Use of Superspace Geometry to find all Supergravity Theories: Case of N = 4 and SO(4) Symmetry.

Thesis by

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to my parents Oscar and Betty

and my wife Gabriela

Abstract

The main subject of this thesis is the study of the N = 4 supergravity theories. Superspace geometry is used to search for all N = 4 supergravities with at least SO(4) global symmetry. It is found that the general solution to the unconstrained Bianchi Identities, with the field content of N = 4 supergravity, are equivalent to the known SO(4) and SU(4) supergravities up to field redefinitions. Therefore the field content determines uniquely the N = 4 theories and further constraints are not necessary.

The SO(4) supergravity is gauged with two coupling constants and a new theory with positive cosmological constant and spontaneous breaking of the four supersymmetries is found. The presence of scalar fields in the kinetic term of the vectors is seen to make the values of the physical gauge coupling constants depend on the choice of vacuum.

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Chapter I. Introduction and Outline

Supergravity theories [1] are extensions of Einstein's theory of gravitation. The graviton, the particle that carries the gravitational force, is related to particles of lower spin by supersymmetry transformations [2]. By having the graviton and the spin three halfs, spin one, spin one half and spin zero particles in the same multiplet (the so called gravitational supermultiplet), extended supergravity theories are candidates for truly unified theories of all interactions.

In Grand Unified Theories [3], the gauge bosons, the matter, and the Higgs particles are in different multiplets of the gauge group. In constrast to this, in extended supergravities one does not only have gauge bosons, matter, and Higgs particles in the same supermultiplet, but gravity, which is left out in Grand Unified Theories, is also included.

Perhaps the greatest advantage of supergravity theories is the cancellation of some of the ultraviolet divergences that appear in Einstein's gravity coupled to matter. Pure gravity is known to be finite at the one loop level but may be infinite at two loops. Gravity coupled to matter, however, shows infinities at the one loop level. Pure supergravity theories are known to be free of ultraviolet divergences at the one loop and two loops level. Infinities, however, may also appear at the three loop level. Supergravity coupled to supermatter seems to be as divergent as ordinary gravity coupled to ordinary matter. Even if supergravity theories turn out to be infinite, they could still be viewed as low energy effective field theories that are the limit of possibly finite superstring theories [4].

The supergravity theories are labelled by the integer N running from one to eight that indicates the number of local supersymmetry transformations under which the theory is invariant. The integer N also indicates the number of real

(Majorana) gravitinos and the global internal symmetry group SO(N) of the theory. The N = 1 supergravity theory contains the graviton and a gravitino. N = 2 supergravity has the graviton, two gravitinos and a photon. It is a unified theory of gravity and electromagnetism. N = 3 supergravity is the first supergravity with matter; it has the graviton, three gravitinos, three vectors and one spinor. The N = 4 supergravity is the first supergravity that has all possible spin fields: a graviton, four gravitinos, six vectors, four spinors and two scalars. The largest supergravity theory is the N = 8 theory, and it contains a graviton, eight gravitinos, twenty-eight vectors, fifty-six spinors and seventy scalars. The N = 8 theory (or its superstring extension), is probably at this time, the best candidate for a unified theory of all interactions.

The internal global SO(N) symmetry of the supergravity theories ($N \geq 2$) can be made local since the supergravitational multiplet contains precisely the required number of vector fields to do this. This leads to theories with two coupling constants, the gravitational coupling constant κ and the internal coupling constant g. The idea is that this gauged SO(N) symmetry is somehow related to the gauge symmetries that describe the strong and electroweak interactions.

Supergravity theories were discovered and formulated as field theories in ordinary four dimensional spacetime. Later on, it was found that certain aspects of these theories could be formulated in a more transparent way in superspace [5]. In addition to the four ordinary bosonic coordinates, superspace has 4N anticommuting Grassmann coordinates. The main advantage of superspace formulations is that it makes supersymmetry manifest. So far no physical meaning has been attached to fermionic coordinates, and superspace is only a mathematical tool for analysis of theories with supersymmetry.

The main subject of this thesis is the study of the N = 4 supergravity theories. There are several reasons for studying the N = 4 supergravities. They are the simplest pure supergravity theories with scalar fields. The presence of scalar fields introduces some complications in the formulation of supergravity theories, thus N = 4 supergravity is the theory in which these complications can be studied with least difficulty. N = 4 supergravity is also very special. There are two versions of this theory, the SO(4) model and the SU(4) model. It is the largest supergravity that can be coupled to supermatter (multiplets with spins that do not exceed one). N = 4 is the largest possible value for conformal supergravity. Finally, the N = 4 supergravity is closely related to the type I superstring theory.

The gauged N = 4 supergravities are special too. Since the group SO(4) is not simple, in fact SO(4) = SU(2) $\otimes$ SU(2), the N = 4 supergravities can be gauged with two coupling constants, in contrast to all other supergravities, which are gauged with one coupling constant. The scalar potentials that appear as a consequence of the gauging are unbounded from below and the critical points are relative maxima. For larger supergravity theories, these features of the scalar potential persist.

In Chapter II of this dissertation basic tools necessary for the superspace formulation of extended supergravity theories are explained. These include a formalism based on the supercovariant derivative and the system of Bianchi identities. The use of these tools is illustrated with N = 3 supergravity. It is shown in detail how to solve the Bianchi identities for this theory. The N = 3 theory is a nontrivial example in which the superspace techniques can be explained easily. This is not the case for N = 4 supergravity, where the amount of calculation required increases by at least an order of magnitude.

Chapter III deals with the ungauged N = 4 supergravities. The system of Bianchi identities is studied in detail. It is shown how the N = 4 supergravities follow uniquely from their known field representations and the system of Bianchi identities with no further constraints.

Chapter IV deals with the gauged N = 4 supergravities and their superspace formulation. The SO(4) supergravity is gauged with two coupling constants and a new theory with positive cosmological constant and spontaneous supersymmetry breaking is found. Effects due to the presence of scalar fields in the kinetic term for the vectors are studied for the N = 4 and N = 5 supergravities.

Chapter II covers some material that is now standard. The particular formalism based on the supercovariant derivative that we will use is due to S. J. Gates, Jr. The section on N = 3 supergravity is based on unpublished work of the author. Chapter III and part of Chapter IV are based on published work coauthored with S. J. Gates Jr., and the rest of Chapter IV is based on published work of the author.

References for Chapter I

[1] For a review of supergravity see: P. Van Nieuwenhuizen Phys. Reports 68 no. 4 (1981).

[2] For a review of supersymmetry see: P. Fayet and S. Ferrara Phys. Reports 32 no. 5 (1977).

[3] For a review of Grand Unified Theories see: P. Langacker Phys. Reports 72 no. 4 (1981).

[4] For a review of Superstring Theory see: J. H. Schwarz Phys. Reports 89 no. 3 (1982).

[5] For a review of Superspace Methods see: S. J. Gates Jr., M. T. Grisaru, M. Rocek and W. Siegel, " Superspace" (New York, Addison Wesley), to be published.

Chapter II. Superspace Techniques

1. Introduction

Superspace is a very convenient tool for the study of theories with supersymmetry. Since the algebra of supersymmetry theories is an extension of the ordinary space-time algebras, it is natural to extend the usual space in which fields are defined to a superspace with additional Grassmann coordinates. Supergravity theories written in superspace have a geometrical interpretation in terms of the torsions and curvatures of the superspace, analogous to the usual interpretation of Einstein's gravity as the theory of curved spacetime.

Superspace methods offer some calculational advantages. The consistency of a superspace formulation implies that the supersymmetry algebra closes. Equations of motion and supersymmetry transformations also follow quite naturally. Finally, supergraph techniques appear to simplify quantum calculations in supersymmetric theories.

In this chapter a brief description of superspace methods relevant to the construction of extended supergravities is given. More comprehensive reviews, with references to the original works, are available in the literature [1,2]. In Section 2 the basic notions of superspace are explained; these include the idea of vielbeins and of the tangent space. In Section 3 a formalism for connecting superspace and component formulations based on the supercovariant derivative [3] is explained. The analogy with usual gauge theories, also based on the construction of covariant derivatives, is emphasized. Section 4 shows how to derive the set of Bianchi identities and how extended supergravities can be naturally described by a field strength superfield. In Section 5 the complete superspace geometry that describes the N = 3 supergravity theory is constructed as an example of the use of Bianchi identities and field strength superfields.

2. Superspace

Superspace is a manifold that has four bosonic coordinates x^m with $m = 0,1,2,3$ and 4N Grassmann coordinates, where N refers to the N-extended supergravity theory. These fermionic coordinates are represented by anticommuting complex parameters $\theta^{\mu i}$, $\bar{\theta}^{\dot{\mu} i}$ where i runs from 1 to N, and μ and $\dot{\mu}$ are spinor indices which can take the values 1 or 2 (using two component spinor notation). The parameters θ and $\bar{\theta}$ are related by conjugation (see Appendix A for superspace conjugation), namely:

$$\overline{\theta^{\mu i}} = \bar{\theta}^{\dot{\mu} i} \ , \ \ \overline{\theta_{\mu i}} = -\bar{\theta}_{\dot{\mu} i} \ . \qquad (2.1)$$

In ordinary Einstein gravity it is convenient to distinguish between curved vector indices and flat vector indices. The vierbein (or tetrad frame) is used to convert between these two types of indices. Letting $m, n, p \ldots$ stand for curved vector indices and $a, b, c \ldots$ stand for flat vector indices, the vierbein $e_m{}^a$ and the inverse vierbein $e_a{}^m$ are defined to satisfy:

$$e_m{}^a \, e_a{}^n = \delta_m^n \ , \ \ e_a{}^m \, e_m{}^b = \delta_a^b \ , \qquad (2.2)$$

and convert the type of indices as follows:

$$A_m = e_m{}^a A_a \ , \ \ A_a = e_a{}^m A_m \ , \qquad (2.3)$$

where flat indices are raised and lowered with the Minkowski metric η_{ab} $(+\,-\,-\,-)$ and curved indices are raised and lowered with the riemannian metric $g_{mn} = e_m{}^a \, e_n{}^b \, \eta_{ab}$.

In superspace curved indices will be denoted by $M = (\mu, \dot{\mu}, m)$, where μ and $\dot{\mu}$ are curved spinorial indices and m is a curved vector index. Flat indices are denoted by $A = (\alpha, \dot{\alpha}, a)$ where α and $\dot{\alpha}$ are flat spinor indices and a is a flat vector index. The geometric object that converts between these indices is the *vielbein* $E_M{}^A$. The vielbein is invertible, and the inverse $E_A{}^M$ is called the *inverse vielbein*. Relations analogous to (2.2) hold:

$$E_M{}^A \, E_A{}^N = \delta_M{}^N \; , \quad E_A{}^M \, E_M{}^B = \delta_A{}^B \; , \tag{2.4}$$

where:

$$\delta_M{}^N = (\delta_\mu^\nu , \delta_{\dot\mu}^{\dot\nu} , \delta_m^n) \; , \quad \delta_A{}^B = (\delta_\alpha^\beta , \delta_{\dot\alpha}^{\dot\beta} , \delta_a^b) \; . \tag{2.5}$$

The vielbein $E_A{}^M$ is a superfield, that is, depends on both bosonic and fermionic coordinates in the superspace manifold. Superfields are usually thought of as a finite power series expansion in the anticommuting θ parameters. The physical fields, which depend only on the bosonic coordinates, are the coefficients of this expansion.

The raising and lowering of spinor indices is done with the Lorentz invariant antisymmetric tensor $C_{\alpha\beta} = -C_{\beta\alpha}$, and its conjugate $C_{\dot\alpha\dot\beta}$:

$$\begin{aligned} A_\alpha &= A^\beta \, C_{\beta\alpha} \; , \quad A^\alpha = C^{\alpha\beta} \, A_\beta \; , \\ A_{\dot\alpha} &= A^{\dot\beta} \, C_{\dot\beta\dot\alpha} \; , \quad A^{\dot\alpha} = C^{\dot\alpha\dot\beta} \, A_{\dot\beta} \; . \end{aligned} \tag{2.6}$$

It follows from equation (2.6) that:

$$C^{\alpha\beta} \, C_{\gamma\beta} = \delta_\gamma^\alpha \; , \quad C^{\dot\alpha\dot\beta} \, C_{\dot\gamma\dot\beta} = \delta_{\dot\gamma}^{\dot\alpha} \; . \tag{2.7}$$

A tensor can be antisymmetric in at most two spinor indices, since spinor indices only run over two values. In this case, the tensor is proportional to the $C_{\alpha\beta}$ symbol:

$$A_{\alpha\beta} - A_{\beta\alpha} = C_{\alpha\beta} \, A_\gamma{}^\gamma \; , \tag{2.8}$$

as can be checked contracting with another $C^{\alpha\beta}$.

In the two component notation that will be used it is often convenient to represent vector indices using spinor indices. As representations of the Lorentz group, a vector $A_a \sim (\frac{1}{2},\frac{1}{2})$ is constructed by taking the direct product of a left handed spinor $\chi_{\dot{\alpha}} \sim (\frac{1}{2},0)$ with a right handed spinor $\chi_\alpha \sim (0,\frac{1}{2})$. Therefore, a vector A_a can be represented as an object $A_{\alpha\dot{\alpha}}$ having one dotted and one undotted spinor index. One uses the Pauli matrices $\sigma^a{}_{\alpha\dot{\alpha}}$ (see Appendix A) to transform a vector index into a pair of spinor indices and vice versa;

$$A_{\alpha\dot{\alpha}} = A_a \, \sigma^a{}_{\alpha\dot{\alpha}} \, , \quad A_a = -\frac{1}{2} \sigma_a{}^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}} \, , \tag{2.9}$$

where

$$\sigma^a{}_{\alpha\dot{\beta}} \, \sigma_a{}^{\gamma\dot{\delta}} = -2 \, \delta_\alpha^\gamma \, \delta_{\dot{\beta}}^{\dot{\delta}} \, , \quad \sigma_{a\,\alpha\dot{\beta}} \, \sigma^{b\,\alpha\dot{\beta}} = -\delta_a^b \, . \tag{2.10}$$

One can define sigma matrices $(\sigma^{ab})_{\alpha\beta}$, and $(\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}$ out of the Pauli matrices

$$\sigma^{[a}{}_{\alpha\dot{\beta}} \, \sigma^{b]}{}_{\gamma\dot{\delta}} = -iC_{\alpha\gamma} (\bar{\sigma}^{ab})_{\dot{\beta}\dot{\delta}} - iC_{\dot{\beta}\dot{\delta}} (\sigma^{ab})_{\alpha\gamma} \, , \tag{2.11}$$

satisfying

$$(\sigma^{ab})_{\alpha\beta} (\sigma_{ab})^{\gamma\delta} = -4 \, \delta_{(\alpha}^\gamma \delta_{\beta)}^\delta \, . \tag{2.12}$$

Any antisymmetric tensor $R_{ab} = -R_{ba}$ can be represented by two tensors $R_{\alpha\beta}$ and $R_{\dot{\alpha}\dot{\beta}}$

$$R_{ab} = \frac{i}{2} R_{\alpha\beta} (\sigma_{ab})^{\alpha\beta} + \frac{i}{2} R_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \, , \tag{2.13}$$

where

$$R_{\alpha\beta} = \frac{i}{4} (\sigma^{ab})_{\alpha\beta} R_{ab} \, , \quad R_{\dot{\alpha}\dot{\beta}} = \frac{i}{4} (\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} R_{ab} \, . \tag{2.14}$$

It should be noted that $R_{\alpha\beta}$ and $R_{\dot{\alpha}\dot{\beta}}$ are conjugates of each other only if R_{ab} is real.

At every point in the superspace manifold one has a tangent space on which superfields are defined. The Lorentz group acts on the tangent space reducibly. Vectors rotate into vectors, and spinors rotate into spinors. In general superfields transform according to the matrix representation appropriate for the Lorentz index they carry. Denoting the Lorentz generators by M_{bc}, we use the following representations for the action on vectors and spinors:

$$[M_{bc}, V_a] = \eta_{ac}\, V_b - \eta_{ab}\, V_c \ ,$$

$$[M_{bc}, V_\alpha] = -\frac{i}{2}(\sigma_{bc})_\alpha{}^\beta\, V_\beta \ . \tag{2.15}$$

It is sometimes simpler to use a two component Lorentz generator $M_{\alpha\beta}$ defined as:

$$M_{\alpha\beta} = \frac{i}{2}(\sigma^{bc})_{\alpha\beta}\, M_{bc} \ . \tag{2.16}$$

Since the tangent space is built with a local Lorentz invariance the description of superspace requires a Lorentz connection. One therefore introduces the Lorentz connection superfield $\Phi_M{}^{bc}$ where the index M is a curved superspace index. In the approach we shall take, this superconnection is an auxiliary field. Therefore, one can find an expression for it in terms of the vielbein $E_M{}^A$. The $\Phi_m{}^{bc}$ superfield, for example, will just be the ordinary spin connection of supergravity $\omega_m{}^{bc}$ plus other fields at higher order in the θ expansion.

One can implement other symmetries in the tangent space, for example, an internal SO(N) invariance that rotates the N gravitinos of N-extended supergravity into each other. To this end, one would just introduce SO(N) generators $t_{ij} = -t_{ji}$ $(i,j = 1,..N)$ and gauge connection superfields $\Phi_M{}^{ij}$. In our discussion we will identify the $A_m{}^{ij}$ vector fields of the supergravity theory with the θ independent part of the $\Phi_m{}^{ij}$ superfield.

Even fake symmetries can be put into the tangent space. For example, in the N = 8 supergravity there are 70 physical scalar fields. It is convenient, however, to work with 133 scalars that parametrize an element of the group $E_{7,7}$. In this case the theory can be formulated with a local SU(8) invariance in the tangent space. This local SU(8) symmetry compensates for the extra 63 degrees of freedom that have been introduced.

3. The Supercovariant Derivative

In this section a superspace formalism based on the supercovariant derivative is explained [3]. This formalism connects the superspace and ordinary space approaches [4] and can be used to derive supersymmetry transformations for the component fields of the theory. The way the supercovariant derivative enters in the construction of supergravity theories is very analogous to the way the ordinary covariant derivative enters in the formulation of gauge theories. It is therefore convenient to review a few basic facts about gauge theories in a notation that will be useful later.

In gauge theories one can parametrize gauge transformations by an element $iK = iK^i T^i$, where the K^i are parameters and T^i are hermitian generators of the gauge group. Under a gauge transformation a matter field Φ_j would transform as:

$$\delta\Phi_j = [iK,\Phi_j] = iK^i[T^i,\Phi_j] = iK^i(T^i)_j{}^k\,\Phi_k \ . \tag{3.1}$$

where $(T^i)_j{}^k$ is the appropriate matrix representation for the T^i generator.

One defines a covariant derivative:

$$D_m = \partial_m + igA^i_m T^i \ , \tag{3.2}$$

by using gauge fields A^i_m. This covariant derivative is thought of as acting according to:

$$D_m = \partial_m + ig\,[A_m,\ ]\ , \tag{3.3}$$

where $A_m = A^i_m T^i$. The requirement that $(D_m\Phi)$ transforms under a gauge transformation the same way as Φ does, implies that one needs:

$$\delta D_m = [iK,D_m]\ . \tag{3.4}$$

Using equations (3.2) and (3.4) one finds the usual transformations law of the gauge fields

$$\delta A^i_m = -\frac{\partial_m K^i}{g} - c_{ijk} K^j A^k_m , \tag{3.5}$$

where $[T^i, T^j] = c_{ijk} T^k$.

One defines torsions and field strengths by commutation of two covariant derivatives

$$[D_m, D_n] = T_{mn}{}^p D_p + igF_{mn} , \tag{3.6}$$

where $F_{mn} = F^i_{mn} T^i$. Using equation (3.2), one finds that the torsion vanishes $T_{mn}{}^p = 0$, and the field strength takes its well-known value:

$$F^i_{mn} = \partial_m A^i_n - \partial_n A^i_m - g\, c_{ijk} A^j_m A^k_n . \tag{3.7}$$

Under a gauge transformation

$$\delta F_{mn} = [iK, F_{mn}] , \tag{3.8}$$

one finds

$$\delta F^i_{mn} = -c_{ijk} K^j F^k_{mn} . \tag{3.9}$$

Finally, one can consider the Jacobi identity for the covariant derivative

$$[D_m,[D_n,D_p]] + [D_n,[D_p,D_m]] + [D_p,[D_m,D_n]] = 0 , \tag{3.10}$$

using equation (3.6), the above equation implies that

$$D_m F_{np} + D_n F_{pm} + D_p F_{mn} = 0 , \tag{3.11}$$

where the derivatives are understood in the sense of equation (3.3). Equation (3.11) is satisfied by the choice of field strength in equation (3.7). Indeed, (3.11) can be rewritten as $D_m \widetilde{F}_{mp} = 0$, where the tilde indicates space-time duality, and is then recognized as the usual form of the Bianchi identity.

Let us now turn to the supercovariant derivative $D_A = (D_{\alpha i}, \overline{D}_{\dot{\alpha} i}, D_a)$ (we follow reference [3]). It will be taken to have the form:

$$D_A = E_A{}^M \partial_M + \varphi_A{}^\Gamma M_\Gamma , \tag{3.12}$$

where $\partial_M = (\partial_\mu, \partial_{\dot{\mu}}, \partial_m)$ are coordinate derivatives and the superfield $\varphi_A{}^\Gamma$ is a gauge field for the tangent space generator M_Γ. This generator can be a Lorentz generator, a central charge generator, an SO(N) generator, among others (say, the generators of local symmetries in the coset space formulations of the scalar sector in N ≥ 4 supergravities). Super-gauge transformations are parametrized by an operator iK :

$$iK = K^M \partial_M + K^\Gamma M_\Gamma , \tag{3.13}$$

which satisfies the reality condition $iK = \overline{iK}$. Both the supercovariant derivative and the operator iK have a θ-expansion (we omit isospin indices for brevity):

$$D_a = \partial_a + \nabla_a^0 + \overline{\theta}^{\dot{\beta}} \nabla_{a\dot{\beta}}^1 + \theta^\beta \nabla_{a\beta}^2 + \cdots , \tag{3.14}$$

$$\begin{aligned} iK = {} & K^0 + \theta^\alpha K_\alpha^1 + \overline{\theta}^{\dot{\alpha}} K_{\dot{\alpha}}^{\cdot 1} + \theta^\alpha \overline{\theta}^{\dot{\beta}} K_{\alpha\dot{\beta}}^2 \\ & + \theta^\alpha \theta^\beta K_{\alpha\beta}^3 - \overline{\theta}^{\dot{\alpha}} \overline{\theta}^{\dot{\beta}} \overline{K_{\alpha\beta}^3} + \cdots , \end{aligned} \tag{3.15}$$

where $K_{\alpha\dot{\beta}}^2 = \overline{K_{\beta\dot{\alpha}}^2}$, so that the reality condition is satisfied, and the operators ∇^i are made of ordinary fields times the ∂_M and M_Γ generators. That is, they have the form indicated in equation (3.12) but with the vielbein and connection superfields replaced by ordinary fields.

The gauge invariance represented by iK is enormous and one uses it to gauge away some fields appearing in the supercovariant derivative. Using:

$$\delta D_A = [iK, D_A] \,, \tag{3.16}$$

one finds that the above supercovariant derivative can be put in the form:

$$D_\alpha = \partial_\alpha + i\bar{\theta}^{\dot{\beta}} \nabla^1_{\alpha\dot{\beta}} + \theta^\beta \nabla^2_{\alpha\beta} + O(\theta^2) \,, \tag{3.17}$$

where:

$$\nabla^1_{\alpha\dot{\beta}} = \overline{\nabla^1_{\beta\dot{\alpha}}} \,, \quad \nabla^2_{\alpha\beta} = \nabla^2_{\beta\alpha} \,, \tag{3.18}$$

making use of all the terms in iK except for K^0. Equation (3.17) is an expression for the supercovariant derivative in the so called 'Wess- Zumino' (WZ) gauge. The remaining transformations, parametrized with K^0 and preserving the form of the WZ gauge, are general coordinate transformations

$$iK_{g.c}(\xi) = \xi^m(x)\,\partial_m \,, \tag{3.19}$$

internal transformations

$$iK_\Gamma(\lambda) = \lambda_\Gamma(x)\,[M_\Gamma + (M_\Gamma)_\alpha{}^\beta \theta^\alpha \partial_\beta + (M_\Gamma)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\theta}^{\dot{\alpha}} \partial_{\dot{\beta}} + \ldots] \,, \tag{3.20}$$

and supersymmetry transformations

$$\begin{aligned} iK_Q(\varepsilon) = {} & \varepsilon^\alpha \partial_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \partial_{\dot{\alpha}} + i(\theta^\alpha \bar{\varepsilon}^{\dot{\beta}} + \bar{\theta}^{\dot{\beta}} \varepsilon^\alpha) \nabla^1_{\alpha\dot{\beta}} \\ & + \theta^\alpha \varepsilon^\beta \nabla^2_{\alpha\beta} - \bar{\theta}^{\dot{\alpha}} \bar{\varepsilon}^{\dot{\beta}} \overline{\nabla^2_{\alpha\beta}} + O(\theta^2) \,. \end{aligned} \tag{3.21}$$

In the usual way, one defines torsions and curvatures by graded commutation:

$$[D_A, D_B\} = T_{AB}{}^C D_C + R_{AB}{}^\Gamma M_\Gamma \,, \tag{3.22}$$

where [..} means anticommutation if A and B are both fermionic and ordinary commutation otherwise.

Our goal is to derive an expression for the supersymmetry generator given in equation (3.21). Using equation (3.17) and its conjugate expression (see Appendix A for conjugation) one finds:

$$2\nabla^2_{\alpha\beta} = \{D_\alpha, D_\beta\} + O(\theta) ,$$

$$2i\,\nabla^1_{\alpha\dot\beta} = \{D_\alpha, D_{\dot\beta}\} + O(\theta) . \qquad (3.23)$$

Using the above equations together with equation (3.17) one finds that the supersymmetry generator becomes:

$$iK_Q(\varepsilon) = \varepsilon^\alpha D_\alpha + \bar\varepsilon^{\dot\alpha} \bar D_{\dot\alpha} + (\theta^\alpha \bar\varepsilon^{\dot\beta} + \bar\theta^{\dot\beta}\varepsilon^\alpha)\{D_\alpha, \bar D_{\dot\beta}\}$$

$$+ \theta^\alpha\varepsilon^\beta\{D_\alpha, D_\beta\} + \bar\theta^{\dot\alpha}\bar\varepsilon^{\dot\beta}\{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} + O(\theta^2) . \qquad (3.24)$$

Now consider the vectorial derivative D_a. To lowest order in θ, we put the ordinary gauge fields:

$$D_a = \nabla_a + O(\theta) , \quad \nabla_a = e_a{}^m \partial_m + \psi^\mu_a \partial_\mu + \bar\psi^{\dot\mu}_a \partial_{\dot\mu} + \varphi^\Gamma_a M_\Gamma , \qquad (3.25)$$

where $e_a{}^m$ is the inverse vierbein, ψ^μ_a is the gravitino, and φ^Γ_a represents the Lorentz connection or any other gauge connection, if present. It is now straightforward to derive forms for the supersymmetry transformations of the gauge fields. From:

$$\delta_Q D_a = [iK_Q, D_a] , \qquad (3.26)$$

one finds, using (3.25):

$$\delta_Q D_a = -(D_a\varepsilon^\beta)D_\beta - (D_a\bar\varepsilon^{\dot\beta})\bar D_{\dot\beta} + \varepsilon^\alpha[D_\alpha, D_a]$$

$$+ \bar{\varepsilon}^{\dot{\alpha}}[\bar{D}_{\dot{\alpha}},D_a] - (\psi_a^\alpha \bar{\varepsilon}^{\dot{\beta}} + \bar{\psi}_a^{\dot{\beta}} \varepsilon^\alpha)\{D_\alpha,\bar{D}_{\dot{\beta}}\}$$

$$- \psi_a^\alpha \varepsilon^\beta \{D_\alpha,D_\beta\} - \bar{\psi}_a^{\dot{\alpha}} \bar{\varepsilon}^{\dot{\beta}} \{\bar{D}_{\dot{\alpha}},\bar{D}_{\dot{\beta}}\} + O(\theta) . \tag{3.27}$$

The supersymmetries follow:

$$\delta e_b{}^m = [\varepsilon^\alpha T_{\alpha b}{}^d + \bar{\varepsilon}^{\dot{\beta}} T_{\dot{\beta} b}{}^d - (\psi_b^\alpha \bar{\varepsilon}^{\dot{\beta}} + \bar{\psi}_b^{\dot{\beta}} \varepsilon^\alpha) T_{\alpha\dot{\beta}}{}^d$$

$$- \psi_b^\alpha \varepsilon^\beta T_{\alpha\beta}{}^d - \bar{\psi}_b^{\dot{\alpha}} \bar{\varepsilon}^{\dot{\beta}} T_{\dot{\alpha}\dot{\beta}}{}^d] e_d{}^m , \tag{3.28}$$

$$\delta \psi_b^\mu = -(D_b \varepsilon^\mu) + \varepsilon^\alpha (T_{\alpha b}{}^\mu + T_{\alpha b}{}^d \psi_d^\mu)$$

$$+ \bar{\varepsilon}^{\dot{\alpha}} (T_{\dot{\alpha} b}{}^\mu + T_{\dot{\alpha} b}{}^d \psi_d^\mu) - (\psi_b^\alpha \bar{\varepsilon}^{\dot{\beta}} + \bar{\psi}_b^{\dot{\beta}} \varepsilon^\alpha)(T_{\alpha\dot{\beta}}{}^\mu + T_{\alpha\dot{\beta}}{}^d \psi_d^\mu)$$

$$- \psi_b^\alpha \varepsilon^\beta (T_{\alpha\beta}{}^\mu + T_{\alpha\beta}{}^d \psi_d^\mu) - \bar{\psi}_b^{\dot{\alpha}} \bar{\varepsilon}^{\dot{\beta}} (T_{\dot{\alpha}\dot{\beta}}{}^\mu + T_{\dot{\alpha}\dot{\beta}}{}^d \psi_d^\mu) . \tag{3.29}$$

$$\delta \varphi_b^\Gamma = \varepsilon^\alpha (T_{\alpha b}{}^d \psi_d^\Gamma + R_{\alpha b}{}^\Gamma) + \bar{\varepsilon}^{\dot{\alpha}} (T_{\dot{\alpha} b}{}^d \varphi_d^\Gamma + R_{\dot{\alpha} b}{}^\Gamma)$$

$$- (\psi_b^\alpha \bar{\varepsilon}^{\dot{\beta}} + \bar{\psi}_b^{\dot{\beta}} \varepsilon^\alpha)(T_{\alpha\dot{\beta}}{}^d \varphi_d^\Gamma + R_{\alpha\dot{\beta}}{}^\Gamma)$$

$$- \psi_b^\alpha \varepsilon^\beta (T_{\alpha\beta}{}^d \varphi_d{}^\Gamma + R_{\alpha\beta}{}^\Gamma) - \bar{\psi}_b^{\dot{\alpha}} \bar{\varepsilon}^{\dot{\beta}} (T_{\dot{\alpha}\dot{\beta}}{}^d \varphi_d^\Gamma + R_{\dot{\alpha}\dot{\beta}}{}^\Gamma) . \tag{3.30}$$

The above expressions are extremely useful. Once the superspace geometry has been worked out, one just has to substitute into the above expressions in order to get the supersymmetry transformations of the gauge fields. Conversely, if the supersymmetries are known one can read off the expressions for the superspace tensors. We shall use these equations in both ways in Chapter III.

For nongauge fields, such as scalars and spinors, or for any other covariant quantity, the component supersymmetry transformation is found by using the basic expression in equation (3.21). For a superfield W one has:

$$\delta_Q W = [iK_Q, W] = \varepsilon^{\alpha} D_{\alpha} W + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} W \ , \tag{3.31}$$

where the limit $\theta = 0$ should be taken. Equation (3.31) shows that superfields contain at power θ^{n+1} the supersymmetry transformation of the field that lies at θ^n. Spinorial derivatives are used, instead of explicit θ expansions, in order to find supersymmetries.

Another result, obtained by manipulating equations (3.17) and (3.25), that is quite useful in connecting component and superspace approaches, is the following:

$$\begin{aligned} T_{ab}{}^{E} D_E + R_{ab}^{\Gamma} M_{\Gamma} = [\nabla_a, \nabla_b] + 2i\psi_{[a}^{\gamma k} \bar{\psi}_{b]k}^{\dot{\delta}} \nabla_{\gamma\dot{\delta}} \\ + (\psi_{[a}^{\underline{\gamma}} \bar{\psi}_{b]}^{\dot{\underline{\alpha}}} T_{\underline{\gamma}\dot{\underline{\alpha}}}{}^{\underline{\delta}} - \varphi_{[a}^{\Gamma} \psi_{b]}^{\underline{\gamma}} (M_{\Gamma})_{\underline{\gamma}}{}^{\underline{\delta}}) D_{\underline{\delta}} \\ + (\psi_{[a}^{\underline{\gamma}} \bar{\psi}_{b]}^{\dot{\underline{\alpha}}} T_{\underline{\gamma}\dot{\underline{\alpha}}}{}^{\dot{\underline{\delta}}} - \varphi_{[a}^{\Gamma} \bar{\psi}_{b]}^{\dot{\underline{\gamma}}} (M_{\Gamma})_{\dot{\underline{\gamma}}}{}^{\dot{\underline{\delta}}}) D_{\dot{\underline{\delta}}} \\ + \psi_{[a}^{\underline{\gamma}} [D_{\underline{\gamma}}, D_{b]}] + + \bar{\psi}_{[a}^{\dot{\underline{\gamma}}} [D_{\dot{\underline{\gamma}}}, D_{b]}] + \psi_{[a}^{\underline{\gamma}} \bar{\psi}_{b]}^{\dot{\underline{\delta}}} R_{\underline{\gamma}\dot{\underline{\delta}}}{}^{\Gamma} M_{\Gamma} \\ + \psi_{a}^{\underline{\gamma}} \psi_{b}^{\underline{\delta}} \{D_{\underline{\gamma}}, D_{\underline{\delta}}\} + + \bar{\psi}_{a}^{\dot{\underline{\gamma}}} \bar{\psi}_{b}^{\dot{\underline{\delta}}} \{\bar{D}_{\dot{\underline{\gamma}}}, \bar{D}_{\dot{\underline{\delta}}}\} + O(\theta) \ , \end{aligned} \tag{3.32}$$

where and underlined spinor index ($\underline{\alpha}$) denotes combined spinor and isospin indices (αi) for extended supersymmetry. Equation (3.32) can be used, for example, to find the expression for the spin connection including torsion terms. Taking $\Phi_a{}^{\Gamma} M_{\Gamma} = \frac{1}{2} \omega_a{}^{bc} M_{cb}$ one finds by equating the coefficient of ∇_a in the expansion of both sides of equation (3.32) ,with $T_{\alpha b}{}^{c} = T_{\alpha\beta}{}^{c} = 0$, that:

$$\begin{aligned} \omega_{abc} = \omega_{abc}(e) + \frac{1}{2}(T_{abc} + T_{cba} + T_{cab}) \\ - i(\psi_{[a}^{\gamma k} \bar{\psi}_{b]k}^{\dot{\delta}} \sigma_{c\gamma\dot{\delta}} + \psi_{[c}^{\gamma k} \bar{\psi}_{b]k}^{\dot{\delta}} \sigma_{a\gamma\dot{\delta}} \psi_{[c}^{\gamma k} \bar{\psi}_{a]k}^{\dot{\delta}} \sigma_{b\gamma\dot{\delta}}) \ , \end{aligned} \tag{3.33}$$

where

$$\omega_{abc}(e) = -\frac{1}{2}(C_{abc} + C_{cba} + C_{cab}) \ , \tag{3.34}$$

and $C_{abc} = \partial_{[a} e_{b]}{}^{m} e_{mc}$. Equation (3.32) can also be used to show that $T_{ab}{}^{\underline{\delta}}$ is the supercovariant gravitino field strength, and in general, it is used to relate supercovariant field strengths to ordinary field strengths.

4. Bianchi Identities and Field Strength Superfields

In this section we shall show how to derive the set of Bianchi identities for the supercovariant derivative. The Bianchi identities are the basic tool for the construction of the superspace geometry.

In extended supergravities without auxiliary fields one can put the non-gauge fields and the field strengths of the theory into a field strenth superfield [5]. We will give the specific forms for the vector, gravitino, and graviton field strengths, and show how they fit into the field strength superfield. The superspace tensors have to be found in terms of this superfield using the Bianchi identities.

The Bianchi identities are nothing else than the Jacobi identities for the supercovariant derivative. Due to the presence of fermionic derivatives the Jacobi identities are formed both with commutators and with anticommutators. Consider forming the triple commutator with $D_{\underline{\alpha}}, D_{\underline{\beta}}$, and $D_{\underline{\gamma}}$. To this end one introduces three constant fermionic parameters $\varepsilon^{\underline{\alpha}}, \eta^{\underline{\beta}}, \xi^{\underline{\gamma}}$ and the ordinary triple commutator:

$$[\varepsilon^{\underline{\alpha}} D_{\underline{\alpha}}, [\eta^{\underline{\beta}} D_{\underline{\beta}}, \xi^{\underline{\delta}} D_{\underline{\delta}}]] + [\eta^{\underline{\beta}} D_{\underline{\beta}}, [\xi^{\underline{\delta}} D_{\underline{\delta}}, \varepsilon^{\underline{\alpha}} D_{\underline{\alpha}}]] + [\xi^{\underline{\delta}} D_{\underline{\delta}}, [\varepsilon^{\underline{\alpha}} D_{\underline{\alpha}}, \eta^{\underline{\beta}} D_{\underline{\beta}}]] = 0. \quad (4.1)$$

Expanding out, using the anticommutation property of the parameters and cancelling them out one obtains:

$$[\{D_{\underline{\alpha}}, D_{\underline{\beta}}\}, D_{\underline{\delta}}] + [\{D_{\underline{\beta}}, D_{\underline{\delta}}\}, D_{\underline{\alpha}}] + [\{D_{\underline{\delta}}, D_{\underline{\alpha}}\}, D_{\underline{\beta}}] = 0. \quad (4.2)$$

Consider the following supercovariant derivative and graded commutator:

$$D_A = E_A{}^M \partial_M + \frac{1}{2} \varphi_A{}^{cd} M_{dc} + \frac{1}{2} \varphi_A{}^{ij} Z_{ji} , \quad (4.3)$$

$$[D_A, D_B\} = T_{AB}{}^C D_C + \frac{1}{2} R_{AB}{}^{cd} M_{dc} + \frac{1}{2} F_{AB}{}^{ij} Z_{ji} , \quad (4.4)$$

where we have introduced the central charge generators $Z_{ij} = -Z_{ji}$, which are required to commute with all other generators:

$$[Z_{ij}, D_A] = [Z_{ij}, M_{cb}] = [Z_{ij}, Z_{kl}] = 0 \ . \tag{4.5}$$

Substituting equation (4.4) into equation (4.2) one finds five Bianchi identities from the requirement that the coefficients of $D_{\underline{\varepsilon}}, \bar{D}_{\underline{\dot\varepsilon}}, D_c, M_\Gamma$ and Z_{mn} vanish. These are respectively:

$$T_{\underline{\alpha\beta}}{}^{\underline{\gamma}} T_{\underline{\gamma\delta}}{}^{\underline{\varepsilon}} - D_{\underline{\delta}} T_{\underline{\alpha\beta}}{}^{\underline{\varepsilon}} + T_{\underline{\alpha\beta}}{}^{\underline{\dot\gamma}} T_{\underline{\delta\dot\gamma}}{}^{\underline{\varepsilon}} + R_{\underline{\alpha\beta\delta}}{}^{\varepsilon} \delta_l^m + P(\alpha i \to \beta j \to \delta l) = 0 \ , \tag{4.6}$$

$$T_{\underline{\alpha\beta}}{}^{\underline{\gamma}} T_{\underline{\gamma\delta}}{}^{\underline{\dot\varepsilon}} + T_{\underline{\alpha\beta}}{}^{\underline{\dot\gamma}} T_{\underline{\delta\dot\gamma}}{}^{\underline{\dot\varepsilon}} - D_{\underline{\delta}} T_{\underline{\alpha\beta}}{}^{\underline{\dot\varepsilon}} + P(\alpha i \to \beta j \to \delta l) = 0 \ , \tag{4.7}$$

$$T_{\underline{\alpha\beta}}{}^{\dot\gamma}{}_l \, C_{\delta\varepsilon} + T_{\underline{\beta\delta}}{}^{\dot\gamma}{}_i \, C_{\alpha\varepsilon} + T_{\underline{\delta\alpha}}{}^{\dot\gamma}{}_j \, C_{\beta\varepsilon} = 0 \ , \tag{4.8}$$

$$T_{\underline{\alpha\beta}}{}^{\underline{\gamma}} R_{\underline{\delta\gamma}}{}^{cd} + T_{\underline{\alpha\beta}}{}^{\underline{\dot\gamma}} R_{\underline{\delta\dot\gamma}}{}^{cd} - D_{\underline{\delta}} R_{\underline{\alpha\beta}}{}^{cd} + P(\underline{\alpha} \to \underline{\beta} \to \underline{\delta}) = 0 \ , \tag{4.9}$$

$$T_{\underline{\alpha\beta}}{}^{\underline{\gamma}} F_{\underline{\delta\gamma}}{}^{mn} + T_{\underline{\alpha\beta}}{}^{\underline{\dot\gamma}} F_{\underline{\delta\dot\gamma}}{}^{mn} - D_{\underline{\delta}} F_{\underline{\alpha\beta}}{}^{mn} + P(\underline{\alpha} \to \underline{\beta} \to \underline{\delta}) = 0 \ , \tag{4.10}$$

where P indicates that to the previous terms two more with the indices cyclically permuted should be added.

Denoting by $I_{ABC}{}^D$ the Bianchi identity that follows from equating to zero the coefficient of the D generator in the expansion of $[D_A,[D_B,D_C\}\}$,we have that the above identities are respectively: $I_{\underline{\alpha\beta\delta}}{}^{\underline{\varepsilon}}, I_{\underline{\alpha\beta\delta}}{}^{\underline{\dot\varepsilon}}, I_{\underline{\alpha\beta\delta}}{}^{c}, I_{\underline{\alpha\beta\delta}}{}^{\Gamma}$ and $I_{\underline{\alpha\beta\delta}}{}^{mn}$. From each triple commutator five identities follow. There are five additional triple commutators beyond the one given in equation (4.2); these are: $[[D_{\underline{\alpha}},D_{\underline{\beta}}\},\bar{D}_{\underline{\dot\delta}}]$,$[[D_{\underline{\alpha}},D_{\underline{\beta}}\},D_d]$, $[[D_{\underline{\alpha}},\bar{D}_{\underline{\dot\beta}}\},D_d]$,$[[D_a,D_b],D_{\underline{\delta}}]$ and $[[D_a,D_b],D_c]$. We therefore have thirty Bianchi identities (this number depends on the number of symmetries that are put into the tangent space).

Let us consider the dimensionality of the various objects. In units of mass, a vector derivative $[D_c]$ has dimension 1, and a spinorial derivative has dimension 1/2, $[D_{\underline{\alpha}}] = [D_{\underline{\dot{\alpha}}}] = \frac{1}{2}$, since spinor derivatives anticommute to give a vectorial derivative. Lorentz generators have dimension zero $[M_\Gamma] = 0$. In superspace it is natural to take bosons to have dimension zero (their canonical dimension is 1) and fermions to have dimension 1/2 (their canonical dimension is 3/2). In this way no explicit κ appears in the formulation. The graviton, however, is left at dimension 1. It then follows from $[D_a, D_b] = \cdots + \frac{1}{2} F_{ab}{}^{ij} Z_{ji}$, where $F_{ab}{}^{ij}$ is a vector field strength, that central charges have dimension one $[Z_{ij}] = 1$.

From the above considerations it follows that a torsion $T_{AB}{}^C$ has dimension $[A] + [B] - [C]$, for example, $T_{\underline{\alpha} b}{}^{\dot{\underline{\gamma}}}$ has dimension $[\underline{\alpha}] + [b] - [\dot{\underline{\gamma}}] = \frac{1}{2} + 1 - \frac{1}{2} = 1$. A curvature $R_{AB}{}^\Gamma$ has dimension $[A] + [B]$, and a field strength $F_{AB}{}^{mn}$ has dimension $[A] + [B] - 1$. Similarly Bianchi identities are classified by their dimensionality. An $I_{ABC}{}^D$ identity has dimensionality $[A] + [B] + [C] - [D]$. The lowest dimension tensors are of dimension zero $(F_{\underline{\alpha}\underline{\beta}}{}^{mn}, T_{\underline{\alpha}\underline{\dot{\beta}}}{}^c, ..)$. The highest dimension tensor is the dimension two curvature $R_{ab}{}^{cd}$. The lowest dimension Bianchi identities are of dimension one-half ($I_{\underline{\alpha}\underline{\beta}\underline{\delta}}{}^c, I_{\underline{\alpha}\underline{\beta}\underline{\delta}}{}^{mn}, \cdots$), and the highest-dimension identity is the $I_{abc}{}^\Gamma$ identity of dimension three. Bianchi identities are conveniently solved in order of increasing dimensionality. It should be emphasized that the expression 'solving a Bianchi identity' is not a misleading one. Bianchi identitites are identities if the field strengths are expressed in terms of the connections. This is precisely what one *does not do*. The field strengths have to be expressed in terms of a field strength superfield (that describes the multiplet of fields) and have to satisfy the Bianchi identities. This is a nontrivial problem.

Let us now turn to a discussion of the field strength superfield. Take the case of N = 4 supergravity, which is a theory that contains fields of all spins less than and equal to two. The N = 4 field strength superfield is a complex scalar superfield W of dimension zero. The non-gauge fields appear directly in the superfield W. Since W is complex it can have the scalar and pseudoscalar A and B at the lowest position of the superfield $W = A + iB + O(\theta)$. The spinors also appear explicitly in the W superfield; they are located in the $O(\theta)$ term. In N = 4 we have four Majorana spinors that can be represented by $\Lambda_{\alpha i}$, where α is the spinor index and i runs from 1 to 4. At order θ, we could have terms such as $\theta^{\alpha i}\Lambda_{\alpha i} + \overline{\theta}^{\dot{\alpha} i}\overline{\chi}_{\dot{\alpha} i}$, where Λ and χ are independent, since there is no reality condition on the superfield. This would mean that there are eight spinors rather than four. We therefore have to impose another condition: the superfield W should contain θ's only and no $\overline{\theta}$'s. This is called a chirality condition, which analytically reads $\overline{D}_{\dot{\underline{\alpha}}} W = 0$, and halves the number of degrees of freedom for the terms at order θ and higher.

Consider now the vector fields of N = 4, there are six real vectors $A_\mu^{[ij]}$ $i,j = 1,..4$. For these vectors one has the real field strengths $F_{\mu\nu}^{[ij]}$ and, according to the discussion of equations (2.13) and (2.14), they can be represented by the tensor $f_{\alpha\beta}{}^{ij}$ ($\overline{f}_{\dot{\alpha}\dot{\beta} ij}$ is its conjugate and carries no new information). The tensor $f_{\alpha\beta}{}^{ij}$ is symmetric in the α,β pair and is complex. Thus it contains six numbers, just as the antisymmetric μ,ν pair does. It is a dimension one object and fits into the θ^2 term of the W superfield in the form $\theta^{\alpha i}\theta^{\beta j} f_{\alpha\beta}{}^{ij}$.

In the same way as the vector fields only enter through their strengths, the gravitino itself $\overline{\psi}_{b,\dot{\alpha}}{}^{i}$ does not appear in the superspace tensors. Consider the gravitino field strength

$$\overline{\psi}_{ab,\dot{\gamma}}{}^{i} = \partial_a \overline{\psi}_{b,\dot{\gamma}}{}^{i} - \partial_b \overline{\psi}_{a,\dot{\gamma}}{}^{i} \,. \tag{4.11}$$

Translating into two component notation and expanding into irreducible pieces, one finds:

$$\overline{\psi}_{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\gamma}i} = \sigma^a{}_{\alpha\dot{\alpha}}\sigma^b{}_{\beta\dot{\beta}}\overline{\psi}_{ab,\dot{\gamma}i}$$

$$= C_{\alpha\beta}(\Sigma^i_{(\dot{\alpha}\dot{\beta}\dot{\gamma})} + C_{\dot{\alpha}\dot{\gamma}}\overline{\psi}^i_{\dot{\beta}} + C_{\dot{\beta}\dot{\gamma}}\overline{\psi}^i_{\dot{\alpha}}) + C_{\dot{\alpha}\dot{\beta}}\psi^i_{(\alpha\beta)\dot{\gamma}} \,. \qquad (4.12)$$

The original field strength represents twelve complex degrees of freedom (without counting the i index). In the above decomposition, we have $\Sigma_{(\dot{\alpha}\dot{\beta}\dot{\gamma})}$ with four complex degrees of freedom, $\overline{\psi}_{\dot{\beta}}$ with two complex degrees of freedom and $\psi_{(\alpha\beta)\dot{\gamma}}$ with six complex degrees of freedom. In the superspace formulation only the $\Sigma^i_{(\dot{\alpha}\dot{\beta}\dot{\gamma})}$ piece, called the 'gravitino Weyl strength,' appears. This piece of the gravitino field strength does not get determined by the equation of motion of the gravitino. Only a Bianchi identity determines it. The Weyl strength has dimension 3/2 and it appears at the power θ^3 in the W superfield in the form $\theta^{\alpha i}\theta^{\beta j}\theta^{\gamma k}C^{ijkp}\Sigma^p_{\alpha\beta\gamma}$.

Consider now the graviton. Its natural field strength is the Riemann curvature. The Riemann tensor R_{abcd} of general relativity has the following symmetries:

$$R_{abcd} = -R_{bacd} = -R_{abdc} \,, \qquad (4.13)$$

$$R_{abcd} = R_{cdab} \,. \qquad (4.14)$$

If the cyclic identity is also satisfied (which requires that the T_{abc} torsion be zero) one has:

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0 \,. \qquad (4.15)$$

These three algebraic restrictions imply that there are only 20 independent components for the Riemann tensor. The decomposition of Riemann into two component tensors was first given by Penrose [6]. We have:

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \sigma^a{}_{\alpha\dot{\alpha}}\sigma^b{}_{\beta\dot{\beta}}\sigma^c{}_{\gamma\dot{\gamma}}\sigma^d{}_{\delta\dot{\delta}}R_{abcd}$$

$$= -\frac{1}{4}\left[C_{\alpha\beta}(\overline{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} + C_{\dot{\alpha}\dot{\beta}}(\sigma^{ab})_{\alpha\beta}\right]\left[C_{\gamma\delta}(\overline{\sigma}^{cd})_{\dot{\gamma}\dot{\delta}} + C_{\dot{\gamma}\dot{\delta}}(\sigma^{cd})_{\gamma\delta}\right] R_{abcd} \,. \tag{4.16}$$

Expanding and using the reality of R one finds:

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = C_{\dot{\alpha}\dot{\beta}}C_{\dot{\gamma}\dot{\delta}}\,\chi_{\alpha\beta\gamma\delta} + C_{\alpha\beta}C_{\gamma\delta}\,\overline{\chi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$$

$$+ C_{\dot{\alpha}\dot{\beta}}C_{\gamma\delta}\,\varphi_{\alpha\beta\dot{\gamma}\dot{\delta}} + C_{\alpha\beta}C_{\dot{\gamma}\dot{\delta}}\,\overline{\varphi}_{\dot{\alpha}\dot{\beta}\gamma\delta} \,. \tag{4.17}$$

Consider first $\chi_{\alpha\beta\gamma\delta}$, it is symmetric in the $(\alpha\beta)$ and $(\gamma\delta)$ pairs, and in the exchange of both. It therefore represents 6 complex degrees of freedom. It can be expanded as:

$$\chi_{\alpha\beta\gamma\delta} = V_{(\alpha\beta\gamma\delta)} + (C_{\alpha\gamma}C_{\beta\delta} + C_{\alpha\delta}C_{\beta\gamma})A \,, \tag{4.18}$$

where $V_{(\alpha\beta\gamma\delta)}$ being totally symmetric represents 5 complex degrees of freedom; it is just the 10 component real Weyl curvature of general relativity. A is a complex scalar.

Consider now the $\varphi_{\alpha\beta\dot{\gamma}\dot{\delta}}$ piece. *A priori*, it represents 9 complex degrees of freedom, but there is a reality condition $\overline{\varphi_{\alpha\beta\dot{\gamma}\dot{\delta}}} = \varphi_{\gamma\delta\dot{\alpha}\dot{\beta}}$. Thus we only have nine real degrees of freedom; it is just the traceless Ricci tensor of general relativity. The scalar curvature, found contracting the Riemann tensor completely, equals $A + \overline{A}$. It is a real function, as one would expect.

So far we have reduced the number of degrees of freedom to 21. Using the constraint (4.15), one obtains one further condition. Following [6], equation (4.15) is equivalent to:

$$S_{abc}{}^{b} = 0 \,, \tag{4.19}$$

where:

$$S_{abcd} = R_{ab}{}^{gf}\,\varepsilon_{gfcd} \,. \tag{4.20}$$

Translating into two-component notation, using equation (A.20), and substituting the results in equation (4.17) and (4.18), one finds that the constraint (4.19) requires that A be real.

The Weyl curvature $V_{(\alpha\beta\gamma\delta)}$ has dimension two and is suitable for the θ^4 term in the W superfield as $\theta^{\alpha i}\theta^{\beta j}\theta^{\gamma k}\theta^{\delta l}\, C^{ijkl}\, V_{(\alpha\beta\gamma\delta)}$. The Weyl curvature is not determined by the equation of motion of the graviton, which just fixes the Ricci curvature. Only the Bianchi identities determine the Weyl curvature. In fact one can think of the Bianchi identities as field equations for the Weyl tensor giving the part of the curvature at a point that depends on the matter distribution elsewhere [7].

In summary, the W superfield contains the nongauge fields and the field strengths for N = 4 supergravity. At successive orders in θ we have: $A + iB$, $\Lambda_{\alpha i}$, $f_{\alpha\beta}{}^{ij}$, $\Sigma^i_{\alpha\beta\gamma}$ and $V_{\alpha\beta\gamma\delta}$. For N = 3 supergravity one needs a W_α spinor superfield, for N = 2, a $W_{(\alpha\beta)}$ superfield, and for N = 1 a $W_{(\alpha\beta\gamma)}$ superfield.

In the case of the maximally extended N = 8 supergravity the analog of the W superfield of N = 4 supergravity is a scalar superfield W^{ijkl} where i, j, k and l are antisymmetric SO(8) indices. At lowest order in θ this superfield would describe 70 scalars plus 70 pseudoscalars. Therefore a duality condition is required to halve this number of degrees of freedom and to obtain the 35 scalars plus 35 pseudoscalars of N = 8 supergravity.

5. The N = 3 Supergravity

The major technical obstacle in formulating extended supergravities in superspace consists in solving the Bianchi identities. In this section we want to show in detail how to solve these identities in the case of N = 3 supergravity. All of the techniques we will describe here are useful for the case of N = 4 supergravity. This latter case is, of course, more complicated, essentially due to the presence of scalar fields. This section is intended as a pedagogical example on how to analyze Bianchi identities.

The spectrum of the N = 3 theory is given by one graviton, three gravitinos, three vector fields and one spinor. The appropriate field strength superfield is a spinor W_α containing at θ^0 the physical spinor Λ_α, at order θ the field strength for the vectors $f_{\alpha\beta j}$, at order θ^2 the Weyl gravitino field strength $\Sigma^i_{\alpha\beta\gamma}$ and at order θ^3 the Weyl curvature $V_{\alpha\beta\gamma\delta}$. Let us also recall what are the available tensors. We have the SO(3) invariant tensors δ_i^j, ε_{ijk} ; and the Lorentz tensors $C_{\alpha\beta}$, $\sigma^a{}_{\alpha\dot\beta}$, $(\sigma^{ab})_{\alpha\beta}$, η_{ab}, ε_{abcd} (not all independent). All the superspace tensors have to be formed out of the fields in W_α and the above tensors.

Let us begin with the dimension zero objects. These tensors have to be constructed without any fields, since the lowest-dimension field available is the dimension 1/2 spinor Λ_α. One easily convinces oneself that it is not possible to construct the dimension zero $F_{\alpha i\dot\beta}{}^{j\,mn}$ and $T_{\alpha i\,\beta j}{}^c$ objects out of the above tensors. One is therefore forced to set:

$$F_{\alpha i\dot\beta}{}^{j\,mn} = 0\,, \quad T_{\alpha i\,\beta j}{}^c = 0\,. \tag{5.1}$$

Now consider the $F_{\alpha i\,\beta j}{}^{mn}$ field strength. It is symmetric in the exchange of the pair of indices (αi) and (βj) and antisymmetric in the pair (mn). We can therefore try to construct the combinations $[\alpha\beta][ij][mn]$ or $(\alpha\beta)(ij)[mn]$. It is not possible to create the $(\alpha\beta)$ term with the requirement of zero dimensionality. The first combination, however, can be easily constructed and we set:

$$F_{\alpha i\,\beta j}{}^{mn} = C_{\alpha\beta}\,\delta_i^{[m}\delta_j^{n]}\,. \tag{5.2}$$

There is one more dimension zero object, it is the $T_{\alpha i\,\dot{\beta}}{}^{j\,c}$ torsion. It is constructed as:

$$T_{\alpha i\,\dot{\beta}}{}^{j\,c} = 2i\,\delta_i^j\,\sigma^c{}_{\alpha\dot{\beta}}\,, \tag{5.3}$$

where the coefficient of 2 has been chosen for convenience. One can also fix the dimension one half torsion $T_{ab}{}^c$. It could only be proportional to a spinor. Since it appears in the supersymmetry of the graviton (see equation (3.28)) and we want to have the standard supersymmetry transformation ($\delta V_{a\mu} = -i\kappa\bar{\varepsilon}\gamma_a\psi_\mu$), we have to require:

$$T_{ab}{}^c = 0\,. \tag{5.4}$$

Let us now start considering the Bianchi identities. We start with the dimension one-half identities, and the first one we choose is $I_{\underline{\alpha}\underline{\beta}\underline{\gamma}}{}^c$:

$$T_{\alpha i\,\beta j}{}^{\dot{\gamma}}{}_l\,C_{\delta\varepsilon} + T_{\beta j\,\delta l}{}^{\dot{\gamma}}{}_i\,C_{\alpha\varepsilon} + T_{\delta l\,\alpha i}{}^{\dot{\gamma}}{}_j\,C_{\beta\varepsilon} = 0\,, \tag{5.5}$$

where only the dimension one half torsion $T_{\alpha i\,\beta j}{}^{\dot{\gamma}}{}_l$ appears. The $(\alpha\beta)\,(ij)$ piece of the torsion is set to zero by equation (5.5). Thus one can only have a $[\alpha\beta]\,[ij]$ piece. Equation (5.5), however, requires that the torsion be antisymmetric in the three isospin indices i, j and l. It is now easy to write an expression for the torsion; it can only be:

$$T_{\alpha i\,\beta j}{}^{\dot{\gamma}}{}_k = C_{\alpha\beta}\,\varepsilon_{ijk}\,\bar{\Lambda}^{\dot{\gamma}}\,. \tag{5.6}$$

Next equation to consider is $I_{\underline{\alpha}\underline{\beta}\underline{\delta}}{}^{mn}$, which reads:

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} F_{\underline{\delta}\underline{\gamma}}{}^{mn} + T_{\underline{\beta}\underline{\delta}}{}^{\underline{\gamma}} F_{\underline{\alpha}\underline{\gamma}}{}^{mn} + T_{\underline{\delta}\underline{\alpha}}{}^{\underline{\gamma}} F_{\underline{\beta}\underline{\gamma}}{}^{mn} = 0 . \qquad (5.7)$$

The field strength is already known from equation (5.2). The most general expression one can write for the torsion is $T_{\alpha i\, \beta j}{}^{\gamma}{}_{k} \sim C_{\alpha\beta}\, \varepsilon_{ijk}\, \Lambda^{\gamma}$. Using this expression, equation (5.7) leads to an inconsistency, so one is forced to set this torsion to zero:

$$T_{\alpha i\, \beta j}{}^{\gamma k} = 0 . \qquad (5.8)$$

Consider now the $I_{\underline{\alpha}\underline{\beta}\dot{\underline{\delta}}}{}^{c}$ identity. Using equations (5.4) and (5.8) one only has:

$$\delta^{\rho}_{\beta}\, T_{\alpha i\, \dot{\delta} l}{}^{\dot{\beta}}{}_{j} + \delta^{\rho}_{\alpha}\, T_{\beta j\, \dot{\delta} l}{}^{\dot{\beta}}{}_{i} = 0 . \qquad (5.9)$$

The only possible expression for this torsion is $T_{\alpha i\, \dot{\delta} l}{}^{\dot{\beta}}{}_{j} = \delta_{\dot{\delta}}^{\dot{\beta}}\, \varepsilon_{ilj}\, \Lambda_{\alpha}$, but the above equation rules it out. We are then forced to put:

$$T_{\alpha i\, \dot{\delta} l}{}^{\dot{\beta}}{}_{j} = 0 . \qquad (5.10)$$

We now turn to the last dimension one-half identity: $I_{\underline{\alpha}\underline{\beta}\dot{\underline{\delta}}}{}^{mn}$. It reads:

$$T_{\alpha i\, \beta j}{}^{\dot{\gamma} k}\, \overline{F}_{\dot{\gamma} k\, \dot{\delta} l}{}^{mn} + 2i\, \delta_{il}\, F_{\alpha\dot{\delta}\, \beta j}{}^{mn} + 2i\, \delta_{jl}\, F_{\beta\dot{\delta}\, \alpha i}{}^{mn} = 0 . \qquad (5.11)$$

Using equations (5.2) and (5.6) one readily finds:

$$F_{\alpha\dot{\delta}\, \beta j}{}^{mn} = \frac{i}{2}\, C_{\alpha\beta}\, \varepsilon^{jmn}\, \overline{\Lambda}_{\dot{\delta}} . \qquad (5.12)$$

This completes the solution of the dimension one half Bianchi identities.

Consider now the dimension one Bianchi identities. We start with $I_{\underline{\alpha}\underline{\beta}\underline{\delta}}{}^{\dot{\varepsilon}}$, which, given the results we have found so far, reads:

$$D_{\underline{\delta}}\, T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\varepsilon}}{}_{m} + D_{\underline{\alpha}}\, T_{\underline{\beta}\underline{\delta}}{}^{\dot{\varepsilon}}{}_{m} + D_{\underline{\beta}}\, T_{\underline{\delta}\underline{\alpha}}{}^{\dot{\varepsilon}}{}_{m} = 0 . \qquad (5.13)$$

Using equation (5.6), one finds:

$$\varepsilon_{ijm}\, D_{\alpha l}\, \overline{\Lambda}^{\dot{\varepsilon}} = \varepsilon_{jlm}\, D_{\alpha i}\, \overline{\Lambda}^{\dot{\varepsilon}} , \tag{5.14}$$

which in turn implies:

$$D_{\alpha i}\, \overline{\Lambda}^{\dot{\varepsilon}} = 0 , \tag{5.15}$$

a chirality condition on the spinor of the theory. Next consider the $I_{\underline{\alpha}\underline{\beta}\underline{\delta}}{}^{\Gamma}$ identity:

$$R_{\alpha i\, \beta j\, \delta\varepsilon}\, \delta_l^m + R_{\beta j\, \delta l\, \alpha\varepsilon}\, \delta_i^m + R_{\delta l\, \alpha i\, \beta\varepsilon}\, \delta_j^m = 0, \tag{5.16}$$

which has the unique solution:

$$R_{\alpha i\, \beta j\, \delta\varepsilon} = 0 . \tag{5.17}$$

Take now the $I_{\underline{\alpha}\underline{\beta} d}{}^{c}$ identity. After some rearrangement it reads:

$$C_{\delta\varepsilon}\, R_{\alpha i\, \beta j\, \dot{\varepsilon}\dot{\delta}} + 2iC_{\beta\varepsilon}\, T_{\delta\dot{\delta}\, \alpha i\, \dot{\varepsilon} j} + 2iC_{\alpha\varepsilon}\, T_{\delta\dot{\delta}\, \beta j\, \dot{\varepsilon} i} = 0 . \tag{5.18}$$

Both the curvature and the torsion are of dimension one. Therefore, to construct them one needs either two spinors or the vector field strength. Only the latter works. Setting:

$$R_{\alpha i\, \beta j\, \dot{\varepsilon}\dot{\delta}} = 2\, C_{\alpha\beta}\, \varepsilon_{ijk}\, \overline{f}_{\dot{\varepsilon}\dot{\delta} k} , \tag{5.19}$$

the torsion $T_{\delta\dot{\delta}\, \alpha i\, \dot{\varepsilon} j}$ can only be proportional to $C_{\delta\alpha}\, \varepsilon_{ijk}\, \overline{f}_{\dot{\varepsilon}\dot{\delta} k}$, and equation (5.18) fixes the constant of proportionality:

$$T_{\delta\dot{\delta}\, \alpha i\, \dot{\varepsilon} j} = -i\, C_{\delta\alpha}\, \varepsilon_{ijk}\, \overline{f}_{\dot{\varepsilon}\dot{\delta} k} . \tag{5.20}$$

Consider now the $I_{\underline{\alpha}\underline{\beta}\underline{\dot{\delta}}}{}^{\dot{\varepsilon}}$ identity. It reads:

$$-\overline{D}_{\dot{\delta} l}\, T_{\alpha i\, \beta j\, \dot{\varepsilon}}{}^{m} + R_{\alpha i\, \beta j\, \dot{\delta}\dot{\varepsilon}}\, \delta_l^m + 2i\delta_{il}\, T_{\alpha\dot{\delta}\, \beta j\, \dot{\varepsilon}}{}^{m} + 2i\delta_{jl}\, T_{\beta\dot{\delta}\, \alpha i\, \dot{\varepsilon}}{}^{m} = 0 . \tag{5.21}$$

Using equations (5.6), (5.19) and (5.20), one finds:

$$\overline{D}_{\dot{\delta} l}\,\overline{\Lambda}_{\dot{\varepsilon}} = 2\overline{f}_{\dot{\delta}\dot{\varepsilon} l}\ . \tag{5.22}$$

The next identity to consider is $I_{\underline{\alpha}\underline{\dot{\beta}}d}{}^{mn}$:

$$2i\delta_{ij}\,F_{\alpha\dot{\beta}\varepsilon\dot{\varepsilon}}{}^{mn} + T_{\varepsilon\dot{\varepsilon}\dot{\beta}j}{}^{\gamma k}\,F_{\gamma k\,\alpha i}{}^{mn} + T_{\varepsilon\dot{\varepsilon}\alpha i}{}^{\dot{\gamma} k}\,\overline{F}_{\dot{\gamma} k\,\dot{\beta} j}{}^{mn}$$

$$+ \overline{D}_{\dot{\beta} j}\,F_{\varepsilon\dot{\varepsilon}\alpha i}{}^{mn} + D_{\alpha i}\,\overline{F}_{\varepsilon\dot{\varepsilon}\dot{\beta} j}{}^{mn} = 0\ . \tag{5.23}$$

This equation determines the field strength $F_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{mn}$:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{mn} = \frac{1}{2}\left(C_{\alpha\beta}\overline{f}_{\dot{\alpha}\dot{\beta}p} + C_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta p}\right)\varepsilon_{pmn}\ . \tag{5.24}$$

We now turn to the $I_{\underline{\alpha}\underline{\beta}d}{}^{mn}$ identity, which after some simplification reads:

$$-\,T_{\alpha i\,\beta j}{}^{\dot{\gamma} k}\,\overline{F}_{\varepsilon\dot{\varepsilon}\dot{\gamma} k}{}^{mn} + T_{\varepsilon\dot{\varepsilon}\alpha i}{}^{\gamma k}\,F_{\gamma k\,\beta j}{}^{mn} + T_{\varepsilon\dot{\varepsilon}\beta j}{}^{\gamma k}\,F_{\gamma k\,\alpha i}{}^{mn} = 0. \tag{5.25}$$

Here the unknown is the torsion $T_{\varepsilon\dot{\varepsilon}\alpha i}{}^{\gamma k}$, which has to be made of two spinors, and since spinors carry no isospin index, there must be a Kronecker delta. We take $T_{\varepsilon\dot{\varepsilon}\alpha i\,\beta}{}^{m} = t_{\varepsilon\dot{\varepsilon}\alpha\beta}\,\delta_i^m$ and find that:

$$t_{\varepsilon\dot{\varepsilon}\alpha\beta} = \frac{i}{4}\Lambda_{\varepsilon}\overline{\Lambda}_{\dot{\varepsilon}}\,C_{\alpha\beta} + a\ C_{\varepsilon(\alpha}\Lambda_{\beta)}\overline{\Lambda}_{\dot{\varepsilon}}\ , \tag{5.26}$$

where a is a number that can take any value. It is convenient to choose $a = -i/4$. One then finds:

$$T_{\varepsilon\dot{\varepsilon}\alpha i\,\beta}{}^{j} = -\frac{i}{2}\,C_{\varepsilon\alpha}\Lambda_{\beta}\overline{\Lambda}_{\dot{\varepsilon}}\,\delta_i^j\ . \tag{5.27}$$

We now consider the $I_{\underline{\alpha}\underline{\beta}\underline{\dot{\delta}}}{}^{\varepsilon}$ identity:

$$T_{\alpha i\,\beta j}{}^{\dot{\gamma} k}\,T_{\dot{\gamma} k\,\dot{\delta}}{}^{l}{}_{\varepsilon}{}^{m} + 2i\delta_i^l\,T_{\alpha\dot{\delta}\,\beta j\,\varepsilon}{}^{m} + 2i\delta_j^l\,T_{\beta\dot{\delta}\,\alpha i\,\varepsilon}{}^{m}$$

$$+ R_{\alpha i\,\dot{\delta}}{}^{l}{}_{\beta\varepsilon}\,\delta_j^m + R_{\beta j\,\dot{\delta}}{}^{l}{}_{\alpha\varepsilon}\,\delta_i^m = 0\ . \tag{5.28}$$

With the previous values for the torsions, one immediately finds that the curvature vanishes:

$$R_{\alpha i\,\dot{\delta}}{}^{l}{}_{\beta\varepsilon} = 0\,. \tag{5.29}$$

This is a consequence of the choice made for the number a in equation (5.26).

The $I_{\alpha\dot{\beta}d}{}^{c}$ identity allows one to find the T_{abc} torsion, which in turn is necessary to find the spin connection ω_{abc} (see equation (3.33)):

$$2\delta_{ij}\, T_{\alpha\dot{\beta}\delta\dot{\delta}\varepsilon\dot{\varepsilon}} - 4C_{\dot{\beta}\dot{\varepsilon}}\, T_{\delta\dot{\delta}\,\alpha i\,\varepsilon j} - 4C_{\alpha\varepsilon}\, T_{\delta\dot{\delta}\,\dot{\beta} j\,\dot{\varepsilon} i} = 0\,. \tag{5.30}$$

Using equation (5.27) the above gives:

$$T_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = i\,[\,\delta_{\alpha}^{\beta}\delta_{\dot{\beta}}^{\dot{\gamma}}\Lambda_{\beta}\bar{\Lambda}_{\dot{\alpha}} - \delta_{\beta}^{\gamma}\delta_{\dot{\alpha}}^{\dot{\gamma}}\Lambda_{\alpha}\bar{\Lambda}_{\dot{\beta}}\,]\,. \tag{5.31}$$

This concludes the solution of the dimension one Bianchi identities. The information found so far enables one to construct all the supersymmetry transformations of the theory (using equations (3.28) to (3.31)). This will be done in the next chapter for the N = 4 theory.

We want to show here how to find all the superspace tensors, so we will continue with the analysis of the higher dimensional (and more complicated) identities. In solving these higher dimensional Bianchi identities we essentially obtain the equations of motion of all the fields in the theory. In getting the remaining information out of superspace, due to the enormous redundancy in the Bianchi identities, some shortcuts are possible. Consider, for example, evaluating a commutator on the spinor:

$$\begin{aligned}\{D_{\alpha i}, D_{\beta j}\}\Lambda_{\varepsilon} &= 2D_{\alpha i}\, f_{\beta\varepsilon j} + 2D_{\beta j}\, f_{\alpha\varepsilon i}\\ &= T_{\alpha i\,\beta j}{}^{A}\, D_{A}\Lambda_{\varepsilon} + \frac{1}{2}R_{\alpha i\,\beta j\,\gamma}{}^{\xi}\,[M_{\xi}{}^{\gamma}, \Lambda_{\varepsilon}]\,,\end{aligned} \tag{5.32}$$

where we have used equations (4.4) and (5.22). Using equation (5.8), (5.15) and (5.17) one arrives at:

$$D_{\alpha i} f_{\beta \varepsilon j} + D_{\beta j} f_{\alpha \varepsilon i} = 0\,, \tag{5.33}$$

which is solved by:

$$D_{\alpha i}\, f_{\beta \varepsilon j} = -\varepsilon_{ijk}\, \Sigma_{\alpha\beta\varepsilon k}\,, \tag{5.34}$$

where the $\Sigma_{\alpha\beta\varepsilon k}$ tensor is symmetric in $(\alpha\beta\varepsilon)$. This tensor corresponds to the Weyl gravitino field strength.

In order to determine the dimension 3/2 curvatures one considers the $I_{\underline{\alpha}\underline{\beta}\dot{\underline{\delta}}}{}^{\Gamma}$ identity. For $\Gamma = \dot{\varepsilon}\dot{\xi}$ one has:

$$-\bar{D}_{\dot{\delta}}\, R_{\alpha i\, \beta j\, \dot{\varepsilon}\dot{\xi}} + 2i\delta_i^j\, R_{\alpha\dot{\delta}\,\beta j\, \dot{\varepsilon}\dot{\xi}} + 2i\,\delta_j^l\, R_{\beta\dot{\delta}\, \alpha i\, \dot{\varepsilon}\dot{\xi}} = 0\,, \tag{5.35}$$

from which one finds:

$$R_{\alpha i\, \beta\dot{\beta}\, \dot{\gamma}\dot{\delta}} = -iC_{\alpha\beta}\Sigma_{\dot{\beta}\dot{\gamma}\dot{\delta} i}\,. \tag{5.36}$$

For $\Gamma = \varepsilon\xi$, the identity reads:

$$T_{\alpha i\, \beta j}{}^{\dot{\gamma}k}\, R_{\dot{\gamma}k\, \dot{\delta}l\, \varepsilon\xi} + 2i\,\delta_{il}\, R_{\alpha\dot{\delta}\,\beta j\, \varepsilon\xi} + 2i\,\delta_{jl}\, R_{\beta\dot{\delta}\, \alpha i\, \varepsilon\xi} = 0\,, \tag{5.37}$$

and yields:

$$R_{\alpha i\, \beta\dot{\beta}\, \gamma\delta} = iC_{\alpha\beta} f_{\gamma\delta i}\, \bar{\Lambda}_{\dot{\beta}}\,. \tag{5.38}$$

The $I_{\underline{\alpha}\underline{\beta}\underline{d}}{}^{\dot{\underline{\varepsilon}}}$ identity reads:

$$\begin{aligned} &- D_{\delta\dot{\delta}}\, T_{\alpha i\, \beta j}{}^{\dot{\varepsilon}m} - T_{\alpha i\, \beta j}{}^{\dot{\gamma}k}\, T_{\delta\dot{\delta}\, \dot{\gamma}k}{}^{\dot{\varepsilon}m} + T_{\delta\dot{\delta}\, \alpha i}{}^{\gamma k}\, T_{\gamma k\, \beta j}{}^{\dot{\varepsilon}m} \\ &+ T_{\delta\dot{\delta}\, \beta j}{}^{\gamma k}\, T_{\gamma k\, \alpha i}{}^{\dot{\varepsilon}m} + D_{\beta j}\, T_{\delta\dot{\delta}\, \alpha i}{}^{\dot{\varepsilon}m} + D_{\alpha i}\, T_{\delta\dot{\delta}\, \beta j}{}^{\dot{\varepsilon}m} = 0\,. \end{aligned} \tag{5.39}$$

After using the values we have for the above tensors, this equation becomes:

$$-C_{\alpha\beta}\varepsilon_{ijm}\, D_{\delta\dot{\delta}}\bar{\Lambda}_{\dot{\varepsilon}} - iC_{\delta\alpha}\,\varepsilon_{imp}\, D_{\beta j}\, \bar{f}_{\dot{\delta}\dot{\varepsilon}p} - iC_{\delta\beta}\,\varepsilon_{jmp}\, D_{\alpha i}\, \bar{f}_{\dot{\delta}\dot{\varepsilon}p} = 0\,. \tag{5.40}$$

The $[\dot{\delta}\dot{\varepsilon}]$ antisymmetric part of the above equation leads to:

$$D_{\delta\dot{\delta}}\overline{\Lambda}^{\dot{\delta}} = 0 \,, \tag{5.41}$$

which is the equation of motion for the spin one half piece. The $(\dot{\delta}\dot{\varepsilon})$ symmetric piece leads to:

$$D_{\delta i}\,\overline{f}_{\dot{\delta}\dot{\varepsilon}j} = \frac{i}{2}\,\delta_{ij}\,D_{\delta(\dot{\delta}}\overline{\Lambda}_{\dot{\varepsilon})} \,. \tag{5.42}$$

The dimension 3/2 torsion gets determined from the $I_{ab\,\underline{a}}{}^{c}$ identity:

$$-4iC_{\delta\varepsilon}\,T_{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\varepsilon}l} - D_{\delta l}\,T_{\alpha\dot{\alpha}\beta\dot{\beta}\varepsilon\dot{\varepsilon}} - 2C_{\dot{\beta}\dot{\varepsilon}}\,R_{\delta l\,\alpha\dot{\alpha}\beta\varepsilon} - 2C_{\beta\varepsilon}\,R_{\delta l\,\alpha\dot{\alpha}\dot{\beta}\dot{\varepsilon}}$$

$$+\, 2C_{\dot{\alpha}\dot{\varepsilon}}\,R_{\delta l\cdot\beta\dot{\beta}\alpha\varepsilon} + 2C_{\alpha\varepsilon}\,R_{\delta l\,\beta\dot{\beta}\dot{\alpha}\dot{\varepsilon}} = 0 \,, \tag{5.43}$$

which after a small calculation yields:

$$T_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma}{}^{k} = -\frac{1}{2}\,C_{\dot{\alpha}\dot{\beta}}\Sigma_{\alpha\beta\gamma}{}^{k} + \frac{1}{2}\,C_{\alpha\beta}\Lambda_{\gamma}\overline{f}_{\dot{\alpha}\dot{\beta}}{}^{k} \,. \tag{5.44}$$

This is the highest dimension torsion and it can be used to derive the gravitino field equation (Chapter III, Section 2).

Evaluating $\{D_{\alpha i}, D_{\beta j}\}\,f_{\gamma\delta k}$ one finds:

$$\varepsilon_{jkp}\,D_{\alpha i}\,\Sigma_{\beta\gamma\delta p} + \varepsilon_{ikp}\,D_{\beta j}\,\Sigma_{\alpha\gamma\delta p} = -\frac{i}{2}\,C_{\alpha\beta}\,\varepsilon_{ijk}\,\overline{\Lambda}^{\dot{\varepsilon}}\,D_{(\gamma\dot{\varepsilon}}\Lambda_{\delta)} \,, \tag{5.45}$$

and using equation (A.13), one finds:

$$D_{\alpha i}\,\Sigma_{\beta\gamma\delta}{}^{j} = \delta_i^j\,[\,V_{\alpha\beta\gamma\delta} + \frac{i}{8}\,C_{\alpha(\beta}\,D_{\gamma\dot{\varepsilon}}\,\Lambda_{\delta)}\overline{\Lambda}^{\dot{\varepsilon}}\,] \,, \tag{5.46}$$

where the symmetrization is over β,γ and δ with no extra factors (six terms), and the totally symmetric Weyl curvature tensor $V_{\alpha\beta\gamma\delta}$ has been introduced.

The $I_{\underline{\alpha}\underline{\beta}d}{}^{\Gamma}$ identity with $\Gamma = \dot{\varepsilon}\dot{\xi}$ reads:

$$T_{\alpha i\,\beta j}{}^{\dot{\gamma}k}\,R_{\dot{\gamma}k\,\delta\dot{\delta}\,\dot{\varepsilon}\dot{\xi}} - D_{\delta\dot{\delta}}\,R_{\alpha i\,\beta j\,\dot{\varepsilon}\dot{\xi}} + T_{\delta\dot{\delta}\,\alpha i}{}^{\gamma k}\,R_{\gamma k\,\beta j\,\dot{\varepsilon}\dot{\xi}} + T_{\delta\dot{\delta}\,\beta j}{}^{\gamma k}\,R_{\gamma k\,\alpha i\,\dot{\varepsilon}\dot{\xi}}$$

$$- D_{\beta j} R_{\alpha i \, \delta \dot{\delta} \dot{\varepsilon} \dot{\xi}} - D_{\alpha i} R_{\beta j \, \delta \dot{\delta} \dot{\varepsilon} \dot{\xi}} = 0 \,, \tag{5.47}$$

after simplification one gets

$$-2C_{\alpha\beta}\varepsilon_{ijk} D_{\delta\dot{\delta}} \overline{f}_{\dot{\varepsilon}\dot{\xi}k} - iC_{\alpha\delta} D_{\beta j} \Sigma_{\dot{\delta}\dot{\varepsilon}\dot{\xi}i} - iC_{\beta\delta} D_{\alpha i} \Sigma_{\dot{\delta}\dot{\varepsilon}\dot{\xi}j} = 0 \,. \tag{5.48}$$

This equation yields two pieces of information. Using equation (A.15) one finds

$$\overline{D}_{\delta}{}^{\dot{\eta}} \overline{f}_{\dot{\eta}\dot{\varepsilon}i} = 0 \,, \tag{5.49}$$

which is the equation of motion for the vector fields, and

$$\overline{D}_{\dot{\alpha} i} \Sigma_{\beta\gamma\delta j} = -\frac{i}{3} \varepsilon_{ijk} D_{(\beta\dot{\alpha}} f_{\gamma\delta)k} \,. \tag{5.50}$$

We can now turn to the highest dimension tensor, the dimension two curvature $R_{\alpha\dot{\alpha}\beta\dot{\beta}\varepsilon\xi}$, which appears in the $I_{\underline{\alpha}\underline{\dot{\beta}}d}{}^{\Gamma}$ identity

$$2i\delta_{ij} R_{\alpha\dot{\beta}\delta\dot{\delta}\varepsilon\xi} + T_{\delta\dot{\delta}\alpha i}{}^{\dot{\gamma}k} R_{\dot{\gamma}k\,\dot{\beta}j\,\varepsilon\xi} - \overline{D}_{\dot{\beta}j} R_{\alpha i\,\delta\dot{\delta}\varepsilon\xi} - D_{\alpha i} R_{\dot{\beta}j\,\delta\dot{\delta}\varepsilon\xi} = 0 \,. \tag{5.51}$$

The only unknown in this equation is the desired curvature. After some calculation, one finds

$$\begin{aligned} R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta} &= -\frac{1}{2} C_{\dot{\alpha}\dot{\beta}} [\, V_{\alpha\beta\gamma\delta} - \frac{i}{8} C_{(\gamma|(\alpha} D_{\beta)\dot{\varepsilon}} \Lambda_{\delta)} \overline{\Lambda}^{\dot{\varepsilon}}] \\ &+ \frac{1}{2} C_{\alpha\beta} [\, 2f_{\gamma\delta m} \overline{f}_{\dot{\alpha}\dot{\beta}m} + \frac{1}{4} D_{(\gamma(\dot{\alpha}} \Lambda_{\delta)} \overline{\Lambda}_{\dot{\beta})}] \,. \end{aligned} \tag{5.52}$$

With this curvature, we have finished the calculation of the superspace tensors. The superspace geometry, however, is complete only if we also know how to take spinorial derivatives on all the covariant tensors since these determine their supersymmetry transformations (see equation (3.31)). We already know how to take $D_{\alpha i}$ or $\overline{D}_{\dot{\alpha} i}$ on Λ_{α}, $f_{\alpha\beta i}$, $\Sigma_{\alpha\beta\gamma i}$, but not yet on the $V_{\alpha\beta\gamma\delta}$ tensor. This information is found from the dimension 5/2 $I_{ab\,\underline{a}}{}^{\Gamma}$ identity and the result is given below.

It is convenient for further reference to list all the superspace tensors together:

A) SO(3) Central Charge Field Strength $F_{AB}{}^{mn}$

$$F_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta}\,\delta_i^{[m}\delta_j^{n]}\,, \quad F_{\underline{\alpha}\underline{\dot\beta}}{}^{mn} = 0\,,$$

$$F_{\alpha\dot\alpha\underline{\beta}}{}^{mn} = \frac{i}{2}\,C_{\alpha\beta}\,\varepsilon_{jmn}\,\overline{\Lambda}_{\dot\alpha}\,,$$

$$F_{\alpha\dot\alpha\beta\dot\beta}{}^{mn} = \frac{1}{2}\left(C_{\alpha\beta}\,\overline{f}_{\dot\alpha\dot\beta p} + C_{\dot\alpha\dot\beta}\,f_{\alpha\beta p}\right)\varepsilon_{pmn}\,. \qquad (5.53)$$

B) Torsion $T_{AB}{}^{C}$

$$T_{\underline{\alpha}\underline{\beta}}{}^{\gamma\dot\gamma} = 0\,, \quad T_{\underline{\alpha}\underline{\dot\beta}}{}^{\gamma\dot\gamma} = i2\delta_i^j\,\delta_\alpha^\gamma\,\delta_{\dot\beta}^{\dot\gamma}\,, \quad T_{\alpha\beta\dot\beta}{}^{\gamma\dot\gamma} = 0\,,$$

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\dot\gamma}} = C_{\alpha\beta}\,\varepsilon_{ijk}\,\overline{\Lambda}^{\dot\gamma}\,, \quad T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = 0\,, \quad T_{\underline{\alpha}\underline{\dot\beta}}{}^{\underline{\gamma}} = 0\,,$$

$$T_{\alpha\dot\alpha\underline{\beta}}{}^{\underline{\dot\gamma}} = -iC_{\alpha\beta}\,\varepsilon_{jkp}\,F_{\dot\alpha}{}^{\dot\gamma}{}_{p}\,, \quad T_{\alpha\dot\alpha\underline{\beta}}{}^{\underline{\delta}} = -\frac{i}{2}\,C_{\alpha\beta}\,\Lambda^\delta\,\overline{\Lambda}_{\dot\alpha}\,\delta_{jk}\,,$$

$$T_{\alpha\dot\alpha\beta\dot\beta}{}^{\gamma\dot\gamma} = i\left[\,\delta_\alpha^\beta\delta_{\dot\beta}{}^{\dot\gamma}\Lambda_\beta\overline{\Lambda}_{\dot\alpha} - \delta_\beta^\gamma\delta_{\dot\alpha}{}^{\dot\gamma}\Lambda_\alpha\overline{\Lambda}_{\dot\beta}\,\right],$$

$$T_{\alpha\dot\alpha\beta\dot\beta}{}^{\underline{\gamma}} = -\frac{1}{2}\,C_{\dot\alpha\dot\beta}\,\Sigma_{\alpha\beta}{}^{\gamma k} + \frac{1}{2}\,C_{\alpha\beta}\,\Lambda^\gamma\,\overline{f}_{\dot\alpha\dot\beta}{}^{k}\,. \qquad (5.54)$$

C) Curvatures $R_{AB\,\gamma\delta}$, $R_{AB\,\dot\gamma\dot\delta}$:

$$R_{\underline{\alpha}\underline{\beta}\gamma\delta} = 0\,, \quad R_{\underline{\alpha}\underline{\beta}\dot\gamma\dot\delta} = 2\,C_{\alpha\beta}\,\varepsilon_{ijk}\,\overline{f}_{\dot\gamma\dot\delta k}\,, \quad R_{\underline{\alpha}\underline{\dot\beta}\gamma\delta} = 0\,,$$

$$R_{\underline{\alpha}\beta\dot\beta\gamma\delta} = iC_{\alpha\beta}\,f_{\gamma\delta i}\,\overline{\Lambda}_{\dot\beta}\,, \quad R_{\underline{\alpha}\beta\dot\beta\dot\gamma\dot\delta} = -iC_{\alpha\beta}\,\Sigma_{\dot\beta\dot\gamma\dot\delta i}\,,$$

$$R_{\alpha\dot\alpha\beta\dot\beta\gamma\delta} = -\frac{1}{2}\,C_{\dot\alpha\dot\beta}\left[\,V_{\alpha\beta\gamma\delta} - \frac{i}{8}\,C_{(\gamma|(\alpha}D_{\beta)\dot\varepsilon}\Lambda_{\delta)}\overline{\Lambda}^{\dot\varepsilon}\right]$$

$$+ \frac{1}{2} C_{\alpha\beta} [2 f_{\gamma\delta m} \bar{f}_{\dot{\alpha}\dot{\beta} m} + \frac{1}{4} D_{(\gamma(\dot{\alpha}} \Lambda_{\delta)} \bar{\Lambda}_{\dot{\beta})}] \,. \qquad (5.55)$$

D) Constituency Equations

$$\bar{D}_{\dot{\alpha}}{}^i \Lambda_\beta = 0 \,, \quad D_{\alpha i} \Lambda_\beta = 2 f_{\alpha\beta i} \,,$$

$$\bar{D}_{\dot{\alpha}}{}^i f_{\beta\gamma j} = \frac{i}{2} \delta_j^i D_{(\beta\dot{\alpha}} \Lambda_{\gamma)} \,, \quad D_{\alpha i} f_{\beta\gamma j} = -\varepsilon_{ijk} \Sigma_{\alpha\beta\gamma k} \,,$$

$$\bar{D}_{\dot{\alpha}}{}^i \Sigma_{\beta\gamma\delta}{}^j = -\frac{i}{3} \varepsilon^{ijk} D_{(\beta\dot{\alpha}} f_{\gamma\delta)}{}^k \,,$$

$$D_{\alpha i} \Sigma_{\beta\gamma\delta j} = \delta_{ij} [V_{\alpha\beta\gamma\delta} + \frac{i}{8} C_{\alpha(\beta} D_{\gamma\dot{\varepsilon}} \Lambda_{\delta)} \bar{\Lambda}^{\dot{\varepsilon}}] \,,$$

$$\bar{D}_{\dot{\alpha}}{}^i V_{\beta\gamma\delta\varepsilon} = \frac{1}{12} [i (D_{(\beta\dot{\alpha}} + \frac{i}{2} \Lambda_{(\beta} \bar{\Lambda}_{\dot{\alpha}}) \Sigma_{\gamma\delta\varepsilon)}{}^i + \varepsilon^{ijk} f_{(\beta\gamma}{}^j f_{\delta\varepsilon)}{}^k \bar{\Lambda}_{\dot{\alpha}}] \,,$$

$$D_{\alpha i} V_{\beta\gamma\delta\varepsilon} = -\frac{1}{12} C_{\alpha(\beta} [i D_{\gamma\dot{\varepsilon}} (f_{\delta\varepsilon) i} \bar{\Lambda}^{\dot{\varepsilon}}) - \frac{1}{2} f_{\gamma\delta i} \Lambda_{\varepsilon)} \bar{\Lambda}_{\dot{\varepsilon}} \bar{\Lambda}^{\dot{\varepsilon}}] \,. \qquad (5.56)$$

This completes our discussion of the N = 3 superspace geometry. The above results can be reproduced by taking the appropriate truncation of the N = 4 superspace geometry, and are therefore a good check on the N = 4 expressions. It is quite easy to truncate the N = 3 superspace geometry to obtain the N = 2 superspace geometry. One just puts the spinors to zero and lets $f_{\alpha\beta i} \to f_{\alpha\beta}$ and $\varepsilon_{ijk} \to \varepsilon_{ij}$ throughout.

References for Chapter II

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Chapter III. Ungauged N = 4 Supergravities

1. Introduction and Summary

The first construction of the full nonlinear N = 4 supergravity was quite difficult [1]. The main complication with respect to N < 4 supergravities was due to the presence of scalar fields which appear nonpolynomially and eliminate the possibility of a step by step construction of the action and transformation laws.

The SO(4) supergravity was constructed requiring U(4) invariance for the equations of motion, parity conservation, and minimal coupling of the scalars to gravity [2]. It is in principle possible that relaxation of the above requirements could lead to other forms of N = 4 supergravity. It is important to understand how restrictive the requirement of local supersymmetry is. One would also like to understand what is the principle that determines the functional form in which the scalars enter into N = 4 supergravity, and whether other forms are possible. When dimensionless coupling constants are introduced into the known forms of N = 4 supergravity, the specific functional forms in which the scalars appear lead to unbounded and inverted potentials [3,4,5]. The same pathology exists in the N > 4 gauged supergravities, simply because they contain the N = 4 theory as a truncation. Although some of the critical points in these potentials have been shown to lead to completely stable background solutions [6,7], investigating how much freedom there is in the functional forms involving the scalars would allow one to determine whether or not other potentials are possible.

Superspace methods provide a convenient framework for dealing with this problem. The most general N = 4 supergravity theory corresponds to the general solution of the superspace Bianchi identities, since these identities are equivalent to the requirement of a consistent local version of the super-Poincare algebra. Moreover, the only assumptions in the superspace construction lie in the choice of constraints and the choice of global symmetry. The rest follows rigorously from the system of Bianchi identities in which every unknown tensor is decomposed into irreducible parts compatible with the available field representations and covariance. By relaxing the constraints one can search for more N = 4 supergravities. With superspace methods one looks for a consistent local algebra or a set of supersymmetry transformations on fields, and for equations of motion. The existence of an action is a separate problem that can be dealt with later, in contrast with component formulations, in which action and transformation laws are generally worked out simultaneously .

The search for other forms of N = 4 supergravity is the main subject of this chapter. We find that the SO(4) and SU(4) supergravities are essentially unique and that all the constraints that have been imposed on the superspace formulation are not necessary. Constraints are just a convenience for N = 4 supergravity, and correspond to a gauge choice. That is, *N = 4 supergravity theories follow from their known field representations and the unconstrained Bianchi identities*. So far, in working with N = 4 supergravity in superspace [8], the idea has been to postulate some constraints in order to reproduce the component formulation and then to use the Bianchi identities. We now show that it is not necessary to postulate or guess constraints. If one solves the Bianchi identities without constraints, the resulting theory, although far more complicated looking, would still be equivalent to the known theories up to some field redefinitions. It is likely that the same will be true for all the extended supergravities including the N = 8 theory.

Theories that are equivalent up to field redefinitions at the ungauged level could in principle become inequivalent or distinct when the internal symmetry group is gauged. This does not happen with field redefinitions involving Weyl rescaling of the graviton, rescaling of the Fermi fields, mixing of the Fermi fields, and scalar manifold redefinitions. Internal duality transformations on the vector fields, however, are transformations that can affect the gauged theories, and they are discussed in Chapter IV. It follows from the above considerations that we are unable to generate new scalar potentials beyond those known now [3,4,5] by a study of the ungauged theories. It is possible, however, that different ways of gauging could lead to new scalar potentials.

In Section 2 the superspace geometries that accommodate the SO(4) and SU(4) supergravities are constructed [8,9]. This is done by postulating some constraints in order to reproduce the component formulation and using the Bianchi identities. It is emphasized that the only essential difference between the two known N = 4 supergravity theories lies in the central-charge field-strength sector of superspace.

In Section 3 we start relaxing the constraints on the superspace formulation. The two functions U and $\bar{V}$ that determine the central charge field strength tensor $F_{\underline{\alpha}\underline{\beta}}{}^{mn}$ are left unspecified. It is shown that the Bianchi identities force them to take their known values for the SO(4) and SU(4) supergravities up to some field redefinitions. For the case of the SO(4) supergravity, scalar field inversions and internal space duality transformations on the vector fields are necessary.

In Section 4 contact is made with the component formulation. The full nonlinear supersymmetry transformations are obtained from the superspace geometry for U and $\bar{V}$ functions slightly more general than those of the known theories. For the known values of U and $\bar{V}$, they reduce to the expressions given before [1,10].

In Section 5 we proceed step by step to relax all the constraints that had been imposed on the superspace geometry. This allows us to complete the proof that the N = 4 theories follow uniquely from their field content and the Bianchi identities with no further constraints. It is shown that constraints only amount to field redefinitions, including Weyl rescaling of the graviton. That is, in the unconstrained forms the scalars are coupled nonminimally to gravity. A particular choice for the Weyl rescaling function allows us to write the SU(4) supergravity in a form in which the scalar fields appear polynomially in the action.

Finally in section 6, we conclude by discussing some possibilities that remain open for attempting to construct a new N = 4 supergravity theory.

2. On-Shell N = 4 Superspace Supergravity

The superspace formulation of the N = 4 supergravity theories was first given in Reference [8]. These geometries were later reformulated [9] in a way that made the relation between the two known N = 4 theories more transparent. In this section we review the superspace construction of these theories.

The SO(4) supergravity was discovered first [1]. Its spectrum consists of a graviton e_μ^a, 4 real gravitinos ψ_μ^i with $i = 1,2..4$, 6 vectors A_μ^{ij}, 4 real spinors χ^i, a scalar A and a pseudoscalar B. The lagrangian is SO(4) invariant. The equations of motion, however, have a larger symmetry, namely SU(4) ⊗ SU(1,1). The scalar fields appear in the lagrangian and transformation laws in a very complicated nonpolynomial fashion, and the theory is only defined in the range $|A|^2 + |B|^2 < 1$.

The SU(4) supergravity was found later [10]. Its spectrum consists of a graviton e_μ^a, 4 real gravitinos ψ_μ^i, 3 vectors A_μ^n, 3 axial vectors B_μ^n, 4 real spinors χ^i, a scalar φ and a pseudoscalar B. The lagrangian is SU(4) invariant, and the equations of motion are SU(4) ⊗ SU(1,1) invariant. The scalar nonpolynomiality is simpler than in the SO(4) theory, since φ appears nonpolynomially only through exponentials $e^{\kappa\varphi}$. The pseudoscalar B enters only through its spacetime derivative.

For the SO(4) theory the supercovariant derivative is taken to be of the form [8]:

$$D_A = E_A{}^M \partial_M + \frac{1}{2}\varphi_{A\gamma}{}^{\delta} M_\delta{}^\gamma + \frac{1}{2}\overline{\varphi}_{A\dot\gamma}{}^{\dot\delta}\, \overline{M}_{\dot\delta}{}^{\dot\gamma} + \frac{1}{2}\varphi_A{}^{ij} Z_{ji} \ , \tag{2.1}$$

where $E_A{}^M$, $\varphi_{A\gamma\delta}$, $\overline{\varphi}_{A\dot\gamma\dot\delta}$ and $\varphi_A{}^{ij}$ are the inverse vielbein, the Lorentz connections and the central-charge connection superfields. The central-charge generator $Z_{ij} = -Z_{ji} = \overline{Z_{ij}}$ annihilates all superfields. Torsion, curvatures and field strengths are defined by graded commutation:

$$[D_A, D_B\} = T_{AB}{}^C D_C + \frac{1}{2} R_{AB\gamma}{}^\delta M_\delta{}^\gamma + \frac{1}{2} \overline{R}_{AB\dot{\gamma}}{}^{\dot{\delta}} \overline{M}_{\dot{\delta}}{}^{\dot{\gamma}} + \frac{1}{2} F_{AB}{}^{ij} Z_{ji} . \tag{2.2}$$

For the SU(4) theory a complex central-charge generator is used [8]:

$$D_A = E_A{}^M \partial_M + \frac{1}{2}\varphi_{A\gamma}{}^\delta M_\delta{}^\gamma + \frac{1}{2}\overline{\varphi}_{A\dot{\gamma}}{}^{\dot{\delta}} \overline{M}_{\dot{\delta}}{}^{\dot{\gamma}} + \frac{1}{2}\varphi_A{}^{ij} Z_{ji} + \frac{1}{2}\overline{\varphi}_{Aij} \overline{Z}^{ji} , \tag{2.3}$$

$$[D_A, D_B\} = T_{AB}{}^C D_C + \frac{1}{2} R_{AB\gamma}{}^\delta M_\delta{}^\gamma + \frac{1}{2} \overline{R}_{AB\dot{\gamma}}{}^{\dot{\delta}} \overline{M}_{\dot{\delta}}{}^{\dot{\gamma}}$$

$$+ \frac{1}{2} G_{AB}{}^{ij} Z_{ji} + \frac{1}{2} \widetilde{G}_{ABij} \overline{Z}^{ij} . \tag{2.4}$$

Here both the central-charge generator Z_{ij} and the central-charge connection superfield $\varphi_A{}^{ij}$ are complex objects. The central charge field strengths G and $\widetilde{G}$ are a priori independent. It would therefore appear that one has 12 vector fields. We shall see, however, that this number is reduced to 6 by a duality condition.

Let us proceed to the construction of the superspace geometry. For the same reasons as in our discussion of the N = 3 supergravity we set:

$$F_{\underline{\alpha}\underline{\dot{\beta}}}{}^{mn} = 0 , \quad T_{\underline{\alpha}\underline{\beta}}{}^{\gamma\dot{\gamma}} = 0 ,$$

$$T_{\underline{\alpha}\underline{\dot{\beta}}}{}^{\gamma\dot{\gamma}} = 2i\delta_i^j \delta_\alpha^\gamma \delta_{\dot{\beta}}^{\dot{\gamma}} , \quad T_{\underline{\alpha}\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = 0 , \tag{2.5}$$

In the same way as we fixed the number a in N = 3 (equation (5.26)), here it is convenient to require:

$$R_{\alpha i\,\dot{\beta}}{}^{i}{}_{\gamma\delta} = -\frac{1}{2} C_{\alpha(\gamma}\Lambda_{\delta)i} \overline{\Lambda}_{\dot{\beta}}{}^{i} , \tag{2.6}$$

where $\Lambda_{\delta i}$ is the spinor of the theory, which first appears in the superspace torsion $T_{\alpha i\,\beta j}{}^{\dot{\gamma}}{}_k$ (see equation (5.6) of Chapter II). We take the expression for this torsion as the definition for the spinor:

$$\bar{\Lambda}^{\dot{\gamma} l} = \frac{1}{12} C^{\alpha\beta} C^{ijkl} T_{\alpha i \beta j}{}^{\dot{\gamma}}{}_{k} \; . \qquad (2.7)$$

Finally, we turn to the central-charge field strength $F_{\underline{\alpha}\underline{\beta}}{}^{mn}$. It is a function of the scalars only and it determines the way the vectors rotate under supersymmetry into the gravitinos (equation (3.30) of Chapter II). One can therefore read off the value for this tensor from the component formulation [1]:

$$F_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta} [\, U \delta_i^{[m} \delta_j^{n]} + \bar{V} C_{ij}{}^{mn} \,] \, , \qquad (2.8)$$

where:

$$U = \frac{1}{\sqrt{1 - W\bar{W}}} \, , \quad \bar{V} = \frac{\bar{W}}{\sqrt{1 - W\bar{W}}} \, , \quad W = -A + iB + O(\theta) \; . \qquad (2.9)$$

For the SU(4) supergravity [10] we read off the following values (the detailed connection with component results is given in Section 4):

$$G_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta} \, U \, \delta_i^{[m} \delta_j^{n]} \, , \quad \tilde{G}_{\underline{\alpha}\underline{\beta}\, mn} = C_{\alpha\beta} \, \bar{V} \, C_{ijmn} \; , \qquad (2.10)$$

where:

$$U = \bar{V} = \frac{1}{\sqrt{1 - W - \bar{W}}} \, , \quad W = \frac{1}{2} (1 - e^{-2\varphi} - 2iB) \; . \qquad (2.11)$$

The above relations fix the starting point for the construction of the superspace formulation. All other tensors follow from the solution of the Bianchi identities. In the following we list the results.

(A.1) SO(4) Central-Charge Field Strength $F_{AB}{}^{mn}$

$$F_{\underline{\alpha},\underline{\beta}}{}^{mn} = C_{\alpha\beta} [\, U \, \delta_i^{[m} \delta_j^{n]} + \bar{V} \, C_{ij}{}^{mn} \,] \, , \qquad F_{\underline{\alpha},\underline{\dot{\beta}}}{}^{mn} = 0 \, ,$$

$$F_{a\underline{\dot{\alpha}},\underline{\beta}}{}^{mn} = -i \, \frac{1}{2} \, C_{\alpha\beta} [\, V \bar{\Lambda}_{\dot{\alpha}}{}^{[m} \delta_j^{n]} + \bar{U} \bar{\Lambda}_{\dot{\alpha}}{}^{i} \, C_{ij}{}^{mn} \,] \, ,$$

$$F_{\alpha\dot{\alpha},\beta\dot{\beta}}{}^{mn} = \frac{1}{2}\, C_{\alpha\beta}[\bar{U}\,\bar{f}_{\dot{\alpha}\dot{\beta}}{}^{mn} + \frac{1}{2}\, V\,\bar{f}_{\dot{\alpha}\dot{\beta}\,ij}\, C^{ijmn}]$$

$$+ \frac{1}{2}\, C_{\dot{\alpha}\dot{\beta}}[U\, f_{\alpha\beta}{}^{mn} + \frac{1}{2}\, \bar{V} f_{\alpha\beta}{}^{ij}\, C_{ij}{}^{mn}]\,,$$

$$W \equiv -A + iB\,, \quad U = \frac{1}{\sqrt{1-|W|^2}}, \quad V = \frac{W}{\sqrt{1-|W|^2}}\,. \tag{2.12}$$

Introducing a matrix $\mathbf{E}_{ij}{}^{mn}$ defined by

$$\mathbf{E}_{ij}{}^{mn} = \frac{1}{2}\,[U\,\delta_i^{[m}\delta_j^{n]} + \bar{V}\, C_{ij}{}^{mn}]\,, \tag{2.13}$$

it can be seen that the internal space SU(4) symmetry breaking in the SO(4) theory occurs only via the presence of $\mathbf{E}_{ij}{}^{mn}$.

(A.2) SU(4) Central-Charge Field Strength Strengths $G_{AB}{}^{mn}$ and $\tilde{G}_{AB\,mn}$

$$G_{\underline{\alpha},\underline{\beta}}{}^{mn} = C_{\alpha\beta} U\, \delta_i^{[m}\delta_j^{n]}, \quad G_{\underline{\alpha},\underline{\dot{\beta}}}{}^{mn} = 0\,,$$

$$\tilde{G}_{\underline{\alpha},\underline{\beta}\,mn} = C_{\alpha\beta} \bar{V}\, C_{ijmn}\,, \quad \tilde{G}_{\underline{\alpha},\underline{\dot{\beta}}}{}^{mn} = 0\,,$$

$$G_{\alpha\dot{\alpha},\underline{\beta}}{}^{mn} = -i\, \frac{1}{2}\, C_{\alpha\beta} V \bar{\Lambda}_{\dot{\alpha}}{}^{[m}\,\delta_j^{n]}\,,$$

$$\tilde{G}_{\alpha\dot{\alpha},\underline{\beta}\,mn} = -i\, \frac{1}{2}\, C_{\alpha\beta} \bar{U}\, \bar{\Lambda}_{\dot{\alpha}}{}^{i}\, C_{ijmn}\,,$$

$$G_{\alpha\dot{\alpha},\beta\dot{\beta}}{}^{mn} = \frac{1}{2}\,[U\, C_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}{}^{mn} + \frac{1}{2}\, V\, C_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}\,ij}\, C^{ijmn}]\,,$$

$$\tilde{G}_{\alpha\dot{\alpha},\beta\dot{\beta}\,mn} = \frac{1}{2}\,[\bar{U}\, C_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}\,mn} + \frac{1}{2}\, \bar{V}\, C_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}{}^{ij}\, C_{ijmn}]\,,$$

$$W \equiv \frac{1}{2}\,(1 - e^{-2\varphi} - i2B)\,, \quad U = V = \frac{1}{\sqrt{1-W-\bar{W}}}\,. \tag{2.14}$$

The equality of U and $\overline{V}$ is very important since it implies that

$$\widetilde{G}_{AB\,ij} = \frac{1}{2}\, C_{ijmn}\, G_{AB}{}^{mn} \; . \tag{2.15}$$

This duality relation between the two *a priori* independent field strengths implies that there are *only* six spin-one fields present in the theory.

Note that without taking index position into account, we find the relation $F_{AB}{}^{mn} = G_{AB}{}^{mn} + \widetilde{G}_{AB\,mn}$. In an SO(4) theory the position of an isospin index is immaterial. This allows one to construct a hybrid version of SO(4) type solution of the Bianchi identities, denoted as the SO(4)′ solution. One choses the field strength as in (2.12), but takes U and $\overline{V}$ as in (2.14). An SO(4)′ solution is a useful formal tool to deal with an SU(4) type solution within the framework of an SO(4) solution, as we shall show in Section 3. An SO(4)′ solution, however, cannot be realized in terms of component fields, since it would require an unacceptable internal space duality condition on the spin-one fields of the theory.

The superspace torsions, curvatures and constituency equations for both the SO(4) and SU(4) theories are the same in terms of U, $\overline{V}$ and W and we can list them together. In order to do so, we introduce the auxiliary variables a, $Q_{\alpha i}$ and $Q_{\alpha\dot{\alpha}}$ defined by

$$a = \frac{1}{|\,U\,|^2} \, , \qquad Q_{\alpha i} = \frac{\overline{V}}{4U}\, \Lambda_{\alpha i} \tag{2.16}$$

and

$$Q_{\alpha\dot{\alpha}} = i \, \mathrm{Im}\Big(\frac{1}{U}\, \frac{\partial U}{\partial W}\, D_{\alpha\dot{\alpha}} W\Big) \, . \tag{2.17}$$

For the SO(4) theory using equation (2.12) we have:

$$SO(4): \quad Q_{\alpha i} = \frac{W}{4}\, \Lambda_{\alpha i} \; , \quad Q_{\alpha\dot{\alpha}} = \frac{1}{4}\, \frac{\overline{W} \overleftrightarrow{D}_{\alpha\dot{\alpha}} W}{1 - W\overline{W}} \; , \tag{2.18}$$

and for the SU(4) theory using equation (2.14) we have:

$$SU(4): \quad Q_{\alpha i} = \frac{1}{4}\Lambda_{\alpha i}\,, \quad Q_{\alpha\dot{\alpha}} = -\frac{i}{2}\, e^{2\varphi} D_{\alpha\dot{\alpha}} B\,. \tag{2.19}$$

It is also convenient to define an operator $\Delta_{\alpha\dot{\alpha}}$ by the equation:

$$\Delta_{\alpha\dot{\alpha}} = D_{\alpha\dot{\alpha}} + 3\, Q_{\alpha\dot{\alpha}}\,, \tag{2.20}$$

where $D_{\alpha\dot{\alpha}}$ acts to the right and $Q_{\alpha\dot{\alpha}}$ acts by simple multiplication. We can now list the torsions, curvatures and constituency equations for the two theories. In the following results, one should use the appropriate U, $\bar{V}$ and W values for each theory.

(B) Torsion $T_{AB}{}^{C}$

$$T_{\underline{\alpha},\underline{\beta}}{}^{\gamma\dot{\gamma}} = 0, \qquad T_{\underline{\alpha},\underline{\dot{\beta}}}{}^{\gamma\dot{\gamma}} = i2\,\delta_i^j \delta_\alpha^\gamma \delta_{\dot{\beta}}^{\dot{\gamma}}\,,$$

$$T_{\underline{\alpha},\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = 0\,, \qquad T_{\underline{\alpha},\underline{\beta}}{}^{\underline{\dot{\gamma}}} = C_{\alpha\beta} C_{ijkl} \bar{\Lambda}^{\dot{\gamma}l}\,,$$

$$T_{\underline{\alpha},\underline{\beta}}{}^{\underline{\gamma}} = Q_{\underline{\alpha}} \delta_\beta^\gamma \delta_j^k + Q_{\underline{\beta}} \delta_\alpha^\gamma \delta_i^k\,,$$

$$T_{\underline{\alpha},\underline{\dot{\beta}}}{}^{\underline{\dot{\gamma}}} = -Q_{\underline{\alpha}} \delta_{\dot{\beta}}{}^{\dot{\gamma}} \delta_k^j\,, \qquad T_{\alpha\dot{\alpha},\underline{\beta}}{}^{\underline{\dot{\gamma}}} = -iC_{\alpha\beta} \bar{f}_{\dot{\alpha}}{}^{\dot{\gamma}}{}_{jk}\,,$$

$$T_{\alpha\dot{\alpha},\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = i[\delta_\beta^\gamma \delta_{\dot{\alpha}}{}^{\dot{\gamma}} \bar{\Lambda}_{\dot{\beta}}{}^{i} \Lambda_{\alpha i} - \delta_\alpha^\gamma \delta_{\dot{\beta}}{}^{\dot{\gamma}} \bar{\Lambda}^{i}{}_{\dot{\alpha}} \Lambda_{\beta i}]\,,$$

$$T_{\alpha\dot{\alpha},\underline{\beta}}{}^{\underline{\delta}} = Q_{\alpha\dot{\alpha}} \delta_\beta^\delta \delta_j^k + \frac{i}{2}\, C_{\alpha\beta} \Lambda^{\delta}{}_{m} \bar{\Lambda}_{\dot{\alpha}}{}^{[k} \delta_j{}^{m]}\,,$$

$$T_{\alpha\dot{\alpha},\beta\dot{\beta}}{}^{\underline{\gamma}} = -\frac{1}{4}\, C_{\dot{\alpha}\dot{\beta}}[2\Sigma_{\alpha\beta}{}^{\gamma k} + i\,\frac{1}{3}\, a^{-1} \delta_{(\alpha}^{\gamma} (D_{\beta)\dot{\gamma}} W) \bar{\Lambda}^{\dot{\gamma}k}]$$

$$+ \frac{1}{4}\, C_{\alpha\beta}[C^{klmn} \Lambda_l^\gamma \bar{f}_{\dot{\alpha}\dot{\beta}mn} + ia^{-1}(D^{\gamma}{}_{(\dot{\alpha}} W) \bar{\Lambda}_{\dot{\beta})}{}^{k}]\,. \tag{2.21}$$

(C) Curvature $R_{AB\gamma\delta}$, $R_{AB\dot\gamma\dot\delta}$

$$R_{\underline{\alpha},\underline{\beta}\gamma\delta} = 0, \quad R_{\underline{\alpha},\underline{\beta}\dot\gamma\dot\delta} = 2\,C_{\alpha\beta}\bar f_{\dot\gamma\dot\delta ab}\ , \quad R_{\underline{\alpha},\dot{\underline{\beta}}\gamma\delta} = -\,\frac{1}{2}\,C_{\alpha(\gamma}\Lambda_{\delta)i}\bar\Lambda_{\dot\beta}{}^{j}\ ,$$

$$R_{\underline{\alpha},\beta\dot\beta\gamma\delta} = i\,\frac{1}{2}\,C_{\alpha\beta}C_{ijkl}\bar\Lambda_{\dot\beta}{}^{j} f_{\gamma\delta}{}^{kl} - \frac{1}{2}\,a^{-1}C_{\alpha(\gamma}\Lambda_{\delta)i}(D_{\beta\dot\beta}\bar W)\ ,$$

$$R_{\underline{\alpha},\beta\dot\beta\dot\gamma\dot\delta} = -i\,C_{\alpha\beta}\Sigma_{\dot\beta\dot\gamma\dot\delta i} + \frac{1}{12}\,a^{-1}C_{\dot\beta(\dot\gamma}(D_{\varepsilon\dot\delta)}\bar W)\Lambda^{\varepsilon}{}_{i}\,C_{\alpha\beta} - \frac{1}{4}\,a^{-1}C_{\dot\beta(\dot\gamma}(D_{(\alpha|\dot\delta)}\bar W)\Lambda_{\beta)i}\ ,$$

$$R_{\alpha\dot\alpha,\beta\dot\beta\gamma\delta} = -\,\frac{1}{2}\,C_{\dot\alpha\dot\beta}[V_{\alpha\beta\gamma\delta} - \frac{i}{8}\,C_{(\gamma|(\alpha}(\Delta_{\beta)\dot\varepsilon}\Lambda_{|\delta)j})\bar\Lambda^{\dot\varepsilon j} + \frac{1}{6}\,a^{-2}C_{\alpha(\gamma}C_{\delta)\beta}(D_{\varepsilon\dot\delta}W)(D^{\varepsilon\dot\delta}\bar W)]$$

$$+\,\frac{1}{2}\,C_{\alpha\beta}[f_{\gamma\delta}{}^{ij}\bar f_{\dot\alpha\dot\beta ij} + \frac{i}{4}\,(\Delta_{(\gamma(\dot\alpha}\Lambda_{\delta)j})\bar\Lambda^{j}_{\dot\beta)} - \frac{1}{2}\,a^{-2}(D_{(\gamma(\dot\alpha}\bar W)(D_{\delta)\dot\beta)}W)]. \qquad (2.22)$$

(D) Constituency Equations

$$\bar D_{\dot\alpha}{}^{i}W = 0, \quad D_{\alpha i}W = a\,\Lambda_{\alpha i}\ ,$$

$$\bar D_{\dot\alpha}{}^{i}\Lambda_{\beta j} = i2\,\delta_j^{\,i}a^{-1}D_{\beta\dot\alpha}W + 3\,\bar Q_{\dot\alpha}{}^{i}\Lambda_{\beta j}\ ,$$

$$D_{\alpha i}\Lambda_{\beta j} = C_{ijkl}f_{\alpha\beta}{}^{kl} - 3\,Q_{\alpha i}\Lambda_{\beta j}\ ,$$

$$\bar D_{\dot\alpha}{}^{i}f_{\beta\gamma}{}^{jk} = 2\,\bar Q_{\dot\alpha}{}^{i}f_{\beta\gamma}{}^{jk} + \frac{1}{2}\,C^{jkmn}[i\delta_m^{\,i}(\Delta_{(\beta\dot\alpha}\Lambda_{\gamma)n}) + \bar\Lambda_{\dot\alpha}{}^{i}\Lambda_{\beta m}\Lambda_{\gamma n}]\ ,$$

$$D_{\alpha i}f_{\beta\gamma}{}^{jk} = \delta_i^{[j}\Sigma_{\alpha\beta\gamma}{}^{k]} - 2\,Q_{\alpha i}f_{\beta\gamma}{}^{jk} + i\,\frac{1}{3}\,\delta_i^{[j}C_{\alpha(\beta}a^{-1}(D_{\gamma)\dot\delta}W)\bar\Lambda^{\dot\delta k]}\ ,$$

$$\bar D_{\dot\alpha}{}^{i}\Sigma_{\beta\gamma\delta}{}^{j} = \bar Q_{\dot\alpha}{}^{i}\Sigma_{\beta\gamma\delta}{}^{j} - \frac{1}{3}\,\bar\Lambda_{\dot\alpha}{}^{i}\Lambda_{(\beta k}f_{\gamma\delta)}{}^{jk} - i\,\frac{1}{6}\,C^{ijkl}a^{-1}(D_{(\beta\dot\alpha}\bar W)\Lambda_{\gamma k}\Lambda_{\delta)l}$$

$$+i\ \frac{1}{3}\ [D_{(\beta\dot{\alpha}}+2\,Q_{(\beta\dot{\alpha}}]f_{\gamma\delta)}{}^{ij}\ ,$$

$$D_{\alpha i}\Sigma_{\beta\gamma\delta}{}^{j} = \delta_i^j V_{\alpha\beta\gamma\delta} - Q_{\alpha i}\Sigma_{\beta\gamma\delta}{}^{j} - \frac{1}{6}\ \delta_i^j a^{-2}(D_{(\beta|\dot{\delta}}\overline{W})(D_{|\gamma|}{}^{\dot{\varepsilon}}W)C_{\delta)\alpha}$$

$$-i\ \frac{1}{8}\ \delta_i^j(\Delta_{(\beta|\dot{\delta}}\Lambda_{|\gamma|k})C_{\delta)\alpha}\overline{\Lambda}^{\dot{\delta}k} + i\ \frac{1}{6}\ (\Delta_{(\beta|\dot{\delta}}\Lambda_{|\gamma|i})C_{\delta)\alpha}\overline{\Lambda}^{\dot{\delta}j}\ ,$$

$$\overline{D}_{\dot{\alpha}}{}^{i}\,V_{\beta\gamma\delta\varepsilon} = \frac{1}{12}\ [i(D_{(\beta\dot{\alpha}}+Q_{(\beta\dot{\alpha}}+i\ \frac{1}{2}\ \Lambda_{(\beta k}\overline{\Lambda}_{\dot{\alpha}}{}^{k})\Sigma_{\gamma\delta\varepsilon)}{}^{i}$$

$$-(C_{klmn}\overline{\Lambda}_{\dot{\alpha}}{}^{l}f_{(\beta\gamma}{}^{mn}+i2\,a^{-1}(D_{(\beta\dot{\alpha}}\overline{W})\Lambda_{\gamma k})f_{\delta\varepsilon)}{}^{ki}+\overline{\Lambda}_{\dot{\alpha}}{}^{i}\Lambda_{(\beta k}\Sigma_{\gamma\delta\varepsilon)}{}^{k}]\ ,$$

$$D_{\alpha i}\,V_{\beta\gamma\delta\varepsilon} = -\ \frac{1}{12}\ C_{\alpha(\beta}[(D_{\gamma\dot{\varepsilon}}-Q_{\gamma\dot{\varepsilon}})(i\ \frac{1}{2}\ C_{ilmn}\overline{\Lambda}^{\dot{\varepsilon}l}f_{\delta\varepsilon)}{}^{mn}-a^{-1}\Lambda_{\delta i}D_{\varepsilon)}{}^{\dot{\varepsilon}}\overline{W})$$

$$+(\frac{1}{4}\ C_{ilmn}\overline{\Lambda}_{\dot{\varepsilon}}{}^{l}f_{\gamma\delta}{}^{mn}-i\ \frac{1}{2}\ a^{-1}(D_{\gamma\dot{\varepsilon}}\overline{W})\Lambda_{\delta i})\Lambda_{\varepsilon)k}\overline{\Lambda}^{\dot{\varepsilon}k}]\ . \qquad (2.23)$$

To conclude this section let us show how the fermionic equations of motion can be derived from the superspace geometry. The fermionic equations of motion were found to be very useful in the construction of the SO(4) supergravity Lagrangian, because they can always be written in terms of supercovariant derivatives [1]. Since our formalism is based on a supercovariant derivative, the derivation of the equations of motion is quite straightforward. In solving for $\overline{D}_{\dot{\alpha}}{}^{i}f_{\beta\gamma}{}^{jk}$ we find that it is necessary to set $\Delta_{\delta\dot{\beta}}\Lambda^{\delta p} = 0$. Explicitly, for the SU(4) theory this implies

$$(D_{\delta\dot{\beta}} - \frac{3}{2}\,i\,e^{2\varphi}D_{\delta\dot{\beta}}B)\Lambda^{\delta}{}_{p} = 0\ . \qquad (2.24)$$

This equation is readily translated to four-component notation (Appendix A) and after numerical rescaling (discussed in Section 4), we get:

$$i\,\gamma^{\mu}\widehat{D}_{\mu}\Lambda^{p} - \frac{3}{2}\,\kappa\, e^{2\kappa\varphi}\widehat{D}_{\mu}B\gamma_5\gamma^{\mu}\Lambda^{p} = 0 \tag{2.25}$$

which is precisely the equation of motion for the spin one-half field given in [10]. For the gravitino we remark that the θ independent part of the superspace torsion $T_{ab}{}^{\gamma}(T_{\alpha\dot{\alpha},\beta\dot{\beta}}{}^{\gamma})$ is by definition the supercovariant gravitino field strength $\psi_{ab}{}^{\gamma}$. Its value is given in equation (2.21). To get the gravitino field equation, we only have to evaluate $\varepsilon^{cd\,ab}\gamma_5\gamma_d T_{ab}{}^{i}$. As expected, the totally symmetric Weyl gravitino field strength $\Sigma_{\alpha\beta\gamma}{}^{i}$ drops out of the equation and we obtain:

$$\varepsilon^{cd\,ab}\gamma_5\gamma_d\psi_{ab}{}^{i} - \frac{i\kappa}{2\sqrt{2}}\, e^{-\kappa\varphi}\widehat{C}^{ij}_{ab}\sigma^{ab}\gamma^{c}\Lambda^{j}$$

$$+ \frac{1}{2}\,\kappa(\widehat{D}_d\varphi + i\,\gamma_5 e^{2\kappa\varphi}\widehat{D}_d B)\gamma^{d}\gamma^{c}\Lambda^{i} = 0\,. \tag{2.26}$$

After rescaling (Section 4), this equation agrees precisely with the one given in [10].

3. Search for more N = 4 Supergravity Theories

Let us examine the constraints that were imposed on the superspace formulation in order to reproduce the known SO(4) supergravity. From Section 2 we have:

$$F_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta}[U\delta_i^{[m}\delta_j^{n]} + \bar{V}C_{ij}{}^{mn}] ,$$

$$F_{\underline{\alpha}\underline{\dot{\beta}}}{}^{mn} = 0 , \quad T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}l} = 0 ,$$

$$T_{\underline{\alpha}\underline{\dot{\beta}}}{}^{\gamma l} = 2i\delta_i^j\delta_\alpha^\gamma\delta_{\dot{\beta}}^{\dot{\gamma}} , \quad T_{\underline{\alpha}\beta\dot{\beta}}{}^{\dot{\gamma}l} = 0 ,$$

$$R_{\alpha i\dot{\beta}}{}^{i}{}_{\gamma\delta} = -\frac{1}{2}C_{\alpha(\gamma}\Lambda_{\delta)i}\bar{\Lambda}_{\dot{\beta}}{}^{i} . \tag{3.1}$$

where the spinor $\bar{\Lambda}^{\dot{\gamma}l}$ entering the above curvature is defined by:

$$\bar{\Lambda}^{\dot{\gamma}l} = \frac{1}{12}C^{\alpha\beta}C^{ijkl}\,T_{\alpha i\,\beta j}{}^{\dot{\gamma}}{}_k , \tag{3.2}$$

and U and $\bar{V}$ are specific functions of the on-shell chiral field strength W:

$$U = \frac{1}{\sqrt{1 - W\bar{W}}} , \quad \bar{V} = \frac{\bar{W}}{\sqrt{1 - W\bar{W}}} . \tag{3.3}$$

An obvious question arises from this: Why must these particular functions be chosen ? One answer to this question has been given by Cremmer and Julia [12]. These authors have asserted that U and $\bar{V}$ are determined by an algebraic principle which assumes that these functions can be used to define a matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} U & \bar{V} \\ V & \bar{U} \end{bmatrix} \tag{3.4}$$

and this matrix is an element of the group SU(1,1) in its two dimensional representation (det $\mathbf{V} = |U|^2 - |V|^2 = 1$). The elements U and $\bar{V}$ of the $\mathbf{V}$

matrix take the values given in (3.3) in the symmetric gauge, that is when only the physical scalars A and B are present. The scalars in N = 4 supergravity are described by an element of SU(1,1) in its two dimensional representation. This element represents three scalars out of which only two are physical due to a local U(1) invariance. Since $\mathbf{V}$ is an element of SU(1,1) it follows that $(D_A \mathbf{V})\mathbf{V}^{-1}$ is invariant under global SU(1,1) transformations. An explicit calculation yields:

$$(D_{\alpha i}\mathbf{V})\mathbf{V}^{-1} = \begin{bmatrix} 2Q_{\alpha i} & 0 \\ \Lambda_{\alpha i} & -2\,Q_{\alpha i} \end{bmatrix} \tag{3.5}$$

$$(D_{\alpha\dot{\alpha}}\mathbf{V})\mathbf{V}^{-1} = \begin{bmatrix} 2Q_{\alpha\dot{\alpha}} & a^{-1}D_{\alpha\dot{\alpha}}\overline{W} \\ a^{-1}D_{\alpha\dot{\alpha}}W & -2\,Q_{\alpha\dot{\alpha}} \end{bmatrix} \tag{3.6}$$

where $Q_{\alpha i}$ and $Q_{\alpha\dot{\alpha}}$ were defined in equations (2.16) and (2.17). According to the general discussion of reference [12] $Q_{\alpha\dot{\alpha}}$ is a gauge field for the local U(1). The presence of this U(1) group is implicit in many of the results of section 2. For example, defining a new supercovariant derivative $\widetilde{D}_A$, given by:

$$\widetilde{D}_A = D_A + Q_A\,Y\,, \tag{3.7}$$

where Y is a generator of U(1) (chirality) transformations, allows for some simplification in the superspace torsions listed in equation (2.21). Note that we have formulated the superspace geometry without a local U(1) symmetry in the tangent space. The local groups observed in N ≥ 4 supergravities are used for convenience and the supergeometries can be constructed without them.

The SU(4) theory, however, does not fit simply in the elegant construction of Cremmer and Julia. When the functions U and $\overline{V}$ are chosen as in equation

(2.14) the matrix **V** is singular. This suggests the need for a more general structure for the construction of N = 4 supergravity. Superspace itself is such a structure.

Starting from equations (3.1) and (3.2), and the spectrum of states for the N = 4 supergravity without auxiliary fields, we look for a more general solution to the Bianchi identities. This well defined problem amounts to finding solutions to a set of differential equations in superspace. We do not assume equations (3.3), instead U and $\bar{V}$ are now completely arbitrary functions of the on-shell chiral field strength W:

$$U = U(W,\bar{W}) \,, \quad \bar{V} = \bar{V}(W,\bar{W}) \,. \tag{3.8}$$

The strategy at this point is to search for differential equations, arising from the Bianchi identities, which force a specific functional dependence on U and $\bar{V}$. It is already known that at least two such solutions exist, namely the SO(4) theory with U and $\bar{V}$ given in equation (3.3) and the SU(4) theory with $U = \bar{V} = \sqrt{1 - W - \bar{W}}$.

It is important to note that, although the constraints in equations (3.1) ((3.2) is only a definition) correspond to SO(4) type theories, we are searching for more general SU(4) theories within the same formalism *simultaneously*. This is because when U equals $\bar{V}$ the solution we have is actually an SO(4)' type solution (see below equation (2.15)), from which the SU(4) theory can be easily found by splitting the central-charge field-strength tensors $F_{AB}{}^{mn}$ into $G_{AB}{}^{mn}$ and $\tilde{G}_{ABmn}$.

In this section we do not attempt to find the most general superspace solution (this is postponed to section 5). We keep the constraints given in equations (3.1), and furthermore, we do not write the most general expression for the $T_{\underline{\alpha}\underline{\beta}}{}^{\gamma}$ torsion; instead we assume:

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = \omega_{\underline{\alpha}}\, \delta_{\beta}^{\gamma}\delta_{j}^{k} + \omega_{\underline{\beta}}\, \delta_{\alpha}^{\gamma}\delta_{i}^{k} \, . \tag{3.9}$$

That is, this torsion keeps the form it has for the known theories (equation (2.21)). The spinor $\omega_{\underline{\alpha}}$, however, is not assumed to be equal to the spinor $Q_{\underline{\alpha}}$.

Analyzing the superspace Bianchi identities up to dimension one, we find the following values for field strengths, torsions and curvatures:

A) Field Strength $F_{AB}{}^{mn}$:

$$F_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta}[U\, \delta_{i}^{[m}\delta_{j}^{n]} + \bar{V}\, C_{ij}{}^{mn}\,] \, , \quad F_{\underline{\alpha}\dot{\underline{\beta}}}{}^{mn} = 0 \, ,$$

$$F_{\alpha\dot{\alpha}\underline{\beta}}{}^{mn} = -\frac{i}{2}\, C_{\alpha\beta}[\, V\bar{\Lambda}_{\dot{\alpha}}{}^{[m}\, \delta_{j}^{n]} + \bar{U}\bar{\Lambda}_{\dot{\alpha}}{}^{i}\, C_{ij}{}^{mn}\,] \, ,$$

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{mn} = \frac{1}{2}\, C_{\alpha\beta}(\overline{Uf}\,_{\dot{\alpha}\dot{\beta}}{}^{mn} + \frac{1}{2}\, V\bar{f}\,_{\dot{\alpha}\dot{\beta}ij}\, C^{ijmn}\,)$$

$$+ \frac{1}{2}\, C_{\dot{\alpha}\dot{\beta}}(Uf\,_{\alpha\beta}{}^{mn} + \frac{1}{2}\, \bar{V}f\,_{\alpha\beta}{}^{ij}\, C_{ij}{}^{mn}) \, . \tag{3.10}$$

B) Torsion $T_{AB}{}^{C}$:

$$T_{\underline{\alpha}\underline{\beta}}{}^{\gamma\dot{\gamma}} = 0 \, , \quad T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{\gamma\dot{\gamma}} = 2i\, \delta_{i}^{j}\delta_{\alpha}^{\gamma}\delta_{\dot{\beta}}^{\dot{\gamma}} \, ,$$

$$T_{\underline{\alpha}\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = 0 \, , \quad T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\underline{\gamma}}} = C_{\alpha\beta}\, C_{ijkl}\, \bar{\Lambda}^{\dot{\gamma}l} \, ,$$

$$T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = \omega_{\underline{\alpha}}\delta_{\beta}^{\gamma}\delta_{j}^{k} + \omega_{\underline{\beta}}\delta_{\alpha}^{\gamma}\delta_{i}^{k} \, ,$$

$$T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{\underline{\gamma}} = -\bar{\omega}_{\dot{\underline{\beta}}}\, \delta_{\alpha}^{\gamma}\delta_{i}^{k} \, , \quad T_{\alpha\dot{\alpha}\underline{\beta}\dot{\gamma}k} = -iC_{\alpha\beta}\bar{f}\,_{\dot{\alpha}\dot{\gamma}jk} \, .$$

$$T_{\alpha\dot{\alpha}\underline{\beta}}{}^{\underline{\delta}} = \frac{i}{8}\delta_{\beta}^{\delta}\delta_{j}^{k}\Pi_{\alpha\dot{\alpha}} + \frac{i}{2}C_{\alpha\beta}\Lambda^{\delta}{}_{p}\bar{\Lambda}_{\dot{\alpha}}{}^{[k}\delta_{j}^{p]} ,$$

$$T_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{\gamma\dot{\gamma}} = i\,[\delta_{\beta}^{\gamma}\delta_{\dot{\alpha}}^{\dot{\gamma}}\bar{\Lambda}_{\dot{\beta}}{}^{i}\Lambda_{\alpha i} - \delta_{\alpha}^{\gamma}\delta_{\dot{\beta}}^{\dot{\gamma}}\bar{\Lambda}_{\dot{\alpha}}{}^{i}\Lambda_{\beta i}\,] . \tag{3.11}$$

C) Curvature $R_{AB\gamma\delta}$, $R_{AB\dot{\gamma}\dot{\delta}}$:

$$R_{\underline{\alpha}\underline{\beta}\gamma\delta} = 0\ , \quad R_{\underline{\alpha}\underline{\beta}\dot{\gamma}\dot{\delta}} = 2C_{\alpha\beta}\bar{f}_{\dot{\gamma}\dot{\delta}ij}\ ,$$

$$R_{\underline{\alpha}\underline{\dot{\beta}}\gamma\delta} = -\frac{1}{2}\,C_{\alpha(\gamma}\Lambda_{\delta)i}\bar{\Lambda}_{\dot{\beta}}{}^{j}\ . \tag{3.12}$$

In the above solution we have introduced the auxiliary variable $\Pi_{\alpha\dot{\alpha}} = \Pi_{\alpha\dot{\alpha}i}{}^{i}$ where:

$$\Pi_{\alpha\dot{\alpha}i}{}^{j} = D_{\alpha i}\bar{\omega}_{\dot{\alpha}}{}^{j} - \bar{D}_{\dot{\alpha}}{}^{j}\omega_{\alpha i} + 2\omega_{\alpha i}\bar{\omega}_{\dot{\alpha}}{}^{j} - \frac{1}{2}\Lambda_{\alpha i}\bar{\Lambda}_{\dot{\alpha}}{}^{j}\ . \tag{3.13}$$

As usual in solving Bianchi identities, there are a number of constituency equations. Up to dimension one-half, we find:

$$D_{\underline{\alpha}}\,U = 2\,U\,\omega_{\underline{\alpha}}\ , \quad D_{\underline{\alpha}}\,\bar{V} = 2\,\bar{V}\omega_{\underline{\alpha}}\ , \tag{3.14a}$$

$$D_{\underline{\alpha}}\,\bar{U} = -2\,\bar{U}\,\omega_{\underline{\alpha}} + \Lambda_{\underline{\alpha}}\,\bar{V}\ , \quad D_{\underline{\alpha}}\,V = -2\,V\omega_{\underline{\alpha}} + \Lambda_{\underline{\alpha}}\,U\ , \tag{3.14b}$$

and at dimension one we get:

$$D_{\underline{\alpha}}\bar{\Lambda}_{\underline{\dot{\beta}}} = 3\,\omega_{\underline{\alpha}}\bar{\Lambda}_{\underline{\dot{\beta}}} + \delta_{i}^{j}\,\bar{S}_{\alpha\dot{\beta}}\ , \tag{3.15a}$$

$$\bar{D}_{\dot{\alpha}}\bar{\Lambda}_{\underline{\dot{\beta}}} = -3\,\bar{\omega}_{\underline{\dot{\alpha}}}\bar{\Lambda}_{\underline{\dot{\beta}}} + C^{ijpq}\,\bar{f}_{\dot{\alpha}\dot{\beta}pq}\ , \tag{3.15b}$$

where the auxiliary variable $\bar{S}_{\alpha\dot{\beta}}$ entering the supersymmetry transformation law for the spin one half fields has been introduced. Finally, there are a few consistency equations that determine $\bar{S}_{\alpha\dot{\beta}}$ and that may place restrictions on the possible values of U and $\bar{V}$:

$$D_{\underline{\alpha}} \omega_{\underline{\beta}} + D_{\underline{\beta}} \omega_{\underline{\alpha}} - C_{\alpha\beta} C_{ijpq} \bar{\Lambda}^{\dot{\gamma}p} \bar{\omega}_{\dot{\gamma}}{}^{q} = 0 , \quad (3.16a)$$

$$\Pi_{\alpha\dot{\alpha}i}{}^{j} = \frac{1}{4} \delta_i^j \Pi_{\alpha\dot{\alpha}} , \quad (3.16b)$$

$$D_{\alpha\dot{\alpha}} U + \frac{i}{2} V \bar{S}_{\alpha\dot{\alpha}} - \frac{i}{4} U \Pi_{\alpha\dot{\alpha}} = 0 , \quad (3.16c)$$

$$D_{\alpha\dot{\alpha}} \bar{V} + \frac{i}{2} \bar{U} \bar{S}_{\alpha\dot{\alpha}} - \frac{i}{4} \bar{V} \Pi_{\alpha\dot{\alpha}} = 0 . \quad (3.16d)$$

We now have to investigate if the above system of equations admits solutions with values of U and $\bar{V}$ different from those for the known theories. It is convenient to begin with equations (3.14). The spinorial derivatives and spinorial objects can be eliminated in order to give the usual type of differential equations. To this end, one uses the fact that W is a chiral superfield : $D_{\underline{\alpha}} \bar{W} = \bar{D}_{\underline{\dot{\alpha}}} W = 0$, and therefore the derivative of any function $h(W, \bar{W})$ is:

$$D_{\underline{\alpha}} h(W, \bar{W}) = \frac{\partial h}{\partial W} D_{\underline{\alpha}} W . \quad (3.17)$$

Furthermore, since spinorial objects have to be related to the first spinor derivative of W, one can define scalar functions ω and Λ by:

$$\omega_{\underline{\alpha}} = \omega D_{\underline{\alpha}} W , \quad \Lambda_{\underline{\alpha}} = \Lambda D_{\underline{\alpha}} W . \quad (3.18)$$

Using equations (3.17) and (3.18), one finds that equations (3.14) reduce to:

$$\frac{\partial U}{\partial W} = 2 \omega U , \quad \frac{\partial \bar{V}}{\partial W} = 2 \omega \bar{V} , \quad (3.19)$$

$$\frac{\partial \bar{U}}{\partial W} = -2 \bar{U} \omega + \Lambda \bar{V} , \quad \frac{\partial V}{\partial W} = -2 V \omega + \Lambda U . \quad (3.20)$$

It should be noted that equations (3.19) and (3.20) *define* the scalar functions ω

and Λ and place constraints on U and $\bar{V}$. From equations (3.19) one obtains:

$$\frac{\partial}{\partial W}\ln\left[\frac{\bar{V}}{U}\right]=0 \quad \rightarrow \quad \frac{\bar{V}}{U}=\bar{f}(\bar{W})\,. \tag{3.21}$$

That is, the ratio of $\bar{V}$ and U should be a function of $\bar{W}$ only. From equations (3.19) and (3.20) one derives:

$$\frac{\partial}{\partial W}(\,|U|^2-|V|^2)=0 \quad \rightarrow \quad |U|^2-|V|^2=\pm c_o{}^2\,,0 \tag{3.22}$$

where c_o is a real number. Equations (3.21) and (3.22) are the only requirements on U and $\bar{V}$ in order to have a consistent determination of the scalar functions ω and Λ. The case $|U|^2-|V|^2=\pm c_o{}^2$ leads to the SO(4) supergravity theory and the case $|U|^2-|V|^2=0$ leads to the SU(4) supergravity. In each case they do so up to some field redefinitions. We shall now consider the two cases separately.

3.1 The $|U|^2 - |V|^2 = \pm c_o^2$ Case

It is straightforward to use equations (3.21) and (3.22) to obtain:

$$\text{If } \; |U|^2 - |V|^2 = c_o^2\!: \; U = \frac{c_o\, e^{i\varphi}}{\sqrt{1 - f\bar{f}}} \,, \quad \bar{V} = \bar{f}\, U \,, \tag{3.1.1}$$

$$\text{If } \; |U|^2 - |V|^2 = -c_o^2\!: \; U = \frac{c_o\, e^{i\varphi}}{\sqrt{f\bar{f} - 1}} \,, \quad \bar{V} = \bar{f}\, U \,, \tag{3.1.2}$$

where we have introduced the real phase $\varphi(W, \bar{W})$. One can now determine the scalar functions ω and Λ from equations (3.19), (3.20) and the values of U and $\bar{V}$ given above. For both cases one obtains:

$$\omega = \frac{i}{2} \frac{\partial \varphi}{\partial W} + \frac{1}{4} \frac{f_W \bar{f}}{1 - f\bar{f}} \,, \quad \Lambda = \frac{f_W\, e^{-2i\varphi}}{1 - f\bar{f}} \,. \tag{3.1.3}$$

where we have defined $f_W = \partial f / \partial W$.

One should now check whether or not the remaining equations in (3.15) and (3.16) restrict further the solution for $U, \bar{V}, \omega$ and Λ that has been found. This does not happen. Let us show briefly how one verifies this.

Consider first equation (3.15a). Using equations (3.18) one derives:

$$\frac{\partial \bar{\Lambda}}{\partial W} D_{\alpha i} W \bar{D}^{\dot{}j}_{\dot{\beta}} \bar{W} + \bar{\Lambda} \{ D_{\alpha i}, \bar{D}^{\dot{}j}_{\dot{\beta}} \} \bar{W} = 3 \omega \bar{\Lambda} D_{\alpha i} W \bar{D}^{\dot{}j}_{\dot{\beta}} \bar{W} + \delta^j_i \bar{S}_{\alpha\dot{\beta}} \,, \tag{3.1.4}$$

expanding the commutator and considering the two irreducible pieces of this equation one finds:

$$\frac{\partial \bar{\Lambda}}{\partial W} = 4 \omega \bar{\Lambda} \,, \tag{3.1.5}$$

which is trivially satisfied by our choice of ω and Λ in equation (3.1.3), and the determination of the $\bar{S}_{\alpha\dot{\beta}}$ tensor:

$$\overline{S}_{\alpha\dot{\beta}} = 2i\overline{\Lambda}D_{\alpha\dot{\beta}}\overline{W} \ . \tag{3.1.6}$$

Equation (3.15b) can be decomposed into four irreducible parts. The $(\dot{\alpha}\dot{\beta})[ij]$ part acts as a definition of $\overline{f}_{\dot{\alpha}\dot{\beta}pq}$, and the $(\dot{\alpha}\dot{\beta})(ij)$ piece is trivially satisfied. The $[\dot{\alpha}\dot{\beta}][ij]$ piece is trivially satisfied, and the $[\dot{\alpha}\dot{\beta}](ij)$ piece defines the value of $[D_{\gamma i}, D^{\gamma}{}_{j}]W$. Equation (3.16a) is readily seen to be just $\{D_{\underline{\alpha}}, D_{\underline{\beta}}\}\ln U = 0$, and expanding the commutator one obtains an identity. Equation (3.16b), upon use of equations (3.18) becomes:

$$\frac{\partial\omega}{\partial\overline{W}} + \frac{\partial\overline{\omega}}{\partial W} - \frac{1}{2}\Lambda\overline{\Lambda} = 0 \ , \tag{3.1.7}$$

which is again trivially satisfied by the functions ω and Λ given in equation (3.1.3). Finally, equations (3.16c) and (3.16d) give the same determination of $\overline{S}_{\alpha\dot{\beta}}$ as in equation (3.1.6) and no further constraints.

As we have seen, none of the relations derived from dimension-one Bianchi identities did restrict further the values of U, $\dot{V}$, ω, and Λ derived from dimension one-half Bianchi identities. One could proceed to analyze all higher dimension identities to see if restrictions arise. Nevertheless, it is more convenient at this stage not to do so. Since dimension one superspace tensors determine completely the supersymmetry transformation laws of all component fields, it is possible now to compare the new expressions with the ones corresponding to the known theories. If they happen to be related by field redefinitions, no further constraints could arise, since, the known theories satisfy all the Bianchi identities. In the event that no field redefinition could be found, an analysis of the higher dimension Bianchi identities would be very important. We recall that for the known theories we have *Form A:*

$$U = \frac{1}{\sqrt{1 - W\overline{W}}} \ , \quad \dot{V} = \frac{\overline{W}}{\sqrt{1 - W\overline{W}}} \ ,$$

$$\omega = \frac{1}{4}\frac{\overline{W}}{1 - W\overline{W}}\,, \quad \Lambda = \frac{1}{1 - W\overline{W}}\,. \tag{3.1.8}$$

One would like to see if there are field redefinitions that transform the values given in equations (3.1.1) and (3.1.3) into those in equation (3.1.8). These field redefinitions would transform the supergravities that follow from (3.1.1) and (3.1.3) into the known SO(4) supergravity. As a first step we make a scalar field redefinition in relations (3.1.1) and (3.1.3). Letting $W' = f\,(W)$, and recalling equation (3.18) we find:

$$U' = \frac{c_0\, e^{i\chi}}{\sqrt{1 - W'\overline{W}'}}\,, \quad \overline{V}' = \overline{W}'\, U'\,,$$

$$\omega' = \frac{i}{2}\frac{\partial\chi}{\partial W'} + \frac{1}{4}\frac{\overline{W}'}{1 - W'\overline{W}'}\,, \quad \Lambda' = \frac{e^{-i2\chi}}{1 - W'\overline{W}'}\,, \tag{3.1.9}$$

where the phase $\chi(W', \overline{W}') = \varphi(W, \overline{W})$ has been introduced. The redefinition $W' = f\,(W)$ corresponds to a coordinate transformation in the manifold of scalar fields.

We shall now show that there is a set of field redefinitions that will take us from equation (3.1.9) to equation (3.1.8). For this purpose we use the set of Weyl redefinitions of the supercovariant derivative [13,14]. The way one uses these redefinitions and the necessary relationships are given in Appendix B. Letting D'_A and D_A be the supercovariant derivatives associated with the tensors in (3.1.9) and (3.1.8), respectively. Equation (B.8) determines e^L:

$$e^L = \sqrt{c_0}\, e^{\frac{i}{2}\chi}\,. \tag{3.1.10}$$

Preservation of $T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{c}$ in equation (B.3) requires that:

$$e^M = c_0\,. \tag{3.1.11}$$

Consideration of the $T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}}$ torsion in (B.4) and the $T_{\underline{\alpha}\underline{\beta}}{}^{\gamma}$ torsion in (B.5) shows that ω' and Λ' do redefine into ω and Λ. Using equations (B.10), (B.13) and (B.15) we find the associated component field redefinitions:

$$e'_a{}^{\mu} = c_0\, e_a{}^m \;,\;\; \psi'_\mu{}^i = \frac{1}{\sqrt{c_0}}\, e^{-\frac{i}{2}\gamma_5\chi}\psi_\mu{}^i \;,\;\; \Lambda'_i = \sqrt{c_0}\, e^{-\frac{3i}{2}\gamma_5\chi}\Lambda_i \;. \qquad (3.1.12)$$

The existence of these redefinitions proves the equivalence between the theory that follows from equation (3.1.1) and the known SO(4) theory (Form A in equation (3.1.8)).

Performing field redefinitions completely analogous to the ones performed above, one shows that the functions given in equations (3.1.2) and (3.1.3) can be redefined into *Form B:*

$$U = \frac{1}{\sqrt{W\overline{W} - 1}} \;,\quad \overline{V} = \frac{\overline{W}}{\sqrt{W\overline{W} - 1}} \;,$$

$$\omega = \frac{1}{4}\,\frac{\overline{W}}{1 - W\overline{W}} \;,\quad \Lambda = \frac{1}{1 - W\overline{W}} \;. \qquad (3.1.13)$$

One can see that this theory is defined for $|W|^2 > 1$, that is, in the outside of the unit disc, in contrast with form A (the known SO(4)), that works in the inside of the unit disc $|W|^2 < 1$. Let us find the relationship between these two theories. It is not just analytic continuation. This can be seen by examining equations (3.1.1) and (3.1.3). In order to continue U and $\overline{V}$ into the outside region we let $(1 - f\overline{f})^{\frac{1}{2}} \to i(f\overline{f} - 1)^{\frac{1}{2}}$, and this implies that $\varphi \to \varphi - \frac{\pi}{2}$. On the other hand, Λ does not change under continuation (it is well defined in the outside region), but in order to have a consistent solution it should change sign due to the above change in φ.

The known SO(4) theory (form A) goes into form B under a scalar inversion, followed by a scalar-field-dependent chiral rotation of the Fermi fields and an internal space duality transformation on the spin one fields. Let us see this in detail.

Starting from theory A in equation (3.1.8) and letting $W = 1/W'$ we find:

$$U' = \frac{\sqrt{W'\bar{W}'}}{\sqrt{W'\bar{W}' - 1}} , \quad \bar{V}' = \left[\frac{W'}{\bar{W}'}\right]^{\frac{1}{2}} \frac{1}{\sqrt{W'\bar{W}' - 1}} ,$$

$$\omega' = \frac{1}{4}\,\frac{1}{W'(1 - W'\bar{W}')} , \quad \Lambda' = \frac{\bar{W}'}{W'}\,\frac{1}{1 - W'\bar{W}'} . \tag{3.1.14}$$

Now a super Weyl rescaling is performed:

$$D''_{\alpha i} = \left[\frac{\bar{W}'}{W'}\right]^{\frac{1}{4}} D'_{\alpha i} , \quad D''_{a} = D'_{a} , \tag{3.1.15}$$

implying (using again Appendix B) that the values given in equation (3.1.14) become:

$$U'' = \frac{\bar{W}'}{\sqrt{W'\bar{W}' - 1}} , \quad \bar{V}'' = \frac{1}{\sqrt{W'\bar{W}' - 1}} ,$$

$$\omega'' = \frac{1}{4}\,\frac{\bar{W}'}{1 - W'\bar{W}'} , \quad \Lambda'' = \frac{1}{1 - W'\bar{W}'} , \tag{3.1.16}$$

under the following component redefinitions:

$$\Lambda_i'' = e^{\frac{3}{4} i\gamma_5\theta} \Lambda_i' , \quad \psi_\mu{}^{i\,''} = e^{\frac{1}{4} i\gamma_5\theta} \psi_\mu{}^{i\,'} , \tag{3.1.17}$$

where $e^{i\theta} = W'/\bar{W}'$. At this stage we have almost complete agreement between the expressions given in equation (3.1.16) and those for form B. The only difference lies in the values of U and $\bar{V}$, which have been interchanged. Inspection of equations (3.10) shows that what we need is an internal-space duality transformation of the central-charge field-strength tensor $F_{AB}{}^{mn}$:

$$F_{AB}{}^{ij}{}_{new} = \frac{1}{2}\, C_{ijpq}\, F''_{AB}{}^{pq} . \tag{3.1.18}$$

This transformation is seen to exchange U and $\bar{V}$ in all the tensors given in equation (3.10). At the component level, choosing the indices A and B in equation (3.1.18) to be spacetime indices we have:

$$\widehat{F}^{ij}_{\mu\nu new} = \frac{1}{2} C^{ijkl} \widehat{F}''^{kl}_{\mu\nu} , \tag{3.1.19}$$

where the hat denotes supercovariant field strength (any superspace field strength is always supercovariant in our formalism). Given that the supersymmetry transformation law for the vectors $A_\mu{}^{ij}$ depends only on the field strengths in equation (3.1.18) it follows that $\delta A^{ij}_{\mu new} = \frac{1}{2} C^{ijkl} \delta A''_\mu{}^{kl}$. Therefore the redefinition in equation (3.1.19) applies to the ordinary field strengths $F_{\mu\nu}{}^{ij} = \partial_{[\mu} A_{\nu]}{}^{ij}$ and to the gauge fields themselves, as long as we are dealing with an ungauged theory. Theories with values of U and $\overline{V}$ exchanged via an internal-space duality transformation can become inequivalent once the internal symmetry is gauged. The possibility of exchanging U and $\overline{V}$ could have been noticed earlier in equations (3.19) and (3.20), as these equations do not change under this replacement.

Given that internal-space dualities on the vector fields are transformations that affect the gauged theories, it is helpful to define the A^* form as the dual of the A form, and the B^* form as the dual of the B form. We summarize below the U and $\overline{V}$ values for each:

$$\textit{Form } A: \quad U = \frac{1}{\sqrt{1 - W\overline{W}}} , \quad \overline{V} = \frac{\overline{W}}{\sqrt{1 - W\overline{W}}} .$$

$$\textit{Form } A^*: \quad U = \frac{\overline{W}}{\sqrt{1 - W\overline{W}}} , \quad \overline{V} = \frac{1}{\sqrt{1 - W\overline{W}}} .$$

$$\textit{Form } B: \quad U = \frac{1}{\sqrt{W\overline{W} - 1}} , \quad \overline{V} = \frac{\overline{W}}{\sqrt{W\overline{W} - 1}} .$$

$$\textit{Form } B^*: \quad U = \frac{\overline{W}}{\sqrt{W\overline{W} - 1}} , \quad \overline{V} = \frac{1}{\sqrt{W\overline{W} - 1}} . \tag{3.1.20}$$

In these four cases:

$$\omega = \frac{1}{4}\frac{\overline{W}}{1 - W\overline{W}}, \quad \Lambda = \frac{1}{1 - W\overline{W}}. \tag{3.1.21}$$

The A and A^* forms are defined in the inside of the unit disc, and the B and B^* forms are defined in the outside of the unit disc. A and B^*, or A^* and B are related by scalar inversion and chiral rotation of the Fermi fields. A and B or A^* and B^* are related by scalar inversion, chiral rotation of the Fermi fields and internal-space dualities. Once gauged, A and B^* remain physically equivalent, and the same is true for A^* and B. Nevertheless, A becomes inequivalent to A^* (or B) and B becomes inequivalent to B^* (or A) as explained in [10].

3.2 The $|U|^2 - |V|^2 = 0$ Case

As we have seen in Section 2, the SU(4) supergravity is characterized by having U equal to $\bar{V}$. It is not therefore too surprising to see that the case when $|U|^2 = |V|^2$ will lead to SU(4) supergravities that can be redefined back into the known SU(4) supergravity. It should be emphasized that when $U = \bar{V}$ the solution to the Bianchi identities, as given in equation (3.10), cannot be implemented, because in this case one cannot solve for $f_{\alpha\beta}{}^{ij}$ in terms of the vector field strength $F_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{ij}$. This precludes the construction of supersymmetries, as one can see in equation (3.15b), in which the above relation is necessary to find the supersymmetry transformation law for the spinors Λ^i. In this subsection it should be understood that the central-charge field-strength tensor is split into its $G_{AB}{}^{mn}$ and $\tilde{G}_{AB}{}^{mn}$ pieces, as was done in equation (2.14).

Let us now find the most general solution for $U, \bar{V}, \omega$ and Λ. Equations (3.19) can be used to show that :

$$\bar{V} = e^{2i\theta_o} U \, , \tag{3.2.1}$$

where θ_o has to be a real constant. Equations (3.20) determine the values of ω and Λ in terms of U and $\bar{V}$ without imposing any further constraint on U and $\bar{V}$. It would therefore appear that the choice of functions U and $\bar{V}$ is totally unrestricted. Nevertheless the remaining constraints, that were trivial in the case of the SO(4) theory, are not trivial now. Equation (3.15a) gives a determination of $\bar{S}_{\alpha\dot{\beta}}$:

$$\bar{S}_{\alpha\dot{\beta}} = 2i\bar{\Lambda} D_{\alpha\dot{\beta}} \bar{W} \, , \tag{3.2.2}$$

and the constraint:

$$\frac{\partial \bar{\Lambda}}{\partial \bar{W}} = 4\omega \bar{\Lambda} \, . \tag{3.2.3}$$

Using equations (3.19) and (3.20), one can show that the above constraint is satisfied only if

$$|U|^2 = \frac{1}{g(W) + \bar{g}(\bar{W})}, \tag{3.2.4}$$

where $g(W)$ is an arbitrary function of W. We therefore have the following solution:

$$U = \frac{e^{i\varphi}}{\sqrt{g(W) + \bar{g}(\bar{W})}}, \quad \bar{V} = e^{2i\theta_0} U,$$

$$\omega = -\frac{1}{2}\frac{\partial}{\partial W}\ln[\bar{U}(g(W) + \bar{g}(\bar{W}))], \quad \Lambda = -\frac{\bar{U}}{\bar{V}}\frac{\partial}{\partial W}\ln[g(W) + \bar{g}(\bar{W})]. \tag{3.2.5}$$

It can be checked, in the same way as was done in Section 3.1, that the remaining constraints (equations (3.15b) and (3.16)) are identically satisfied by the solution given in equation (3.2.5).

We recall that for the known SU(4) supergravity one has

$$U = \frac{1}{\sqrt{1 - W - \bar{W}}}, \quad \bar{V} = U,$$

$$\omega = \frac{1}{4}\frac{1}{1 - W - \bar{W}}, \quad \Lambda = \frac{1}{1 - W - \bar{W}}. \tag{3.2.6}$$

Although the expressions in equation (3.2.5) appear to be more general than the ones given above for the known SU(4) theory, they would lead to theories equivalent to the known one. One can show that this is the case by giving the field redefinitions that turn one set of expressions into the other one. As a first step one needs the scalar field redefinition $g(W) = \frac{1}{2} - W'$ that transforms equations (3.2.5) into:

$$U' = \frac{e^{i\chi}}{\sqrt{1 - W' - \bar{W}'}}, \quad \bar{V}' = e^{2i\theta_0} U',$$

$$\omega' = \frac{i}{2}\frac{\partial\chi}{\partial W'} + \frac{1}{4}\frac{1}{1 - W' - \overline{W}'}\,, \quad \Lambda' = \frac{e^{-2i(\chi + \theta_o)}}{1 - W' - \overline{W}'}\,, \tag{3.2.7}$$

where $\chi(W',\overline{W}') = \varphi(W,\overline{W})$. As a second step, we perform a chiral rotation in order to eliminate the phase χ from the above expressions and to reshuffle the phase $2\theta_o$ between U' and $\overline{V}'$:

$$D''_{ai} = e^{-\frac{i}{2}(\chi + \theta_o)} D'_{ai}\,, \quad D''_{a} = D'_{a}\,. \tag{3.2.8}$$

This chiral rotation implies the following component redefinitions (Appendix B):

$$\Lambda''_i = e^{\frac{3}{2}i\gamma_5(\chi + \theta_o)} \Lambda'_i\,, \quad \psi''^{i}_{\mu} = e^{\frac{i}{2}\gamma_5(\chi + \theta_o)} \psi'^{i}_{\mu}\,, \tag{3.2.9}$$

and transforms the expressions in (3.2.7) into:

$$U'' = \frac{e^{-i\theta_o}}{\sqrt{1 - W' - \overline{W}'}}\,, \overline{V}'' = \frac{e^{i\theta_o}}{\sqrt{1 - W' - \overline{W}'}}\,,$$

$$\omega'' = \frac{1}{4}\frac{1}{1 - W' - \overline{W}'}\,, \quad \Lambda'' = \frac{1}{1 - W' - \overline{W}'}\,. \tag{3.2.10}$$

At this stage, we have almost complete agreement with equation (3.2.6). The only difference lies in the phases $e^{-i\theta_o}$ and $e^{i\theta_o}$ present in the U'' and $\overline{V}''$ functions respectively. Actually, the expressions given in equation (3.2.10) lead to supersymmetry transformation formulas identical to those that follow from equation (3.2.6), as we will see in Section 4. There is therefore no necessity to redefine the expressions in (3.2.10). It can be done, however, by letting:

$$\varphi^{ij}_{A\,new} = e^{i\theta_o} \varphi^{ij}_{A}\,, \quad \overline{\varphi}_{Aij\,new} = e^{-i\theta_o} \overline{\varphi}_{Aij}\,, \tag{3.2.11}$$

where φ^{ij}_{A} and $\overline{\varphi}_{Aij}$ are the central-charge connection superfields that appear in the definition of the supercovariant derivative for the SU(4) supergravity in equation (2.3). Nevertheless, equations (3.2.11) in terms of the real gauge fields A^n_μ and B^n_μ are not redefinitions at all. If the relation between the central-charge connection φ_a^{ij} and the gauge fields A_μ^n and B_μ^n is set up consistently (Section 4), one has $A_{\mu new}^n = A_\mu^n$ and $B_{\mu new}^n = B_\mu^n$.

In summary, the analysis of this section has shown that the relaxation of the constraint that the U and $\overline{V}$ functions be restricted to their known expressions has led in the case of the SO(4) supergravity to forms that are defined outside the unit disc and to forms related by internal-space dualities on the vector fields. In the case of the SU(4) supergravities we have seen that trivial field redefinitions reduce the more general forms into the known ones.

4. Supersymmetry Transformations

In this section we study at the component level the implications of the superspace solutions given in Section 3. Since the analysis of the Bianchi identities has been carried out up to quantities of dimension one, we are now in a position to derive the full nonlinear supersymmetry transformations that the solutions imply. We will obtain general forms for the supersymmetries applicable for the most general values of U and $\bar{V}$ found in Section 3. In particular they are applicable for the forms A, A^*, B and B^* given in equation (3.1.20). These forms are of interest because, depending on the way they are gauged they can lead to different theories. In the case of form A (that is, the "known" theory), we reproduce precisely the results given by Cremmer and Scherk [1]. For the case of SU(4) supergravities, we also give the complete supersymmetries. Again, for the particular values of U and $\bar{V}$ corresponding to the known theory, we reproduce precisely the results of Cremmer, Scherk and Ferrara [10].

The relation between superspace geometry and component formulations has been known for some time [15]. This relation has also been worked out for a formalism based on the supercovariant derivative in reference [16]. This formalism was explained in Section 3 of Chapter II. For our choice of constraints in Section 3 ($T_{\underline{\alpha}\underline{\beta}}{}^{c} = T_{\underline{\alpha}b}{}^{c} = 0$, among others), the results of Chapter II, giving the supersymmetry transformation of the component fields in terms of the superspace tensors, can be written as (a, b, c ... are flat vector indices and μ, ν, ρ ... are curved vector indices):

$$\delta e_{a}{}^{\mu} = (\bar{\varepsilon}^{\dot{\underline{\beta}}} \psi_{a}{}^{\underline{\alpha}} + \varepsilon^{\underline{\alpha}} \bar{\psi}_{a}{}^{\dot{\underline{\beta}}}) \, T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{b} \, e_{b}{}^{\mu} \, , \tag{4.1}$$

$$\delta \psi_{\mu}{}^{\underline{\alpha}} = -D_{\mu}\varepsilon^{\underline{\alpha}} - \varepsilon^{\underline{\beta}} \, T_{\mu\underline{\beta}}{}^{\underline{\alpha}} - \bar{\varepsilon}^{\dot{\underline{\beta}}} \, T_{\mu\dot{\underline{\beta}}}{}^{\underline{\alpha}}$$

$$+ (\bar{\varepsilon}^{\dot{\beta}} \psi_{\mu}^{\beta} + \varepsilon^{\beta} \bar{\psi}_{\mu}^{\dot{\beta}}) T_{\beta\dot{\beta}}{}^{\underline{\alpha}} + \varepsilon^{\beta} \psi_{\mu}^{\gamma} T_{\beta\gamma}{}^{\underline{\alpha}} + \bar{\varepsilon}^{\dot{\beta}} \bar{\psi}_{\mu}^{\dot{\gamma}} T_{\dot{\beta}\dot{\gamma}}{}^{\underline{\alpha}} , \qquad (4.2)$$

for the graviton and the gravitinos respectively. For the vector fields φ_{μ}^{ij} of the SO(4) type theories entering in the supercovariant derivative as shown in equation (2.1), the results of Chapter II imply that:

$$\delta \varphi_{\mu}^{ij} = -\varepsilon^{\underline{\alpha}} F_{\mu\underline{\alpha}}{}^{ij} - \bar{\varepsilon}^{\underline{\dot{\alpha}}} F_{\mu\underline{\dot{\alpha}}}{}^{ij} + \varepsilon^{\underline{\alpha}} \psi_{\mu}^{\underline{\beta}} F_{\underline{\alpha}\underline{\beta}}{}^{ij} + \bar{\varepsilon}^{\underline{\dot{\alpha}}} \bar{\psi}_{\mu}^{\underline{\dot{\beta}}} F_{\underline{\dot{\alpha}}\underline{\dot{\beta}}}{}^{ij} . \qquad (4.3)$$

For the vector fields of the SU(4) type theories the situation is somewhat more delicate. One should first notice that the duality condition in equation (2.15) has to be modified because U no longer equals $\bar{V}$. Instead, we now have $\bar{V} = e^{2i\theta_0} U$, therefore the appropriate duality condition reads:

$$\tilde{G}_{ABij} = \frac{1}{2} e^{2i\theta_0} C_{ijmn} G_{AB}{}^{mn} . \qquad (4.4)$$

This duality condition constrains the way the gauge connections φ_{μ}^{ij} and $\bar{\varphi}_{\mu ij}$ are defined in terms of the vector fields A_{μ}^{n} and B_{μ}^{n}. Since $G_{\mu\nu}{}^{ij}$ is the field strength for φ_{μ}^{ij} and $\tilde{G}_{\mu\nu ij}$ is the field strength for $\bar{\varphi}_{\mu ij}$, it is consistent to take:

$$\varphi_{\mu}^{ij} = \frac{1}{2\sqrt{2}} (\alpha_{ij}^{n} A_{\mu}^{n} - i\beta_{ij}^{n} B_{\mu}^{n}) e^{-i\theta_0} , \qquad (4.5)$$

$$\bar{\varphi}_{\mu ij} = \frac{1}{2\sqrt{2}} (\alpha_{ij}^{n} A_{\mu}^{n} + i\beta_{ij}^{n} B_{\mu}^{n}) e^{i\theta_0} , \qquad (4.6)$$

where α_{ij}^{n} and β_{ij}^{n} with $n = 1, 2, ..4$ are the 4×4 antisymmetric matrices of reference [9] . One also has to be careful with conjugation. Conjugation of the G tensor gives the $\tilde{G}$ tensor and vice versa. For example, from equation (2.4) it follows that :

$$\{D_{\underline{\alpha}}, D_{\underline{\beta}}\} = \cdots + \frac{1}{2} G_{\underline{\alpha}\underline{\beta}}{}^{ij} Z_{ji} + \frac{1}{2} \tilde{G}_{\underline{\alpha}\underline{\beta}ij} Z^{ij} , \qquad (4.7)$$

$$\{D_{\dot{\underline{\alpha}}}, D_{\dot{\underline{\beta}}}\} = \cdots + \frac{1}{2} G_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{ij} Z_{ji} + \frac{1}{2} \widetilde{G}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}ij} \overline{Z}^{ji} . \tag{4.8}$$

Since equation (4.8) follows from equation (4.7) by conjugation one has:

$$G_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{ij} = -\overline{\widetilde{G}_{\underline{\alpha}\underline{\beta}ij}} = \overline{\widetilde{G}}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{ij} , \tag{4.9}$$

$$\widetilde{G}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}ij} = -\overline{G_{\underline{\alpha}\underline{\beta}}{}^{ij}} = \overline{G}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}ij} . \tag{4.10}$$

Using equation (4.4) and the above results, one sees that complex conjugation involves the antisymmetric tensor C_{ijmn}. Equations analogous to (4.9) and (4.10) hold for the conjugation of G tensors with both spinorial and vectorial indices.

Taking into account the above remarks the general results in Chapter II can be used to derive the following transformation laws:

$$\delta\varphi_{\mu}^{ij} = -\varepsilon^{\underline{\alpha}} G_{\mu\underline{\alpha}}{}^{ij} - \overline{\varepsilon}^{\dot{\underline{\alpha}}} \overline{\widetilde{G}}_{\mu\dot{\underline{\alpha}}}{}^{ij} + \varepsilon^{\underline{\alpha}} \psi_{\mu}^{\underline{\beta}} G_{\underline{\alpha}\underline{\beta}}{}^{ij} + \overline{\varepsilon}^{\dot{\underline{\alpha}}} \overline{\psi}_{\mu}^{\dot{\underline{\beta}}} \overline{\widetilde{G}}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}}{}^{ij} , \tag{4.11}$$

$$\delta\overline{\varphi}_{\mu ij} = -\varepsilon^{\underline{\alpha}} \widetilde{G}_{\mu\underline{\alpha}ij} - \overline{\varepsilon}^{\dot{\underline{\alpha}}} \overline{G}_{\mu\dot{\underline{\alpha}}ij} + \varepsilon^{\underline{\alpha}} \psi_{\mu}^{\underline{\beta}} \widetilde{G}_{\underline{\alpha}\underline{\beta}ij} + \overline{\varepsilon}^{\dot{\underline{\alpha}}} \psi_{\mu}^{\dot{\underline{\beta}}} \overline{G}_{\dot{\underline{\alpha}}\dot{\underline{\beta}}ij} . \tag{4.12}$$

One can use either equation (4.11) or (4.12) together with equations (4.5) or (4.6) to find the supersymmetries for the vector fields A_{μ}^{n} and B_{μ}^{n}. Both give the same result.

Finally, for the scalars and the spinors (non-gauge fields), the supersymmetry transformation are found using:

$$\delta Y = \varepsilon^{\underline{\alpha}} D_{\underline{\alpha}} Y + \overline{\varepsilon}^{\dot{\underline{\alpha}}} \overline{D}_{\dot{\underline{\alpha}}} Y , \tag{4.13}$$

where Y denotes any of these fields, together with the constituency equations (3.14) and (3.15) that give the spinorial derivatives for the scalars and the spinors, respectively.

In the following our results are in four-component notation and follow the conventions of References [1] and [10]. The transcription from two-component to four-component notation makes use of the translation table given in Appendix A.

We first consider the supersymmetries for the SO(4) type theories. Some numerical rescaling of fields and parameters is needed in order to obtain full agreement with the known SO(4) theory when the functions U and $\overline{V}$ assume their standard values. We let $\Lambda^i \to -2\Lambda^i$, $\psi^i_\mu \to -\psi^i_\mu/\sqrt{2}$, $\bar{\varepsilon}^i \to \bar{\varepsilon}^i/\sqrt{2}$, $\varphi^{ij}_\mu = -A^{ij}_\mu/2$ and $\gamma_\mu \to -\gamma_\mu$. Using equations (4.1), (4.13) and the relation $\Lambda_{\alpha i} = \Lambda D_{\alpha i} W$, we derive the supersymmetries for the graviton and the scalar fields:

$$\delta V_{a\mu} = -i\,\kappa\,\bar{\varepsilon}^i \gamma_a \psi^i_\mu$$

$$\delta A = \frac{1}{\sqrt{2}}\,\bar{\varepsilon}^i \|\frac{1}{\Lambda}\|\Lambda^i, \qquad \delta B = \frac{i}{\sqrt{2}}\,\bar{\varepsilon}^i\gamma_5\|\frac{1}{\Lambda}\|\Lambda^i\ , \tag{4.14}$$

where the $\|..\|$ notation is defined by $\|f\| = Re(f) + i\gamma_5 Im(f)$. The transformation law for the vectors follows from equations (4.3) and (3.10):

$$\delta A^{ij}_\mu = \frac{i}{\sqrt{2}}\,\{C^{ijke}\,\bar{\varepsilon}^k\|\overline{U}\|\gamma_\mu\Lambda^e + \bar{\varepsilon}^{[i}\|V\|\gamma_\mu\Lambda^{j]}\}$$

$$+\ \bar{\varepsilon}^{[j}\|U\|\psi_\mu{}^{i]} - C^{ijke}\,\bar{\varepsilon}^k\|\overline{V}\|\psi^e_\mu\ . \tag{4.15}$$

In order to obtain the supersymmetry transformation for the spinors, we first have to express the $\bar{f}_{\dot{\alpha}\dot{\beta}pq}$ tensor appearing in equation (3.15b) in terms of the spin one field strength $F_{\alpha\dot{\alpha}\beta\dot{\beta}}{}^{mn}$. Using equation (3.10) one readily obtains:

$$-i\,(\sigma^{ab})_{\alpha\beta}F_{ab}{}^{mn} = \frac{1}{2}\,f_{\alpha\beta}{}^{ij}\,[\,U\delta_i^{[m}\delta_j^{n]} + \overline{V}\,C_{ij}{}^{mn}\,]\ . \tag{4.16}$$

Solving for $f_{\alpha\beta}{}^{ij}$ and rearranging somewhat the result one finds:

$$f_{\alpha\beta}{}^{ij} = -\frac{i}{2U}(\sigma^{\mu\nu})_{\alpha\beta}[\widehat{F} + \widehat{H}]^{ij}_{\mu\nu}, \tag{4.17}$$

where the field strength $H^{ij}_{\mu\nu}$ is given by the same expression as in the known SO(4) theory, namely:

$$H^{ij}_{\mu\nu} = g_1 F^{ij}_{\mu\nu} - g_2 F^{*ij}_{\mu\nu} - g_3 \widetilde{F}^{*ij}_{\mu\nu} - g_4 \widetilde{F}^{ij}_{\mu\nu}, \tag{4.18}$$

but with the following values for the functions g_i:

$$g_1 - ig_4 = \frac{1+f^2}{1-f^2}, \quad g_3 - ig_2 = -\frac{2if}{1-f^2}, \tag{4.19}$$

where $f = \overline{V}/U$. Use of equations (3.15), (3.1.6), (4.13) and (4.17) leads to the supersymmetry transformation of the spin one-half fields:

$$\delta\overline{\Lambda}^i = \frac{i}{\sqrt{2}}\,\overline{\varepsilon}^i||\overline{\Lambda}||\widehat{D}_\mu(A+i\gamma_5 B)\gamma^\mu + \frac{1}{2\sqrt{2}}\,[\widehat{F}^*+\widehat{H}^*]^{ij}_{\alpha\beta}\overline{\varepsilon}^j||\frac{1}{U}||\sigma^{\alpha\beta}$$

$$+ \frac{3\kappa}{2\sqrt{2}}\overline{\varepsilon}^j\gamma_5||\frac{4\omega}{\Lambda}||\Lambda^j\overline{\Lambda}^i\gamma_5\,. \tag{4.20}$$

We now turn to the gravitinos. In order to use equation (4.2) we only need to simplify somewhat the expression for $\Pi_{a\dot{a}}$ given in equation (3.13). Equation (3.18), (3.19) and (3.1.7) can be used to show that:

$$\Pi_{a\dot{a}} = 8\,\mathrm{Im}\left[\frac{1}{U}\frac{\partial U}{\partial W}D_{a\dot{a}}W\right]. \tag{4.21}$$

With this expression, and the expressions for the other torsions in equation (3.11) one finds:

$$\delta\overline{\psi}^i_\mu = \frac{1}{\kappa}\,\overline{\varepsilon}^i\overleftarrow{D}_\mu + i\,\overline{\varepsilon}^i\gamma_5\mathrm{Im}[\frac{1}{U}\,\frac{\partial U}{\partial W}\,\widehat{D}_\mu W] - \frac{i}{4}\,[\widehat{F}+\widehat{H}]^{ij}_{\alpha\beta}\overline{\varepsilon}^j\gamma_\mu\sigma^{\alpha\beta}||\frac{1}{U}||$$

$$+ \delta\overline{\psi}^i_\mu(1) + \delta\overline{\psi}^i_\mu(2)\,, \tag{4.22}$$

where $\delta\overline{\psi}^i_\mu(1)$ contains spinor-spinor terms:

$$\delta\overline{\psi}^i_\mu(1) = \frac{i\kappa}{4}\left[\overline{\Lambda}^j\gamma_5\gamma^a\Lambda^j\overline{\varepsilon}^i\gamma_5\gamma_\mu\gamma_a + \overline{\Lambda}^i\gamma^a\Lambda^j\overline{\varepsilon}^j\gamma_\mu\gamma_a - \overline{\Lambda}^i\gamma_5\gamma^a\Lambda^j\overline{\varepsilon}^j\gamma_5\gamma_\mu\gamma_a\right], \quad (4.23)$$

and $\delta\overline{\psi}^i_\mu(2)$ contains the spinor-gravitino terms:

$$\delta\overline{\psi}^i_\mu(2) = -\frac{\kappa}{2\sqrt{2}}\left[\overline{\psi}^j_\mu\gamma_5\left\|\frac{4\omega}{\Lambda}\right\|\Lambda^j\overline{\varepsilon}^i\gamma_5 - \overline{\varepsilon}^j\gamma_5\left\|\frac{4\omega}{\Lambda}\right\|\Lambda^j\overline{\psi}^i_\mu\gamma_5\right.$$

$$\left. + 2\, C^{ijkp}(\overline{\varepsilon}^k\psi^j_\mu\overline{\Lambda}^p - \overline{\varepsilon}^k\gamma_5\psi^j_\mu\overline{\Lambda}^p\gamma_5)\right]. \quad (4.24)$$

The value of the spin connection, which appears in the supersymmetry transformation of the gravitino, can be determined using equation (3.33) of Chapter II. We find:

$$\varphi_{\mu ab} = \varphi_{\mu ab}(e) + \frac{i\kappa^2}{2}\,(\overline{\psi}^i_\mu\gamma_b\psi^i_a + \overline{\psi}^i_b\gamma_\mu\psi^i_a + \overline{\psi}^i_b\gamma_a\psi^i_\mu)$$

$$- \frac{\kappa^2}{4}\, e^e_\mu\,\varepsilon_{eabc}\overline{\Lambda}^i\gamma_5\gamma^c\Lambda^i\ . \quad (4.25)$$

It is a simple matter to verify that the supersymmetries given above reduce to the ones given before [1] when U and $\overline{V}$ take their known values. For the four cases considered in equation (3.1.20), namely the A,A^*,B and B^* cases, the full supersymmetries are obtained immediately by substituting the appropriate values of U and $\overline{V}$, and the values of ω and Λ into the above equations. Since we have given in the previous section the field redefinitions that turn any of the four cases into one another, the supersymmetries could have been found, in a much more involved way, by direct redefinition. The above expressions are a check of the known case and can be used for values of U, $\overline{V}$,ω and Λ of the form given in equations (3.1.1), (3.1.2) and (3.1.3).

We now turn to the SU(4) theories. Here too, numerical rescaling is necessary to obtain full agreement with the known SU(4) supersymmetries. We put $\Lambda^i \to 2\Lambda^i$, $\psi^i_\mu \to -\psi^i_\mu/\sqrt{2}$, $\overline{\varepsilon}^i \to \overline{\varepsilon}^i/\sqrt{2}$, and $\gamma_\mu \to -\gamma_\mu$. The supersymmetry transformations for the graviton and the scalars are:

$$\delta V_{a\mu} = -i\kappa\bar{\varepsilon}^i\gamma_a\psi^i_\mu$$

$$\delta\varphi = \frac{1}{\sqrt{2}}\, e^{2\kappa\varphi}\bar{\varepsilon}^i\|\frac{1}{\Lambda}\|\Lambda^i\,, \qquad \delta B = \frac{i}{\sqrt{2}}\,\bar{\varepsilon}^i\gamma_5\|\frac{1}{\Lambda}\|\Lambda^i\,. \tag{4.26}$$

In order to find the supersymmetry transformations for the vectors A^n_μ and the axial vectors B^n_μ, we use equations (4.5) or (4.6) and (4.11) or (4.12), where the G and $\tilde{G}$ tensors are read off from equations (3.10), performing the splitting discussed in Section 2. We find:

$$\delta A^n_\mu = \frac{1}{\sqrt{2}}\,\alpha^n_{ij}[\bar{\varepsilon}^i\|U'\|\psi^j_\mu + \frac{i}{\sqrt{2}}\,\bar{\varepsilon}^i\gamma_\mu\|U'\|\Lambda^j]\,,$$

$$\delta B^n_\mu = \frac{i}{\sqrt{2}}\,\beta^n_{ij}[\bar{\varepsilon}^i\gamma_5\|U'\|\psi^j_\mu + \frac{i}{\sqrt{2}}\,\bar{\varepsilon}^i\gamma_5\gamma_\mu\|U'\|\Lambda^j]\,, \tag{4.27}$$

where we have introduced the variable U' defined as:

$$U' = e^{i\theta_0}\,U = (\,U\,\bar{V}\,)^{\frac{1}{2}}\,. \tag{4.28}$$

For the spinors one finds:

$$\delta\bar{\Lambda}^i = \frac{i}{\sqrt{2}}\, e^{-2\kappa\varphi}\bar{\varepsilon}^i\|\bar{\Lambda}\|\,(\hat{D}_\mu\varphi + i\gamma_5 e^{2\kappa\varphi}\hat{D}_\mu B)\gamma^\mu + \frac{1}{2}\,\bar{\varepsilon}^j\,\hat{C}^{ij}_{\alpha\beta}\|\frac{1}{U'}\|\sigma^{\alpha\beta}$$

$$- \frac{3\kappa}{2\sqrt{2}}\,\bar{\varepsilon}^j\gamma_5\|\frac{4\omega}{\Lambda}\|\Lambda^j\bar{\Lambda}^i\gamma_5\,, \tag{4.29}$$

and for the gravitino:

$$\delta\bar{\psi}^i_\mu = \frac{1}{\kappa}\,\bar{\varepsilon}^i\overleftarrow{D}_\mu + i\bar{\varepsilon}^i\gamma_5\,\mathrm{Im}[\frac{1}{U}\,\frac{\partial U}{\partial W}\,\hat{D}_\mu W] + \frac{i}{2\sqrt{2}}\,\bar{\varepsilon}^j\,\hat{C}^{ij}_{\alpha\beta}\gamma_\mu\sigma^{\alpha\beta}\|\frac{1}{U'}\|$$

$$+ \,\delta\bar{\psi}^i_\mu(1) - \delta\bar{\psi}^i_\mu(2)\,, \tag{4.30}$$

where $\delta\bar{\psi}^i_\mu(1)$ and $\delta\bar{\psi}^i_\mu(2)$ were given in equations (4.23) and (4.24) respectively.

The spin connection is the same as for the SO(4) type theories, namely the one given in equation (4.25). Again, the above expressions reduce to the known ones when $U = \bar{V} = e^{\kappa\varphi}$. They are valid for U and $\bar{V}$ functions of the type given in equation (3.2.5). Once gauged, these forms do not lead to physically inequivalent theories, because, as shown in Section 3, trivial field redefinitions turn them into the known SU(4) supergravity theory.

5. Relaxing the Constraints

The search for the most general solution of the system of Bianchi identities is a formidable problem. Fortunately, it can be studied systematically. One first writes the known theory in superspace, imposing some constraints on the geometry. The known theory defines a consistent solution to the Bianchi identities. Then one proceeds to a step by step relaxation of the previously imposed constraints in a way compatible with the known field content of the theory. At each step the Bianchi identities are studied to find the most general solution compatible with the remaining constraints. One must also use the set of Weyl redefinitions of the supercovariant derivative (Appendix B) to see whether or not the solutions are trivially related to the known solutions.

There are only two possible outcomes in this step by step relaxation of constraints. Either one finds new theories (or theories that relate back nontrivially so that they become inequivalent once gauged), or one shows that all the constraints were actually irrelevant to the construction of the theory. In Section 3 we relaxed the constraint that fixed U and $\bar{V}$ to have their known values. We found that the more general solutions could be trivially related to the known solution for the case of SU(4) supergravities. For the case of SO(4) type solutions, internal-space dualities and scalar field inversions were necessary, leading us to consider the A,A^*,B and B^* solutions.

In this Section we show that the relaxation of the remaining constraints does not lead to anything new beyond the results of Section 3. We therefore conclude that constraints are not necessary for the construction of N = 4 supergravities. *The well-known field representations of $N = 4$ supergravity* (one graviton, four gravitinos, six vectors, four spinors and two scalars), *together with the Bianchi identities is all one needs to know to construct the theories, and the theories follow uniquely from this knowledge.* A particular set of constraints corresponds to a choice of super Weyl gauge. It should be emphasized that the above comments do not apply for extended supergravities with auxiliary fields, because in this case the field representations are not known in general and constraints have the role of determining which representations may or may not enter into the theory.

As we remarked in Section 3, the solutions found there do not correspond to the most general solutions to the Bianchi identities with the constraints given in equations (3.1). They were not so because equation (3.9) is not the most general possibility for the $T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}}$ torsion. Let us now consider the complete decomposition of this torsion into irreducible pieces:

$$T_{\alpha i\,\beta j}{}^{\gamma k} = \omega_{\alpha i}\delta_\beta^\gamma\delta_j^k + \omega_{\beta j}\delta_\alpha^\gamma\delta_i^k + \Gamma_{\alpha j}\delta_\beta^\gamma\delta_i^k + \Gamma_{\beta i}\delta_\alpha^\gamma\delta_j^k$$

$$+ g_{\alpha(ij)}{}^k\delta_\beta^\gamma + g_{\beta(ij)}{}^k\delta_\alpha^\gamma + g_{\alpha[ij]}{}^k\delta_\beta^\gamma - g_{\beta[ij]}{}^k\delta_\alpha^\gamma$$

$$+ h_{i(\alpha\beta)}{}^\gamma\delta_j^k + h_{j(\alpha\beta)}{}^\gamma\delta_i^k + z_{(\alpha\beta)(ij)}{}^{\gamma k} , \tag{5.1}$$

where any contraction of a lower index with an upper index in the functions g, h and z gives zero. Nevertheless, it is not possible to construct the dimension one-half tensors $g_{\alpha(ij)}{}^k$, $h_{i(\alpha\beta)}{}^\gamma$ and $z_{(\alpha\beta)(ij)}{}^{\gamma k}$ out of the available fields, and therefore they have to be set to zero. Again, field representations and dimensionality force one to set $g_{\alpha[ij]}{}^k = g_{\alpha p}\, C_{ij}{}^{kp}$, where $g_{\alpha p}$ is a dimension one-half spinor and C_{ijkp} is the totally antisymmetric tensor. Evaluation of the $I_{\underline{\alpha}\underline{\beta}\underline{\gamma}}{}^{mn}$ Bianchi identity, however, requires that $g_{\alpha p} = 0$. One is therefore left with:

$$T_{\alpha i\,\beta j}{}^{\gamma k} = \omega_{\underline{\alpha}}\delta_\beta^\gamma\delta_j^k + \omega_{\underline{\beta}}\delta_\alpha^\gamma\delta_i^k + \Gamma_{\alpha j}\delta_\beta^\gamma\delta_i^k + \Gamma_{\beta i}\delta_\alpha^\gamma\delta_j^k , \tag{5.2}$$

as the most general expression for the torsion.

Let us call theories I the theories that follow from constraints (3.1) and (3.9), and theories II the theories that follow from constraint (3.1) only (equation (5.2) should not be thought of as a constraint since it is the most general expression one can write). Before solving the Bianchi identities again, it is convenient to find out first under what conditions theories II could be related trivially to theories I. In the notation of Appendix B we take theories II to be described by the unprimed tensors, and we want to redefine them into the primed tensors, that refer to theories I. Preservation of the value of $T_{\alpha i\, \dot{\beta} j}{}^{c}$ requires ((B.3)):

$$M = L + \bar{L} \,, \tag{5.3}$$

and preservation of $T_{\underline{a} b}{}^{c} = 0$ requires ((B.6)):

$$(D_{\alpha i} M)\, \delta^{c}_{\underline{a}} - 2i \sigma^{c}_{\alpha \dot{\beta}} \bar{f}_{a}{}^{\dot{\beta}}{}_{i} + f_{\alpha i\, a}{}^{c} = 0 \,. \tag{5.4}$$

The elimination of the last two terms in equation (5.2), in order that the $T_{\underline{\alpha}\underline{\beta}}{}^{\gamma}$ torsion takes the form it has in theories I, requires that ((B.5)):

$$f_{\alpha i \gamma}{}^{\delta} = \Gamma\left(\frac{1}{2} D_{\alpha i} W\, \delta^{\delta}_{\gamma} - D_{\gamma i} W\, \delta^{\delta}_{\alpha}\right) , \tag{5.5}$$

where $\Gamma_{\alpha i} = \Gamma D_{\alpha i} W$. Substituting the result in equation (5.5) into equation (5.4) (with $f_{\alpha i \dot{\gamma}}{}^{\dot{\delta}} = 0$), one determines:

$$\bar{f}_{a}{}^{\dot{\beta}}{}_{j} = -\frac{i}{4} \Gamma \sigma_{a}{}^{\alpha \dot{\beta}} D_{\alpha j} W \,, \tag{5.6}$$

$$\frac{\partial M}{\partial W} = -\frac{1}{2} \Gamma \,. \tag{5.7}$$

The value of $f_{ac}{}^{d}$ can be found by requiring that the constraint on the curvature $R_{\underline{\alpha}\dot{\underline{\beta}}\gamma\delta}$ be preserved.

Equations (5.3) and (5.7) imply that Γ is the derivative with respect to W of a real function of W and $\overline{W}$, that is:

$$\Gamma = \frac{\partial H}{\partial W} , \tag{5.8}$$

where $H = H(W, \overline{W})$ is a real function. The above analysis implies that if the function Γ introduced in equation (5.2) satisfies the constraint (5.8), then theories II can be redefined trivially (via super Weyl rescalings) into theories I.

One can solve again the Bianchi identities using equations (3.1) and (5.2). Up to dimension one-half level, we find the following constituency equations:

$$D_{\underline{\alpha}} U = (2\omega_{\underline{\alpha}} + \Gamma_{\underline{\alpha}})\, U \,, \quad D_{\underline{\alpha}} \overline{V} = (2\omega_{\underline{\alpha}} + \Gamma_{\underline{\alpha}})\, \overline{V} \,. \tag{5.9}$$

$$D_{\underline{\alpha}} \overline{U} = -2\omega_{\underline{\alpha}} \overline{U} + \Lambda_{\underline{\alpha}} \overline{V} \,, \quad D_{\underline{\alpha}} V = -2\omega_{\underline{\alpha}} V + \Lambda_{\underline{\alpha}} U \,. \tag{5.10}$$

A little calculation, using the scalar functions Λ, ω and Γ, gives:

$$\Gamma = \frac{\partial}{\partial W} \ln \big|\, |U|^2 - |V|^2 \,\big| \,, \tag{5.11}$$

for $|U|^2 - |V|^2 \neq 0$, that is for SO(4)-type theories. Equation (5.11) indicates that the constraint in equation (5.8) is satisfied. Therefore all SO(4)-type theories relate trivially to the known solutions. For the case of SU(4) supergravities, $|U|^2 = |V|^2$, and equations (5.9) and (5.10) do not imply that Γ is the derivative of a real function. One has to go beyond the analysis of dimension one-half identities. Among the constraints that one finds analyzing the dimension-one Bianchi identities one has:

$$D_{\underline{\alpha}} \overline{\Lambda}_{\underline{\dot{\beta}}} = (3\omega_{\underline{\alpha}} + \Gamma_{\underline{\alpha}}) \overline{\Lambda}_{\underline{\dot{\beta}}} + \delta_i^j \, \overline{S}_{\alpha\dot{\beta}} \,, \tag{5.12}$$

$$(D_{\widehat{\alpha i}} + \omega_{\widehat{\alpha i}})\, \Gamma_{\beta}^{\widehat{j}} + (D_{\beta}^{\widehat{j}} + \overline{\omega}_{\beta}^{\widehat{j}})\, \Gamma_{\widehat{\alpha i}} = 0 \,, \tag{5.13}$$

where the hat on the isospin indices indicates that the trace has been removed: $f_{\hat{i}}{}^{\hat{j}} = f_i{}^j - \frac{1}{4}\delta_i^j f_p{}^p$. From the traceless part of equation (5.12), and equation (5.9), one finds:

$$\Gamma = -\frac{\partial}{\partial W} \ln \frac{\overline{\Lambda}}{U\overline{V}} . \tag{5.14}$$

From equation (5.13) one finds:

$$\frac{\partial \Gamma}{\partial \overline{W}} = \frac{\partial \overline{\Gamma}}{\partial W} . \tag{5.15}$$

Equations (5.14) and (5.15) imply that Γ *can* be written as the derivative with respect to W of a real function. We therefore conclude that SU(4)-type theories can also be related trivially to the type I forms.

At this point we have already succeeded in relaxing most of the constraints and it will take little effort to consider the remaining ones. One should first realize that it is not possible to alter the superspace field strengths $F_{\underline{\alpha}\underline{\beta}}{}^{mn}$ and $F_{\underline{\alpha}\dot{\underline{\beta}}}{}^{mn}$ or the $T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}\gamma}$ torsion with the available field representations for N = 4 supergravity. This is not the case for the remaining torsions and the curvature in equation (3.1). It is possible to relax the constraints that fix those tensors. The most general expressions one can write are:

$$T_{\underline{\alpha}\dot{\underline{\beta}}}{}^{\dot{\gamma}\gamma} = 2i\, e^{H}\, \delta_i^j \delta_\alpha^\gamma \delta_{\dot{\beta}}^{\dot{\gamma}} ,$$

$$T_{\underline{\alpha} b}{}^{c} = N\, \delta_b^c\, D_{\underline{\alpha}} W + P(\sigma_b{}^c)_\alpha{}^\beta D_{\beta i} W ,$$

$$R_{\alpha i \dot{\beta}}{}^{i}{}_{\gamma\delta} = Q\, C_{\alpha(\gamma} \Lambda_{\delta) i}\, \overline{\Lambda}_{\dot{\beta}}{}^{i} , \tag{5.16}$$

where H,N,P and Q are arbitrary functions of the scalar fields. Hermiticity of the supercovariant derivative, however, requires that H be a real function. Let us call type III theories, the theories that follow from the above expressions. It is not possible to write a more general expression for an N = 4 supergravity without auxiliary fields. We shall now show that Weyl rescalings transform trivially type III theories into the type II theories considered earlier. Since type II theories are equivalent to the theories considered in Section 3 we conclude that relaxation of the constraints does not lead to new theories.

As a first step we do the following redefinition:

$$D'_a = D_a + \frac{1}{2} f_{ac}{}^d M_d{}^c , \qquad (5.17)$$

which is just a redefinition of the spin connection. One can show that, under this redefinition

$$R'_{\alpha i \dot{\beta}}{}^{i}{}_{\gamma\delta} = R_{\alpha i \dot{\beta}}{}^{i}{}_{\gamma\delta} - 8i f_{\alpha\dot{\beta}\gamma\delta} . \qquad (5.18)$$

Therefore, choosing $f_{\alpha\dot{\beta}\gamma\delta}$ appropriately, one can restore the value for the trace of this curvature given in equation (3.1). None of the remaining superspace tensors considered in equation (3.1) is affected by this redefinition.

We now need a further Weyl rescaling to eliminate the functions H,N and P from equation (5.16). Letting the unprimed derivatives of Appendix B refer to type III theories, we take $M = H$ and $L = 0$. This sets $T_{\underline{\alpha}\underline{\dot{\beta}}}{}^{\gamma\dot{\gamma}}$ back to the standard value. Using equation (B.6), one finds that it is possible to set $T_{\underline{\alpha} b}{}^c$ equal to zero by choosing:

$$f_{\alpha i \gamma}{}^{\delta} = [-2iP + 2(N + \frac{\partial H}{\partial W})](\frac{1}{2} D_{\alpha i} W \delta^{\delta}_{\gamma} - D_{\gamma i} W \delta^{\delta}_{\alpha}) ,$$

$$f_{\alpha i \dot{\gamma}}{}^{\dot{\delta}} = 0 , \quad \bar{f}_{\gamma\dot{\delta}}{}^{\dot{\beta}}{}_{i} = -i\, e^{-H} (N + \frac{\partial H}{\partial W}) D_{\gamma i} W \delta^{\dot{\beta}}_{\dot{\delta}} , \qquad (5.19)$$

Even though the torsion $T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}}$ is modified, it remains of the general form given in equation (5.2). Therefore the redefinition of type III theories into type II theories is complete, and *the most general superspace solutions can be redefined into the known theories*.

We have thus arrived at the conclusion that the SO(4) and SU(4) supergravities are unique. They can be rewritten in different ways, all of them physically equivalent, by a choice of a Weyl gauge, which corresponds to the freedom to perform super Weyl field redefinitions.

It is interesting to note that both theories II and III discussed above, if implemented at the component level, would show the graviton coupled nonminimally to the scalars. This occurs because the transformation that relates those theories to the known supergravities, which have the scalars coupled minimally, involve scalar-field dependent Weyl rescalings of the graviton. This nonminimal coupling happens in type II theories despite the constraints $T_{\underline{a}b}{}^{c} = 0$ and $T_{\alpha i \dot{\beta}}{}^{j\, c} = 2i\, \delta_i^j \sigma^c{}_{\alpha\dot{\beta}}$ that insure that the graviton has the standard supersymmetry transformation into the gravitino. One can trace the origin of the nonminimal coupling of the scalars to the Γ terms in equation (5.2). This term has been seen to produce supersymmetry transformations that require a term of the type $\bar{\chi}\sigma^{\mu\nu}D_\mu\psi_\nu$ to be present in the lagrangian, and this term in turn requires that scalars appear in front of the Einstein part of the action.

It has been emphasized in the literature that the scalar fields enter nonpolynomially in the N $\geq$ 4 supergravity theories. It is also known that the nonpolynomiality of the SU(4) model is much simpler that the one present in the SO(4) model. One can show that in the case of the SU(4) supergravity the scalar nonpolynomiality can be avoided altogether in the lagrangian if one allows nonminimal coupling of the scalars to gravitation, that is, if one allows the presence of a function $f\,(\varphi)$ in front of the scalar curvature.

The polynomial form for the lagrangian is found by first rescaling some fields by powers of $e^{\kappa\varphi}$ and then redefining the scalar field φ itself in order to eliminate the exponentials. These steps are given in Appendix C where the polynomial form of the lagrangian and the supersymmetry transformation laws are given. It should be noted, however, that the supersymmetry transformations have nonpolynomiality in expressions such as $(\kappa\varphi)^{-1}$. It does not appear to be possible to eliminate the nonpolynomiality from both the supersymmetries and the lagrangian. There is no difficulty in writing a polynomial form for the gauged SU(2) $\otimes$ SU(2) model [4]. In this case the scalar potential is of the form $(-\varphi^2)$.

6. Conclusions

We have searched for the most general N = 4 supergravity without auxiliary fields. Our results indicate that the SO(4) and SU(4) supergravities are unique. The different forms that one can obtain for these theories are always redefinable back into the known forms. In the case of the SO(4) supergravities the redefinitions include internal-space duality transformations on the vector fields, and can lead to inequivalence when the theories are gauged .

It has been shown that N = 4 supergravity theories follow from the Bianchi identities and the known field representations. In particular, further constraints on torsions, curvatures or field strengths are not necessary. They are just a convenience in order to reproduce the particular forms of the component formulations.

Our search for more N = 4 supergravities assumed that there is at least a global SO(4) symmetry. The SU(4) supergravity was obtained as a particular solution of the superspace equations, which could have more symmetry than SO(4). We have also assumed that the gravitational constant κ does not appear explicitly in the superspace tensors if bosons are taken to have dimension zero and fermions dimension one-half. Without this requirement the dimensionality of the fields could be easily altered to any value by inserting appropriate factors of κ, and the expansion of any superspace tensor could have an infinite number of terms. Although it may be possible to relax the above assumptions it appears unlikely that this will lead to new forms of N = 4 supergravity.

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Chapter IV. Gauged N = 4 Supergravities

1. Introduction and Summary

It is well known that extended supergravity theories with N local supersymmetries have a global SO(N) invariance which can be made local using the spin-one fields of the theory as gauge fields. In this way, supergravity theories have a Yang-Mills-type gauge invariance which might be related in some way to the Yang-Mills-type theories that describe the electroweak and strong interactions.

The gauged supergravities, however, seem to have some problematic features. Not even the largest group, namely SO(8), contains the standard SU(3) x SU(2) x U(1) of phenomenology as a subgroup [1]. A cosmological constant $\Lambda \sim -g^2/\kappa^2$ where g is the SO(N) gauge coupling constant and κ is Newton's constant, is present in the theory. For a typical value of g (~ 0.1) Λ is an unacceptably large negative number.

Despite the above and other difficulties, encouraging progress has been made recently. The inverted scalar potentials of the gauged SO(N $\geq$ 4) theories are now known to have at least one stable ground state [2,3]. For N $\geq$ 5 it was also observed that the scalar potential can break both supersymmetry and the Yang-Mills symmetry spontaneously [2,3,4]. This is an attractive possibility since the scalars that trigger the Higgs and super-Higgs effects are the ones present in the original supermultiplet.

The case of N = 4 supergravity is quite special. As we have seen in Chapter III there are two versions of this theory, namely the SO(4) [5] and the SU(4) [6] models. Moreover, the so(4) algebra is not simple. In fact so(4) = su(2) $\oplus$ su(2), and therefore gauged N = 4 supergravities should have two independent coupling constants, each associated with an su(2) subalgebra. This was known to be the case for the gauged SU(4) model [7] but not for the gauging of the SO(4) model

[8]. We have derived the gauged SO(4) model with two coupling constants [9], and it shows some surprising features. For the first time a positive cosmological constant is obtained in gauged extended supergravity. The positive cosmological constant or de-Sitter background (the background of inflationary universe scenarios) breaks the four supersymmetries spontaneously. The super Higgs effect takes place and the four gravitinos become massive by eating the four spin one half fields. The gauge group, however, remains unbroken, since the scalars in N = 4 supergravity do not transform under SO(4). That is, the Higgs effect does not take place, the vacuum does not break SO(4), and the six vectors remain massless.

The cosmological constant in the gauged SO(4) model with two parameters g_1 and g_2 is proportional to the product of these two parameters $\Lambda \sim -g_1 g_2 / \kappa^2$. One could conclude that the cosmological constant is adjustable. This conclusion would be incorrect, *the cosmological constant cannot be adjusted.* It is shown here that the g_1 and g_2 constants introduced into the SO(4) model cannot be readily identified with gauge coupling constants because of the presence of scalar fields in front of the kinetic term for the vector fields. If at the critical point of the potential the scalar fields acquire some vacuum expectation value, the canonical normalization for the vector fields can be lost. It is then necessary to rescale the vector fields to recover the standard normalization, and the rescaling of the gauge fields implies a rescaling of the gauge coupling constant. This is precisely what happens in the gauged SO(4) model, as will be shown here. For g_1 and g_2 of the same sign but different magnitude, the scalar field acquires a vacuum expectation value. If the vector fields are then rescaled to the standard normalization, one finds the effective coupling constants to be $g_{1eff} = g_{2eff} = \sqrt{g_1 g_2}$, that is, the theory becomes a one coupling constant theory with negative cosmological constant $\Lambda \sim -g_{eff}^2 / \kappa^2$. If g_1

and g_2 have different signs and magnitudes, one finds $g_{1eff} = -g_{2eff} = \sqrt{-g_1 g_2}$, that is, a one coupling constant theory with positive cosmological constant $\Lambda \sim g_{eff}^2/\kappa^2$. The case when either g_1 or g_2 is zero is inequivalent to the above two cases and is a particular case of the gauged SU(2) ⊗ SU(2) model[7].

The presence of scalars in the kinetic terms for the vectors has implications for the gauged SU(2) ⊗ SU(2) model [7] and for the gauged N = 5 supergravity and they are discussed here too. The case of N = 8 supergravity and the case of dimensionally-reduced theories of gravity, in which the scalar fields appear as an internal metric for the vector fields, will not be discussed.

In the second section of this Chapter, the gauged N = 4 supergravities are formulated in superspace. It is shown that in order to obtain the gauged theories it is sufficient to replace the central charge (commuting) generators of the ungauged theories by generators of SO(4) rotations. The methods described in this section are used to derive the gauged SO(4) model with two parameters and to rederive the gauged SU(2) ⊗ SU(2) model. They are explained here in some detail, for they could be useful for other problems.

In Section 3 the scalar potentials are obtained from the superspace geometry in two different ways. As was conjectured in ref. [10], the scalar potentials are seen to be quadratic expressions in terms of the U and $\overline{V}$ functions that determine the superspace central charge field strength $F_{\underline{\alpha}\beta}{}^{ij}$ (a dimension zero tensor that specifies the way scalars appear in the term that rotates the vectors into the gravitinos under supersymmetry). Properties of the potentials, such as critical points and global limits, are examined.

In Section 4 the class of potentials of the gauged SO(4) model with two coupling constants are shown to fall into the three inequivalent cases mentioned above. The role of internal-space duality transformations is clarified. It is shown that if one performs an internal duality transformation on the vector fields of the known ungauged SO(4) theory (obtaining the form A^* of reference [11]), and then gauges the theory with $g_1 = g_2$ one obtains the positive cosmological constant model. Scalar inversions are also discussed.

In Section 5 the gauged N = 5 supergravity is examined in the vector and scalar sectors. The functions of the scalars that appear in the kinetic term for the vectors are obtained explicitly and are seen to be necessary to find the mass parameters for the vectors in the broken phase. The remaining SO(3) symmetry, however, turns out to be gauged with the same coupling constant as the original SO(5) symmetry.

Finally, some comments are made on issued concerning the stability of the potentials, charge renormalization, and gauging of subgroups.

2. Superspace Formulation of the Gauged N = 4 Supergravity Theories

In this section the superspace geometries necessary to describe the gauged N = 4 supergravities are derived. It is found that their description in superspace requires ***very few changes*** in the torsions, curvatures and constituency equations of the globally symmetric theories. In fact, the constraints for the gauged and ungauged theories are ***exactly*** the same. Furthermore, the changes in the superspace geometry are of the same type for gauging of the SO(4) and SU(4) models when expressed in terms of the functions U and $\overline{V}$ entering the central charge field strength $F_{\underline{\alpha}\,\underline{\beta}}{}^{ij}$.

Both in the SO(4) model and in the SU(4) model, the SO(4) global symmetry can be gauged. At the global level the SO(4) group has the direct product group SU(2) x SU(2) as covering group; in fact SO(4) is isomorphic to SU(2) x SU(2)/Z_2. At the level of the algebras, the so(4) algebra is the direct sum of two su(2)'s, so(4) = su(2) ⊕ su(2), this implies that the two su(2) subalgebras can be gauged with different coupling constants. Throughout this paper the gauged theories will be referred to as the gauged SO(4) or gauged SU(4) models implying that they have a local SO(4) invariance implemented with two coupling constants.

We discuss the gauging of the SO(4) model first. In the superspace formulation of the ungauged SO(4) theory (Chapter III) a set of real central charges Z_{ij} together with their associated gauge connection superfields $\varphi_A{}^{ij}$ were introduced in the supercovariant derivative and in the graded commutator:

$$D_A = E_A{}^M \partial_M + \frac{1}{2}\,\varphi_{A\gamma}{}^{\delta} M_{\delta}{}^{\gamma} + \frac{1}{2}\,\overline{\varphi}_{A\dot{\gamma}}{}^{\dot{\delta}} \overline{M}_{\dot{\delta}}{}^{\dot{\gamma}} + \frac{1}{2}\,\varphi_A{}^{ij}\, Z_{ji}\;, \tag{2.1}$$

$$[D_A, D_B\} = T_{AB}{}^C D_C + \frac{1}{2} R_{AB\gamma}{}^{\delta} M_{\delta}{}^{\gamma} + \frac{1}{2}\overline{R}_{AB\dot{\gamma}}{}^{\dot{\delta}} \overline{M}_{\dot{\delta}}{}^{\dot{\gamma}} + \frac{1}{2} F_{AB}{}^{ij}\, Z_{ji}\;, \tag{2.2}$$

In order to make contact with the component formulation, one adopts a Wess-Zumino gauge in which the above supercovariant derivative takes the form:

$$D_{\alpha i} = \partial_{\alpha i} + O(\theta), \quad \overline{D}_{\dot{\alpha}}^{i} = \partial_{\dot{\alpha}}^{i} + O(\theta),$$

$$D_a = e_a{}^m \partial_m + \frac{1}{2} \varphi_{a\gamma}{}^{\delta} M_{\delta}{}^{\gamma} + \frac{1}{2} \overline{\varphi}_{a\dot{\gamma}}{}^{\dot{\delta}} M_{\dot{\delta}}{}^{\dot{\gamma}}$$

$$+ \psi_a{}^{\mu i} \partial_{\mu i} + \overline{\psi}_a{}^{\dot{\mu}}{}_i \, \partial_{\dot{\mu}}^{i} + \frac{1}{2} \varphi_a{}^{ij} Z_{ji} + O(\theta). \tag{2.3}$$

As it can be seen in equation (2.3), the vector fields $\varphi_a{}^{ij}$ appear in the supercovariant derivative. Nevertheless, the theory is ungauged because the central charge generators are defined to annihilate any superfield. No field transforms under a central-charge transformation except the vector fields themselves. Indeed, under a central-charge transformation parametrized by $K = \frac{1}{2} \varepsilon^{ij} Z_{ij}$:

$$\delta D_a = [K, D_a] = -\frac{1}{2} \partial_a \varepsilon^{ij} Z_{ji},$$

$$\delta \varphi_a{}^{ij} = -\partial_a \varepsilon^{ij}(x). \tag{2.4}$$

Therefore the Z_{ij} are seen to be six generators for the $[\mathrm{U}(1)]^6$ group associated with the gauge invariance of the spin-one fields of the theory.

In order to generate the gauged theories one replaces the central charge generators Z_{ij} by generators t_{ij} of the group that is to be gauged. This replacement is done both in the supercovariant derivative and in the graded commutation:

$$\varphi_A{}^{ij} Z_{ji} \to \varphi_A{}^{ij} t_{ji}, \quad F_{AB}{}^{ij} \to F_{AB}{}^{ij} t_{ji}. \tag{2.5}$$

Suppose the t_{ij}'s generate SO(N), namely they satisfy $[t_{ij}, t_{kl}] = \delta_{jk} t_{il} - \delta_{ik} t_{jl} - (k \leftrightarrow l)$. The defining or vector representation of SO(N) is: $(t_{ij})_{kl} = \delta_{k[i} \delta_{j]l}$. Thus, in order to have the N gravitinos transforming under this representation, one has to impose:

$$[t_{ij}, D_{\alpha k}] = g \kappa^{-1} \delta_{k[i} \delta_{j]l} D_{\alpha l},$$

$$[t_{ij},\overline{D}_a^{\cdot k}] = g\kappa^{-1}\delta_{k[i}\delta_{j]l}\overline{D}_a^{\cdot l},$$

$$[t_{ij},D_a] = 0. \tag{2.6}$$

In equation (2.6) g is a dimensionless coupling constant and κ is the gravitational constant, which is needed to give the t_{ij} generators a dimension of mass (equal to that of the Z_{ij} generators). Equations (2.3) and (2.6) imply that the gravitinos transform under the desired representation, that the graviton is a singlet, and that the vectors transform according to the adjoint representation of SO(N).

Equations (2.6) are applicable for all extended supergravities. For SO(4) supergravity, however, they are only a particular case, since they lead to gauged theories with one coupling constant. In the case of N = 4, it is necessary to untangle the two SU(2) subalgebras. Let X_Γ and Y_Γ with Γ = 1,2,3 be antihermitian generators of SU(2) $\otimes$ SU(2). In terms of the SO(4) generators they are given by:

$$X_\Gamma = \frac{1}{2}\,\varepsilon_{\Gamma\Pi\Lambda}\,t_{\Gamma\Lambda} + t_{\Gamma 4},$$

$$Y_\Gamma = \frac{1}{2}\,\varepsilon_{\Gamma\Pi\Lambda}\,t_{\Pi\Lambda} - t_{\Gamma 4}, \tag{2.7}$$

where $\varepsilon_{\Gamma\Pi\Lambda}$ is the totally antisymmetric Levi-Civita tensor with $\varepsilon_{123} = 1$. Equation (2.7) implies that the α_{ij}^Γ and β_{ij}^Γ matrices of ref. [6] are 4 x 4 real antisymmetric matrices representing the generators X_Γ and Y_Γ respectively. Inverting equation (2.7) one finds:

$$t_{ij} = \kappa^{-1}[\alpha_{ij}^\Gamma X_\Gamma + \beta_{ij}^\Gamma Y_\Gamma]. \tag{2.8}$$

where κ has been inserted for dimensional reasons, and a normalization for the t_{ij} generators has been chosen. Equation (2.8) is the desired expression for the SO(4) generators. Now the action of the X^Γ and Y^Γ generators on the supercovariant derivative has to be given:

$$[X^\Gamma, D_{\alpha i}] = g_1\, \alpha^\Gamma_{ij} D_{\alpha j}, \quad [X^\Gamma, \overline{D}^i_{\dot\alpha}] = g_1\, \alpha^\Gamma_{ij}\, \overline{D}^{\cdot j}_{\dot\alpha},$$

$$[Y^\Gamma, D_{\alpha i}] = g_2\, \beta^\Gamma_{ij} D_{\alpha j}, \quad [Y^\Gamma, \overline{D}^i_{\dot\alpha}] = g_2\, \beta^\Gamma_{ij} \overline{D}^{\cdot j}_{\dot\alpha},$$

$$[X^\Gamma, D_a] = [Y^\Gamma, D_a] = 0. \tag{2.9}$$

$$[X^\Gamma, X^\Pi] = -2g_1\, \varepsilon^{\Gamma\Pi\Lambda} X^\Lambda, \quad [Y^\Gamma, Y^\Pi] = -2g_2\, \varepsilon^{\Gamma\Pi\Lambda} Y^\Lambda,$$

$$[X^\Gamma, Y^\Pi] = 0, \tag{2.10}$$

where g_1 and g_2 are the dimensionless coupling constants associated with the two SU(2)'s. (Rigorously speaking, the X_Γ and X_Γ generators in equations (2.8) to (2.10) have been rescaled.) From equations (2.8) and (2.9) one derives:

$$[t_{ij}, D_{\alpha k}] = \kappa^{-1}[g_+ \delta_{k[i}\delta_{j]l} + g_- C_{ijkl}] D_{\alpha l},$$

$$[t_{ij}, \overline{D}^{\cdot k}_{\dot\alpha}] = \kappa^{-1}[g_+ \delta_{k[i}\delta_{j]l} + g_- C_{ijkl}] \overline{D}^{\cdot l}_{\dot\alpha},$$

$$[t_{ij}, D_a] = 0. \tag{2.11}$$

where $g_\pm = g_1 \pm g_2$. Equation (2.6) for N = 4 is a particular case of equation (2.11), namely the case in which $g_1 = g_2$. Using equations (2.8) and (2.10), one derives the following commutator for the SO(4) generators:

$$[t_{ij}, t_{kl}] = \frac{g_+}{2\kappa}(\delta_{jk} t_{il} - \delta_{ik} t_{jl} - (k \leftrightarrow l)) + \frac{g_-}{2\kappa}(\delta_{jk} t^*_{il} - \delta_{ik} t^*_{jl} - (k \leftrightarrow l)), \tag{2.12}$$

where $t^*_{ij} = \frac{1}{2} C_{ijkl} t_{kl}$.

Substituting equation (2.2) into the commutators given in equation (2.9), one derives the way component fields rotate under the two SU(2)'s:

$$\delta\psi^i_\mu = [\Lambda^\Gamma \alpha^\Gamma_{ij} + \Lambda'^\Gamma \beta_{ij}{}^\Gamma]\psi_\mu{}^j,$$

$$\delta\chi^i = [\Lambda^\Gamma \alpha_{ij}{}^\Gamma + \Lambda'^\Gamma \beta_{ij}{}^\Gamma]\chi^j,$$

$$\delta\varphi_\mu{}^\Sigma = 2\varepsilon^{\Sigma\Gamma\Pi}\,\Lambda^\Gamma \varphi'_\mu{}^\Pi,$$

$$\delta\varphi'_\mu{}^\Sigma = 2\varepsilon^{\Sigma\Gamma\Pi}\Lambda^\Gamma \varphi'_\mu{}^\Pi, \tag{2.13}$$

where $\varphi_\mu{}^\Gamma$ and $\varphi'_\mu{}^\Gamma$ are linear combinations of the gauge vectors $A_\mu{}^{ij}$:

$$\varphi_\mu^\Gamma = \frac{1}{2\sqrt{2}}\,\alpha_{ij}{}^\Gamma A_\mu{}^{ij}\ ,\quad \varphi'_\mu{}^\Gamma = \frac{1}{2\sqrt{2}}\,\beta_{ij}{}^\Gamma\, A_\mu{}^{ij}. \tag{2.14}$$

The above transformations under the two SU(2)'s are just $\varphi^i, \chi^i \sim (2,2)$, $A_\mu{}^{ij} \sim (3,1) + (1,3)$ and the graviton and scalars are singlets. The superspace geometry that describes the ungauged theories has to be modified in order to accommodate the gauged theories. Once the Z_{ij} generators are replaced by the t_{ij} generators, the Bianchi identities are modified. If we denote by $I_{ABC}{}^D$ the Bianchi identity obtained by equating to zero the coefficient of D_D in the expansion of $[[D_{[A}, D_B\}, D_{C)}\}$, then only the $I_{ABC}{}^D$ identities, where D is a spinorial index, are changed. Explicitly, these identities are changed from their ungauged values by an amount $\Delta_{gauge}\ I_{ABC}{}^D$ where:

$$\Delta_{gauge} I_{ABC}{}^D = \sum_{(ABC)} \frac{1}{2}\, F_{AB}{}^{ij}\,(t_{ji})_C{}^D, \tag{2.15}$$

and $(t_{ij})_C{}^D$ denotes the appropriate matrix representation of the generators t_{ij} which can be obtained from equation (2.11). Since $F_{\underline{\alpha}\dot{\underline{\beta}}}{}^{ij} = 0$, and $(t_{ij})_C{}^D$ vanishes unless C and D are spinor indices of the same type, the only identities that are modified are $I_{\underline{\alpha}\underline{\beta}\underline{\gamma}}{}^{\underline{\delta}}$, $I_{\underline{\alpha}\underline{\beta}\dot{\underline{\gamma}}}{}^{\dot{\underline{\delta}}}$, $I_{\alpha\dot{\alpha}\underline{\beta}\underline{\gamma}}{}^{\underline{\delta}}$, $I_{\alpha\dot{\alpha}\dot{\underline{\beta}}\underline{\gamma}}{}^{\underline{\delta}}$ and $I_{\alpha\dot{\alpha}\beta\dot{\underline{\beta}}\underline{\gamma}}{}^{\underline{\delta}}$. The modified Bianchi identities have to be solved in order to obtain the superspace geometry for the gauged SO(4) theory. This requires considerably less work than that necessary to solve the Bianchi identities for the ungauged theory. One only needs to find the modifications of the previously derived results.

Since the Bianchi identities are solved in order of increasing dimensionality, and the lowest dimension identities which change are of dimension one, all dimension zero and one-half results remain unchanged. In particular, none of the field strengths of dimension less than one can be modified. An explicit calculation shows that the dimension one field strength $F_{ab}{}^{ij}$ also remains unchanged. Therefore, ***the field strength sector of the theories is not explicitly altered by the gauging***. At the component level, this implies that neither the supersymmetry transformation of the vector fields nor the way the Fermi fields rotate into the vectors is altered.

Only two of the superspace torsions are changed from their ungauged values given in Chapter III. A calculation gives:

$$\Delta_{gauge}\; T_{a\dot{\alpha}\underline{\beta}}{}^{\dot{\underline{\gamma}}} = -\frac{i}{2}\,\kappa^{-1}\,\mathbf{U}\,C_{\alpha\beta}\,\delta_{\dot{\alpha}}{}^{\dot{\gamma}}\,\delta_{ij}\,,$$

$$\Delta_{gauge}\; T_{a\dot{\alpha}\beta\dot{\beta}\dot{\underline{\gamma}}} = \frac{1}{12}\,\kappa^{-1}\mathbf{V}\,C_{\dot{\alpha}\dot{\beta}}C_{\gamma(\alpha}\Lambda_{\beta)i}\,, \tag{2.16}$$

where we have defined for the SO(4) model:

$$SO(4): \qquad \mathbf{U} = Ug_{+} + \bar{V}g_{-},$$

$$\mathbf{V} = Vg_{+} + \bar{U}g_{-}. \tag{2.17}$$

The U and $\bar{V}$ functions determine the $F_{\underline{\alpha}\underline{\beta}}{}^{mn}$ tensor:

$$F_{\underline{\alpha}\underline{\beta}}{}^{mn} = C_{\alpha\beta}\,[\,U\delta_i^{[m}\delta_j^{n]} + \bar{V}C_{ij}{}^{mn}\,]\,, \tag{2.18}$$

and their standard values in terms of the chiral superfield W are:

$$U = \frac{1}{\sqrt{1 - W\bar{W}}}\,,\quad \bar{V} = \frac{\bar{W}}{\sqrt{1 - W\bar{W}}}\,,\quad W = -A + iB\,. \tag{2.19}$$

The changes in the curvature supertensor are found to be:

$$\Delta_{gauge}\; R_{\underline{\alpha}\underline{\beta}\gamma\delta} = -\kappa^{-1}\mathbf{U}C_{\alpha(\gamma}C_{\delta)\beta}\,\delta_{ij}\,,$$

$$\Delta_{gauge}\ R_{\underline{\alpha}\beta\dot{\beta}\dot{\gamma}\dot{\delta}} = \frac{i}{6}\,\kappa^{-1}\mathbf{V}C_{\alpha\beta}C_{\dot{\beta}(\dot{\gamma}}\bar{\Lambda}_{\dot{\delta})i},$$

$$\Delta_{gauge}\ R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta} = \frac{1}{12}\,\kappa^{-2}C_{\dot{\alpha}\dot{\beta}}C_{\alpha(\gamma}C_{\delta)\beta}(3|\mathbf{U}|^2 - |\mathbf{V}|^2), \tag{2.20}$$

while no other curvature gets modified. The constituency equations are also modified by the gauging. The constituency equations determine the supersymmetry transformation laws for covariant quantities. They are necessary to find the supersymmetries for the scalars and for the spinors, since these are not gauge fields appearing in the supercovariant derivative. The extra terms with respect to the ungauged values are:

$$\Delta_{gauge}\ D_{\underline{\alpha}}\Lambda_{\underline{\beta}} = \kappa^{-1}\mathbf{V}\ C_{\alpha\beta}\,\delta_{ij},$$

$$\Delta_{gauge}\ D_{\underline{\alpha}}\ f_{\beta\gamma}{}^{jk} = \frac{1}{6}\,\kappa^{-1}\bar{\mathbf{V}}\ C_{\alpha(\beta}\Lambda_{\gamma)}{}^{[j}\delta_i^{k]} + \frac{1}{2}\,\kappa^{-1}\mathbf{U}\ C_i{}^{jkl}\,C_{\alpha(\beta}\Lambda_{\gamma)l},$$

$$\Delta_{gauge}\ D_{\underline{\alpha}}\Sigma_{\beta\gamma\delta}{}^{j} = \frac{1}{12}\,\kappa^{-1}\bar{\mathbf{V}}\ C_{\alpha(\beta}f_{\gamma\delta)}{}^{kl}\,C_{kli}{}^{j} + \frac{1}{2}\,\kappa^{-1}\mathbf{U}\ C_{\alpha(\beta}f_{\gamma\delta)i}{}^{j},$$

$$\Delta_{gauge}\ D_{\underline{\alpha}}\ V_{\beta\gamma\delta\varepsilon} = \frac{1}{6}\,\kappa^{-1}\mathbf{U}\ C_{\alpha(\beta}\Sigma_{\gamma\delta\varepsilon)i}. \tag{2.21}$$

Equations (2.16) to (2.21) indicate all the changes in the superspace necessary to describe the gauged SO(4) model. It is straightforward to find the changes in the supersymmetry transformation laws for the component fields. For the spinors one has (see equation (3.31) of Chapter II):

$$\delta'\Lambda_{\beta j} = \varepsilon^{\alpha i}\Delta_{gauge}\ D_{\alpha i}\Lambda_{\beta j}. \tag{2.22}$$

Using the first equation in (2.16) and translating into four-component notation (using the rescalings of Chapter III, Section 4):

$$\delta'\bar{\chi}^i = -\frac{\bar{\varepsilon}^i}{\sqrt{2}\,\kappa^2}\,||\mathbf{V}||, \tag{2.23}$$

where, as before the $||..||$ notation is defined by $||f|| = Re(f) + i\gamma_5 Im(f)$. For the gravitino, using equation (3.29) of Chapter II, one has:

$$\delta'\psi_{\mu\alpha}{}^{i} = \frac{1}{2}\,\sigma_{\mu}{}^{\delta\dot{\delta}}\,\bar{\varepsilon}^{\dot{\gamma}k}\,\Delta_{gauge}\;T_{\delta\dot{\delta}\dot{\gamma}}{}^{k}{}_{\alpha}{}^{i}. \tag{2.24}$$

Making use of the first equation in (2.16), one finds:

$$\delta'\bar{\psi}^{i}_{\mu} = \frac{i}{2\kappa^2}\,\bar{\varepsilon}^{i}\gamma_{\mu}\|\overline{\mathbf{U}}\|. \tag{2.25}$$

The expressions for the supersymmetry transformation laws given in eqs. (2.23) and (2.25) are quite general since they do not assume the standard values for U and $\bar{V}$. The present expressions will be used to study the effect of internal space dualities of the vector fields on the scalar potential. They will also enable us to study the gauging of the SO(4) theory working in the exterior of the unit disc $|W|^2 \leq 1$.

The explicit expression for the supercovariant derivative acting on the Fermi fields is found using eqs. (2.3), (2.5), (2.8) and (2.9):

$$\widehat{D}_{\mu}\psi_{\nu}{}^{k} = (\partial_{\mu} + \frac{1}{2}\,\omega_{\mu ab}\,\sigma^{ab})\,\psi_{\nu}{}^{k} + (g_{+}A_{\mu}{}^{kl} + g_{-}A^{*kl}_{\mu})\,\psi_{\nu}{}^{l}, \tag{2.26}$$

where $A^{*kl}_{\mu} = \frac{1}{2}\,\varepsilon^{klpq}\,A^{pq}_{\mu}$. Analogous expressions hold for $\widehat{D}_{\mu}\chi^{i}$ and $\widehat{D}_{\mu}\varepsilon^{i}$. The Yang-Mills field strength is defined from the commutator of two spacetime derivatives:

$$[D_a, D_b] = \frac{1}{2}\,F_{ab}{}^{ij}\,t_{ji} + (\text{more terms}). \tag{2.27}$$

From this one derives:

$$F^{ij}_{\mu\nu} = \partial_{\mu}A^{ij}_{\nu} - \partial_{\nu}A^{ij}_{\mu} - 2g_1\vec{\alpha}_{ij}\cdot(\vec{\Phi}_{\mu}\times\vec{\Phi}_{\nu}) - 2g_2\vec{\beta}_{ij}\cdot(\vec{\Phi}'_{\mu}\times\vec{\Phi}'_{\nu}). \tag{2.28}$$

or using equation (2.12)

$$F^{ij}_{\mu\nu} = \partial_{\mu}A^{ij}_{\nu} - \partial_{\nu}A^{ij}_{\mu} + g_{+}A_{[\mu}{}^{ip}\,A_{\nu]}{}^{pj} + g_{-}C_{ijmn}A^{mp}_{\mu}A^{pn}_{\nu}\,. \tag{2.29}$$

One can also define field strengths for the φ^{Γ}_{μ} and φ'^{Γ}_{μ} fields. From equation (2.14) one has:

$$\varphi_{\mu\nu}^{\Gamma} = \partial_\mu \varphi_\nu^\Gamma - \partial_\nu \varphi_\mu^\Gamma - 2\sqrt{2} g_1 (\vec{\varphi}_\mu \times \vec{\varphi}_\nu)^\Gamma ,$$

$$\varphi'^{\Gamma}_{\mu\nu} = \partial_\mu \varphi'^{\Gamma}_\nu - \partial_\nu \varphi'^{\Gamma}_\mu - 2\sqrt{2} g_2 (\vec{\varphi}'_\mu \times \vec{\varphi}'_\nu)^\Gamma . \tag{2.30}$$

In order to preserve the local supersymmetry invariance of the SO(4) model the following terms have to be added to the Lagrangian:

$$L' = \frac{e}{\kappa} \bar{\psi}_\mu{}^i \sigma^{\mu\nu} ||\mathbf{U}|| \psi_\nu{}^i - \frac{ie}{\sqrt{2}\kappa} \bar{\psi}^i \cdot \gamma \, ||\mathbf{V}|| \chi^i$$

$$+ \frac{e}{2\kappa^4} (3|\mathbf{U}|^2 - |\mathbf{V}|^2), \tag{2.31}$$

where $e = \det(e_{a\mu})$. Again, equation (2.31) is quite general. The last term is the scalar potential. Its properties will be discussed in the next section.

The gauging of the SU(4) model [7] can be done in superspace in complete analogy to the gauging of the SO(4) model. For the SU(4) theory the following replacements have to be made in the supercovariant derivative and in the graded commutator of Chapter III:

$$\varphi_A{}^{ij} Z_{ij} + \bar{\varphi}_{A\,ij} Z^{ji} \to \varphi_A{}^{ij} t'_{ji} + \bar{\varphi}_{A\,ij} \bar{t}'^{ji},$$

$$G_{AB}{}^{ij} Z_{ji} + \tilde{G}_{AB\,ij} Z^{ji} \to G_{AB}{}^{ij} t'_{ji} + \tilde{G}_{AB\,ij} \bar{t}'^{ji} . \tag{2.32}$$

Here, $\varphi_a{}^{ij}$ and t'_{ij} are defined to be:

$$\varphi_a{}^{ij} = \frac{1}{2\sqrt{2}} \left[\alpha_{ij}^\Gamma A_a^\Gamma - i \beta_{ij}^\Gamma B_a^\Gamma\right],$$

$$t'_{ij} = \kappa^{-1}\left[\alpha_{ij}^\Gamma X^\Gamma + i \beta_{ij}^\Gamma Y^\Gamma\right], \tag{2.33}$$

where A_a^Γ and B_a^Γ are the three vectors and the three axial vectors of the SU(4) model, and the X^Γ and Y^Γ generators were defined in equation (2.9). The objects $\bar{\varphi}_{a\,ij}$ and $\bar{t}'^{ij}$ are just complex conjugates of φ_a^{ij} and t'_{ij}.

Using equations (2.9) and (2.33) one derives:

$$[t'_{ij}, D_{\delta l}] = \kappa^{-1}[\lambda\, \delta_i^{[l}\delta_j^{m]} + \bar{\lambda} C_{ij}{}^{lm}]D_{\delta m},$$

$$[t'_{ij}, \bar{D}_\delta^l] = \kappa^{-1}[\lambda\, \delta_{i[l}\delta_{m]j} + \bar{\lambda}\, C_{ijlm}]\bar{D}_\delta^m,$$

$$[t'_{ij}, D_{a\dot{a}}] = 0, \tag{2.34}$$

where $\lambda = g_1 + ig_2$. It also follows that the action of $\bar{t}'_{ij}$ on the supercovariant derivative can be found by interchanging λ and $\bar{\lambda}$ in the above equations.

Exactly the same Bianchi identities that were modified in the SO(4) case now get modified in the SU(4) theory, but now one finds:

$$\Delta_{gauge}\, I_{ABC}{}^D = \sum_{(ABC)} \left[\frac{1}{2}\, G_{AB}{}^{ij}\, (t'_{ji})_C{}^D + \frac{1}{2}\, \tilde{G}_{ABji}\, (\bar{t}'^{ji})_C{}^D \right]. \tag{2.35}$$

In this expression the representation matrices t'_{ij} and $\bar{t}'^{ij}$ can be obtained from the results in equation (2.34) and their conjugate equations.

As a consequence of the above changes the superspace tensors that describe the SU(4) theory are modified. These changes are essentially the same as in the SO(4) case with a slight change in the definition of the **U** and $\bar{\mathbf{V}}$ functions:

$$SU(4): \qquad \mathbf{U} = 2\lambda U = 2\lambda e^{\kappa\varphi},$$

$$\mathbf{V} = 2\lambda V = 2\lambda e^{\kappa\varphi}. \tag{2.36}$$

With these definitions, equations (2.16), (2.20) and (2.21) give the superspace tensors for the gauging of SO(4) in the SU(4) model. Furthermore, with the values of **U** and **V** given in equation (2.36), equations (2.23) and (2.25) give the extra terms in the supersymmetry transformations of the Fermi fields, and equation (2.31) gives the extra terms needed in the Lagrangian. These results are in complete agreement with the ones given by Freedman and Schwarz [7] after a numerical rescaling of the gauge coupling constants $(\lambda = \frac{1}{2\sqrt{2}}\,(e_A + ie_B))$.

3. The Scalar Potentials

3.1 Scalar Potentials from Superspace

In the previous section the superspace formulation was used to derive the extra terms in the supersymmetry transformations of the Fermi fields. Then, the Noether procedure was used to find the extra terms needed in the Lagrangian, in particular the scalar potential. In the gauging of the SO(8) supergravity in superspace [13], superspace was also used to derive the supersymmetries, but not the scalar potential. In this section two ways of obtaining the scalar potential from the superspace formulation are discussed.

The first way to find the scalar potential out of the superspace formulation is to get it from $\Delta_{gauge}\ R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$:

$$\Delta_{gauge}\ R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta} = -\frac{2}{3}\kappa^2\, C_{\dot{\alpha}\dot{\beta}}\, C_{\alpha(\gamma}C_{\delta)\beta}\, P(W,\overline{W}),$$

$$P(W,\overline{W}) = -\frac{1}{8\kappa^4}\,(3|\mathbf{U}|^2 - |\mathbf{V}|^2). \qquad (3.1.1)$$

$P(W,\overline{W})$ is the scalar potential, and the reason for this can be easily understood. It is known from the component formulation that, in order to maintain the supersymmetry of the gauged theories, it is necessary to add a scalar potential to the action of the ungauged theories. The variation of this action with respect to $V_{a\mu}$ produces the equation of motion of the graviton $R_{ab} - \frac{1}{2}\delta_{ab}\,R = \kappa^2\,T_{ab}$, where R_{ab} is the Ricci tensor and T_{ab} is the energy-momentum tensor. Since the superspace formulation is on-shell, the calculation of the superspace torsions and curvatures implicitly uses the equations of motion for all the component fields. In particular, by contracting the x-space supercovariant Riemann curvature $R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$, one can reconstruct the energy-momentum tensor T_{ab} and form the equation of motion for the graviton. Thus, the addition of the scalar potential to the action is manifest in superspace by the addition of the potential to $R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$. In other words, in a superspace formulation the scalar potential can be read off from $\Delta_{gauge}\, R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$. Let us show explicitly that $P(W,\overline{W})$ is the potential for the scalars that appears in the component Lagrangian. From equation (3.1), we derive:

$$\Delta_{gauge}\, R_{ab} = \Delta_{gauge}\, R_{abd}{}^{d} = -\,2\kappa^2\, P(W,\overline{W})\eta_{ab}. \tag{3.1.2}$$

Therefore, the equation of motion of the graviton reads:

$$G_{ab} = R_{ab} - \frac{1}{2}\,\eta_{ab}R = 2\kappa^2 P(W,\overline{W})\,\eta_{ab} + (\text{more terms}), \tag{3.1.3}$$

and this equation of motion follows from the Lagrangian:

$$L = -\,\frac{eR}{4\kappa^2} - eP(W,\overline{W}) + (\text{more terms}), \tag{3.1.4}$$

in which $P(W,\overline{W})$ is seen to enter as a potential for the scalars.

The method of deriving the potential discussed above involves carrying out the analysis of the modified Bianchi identities up to dimension two, that is, the dimension of the x-space curvature $R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$. This involves a fair amount of work, since the dimension 3/2 identities are somewhat complicated. It is possible, however, to obtain some partial information about the potential for the scalars at an earlier stage of the calculation. Once the dimension one identities have been worked out one can produce some of the terms in the equation of motion for the scalars. In analogy to the case in ordinary field theory, where the equation of motion for a scalar field derived from the Lagrangian density $L = \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$ is $\Box\varphi + \frac{dV(\varphi)}{d\varphi} = 0$, one is able to determine the potential $V(\varphi)$ from the equation of motion up to an integration constant. Let us show the details of the calculation.

From the third constituency equation in equation (2.23) of Chapter III one has:

$$8ia^{-2}D_{\beta\dot{\alpha}}W = a^{-1}\overline{D}_{\dot{\alpha}}^{\cdot i}\Lambda_{\beta i} - 3a^{-1}\overline{Q}_{\dot{\alpha}}^{\cdot i}\Lambda_{\beta j}. \tag{3.1.5}$$

Applying another spacetime derivative to this expression and keeping only terms that can contribute to the potential, one finds:

$$8i\, a^{-2}D^{\beta\dot{\alpha}}D_{\beta\dot{\alpha}}W = a^{-1}[D^{\beta\dot{\alpha}},\overline{D}_{\dot{\alpha}}^{\cdot i}]\Lambda_{\beta i} + \ldots$$

$$= a^{-1}T^{\beta\dot{\alpha}}{}_{\dot{\alpha}}^{\cdot i\,\gamma k}\, D_{\gamma k}\Lambda_{\beta i} + \ldots \tag{3.1.6}$$

Both $T^{\beta\dot{\alpha}}{}_{\dot{\alpha}}^{\cdot i\gamma k}$ and $D_{\gamma k}\Lambda_{\beta i}$ are altered by the gauging. Therefore, using the modifications indicated in equations (2.16) and (2.21), one obtains:

$$\frac{1}{2\kappa^2}\, a^{-2}D^a D_a W = \frac{1}{4}\,\frac{a^{-1}}{\kappa^4}\,\overline{\mathbf{U}}\mathbf{V} + \ldots \tag{3.1.7}$$

Note that it is necessary to keep the factor a^{-2} in the left-hand side of eq. (3.1.7) because this factor appears in front of the kinetic term for the scalars in the ungauged Lagrangian. This is also apparent in the way the scalars enter into $R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\delta}$ in equation (2.22) of Chapter III. Equation (3.1.7) is the desired result. In its right-hand side there is a term containing only scalars. This term is just $(-\frac{\partial P}{\partial \overline{W}})$, where P is the potential for the scalars. For the gauging of the SO(4) model with:

$$U = \frac{1}{\sqrt{1-|W|^2}}, \quad \overline{V} = \frac{\overline{W}}{\sqrt{1-|W|^2}}, \tag{3.1.8}$$

one obtains:

$$-\frac{\partial P}{\partial \overline{W}} = \frac{(g_+^2 + g_-^2)}{4\kappa^4}\,\frac{W}{(1-W\overline{W})^2} + \frac{g_+g_-(1+WW)}{4\kappa^4(1-W\overline{W})^2}, \tag{3.1.9}$$

which upon integration becomes:

$$P(W,\overline{W}) = -\frac{1}{8\kappa^4}\left[(g_+^2 + g_-^2)(\frac{3 - W\overline{W}}{1 - W\overline{W}}) + 2g_+g_-\frac{(W + \overline{W})}{1 - W\overline{W}} + (const.)\right]. \quad (3.1.10)$$

The potential given in eq. (3.1.10) agrees precisely with the one obtained in equation (3.1.1), once we set the constant equal to $4g^2$. (The potential $P(W,\overline{W})$ determined from superspace agrees with the one given in eq. (2.31), once coupling constants are rescaled. This is a consequence of the rescalings necessary to go from the superspace formulation to the component formulation. For details see Chapter III, Section 4).

For the SU(4) theory

$$U = \overline{V} = \frac{1}{\sqrt{1-W-\overline{W}}}\ , \quad W = \frac{1}{2}(1 - e^{-2\kappa\varphi} - 2i\kappa B)\ , \qquad (3.1.11)$$

and one finds

$$P(W,\overline{W}) = -\frac{|\lambda|^2}{\kappa^4}\left[\frac{1}{1-W-\overline{W}} + (\text{constant})\right], \qquad (3.1.12)$$

which is just the potential obtained by Freedman and Schwarz [7] in their gauging of the SU(4) model. It should be remarked that the two methods discussed above to find the scalar potential from the superspace formulation require some modification in case there is a conformal factor in front of Einstein's term in the component Lagrangian.

3.2 Properties of the Scalar Potentials

Let us now discuss the properties of the scalar potentials of the gauged N = 4 supergravities. The gauged SO(4) model with two equal coupling constants [8], and the gauged SU(4) model [7] have the following scalar potentials:

$$P_{SO(4)} = -\frac{g^2}{2\kappa^4}\left[\frac{3-|W|^2}{1-|W|^2}\right], \tag{3.2.1}$$

$$P_{SU(4)} = -\frac{(e_A{}^2+e_B{}^2)}{8\kappa^4}\, e^{2\kappa\varphi}. \tag{3.2.2}$$

Both potentials are inverted and unbounded from below, as can be seen in figures 1 and 2. The SO(4) potential has a critical point for W = 0. This critical point defines a stable background state [2, 3]. The potential for the gauged SU(4) model does not have a critical point. Stability in this theory is achieved by giving some vacuum expectation value to the spin-one states of the theory [14].

Let us now turn to the gauged SO(4) model with two coupling constants, whose superspace formulation was given in Section 2. It will be shown in the next section that the class of potentials for this theory reduces to three inequivalent models. It is convenient, however, to display the form with the two parameters g_1 and g_2 for several reasons. It is a single expression applicable for the three inequivalent models. Its properties are necessary in order to motivate the transformations required to prove the equivalence of some of the theories. Finally, it is conceivable that once the SO(4) supergravity will be coupled to extra matter multiplets the situation could change completely. In what follows, the properties of the SO(4) model with two coupling constants are discussed.

The modified supersymmetry transformation laws and the extra terms required in the action are obtained using $U = (1-W\bar{W})^{-\frac{1}{2}}$ and $\bar{V} = \bar{W}U$ in

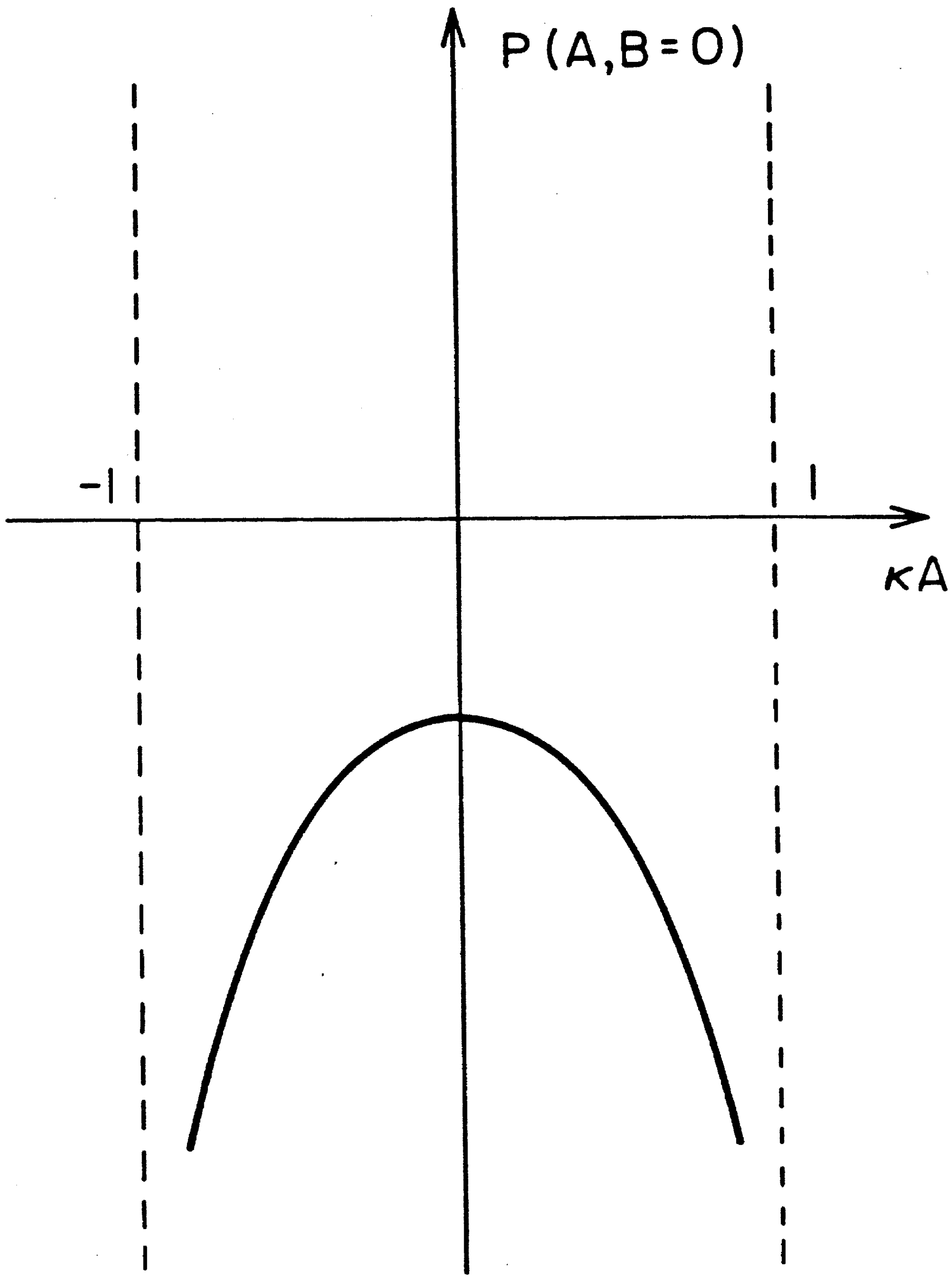

Figure 1. Scalar Potential for the gauged SO(4) supergravity with $g_1 = g_2 = g/2$.

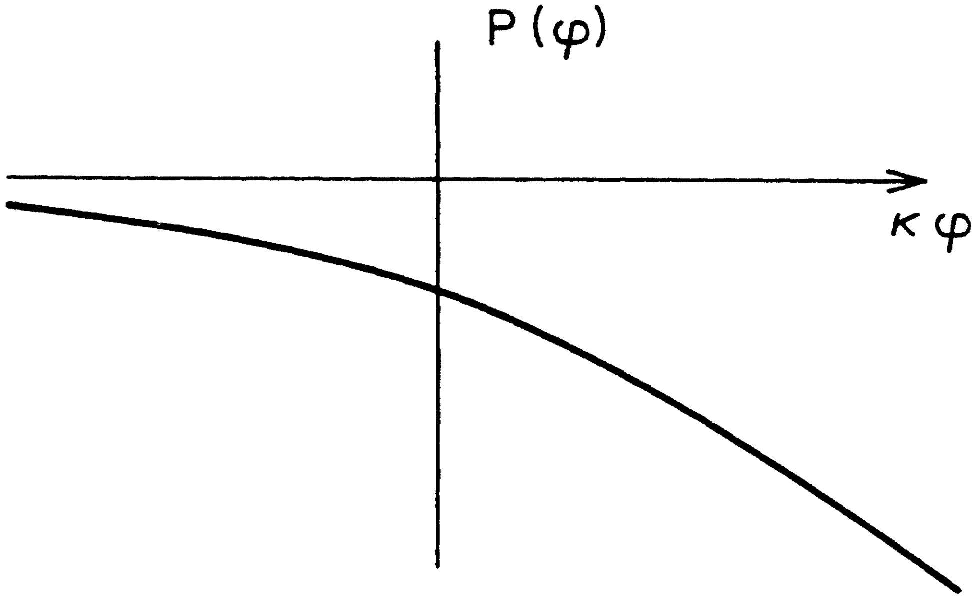

Figure 2. Scalar Potential for the SU(4) supergravity with local SU(2)⊗SU(2).

equations (2.23), (2.25), and (2.31):

$$\delta'\overline{\psi}^i_\mu = \frac{i}{2\kappa^2}\,\overline{\varepsilon}^i\gamma_\mu\,\frac{[g_+ + \kappa(-A+i\gamma_5 B)g_-]}{(1-|W|^2)^{\frac{1}{2}}}\,, \tag{3.2.3}$$

$$\delta'\overline{\chi}^i = \frac{\overline{\varepsilon}^i}{\sqrt{2}\kappa^2}\,\frac{[\kappa(A-i\gamma_5 B)g_+ - g_-]}{(1-|W|^2)^{\frac{1}{2}}}\,, \tag{3.2.4}$$

$$L' = \frac{e}{\kappa(1-|W|^2)^{\frac{1}{2}}}\,\overline{\psi}^i_\mu\sigma^{\mu\nu}\,[g_+ - \kappa(A+i\gamma_5 B)g_-]\psi^i_\nu$$

$$+ \frac{ie}{\sqrt{2}\kappa(1-|W|^2)^{\frac{1}{2}}}\overline{\psi}^i\cdot\gamma\,[\kappa(A+i\gamma_5 B)g_+ - g_-]\chi^i$$

$$+ \frac{e}{2\kappa^4}\left[g_+^2\left(\frac{3-|W|^2}{1-|W|^2}\right) - g_-^2\left(\frac{1-3|W|^2}{1-|W|^2}\right) - \frac{4g_+g_-(\kappa A)}{1-|W|^2}\right]. \tag{3.2.5}$$

For the special case of $g_1 = g_2 = g/2$, the scalar potential, given in the last term of equation (3.2.5) agrees with the one given in equation (3.2.1). The critical point of the potential and its associated cosmological constant Λ depend on the values of the parameters g_1 and g_2. The critical points are found to be:

For $g_1 g_2 > 0$:

$$\kappa A = \frac{g_-}{g_+},\quad B = 0,\quad \Lambda = \kappa^2 P(W_{crit}) = -\frac{6}{\kappa^2}\,g_1 g_2. \tag{3.2.6}$$

For $g_1 g_2 < 0$:

$$\kappa A = \frac{g_+}{g_-},\quad B = 0,\quad \Lambda = \kappa^2 P(W_{crit}) = -\frac{2}{\kappa^2}\,g_1 g_2. \tag{3.2.7}$$

If the two coupling constants have the same sign, the cosmological constant is negative. If they have opposite signs, the cosmological constant is positive. The cosmological constant, however, cannot be zero. For the case $g_+ = \pm g_-$ which would at first appear to give zero cosmological constant, the scalar potential reduces to:

$$P(W) = -\frac{g_+^2}{\kappa^4}\left[\frac{(1-\kappa A)^2 + (\kappa B)^2}{1-\kappa^2(A^2+B^2)}\right]. \tag{3.2.8}$$

For $\kappa A = 1$ and $B = 0$, $P(W) = 0$, but this point is not a critical point of the potential, therefore the above values do not satisfy the equation of motion of the scalars and it does not make sense to associate with this point a cosmological constant.

For negative cosmological constant the background state defined by:

$$\psi_\mu^i = \chi^i = 0\,,\quad A_\mu^{ij} = 0\,,\quad \kappa A = \frac{g_-}{g_+}\,,\quad B = 0\,, \tag{3.2.9}$$

is invariant under the global supersymmetry of OSp(4,4). The bosonic fields are clearly invariant because the fermionic fields are initially zero. The spinors χ^i are invariant as one can see by inspection of equation (3.14b). Finally the gravitinos are invariant if:

$$\delta\overline{\psi}_\mu^i = \frac{1}{\kappa}\overline{\varepsilon}^i D_\mu + \frac{i}{\kappa^2}\overline{\varepsilon}^i\gamma_\mu\sqrt{g_1 g_2} = 0\,, \tag{3.2.10}$$

that is, if there exist four linearly-independent covariantly-constant spinors $\varepsilon^i(x)$, called Killing spinors, satisfying equation (3.2.10). Such solutions are well-known to exist [15,2]. Being invariant under global supersymmetry, it follows [2,3] that the negative-cosmological constant backgrounds are stable to all orders against fluctuations.

The supersymmetric background, given in equation (3.2.9), can be used to derive the rigid superalgebra for the gauged SO(4) theories. The superspace formulation for the theory, that was obtained by solving the Bianchi identities, is nothing else than a consistent local supersymmetry algebra. The commutators that define the local algebra are just the commutators of supercovariant derivatives:

$$[D_A, D_B\} = T_{AB}{}^C D_C + \frac{1}{2}R_{AB\gamma}{}^\delta M_\delta{}^\gamma + \frac{1}{2}\overline{R}_{AB\dot\gamma}{}^{\dot\delta}\overline{M}_{\dot\delta}{}^{\dot\gamma} + \frac{1}{2}F_{AB}{}^{ij}t_{ji}\,. \tag{3.2.11}$$

In order to construct the global limit, one identifies the spinorial derivative $D_{\alpha i}$ with the generator of supersymmetry $Q_{\alpha i}$ and the vectorial derivative $D_{\alpha\dot{\alpha}}$ with the translation generator $P_{\alpha\dot{\alpha}}$. Furthermore, one has to take the global limit of the torsions, curvatures and field strengths appearing in the right hand side of equation (3.2.11), using the values given in equation (3.2.9). Consider the $\{D_{\alpha i}, D_{\beta j}\}$ commutator. $T_{\underline{\alpha}\underline{\beta}}{}^{c}$ is zero by constraint, $T_{\underline{\alpha}\underline{\beta}}{}^{\gamma}$ and $T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}}$ are proportional to the spinors (see Chapter III), and therefore vanish in the background, $R_{\underline{\alpha}\underline{\beta}\dot{\gamma}\dot{\delta}}$ is proportional to the spin one field strength (Chapter III), and it vanishes too. One is left with:

$$\{Q_{\alpha i}, Q_{\beta j}\} = -\frac{1}{2} R_{\alpha i\, \beta j\, \gamma\delta} M^{\gamma\delta} + \frac{1}{2} F_{\alpha i\, \beta j}{}^{mn}\, t_{nm} \,,$$

$$= -\frac{1}{\kappa} \mathrm{U} \delta_{ij}\, M_{\alpha\beta} + \frac{1}{2} C_{\alpha\beta} (U \delta_i^{[m} \delta_j^{n]} + \overline{V} C_{ij}{}^{mn}) t_{nm} \,, \qquad (3.2.12)$$

where use has been made of equations (2.18), (2.19) and (2.20). Use of equations (2.8), (2.17) and (3.2.9) finally gives:

$$\{Q_{\alpha i}, Q_{\beta j}\} = -\frac{2}{\kappa} \sqrt{g_1 g_2}\, \delta_{ij}\, M_{\alpha\beta} - \frac{C_{\alpha\beta}}{\kappa} \left[\left(\frac{g_2}{g_1}\right)^{\frac{1}{2}} \alpha_{ij}^{\Gamma} X^{\Gamma} + \left(\frac{g_1}{g_2}\right)^{\frac{1}{2}} \beta_{ij}^{\Gamma} Y^{\Gamma} \right] . \quad (3.2.13)$$

The remaining commutators of the superalgebra are found in a completely analogous way. They are:

$$\{Q_{\alpha i}, \overline{Q}_{\dot{\beta}}^{j}\} = 2i \delta_i^j P_{\alpha\dot{\beta}} \,, \qquad (3.2.14)$$

$$[Q_{\alpha i}, P_{\beta\dot{\beta}}] = -\frac{i}{\kappa} \sqrt{g_1 g_2}\, C_{\alpha\beta}\, \overline{Q}_{\dot{\beta} i} \,, \qquad (3.2.15)$$

$$[P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = \frac{g_1 g_2}{\kappa^2} (C_{\alpha\beta} \overline{M}_{\dot{\alpha}\dot{\beta}} + C_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta}) \,, \qquad (3.2.16)$$

$$[M_{\alpha\beta}, Q_{\gamma i}] = C_{\gamma(\alpha} Q_{\beta) i} \,, \quad [M_{\alpha\beta}, P_{\gamma\dot{\gamma}}] = C_{\gamma(\alpha} P_{\beta)\dot{\gamma}} \,, \qquad (3.2.17)$$

together with relations (2.9) and (2.10). All the Jacobi identities for this superalgebra have been explicitly verified. The fact that the background values in equation (3.2.9) led to a consistent superalgebra is a verification of the supersymmetry invariance of the background configuration. It should be noted that the above algebra is consistent only if $g_1 g_2 > 0$, that is, when the cosmological constant is negative. In the case of positive cosmological constant the background state that satisfies the equations of motion is

$$\psi^i_\mu = \chi^i = 0 \, , \quad A^{ij}_\mu = 0 \, , \quad \kappa A = \frac{g_+}{g_-} \, , \quad B = 0 \, . \tag{3.2.18}$$

It is seen from equation (3.2.4) that under supersymmetry none of the four spinors χ^i is left invariant at zero value. This indicates that the above background breaks the four supersymmetries spontaneously. Since the scalar potential has only one critical point, it follows that for the positive cosmological constant potentials there is no scalar-gravitational background with unbroken supersymmetry. Indeed, it is not possible to obtain a consistent global algebra starting from the consistent local algebra that describes the positive cosmological constant N = 4 supergravity (global limits of spontaneously broken theories have been considered in [16]). Trying to construct this global algebra one finds $\mathbf{U} = 0$, implying that the commutator of two supersymmetry charges is $\{Q, Q\} \propto \alpha X + \beta Y$. The absence of a term with a Lorentz generator such as in equation (3.2.13) leads to the failure of the (QQQ) Jacobi identity.

The theorem of Ferrara [17] requiring negative cosmological constant in gauged extended supergravity was based on the impossibility of constructing a global graded deSitter algebra. As it has been shown there is no such global superalgebra, but there is a consistent local superalgebra leading to a consistent supergravity theory with positive cosmological constant. In conclusion, absence of global limits do not forbid the existence of spontaneously broken realizations. Since the Bianchi identities only imply the existence of a local algebra realized on fields, they can be used to study spontaneously broken theories.

4. The Inequivalent Models

4.1 Reduction from the Gauged SO(4) Theory

In this section a better understanding of the unusual properties of the gauged SO(4) model is pursued. It will be shown that the class of theories that is generated by introducing two parameters into the ungauged SO(4) theory consists of only three inequivalent models. That is, one can find field redefinitions that map the continuum of theories with two parameters into three discrete cases. This unusual circumstance takes place because of the presence of scalar fields in the kinetic term for the vector fields.

Suppose at the critical point of the scalar potential the scalar fields acquire a vacuum expectation value φ_o such that the kinetic term for the vectors reads (schematically):

$$-\frac{1}{4}\,|f(\varphi_o)|^2 F_{\mu\nu}\cdot F^{\mu\nu}\,, \tag{4.1.1}$$

where $F_{\mu\nu}$ is a gauge covariant field strength:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu\,, \tag{4.1.2}$$

c_{ijk} are the structure constants of the gauge group and g is the gauge coupling constant. If $f(\varphi_o)$ is not equal to ± 1, then the kinetic term for the vectors does not have the canonical normalization, and the gauge field A^i_μ has to be rescaled. Letting:

$$A^i_\mu = \frac{\pm 1}{|f(\varphi_o)|}\,A'^i_\mu\,, \tag{4.1.3}$$

one finds that the kinetic term in equation (4.1.1) recovers the standard normalization $-\frac{1}{4}(F'_{\mu\nu})^2$, with:

$$F'^{i}_{\mu\nu} = \partial_\mu A'^{i}_{\nu} - \partial_\nu A'^{i}_{\mu} + g_{eff}\, c_{ikj} A'^{j}_{\mu} A'^{k}_{\nu} , \tag{4.1.4}$$

where:

$$g_{eff} = \frac{\pm 1}{|f(\varphi_o)|} g . \tag{4.1.5}$$

The g_{eff} is the real gauge coupling constant for the perturbation theory based on the vacuum $\varphi = \varphi_o$. The g introduced in equation (4.1.2) is only a parameter, which equals the gauge coupling constant if the vector fields have the canonical normalization. Scalar fields appear in the kinetic term for the vectors for N ≥ 4 supergravity theories. In the SO(4) supergravity the kinetic term for the vectors takes the following form [5]:

$$L^1 = -\frac{e}{8} F^{ij}_{\mu\nu} \left[g_1 F^{\mu\nu}_{ij} - g_2 F^{*\mu\nu}_{ij} - g_3 \tilde{F}^{*\mu\nu}_{ij} - g_4 \tilde{F}^{\mu\nu}_{ij} \right] , \tag{4.1.6}$$

where the star denotes internal duality transformation and the tilde denotes spacetime duality transformation. The g_i functions (not to be confused with the coupling constants) are given by:

$$g_1 - ig_4 = \frac{1 + W^2}{1 - W^2} , \quad g_3 - ig_2 = \frac{-2iW}{1 - W^2} , \quad W = \kappa(-A + iB) . \tag{4.1.7}$$

For the critical point of the SO(4) scalar potential given in equation (3.2.1), $W = 0$. Therefore $g_1 = 1$, $g_2 = g_3 = g_4 = 0$, and the vectors in equation (4.1.6) have canonical normalization. For the critical points of the potential in the gauged SO(4) model, given in equations (3.2.6) and (3.2.7), $g_1 \neq 1$, and g_2 acquires a vacuum expectation value. The vectors then lose the canonical normalization. In order to see this clearly, it is convenient to exhibit a different rewriting of the kinetic term of the vectors. Instead of using the A^{ij}_{μ} vectors, it is better to use the gauge fields for the two SU(2) subgroups, namely the φ^{Γ}_{μ} and φ'^{Γ}_{μ} defined in equation (2.14). In terms of these fields, the kinetic term for the vectors takes the form:

$$L^1 = -\frac{e}{4}[h_1(\varphi^n_{\mu\nu})^2 + h_2(\varphi'^n_{\mu\nu})^2 + h_3\varphi^n_{\mu\nu}\tilde{\varphi}^{n\mu\nu} + h_4\varphi'^n_{\mu\nu}\tilde{\varphi}'^{n\mu\nu}] , \qquad (4.1.8)$$

where the scalar functions h_1, h_2, h_3 and h_4 are defined by:

$$h_1 + ih_3 = \frac{1}{h_2 + ih_4} = \frac{1 - W}{1 + W} . \qquad (4.1.9)$$

The scalar functions h_i are useful in order to get a simple expression for the scalar potential. From equations (3.2.5) and (4.1.9) one finds:

$$P(g_1, g_2, W) = -\frac{1}{\kappa^4}\left[\frac{g_1^2}{h_1} + \frac{g_2^2}{h_2} + 4g_1g_2\right] . \qquad (4.1.10)$$

In this expression the square of each coupling constant is divided by the scalar function that appears in the kinetic term of the respective gauge field. Equation (4.1.10) can be used to relate the physical coupling constants to the cosmological term Λ. At the critical point W_0, the scalar functions attain the vacuum expectation values h_1^0 and h_2^0 where $h_1^0 h_2^0 = 1$ (using equation (4.1.9) with W_0 real). Therefore equations (4.1.1) and (4.1.5) imply that the cosmological constant, that is, the vacuum expectation value of the potential P, can be written as:

$$\Lambda = \kappa^2 P(W_0) = -\frac{1}{\kappa^2}[g^2_{1eff} + g^2_{2eff} \pm 4g_{1eff}g_{2eff}] . \qquad (4.1.11)$$

Neither vacuum expectation values of scalars nor the parameters g_1 and g_2 appear in this expression, only the effective coupling constants do. Equation (4.1.10) also shows that the exchange of g_1 and g_2 is a symmetry of the potential, if at the same time one lets $W \to -W$ (see equation (4.1.9)). Therefore the exchange of g_1 and g_2 leads to a physically equivalent potential, and thus the two SU(2)'s are physically equivalent. The inequivalent potentials arise because

h_1 and h_2, in spite of being related by reflection ($W \rightarrow -W$), are different functions, and because of the presence of the term $4g_1g_2$, which alters the value of the cosmological constant.

Consider first the negative cosmological constant theories ($g_1g_2 > 0$), as defined in equations (3.2.3) to (3.2.6). Here, at the critical point one finds:

$$h_1 = \frac{g_1}{g_2}, \quad h_2 = \frac{g_2}{g_1}, \tag{4.1.13}$$

implying that we have:

$$g_{1eff} = g_{2eff} = \sqrt{g_1g_2}, \tag{4.1.14}$$

after rescaling the gauge fields with $+\sqrt{h_1}$ and $+\sqrt{h_2}$, respectively. It therefore appears that about the critical point the theory behaves as a single coupling constant theory. It can be shown, however, that the full theory with parameters g_1 and g_2 can be shown to be equivalent to the gauged SO(4) model with one coupling constant $g = \sqrt{g_1g_2}$. As a first step, a redefinition of the scalar field W is required. This redefinition should leave the form of the scalar kinetic term invariant and should recast the potential in equation (4.1.10) into the known SO(4) potential of equation (3.2.1). The required transformation is:

$$W = \frac{g_+W' - g_-}{-g_-W' + g_+}. \tag{4.1.15}$$

In terms of the new scalar field W', the scalar potential reads:

$$P(W') = -\frac{2g_1g_2}{\kappa^4}\left[\frac{3 - |W'|^2}{1 - |W'|^2}\right], \tag{4.1.16}$$

and the functions h_i become:

$$h_1(W) + ih_3(W) = \frac{g_1}{g_2}(h_1(W') + ih_3(W')),$$

$$h_2(W) + ih_4(W) = \frac{g_2}{g_1}(h_2(W') + ih_4(W')). \tag{4.1.17}$$

A rescaling of the gauge fields is now necessary in order to preserve the form of the kinetic term for the vectors (equation (4.1.8)), letting:

$$\varphi_\mu^n \to \left[\frac{g_2}{g_1}\right]^{\frac{1}{2}} \varphi_\mu^n \quad \varphi'^n_\mu \to \left[\frac{g_1}{g_2}\right]^{\frac{1}{2}} \varphi'^n_\mu , \tag{4.1.18}$$

one finds that the gauge covariant field strengths become:

$$\varphi_{\mu\nu}^n(g_1) \to \left[\frac{g_2}{g_1}\right]^{\frac{1}{2}} \varphi_{\mu\nu}^n(\sqrt{g_1 g_2}) \, , \; \varphi'^n_{\mu\nu}(g_2) \to \left[\frac{g_1}{g_2}\right]^{\frac{1}{2}} \varphi'^n_{\mu\nu}(\sqrt{g_1 g_2}) \, . \tag{4.1.19}$$

In order to complete the redefinition of the whole theory, it is convenient to use superspace techniques. At the level of superfields, equation (4.1.18) reads

$$\varphi_A^n \to \left[\frac{g_2}{g_1}\right]^{\frac{1}{2}} \varphi_A^n \, , \; \varphi'^n_A \to \left[\frac{g_1}{g_2}\right]^{\frac{1}{2}} \varphi'^n_A \, , \tag{4.1.20}$$

and equations (2.1), (2.2), (2.8) and (2.14) then imply that:

$$U + \overline{V} \to \left[\frac{g_1}{g_2}\right]^{\frac{1}{2}} (U + \overline{V}) \, , \quad U - \overline{V} \to \left[\frac{g_2}{g_1}\right]^{\frac{1}{2}} (U - \overline{V}) \, , \tag{4.1.21}$$

or, simplifying:

$$U \to \frac{1}{2\sqrt{g_1 g_2}} (g_+ U + g_- \overline{V}) \, , \; \overline{V} \to \frac{1}{2\sqrt{g_1 g_2}} (g_+ \overline{V} + g_- U) \, . \tag{4.1.22}$$

Using the values for U and $\overline{V}$ given in equation (2.19), and equation (4.1.15), one finds:

$$U \to e^{i\theta/2} \frac{1}{\sqrt{1 - W'\overline{W}'}} \, , \quad \overline{V} \to e^{i\theta/2} \frac{\overline{W}'}{1 - W'\overline{W}'} \, , \tag{4.1.23}$$

with:

$$e^{i\theta} = \frac{-g_- W' + g_+}{-g_- \overline{W}' + g_+} \, . \tag{4.1.24}$$

We see that the functions U and $\overline{V}$ get replaced by their standard values in terms of the redefined scalar field W', apart from a phase factor. This phase is eliminated by performing a Weyl rescaling of the supercovariant derivative (the way to do this is explained in Appendix B). The above rescaling implies the following redefinition of the Fermi fields:

$$\psi^i_\mu \to e^{-\frac{i\theta\gamma_5}{4}} \psi^i_\mu , \quad \chi_i \to e^{-\frac{3i\theta\gamma_5}{4}} \chi_i , \tag{4.1.25}$$

and that finally we get:

$$U(W,\overline{W}) \to U(W',\overline{W}') , \quad \overline{V}(W,\overline{W}) \to \overline{V}(W',\overline{W}') . \tag{4.1.26}$$

It is possible to verify now that letting $g = 2\sqrt{g_1 g_2}$, with the above redefinitions, the general $g_1 g_2 > 0$ theories of Section 2 become the standard gauged SO(4) model [8], described by the following equations:

$$F^{ij}_{\mu\nu} = \partial_\mu A^{ij}_\nu - \partial_\nu A^{ij}_\mu + g\, A_{[\mu}{}^{ip} A_{\nu]}{}^{pj} , \tag{4.1.27}$$

$$\widehat{D}_\mu \psi^k_\nu = (\partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi^k_\nu + g\, A^{kl}_\mu \psi^l_\nu , \tag{4.1.28}$$

$$\delta'\overline{\chi}^i = \frac{g}{\sqrt{2}\kappa^2}\,\overline{\varepsilon}^i \frac{\kappa(A - i\gamma_5 B)}{(1 - |W|^2)^{\frac{1}{2}}} , \tag{4.1.29}$$

$$\delta'\overline{\psi}^i_\mu = \frac{ig}{2\kappa^2}\,\overline{\varepsilon}^i \gamma_\mu \frac{1}{(1 - |W|^2)^{\frac{1}{2}}} , \tag{4.1.30}$$

$$L' = \frac{eg}{\kappa(1 - |W|^2)^{\frac{1}{2}}}\overline{\psi}^i_\mu \sigma^{\mu\nu}\psi^i_\nu + \frac{ieg}{\sqrt{2}\kappa(1 - |W|^2)^{\frac{1}{2}}}\overline{\psi}^i . \gamma\kappa(A + i\gamma_5 B)\chi^i$$
$$+ \frac{eg}{2\kappa^4}\left[\frac{3 - |W|^2}{1 - |W|^2}\right] . \tag{4.1.31}$$

Consider now the positive-cosmological-constant theories ($g_1 g_2 < 0$), as defined in equations (3.2.3) to (3.2.5), and (3.2.7). Let us briefly discuss how these theories are all equivalent to the positive-cosmological-constant theory given by $g_1 = -g_2$. Again, the key point is that at the critical point of the potential the vector fields do not have the standard normalization. Now we introduce the following redefinition of the scalar fields:

$$W = \frac{g_- W' - g_+}{-g_+ W' + g_-} . \qquad (4.1.32)$$

As a consequence of this redefinition, the h_i functions rescale and one has to redefine the gauge fields in order to preserve the form of the kinetic term for the vectors:

$$\varphi_A^n \to \left(\frac{g_2}{g_1}\right)^{\frac{1}{2}} \varphi_A^n , \quad \varphi'^n_A \to -\left(\frac{g_1}{g_2}\right)^{\frac{1}{2}} \varphi'^n_A . \qquad (4.1.33)$$

The gauge-covariant field strengths are then redefined as follows:

$$\varphi_{\mu\nu}^n(g_1) \to \left(-\frac{g_2}{g_1}\right)^{\frac{1}{2}} \varphi_{\mu\nu}^n[sgn(g_1)\sqrt{-g_1 g_2}] ,$$

$$\varphi_{\mu\nu}^n(g_2) \to \left(-\frac{g_1}{g_2}\right)^{\frac{1}{2}} \varphi'^n_{\mu\nu}[sgn(g_2)\sqrt{-g_1 g_2}] . \qquad (4.1.34)$$

where the sgn function gives the sign of the argument. As before this redefinition implies that:

$$U + \overline{V} \to \left(-\frac{g_1}{g_2}\right)^{\frac{1}{2}} (U - \overline{V}) , \quad U - \overline{V} \to \left(\frac{g_2}{g_1}\right)^{\frac{1}{2}} (U + \overline{V}) . \qquad (4.1.35)$$

Using the values for U and $\overline{V}$ given in equation (2.19), and equation (4.1.32), one finds

$$U \to e^{i\theta/2} \frac{sgn(g_1)}{\sqrt{1 - W'\overline{W}'}} , \quad \overline{V} \to e^{i\theta/2} \frac{sgn(g_1)\overline{W}'}{\sqrt{1 - W'\overline{W}'}} , \qquad (4.1.36)$$

where:

$$e^{i\theta} = \frac{-g_+ W' + g_-}{-g_+ \overline{W}' + g_-} . \qquad (4.1.37)$$

Again, in order to eliminate the phases from the U and $\overline{V}$ functions, one requires a rescaling of the Fermi fields. These are precisely the ones given in equation (4.1.25), but with the value of $e^{i\theta}$ given in equation (4.1.37). One then has $U(W) \to sgn(g_1)\, U(W')$ and $\overline{V}(W) \to sgn(g_1)\, \overline{V}(W')$, as in equation (4.1.26).

The above redefinitions turn the $g_1 g_2 < 0$ theories into the positive-cosmological-constant theory with $g_1 = -g_2$. Letting $g = 2\sqrt{-g_1 g_2}$, one obtains:

$$F_{\mu\nu}^{ij} = \partial_\mu A_\nu^{ij} - \partial_\nu A_\mu^{ij} + g C_{ijkl} A_\mu^{kp} A_\nu^{ln} \,, \tag{4.1.38}$$

$$\widehat{D}_\mu \psi_\nu^k = (\partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi_\mu^k + g\, A_\mu^{*kl}\, \psi_\nu^l \,, \tag{4.1.39}$$

$$\delta' \overline{\chi}^i = -\frac{g}{\sqrt{2}\kappa^2}\, \overline{\varepsilon}^i\, \frac{1}{(1 - |W|^2)^{\frac{1}{2}}} \,, \tag{4.1.40}$$

$$\delta' \overline{\psi}_\mu^i = \frac{ig}{2\kappa^2}\, \overline{\varepsilon}^i\, \gamma_\mu\, \frac{\kappa(-A + i\gamma_5 B)}{(1 - |W|^2)^{\frac{1}{2}}} \,, \tag{4.1.41}$$

$$L' = -\frac{eg}{\kappa(1 - |W|^2)^{\frac{1}{2}}}\, \overline{\psi}_\mu^i \sigma^{\mu\nu} \kappa(A + i\gamma_5 B)\psi_\nu^i$$

$$-\frac{ieg}{\sqrt{2}\kappa(1 - |W|^2)^{\frac{1}{2}}}\, \overline{\psi}^i \cdot \gamma \chi^i - \frac{eg^2}{2\kappa^4}\left[\frac{1 - 3|W|^2}{1 - |W|^2}\right] . \tag{4.1.42}$$

The above equations indicate the gauge covariant objects, and the extra terms in the supersymmetry transformation formulas ies and in the lagrangian for the gauged SO(4) theory with positive cosmological constant. The scalar potential is shown in figure 3.

Finally, let us consider the third inequivalent theory. It is the theory in which one of the two coupling constants is set equal to zero. Take $g_2 = 0$ (the

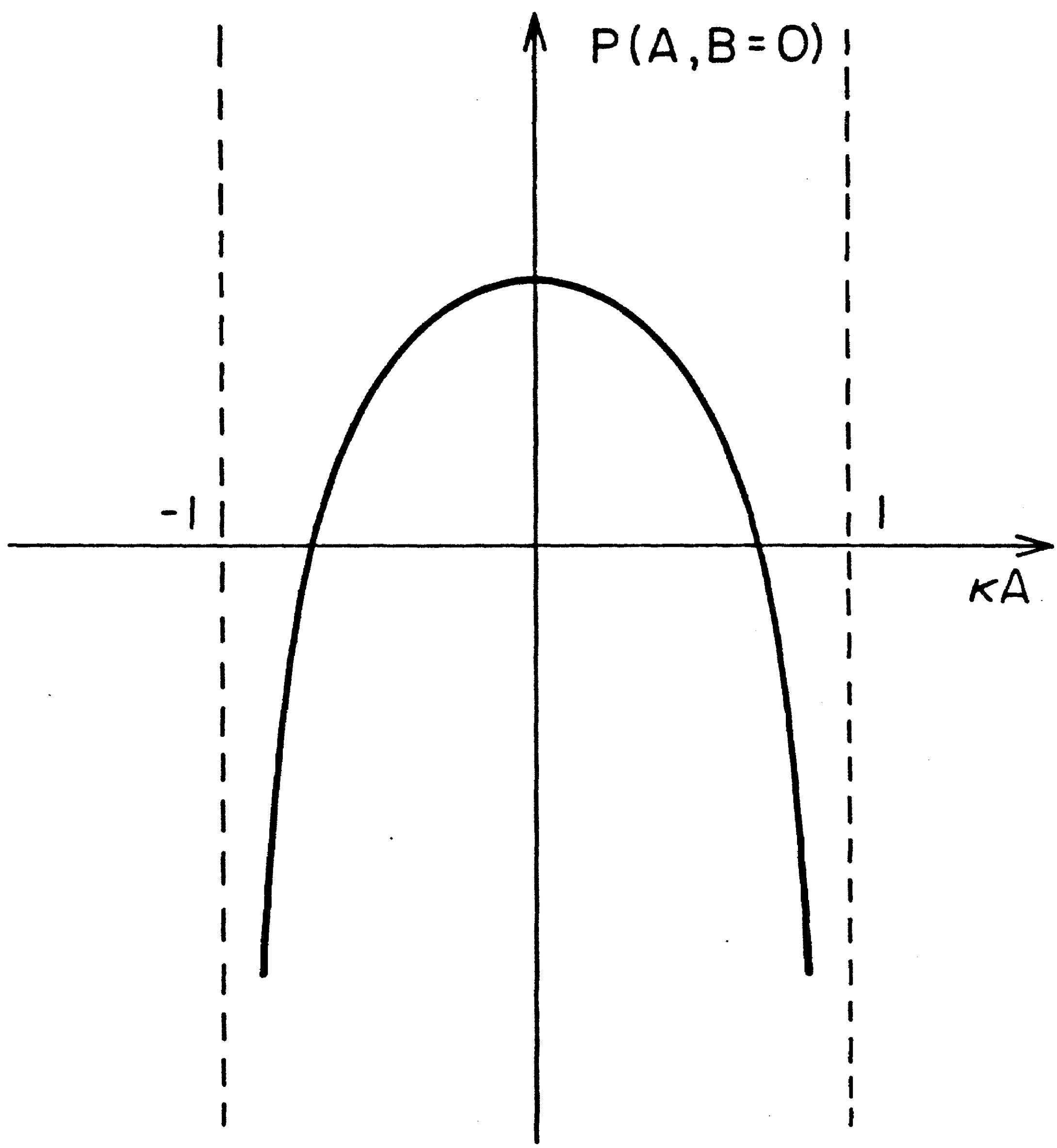

Figure 3. Scalar Potential for the gauged SO(4) supergravity with $g_1 = -g_2 = g/2$.

case where $g_1 = 0$ can be shown to be redefinable into the above case), or equivalently, $g_+ = g_- = g$. The relations for this model are then

$$\varphi^n_{\mu\nu} = \partial_\mu \varphi^n_\nu - \partial_\nu \varphi^n_\mu - 2\sqrt{2} g\, (\vec{\varphi}_\mu \times \vec{\varphi}_\nu)^n \,, \tag{4.1.43}$$

$$\varphi'^n_{\mu\nu} = \partial_\mu \varphi'^n_\nu - \partial_\nu \varphi'^n_\mu \,, \tag{4.1.44}$$

$$\widehat{D}_\mu \psi^k_\nu = (\partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi^k_\nu + \sqrt{2} g\, \alpha^n_{kl}\, \varphi^n_\mu \psi^l_\nu \,, \tag{4.1.45}$$

$$\delta' \overline{\chi}^i = \frac{g}{\sqrt{2}\kappa^2}\, \overline{\varepsilon}^i \frac{[\kappa(A - i\gamma_5 B) - 1]}{(1 - |W|^2)^{\frac{1}{2}}} \,, \tag{4.1.46}$$

$$\delta' \overline{\psi}^i_\mu = \frac{ig}{2\kappa^2}\, \overline{\varepsilon}^i \gamma_\mu \frac{[1 + \kappa(-A + i\gamma_5 B)]}{(1 - |W|^2)^{\frac{1}{2}}} \,, \tag{4.1.47}$$

$$L' = \frac{eg}{\kappa(1 - |W|^2)^{\frac{1}{2}}}\, \overline{\psi}^i_\mu \sigma^{\mu\nu}[1 - \kappa(A + i\gamma_5 B)]\, \psi^i_\nu$$

$$+ \frac{ieg}{\sqrt{2}\kappa(1 - |W|^2)^{\frac{1}{2}}}\, \overline{\psi}^i . \gamma\, [\kappa(A + i\gamma_5 B) - 1]\, \chi^i + \frac{eg^2}{\kappa^4}\left[\frac{(1 - \kappa A)^2 + (\kappa B)^2}{1 - \kappa^2(A^2 + B^2)}\right] \tag{4.1.48}$$

This theory, whose potential is shown in figure 4, is actually the Freedman - Schwarz SU(2) ⊗ SU(2) model with $e_B = 0$. In both theories the potential has no critical points. One can check that the redefinitions of the scalars, gravitinos and spinors given in reference [6] turn the above expressions into the ones given for the SU(2) ⊗ SU(2) theory [7]. The SU(4) supergravity has three vectors A^n_μ and three axial vectors B^n_μ. The vectors A^n_μ are obtained by ordinary field redefinitions from the vectors of the SO(4) model, namely $A^n_\mu = \varphi^n_\mu$ (see equation (2.14)). The B^n_μ fields are obtained using duality transformations. These dualities cannot be implemented once the theories are gauged, but since e_B is zero the B^n_μ fields remain abelian and the dualities can still be performed.

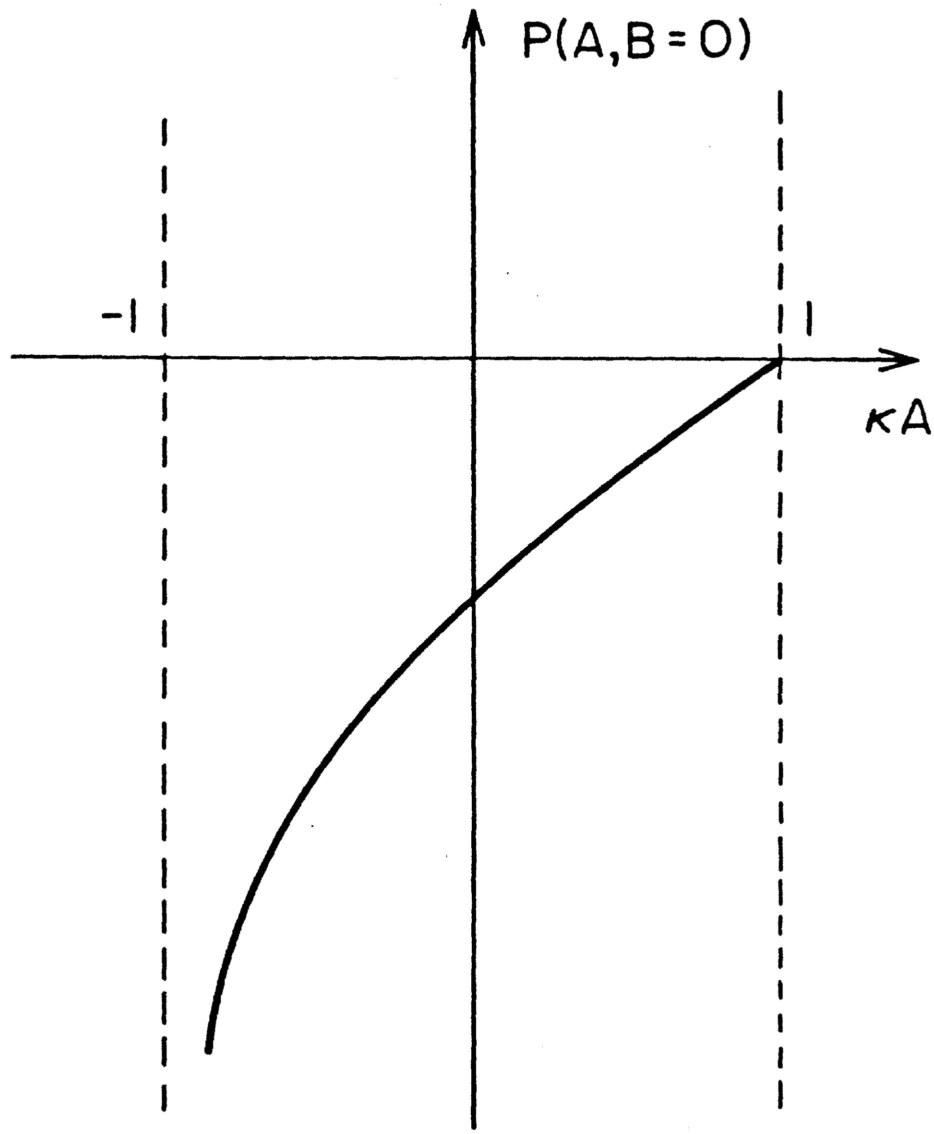

Figure 4. Scalar Potential for the gauged SO(4) supergravity with $g_1 = g$, $g_2 = 0$.

In the SU(2) ⊗ SU(2) theory [7], the kinetic term for the vectors takes the form:

$$-\frac{1}{4} V e^{-2\kappa\varphi} (\vec{A}_{\mu\nu}.\vec{A}^{\mu\nu} + \vec{B}_{\mu\nu}.\vec{B}^{\mu\nu}) \,. \tag{4.1.49}$$

Therefore, according to the discussion about gauge coupling constants, one sees that the e_A, and e_B are parameters and the physical coupling constants are $e_A\, e^{\kappa\varphi_0}$ and $e_B\, e^{\kappa\varphi_0}$, where φ_0 is the background value for the scalar field φ. The scalar potential, however, has no critical points. As φ goes to $-\infty$, $dV/d\varphi$ goes to zero and the cosmological constant goes to zero, but at the same time the physical coupling constants go to zero ! Indeed, the vacuum expectation value of the scalar potential corresponds to a cosmological constant of value $\Lambda \propto -(e_{A\,eff}^2 + e_{B\,eff}^2)/\kappa^2$. It therefore does not appear to be possible to obtain reasonable gauge coupling constants and a small cosmological term in this theory.

4.2 Dualities and Inversions

It is interesting to clarify the role of internal-space dualities on the vector fields. As is well known, x-space duality transformations on the vector field strengths are nonlocal field redefinitions on the gauge fields and lead to theories with inequivalent potentials, once the internal symmetry is gauged [7]. Then the x-space dualities cannot be implemented any more. Something analogous happens with internal-space dualities.

If in the ungauged SO(4) theory one lets:

$$F^{ij}_{\mu\nu,new} = \frac{1}{2} C^{ijkl} F^{kl}_{\mu\nu} , \tag{4.2.1}$$

this transformation can be implemented (in contrast with x-space dualities) as a local transformation of the vector potentials, namely:

$$A^{ij}_{\mu,new} = \frac{1}{2} C^{ijkl} A^{ij}_{\mu} = A^{*ij}_{\mu} . \tag{4.2.2}$$

Nevertheless, once the two SU(2)'s are gauged, from equation (1.22) one has:

$$F^{ij}_{\mu\nu} = \partial_\mu A^{ij}_\nu - \partial_\nu A^{ij}_\mu - \frac{1}{4}[g_1 \vec{\alpha}_{ij} \cdot (\vec{\alpha}_{pq} \times \vec{\alpha}_{rs}) + g_2 \vec{\beta}_{ij} \cdot (\vec{\beta}_{pq} \times \vec{\beta}_{rs})] A^{pq}_\mu A^{rs}_\nu , \tag{4.2.3}$$

and using the self-duality of the α's and the antiself-duality of the β's, one sees that the transformation (4.2.1) cannot be implemented any more as a transformation of the gauge fields, since the sign of g_2 cannot be changed.

Consider the gauging of the SO(4) model with U and $\overline{V}$ functions given in equation (2.19), that is, the standard SO(4) theory, or form A, as defined in Chapter III. Take $g_1 = g_2 = g/2$ ($g_+ = g$, $g_- = 0$) and denote the scalar potential by $P_A(g,g)$. Using equation (2.31), one finds that it is given by

$$P_A(g,g) = -\frac{1}{2\kappa^4}(3|U|^2 - |V|^2) = -\frac{g^2}{2\kappa^4}\left[\frac{3 - W\overline{W}}{1 - W\overline{W}}\right] . \tag{4.2.4}$$

This is the well-known potential of the SO(4) theory with negative cosmological constant and unbroken supersymmetry. Performing an internal duality transformation at the ungauged level implies exchanging the two functions U and $\bar{V}$ as was shown in Chapter III. The dual form was called A^*. Gauging the form A^* with $g_1 = g_2 = g/2$, and using the general result in equation (2.31) with the interchanged values of U and $\bar{V}$ one finds:

$$P_{A^*}(g,g) = -\frac{1}{2\kappa^4}(3|U|^2 - |V|^2) = \frac{g^2}{2\kappa^4}\left[\frac{1-3W\overline{W}}{1-W\overline{W}}\right]. \tag{4.2.5}$$

This is the positive cosmological constant potential, clearly inequivalent to the one given in equation (4.2.4). Thus the gauging of the form A^* leads to the spontaneously broken model.

In general, it can be seen from equations (2.17) that the exchange of U and $\bar{V}$ (duality) is just equivalent to the exchange of g_+ and g_-, which in turn implies that $g_1 \to g_1$ and $g_2 \to -g_2$. One therefore concludes that if a theory leads to a potential $P(g_1, g_2)$, the dual theory leads to a potential $P(g_1, -g_2)$. If the original theory once gauged is unbroken (g_1 and g_2 have the same sign) then the dual theory once gauged is broken (g_1 and $-g_2$ have opposite signs) and vice versa. Internal dualities at the ungauged level can therefore lead to inequivalent gauged theories.

Gauging the standard SO(4) theory (form A) with $g_1 = g_2$ (that is using the prescription in equation (2.6)) leads to the unbroken theory, but gauging it with $g_1 = -g_2$ leads to the broken one. Gauging the SO(4) theory with U and $\bar{V}$ interchanged (form A^*) with $g_1 = g_2$ (using equation (2.6)) leads to the broken theory, while gauging with $g_1 = -g_2$ leads to the unbroken one. This could have also been seen in Section 4.1, when the $g_1 g_2 < 0$ models were discussed. There it was shown that the positive cosmological constant forms could be redefined into a theory with the standard values for the U and $\bar{V}$ functions and two opposite sign coupling constants. Had one redefined the vectors in equation (4.1.33) as:

$$\varphi_A^n \to \left(-\frac{g_2}{g_1}\right)^{\frac{1}{2}} \varphi_A^n \, , \quad \varphi'^n_A \to -\left(-\frac{g_1}{g_2}\right)^{\frac{1}{2}} \varphi'^n_A \, , \tag{4.2.6}$$

then the gauge covariant objects would have turned into:

$$\varphi^n_{\mu\nu}(g_1) \to \left(-\frac{g_2}{g_1}\right)^{\frac{1}{2}} \varphi^n_{\mu\nu}[sgn(g_1)\sqrt{-g_1 g_2}] \, ,$$

$$\varphi'^n_{\mu\nu}(g_2) \to -\left(-\frac{g_1}{g_2}\right)^{\frac{1}{2}} \varphi'^n_{\mu\nu}[-sgn(g_2)\sqrt{-g_1 g_2}] \, , \tag{4.2.7}$$

where the effective coupling constants have the same sign. One would find at the end of the redefinition process that $U(W) \to \overline{V}(W')$ and $\overline{V}(W) \to U(W')$, indicating, in accordance with the above discussion, that a $g_1 g_2 < 0$ theory can be redefined into a theory with two equal coupling constants if the functions U and $\overline{V}$ are interchanged.

Consider now the theories defined outside the unit disc $|W|^2 < 1$. As was shown in Chapter III, it is not possible to continue the interior solution into the exterior. The theories defined on the exterior have been shown to be related by scalar field inversion plus other redefinitions possibly including dualities. If only a scalar field inversion is performed, the gauged exterior theory has a scalar potential which is just a copy of the scalar potential of the inside disc and the two theories are equivalent. Consider, however, form B in equation (3.1.20) of Chapter III:

$$U = \frac{1}{\sqrt{W\overline{W} - 1}} \, , \quad \overline{V} = \frac{\overline{W}}{\sqrt{W\overline{W} - 1}} \, . \tag{4.2.8}$$

This theory was shown to be related to the standard inside theory by an inversion, a chiral rotation of the Fermi fields, and an internal duality on the spin one states. The above discussion on internal dualities implies that if the inside theory works in the unbroken phase, the exterior theory works in the broken supersymmetric phase. Indeed it follows from equations (4.2.8) and (2.31) that gauging with $g_1 = g_2 = g/2$ the analytic expression for the potential in the exterior region is just:

$$P_B(g,g) = \frac{g^2}{2\kappa^4}\left[\frac{3 - W\overline{W}}{1 - W\overline{W}}\right], \tag{4.2.9}$$

For the form B^* the U and $\overline{V}$ functions of equation (4.2.8) are interchanged, and therefore one has:

$$P_{B^*}(g,g) = -\frac{g^2}{2\kappa^4}\left[\frac{1 - 3W\overline{W}}{1 - W\overline{W}}\right]. \tag{4.2.10}$$

Sketches of the potentials are shown in figure 5.

It has been seen that once gauged A and A^* are inequivalent, the same is true for B and B^* as can be seen from equations (4.2.5) and (4.2.10). A and B^* or A^* and B remain physically equivalent once gauged. Their scalar potentials are just related by inversion and their Fermi fields by a chiral rotation.

As we have seen, it is possible to construct physically inequivalent theories on the inside unit disc and on the outside. It would be interesting to understand whether or not these two theories could be thought of as a single theory defined throughout the W plane.

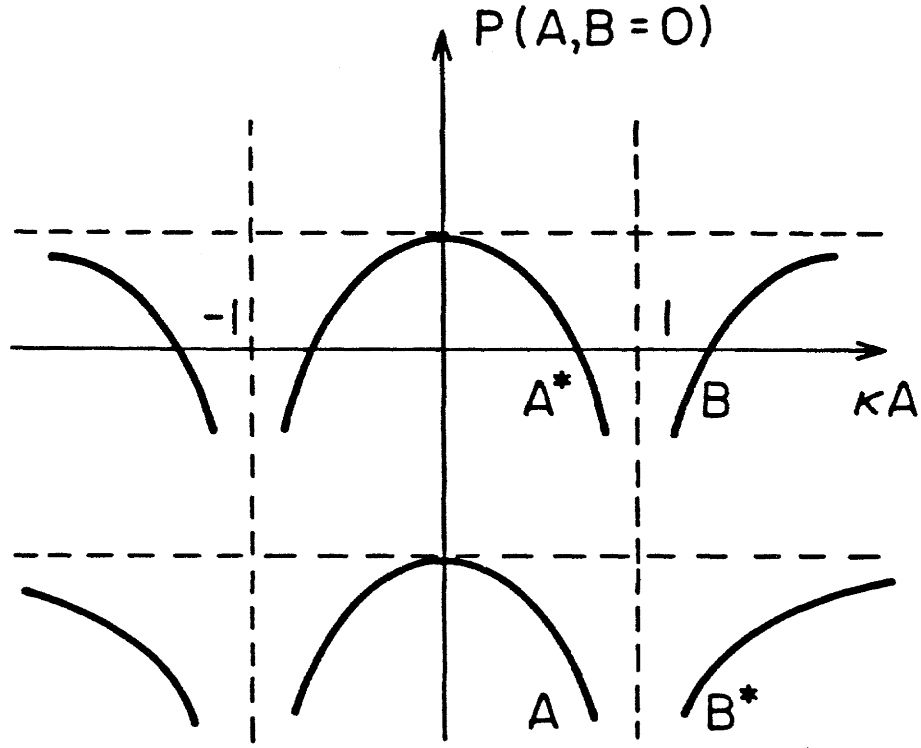

Figure 5. Scalar Potentials for the gauged SO(4) supergravity in the inside and outside of the unit disc $|W|^2 = 1$.

5. Higgs Problem in Gauged SO(5) Supergravity

As has been shown in the previous section, the presence of scalar fields in the kinetic term of the vectors can affect the values of the physical coupling constants in N = 4 supergravities. The case of N = 5 supergravity can be studied without great difficulty. The gauged N = 5 theory [18] has a potential with two possible vacua. In the first one neither supersymmetry nor the gauge symmetry is broken, and in the other one the five supersymmetries are spontaneously broken and the gauge group is broken from SO(5) down to SO(3). In the unbroken vacuum the vector fields have the canonical normalization, and therefore the gauge coupling constant equals the g parameter. It is not *a priori* clear that the same should hold for the broken vacuum, because the nonzero vacuum expectation values for the scalars could alter the normalization for the vectors. This does not happen. As will be shown here, the remaining SO(3) symmetry is gauged with gauge coupling constant g . The function of the scalars appearing in the vector kinetic term, however, is necessary in order to read correctly the mass parameters for the vector fields in the broken vacuum.

Consider first the scalar sector of the gauged N = 5 theory. The scalar potential takes the following form [18]:

$$V(\varphi) = -g^2(2 + 4e_1^2 - \frac{1}{2}e_1^4\,[\,|\varphi|^4 - (\varphi_i)^2(\varphi^j)^2\,]\,)\,, \tag{5.1}$$

where $\varphi_i = A_i + iB_i$ with $i = 1,2,\cdots 5$, $\varphi^i = (\varphi_i)^*$, and $e_1 = (1 - |\varphi|^2)^{-\frac{1}{2}}$. The scalar kinetic term is:

$$L^o = -\frac{1}{2}\,Vg^{\mu\nu}a_\mu^i\,a_{\nu i}\,, \tag{5.2}$$

where:

$$a_\mu^i = -\sqrt{2}\,e_1(\delta_j^i - e_2\varphi^i\varphi_j)\,\partial_\mu\varphi^j\,, \tag{5.3}$$

and $e_2 = (1 - e_1)/\,|\varphi|^2$.

One can find that the potential has a seven dimensional extremal surface described by:

$$|\varphi|^2 = \frac{4}{5}\,, \quad (\varphi_i)^2 = 0\,. \tag{5.4}$$

Choosing the vacuum value for φ_i to be:

$$<\varphi_i>_o \, = \sqrt{\frac{2}{5}}(0\,,0\,,0\,,1\,,i\,)\,, \tag{5.5}$$

one can see that SO(5) is spontaneously broken to SO(3) [2,3], and seven out of the ten scalars of the theory can be gauged away. Choosing the following parametrization for the three surviving scalars:

$$\varphi_1 = \varphi_2 = \varphi_3 = 0\,,$$

$$\varphi_4 = \sqrt{\frac{2}{5}} + \frac{\rho}{\sqrt{50}} + \frac{\eta}{\sqrt{10}} + i\frac{\xi}{\sqrt{10}}\,,$$

$$\varphi_5 = i\sqrt{\frac{2}{5}} + i\frac{\rho}{\sqrt{50}} - i\frac{\eta}{\sqrt{10}} + \frac{\xi}{\sqrt{10}}\,, \tag{5.6}$$

one finds that the kinetic term for the scalars in equation (5.2) becomes:

$$-V((\partial_\mu\rho)^2 + (\partial_\mu\eta)^2 + (\partial_\mu\xi)^2) + (\mathit{higher\ order})\,, \tag{5.7}$$

while the scalar potential turns into:

$$-g^2V(14 - 40\rho^2 + 8\eta^2 + 8\xi^2) + (\mathit{higher\ order})\,. \tag{5.8}$$

It follows from equations (5.7), (5.8), and the results of reference [2] that the mass eigenvalue parameters for the surviving scalars are $(-\frac{60}{7}\,, \frac{12}{7}\,, \frac{12}{7})$ (the first of these values differs from the one quoted in [2]). Since none of these values exceeds the critical value of 9/4, this background is recognized to be at least perturbatively stable [2].

Consider now the vector sector of the theory. The kinetic term for the vector fields is given by [18]:

$$-\frac{1}{8}V[\,(2S^{ij,kl}-\delta^{ik}\delta^{jl})\,F^{+}_{\mu\nu ij}\,F^{+\mu\nu}{}_{kl}+h.c.\,]\,, \tag{5.9}$$

where $F^{+\mu\nu ij}$ is a self dual field strength, and $S^{ij,kl}$ is a nonpolynomial function for the scalars defined to satisfy the following equation:

$$(\,\delta^{ij}_{kj}-\overline{S}^{ij,kl}\,)\,S^{kl,mn}=\delta^{ij}_{mn}\,, \tag{5.10}$$

where $\overline{S}$ is given by:

$$\overline{S}^{ij,kl}=-\frac{1}{2}\varepsilon^{ijklm}\,\varphi_m\,. \tag{5.11}$$

Surprisingly enough it is possible to solve equation (5.10) explicitly for S, which turns out to be only slightly more complicated than in the N = 4 theory:

$$S^{ij,kl}=\frac{1}{1-(\varphi_p)^2}\,(\,\delta^{ij}_{kl}-2\varphi_{[i}\varphi_{[k}\delta_{l]j]}-\frac{1}{2}\varepsilon^{ijklp}\,\varphi_p\,)\,. \tag{5.12}$$

where following the conventions of [18], antisymmetrization is with unit strength: $\delta^{ij}_{kl}=\frac{1}{2}(\delta^i_k\delta^j_l-\delta^i_l\delta^j_k)$. Truncating the above result to the N = 4 theory one gets

$$S^{ij,kl}=\frac{1}{1-W^2}\,(\,\delta^{ij}_{kl}-\frac{1}{2}\,\varepsilon^{ijkl}\,W\,)\,, \tag{5.13}$$

where $W=\kappa(-A+iB)$. This form for S together with equation (5.9) leads to the well known g_i functions of N = 4 supergravity (see equations (4.1.6) and (4.1.7)).

For the critical point given in equation (5.5), the kinetic term for the vectors reduces to :

$$-\frac{1}{4}V\Big[(F_{12})^2+(F_{13})^2+(F_{23})^2+(F_{45})^2$$

$$+\frac{1}{5}[\,(F_{14})^2+(F_{24})^2+(F_{34})^2\,]+\frac{9}{5}[\,(F_{15})^2+(F_{25})^2+(F_{35})^2\,]$$

$$+4\sqrt{\frac{2}{5}}[F_{12}F_{35}-F_{13}F_{25}+F_{23}F_{15}\,]\Big|+(\textit{higher order})\,, \qquad (5.14)$$

where the contraction of spacetime indices is not shown, and the $F\tilde{F}$ terms have been dropped, since to this order of approximation they are pure divergences. The vectors that gauge the surviving SO(3) symmetry, namely A_{12}, A_{13} and A_{23}, have the standard normalization, but are coupled to the A_{35}, A_{25} and A_{15} vectors. This is somewhat unusual. Letting:

$$A_{12}=A'_{12}-2\sqrt{\frac{2}{5}}A_{35}\,,$$

$$A_{13}=A'_{13}+2\sqrt{\frac{2}{5}}A_{25}\,,$$

$$A_{23}=A'_{23}-2\sqrt{\frac{2}{5}}A_{15}\,, \qquad (5.14)$$

one finds that the A'_{12}, A'_{13} and A'_{23} vectors have the standard normalization and are uncoupled. Therefore, they gauge SO(3) with coupling constant g, *the same* as the coupling constant in the unbroken vacuum. Further rescaling is necessary for the other vectors. One lets $A_{ij}=\sqrt{5}A'_{ij}$ for the $(ij)=14,24,34,15,25,35$ vectors, $A_{45}=A'_{45}$, and finally the vector kinetic term reads $-\frac{1}{4}V(F'_{ij})^2$. The vector mass terms are found from the gauge covariant derivatives in the kinetic term for the scalars. Taking into account the above rescalings one gets:

$$-10g^2V[\,(A'_{14})^2+(A'_{24})^2+(A'_{34})^2+(A'_{15})^2+(A'_{25})^2+(A'_{35})^2+2(A'_{45})^2]. \quad (5.15)$$

In summary, we have seen that in N = 5 supergravity the remaining gauge symmetry in the broken vacuum is gauged with the same coupling constant as the original theory in the unbroken vacuum, and for the massive vectors we get two mass parameters. It would be interesting to see what the situation is for the extrema [4] of the gauged N = 8 supergravity theory.

8. Discussion and Open Questions

The existence of a gauged SO(4) supergravity with positive cosmological constant raises a few questions. One of them is the issue of stability. It would be interesting to know whether the positive cosmological constant inverted potential admits a stable background state. The methods developed by Breitenlohner and Freedman [2] are not applicable in this case because the background state is deSitter rather than anti-deSitter. DeSitter spacetime has an event horizon enclosing a finite volume. Therefore the key element of the stability proof of reference [2] for anti-deSitter spacetimes, namely, the fact that a scalar fluctuation $\varphi(x)$ of finite energy should vanish at spatial infinity, cannot be used here. The stability argument of Gibbons, Hull and Warner [3] is not applicable either. This is because in the deSitter background all the supersymmetries are broken. Even in case these potentials turn out to be unstable, a more complicated background state such as an electro-vac solution could be stable. Electro-vac solutions appear to provide a stable background state for the gauged SU(4) model [14].

Consider now the issue of charge renormalization. With positive cosmological constant the four supersymmetries are spontaneously broken. It would be interesting to know whether or not the charge renormalization is still related to the corrections to the cosmological constant [19].

A question that remains open is that of gauging subgroups. The N = 4 theory is the only extended supergravity theory in which this has been shown so far to be possible. Nevertheless the two SU(2) groups in N = 4 are commuting and it remains to be seen whether or not something less trivial such as SU(2)×U(1) or U(1)×U(1) could be gauged without explicit breakdown of supersymmetry. The methods developed in this chapter might be helpful in resolving this problem.

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Appendix A. Notation and Identities

In this appendix conventions are presented together with some useful identities and a few relations that allow direct transcription from two-component to four-component formalism.

We use the Lorentz metric $\eta_{ab} = diag(+---)$ with signature -2. The Lorentz generators M_{bc} act according to the following commutators:

$$[M_{bc}, V_a] = \eta_{ac} V_b - \eta_{ab} V_c \qquad \text{(A.1)}$$

$$[M_{bc}, V_\beta] = -i \, \frac{1}{2} (\sigma_{bc})_\beta{}^\delta V_\delta \, . \qquad \text{(A.2)}$$

The two-component Lorentz generator $M_\alpha{}^\gamma$ is defined by:

$$M_\alpha{}^\gamma = \frac{i}{2} (\sigma^{bc})_\alpha{}^\gamma M_{bc} \, . \qquad \text{(A.3)}$$

We raise and lower indices with the invariant antisymmetric tensor $C_{\alpha\beta}$:

$$A^\alpha C_{\alpha\beta} = A_\beta \, , \qquad C^{\alpha\beta} A_\beta = A^\alpha \qquad \text{(A.4)}$$

$$C_{\alpha\beta} = C_{\dot\alpha\dot\beta} = -C^{\alpha\beta} = -C^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \qquad \text{(A.5)}$$

An underlined spinor index ($\underline{\alpha}$) denotes combined spinor and isospin indices. A pair of spinor indices ($\alpha\dot\alpha$) corresponds to a vector index:

$$V_{\alpha\dot\alpha} = V_a \sigma^a{}_{\alpha\dot\alpha} \qquad V_a = -\frac{1}{2} \sigma_a{}^{\alpha\dot\alpha} V_{\alpha\dot\alpha}$$

$$V^a V_a = -\frac{1}{2} V^{\alpha\dot\alpha} V_{\alpha\dot\alpha} \, . \qquad \text{(A.6)}$$

For an antisymmetric tensor $R_{ab} = -R_{ba}$ we use

$$R_{ab} = \frac{i}{2} R_{\alpha\beta} (\sigma_{ab})^{\alpha\beta} + \frac{i}{2} \bar R_{\dot\alpha\dot\beta} (\bar\sigma_{ab})^{\dot\alpha\dot\beta} \qquad \text{(A.7)}$$

$$R_{\alpha\beta} = \frac{i}{4} (\sigma^{ab})_{\alpha\beta} R_{ab} \, , \qquad \bar R_{\dot\alpha\dot\beta} = \frac{i}{4} (\bar\sigma^{ab})_{\dot\alpha\dot\beta} R_{ab} \qquad \text{(A.8)}$$

$$R_{\alpha\dot{\alpha}\beta\dot{\beta}} = \sigma^{a}_{\alpha\dot{\alpha}}\sigma^{b}_{\beta\dot{\beta}}R_{ab} = -2(C_{\dot{\alpha}\dot{\beta}}R_{\alpha\beta}+C_{\alpha\beta}R_{\dot{\alpha}\dot{\beta}}) \ . \tag{A.9}$$

Note that for the Lorentz generator $M_{\alpha}{}^{\gamma}$ in (A.3), we use a factor of 1/2 rather than the 1/4 given in (A.8).

The following identities are used extensively in calculations in two-component notation:

$$C^{\alpha\beta}C_{\gamma\beta} = \delta^{\alpha}{}_{\gamma}$$

$$\sigma^{a}{}_{\alpha\dot{\beta}}\sigma_{a}{}^{\gamma\dot{\delta}} = -2\,\delta_{\alpha}^{\gamma}\delta_{\dot{\beta}}{}^{\dot{\delta}}$$

$$\sigma_{a\,\alpha\dot{\beta}}\sigma_{b}{}^{\gamma\dot{\beta}} = -\eta_{ab}\,\delta_{\alpha}^{\gamma} - i(\sigma_{ab})_{\alpha}{}^{\gamma}$$

$$\sigma^{[a}{}_{\alpha\dot{\beta}}\sigma^{b]}{}_{\gamma\dot{\delta}} = -i\ C_{\alpha\gamma}(\bar{\sigma}^{ab})_{\dot{\beta}\dot{\delta}} - i\ C_{\dot{\beta}\dot{\delta}}(\sigma^{ab})_{\alpha\gamma}$$

$$(\sigma^{ab})_{\alpha\beta}(\sigma_{ab})^{\gamma\delta} = -4\,\delta_{(\alpha}^{\gamma}\delta_{\beta)}^{\delta}$$

$$(\sigma^{ab})_{\alpha\gamma}(\sigma_{cd})^{\alpha\gamma} = -4[\,\frac{1}{2}\,\delta_{c}^{[a}\delta_{d}^{b]} + \frac{i}{2}\,\varepsilon^{ab}{}_{cd}\,]$$

$$(\sigma^{ab})_{\alpha\beta} = \frac{i}{2}\,\varepsilon^{ab}{}_{cd}\,(\sigma^{cd})_{\alpha\beta} \ . \tag{A.10}$$

Here $\varepsilon^{0123} = 1$.

In dealing with symmetrization in various indices the following identities are useful:

$$C_{\alpha\beta}A_{\gamma} = C_{\gamma\beta}A_{\alpha} - C_{\gamma\alpha}A_{\beta}\ , \tag{A.11}$$

$$C_{\alpha\beta}A_{\gamma} = \frac{1}{3}[\,C_{\alpha(\beta}A_{\gamma)} - C_{\beta(\alpha}A_{\gamma)}\,]\ , \tag{A.12}$$

$$C_{\alpha\beta}A_{(\gamma\delta)} = \frac{1}{4}[\,C_{\alpha(\beta}A_{\gamma\delta)} - C_{\beta(\alpha}A_{\gamma\delta)}\,]\ , \tag{A.13}$$

$$C_{\alpha\beta}A_{(\gamma\delta)} = \frac{1}{3}\,C_{\alpha(\beta}A_{\gamma\delta)} - \frac{1}{3}\,C_{\beta\gamma}A_{(\alpha\delta)} - \frac{1}{3}\,C_{\beta\delta}A_{(\alpha\gamma)}\ . \tag{A.14}$$

Using equation (A.14) for a symmetric tensor $S_{\beta\gamma}$, one can derive:

$$A_\alpha S_{\beta\gamma} = \frac{1}{6} A_{(\alpha} S_{\beta\gamma)} + \frac{1}{3} A^\varepsilon S_{\varepsilon(\gamma} C_{\beta)\alpha} \, . \tag{A.15}$$

Superspace conjugation is defined as follows:

$$\overline{\theta^\alpha} = \overline{\theta}^{\dot\alpha} \, , \quad \overline{\theta_\alpha} = -\overline{\theta}_{\dot\alpha} \, . \tag{A.16}$$

For more complicated tensors one replaces the tensor by a string of θ's and conjugates those. For example, for a tensor $V_{\alpha\dot\beta\gamma}{}^{\delta}$ one takes:

$$\overline{\theta_\alpha \theta_{\dot\beta} \theta_\gamma \theta^\delta} = \overline{\theta^\delta}\,\overline{\theta_\gamma}\,\overline{\theta_{\dot\beta}}\,\overline{\theta_\alpha} = -\overline{\theta}^{\dot\delta}\,\overline{\theta}_{\dot\gamma}\theta_\beta\overline{\theta}_{\dot\alpha} = -\overline{\theta}_{\dot\alpha}\theta_\beta\overline{\theta}_{\dot\gamma}\overline{\theta}^{\dot\delta} \, . \tag{A.17}$$

One therefore concludes that $\overline{V_{\alpha\dot\beta\gamma}{}^{\delta}} = -V_{\dot\alpha\beta\dot\gamma}{}^{\dot\delta}$.

In order to translate from two-component notation to four-component notation and vice versa, we first choose a representation for the γ-matrices:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{A.18}$$

where the σ^i, $i = 1, 2, 3$ are the ordinary Pauli matrices which we identify with $\sigma^{i\,\alpha\dot\beta}$ or $\sigma^i_{\alpha\dot\beta}$. $\sigma^0_{\alpha\dot\beta}$ is defined to be a unit two by two matrix. Next consider a four-component Majorana spinor ψ:

$$\psi = \begin{pmatrix} \psi^\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix} \tag{A.19}$$

here $\overline{\psi}^{\dot\alpha} = (\psi^\alpha)^+$ and $\overline{\psi}_{\dot\alpha} = -(\psi_\alpha)^+$ with $\alpha, \dot\alpha = 1,2$. We also have

$$\overline{\varepsilon} \equiv \varepsilon^+ \gamma_0 = (-\varepsilon_\alpha, \overline{\varepsilon}^{\dot\alpha}) \, . \tag{A.20}$$

Then one finds:

$$\overline{\varepsilon}\psi = \varepsilon^\alpha \psi_\alpha + \overline{\varepsilon}^{\dot\alpha}\overline{\psi}_{\dot\alpha}$$

$$\overline{\varepsilon}\gamma_5\psi = \varepsilon^\alpha \psi_\alpha - \overline{\varepsilon}^{\dot\alpha}\overline{\psi}_{\dot\alpha}$$

$$\overline{\varepsilon}\gamma^a\psi = \sigma^a_{\alpha\dot\alpha}(\overline{\varepsilon}^{\dot\alpha}\psi^\alpha + \varepsilon^\alpha\overline{\psi}^{\dot\alpha})$$

$$\bar{\varepsilon}\gamma_5\gamma^a\psi = \sigma^a_{\alpha\dot{\alpha}}(\varepsilon^\alpha\bar{\psi}^{\dot{\alpha}} - \bar{\varepsilon}^{\dot{\alpha}}\psi^\alpha)$$

$$\bar{\varepsilon}\sigma_{ab}\psi = -\frac{i}{2}\left((\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\bar{\varepsilon}^{\dot{\alpha}}\bar{\psi}^{\dot{\beta}} + (\sigma_{ab})_{\alpha\beta}\varepsilon^\alpha\psi^\beta\right) \tag{A.21}$$

with:

$$\sigma_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right]. \tag{A.22}$$

Relations (A.19), (A.20) and (A.21) are sufficient to translate expressions from two-component to four-component notation and vice versa. More complicated relations involving products of several γ-matrices can be worked out by repeated use of (A.19) to (A.21).

Appendix B. Weyl Redefinitions

In this appendix we study the subject of field redefinitions. Suppose we have two sets of superspace tensors, the unprimed set and the primed set. Each of these two sets leads to a different set of supersymmetry transformation laws. The problem is to find, if possible, a set of field redefinitions that turns one set of supersymmetries into the other. If there is one, the two sets of superspace tensors describe theories that are equivalent up to field redefinitions.

Consider the following set of redefinitions of the supercovariant derivative:

$$D'_{\alpha i} = e^{L}[D_{\alpha i} + \frac{1}{2} f_{\alpha i\, b}{}^{c}\, M_c{}^{b}]\,,$$

$$\bar{D}'{}^{\cdot i}_{\dot\alpha} = e^{\bar{L}}[\bar{D}^{\cdot i}_{\dot\alpha} + \frac{1}{2}\bar{f}^{\cdot i}_{\dot\alpha}{}_{b}{}^{c}\, M_c{}^{b}]\,,$$

$$D'_{a} = e^{M}\,[D_a + f_a{}^{\alpha i} D_{\alpha i} + \bar{f}_a{}^{\dot\alpha}{}_{i}\bar{D}^{\cdot i}_{\dot\alpha} + \frac{1}{2} f_{ac}{}^{d} M_d{}^{c}]\,. \tag{B.1}$$

These redefinitions have been considered some time ago [1,2] in order to understand the various types of constraints that can be imposed on the superspace geometry. Using relations [B.1] and the definition of torsions, curvatures and field strength by graded commutation (equation (2.2) of Chapter III) one can relate the primed and unprimed superspace tensors. Assuming $T_{\alpha i\,\beta j}{}^{c} = 0$ one finds [1]:

$$T'_{\alpha i\,\beta j}{}^{c} = 0\,, \tag{B.2}$$

$$T'_{\alpha i\,\dot\beta j}{}^{c} = e^{L+\bar{L}-M}\, T_{\alpha i\,\dot\beta j}{}^{c}\,, \tag{B.3}$$

$$T'_{\alpha i\,\beta j}{}^{\dot\gamma k} = e^{2L-\bar{L}}\, T_{\alpha i\,\beta j}{}^{\dot\gamma k}\,, \tag{B.4}$$

$$T'_{\alpha i\,\beta j}{}^{\gamma k} = e^{L}[T_{\alpha i\,\beta j}{}^{\gamma k} + D_{\alpha i}L\delta^{\gamma}_{\beta}\delta^{k}_{j} + D_{\beta j}L\delta^{\gamma}_{\alpha}\delta^{k}_{i} + f_{\beta j\,\alpha}{}^{\gamma}\delta^{k}_{i} + f_{\alpha i\,\beta}{}^{\gamma}\delta^{k}_{j}]\,, \tag{B.5}$$

$$T'_{\alpha i\, a}{}^{c} = e^{L}[T_{\alpha i\, a}{}^{c} + (D_{\alpha i}M)\delta^{c}_{a} - \bar{f}_a{}^{\dot\beta j}\, T_{\alpha i\,\dot\beta j}{}^{c} + f_{\alpha i\, a}{}^{c}]\,, \tag{B.6}$$

and for the field strengths one has:

$$F'_{\alpha i\,\dot\beta j}{}^{mn} = e^{L+\bar L}\,F_{\alpha i\,\dot\beta j}{}^{mn}\ , \tag{B.7}$$

$$F'_{\alpha i\,\beta j}{}^{mn} = e^{2L}\,F_{\alpha i\,\beta j}{}^{mn}\ . \tag{B.8}$$

The superspace tensors listed above are the only ones needed to study field redefinitions in N = 4 supergravity since all the other tensors follow uniquely from these and the Bianchi identities. If a set of redefinitions works for the above tensors, it will work for all the superspace tensors. Given two sets of superspace tensors one uses equations (B.2) to (B.8) to determine $L, M, f_{\alpha i\, b}{}^{c}$ and $f_{a}{}^{\alpha i}$. We shall not determine $f_{ac}{}^{d}$. Its value can be obtained requiring that the constraint imposed on $R_{\underline{\alpha}\underline{\dot\beta}\gamma\delta}$ (equation (3.1) of Chapter III) be preserved. Its value, however, is not necessary in the following discussion.

The redefinitions of the supercovariant derivative imply some redefinitions for the component fields of the supergravity theory. We now derive the component field redefinitions that follow from equations (B.1). In a Wess - Zumino gauge we have:

$$D_{\alpha i} = \partial_{\alpha i} + O(\theta)\ ,\quad \bar D_{\dot\alpha}^{\ i} = \partial_{\dot\alpha}^{\ i} + O(\theta)\ ,$$

$$D_{a} = e_a^m\,\partial_m + \psi_a^{\mu i}\,\partial_{\mu i} + \bar\psi_a{}^{\dot\mu}{}_i\,\partial_{\dot\mu}^{\ i}$$

$$+\frac{1}{2}\varphi_{a\gamma}{}^{\delta}M_{\delta}{}^{\gamma} + \frac{1}{2}\bar\varphi_{a\dot\gamma}{}^{\dot\delta}M_{\dot\delta}{}^{\dot\gamma} + \frac{1}{2}\varphi_a^{ij}\,Z_{ji} + O(\theta)\ , \tag{B.9}$$

where e_a^m is the vierbein, $\psi_a^{\mu i}$ are the gravitinos, $\varphi_{a\gamma}{}^{\delta}$ is the spin connection, and φ_a^{ij} are the spin one fields. For the primed supercovariant derivatives, relations analogous to (B.9) hold. We take $\partial_m = \partial'_m$ since coordinate transformations should not have any role in the redefinitions, but $\partial_{\alpha i} \neq \partial'_{\alpha i}$ because of equations (B.1) and (B.9). We therefore have:

$$e'_a{}^m = e^M e_a{}^m \,, \tag{B.10}$$

for the redefined vierbein. For the gravitinos:

$$\psi'_a{}^{\alpha i} = e^{M-L}(\psi_a{}^{\alpha i} + f_a{}^{\alpha i}) \,, \tag{B.11}$$

using curved indices one has:

$$\psi'_\mu{}^{\alpha i} = e^{-L}(\psi_\mu{}^{\alpha i} + f_\mu{}^{\alpha i}) \,, \tag{B.12}$$

and finally translating to four component notation:

$$\psi'_\mu{}^i = ||e^{-L}||(\psi_\mu{}^i + f_\mu{}^i) \,, \tag{B.13}$$

where the $||..||$ notation is defined by $||f|| = Re(f) + i\gamma_5 Im(f)$. It should be noted that in equation (B.13) curved indices refer to the respective vierbeins. In order to find the redefinition of the spin one-half fields, we consider the torsion $T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}}$ in which they appear (see equation (2.21) of Chapter III). Using equation (B.4), we find:

$$\bar{\Lambda}'^{\dot{\alpha} i} = e^{2L-\bar{L}} \bar{\Lambda}^{\dot{\alpha} i} \,, \tag{B.14}$$

and translating to four component notation:

$$\Lambda'_i = ||e^{2\bar{L}-L}||\Lambda_i \,. \tag{B.15}$$

Consider now the vector fields, from equations (B.1), (B.9) and (B.10) one derives:

$$\varphi'_\mu{}^{ij} = \varphi_\mu{}^{ij} \,. \tag{B.16}$$

The redefinition of the scalar fields cannot be derived from the relations given above. The explicit forms of the $F_{\underline{\alpha}\underline{\beta}}{}^{mn}$ tensors entering equation (B.8) is necessary for this purpose.

Equations (B.10), (B.13), (B.15) and (B.16) are the desired relations. They give the field redefinitions that turn the primed theory into the unprimed theory. We have not given the redefinition of the spin connection, because it is not needed for our discussion. We should also remark that the above set of redefinitions turn the supersymmetry transformation formulas of one theory into those of the other one, up to a Lorentz rotation, whose parameter is proportional to the $f_{\alpha i b}{}^{c}$ tensor appearing in equation (B.1).

Appendix C. Polynomial Form

To obtain a form of the SU(4) supergravity in which the scalars appear polynomially, we first rescale the following fields:

$$V_{a\mu} \to e^{-\kappa\varphi} V_{a\mu}, \quad \psi_\mu^i \to e^{-\kappa\varphi/2} \psi_\mu^i,$$

$$\chi^i \to e^{\kappa\varphi/2} \chi^i, \quad B \to e^{-2\kappa\varphi} B . \tag{C.1}$$

Once these substitutions are made in the lagrangian of the SU(4) model [3], one obtains a lagrangian in which the only nonpolynomial appearance of the field φ occurs through a factor $e^{-2\kappa\varphi}$ multiplying all the terms in the lagrangian. In the supersymmetry transformations, however, the scalar fields appear entirely polynomially.

We now redefine the scalar field φ:

$$e^{-\kappa\varphi} \to \kappa\varphi , \tag{C.2}$$

and finally obtain:

$$L = \kappa^2 \varphi^2 L_o$$

where L_o is:

$$L_o = -\frac{VR(\omega)}{4\kappa^2} - \frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\overline{\psi}_\lambda^i\gamma_5\gamma_\mu D_\nu\psi_\rho^i + \frac{i}{2}V\overline{\chi}^i\gamma^\mu\widehat{D}_\mu\chi^i$$

$$-\frac{1}{4}V(A_{\mu\nu}^n A^{n\,\mu\nu} + B_{\mu\nu}^n B^{n\,\mu\nu}) + \frac{1}{2}Vg^{\mu\nu}\left(\frac{\partial_\mu\varphi\,\partial_\nu\varphi}{\kappa^2\varphi^2} + \widetilde{D}_\mu B\,\widetilde{D}_\nu B\right)$$

$$-\frac{1}{2}\kappa B(\widetilde{A}^{n\,\mu\nu}A_{\mu\nu}^n + \widetilde{B}^{n\,\mu\nu}B_{\mu\nu}^n) + \frac{\kappa}{4\sqrt{2}}[V\overline{\psi}_\mu^i(C_{ij}^{\mu\nu} + \widehat{C}_{ij}^{\mu\nu})\psi_\nu^j - i\overline{\psi}_\mu^i\gamma_5(\widetilde{C}_{ij}^{\mu\nu} + \widetilde{\widetilde{C}}_{ij}^{\mu\nu})\psi_\nu^j]$$

$$+ \frac{\kappa V}{4i}\overline{\psi}_\rho^i C_{ij}^{\mu\nu}\sigma_{\mu\nu}\gamma^\rho\chi^j - \frac{i\kappa}{4}\widetilde{D}_\sigma B\,\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu^i\gamma_\rho\psi_\nu^i - \frac{3}{4}\kappa V\widehat{\widetilde{D}}_\mu B\,\overline{\chi}^i\gamma_5\gamma^\mu\chi^i$$

$$-\frac{\kappa V}{2\sqrt{2}}\bar{\psi}^i_\mu\,[-\frac{\partial_\nu\varphi}{\kappa\varphi} + i\gamma_5\widetilde{D}_\nu B]\,\gamma^\nu\gamma^\mu\chi^i\ , \tag{C.3}$$

where we have defined:

$$\widetilde{D}_\mu B = \partial_\mu B + \frac{2B}{\varphi}\partial_\mu\varphi\ . \tag{C.4}$$

For the above lagrangian the supersymmetry transformation laws are:

$$\delta V^\mu_a = -i\kappa\bar{\varepsilon}^i\gamma_a\psi^i_\mu + \frac{\kappa}{\sqrt{2}}\bar{\varepsilon}^i\chi^i\,V_{a\,\mu}\,,$$

$$\delta\varphi = -\frac{1}{\sqrt{2}}\kappa\varphi\,\bar{\varepsilon}^i\chi^i\ ,\quad \delta B = \frac{i}{\sqrt{2}}\bar{\varepsilon}^i\gamma_5\chi^i + \sqrt{2}\kappa B\,\bar{\varepsilon}^i\chi^i\ ,$$

$$\delta A^n_\mu = \frac{1}{\sqrt{2}}\alpha^n_{ij}\,[\bar{\varepsilon}^i\psi^j_\mu + \frac{i}{\sqrt{2}}\bar{\varepsilon}^i\gamma_\mu\chi^j]\,,$$

$$\delta B^n_\mu = \frac{i}{\sqrt{2}}\beta^n_{ij}\,[\bar{\varepsilon}^i\gamma_5\psi^i_\mu + \frac{i}{\sqrt{2}}\bar{\varepsilon}^i\gamma_5\gamma_\mu\chi^j]\,,$$

$$\delta\bar{\chi}^i = \frac{i}{\sqrt{2}}[-\frac{\widehat{D}_\mu\varphi}{\kappa\varphi} + i\gamma_5\widehat{D}_\mu B]\gamma^\mu + \frac{1}{2}\bar{\varepsilon}^j\,\widehat{C}^{ij}_{\alpha\beta}\sigma^{\alpha\beta}$$

$$-\frac{\kappa}{2\sqrt{2}}\bar{\varepsilon}^j\chi^j\bar{\chi}^i - \frac{3\kappa}{2\sqrt{2}}\bar{\varepsilon}^j\gamma_5\chi^j\bar{\chi}^i\gamma_5\,,$$

$$\delta\bar{\psi}^i_\mu = \frac{1}{\kappa}\bar{\varepsilon}^i\overleftarrow{D}_\mu - \frac{1}{2}\bar{\varepsilon}^i[-\frac{\partial_\mu\varphi}{\kappa\varphi} + i\gamma_5\widetilde{D}_\mu B] + \frac{i}{2\sqrt{2}}\bar{\varepsilon}^j\,\widehat{C}^{ij}_{\alpha\beta}\gamma_\mu\sigma^{\alpha\beta}$$

$$-\frac{\kappa}{2\sqrt{2}}[\bar{\varepsilon}^j\gamma_5\chi^j\,\bar{\psi}^i_\mu\gamma_5 - \bar{\varepsilon}^j\chi^j\,\bar{\psi}^i_\mu] + \frac{\kappa}{\sqrt{2}}\,\varepsilon^{ijkl}[\bar{\varepsilon}^k\psi^j_\mu\bar{\chi}^l - \bar{\varepsilon}^k\gamma_5\psi^j_\mu\bar{\chi}^l\gamma_5]$$

$$+\frac{i\kappa}{4}[\chi^j\gamma_5\gamma^a\chi^i\,\bar{\varepsilon}^i\gamma_5\gamma_\mu\gamma_a + \bar{\chi}^i\gamma^a\chi^j\,\bar{\varepsilon}^j\gamma_\mu\gamma_a - \bar{\chi}^i\gamma_5\gamma^a\chi^j\,\bar{\varepsilon}^j\gamma_5\gamma_\mu\gamma_a]\,. \tag{C.5}$$

The spin connection is the one obtained solving its nonpropagating field equation from the lagrangian:

$$\omega_{\mu ab} = \omega_{\mu ab}(e) + \frac{i\kappa^2}{2}(\overline{\psi}_\mu \gamma_b \psi_a + \overline{\psi}_b \gamma_\mu \psi_a + \overline{\psi}_b \gamma_a \psi_\mu)$$

$$-e_{[a\mu} e_{b]}{}^{\nu} \frac{\partial_\nu \varphi}{\varphi} - \frac{\kappa^2}{4} e_\mu^a \, \varepsilon_{abcd} \, \overline{\chi}^i \gamma_5 \gamma^d \chi^i \, . \qquad \text{(C.6)}$$

The lagrangian given in equation (C.3) is polynomial in both the scalar fields φ and B, but the supersymmetry transformations, as given in equation (C.5), are nonpolynomial in φ.

References for Appendices

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