THE INFRARED BEHAVIOUR OF QUANTUM CHROMODYNAMICS

Thesis by

David Philip Crewther

In Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1978

(Submitted May 16, 1978)

ACKNOWLEDGMENTS

It has been my pleasure and privilege to have worked for Professor Gell-Mann whose insights into this field have been far-reaching.

I thank Professors F. Zachariasen and J. M. Cornwall for detailed comments on infrared phenomena. Remarks by members of the high energy group at Caltech, especially the graduate students, have been helpful. I would also like to acknowledge the hospitality of the theoretical group at Brookhaven National Laboratory, where a very pleasant summer was spent as a Junior Research Scientist.

For financial support, I am indebted to the Aitchison-Myer Foundation, the Department of Energy, and the California Institute of Technology.

Most importantly, I thank Sheila, whose love and encouragement made all of this possible.

ABSTRACT

The infrared properties of scattering matrix elements in Quantum Chromodynamics are studied from the point of view of finding a systematization of perturbation theory that goes beyond the actual order-by-order results. An integro-differential equation for the matrix elements of Quantum Chromodynamics is derived. It is shown (using the ghost-free gauges and an assumption that seems reasonable in those gauges) that all the infrared singularities arising in on mass-shell scattering amplitudes may be collected by a reorganization of the perturbation theory into the iteration of the sum of all insertions of a single gluon between the external states, where the coupling constant at the point of insertion is replaced by the effective coupling constant $g(k^2)$ where k is the momentum of the inserted gluon. It is also shown that colour-averaged cross-sections are finite in the infrared limit to all orders of perturbation theory in a fashion that resembles the low order results that have appeared in the literature.

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CHAPTER I

Quantum Chromodynamics is the most promising field theory of the strong interactions known at present. It describes the interaction between quarks mediated by non-Abelian gauge fields called gluons. Just as Quantum Electrodynamics (QED) has provided a complete description of the interaction of photons and electrons to the precision obtained by experimentalists so far, it is the hope of many that Quantum Chromodynamics (QCD) will provide a complete description of the strong interactions. However, the large bulk of experimental strong interaction data deals solely with bound states of quarks and gluons and only indirectly, as a result of theoretical demands, has the inner structure of these bound states been probed. Thus QCD confronts theorists with two embarrassing problems: firstly, the inability to calculate properties of the bound states using the quarks and gluons as building blocks, and secondly, to explain why these fundamental fields are not physically observable. The quark model of Gell-Mann and Zweig(1) led to problems with the Pauli exclusion principle. Greenberg(2) introduced the notion of quark parastatistics to give the hadrons their correct symmetry properties, but did not point out that the para particles should be suppressed. Han and Nambu(3) suggested that three types or colours of quark should exist. However, their quarks possessed integral electric charge and the colour gauge symmetry was broken. Within the Han-Nambu scheme, Nambu(3) actually formulated Quantum Chromodynamics. In 1971, the proposal that quarks should be confined was made clear(4), and the theory of QCD with confinement was established and refined over the next two years(5). QCD possesses an exact SU(3) colour symmetry which has a corollary effect of requiring there to be three valence quarks in colour neutral baryons (those observed in nature). Thus, this feature of the phenomenologically successful quark model is a natural consequence of QCD. The $\pi^0 \to 2\gamma$ decay rate supports the existence of three colours of each flavour of quark. The only dynamical feature of the model which has been tested to any extent is the asymptotic freedom property(6) which enables some predictions to be made regarding e^+e^- annihilation, ep and νp interactions.

The consistency between theory and experiment and the beauty and simplicity of the theory, suggesting a unification of the strong, weak and electromagnetic interactions and possibly gravity are compelling reasons to look more closely at QCD.

The term confinement used in conjunction with QCD means that all physical states of the theory are colour SU(3) singlet states; that is, quarks and gluons do not exist as free entities and furthermore, there are no thresholds for the excitation of coloured bound states of quarks and gluons. Thus if confinement is the correct behaviour of the strong interaction theory, the historical process of investigating nature at shorter and shorter distances (or alternately at higher and higher energies) prompting the discovery of molecules, atoms, nuclei and nucleons (that is, protons and neutrons) has reached a logical endpoint as that process required the separation of the new entities into the experimental detector. This is not to say, of course, that further experiments at higher energies would be useless, merely that strong interaction experiments will continue to be indirect, relying on probes such as electrons, neutrinos and protons to investigate such topics as the nature of the strong force and the behaviour of the flavour interaction at high energies.

The term "infrared slavery" should be contrasted with confinement. Infrared slavery is a statement about the effective colour charge $g(k^2)$ in the infrared region specified by $k^2 \to 0$. The renormalization group enables such a definition of effective charge to be made. In the deep Euclidean region $k^2 \to -\infty$, the QCD effective charge $g(k^2) \to 0$ demonstrating the well-known "asymptotic freedom" property of QCD. As one moves away from the deep Euclidean region, the effective charge certainly increases in magnitude, but its behaviour in the infrared region is unknown as perturbation theory is inapplicable due to the large value of the coupling constant. The property of infrared slavery requires that $g(k^2)$ is singular in the limit $k^2 \to 0$. The exact connection between infrared slavery and confinement is unknown, but the consequences of these two hypotheses will be discussed in much more detail below.

There are well-known examples of field theories which actually do possess the confinement property. Electrodynamics in one space and one time dimension and also in two space and one time dimension are examples. However the confinement mechanism in lower dimensional systems may be completely distinct from that which is supposed to operate in QCD in three space and one time dimension.

Several approaches to the study of the infrared problem in QCD have been attempted. These include the approach of Wilson[7], who establishes the field theory on a discrete lattice and employs methods invented for use in statistical mechanics to search for phase transitions in the theory at some critical value of the coupling constant. Such phases might encompass the confined phase with colour singlet bound states as the only physical entities, a free QED-like phase with coloured gluons radiating in a fashion similar to the radiation of photons, or perhaps a dielectric phase with massive gluons which comes from some sort of dynamical Higgs mechanism. Of course, the hope is to show that the theory is confining for strong coupling values and does not possess a phase transition as the coupling constant tends to zero, that is, the renormalization group function $\beta(g)$ ¶ does not possess a fixed point for non-zero values of the coupling constant. Using these methods, Wilson has demonstrated that for sufficiently large values of the coupling, both the lattice theories of QED and QCD have confining phases.

Another approach deals with classical solutions of QCD such as instantons and merons[8] and investigates the quantum corrections around the classical solutions associated with the relativistic field theory. The vacuum state is then described as a dilute gas of soliton-antisoliton pairs; such vacua have been shown to be unstable. Instantons alone have been shown not to give confinement, at least in the dilute gas approximation[10], however it is quite likely that considerations based upon perturbation theory may miss some important features of QCD.

¶ related to the function $\psi(g^2)$ of Gell-Mann and Low[9] (in the original renormalization group paper) by $\beta(g) = \psi(g^2)/g$.

The last approach, and the most conventional, is based upon perturbation theory, although the results of such investigations may transcend the limitation of small coupling constant required for the convergence of the perturbation series. Quantum Electrodynamics, although not a confining theory (at least for small coupling), does provide a good analogy from which to work; it possesses charged fermions with the electric force mediated by massless vector particles, but as the gauge group U(1) is Abelian, the photons do not carry electric charge. The gluons in QCD do carry colour charge precisely because the gauge group of which they are manifestations is non-Abelian. Furthermore, low order perturbation theory calculations have shown a striking similarity[11] in the infrared behaviour of QED and QCD. In spite of the greater topological complexity of the Feynman graphs in QCD, there are many almost miraculous cancellations leading to a situation similar to QED: the infrared singularities associated with an on mass-shell amplitude factor out from the amplitude leaving a piece completely free from infrared singularities. The form of the infrared singular factor is (to the order calculated) equal to the exponential of the infrared singular piece of the one-loop correction to the Born term.

This dissertation comprises a study of the differences in infrared behaviour between QCD and QED, and the ramifications of such differences. So far, two important differences have emerged. The first is that in QCD with massive quarks and massless gluons, the vertex and self-energy contributions modify the one-loop correction mentioned above, replacing g^2, the perturbation expansion parameter, by $g^2(k^2)$, the square of the effective charge, which receives contributions only from those fields which are massless. This does not occur in QED since photons couple to each other only via electrons which are massive. The other difference between QED and QCD in the infrared region is that because gluons carry charge, virtual radiative corrections to processes involving external on-shell gluons include a separate infrared singular factor for each emitted gluon. This is distinct from QED where the exponential factor containing the infrared singularities is independent of the number of emitted photons.

Although a definition of confinement has been established, the manner in which one considers the theory in order to prove the confinement property is not straightforward. A knowledge of $\beta(g)$ or equivalently of the effective coupling constant in the infrared region (i.e. the infrared slavery question) although a significant step forward is not a proof or disproof.

From a phenomenological point of view, confining potentials have been invented, such as bag models of hadrons[12], string models[13], and the quark-antiquark Schrödinger potentials of the form $V(r) = ar + br^{-1}$ which have been successfully applied to modelling the spectra of the charmonium family of mesons[14]. All of these phenomenological attempts are in some sense approximations to QCD; for example, if QCD confines, then the gluons exchanged between quarks and antiquarks probably form flux tubes, which, if the quark and antiquark are separated resemble elastic strings, the energy in the string being proportional to separation. Thus, once confinement is assumed QCD becomes a very plausible theory.

A stringent requirement for confinement has been invented by K.Wilson[15]. If one considers the vacuum expectation value

$$C = \langle \exp[\ ig \oint_L A_\mu dx^\mu\ ]\rangle_0 = \int [dA\ ..]\exp(-S(A))\exp(ig\oint_L A_\mu dx^\mu)$$

where S(A) is the action, and L is a loop of spatial extent R and temporal extent T, and for the non-Abelian gauge field A there is an ordering operation on the line integral. If $\ln(C)$ varies as the area of the loop (i.e. RT) then the theory confines since the energy of two quarks separated by distance R, $E_q(R)$ is linear in R — the well-known confining potential. However such a calculation in three space and one time dimension has proved inordinately difficult except when the theory is defined on a lattice of points. It is hoped that the limit in which the lattice spacing tends to zero restores the full theory.

If one follows the approach to the infrared problem suggested by QED, one also needs a signal or discriminant for the confinement property. To follow the QED approach it is easiest to assume an S-matrix to exist for processes involving external quarks and gluons. Then, as indicated by perturbation theory calculations and in analogy with QED, all QCD matrix elements (exclusive) vanish. The signal for confinement will be to show that the inclusive cross-sections, allowing for the emission of soft gluons accompanying the basic process under consideration are also zero. This is not the case in QED where infrared singularities arising from the phase space integrations exactly cancel the virtual infrared singularities. Actually, a proof of confinement should also disprove the existence of a coloured threshold for the production of bound states of quarks and gluons not in the colour singlet representation of SU_3^c. However, for the purposes here, confinement shall refer only to the non-emergence of the elementary fields.

The assumption of the existence of an S-matrix in QCD allows for a conventional definition of the quark mass — that is, the location of the branch point of the quark two-point function. In terms of the program set out (that is to test whether inclusive cross-sections vanish) such a definition of quark mass is quite consistent. However, if the theory does confine, then the quark "masses" are no more than parameters of the theory describing the breaking of the flavour symmetry. In this case, a definition of mass based on a sliding scale renormalization scheme would seem more appropriate. Thus the inverse quark propagator $S_F^{-1}(\not{p})$ may be written

$$S_F^{-1}(\not{p}) = A(p^2)\not{p} + B(p^2)$$

where at some point $p^2 = M^2$,

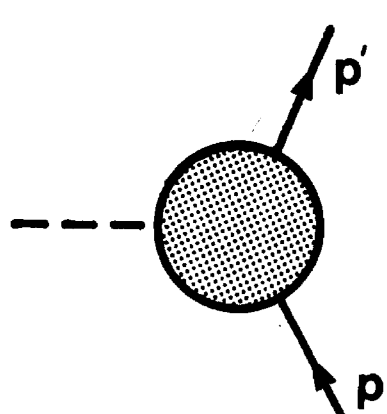
$$A(M^2) = 1 \quad , \quad B(M^2) = m.$$

Thus we have defined a mass which is dependent on M, m(M). It should be noted that the M used here is related to the renormalization group invariant mass μ :

$$M^2 e^{-1/bg^2} = \mu^2$$

However, such a definition of mass does not lend itself to a discussion of the infrared singularities which might arise in the Green's functions.

As stated above, in terms of their perturbative infrared behaviour QED provides an excellent comparison with QCD. In order to make this comparison a detailed knowledge of the infrared behaviour in QED is necessary. For an example consider the scattering of an electron with momentum p from an external potential producing an outgoing electron with momentum p'. Infrared divergences arise when one considers virtual corrections to the basic process (see below) because both the photon and the two electron propagators can approach their mass shells simultaneously.



The Feynman integral which contains this infrared divergence in the one loop graph is of the form

$$B = \int \frac{d^4k}{(2\pi)^4} \frac{4p.p'}{(k^2-2p.k)(k^2-2p'.k)k^2} \qquad \text{(I-1)}$$

Because of an intrinsic fear of infinities, various devices have been employed to regulate the infrared divergence. This may be done by introducing a small mass λ for the photon[16] in which case

$$B = -\frac{1}{2\pi}\left[(\ell n\,(\frac{2p.p'}{m^2}) - 1)\,\ell n\,(\frac{m^2}{\lambda^2}) + \frac{1}{2}(\ell n\,(\frac{2p.p'}{m^2}))^2 - \frac{1}{2}\,\ell n\,(\frac{2p.p'}{m^2})\right] \qquad \text{(I-2)}$$

or by changing the number of dimensions d of space-time in a method known as dimensional regularization[17] in which case

$$B = -\frac{1}{2\pi}\left[(\ell n(\frac{2p.p'}{m^2}) -1)\,(\frac{2}{d-4} + \ell n(\frac{m^2}{\mu^2})) + \right.$$
$$\left. + \frac{1}{2}\left(\ell n\left(\frac{2p.p'}{m^2}\right)\right)^2 - \frac{1}{2}\,\ell n\left(\frac{2p.p'}{m^2}\right)\right] \qquad \text{(I-3)}$$

where μ is an arbitrary mass scale introduced in order to maintain the dimensionlessness of g in other than four dimensions. The correspondence between logarithmic singularities of the form $\ell n\,\frac{m^2}{\lambda^2}$ in the photon mass regulation method and poles of the form $\frac{2}{d-4}$ in the dimensional regularization scheme occurs in all infrared calculations (at least at the leading singularity level)[18].

As shown by Yennie, Frautschi and Suura[16] if the elastic scattering amplitude for electron scattering $M(p,p')$ is represented by a sum over diagrams containing n virtual photons,

$$M(p,p') = \sum_{n=0}^{\infty} M_n(p,p') \qquad \text{(I-4)}$$

the amplitudes M_n have the infrared structure

$$M_o = m_0$$

$$M_1 = m_0 \, (\alpha B) + m_1$$

$$M_2 = m_0 \, \frac{(\alpha B)^2}{2!} + m_1 \, (\alpha B) + m_2$$

$$\vdots$$

$$M_n = \sum_{r=0}^{n} m_{n-r} \, \frac{(\alpha B)^r}{r!}$$

where $\alpha = \frac{e^2}{4\pi}$, and the m_n are infrared finite and of order α^n with respect to m_0. Summing the series (eq. I-4) produces the result

$$M(p,p') = \exp\,(\alpha B) \sum_{n=0}^{\infty} m_n \qquad \text{(I-5)}$$

This form has several features: firstly, it exhibits factorization of infrared singular and non-singular parts. Secondly, the elastic scattering cross-section vanishes because $B \to -\infty$ as either $\lambda \to 0$ or $d-4 \to 0$ (depending on the type of regulation). This is not, however a proof of confinement of massive QED in four dimensions! The vanishing elastic cross-section is a consequence of a problem that always arises in the definition of an S-matrix for a theory involving massless particles. Basically, it does not cost very much energy to emit a long wavelength photon along with the outgoing electron. An experimenter who is trying to measure the differential cross-section possesses a detector which has but a finite energy resolution. He thus cannot distinguish between an electron and an electron with a little less energy, but in concert with any number of soft photons. Now, the amplitude for the scattering of an electron with the emission of n soft photons again factorizes, however the infrared singularity is precisely the same as the factor written in eq.(I-3) for the process with no accompanying soft photons. When the differential cross-section $\frac{d\sigma}{d\varepsilon}$ for the scattering of the electron with the emission of any number of undetected photons with energy ε (that is, the energy lost by the electron) is computed, infrared divergences arise from the integration over the photon phase space. Yennie, Frautschi and Suura found that these infrared divergences from soft photon emission also factorize and exponentiate:

$$\frac{d\sigma}{d\varepsilon} = \exp\left[\, 2\alpha(B+\tilde{B})\,\right] \cdot \frac{d\sigma^{c}}{d\varepsilon}$$

where $\frac{d\sigma^{c}}{d\varepsilon}$ is infrared convergent and $\tilde{B}$ is given by (using the photon mass regulation):

$$\tilde{B} = \frac{1}{2\pi}\left[\left(\ell n\left(\frac{2p.p'}{m^2}\right) - 1\right)\ell n\,\frac{m^2}{\lambda^2} + \frac{1}{2}\left(\ell n\left(\frac{2p.p'}{m^2}\right)\right)^2 + \left(1 - \ell n\left(\frac{EE'}{\varepsilon^2}\right)\right)\ell n\left(\frac{EE'}{\varepsilon^2}\right)\right] \qquad \text{(I-6)}$$

in the region of high energy and small ε, that is,

$\varepsilon \ll m \ll E, E'$.

In the dimensional regularization scheme,

$$\tilde{B} = \frac{1}{2\pi}\left[\left(\ell n\left(\frac{2p.p'}{m^2}\right) - 1\right)\left(\frac{2}{d-4} + \ell n\left(\frac{m^2}{\mu^2}\right)\right.\right.$$
$$\left.+ \frac{1}{2}\left(\ell n\left(\frac{2p.p'}{m^2}\right)\right)^2 + \left(1 - \ell n\left(\frac{EE'}{\varepsilon^2}\right)\right)\ell n\left(\frac{EE'}{\varepsilon^2}\right)\right] \qquad \text{(I-7)}$$

Thus there is no infrared divergence in the differential cross-section. However, some important non-perturbative information is still contained in the now infrared convergent exponential

$$\exp\ [\ 2\alpha(B+\tilde{B})\ ]$$

such as radiation damping information and large momentum transfer behaviour for the region where $p.p' \gg m^2$.

The absence of infrared singularities in physically observable processes suggests a different description of the asymptotic states in Quantum Electrodynamics. Thus if coherent states describing charged particles along with an indefinite number of soft photons are used, the resulting S-matrix is free of any infrared difficulties[(19)]. It follows that even in QED the concept of a charged particle has to be modified. In QCD, the conceptual difficulties associated with charged particles are even greater. One must confront the property of "twinkling" which would occur if quarks and gluons were free entities. A quark accompanied by a cloud of gluons could continually change its colour charge by emitting and absorbing gluons which themselves carry colour charge.

Perturbation Theory and Quantum Chromodynamics

Cornwall and Tiktopoulos[20] studied the leading infrared singularities arising in QCD in three kinematic regimes. The infrared regime refers to the limit in which the infrared regulator (whether it be the gluon mass λ or the dimensional regulator $\varepsilon = \frac{d-4}{2}$) tends to zero while all external momenta are held fixed and the external particles are on their mass-shells; also, no momentum transfer is allowed to vanish.

The second regime is the fixed angle situation where all invariants, that is, squared energies and momentum transfers (s,t,u,...) are much larger than any of the masses in the theory. The third regime is the Sudakov region where external particles are off mass-shell and invariant momenta squared of the form $(p_i-p_j)^2$ where the p_i are the external momenta, are much greater than the degree to which the particles are off mass-shell, that is, $(p_i-p_j)^2 \gg p_i^2 - m_i^2$, for all i,j. In this region artificial regulation is unnecessary as the singularities appear in the limit as $p_i^2 \to m_i^2$.

In all three regimes the leading infrared singularities for, as an example, the colour singlet form factor F of a quark (up to $O(g^6)$) take on a very similar appearance.

$$F = \exp -\{\frac{g^2 C_F}{8\pi^2} H(t)\} .\ F_B$$

where F_B is the Born approximation to F, C_F is the quadratic Casimir eigenvalue for the fermion representation of $SU(3)^c$ and $H(t)$ is the one-loop Feynman integral associated with the lowest order correction to the Born approximation.

For the fixed-angle regime, the amplitude T for a process with any number of quark and gluon legs can be written as

$$T = \prod_i F_i^{1/2}(t) \quad . \; T_B$$

where $F_i(t)$ is the asymptotic form factor for the i^{th} particle (which carries colour charge C_i). The simplification in this region is due to the fact that all the invariants have fixed ratios with respect to one another and thus logarithms of different invariants can be collected into a logarithm of a common scale plus terms which contribute to the non-leading singularities.

There are several very attractive indications that might be drawn from this picture of the leading singularities of QCD. Since each external gluon has associated with it a form factor, that is, the exponential of something large and negative, the calculation of cross-sections involving the emission of Bremsstrahlung radiation will not follow the pattern of QED. In fact, instead of the usual infrared divergent phase space integral associated with a photon of momentum k, which takes the general form

$$P_{QED} \sim \int \frac{d^{d-1}k}{2k_0} \; \frac{p.p'}{p.k \; p'.k} \quad \sim \quad \frac{1}{d-4}$$

there is now an exponential damping factor involved provided all the virtual radiative corrections associated with the emission of the gluon of momentum k are summed before the phase space integral is carried out, and

$$P_{QCD} \sim \int \frac{d^{d-1}k}{2k_0} \frac{p.p'}{(p.k)(p'.k)} \exp-\left(\frac{k^{d-4}}{(d-4)^2}\right) \sim (d-4)^0$$

Thus the situation in QCD with massive quarks is as follows: the Bremsstrahlung corrections to the cross-section for the scattering of a quark by a colour singlet potential do not contain infrared divergences arising from phase space integrations due to the factor which is the result of summing the virtual infrared divergences to arbitrarily high order in the coupling constant. It appears, then, that the inclusive cross-section still contains the virtual infrared divergences due to the radiative corrections to the basic process (quark + photon $\rightarrow$ quark) and thus that the cross-section in the limit as the infrared regulator tends to zero, is itself zero.

This "indication of confinement" is however based on a certain order of summing divergences and this double summation over the virtual and Bremsstrahlung divergences is manifestly non-uniform in its convergence. This fact has been confirmed by several authors who have reported that order-by-order in QCD the same inclusive cross-section (as would be measured by a real colour-blind detector with finite energy resolution) is infrared finite[(21)]. Indeed, there are proofs that such a statement is true to <u>all</u> orders in perturbation theory[(22)]. The result is perhaps not surprising in the light of the theorems of Kinoshita[23] and Lee and Nauenberg[24]. Kinoshita's theorem states that whenever infrared divergences arise in a process to any order of perturbation theory, they can be eliminated at the transition rate level by summing over the set of all initial and final states which are degenerate in energy, where by degenerate, it is meant that, for example, a massless electron with three momentum $\vec{p}$ is degenerate with a state containing a massless electron with momentum $\vec{p}-\vec{k}$ and a photon with momentum k when the angle between $\vec{p}-\vec{k}$ and $\vec{k}$ is zero. It should be noted that this theorem is stated in terms of the "bare" parameters of the theory because renormalization may introduce mass singularities not covered by the theorem.

At present no-one has created coherent states of quarks and gluons in a fashion similar to that employed by several authors[19] in QED, so that the question of which is the "right" way to sum the double series is quite unresolved.

Chapters II and III and all but the last two pages of Chapter IV are devoted to the development of the differential equation for QCD matrix elements, the problem of the separation of overlapping infrared divergences, and ultimately, the solution of the equation. The end of Chapter IV comprises some comments on different renormalization procedures and their effects on the infrared singularities. Chapter V contains an application of the contents of the previous chapters to the study of the infrared singularities associated with semi-inclusive cross-sections in QCD.

The considerations of Chapter IV are rather formal, and as such, need to be supplemented by low order perturbation theory calculations in the axial gauge before the situation can be resolved.

CHAPTER II

A Differential Equation for Matrix Elements in QCD

The need for information about QCD in a region where the effective coupling becomes large, such as the infrared region, requires methods more powerful than perturbation theory. Such methods do exist, coming from general consideration of field theory; for example, the Dyson equations[25] by which Green's functions can be implicitly related by an integral equation. Also to be included in this category is the Bethe-Salpeter[26] equation which is useful in the solution of bound state and scattering problems. A difficulty which arises with such powerful equations is that the kernel is usually as hard to calculate as the amplitude in question; thus, a solution entails firstly an approximation to the kernel at which point the integral equation becomes tractable. However, in terms of Feynman diagrams, although diagrams of infinite order are summed by this method, there are an infinite number of diagrams which are not taken into account.

Since infrared divergent corrections to a scattering process have a length scale much greater than that associated with hard scattering events (thatis, the scale is that associated with the transferred momentum) it is plausible that such divergences should ignore the intricacies of the actual scattering process and depend only on the external charged objects participating. Following along this line of reasoning would lead one to suspect that the only infrared divergences arising from the insertion of an extra gluon (as when calculating higher order virtual radiative corrections) should come from insertions on the external lines. Indeed, in QED, such is the case. Yennie, Frautschi and Suura[(16)] showed that the overlapping infrared divergences arising from the insertion of a photon could be resolved leaving only those divergences arising from insertions on external charged lines.

From a study of the combinatorics of QED, Caianiello and Okubo[(27)] derived a differential equation with respect to coupling constant of any S-matrix element. The effect of differentiating with respect to the coupling is to insert a photon propagator in all possible ways into the diagrams which make up the matrix element. They proceeded to separate the insertions into infrared divergent and infrared finite contributions. However, it was not made clear that the problem of overlapping infrared divergences had been dealt with[(28)]. More recently, Korthals-Altes and de Rafael[(29)] used a differential equation with respect to photon mass (inserted by hand into each photon propagator) to study the QED infrared problem. The overlapping divergences were treated by using a modification of the technique of Grammer and Yennie[(30)]. This involves separating the photon propagator into two pieces (so-called "gentle" and "hard") whose separate contributions can be analysed systematically.

The complications that arise from similar studies in QCD are due basically to the non-commutation of the colour charge and the increased combinatorical complexity brought on by gluon self-couplings and the existence of the (massless) ghost fields. Since the Grammer and Yennie approach to the overlapping divergences was based on an expansion in the number of photons involved in correcting the basic process, this method cannot be applied to QCD since the number of gluons in diagrams of one order of perturbation theory is not uniform.

In the following, all calculations are performed in ghost-free gauges (e.g., the axial gauge, the timelike gauge, the light-cone gauge (if it exists)) because the combinatorics are simpler (there being no ghost fields) and more importantly, because the overlapping divergences can be simply treated due to the simplicity of the Ward identities in these gauges.

Thus, the QCD Lagrangian is written

$$\mathcal{L} = -\frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a} + i\bar{\psi}_{i}\gamma_{\mu}D^{\mu}_{ij}\psi_{j} - m\bar{\psi}_{i}\psi_{i} - \frac{1}{2\kappa}(n^{\mu}A^{a}_{\mu})^{2}$$

where A^{a}_{μ} is the gluon field, ψ_{j} is the quark field and the field strength $G^{a}_{\mu\nu}$ is given by

$$G^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}A^{c}_{\nu}$$

and the covariant derivative D^{μ}_{ij} (in the fermion representation) is

$$D^{\mu}_{ij} = \partial^{\mu}\delta_{ij} - ig A^{\mu}_{a} T^{a}_{ij} .$$

Notice that because the colour force is being investigated, all reference to quark flavour and flavour dynamics is suppressed. The term $\frac{-1}{2\kappa}(n^{\mu}A^{a}_{\mu})^{2}$ fixes the gauge.

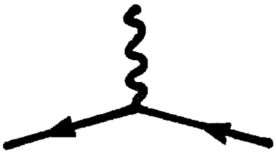
The Feynman rules corresponding to this choice of gauge are as follows:

Quark propagator

$$S_F = \frac{-i\,\delta_{ij}}{\not p - m}$$

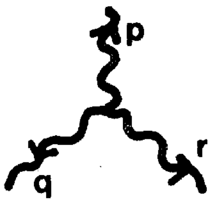
Gluon propagator

$$D^{\mu\nu}_{ab} = \frac{-i\delta_{ab}}{k^2}\left(g^{\mu\nu} + \frac{n^2 k^\mu k^\nu}{(n.k)^2} - \frac{n^\mu k^\nu + k^\mu n^\nu}{n.k}\right)$$



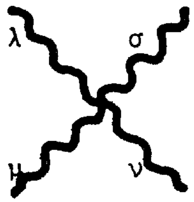
Quark-gluon vertex

$$= -i\,g\,\gamma^\mu T^a_{ij}$$



Three gluon vertex

$$V^{abc}_{\lambda\mu\nu} = g\,f^{abc}[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}]$$



Four gluon vertex

$$\begin{aligned} V^{abcd}_{4\lambda\mu\nu\sigma} = & -ig^2\, f^{abe} f^{cde}\,(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) \\ & -ig^2\, f^{ace} f^{bde}\,(g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) \\ & -ig^2\, f^{ade} f^{cbe}\,(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\mu}g_{\sigma\nu}) \end{aligned}$$

The gauge-fixing vector n^μ is an arbitrary (fixed) Lorentz four-vector and the form of the gluon propagator tabulated above corresponds to the selection $\kappa=0$. For $\kappa \neq 0$ there is an additional term in the gluon propagator $\dfrac{i\kappa\,\delta_{ab}}{k^2}\;\dfrac{k^2\,k^\mu k^\nu}{(n.k)^2}$. With the inclusion of this term, however, power-counting arguments relating to the degree of divergence are difficult to formulate.

We wish to discuss manipulations carried out on the full perturbation series for S-matrix elements in QCD (assuming such matrix elements to exist). The S-matrix element for a process involving N quarks and antiquarks and M gluons as the external particles is conventionally written as the quotient of two perturbative expansions, the numerator yielding the set of all graphs (with correct incoming and outgoing states) including the disconnected graphs, while the denominator is the vacuum-to-vacuum amplitude. The quotient yields, as is well known, the set of all one particle irreducible (1PI) graphs. This decomposition is adequately explained in many texts (e.g., Bjorken and Drell [31]).

For simplicity, all colour, Dirac and Lorentz indices are omitted unless required for clarification. Also, the only label required to keep track of the vertices which occur in the expansion of the exponential of the integral of the interaction Lagrangian over all space and time sandwiched between the initial and final states (i.e., the numerator referred to above) is their position four-vector. Thus, a quark-gluon vertex at position x_1^μ is written as $\lambda\gamma_1 T_1$, a three gluon vertex at position $x_1'^\mu$ is written as $\lambda' f_1 \partial_1$ and a four gluon vertex at $x_1''^\mu$ is written as $\lambda'' f_1^2$. Different symbols λ, λ' and λ'' are used for the couplings of the quark-

-gluon, three gluon and four gluon vertices respectively to aid in the identification of terms in the expansion. Actually,

$$\lambda = -i\,\lambda' = -ig, \qquad \lambda'' = -ig^2.$$

The expanded form of the matrix element can be summarized as

$$M\begin{pmatrix} p' \\ p \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k,\ell} \binom{n}{k,\ell} \int d^4x_1 \ldots d^4x_k d^4x'_1 \ldots d^4x'_\ell$$

$$\cdot\; d^4x''_1 \ldots d^4x''_{n-k-\ell} \;\; \lambda^k \lambda'^{\ell} \lambda''^{(n-k-\ell)} \; \gamma_1 T_1 \ldots \gamma_k T_k$$

$$\cdot\; f_1 \partial_1 \ldots f_\ell \partial_\ell \;\; f_1^2 \ldots f^2_{(n-k-\ell)} \;\cdot$$

$$\begin{pmatrix} p' & x_1 \ldots x_k \\ p & x_1 \ldots x_k \end{pmatrix} [\, x_1 \ldots x_k \;\; 3x'_1 \ldots 3x'_\ell \;\; 4x''_1 \ldots 4x''_{n-k-\ell}] \tag{II-1}$$

where $\binom{n}{k,\ell}$ is the trinomial coefficient $\dfrac{n!}{k!\;\ell!\,(n-k-\ell)!}$

The expression under the integral sign is the n^{th} term in the expansion of the exponential of the action sandwiched between the initial and final states. The square bracket is a shorthand notation (referred to sometimes as a "hafnian" (27)) for the vacuum expected value of a time-ordered product of gluon fields. The expansion of such a square bracket is carried out using Wick's theorem in terms of ordered pairs $[x_1 x_2]$ which are precisely the gluon propagators:

$$[x_1 x_2] \equiv iD^{\mu_1\mu_2}_{a_1a_2}(x_1 - x_2)$$

The square bracket of eq.(II-1) (written explicitly in terms of the gluon fields it represents) is

$$[x_1 \ldots x_k 3x'_1 \ldots 3x' 4x''_1 \ldots 4x''_{n-k-\ell}]$$

$$\equiv \langle 0|\ T(A_{a_1}^{\mu_1}(x_1)\ldots A_{a_k}^{\mu_k}(x_k)A_{b_1^1}^{\nu_1^1}(x'_1)A_{b_1^2}^{\nu_1^2}(x'_1)A_{b_1^3}^{\nu_1^3}(x'_1)\ldots$$

$$\ldots A_{b_\ell^1}^{\nu_\ell^1}(x')A_{b_\ell^2}^{\nu_\ell^2}(x')A_{b_\ell^3}^{\nu_\ell^3}(x')A_{c_1^1}^{\sigma_1^1}(x''_1)A_{c_1^2}^{\sigma_1^2}(x''_1)A_{c_1^3}^{\sigma_1^3}(x''_1)A_{c_1^4}^{\sigma_1^4}(x''_1)\ldots$$

$$\ldots A_{c_{n-k-\ell}^1}^{\sigma_{n-k-\ell}^1}(x''_{n-k-\ell})A_{c_{n-k-\ell}^2}^{\sigma_{n-k-\ell}^2}(x_{n-k-\ell})A_{c_{n-k-\ell}^3}^{\sigma_{n-k-\ell}^3}(x_{n-k-\ell})A_{c_{n-k-\ell}^4}^{\sigma_{n-k-\ell}^4}(x_{n-k-\ell}))|0\rangle$$

The curved bracket $\begin{pmatrix} p' x_1 \ldots x_k \\ p\ x_1 \ldots x_k \end{pmatrix}$ is similarly the vacuum expected value of the time-ordered products of k quark fields and k antiquark fields and also the external quark wave functions specified here by their momenta p and p'. The curved bracket is nothing more than the determinant of the matrix of ordered pairs formed from the top and bottom rows of the bracket, and these ordered pairs are, of course, the Dirac propagators

$$(x_1 x_2) \equiv iS_{Fi_1i_2}^{\alpha^1\alpha^2}(x_1 - x_2).$$

An ordered pair that involves one of the external momenta and one of the internal position vectors is simply the quark spinor wave function at that position; for example,

$$(x_1\, p) = u_p^{i_1}(x_1) ,$$

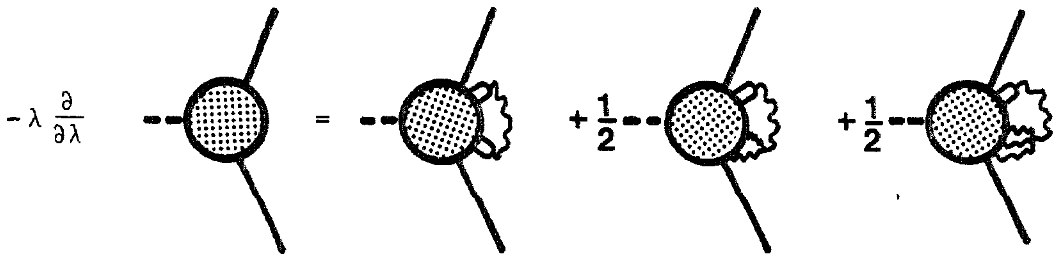
the incoming quark spinor carrying momentum and colour index i_1.

Finally, the pair (p p') is zero corresponding to the selection of the non-forward scattering problem for consideration, that is, the incoming quark must suffer interactions before emerging. Such a term remains, however in forward scattering giving a contribution from the disconnected graphs.

Differentiating eq.(II-1) with respect to λ and expanding the square bracket about its first entry (using the relabelling invariance of the integration variables to recompose the matrix elements) yields

$$- \lambda \frac{\partial}{\partial \lambda} M = \int d^4x_1 \int d^4x_2 \; [x_1 x_2] \left[(g\gamma_1 T_1)(g\gamma_2 T_2) \; M\begin{pmatrix} p' x_1 x_2 \\ p \; x_1 x_2 \end{pmatrix} + (g\gamma_1 T_1)(g f_2 \partial_2) \; M\left(\begin{matrix} p' x_1 \\ p \; x_1 \end{matrix} \middle| \; x_2 x_2 \right) + (g\gamma_1 T_1)(g \; f_2^2) \; M\left(\begin{matrix} p' x_1 \\ p \; x_1 \end{matrix} \middle| \; x_2 x_2 x_2 \right) \right] \tag{II-2}$$

Diagrammatically, then



Similar integro-differential equations may be written with respect to λ' and λ''. The final form of the sought-after equation is found by combining the three equations using the relation

$$g^2 \frac{\partial}{\partial g^2} = \frac{1}{2} [\lambda \frac{\partial}{\partial \lambda} + \lambda' \frac{\partial}{\partial \lambda'} + 2\lambda'' \frac{\partial}{\partial \lambda''}]$$

It is:

$$g^2\frac{\partial}{\partial g^2}\ M(p,p') = -\int d^4x_1 d^4x_2\ [x_1x_2]\Big[\tfrac{1}{2}(-ig\gamma_1T_1)(-i\gamma_2T_2)$$

$$\cdot M\begin{pmatrix} p'x_1x_2 \\ p\ x_1x_2 \end{pmatrix} + (-ig\gamma_1T_1)(-igf_2\partial_2)M\left(\begin{matrix} p'x_1 \\ p\ x_1 \end{matrix}\middle|\ x_2x_2\right)$$

$$+ \tfrac{1}{2}(-igf_1\partial_1)(-igf_2\partial_2)M\left(\begin{matrix} p' \\ p \end{matrix}\middle| x_1x_1x_2x_2\right)$$

$$+ (-ig\gamma_1T_1)(-ig^2f_2^2)M\left(\begin{matrix} p'x_1 \\ p\ x_1 \end{matrix}\middle|\ x_2x_2x_2\right)$$

$$+ (-igf_1\partial_1)(-ig^2f_2^2)M\left(\begin{matrix} p' \\ p \end{matrix}\middle| x_1x_1x_2x_2x_2\right)$$

$$+ \tfrac{1}{2}(-ig^2f_1^2)(-ig^2f_2^2)M\left(\begin{matrix} p' \\ p \end{matrix}\middle| x_1x_1x_1x_2x_2x_2\right)\Big]$$

$$+\int d^4x_1\ [x_1x_1]\left[(-igf_1\partial_1)M\left(\begin{matrix} p' \\ p \end{matrix}\middle| x_1\right) + (-ig^2f_1^2)M\left(\begin{matrix} p' \\ p \end{matrix}\middle| x_1x_1\right)\right] \tag{II-3}$$

This equation is represented diagrammatically in Fig.1 where each term is shown explicitly. The last two terms on the right hand side of eq. (II-3) could be dispensed with: the tadpole  disappears because of the antisymmetric nature of the coupling, that is,

$$f^{abc}\ D^{\mu\nu}_{bc}(k) = 0 \ ,$$

while the gluon propagator self-energy term can be made to vanish in the dimensional regularization scheme. If the gluon were given a mass λ then the integral involved in the gluon self-energy correction would have the form $\int \frac{d^{4-2\varepsilon}k}{(k^2-\lambda^2)} \sim (\lambda^2)^{2-2\varepsilon}$.

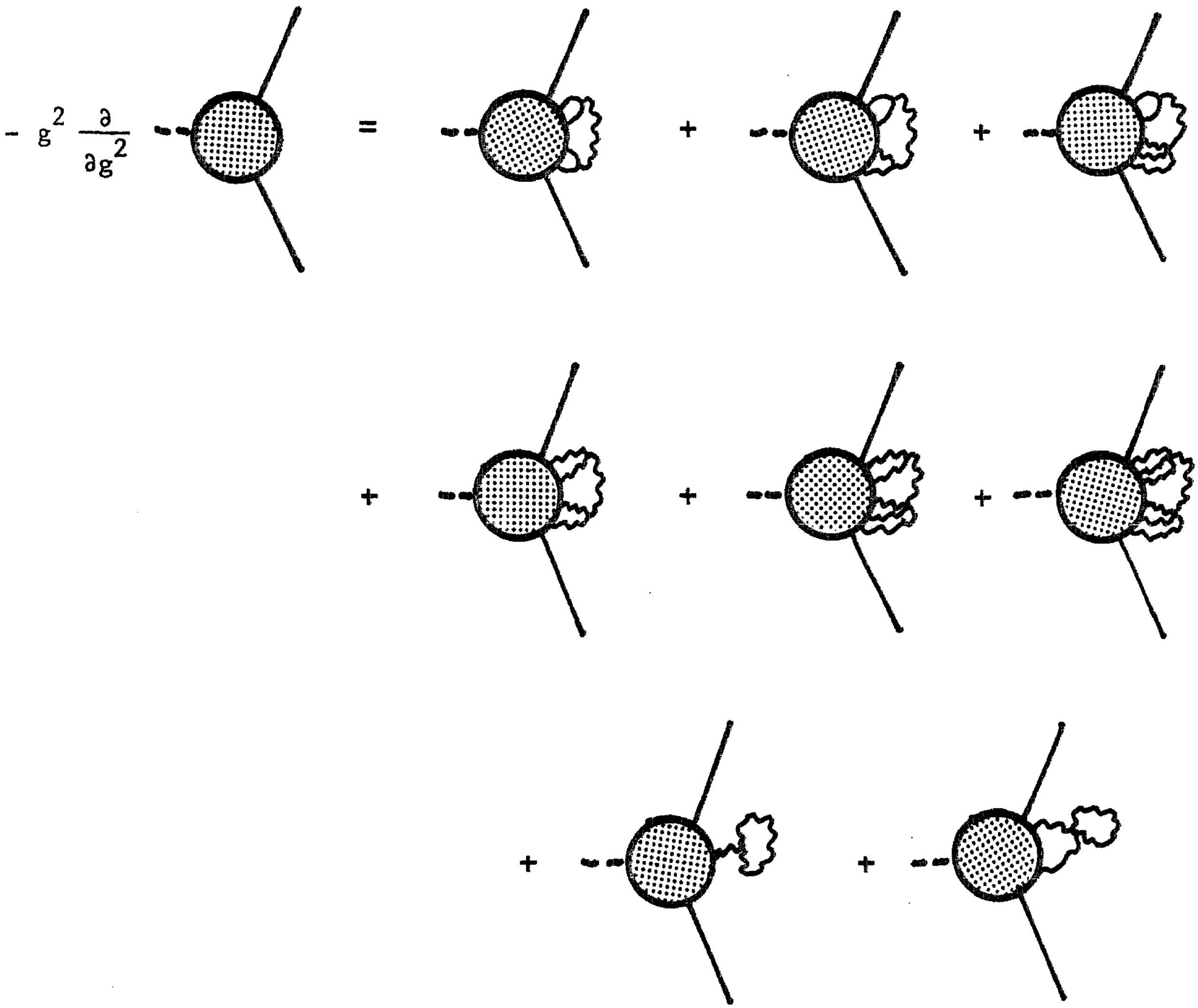


Fig.1

which vanishes in the limit of zero mass. However, from the point of view of constructing the equation order-by-order in perturbation theory, it is a convenient reminder that non-zero terms may be formed by gluon insertions into tadpole diagrams.

In momentum space, eq.(II-3) can be written very concisely:

$$-g^2 \frac{\partial}{\partial g^2} M(p,p') = \int Dk \; D^{\mu\nu}(k) \; M_{\mu\nu}(p,p';k,-k) \qquad \text{(II-4)}$$

where $M_{\mu\nu}(p,p';k,-k)$ is the matrix element for quark scattering accompanied by the emission of a gluon with momentum k and a gluon with momentum $-k$ (without the gluon wave functions).

It seems surprising that such a concise result as eq. (II-4) requires as complicated a derivation. In fact, in QED, the equation analogous to eq. (II-4) can be derived rather elegantly using functional techniques, as demonstrated in appendix A. Attempts at a similar derivation for the equation in QCD are in progress.

CHAPTER III

Ghost-Free Gauges and the Ward Identities

The class of ghost-free gauges to be considered are those which are formed by the addition of the term $-\frac{1}{2\kappa}(n_\mu A^\mu)^2$ to the symmetric Lagrangian (or alternately, by the employment of the subsidiary condition $n_\mu A^\mu = 0$). The introduction of a ghost field is not necessary mainly because the gauge-fixing term does not involve derivatives of A^a_μ, the gluon field. An elegant discussion of this gauge using functional methods is given in the 1973 Erice summer school lectures by S.Coleman[(32)].

The Lorentz four-vector n^μ is arbitrary for the purposes of this discussion, however conventionally, gauges for which $n^2 < 0$ are called axial gauges, gauges for which $n^2 > 0$ are known as timelike. The light-cone gauge, specified by $n^2 = 0$ has come under some criticism recently[(33)] as being not well-defined. This is unfortunate, as it is the only ghost-free gauge in which calculations of Feynman graphs are as easy as in a covariant gauge (see Cornwall[(34)] for theorems).

The only difference (in terms of Feynman rules) between these non-covariant gauges and the covariant gauges lies in the gluon propagator. In a gauge with arbitrary n^2 and κ the free gluon propagator of the theory is

$$D^{ab}_{\mu\nu}(k) = -\frac{i\delta^{ab}}{k^2}\left[\, g_{\mu\nu} + \frac{(n^2-\kappa k^2)}{(n.k)^2}\, k_\mu k_\nu - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n.k)}\,\right] \qquad \text{(III-1)}$$

In order to study the Ward identities in these gauges, consider the generating functional $Z[J,\bar{\eta},\eta]$ for the Green's functions:

$$Z[J,\bar{\eta},\eta] = Z_o \int\!\!\!\int (\mathcal{D} A^a)(\mathcal{D}\bar{\psi})(\mathcal{D}\psi) \exp i\left(\int d^4x \;\; \mathcal{L}_{QCD} - \frac{1}{2\kappa}(n_\mu A^\mu)^2 + J^a_\mu A^\mu_a + \bar{\eta}\psi + \bar{\psi}\eta\right) \qquad \text{(III-2)}$$

where $\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}\,(i\not{D} - m)\psi$

and D^μ_{ab} is the gauge covariant derivative:

$$D^\mu_{ab} = \partial^\mu \delta_{ab} + gf_{abc} A^\mu_c \quad \text{for the adjoint representation}$$

and $$D^\mu_{ij} = \partial^\mu \delta_{ij} - igT^a_{ij} A^\mu_a \quad \text{for the spinor representation.}$$

J_a , $\bar{\eta}$ and η are the source functions for the gluon, quark and antiquark fields respectively. Under an infinitesimal gauge transformation, the QCD fields transform according to the variations

$$(\delta A^a) = (D_\mu \omega)^a = \partial_\mu \omega^a - igf^{abc} \frac{\delta}{\delta J^\mu_b} \omega^c$$

$$\delta\psi = igT^a \psi\, \omega^a$$

$$\delta\bar{\psi} = -ig\bar{\psi}\, T^a\, \omega^a$$

where ω^a is an arbitrary gauge function. Since the integration measure is invariant under such a gauge transformation as is $G^a_{\mu\nu} G^{\mu\nu}_a$, and because ω^a is arbitrary and redefinition of internal integration variables cannot affect the value of $Z[J,\bar{\eta},\eta]$,

$$\delta\, Z[J,\bar{\eta},\eta] = 0.$$

That is,

$$\left[\frac{i}{\kappa} n_\mu \partial^\mu n_\nu \frac{\delta}{\delta J^a_\nu} - igf^{abc} J^b_\mu \frac{\delta}{\delta J^c_\mu} + \partial^\mu J^a_\mu + ig\bar{\eta} T^a \frac{\delta}{\delta \bar{\eta}} - igT^a \eta \frac{\delta}{\delta \eta}\right] Z[J^a,\bar{\eta},\eta]=0. \tag{III-3}$$

Relations between Green's functions (which are the derivatives of Z with respect to the source functions) may be found by carrying out functional differentiation on eq.(III-3) with respect to source terms the requisite number of times and evaluating with the sources set to zero. Thus, taking one functional derivative with respect to J^a_ν of eq. (III-3) yields (after transforming to momentum space)

$$n_\mu \, \Delta^{\mu\nu}(k) = - \frac{\kappa k^\nu}{n.k} \tag{III-4}$$

where κ is the gauge parameter, and $\Delta^{\mu\nu}(k)$ is the fully dressed unrenormalized gluon propagator.

In a similar fashion, taking two functional derivatives of eq. (III-3) with respect to $\bar{\eta}$ gives

$$\left[\frac{i}{\kappa} n_\mu \partial^\mu n_\nu \frac{\delta}{\delta J^a_\nu} \frac{\delta}{\delta \bar{\eta}} \frac{\delta}{\delta \bar{\eta}} + igT^a \frac{\delta}{\delta \bar{\eta}} \frac{\delta}{\delta \bar{\eta}}\right] Z[J^a,\bar{\eta},\eta] = 0 \tag{III-5}$$

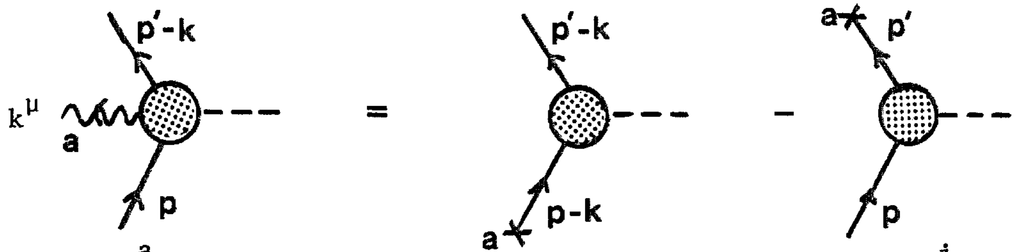
after evaluating at $J = \bar{\eta} = \eta = 0$. This leads, upon Fourier transforming and amputating the external legs, to

$$k^\mu \Lambda^a_{ij\,\mu}(p,p-k,k) = ig \left(T^a S_F^{-1}(p-k) - S_F^{-1}(p) T^a\right)_{ij} \tag{III-6}$$

where Λ^a_{ij} is the vertex part associated with the quark-gluon vertex and $S_F^{-1}(p)$ is the inverse quark propagator with momentum p. This is just the naïve Ward identity of the type one finds in QED[35]. Note that the Ward identity for a quark scattering from a colour singlet potential takes on a form very similar to eq. (III-6):

$$k_\mu \Lambda^{a\,\mu}_{ij}(p,p'-k,k) = ig\left[(T^a)^{i'}_{i}\,\Gamma^{j}_{i'}(p-k,p'-k) - \Gamma^{j'}_{i}(p,p')(T^a)^{j}_{j'}\right] \tag{III-7}$$

Diagrammatically,



where $\Lambda^{a}_{ij}(p,p'-k,k)$ is the associated vertex part and $\Gamma^{j}_{i}(p,p')$ is the amputated Green's function for an incoming quark with momentum p and an outgoing quark with momentum p' .

The last Ward identity that is of use in the discussion of infrared phenomena is that found by operating with $(\frac{\delta}{\delta J})(\frac{\delta}{\delta J})$ on eq. (III-3). Here again, there is little complication, resulting in the coordinate space expression

$$\frac{i}{\kappa} n^\mu \partial_\mu n_\lambda \langle 0|\, T(A^\lambda_a(x)A^\mu_b(y)A^\nu_c(z)\exp i\int d^4w\, \mathcal{L}(w)\,)\,|0\rangle$$

$$= gf^{abd}\,\delta^4(x-y)\,\langle 0|\, T(A^\mu_d(y)A^\nu_c(z)\exp i\int d^4w\,\mathcal{L}\,)|0\rangle$$

$$+ gf^{acd}\,\delta^4(x-z)\,\langle 0|\, T(A^\mu_b(y)A^\nu_d(z)\exp i\int d^4w\,\mathcal{L}\,)|0\rangle \tag{III-8}$$

or, in momentum space, in terms of amputated Green's functions

$$k^\lambda \Gamma^{abc}_{\lambda\mu\nu}(p,p-k,k) = gf^{abd}\,\Delta^{-1\,\mu\nu}_{dc}(p-k) + gf^{acd}\,\Delta^{-1\,\mu\nu}_{bd}(p) \tag{III-9}$$

There is one relation of great use in analysis of infrared singularities which follows from eq. (III-4). In the gauges specified by $\kappa = 0$,

$$n_\mu \, D^{\mu\nu}_{ab}(k) = 0 \tag{III-10}$$

a result that holds for both the bare and the fully dressed propagators.

Another simplifying feature of the ghost-free gauges is that the gluon field renormalization coefficient Z_A and the coupling constant renormalization coefficient Z_g are related by

$$Z_g = Z_A^{-1/2} \tag{III-11}$$

where A_a(unrenormalized) $= Z_A^{1/2}$ A(renormalized)

and g(unrenormalized) $= Z_g$ g(renormalized), provided a gauge invariant regularization and renormalization procedure is employed. This implies in particular that the renormalization group coefficients $\beta(g)$ and $\gamma_A(g)$ are related (γ_A is the anomalous dimension of the vector gluon field) by

$$\beta(g) = -g\gamma_A(g) = \psi(g^2)/g \tag{III-12}$$

The Lorentz structure of the gluon self-energy is also considered here. If all the radiative corrections to the inverse propagator $\Delta^{-1}_{\mu\nu}(k)$ are denoted by $\Pi_{\mu\nu}$, that is,

$$\Delta^{-1}_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu}) - \frac{1}{\kappa}\, n_\mu n_\nu + \Pi_{\mu\nu}$$

then $\Pi_{\mu\nu}$ has the explicit Lorentz structure

$$\Pi_{\mu\nu} = (k_\mu k_\nu - k^2 g_{\mu\nu})\,\Pi_1\left(\frac{(n.k)^2}{n^2k^2}\right) + (k_\mu - n_\mu \frac{k^2}{n.k})(k_\nu - n_\nu \frac{k^2}{n.k})\,\Pi_2\left(\frac{(n.k)^2}{n^2k^2}\right) \qquad \text{(III-13)}$$

As Curtright points out[33], because the Lorentz tensor expression preceding $\Pi_2\left(\frac{(n.k)^2}{k^2n^2}\right)$ in eq. (III-13) does not appear in the Lagrangian, then Π_2 must be ultra-violet finite in four space-time dimensions or else the theory would not be renormalizable.

For technical details regarding such topics as renormalizability and unitarity in the axial gauge, the reader is referred to the various papers of Konetschny and Kummer[36].

CHAPTER IV

Non-Leading Infrared Divergences in Quark Scattering

As presented in Chapter I, the leading infrared singularities in QCD combine to give a simple picture, not unlike QED in form. Although this picture is appealing, it would not be wise to regard the leading singularities as providing the dominant large-distance behaviour of QCD because this large-distance region is precisely that in which the effective coupling constant of the theory becomes large. Thus, in this chapter, the complete infrared singularity structure for the scattering of a quark by a colour-singlet potential is investigated using the differential equation derived in chapter II. It is found that the self-coupling of the massless gluons contributes infrared divergences not found in QED and it is argued that these extra singularities have the effect of changing the insertion of a bare propagator into the insertion of a fully-dressed propagator, or alternately, the replacement of the perturbation expansion parameter g by the effective coupling constant $g(k)$ at the ends of the inserted gluon.

A comparison of the literature on leading infrared singularities demonstrates some differences in the form of the singular factor which may be traced back to differences in the renormalization procedure. If one renormalizes the charge at a point far off-shell (as is the case with the usual asymptotic freedom calculations and also with the calculations of McCoy and Wu[37]), then the leading infrared singularities are independent of the renormalization point. If, however, one renormalizes the charge at a mass equal to that given to the gluon to regulate the infrared singularities(38) or if one regulates both the ultraviolet and infrared singularities using dimensional regularization and on-shell renormalization(39), the renormalization group function β appears explicitly in the expression for the leading singularities. Thus, Korthals-Altes and de Rafael(40) and Cvitanović(41) have shown that the anomalous magnetic moment of a coloured quark (infrared finite in unrenormalized perturbation theory and infrared finite for the analogous QED process - the anomalous magnetic moment of the electron) develops infrared singularities due to charge renormalization when that renormalization is carried out near or at the mass-shell of the quark. In Ref.(41) it is conjectured that the infrared behaviour of the renormalized anomalous magnetic moment $a_q(\tau,\alpha)$ is governed by the equation:

$$[\frac{\partial}{\partial\tau} + \beta_\lambda(\alpha)\alpha\,\frac{\partial}{\partial\alpha}]\;a_q(\tau,\alpha) = 0$$

where $\tau = \log\frac{\lambda}{m}$ (m is the mass of the quark), and $\beta_\lambda(\alpha) = \lambda\,\frac{\partial\alpha}{\partial\lambda}$, λ being the gluon mass inserted to regulate infrared divergences and is also the renormalization point for the charge $\alpha = \frac{g^2}{4\pi}$.

From the results of perturbation theory calculations to order g^4 the leading infrared singular corrections to the cross-section for the scattering of a quark by an external colour-singlet potential have the form[42] (when calculated in $d = 4 + 2\varepsilon$ dimensions, using dimensional regularization):

$$\sigma = F(t,q^2) \cdot \sigma_{Born}$$

where $t = -\frac{1}{2\varepsilon}$ for the exclusive cross-section, and $t = \log \Delta$ for the inclusive processes, Δ being the upper bound on the energy of the emitted gluon Bremsstrahlung. The function $F(t,q^2)$ to their order of approximation was found to be consistent with

$$F(t,q^2) = \exp \int_0^t \bar{g}^2(t') \, C_F \, M(q^2) \, dt'$$

where $$M(q^2) = \frac{1}{2\pi^2} \left[\frac{1+r^2}{2r} \log \left(\frac{1+r}{1-r}\right) - 1 \right],$$

$$r = \left(1 + \frac{4m^2}{q^2} \right)^{-1/2} \qquad \text{and}$$

$$\bar{g}^2(t) = g^2 \left[1 - \frac{11}{24\pi^2} g^2 \, C_V t + \ldots . \right]$$

where C_F and C_V are the quark and gluon Casimir operator eigenvalues.

The situation with regard to the non-leading infrared divergences associated with the scattering of a quark by a colour-singlet potential is not nearly as clear. Apart from the complexity of actually calculating non-leading effects in two-loop virtual corrections to the basic scattering process there are more subtleties that arise in definition of the charge and the use of an infrared regulation that depends on the renormalization point. It is due to difficulties such as these that render comparison of results somewhat awkward. Frenkel and Taylor[43] have questioned the simplicity of the infrared behaviour of QCD as indicated by the leading singularities. They claim (from investigations of quark scattering to $O(g^4)$) that the infrared singular part of a matrix element should satisfy a differential equation with respect to $\ell = \frac{1}{d-4}$ which includes a momentum dependent infrared anomalous dimension:

$$[\frac{\partial}{\partial \ell} + \beta(g) \frac{\partial}{\partial g} + g^2 \; G(\frac{q^2}{m^2}, \ell, g^2) \;] \; A(\frac{q^2}{m^2}, \ell, g^2) = 0$$

These results have been confirmed by Tyburski[44]. Their renormalization scheme involved renormalization at a point $(-\mu^2)$ far off-shell $(\mu^2 >> m^2)$. Poggio[45] has found that simplifications arise in the form of the infrared singular factor by choosing a subtraction procedure that carefully plays off infrared and ultraviolet effects for the near mass-shell behaviour of the theory. Thus, he found that if subtractions are carried out for the massless gluons at $q^2 = -\lambda^2$ and for massive quarks at $p^2 - m^2 = -m\lambda$ where $\lambda^2 << m^2$, the colour-singlet form factor F_1 is consistent with the form (up to $O(g^4)$)

$$F_1 = \exp \; [B_1([g(k^2)]; \; G = -1)]$$

where $B_1([g(k^2)]; \; G = -1)]$ is the one loop contribution calculated in the Landau gauge $(G = -1)$ with the effective coupling $g(k^2)$

substituted for the perturbation expansion coupling g_λ and $g(k^2) \equiv g_\lambda$ for $k^2 = -\lambda^2$. There are renormalization group arguments which suggest that this is an exact result[46].

The approach to the infrared problem presented in this chapter involves the differentiation with respect to coupling constant of the matrix element under consideration (quark scattering by a colour-singlet potential) which is equivalent (from the discussion in Chapter II) to the insertion of a gluon propagator in all possible ways into the complete set of Feynman diagrams which represent the matrix element. An analysis similar to that employed by Yennie, Frautschi and Suura[16] in their study of QED is used to separate the overlapping infrared divergences.

In Chapter II it was shown that the matrix element for quark scattering $M(p,p')$ satisfies the differential equation

$$\alpha \frac{\partial}{\partial \alpha} M(p,p') = \int \frac{d^d k}{(2\pi)^d} D^{\mu\nu}(k^2) M_{\mu\nu}(p,p',k,-k) \qquad \text{(IV-1)}$$

where $M(p,p')$ is the matrix element for the scattering of a quark of momentum p by an electromagnetic potential (acting once) supplying momentum $q = p' - p$, while $M_{\mu\nu}(p,p',k,-k)$ is the matrix element with the emission of two additional gluons (of momenta k_μ and $-k_\nu$) with the gluon legs amputated. The conversion from disconnected to connected Green's functions in general involves division by the vacuum-to-vacuum amplitude which cannot affect the infrared behaviour of the matrix element since the infrared singularities are a function only of the external momenta. The integrations over gluon momentum k in eq.(IV-1) can be separated into two classes - the first yields an infrared divergent factor which multiplies the original matrix element M(p,p'), while the second part is a sum of integrations each of which is infrared finite in a manner to be described below. Infrared singularities arise in d dimensions as powers of $\frac{1}{d-4}$ in the limit $d \to 4$; for example, the one-loop correction to the basic scattering process yields an integral of the form

$$4p.p' \, \bar{u}(p')\gamma^{\mu}u(p) \cdot \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k^2-2p'.k} \frac{1}{k^2-2p.k} \qquad + O((d-4)^0)$$

and in the limit $k \to 0$, the integral has a singular part given by

$$\int \frac{d^d k}{k^4} \sim \frac{1}{d-4}$$

The extraction of infrared singular contributions is carried out in a fashion similar to that employed by Yennie et al.[16] in their study of QED; one separates that part which is supposed to be the total infrared singular contribution due to the insertion of one end of the gluon and shows that a modified perturbation theory constructed from the remainder contains no infrared singularities. In order to follow the treatment of QED as closely as possible, and to avoid the combinatorical problems associated with zero mass ghost fields, a ghost-free gauge (GFG) is used, specified by the gauge-fixing term $\frac{-1}{2\kappa}(n_{\mu}A^{\mu})^2$ in the Lagrangian, where A^{μ} is the gluon field and n_μ is some fixed Lorentz vector. The gluon propagator (as discussed in Chapter III) thus defined is

$$D^{\mu\nu}(k^2) = \frac{i}{k^2}\left[-g^{\mu\nu} - n^2 \frac{k^\mu k^\nu}{(n.k)^2} + \frac{n^\mu k^\nu + k^\mu n^\nu}{n.k}\right],$$

corresponding to the choice $\kappa = 0$.

The matrix element is further specified by factoring out the spinor dependence

$$M(p,p') \equiv \bar{u}(p')\, \Gamma_i^j\, (p,p')\, u(p) \qquad \text{(IV-2)}$$

corresponding to the scattering of a quark of momentum p and colour index i, leaving an outgoing quark of momentum p' and colour index j. In the case under consideration of scattering by a photon,

$$\Gamma_i^j\, (p,p') = \Gamma(p,p')\, \delta_i^j$$

All reference to the photon is suppressed - indeed, scattering from a colour singlet scalar field may be treated entirely analogously.

Consider the contribution from inserting one end of the additional gluon on to the incoming quark line before the virtual gluon exchange interaction region. Suppose the gluon removes momentum k^λ and possesses the colour index a, then the factor in the matrix element is (keeping q fixed)

$$\begin{aligned} &\bar{u}(p'-k)\ \Gamma_{i'}^j(p-k,p'-k)\ \frac{1}{\not p - \not k - m}\ (T^a)_i^{i'}\ \gamma^\lambda u(p) \\ &= \bar{u}(p'-k)\ \Gamma_{i'}^j\ (p-k,p'-k)\ u(p)\ \frac{(2p-k)^\lambda}{k^2 - 2p.k}\ (T^a)_i^{i'} \\ &\quad - \bar{u}(p'-k)\ \Gamma_{i'}^j\ (p-k,p'-k)\ \frac{1/2[\not k,\gamma^\lambda]}{k^2 - 2p.k}\ (T^a)_i^{i'}\ u(p) \end{aligned} \qquad \text{(IV-3)}$$

The latter term on the right-hand side of eq. (IV-3) has the form of a magnetic moment interaction and as both numerator and denominator are proportional to k (for small k) this term is infrared finite with respect to k in the sense that no k-dependent singularity arises as $k \to 0$. Approximating the former term in eq.(IV-3) by

$$\frac{(2p-k)^{\lambda}}{k^2 - 2p.k} (T^a)_i^{i'} \bar{u}(p'-k) \Gamma^j_{i'}(p,p') u(p)$$

leaves a remainder:

$$\frac{(2p-k)^{\lambda}}{k^2 - 2p.k} (T^a)_i^{i'} \bar{u}(p'-k) \{ \Gamma^j_{i'}(p-k,p'-k) - \Gamma^j_{i'}(p,p') \} u(p) \tag{IV-4}$$

One of the reasons for using the ghost-free gauges now becomes apparent, as the Ward identity involving $\Gamma^j_i(p,p')$ is the naïve one (see eq.(III-7)) and so the above expression (eq.(IV-4)) is identically equal to:

$$\frac{(2p-k)^{\lambda}}{k^2 - 2p.k} \bar{u}(p'-k) k^{\rho} (\Lambda^a_{\rho})^j_i(p,p'-k,k) u(p)$$

where $(\Lambda^a)^j_i$ is the vertex part associated with the insertion of a gluon (colour index a) into $\Gamma^j_i(p,p')$. Thus the total contribution from this remainder plus the insertions of the gluon in all possible ways into the interaction region can be represented as

$$\bar{u}(p'-k) [\Lambda^a_{\lambda} + \frac{(2p-k)_{\lambda}}{k^2-2p.k} k^{\rho} \Lambda^a_{\rho}]^j_i u(p) \tag{IV-5}$$

The perturbation expansion of the formula (IV-5) results in the replacement of the normal Feynman rules for vertices in QCD by those for modified vertices; for example, the quark-gluon vertex suffers the replacement

$$-ig(T^a)^j_i\,\gamma^\mu \rightarrow -ig(T^a)^j_i\,[\gamma^\mu + \frac{(2p-k)^\mu}{k^2-2p.k}\,\not{k}]$$

and the three gluon vertex is modified according to

$$V^{abc}_{\lambda\mu\nu}(Q,k) = [(2k-Q)_\nu g_{\lambda\mu} + (2Q-k)_\lambda g_{\mu\nu} + (-Q-k)_\mu g_{\nu\lambda}]gf^{abc}$$

$$\tilde{V}^{abc}_{\lambda\mu\nu}(Q,k) = [(2k-Q)_\nu g_{\lambda\mu} + (2Q-k)_\lambda g_{\mu\nu} + (-Q-k)_\mu g_{\nu\lambda} + \frac{(2p-k)_\lambda}{k^2-2p.k}\{(k-Q)_\nu k_\mu + (2Q.k-k^2)g_{\mu\nu} + (-Q-k)_\mu k_\nu\}]gf^{abc}$$

A similar modification is made for the four gluon vertex. It should be noted that all of the modified vertices are gauge invariant in the sense that a longitudinal polarization vector will give zero contribution to the matrix element; thus, for example

$$k^\lambda\,\tilde{V}^{abc}_{\lambda\mu\nu}(Q,k) = 0.$$

It is clear that when a gluon line of momentum k^λ is inserted into a quark line or gluon line somewhere inside the interaction region, the number of denominators which can become small will, in general increase, raising the possibility of a higher degree of infrared divergence. However, and this is the crux of the matter, it is possible to show that the effect of using a modified vertex at the point of a gluon insertion is to cause no increase in the degree of infrared divergence, except when the other end of the gluon is inserted in such a way as to contribute a gluon self-energy correction. Were it not for this self-energy term, all possible insertions of a gluon of momentum k^λ into the $O(g^{2n})$ approximation to the matrix element would result in a factor (infrared singular) times the original matrix element (in its $O(g^{2n})$ approximation), that is,

$$ig \left[\frac{(2p-k)^\lambda}{k^2-2p.k} - \frac{(2p'-k)^\lambda}{k^2-2p'.k} \right] (T^a)^j_i \, \bar{u}(p'-k) \, \Gamma(p,p') \, u(p), \tag{IV-6}$$

plus a part which is not singular in k and which is no more singular in any of the other loop momenta than beforehand - a result completely analogous to QED. However, as will be explained below, this is not the whole story with QCD.

Following Yennie et al.[16], the insertion of a gluon of momentum k^λ into a quark line with momentum $p + Q$ (using a modified quark-gluon vertex) where Q is the momentum transferred to the quark since its entry into the interaction region, is effected by the replacement

$$\frac{1}{\not{p}+\not{Q}-m} \rightarrow \frac{1}{\not{p}+\not{Q}+\not{k}-m} \left[\gamma^\lambda + \frac{(2p-k)^\lambda}{k^2-2p.k} \not{k} \right] T^a \frac{1}{\not{p}+\not{Q}-m} \tag{IV-7}$$

An increased degree of infrared singularity is possible due to the two denominators which may vanish simultaneously. The right hand side of eq.(IV-7) may be rewritten by anti-commuting the first propagator through the vertex term, yielding

$$\frac{T^a}{(Q-k)^2+2p.(Q-k)}\left(-\left[\gamma^\lambda+\frac{(2p-k)^\lambda}{k^2-2p.k}\not{k}\right](\not{p}+\not{Q}-m)\right.$$
$$\left.+2\left[Q^\lambda+\frac{(2p-k)^\lambda}{k^2-2p.k}Q.k\right]-\frac{1}{2}[\not{k},\gamma^\lambda]\right)\frac{1}{\not{p}+\not{Q}-m}\quad.$$
(IV-8)

The first term in curly brackets in eq.(IV-8) contains an inverse quark propagator which cancels the second propagator, rendering this part of the expression no more singular than before the gluon was inserted. The second term vanishes as $Q \to 0$, thus preventing an extra singularity from arising since for Q^λ small, $\frac{Q^\lambda}{Q^2-2p.Q}$ is not singular in general. For the third term, if Q is set equal to zero in the first propagator then the k integration is regular in the region around $k = 0$. Thus the use of a modified quark-gluon vertex at the point of insertion of a gluon onto a quark line prevents an increase in the degree of infrared divergence. This is to be expected as the situation is completely analogous to the insertion of a photon into an electron line in QED.

The insertion of a gluon of momentum k^λ into another gluon line with momentum Q (using the modified three gluon vertex) is rather more complicated. The insertion can be represented by the replacement

$$\frac{1}{Q^2}\left[-g^{\mu\nu}-\frac{n^2}{(n.Q)^2}Q^\mu Q^\nu+\frac{Q^\mu n^\nu+n^\mu Q^\nu}{n.Q}\right]$$
$$\to\frac{1}{(Q-k)^2}\left[-g^{\mu\rho}-\frac{n^2(Q-k)^\mu(Q-k)^\rho}{[n.(Q-k)]^2}+\frac{n^\mu(Q-k)^\rho+(Q-k)^\mu n^\rho}{n.(Q-k)}\right].$$

$$\cdot \Bigg((k-Q)^{\sigma} g^{\lambda\rho} + (-Q)^{\rho} g^{\lambda\sigma} + \bar{V}^{\rho\lambda\sigma}(Q,k) +$$

$$+ \frac{(2p-k)^{\lambda}}{k^2-2p.k} \left[(Q-k)^{\rho}(Q-k)^{\sigma} - Q^{\rho}Q^{\sigma} + k_{\tau}\bar{V}^{\rho\tau\sigma}(Q,k) \right] \Bigg) \cdot$$

$$\cdot \frac{1}{Q^2} \left[-g^{\sigma\nu} - \frac{n^2 Q^{\sigma} Q^{\nu}}{(n.Q)^2} + \frac{n^{\sigma}Q^{\nu} + Q^{\sigma}n^{\nu}}{n.Q} \right] \qquad \text{(IV-9)}$$

where $\bar{V}^{\lambda\rho\sigma}(Q,k) = V^{\lambda\rho\sigma}(Q,k) + (Q-k)^{\sigma} g^{\lambda\rho} + Q^{\rho} g^{\lambda\sigma}$ and

$V^{\lambda\rho\sigma}(Q,k) = [\ (2k-Q)^{\sigma} g^{\lambda\rho} + (2Q-k)^{\lambda} g^{\rho\sigma} + (-Q-k)^{\rho} g^{\sigma\lambda}\ ]$. The terms containing $\bar{V}$ have a form reminiscent of the insertions into the quark line:

$$\bar{V}^{\lambda\rho\sigma}(Q,k) + \frac{(2p-k)^{\lambda}}{k^2-2p.k} k_{\tau}\bar{V}^{\tau\rho\sigma}(Q,k)$$

$$= k^{\sigma} g^{\lambda\rho} - k^{\rho} g^{\lambda\sigma} + g^{\rho\sigma}(2Q-k)^{\lambda} + \frac{(2p-k)^{\lambda}}{k^2-2p.k}(2Q.k-k^2) \quad g^{\rho\sigma}$$

The remainder, upon expanding out the product of (propagator)$\cdot$ (vertex)$\cdot$(propagator) and carrying out a plethora of cancellations, may be written as (leaving out the propagator poles $\frac{1}{Q^2}$ and $\frac{1}{(Q-k)^2}$)

$$k^{\nu} g^{\lambda\nu} - k^{\mu} g^{\lambda\nu} + \frac{1}{n.Q}[n^{\nu} g^{\lambda\mu} Q\cdot(Q-k) - Q^{\nu} g^{\lambda\mu} n.k + n^{\nu} Q^{\lambda} k^{\mu} + Q^{\nu} n^{\lambda} k^{\mu}$$

$$+ \frac{(2p-k)^{\lambda}}{k^2-2p.k} \{ n^{\nu} k^{\mu} Q^2 - n^{\nu}(Q-k)^{\mu} Q\cdot(Q-k) \}]$$

$$+ \frac{1}{n.(Q-k)} [n^{\mu} g^{\lambda\nu} Q\cdot(Q-k) + (Q-k)^{\mu} g^{\lambda\nu} n.k - n^{\mu}(Q-k)^{\lambda} k^{\nu} - (Q-k)^{\mu} n^{\lambda} k^{\nu}$$

$$+ \frac{(2p-k)^{\lambda}}{k^2-2p.k} \{-n^{\mu}(Q-k)^2k^{\nu}+n^{\mu}Q^{\nu}Q\cdot(Q-k)\}]$$

$$+ \frac{1}{n.Q}\,\frac{1}{n.(Q-k)}[-n^{\mu}n^{\nu}Q\cdot(Q-k)\ \{\ (2Q-k)^{\lambda}+ \frac{(2p-k)^{\lambda}}{k^2-2p.k}\ (2Q.k-k^2)\}$$

$$+ n^{\lambda}n^{\mu}Q^{\nu}Q\cdot(Q-k)\ -\ n^{\lambda}n^{\nu}Q^{\mu}Q\cdot(Q-k)\ +\ n^{\mu}Q^{\nu}(Q-k)^{\lambda}n.k-n^{\nu}Q^{\lambda}(Q-k)^{\mu}n.k$$

$$+ \frac{(2p-k)^{\lambda}}{k^2-2p.k} \{n^{\mu}Q^{\nu}n.k(Q-k)^2\ -\ n^{\nu}(Q-k)^{\mu}n.k\ Q^2\}]$$

$$+ \frac{n^2}{[n.Q]^2}\ [-Q^{\nu}g^{\lambda\mu}Q\cdot(Q-k)-k^{\mu}Q^{\nu}\{Q^{\lambda}+ \frac{(2p-k)^{\lambda}}{k^2-2p.k}\ Q^2\ \}]$$

$$+ \frac{n^2}{[n.(Q-k)]^2}\ [-(Q-k)^{\mu}g^{\lambda\nu}Q\cdot(Q-k)+k^{\nu}(Q-k)^{\mu}[(Q-k)^{\lambda}+\frac{(2p-k)^{\lambda}}{k^2-2p.k}(Q-k)^2]]$$

$$+ \frac{n^2}{[n.Q]^2n.(Q-k)}\ [Q^{\nu}(Q-k)^{\mu}n.k(Q^{\lambda}+ \frac{(2p-k)^{\lambda}}{k^2-2p.k}\ \ Q^2)]$$

$$+ \frac{n^2}{[n.(Q-k)]^2n.Q} [-(Q-k)^{\mu}Q^{\nu}n.k\ \{(Q-k)^{\lambda}- \frac{(2p-k)^{\lambda}}{k^2-2p.k}\ (Q-k)^2\ \}]$$

$$+ \frac{n^2}{[n.Q]^2}\frac{n^2}{[n.(Q-k)]^2}\ (Q-k)^{\mu}Q^{\nu}Q\cdot(Q-k)[-(2Q-k)^{\lambda}- \frac{(2p-k)^{\lambda}}{k^2-2p.k}(2Q.k-k^2)]$$

(IV-10)

This horrifying collection of terms can, however, be classified into three genera. The first kind of term is one proportional to Q^2 or $(Q-k)^2$, the second type consists of terms proportional to k and the third type consists of a grouping of the form proportional to $[Q^{\lambda}+ \frac{(2p-k)^{\lambda}}{k^2-2p.k}\ Q.k]$.

The first type is characterized by a cancellation of one of the propagator poles, that is, either of $\frac{1}{Q^2}$ or $\frac{1}{(Q-k)^2}$ and leaves in its place a term such as $\frac{1}{n.Q}$ or $\frac{1}{n.(Q-k)}$. That such an action reduces the degree of infrared divergence can be seen by comparing Feynman integrals, for instance

$$I_1 = \int d^dk \frac{1}{k^2(k^2-2p.k)(k^2-2p'-k)}$$

$$= \frac{1}{\varepsilon} \frac{-i}{16\pi^2} \frac{1}{\sqrt{(p.p')^2-m^2}} \ln\left(\frac{(p.p') - m^2 + \sqrt{(p.p')^2-m^2}}{(p.p') - m^2 - \sqrt{(p.p')^2-m^2}}\right) + 0(\varepsilon^0)$$

and $$I_2 = \int \frac{d^dk}{(n.k)^2(k^2-2p.k)(k^2-2p'.k)}$$

$$= 4\pi^2 \left(\frac{-i}{16\pi^2}\right) \cdot I + 0(\varepsilon^1)$$

where $$I = \int_0^1 dy \left[\frac{1}{\Delta}\left(\frac{b+2c(1-y)}{\sqrt{-\Delta}} \ln\left(\frac{b+2c(1-y)-\sqrt{-\Delta}}{b+2c(1-y)+\sqrt{-\Delta}} \frac{b+\sqrt{-\Delta}}{b-\sqrt{-\Delta}}\right) - \frac{b(1-y)}{a}\right)\right]$$

where $$\Delta = 4\{m^4y^2+m^2n^2[(1-y)^2+y^2]-2m^2n.p'\,y(1-y)+2m^2n.py^2\}$$

$$a = m^2y^2 + n^2(1-y)^2 - 2n.p'\,y(1-y)$$

$$b = 2n^2(1-y) - 2n.p(1-y) + 2n.p'\,y + 2p.p'\,y$$

$$c = m^2 + n^2 + 2n.p$$

and the parametric integral I is certainly of order ε^0.

The second group of terms, those proportional to k can be further sub-classified. One subclass concerns integrals such as

$$n.k\int\frac{d^dQ}{Q^2(Q^2-2p.Q)n.Q\ n.(Q-k)}$$

which, due to the identity

$$\frac{n.k}{n.Q\ n.(Q-k)} = \frac{1}{n.(Q-k)} - \frac{1}{n.Q}$$

can be shown to be finite in the limit $k \to 0$. The other subclass exhibits a basic antisymmetry of the form $(k^{\nu}g^{\lambda\mu} - k^{\mu}g^{\lambda\nu})$ or variations thereupon. The nature of the rest of the matrix element $M_{\mu\nu}$ is symmetric in terms of its Lorentz structure and thus these terms do not contribute infrared singularities.

The last type of remainder term

$$\left(Q^{\lambda} + \frac{(2p-k)^{\lambda}}{k^2 - 2p.k}\, Q.k\right) g^{\rho\sigma} \qquad \text{(IV-11)}$$

has the property of making any integral over Q involving such a term deficient in one power of p. Now, if one considers a general graph representative of colour-singlet quark scattering (see Fig. 2), in most cases, the gluon of momentum Q into which the new gluon is inserted can be traced back the p-line, forming a loop that does not intersect the p'-line (i.e., the outgoing quark line). In this case, the effect of the integral over Q will be to replace Q by a linear combination of the other momenta in the diagram:

$$Q^{\mu} \to \alpha_1 p^{\mu} + \alpha_2 n^{\mu} + \alpha_3 k^{\mu} + \Sigma \beta_i k_i^{\mu}$$

where $\{k_i\}$ is the set of internal momenta intersected by the Q loop. The replacement of Q^{μ} by $\alpha_1 p^{\mu}$ renders the expression

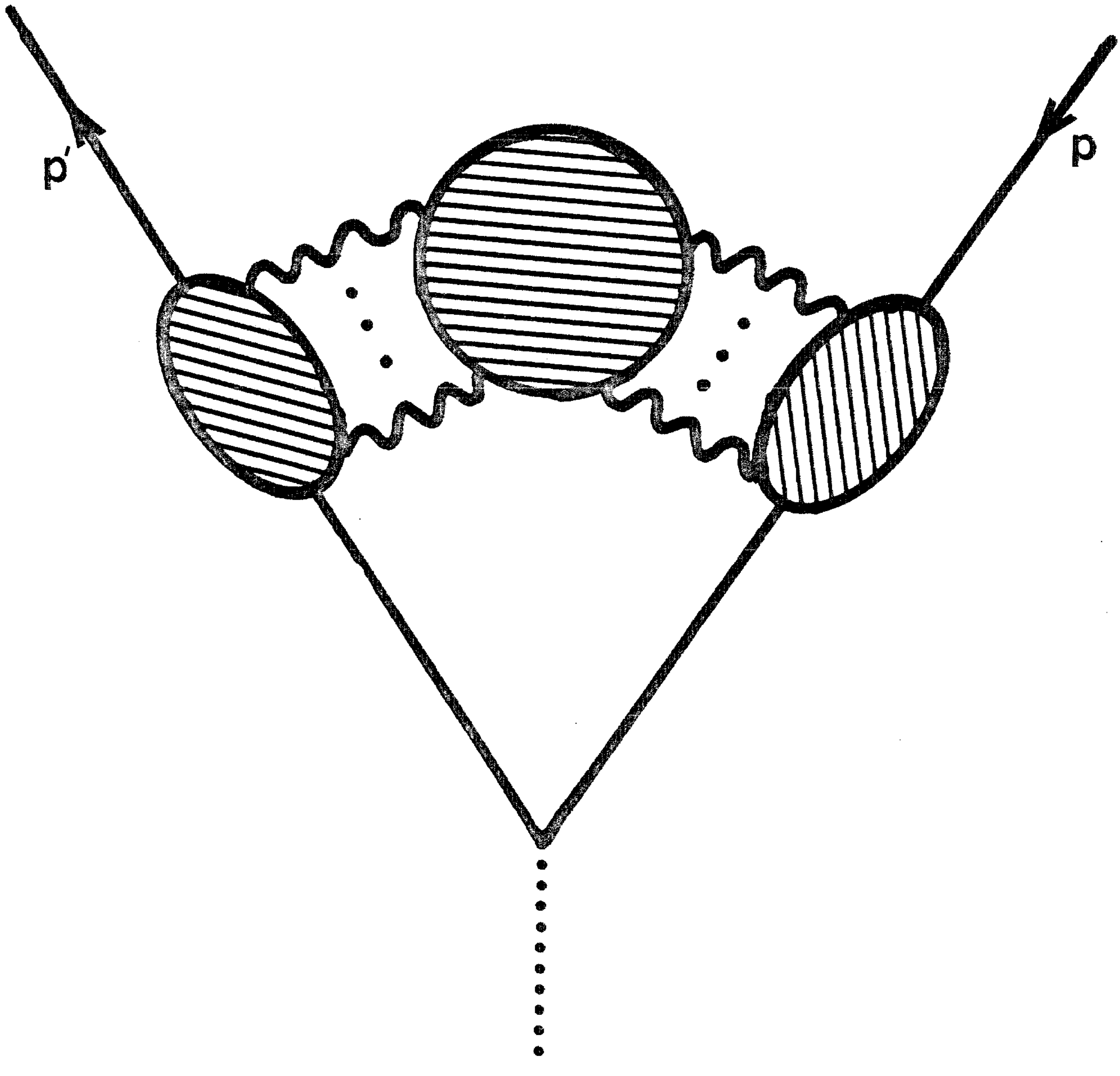


Fig. 2

(IV-11) of order $(k)^1$ rather than $O(k^0)$ and removes any problems of divergence associated with this vertex. Similarly, the replacement of Q^μ by $\alpha_3 k^\mu$ gives a finite contribution in the limit $k^\mu \to 0$. Replacement of Q^μ by any of the internal momenta $\{k_i\}$ also reduces the degree of infrared divergence. In the case of small momentum transfer that is being considered at present, one may choose n^μ to be parallel to p^μ, that is, $n^\mu = \xi p^\mu$ (where ξ is a fixed complex number). The replacement $Q^\mu \to \alpha_2 n^\mu$ then enables the utilization of arguments for the replacement of Q^μ by $\alpha_1 p^\mu$. However, if one wishes to study a general matrix element with several external momenta, the restriction of n^μ parallel to p^μ is not adequate. The terms for which $Q^\lambda g^{\rho\sigma} \to \alpha_2 n^\lambda g^{\rho\sigma}$ do not contribute to the integral over the inserted gluon propagator because of the Ward identity

$$n^\lambda D_{\lambda\zeta}(k) = 0. \qquad \text{(IV-12)}$$

Also the second term of eq.(IV-11) may be decomposed as a sum of terms of the first type which were discussed above, since

$$g^{\rho\sigma} \frac{(2p-k)^\lambda}{k^2-2p.k} Q.k = -\frac{1}{2} g^{\rho\sigma} \frac{(2p-k)^\lambda}{k^2-2p.k} [(Q-k)^2 - Q^2 - k^2]$$

and hence do not contribute to the infrared divergences. Thus the replacement of Q by p or n leads to the highest power of p or n to be cancelled by virtue of the form of the modified vertex, the resulting expression containing one more power of k or one of the internal momenta $\{ k_i \}$.

The radiative corrections associated purely with the p'-line are independent of the radiative corrections associated purely with the p-line as in QED, and so if a gluon of momentum k is inserted into the p-line blob (see Fig.2) there will be no infrared singularities introduced in the p'-line blob as the hard momentum transfer $q = p'-p$ can absorb the small momentum k without singular adjustment.

The remaining class of diagrams are those in which the insertion point inside the gluonic blob (see Fig.2) is not multiply connected with the p-line, that is, there is no closed Q-loop that connects the point of insertion with the p-line that does not also intersect the p'-line. The only way in which this can happen is when the gluonic blob is disconnected (see Fig.3). It now becomes important where the other end of the gluon is inserted. Insertion into either the p-line blob or the p'-line blob or insertion into a part of the gluonic blob already multiply connected with the p-line will not yield an extra infrared divergence as demonstrated above. Also, insertion between disconnected parts of the gluonic blob (as shown in Fig. 4) do not introduce new infrared divergences because one creates a k-loop that connects with the p-line solely and arguments can be made showing that the use of the modified vertices at the points of insertion prevent divergences arising

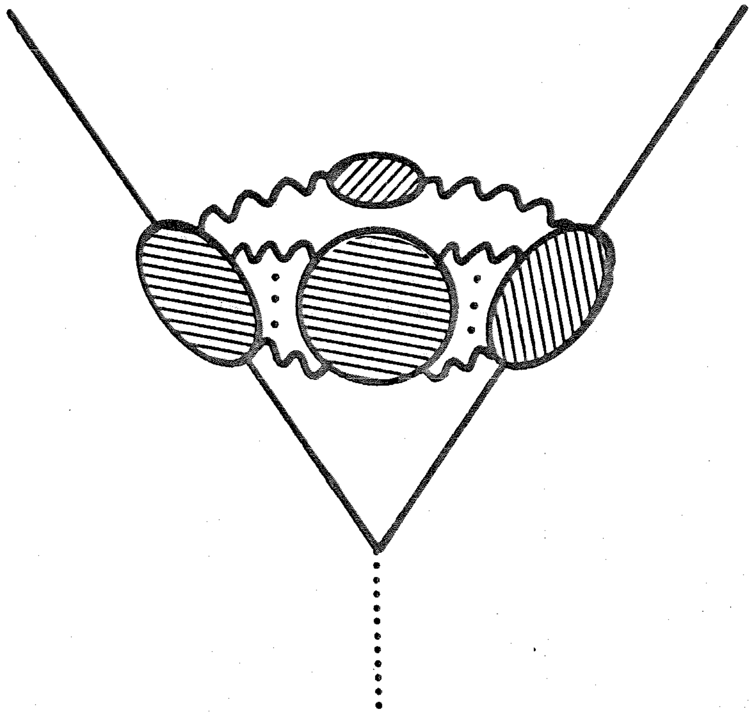


Fig. 3

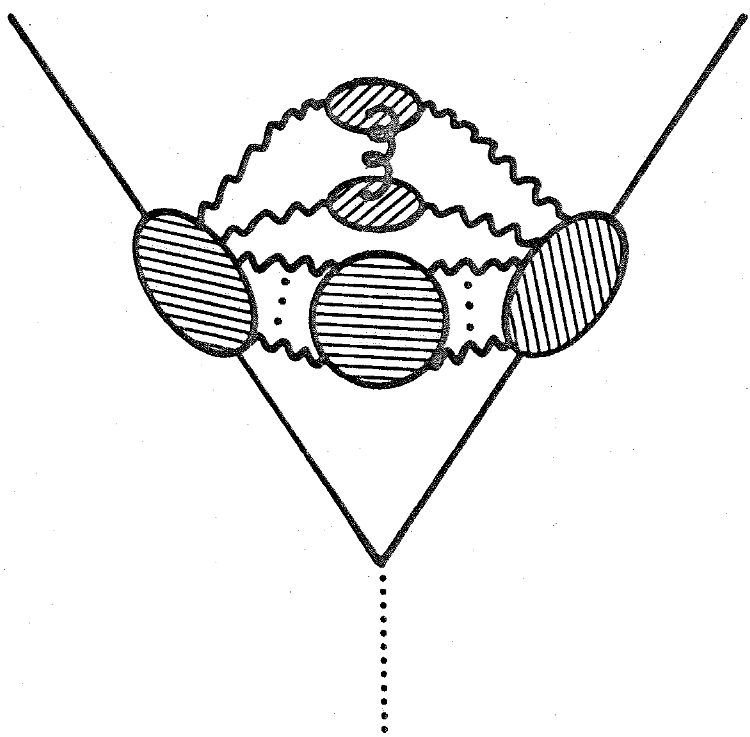


Fig.4

in the k-loop. Then the only place where a divergence may arise is from the insertion of both ends of the gluon into the same disconnected part of the gluonic blob, in other words, the modified vertices help to remove all infrared divergences except for those which arise from gluon self-energy insertions. Even then, an infrared divergence will only occur provided the underlying dressed gluon onto which the insertion is made itself contributes to the infrared divergences.

The last type of modified vertex, that is, the modified four gluon vertex contributes no extra infrared divergences in terms of the program set out above. The insertion of one end of a gluon into a three gluon vertex to form a four gluon vertex does not increase the number of denominators which may become small (compared with insertions into a quark or gluon propagator which do increase the possible number of small denominators). This is also the case in scalar quantum electrodynamics[16] where there is a scalar-scalar-vector-vector vertex. Also, a study of the leading logarithmically divergent graphs in QCD demonstrates nicely the absence of individual graphs containing four gluon vertices. Thus a graph that is leading at one order of perturbation theory will always be non-leading at the next higher order if the higher order graph is constructed by an insertion of a gluon which produces a four gluon vertex.

Were it not for the gluon self-energy corrections, the insertion of the other end of the gluon everywhere inside and outside the interaction region would then yield a similar infrared divergent factor multiplying the matrix element plus an innocuous remainder. Sewing the insertions together with a gluon propagator and integrating over k, at the same time symmetrizing over the two gluon insertion points to allow for the consequences of double counting, gives on the right hand side of eq.(IV-1) an infrared divergent integral

$$I(q^2) = -\frac{g^2 C_F}{2} \int \frac{d^d k}{(2\pi)^d} D_{\mu\nu}(k) \left[\frac{(2p-k)^\mu}{k^2-2p.k} - \frac{(2p'-k)^\mu}{k^2-2p'.k}\right] \cdot \left[\frac{(2p-k)^\nu}{k^2-2p.k} - \frac{(2p'-k)^\nu}{k^2-2p'.k}\right] \qquad \text{(IV-13)}$$

$$= -\frac{g^2 C_F}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2} \left[\frac{(2p-k)^\mu}{k^2-2p.k} - \frac{(2p'-k)^\mu}{k^2-2p'.k}\right]^2$$

times the original matrix element $\bar{u}(p') \Gamma_i^j(p,p') u(p)$ plus integrals over k which are non-singular as $d \to 4$. That $I(q^2)$ is independent of n , the gauge fixing vector, and is in fact gauge invariant may easily be checked.

However, the effect of infrared divergences arising from gluon self-energy corrections is to change the insertion from that of the bare propagator into the insertion of a fully dressed propagator, or alternately, the replacement of g^2, the perturbation theory expansion parameter, by $g^2(k^2)$, the effective coupling at the scale given by k^2. An inspection of the differential equation order by order in perturbation theory is useful for clarification of this statement. For the analogous QED process (electron scattering, Yennie et al.[16] showed that if the perturbation expansion is written as

$$M(p,p') = \sum_{n=0}^{\infty} M_n(p,p')$$

where n is the number of virtual photons involved in the radiative correction to the basic process, then

$$M_n(p,p') = \sum_{r=0}^{n} \frac{(\alpha B)^r}{r!} m_{n-r}$$

where the m_j are infrared finite functions of order α^j relative to m_0 and αB is the one-loop virtual correction to the basic scattering process. Summing the series yields

$$M = \exp(\alpha B) \sum_{n=0}^{\infty} m_n$$

In contrast, for QCD, if the perturbation expansion for quark scattering is written as

$$M(p,p') = \sum_{n=0}^{\infty} M_n(p,p') \qquad \text{(IV-14)}$$

where n now refers to the order (in α) of the correction to the basic scattering of a quark by a colour-singlet current.

The lowest order term is infrared finite, i.e., $M_0 \equiv m_0$, and in the particular process chosen is independent of α. At the next order, the differential equation reads

$$\alpha \frac{\partial}{\partial\alpha} (M_0 + M_1) = (\alpha B_0)\ M_0 + \alpha\frac{\partial}{\partial\alpha}\ m_1 \tag{IV-15}$$

Inverting this equation gives

$$M_1 = (\alpha B_0)\ m_0 + m_1 \tag{IV-16}$$

where B_0 is the one loop correction using the bare gluon propagator. For $O(\alpha^2)$, the differential equation is

$$\alpha \frac{\partial}{\partial\alpha} (M_0 + M_1 + M_2) = (\alpha B_0)\ (M_0 + M_1) + \alpha \frac{\partial}{\partial\alpha}\ m_2 + \alpha \frac{\partial}{\partial\alpha}\ m_1 + (\alpha B_0)\ \alpha\frac{\partial}{\partial\alpha}\ m_1 + (\alpha B_1)\ m_0 \tag{IV-17}$$

where αB_1 is the correction to the basic process where the gluon propagator itself has a self-energy insertion (to order α^1). The diagrammatic forms of eqs.(IV-15) and (IV-17) are shown in Fig.(5). The hatched circle represents the matrix element to the order of concern while the filled (black) circle indicates an infrared finite function and the triangular objects with curly tops represent the infrared divergent integrals and the fact that they factor out of the rest of the matrix element.

Eq. (IV-17) may now be used to extract the equation for M_2 which is

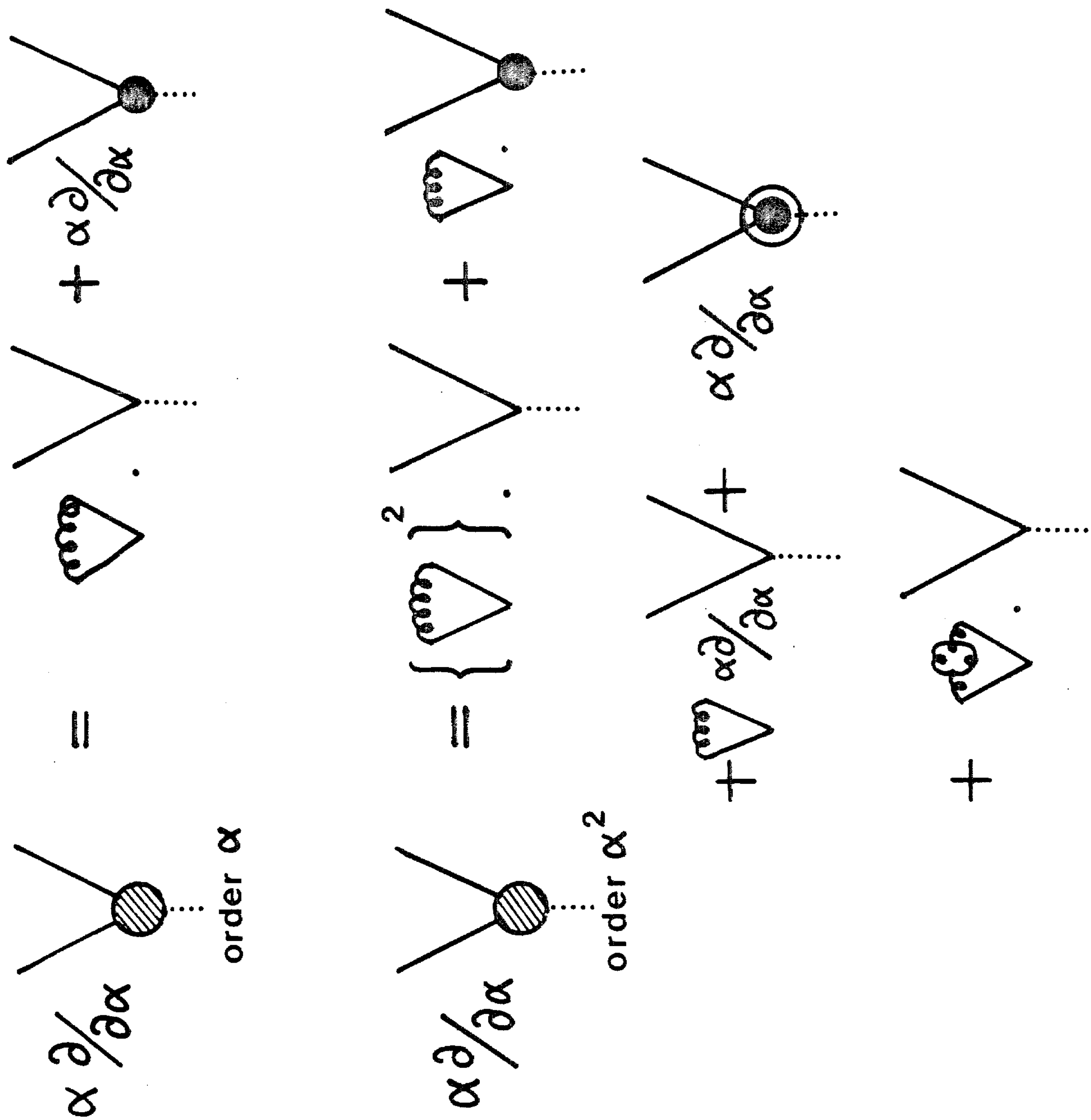


Fig.5

$$\alpha \frac{\partial}{\partial\alpha} M_2 = (\alpha B_0)^2 m_0 + (\alpha B_0)\alpha \frac{\partial}{\partial\alpha} m_1 + (\alpha B_0)\ m_1 + \alpha\frac{\partial}{\partial\alpha} m_2 + (\alpha B_1) m_0 \tag{IV-18}$$

which has the solution

$$M_2 = \frac{(\alpha B_0)^2}{2!} \cdot m_0 + (\alpha B_0)\ m_1 + m_2 + [\int d\alpha B_1\ ]\ m_0 \tag{IV-19}$$

The result of this separation of infrared divergences is that the differential equation eq.(IV-1) can be rewritten as

$$-\alpha \frac{\partial}{\partial\alpha} M(p,p') = [\alpha \frac{\partial}{\partial\alpha} I(\bar{\alpha},q^2)] \cdot M(p,p') + F(p,p') \tag{IV-20}$$

where $F(p,p')$ may be reconstructed from perturbation theory, and $I(\bar{\alpha},q^2)$ is given by

$$I(\bar{\alpha},q^2) = -\frac{ig^2C_F}{2} \int \frac{d^dk}{(2\pi)^d}\ \Delta_{\mu\nu}(k^2)\left[\frac{(2p-k)^\mu}{k^2-2p.k} - \frac{(2p'-k)^\mu}{k^2-2p'.k}\right]$$

$$\cdot \left[\frac{(2p-k)^\nu}{k^2-2p.k} - \frac{(2p'-k)^\nu}{k^2-2p'.k}\right] \tag{IV-21}$$

where $\Delta_{\mu\nu}(k^2)$ is the fully dressed gluon propagator (in the purely gluonic sector of the theory) which may be written in terms of the Lorentz scalars Π_1 and Π_2 [(33)]:

$$\Delta_{\mu\nu} = \frac{-1}{k^2[1+\Pi_1]}\left\{g_{\mu\nu} + \frac{[k_\mu k_\nu n^2 - (n_\mu k_\nu + k_\mu n_\nu)n.k][1+\Pi_1+\Pi_2] + n_\mu n_\nu k^2\Pi_2}{[(n.k)^2(1+\Pi_1+\Pi_2) - n^2k^2\Pi_2]}\right\}$$

The term $\dfrac{n_\mu n_\nu k^2\ \Pi_2}{[(n.k)^2(1+\Pi_1+\Pi_2) - n^2k^2\Pi_2]}$ appears not to contribute to the infrared divergences because it is proportional to k^2 and thus removes the pole of the propagator. In this case, $I(\bar{\alpha},q^2)$ may be rewritten as

$$I(\bar{\alpha},q^2) = -\frac{iC_F}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{g^2(k^2)}{k^2}\left[\frac{(2p-k)^\mu}{k^2-2p.k}-\frac{(2p'-k)^\mu}{k^2-2p'.k}\right]^2 \tag{IV-22}$$

where $g^2(k^2)$ receives contributions only from the massless fields (i.e., gluons). Equation (IV-20) may be solved[27] or alternately, eq.(IV-14) may be summed to give the complete form of the infrared singularities for this QCD scattering process:

$$M(p,p') = \exp -I(\bar{\alpha},q^2)\;\cdot M^f(p,p') \tag{IV-23}$$

where $M^f(p,p')$ contains no infrared singularities.

If renormalization of the colour charge is carried out in a manner which is independent of the infrared regulation, eq.(IV-23) expressed in terms of renormalized quantities will contain no hidden infrared singularities. If, however, the renormalization point as connected to the infrared cutoff, then there will be an interference of infrared and ultraviolet singularities. A comparison of the perturbation theory results of Frenkel et al.[42], Cvitanović[41] and Korthals-Altes and de Rafael[40] on the one hand and Cornwall and Tiktopoulos[20], McCoy and Wu[37] and Carazzone et al.[47] on the other will convince the reader of this distinction.

The effect of this interference of singularities is felt only in the colour charge. This can be seen if the charge is renormalized on-shell. In the ghost-free gauges, the coupling constant renormalization coefficient Z_α is simply equal to the gluon wave function renormalization coefficient and can be expressed rather elegantly (using dimensional regularization with dimension $d = 4 + 2\varepsilon$) in terms of the renormalization group function $\beta(\alpha)$

$$Z_\alpha = \exp\int_0^\alpha \frac{dx}{x} \frac{\beta(x)}{\beta(x)+2\varepsilon} \qquad \text{(IV-24)}$$

The ghost-free gauge is necessary in order that the work of Lautrup[48] in QED can be carried over without modification to QCD. Using eq.(IV-24), with $\alpha_U = Z_\alpha^{-1}\alpha_R$

$$\alpha_U \frac{\partial}{\partial\alpha_U} = [\ 1 + \frac{\beta(\alpha)}{2\varepsilon}\ ]\ \alpha_R \frac{\partial}{\partial\alpha_R}$$

where α_U and α_R are the unrenormalized and renormalized couplings respectively. One can still write down the solution to the differential equation in the form of eq.(IV-23), but the coupling constant must be modified to include the extra dependence on $\frac{\beta(\alpha)}{2\varepsilon}$ indicated by eq.(IV-24), and the solution now has hidden singularities characterized by a singular expansion parameter.

The fact that the complete form of the infrared singularities in QCD (though simple in comparison to the ugly combinatorics involved in sorting out the order-by-order perturbation theory) is more complicated than that of the other gauge theory with massless gauge bosons in current use - QED - is perhaps not surprising when one compares the behaviour of the renormalization group functions for the two theories. In Quantum Electrodynamics on the one hand, $\alpha = 0$ is an infrared stable fixed point of the theory (and the existence of other fixed points is unknown in $d = 4 + 2\varepsilon$ dimensions) while on the other hand, for QCD, in $d = 4 + 2\varepsilon$ dimensions, low order perturbation theory indicates the existence of another fixed point in the theory (to $O(\alpha^2)$), at $\alpha = \frac{\varepsilon}{b_0}$ where $\beta(\alpha) = 2\varepsilon\alpha - 2b_0\alpha^2 + \ldots$. Thus QCD in greater than four dimensions has an infrared stable fixed point at $\alpha = 0$ indicating perhaps a "free" phase; however, it is the region $\alpha > \frac{\varepsilon}{b_0}$ that is presumably of physical significance. The claim to a knowledge of the complete structure of the infrared singularities of the quark scattering matrix element is made modulo a knowledge of the β function as the coupling constant becomes large; this requires further non-perturbative calculation. However, some progress has been made in that speculations based only on leading log infrared results can now be justified - the deeper structure of the theory produces no real surprises. It is gratifying to note that the result for the corresponding situation in QED can be obtained simply by setting $C_F = 1$ and $C_V = 0$ in eq.(IV-23).

This approach to the analysis of the infrared singularities in QCD is quite flexible as the separation of overlapping divergences does not depend on the method of renormalization (provided the regularization procedure preserves gauge invariance) nor does it depend at first sight on the manner in which one regulates the infrared divergences. For example, one could consider an off-shell process and separate the contributions from the inserted gluon in eq.(IV-1) which are infrared singular and non-singular in the limit as the external quarks tend to their mass-shells.

It is interesting that only the gluonic sector contributes to $g^2(k^2)$ in Eq.(IV-22) for the theory with massive fermions. This is because, as with QED[30], insertions into a closed massive fermion loop do not give rise to infrared singularities. Thus, it is possible in QCD with the number of flavours $n_f \geq 17$ (too large for asymptotic freedom), to have an infrared behaviour that has been suspected of having a relation to confinement.

The discussion of infrared singularities presented here is completely formal - low order perturbation theory calculations (to order g^4) in the axial gauge need to be carried out before the result can be fully believed. This is especially so in light of the contradictory calculations carried out by Frenkel and Taylor[43], albeit in a different gauge. The fact that Poggio[45] explicitly agrees with the result formulated here (to order g^4) does not carry much weight due to his unorthodox renormalization prescription.

CHAPTER V

Infrared Divergences in Inclusive Cross-Sections

It has been known for a long time that the infrared divergences in QED arising from phase space integrations over radiated soft photons contribute an exponential divergent factor which cancels the divergences due to virtual corrections to the process under consideration[49],[50].

In QCD, several authors[21] have found that the Bloch-Nordsieck program order-by-order in perturbation theory produces a cancellation between virtual corrections and soft gluon emission and hence leads to finite transition rates provided that no colour charge is detected - the so-called "colour-blind" experiments. Appelquist et al.[21] studied the production of a quark by a colour-singlet current and its detection by a colour-blind quark detector (triggering, for instance, on the fractional elctric charge of the quark) with energy resolution ΔE. To the three-loop level in perturbation theory, they showed that the inclusive transition probability was infrared finite. Inclusive here means that the transition probability includes contributions from processes with soft gluons in the final state, as a quark and a long-wavelength gluon are almost degenerate in energy and hence will not be resolved by the detector.

Poggio and Quinn[22] and Sterman[22] have shown that the totally unrestricted quark production rate is infrared finite to all orders of perturbation theory. This statement is not really surprising when one considers that the cross-section is found from taking the imaginary part of the vacuum polarization of the electromagnetic current and that for quark production this current has a momentum squared that is different from zero (i.e., off mass-shell for the photon) which provides an infrared cut-off.

Consider the quark scattering process studied in Chapter IV. If the outgoing quark is detected by a colour-blind apparatus with a finite energy resolution ΔE, then the inclusive cross-section (allowing for the emission of soft gluons up to a combined energy of ΔE) can be found by summing the various unitarity cuts associated with the process of forward elastic quark scattering by a colour-singlet current. Naturally, because of the phase space restriction, it is only a part of the full unitarity cut that contributes to the desired cross-section. Now the effects of the operation $\alpha \frac{\partial}{\partial\alpha}$ on the elastic scattering amplitude of a photon with a quark are described in Chapter IV in some detail. In summary, the analysis shows that the insertion of an extra gluon into the collection of diagrams that make up the amplitude yields infrared divergences when the insertion is between external charged lines. Thus, if the cross-section under consideration was of the form

photon + quark $\rightarrow$ anything

the only insertion that could possibly yield an infrared divergence in the corresponding elastic amplitude

photon + quark $\rightarrow$ photon + quark

comes from the external insertion between the incoming quark of momentum p and the outgoing quark also with momentum p (see Fig.6). However, the integrand corresponding to this insertion is identically zero. Explicitly, the insertion gives a contribution (from Eq.IV-22)

$$\alpha I(p,p) = \int \frac{d^d k}{(2\pi)^d} \left[\frac{(2p-k)^\mu}{k^2-2p.k} - \frac{(2p-k)^\mu}{k^2-2p.k}\right]^2 \frac{g^2(k^2)}{k^2}$$

$$= 0.$$

Since the imaginary part of this amplitude is (apart from trivial kinematic factors) the desired total cross-section, iteration of Eq.(IV-1) shows immediately, to all orders (leading and non-leading) that the total production rate is infrared finite.

As for the process

quark + photon $\rightarrow$ quark + soft gluons,

with a restriction on the gluon phase space, one may no longer ignore the fact that taking the unitarity cut places all particles in the intermediate states on their mass-shells.

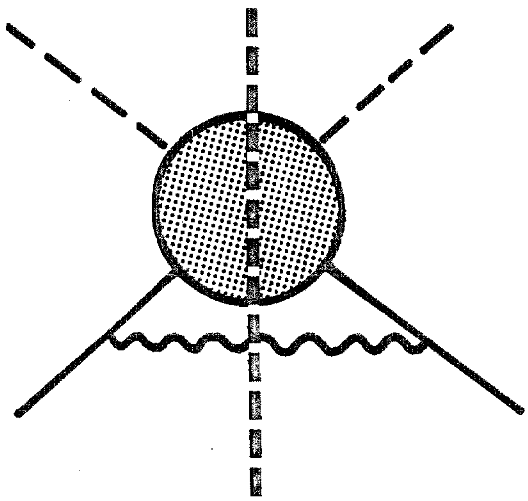


Fig.6

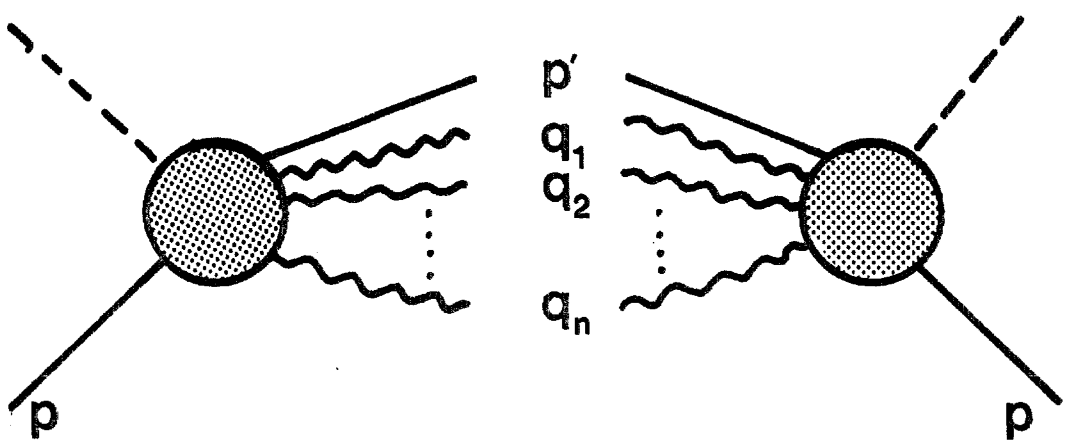


Fig. 7

Let the differential cross-section for semi-inclusive quark scattering be written as

$$\frac{d\sigma}{d\varepsilon} = \sum \int dk_1 \ldots dk_n \; |M_n|^2 \qquad \text{(V-1)}$$

where M_n is the amplitude for quark scattering with n soft gluons in the final state (see Fig.(7)) as well as the final state quark. Extra gluons in the final state are the only particles being considered as ΔE can be adjusted to be below the production threshold for $q\bar{q}$. The gluon insertions which may produce infrared divergences are of three types (shown in Fig.(8)). There is the insertion between the initial colour-charged quark and its counterpart (outgoing in the elastic forward amplitude) with the same momentum. This insertion, as mentioned above gives zero contribution. The second type of insertion consists of one termination of the gluon on the incoming quark of momentum p and the other termination of the gluon on one of the final state particles (intermediate states in terms of the elastic forward amplitude). Lastly, there are the insertions between final state particles.

The insertions of the second type can most easily be considered in terms of the insertions into the elastic amplitude and the subsequent cutting of the graphs to yield the square of the inelastic amplitude. Figure (9) represents the insertion of a gluon into the ℓ^{th} gluon of M_n (the other gluons are not drawn for

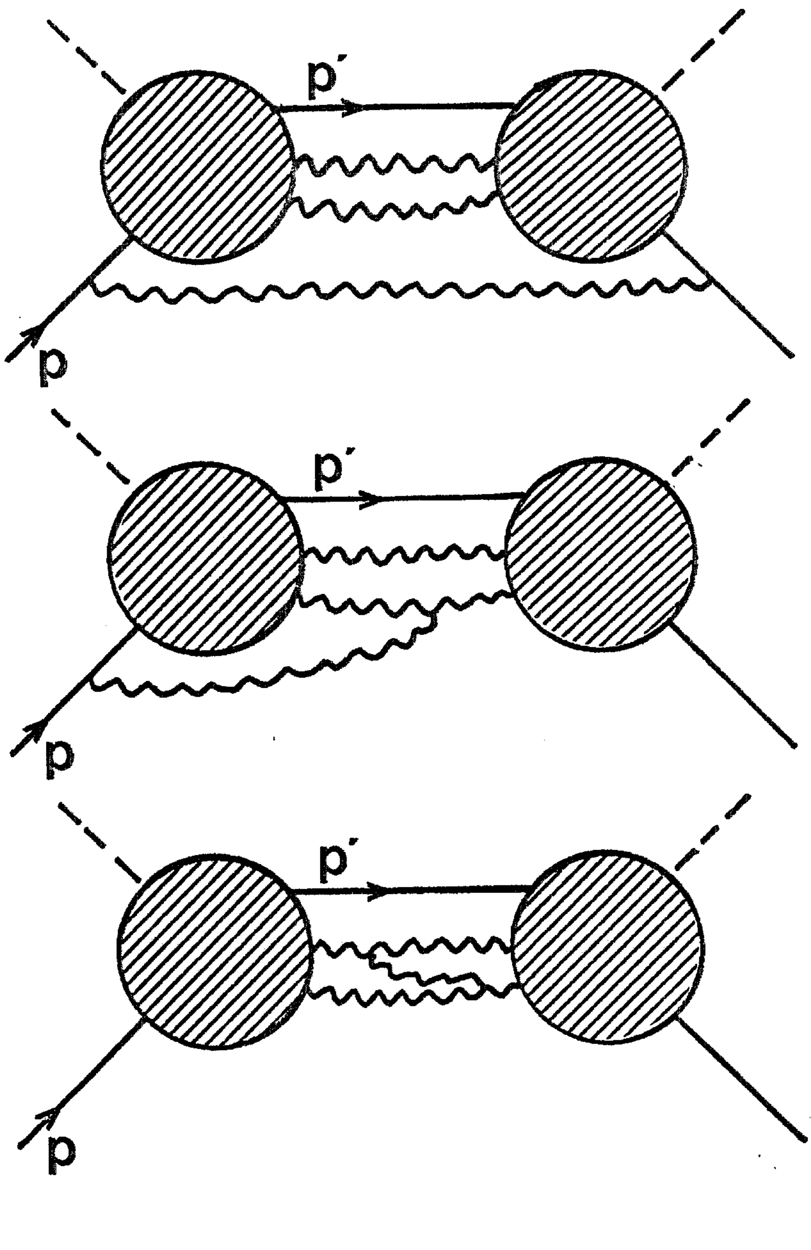


Fig.8

simplicity sake) and the incoming quark line. It shows the two possible unitarity cuts associated with such an insertion; one cut (Fig.(9-a)) giving an infrared divergent virtual correction to one of the M_n, while the other (see Fig.(9-b)) gives a contribution to the partial cross-section with $n+1$ gluons in the final state. However, the contribution of this inserted real gluon in the final state may be simply integrated to give an infrared singular contribution to the partial cross-section containing n gluons in the final state. This is because the inserted gluon is strictly external by construction, which means that the cross-hatched circles in Fig. (9) are not dependent on the momentum k of the inserted gluon. Specifying the colour group structure of the inelastic amplitude by the colour indices of the initial and final state particles, that is, representing the amplitude by $M_{n\,i}^{\,j\,a_1 \cdot\cdot a_n}$, the effect of the insertion shown in Fig. (9-a) on the partial cross-section is given by

$$\left(M_{n_i}^{\;j\;a_1 \cdot\cdot\, a_n}\right)^* M_{n_{i'}}^{\;j\;a_1 \cdots a_{\ell'} \cdot\cdot a_n} f^{a_{\ell'} b a_\ell} (T^b)_i^{\;i'} \; I(p,q_\ell) \tag{V-2}$$

and the effect of the insertion shown in Fig. (9-b) on the partial cross-section is given by

$$\left(M_{n_i}^{\;j\;a_1 \cdot\cdot a_{\ell'} \cdot\cdot a_n}\right)^* M_{n_{i'}}^{\;j\;a_1 \cdots a_n} (T^b)_i^{\;i'} \; f^{a_\ell b a_{\ell'}} \; \tilde{I}(p,q_\ell) \tag{V-3}$$

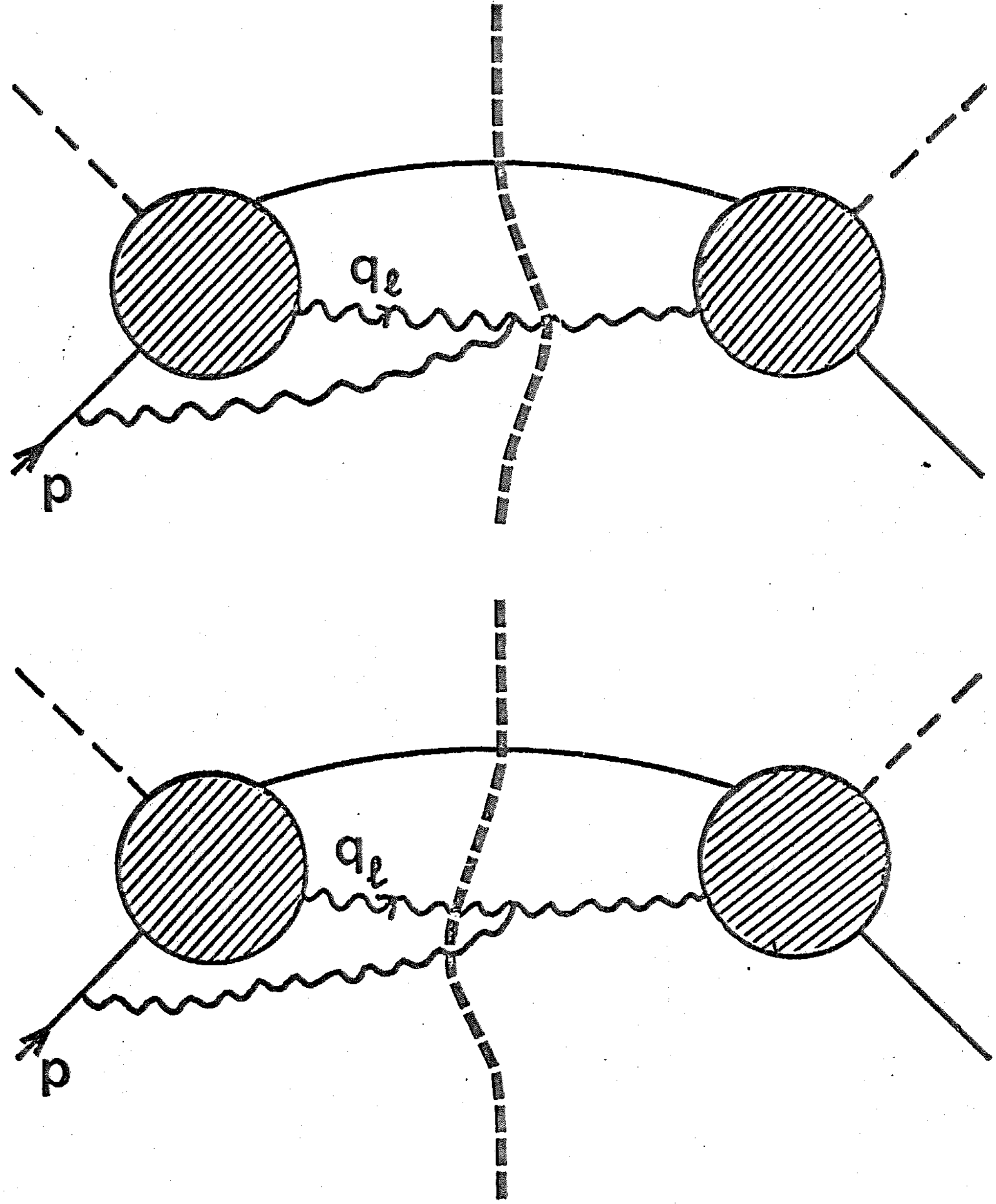


Fig.9

The integrals $I(p,q_\ell)$ and $\tilde{I}(p,q_\ell)$ have the form

$$I(p,q_\ell) = \int \frac{d^d k}{(2\pi)^d} \frac{(2p-k)^\mu}{k^2-2p.k} D^{\rho\rho'}(q_\ell-k)\, V^{\mu\sigma_\ell\rho}(q_\ell,k) \frac{g^2(k^2)}{k^2}$$

$$\tilde{I}(p,q_\ell) = \int \frac{d^d k}{(2\pi)^d} \delta(k^2) \frac{(2p-k)^\mu}{2p.k} D^{\rho\rho'}(q_\ell-k) V^{\mu\sigma_\ell\rho}(q_\ell,k) \frac{g^2(k^2)}{k^2}$$

(V-4)

where $V^{\mu\sigma_\ell\rho}(q_\ell,k)$ is the Lorentz tensor part of the three gluon vertex. Similarly, for the third type of insertion, that is, insertion of the gluon between two particles in the intermediate state, there are four possible unitarity cuts that may be made. Figure (10) shows the possible cuts associated with the insertion between the i^{th} and j^{th} gluons of M_n (the other gluons are not shown). The contribution from Fig.(10-a) in terms of its group structure is

$$(M_{n_i}^{j\; a_1..a_{i'}..a_{j'}..a_n} f^{a_{i'} a_i b} f^{a_j a_{j'} b})^* M_{n_i}^{j\; a_1..a_n} I(q_i,q_j)$$

The contribution from Fig. (10-b) is

$$(M_{n_i}^{j\; a_1..a_n})^* M_{n_i}^{j a_1..a_{i'}..a_{j'}..a_n} f^{a_i a_{i'} b} f^{a_{j'} a_j b} I(q_i,q_j)$$

The contribution from Fig.(10-c) is

$$(M_{n_i}^{j a_1..a_{i'}..a_n} f^{a_{i'} a_i b})^* M_{n_i}^{j a_1..a_{j'}..a_n} f^{a_{j'} a_j b} \tilde{I}(q_i,q_j)$$

The contribution from Fig.(10-d) is

$$(M_{n_i}^{j a_1..a_{j'}..a_n} f^{a_j a_{j'} b})^* M_{n_i}^{j a_1..a_{i'}..a_n} f^{a_i a_{i'} b} \tilde{I}(q_i,q_j)$$

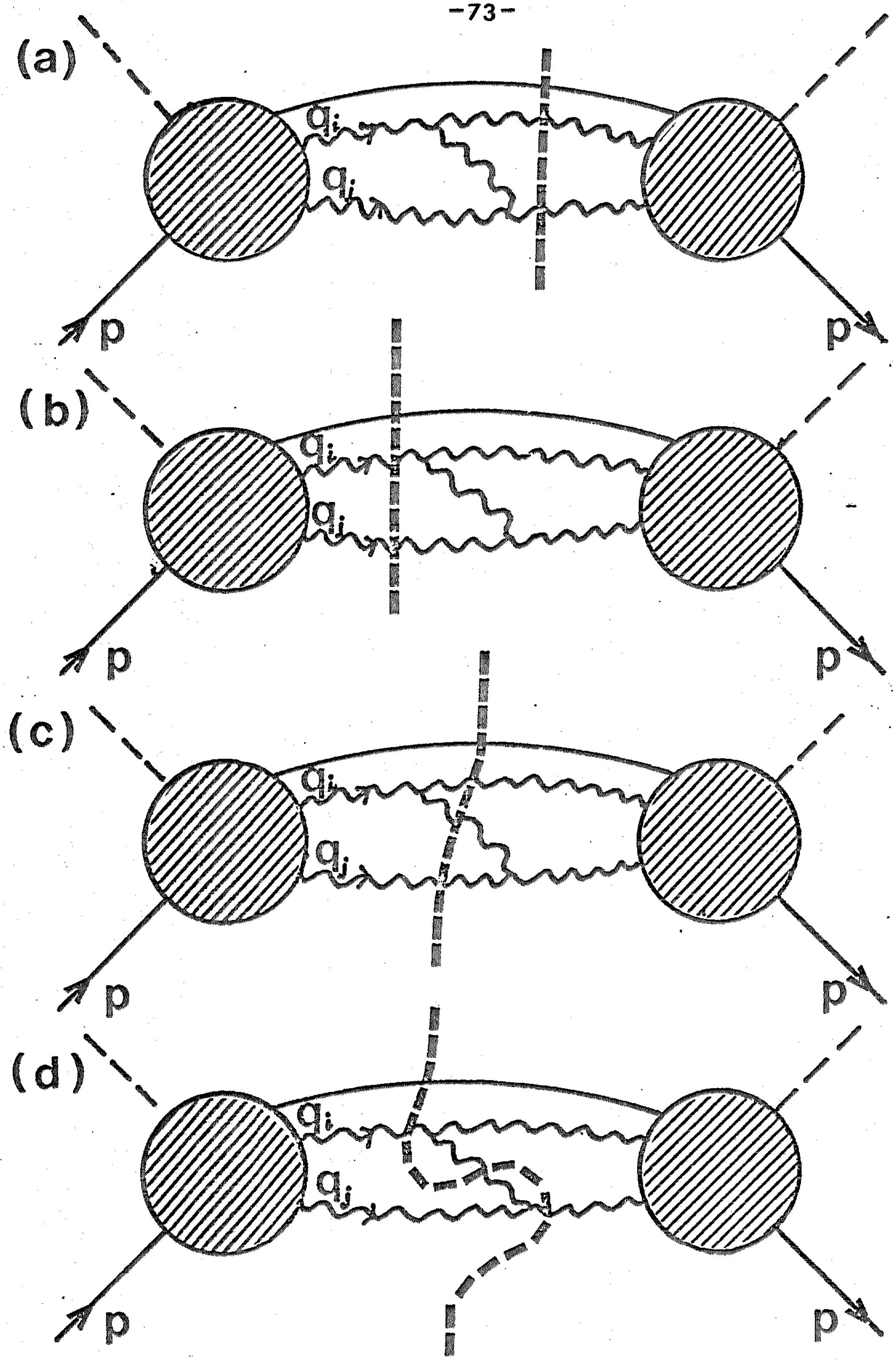


Fig.10

For the virtual insertions,

$$I(q_i,q_j) = \int \frac{d^dk}{(2\pi)^d} V^{\nu_i\mu_i\lambda}(q_i,k) D_{\mu_{i'}\nu_i}(q_i+k) \cdot \cdot V^{\nu_j\mu_j\lambda}(q_j,k)\, D_{\nu_j\mu_{j'}}(q_j-k)\, \frac{g^2(k^2)}{k^2}$$

and for the insertions yielding an extra gluon in the final state,

$$\tilde{I}(q_i,q_j) = \int \frac{d^dk}{(2\pi)^d} \delta(k^2) V^{\nu_i\mu_i\lambda}(q_i,k) D_{\mu_{i'}\nu_i}(q_i+k) \cdot \cdot V^{\nu_j\mu_j\lambda}(q_j,k)\, D_{\nu_j\mu_{j'}}(q_j+k)\, \frac{g^2(k^2)}{k^2} \tag{V-5}$$

The generalization of the SU(2) relation

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

to SU(3) is not so simple, due to the existence of the symmetric coefficients in SU(N) for N > 2:

$$f_{abe}f_{cde} = \frac{2}{3}\{\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\} + \{d_{ace}d_{bde} - d_{bce}\, d_{ade}\} \tag{V-6}$$

and there is no benefit to be derived from such a substitution. Thus the differential equation for the cross-section $\frac{d\sigma}{d\varepsilon}$ may now be written as

$$\alpha\frac{\partial}{\partial\alpha}\left(\frac{d\sigma}{d\varepsilon}\right) = \alpha\frac{\partial}{\partial\alpha} \sum \left| M_{n_i}^{ja_1\cdot\cdot a_n} \right|^2$$

$$= 2\,\{I(p,p') + \tilde{I}(p,p')\}\,\frac{1}{2}\,\left(M_{n_i}^{ja_1\cdot\cdot a_n}\right)^* M_{n_i}^{ja_1\cdot\cdot a_n} - \frac{1}{3}\left|M_{n_i}^{ia_1\cdot\cdot a_n}\right|^2$$

$$+ 2\sum_j \{I(p,q_j)+\tilde{I}(p,q_j)\}\,[T^{a_{j'}},T^{a_j}]\,\left(M_{n_i}^{ja_1\cdot\cdot a_j\cdot a_n\,*} M_{n_i}^{j\cdot\cdot a_{j'}\cdot a_n}\right)$$

$$+ 2\sum_{i<j}\{\, I(p,p') + \tilde{I}(p,p')\}\, M_{n_i}^{ja_1\cdot\cdot a_{i'}\cdot a_{j'}\cdot a_n\,*}\, M_{n_i}^{ja_1\cdot a_i\cdot a_j\cdot a_n}.$$

$$\cdot\, f^{\hat{a}_{i'} a_i b}\, f^{a_j a_{j'} b}$$

$$+ \text{ finite insertions.} \qquad \text{(V-7)}$$

It is quite evident that the sum of the virtual and Bremsstrahlung contributions shown above in Eq.(V-7) does not possess an infrared singular part in the limit as $d \to 4$, that is,

$$I(q_i,q_j) + \tilde{I}(q_i,q_j) = O[\,(\frac{1}{d-4})^{o}\,] \text{ in the limit } d \to 4.$$

Similar statements hold for the sum of $I(p,q_j)$ and $\tilde{I}(p,q_j)$ and for the sum of $I(p,p')$ and $\tilde{I}(p,p')$.

Since the exchange of a gluon causes the various colour channels to become mixed, it is necessary to write Eq.(V-7) in terms of the colour triplet channel for the calculation of the differential cross-section if the quark colour is the experimental trigger, or in terms of the colour singlet $tr|M^2|$ for the case of a colour-blind experiment. The treatment as presented above implicitly assumes that the colours of the final state particles are averaged over since the insertions are made into the intermediate states of the elastic quark scattering amplitude. However, the modification required to study the semi-inclusive cross-section for the detection of a quark with energy resolution ΔE when the experimental trigger is the quark colour, that is, when it is only the colours of the quark and the gluons in the resticted phase space specified by the detector which are fixed and all colours of the other particles in the final state are averaged over, is minor. Consider only those gluons contributing to the coloured state. As the group structure changes caused by the virtual insertion of an extra gluon are precisely the same as those brought about by the emission of an extra gluon into the final state, provided the colour index of the emitted gluon is summed over, it is possible to show that the partial cross-section involving a quark with colour index j plus n gluons with indices $a_1, \ldots\ldots, a_n$ plus any number of soft gluons whose colour indices are averaged over is also finite. This follows from an equation almost identical to Eq.(V-7) but for one particular M_n, rather than the sum. But the desired cross-section for colour detection is just the sum of the integral (phase space) over those particular $|M_n|^2$ which have the required colour.

Thus, as a direct result of the analysis of Chapter IV it has been shown by a simple and fairly elegant argument that:

(i) The totally unrestricted rate for the production of quarks and gluons from the scattering of an incoming massive quark on a photon is infrared finite. This result was already known for a related process[(21),(22)].

(ii) The restricted rate for the scattering of a quark and a photon giving a quark and many soft gluons in the final state up to a total energy ΔE where the trigger for the quark detector does not depend on the colour of the quark, is infrared finite.

(iii) The semi-inclusive rate for quark plus photon gives quark plus anything where the quark detector has energy resolution

ΔE and triggers on a particular colour of quark (massive) also appears to be free from infrared singularities.

There has been a lengthy discussion in the literature about whether the cross-sections mentioned above are, in fact, infrared finite[(20),(21),(22)]. The consensus of opinion was that the double series consisting of the summation over the virtual corrections and the summation over the Bremsstrahlung contributions does not possess uniform convergence. Thus, Cornwall and Tiktopoulos[(20)] showed that if all the virtual (leading) infrared singular corrections to a scattering process were summed first, then the exponential damping factor associated with the emission of a real gluon prevented the emergence of an infrared singularity in the phase space integral. This is contrasted with the expectations of the Kinoshita theorem and the results of low order perturbation theory calculations which show that order-by-order, the cross-sections are free of singularities.

It is interesting that the order of summation of infrared singularities used in this chapter is again different. The amplitudes M_n contain all orders of perturbation theory, and yet the effect of the insertion of a gluon is to donate singularities to the virtual sum and to the Bremsstrahlung sum at equal rates. Thus the differential equation aproach when applied directly to the cross-section gives a hybrid order of the double summation, although more closely related to the order-by-order method.

CONCLUSIONS

This thesis asserts that all the infrared singularities arising in on mass-shell scattering amplitudes in Quantum Chromodynamics can be collected by a reorganization of the perturbation theory into the iteration of the sum of all insertions of a single gluon between the external (asymptotic) states with the effective coupling $g(k^2)$, where k is the momentum of insertion, replacing the perturbation expansion coupling g (renormalized at some off mass-shell point). This statement justifies previous statements made in the context of the leading singularities at each order of perturbation theory; an important step, since leading singularity results often do not reflect the true nature of the theory in a strong coupling region such as the infrared region. It should be noted that the ghost-free gauges used here are essential to the simplicity of the derivation of the result, and it is possible that the simple form of the solution in these gauges corresponds to the rather more complicated results of Frenkel and Taylor in a covariant gauge. The resolution of this situation requires the perturbation theory calculation (at the two loop level) of the quark colour-singlet form factor in the axial gauge.

The work presented here has not added directly to the question of whether the quarks and gluons of Quantum Chromodynamics are permanently bound within hadrons; however it does indicate the single most important question that must be answered before the problem of confinement can be settled. This is the behaviour of the effective coupling constant $g^2(k^2)$ in the limit as $k^2 \to 0$.

The differential equation studied in Chapter II may be of some use in determining this quantity. For example, this differential equation may be written for the inverse gluon propagator in the axial gauge and a separation at least of the soft infrared singularities can be made. The resulting form is much simpler than the Dyson equation approach(51) which requires several strong assumptions before the problem becomes tractable. However, since the gluon is off mass-shell, it is not obvious that the hard divergences which occur for massless particles with parallel three momenta are actually controlled by the separation of divergences.

The problem of hard divergences associated with a totally massless theory also occurs with the extension of the material presented in Chapter V to the consideration of massless quarks. Such an extension would allow an explicit (and simple) verification of the Kinoshita-Lee-Nauenberg theorem(23),(24) and also a probable solution to the question of factorization of the dependence on small momenta squared in semi-inclusive quark scattering processes. Mueller(52) has shown that the required factorization does take place to all orders in perturbation theory but only in the physically less interesting theory of ϕ^4 .

APPENDIX A

The derivation of the differential equation described in Chapter II, although straightforward, is somewhat clumsy. Since the result is independent of perturbation theory, one might expect that functional techniques may be applied to yield the same result in a more elegant fashion. In QED this is certainly true[¶].

The Lagrangian in QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$$

may be rewritten as

$$\mathcal{L} = A^{\mu}\Delta^{-1}_{\mu\nu} A^{\nu} + e\, J^{\mu}A_{\mu} + j_{\psi}\psi + j_A{}^{\mu} A_{\mu}$$

where $\Delta_{\mu\nu}$ is the photon propagator (bare) and J^{μ} is the source for the interaction Lagrangian (i.e., $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$). The sources j_{ψ} and j_A enable the formation of the external fields by functional differation of the generating functional for the Green's functions. Consider, then, this generating functional $Z[J,j_{\psi},j_A]$:

$$Z[J,j_{\psi},j_A] = \int [\mathcal{D} A][\mathcal{D} \psi]\; e^{i \int d^4x(A\Delta^{-1}A + eJA + j_{\psi}\psi + j_A A)} \tag{A-1}$$

¶ This appendix arose from conversations with J. Schonfeld whose comments the author wishes to acknowledge.

Since J does not depend explicitly on the photon field A^{μ} and and since the form of the exponent is quadratic, the Gaussian integral over A may be carried through exactly. After completing the square (and setting j_A to zero since our interest is only in Green's functions with external electrons, at present) the result is:

$$Z[J,j_{\psi}] = \int [\mathcal{D}\psi] \exp\left(i\int d^4x \, (-\alpha J^{\mu}\Delta_{\mu\nu}J^{\nu} + j_{\psi}\psi \right) \tag{A-2}$$

Green's functions may be formed by functional differentiation with respect to j_{ψ}, and J^{μ}. Thus, for example, the electron-electron scattering Green's function would be formed by evaluating

$$G^{e^-e^-}(x_1,x_2,x_3,x_4) = \left(-i\frac{\partial}{\partial j_{\psi}}\right)\left(-i\frac{\partial}{\partial j_{\psi}}\right)\left(-i\frac{\partial}{\partial j_{\psi}}\right)\left(-i\frac{\partial}{\partial j_{\psi}}\right) Z[J,j_{\psi}]\Big|_{J=j_{\psi}=0}$$

Now the operation $\alpha\frac{\partial}{\partial\alpha}$ on such a Green's function brings down one factor of $\alpha\int J\Delta J \, d^4x$. This is precisely the required result: that is, differentiation of a Green's function with respect to coupling constant yields an integral of the Green's function with two extra $\bar{\psi}\gamma^{\mu}\psi$ sources integrated over all points of insertion.

$$\alpha\frac{\partial}{\partial\alpha} G^{e^-e^-}(x_1,x_2,x_3,x_4) = \int d^4z_1 d^4z_2 \, \Delta^{\mu\nu}(z_1-z_2)G^{e^-e^-}_{\mu\nu}(x_1..x_4;z_1,z_2) \tag{A-3}$$

This formulation cannot, however, be carried over trivially to derive the differential equation for QCD.

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