Then a change of variable shows that

$$U^{(1)}\left(\frac{\nu}{[\lambda_1]^{\alpha(0)+1/2}}\right) = U^{(2)}\left(\frac{\nu}{[\lambda_2]^{\alpha(0)+1/2}}\right) \qquad (6.18)$$

and as a result

$$\alpha^{(1)}\left(\frac{\nu}{[\lambda_1]^{\alpha(0)+1/2}}\right) = \alpha^{(2)}\left(\frac{\nu}{[\lambda_2]^{\alpha(0)+1/2}}\right) \qquad (6.19)$$

In particular, we have

$$\alpha^{(1)}(\infty) = \alpha^{(2)}(\infty) \quad . \qquad (6.20)$$

Thus we see that $\alpha(\infty)$ is determined by the subtraction constant $\alpha(0)$ and is independent of λ. Similarly the solutions of (6.16b) give the same $\alpha(0)$, if $\alpha(\infty)$ is fixed and λ is varied, and equalities similar to (6.18) and (6.19) hold.

Now let us turn to the question of the existence of a solution of Equation (6.16a) or (6.16b). First, it is clear that because of Equation (6.18) and (6.19), if there is a solution of Equation (6.16a) for a certain λ and $\alpha(0)$, then there is always a solution of Equation (6.16a) for an arbitrary λ and the same $\alpha(0)$. The same is true for (6.16b). The question of existence and uniqueness of a solution depends on the subtraction constant $\alpha(0)$ (or $\alpha(\infty)$) only. Secondly, (6.16a) does <u>not</u> have a solution for an arbitrary $\alpha(0)$. A necessary condition for the existence of a solution of Equation (6.16a) is Equation (6.17a). For, if there is a solution of Equation (6.16a), then $U(0) = 0$, and the integral on the side of (6.16a) does not converge at the end point $\nu' = 0$ unless (6.17a) is satisfied. Similarly, a necessary condition for the existence of a solution of Equation (6.16b) is (6.17b).

Some precise theorems on the existence and uniqueness of the solutions of Equations (16a,b) can be proved (51) if certain conditions on the subtraction constants are satisfied.

7. REGGE POLE PARAMETERS FOR A SINGLE YUKAWA POTENTIAL. PRESENTATION AND DISCUSSION OF RESULTS

The Regge pole parameters associated with a single Yukawa potential of unit range have been obtained by Ahmadzadeh, Burke, and Tate (40) and by Lovelace and Masson (41) for several potential strengths. Ahmadzadeh *et al*. (40) obtained their results by solving the Schrödinger equation numerically, while Lovelace and Masson (41) used a continued fraction technique applied to the known (8, 37, 41) form (in *potential* theory) of the asymptotic ($k^2 \to \infty$) expansions of the Regge parameters.

A comparison of the Regge parameters as calculated using Equation (6.2) of the preceding section with the results of Ahmadzadeh *et al*. (40) and Lovelace and Masson (41) provides an important test of the accuracy of our approximation. This section contains such a comparison.

Our procedure was to use the results of references (40) and (41) to supply the value of $\alpha(\nu)$ and $\beta(\nu)$ at a subtraction point ν_0. (We actually obtain $\beta(\nu_0)$ from $\text{Im}\,\alpha(\nu_0)$ and the unitarity condition (6.1).) Then we solve Equations (6.2) and (5.1) numerically* for the

*In this regard, see the footnote on page 13 and also reference (13).

functions $\mathrm{Im}\,\alpha(\nu)$ and $\mathrm{Re}\,\alpha(\nu)$ as functions of ν for ν in the range $-\infty < \nu < \infty$. A solution could always be obtained after a few iterations if we used the average value of the input and output functions as the next input.

The results for several values of the potential strength A are presented and compared with the results of references (40) and (41) in Figures (3) - (10). Results for $\mathrm{Im}\,\alpha(\nu)$ are presented for the range $\nu = 0 \to \infty$ and for $\mathrm{Re}\,\alpha(\nu)$ for the range $\nu = -2 \to +\infty$.

Let us consider some individual curves. In the case of strong coupling, A = 5 and 15, we see from Figures (3) - (6) that we obtain quite good qualitative and quantitative agreement between our results and those of references (40) and (41) over the entire range of energy. Note that in the case A = 5, results are presented for two different subtraction points. The solutions are essentially the same. We would like to mention also that if we have obtained $\mathrm{Im}\,\alpha(\nu)$ correctly, $\mathrm{Re}\,\alpha(\nu)$ must also be given correctly, subject to the assumptions: (a) that $\alpha(\nu)$ is a real analytic function; (b) that the trajectory considered does not cross with other trajectories; and (c) that the Hilbert transform has been performed without significant numerical error.

Next we consider curves in the regime of intermediate coupling, A = 1, 1.8, and 3. For the case A = 3, Figures (7) and (8), the solution obtained has an accuracy comparable to that found in the strong coupling cases. For the case A = 1.8, Figures (9) and (10), we find a case in which we obtain our poorest agreement, but the solution still possesses the correct qualitative shape, and is quantitatively accurate in a region around the subtraction point. The case A = 1 shows the same general features (Figures (7) and (8).

FIGURE 3: Im $\alpha(\nu)$ vs. $\nu/1+\nu$, $0 \leq \nu < \infty$. The results of this work are compared with those of Ahmadzadeh et al. (40) for single attractive Yukawa potentials of unit range and strengths A = 0.05, 0.30, and 5. In our equations, subtractions have been made at $\nu_o = 1.0$ and 0.1 for the case A = 5, and at $\nu_o = 0.1$ for the case A = 0.05 and 0.3. Here $\nu = k^2$.

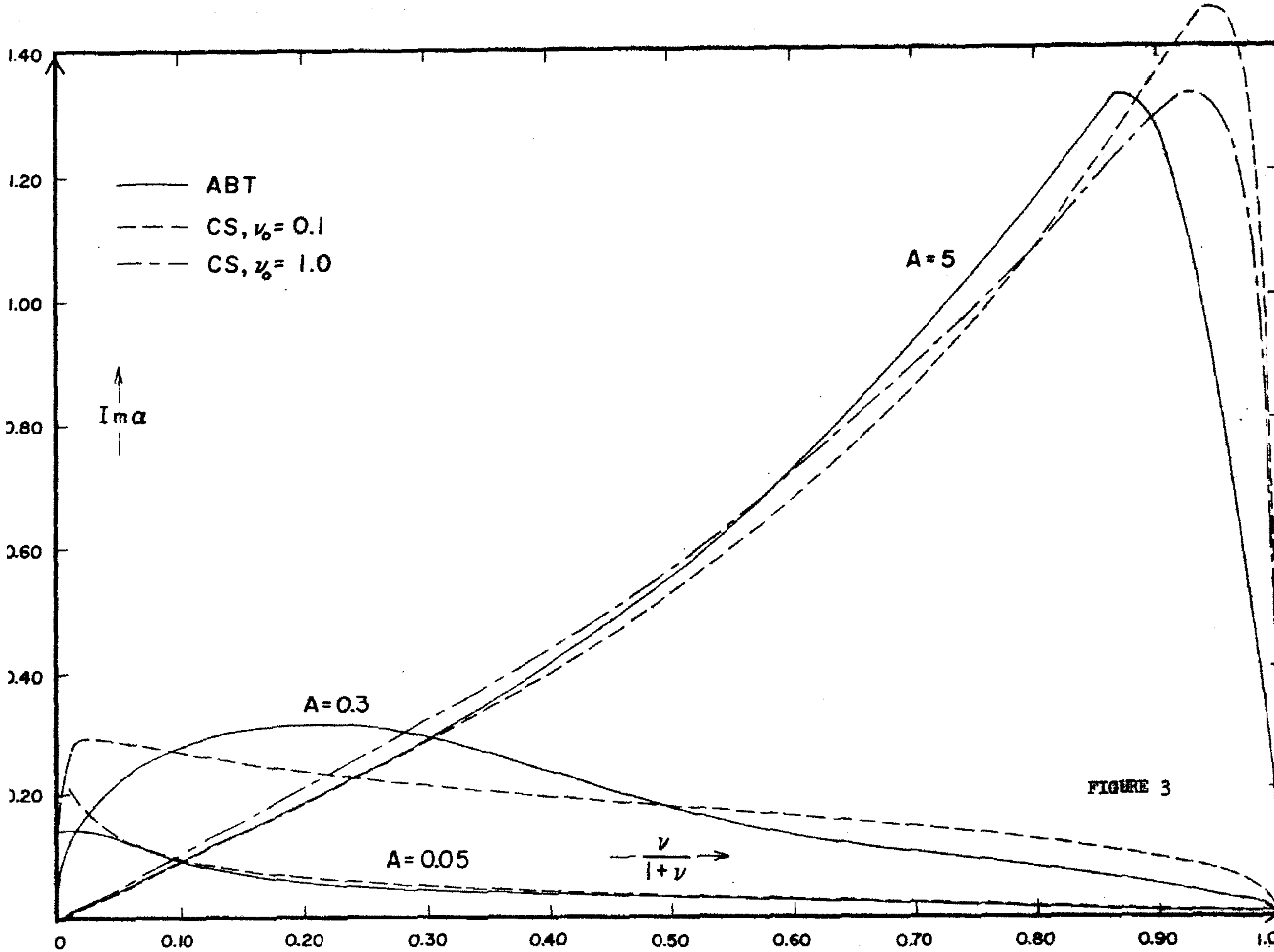

ABT
CS, ν_o = 0.1
CS, ν_o = 1.0
Im α
A = 5
A = 0.3
A = 0.05
$\frac{\nu}{1+\nu}$
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
1.40
1.20
1.00
0.80
0.60
0.40
0.20

FIGURE 3

FIGURE 4: Re $\alpha(\nu)$ vs. $\nu/1+\nu$, $-2 \lesssim \nu < \infty$. Results for potential strength $A = 5$ compared with those of Ahmadzadeh *et al.* (40). See caption of Figure (3).

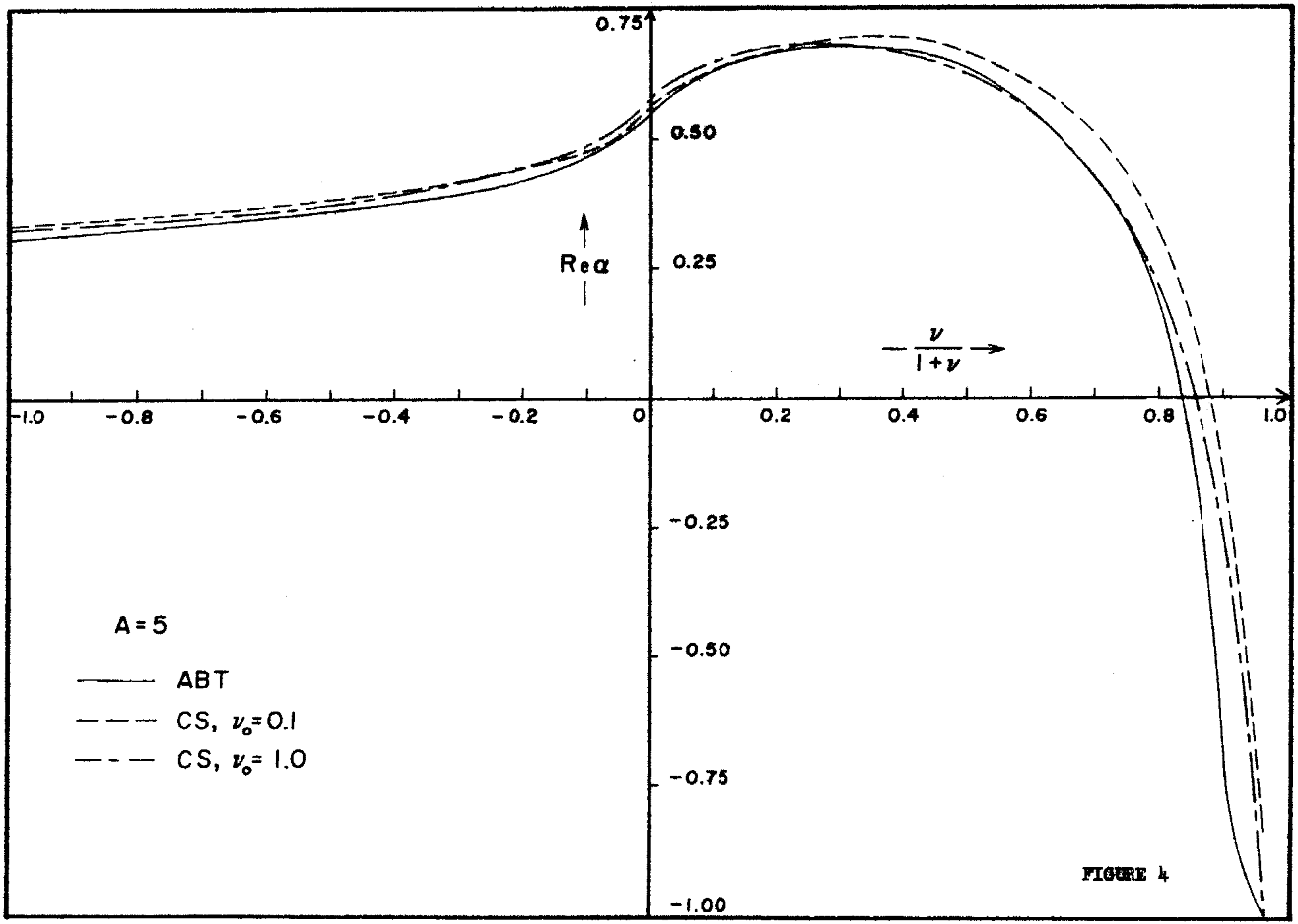

0.75
0.50
0.25
-0.25
-0.50
-0.75
-1.00
Re α
$\frac{\nu}{1+\nu}$
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
A = 5
ABT
CS, ν_o = 0.1
CS, ν_o = 1.0

FIGURE 4

FIGURE 5: Im $\alpha(\nu)$ vs. $\nu/1+\nu$, $0 \lesssim \nu < \infty$. Comparison of the results of this work with those of Lovelace and Masson (41) for a single Yukawa potential of unit range and strength $A = 15$. The point of subtraction was $\nu_o = 1.0$. Here $\nu = k^2$.

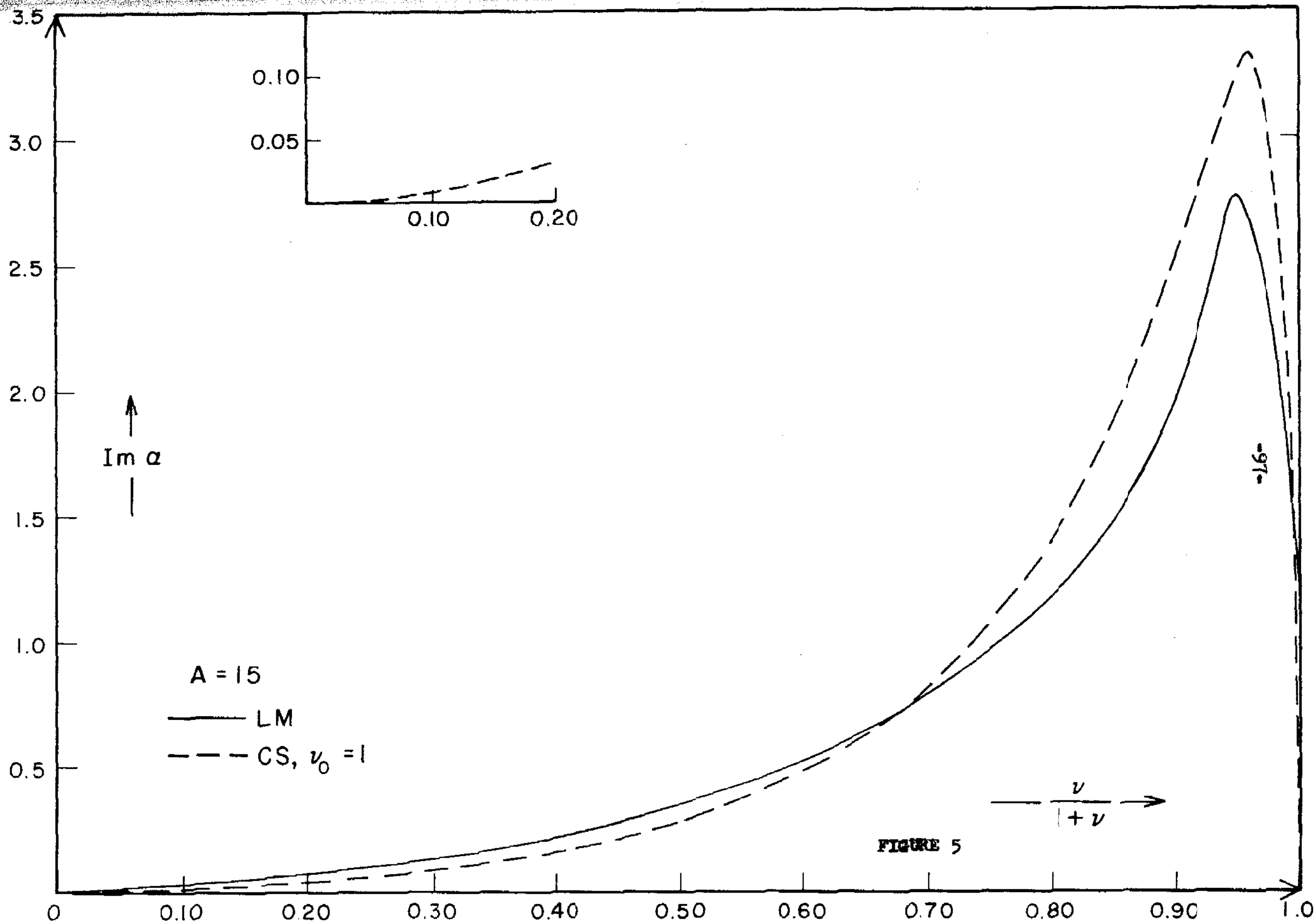

Im α
A = 15
LM
CS, ν_0 = 1
$\frac{\nu}{1+\nu}$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.0
0.10
0.05
0.10
0.20

FIGURE 5

FIGURE 6: Re $\alpha(\nu)$ vs. $\nu/1+\nu$, $-2 \lesssim \nu < \infty$. Results for potential strength $A = 15$ compared with those of Lovelace and Masson (41). See caption of Figure (5).

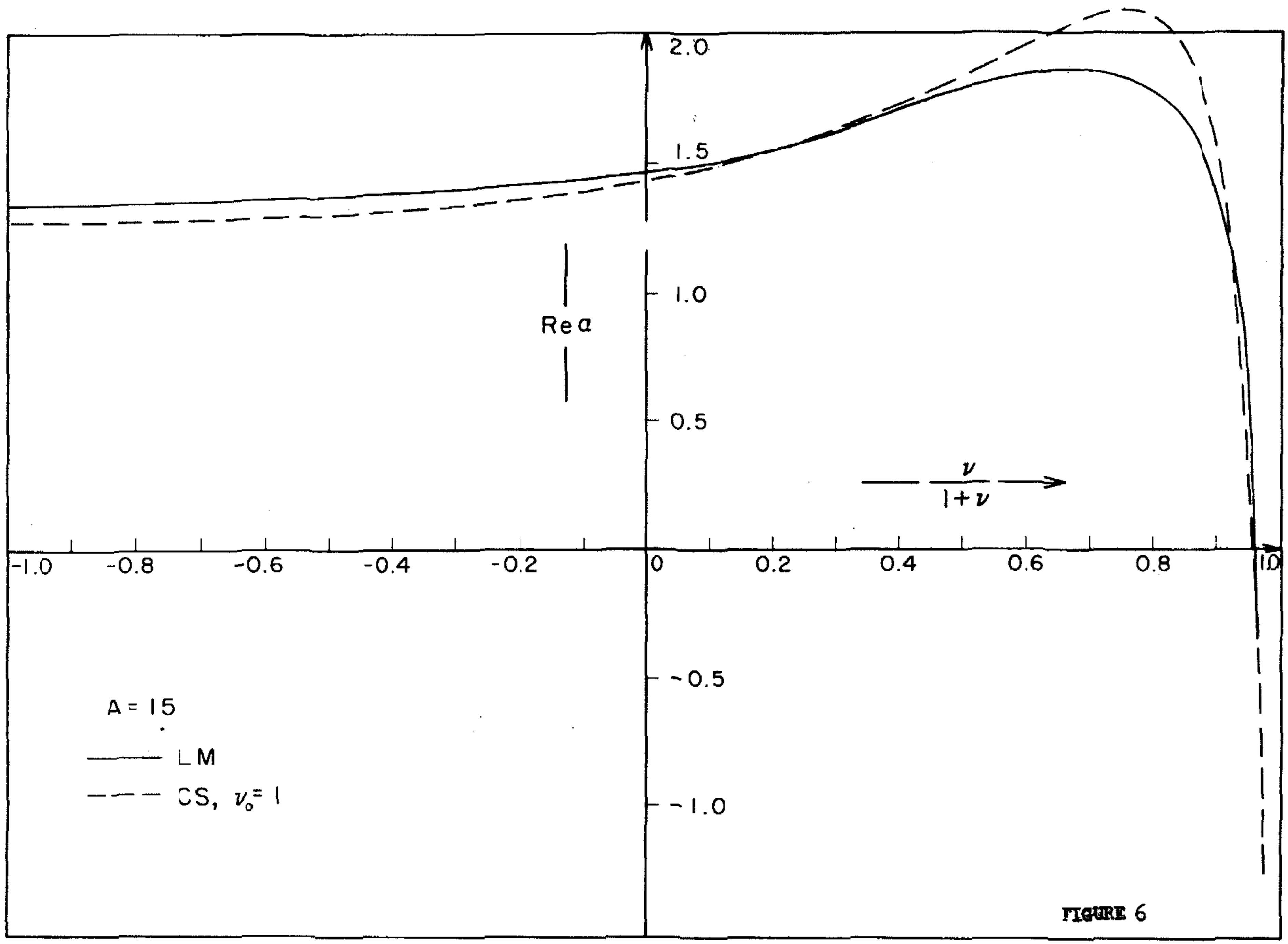


FIGURE 6

FIGURE 7: Im $\alpha(\nu)$ vs. $\nu/1+\nu$, $0 \lesssim \nu < \infty$. Results of this work compared with those of Lovelace and Masson (41) for single Yukawa potentials of unit range and strengths $A = 1$ and $A = 3$. The point of subtraction was $\nu_0 = 1$. Here $\nu = k^2$.

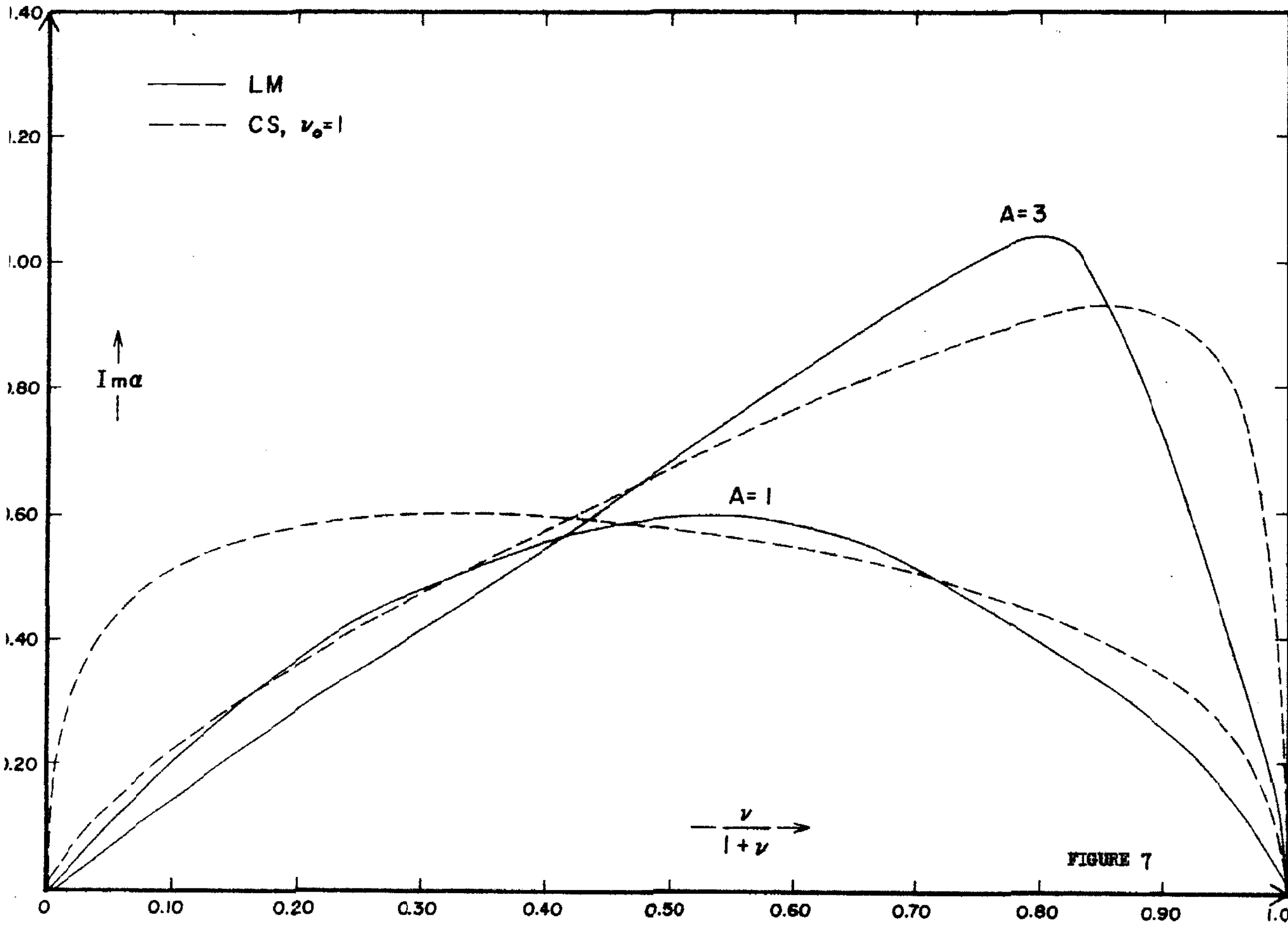

LM
CS, $\nu_e = 1$
A= 3
A= 1
Im α
$\frac{\nu}{1+\nu}$
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00

FIGURE 7

FIGURE 8: Re $\alpha(\nu)$ vs. $\nu/1+\nu$, $-2 \lesssim \nu < \infty$. Results for potential strengths $A = 1$ and 3 compared with those of Lovelace and Masson (41). See caption of Figure (7).

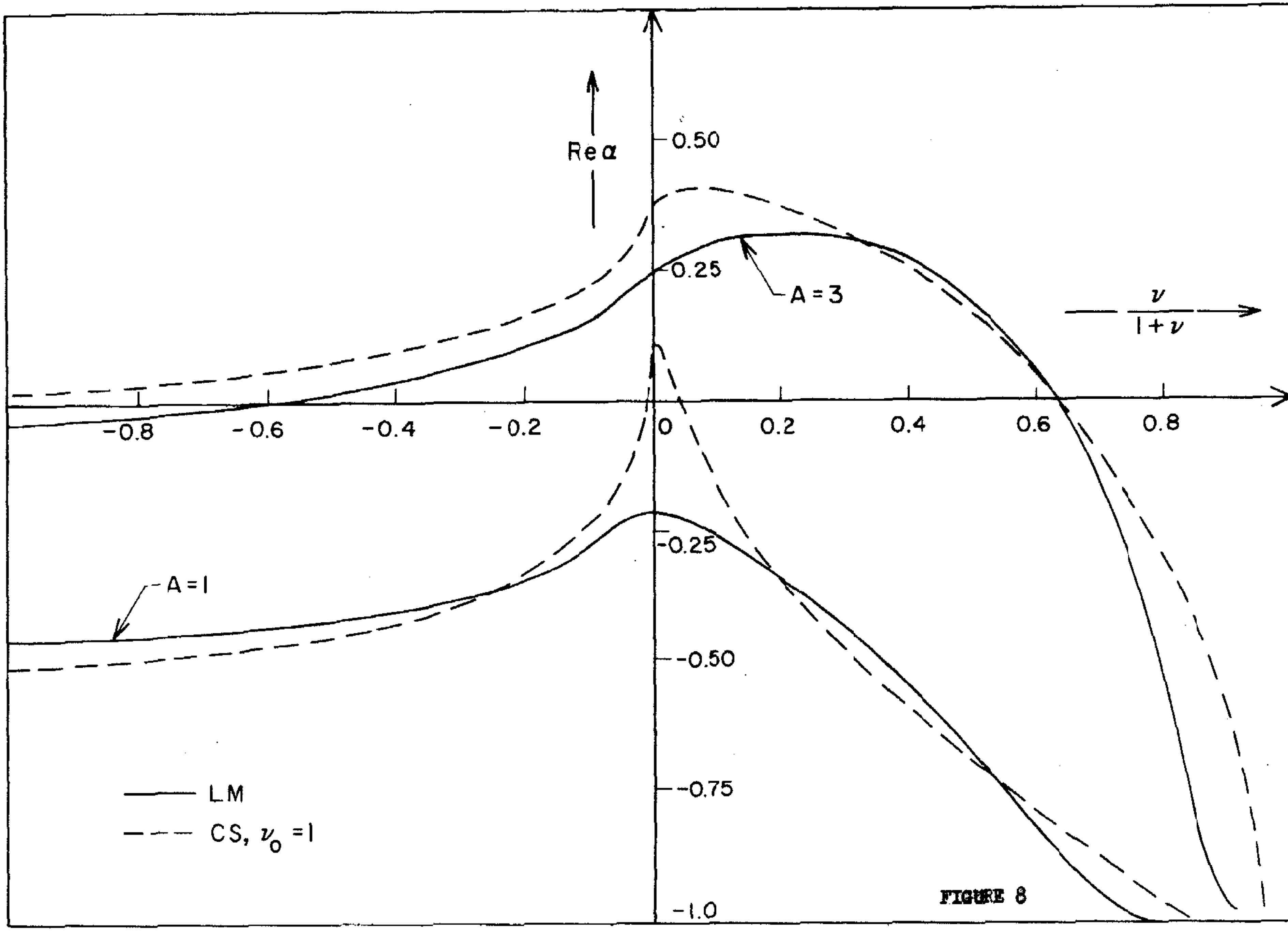

Re α
ν / (1+ν)
0.50
0.25
-0.25
-0.50
-0.75
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
A = 3
A = 1
LM
CS, ν₀ = 1

FIGURE 8

FIGURE 9: Im $\alpha(\nu)$ vs. $\nu/1+\nu$, $0 \leq \nu < \infty$. Results of this work compared with those of Ahmadzadeh <u>et</u> <u>al</u>. (40) for a single Yukawa potential of unit range and strength $A = 1.8$. Subtractions were made at the energies $\nu_o = 0.1$ and $\nu_o = 0.4$. Here $\nu = k^2$.

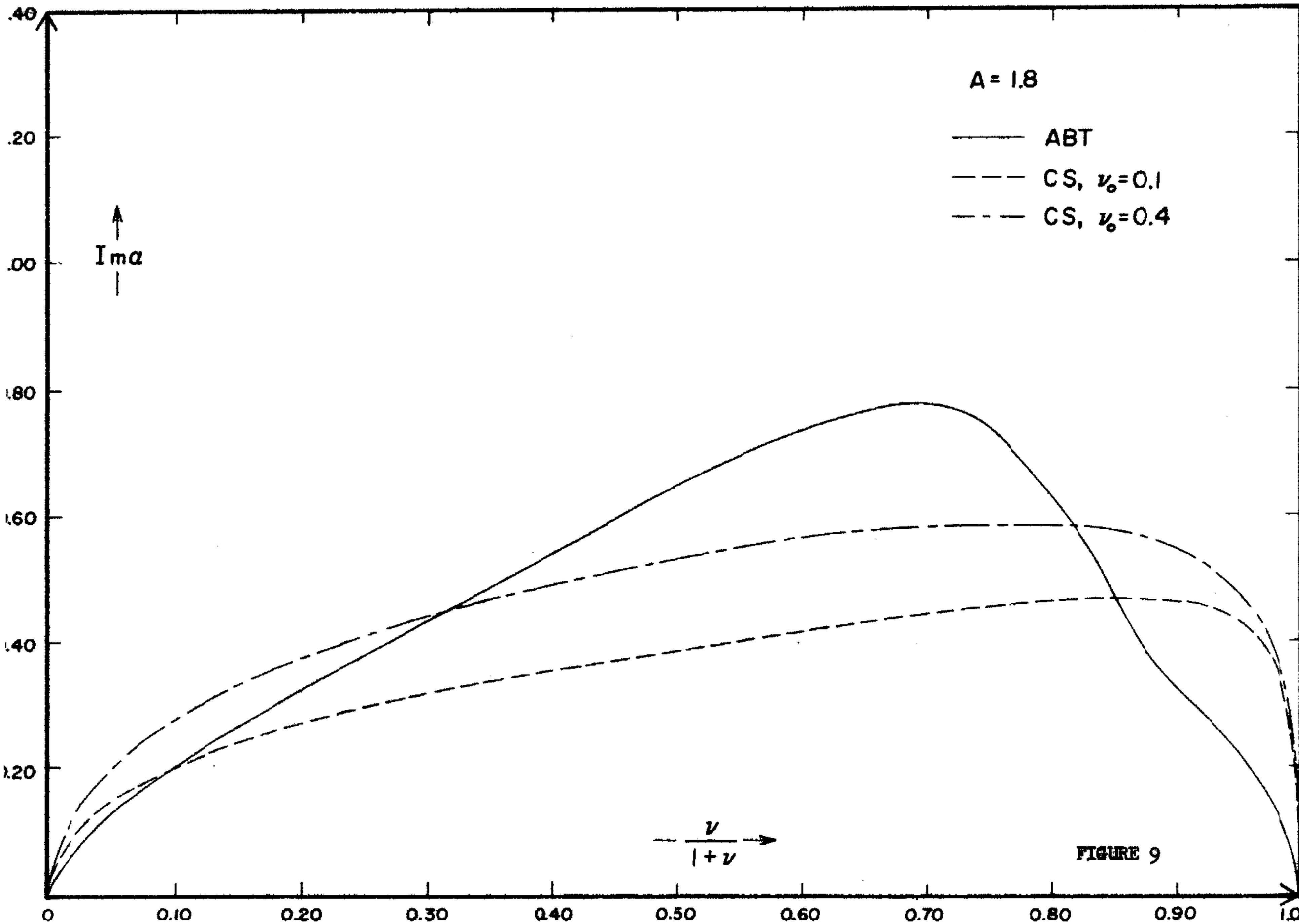

A = 1.8
ABT
CS, ν_o=0.1
CS, ν_o=0.4
Im α
$\frac{\nu}{1+\nu}$
.40
.20
.00
.80
.60
.40
.20
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00

FIGURE 9

FIGURE 10: Re $\alpha(\nu)$ vs. $\nu/1+\nu$, $-2 \lesssim \nu < \infty$. Our results for a potential strength $A = 1.8$ compared with those of Ahmadzadeh *et al.* (40). See caption of Figure (9).

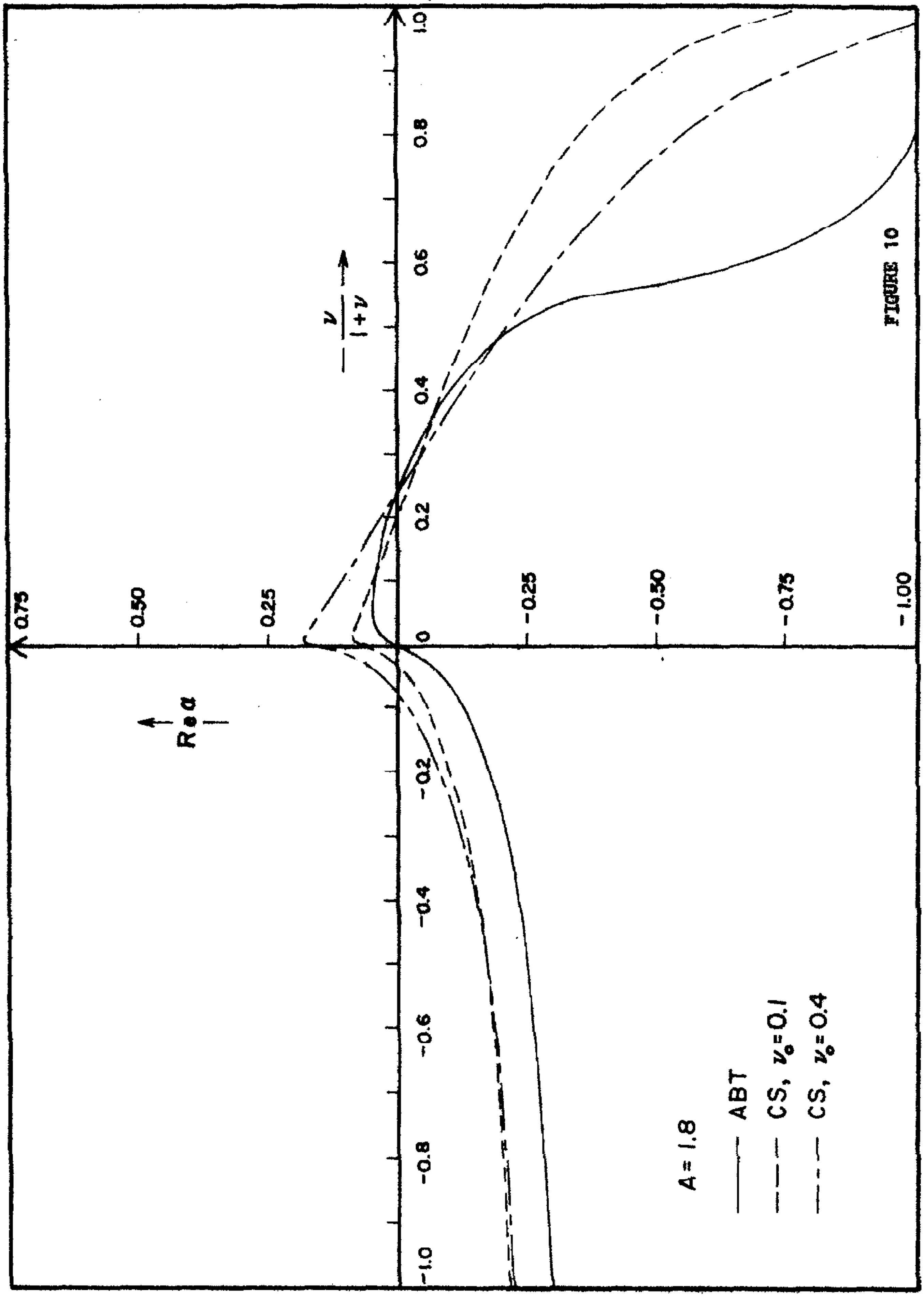

Re α
ν / (1+ν)
0.75
0.50
0.25
0
-0.25
-0.50
-0.75
-1.00
1.0
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1.0
A = 1.8
ABT
CS, ν_o=0.1
CS, ν_o=0.4

FIGURE 10

Turning now to the weak coupling cases, A = 0.05 and 0.30, we find the agreement much improved. In the case A = 0.05, Figure (3), the agreement for the Im α curve is fully comparable to that obtained in the strong coupling cases, with the exception of the region $0 < \frac{\nu}{1+\nu} < 0.05$. We understand the departure of the curves in this region from the correct ones to be due to limitations on the numerical accuracy with which we carried out the integral transforms involved. This problem is extreme in these cases, because we find from Reference (40) that in the case A = 0.05, Im $\alpha(\nu)$ goes from 0 to ~ 0.10 while ν ($\approx \frac{\nu}{1+\nu}$ for $\nu \ll 1$) goes from 0 to 10^{-3}. The value 0.10 represents ~ 60% of its peak value. Our program could not handle such rapid changes accurately, although this situation could no doubt be improved. Similar remarks apply to the case A = 0.3 (Figure (3)).

We do not understand why our approximation should be so much better in the strong and weak coupling cases than in the intermediate coupling case. One possibility is that the intermediate coupling region is one in which the first and second Regge trajectories for a given coupling strength cross. If this is the case, then the dynamical equations for the Regge parameters in the form we have used them here are not correct (37). However, we have no evidence that such crossings are in fact responsible for the disagreement.

An interesting fact is that the curves Re $\alpha(\nu)$, for $\nu \to \infty$, all approach negative values fairly close to -1 in good agreement with the asymptotic behavior of Re $\alpha(\nu)$ which has been proven rigorously (8, 37, 41). (A = 5, Re $\alpha(\infty) \approx -1.39$; A = 3, Re $\alpha(\infty) \approx -1.29$; A = 1.8, Re $\alpha(\infty) \approx -0.75, -1.05$; A = 1.0, Re $\alpha(\infty) \approx -1.19$; exception

$A = 15$, Re $\alpha(\infty) \sim -2.66$) This result was obtained without making any explicit assumption about the asymptotic behavior of Re α.

The function $r(\nu)$ obtained from our equation, in the case $A = 5$, was compared with that obtained by Ahmadzadeh, Burke, and Tate (40). The curves have a reasonably correct shape, and there is quantitative agreement in the low energy region.

To summarize, we feel that the results presented here support the following general conclusions in the case of potential theory:

1) Our equations provide a dynamical determination of the Regge parameters which always gives the correct shape of the curves, and gives quantitatively correct results in the neighborhood of the subtraction point.

2) In case the coupling of the Regge poles is strong or weak, we get good quantitative agreement along the entire length of the curve.

Finally, we emphasize that our equation has been tested for the leading trajectories only, which have no zeroes. It will be interesting to see if our equation yields accurate solutions for other trajectories which have zeroes and for which $\alpha(0) > -\frac{1}{2}$.

8. THE REGGE POLE PARAMETERS IN RELATIVISTIC $\pi\pi$-SCATTERING

In this section we shall apply the equations derived in Section 5 to discuss elastic $\pi\pi$-scattering at high energies. We will consider the contributions to this scattering of the Pomeranchuk trajectory, which has the quantum numbers of the vacuum and $\alpha_p(0) = 1$, and the ρ-trajectory which gives a $2\pi(J = 1, I = 1)$ resonance at 750 MeV. We shall also briefly discuss the second vacuum trajectory P' introduced by Igi (22). Since direct measurements of the $\pi\pi$-scattering cross sections are not yet available, we have concentrated here on obtaining the positions $\alpha(t)$ of these trajectories, which functions will then occur in all reactions having the proper quantum numbers.

We shall first discuss the Pomeranchuk trajectory. Its Regge pole parameters will be determined by Equations (5.1) and (5.15) which, as we have mentioned, couple the Pomeranchuk trajectory only to itself.

Our procedure was to supply as input parameters the quantities $\alpha_p(0) = 1$ and (46) $\sigma_{\pi\pi}(\infty) = -(4\pi^2/3)\left[2\alpha_p(0) + 1\right] r_p(0)$, and solve the equation for $\operatorname{Im}\alpha_p$ by an iteration procedure which takes the average value of the input and output functions as the next input. Then $\operatorname{Re}\alpha_p(t)$ was obtained from the dispersion relation (5.1).

We have obtained solutions for $\sigma_{\pi\pi}(\infty)$ in the range 3mb → 30mb. The results for $\operatorname{Re}\alpha_p(t)$ and $\operatorname{Im}\alpha_p(t)$ for $\sigma_{\pi\pi}(\infty)$ = 10 mb, 15 mb, and 20 mb are shown in Figures (11) and (12). In Figure (13), our results for $\operatorname{Re}\alpha_p(t)$, for $-0.8(\text{BeV/c})^2 < t < 0$, are compared with those obtained by Foley et al. (35) from an analysis of the $\pi^- p$ angular distributions measured at incident momenta in the range 7 to 17 BeV/c, and for the above-mentioned range of t.

It will be noted that our $\operatorname{Re}\alpha_p$ curves fall within the error flags around the experimental points measured by Foley et al. (35). We feel that this agreement is reasonably significant because the region where the comparison is made is very close to the subtraction point ($t = 0$), which is of course the region in which our results are most reliable. Secondly, the results are not extremely sensitive to the value of the input parameter $\sigma_{\pi\pi}(\infty)$.

FIGURE 11: Pomeranchuk trajectory. Re $\alpha_p(t)$ vs. t, $-0.80\ (\mathrm{BeV}/c)^2 < t < \infty$. The three curves shown are calculated using the input parameters: (a) $\alpha_p(0) = 1$, $\sigma_{\pi\pi}(\infty) = 10$ mb; (b) $\alpha_p(0) = 1$, $\sigma_{\pi\pi}(\infty) = 15$ mb; and (c) $\alpha_p(0) = 1$, $\sigma_{\pi\pi}(\infty) = 20$ mb.

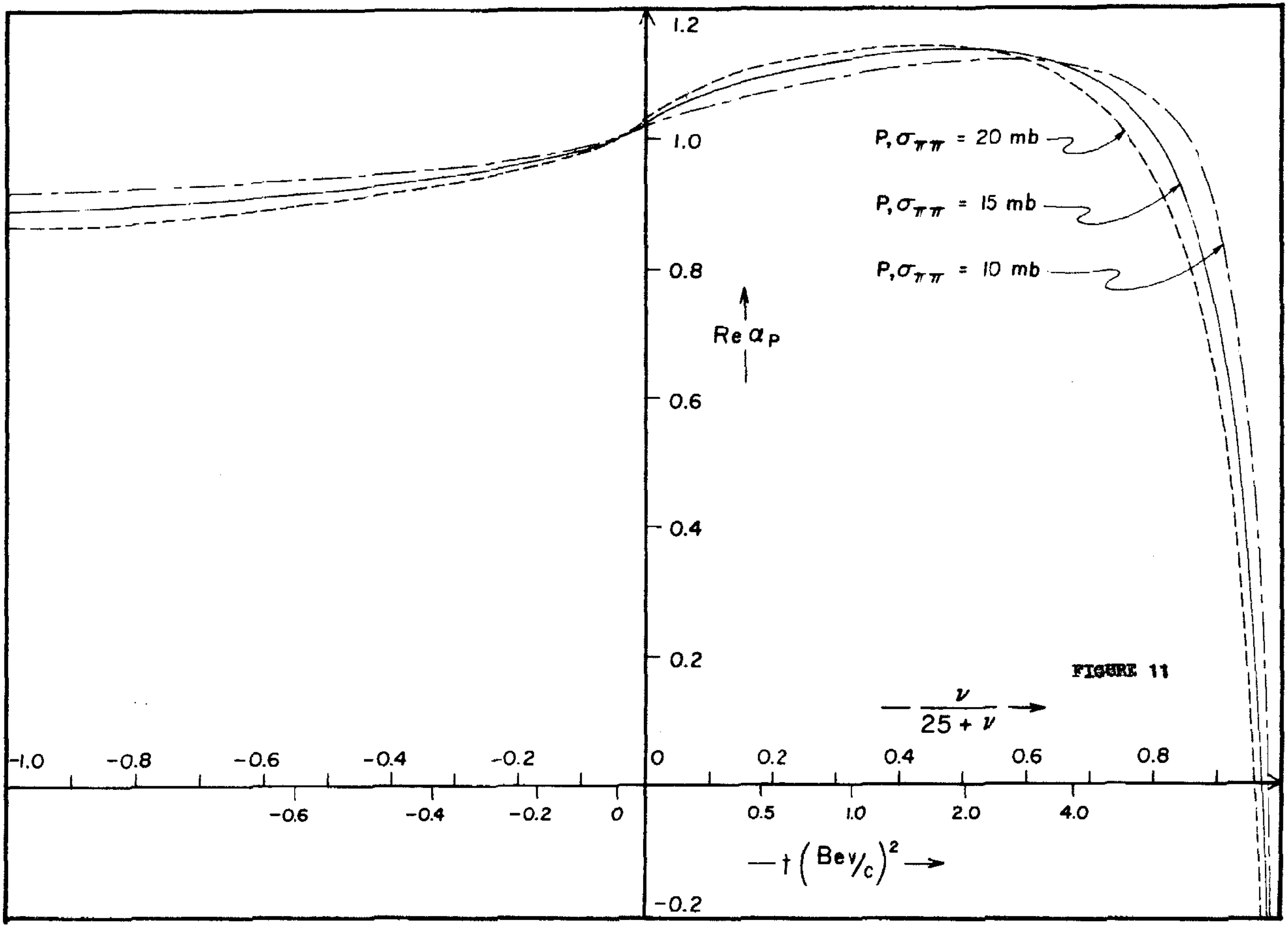
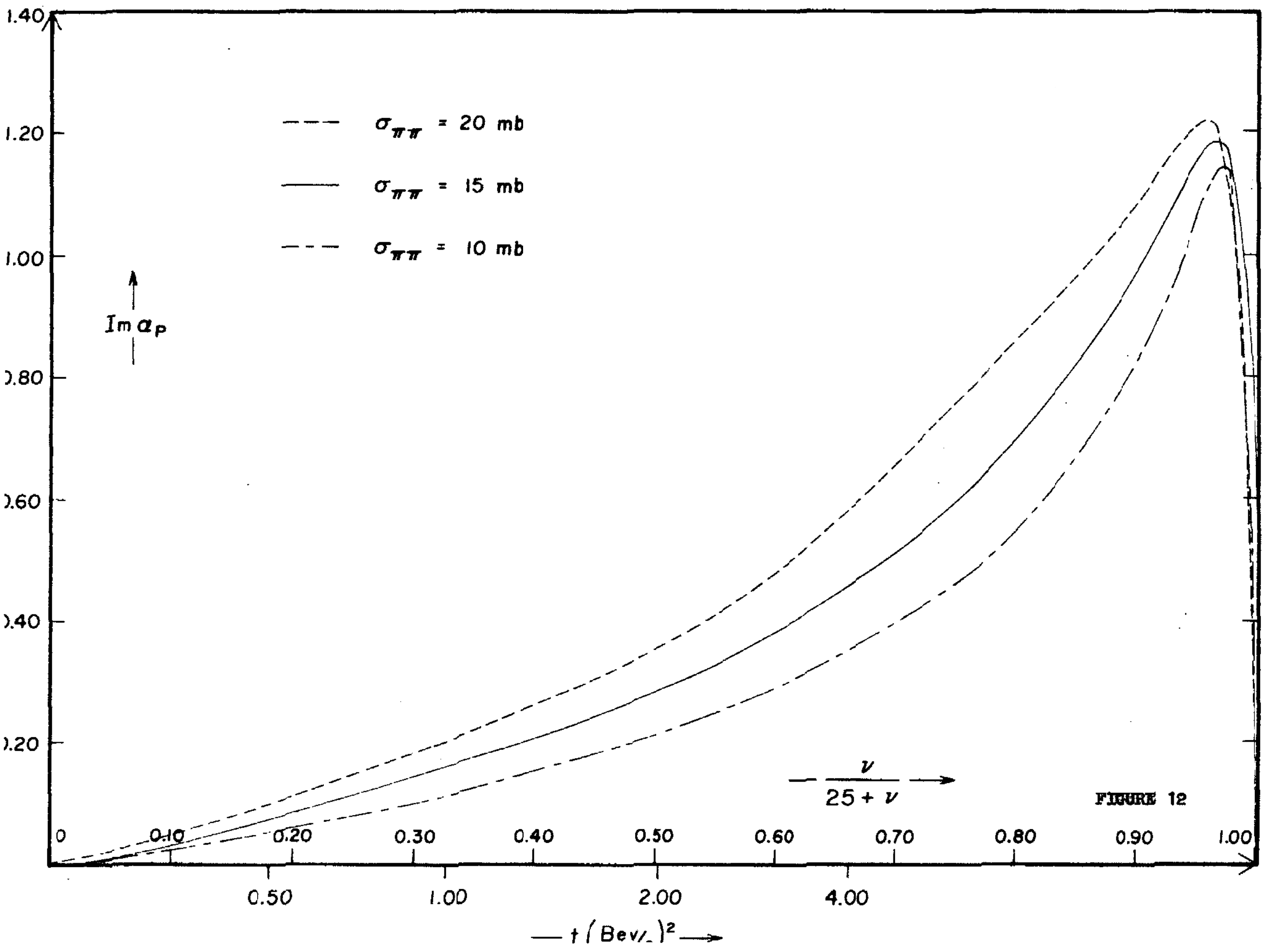

$P, \sigma_{\pi\pi}$ = 20 mb
$P, \sigma_{\pi\pi}$ = 15 mb
$P, \sigma_{\pi\pi}$ = 10 mb
Re α_P
1.2
1.0
0.8
0.6
0.4
0.2
-0.2
$\frac{\nu}{25 + \nu}$
-1.0 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8
-0.6 -0.4 -0.2 0 0.5 1.0 2.0 4.0
$-t\,(\mathrm{Bev}/c)^2$

FIGURE 11

FIGURE 12: Pomeranchuk trajectory. Im $\alpha_p(t)$ vs. t, $0.08\ (\mathrm{BeV/c})^2 < t < \infty$. The three curves shown were calculated using the input parameters listed in the caption of Figure (11).



$\sigma_{\pi\pi}$ = 20 mb
$\sigma_{\pi\pi}$ = 15 mb
$\sigma_{\pi\pi}$ = 10 mb
Im α_P
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.50
1.00
2.00
4.00
$\frac{\nu}{25+\nu}$
t (Bev/c)^2

FIGURE 12

FIGURE 13: Comparison of Re $\alpha(t)$ for the Pomeranchuk trajectory as computed in this thesis (input parameters; $\alpha_p(0) = 1$, $\sigma_{\pi\pi}(\infty) = 10, 15, 20$ mb) with Re $\alpha(t)$ as determined by Foley _et al_. (35) from an analysis of $\pi^- p$ angular distributions for incident momenta in the range 7 BeV/c to 17 BeV/c and $-0.80\ (\mathrm{BeV/c})^2 < t < -0.20\ (\mathrm{BeV/c})^2$.

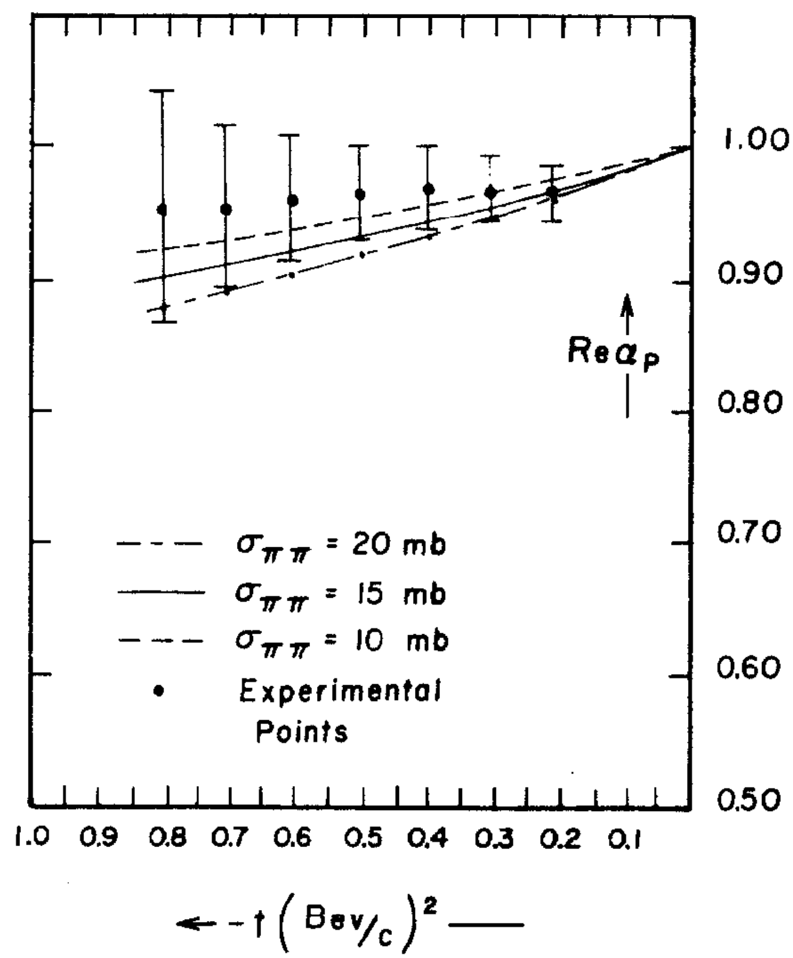

1.00
0.90
0.80
0.70
0.60
0.50
Re α_P
$\sigma_{\pi\pi}$ = 20 mb
$\sigma_{\pi\pi}$ = 15 mb
$\sigma_{\pi\pi}$ = 10 mb
Experimental Points
1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
$\leftarrow -t\ (\text{Bev}/c)^2$

FIGURE 13

The fact that our result for Re α_p agrees with that of Foley et al. (35) naturally implies that it disagrees with Re α_p as it has so far been determined from an analysis of NN scattering data (15, 35).

We do not have a resolution of this puzzle. However, we do feel that it is more likely that the πN rather than the NN angular distributions are dominated by the Pomeranchuk trajectory. The reason for this is that the statement that the Pomeranchuk trajectory dominates NN scattering, which depends on the assumption of a cancellation of large contributions from the P' and ω (or perhaps ϕ) trajectories (21, 24) is much more model dependent than the conjecture that it dominates πN scattering.

We note from Figure (11) that Re $\alpha_p(t)$ does not pass through 2 for any value of t. This implies that there is no spin 2 resonance on the Pomeranchuk trajectory. However, it may well be that the inclusion of inelastic states could change this conclusion. Moreover, the region where the curves peak $(t \sim 2 \text{ or } 3 \ (\text{BeV/c})^2)$ is rather far away from the subtraction point, which may result in further inaccuracies.

We have also obtained solutions for the P' trajectory (22) assuming $\alpha_{p'}(0) = 1/2$ and, quite arbitrarily, that at $t = 0$ and $s \sim 20 \ (\text{BeV})^2$ it contributed 5 mb to the total $\pi\pi$ cross section. Results are shown in Figures (14) and (15). It is of interest to note that Re $\alpha_{p'}$ falls off considerably faster for negative t than does the Pomeranchuk trajectory, and that it reaches its peak value at a much lower energy $(t \approx 0.15 \ (\text{BeV/c})^2)$.

FIGURE 14: P' trajectory and ρ-meson trajectory. Re $\alpha_{P'}(t)$ vs. t; Re $\alpha_\rho(t)$ vs. t, $-0.8\ (\mathrm{BeV/c})^2 < t < \infty$. The input parameters were:

(a) for the P'; $\alpha_{P'}(0) = 0.50$, $\sigma_{\pi\pi}^{P'}(s)\ (s \approx 20\ (\mathrm{BeV})^2) = 5$ mb;

(b) for the ρ, Re $\alpha_\rho(m_\rho^{\ 2}) = 1$, Im $\alpha_\rho(m_\rho^{\ 2}) = 0.10$.

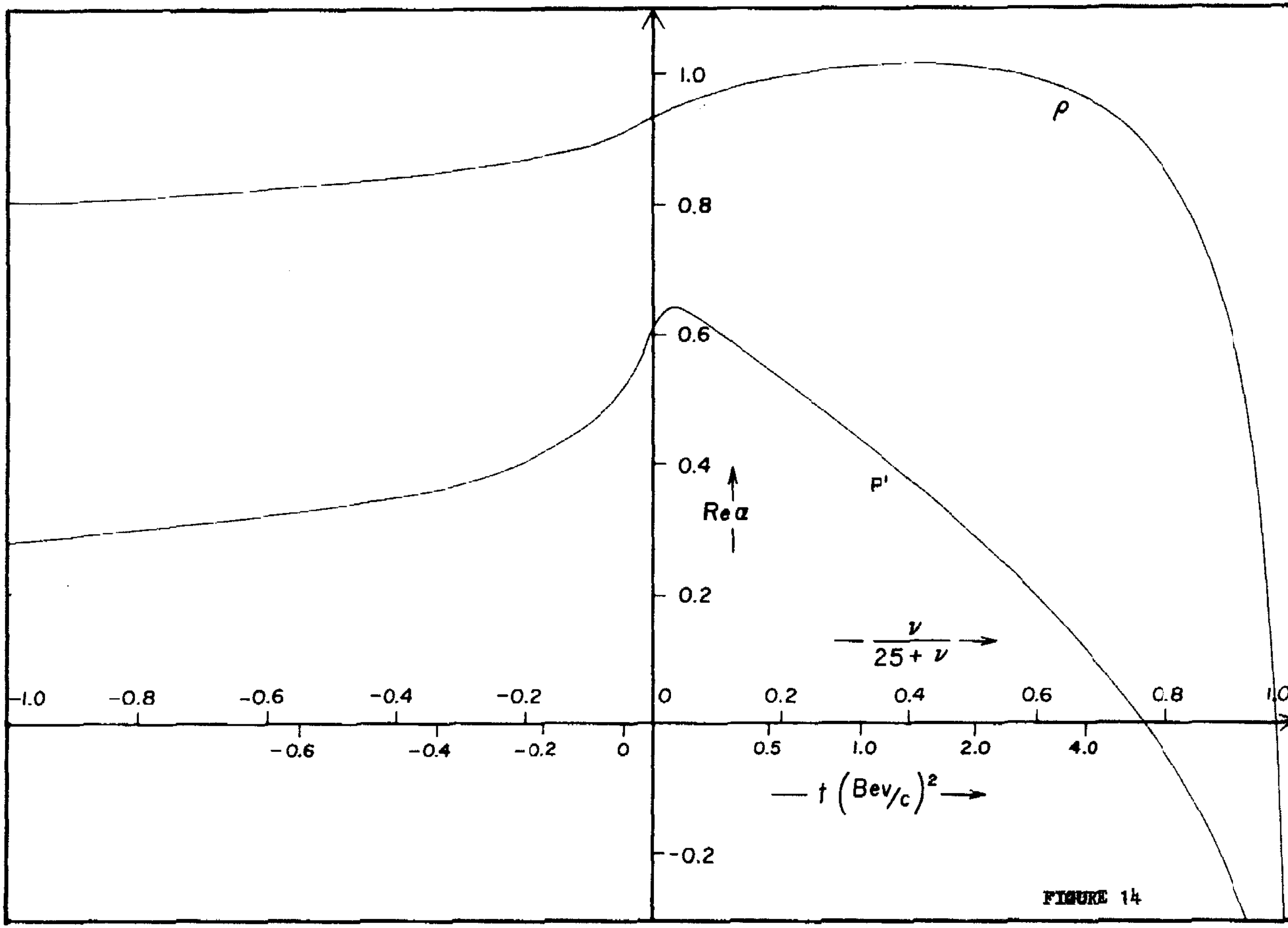

1.0
0.8
0.6
0.4
0.2
-0.2
Re α
ρ
P'
ν / (25 + ν)
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
0.5
1.0
2.0
4.0
— t (Bev/c)²
FIGURE 14

FIGURE 15: P' trajectory and ρ-meson trajectory. Im $\alpha_{p'}(t)$ vs. t; Im $\alpha_{\rho}(t)$ vs. t, $0.08\ (\mathrm{BeV/c})^2 < t < \infty$. These curves were calculated with the input parameters listed in the caption of Figure (14).

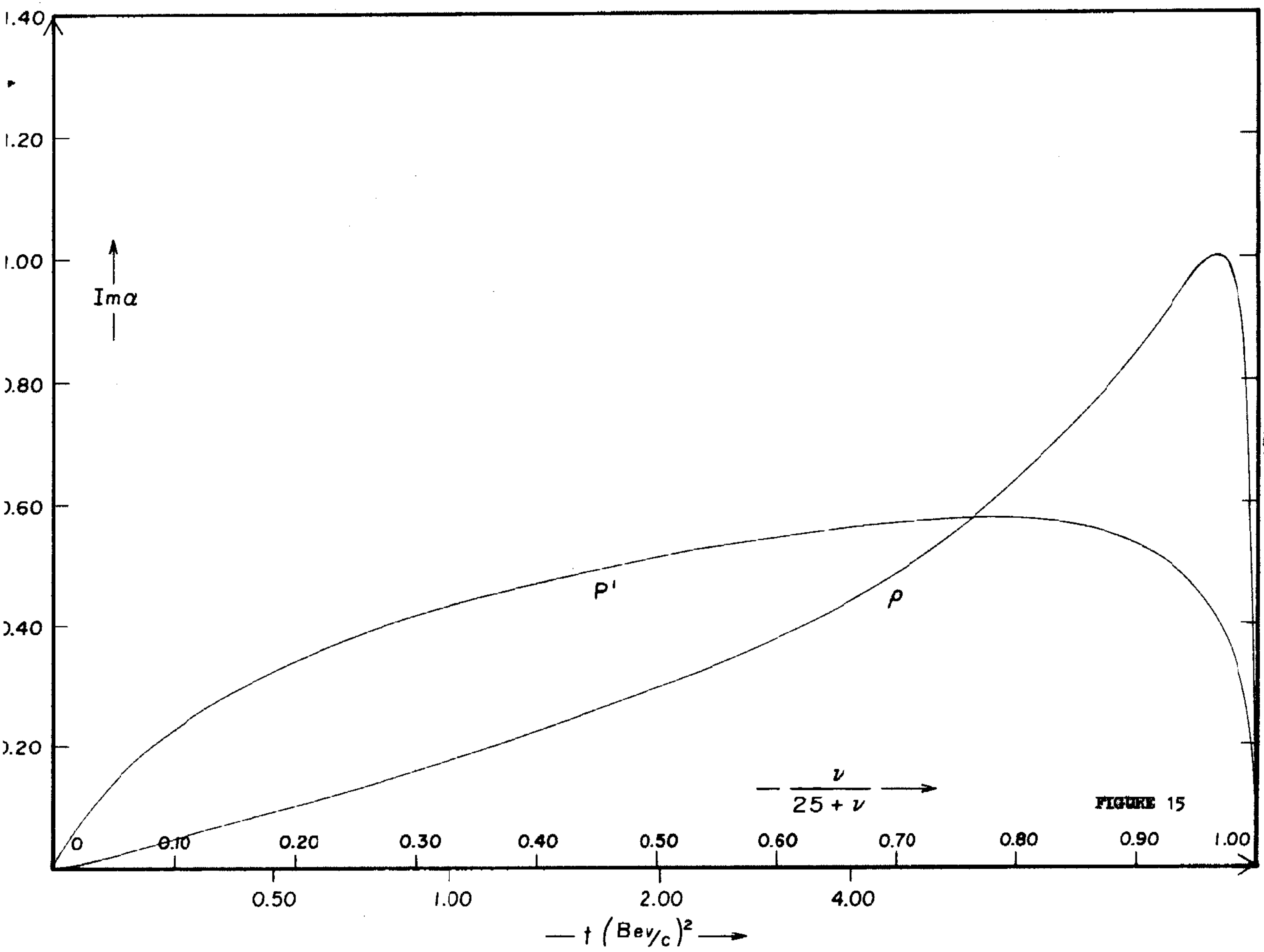

1.40
1.20
1.00
0.80
0.60
0.40
0.20
Im α
P'
ρ
ν / (25 + ν)
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.50
1.00
2.00
4.00
— t (Bev/c)²

FIGURE 15

Lastly, we have obtained solutions for the ρ-trajectory.

In this case, we solved for the trajectories in the following way. We used the fact that $\mathrm{Re}\,\alpha_\rho(m_\rho^2) = 1$ and then we chose a reasonable corresponding value of $\mathrm{Im}\,\alpha_\rho(m_\rho^2)$. We then obtained a set of solutions corresponding to these parameters, computed $\epsilon_\rho(m_\rho^2)$ and checked to see if the width as given by

$$m_\rho \Gamma_\rho = \mathrm{Im}\,\alpha_\rho(m_\rho^2)/\epsilon_\rho(m_\rho^2) \tag{8.1}$$

where

$$\epsilon_\rho(m_\rho^2) = d\,\mathrm{Re}\,\alpha_\rho/dt \Big|_{t = m_\rho^2} \tag{8.2}$$

came out correctly. Using this trial and error procedure, we were not able to find a set of parameters which gave a precisely correct value for the ρ-width.

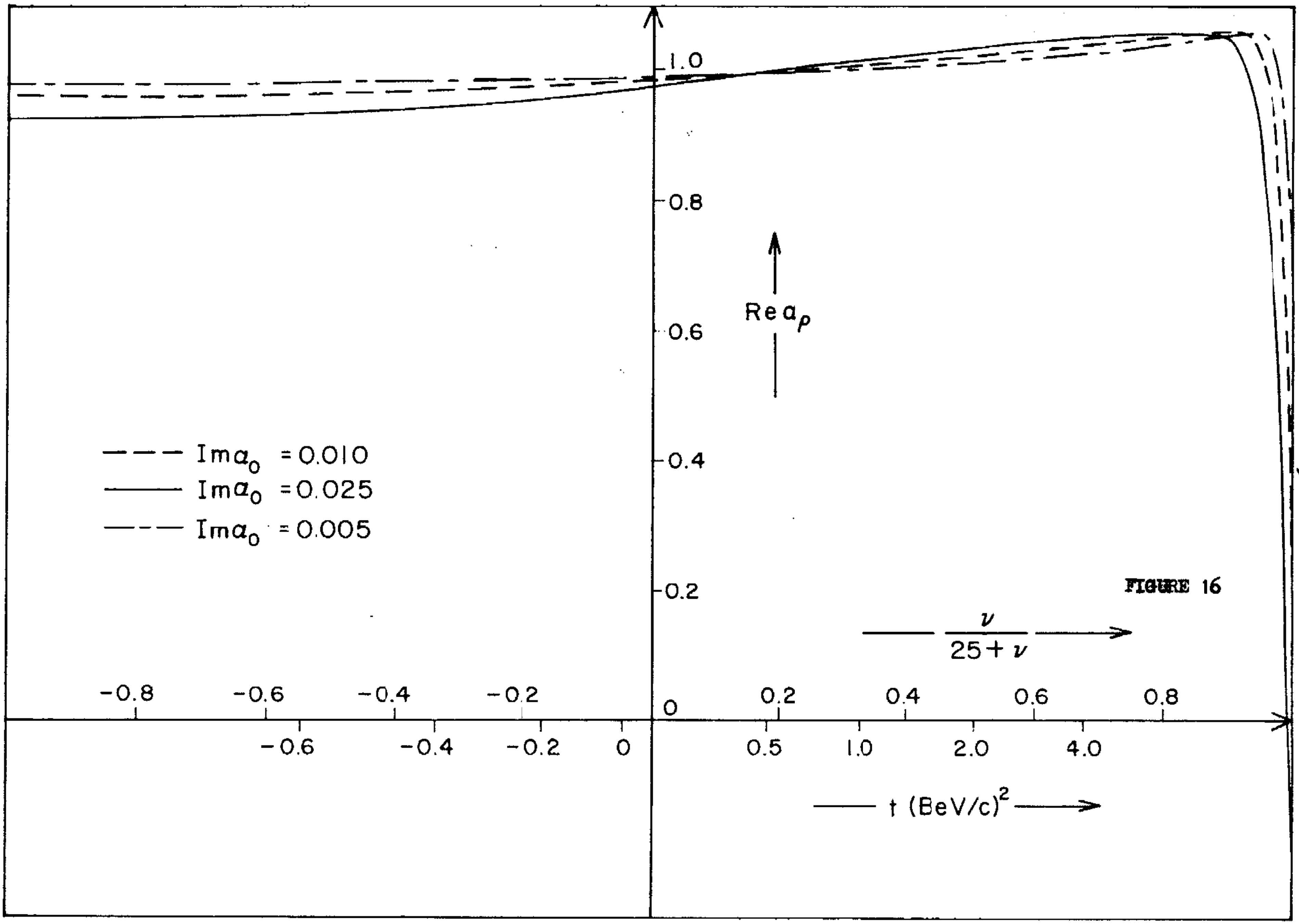
In Figures (14) - (17), we display our results for $\mathrm{Re}\,\alpha_\rho(t)$ and $\mathrm{Im}\,\alpha_\rho(t)$ for several values of the input parameter $\mathrm{Im}\,\alpha_\rho(m_\rho^2)$. The corresponding values of the width and $\alpha_\rho(0)$ are summarized in Table X.

It is to be noted that we obtain a very large value of $\alpha_\rho(0)$, and that we find $\alpha_\rho(t) > 0.90$ for $-0.80\ (\mathrm{BeV/c})^2 < t < 0$ (Figure 16). This fact is quite insensitive to the magnitude of the input parameter $\mathrm{Im}\,\alpha_\rho(m_\rho^2)$. Thus we feel that the numbers we obtain for $\alpha_\rho(t)$, $t = 0$ or $t \lesssim 0$, may not be modified greatly by the inclusion of inelastic states. The value of $\alpha_\rho(t)$, $-0.80\ (\mathrm{BeV/c})^2 < t < 0$, that we find seems to be consistent with the recent observations of S. J. Lindenbaum et al. (52), who find little or no energy dependence of the $\pi^\pm p$ angular distributions. This suggests that $0.80 < \alpha_\rho(0) < 1$, while we find typically

TABLE X. A list of values of Γ_ρ and $\alpha_\rho(0)$ for input parameters $\alpha_\rho(m_\rho^2) = 1$ and $\text{Im}\,\alpha_\rho(m_\rho^2)$.

$\text{Im}\,\alpha_\rho(m_\rho^2)$	$\Gamma_\rho = \dfrac{\text{Im}\,\alpha_\rho(m_\rho^2)}{m_\rho\,\epsilon_\rho(m_\rho^2)}$	$\alpha_\rho(0)$
0.005	3.79 m_π	0.990
0.010	4.45 m_π	0.983
0.025	6.35 m_π	0.966
0.100	18.9 m_π	0.913

FIGURE 16: ρ-meson trajectory. Re $\alpha_\rho(t)$ vs. t, $-0.8\ (\text{BeV}/c)^2 < t < \infty$. The three curves shown were calculated from the input parameters: (a) Re $\alpha_\rho(m_\rho^{\ 2}) = 1$, Im $\alpha_\rho(m_\rho^{\ 2}) = 0.005$; (b) Re $\alpha_\rho(m_\rho^{\ 2}) = 1$, Im $\alpha_\rho(m_\rho^{\ 2}) = 0.010$; (c) Re $\alpha_\rho(m_\rho^{\ 2}) = 1$, Im $\alpha_\rho(m_\rho^{\ 2}) = 0.025$.

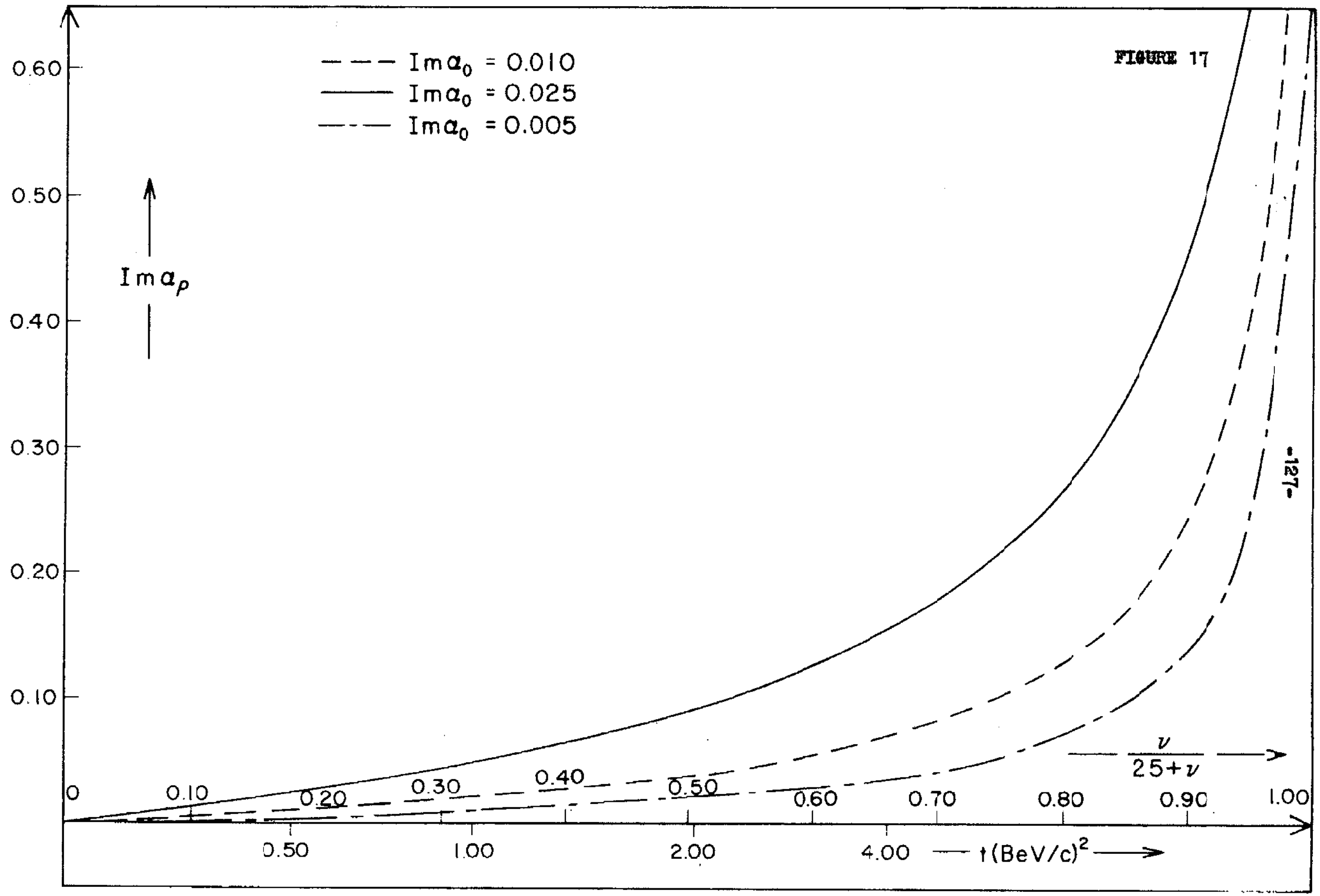


Re a_ρ
1.0
0.8
0.6
0.4
0.2
0
$Im\, a_0$ = 0.010
$Im\, a_0$ = 0.025
$Im\, a_0$ = 0.005
−0.8
−0.6
−0.4
−0.2
0.2
0.4
0.6
0.8
$\frac{\nu}{25+\nu}$
−0.6
−0.4
−0.2
0
0.5
1.0
2.0
4.0
t $(BeV/c)^2$

FIGURE 16

FIGURE 17: ρ-meson trajectory. $\text{Im}\,\alpha_\rho(t)$ vs. t, $0.08\ (\text{BeV/c})^2 < t < \infty$. The three curves shown were calculated using the input parameters listed in the caption of Figure (16).

FIGURE 17



Im α_0 = 0.010
Im α_0 = 0.025
Im α_0 = 0.005
Im α_p
0.60
0.50
0.40
0.30
0.20
0.10
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
$\frac{\nu}{25+\nu}$
0.50
1.00
2.00
4.00
t (BeV/c)^2

$\alpha_\rho(0) \sim 0.98$. An analysis of earlier date (53) on the $\pi^{\pm}p$ total cross sections, restricted to incident momenta greater than 10 BeV/c may also support the conclusion that $\alpha_\rho(0)$ is larger than ~ 0.80 (52). Moreover, one should bear in mind that the σ_{np} data is so poor that a determination of $\alpha_\rho(0) \sim 0.4$ from that data is without much statistical significance (54).

The width of the ρ-meson comes out too large by a factor of ~ 5, assuming $\Gamma_\rho \sim 100$ MeV. This, no doubt, indicates that inelastic states must be included in order to obtain the ρ-width correctly. This is probably not surprising in view of the results of other attempts to determine the ρ-width dynamically (55).

There is an additional complication that enters the determination of the widths from our Regge parameters. This is the fact that $\epsilon_\rho(m_\rho^2)$ is a small difference of large quantities, and a very small percentage error in Re α_ρ(~ 1/2 $^o/_o$) may result in very large errors in ϵ_ρ(~ 100 $^o/_o$). This may account for some of the error in our value of the ρ-width.

Finally, we sould like to record that we found Re $\alpha_P(\infty) \sim -0.66$ ($\sigma_{\pi\pi} = 15$ mb); Re $\alpha_{P'}(\infty) \sim -0.63$ and Re $\alpha_\rho(\infty) \sim -0.56$ (Im $\alpha_\rho(m_\rho^2) = 0.10$). We have made no explicit assumption about the asymptotic behavior except that Im $\alpha(t) \to 0$ as $t \to \infty$.

9. CONCLUSION

We have presented in this thesis an approximate method for the dynamically determination of the Regge pole parameters. The equations we have derived for this purpose are simple in structure and rather easy to solve numerically.

In the potential theory case, where a comparison with an exact solution is possible, the agreement is gratifying in most instances.

We do not understand why in the non-relativistic case the accuracy of the solution obtained appears to be poorest when the potential strength is in the range $1 \lesssim A \lesssim 3$. It may mean that for A in this range the one pole approximation is not adequate. Alternatively, this trajectory may cross another, in which case the equations must be formulated differently (37).

In the relativistic case, the solutions obtained for the Pomeranchuk trajectory agree quite well with the experimental results of Foley et al. (35). Our solutions for the ρ-trajectory give a value of $\alpha_\rho(0)$ which seems to be consistent with recent measurements of Lindenbaum et al. (52). However, we find that the width of the ρ-resonance comes out too large. The inclusion of inelastic channels should improve the results. But whether we can achieve quantitatively accurate solutions by including just the two-body inelastic channels remains to be seen.

The work carried out in this thesis suggests a number of interesting problems, both analytical and numerical, for further investigation.

We have mentioned the problem of including the inelastic channels in the equations, and finding their effect on, for example, the ρ-width.

A critical test of our equations can come from a determination of the Fermion trajectories. For example, if we supply the mass of the nucleon and the πNN coupling strength, can we predict the position and width of the $f_{5/2}$ resonance that is believed to lie on the nucleon trajectory? If so, the same method can be used to discuss all the meson-baryon resonances.

We have noted (Section 8) that the Pomeranchuk trajectory $\text{Re}\,\alpha_p(t)$ that we obtain is in agreement with that obtained from πN-scattering, but not with the results from NN-scattering. This probably means that several Regge poles contribute in an important way to NN-scattering at presently explored energies. To achieve a correct understanding of high energy NN and $N\bar{N}$ scattering, which because of the spin structure of the amplitudes will involve the application of our equations in the many channel case, forms another interesting and important problem.

Turning now to analytical problems, it is clear that an improvement of the one pole approximation for the partial wave amplitude is very desirable. By including the correct contribution of a few nearby poles in the partial wave amplitude, one could probably obtain satisfactory solutions in all instances for the potential case. A representation of the partial wave amplitudes solely in terms of Regge pole parameters should help such a formulation.

It would be interesting to learn if the zeroes of the residue functions, which appear as input parameters in our equations in their present formulation, can be determined if several poles are coupled together. If this is not the case, how can one determine the number and location of the zeroes of a given trajectory? The residue functions of the Pomeranchuk trajectory have a zero when α_p passes through zero. Since we have not taken account of this fact in the numerical work carried out here, it will be interesting to see how the solutions are modified if a zero is supplied.

Finally, we wish to repeat that one feature of dispersion theory, the crossing symmetry, has so far been totally neglected in our method. An application of the crossing theorem may enable one to determine many of the subtraction constants in a self-consistent manner. Work in this direction is still lacking.

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