THE THERMAL RAYLEIGH PROBLEM IN PERFECT AND IONIZED GASES

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ABSTRACT

The problem of a heated gas at rest in contact with a cold wall has applications in shock tubes when a shock wave reflects at the end wall. A boundary layer arises that can be solved for constant conditions outside the boundary layer. In this paper, this condition is relaxed and the gas is permitted to undergo variations of pressure.

The equation has been derived for the boundary layer under those conditions, and a similarity solution was found for isentropic variations of a perfect gas outside the boundary layer. This solution is only valid for a temperature of the gas much greater than the temperature of the wall, unless the latter would follow a determined variation with time. Two cases were studied; one with thermal conductivity of the gas proportional to a power of the temperature, and the other with conductivity proportional to temperature. In the first case, an integral method has to be used, which gives an excellent approximation. For the second case, the solution is worked out either by similarity procedures or by series expansions.

Heat transfer and temperature in the boundary layer are seen to depend on the pressure, which acts like a weighting factor on time.

For ionized gases, two cases were considered; frozen and equilibrium flow. Similarity solutions do not exist for variable pressure outside the boundary layer unless the variations of pressure with time are small enough. In this case, pressure acts again like a weighting factor on time, although the variations outside the boundary layer cannot be assumed isentropic anymore. At the same time, new equations for the equilibrium boundary layer were derived, in which the excited state of ions and atoms is taken into account.

The effect of the weighting factor of the pressure is applied to explain the experimental results obtained at the end wall of a shock tube, giving explanation to certain differences between experiment and theory.

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PART I.

THE THERMAL RAYLEIGH PROBLEM IN PERFECT GASES

1. Introduction

If we have a gas initially at rest and in contact with a wall, both at the same temperature, a problem of a Rayleigh type arises when the gas is suddently heated. For instance, this happens at the end wall of a shock tube when the shock wave traveling along it arrives at the end wall and reflects. In this case, the gas is heated while the wall keeps the ambient temperature, and a boundary layer arises in which the thermal conductivity plays an important role. It is in that boundary layer that the transport of heat takes place, causing the cooling of the gas. Solutions for this problem have been worked out [1, 2] for a perfect gas with different relations between the thermal conductivity and the temperature, but maintaining the pressure in the gas constant. In the first part of this paper then, solutions for this case will be obtained for two different relations between thermal conductivity and temperature; namely, thermal conductivity proportional to the temperature,and thermal conductivity proportional to a power of the temperature. Finally, a solution in series similar to the one used by Blasius and Howarth [3] will be developed.

2. Derivation of the Equations

We follow essentially the work of Goldsworthy [1], but keeping the pressure outside the boundary layer a function of time. We assume then that the gas, suddenly heated to a uniform temperature at $t = 0$, and the wall represent a one-dimensional problem in which changes occur only in the y-direction normal to the wall, represented by $y = 0$, and where the gas is indicated by $y > 0$. In that case, the equation for the boundary layer reduces to the energy equation with a von Mises type of transformation

$$\psi = \int_0^y \rho \, dy \,, \tag{2.1}$$

which satisfies automatically the equation of conservation of mass. Then the boundary layer energy equation is, for a perfect gas with constant specific heats,

$$\frac{\partial T}{\partial t} - \frac{\gamma-1}{\gamma P} \frac{dP}{dt} T = \frac{P}{c_p R} \frac{\partial}{\partial \psi} \left(\frac{k}{T} \frac{\partial T}{\partial \psi} \right), \tag{2.2}$$

where T is the temperature and P is the pressure at the edge of the boundary layer and a function of time. The boundary conditions for this equation are

$$\psi = 0 : \qquad T = T_w \,, \tag{2.3a}$$

$$\psi \rightarrow \infty : \qquad \frac{T}{P^{\frac{\gamma-1}{\gamma}}} \rightarrow \left(\frac{T_e}{P^{\frac{\gamma-1}{\gamma}}} \right)_{t=0} \quad \text{(constant)} \,. \tag{2.3b}$$

3. Solution for a Perfect Gas with Thermal Conductivity Proportional to Temperature

If we assume a linear dependence between the thermal conductivity of the gas and temperature

$$k = k_1 T \,, \tag{3.1}$$

equation (2.2) follows the simplest form, that is

$$\frac{\partial T}{\partial t} - \frac{\gamma-1}{\gamma P}\frac{dP}{dt}\, T = \frac{Pk_1}{c_p R}\frac{\partial^2 T}{\partial \psi^2} \,. \tag{3.2}$$

Solutions of the type

$$T(\psi, t) = \sigma(t)\tau[\psi\xi(t)] = \sigma(t)\tau(\eta) \tag{3.3}$$

can be found by substituting into the equation (3.2) the expression (3.3),

$$\frac{\dot{\sigma}}{\sigma P\xi^2}\,\tau + \frac{\dot{\xi}}{\xi P\xi^2}\,\eta\frac{d\tau}{d\eta} - \frac{\dot{P}}{PP\xi^2}\,\frac{\gamma-1}{\gamma}\,\tau = \frac{k_1}{c_p R}\frac{d^2\tau}{d\eta^2} \,. \tag{3.4}$$

A solution of the form (3.3) will exist if we make

$$\frac{\dot{\sigma}}{\sigma P\xi^2}\,\tau - \frac{\dot{P}}{PP\xi^2}\,\frac{\gamma-1}{\gamma}\,\tau = 0 \,, \tag{3.5}$$

$$\dot{\xi}/(\xi P\xi^2) = -1 \,, \tag{3.6}$$

which gives us finally

$$T(\psi, t) = P^{\frac{\gamma-1}{\gamma}}\,\tau\left[\psi\Big/\Big(2\int_0^t P\,dt\Big)^{\frac{1}{2}}\right] \tag{3.7}$$

and reduces equation (3.4) to

$$\eta\frac{d\tau}{d\eta} + \frac{k_1}{c_p R}\frac{d^2\tau}{d\eta^2} = 0 \,, \tag{3.8}$$

which has the solution

$$\tau = A\int_0^\eta \exp\Big(-\frac{c_p R}{2k_1}\,\eta^2\Big)d\eta + B \,, \tag{3.9}$$

where A and B are constants to be determined by the boundary conditions (2. 3), which in this particular case take the form

$$\eta = 0: \qquad \tau_w = \frac{T_w}{P^{\frac{\gamma-1}{\gamma}}}, \tag{3. 10a}$$

$$\eta \to \infty: \qquad \tau \to \left(\frac{T_e}{P^{\frac{\gamma-1}{\gamma}}} \right)_{t=0} \quad \text{(constant)} . \tag{3. 10b}$$

The values of τ at $\eta = 0$ and $\eta \to \infty$ must be constant. This is fulfilled in the boundary condition (3. 10b) under any circumstance and in (3. 10a) only for the case

$$\frac{T_w}{P^{\frac{\gamma-1}{\gamma}}} = \text{constant}. \tag{3. 11}$$

For this special relationship with time of the temperature at the wall, the solution for the boundary layer from equations (3. 7) and (3. 9), applying boundary conditions (3. 10), is

$$\frac{T-T_w}{T_e-T_w} = \operatorname{erf} \left\{ \left(\frac{c_p R}{2k_1} \right)^{\frac{1}{2}} \frac{\psi}{\left(2 \int_0^t P dt \right)^{\frac{1}{2}}} \right\} . \tag{3. 12}$$

From this solution we can obtain, for the case

$$T_e >> T_w , \tag{3. 13}$$

an approximate expression for the temperature

$$T = T_e \operatorname{erf} \left\{ \left(\frac{c_p R}{2k_1} \right)^{\frac{1}{2}} \frac{\psi}{\left(2 \int_0^t P dt \right)^{\frac{1}{2}}} \right\} , \tag{3. 14}$$

which is not valid near the wall. So this solution can be used when T_w does not satisfy the relationship (3. 11) only if we keep away from the wall.

The heat transfer rate to the wall

$$q_w = (-q)_{\eta=0} = \left(k \frac{\partial T}{\partial y}\right)_{\eta=0} , \qquad (3.15)$$

when calculated from (3.12) is

$$q_w \frac{\left(\int_0^t P dt\right)^{\frac{1}{2}}}{P(T_e - T_w)} = \left(\frac{\gamma}{\gamma-1} \frac{k_1}{\pi}\right)^{\frac{1}{2}} . \qquad (3.16)$$

When the inequality (3.13) is valid, (3.16) reduces to

$$q_w \frac{\left(\int_0^t P dt\right)^{\frac{1}{2}}}{P T_e} = \left(\frac{\gamma}{\gamma-1} \frac{k_1}{\pi}\right)^{\frac{1}{2}} , \qquad (3.17)$$

which is independent of the boundary condition (3.10a).

4. Solution for a Perfect Gas with Thermal Conductivity Proportional to a Power of the Temperature

For a linear relationship between the thermal conductivity of the gas and a power of the temperature

$$k = k_1 T^{\nu} , \tag{4. 1}$$

equation (2. 2) becomes

$$\frac{\partial T}{\partial t} - \frac{\gamma-1}{\gamma P}\frac{dP}{dt} T = \frac{P k_1}{c_p R}\frac{\partial}{\partial \psi}\left(T^{\nu-1}\frac{\partial T}{\partial \psi}\right) \tag{4. 2}$$

and when expression (3. 3) is substituted in (4. 2)

$$\frac{\dot{\sigma}}{\sigma P \xi^2 \sigma^{\nu-1}}\tau + \frac{\dot{\xi}}{\xi P \xi^2 \sigma^{\nu-1}}\eta\frac{d\tau}{d\eta} - \frac{P}{P P \xi^2 \sigma^{\nu-1}}\frac{\gamma-1}{\gamma}\tau = \frac{k_1}{c_p R}\frac{d}{d\eta}\left(\tau^{\nu-1}\frac{d\tau}{d\eta}\right) . \tag{4. 3}$$

A solution is obtained when we make

$$\frac{\dot{\sigma}}{\sigma P \xi^2 \sigma^{\nu-1}} - \frac{\dot{P}}{P P \xi^2 \sigma^{\nu-1}}\frac{\gamma-1}{\gamma} = 0 , \tag{4. 4}$$

$$\frac{\dot{\xi}}{\xi P \xi^2 \sigma^{\nu-1}} = -1 , \tag{4. 5}$$

which gives the temperature the expression

$$T(\psi, t) = P^{\frac{\gamma-1}{\gamma}}\tau\left\{\frac{\psi}{\left[2\int_0^t P^{\left(\frac{1-\nu}{\gamma}+1\right)} dt\right]^{\frac{1}{2}}}\right\} \tag{4. 6}$$

and reduces equation (4. 3) to

$$\eta\frac{d\tau}{d\eta} + \frac{k_1}{c_p R}\frac{d}{d\eta}\left(\tau^{\nu-1}\frac{d\tau}{d\eta}\right) = 0 . \tag{4. 7}$$

An integral method introduced by Jepson [2] can be used to solve (4.7) that gives the heat transfer rate to the wall. Using formulae (3. 15) and (4. 6), this takes the form

$$q_w = \frac{k_1 T_w^{\nu-1} P^{\frac{2\gamma-1}{\gamma}}}{R\left[2\int_0^t P^{\left(\frac{1-\nu}{\gamma}+1\right)} dt\right]^{\frac{1}{2}}} \left(\frac{d\tau}{d\eta}\right)_w \quad . \tag{4.8}$$

If we define a heat flux potential ϕ such that

$$\frac{d\phi}{d\eta} = \tau^{\nu-1} \frac{d\tau}{d\eta} , \tag{4.9}$$

that is,

$$\phi - \phi_w = \frac{\tau^{\nu} - \tau_w^{\nu}}{\nu} , \tag{4.10}$$

equation (4.8) takes the form, when we use (4.6),

$$q_w = \frac{k_1 P^{\left(\frac{\gamma-1}{\gamma}\nu+1\right)}}{R\left[2\int_0^t P^{\left(\frac{1-\nu}{\gamma}+1\right)} dt\right]^{\frac{1}{2}}} \left(\frac{d\phi}{d\eta}\right)_w \quad . \tag{4.11}$$

Furthermore, the integration of equation (4.7) with respect to η from $\eta = 0$ to $\eta \to \infty$ yields, if we impose the condition

$$\eta \to \infty \;:\; \frac{d\tau}{d\eta} \to 0 , \tag{4.12}$$

the following results:

$$\left(\frac{d\phi}{d\eta}\right)_w = \frac{c_p R}{k_1} \int_{\tau_w}^{\tau_\infty} \eta \, d\tau \quad . \tag{4.13}$$

Equation (4.7) also establishes that near the wall, where $\eta \simeq 0$,

$$\frac{d}{dy}\left(\frac{d\phi}{d\eta}\right) = 0 \quad , \quad \frac{d\phi}{d\eta} = \left(\frac{d\phi}{d\eta}\right)_w \quad . \tag{4.14}$$

We then assume that between τ_w and τ_∞ (4.14) will give a good approximation for the relationship between η and τ ,

$$\phi - \phi_w = \left(\frac{d\phi}{d\eta}\right)_w \eta \,, \tag{4.15a}$$

$$\eta = \frac{\tau^{\nu} - \tau_w^{\nu}}{\nu \left(\frac{d\phi}{d\eta}\right)_w} \,,$$

that substituted into expression (4. 13) yields

$$\left(\frac{d\phi}{d\eta}\right)_w = \left\{ \frac{c_p R}{k_1} \left(\frac{\tau_\infty^{\nu+1} - \tau_w^{\nu+1}}{\nu(\nu+1)} - \tau_w^{\nu} \frac{\tau_\infty - \tau_w}{\nu} \right) \right\}^{\frac{1}{2}} \,, \tag{4.16}$$

where the values of τ at $\eta = 0$ and $\eta \to \infty$ are given by equations (3.10) and (3. 11). We finally obtain for the heat transfer rate at the wall from (4. 10) by substitution of (4. 16)

$$q_w \frac{\left[\int_0^t P^{\left(\frac{1-\nu}{\gamma} + 1\right)} dt \right]^{\frac{1}{2}}}{P^{\left[\frac{\gamma-1}{2\gamma}(\nu-1)+1\right]}} = \left\{ \frac{c_p k_1}{2R} \left(\frac{T_e^{\nu+1} - T_w^{\nu+1}}{\nu(\nu+1)} - T_w^{\nu} \frac{T_e - T_w}{\nu} \right) \right\}^{\frac{1}{2}} . \tag{4.17}$$

For $\nu = 1$, (4. 17) yields

$$q_w \frac{\left(\int_0^t P dt \right)^{\frac{1}{2}}}{P(T_e - T_w)} = \left(\frac{\gamma}{\gamma - 1} \frac{k_1}{4} \right)^{\frac{1}{2}} \,, \tag{4.18}$$

which, compared with the exact solution (3. 16), gives an error of 13 per cent. When the inequality (3. 13) is valid, (4. 17) reduces to

$$q_w \frac{\left[\int_0^t P^{\left(\frac{1-\nu}{\gamma} + 1\right)} dt \right]^{\frac{1}{2}}}{P^{\left[\frac{\gamma-1}{2\gamma}(\nu-1)+1\right]}} = \left(\frac{\gamma}{\gamma-1} \frac{k_1}{2} \frac{T_e^{\nu+1}}{\nu(\nu+1)} \right)^{\frac{1}{2}} \,, \tag{4.19}$$

which is independent of the boundary condition (3. 10a).

5. Solution in Series for a Perfect Gas with Thermal Conductivity Proportional to Temperature

When a linear dependence between the thermal conductivity of the gas and temperature was assumed, and when the temperature at the wall satisfied equation (3. 11), we obtained a solution (3. 7) for the boundary layer in which temperature was not a function of time but of

$$t' = \int_0^t p \, dt \, , \tag{5. 1}$$

where

$$p = \frac{P(t)}{P(0)} = \frac{P(t)}{P_o} \, . \tag{5. 2}$$

If the temperature at the wall does not satisfy (3. 11), a solution in series similar to the one used by Blasius and Howarth [3] could be worked out, but using, instead of the variable time, the new variable t' , which will give a faster convergence and more accurate information.

A change of variables is then done according to the transformation (5. 1) which yields

$$\left(\frac{\partial}{\partial t}\right)_\psi = p\left(\frac{\partial}{\partial t'}\right)_\psi \, , \tag{5. 3a}$$

$$\left(\frac{\partial}{\partial \psi}\right)_t = \left(\frac{\partial}{\partial \psi}\right)_{t'} \, . \tag{5. 3b}$$

Equation (3. 2) will be transformed by (5. 3) to a new expression

$$p \frac{\partial \theta}{\partial t'} - \frac{k_1 P_o}{c_p R} p \frac{\partial^2 \theta}{\partial \psi^2} - \frac{\gamma - 1}{\gamma} \frac{dP}{dt'} \theta = 0 \, , \tag{5. 4}$$

where θ is a non-dimensional temperature. That is, if T_o is a reference temperature,

$$\theta = T/T_o \, . \tag{5. 5}$$

The pressure is represented then by a power series

$$p = 1 + a_1 t' + a_2 t'^2 + \dots \tag{5.6}$$

and it is assumed that the temperature is given by the following expression

$$\theta = f(\eta, t') = f_o(\eta) + a_1 f_1(\eta) t' + \{a_2 f_2(\eta) + a_1^2 f_{11}(\eta)\} t'^2 +$$

$$+ \{a_3 f_3(\eta) + a_2 a_1 f_{21}(\eta) + a_1^3 f_{111}(\eta)\} t'^3 + \dots \quad , \tag{5.7}$$

where

$$\eta = \frac{\psi}{(2t')^{\frac{1}{2}}} = \frac{\psi}{\left(2\int_0^t p\,dt\right)^{\frac{1}{2}}} \; . \tag{5.8}$$

With (5.8), the equation (5.4) for the temperature becomes

$$-p\frac{\partial f}{\partial \eta}\frac{\eta}{2t'} + p\frac{\partial f}{\partial t'} - \frac{k_1 P_o}{c_p R}\, p\frac{\partial^2 f}{\partial \eta^2}\frac{1}{2t'} - \frac{\gamma-1}{\gamma}\frac{dp}{dt'} f = 0 \tag{5.9}$$

or using (5.7),

$$\left\{-f_o'\frac{\eta}{2} - \frac{k_1 P_o}{2c_p R} f_o''\right\}\frac{1}{t'} + \left\{-(f_o'+f_1')\frac{\eta}{2} + f_1 - \frac{k_1 P_o}{2c_p R}(f_1''+f_o'') - \frac{\gamma-1}{\gamma} f_o\right\} a_1 +$$

$$+\left\{-(f_o'+f_2')\frac{\eta}{2} + 2f_2 - \frac{k_1 P_o}{2c_p R}(f_2''+f_o'') - \frac{\gamma-1}{\gamma}\, 2f_o\right\} a_2 t' +$$

$$+\left\{-(f_1'+f_{11}')\frac{\eta}{2} + f_1 + 2f_{11} - \frac{k_1 P_o}{2c_p R}(f_1''+f_{11}'') - \frac{\gamma-1}{\gamma} f_1\right\} a_1^2 t' + \dots = 0 . \tag{5.10}$$

If we make every coefficient in (5.10) equal to zero, we arrive at

$$\eta f_o' + \frac{k_1 P_o}{c_p R} f_o'' = 0 , \tag{5.11a}$$

$$\eta f_1' + \frac{k_1 P_o}{c_p R} f_1'' - 2f_1 = -2\frac{\gamma-1}{\gamma} f_o , \tag{5.11b}$$

$$\eta f_2' + \frac{k_1 P_o}{c_p R} f_2'' - 4f_2 = -4\frac{\gamma-1}{\gamma} f_o , \tag{5.11c}$$

$$\eta f_{11}' + \frac{k_1 P_o}{c_p R} f_{11}'' - 4f_{11} = -2\frac{\gamma-1}{\gamma} f_1 \ . \tag{5.11d}$$

As boundary conditions we have

$$\eta = 0 \ : \qquad T = T_w \ \text{(assumed now constant)} \tag{5.12a}$$

$$\eta \to \infty \ : \qquad \frac{T}{P^{\frac{\gamma-1}{\gamma}}} \to \left(\frac{T_e}{P^{\frac{\gamma-1}{\gamma}}}\right)_{t=0} \quad \text{(constant)} \ , \tag{5.12b}$$

that can be written alternatively

$$\eta = 0 \ : \qquad \theta = \theta_w \ \text{(constant)} , \tag{5.13a}$$

$$\eta \to \infty \ : \qquad \theta = p^{\frac{\gamma-1}{\gamma}} \ , \tag{5.13b}$$

if T_o is the temperature outside the boundary layer for $t = 0$, that is

$$T_o = (T_e)_{t=0} \ . \tag{5.14}$$

In series expansions, the boundary conditions are given from (5.7) as

$$\eta = 0 \ : \qquad (f)_{\eta=0} = \theta_w \ , \tag{5.15a}$$

$$\eta \to \infty \ : \quad (f)_{\eta\to\infty} = \left(1+a_1t'+a_2t'^2+\ldots\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{\gamma-1}{\gamma}a_1t' + \ldots \tag{5.15b}$$

The solution for the first approximation $f_o(\eta)$ is readily obtained from (5.11a) and (5.15), that is, from

$$\eta \frac{df_o}{d\eta} + \frac{k_1 P_o}{c_p R}\frac{d^2 f_o}{d\eta^2} = 0 \tag{5.16}$$

with boundary conditions

$$\eta = 0 \ : \qquad (f_o)_{\eta=0} = \theta_w \ , \tag{5.17a}$$

$$\eta \to \infty \ : \qquad (f_o)_{\eta\to\infty} = 1 \ . \tag{5.17b}$$

The value for f_o is

$$\frac{T_o f_o - T_w}{T_o - T_w} = \mathrm{erf}\left\{\left(\frac{c_p R}{2k_1}\right)^{\frac{1}{2}} \frac{\psi}{\left(2\int_0^t P dt\right)^{\frac{1}{2}}}\right\} , \qquad (5.18)$$

and taking $T_o f_o$ as a first approximation of T ,

$$\frac{T - T_w}{(T_e)_{t=0} - T_w} = \mathrm{erf}\left\{\left(\frac{c_p R}{2k_1}\right)^{\frac{1}{2}} \frac{\psi}{\left(2\int_0^t P dt\right)^{\frac{1}{2}}}\right\} , \qquad (5.19)$$

which has the same form as the solution (3. 12) for the similarity solution.

The heat transfer rate to the wall,

$$q_w = (-q)_{\eta=0} = \left(k\frac{\partial T}{\partial y}\right)_{\eta=0} , \qquad (5.20)$$

when calculated from (5. 19) is

$$q_w = \frac{\left(\int_0^t P dt\right)^{\frac{1}{2}}}{P[(T_e)_{t=0} - T_w]} = \left(\frac{\gamma}{\gamma - 1}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} , \qquad (5.21)$$

that resembles again expression (3. 16) for the exact case.

6. Concluding Remarks

If we solve the thermal Rayleigh problem and we keep the pressure outside of the boundary layer constant, the solutions [4] will be a function of only one variable, namely:

$$\eta = \left(\frac{c_{p_o}}{2t\rho_o k_o}\right)^{\frac{1}{2}} \int_0^y \rho \, dy \; . \tag{6.1}$$

This indicates that the different dependent variables in the boundary layer reach the same values at different times for the same η. Thus we have a thickness of the boundary layer as

$$\delta \sim \left(\frac{k_o t}{\rho_o c_{p_o}}\right)^{\frac{1}{2}} . \tag{6.2}$$

If we compare with the classical Rayleigh problem, in which the density is constant, and where the similarity variable is

$$\eta = y/(\nu t)^{\frac{1}{2}} , \tag{6.3}$$

therefore, a thickness of the boundary layer

$$\delta \sim (\nu t)^{\frac{1}{2}} , \tag{6.4}$$

we see that there is only a change in the factor of time. This is because the first is a thermal boundary layer, so its thickness depends on the thermal conductivity, and the second is a velocity boundary layer, so its thickness depends on the viscosity. The relation between both lengths is the Prandtl number.

In the thermal Rayleigh problem, if we allow the pressure to be a function of time, the similarity variable is

$$\eta = \left(\frac{c_{p_o}}{2\rho_o k_o \int_0^t p \, dt}\right)^{\frac{1}{2}} \int_0^y \rho \, dy \; . \tag{6.5}$$

The width of the boundary layer is now

$$\delta \sim \left(\frac{k_o \int_0^t p\,dt}{\rho_o c_{p_o}} \right)^{\frac{1}{2}} . \qquad (6.6)$$

We then arrive at the conclusion that an increasing of the pressure with time will result in an increase of the thickness of the boundary layer, and vice versa. Consequently, this will result in less or more heat transfer to the wall.

PART II.

THE THERMAL RAYLEIGH PROBLEM IN IONIZED GASES

7. Introduction

In this section we study the same problem as in the first part, but now we consider shock waves strong enough to produce ionization of the gas,argon, which is in contact with the cold wall. For constant conditions outside the boundary layer, catalytic wall, equal temperature for all species, ambipolar diffusion, and omitting the plasma sheath at the wall, numerical integrations have been made of the boundary layer equations by Fay and Kemp [4] for two limiting cases: one case in which there is complete thermodynamic equilibrium behind the reflected shock and within the boundary layer; and another in which there is equilibrium behind the reflected shock, but no gas-phase recombination in the boundary layer.

In spite of the complexity of the boundary layer equations, we will be able to estimate the effect of the variable pressure for short times on the heat transfer rate to the wall given in [4]. Also, a more formal derivation is made of the equilibrium boundary layer equations in which the electronic excitation of ions and atoms is taken into account.

8. Derivation of the Equations for a Gas in Frozen State

Following Fay and Kemp, but keeping the pressure outside the boundary layer a function of time, we examine an ionized gas with an equilibrium free stream and a non-gas-phase recombination boundary layer. We assume again that the gas, suddenly heated to a uniform temperature at $t = 0$, and the wall represent a one-dimensional problem in which changes occur only in the y-direction normal to the wall, represented by $y = 0$, and where the gas is indicated by $y > 0$. If v and V_i represent respectively the mean mass velocity and the diffusion velocity for the i^{th} species, the boundary layer equations [4] are then for a frozen gas

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial y}\rho_i(v + V_i) = 0 \,, \tag{8.1a}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - \Sigma\,\rho_i V_i c_{p_i}\frac{\partial T}{\partial y} + \frac{dP}{dt} \,, \tag{8.1b}$$

where P is the pressure at the edge of the boundary layer and a function of time. Equation (8.1a) represents conservation of mass and can be reduced to [4]

$$\frac{\partial s}{\partial t} - \frac{\partial}{\partial \psi}\left(\frac{k_A\rho/c_p}{k_o\rho_o/c_{p_o}}\,L_A\frac{\partial s}{\partial \psi}\right) = 0 \,, \tag{8.2}$$

where

$$s = \alpha/\alpha_o \tag{8.3}$$

and α is the degree of ionization of the gas,

$$k_A = 5.8 \times 10^{-7}\,T^{3/4}\ \text{cal/cm sec }^{o}\text{K} \tag{8.4}$$

is the thermal conductivity of argon [4],

$$L_A = 1.11\,T^{-0.16} \tag{8.5}$$

is the Lewis number for T in degrees Kelvin, and

$$\psi = \left(\frac{c_{p_o}}{\rho_o k_o}\right)^{\frac{1}{2}} \int_0^y \rho dy \,, \tag{8.6}$$

indicating by the subscript o reference quantities. Equation (8. 1b), representing conservation of energy, is likewise reduced to [4]

$$\frac{c_p}{c_{p_o}} \frac{\partial\theta}{\partial t} - \frac{1}{c_{p_o} \rho T_o} \frac{dP}{dt} = \frac{\partial}{\partial\psi}\left(\frac{k\rho}{k_o\rho_o}\frac{\partial\theta}{\partial\psi}\right) + \frac{k_A\rho/c_p}{k_o\rho_o/c_{p_o}} L_A \frac{c_{p_A}}{c_{p_o}} \alpha_o \frac{\partial s}{\partial\psi}\frac{\partial\theta}{\partial\psi} \,, \tag{8.7}$$

where

$$\theta = T/T_o \,. \tag{8.8}$$

It is convenient to define a non-dimensional pressure

$$p = P/P_o \,, \tag{8.9}$$

and to choose the reference quantities T_o and P_o as the temperature and pressure at the edge of the boundary layer for $t = 0$, that is,

$$P_o = P(0) \,, \tag{8.10a}$$

$$T_o = (T_e)_{t=0} \,. \tag{8.10b}$$

With these definitions and the expression for the density

$$\rho = \frac{Pm_A}{(1+\alpha_o s)\varkappa T_o\theta} \tag{8.11}$$

with m_A representing the mass of the argon atom, we can write

$$\rho = \frac{P}{P_o} \frac{P_o m_A}{(1+\alpha_o s)\varkappa T_o\theta} = p \frac{P_o m_A}{(1+\alpha_o s)\varkappa T_o\theta} = p\rho_r \,, \tag{8.12}$$

where

$$\rho_r = \frac{P_o m_A}{(1+\alpha_o s)\varkappa T_o\theta} \tag{8.13}$$

would be the density if the pressure outside the boundary layer didn't change. The equations for the frozen boundary layer are then, from (8. 2) and (8. 7) with (8. 12),

$$\frac{\partial s}{\partial t} - p\,\frac{\partial}{\partial\psi}\left(\frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{\partial s}{\partial\psi}\right) = 0\,, \tag{8.14a}$$

$$\frac{c_p}{c_{p_o}}\frac{\partial\theta}{\partial t} - \frac{(1+\alpha_o s)\varkappa\theta}{m_A c_{p_o}}\,\frac{1}{p}\frac{dp}{dt} = p\,\frac{\partial}{\partial\psi}\left(\frac{k\rho_r}{k_o\rho_o}\,\frac{\partial\theta}{\partial\psi}\right) + p\,\frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{c_{p_A}}{c_{p_o}}\,\alpha_o\,\frac{\partial s}{\partial\psi}\,\frac{\partial\theta}{\partial\psi}\,. \tag{8.14b}$$

Solutions are tried of the form

$$s = s[\psi\xi(t)] = s(\eta)\,, \tag{8.15a}$$

$$\theta = \theta[\psi\xi(t)] = \theta(\eta)\,, \tag{8.15b}$$

that, substituted into (8.14), give

$$\eta = \frac{\psi}{\left(2\int_0^t p\,dt\right)^{\frac{1}{2}}} \tag{8.16}$$

and reduce the equations of the boundary layer to

$$\frac{d}{d\eta}\left(\frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{ds}{d\eta}\right) + \eta\,\frac{ds}{d\eta} = 0\,, \tag{8.17a}$$

$$\frac{d}{d\eta}\left(\frac{k\rho_r}{k_o\rho_o}\,\frac{d\theta}{d\eta}\right) + \left(\eta\,\frac{c_p}{c_{p_o}} + \frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{c_{p_A}}{c_{p_o}}\,\alpha_o\,\frac{ds}{d\eta}\right)\frac{d\theta}{d\eta} = - \frac{(1+\alpha_o s)\varkappa\theta}{m_A c_{p_o}}\,\frac{\dot{p}}{p^2\xi^2}\,. \tag{8.17b}$$

The boundary conditions are given by

$$y = 0: \qquad T = T_w \ \text{(constant)}\,,\ \alpha = 0\,, \tag{8.18a}$$

$$y \to \infty: \qquad T \to T_e\,, \tag{8.18b}$$

which, in our case, become

$$\eta = 0: \qquad \theta = \theta_w \ \text{(constant)}\,,\ s = 0\,, \tag{8.19a}$$

$$\eta \to \infty: \qquad \theta = \theta_e\,, \tag{8.19b}$$

assuming a catalytic wall.

Solutions of type (8. 15) are then invalidated by the last term of equation (8. 17b), which depends on time. However, looking at equation (8. 14b), we can see that this term is negligible for the case

$$\frac{t}{p}\,\frac{dp}{dt} << 1 \,, \tag{8. 20}$$

which corresponds to short periods of time. Solutions of type (8. 15) are then possible if the temperature and degree of ionization outside the boundary layer have a much slower variation with time than the pressure, and we can assume for the range of values of time for which (8. 20) applies that temperature and degree of ionization outside the boundary layer remain constant and equal to the initial value. In this case, however, the gas outside would not be isentropic. In these circumstances, equations (8. 17) become

$$\frac{d}{d\eta}\left(\frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{ds}{d\eta}\right) + \eta\,\frac{ds}{d\eta} = 0 \,, \tag{8. 21a}$$

$$\frac{d}{d\eta}\left(\frac{k\rho_r}{k_o\rho_o}\,\frac{d\theta}{d\eta}\right) + \left(\eta\,\frac{c_p}{c_{p_o}} + \frac{k_A\rho_r/c_p}{k_o\rho_o/c_{p_o}}\,L_A\,\frac{c_{p_A}}{c_{p_o}}\,\alpha_o\,\frac{ds}{d\eta}\right)\frac{d\theta}{d\eta} = 0 \,, \tag{8. 21b}$$

which are equivalent to the ones with outside pressure constant and solved by Fay and Kemp [4]. The influence of the variation of pressure with time appears in the variable η, as we can see in equation (8. 16). This influence will be explicit when we calculate the heat transfer rate to the wall.

$$q_w \frac{\left(\int_0^t p\,dt\right)^{\frac{1}{2}}}{p(t)^{\frac{1}{2}}} = \left[\frac{c_{p_o} k_o \rho_o}{2t}\right]^{\frac{1}{2}} T_o \left[\frac{k\rho_r}{k_o\rho_o}\frac{d\theta}{d\eta} + \frac{k_A\rho_r}{k_o\rho_o}\frac{c_{p_A}}{c_p} L_A\left(\theta + \frac{h_I^o}{c_{p_A} T_o}\right)\frac{d\alpha}{d\eta}\right]_{\eta=0} , \qquad (8.22)$$

which only differs on the left hand side with the result of reference [4] by the correction factor

$$c_f = \frac{\left(\int_0^t p\,dt\right)^{\frac{1}{2}}}{p(t)^{\frac{1}{2}}} \quad . \qquad (8.23)$$

9. Derivation of the Equations for a Gas in Equilibrium State

The boundary layer equations at the end wall for an equilibrium gas [4] are

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial y} = 0, \tag{9.1a}$$

$$\rho\,\frac{\partial h}{\partial t} + \rho v\,\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\,(-q_y) + \frac{dP}{dt}\,, \tag{9.1b}$$

which represent respectively conservation of mass and conservation of energy. They are reduced [4] to

$$\frac{c_{p_A}}{c_{p_o}}\,\frac{\partial}{\partial t}\left(\frac{h}{c_{p_A}T_o}\right) - \frac{1}{c_{p_o}T_o\rho}\,\frac{dP}{dt} =$$

$$= \frac{\partial}{\partial\psi}\left[\frac{k\rho}{k_o\rho_o}\,\frac{\partial\theta}{\partial\psi} + \frac{k_A\rho}{k_o\rho_o}\,\frac{c_{p_A}}{c_p}\,L_A\left(\theta + \frac{h_I^o}{c_{p_A}T_o}\right)\frac{\partial\alpha}{\partial\psi}\right], \tag{9.2}$$

where h_I^o is the ionization energy per unit mass of atoms.

By the considerations made in (8.12) we can write (9.2) alternatively

$$\frac{c_{p_A}}{c_{p_o}}\,\frac{\partial}{\partial t}\left(\frac{h}{c_{p_A}T_o}\right) - \frac{(1+\alpha)\varkappa\theta}{m_A c_{p_o}}\,\frac{1}{p}\,\frac{dP}{dt} =$$

$$= p\,\frac{\partial}{\partial\psi}\left[\frac{k\rho_r}{k_o\rho_o}\,\frac{\partial\theta}{\partial\psi} + \frac{k_A\rho_r}{k_o\rho_o}\,\frac{c_{p_A}}{c_p}\,L_A\left(\theta + \frac{h_I^o}{c_{p_A}T_o}\right)\frac{\partial\alpha}{\partial\psi}\right]. \tag{9.3}$$

For equilibrium, α is related to θ through the Saha equation,

$$\alpha = \left[1 + Ph_P^3(\varkappa T)^{-5/2}(2\pi m_E)^{-3/2}\,\frac{Qe\ell_A}{Qe\ell_I\,Qe\ell_E}\,\exp(m_A h_I^o/\varkappa T)\right]^{-1/2}, \tag{9.4}$$

where $Qe\ell_i$ is the electronic partition function of species i, and h_P is Planck's constant, so $\partial\alpha/\partial\psi$ can be related to $\partial\theta/\partial\psi$. If excitation above the ground level is ignored in (9.4),

$$\frac{\partial\alpha}{\partial T} = \frac{1}{2}\alpha^3\left[\frac{5}{2T_o\theta}(\alpha^{-2}-1)+\frac{m_A h_I^o}{\kappa T_o^2\theta^2}(\alpha^{-2}-1)\right], \tag{9.5}$$

and substituting

$$\frac{m_A}{\kappa} = \frac{5}{2}\,\frac{1}{c_{p_A}} \tag{9.6}$$

we arrive at

$$\frac{\partial\alpha}{\partial T} = \frac{5}{4}\,\frac{\alpha(1-\alpha^2)}{\theta^2 T_o}\left(\theta+\frac{h_I^o}{c_{p_A}T_o}\right), \tag{9.7a}$$

$$\frac{\partial\alpha}{\partial P} = -\frac{1}{2}\alpha^3\,\frac{\alpha^{-2}-1}{P}, \tag{9.7b}$$

$$\frac{\partial\alpha}{\partial\psi} = T_o\,\frac{\partial\alpha}{\partial T}\,\frac{\partial\theta}{\partial\psi} = \frac{5}{4}\,\frac{\alpha(1-\alpha^2)}{\theta^2}\left(\theta+\frac{h_I^o}{c_{p_A}T_o}\right)\frac{\partial\theta}{\partial\psi}\,. \tag{9.7c}$$

We call

$$S = \frac{5}{4}\,\frac{\alpha(1-\alpha^2)}{\theta^2 c_p/c_{p_A}}\left(\theta+\frac{h_I^o}{c_{p_A}T_o}\right)^2 \tag{9.8}$$

and substitute (9.7c) into (9.3), arriving finally [4] at

$$\frac{c_{p_A}}{c_{p_o}}\,\frac{\partial}{\partial t}\left(\frac{h}{c_{p_A}T_o}\right)-\frac{(1+\alpha)\kappa\theta}{m_A c_{p_o}}\,\frac{1}{p}\,\frac{dP}{dt} = p\,\frac{\partial}{\partial\psi}\left[\frac{k\rho_r}{k_o\rho_o}\,\frac{\partial\theta}{\partial\psi}\left(1+L_A\,\frac{k_A}{k}\,S\right)\right]. \tag{9.9}$$

The values of the enthalpy and the constant-pressure specific heat of all particles,

$$h = c_p T + \alpha h_I^o\,, \tag{9.10a}$$

$$c_p = c_{p_A}(1+\alpha)\,, \tag{9.10b}$$

are used to show that

$$\frac{c_{p_A}}{c_{p_o}}\,\frac{\partial}{\partial t}\left(\frac{h}{c_{p_A}T_o}\right) = \frac{c_p}{c_{p_o}}\,\frac{\partial\theta}{\partial t}+\frac{c_{p_A}}{c_{p_o}}\left(\theta+\frac{h_I^o}{c_{p_A}T_o}\right)\frac{\partial\alpha}{\partial t}\,. \tag{9.11}$$

The local quasi-equilibrium rate of ionization can be estimated from

(9.7a) and (9.7b)

$$\frac{\partial\alpha}{\partial t} = \frac{\partial\alpha}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\alpha}{\partial P}\frac{dP}{dt} = \frac{5}{4}\,\frac{\alpha(1-\alpha^2)}{\theta^2 T_o}\left(\theta + \frac{h_I^o}{c_{p_A} T_o}\right)\frac{\partial\theta}{\partial t} - \frac{1}{2}\,\alpha^3\,\frac{\alpha^{-2}-1}{P}\,\frac{dP}{dt}\ , \tag{9.12}$$

that substituted [4] into (9.11) and (9.9) yields

$$\frac{c_p}{c_{p_o}}(1+S)\frac{\partial\theta}{\partial t} - \left[\frac{(1+\alpha)\varkappa\theta}{m_A c_{p_o}} + \frac{c_{p_A}}{c_{p_o}}\left(\theta + \frac{h_I^o}{c_{p_A} T_o}\right)\frac{\alpha}{2}(1-\alpha^2)\right]\frac{1}{p}\frac{dP}{dt} =$$

$$= p\,\frac{\partial}{\partial\psi}\left[\frac{k\rho_r}{k_o\rho_o}\,\frac{\partial\theta}{\partial\psi}\left(1 + L_A\,\frac{k_A}{k}\,S\right)\right]\ . \tag{9.13}$$

Solutions of the form

$$\theta = \theta[\psi, \xi(t)] = \theta(\eta) \tag{9.14}$$

are tried, that when substituted into (9.13) give

$$\eta = \frac{\psi}{\left(2\int_0^t P\,dt\right)^{\frac{1}{2}}}\ , \tag{9.15}$$

and reduce the equation of the boundary layer to

$$\frac{d}{d\eta}\left[\frac{k\rho_r}{k_o\rho_o}\,\frac{d\theta}{d\eta}\left(1 + L_A\,\frac{k_A}{k}\,S\right)\right] + \eta\,\frac{c_p}{c_{p_o}}(1+S)\frac{d\theta}{d\eta} =$$

$$= -\left[\frac{(1+\alpha)\varkappa\theta}{m_A c_{p_o}} + \frac{c_{p_A}}{c_{p_o}}\left(\theta + \frac{h_I^o}{c_{p_A} T_o}\right)\frac{\alpha}{2}(1-\alpha^2)\right]\frac{\dot{P}}{P^2\xi^2}\ . \tag{9.16}$$

The boundary conditions are given by

$y = 0$: $T = T_w$ (constant) , (9.17a)

$y \to \infty$: $T \to T_e$, (9.17b)

which in our case becomes

$\eta = 0$: $\theta = \theta_w$ (constant) , (9.18a)

$\eta \to \infty$: $\theta = \theta_e$, (9.18b)

the degree of ionization being given by the Saha equation.

Solutions of type (9. 14) are then invalidated by the last term of equation (9. 16), which depends on time, and by the pressure factor appearing in the Saha equation (9. 4). But, again, when (8. 20) applies and the temperature outside the boundary layer has a much slower variation with time than the pressure, a solution of type (9. 14) is possible. In this event, we will not have an isentropic flow. A further approximation is done if we keep the pressure in (9. 4) equal to the initial pressure. This will give a 10 per cent error in the degree of ionization for our variations of pressure. In these circumstances, equation (9. 16) becomes

$$\frac{d}{d\eta}\left[\frac{k\rho_r}{k_o\rho_o}\frac{d\theta}{d\eta}\left(1+L_A\frac{k_A}{k}S\right)\right]+\eta\frac{c_p}{c_{p_o}}(1+S)\frac{d\theta}{d\eta}=0\,, \qquad (9.19)$$

which is again equivalent to the one with constant pressure and solved by Fay and Kemp [4]. The variation of pressure appears when we calculate the heat transfer rate to the wall

$$q_w\,\frac{\left(\int_0^t P\,dt\right)^{\frac{1}{2}}}{p(t)^{\frac{1}{2}}}=\left[\frac{c_{p_o}k_o\rho_o}{2t}\right]^{\frac{1}{2}}T_o\left[\frac{k\rho_r}{k_o\rho_o}\frac{d\theta}{d\eta}\left(1+L_A\frac{k_A}{k}S\right)\right]_{\eta=0}\,, \qquad (9.20)$$

which only differs on the left hand side with the result of [4] by the correction factor

$$c_f=\frac{\left(\int_0^t p\,dt\right)^{\frac{1}{2}}}{p\,(t)^{\frac{1}{2}}}\,.$$

10. Quasi-equilibrium Transfer Theory for an Ionized Gas

In the paper by Fay and Kemp [4], equations for the equilibrium boundary layer were derived without much consideration for the formality of the approach. Here, we derive the equations following a rigorous method using kinetic theory concepts, as a natural extension of Lighthill's paper [5] for a dissociating gas. Along this development we can see the meaning of the different approximations made by Fay and Kemp, as well as the limitations that these approximations impose. We consider then a fluid composed of atoms, ions, and electrons, with masses m_A, m_I, and m_E, respectively. Due to the small mass of electrons we can write

$$\frac{m_E}{m_A} \ll 1 \ , \quad m_A = m_I \ . \tag{10.1}$$

We again assume ambipolar diffusion and consequently that the number density of ions and electrons is the same. That is,

$$\langle \underline{v}_I \rangle = \frac{\int f_I \underline{v}_I d\underline{v}_I}{\int f_I d\underline{v}_I} = \langle \underline{v}_E \rangle = \frac{\int f_E \underline{v}_E d\underline{v}_E}{\int f_E d\underline{v}_E} \ , \tag{10.2a}$$

$$n_I = \int f_I d\underline{v}_I = n_E = \int f_E d\underline{v}_E \ . \tag{10.2b}$$

The mass velocity of the gas will then be

$$\underline{v} = \frac{m_A \int \underline{v}_A f_A d\underline{v}_A + m_I \int \underline{v}_I f_I d\underline{v}_I + m_E \int \underline{v}_E f_E d\underline{v}_E}{m_A \int f_A d\underline{v}_A + m_I \int f_I d\underline{v}_I + m_E \int f_E d\underline{v}_E} \ , \tag{10.3}$$

that, because of (10.1) and (10.2) becomes

$$\underline{v} = \frac{\int \underline{v}_A f_A d\underline{v}_A + \int \underline{v}_I f_I d\underline{v}_I}{n_A + n_I} \ . \tag{10.4}$$

Due to the ambipolar diffusion, the diffusion velocity of ions and electrons will be equal, that is,

$$V_A = \langle \underline{v}_A \rangle - \underline{v} = V_I = \langle \underline{v}_I \rangle - \underline{v} \ . \tag{10.5}$$

For monatomic gases, with electronic excitation, we still have spherical symmetry for atoms and ions, and the Boltzmann equation still applies [6]; that is,

$$\frac{\partial f_\ell}{\partial t} + v_{\ell x}\frac{\partial f_\ell}{\partial x} + v_{\ell y}\frac{\partial f_\ell}{\partial y} + v_{\ell z}\frac{\partial f_\ell}{\partial z} = \frac{\partial_e f_\ell}{\partial t} \tag{10.6}$$

for atoms, ions, and electrons. From (10.6) the continuity equation for the gas is readily obtained [5],

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \ , \tag{10.7}$$

as well as the equation of momentum,

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j + p_{ij}) = 0 \ , \tag{10.8}$$

where

$$p_{ij} = \rho_A \langle(\underline{v}_A - \underline{v})^2\rangle + \rho_I \langle(\underline{v}_I - \underline{v})^2\rangle + \rho_E \langle(\underline{v}_E - \underline{v})^2\rangle \ . \tag{10.9}$$

The equation of energy will be likewise

$$\frac{\partial[\rho(u+\frac{1}{2}v_i^2)]}{\partial t} + \frac{\partial}{\partial x_j}[\rho v_j(u+\tfrac{1}{2}v_i^2) + p_{ij}v_i + q_j] = 0 \ , \tag{10.10}$$

where

$$\rho u = \rho_A u_A + \rho_I u_I + \rho_E u_E \ , \tag{10.11}$$

$$u_\ell = \frac{\langle E_\ell \rangle}{m_\ell} \ , \tag{10.12}$$

indicating by E_ℓ the total energy of the particle of species ℓ. In the case of the electron, it will consist of translational energy. In the case of atoms, it will consist of translational and excitation energies; and in the case of ions, of translational, excitation, and ionization energies. That is,

$$E_A = \tfrac{1}{2}m_A(\underline{v}_A - \underline{v})^2 + e_A \ , \qquad (10.13a)$$

$$E_I = \tfrac{1}{2}m_I(\underline{v}_I - \underline{v})^2 + e_I + I \ , \qquad (10.13b)$$

$$E_E = \tfrac{1}{2}m_E(\underline{v}_E - \underline{v})^2 \ , \qquad (10.13c)$$

the excitation energy, represented by e_ℓ. Therefore,

$$\underline{q} = \tfrac{1}{2}[\rho_A\langle(\underline{v}_A-\underline{v})^2(\underline{v}_A-\underline{v})\rangle + \rho_I\langle(\underline{v}_I-\underline{v})^2(\underline{v}_I-\underline{v})\rangle + \rho_E\langle(\underline{v}_E-\underline{v})^2(\underline{v}_E-\underline{v})\rangle] + n_A\langle e_A(\underline{v}_A-\underline{v})\rangle + n_I\langle(I+e_I)(\underline{v}_I-\underline{v})\rangle \ . \qquad (10.14)$$

In these circumstances, a mechanical pressure and a translational temperature can be defined and related in the following way:

$$P = \frac{1}{3}[\rho_A\langle(\underline{v}_A-\underline{v})^2\rangle + \rho_I\langle(\underline{v}_I-\underline{v})^2\rangle + \rho_E\langle(\underline{v}_E-\underline{v})^2\rangle] = n\kappa T \ . \qquad (10.15)$$

Further simplifications are possible if we take into account, as pointed out by Fay [7], that the electrons make no contribution to the viscosity because of their extremely small mass. Thus, the momentum flow in equation (10.9) will depend only on the translational motion of the atoms and ions. We assume, then, that the results of kinetic theory for a binary mixture of atoms and ions applies and expresses p_{ij} as

$$p_{ij} = P\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) + \frac{2}{3}\mu\frac{\partial x_k}{\partial x_k}\delta_{ij} \ , \qquad (10.16)$$

where δ_{ij} is 1 when $i = j$ and 0 otherwise. Here, μ is the viscosity of the binary mixture.

Following now Lighthill [5], we assume in the quasi-equilibrium theory of the gaseous transport properties, that deviations from thermodynamic equilibrium are small enough to neglect the square of such deviations. So we can use equilibrium relations in the equation (10.14), which is already an effect of the deviation from equilibrium. But it could not be done in (10.15) with the mechanical pressure, because this is not an effect of deviations from equilibrium. The values e_A and e_I are then substituted by their expressions in thermodynamic equilibrium, as can be found in [8] and [9],

$$e_A = e_A(T) \;, \quad e_I = e_I(T) \,. \tag{10.17}$$

Equation (10.14) becomes now

$$\begin{aligned} \underline{q} = \tfrac{1}{2}[\rho_A\langle(\underline{v}_A-\underline{v})^2(\underline{v}_A-\underline{v})\rangle + \rho_I\langle(\underline{v}_I-\underline{v})^2(\underline{v}_I-\underline{v})\rangle + \rho_E\langle(\underline{v}_E-\underline{v})^2(\underline{v}_E-\underline{v})\rangle] + \\ + n_A e_A(T)\langle\underline{v}_A-\underline{v}\rangle + n_I[e_I(T)+I]\langle\underline{v}_I-\underline{v}\rangle \,. \end{aligned} \tag{10.18}$$

Since we are assuming ambipolar diffusion, the results of kinetic theory [6] are applied for atoms diffusing through pairs of ions and electrons. If D is the binary diffusion coefficient, we can write

$$V_{A_i} - V_{I_i} = \frac{-D}{\alpha(1-\alpha)}\left[\frac{\partial(1-\alpha)}{\partial x_i} + \alpha_T\alpha(1-\alpha)\frac{1}{T}\frac{\partial T}{\partial x_i}\right] \,, \tag{10.19}$$

where α_T is the thermal diffusion factor. On the other hand, equation (10.5) says

$$V_{A_i} - V_{I_i} = 0 \,, \tag{10.20}$$

so we can get

$$V_{A_i} = \frac{-D}{1-\alpha}\left[\frac{\partial(1-\alpha)}{\partial x_i} + \alpha_T\alpha(1-\alpha)\frac{1}{T}\frac{\partial T}{\partial x_i}\right] \,, \tag{10.21a}$$

$$V_{I_i} = \frac{D}{\alpha}\left[\frac{\partial(1-\alpha)}{\partial x_i} + \alpha_T\alpha(1-\alpha)\frac{1}{T}\frac{\partial T}{\partial x_i}\right] \,. \tag{10.21b}$$

Similarly, the same kinetic theory [6] for ambipolar diffusion will lead us to

$$\tfrac{1}{2}[\rho_A\langle(\underline{v}_A-\underline{v})^2(\underline{v}_{A_i}-\underline{v}_i)\rangle + \rho_I\langle(\underline{v}_I-\underline{v})^2(\underline{v}_{I_i}-\underline{v}_i)\rangle + \rho_E\langle(\underline{v}_E-\underline{v})^2(\underline{v}_{E_i}-\underline{v}_i)\rangle] =$$

$$= -k\frac{\partial T}{\partial x_i} + \frac{5}{2}\varkappa T(n_A V_{A_i} + n_I V_{I_i}) + \varkappa(n_A+n_I)T\alpha_T\alpha(1-\alpha)(V_{A_i}-V_{I_i}) \quad . \qquad (10.22)$$

With (10.21) and (10.22), the equation (10.18) becomes

$$q_i = -k\frac{\partial T}{\partial x_i} + (n_A+n_I)D[\frac{\partial(1-\alpha)}{\partial x_i} + \alpha_T\alpha(1-\alpha)\frac{1}{T}\frac{\partial T}{\partial x_i}][e_I(T)+I-e_A(T)-\varkappa T\alpha_T] \, .$$

(10.23)

If, in this expression, we assume for α the value in function of P and T that corresponds to the thermodynamic equilibrium, we will have, as we said before, an error of the second order. That expression is given by the Saha equation (9.4), that we write now as

$$\alpha = [1 + P\,F(T)]^{-\frac{1}{2}} , \qquad (10.24)$$

where

$$F(T) = h_P^3(\varkappa T)^{-5/2}(2\pi m_E)^{-3/2}\frac{Qe\ell_A}{Qe\ell_I Qe\ell_E}\exp(m_A h_I^{\,o}/\varkappa T). \qquad (10.25)$$

If

$$P = P(t) , \qquad (10.26)$$

as happens in the thermal boundary layer that we are studying, then

$$\frac{\partial\alpha}{\partial x_i} = -\tfrac{1}{2}\alpha^3\frac{dF}{dT}\frac{\partial T}{\partial x_i} , \qquad (10.27)$$

and equation (10.23) becomes

$$q_i = -\left\{k - (n_A+n_I)D[\tfrac{1}{2}\alpha^3\frac{dF}{dT} + \alpha_T\alpha(1-\alpha)\frac{1}{T}][e_I(T)+I-e_A(T)-\varkappa T\alpha_T]\right\}\frac{\partial T}{\partial x_i} \; .$$

(10.28)

Defining now the enthalpy as

$$h = u + \frac{P_m}{\rho} , \qquad (10.29)$$

and calling

$$k_T = k-(n_A+n_I)D[\tfrac{1}{2}\alpha^3 \frac{dF}{dT}+\alpha_T\alpha(1-\alpha)\frac{1}{T}][e_I(T)+I-e_A(T)-\varkappa T\alpha_T] , \quad (10.30)$$

equation (10. 10) becomes, for the case of the end wall,

$$\rho \frac{Dh}{Dt} - \frac{DP}{Dt} = \frac{\partial}{\partial y}\left\{k_T \frac{\partial T}{\partial y}\right\} + \frac{4}{3}\mu\left(\frac{\partial v}{\partial y}\right)^2 . \quad (10.31)$$

We can now replace the temperature in the third term of (10. 31) by its value in thermodynamic equilibrium, because it will produce only an error of a second order; that is,

$$h = h(P, T) , \quad (10.32)$$

and because P remains constant through the boundary layer [2], it can be written

$$\frac{\partial T}{\partial y} = \frac{1}{c_p}\frac{\partial h}{\partial y} . \quad (10.33)$$

Furthermore, the dissipative term is neglected in the thermal boundary layer [2], so we get finally for (10. 31)

$$\frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{dP}{dt} = \frac{\partial}{\partial \psi}\left(\frac{\mu\rho}{\sigma_T}\frac{\partial h}{\partial \psi}\right) , \quad (10.34)$$

where

$$\psi = \int_0^y \rho \, dy , \quad (10.35)$$

and

$$\sigma_T = \frac{\mu c_p}{k_T} , \quad (10.36)$$

which is called the effective Prandtl number. When P is independent of time,

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial \psi}\left(\frac{\mu\rho}{\sigma_T}\frac{\partial h}{\partial \psi}\right) , \quad (10.37)$$

where now we can use for ρ the equilibrium value

$$\rho = \rho(P, h, \alpha) , \quad (10.38)$$

where α is given in function of the pressure and enthalpy by the Saha equation, since that gives only a second order error in (10.37). This equation is similar to the one obtained in [4], equation (9.9), for

$$dp/dt = 0 . \tag{10.39}$$

However, we have considered in this case electronic excitation, as we can see in (10.30), and our approximations have been analyzed in a consistent way.

11. Comparison of the Theory with the Experimental Results

Goldsworthy [1] has solved in his paper the boundary layer problem at the end wall of the shock tube after a shock is reflected from the face of the wall. It is assumed, in this problem, that there is a constant pressure across the boundary layer and it is equal to the Rankine-Hugoniot value, that is

$$P(t) = P(0) = P_o , \tag{11.1}$$

and also the thermal conductivity is proportional to the temperature,

$$k = k_1 T . \tag{11.2}$$

The complete problem for the gas and the wall is then worked out, assuming that at the face of the wall the temperature and the flux of heat are continuous. This temperature at the face of the wall has the constant value

$$T_{wo} = \frac{mT_{eo} - T_1}{1 + m} , \tag{11.3}$$

where T_{eo} is the temperature outside of the boundary layer, that is, the Rankine-Hugoniot value, and hence constant; T_1 is the initial temperature at the wall, and m is given by

$$m = \frac{\gamma}{\gamma - 1} \frac{k_1 P_o}{k_w \rho_w c_w} , \tag{11.4}$$

representing the thermal conductivity, density, and specific heat of the wall by k_w, ρ_w, and c_w, respectively.

Formula (3.17) for the heat transfer can then be applied to this case, namely:

$$q_w \frac{\left(\int_0^t P\, dt \right)^{\frac{1}{2}}}{P\, T_e} = q_{wo} \frac{(t)^{\frac{1}{2}}}{T_{eo}(P_o)^{\frac{1}{2}}} = \left(\frac{\gamma}{\gamma - 1} \frac{k_1}{\pi} \right)^{\frac{1}{2}} , \qquad (11.5)$$

that gives for q_{wo} the final value of

$$q_{wo} = \left(\frac{\gamma}{\gamma - 1} \frac{k_1}{\pi} \right)^{\frac{1}{2}} \frac{T_{eo}(P_o)^{\frac{1}{2}}}{(t)^{\frac{1}{2}}} \quad . \qquad (11.6)$$

We have assumed in using (11.5) that

$$T_{wo}/T_{eo} \ll 1 \quad , \qquad (11.7)$$

or, what is equivalent,

$$m \ll 1 \quad , \qquad (11.8)$$

since in formula (11.3)

$$T_1 \ll T_{eo} \quad . \qquad (11.9)$$

A first order correction for the heat transfer would be desirable, and it can be obtained by using the first order correction for the pressure, given by Goldsworthy as

$$\frac{P'}{P_o} = - B(M_s)\left(1 - \frac{T_1}{T_{eo}} \right)(\tilde{t})^{\frac{1}{2}} , \qquad (11.10)$$

in which the notation of Baganoff [10] has been used, and where $B(M_s)$ is a weak function of the incident-shock Mach number M_s and the ratio of specific heats γ. For monatomic gases

$$0.928 < B(M_s) \le 1 \quad . \qquad (11.11)$$

On the other hand,

$$T_1 \ll T_{eo} \quad , \qquad (11.12)$$

and finally

$$\tilde{t} = \frac{t}{\tau_{eo}} \quad , \qquad (11.13)$$

where

$$\tau_{eo} = (\gamma^2 k/\pi c_p \rho a^2)_o \ , \tag{11.14}$$

where the subscript indicates that the quantities are the ideal values in the reflected region. The constant given by (11.14) is approximately equal to the collision time in that region, which stands at the edge of the boundary layer. With these values, equation (11.10) become

$$\frac{P'}{P_o} = -\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}} , \tag{11.15}$$

which is only valid for

$$t \gg \tau_{eo} \ , \tag{11.16}$$

the range for which the boundary layer theory is valid.

The first order perturbation for the temperature due to this correction of the pressure can be obtained by the isentropic relationship

$$\left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_e}{T_{eo}} \ , \tag{11.17}$$

and if we keep only first order perturbations,

$$T_e = T_{eo}\left(1 - \frac{\gamma-1}{\gamma}\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} . \tag{11.18}$$

Substituting (11.15) and (11.18) into (3.17), and keeping only first order terms, we get

$$q_w = \left(\frac{\gamma}{\gamma-1}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} \frac{P\,T_e}{\left(\int\limits_0^t P\,dt\right)^{\frac{1}{2}}} =$$

$$= \left(\frac{\gamma}{\gamma-1}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} P_o\left(1-\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right) T_{eo}\left(1-\frac{\gamma-1}{\gamma}\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right)\frac{1}{(P_o t)^{\frac{1}{2}}}\left(1-2\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \simeq$$

$$\simeq \left(\frac{\gamma}{\gamma-1}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} \frac{T_{eo}(P_o)^{\frac{1}{2}}}{(t)^{\frac{1}{2}}}\left(1-\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}-\frac{\gamma-1}{\gamma}\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}+\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right) =$$

$$= q_{wo}\left(1-\frac{\gamma-1}{\gamma}\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}}\right), \tag{11. 19}$$

which gives the first order correction for the heat transfer to the wall. We can see from (11. 19) that the pressure perturbation does not give any contribution to the heat transfer by itself, but only through the corresponding perturbation of the temperature.

The temperature at the face of the wall T_{wo} was given by equation (11. 3) and its value is determined by the heat transfer q_{wo}. Its value is constant due to the $t^{-\frac{1}{2}}$ variation of q_{wo} with time. Now it is possible to deduce the first order correction to the temperature T_{wo} from (11. 19). We consider then a semi-infinite solid at zero temperature, which gets a flux of heat at the face per unit time and unit area equal to

$$q'_w = -\,q_{wo}\frac{\gamma-1}{\gamma}\left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}} = -\left(\frac{\gamma-1}{\gamma}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} T_{eo}(P_o)^{\frac{1}{2}}\frac{(\tau_{eo})^{\frac{1}{2}}}{t}\,. \tag{11. 20}$$

The temperature at the face of the wall is then*

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* See page 57, reference 11.

$$T'_w = \frac{1}{(\rho_w c_w k_w \pi)^{\frac{1}{2}}} \int_0^t q'_w(t-\tau) \frac{d\tau}{\tau^{\frac{1}{2}}} \ . \tag{11.21}$$

However, the value (11.20) for q'_w is not integrable, so we assume that during a time $0 < t < t_o$ the heat transfer q'_w is zero, that is, for the time in which the boundary layer approximation is not valid. The heat transfer perturbation is now

$$q'_w = -\left(\frac{\gamma-1}{\gamma}\frac{k_1}{\pi}\right)^{\frac{1}{2}} T_{eo}(P_o)^{\frac{1}{2}} \frac{(\tau_{eo})^{\frac{1}{2}}}{t} u(t-t_o) \ , \tag{11.22}$$

where $u(t-t_o)$ is the unit step function, defined as

$$u(t-t_o) = 0 \qquad \text{if} \qquad t-t_o < 0 \ , \tag{11.23a}$$

$$u(t-t_o) = 1 \qquad \text{if} \qquad t-t_o > 0 \ . \tag{11.23b}$$

Expression (11.22) substituted into (11.21) gives, for the perturbation wall temperature:

$$T'_w = -\left(\frac{\gamma-1}{\gamma}\frac{k_1}{\pi}\right)^{\frac{1}{2}} \frac{2T_{eo}(P_o)^{\frac{1}{2}}}{(\rho_w c_w k_w \pi)^{\frac{1}{2}}} \left(\frac{\tau_{eo}}{t}\right)^{\frac{1}{2}} \log \frac{\left(1+\left(1-\frac{t_o}{t}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}}{\left(1-\left(1-\frac{t_o}{t}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}} \ . \tag{11.24}$$

Because this value for the temperature is only valid for

$$t \gg \tau_{eo} \ , \tag{11.25}$$

and also because we want to have

$$t \gg t_o \ , \tag{11.26}$$

we can make

$$t_o = \tau_{eo} \ , \tag{11.27}$$

and then to assume in (11.24)

$$\tilde{t} = t/\tau_{eo} \gg 1 \ . \tag{11.28}$$

In these circumstances,

$$\frac{1}{(\tilde{t})^{\frac{1}{2}}}\log\left(\frac{1+(1-\tilde{t}^{-1})^{\frac{1}{2}}}{1-(1-\tilde{t}^{-1})^{\frac{1}{2}}}\right)^{\frac{1}{2}} \sim \frac{\log(\tilde{t})^{\frac{1}{2}}}{(\tilde{t})^{\frac{1}{2}}} \quad , \tag{11.29}$$

and finally we get

$$T'_w = -\left(\frac{\gamma-1}{\gamma}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} \frac{2T_{eo}(P_o)^{\frac{1}{2}}}{(\rho_w c_w k_w \pi)^{\frac{1}{2}}}\,\frac{\log(\tilde{t})^{\frac{1}{2}}}{(\tilde{t})^{\frac{1}{2}}} \quad . \tag{11.30}$$

The total temperature at the face of the wall is then given by

$$T_w = T_{wo} + T'_w = \frac{mT_{eo}-T_1}{1+m} - \left(\frac{\gamma-1}{\gamma}\,\frac{k_1}{\pi}\right)^{\frac{1}{2}} \frac{2T_{eo}(P_o)^{\frac{1}{2}}}{(\rho_w c_w k_w \pi)^{\frac{1}{2}}}\,\frac{\log(\tilde{t})^{\frac{1}{2}}}{(\tilde{t})^{\frac{1}{2}}} \quad , \tag{11.31}$$

valid for

$$\tilde{t} \gg 1 \quad , \tag{11.32}$$

which can be compared with experimental results obtained at the end wall of the shock tube. With result (11.31) a jump in the temperature at the end wall is not necessary, and a transition temperature appears until the asymptotic constant value is reached.

If we compare the temperature perturbation (11.31) with the pressure perturbation (11.15), we can assert that the pressure reaches the asymptotic constant value faster than the temperature, since the $\tilde{t}^{-\frac{1}{2}}$ correction for the first goes to zero faster than the $\tilde{t}^{-\frac{1}{2}}\log(\tilde{t})^{\frac{1}{2}}$ correction for the second. This is in agreement with experimental observations.

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