Normal Modes in Damped Systems

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Abstract

A general review of normal mode theory as applied to the vibration of linear damped lumped parameter bilateral systems is presented. It is shown that systems possessing classical damping may always be solved by the method developed by Rayleigh. However, for more general type non-classical damping the method proposed by F. A. Foss must be used. The main differences between classical and non-classical normal modes are noted. A non-classically damped system which does not possess a mode type solution is solved by La place Transform techniques.

The effect of damping on the natural frequencies of a linear system is discussed. It is shown that in classically damped systems increasing the damping decreases the natural frequencies of the system. With non-classical damping some of the natural frequencies of the damped system may be greater than the corresponding natural frequencies of the undamped system. From the perturbation analysis, used in determining the effect of damping on the natural frequencies of the system, the concept of equivalent classical damping for non-classically damped systems is derived.

Experimental techniques needed to determine the mode shapes, natural frequencies, mass spring and damping matrices of classically damped systems are presented.

By the use of the concept of equivalent classical damping an estimate of the damping matrix in non-classical systems may be obtained. Experimental results supporting the theory are presented.

Table of Contents

Abstract	
Introduction	1
Chapter I- General Theory	4
Chapter II - Effect of Damping on the Natural Frequencies of Linear Dynamic Systems	100
Chapter III - Experimental Investigations	115
Discussion and Conclusions	144
Appendix - Experimental Results	146
Figures	163
References	179

Introduction

The underlying theory of normal modes was developed to a high degree by the classical physicists and early mathematicians. In 1753 Daniel Bernoulli introduced the idea of superposition of motion and later in 1762-65 Lagrange developed the general theory of undamped linear vibrating systems. Routh, in his Adams Prize Essay of 1877 and in his text Rigid Dynamics, appears to have been one of the first to give a systematic treatment of small oscillations using normal mode techniques. The climax in the early development was reached by Lord Rayleigh when he introduced the concept of the dissipation function and thereby laid the foundations of normal modes in damped systems.

At the turn of the century most physicists felt that enough work had been done on the analysis of multi-degree of freedom systems. However, with the introduction of high speed rotating machinery and the associated problems of whirling of shafts and vibration isolation, engineers began to analyze practical systems. As it is extremely tedious to solve systems of greater than three degrees of freedom exactly by hand calculators, many iterative numerical methods were developed at this time to rapidly approximate the normal modes of the system.

Up to this time most of the work was concerned with what is now called classically damped systems. In point of fact the damping was further restricted to Rayleigh type uniform damping. With the rapid advances in structural dynamics initiated by the aircraft industry, it was inevitable that the methods of normal modes as used by Rayleigh would be found to be inadequate to solve some interesting physical problems. T. K. Caughey showed that Rayleigh's approach has much wider application than had been supposed. K. A. Foss developed a method of deriving normal mode solutions for systems possessing non-classical damping.

Today normal mode techniques are in common use in engineering and science. Whereas in principal any linear dynamic system may be solved by integral transform techniques the use of normal modes will greatly reduce the computational work. In electrical lumped parameter circuit theory the normal mode approach is rarely used whereas in mechanical lumped parameter system this approach is practically standard. This difference in the approach to lumped parameter systems is largely due to the standardized forms of electrical networks, e.g., filter sections which are much more readily treated by such specialized techniques as the four pole parameter methods. However, electrical engineers use normal mode solutions in many electro-magnetic problems. Although the theory needed to find the normal modes of either classical or non-classical systems has been developed there is a need to critically analyze some of this theory and to correlate the work of the main researches in the field. This work attempts to point out some of the difficulties that may arise in solving systems by the normal mode approach particularly if these systems are non-classically damped. As there is a great need for the development of practical tests to determine the parameters of a linear system some work in this area is presented. Lastly the work is interested in the general physical effects of non-classical damping and in the development of equivalent classical systems.

This work is limited to physically realizable passive systems. Consequently the systems considered are bilateral as well as linear. Bilateral systems are systems such that no unsymmetrical coupling terms that would violate the reciprocity theorem are introduced. This, of course, prevents the use of the results to either aircraft flutter or systems possessing gyroscopic motion.

Chapter I

Not all methods of deriving the equations of motion for a multi-degree of freedom system lead to expressions which are symmetric in the co-ordinates of the system. It is always possible, when dealing with physically realizable passive non-gyroscopic systems, to transform a set of unsymmetric equations of motion into a set of symmetric equations. However, the use of either energy or variational methods of derivation results directly in a symmetric set of equations. A further feature of the energy and variational methods is the possibility of selecting a set of co-ordinates which may considerably simplify the algebraic and numerical work involved in solving for the displacements and velocities of the system.

There are many advantages in using Lagrange's Equations, the energy method most frequently encountered in engineering analysis, to describe the motion of systems. The equations of motion are derived in exactly the same way for every possible set of co-ordinates. As only the potential and kinetic energies are involved there is no possibility of difficulty with the algebraic signs of the displacements and velocities, and there is no need to determine the accelerations. Whereas, in fact, many of the systems treated later may be solved by Newton's Second Law of Motion, some difficulty could exist regarding the signs of the displacements and velocities while the acceleration of each mass would have to be determined.

To use Lagrange's Equations it is necessary to define:

1) Holonomic System: A system such that the number of degrees of freedom equals the required number of co-ordinates to completely describe it.
2) Non-Holonomic System: A system, so constrained, that the number of degrees of freedom is less than the required number of co-ordinates to completely describe it.
3) Generalized Co-ordinates $q_i, (q_1, q_2 \cdots q_n)$: A set of independent co-ordinates used to completely describe the motion of the system.

As non-holonomic systems rarely occur in practise, this work will be restricted to holonomic systems. The co-ordinates may be chosen in any suitable way consistent with the geometry of the problem.

<u>Lagrange's Equations for a Holonomic System with n degrees of freedom in the usual form are</u>

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} T - \frac{\partial}{\partial q_i}(T) = Q_i \qquad i = 1, 2 \ldots n \tag{1}$$

where T = kinetic energy of system

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\, \dot{q}_i\, \dot{q}_j \tag{2}$$

and Q_i = Generalized Force at q_i

A more convenient form for our purpose may be obtained by defining $Q_i = Q_{ic} + F_i$

where Q_{ic} can be expressed as a potential function, i.e.

$$Q_{ic} = -\frac{\partial V}{\partial q_i} \tag{3}$$

V = Potential Energy of system

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} K_{ij}\, \dot{q}_i\, \dot{q}_i \tag{4}$$

and F_i is the non-conservative part of the generalized force Q_i.

As V is independent of $\dot{q}_i$ Equation (1) may be rewritten as follows:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}(T-V) - \frac{\partial}{\partial q_i}(T-V) = F_i \tag{5}$$

Let $T - V = L$, Lagrange's Function

$$\therefore \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}(L) - \frac{\partial}{\partial q_i}(L) = F_i \tag{6}$$

or for a conservative system

$$\frac{d}{dt}\frac{\partial}{\partial q_i}(L) - \frac{\partial}{\partial q_i}(L) = 0 \tag{7}$$

<u>Derivation of Equations of Motion of multi-degree of freedom system.</u>

Consider a system of n discrete masses m_i coupled together through springs and dashpots as shown in Figure 1. Choose the generalized co-ordinates x_i to specify the motion of the system. Let $x_i = 0$ all i when system is in stable equilibrium.

As this is clearly a holonomic system

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i} L - \frac{\partial L}{\partial x_i} = F_i \tag{8}$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{x}_i^2 \tag{9}$$

$$V = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_{ij} x_i x_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\dot{x}_i \dot{x}_j \tag{10}$$

where m_i are the masses $i = 1, 2, \cdots N$

K_{ij}=force exerted on m_i by the spring system when $x_i = \dot{x}_i = 0$ all i except $i = j$, $x_j = 1$, $\dot{x}_j = 0$.

It should be noted that $K_{ij} = K_{ji}$ as can easily be shown by performing the following two experiments:

Place system in stable equilibrium

Displace m_i so that $x_i = \bar{x}_i \neq 0$

$x_j = 0$ all $j \neq i$

Holding m_i so that $x_i = \bar{x}_i \neq 0$

Displace m_ℓ so that $x_\ell = \bar{x}_\ell \neq 0$

$x_j = 0 \quad j \neq i, \neq \ell$

Total energy stored in the spring system $= \frac{1}{2}\sum_{i=1}^{N} K_{i\ell}\bar{x}_\ell^2$

$$+ \frac{1}{2}\sum_{j=1}^{N} K_{ji}\bar{x}_j^2 + K_{\ell i}\bar{x}_i \bar{x}_\ell \tag{11}$$

Now reverse sequence of the above experiment.

Place system in stable equilibrium

Displace m_ℓ so that $x_\ell = \bar{x}_\ell = 0$

$x_j = 0$ all $j \neq \ell$

Holding m_ℓ so that $x_\ell = \bar{x}_\ell \neq 0$

$x_j = 0$ all $j \neq \ell$

Displace m_i so that $x_i = \bar{x}_i \neq 0$

$x_j = 0 \quad j \neq i, \; \neq \ell$

$x_\ell = \bar{x}_\ell \neq 0$

Total energy stored in spring system

$$= \frac{1}{2}\sum_{j=1}^{N} K_{j\ell}\,\bar{x}_\ell^2 + \frac{1}{2}\sum_{j=1}^{N} K_{ji}\,\bar{x}_i^2 + K_{i\ell}\,\bar{x}_i\,\bar{x}_\ell \qquad (12)$$

The final configuration of the system is the same in these two experiments. As there is no energy sink the energy stored must be the same.

Equation (11) $\equiv$ Equation (12)

$\therefore \; K_{\ell i} = K_{i\ell}$.

Similarly it may be shown that

$C_{ij} = C_{ji}$.

Returning to Equation (8) and on substituting Equation (9) and Equation (10) for $L = T - V$ the equations of motion for the system may be written

$$m_i\ddot{x}_i + \sum_{j=1}^{N} K_{ij}x_j + \sum_{j=1}^{N} C_{ij}x_j = F_i, \quad i = 1, 2, \ldots\ldots n \qquad (13)$$

For simplicity of notation and elegance of presentation, matrices will be used whenever possible. Define

mass matrix

$$[M]=\begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & & m_n \end{bmatrix} = [M_i] \qquad (14)$$

Spring matrix

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1N} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2N} \\ \vdots & & & & \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} = [K_{ij}] \qquad (15)$$

Damping matrix

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1N} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2N} \\ \vdots & & & & \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{bmatrix} = [C_{ij}] \qquad (16)$$

∴ on substituting Equation (14), Equation (15) and Equation (16) into Equation (13)

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \qquad (17)$$

where

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}$$ a column vector of order $N \times 1$

and

$$\{F\} = \begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{Bmatrix}$$ a column vector of order $N \times 1$.

It should be noted that for a physically realizable passive system $[M]$, $[K]$ and $[C]$ are symmetric matrices. $[M]$ and $[K]$ are positive definite and $[C]$ is at least non-negative definite.

In this example $[M]$ is also a diagonal matrix, but this is not necessarily so in all multi-degree of freedom problems.

Now if $[C]$ is a null matrix, i.e. in the absence of viscous damping, Equation (17) reduces to

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\} \qquad (18)$$

To solve this set of equations by classical methods it is necessary to first solve the inhomogenous equation

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \tag{19}$$

This equation is also known as the equation of free vibrations of the undamped system. As some type of vibrational motion is expected, assume

$$\{x\} = \{q\} e^{i\omega t} \tag{20}$$

where $\{q\}$ is a column vector of order $N \times 1$ the elements of which are independent of time.

On substituting Equation (20) into Equation (19)

$$\left(-\omega^2 [M]\{q\} + [K]\{q\}\right) e^{i\omega t} = 0 \tag{21}$$

$$\therefore \left[-\omega^2 [M] + [K]\right]\{q\} = 0 \tag{22}$$

For non-trivial solutions of Equation (22)

$$\left\|\left[-\omega^2 [M] + [K]\right]\right\| = 0 \tag{23}$$

Equation (23), known as the frequency equation, reduces to a poly-nomial of degree n in ω^2 as the determinent is of order n. $[M]$ and $[K]$ being symmetric and positive definite the roots of this equation are all real and positive. Neglecting for the present the case of repeated roots, there exists $2n$ distinct values of ω^2 that satisfy Equation (23)

$$\omega_i = +\sqrt{\omega_i^2} \qquad i = 1, 2, \cdots\cdots n$$

$$\omega_{i+n} = -\sqrt{\omega_i^2} \qquad i = 1, 2, \cdots\cdots n \tag{24}$$

where the ω_i^2 's are the roots of Equation (23).

For each distinct ω_i^2 there exists a vector $\{q^i\}$ which satisfies the following equation:

$$\Big[-\omega^2[M] + [K]\Big]\{q^i\} = 0$$

The vectors $\{q^i\}$ form an linearly independent set.

In order to formalize the above procedure it is necessary to note that as $[M]$ and $[K]$ are symmetric and positive definite there exists a transformation $[Q]$ such that

$$[Q]^T[M][Q] = [\bar{M}] \quad \text{a diagonal matrix.}$$

$$[Q]^T[K][Q] = [\bar{K}] \quad \text{a diagonal matrix.}$$

Let

$$\{x\} = [Q]\{\eta(t)\} \tag{25}$$

where $\{\eta(t)\}$ is a column vector of order $N \times 1$ with elements $\eta_i(t)$.

On substituting Equation (25) into Equation (19)

$$[M][Q]\{\ddot{\eta}(t)\} + [K][Q]\{\eta(t)\} = 0 \tag{26}$$

Pre-multiply Equation (26) by $[Q]^T$.

$$\therefore [Q]^T[M][Q]\{\ddot{\eta}(t)\} + [Q]^T[K][Q]\{\eta(t)\} = 0 \tag{27}$$

$$\therefore [\bar{M}]\{\ddot{\eta}(t)\} + [\bar{K}]\{\eta(t)\} = 0 \tag{28}$$

where $[\bar{M}]$ and $[\bar{K}]$ are diagonal matrices. Equation (28) is now a set of uncoupled linear equations of type

$$\bar{M}_i \ddot{\eta}_i(t) + \bar{K}_i \eta_i(t) = 0 \tag{29}$$

with solution

$$\eta_i(t) = A \sin \omega_i t + B \cos \omega_i t \tag{30}$$

where $\omega_i = \sqrt{\frac{\bar{K}_{ii}}{\bar{M}_i}}$ is the natural frequency of the i^{th} normal mode, A and B are arbitrary constants which may be determined from the initial conditions.

$$\therefore \{x\} = [Q]\{\eta(t)\} \tag{31}$$

Thus any x_i is composed of the sum of n quantities of type $q_i^{\,j}\eta_j(t)$ where $[Q] = [q_i^{\,j}]$. For this reason the columns of $[Q]$ are looked upon as vectors and $\{x\}$ is said to be a linear combination of these vectors. As these vectors possess the property of orthogonality in $[M]$ and $[K]$ they are commonly called normal modes.

From the above analysis it is immediately apparent that a set of generalized co-ordinates will exist which will give uncoupled equations of motion.

For let

$$\{x\} = [Q]\{\eta(t)\} \tag{32}$$

From Equation (9)

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{x}_i^2 \tag{33}$$

$$= \frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\} \tag{34}$$

On substituting Equation (32) into Equation (34)

$$T = \{\eta(t)\}^T[Q]^T[M][Q]\{\eta(t)\} \tag{35}$$

$$= \{\eta(t)\}^T[\bar{M}]\{\eta(t)\} \tag{36}$$

where $[\bar{M}]$ is diagonal.

Again as $[C]$ is assumed to the null matrix from

Equation (10)

$$V = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} K_{ij}\, x_i\, x_j$$

$$= \frac{1}{2} \{x\}^T [K] \{x\} \qquad (37)$$

$$= \frac{1}{2} \{\eta(t)\}^T [\bar{K}] \{\eta(t)\}$$

where $[\bar{K}]$ is diagonal.

Now on applying Lagrange's Equations to T and V given by Equation (36) and Equation (37) an uncoupled set of equations results.

Orthogonality of Normal Modes

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \qquad (38)$$

Let (39)

$$\{x\} = \{q^j\}\, e^{i\omega_j t}$$

where $\{q^j\}$ is the j^{th} normal mode, i.e.

$$[Q] = [\{q^1\}, \{q^2\}, \cdots\cdots, \{q^n\}] \qquad (40)$$

On substituting Equation (39) into Equation (38), rearranging and dividing both sides by $e^{i\omega_j t}$

$$-\omega_j^2 [M]\{q^j\} + [K]\{q^j\} = 0 \qquad (41)$$

Again let

$$\{x\} = \{q^l\} e^{i\omega_l t} \qquad (42)$$

On substituting Equation (42) into Equation (38), rearranging and dividing both sides by $e^{i\omega_l t}$

$$-\omega_l^2 [M]\{q^l\} + [K]\{q^l\} = 0 \qquad (43)$$

Pre-multiply Equation (41) by $\{q^l\}^T$

$$\therefore -\omega_j^2 \{q^l\}^T[M]\{q^j\} + \{q^l\}^T[K]\{q^j\} = 0 \qquad (44)$$

Transpose Equation (43) and post-multiply by $\{q^j\}$

$$-\omega_l^2 \{q^l\}^T[M]^T\{q^j\} + \{q^l\}^T[K]^T\{q^j\} = 0 \qquad (45)$$

As $[M]$ and $[K]$ are symmetric

$[M]^T = [M]$

$[K]^T = [K]$

Equation (45) reduces to

$$-\omega_l^2 \{q^l\}^T[M]\{q^j\} + \{q^l\}^T[K]\{q^j\} = 0 \qquad (46)$$

On substituting Equation (45) from Equation (44)

$$(\omega_l^2 - \omega_j^2)\{q^l\}^T[M]\{q^j\} = 0 \qquad (47)$$

If

$$\omega_\ell \neq \omega_j \qquad \ell \neq j \tag{48}$$

$$\left\{q^\ell\right\}^T \left[M\right] \left\{q^j\right\} = 0 \tag{49}$$

Equation (48) will be satisfied for all $\ell \neq j$ provided the roots of the frequency equation are distinct. On substituting Equation (49) into Equation (46)

$$\left\{q^\ell\right\}^T \left[K\right] \left\{q^j\right\} = 0 \qquad \ell \neq j \tag{50}$$

Equation (49) and Equation (50) are known as the orthogonality conditions for the normal modes $\left\{q^j\right\}$ and $\left\{q^\ell\right\}$.

At this stage it is necessary to note that the $\left\{q^i\right\}$ form a complete set, i.e., any $\left\{X\right\}$ can be represented as a linear combination of these N independent vectors. In other words the normal modes span the N vector space.

Force Vibration of Undamped System:

Again assuming distinct roots of the frequency equation, the solution to the forced vibrations of an undamped system is presented

$$\left[M\right]\left\{\ddot{x}\right\} + \left[K\right]\left\{x\right\} = \left\{f(t)\right\} \tag{51}$$

where $\{f(t)\} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ \vdots \\ f_n(t) \end{Bmatrix}$ a column vector of order x 1 (52)

$f_i(t)$=force on m_i due to external forcing system.

Let $\{x\} = [Q]\{\eta(t)\}$ (53)

where $[Q]^T[M][Q] = [\bar{M}]$ a diagonal matrix

$[Q]^T[K][Q] = [\bar{K}]$ a diagonal matrix

Substituting Equation (53) into Equation (51) and pre-multiplying by $[Q]^T$

$$[Q]^T[M][Q]\{\ddot{\eta}(t)\} + [Q]^T[K][Q]\{\eta(t)\} = [Q]^T\{F(t)\} \quad (54)$$

$$\therefore [\bar{M}]\{\ddot{\eta}(t)\} + [\bar{K}]\{\eta(t)\} = [Q]^T\{F(t)\} \quad (55)$$

This is a system of uncoupled equations of type

$$\bar{M}_{ii}\ddot{\eta}_i(t) + \bar{K}_{ii}\eta_i(t) = \{g_i(t)\} \quad (56)$$

where $[Q]^T\{F(t)\} = \{G(t)\}$ (57)

and $\{G(t)\} = \begin{Bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ \vdots \\ g_n(t) \end{Bmatrix}$ a column vector of order $N \times 1$ (58)

From Equation (56)

$$\eta_i(t) = \frac{1}{\omega_i}\int_0^t \frac{g_i(\tau)}{\bar{M}_{ii}} \sin \omega_i (t-\tau)\, d\tau + A_i \sin(\omega_i t + \alpha_i) \tag{59}$$

where $\omega_i = \sqrt{\dfrac{k_{ii}}{\bar{M}_{ii}}}$ is the natural frequency of (60) the i^{th} mode. $\alpha_i =$ constant phase angle of the mode and $A_i =$ constant coefficient of complimentary function are both determined from initial conditions. Again the complete solution

$$\{x\} = [Q]\{\eta(t)\} \tag{61}$$

is a linear combination of normal modes $\{q^i\}$.

From the above analysis it is evident that any undamped system forced or free can be solved by normal mode techniques provided the roots of the frequency equation are distinct. The case of equal roots will be taken up later.

Damped Systems:

Now viscous damping is introduced into the system. The equations of motion for a linear damped system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \tag{62}$$

where $[C]$ is symmetric and at least non-negative definite. As before a vibrational solution of type

$$\{x\} = \{q\} e^{\alpha t} \qquad (62)$$

is assumed. On substituting Equation (62) into Equation (62)

$$\alpha^2 [M]\{q\} e^{\alpha t} + \alpha [C]\{q\} e^{\alpha t} + [K]\{q\} e^{\alpha t} = \{F\} \qquad (63)$$

To solve Equation (63) by classical methods it is necessary to first solve the inhomogeneous equation

$$\alpha^2 [M]\{q\} e^{\alpha t} + \alpha [C]\{q\} e^{\alpha t} + [K]\{q\} e^{\alpha t} = 0 \qquad (64)$$

on dividing through by $e^{\alpha t}$

$$\left[\alpha^2 [M] + \alpha [C] + [K] \right] \{q\} = 0 \qquad (65)$$

For non-trivial $\{q\}$

$$\left\| \left[\alpha^2 [M] + \alpha [C] + [K] \right] \right\| = 0 \qquad (66)$$

Equation (66) is the frequency equation for the damped system. In general this equation, being a polonomial of degree $2N$ in α, has $2N$ roots.

For a stable system these roots, if not completely imaginary, must contain a negative real part. In principle it is possible to solve this frequency equation for the α_i 's and to calculate the corresponding $\{q^i\}$'s and thereby form a complete solution for the free vibrations of damped linear systems. The details of this procedure will be presented later and now a more eloquent and simpler method will be reviewed. In passing it should be noted that as the α_i 's are in general complex the corresponding $\{q^i\}$'s may have complex components.

Equations of motion for free vibrations of a damped linear system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{67}$$

As $[M]$ and $[K]$ are symmetric and positive definite there exists a transformation $[Q]$ such that

$[Q]^T[M][Q] = [\bar{M}]$ is a diagonal matrix.

$[Q]^T[K][Q] = [\bar{K}]$ is a diagonal matrix

where $[Q]$ is an $N \times N$ matrix. Now if $[C]$ is such that

$[Q]^T[C][Q] = [\bar{C}]$ is a diagonal matrix (68)

then it is possible to completely uncouple the above equations of motion for

Let $\{x\} = [Q]\{\eta(t)\}$ (69)

where $\{\eta(t)\}$ is a column vector of order $N \times 1$.

On substituting Equation (69) into Equation (68)

$$[M][Q]\{\ddot{\eta}(t)\} + [C][Q]\{\dot{\eta}(t)\} + [K][Q]\{\eta(t)\} = 0 \quad (70)$$

Premultiply Equation (70) by $[Q]^T$

$$\therefore [Q]^T[M][Q]\{\ddot{\eta}\} + [Q]^T[C][Q]\{\dot{\eta}\} + [Q]^T[K][Q]\{\eta\} = 0 \quad (71)$$

$$\therefore [\bar{M}]\{\ddot{\eta}\} + [\bar{C}]\{\dot{\eta}\} + [\bar{K}]\{\eta\} = 0 \quad (72)$$

This is a set of uncoupled equations of type

$$\bar{M}_{ii}\ddot{\eta}_i + \bar{C}_{ii}\dot{\eta}_i + \bar{K}_{ii}\eta_i = 0 \quad (73)$$

Solving for $\eta_i(t)$, assuming the i^{th} mode is underdamped

$$\eta_i(t) = A_i e^{-\frac{\bar{C}_{ii}}{2\bar{M}_{ii}}t} \sin\left(\sqrt{\frac{\bar{K}_{ii}}{\bar{M}_{ii}} - \left(\frac{\bar{C}_{ii}}{2\bar{M}_{ii}}\right)^2}\, t + \alpha_i\right) \quad (74)$$

where $\sqrt{\frac{\bar{K}_{ii}}{\bar{M}_{ii}} - \left(\frac{\bar{C}_{ii}}{2\bar{M}_{ii}}\right)^2}$ is the damped natural frequency of the i^{th} mode. A_i, α_i are arbitrary constants depending on the initial conditions.

$$\therefore \{x\} = [Q]\{\eta(t)\} \quad (75)$$

In this case the solution consists of a linear combination of normal modes.

The above analysis is possible only if $[C]$ is such that the transformation which uncouples the undamped system will also uncouple the damped system. Dr. T. K. Caughey realizing this fact derived sufficient conditions for $[C]$ such that the equations can be uncoupled as above. Unfortunately necessary conditions on the dampingmatrix have not been developed. A brief review of Caughey's work is presented.

The equations of motion of a damped linear system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{76}$$

$$\{x\} = [N]^{-1}\{p\} \tag{77}$$

where $[N] = [\sqrt{M}]$ is a diagonal matrix of order $n \times n$. Substituting Equation (77) into Equation (76)

$$[M][N]^{-1}\{\ddot{p}\} + [C][N]^{-1}\{\dot{p}\} + [K][N]^{-1}\{p\} = 0 \tag{78}$$

Premultiply Equation (78) by $[N]^{-1}$

$$\therefore [N]^{-1}[M][N]^{-1}\{\ddot{p}\} + [N]^{-1}[C][N]^{-1}\{\dot{p}\} + [N]^{-1}[K][N]^{-1}\{p\} = 0 \tag{79}$$

$$\therefore [I]\{\ddot{p}\} + [A]\{\dot{p}\} + [B]\{p\} = 0 \tag{80}$$

where $[I]$ is the unit matrix of order $n \times n$.

$$A = [N]^{-1}[C][N]^{-1} \tag{81}$$

$$B = [N]^{-1}[K][N]^{-1} \tag{82}$$

As $[C]$ and $[K]$ are symmetric and positive definite it follows that $[A]$ and $[B]$ are also symmetric and positive definite. A system that can be completely uncoupled by the $[Q]$ matrix as shown above is said to possess classical normal modes and to be classically damped.

Caughey shows that a sufficient though not necessary condition that the original system possesses classical normal modes is that

$$[A] = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} Q_{n\ell} [B]_1^{\ell/N} \tag{83}$$

where $Q_{n\ell}$ are a set of arbitrary constants and $[B]_1^{\ell/N}$ is some ℓ/n th root of $[B]$. To express Equation (83) in terms of matrices of the original system substitute Equation (81) and Equation (82) into Equation (83)

$$\therefore [N]^{-1}[C][N]^{-1} = \sum_{n=0}^{\infty} \sum_{\ell=1}^{n-1} Q_{n\ell} \left[[N]^{-1}[K][N]^{-1} \right]_{1}^{\ell/N} \tag{84}$$

where $[N] = [\sqrt{M}\,]$

It is of interest to note some of the properties of classical normal modes. If a system possesses classical normal modes and the system is vibrating freely in this mode, which is assumed to be underdamped, all the m_i's pass through their points of stable equilibrium at the same instant. This implies that for classical normal modes the ϕ^i's can be expressed as vectors with real components. If in a classically damped system the damping is scalarly increased throughout the system the same normal modes exist as previously. Should any mode be overdamped solution of type

$$\{x\} = \left(Ae^{\alpha_i^{(1)}t} + Be^{\alpha_i^{(2)}t}\right)\{q^i\} \tag{85}$$

exists, where $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ are negative real numbers and $\left(Ae^{\alpha_i^{(1)}t} + Be^{\alpha_i^{(2)}t}\right)$ is the solution to the i^{th} overdamped uncoupled equation:

$$\bar{M}_{ii}\ddot{\eta}_i(t) + \bar{C}_{ii}\dot{\eta}_i(t) + \bar{K}_{ii}\eta_i(t) = 0$$

$$\alpha_i^{(1)}, \alpha_i^{(2)} = -\frac{\bar{C}_{ii}}{2\bar{m}_{ii}} \pm \sqrt{\left(\frac{\bar{C}_{ii}}{2\bar{m}_{ii}}\right)^2 - \frac{\bar{K}_{ii}}{\bar{m}_{ii}}} \tag{86}$$

overdamped if $\left(\frac{\overline{C_{ii}}}{2\overline{m_{ii}}}\right)^2 > \frac{\overline{K_{ii}}}{\overline{m_{ii}}}$ (87)

These properties of classical normal modes will be contrasted to the properties of the normal modes of systems which are not classically damped.

Forced Oscillations of classically damped systems:

In quite an analogous fashion to the previous work it is possible to extend these ideas to the forced vibration of classically damped systems.
Equation of motion of forced vibration of damped systems

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \tag{88}$$

As before let

$$\{x\} = [Q]\{\eta\} \tag{89}$$

Where $[M]$ and $[K]$ are symmetric and positive definite and $[Q]^T[M][Q] = [\bar{M}]$ is a diagonal matrix of order N. $[Q]^T[K][Q] = [\bar{K}]$ is a diagonal matrix of order N and as the system is classically damped $[Q]^T[C][Q] = [\bar{C}]$ is a diagonal matrix of order N

On substituting Equation (89) into Equation (88)

$$[M][Q]\{\ddot{\eta}\}+[C][Q]\{\dot{\eta}\}+[K][Q]\{\eta\}=\{F(t)\} \qquad (90)$$

Premultiply Equation (90) by $[Q]^T$

$$\therefore [Q]^T[M][Q]\{\ddot{\eta}\}+[Q]^T[C][Q]\{\dot{\eta}\}+[Q]^T[K][Q]\{\eta\}=[Q]^T\{F(t)\} \qquad (91)$$

$$\therefore [\bar{M}]\{\ddot{\eta}\}+[\bar{C}]\{\dot{\eta}\}+[\bar{K}]\{\eta\}=[Q]^T[F(t)] \qquad (92)$$

This is a system of uncoupled equations of type

$$\overline{M_{ii}}\,\ddot{\eta}_i+\overline{C_{ii}}\,\dot{\eta}_i+\overline{K_{ii}}\,\eta_i=g_i(t) \qquad (93)$$

where $[Q]^T[F(t)]=\{G(t)\}$

and $\{G(t)\}$ is a column vector $N \times 1$ with elements $g_i(t)$.

Solving Equation (93)

$$\eta_i=-\frac{1}{\overline{M_{ii}}\sqrt{\frac{\overline{K_{ii}}}{\overline{M_{ii}}}-\left(\frac{\overline{C_{ii}}}{2\overline{M_{ii}}}\right)^2}}\int_0^t g_i(\tau)e^{-\frac{\overline{C_{ii}}}{2\overline{M_{ii}}}(t-\tau)}\sin\sqrt{\frac{\overline{K_{ii}}}{\overline{M_{ii}}}-\left(\frac{\overline{C_{ii}}}{2\overline{M_{ii}}}\right)^2}\,(t-\tau)\,d\tau \qquad (94)$$

$$+A_i e^{-\frac{\overline{C_{ii}}}{2\overline{M_{ii}}}t}\sin\left(\sqrt{\frac{\overline{K_{ii}}}{\overline{M_{ii}}}-\left(\frac{\overline{C_{ii}}}{2\overline{M_{ii}}}\right)^2}\;t+\alpha_i\right)$$

where $\sqrt{\frac{\bar{K}_{ii}}{\bar{m}_{ii}} - \left(\frac{\bar{C}_{ii}}{2\bar{m}_{ii}}\right)^2}$ is the natural frequency, A_i and α_i are arbitrary constants determined from the initial conditions.

$$\therefore \{x\} = [Q]\{\eta(t)\} \tag{95}$$

Before going on to the discussion of more general types of damping a brief summary may be in order.

In a classically damped system, which by definition includes the undamped case, it is always possible to obtain a solution of type

$$\{x\} = [Q]\{\eta\} \tag{96}$$

where $[Q]$ is such that it simultaneously diagonalizes $[M]$, $[K]$ and $[C]$. The columns of $[Q]$ are the normal modes of the system and are the eigenvectors of $[M]^{-1}[K]$, for each $\{q^i\}$ satisfies

$$\left[\lambda_i^2 [M] + [K]\right]\{q_i\} = 0 \tag{97}$$

or

$$\left[\lambda_i^2 I + [M]^{-1}[K]\right]\{q_i\} = 0 \tag{98}$$

where Equation (98) is the usual form of an eigenvalue problem. It should be noted that each $\{q^i\}$ may only be determined to an arbitrary scalar multiplier and may be reduced to a vector of order $N \times 1$ with real components when this scalar multiplier, if imaginary, is neglected. In the next section it will be shown that the restriction of distinct roots of the frequency equation is not necessary for the existence of a normal mode solution to classically damped systems.

It is interesting to note that Lord Rayleigh stated that if $[C] = \alpha [M] + \beta [K]$

$$\alpha, \beta \text{ constants} \tag{99}$$

then the system has classical normal modes which are of course identical with the modes of the undamped system. However, Equation (99) may be obtained directly from Equation (84) which is Caughey's sufficient condition for classical normal modes

$$[N]^{-1}[C][N] = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} a_{n\ell} \Big[[N]^{-1}[K][N]^{-1} \Big]^{\ell/N} \tag{100}$$

by letting $a_{n\ell} = 0$ all $\ell \neq 0, 1$, all $n \neq 1$

$$a_{10} = \alpha$$

$$a_{11} = \beta$$

$\therefore$ Equation (100) reduces to

$$[N]^{-1}[C][N]^{-1} = \alpha \Big[[N]^{-1}[K][N]^{-1} \Big]^{0} + \beta \Big[[N]^{-1}[K][N]^{-1} \Big]^{1}$$

$$= \alpha [I] + \beta [N]^{-1}[K][N]^{-1}$$

On premultiplying and postmultiplying Equation (101) by $[N]$

$$[N][N]^{-1}[C][N]^{-1}[N] = \alpha[N][N] + \beta[N][N]^{-1}[K][N]^{-1}[N] \qquad (102)$$

as $[N] = [\sqrt{M}\,]$

$$\therefore [C] = \alpha[M] + \beta[K] \qquad (103)$$

which is Rayleigh's condition for classical normal modes. Although Rayleigh's result appears to be obvious when expressed in matrix notation it should be remembered that Rayleigh did not have the benefit of the use of this concise and highly suggestive notation. As a matter of fact Rayleigh introduced a Dissipation Function of type

$$D = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\,\dot{x}_i\,\dot{x}_j \qquad (104)$$

$$= \frac{1}{2}\{\dot{x}\}^T[C]\{\dot{x}\} \qquad (105)$$

to study damped linear systems.

Equal roots in classically damped systems:

In this section the theorems of linear algebra of interest in vibrational analysis are presented as well as a discussion of the case of equal roots of the frequency equation. Some of the theorems have already been used but it is felt that a formal presentation is desirable.

As shown previously the determination of the classical normal modes of a system is essentially an eigenvalue and eigenvector problem of type

$$\left[\lambda^2 [M] + [K]\right]\{q\} = 0 \tag{106}$$

A well known theorem in linear algebra can be stated as follows:

Let $[A]$ and $[B]$ be $N \times N$ real symmetric matrices $[A]$ being positive definite. Then there is a nonsingular real matrix such that

$[R]^T [A] [R] = I$ a unit matrix of order $N \times N$

$[R]^T [B][R] = [D]$ a diagonal matrix of order $N \times N$. As in most practical cases $[M]$ and $[K]$ will be positive definite as well as symmetric, define $[R]$ as a matrix of order $N \times N$ such that

$$[R]^T [M] [R] = I \tag{107}$$

is the unit matrix of order N

$$[R]^T [K][R] = [D] \tag{108}$$

$[D]$ is a diagonal matrix of order N where $[R]$ is non-singular. Define $\{\rho\}$ a column vector of order $N \times 1$ such that

$$\{\rho\} = [R]^{-1}\{q\} \tag{109}$$

As $[R]$ is non-singular $[R]^{-1}$ exists

$$\therefore \{q\} = [R]\{\rho\} \tag{110}$$

Substituting Equation (110) into Equation (106)

$$\left[\lambda^2[M][R] + [K][R]\right]\{\rho\} = 0 \tag{111}$$

Premultiply Equation (111) by $[R]^T$

$$\left[\lambda^2[R]^T[M][R] + [R]^T[K][R]\right]\{\rho\} = 0 \tag{112}$$

On substituting Equation (107) and Equation (108)

$$\left[\lambda^2 I + [D]\right]\{\rho\} = 0 \tag{113}$$

As $[D]$ is symmetric a theorem in linear algebra states that the eigenvalues of Equation (113) are all real and in fact are the diagonal elements of $[D]$. Because $[D]$ is similiar to $[K]$, a positive definite matrix, these eigenvalues are all negative. Again for any pair of distinct eigenvalues λ_i and λ_j there exists corresponding eigenvectors $\{\rho^i\}$ and $\{\rho^j\}$ such that $\{\rho^i\}$ and $\{\rho^j\}$ are orthogonal. This is consistent with the previously derived orthogonality conditions Equation (49) and Equation (50). Thus if the eigenvalues are all distinct the eigenvectors form an orthogonal base for the N space.

Suppose now that the eigenvalues are not all distinct, i.e., there exists a particular λ_i^2, a root of

$$\left\| \left[\lambda^2 I + [D] \right] \right\| = 0 \tag{114}$$

with multiplicity m. It can be shown that there exists precisely M linearly independent eigenvectors associated with this eigenvalue λ_i^2 of multiplicity M. These eigenvectors need not be orthogonal, however. But by the Gram Schmidt orthogonalization process it is possible to construct orthonormal vectors which are linear combinations of these M linearly independent eigenvectors.

But any linear combination of these eigenvectors is a solution to Equation (113) when $\lambda^2 = \lambda_i^2$. Hence, each vector of the orthonormal set derived by the Gram Schmidt process is in fact an eigenvector.

Thus if $[M]$ and $[K]$ are symmetric and positive definite there exists N eigenvectors which form a complete orthogonal set. Hence, there is no difficulty attached to equal roots of the frequency equation in classically damped systems for the resulting set of eigenvectors can be made to span the complete N space. Now consider an eigenvalue and eigenvector problem of type:

$$\left[\lambda_i I + [B]\right]\{x^i\} = 0 \qquad (115)$$

where λ_i are the eigenvalues. $[B]$ is a nonsymmetric matrix of order $N \times N$, $\{x^i\}$ are the eigenvectors of order $N \times 1$. There are N values of λ which satisfy the following equation

$$\left\|\left[\lambda I + [B]\right]\right\| = 0 \qquad (116)$$

Should these eigenvalues be distinct a complete set of eigenvectors exist. For a theorem in linear algebra states that for each distinct eigenvalue there exists a vector independent of all other eigenvectors.

As there are N distinct roots there exists a set of N independent eigenvectors. Unfortunately, if one eigenvalue has multiplicity, $M > 1$, a complete set of eigenvectors may or may not exist. The criterion for the existence of a complete set of eigenvectors in this case is the degeneracy of the matrix $[\lambda_i I + B]$ where λ_i is the eigenvalue of multiplicity M. If the rows of a matrix are linearly connected by more than one relation the matrix is multiply degenerate and in fact the degeneracy is N if there are N such relations. The degeneracy of $[\lambda_i I + B]$ must be equal to M, the multiplicity of the root λ_i for M distinct eigenvectors to exist corresponding to this root.

The treatment of generalized damping by the method of K. A. Foss:

If $[C]$ is such that it cannot be diagonalized by the same transformation as simultaneously diagonalized $[M]$ and $[K]$ the system is said to possess non-classical or generalized damping and the methods of solution presented above are not applicable. K. A. Foss has developed an interesting method for treating some cases of generalized damping. The essence of this method is to introduce a $2N$ space in which the equations of motion of the system can be uncoupled. A review of Foss's method for forced oscillations of linear damped systems is presented.

Equations of motion of linear damped systems:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{117}$$

By the introduction of a pair of $2N \times 1$ column vectors

$$\{Z\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \tag{118}$$

$$\{F\} = \begin{Bmatrix} \{0\} \\ f(t) \end{Bmatrix} \tag{119}$$

and the following set of matrices of order $2N \times 2N$

$$[R] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \tag{120}$$

$$[P] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \tag{121}$$

where $\{x\}$, $\{\dot{x}\}$, $\{0\}$ and $\{f(t)\}$ are column vectors of order $N \times 1$ associated with Equation (117). $[M]$, $[K]$ and $[C]$ are matrices of order $N \times N$ of the original linear damped system.

Equation (117) can now be reduced to

$$[R]\{\dot{z}\} + [P]\{z\} = \{F(t)\} \tag{122}$$

as Equation (122) on performing the matrix multiplications reduces to

$$\begin{Bmatrix} [M]\{\dot{x}\} \\ \\ [M]\{\ddot{x}\}+[C]\{\dot{x}\} \end{Bmatrix} + \begin{Bmatrix} -[M]\{\dot{x}\} \\ \\ [K]\{x\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \\ \{f(t)\} \end{Bmatrix} \tag{123}$$

Equation (123) is equivalent to 2 equations.

$$[M]\{\dot{x}\} - [M]\{\dot{x}\} = \{0\} \tag{124}$$

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{125}$$

Equation (124) is an identity while Equation (125) is the original equation of motion Equation (117). To solve Equation (123) by classical methods first solve the inhomogeneous Equation

$$[R]\{\dot{z}\} + [P]\{z\} = \{0\} \tag{126}$$

This is also the free vibration equation for the damped linear system transformed to $2N$ space. As vibrational solutions are expected

Let

$$\{Z\} = e^{\alpha t}\{\Phi\} \tag{127}$$

where $\{\Phi\}$ is a column vector of order $2N \times 1$. On substituting Equation (127) into Equation (126)

$$\alpha[R]\{\Phi\}e^{\alpha t} + [P]\{\Phi\}e^{\alpha t} = \{0\} \tag{128}$$

On dividing Equation (128) through by $e^{\alpha t}$ and rearranging

$$\Big[\alpha[R] + [P]\Big]\{\Phi\} = \{0\} \tag{129}$$

Equation (129) may be further reduced by premultiplying by $[P]^{-1}$, provided it exists, and dividing through by α

$$\Big[[P]^{-1}[R] + \frac{1}{\alpha}I\Big]\{\Phi\} = 0 \tag{130}$$

Equation (130) is in the usual form of an eigenvalue problem. That $[P]^{-1}$ exists may be seen from the following

Let $$[P]^{-1} = \begin{bmatrix} [a] & [b] \\ [c] & [d] \end{bmatrix} \tag{131}$$

where $[a]$, $[b]$, $[c]$ and $[d]$ are matrices of order $N \times N$. If $[P]^{-1}$ exists then on using Equation (131) and Equation (121)

$$[P][P]^{-1} = I = \begin{bmatrix} -[a][M] & [b][k] \\ -[c][M] & [d][k] \end{bmatrix} \tag{132}$$

$$\therefore \begin{aligned} [a] &= -[M]^{-1} \\ [b] &= [0] \\ [c] &= [0] \\ [d] &= [k]^{-1} \end{aligned} \tag{133}$$

$\therefore [P]^{-1}$ exists as $[M]^{-1}$ and $[K]^{-1}$ exist if $[M]$ and $[K]$ are nonsingular. It is not necessary that $[M]$ and $[K]$ be nonsingular in a physically realizable system. However, those cases where either $[M]$ or $[K]$ are singular, generally arise in practice from over-simplification of the physical system, and will be neglected.

$$\therefore [P]^{-1} = \begin{bmatrix} -[M]^{-1} & [0] \\ [0] & [K]^{-1} \end{bmatrix} \tag{134}$$

$$[P]^{-1}[R] = \begin{bmatrix} 0 & I \\ -[k]^{-1}[M] - [k]^{-1}[c] \end{bmatrix} \equiv [u] \tag{135}$$

where $[u]$ is a matrix of order $2N \times 2N$. Substituting Equation (124) into Equation (119) and rearranging to a form more suitable for numerical work

$$[u]\{\Phi\} = -\frac{1}{\alpha}\{\Phi\} \tag{136}$$

This is an eigenvalue problem and so as $[u]$ is non-symmetric, except in the trivial case when

$$\left.\begin{array}{l} [K]^{-1}[M] = [I] \\ [K]^{-1}[C] \text{ is a symmetric matrix} \end{array}\right. \tag{137}$$

e.g., if $\quad [K]^{-1} = [M]$

$\qquad [K]$ and $[C]$ are diagonal matrices

$2N$ independent eigenvectors may exist only if there are $2N$ distinct roots of the frequency equation in $2N$ space

$$\left\| \left[[u] + \frac{1}{\alpha}[I] \right] \right\| = 0 \tag{138}$$

For each distinct root $1/\alpha_i$ of Equation (127) there exists an independent vector $\{\Phi^i\}$. As stated previously a root of multiplicity M may or may not have M associated linearly independent eigenvectors. The eigenvalues $1/\alpha_i$ of Equation (125) may be purely real, complex or purely imaginary, but for a stable system the real part of each $1/\alpha_i \leq 0$. As $[u]$ contains only real elements the complex roots must form sets of complex conjugate roots. For each complex conjugate root pair the elements of the corresponding eigenvectors are also complex conjugates.

Orthogonality conditions in $2N$ space:

Assume that $[u]$ is such that there exists a complete set of $2N$ eigenvectors $\{\Phi^i\}$ and distinct eigenvalues $1/\alpha_i$

Let $\{z\} = \{\Phi^j\} e^{\alpha_j t}$ (139)

On substituting Equation (139) into Equation (126) and dividing both sides by $e^{\alpha_j t}$

$$\alpha_j [R]\{\Phi^j\} + [P]\{\Phi^j\} = \{0\} \tag{140}$$

Again let

$$\{z\} = \{\Phi^\ell\} e^{\alpha_\ell t} \tag{141}$$

On substituting Equation (141) into Equation (126) and dividing both sides by $e^{\alpha_\ell t}$

$$\alpha_\ell [R]\{\Phi^\ell\} + [P]\{\Phi^\ell\} = \{0\} \tag{142}$$

Transpose Equation (140) and postmultiply by $\{\Phi^{\ell}\}$

$$\alpha_j \{\Phi^j\}^T [R]^T \{\Phi^{\ell}\} + \{\Phi^j\}^T [P]^T \{\Phi^{\ell}\} = 0 \tag{143}$$

Premultiply Equation (142) by $\{\Phi^{\ell}\}^T$

$$\alpha_{\ell} \{\Phi^j\}^T [R] \{\Phi^{\ell}\} + \{\Phi^j\}^T [P] \{\Phi^{\ell}\} = \{0\} \tag{144}$$

On subtracting Equation (144) from Equation (143)

$$(\alpha_j - \alpha_{\ell}) \{\Phi^j\}^T [R] \{\Phi^{\ell}\} = \{0\} \tag{145}$$

if $\alpha_j \neq \alpha_{\ell} \quad j \neq \ell$

$$\{\Phi^j\}^T [R] \{\Phi^{\ell}\} = 0 \tag{146}$$

$$j \neq \ell$$

On substituting Equation (144)

$$\{\Phi^j\} [P] \{\Phi^{\ell}\} = 0 \quad j \neq \ell \tag{147}$$

Equation (146) and Equation (147) are analogous to the orthogonality conditions previously obtained with classical normal modes.

Basis of Foss's method

It is well to note the whole basis of Foss's method rests on the assumption that

$$\{Z\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = e^{\alpha t} \{\Phi\} \tag{148}$$

where $\{\Phi\}$ is a column vector of order $2N \times 1$. From Equation (148)

$$\{x\} = e^{\alpha t} \{\phi\} \tag{149}$$

where $\{\phi\}$ is a column vector of order $N \times 1$

$$\therefore \quad \{\dot{x}\} = e^{\alpha t} \{\alpha\phi\} \tag{150}$$

On substituting Equation (149) and Equation (150) into Equation (148)

$$\{\dot{z}\} = \alpha e^{\alpha t} \begin{Bmatrix} \alpha\phi \\ \phi \end{Bmatrix} \tag{151}$$

Now not all systems are such that

$$\{\dot{x}\} = \alpha e^{\alpha t} \{\phi\} \tag{152}$$

for take the case of a classically damped system such that the i^{th} uncoupled equation

$$\bar{M}_{ii}\ddot{\eta}_i + \bar{C}_{ii}\dot{\eta}_i + \bar{K}_{ii}\eta_i = 0 \tag{153}$$

has solution of type

$$(A_i + B_i t)e^{\alpha_i t} \tag{154}$$

where $\alpha_i = -\frac{\bar{C}_{ii}}{2\bar{M}_{ii}}$

$$\left(\frac{\bar{C}_{ii}}{2\bar{M}_{ii}}\right)^2 = \frac{\bar{K}_{ii}}{\bar{M}_{ii}} \tag{155}$$

Here $\{x\} = (A_i + B_i t)e^{\alpha_i t}\{\phi_i\}$ (156)

$$\therefore \{\dot{x}\} = (A_i + B_i t)e^{\alpha_i t}\{\alpha_i \phi_i\} + B_i e^{\alpha_i t}\{\phi_i\} \tag{157}$$

$\therefore$ in this case

$$\{Z\} = (A_i + B_i t)e^{\alpha_i t}\begin{Bmatrix}\{\alpha_i\phi_i\}\\ \{\phi_i\}\end{Bmatrix} + B_i e^{\alpha_i t}\begin{Bmatrix}\{\phi_i\}\\ \{0\}\end{Bmatrix} \tag{158}$$

It is not possible to express Equation (158) in the form of Equation (151) and so Foss's method does not give a solution in this case.

Expanding the orthogonality conditions in terms of matrices of the system:

From Equation (151) and Equation (148)

$$\{\Phi\} = \begin{Bmatrix} \alpha\{\phi\} \\ \{\phi\} \end{Bmatrix} \tag{159}$$

Substituting Equation (120) and Equation (159) into Equation (146)

$$(\alpha_\ell + \alpha_j)\,\{\phi^j\}^T[M]\{\phi^\ell\} + \{\phi^j\}^T[C]\{\phi^\ell\} = 0 \qquad \ell \neq j \tag{160}$$

Again on expanding the 2^{nd} orthogonality condition Equation (147)

$$-(\alpha_\ell\,\alpha_j)\{\phi^j\}^T[M]\{\phi^\ell\} + \{\phi^j\}^T[K]\{\phi^\ell\} = 0 \qquad \ell \neq j \tag{161}$$

Complete Solution of Generalized Damped Linear Systems by Foss's Method:

Making use of the orthogonality conditions Equation (146) and Equation (147) it is now possible to construct the complete solution to the generalized damping problem by Foss's method, provided the roots of the frequency equation are distinct.

The equation of motion in $2N$ space

$$[R]\{\dot{Z}\} + [P]\{Z\} = \{F(t)\} \tag{162}$$

Let

$$\{Z\} = [\Omega]\{\xi\} \qquad (163)$$

where $[\Omega]$ is a matrix of order $2N \times 2N$ the columns of which are the $\{\Phi^i\}$'s. Each

$$\{\Phi^i\} = \begin{Bmatrix} \alpha_i\{\phi^i\} \\ \{\phi^i\} \end{Bmatrix} \qquad (164)$$

and $\{\xi\}$ is a column vector of order $2N \times 1$. On substituting Equation (163) into Equation (162)

$$[R][\Omega]\{\dot{\xi}\} + [P][\Omega]\{\xi\} = \{F(t)\} \qquad (165)$$

Premultiply Equation (165) by $[\Omega]^T$

$$[\Omega]^T[R][\Omega]\{\dot{\xi}\} + [\Omega]^T[P][\Omega]\{\xi\} = [\Omega]^T\{F(t)\} \qquad (166)$$

Now the ij^{th} element of

$$[\Omega]^T[R][\Omega] = \{\Phi^i\}^T[R]\{\Phi^j\} \qquad (167)$$

the ij^{th} element of

$$[\Omega]^T[P][\Omega] = \{\Phi^i\}^T[P]\{\Phi^j\} \qquad (168)$$

Hence on using the orthogonality conditions Equation (146) and Equation (147), Equation (166) reduces to a set of uncoupled equations of type

$$\{\Phi^i\}^T[R]\{\Phi^i\}\dot{\xi}_i + \{\Phi^i\}^T[P]\{\Phi^i\}\xi = \{\Phi^i\}^T\{F(t)\} \tag{169}$$

Now

$$\{\Phi^i\}^T[R]\{\Phi^i\} = 2\alpha_i\{\Phi^i\}^T[M]\{\Phi^i\} + \{\Phi^i\}^T[C]\{\Phi^i\} \tag{170}$$

and

$$\{\Phi^i\}^T[P]\{\Phi^i\} = -\alpha_i^2\{\Phi^i\}^T[M]\{\Phi^i\} + \{\Phi^i\}^T[P]\{\Phi^i\} \tag{171}$$

From Equation (140)

$$\alpha_i\,[R]\{\Phi^i\} = -\,[P]\{\Phi^i\} \tag{172}$$

On premultiplying Equation (172) by $\{\Phi^i\}^T$

$$\alpha_i\{\Phi^i\}^T[R]\{\Phi^i\} = -\{\Phi^i\}^T[P]\{\Phi^i\} \tag{173}$$

On substituting Equation (173) into Equation (169)

$$R_i\dot{\xi}_i - \alpha_i R_i\xi_i = \{\Phi^i\}^T\{F(t)\} \tag{174}$$

where $\mathcal{R}_i$ a scalar, is defined

$$\mathcal{R}_i = \{\Phi^i\}^T [R] \{\Phi^i\} \tag{175}$$

As the $\{\Phi^i\}$'s form a complete set it may be assumed that $\{F(t)\}$ can be expanded in terms of the modal vectors as follows

$$\{F(t)\} = \sum_{n=1}^{2N} \lambda_n [R] \{\Phi^n\} \tag{176}$$

Premultiply Equation (176) by $\{\Phi^m\}^T$

$$\therefore \{\Phi^m\}^T \{F(t)\} = \sum_{n=1}^{2N} \lambda_n \{\Phi^m\}^T [R] \{\Phi^n\} \tag{177}$$

On using the orthogonality condition Equation (146) Equation (177) reduces to

$$\{\Phi^m\}^T \{F(t)\} = \lambda_n \{\Phi^m\}^T [R] \{\Phi^m\} \tag{178}$$

$$\therefore \lambda_m = \frac{\{\Phi^m\}^T \{F(t)\}}{\{\Phi^m\}^T [R] \{\Phi^m\}} \tag{179}$$

$$= \frac{\{\Phi^m\}^T \{F(t)\}}{\mathcal{R}_m} \tag{180}$$

On substituting Equation (180) into Equation (176)

$$\{F(t)\} = \sum_{n=1}^{2N} \frac{\{\Phi^n\}\{F(t)\}}{R_n} [R]\{\Phi^n\} \tag{181}$$

On expanding Equation (181)

$$\begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} = \sum_{n=1}^{2N} \frac{\{\phi^n\}^T\{f(t)\}}{R_n} \begin{Bmatrix} [M]\{\phi^n\} \\ \alpha_n [M]\{\phi^n\} + [C]\{\phi^n\} \end{Bmatrix} \tag{182}$$

$$\therefore \{0\} = \sum_{n=1}^{2N} \frac{\{\phi^n\}^T\{f(t)\}}{R_n} [M]\{\phi^n\} \tag{183}$$

$$\{f(t)\} = \sum_{n=1}^{2N} \frac{\{\phi\}^T\{f(t)\}}{R_n} \{\alpha_n [M]\{\phi^n\} + [C]\{\phi^n\}\} \tag{184}$$

From Equation (183) on premultiplying by $[M]^{-1}$

$$\{0\} = \sum_{n=1}^{2N} \frac{\{\phi^n\}^T\{f(t)\}}{R_n} \{\phi^n\} \tag{185}$$

Now $\{Z\} = \sum_{i=1}^{2N} \{\phi^i\} \xi_i$

where ξ_i is the solution of equations of Equation (174)

$$\therefore \xi_i = \frac{1}{R_i} \int_0^t e^{\alpha_i (t-Z)} \{\phi^i\}^T \{F(Z)\} dZ + A_i e^{\alpha_i t} \tag{186}$$

where A_i is an arbitrary constant depending on the initial conditions. From Equation (186)

$$\dot{\xi}_i = \frac{\alpha_i}{R_i}\int_0^t e^{\alpha_i(t-Z)}\{\phi^i\}^T\{F(Z)\}\,dZ + A_i\alpha_i e^{\alpha_i t} + \frac{\{\phi^i\}\{F(t)\}}{R_i} \qquad (187)$$

From Equation (186)

$$\{Z\} = \sum_{n=1}^{2N} \frac{1}{R_n}\{\phi^n\}\left(\int_0^t e^{\alpha_n(t-Z)}\{\phi^n\}^T\{F(Z)\}\,dZ + A_n e^{\alpha_n t}\right) \qquad (188)$$

On expanding Equation (188)

$$\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} = \sum_{n=1}^{2N} \frac{1}{R_n}\left(\int_0^t e^{\alpha_n(t-Z)}\{\phi^n\}^T\{f(Z)\}\,dZ + A_n e^{\alpha_n t}\right)\begin{Bmatrix}\alpha_n\{\phi^n\}\\ \{\phi^n\}\end{Bmatrix} \qquad (189)$$

On separating out Equation (189)

$$\{\dot{x}\} = \sum_{n=1}^{2N} \frac{\alpha_n}{R_n}\left(\int_0^t e^{\alpha_n(t-Z)}\{\phi^n\}^T\{f(Z)\}\,dZ + A_n e^{\alpha_n t}\right)\{\phi^n\} \qquad (190)$$

$$\{x\} = \sum_{n=1}^{2N} \frac{1}{R_n}\left(\int_0^t e^{\alpha_n(t-Z)}\{\phi^n\}^T\{f(Z)\}\,dZ + A_n e^{\alpha_n t}\right)\{\phi^n\} \qquad (191)$$

As a check calculate $\{\dot{x}\}$ from Equation (191)

$$\{\dot{x}\} = \sum_{n=1}^{2N} \frac{\alpha_n}{R_n}\left(\int_0^t e^{\alpha_n(t-Z)}\{\phi^n\}^T\{f(Z)\}\,dZ + A_n e^{\alpha_n t}\right)\{\phi^n\} + \sum_{N=1}^{2N}\frac{1}{[R_n]}\{\phi^n\}^T\{f(t)\}\{\phi^n\} \qquad (192)$$

On substituting Equation (185) into Equation (192)

$$\{\dot{x}\} = \sum_{n=1}^{2N} \frac{\alpha_n}{R_n} \left(\int_0^t e^{\alpha_n (t-Z)} \{\phi^n\}^T \{f(z)\} dZ + A_n e^{\alpha_n t} \right) \{\phi^n\} \quad (193)$$

Equation (193) is the same as Equation (190) and thus if the $\{\Phi^i\}$'s form a complete set Foss's method does in fact give the solution to a generalized damped system.

Complex Roots and Eigenvectors give real solutions:

Take

$$\alpha_i = \overline{\alpha_{i+N}} \quad (194)$$

$\{\Phi^i\}$ and $\{\Phi^{i+n}\}$ are the modal columns corresponding to α_i and α_{i+n}, respectively. Equations of motion in $2N$ space

$$[R]\{\dot{Z}\} + [P]\{Z\} = 0 \quad (195)$$

Let

$$\{Z\} = e^{\alpha_i}\{\Phi^i\} \quad (196)$$

On substituting Equation (196) into Equation (195)

$$\left[\alpha_i \, [R] + [P]\right]\{\Phi^i\} = 0 \quad (197)$$

Let

$$\{Z\} = e^{\alpha_{i+n}}\{\Phi^{i+n}\} \tag{198}$$

On substituting Equation (198) into Equation (195)

$$\left[\alpha_{i+n}[R] + [P]\right]\{\Phi^{i+n}\} = \{0\} \tag{199}$$

Taking complex conjugates of Equation (199) and substituting Equation (194)

$$\left[\alpha_i[R] + [P]\right]\{\overline{\Phi^{i+n}}\} = \{0\} \tag{200}$$

On comparing Equation (197) and Equation (200)

$$\{\Phi^i\} \propto \{\overline{\Phi^{i+n}}\} \tag{201}$$

On normalizing $\{\Phi^i\}$ and $\{\Phi^{i+n}\}$ Equation (201) reduces to

$$\{\bar{\Phi}^i\} = \{\Phi^{i+n}\} \tag{202}$$

From Equation (186)

$$\xi_i = A_i e^{\alpha_i t} + \frac{1}{R_i}\int_0^t e^{\alpha_i(t-z)}\{\Phi^i\}^T\{F(z)\}\,dz \tag{203}$$

Taking complex conjugates of Equation (203) and substituting Equation (202)

$$\bar{\xi}_i = \bar{A}_i e^{\alpha_{i+n} t} + \frac{1}{R_{i+n}} \int_0^t e^{\alpha_{i+n}(t-Z)} \left\{ \Phi^{i+n} \right\}^T \left\{ F(Z) \right\} dZ \qquad (204)$$

From Equation (203)

$$\xi_{i+n} = A_{i+n} e^{\alpha_{i+n} t} + \frac{1}{R_{i+n}} \int_0^t e^{\alpha_{i+n}(t-Z)} \left\{ \Phi^{i+n} \right\}^T \left\{ F(Z) \right\} dZ \qquad (205)$$

$$\therefore \bar{\xi}_i = \xi_{i+n} \qquad (206)$$

if $\quad \bar{A}_i = A_{i+n} \qquad (207)$

But $$\left\{ Z(0) \right\} = \sum_{j=1}^{N} \left\{ \Phi^j \right\} A_j \qquad (208)$$

where $\left\{ Z(0) \right\} = \left\{ Z(t) \right\}$ at $t = 0$.

On premultiplying Equation (208) by $\left\{ \Phi^\ell \right\}^T [R]$

$$\left\{ \Phi^\ell \right\}^T [R] \left\{ Z(0) \right\} = \left\{ \Phi^\ell \right\}^T [R] \sum_{j=1}^{\ell N} \left\{ \Phi^j \right\} A_j \qquad (209)$$

On using the orthogonality condition Equation (146)

$$\{\Phi^\ell\}^T [R] \{Z(0)\} = \{\Phi^\ell\}^T [R] \{\Phi^\ell\} A_\ell \tag{210}$$

From Equation (210)

$$A_i = \frac{\{\Phi^i\}^T [R] \{Z(0)\}}{\{\Phi^i\}^T [R] \{\Phi^i\}} \tag{211}$$

$$A_{i+n} = \frac{\{\Phi^{i+n}\}^T [R] \{Z(0)\}}{\{\Phi^{i+n}\}^T [R] \{\Phi^{i+n}\}} \tag{212}$$

From Equation (210)

$$\bar{A}_i = \frac{\{\Phi^{i+n}\}^T [R] \{Z(0)\}}{\{\Phi^{i+n}\}^T [R] \{\Phi^{i+n}\}} = A_{i+n} \tag{213}$$

By Equation (207)

$$\bar{\xi}_i = \xi_{i+n}$$

$\therefore$ contribution to $Z(t)$ from α_i and α_{i+n}

where $\bar{\alpha}_i = \alpha_{i+n}$

$$\{Z(t)\}_i = \xi_i \{\Phi^i\} + \xi_{i+n} \{\Phi^{i+n}\} \tag{214}$$

$$= \xi_i \{\Phi^i\} + \bar{\xi}_i \{\bar{\Phi}^i\}$$

$$= 2 \, \mathrm{Re} \, \xi_i \{\Phi^i\} \tag{215}$$

$\therefore$ contribution of α_i and α_{i+n} to x_j

$$x_j = 2\mathcal{R}e\left(A_i e^{\alpha_i t} + \frac{1}{R_i}\int_0^t e^{\alpha_i(t-Z)}\{\Phi^i\}^T\{F(Z)\}\,dZ\right)\phi_j^i \quad (216)$$

$$= 2\left(|A_i||\Phi_j^i| e^{\rho_i t}\cos(\omega_i t + \delta_i + \beta_i)\right) + \frac{2|\phi_j^i|}{|R_i|}\int_0^t e^{\rho_i(t-Z)} \left|\{\Phi^i\}^T\{F(Z)\}\right| \cos(\omega_i(t-Z) + \gamma_i + \delta_i - \Gamma_i)\,dZ$$

where

$$A_i = |A_i| e^{i\beta_i}$$

$$\alpha_i = \rho_i + i\omega_i$$

$$R_i = |R_i| e^{i\Gamma_i} \quad (217)$$

$$\{\Phi^i\}^T\{F(Z)\} = \left|\{\Phi^i\}^T\{F(Z)\}\right| e^{i\gamma_i}$$

$$\phi_j^i = |\phi_j^i| e^{i\delta_i}$$

Derivation of Caughey's conditions from Foss's orthogonality conditions:

It has already been remarked that any root α_i can be either purely real or a complex conjugate of another root. The case of purely real roots corresponds to overdamped vibrations and will be neglected in the present section. If only complex conjugate roots are considered, it is shown that the necessary and sufficient conditions for real $\{\phi^i\}$ in N space is that $[C]$ be diagonalized by the same transformation as diagonalizes $[M]$ and $[K]$. Real $\{\phi^i\}$ with complex roots implies classical normal modes as discussed above.

Orthogonality condition derived in Foss's method Equation (160)

$$\left(\alpha_i + \alpha_j\right)\left\{\phi^j\right\}^T [M] \left\{\phi^\ell\right\} + \left\{\phi^j\right\}^T [C] \left\{\phi^\ell\right\} = 0 \qquad (218)$$

Take 2 pairs of complex conjugate roots α_ℓ, $\alpha_{\ell+n}$, α_j and α_{j+n} such that $\left.\begin{array}{l}\bar{\alpha}_\ell = \alpha_{\ell+n} \\ \bar{\alpha}_j = \alpha_{j+n}\end{array}\right\}$

The vectors $\left\{\Phi^i\right\}$ corresponding to two complex conjugate roots α_i and α_{i+n} are such that

$$\left\{\phi^i\right\} = \left\{\overline{\phi^{i+n}}\right\} \qquad (219)$$

Now if $\left\{\phi^\ell\right\}$ and $\left\{\phi^j\right\}$ are real vectors

$$\left\{\phi^\ell\right\} = \left\{\phi^{\ell+n}\right\} \qquad (220)$$

$$\left\{\phi^j\right\} = \left\{\phi^{j+n}\right\} \qquad (221)$$

From Equation (218) on letting $\begin{array}{l}\ell = \ell + n \\ j = j + n\end{array}$

$$\left(\alpha_{\ell+n} + \alpha_{j+n}\right)\left\{\phi^{j+n}\right\}^T [M] \left\{\phi^{\ell+n}\right\} + \left\{\phi^{j+n}\right\} [C] \left\{\phi^{\ell+n}\right\} = 0 \qquad (222)$$

On taking the complex conjugate of Equation (222)

$$\left(\alpha_{\ell}+\alpha_{j}\right)\left\{\phi^{j+n}\right\}^{T}[M]\left\{\phi^{\ell+n}\right\}+\left\{\phi^{j+n}\right\}^{T}[C]\left\{\phi^{\ell+n}\right\}=0 \qquad (223)$$

On substracting Equation (223) from Equation (220)

$$\left(\alpha_{\ell+n}+\alpha_{j+n}-\alpha_{d}-\alpha_{j}\right)\left\{\phi^{j+n}\right\}[M]\left\{\phi^{\ell+n}\right\}=0 \qquad (224)$$

as α_{ℓ} and α_{j} are distinct complex roots Equation (224) reduces to

$$\left\{\phi^{j+n}\right\}^{T}[M]\left\{\phi^{\ell+n}\right\}=0 \quad j \neq \ell \qquad (225)$$

or

$$\left\{\phi^{j}\right\}^{T}[M]\left\{\phi^{\ell}\right\}=0 \quad j \neq \ell \qquad (226)$$

On substituting Equation (226) into Equation (218)

$$\left\{\phi^{j}\right\}^{T}[C]\left\{\phi^{\ell}\right\}=0 \quad j \neq \ell \qquad (227)$$

On substituting Equation (227) into the 2^{nd} orthogonality condition (Equation (161))

$$\left(-\alpha_{i}\alpha_{j}\right)\left\{\phi^{j}\right\}^{T}[M]\left\{\phi^{\ell}\right\}+\left\{\phi^{j}\right\}^{T}[K]\left\{\phi^{\ell}\right\}=0 \quad i \neq j \qquad (228)$$

it follows that

$$\left\{\phi^{j}\right\}^{T}[K]\left\{\phi^{\ell}\right\}=0_{\;j \neq \ell.} \tag{229}$$

Equation (226), Equation (227) and Equation (229) are the conditions for classical normal modes. Hence, with distinct complex roots of the frequency equation all real modal columns are possible only if the system is classically damped. It should be noted that it is possible for any two modal columns to satisfy Equation (226), Equation (227) and Equation (229) even though the system is not classically damped. This is so because for classically damped systems all the roots must satisfy relations such as Equation (206), Equation (207) and Equation (208).

<u>Excitation of pure normal modes in classically damped systems</u>:

At this point it is of interest to discuss one remarkable difference between classical and non-classical damping. The equations of motion of linear damped systems

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{0\} \tag{230}$$

Now examine under what conditions it is possible to excite a pure normal mode. If the system is classically damped let

$$\{x\} = [Q]\{\eta\} \tag{231}$$

where $[Q]^T[M][Q]$ is a diagonal matrix of order $N \times N$

$[Q]^T[K][Q]$ is a diagonal matrix of order $N \times N$

$[Q]^T[C][Q]$ is a diagonal matrix of order $N \times N$

On substituting Equation (231) into Equation (230) and premultiplying by $[Q]^T$

$$[Q]^T[M][Q]\{\ddot{\eta}\} + [Q]^T[C][Q]\{\dot{\eta}\} + [Q]^T[K][Q]\{\eta\} = 0 \tag{232}$$

$$\therefore [\bar{M}]\{\ddot{\eta}\} + [\bar{C}]\{\dot{\eta}\} + [\bar{K}]\{\eta\} = 0 \tag{233}$$

If the j^{th} mode has to be excited

$$\{x\} = [Q]\{\eta(t)\} \tag{234}$$

where $\{\eta(t)\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ \eta_j(t) \\ 0 \\ 0 \end{Bmatrix}$ a column vector of order $N \times 1$ with all zero elements except the j^{th} element. (235)

The system of equations Equation (233) are of type

$$\bar{M}_{ii}\ddot{\eta}_i + \bar{C}_{ii}\dot{\eta}_i + \bar{K}_{ii}\eta_i = 0$$

To satisfy Equation (235) these equations must have solutions

$$\eta_i(t) = 0 \text{ all } i \neq j$$
$$\eta_j(t) \neq 0$$
$$\therefore \eta_i(0) = \dot{\eta}_i(0) \quad \text{all } i \neq j \tag{236}$$

but

$$\{x(0)\} = [Q]\{\eta(0)\} = \{q^j\}\eta_j(0) \tag{237}$$

$$\{\dot{x}(0)\} = [Q]\{\dot{\eta}(0)\} = \{q^j\}\{\dot{\eta}_j(0)\} \tag{238}$$

Equation (237) and Equation (238) give the initial displacement and velocity distributions necessary to excite a pure normal mode in a system with classical damping. It may be noted that it is necessary to specify both the velocity and the displacement distributions but that the relative magnitudes of the velocities and the displacements are arbitrary. Thus, if a system possesses classical damping it is always possible to excite a pure normal mode by an initial distribution of displacements and velocities.

<u>To force excite a pure normal mode in classically damped systems:</u>

The equations of motion of forced vibrations of damped systems

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \tag{239}$$

Let
$$\{x\} = [Q]\{\eta(t)\} \tag{240}$$

On substituting Equation (240) into Equation (239) and premultiplying by $[Q]^T$

$$[Q]^T[M][Q]\{\ddot{\eta}\} + [Q]^T[C][Q]\{\dot{\eta}\} + [Q]^T[K][Q]\{\eta\} = [Q]^T\{F(t)\} \tag{241}$$

$$\therefore [\bar{M}]\{\ddot{\eta}\} + [\bar{C}]\{\dot{\eta}\} + [\bar{K}]\{\eta\} = [Q]^T\{F(t)\} \tag{242}$$

To excite the j^{th} normal mode assume zero initial conditions $\therefore \eta_j(0) = \dot{\eta}_j(0) = 0$ all j.

Let
$$[Q]^T\{F(t)\} = \{G(t)\} \tag{243}$$

where $\{G(t)\}$ is a column vector of order $N \times 1$ and with elements $g_i(t)$

If

$$g_i(t) = 0 \qquad \text{all} \quad i \neq j \tag{244}$$
$$g_j(t) \neq 0$$

then

$$\eta_i(t) = 0 \qquad \text{all} \quad i \neq j \tag{245}$$
$$\eta_j(t) \neq 0$$

Thus to excite the j^{th} normal mode

$$[Q]^T \{F(t)\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ g_j(t) \\ \vdots \\ 0 \\ 0 \end{Bmatrix} = \{G(t)\} \tag{246}$$

From Equation (246) as $\left([Q]^T\right)^{-1}$ exists

$$\{F(t)\} = \left([Q]^T\right)^{-1} \{G(t)\} \tag{247}$$

where $g_j(t)$ may be any function of time. It is interesting to note that it is possible to force excite the system to vibrate in a pure mode at a frequency other than that of the natural frequency of the mode. However, for a given force level the largest response will be obtained from the system when the forcing frequency is approximately the natural frequency of the mode.

Excitation of normal modes in non-classically damped systems:

As was already discussed if the roots of the frequency equation are distinct in a non-classically damped system normal mode solutions will exist. However, due to the non-symmetric matrices involved little can be said in the case of repeated roots. Although a system may not be solvable in terms of normal modes for arbitrary initial conditions and forcing functions a solution of type

$$\{x\} = R\ell\{\phi\}\eta(t) \tag{248}$$

where $\{\phi\}$ is a eigenvector, may exist for certain initial conditions and forcing functions. In the present context such solutions will be looked upon as normal mode solutions. To discuss the excitation of pure normal modes in non-classically damped systems it is necessary to distinquish three cases corresponding to the roots of the frequency equation

1) complex conjugate roots
2) real roots
3) equal roots.

In passing it should be noted that any physical system may have roots in each of these three categories. In these cases it is necessary to apply the following theory to each type of root separately.

Complex conjugate roots:

In the $2N$ space the equations of motion are

$$[R]\{\dot{z}\}+[P]\{z\} = \{F(t)\} \tag{249}$$

As shown previously if the j^{th} mode is excited

$$\{z\}_j = 2Rl\ \{\Phi^j\}\ \eta_j(t) \tag{250}$$

In the free vibration case $\{F(t)\} = 0$ Equation (250) reduces to

$$\{z\}_j = 2Rl\ \{\Phi^j\}\ A_j\, e^{\alpha_i t} \tag{251}$$

On substituting Equation (216) and expanding

$$\begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = 2Rl \begin{Bmatrix} \alpha_j \{\phi^j\} A_j\, e^{\alpha_j t} \\ \{\phi^j\} A_j\, e^{\alpha_j t} \end{Bmatrix} \tag{252}$$

$$\{x\} = 2Rl\left(\alpha_j \{\phi^j\} A_j e^{\alpha_j t}\right) \tag{253}$$

$$\{x\} = 2Rl\ \left(\{\phi^j\} A_j\, e^{\alpha_j t}\right) \tag{254}$$

From Equation (253) and Equation (254) the initial conditions necessary to allow the system to vibrate in a pure normal mode are

$$\{\dot{x}(0)\} = 2\mathcal{R}\ell(\alpha_i\{\phi^j\}A_j) \tag{255}$$

$$\{x(0)\} = 2\mathcal{R}\ell(\{\phi^j\}A_j) \tag{256}$$

Equation (255) and Equation (256) show that in the case of non-classical damping with complex roots of the frequency equation the relative magnitude of the initial displacement $\{x(0)\}$ and initial velocity $\{\dot{x}(0)\}$ necessary to excite a pure normal mode is fixed. This should be contrasted with the classically damped case where although given initial displacement and velocity distributions are necessary to excite a pure mode the relative magnitude of these distributions is arbitrary.

Having established that a pure mode may be excited by a suitable choice of initial conditions it is now of interest to determine if it is possible to force excite a pure mode.

The impossibility of exciting a pure normal mode in a non-classically damped system, α_i complex, by any arbitrary distribution of force and zero initial conditions will now be demonstrated. Equations of motion in $2N$ space

$$[R]\{\dot{z}\} + [P]\{z\} = \{F(t)\} \tag{257}$$

If the roots of the frequency equation are distinct the $\{\Phi^i\}$'s form a complete set. Let

$$\{Z\} = [\Phi]\{\eta_i\} \tag{258}$$

On substituting Equation (258) into Equation (257) and premultiplying by $[\Phi]^T$

$$[\Phi]^T[R]\,[\Phi]\{\dot{\eta}\}+[\Phi]^T[P]\,[\Phi]\{\eta\} = [\Phi]^T\{F(t)\} \tag{259}$$

By the orthogonality conditions Equation (146) and Equation (147), Equation (259) reduces to

$$[\bar{R}]\{\dot{\eta}\} + [\bar{P}]\{\eta\} = [\Phi]^T\{F(t)\} \tag{260}$$

To excite the j^{th} mode it is necessary that

$$\eta_j(t) \neq 0 \quad \eta_{j+n}(t) \neq 0 \quad \eta_i(t) = 0 \text{ all } i \neq j, j+n \tag{261}$$

Let $$[\Phi]^T\{F(t)\} = \{G(t)\} \tag{262}$$

where

$$\{G(t)\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ g_j(t) \\ \vdots \\ g_{j+n}(t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{263}$$

Now as the $\{\Phi^i\}$'s span the $2N$ space

$$\{F(t)\} = \sum_{i=1}^{2N} B_i [R]\{\Phi^i\} \tag{264}$$

premultiply Equation (264) by $\{\Phi^j\}^T$

$$\therefore \{\Phi^j\}^T\{F(t)\} = \{\Phi^j\}^T \sum_{i=1}^{2N} B_i [R]\{\Phi^i\} \tag{265}$$

On using the orthogonality conditions

$$\{\Phi^j\}^T\{F(t)\} = B_j \{\Phi^j\}^T[R]\{\Phi^j\} \tag{266}$$

$$\therefore B_j = \frac{\{\Phi^j\}^T\{F(t)\}}{\{\Phi^j\}^T[R]\{\Phi^j\}} \tag{267}$$

On substituting Equation (267) into Equation (264)

$$\{F(t)\} = \sum_{i=1}^{2N} \frac{\{\Phi^i\}^T\{F(t)\}}{\{\Phi^i\}^T[R]\{\Phi^i\}} [R]\{\Phi^i\} \tag{268}$$

Using Equation (260) on separating out the j^{th} equation

$$\bar{R}_{jj}\,\dot{\eta}_j + \bar{P}_{jj}\,\eta_j = g_j(t) \tag{269}$$

But

$$\bar{R}_{jj} = \{\Phi^j\}^T[R]\{\Phi^j\} \tag{270}$$

$$\bar{P}_{jj} = \{\Phi^j\}^T[P]\{\Phi^j\} \tag{271}$$

$$g_j(t) = \{\Phi^j\}^T\{F(t)\} \tag{272}$$

Substituting these equations into Equation (269) and rearranging

$$\dot{\eta}_i + \frac{\{\Phi^j\}^T[P]\{\Phi^i\}}{\{\Phi^i\}^T[R]\{\Phi^i\}}\eta_i = \frac{\{\Phi^j\}^T\{F(t)\}}{\{\Phi^j\}^T[R]\{\Phi^j\}} \tag{273}$$

It follows from Equation (273) that if the j^{th} mode is force excited

$$\frac{\{\Phi^i\}^T\{F(t)\}}{\{\Phi^i\}^T[R]\{\Phi^i\}} = 0 \qquad (273\text{-})$$

all $i \neq j, j+N$

From Equation (267) condition Equation (273) reduces to

$$\mathcal{B}_i = 0 \tag{274}$$

all $i \neq j, j+N$

Using Equation (264) and Equation (274)

$$\{F(t)\} = \sum_{i=1}^{2N} \mathcal{B}_i [R]\{\Phi^i\} \tag{275}$$

$$= 2\mathcal{R}\ell\, \mathcal{B}_j\, [R]\{\Phi^j\} \tag{276}$$

As $\mathcal{B}_i = \overline{\mathcal{B}_{i+N}}$ (277)

From Equation (276) on expanding

$$\begin{Bmatrix} \{0\} \\ \{f\} \end{Bmatrix} = 2\mathcal{R}\ell\, \mathcal{B}_j \begin{Bmatrix} [M]\{\phi^j\} \\ \alpha_j [M]\{\phi^j\} + [C]\{\phi^j\} \end{Bmatrix} \tag{278}$$

$$\therefore \{0\} = 2\mathcal{R}\ell\left(\beta_j [M]\{\phi^j\}\right) \tag{279}$$

$$\{f(t)\} = 2\mathcal{R}\ell\, \mathcal{B}_j \left(\mathcal{L}_j [M]\{\phi^j\} + [C]\{\phi^j\}\right) \tag{280}$$

As $[M]$ is generally non-singular Equation (279) reduces to

$$\{0\} = 2\mathcal{R}\ell\left(\beta_j \{\phi^j\}\right) \tag{281}$$

Let $\beta_j = a + i b$ (282)

$$\{\phi^j\} = \{\phi^j_{Re}\} + i\{\phi^j_I\} \tag{283}$$

where $\{\phi^j_{Re}\}$ and $\{\phi^j_I\}$ are real vectors

Substitute Equation (282) and Equation (283) into Equation (281)

$$\{0\} = a \{\phi^j_{Rl}\} - b \{\phi^j_I\} \tag{284}$$

From Equation (284)

$$\text{either } a = b = 0 \tag{285}$$

$$\text{or } \{\phi^j_{Rl}\} = \frac{b}{a} \{\phi^j_I\} \tag{286}$$

Equation (285) implies that $\{f(t)\} = 0$ and if this equation is satisfied initial conditions are the only possibility of exciting a pure normal mode.

Equation (286) implies that $\{\phi^j\}$ as given by Equation (283) is merely a complex scalar times a real vector. This is the case for classical damping and so it is impossible to force excite a pure normal mode in non-classically damped systems with complex roots to the frequency equation.

Real Roots:

If α_i is real then the corresponding eigenvector $\{\phi^i\}$ is also real. Equations of motion in $2N$ space

$$[R]\{\dot{z}\} + [P]\{z\} = 0 \tag{284}$$

Now

$$\{z\} = [\Phi]\{\eta(t)\} \tag{285}$$

assuming the $\{\Phi^i\}$'s span the space. To excite the j^{th} mode $\{\eta(t)\}$ must have form

$$\{\eta(t)\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \eta_j(t) \\ \vdots \end{Bmatrix} \tag{286}$$

From Equation (285) and Equation (286)

$$\{z(0)\} = \{\Phi^j\}\,\eta_j(0) \tag{287}$$

Premultiply Equation (287) by $\{\Phi^j\}^T[R]$

$$\{\Phi^j\}^T[R]\{z(0)\} = \{\Phi^j\}^T[R]\{\Phi^j\}\,\eta_j(0) \tag{288}$$

$$\therefore\ \eta_j(0) = \frac{\{\Phi^j\}^T[R]\{z(0)\}}{\{\Phi^j\}^T[R]\{\Phi^j\}} \tag{289}$$

$$\eta_i(0) = 0 \qquad i \neq j \tag{290}$$

$\therefore$ if

$$\{z(0)\} = \{\Phi^j\}\,\eta_j(0) \tag{291}$$

$$\eta_j(0) \neq 0 \tag{292}$$

the j^{th} mode is excited.
Equation (291) on expanding

$$\{\dot{x}(0)\} = \alpha_j \{\phi^j\} \eta_j(0) \tag{293}$$

$$\{x(0)\} = \{\phi^j\} \eta_j(0) \tag{294}$$

Again it is seen that to excite a pure normal mode by initial conditions there is a required relationship between the relative magnitude of the velocity and displacement distributions. The impossibility of force exciting a pure mode in non-classically damped systems, with real α_i, will now be demonstrated. Equations of motion

$$[R]\{\dot{z}\} + [P]\{z\} = \{F(t)\} \tag{295}$$

Let

$$\{z\} = [\Phi]\{\eta(t)\} \tag{296}$$

On substituting Equation (296) into Equation (295) and premultiplying by $[\Phi]^T$

$$[\Phi]^T[R]\,[\Phi]\{\dot{\eta}(t)\}+[\Phi]^T[P]\,[\Phi]\{\eta(t)\}=[\Phi]^T\{F(t)\} \tag{297}$$

On using the orthogonality conditions Equation (297) reduces to

$$[\bar{R}]\,\{\dot{\eta}(t)\}+[\bar{P}]\,\{\eta(t)\}=[\Phi]^T\{F(t)\} \tag{298}$$

This is a set of uncoupled equations and to excite the j^{th} mode with zero initial conditions

$$[\Phi]^T\{F(t)\}=\{G(t)\} \tag{299}$$

where

$$\{G(t)\}=\begin{Bmatrix}0\\0\\0\\\vdots\\g_j(t)\\\vdots\\0\end{Bmatrix} \tag{300}$$

$$g_j(t)\neq 0 \tag{301}$$

But

$$[\Phi]^T\{F(t)\}=\left[[\alpha\,\phi]^T,[\phi]^T\right]\begin{Bmatrix}\{0\}\\\{f(t)\}\end{Bmatrix} \tag{302}$$

where $[\alpha\phi]$ and $[\phi]^T$ are $2N \times N$ matrices of type

$$[\phi]^T = \begin{bmatrix} \{\phi^1\}^T \\ \{\phi^2\}^T \\ \vdots \\ \{\phi^{1+N}\}^T \\ \vdots \\ \{\phi^{2N}\}^T \end{bmatrix} \tag{303}$$

On substituting Equation (299) and Equation (303) into Equation (302)

$$\{G(t)\} = \begin{Bmatrix} \{\phi^1\}^T\{f(t)\} \\ \vdots \\ \{\phi^{1+N}\}^T\{f(t)\} \\ \vdots \\ \{\phi^{2N}\}^T\{f(t)\} \end{Bmatrix} \tag{304}$$

On comparing Equation (300) with Equation (304)

$$\{\phi^i\}^T\{f(t)\} = g_j(t)\,\delta_{ij} \tag{305}$$

where δ_{ij} is Kroneckers delta.

The $2N$ elements of $\{G(t)\}$ may be split into two groups of N elements as follows:

$$\left.\begin{matrix} \{\phi^1\}^T\{f\} \\ \vdots \\ \{\phi^n\}^T\{f\} \end{matrix}\right\} \quad N \text{ elements} \tag{306}$$

$$\left.\begin{matrix} \{\phi^{1+N}\}^T\{f\} \\ \vdots \\ \{\phi^{2N}\}^T\{f\} \end{matrix}\right\} \quad N \text{ elements} \tag{307}$$

Suppose that the element $g_j(t) \neq 0$ is in the first set of N elements Equation (306). Now the $\{\phi^{\ell}\}$'s, $\ell = 1, 2 \cdots N$ being N dimensional vectors may form a complete set for the N space. Suppose the $\{\phi^{\ell}\}$'s do in fact form a complete set. Then any

$$\{\phi^{\ell+N}\}^T = \sum_{k=1}^{N} a_{\ell k} \{\phi^k\}^T \tag{308}$$

On postmultiplying Equation (308) by $\{f(t)\}$

$$\{\phi^{\ell+N}\}^T \{f(t)\} = \sum_{K=1}^{N} a_{\ell k} \{\phi^k\}^T \{f(t)\} \tag{309}$$

If the $\{\phi^{\ell+N}\}$'s $i = 1, 2 \cdots N$ form a complete set it is clearly possible to satisfy Equation (305) for $j = 1, 2, \cdots N$. But from Equation (309) on substituting Equation (305)

$$\{\phi^{\ell+N}\}^T \{f(t)\} = a_{\ell j}\, g_i(t) \tag{310}$$

$\therefore$ as $a_{\ell j} \neq 0$ in general $\{\phi^{\ell+N}\}^T \{f(t)\} \neq 0$

$$\tag{311}$$

Therefore, if the $\{\phi^i\}$'s $i = 1, 2 \cdots N$ span the N space it is not possible to select $\{f(t)\}$ so that $\{G(t)\}$ will have the form given by Equation (305). Thus it is not possible to force excite a pure normal mode in non-classically damped systems with distinct real roots if the $\{\phi^i\}$'s $_{1,2\cdots\cdots N}$ span the N space.

Equal Roots:

As was mentioned previously the case of equal roots of the frequency equation in non-classical damping cannot be readily analyzed due to the non-symmetric matrices involved. As $[u]$ has form given by Equation (135)

$$[u] = \begin{bmatrix} 0 & I \\ \\ -[K]^{-1}[M] & -[K]^{-1}[C] \end{bmatrix} \tag{312}$$

the rank of $[u-\lambda I]$ must be at least N as the I matrix in $[u]$ is non-singular. Thus for $2N$ distinct eigenvectors in the $2N$ space at most N equal roots of the frequency equation can exist. However, there is no guarantee that with roots of any multiplicity $M \leq N$ a complete set of eigenvectors will exist. It is interesting to note that the above facts fit in rather nicely with the classically damped system. In this system it is possible to have N equal roots, these roots corresponding to a solution to each uncoupled equation in the N space, and still obtain a normal mode solution of the type Foss assumes. However, should a classically damped system possess equal roots which correspond to two equal roots of an uncoupled equation a normal mode solution of the Foss type will not exist.

Excitation of pure mode in case of equal roots:

If a complete set of eigenvectors exist in the $2N$ space then the treatment of equal roots is similiar to the cases already discussed. If, however, the number of possible independent eigenvectors is less than $2N$, the discussion given above does not apply.

It will be shown that it is possible to excite a pure mode by initial conditions in a system with equal roots. The equation of motion in $2N$ space is

$$[R]\{\dot{Z}\} + [P]\{Z\} = 0 \qquad (313)$$

if α_i the repeated root is completethere exists α_{i+N} such that

$$\alpha_i = \overline{\alpha_{i+N}} \qquad (314)$$

for each α_i repeated. Solution to Equation (313)

$$\{Z\} = 2\,Re\,\{\Phi^i\}\,A_i\,e^{\alpha_i t} \qquad (315)$$

$$A_i \neq 0 \qquad (316)$$

From Equation (315) if

$$\{Z(0)\} = 2\,Rl\,A_i\,\{\Phi^i\} \qquad (317)$$

a normal mode solution will exist.

Equation (317) reduces to

$$\{\dot{x}(0)\} = 2Re\left(\alpha_i \{\phi^i\}\right) A_i \tag{318}$$

$$\{x(0)\} = 2Re\left(\{\phi^i\}\right) A_i \tag{319}$$

Thus, if the initial conditions are given by Equation (318) and Equation (319) a pure mode will be excited. Now in the case of equal real roots a solution of type

$$\{Z\} = A_i \{\Phi^i\} e^{\alpha_i t} \tag{320}$$

$$A_i \neq 0 \tag{321}$$

where α_i is real, exists. Again if

$$\{Z(0)\} = A_i \{\Phi^i\} \tag{322}$$

the i^{th} mode is excited by initial conditions. Equation (322) reduces to

$$\{\dot{x}(0)\} = A_i \alpha_i \{\phi^i\} \tag{323}$$

$$\{x(0)\} = A_i \{\phi^i\} \tag{324}$$

Unfortunately in the case of equal roots and an incomplete set of eigenvalues nothing can be said about the possibility of forced excitation of a normal mode.

Other methods of solving generalized damping problems:

As was noted previously a normal mode solution will always exist in classically damped systems. A mode solution may not exist in the case of equal roots of the frequency equation in non-classically damped systems.

If a normal mode solution does in fact exist it is always possible to solve the problem in N space without transforming to the $2N$ space. Equation of motion of damped system in N space

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\} = 0 \tag{325}$$

As a mode solution exists

$$x = e^{\lambda t}\{\phi\} \tag{326}$$

On substituting Equation (326) into Equation (325)

$$\lambda^2[M]\{\phi\}+\lambda[C]\{\phi\}+[K]\{\phi\} = 0 \tag{327}$$

On rearranging Equation (327)

$$\left[\lambda^2[M]+\lambda[C]+[K]\right]\left\{\phi\right\} = 0 \tag{328}$$

For non-trivial solutions of Equation (328)

$$\left|\left|\left[\lambda^2[M] + \lambda[C] + [K]\right]\right|\right| = 0 \tag{329}$$

This equation reduces to a polynomial of degree $2N$ in λ and thus there are $2N$ values of λ which will satisfy the equation. Corresponding to each λ_i there exists a $\{\phi^i\}$ such that Equation (328) is satisfied.

To determine $\{\phi^i\}$ corresponding to a particular λ_i proceed as follows:

On expanding Equation (328) with $\lambda = \lambda_i$, $\{\Phi\} = \{\Phi^i\}$

$$\begin{bmatrix} \lambda_i^2 M_{11} + \lambda_i C_{11} + K_{11} & \lambda_i^2 M_{12} + \lambda_i C_{12} + K_{12} & \cdots & \lambda_i^2 M_{1N} + \lambda_i C_{1N} + K_{1N} \\ \lambda_i^2 M_{21} + \lambda_i C_{21} + K_{21} & \lambda_i^2 M_{22} + \lambda_i C_{22} + K_{22} & \cdots & \lambda_i^2 M_{2N} + \lambda_i C_{2N} + K_{2N} \\ \vdots & & & \vdots \\ \lambda_i^2 M_{N1} + \lambda_i C_{N1} + K_{N1} & \lambda_i^2 M_{N2} + \lambda_i C_{N2} + K_{N2} & \cdots & \lambda_i^2 M_{NN} + \lambda_i C_{NN} + K_{NN} \end{bmatrix} \begin{Bmatrix} \phi_1^i \\ \phi_2^i \\ \vdots \\ \phi_N^i \end{Bmatrix} \tag{330}$$

Move column 1 to the right and omit row 1

$$-\begin{bmatrix} \lambda_i^2 M_{22} + \lambda_i C_{22} + K_{22} & \cdots & \lambda_i^2 M_{2N} + \lambda_i C_{2N} + K_{2N} \\ & \vdots & \\ \lambda_i^2 M_{N2} + \lambda_i C_{N2} + K_{N2} & \cdots & \lambda_i^2 M_{NN} + \lambda_i C_{NN} + K_{NN} \end{bmatrix} \begin{Bmatrix} \phi_2^i \\ \vdots \\ \phi_N^i \end{Bmatrix} = \begin{Bmatrix} \phi_1^i(\lambda_i^2 M_{21} + \lambda_i C_{21} + K_{21}) \\ \vdots \\ \phi_1^i(\lambda_i^2 M_{N1} + \lambda_i C_{N1} + K_{N1}) \end{Bmatrix} \tag{331}$$

This set of $n-1$ equations may now be solved by Cramer's method. For each λ_i there exists a corresponding $\{\phi^i\}$ and so

$$\{x\} = [Q]\{\eta(t)\} \tag{332}$$

where $[Q]$ is a matrix of order $N \times 2N$ formed of the modal columns $\{\phi^i\}$

$\{\eta(t)\}$ is a column vector of order $2N \times 1$

$$\eta_i(t) = A_i e^{\lambda_i t} \tag{333}$$

To complete the solution to Equation (325) it is necessary to prescribe a value to $\{x(t)\}$ and $\{\dot{x}(t)\}$ at some point in time. Assume

$$\{x(t)\} = \{x(0)\} \qquad \text{at } t=0 \tag{334}$$

$$\{\dot{x}(t)\} = \{\dot{x}(0)\} \qquad \text{at } t=0 \tag{335}$$

On substituting Equation (334) and Equation (335) into Equation (333)

$$\{x(0)\} = [Q]\{\eta(0)\} \tag{336}$$

$$= [Q]\{A_i\} \tag{337}$$

$$\{\dot{x}(0)\} = [Q]\{A_i \lambda_i\} \tag{338}$$

There are $2N$ unknown A_i's and there are $2N$ equations in these unknowns, Equation (337) and Equation (338).

<u>The orthogonality relations of generalized damped systems in N space</u>:

The equations of motion in N space

$$[M]\{\ddot{x}\}+[K]\{\dot{x}\}+[K]\{x\}=0 \tag{339}$$

Let $$\{x\}=\{\phi^j\}e^{\lambda_j t} \tag{340}$$

On substituting Equation (340) into Equation (339)

$$\lambda_j^2\,[M]\{\phi^j\}+\lambda_j[C]\{\phi^j\}+[K]\{\phi^j\}=0 \tag{341}$$

Let $$\{x\}=\{\phi^i\}e^{\lambda_i t} \tag{342}$$

On substituting Equation (342) into Equation (339)

$$\lambda_i^2[M]\{\phi^i\}+\lambda_i[C]\{\phi^i\}+[K]\{\phi^i\}=0 \tag{343}$$

Transpose Equation (343 and postmultiply by

$$\lambda_i^2\{\phi^i\}^T[M]^T\{\phi^j\}+\lambda_i\{\phi^i\}^T[C]^T\{\phi^j\}+\{\phi^i\}^T[K]^T\{\phi^j\}=0 \tag{344}$$

As $[M]$, $[C]$ and $[K]$ are symmetric, Equation (344) reduces to

$$\lambda_i^2 \{\phi^i\}^T[M]\{\phi^j\} + \lambda_i\{\phi^i\}^T[C]\{\phi^j\} + \{\phi^i\}^T[K]\{\phi^j\} = 0 \tag{345}$$

On premultiplying Equation (341) by $\{\phi^i\}^T$

$$\lambda_j^2 \{\phi^i\}^T[M]\{\phi^j\} + \lambda_j\{\phi^i\}^T[C]\{\phi^j\} + \{\phi^i\}^T[K]\{\phi^j\} = 0 \tag{346}$$

On subtracting Equation (346) from Equation (345)

$$(\lambda_i^2-\lambda_j^2)\{\Phi^i\}^T[M]\{\Phi^j\}+(\lambda_i-\lambda_j)\{\Phi^i\}^T[C]\{\Phi^j\} = 0 \tag{347}$$

On dividing through by $(\lambda_i - \lambda_j)$

$$(\lambda_i+\lambda_j)\{\phi^i\}^T[M]\{\phi^j\} + \{\phi^i\}^T[C]\{\phi^j\} = 0 \tag{348}$$

Multiply Equation (345) by

$$\lambda_i^2\lambda_j\{\phi^i\}^T[M]\{\phi^j\}+\lambda_i\lambda_j\{\phi^i\}^T[C]\{\phi^j\}+\lambda_j\{\phi^i\}^T[K]\{\phi\} = 0 \tag{349}$$

Multiply Equation (346) by λ_i

$$\lambda_i\lambda_j^2\{\phi^i\}^T[M]\{\phi^j\}+\lambda_i\lambda_j\{\phi^i\}^T[C]\{\phi^j\}+\lambda_i\{\phi^i\}[K]\{\phi^j\} = 0 \tag{350}$$

Subtracting Equation (350) from Equation (349)

$$(\lambda_i^2\lambda_j - \lambda_i\lambda_j^2)\{\phi^i\}^T[M]\{\phi^j\} + (\lambda_j - \lambda_i)\{\phi^i\}^T[K]\{\phi^j\} = 0 \qquad (351)$$

Dividing Equation (351) by

$$-\lambda_i\lambda_j\{\phi^i\}^T[M]\{\phi^j\} + \{\phi^i\}^T[K]\{\phi^j\} = 0 \qquad (352)$$

Equation (348) and Equation (352) have been derived previously by expanding the orthogonality conditions in $2N$ space associated with Foss's method.

To solve the forced vibration problem it is much simpler to use Foss's method than to solve the problem in N space. The main advantage of using N space for the solution of the homogeneous equation is that the matrices are of order $N \times N$, whereas, in $2N$ space the order of the matrices is $2N \times 2N$. In dealing with systems with many degrees of freedom this consideration may be important, especially when using a limited storage digital computer.

Integral Transform Techniques:

Until Foss developed his method for handling generalized damping most non-classically damped systems were solved by transform methods. Although straightforward in application the use of transform techniques leads to tedious algebraic work. If a system does not possess a mode solution it may be very difficult to obtain a solution without the use of transform methods. In the next section a system that does not possess a mode solution will be solved by Laplace transform, the integral transform technique most frequently used in engineering analysis.

Numerical Examples to Illustrate Some of the Theory Developed Above:

In a later section of this thesis two fully worked examples of classically and non-classically damped systems will be presented. Here, a few examples of systems that highlight some of the above theory, are given.

Example 1:

Non-classically damped system with degrees of freedom. Equation of motion

$$\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{Bmatrix} 2 & -1 \\ -1 & 3 \end{Bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (353)$$

Here

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (354)$$

$$[K] = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \tag{355}$$

$$[C] = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \tag{356}$$

Let $[Q]_{M,K}$ be the matrix of order 2×2 which simultaneously diagonalizes $[M]$ and $[K]$. It is easy to show that

$$[Q]_{M,K} = \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{bmatrix} \tag{357}$$

and

$$[Q]^T_{M,K}\,[C]\,[Q]_{M,K} = \begin{bmatrix} 7+2\sqrt{2} & -1 \\ -1 & 7-2\sqrt{2} \end{bmatrix} \tag{358}$$

From Equation (358) it is seen that this system possesses non-classical damping. In the notation used previously

$$[U] = \begin{bmatrix} [0] & I \\ -[K]^{-1}[M] & -[K]^{-1}[C] \end{bmatrix} \tag{359}$$

where $[0]$ and I are matrices of order 2×2. From Equation (355)

$$[K]^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \tag{360}$$

To determine α

$$[u]\{\Phi\} = \frac{1}{\alpha}\{\Phi\} \tag{361}$$

where $\{\Phi\}$ is a modal column of order 4×1. From Equation (361)

$$\left[[u] - \frac{1}{\alpha}[I]\right]\{\Phi\} = 0 \tag{362}$$

Let $\frac{1}{\alpha} = \lambda$ (363)

On substituting Equation (359), Equation (360) and Equation (363) into Equation (362)

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & -2\frac{1}{2}-\lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1-\lambda \end{bmatrix} \left\{ \Phi \right\} = 0 \tag{364}$$

From Equation (364) for non-trivial $\{\Phi\}$

$$\left\| \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & -2\frac{1}{2}-\lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1-\lambda \end{bmatrix} \right\| = 0 \tag{365}$$

Equation (365) reduces to

$$2\lambda^4 + 7\lambda^3 + 9\lambda^2 + 5\lambda + 1 = 0 \tag{366}$$

There are four roots to Equation (366)

$$\lambda = -1, -1, -1, -\frac{1}{2} \tag{367}$$

$$\therefore \alpha = -1, -1, -1, -2 \tag{368}$$

Thus this system has a root of multiplicity three of the frequency equation. As the system is non-classical, a mode solution may not exist. The following calculations show that only two independent eigenvectors are possible and thus the system does not possess a mode solution. From Equation (364)

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\frac{3}{2} & -\frac{1}{2} & -2\frac{1}{2}-\lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1-\lambda \end{bmatrix} \begin{Bmatrix} 1 \\ a \\ b \\ c \end{Bmatrix} = \{0\} \tag{369}$$

where

$$\{\Phi\} = \begin{Bmatrix} 1 \\ a \\ b \\ c \end{Bmatrix} \tag{370}$$

Taking $\lambda = -1$ and on performing the matrix multiplication Equation (369) reduces to

$$\begin{Bmatrix} 1 + b \\ a + c \\ -\frac{3}{2} - \frac{1}{2}a - \frac{3}{2}b \\ -\frac{1}{2} - \frac{a}{2} - \frac{b}{2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{371}$$

For Equation (371) to hold it is easy to verify that the following values of a , b and c are required.

$$a = 0 \tag{372}$$

$$b = -1 \tag{373}$$

$$c = 0 \tag{374}$$

$$\therefore \left\{\Phi\right\}_{-1} = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \lambda \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \end{Bmatrix} \qquad \lambda = -1 \tag{375}$$

Thus there is only one eigenvector corresponding to the root, $\lambda = -1$, of multiplicity three.

Similarly it may be shown that the eigenvector corresponding to the root $\lambda = -\frac{1}{2}$ is

$$\left\{\Phi\right\}_{-\frac{1}{2}} = \begin{Bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{Bmatrix} \tag{376}$$

Therefore in this system only two independent modal columns can be obtained and in general a mode solution is not possible. It is of interest to pursue some of the properties of this example a little further. The equation of motion of a damped system in N space

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{377}$$

As $\{x\} = \{q^i\} e^{\alpha_i t}$ is a solution (378) substituting Equation (378) into Equation (377)

$$\alpha^2 [M] \{q^i\} + \alpha [C] \{q^i\} + [K] \{q^i\} = 0 \tag{379}$$

Premultiply Equation (379) by $\{q^i\}^T$

$$\alpha^2 \{q^i\}^T [M] \{q^i\} + \alpha \{q\}^T [C] \{q^i\} + \{q^i\}^T [K] \{q^i\} = 0 \tag{380}$$

Now in classically damped systems Equation (380) which is a quadratic in α_i gives the two values of α_i which correspond to the eigenvector $\{q^i\}$. In the present system it will be shown that equations of type Equation (388) bring in an extraneous root which does not satisfy the frequency equation and hence is not an eigenvalue. Substitute Equation (354), Equation (355) and Equation (356) and

$$\{q\}_{-1} = \left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\} \tag{381}$$

into Equation (380) and on simplifying

$$\alpha^2 + 2\alpha + 1 = 0 \tag{382}$$

From Equation (382) $\alpha_{-1} = -1, -1$ (383)

Similiarly on substituting Equation (354), Equation (355) and Equation (356) and

$$\{q\}_{-2} = \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\} \tag{384}$$

into Equation (380) and on simplifying

$$2\alpha^2 + 7\alpha + 6 = 0 \tag{385}$$

From Equation (385)

$$\alpha_{-2} = -2, -\frac{3}{2} \tag{386}$$

The root, $\alpha = -3/2$, which does not satisfy the frequency equation,

$$\left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha^2 + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \alpha + \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \right\| = 0 \tag{387}$$

is extraneous in this case.

<u>Example 2</u>:

This example is a case of a non-classically damped system with 2 degrees of freedom and 2 equal roots of the frequency equation. As there are only 2 equal roots in this case the theory presented above predicts that a mode solution may be possible. However, as is shown below a mode solution does not in fact exist. The equation of motion

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{388}$$

In this case let

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{389}$$

$$[K] = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \tag{390}$$

$$[C] = \begin{bmatrix} 6 & -1 \\ -1 & 1 \end{bmatrix} \tag{391}$$

The matrix of order 2 x 2 that simultaneously diagonalizes matrices $[M]$ and $[K]$ is

$$[Q]_{M,K} = \begin{bmatrix} \frac{-5-\sqrt{61}}{2} & \frac{-5+\sqrt{61}}{2} \\ 3 & 3 \end{bmatrix} \tag{392}$$

It is easy to show that

$[Q]^T_{M,K} [C][Q]_{M,K}$ is not a diagonal matrix

and therefore this system is non-classically damped. In the notation previously used

$$[U] = \begin{bmatrix} 0 & I \\ -[K]^{-1}[M] & -[K]^{-1}[C] \end{bmatrix} \tag{393}$$

From Equation (390)

$$[K]^{-1} = \frac{1}{27}\begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \tag{394}$$

To determine λ solve the frequency equation

$$\left\| \left[[u] - \lambda I \right] \right\| = 0 \tag{395}$$

On substituting Equation (389), Equation (391) and Equation (394) into Equation (395) and expanding

$$27\lambda^4 + 27\lambda^3 + 18\lambda^2 + 7\lambda + 1 = 0 \tag{396}$$

$$\lambda = -1/3\,,\ -1/3\,,\ -\frac{1 \pm \sqrt{11}}{2} \tag{397}$$

This is a system with a root, $\lambda = -1/3$, of multiplicity 2 and on performing similiar calculations to those of Example 1, it can be shown that

$$\left\{ \Phi \right\}_{-1/3} = \frac{3}{\sqrt{10}} \begin{Bmatrix} 1 \\ 0 \\ 1/3 \\ 0 \end{Bmatrix}$$

is the unique nomalized eigenvector corresponding to this root. As there are less than 4 distinct eigenvectors in this case a mode solution will not exist.

<u>Example 3</u>:

As an illustration of the use of transform techniques in solving multi-degree of freedom systems the solution to Example 1 which cannot be solved by the normal mode approach is now presented. Here Laplace transform is used.

Given

$$f(t), -\infty \leq t \leq \infty \quad f(t) = 0 \quad t < 0 \tag{398}$$

Define $$\mathcal{L} f(t) = \int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p) \tag{399}$$

p complex as the Laplace transform of $f(t)$. It can be shown

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \bar{f}(p)\,dp \tag{400}$$

Equation (400) is known as the Inversion Formula. In Equation (399) p is a complex variable, $p = a + i\omega$, with the real part sufficiently large that the integral exists. In Equation (400) the integral is a line integral in the complex plane carried out on the contour shown in Figure 2. The contour is parallel to the imaginary axis, but with γ chosen sufficiently large that the contour is to the right of all singularities (poles and branch points of $\bar{f}(p)$). The reason for this is that the integral represents zero for $t < 0$ which can easily be shown by closing the contour with a large semi-circle on the left and using Cauchy 's theorem. For $t < 0$ it is necessary to evaluate Equation (400) by the standard complex integration techniques.

Solution by Laplace Transform Method:

Equations of motion in N space

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\} = 0 \tag{401}$$

where $[M]$, $[C]$ and $[K]$ are given by Equation (354), Equation (355) and Equation (356) respectively. Taking Laplace transforms of Equation (401) noting

$$\mathcal{L}\{\dot{x}\} = p\mathcal{L}\{x\} - \{x(0)\} \tag{402}$$

$$= p\{\bar{x}\} - \{x(0)\} \tag{403}$$

where $$\{\bar{x}\} = \mathcal{L}\{x\} \tag{404}$$

$$\mathcal{L}\{\ddot{x}\} = p^2\mathcal{L}\{x\} - p\{x(0)\} - \{\dot{x}(0)\} \tag{405}$$

$$= p^2\{\bar{x}\} - p\{x(0)\} - \{\dot{x}(0)\} \tag{406}$$

where $\{x(0)\}$ and $\{\dot{x}(0)\}$ are the initial conditions at $t = 0$.

$$p^2[M]\{\bar{x}\}+p[C]\{\bar{x}\}+[K]\{x\} = p[M]\{x(0)\}+[C]\{x(0)\}+[M]\{\dot{x}(0)\} \tag{407}$$

On rearranging Equation (407)

$$\left[p^2[M]+p[C]+[K]\right]\left\{\bar{x}\right\} = \left[p[M]+[C]\right]\left\{x(0)\right\}+[M]\{\dot{x}(0)\} \tag{408}$$

On substituting Equation (354), Equation (355) and Equation (356) into Equation (408)

$$\begin{bmatrix} p^2+2p+1 & -p-1 \\ -p-1 & p^2+3p+3 \end{bmatrix} \left\{ \bar{X} \right\} = \begin{bmatrix} p+2 & -1 \\ -1 & p+3 \end{bmatrix} \left\{ X(0) \right\} + \left\{ \dot{X}(0) \right\} \qquad (409)$$

From Equation (409)

$$\left\{ \bar{X} \right\} = \begin{bmatrix} p^2+2p+1 & -p-1 \\ -p-1 & p^2+3p+3 \end{bmatrix}^{-1} \begin{bmatrix} p+2 & -1 \\ -1 & p+3 \end{bmatrix} \left\{ X(0) \right\} + \begin{bmatrix} p^2+2p+1 & -p-1 \\ -p-1 & p^2+3p+3 \end{bmatrix}^{-1} \left\{ \dot{X}(0) \right\} \qquad (410)$$

From Equation (410)

$$\left\{ \bar{X} \right\} = \frac{1}{(p+1)^3(p+2)} \begin{bmatrix} p^3+5p^2+8p+5 & p \\ p+1 & p^3+5p^2+6p+2 \end{bmatrix} \left\{ X(0) \right\} + \frac{1}{(p+1)^3(p+2)} \begin{bmatrix} p^2+3p+2 & p+1 \\ p+1 & p^2+2p+1 \end{bmatrix} \left\{ \dot{X}(0) \right\} \qquad (411)$$

To evaluate $\{X\}$ it is necessary to know

$$\mathcal{L}^{-1} \frac{1}{(p+1)^3(p+2)} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{1}{(p+1)^3(p+2)} \, dp \qquad (412)$$

By Cauchy's Theorem it can be shown that

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\frac{1}{(p+1)^3(p+2)}\,dp + \oint_{\Gamma} e^{pt}\frac{1}{(p+1)^3(p+2)}\,dp \tag{413}$$

$$= 2\pi i \sum \text{Residues in } D$$

where Γ is contour shown in Figure 2, D is domain enclosed by Γ and the line joining $\gamma - i\infty$ to $\gamma + i\infty$.

By Jordan's lemma it is easy to show that

$$\int e^{pt}\frac{1}{(p+1)^3(p+2)}\,dp = 0 \tag{414}$$

By well known techniques it is possible to evaluate the residues at the poles of $\dfrac{e^{pt}}{(p+1)^3(p+2)}$. For this particular function there are singular points at $p=-1$ and $p=-2$.

Residue due to 1st order Singular point at $p=-2 \quad = -e^{-2t}$ (415)

Residue due to 3rd order singular point at $p=-1$

$$= e^{-t}\left(\tfrac{1}{2}t^2 - t + 1\right) \tag{416}$$

On using Equation (413), Equation (414), Equation (415) and Equation (416)

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\frac{1}{(p+1)^3(p+2)}\,dp = 2\pi i\left[-e^{-2t} + e^{-t}\left(\tfrac{1}{2}t^2 - t - 1\right)\right] \tag{417}$$

$$\therefore \mathcal{L}^{-1}\frac{1}{(p+1)^3(p+2)} = e^{-2t} + e^{-t}\left(\tfrac{1}{2}t^2 - t - 1\right) \tag{418}$$

Using the well-known result

$$\mathcal{L}^{-1} p^N \bar{f}(p) = \frac{d^N}{dt^N} \mathcal{L}^{-1} \overline{f(p)} \tag{419}$$

$\mathcal{L}^{-1} \dfrac{p^N}{(p+1)^3 (p+2)}$ $\quad N = 1, 2, 3$ can be evaluated from Equation (418) and on substituting these functions of t into the inverse Laplace transform of Equation (411)

$$\begin{aligned} \{x\} &= \begin{bmatrix} e^{-2t} + e^{-t}(\tfrac{1}{2}t^2 + 2) & 2e^{-2t} + e^{-t}(\tfrac{1}{2}t^2 + 2t + 2 \\ e^{-2t} + e^{-t}(t-1) & -2e^{-2t} + e^{-t}(-t+3) \end{bmatrix} \left\{ x(0) \right\} \\ &+ \begin{bmatrix} -e^{-2t} + e^{-t}(\tfrac{1}{2}t^2 + 1) & e^{-2t} + e^{-t}(t-1) \\ e^{-2t} + e^{-t}(t-1) & e^{-t} - e^{-2t} \end{bmatrix} \left\{ \dot{x}(0) \right\} \end{aligned} \tag{420}$$

As a check on previous work

if

$$\{x(0)\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} ; \{\dot{x}(0)\} = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \tag{421}$$

$$\{x\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^{-t}$$ as shown previously.

Again if

$$\{x(0)\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} ; \{\dot{x}(0)\} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} \tag{422}$$

$$\{x\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-2t}$$ as shown previously.

It is easy to see from the form of $\{x\}$ as given by Equation (420) that mode solutions do not in general exist as

$$\{x\} + [Q]\{\eta(t)\}$$

where $[Q]$ is any matrix of order 2×4 and $\{\eta(t)\}$ is a column vector of order 4×1.

If the above system is force excited it is necesssary to determine the solution to the inhomogenous equation by transform methods. In principle this is similar to the homogeneous case and the solution is not presented. In passing it may be noted that one advantage of transform methods is that the initial conditions must be incorporated into the calculations before any solution is obtained. This is not so with other methods of solution where it is necessary to solve first the homogeneous problem with the given initial conditions and then the inhomogeneous problem with zero initial conditions.

Chapter II

Effect of Damping on the Natural Frequencies of Linear Dynamic Systems *

An analysis is presented of the effect of weak damping on the natural frequencies of linear dynamic systems. It is shown that for certain damping matrices, some of the damped natural frequencies of a dynamic system may be larger than the corresponding frequencies for the undamped systems.

Introduction

In his Doctoral thesis, Berg[8] considered the vibration of a dynamic system with generalized linear damping, and showed numerically that the damped natural frequency of the lower mode was larger than the corresponding frequency of the undamped system.

It is well known that in a single degree of freedom system, the damped natural frequency is always less than the undamped natural frequency. In the case of multi-degree of freedom systems with classical normal modes[1] it may be shown that the damped natural frequencies are always less than, or equal to, the corresponding

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* The author is indebted to Dr. T. K. Caughey for suggesting the analytical approaches used in this chapter.

undamped frequencies.

Here it is intended to study the effects of weak damping on the natural frequencies of linear dynamic systems, and to show under what conditions Berg's anomalous results are obtained.

Analysis:

The equations of motion of an N degree of freedom linear dynamic system with lumped parameters may be written in matrix notation as:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\} = \{f(t)\} \tag{423}$$

For passive systems the $N \times N$ matrices $[M]$ and $[K]$ are symmetric and positive definite, and the matrix $[C]$ is symmetric and non-negative definite. Consider the homogeneous system obtained by setting $\{f(t)\}=0$ in Equation (423).

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\} = 0 \tag{424}$$

Classical Normal Modes:

The system defined by Equation (424) possesses classical normal modes, if and only if the matrix $[C]$

is diagonalized by the same transformation which simultaneously diagonalizes $[M]$ and $[K]$. Let

$$\{x\} = [Q]\{\eta\} \tag{425}$$

where $[Q]$ is the normalized matrix which simultaneously diagonalizes $[M]$ and $[K]$. If $[C]$ is such that classical normal modes exist, then $[Q]^T[C][Q] = [\bar{C}]$ a diagonal matrix with elements

$$\bar{C}_i = \{q^i\}^T [C] \{q^i\} \tag{426}$$

If Equation (425) is substituted into Equation (424) and then premultiplied by $[Q]^T$, there results the system of equations:

$$\bar{M}_i \ddot{\eta}_i + \bar{C}_i \dot{\eta}_i + \bar{K}_i \eta_i = 0 \tag{427}$$

where

$$\left.\begin{aligned} \bar{M}_i &= \{q^i\}^T [M] \{q^i\} \\ \bar{C}_i &= \{q^i\}^T [C] \{q^i\} \\ \bar{K}_i &= \{q^i\}^T [K] \{q^i\} \end{aligned}\right\} \tag{428}$$

Let $\eta_i(t) = \tilde{\eta}_i e^{\lambda_i t}$ (429)

Then

$$\lambda_i = -\frac{\bar{C}_i}{2\bar{M}_i} \pm i\sqrt{\frac{\bar{K}_i}{\bar{M}_i} - \left(\frac{C_i}{2\bar{M}_i}\right)^2} \tag{430}$$

Hence, the damped natural frequency is given by:

$$\omega_i d = \sqrt{\omega_i^2 - \left(\frac{\bar{C}_i}{2M_i}\right)^2} \leq \omega_i \qquad i = 1, 2 \cdots\cdots N \tag{431}$$

Thus, if a system possesses classical normal modes, the damped natural frequencies are always less than, or equal to, the corresponding undamped frequencies.

Non-Classical Normal Modes:

If the matrix $[C]$, in Equation (424) is such that it cannot be diagonalized by the transformation which simultaneously diagonalizes $[M]$ and $[K]$, the system is said to possess non-classical normal modes and must be treated by Foss's method[3].

To analyze the effect of weak damping on the frequencies in this case, rewrite Equation (424) in the following manner:

$$[M]\{\ddot{x}\} + \varepsilon[C]\{\dot{x}\} + [K]\{x\} = 0 \tag{432}$$

where ε is a small parameter, the problem can now be treated by perturbation analysis [13]. Let

$$\{X\} = \bar{\phi}^n e^{\lambda_n t} \tag{433}$$

Substituting Equation (433) into Equation (432)

$$\lambda_n^2 [M] \bar{\phi}^n + \varepsilon \bar{\lambda}_n [C'] \{\bar{\phi}_n\} + [K] \{\bar{\phi}^n\} = 0 \tag{434}$$

Let $\bar{\phi}_n = \phi^n + \varepsilon \psi^n + \varepsilon^2 \theta^n$ (435)

$$\bar{\lambda}_N = \lambda_n + \varepsilon \mu_N + \varepsilon^2 \nu_N \tag{436}$$

where ϕ^N and λ_N are the n^{th} eigenvector and eigenvalue for the undamped problem, $\varepsilon = 0$. Inserting Equation (435) and Equation (436) into Equation (432), leads to the following system of equations on separating out the various orders in

$$\left(\lambda_n^2 [M] + [K]\right) \{\phi^N\} = 0 \tag{437}$$

$$(\lambda_N^2 [M] + [K])\{\psi^N\} = -(2[M]\lambda_n \mu_N + \lambda_n [C'])\{\phi^N\} \tag{438}$$

$$(\lambda_N^2 [M] + [K])\{\theta^N\} = -\left((\mu_N^2 + 2\lambda_N \nu_N)[M] + \mu_n [C']\right)\{\phi^n\} \tag{439}$$

$$- (2\lambda_n \mu_N [M] + \lambda_n [C']) \{\psi^N\}$$

From these equations, the perturbations of various orders may be calculated.

Zeroth Order Solutions:

The zeroth order solutions are obtained from Equation (437)

$$\left(\lambda_n^2 [M] + [K]\right)\{\phi^n\} = 0 \qquad (440)$$

$$n = 1, 2, \ldots N$$

Since $[M]$ and $[K]$ are symmetric and positive definite:

1) $\lambda_n^2 < 0$ all n. That is, the eigenvalues are purely imaginary.
2) The $\{\phi^n\}$'s are real.
3) The $\{\phi^n\}$'s are orthogonal in $[M]$ and $[K]$. That is $\{\phi^\ell\}^T[M]\{\phi^k\} = 0 \quad \ell \neq k$

In the analysis which follows it will be assumed for simplicity that the λ_n's are distinct.

First Order Perturbations:

The first order perturbations are obtained from Equation (438)

$$\left(\lambda_n^2 [M] + [K]\right)\{\psi^n\} = -\left(2[M]\lambda_n \mu_n + \lambda_n [C']\right)\{\phi^n\} \qquad (441)$$

In order to evaluate the first order perturbations, express

$\{\psi^n\}$ in terms of the $\{\phi^j\}$'s. Thus

$$\{\psi^n\} = \sum_{j=1}^{N} a_{nj} \{\phi^j\} \tag{442}$$

Premultiply Equation (441) by $\{\phi^\ell\}^T$

$$\therefore \lambda_n^2 \{\phi^\ell\}^T [M]\{\psi^n\} + \{\phi^\ell\}^T [K]\{\psi^n\} =$$

$$-2\lambda_n \mu_n \{\phi^\ell\}^T [M]\{\phi^n\} - 2\lambda_n \mu_n \{\phi^\ell\}^T [C']\{\phi^n\} \tag{443}$$

Now

$$\{\phi^\ell\}^T [M]\{\phi^n\} = \delta_{\ell n}\{\phi^n\}^T [M]\{\phi^n\} = \delta_{\ell n} \tag{444}$$

where $\left.\begin{array}{ll} \delta_{\ell n} = 0 & \ell \neq n \\ \quad\;\; = 1 & \ell = n \end{array}\right\}$ is Kronecker's delta.

The $\{\phi^n\}$ may be normalized such that

$$\{\phi^n\}^T [M]\{\phi^n\} = 1 \tag{445}$$

$$n = 1, 2 \ldots\ldots N$$

If in Equation (440), n is replaced by ℓ and the resulting equation transposed, and then postmultiplied by $\{\psi^n\}$: Then

$$\lambda_\ell^2 \{\phi^\ell\}^T [M]\{\psi^n\} + \{\phi^\ell\}^T [K]\{\psi^n\} = 0 \tag{446}$$

Hence, Equation (443) becomes:

$$(\lambda_n^2 - \lambda_\ell^2)\{\phi^\ell\}^T[M]\{\psi^n\} = -(2\lambda_n \mu_n \delta_{\ell n} + \lambda_n\{\phi^\ell\}^T[C']\{\phi^n\}) \quad (447)$$

If $\lambda_n \neq \lambda_i$, i.e. $\ell \neq n$

$$\{\phi^\ell\}^T[M]\{\psi^n\} = -\frac{\lambda_n}{\lambda_n^2 - \lambda_\ell^2}\{\phi^\ell\}^T[C']\{\phi^n\} \quad (448)$$

If $\ell = n$, then

$$\mu_n = -\tfrac{1}{2}\{\phi^n\}^T[C']\{\phi^n\} \quad (449)$$

Now premultiply Equation (442) by $\{\phi^\ell\}^T[M]$:

Thus

$$\{\phi^\ell\}^T[M]\{\psi^n\} = \sum_{j=1}^{N} a_{nj}\{\phi^\ell\}^T[M]\{\phi^j\} \quad (450)$$

$$= a_{n\ell}$$

Thus

$$a_{n\ell} = \{\phi^\ell\}^T[M]\{\psi^n\}$$

$$\therefore \underset{n \neq \ell}{a_{n\ell}} = -\frac{\lambda_n}{\lambda_n^2 - \lambda_\ell^2}\{\phi^\ell\}^T[C']\{\phi^n\} \quad (451)$$

The quantity a_{nn} is found from the normalization condition

$$\{\bar{\phi}^n\}^T[M]\{\bar{\phi}^n\} = 1 \quad (452)$$

Hence, $a_{nn} = 0$ (453)

Therefore, if ψ^n can be expanded in terms of ϕ^j

$$\psi^n = \sum_{j=1}^{N}{}' - \frac{\lambda_n}{\lambda_n^2 - \lambda_j^2}\left(\{\phi^j\}^T[C']\{\phi^n\}\right)\{\phi^j\} \qquad (454)$$

Where the symbol $\sum'$ denotes summation of the indicated values of j , omitting the term for which $j = n$.

Second Order Perturbations:

Having determined the first order perturbations, the second order terms may be found in a similar manner. Let

$$\theta^n = \sum_{j=1}^{N} b_{nj}\,\phi^j \qquad (455)$$

Using the same technique as used above:

$$\beta_{n\ell} = \frac{1}{\lambda_n^2 - \lambda_\ell^2}\left[\tfrac{1}{2}\left(\{\phi^n\}^T[C']\{\phi^n\}\right)\left(\{\phi\}^T[C']\{\phi^n\}\right) + \sum_{j=1}^{N}{}' \frac{\lambda_n^2}{\lambda_n^2 - \lambda_j^2}\left(\{\phi^j\}^T[C']\{\phi^n\}\right)\left(\{\phi^\ell\}^T[C']\{\phi^j\}\right)\right] \qquad (456)$$

$$n \neq \ell$$

$$\beta_{nn} = \{\phi^n\}^T[M]\{\theta^n\} = \tfrac{1}{2}\sum_{j=1}^{N} a_{nj}^{\,2} \qquad (457)$$

$$\nu_n = \frac{1}{8\lambda_m}\left(\{\phi^n\}^T[C']\{\phi^n\}\right)^2 + \tfrac{1}{2}\sum_{j=1}^{N}{}' \frac{\lambda_n}{\lambda_n^2 - \lambda_j^2}\left(\{\phi^j\}^T[C']\{\phi^n\}\right)^2 \qquad (458)$$

Eigenvectors in Damped Systems:

The eigenvectors for the damped system are to terms of order ε^2 :

$$\begin{aligned}\bar{\phi}^n &= \phi^n + \varepsilon\psi^n + \varepsilon^2\theta^n + o(\varepsilon^3) \\ &= \phi^n + \varepsilon\sum_{j=1}^{N} a_{nj}\phi^j + \varepsilon^2\sum_{j=1}^{N} \beta_{nj}\phi^j\end{aligned} \tag{459}$$

Where a_{nj} is given by Equation (451) and Equation (452), β_{nj} is given by Equation (456) and Equation (457).

Some Interesting Properties of Equation (459)

1) If the matrix $[C']$ is such as to admit classical normal modes, then

$$\{\phi^j\}^T[C']\{\phi^n\} = 0 \qquad n \neq j \tag{460}$$

Hence,

$$\left.\begin{aligned} a_{nj} &= 0 \qquad \begin{aligned} j &= 1, 2, 3 \ldots\ldots N \\ n &= 1, 2, 3 \ldots\ldots N \end{aligned} \\ \beta_{nj} &= 0 \end{aligned}\right\} \tag{461}$$

$$\therefore \bar{\phi}^n = \phi^n \tag{462}$$

That is, the eigenvectors are identical with those for the undamped problem.

2) If the matrix $[C']$ is non-classical, then in general

$$\{\phi^j\}^T[C']\{\phi^n\} \neq 0 \tag{463}$$

Now $\lambda_i = \sqrt{-1}\ \omega_i$ (464)

$\therefore \bar{\phi}^n = \phi^n + \varepsilon\sqrt{-1}$ (real vector) $+ \varepsilon^2$ (real vector) (465)

Thus, the eigenvectors are, in general, complex.

Eigenvalues in Damped Systems:

The eigenvalues for the damped system are to terms of order ε^2:

$$\bar{\lambda}_n = \lambda_n - \varepsilon/2\{\phi^n\}^T[C']\{\phi^n\} + \varepsilon^2\Big\{ 1/8\ \frac{(\{\phi^n\}^T[C']\{\phi^n\})^2}{\lambda_n}$$

$$+ 1/2 \sum_{j=1}^{N}{}' \frac{\lambda_n}{\lambda_n^2 - \lambda_j^2}\left(\{\phi^j\}^T[C']\{\phi^n\}\right)^2\Big\} \tag{466}$$

Now $\lambda_n = \sqrt{-1}\ \omega_n$

$$n = 1, 2 \ \cdots\cdots\ N \tag{467}$$

Thus

$$\bar{\lambda}_n = \sqrt{-1}\ \omega_n \Big\{ 1 - \varepsilon^2/2 \sum_{j=1}^{N}{}' \left(\{\phi\}^{jT} [C'] \{\phi^n\}^2 (\omega_n^2 - \omega_j^2)^{-1} \right)$$

$$- \frac{\varepsilon^2}{8\omega_n^2}\left(\{\phi^n\}^T[C']\{\phi^n\}\right)^2 \Big\} - \varepsilon/2\{\phi^n\}^T[C']\{\phi^n\} \tag{468}$$

Damped Natural Frequencies:

The damped natural frequency for the system is given by:

$$\omega_{nd} = \omega_n \left\{ 1 - \varepsilon^2/2 \sum_{j=1}^{N}{}' \left(\{\phi^j\}^T [C'] \{\phi^n\} \right)^2 \left(\omega_n^2 - \omega_j^2 \right)^{-1} - \varepsilon^2/8\omega_n^2 \left(\{\phi^n\}^T [C'] \{\phi^n\} \right)^2 \right\} + o(\varepsilon^4) \tag{469}$$

The $o(\varepsilon^4)$ term arises due to the fact that the *3rd* order term in $\bar{\lambda}_n$ is purely real.

Some Interesting Properties of Equation (469):

1) If $[C']$ is such as to admit classical normal modes:

Then

$$\{\phi^j\}^T [C'] \{\phi^n\} = 0 \quad n \neq j$$

Thus $$\omega_{nd} = \omega_n \left\{ 1 - \varepsilon^2/8\omega_n^2 \left(\{\phi^n\}^T [C'] \{\phi^n\} \right)^2 \right\} \tag{470}$$

$$n = 1, 2 \ldots\ldots N$$

Hence, $\omega_{nd} \leq \omega$ (471)

Equation (471) is in agreement with Equation (431).

2) If $[C']$ is non-classical, then, in general

$$\{\phi^j\}^T [C'] \{\phi^n\} \neq 0 \qquad n \neq j$$

If in Equation (469) n is set equal to N. Now

$$\omega_n > \omega_{n-1} > \cdots\cdots > \omega_j > \cdots\cdots > \omega_1$$

$$\therefore \omega_n^2 - \omega_j^2 > 0$$

$$\therefore \omega_{nd} = \omega_N \left\{ 1 - \varepsilon^2/2 \sum_{j=1}^{N}{}' \left(\{\phi^j\}^T [C'] \{\phi^N\} \right)^2 \left(\omega_N^2 - \omega_j^2 \right)^{-1} - \varepsilon^2/8\omega_N^2 \left(\{\phi^N\}^T [C'] \{\phi^N\} \right)^2 \right\} \tag{472}$$

From Equation (472) it will be seen that

$$\omega_{nd} \leq \omega_N \tag{473}$$

3) If $n=1$ then $\omega_j > \omega_n \quad j \neq n$

$$\therefore \omega_{id} = \omega_i \left\{ 1 + \varepsilon^2/2 \sum_{j=1}^{N}{}' \left(\{\phi^j\}^T [C'] \{\phi'\} \right)^2 \left(\omega_j^2 - \omega_1^2 \right)^{-1} - \varepsilon^2/8\omega_1^2 \left(\{\phi'\}^T [C'] \{\phi'\} \right)^2 \right\} \tag{474}$$

If

$$4\omega_1^2 \sum_{j=1}^{N}{}' \left(\{\phi^j\}^T [C'] \{\phi'\}^2 \right) \left(\omega_j^2 - \omega_1^2 \right)^{-1} > \left(\{\phi'\}^T [C'] \{\phi'\} \right)^2 \tag{475}$$

Then $\omega_{id} > \omega_1$, and Berg's anomalous result is proved.

Conclusions:

From the above analysis the following conclusions may be drawn:

1) In a linear dynamic system with weak damping, the damped natural frequency of the highest mode is always less than, or equal to, the undamped frequency, no matter what form of damping matrix is used.
2) The damped natural frequency of the lowest mode may be higher than the corresponding undamped frequency, depending on the choice of damping matrix.
3) In a system with classical normal modes, the damped natural frequencies are always less than, or equal to the corresponding undamped frequencies.

Example: To illustrate the results of the above analysis, consider the following system:

$$(476)$$

where $[M] = I$

$$\left.\begin{aligned} [K] &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ [C'] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \varepsilon = 0.1 \end{aligned}\right\} \qquad (477)$$

Undamped Systems:

For the undamped system

$$\left.\begin{aligned} \{\phi^1\} &= \tfrac{1}{2}\begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} \quad \omega_1 = 0.765366 \\ \{\phi^2\} &= \tfrac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \omega_2 = 1.414214 \\ \{\phi^3\} &= \tfrac{1}{2}\begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} \quad \omega_3 = 1.847759 \end{aligned}\right\} \qquad (478)$$

Using Equation (469), the damped natural frequencies are:

$$\left.\begin{aligned} \omega_{1d} &\simeq 0.765687 & > \omega_1 \\ \omega_{2d} &\simeq 1.413993 & < \omega_2 \\ \omega_{3d} &\simeq 1.846696 & < \omega_3 \end{aligned}\right\} \qquad (479)$$

The exact values obtained by solving Equation (476) are:

$$\left.\begin{aligned} \omega_{1d} &= 0.765688 \\ \omega_{2d} &= 1.413990 \\ \omega_{3d} &= 1.846696 \end{aligned}\right\} \qquad (480)$$

Comparison of Equation (479) and Equation (480) shows excellent numerical agreement. It should be noted that, the damped natural frequency of the first mode is higher than that for the undamped system, while the damped frequencies for the second and third modes are lower than the corresponding values for the undamped system.

Chapter 3

Experimental Investigations

In experimental work it is often desirable to excite a system predominantly in a pure mode. Frequently, the exact parameters of the system are not known and hence, it is not possible to set up a force distribution that will excite a pure mode. However, the following iterative procedure for selecting the force distribution can be applied to classically damped systems. If the natural frequencies of the system are well separated this procedure will converge rapidly to a force distribution that will excite a pure mode.

Equation of motion in N space

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\} = \{F(t)\} \tag{481}$$

where $\{F(t)\}$ is an arbitrary force distribution. Let

$$\{p\} = [n]\{x\} \tag{482}$$

where $[n] = [\sqrt{M}]$

$[M]$ is a diagonal matrix.

$\{p\}$ is a column vector of order $N \times 1$.

From Equation (482)

$$\{x\} = [n]^{-1}\{p\} \tag{483}$$

Substitute Equation (483) into Equation (481) and pre-multiply by $[n]^{-1}$

$$[n]^{-1}[M][n]^{-1}\{\ddot{p}\}+[n]^{-1}[C][n]^{-1}\{\dot{p}\}+[n]^{-1}[K][n]^{-1}\{p\}=[n]^{-1}\{F(t)\} \tag{484}$$

Let $[\bar{\bar{C}}]=[n]^{-1}[C][n]^{-1}$ a symmetric matrix

$[\bar{\bar{K}}]=[n]^{-1}[K][n]^{-1}$ a symmetric and positive definite matrix. (485)

Substituting Equation (485) into Equation (484) and simplifying

$$I\{\ddot{p}\}+[\bar{\bar{C}}]\{\dot{p}\}+[\bar{\bar{K}}]\{p\}=[n]^{-1}\{F(t)\} \tag{486}$$

Let $\{p\}=[Q]\{\xi\}$ (487)

where as the system is classically damped

$[Q]^T[\bar{\bar{C}}][Q]=[\bar{C}]$ a diagonal matrix

$[Q]^T[\bar{\bar{K}}][Q]=[\bar{K}]$ a diagonal matrix

On substituting Equation (487) into Equation (486) and premultiplying by $[Q]^T$

$$I\{\ddot{\xi}\}+[\bar{C}]\{\dot{\xi}\}+[\bar{K}]\{\xi\}=[Q]^T[n]^{-1}\{F(t)\} \tag{488}$$

Let $\xi=\mathcal{R}\ell\,\bar{\xi}e^{i\omega t}$ (489)

On substituting Equation (489) into Equation (488) and rearranging

$$\mathcal{R}\ell\left[-\omega^2 I+i\omega[\bar{C}]+[\bar{K}]\right]\bar{\xi}e^{i\omega t}=[Q]^T[n]^{-1}\{F(t)\} \tag{490}$$

From Equation (490)

$$\bar{\xi} = \left[\frac{1}{Z(\omega)}\right] [Q]^T [n]^{-1} \{\bar{F}(t)\} \tag{491}$$

where $\{F(t)\} = \mathcal{R}\ell \{\bar{F}(t)\} e^{i\omega t}$

$$\left[\frac{1}{Z(\omega)}\right] = \left[-\omega^2 I + i\omega [\bar{C}] + [\bar{K}]\right]^{-1}$$

From Equation (487) and Equation (482)

$$\{p\} = [n]\{x\} = [Q]\{\xi\} \tag{492}$$

$$\therefore \{x\} = [n]^{-1}[Q]\{\xi\}$$

From Equation (491)

$$\{x\} = \mathcal{R}\ell\, [n]^{-1}[Q] \left[\frac{1}{Z(\omega)}\right] [Q]^T [n]^{-1} \{\bar{F}(t)\} e^{i\omega t} \tag{493}$$

Now let an iterative procedure be set up such that

$$\{F_0(t)\} = \mathcal{R}\ell \{\bar{F}(t)\} e^{i\omega t}$$

where $\{\bar{F}(t)\} e^{i\omega t}$ is any arbitrary force distribution made up of forces which have the same frequency ω.

$$\{F_1(t)\} = [M]\{X_0\}$$

where $\{X_0\}$ is the response of the system to

$\vdots$

$$\{F_n(t)\} = [M]\{X_{n-1}\}$$

where $\{X_{n-1}\}$ is the response of the system to $\{F_{n-1}(t)\}$

From Equation (493) on using the iterative procedure

$$\{X_1\} = \mathcal{R}\ell\, [n]^{-1}[Q] \left[\frac{1}{Z(\omega)}\right] [Q]^T [n]^{-1} [M] [n] \left[\frac{1}{Z(\omega)}\right] [Q]^T [n]^{-1} \{\bar{F}(t)\} e^{i\omega t} \tag{494}$$

Now $[n]^{-1}[n]^{-1} = [M]^{-1}$

$$= \left[\frac{1}{M_{ii}}\right] \quad \text{a diagonal matrix} \tag{495}$$

Again $[Q]^T[Q] = I$

But $[Q]^T[M][Q] = [Q]^T[n][n][Q]$

$$\therefore [Q]^T[n]^{-1}[M][n]^{-1}[Q] = I \tag{496}$$

Now $\left([Q][n]^{-1}\right)^T = \left([n]^{-1}\right)^T[Q]^T$

$$= [n]^{-1}[Q]^T \tag{497}$$

$[n]$ being a diagonal matrix. From Equation (492) and Equation (493) it is seen that $[M]$ is diagonalized by $[Q][n]^{-1}$. It is easy to show that $[K]$ and $[C]$ are also diagonalized by $[Q][n]^{-1}$. Therefore, the columns of $[Q][n]^{-1}$ are the normal modes of the system. On substituting Equation (496) into Equation (494)

$$\{x_i\} = \mathcal{R}\ell\,[n]^{-1}[Q]\left[\frac{1}{Z(\omega)}\right]^2[Q]^T[n]^{-1}\{F(t)\}e^{i\omega t} \tag{498}$$

But $\left[\frac{1}{Z(\omega)}\right]$ is a diagonal matrix whose ii^{th} element is

$$\frac{1}{(-\omega^2 + i\omega\bar{C}_{ii} + \bar{k}_{ii})} \tag{499}$$

where $\bar{k}_{ii} = \omega_{ii}$ is the natural frequency of the i^{th} mode. On continuing on with the iterative procedure,

$$\{X_n\} = Rl\,[n]^{-1}[Q]\left[\frac{1}{Z(\omega)}\right]^{n+1}[Q]^T[n]^{-1}\{\overline{F(t)}\}e^{i\omega t} \tag{500}$$

Let $Rl\,\{\bar{F}(t)\}e^{i\omega t} = Rl\,[n][Q][a]e^{i\omega t}$

$$\{X_n\} = Rl\,\{\bar{X}_n\}e^{i\omega t} \tag{501}$$

where $\{a\}$ is a column vector of order $N \times 1$. Substituting Equation (501) into Equation (500)

$$\{\bar{X}_n\} = [n]^{-1}[Q]\left[\frac{1}{Z(\omega)}\right]^{n+1}\{a\} \tag{502}$$

From Equation (499) if

$$\left|-\omega^2 + i\omega\,\bar{C}_{ii} + \omega_{ii}\right| < \left|-\omega^2 + i\omega\,\bar{C}_{jj} + \omega_{jj}\right| \tag{503}$$

$$\text{all } j \neq i$$

Equation (502) reduces to

$$\{\bar{X}_n\} \simeq \left([n]^{-1}[Q]\right)_i \left(\frac{1}{-\omega^2 + i\omega\,\bar{C}_{ii} + \omega_{ii}}\right)^{n+1} a_i \tag{504}$$

where $\{a_i\} = \{a\}$ $\left([n]^{-1}[Q]\right)_i = i^{th}$ column of $\left([n]^{-1}[Q]\right)$ $= i^{th}$ normal mode.

From Equation (504)

$$\{X_n\} \longrightarrow A\left([n]^{-1}[Q]\right)_i \tag{505}$$

A:constant

As shown above the convergence of this iterative procedure is conditional on Equation (503) being satisfied.

If all $\bar{C}_{jj}$ are approximately the same value this condition may be reduced to

$$\left|-\omega^2 + \omega_{ii}\right| < \left|-\omega^2 + \omega_{jj}\right| \tag{506}$$

all $j \neq i$

From Equation (506) it can be seen that if the natural frequencies of the system are well separated from one another the above iterative procedure will converge on the mode corresponding to that natural frequency ω_{ii} nearest the forcing frequency ω.

<u>To experimentally determine the mass matrix</u>:

The equations of motion of a classically damped system excited by an impulse forcing function

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\} = \{\delta F\} \tag{507}$$

where $\{\delta F\}$ is a column vector of order $N \times 1$, the elements of which are impulses of amplitude a_i .

Let $\{x\} = [Q]\{\xi\}$ (508)

Substitute Equation (508) into Equation (507) and premultiply by $[Q]^T$.

$$[Q]^T[M][Q]\{\ddot{\xi}\} + [Q]^T[C][Q]\{\dot{\xi}\} + [Q]^T[K][Q]\{\xi\} = [Q]^T[\delta F] \qquad (509)$$

Equation (509) is a set of uncoupled equations of type

$$\bar{M}_{ii}\,\ddot{\xi}_i + \bar{C}_{ii}\,\dot{\xi}_i + \bar{K}_{ii}\,\xi_i = \{q_i\}^T\{\delta F\} \qquad (510)$$

where $\{q^i\}$ is the i^{th} normal mode. Now $\{q^i\}^T\{\delta F\}$, the forcing function of the i^{th} uncoupled equation, is merely a sum of impulses. Suppose the system has zero velocity before the application of the distribution of impulses: From Equation (508) if

$$\{\dot{x}(0)\} = \{0\}$$
$$\{\dot{\xi}(0)\}_{0-} = \{0\} \qquad \text{at } t = 0_-$$

From Equation (510) after the application of the impulses

$$\dot{\xi}_i(0)_{0+} = \frac{1}{\bar{M}_{ii}}\{q^i\}^T\{\delta F\} \qquad (511)$$
$$\text{at } t = 0_+$$

$$\therefore \left\{\dot{\xi}(0)\right\}_{0+} = \left([Q]^T[M][Q]\right)^{-1}[Q]^T\{\delta F\} \qquad (512)$$

From Equation (508)

$$\{\ddot{x}(0)\}_{0+} = [Q]\,[Q]^{-1}\,[M]^{-1}\left([Q]^{T}\right)^{-1}[Q]^{T}\,\{\delta F\}$$
$$= [M]^{-1}\,\{\delta F\} \qquad (513)$$

Hence, if the velocity of the system is measured immediately after the application of the impulse force, the mass matrix can be determined by using Equation (513). From Equation (513)

$$[M]\,\{\ddot{x}(0)\}_{0+} = \{\delta F\} \qquad (514)$$

In $[M]$ there are N^2 elements.

Allowing for mutual masses there are N^2 unknowns M_{ij} in Equation (514). Each particular force distribution $\{\delta F\}$ gives rise to N equations as in Equation (514). Hence, to completely specify $[M]$ it is necessary to perform N experiments of the type described above.

<u>To experimentally determine the spring matrix $[K]$</u> :

If a spring mass system is acted on by a static force distribution the inertia and damping terms may be omitted from the equation of motion

$$[K]\,\{x\} = \{F\} \qquad (515)$$

where $\{F\}$ is the static force distribution.

Now to apply this force distribution it is necessary to slowly vary the force on each mass until the desired force distribution is achieved. If the force is applied instantaneously, then the problem is not a static one as treated here, but a dynamic one. However, in actual fact, if a steady force distribution is rapidly applied to the system, the solution for large time tends to the static solution as the vibrational motion is damped out. From Equation (515)

$$\{x\} = [K]^{-1}\{F\} \tag{516}$$

From Equation (516) it can be seen that it is possible to determine $[K]$ provided N distinct experiments are carried out. It may be noted that as $\{F\}$ is a column vector in N space it is possible to specify N independent force distributions.

It was shown above that with a series of experiments the natural frequencies, the normal modes, the mass matrix and the spring matrix can be determined for any classically damped system. The impulse test for determining the mass matrix is not of great practical significance, because of the measuring difficulties. However, it is possible to get an estimate of mass matrix knowing the natural frequencies, mode shapes and spring matrix of a lightly damped system.

For $$\omega_i \simeq \sqrt{\frac{\overline{K_{ii}}}{\overline{M_{ii}}}} \tag{517}$$

But $$\overline{K_{ii}} = \{\phi^i\}^T [K] \{\phi^i\}$$
$$\overline{M_{ii}} = \{\phi^i\}^T [M] \{\phi^i\} \tag{518}$$

as $\{\phi^i\}$, $[K]$ and ω_i are known an estimate of the value of M_{ii} can be made.

But
$$[\overline{M_{ii}}] = [Q]^T [M] [Q] \tag{519}$$

$$\therefore [M] = [Q]^{-1} [\overline{M_{ii}}] \left([Q]^T\right)^{-1} \tag{520}$$

Later a technique will be presented to correct for the damping of the system.

Non-classical Systems:

It should be noted that the experiments described above for determining the spring matrix can be used with either classical or non-classical systems. However, the experiments for determining the mass matrix have been justified in the case of classical systems only. Here it will be shown that a similar set of experiments are in fact sufficient to determine the mass matrix of a non-classical system.

Equations of motion of matrix of non-classically damped system in $2N$ space:

$$[R]\{\dot{z}\}+[P]\{z\} = \{F(t)\} \tag{521}$$

where

$$\{F(t)\} = \begin{Bmatrix} \{0\} \\ \{f(t)\} \end{Bmatrix} \tag{522}$$

Let $\{f(t)\}$ be a distribution of impulses of magnitude f_i . Assume that a mode solution is possible.

Let

$$\{z\} = [\Phi]\{\xi\} \tag{523}$$

Substitute Equation (523) into Equation (521) and premultiply by $[\Phi]^T$

$$[\Phi]^T[R][\Phi]\{\dot{\xi}\}+[\Phi]^T[P][\Phi]\{\xi\} = [\Phi]^T\{F(t)\} \tag{524}$$

Equation (524) is a set of uncoupled equations of type

$$R_{ii}\,\dot{\xi}_i + P_{ii}\,\xi_i = \{\phi^i\}^T\{f\} \tag{525}$$

Now as $\{f\}$ is an impulse force distribution

$$\dot{\xi}_i(0)_{0+} = \frac{1}{R_{ii}}\{\phi^i\}^T\{f\} \tag{526}$$

provided $\dot{f}_i(o)_{o-} = 0$ (527)

$$\therefore \{\dot{f}(o)\}_{o+} = \left([\Phi]^T[R][\Phi]\right)^{-1}[\Phi]^T[F] \tag{528}$$

From Equation (523) and Equation (528)

$$\begin{aligned}\{\dot{z}(o)\}_{o+} &= [\Phi][\Phi]^{-1}[R]^{-1}\left([\Phi]^T\right)^{-1}[\Phi]^T\{F\}\\ &= [R]^{-1}\{F\}\end{aligned} \tag{529}$$

provided $[R]^{-1}$ exists. But

$$[R] = \begin{bmatrix} [o] & [M] \\ [M] & [C] \end{bmatrix} \tag{530}$$

It is easy to show that

$$[R]^{-1} = \begin{bmatrix} -[M]^{-1}[C][M]^{-1} & [M]^{-1} \\ [M]^{-1} & [o] \end{bmatrix} \tag{531}$$

$\therefore$ as $[M]^{-1}$ exists $[R]^{-1}$ exists.

On reducing Equation (529)

$$\{\ddot{x}(o)\}_{o+} = [M]^{-1}\{f\} \tag{532}$$

$$\{\dot{x}(o)\}_{o+} = 0 \tag{533}$$

Equation (532) is exactly analogous to that previously obtained in the case of classical systems. Hence it is possible to determine $[M]$ from a series of N independent experiments.

Determination of damping in classically damped systems:

The essential feature of a classically damped system is the reduction of the equations of motion to a set of N uncoupled equations in the transformed plane. Consider such an uncoupled equation.

$$\bar{M}_{ii} \ddot{\xi}_i + \bar{C}_{ii} \dot{\xi}_i + \bar{K}_{ii} \xi_i = f_i(t) \tag{534}$$

where

$$f_i(t) = \mathcal{R}\ell \, \bar{F} e^{i\omega t} \tag{535}$$

Let $$\xi_i = \mathcal{R}\ell \, \tilde{A} e^{i\omega t} \tag{536}$$

On substituting Equation (536) into Equation (534)

$$\tilde{A} \left(-\omega^2 \bar{M}_{ii} + \bar{K}_{ii} + i\omega \bar{C}_{ii} \right) = \bar{F} \tag{536-}$$

$$\therefore \tilde{A} = \frac{\bar{F}}{(-\omega^2 \bar{M}_{ii} + \bar{K}_{ii} + i\omega \bar{C}_{ii})} \tag{537}$$

$$= \frac{\bar{F}/\bar{K}_{ii}}{\omega^2 \, 1/\omega_{ii}^2 + 1 + i \, \omega/\bar{K}_{ii} \, \bar{C}_{ii}} \tag{538}$$

$$= \frac{\bar{F}/\bar{K}_{ii}}{(1-\beta^2) + 2i\beta\gamma} \tag{539}$$

where $\beta = \omega/\omega_{ii}$

$$\omega_{ii} = \sqrt{\bar{K}_{ii}/\bar{m}_{ii}}$$

$$\gamma = \bar{C}_{ii}/2\sqrt{\bar{K}_{ii}\,\bar{m}_{ii}}$$

For simplicity let $F/\bar{K}_{ii} = 1$

It is of interest to calculate the locus of $\tilde{A}$ considered as a complex number with amplitude $|A|$ and plane ϕ.

$$A = |A| e^{i\phi} = x + iy \tag{540}$$

where $x = |A| \cos\phi$

$$y = |A| \sin\phi$$

On rationalizing the denominator of Equation (539)

$$A = \frac{(1-\beta^2) - 2i\beta\gamma}{(1-\beta^2)^2 + 4\beta^2\gamma^2} \tag{541}$$

From Equation (540)

$$X = \frac{1-\beta^2}{(1-\beta^2)^2 + 4\beta^2\gamma^2} \tag{542}$$

$$y = \frac{-2\beta\gamma}{(1-\beta^2)^2 + 4\beta^2\gamma^2} \tag{543}$$

From Equation (542) and Equation (543)

$$x^2 + y^2 = \frac{1}{(1-\beta^2)^2 + 4\gamma^2\beta^2} \tag{544}$$

Substituting Equation (543) into Equation (544)

$$x^2 + y^2 = -\ y/2\beta\gamma \tag{545}$$

$$\therefore\ x^2 + y^2 + y/2\beta\gamma + 1/16\beta^2\gamma^2 = 1/16\beta^2\gamma^2 \tag{546}$$

$$\therefore\ (\beta x)^2 + (\beta y + 1/4\gamma)^2 = (1/4\gamma)^2 \tag{547}$$

The geometrical interpretation of Equation (547) is a circle drawn in the βx, βy plane of radius $1/4\gamma$ and with center at $(0, -1/4\gamma)$. Noting that

$$\beta = \omega/\omega_{ii} \qquad \text{and} \quad \dot{\xi}_i = \omega \xi_i \tag{548}$$

It is easy to see that Equation (547) represents a circle in the velocity plane. Thus the polar plot of the velocity response over the entire frequency range $0 \longrightarrow \infty$, of an uncoupled equation of motion is a circle. The angle of lag $\phi(\mp)$ in this plot is the plane angle between the response and the forcing function.

Should a multi-degree of freedom system be vibrating in a pure mode then the velocity response locus of any mass is a circle. For assume that the system is vibrating in the i^{th} normal mode.

$$\therefore\ \xi_i = \mathcal{R}e\ \bar{A}e^{i\omega t}$$

$$\xi_j = 0 \qquad \text{all } j \neq i \tag{547'}$$

$$\therefore \{x\} = [Q]\{\xi\} \tag{548'}$$

where $\{x\}$ is the response of the, $[Q]$ is the matrix whose columns are the normal modes of the system.

Hence any

$$x_\ell = \sum_{j=1}^{N} q_\ell^{\,j}\, \xi_j \tag{549}$$

where $(q_\ell^{\,j})$ is the ℓ^{th} element of the j^{th} mode.

From Equation (548)'

$$x_\ell = q_\ell^{\,i}\, \xi_i \tag{550}$$

$$\therefore\ \dot{x}_\ell = q_\ell^{\,i}\, \dot{\xi}_i \tag{551}$$

As the locus of velocity response $\dot{\xi}_i$ is a circle it follows that $\dot{x}_\ell$ also has a circular response locus.

Again

$$\tan\phi = y/x = 2\beta\gamma/(1-\beta^2) \tag{552}$$

$$\frac{d(\tan\phi)}{d\beta} = -\left[\frac{(1-\beta^2)2\gamma - 4\beta^2\gamma}{(1-\beta^2)^2}\right] \tag{553}$$

Hence $\dfrac{d(\tan\phi)}{d\beta}$ is a maximum at $\beta = 1$

Therefore the rate of change of ϕ is more rapid around $\beta = 1$ than at any other value of β . This fact is of great use experimentally as will now be discussed.

Suppose that a classically damped system is excited by a force distribution of given frequency. By changing the frequency of the force distribution, but keeping the magnitudes and phase of the forces constant a velocity response locus may be obtained for a particular mass. Should it happen that the force distribution was such as to excite a pure normal mode, the above theory predicts that the velocity response locus would be circular. However, if the natural frequencies of the system are fairly well separated, then around each natural frequency the velocity response will be a circular arc. This is so because the response of the mass is a superposition of the responses in each normal mode and around the natural frequency ω_n of one mode the phase angle ϕ_n corresponding to that mode is varying much more rapidly than the phase angles corresponding to the other modes. The response of the system may then be approximated as a summation of two vectors:

A constant vector in magnitude and phase corresponding to the contribution to the response of the mass from the modes whose natural frequencies are not near the operating frequency.

A vector, the tip of which describes a circular path, which corresponds to the contribution of that mode whose natural frequency is near the operating frequency.

As further assistance in identifying the natural frequency of a particular mode it should be remembered that a maximum value of the velocity often occurs near a natural frequency. This is so particularly if the force distribution is arranged to be favorable to the excitation of this mode.

From this discussion it is clear that the concept of the velocity response locus is of great interest in experimental work. A possible procedure for use with an N degree of freedom system in an effort to experimentally determine the natural frequencies and damping of each mode may now be described.

Choose a force distribution that is probably favorable to some particular mode. Plot the velocity response of one mass as the frequency of this force distribution is changed. Sketch in circular arcs where the phase shift of the response changes rapidly with frequency. An estimation of the natural frequency may be obtained by noting where the rate of change of the phase shift with frequency is a maximum. This frequency will, if the modes are well separated, be at the intersection of a line parallel to the axis with $\phi = \pi/2$ and the velocity response locus. To estimate the damping corresponding to a particular mode the simplest procedure is to plot a curve of the amplitude of the response in this mode, as given by the circle, against the frequency. Of course, only a partial amplitude frequency response is possible as the circle may only coincide with the response locus over a short frequency range. However, it is necessary to draw the complete circle as the point from which the amplitudes are measured is the intersection of a line from the natural frequency on the response curve through the center of the circle and the circle itself (Figures 11 and 12). From this velocity amplitude frequency curve a displacement amplitude frequency curve may be plotted by dividing each velocity amplitude by $\beta = \omega/\omega_n$ for this particular circle.

Having the displacement amplitude curve it is easy to determine the damping of this mode. The usual $\frac{1}{2}$ power law is generally accurate enough considering the inherent inaccuracies of drawing the circle and plotting the curves.

Sometimes a very small circular arc can only be obtained. This frequently occurs when the experimentor has limited selectivity as far as the magnitudes of the forces to be applied are concerned. Occasionally in rotating mass shaking machines the magnitudes of the force distribution applied are not constant over the frequency range. In the latter case it is necessary to correct for the changing force amplitude by dividing the magnitude of the responses by a suitable scaling factor. If the phase of the elements of the force distribution changes relatively to one another, the problem of determining the damping corresponding to any mode is much more difficult. However, if the phase shifts can be assumed to be slowly varying it may be possible to estimate the damping directly from the velocity response locus. This procedure may also be used when only a small arc can be obtained due to some limit of the experimental equipment. Assume the mode separation is such that the diameter through the natural frequency point on the response locus is approximately parallel to axis with $\phi = \pi/2$. Take two frequencies close to the natural frequency one above and one below. Measure the respective value of X from the response locus for each frequency. Denote these values by X_1 and X_2. From Equation (542)

$$X_1 = \alpha \frac{1-\beta_1^2}{(1-\beta_1^2)^2+4\beta_1^2\gamma^2} \tag{554}$$

$$X_2 = \alpha \frac{1-\beta_2^2}{(1-\beta_2)^2+4\beta_2^2\gamma^2} \tag{555}$$

where α is a constant of the system

$$\therefore \; X_1/X_2 = \frac{(1-\beta_1^2)}{(1-\beta_2^2)} \frac{\left[(1-\beta_2^2)^2+4\beta_2^2\gamma^2\right]}{\left[(1-\beta_1^2)^2+4\beta_1^2\gamma^2\right]} \tag{556}$$

From Equation (556) as X_1, X_2 may be measured β_1, β_2 may be calculated and thus an estimate of γ may be made.

Naturally the above procedure is only recommended in extreme cases. The ½ power determination of damping is to be preferred as in this method an experimental error is not very serious. In point of fact experimental errors are easily located by the distorted shape of the curve. Mode interference can be seen readily.

Structural Damping:

In passing an interesting phenomena may be noted. The equations of motion of a system with structural damping is often written as follows:

$$m\ddot{x} + igx + Kx = \tilde{F}e^{i\omega t} \tag{557}$$

where the spring constant ig implies that this force is in fact $\pi/2$ radians out of phase with the displacement x. It is not intended to discuss structural damping here but to obtain a solution to Equation (557)

Let $x = Re\,\tilde{A}e^{i\omega t}$ (558)

On substituting Equation (558) into Equation (557)

$$\tilde{A}(-\omega^2 M + ig + K) = \tilde{F} \tag{559}$$

$$\therefore \tilde{A} = \frac{\tilde{F}}{-\omega^2 M + K + ig} \tag{560}$$

$$= \frac{\tilde{F}/M}{\omega_n^2 - \omega^2 + i\, g/M}$$

$$= \frac{\tilde{F}/K}{(1-\beta^2) + i\, g/K} \tag{561}$$

To simplify the algebraic work let $\tilde{F}/K = 1$

$$\therefore \tilde{A} = \frac{1}{(1-\beta^2) + i\, g/K} \tag{562}$$

Let $\tilde{A}$ a complex number be defined

$$\tilde{A} = x + iy = |A| e^{i\phi} \tag{563}$$

From Equation (562)

$$x = \frac{(1-\beta^2)}{(1-\beta^2)^2 + g^2/K^2} \tag{564}$$

$$y = \frac{-g/K}{(1-\beta^2)^2 + g^2/K^2} \tag{565}$$

$$\therefore\ x^2 + y^2 = \frac{1}{(1-\beta^2)^2 + g^2/K^2} = -K/g\, y \qquad (566)$$

$$\therefore\ x^2 + y^2 + K/g\, y + \left(K/2g\right)^2 = \left(K/2g\right)^2 \qquad (567)$$

$$\therefore\ x^2 + \left(y^2 + K/2g\right)^2 = \left(K/2g\right)^2 \qquad (568)$$

Geometrically Equation (568) is a circle and thus the displacement response locus of a system with structural damping is circular. Whigley and Lewis [(6)] were the first to use the displacement response circle in connection with vibration tests on aircraft structures.

This is an interesting fact because it should be possible to experimentally determine the type of damping a structure actually possesses by plotting the response vector locus in both the velocity and displacement planes. However, the detailed mechanism of structural damping is in considerable doubt and much work needs to be done in this area before any such tests on structures would be justified. In actual fact structural damping may well be a type of hysteresis effect idealized by the bi-linear modal. It is difficult to say whether or not viscous damping in structures is even physically reasonable. To date few structures seem to have been tested beyond the linear range of Hooke's law and there appears to be a great lack of experimental data in this area of structural dynamics. One pressing need is better instrumentation to measure the relatively small changes in the stresses and the strains produced by modern shaking machines. Actual buildings are further complicated by the difficulty of an exact description of soil and foundation conditions.

Experimental Determination of Damping in Non-Classical Systems:

In non-classically damped systems the velocity response locus may again be used to give some estimate of the damping in each mode. As shown in Chapter 2, the first order approximation to the damping of any mode in these systems is the corresponding diagonal element of the reduced matrix $[Q]^T_{M,K}\,[C]\,[Q]_{M,K}$. Thus, to a certain approximation, non-classical systems may be represented by an equivalent classical system with damping in each uncoupled equation equal to the corresponding diagonal element of $[Q]^T_{M,K}\,[C]\,[Q]_{M,K}$. Later, some analogue work will be presented to show that this procedure is justified for systems with damping of from 0% to 20% critical in each mode.

Although it is not possible to force excite a pure mode in non-classical systems a resonance condition is achieved near each natural frequency. In an effort to determine the damping, in non-classically damped systems, experimentally with greater accuracy than the above method gives, some work has been done on the response loci of these systems. By expanding the expression for the contribution of one mode to the displacement of a particular mass it may be shown that this displacement response has a locus which is the sum of four vectors. The tips of these vectors have circular loci over the entire frequency range $(0 \rightarrow \infty)$. In point of fact these four circular loci are contained in two circles cutting one another at right angles. As this displacement response locus of non-classically damped systems has little practical significance, the details of the locus are not presented.

Analogue Computer Investigations:

There is no need to elaborate on the use of the electric analogue computer to simulate problems in mechanical vibrations. The nodal and loop analogies are the basic electrical-mechanical analogies and their properties are summarized in the following table:

Nodal Analogy

Electrical System	Mechanical System
Capacitance (C)	Mass (M)
1/Inductance (1/L)	Spring Constant (K)
1/Resistance (1/R)	Damping C
Currents (I)	Forces (F)
Voltages (E)	Velocities (V)

Loop Analogy

Electrical System	Mechanical System
Inductance (L)	Mass (M)
1/Capacitance (1/C)	Spring Constant (K)
Resistance (R)	Damping (C)
Voltages (E)	Forces (F)
Currents (I)	Velocities (V)

In the appendix a typical three degree of freedom system is represented by both the nodal and mesh analogies.

The analogue computer was used mainly to check the difficulty of determining the damping of a typical three degree of freedom system. At first a classical system was simulated on the computer and the velocity response was plotted. It was found that good circular loci were obtained around each natural frequency provided the force distribution was adjusted to suit the mode being excited. In fact it was relatively easy to miss a natural frequency by using a force distribution that practically excited a pure mode and so effectively damped out the contributions of the other modes. Later a non-classical system was simulated and the velocity response locus determined. The spring and mass matrix of this non-classical system were taken to be the same as the classical system previously examined. The damping matrix was so arranged that $[Q]^T_{M,K} [C] [Q]_{M,K}$ had the same diagonal terms as in the classical case. The forces applied were the same exactly as in the classical case which had been determined to suit the excitation of a certain mode. From these response curves the damping in each mode was determined and the values obtained give assurance that the results of the concept of equivalent classical damping have sufficient accuracy for engineering purposes.

Root Locus:

Suppose there exists a damped system characterized by $[M]$, $[K]$ and $\beta[C]$ where β is a scalar constant that varies continuously from $0 \to \infty$. If $[M]$, $[K]$ and $[C]$ remain fixed the locus of the eigenvalues of the system as β varies is said to be the root locus of the system. Now if the system is classically damped each uncoupled equation has the form:

$$\bar{M}_{ii}\ddot{\xi} + \beta C_{ii}\dot{\xi} + K_{ii}\xi = 0 \qquad (569)$$

eigenvalue corresponding to this equation

$$\alpha_i = \frac{-\beta\bar{C}_{ii} \pm \sqrt{(\beta\bar{C}_{ii})^2 - 4\bar{m}_{ii}\bar{K}_{ii}}}{2\bar{m}_{ii}} \qquad (570)$$

$$= \frac{-\beta\bar{C}_{ii} \pm i\sqrt{4\bar{m}_{ii}\bar{K}_{ii} - (\beta\bar{C}_{ii})^2}}{2\bar{m}_{ii}} \qquad (571)$$

From Equation (571) it is easy to see that the locus of α_i is a semi-circle on the left half plane of radius $\sqrt{\bar{K}_{ii}/\bar{m}_{ii}} = \omega_{ii}$. In passing it may be noted that in classical systems the set of eigenvectors remain the same independent of the value of β.

From consideration of $[u]$ the $2N \times 2N$ matrix it is easily seen that eigenvalues in non-classically damped systems have in general no simple locus. It is clear that the eigenvectors of such a system change continuously as β changes.

In an effort to show the great differences between classical and non-classical systems as far as root locus is concerned a plot of the roots of the system treated on the analogue computer is shown in the appendix. This plot shows that each α_i has a negative real part for all β and for particular values of β these roots may be purely negative.

Discussion and Conclusions

In this work an effort has been made to show the wide range of application of normal mode techniques to the solution of linear damped parameter systems. The only case where a normal mode solution cannot be obtained is if there are less than 2N independent eigenvectors in the case of repeated roots of the frequency equation of non-classically damped systems solved by Foss's method. This latter case may best be solved by integral transform methods. It is always possible to excite a system to vibrate in a pure mode by a suitable choice of initial conditions. However, it is not possible to force excite a pure mode with zero initial conditions in non-classically damped systems. The normal modes of both a classically and non-classically damped system satisfy certain orthogonality conditions. The condition for the existence of classical damping is that derived by Caughey, namely, the damping matrix must be diagonalized by the same transformation as simultaneously diagonalizes both the mass and spring matrices.

In classical systems increasing the damping decreases the natural frequency of the system. In the case of non-classical damped systems it is possible that the introduction of damping will increase the natural frequencies of the system. However, the highest frequency of the damped non-classical case must be less than, or equal to the highest natural frequency of the undamped system.

The experimental work shows that the concept of equivalent classical damping for non-classically damped systems is accurate enough for engineering purposes. By performing the experiments that are presented above, it is possible to completely specify a multi-degree of freedom system. These experiments should be of use when trying to determine an approximate spring mass dash pot equivalent circuit for a complex structure.

As regards future work there is little need to stress the desirability of attempting to extend some of these ideas to continuous systems. However, in such an attempt certain difficulties arise immediately. Matrices are replaced by functions and in general it is much more difficult to work in function space than in N dimensional space. The real advantage of working with discrete systems is that elementary matrix theory is sufficient for most phases of the analysis.

Appendix

Experimental Results

Before investigating a three degree of freedom system it was desirable to ascertain the accuracy of the analogue computer under normal operation. As a check on this accuracy the velocity response spectrum of a single degree of freedom system was determined. At this time the nodal analogy was used, but in later experimental work the loop analogy had to be adopted because the current generators developed some instability in operation.

The circuit used is as shown in Figure 4. At a given frequency the magnitude and phase of the current I fed into the circuit and of the voltage E across the capacitor C_{14} were recorded. The phase angles were measured relative to a fixed standard and so the phase angle between the current and the voltage could be obtained at any frequency. In the nodal analogy the voltage across a capacitor corresponds to the velocity of a mass in the analogous mechanical system. The current flowing into the circuit corresponds to the force applied at the mass in the single degree of freedom system.

As can be seen from the circular velocity response locus plot Figure 4 the accuracy of the computer justifies its use in further experimental work.

The main experimental effort was devoted to the determination of the natural frequencies and the percentage of critical damping in each mode of classically and non-classically damped three degree of freedom systems. In this case the non-classical damping matrix $[C']$ was so chosen that the diagonal terms of

$$[Q]^T_{M,K}\,[C']\,[Q]_{M,K} \tag{572}$$

were identical with the corresponding diagonal terms of

$$[Q]^T_{M,K}\,[C]\,[Q]_{M,K} \tag{573}$$

where $[C]$ is the classically damped matrix, $[Q]_{M,K}$ is the matrix which simultaneously diagonalizes $[M]$ and $[K]$. The parameters of the system were selected so that the computer was operating at a current level where parasitic resistence was negligible.

$$[M] = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{574}$$

$$[K] = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & -0.5 \\ -2 & -0.5 & 3 \end{bmatrix} \tag{575}$$

and $[C]$ the classically damped matrix is chosen to satisfy Rayleigh's condition

$$[C] = \alpha [M] + \beta [K]$$

$$[C] = 20 \begin{bmatrix} 14 & -3 & -6 \\ -3 & 10 & -1.5 \\ -6 & -1.5 & 11 \end{bmatrix} \tag{576}$$

Here $\alpha = 160$; $\beta = 6 \times 10^{-5}$

To obtain the natural frequencies of this system it is necessary to solve the frequency equation

$$\left\| \left[-\omega^2 I + [M]^{-1} [K] \right] \right\| = 0 \tag{577}$$

From Equation (574) and Equation (575)

$$[M]^{-1} [K] = 2 \begin{bmatrix} 8 & -2 & -4 \\ -1 & 2 & -0.5 \\ -4 & -1 & 6 \end{bmatrix} 10^6 \tag{578}$$

On substituting Equation (578) into Equation (577) and after solving the resulting cubic equation in ω^2 it is easy to show that the three natural frequencies of the system are

$$\begin{aligned} \omega_1 &= 215 \text{ cycles per second} \\ \omega_2 &= 445 \text{ cycles per second} \\ \omega_3 &= 752 \text{ cycles per second} \end{aligned} \tag{579}$$

Letting $\{\phi^i\}$ = normal mode corresponding to ω_i

$$\{\phi^i\} = \begin{Bmatrix} a_i \\ b_i \\ 1 \end{Bmatrix} \tag{580}$$

From Equation (577) and Equation (578) these $\{\phi^i\}$'s must satisfy equations of the following type:

$$\begin{bmatrix} 4\times10^6-\omega_i & 10^6 & -2\times10^6 \\ -5\times10^5 & 10^6-\omega_i & -2.5\times10^6 \\ -2\times10^6 & -5\times10^5 & 3\times10^6-\omega_i \end{bmatrix} \begin{Bmatrix} a_i \\ b_i \\ 1 \end{Bmatrix} = \{0\} \tag{581}$$

From Equation (581)

$$a_i = \frac{8\times10^6-2\omega_i}{8\times10^6-\omega_i} \tag{582}$$

$$b_i = \frac{4\times10^6-\omega_i}{4.5\times10^6-4\omega_i} \tag{583}$$

On substituting the values of ω_i derived from Equation (577) into Equation (582) and Equation (583) $[Q]_{M,K}$ may be determined

$$[Q]_{M,K} = \begin{bmatrix} 1 & 0.67663 & 0.70831 \\ -0.06761 & 0.61616 & 1 \\ -0.75983 & 1 & 0.75421 \end{bmatrix} \tag{584}$$

It may be shown that

$$[Q]_{M,K}[C][Q]_{M,K} = 20 \begin{bmatrix} 29.76603 & 0 & 0 \\ 0 & 17.43652 & 0 \\ 0 & 0 & 10.35794 \end{bmatrix} \tag{585}$$

$$[Q]^T_{M,K}[M][Q]_{M,K} = \frac{1}{4}\begin{bmatrix} 1.58648 & 0 & 0 \\ 0 & 2.21713 & 0 \\ 0 & 0 & 3.07054 \end{bmatrix} \tag{586}$$

$$[Q]^T_{M,K}[K][Q]_{M,K} = \begin{bmatrix} 8.86434 & 0 & 0 \\ 0 & 4.33408 & 0 \\ 0 & 0 & 1.40563 \end{bmatrix} \tag{587}$$

A single degree of freedom system is said to be critically damped if

$$C = C_c = 2\sqrt{KM} \tag{588}$$

Percentage of critical damping possessed by any single degree of freedom system is defined to be

$$C/C_c \times 100\% \tag{589}$$

As each uncoupled equation is in effect a single degree of freedom system these ideas may be carried over to multi-degree of freedom classically damped systems. Let C_{c_i} = critical damping of i^{th} mode. From Equation (586) and Equation (587)

$$\begin{aligned} C_{c_1} &= 3.7481 \times 10^3 \\ C_{c_2} &= 3.0998 \times 10^3 \\ C_{c_3} &= 2.0775 \times 10^3 \end{aligned} \tag{590}$$

From Equation (585), Equation (589) and Equation (590)

Percentage critical damping in 1st mode   9.97%

Percentage critical damping in 2nd mode   11.25%

Percentage critical damping in 3rd mode   15.88%

For the non-classically damped case the following damping matrix was used

$$[C'] = 20 \begin{bmatrix} 13.63444 & -4.56722 & -6.84108 \\ -4.56722 & 14.64898 & -1.65492 \\ -6.84108 & -1.65492 & 9.04290 \end{bmatrix} \qquad (587)$$

It may be shown

$$[Q]^T_{M,K} [C'] [Q]_{M,K} = 20 \begin{bmatrix} 29.7660 & 2.0112 & -1.0064 \\ 2.0112 & 17.4365 & -5.9935 \\ -1.0064 & -5.9935 & 10.3579 \end{bmatrix} \qquad (588)$$

and so on comparing Equation (588) with Equation (585) it is seen that the corresponding diagonal terms are the same. In passing it should be noted that although most of the off diagonal terms are small the ratio of the largest off diagonal term to the smallest diagonal term is 0.58. This means that there is considerable coupling between the second and third modes.

To find the normal modes in the non-classical case it is necessary to determine the eigenvalues of

$$\begin{bmatrix} 0 & 0 & 0 & 10^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^6 \\ -0.14375 & -0.20000 & -0.11250 & -58.68840 & -49.77454 & 10.52568 \\ -0.1000 & -0.40000 & -0.10000 & 18.72880 & -185.60656 & 8.8646 \\ -0.11250 & -0.2000 & -0.17500 & 9.60320 & -52.91798 & -51.79152 \end{bmatrix} = [u] \qquad (589)$$

Solving the frequency equation

$$\left\|\left[[u] - \lambda I\right]\right\| = 0 \tag{590}$$

$$\begin{aligned}
\lambda_{1,4} &= (-0.33605 \pm i\,2.08947) \times 10^{-4} \\
\lambda_{2,5} &= (-0.40398 \pm i\,3.56257) \times 10^{-4} \\
\lambda_{3,6} &= (-0.73536 \pm i\,7.33178) \times 10^{-4}
\end{aligned} \tag{591}$$

From Equation (591) $\alpha_{1,4}$, $\alpha_{2,5}$ and $\alpha_{3,6}$ may be determined. It should be noted that each α_i has a negative real part which corresponds to stable motion. The imaginary part of each α_i corresponds to the natural frequency. There are three natural frequencies for this system.

$$\begin{aligned}
f_1 &= 215 \text{ cycles/second} \\
f_2 &= 441 \text{ cycles/second} \\
f_3 &= 742 \text{ cycles/second}
\end{aligned} \tag{592}$$

On comparing Equation (592) with Equation (579) it is seen that the non-classical natural frequencies are very close to the classical natural frequencies. This fact will later be borne out by the velocity response plots for the two cases.

As is known from the theory the contribution of any mode to the response of the system with a constant amplitude forcing function is largest when the frequency of the forcing function is near the natural frequency of the mode and the forcing function distribution is so adjusted as to predominately excite this mode. Bearing these facts in mind the forcing function was changed in going from one frequency range to another to facilitate the determination of the damping in each mode.

In order to excite a pure mode in a classically system the forcing function $\{F(t)\}$ is given by

$$\{F_i(t)\} = g(t)\,[M]\,\{q^i\} \tag{593}$$

where $g(t)$ is an arbitrary scalar function of time, $\{q^i\}$ is the mode being excited. From Equation (584) and Equation (574)

$$\{F_1\} = g_1(t)\begin{Bmatrix} 0.70831 \\ 2 \\ 0.75421 \end{Bmatrix} \tag{594}$$

$$\{F_2(t)\} = g_2(t)\begin{Bmatrix} 0.67663 \\ -1.23232 \\ 1 \end{Bmatrix} \tag{595}$$

$$\{F_3(t)\} = g_3(t)\begin{Bmatrix} 1 \\ -0.13522 \\ -0.75982 \end{Bmatrix} \tag{596}$$

The transformer settings were arranged to give the relative amplitude of each force in the force distributions. The scalar functions of time $g_i(t)$, were of type $A_i \sin \omega t$.

Where the A_i 's were determined experimentally on the computer by adjusting the output of the power amplifier until a desired level of response was obtained.

Although there is no possibility of force exciting a pure mode in the non-classically damped system the forcing functions given by Equation (594), Equation (595) and Equation (596) were used.

The experimental results for the classical system are shown in Figures 5, 6 and 7. As can be seen practically pure modes were excited in each case. From these curves the damping was estimated by the circle method developed above. Comparisons between the natural frequency and damping derived by the circle method and the exact calculated valuesare shown below. Generally the peak amplitude method of calculation of the damping has been used in the past. It should be noted that in the cases shown due to the excitation of practically pure modes the peak amplitude method has a better chance of being accurate than in cases where a large contribution from interfering modes is present. In the latter cases the circle method will quickly show up the interfering modes and much more accurate results will be obtained by the use of the circle method than the peak amplitude method.

The experimental results for the non-classical case is shown in Figures 8, 9 and 10. As can be seen these response loci are practically circular and the concept of equivalent classical damping appears to be justified. These loci are treated by the circle method to determine the equivalent damping in each mode. Comparison with experimental results and actual calculated values are shown below.

Single Degree of Freedom System

Nodal Analogy

	Input			Output		
freq.	I	ϕ_I	E	ϕ_E	E/I	$\phi_E - \phi_I$
100	5.0	241.5	3.57	323	.714	81.5
120	5.0	242	4.50	323	.90	81.0
140	5.0	246	5.65	322.5	1.13	76.5
160	5.0	248	6.95	322	1.39	74.0
180	5.0	251.5	8.70	322	1.74	70.5
200	5.0	259	10.9	322	2.18	63.0
200	5.0	259.5	10.6	322	2.12	62.5
220	5.0	266.0	13.3	322	2.66	56.0
240	5.0	276.0	17.2	322	3.44	46.0
260	5.0	293.0	21.2	321.5	4.24	28.5
280	5.0	313.0	24.0	321.5	4.80	8.5
300	5.0	327.5	23.5	321.5	4.70	- 6.0
320	5.0	351.0	21.2	321.5	4.24	-29.5
340	5.0	3.0	18.0	321.5	3.60	-41.5
360	5.0	12.5	15.3	321.5	3.06	-51.0
380	5.0	18.0	13.1	321.5	2.62	-56.5
380	5.0	16.5	13.2	322.0	2.64	-54.5
400	5.0	22.0	11.7	322.0	2.34	-60.0
420	5.0	25.0	10.3	322.0	2.06	-63.0
440	5.0	26.5	9.5	322	1.90	-64.5
460	5.0	29.0	8.6	321.5	1.72	-67.5
480	5.0	29.1	7.95	321.5	1.59	-67.6
500	5.0	31.1	7.3	321.5	1.46	-69.6
550	5.0	33.0	6.3	321.5	1.26	-71.5
600	5.0	35.0	5.4	321.5	1.08	-73.5
700	5.0	38.0	4.4	321.5	.88	-76.5
800	5.0	41.5	3.7	321.5	.74	-80.0
900	5.0	41.5	3.15	321.5	.63	-80.0
1000	5.0	43.0	2.8	321	.56	-82.0

Force Distribution $\begin{Bmatrix} 3.4 \\ 10.0 \\ 3.7 \end{Bmatrix}$ to Excite 1st Mode   Loop Analogy

Damping: Classical   3 Degree of Freedom System

Frequency	Input		Response		
	E	ϕ_E	I	ϕ_I	$360 - \phi_I + \phi_E$
100	10	21	8.7	288	93
120	10	20	11.3	284	96
140	10	20	15.6	279	101
160	10	19	22.2	272	107
180	10	17	34.7	258	119
200	10	15	55.9	230	145
220	10	17.0	65.2	186	191
240	10	21.0	47.0	156	225
260	10	22.0	33.3	143	239
280	10	22.0	25.0	135	247
300	10	21.0	20.0	131	250
320	10	21.0	17.2	128	253
340	10	21	14.8	125	256
360	10	21	13.0	123	258
380	10	20	11.8	122	258
400	10	20	10.6	121	259
420	10	20.5	10.0	120	260.5
440	10	20.0	9.6	119.5	260.5
460	10	20.5	8.9	119	261.5
480	10	20	8.4	118	262
500	10	19	7.9	117	262
520	10	19	7.42	117	262
540	10	20	7.1	116	264
560	10	20	6.8	116	264
580	10	20	6.4	115	265
600	10	20	6.2	115	265
620	10	20	5.9	115	265
640	10	20	5.65	115	265
660	10	20	5.5	114	266
680	10	19	5.15	115	264
700	10	18.5	5.05	114.5	264
750	10	20	4.56	114.0	266
800	10	20	4.33	113.5	266.5
850	10	20	4.03	113.0	267
900	10	19	3.80	112.5	266.5
950	10	19	3.55	112.5	266.5
1000	10	19	3.25	112.5	266.5

Force Distribution $\begin{Bmatrix} 6.6 \\ -12.5 \\ 10 \end{Bmatrix}$ to Excite 2nd Mode   Loop Analogy

Damping: Classical   3 Degree of Freedom System

Frequency	Input	Response			
	E	ϕ_E	I	ϕ_I	$360 - \phi_I + \phi_E$
100	10	22	3.55	291	91
120	10	21.5	3.95	289	92.5
140	10	21.0	4.70	288	93
160	10	21.0	5.6	287.5	93.5
180	10	20.5	6.7	291.0	89.5
200	10	21	8.0	290.0	91
220	10	19	10.0	288.0	91
240	10	18.5	11.0	282	96.5
260	10	19.0	12.5	278.5	100.5
280	10	19.0	14.1	276.5	102.5
300	10	18.5	16.7	273.0	105.5
320	10	17.0	19.2	269	108
340	10	17.0	23.2	264	113
360	10	14.5	28.4	257	117.5
380	10	12.0	34.8	247	125
400	10	10.0	42.5	234	136
420	10	8.0	52.5	216	152
440	10	10	58.2	196	174
460	10	14.5	56.2	177	197.5
480	10	17.0	50.0	165	212
500	10	20.0	41.3	155	225
520	10	20.0	35.7	148	232
540	10	21.0	30.9	143	238
560	10	23.0	26.7	139	244
580	10	21.0	23.8	135	246
600	10	22.5	21.3	132	250.5
620	10	22.5	19.5	130	252.5
640	10	21.5	17.8	129	252.5
660	10	21.0	16.5	128	253
700	10	21.0	14.5	126	255
750	10	21.0	12.3	125	256
800	10	20.0	11.0	123	257
850	10	21.0	10.0	122	259
900	10	21.0	9.3	121	260
950	10	21.0	8.6	120	261
1000	10	20.0	7.9	119	261

Force Distribution $\begin{Bmatrix} 10 \\ -1.36 \\ -7.6 \end{Bmatrix}$ to Excite 3rd Mode   Loop Analogy

Damping: Classical   3 Degree of Freedom System

Frequency	Input		Response		
	E	ϕ_E	I	ϕ_I	$360 - \phi_I + \phi_E$
100	10	21	1.24	291	90
125	10	22	1.59	289	93
150	10	21	1.98	288	93
175	10	21.5	2.56	283	99
200	10	21	2.80	277.5	103.5
225	10	21.0	2.61	273	101
250	10	20	2.82	283.0	97
275	10	20.0	3.31	283	97
300	10	20	3.80	283.5	96.5
340	10	20	4.62	282.0	98
360	10	19	5.08	281.0	98
380	10	19	5.55	278.5	100.5
400	10	18	6.00	278.5	99.5
420	10	18.5	6.61	278.5	100
440	10	18.5	7.20	277.0	101.5
460	10	18.0	7.80	275.0	103
480	10	17.5	8.5	275.0	102.5
500	10	17.0	9.3	273.0	104
520	10	16.0	10.0	272	104
540	10	16.5	10.8	269	107.5
560	10	17.0	11.8	266.5	110.5
580	10	15.5	13.2	264.0	111.5
600	10	15.0	14.4	261.0	114
620	10	14.0	16.0	256.0	118
640	10	14.0	17.5	251.0	123
660	10	12.0	19.2	243.0	129
680	10	11.0	21.0	236.0	135
700	10	12.0	22.5	224.0	148
720	10	10.0	23.7	214.0	156
740	10	8.0	24.1	203.0	165
760	10	8.0	24.0	191.0	177
780	10	8.0	23.7	183.0	185
800	10	9.0	22.7	178.0	191
820	10	8.0	21.2	170.0	198
840	10	8.5	20.0	165.0	203.5
860	10	10.0	18.8	160	210
880	10	11.0	17.3	156	215
900	10	11.0	16.1	154	217
920	10	10.0	15.2	150	220
940	10	11.0	14.2	148	223
960	10	10.0	13.3	147	223
1000	10	10.0	12.9	145	225

Force Distribution $\begin{Bmatrix} 3.5 \\ 10 \\ 8.7 \end{Bmatrix}$ to Excite 1st Mode   Loop Analogy

Damping: Generalized   3 Degree of Freedom System

Frequency	Input		Response		
	E	ϕ_E	I	ϕ_I	$360-\phi_I+\phi_E$
100	10	22	8.7	299	83
120	10	21	11.3	293	88
140	10	20	15.8	286	94
160	10	19	22.5	270	109
180	10	17.5	34.0	257	120.5
200	10	15	54.3	229	146
220	10	18	66.2	188	190
240	10	22	48.0	156	226
260	10	23	33.8	140	243
280	10	21	25.1	131	250
300	10	22	20.0	127	255
320	10	21	17	125	256
340	10	22	14.6	120	262
360	10	21	12.7	117	264
380	10	20	11.2	116	264
400	10	20	9.9	115	265
420	10	20	8.6	114.5	265.5
440	10	20	7.6	117.5	262.5
460	10	19	7.3	123.0	256
480	10	19	7.2	124.5	254.5
500	10	20	7.1	123.0	257
520	10	19	6.95	123.0	256
540	10	20	6.7	122	258
560	10	20	6.42	121	259
580	10	19	6.10	120	259
600	10	19	5.95	119	260
620	10	19	5.78	119	260
640	10	20	5.58	118.5	261.5
660	10	19	5.4	117	262
680	10	19	5.1	117	262
700	10	20	5.0	117	263
750	10	19	4.6	115	264
800	10	18	4.4	116	262
850	10	19	4.0	116	263
900	10	19	3.75	114.5	264.5
950	10	19.5	3.58	115	264.5
1000	10	19.0	3.35	114	265

Force Distribution $\begin{Bmatrix} 6.6 \\ -12.6 \\ 10 \end{Bmatrix}$ to Excite 2nd Mode   Loop Analogy

Damping: Generalized   3 Degree of Freedom System

Frequency	Input		Response		
	E	ϕ_E	I	ϕ_I	$360 - \phi_I + \phi_E$
100	10	22	3.28	294	88
120	10	22	4.05	294	88
140	10	22.5	4.90	295	87.5
160	10	22.0	6.0	296	86
180	10	21	7.5	300	81
200	10	21	10.1	294	87
220	10	20	12.8	285	95
240	10	20	13.3	275	105
260	10	19	14.1	271	108
280	10	19	16.0	270	109
300	10	17	17.8	267	110
320	10	17	20.5	265	112
340	10	16	24.3	261	115
360	10	15	29.5	252	123
380	10	12	35.5	244	128
400	10	10	46.5	229	141
420	10	8	56.2	213	155
440	10	10	62.0	191.5	178.5
460	10	14	59.0	173.0	201
480	10	19	50.7	160.0	219
500	10	20	42.5	150.0	230
520	10	21	36.5	145.0	236
540	10	21	31.2	140	241
560	10	23	27.0	137	246
580	10	23	24.2	133	250
600	10	22	21.8	132	250
620	10	21	19.9	129	252
640	10	22	18.3	127	255
660	10	23	17.0	126	257
680	10	23	16.0	125	258
700	10	22	15.0	125	257
750	10	23	12.8	121	262
800	10	21	11.3	119	262
850	10	21	10.2	118	263
900	10	21	9.4	118	263
950	10	22	8.7	117	265
1000	10	22	8.0	116	266

Force Distribution $\begin{Bmatrix} 10 \\ -1.25 \\ -7.4 \end{Bmatrix}$ to Excite 3rd Mode   Loop Analogy

Damping: Generalized   3 Degree of Freedom System

Frequency	Input		Response		
	E	ϕ_E	I	ϕ_I	$360 - \phi_I + \phi_E$
100	10	22	1.23	289	93
125	10	22	1.60	289	93
150	10	21.5	1.96	289	92.5
175	10	22.0	2.42	286	96
200	10	20	2.95	279	101
225	10	20	2.87	272	108
250	10	20	2.93	277	103
275	10	20	3.35	279	101
300	10	19	3.80	279	100
320	10	19	4.15	278	101
340	10	19	4.6	278	101
360	10	19	5.0	276.5	102.5
380	10	18	5.4	275	103
400	10	18	5.8	274	104
420	10	18	6.2	273	105
440	10	18	6.67	273	105
460	10	18	7.30	273	105
480	10	18	8.15	273	105
500	10	17	8.90	270	107
520	10	16	9.90	267	109
540	10	16	10.07	265	111
560	10	16	11.8	261	115
580	10	16	12.8	257.5	118.5
600	10	15	14.3	254	121
620	10	15	16.0	248	127
640	10	15	17.7	241	134
660	10	14	19.2	235	139
680	10	13	21.0	227.5	145.5
700	10	12	22.5	220	152.0
720	10	11	23.5	210	161.0
740	10	11.5	23.8	204	167.5
760	10	10.5	24.0	193	177.5
780	10	11.5	23.5	185	186.5
800	10	11.0	22.5	179	192
820	10	10.0	21.3	173	197
840	10	10.0	19.8	167	203
860	10	12.0	18.8	164	208
880	10	12.0	17.5	158	214
900	10	12.0	16.3	154	218
920	10	12.0	15.2	152	220
940	10	12.5	14.0	149	223.5
960	10	12.5	13.5	148	224.5
980	10	13.0	12.6	145	228
1000	10	13.0	12.7	145	228

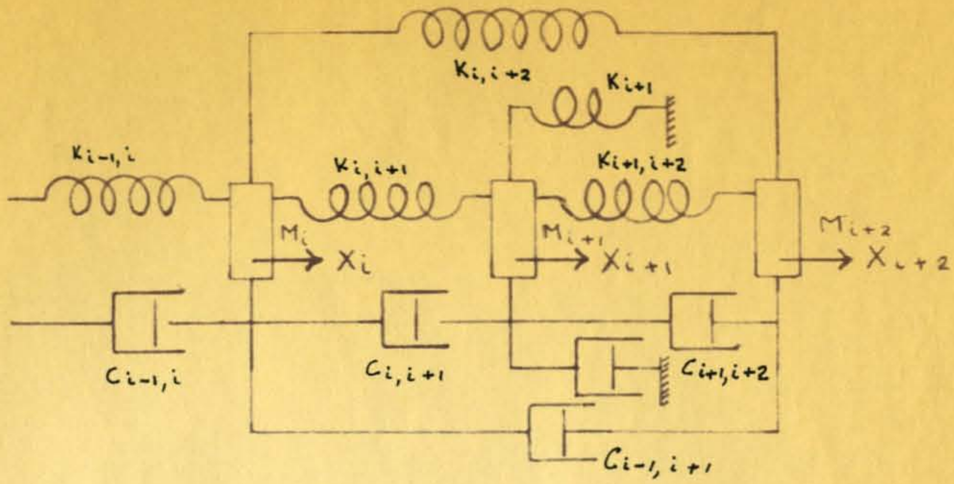


Figure I

General Mass, Spring and Dashpot System.

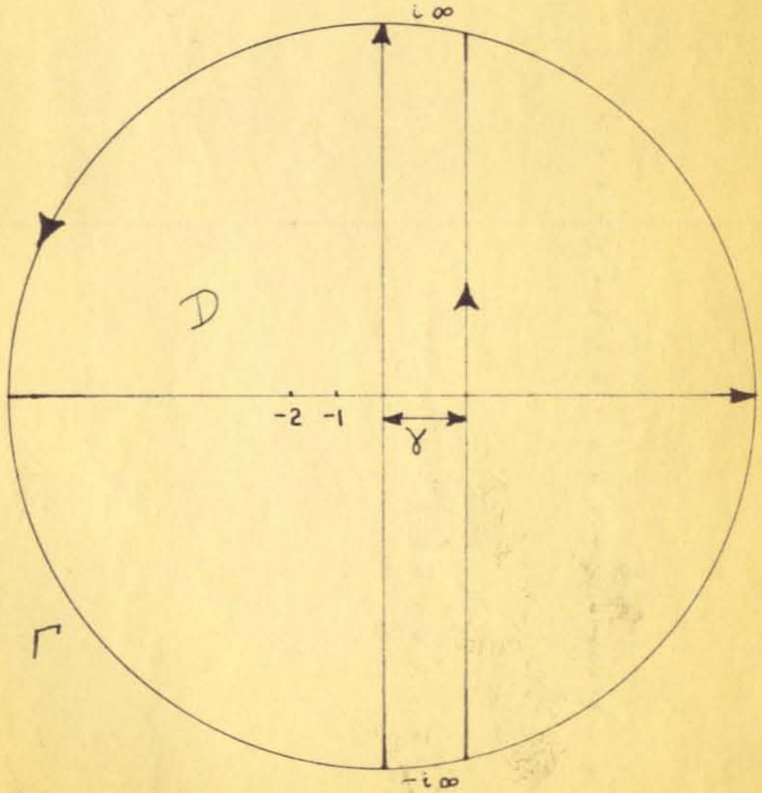
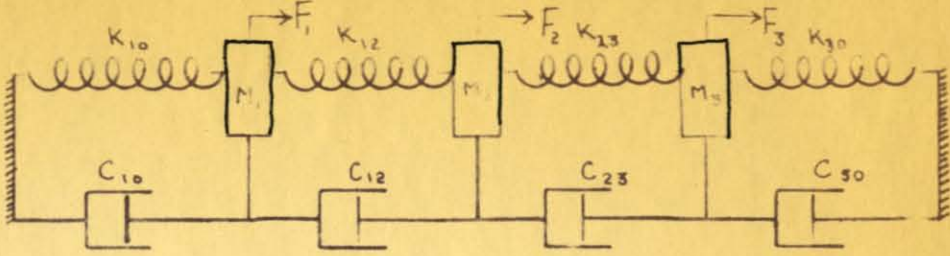
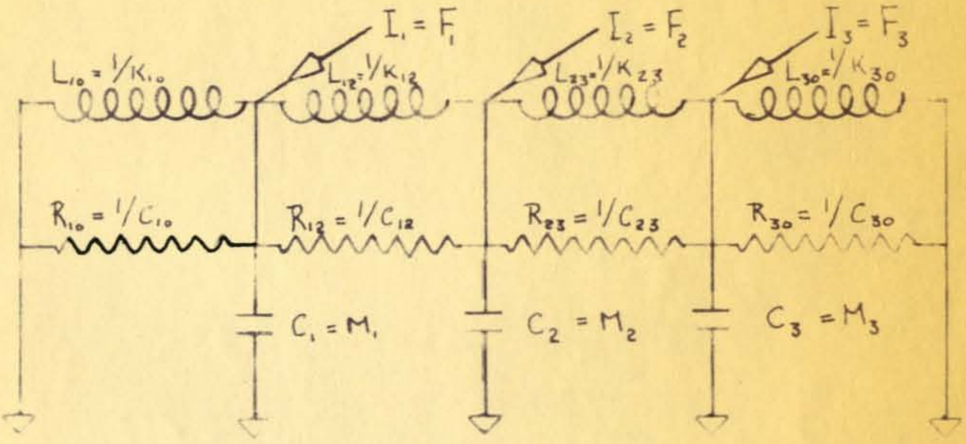


Figure 2

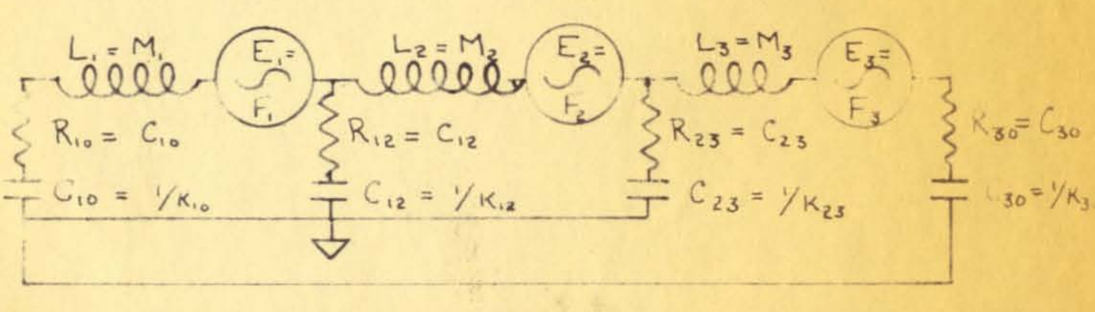
Complex p plane



Mechanical System



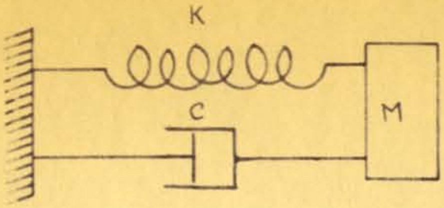
Nodal Analogy



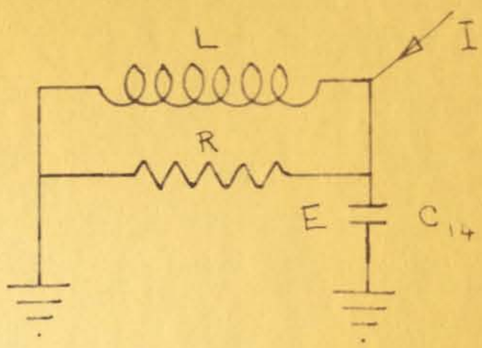
Loop Analogy

Figure 3

3 Degrees of Freedom System



Mechanical System



Electrical Nodal Analogy

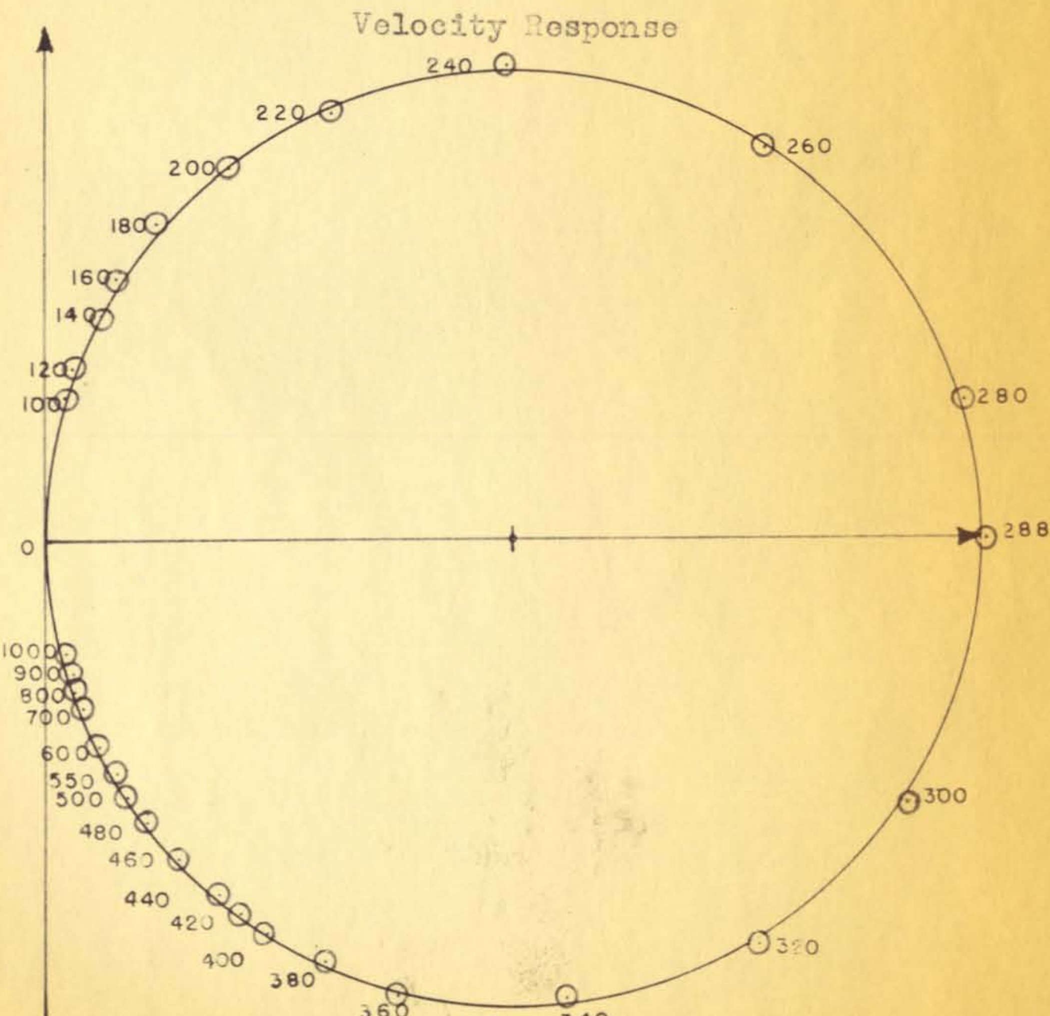
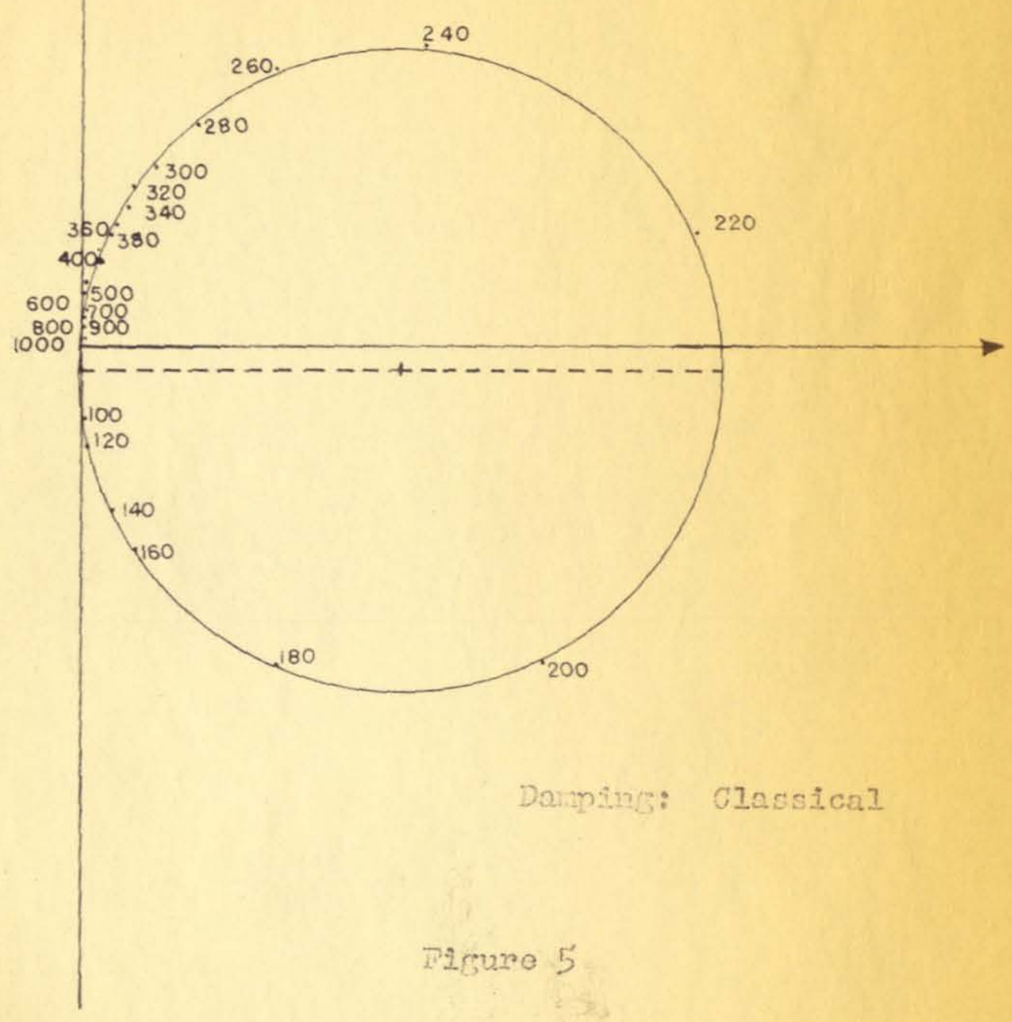


Figure 4

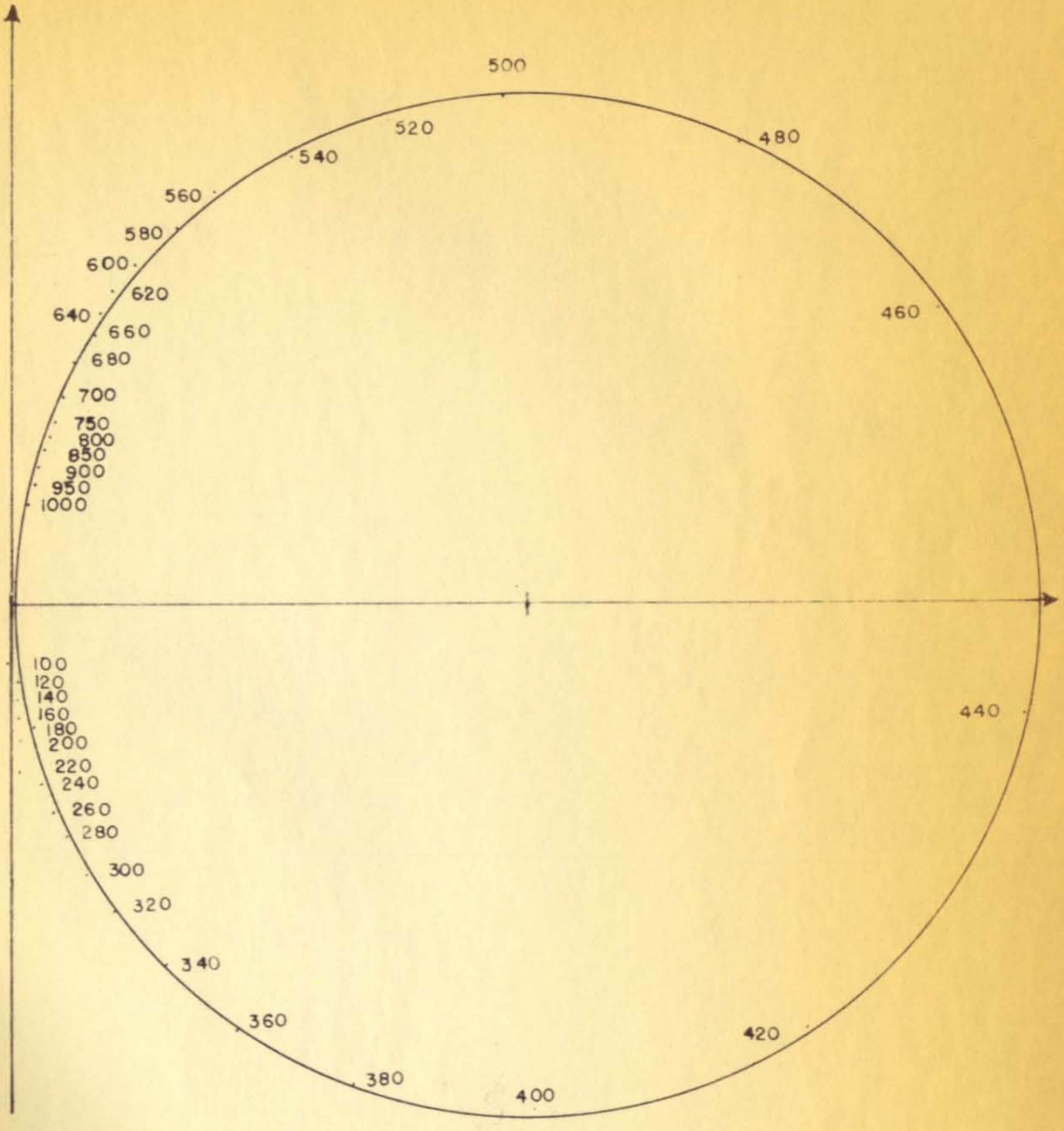
Single Degree of Freedom System



Damping: Classical

Figure 5

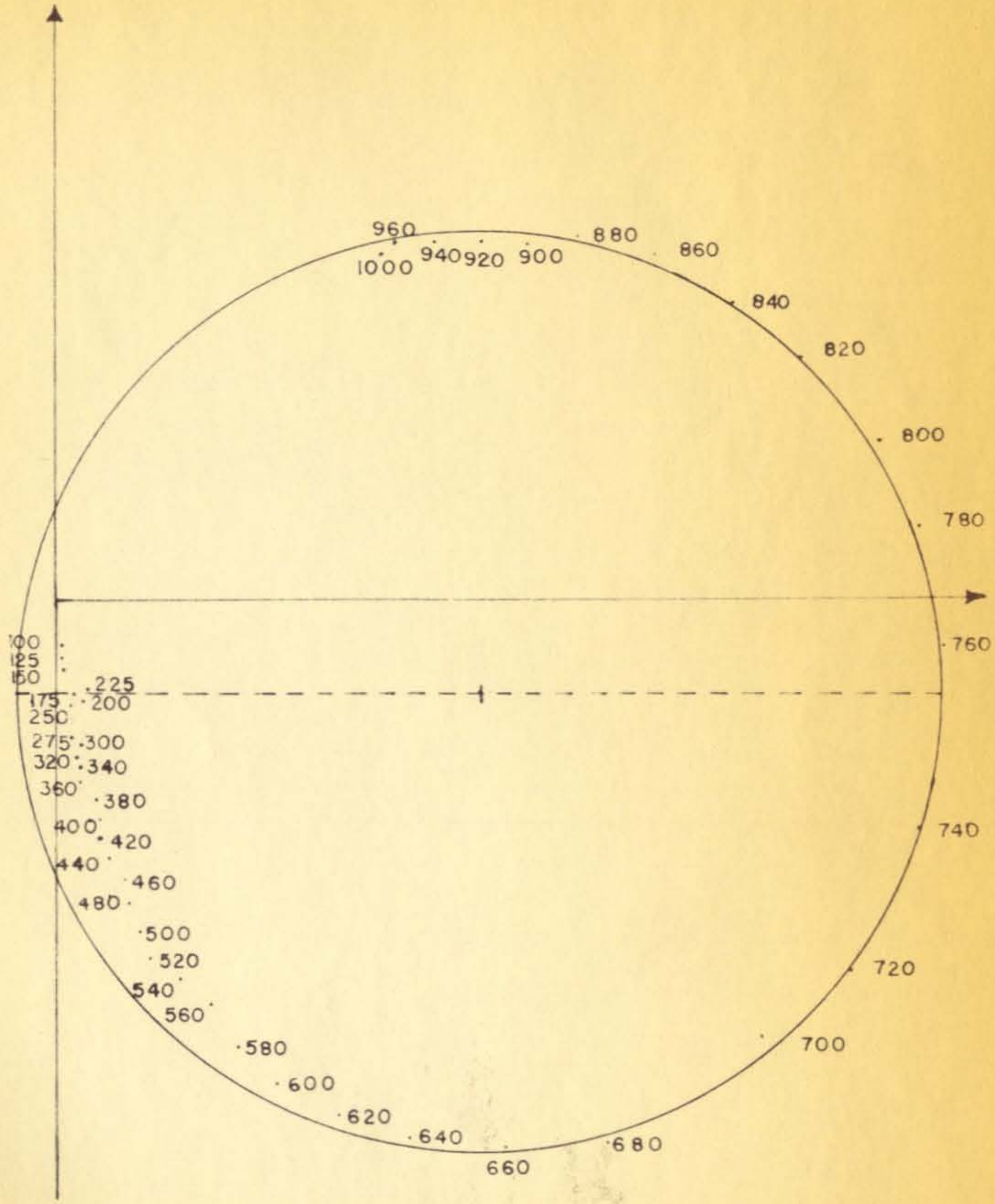
Velocity Response of a 3 Degree of Freedom System so excited as to be predominantly vibrating in its 1st mode.



Damping: Classical

Figure 6

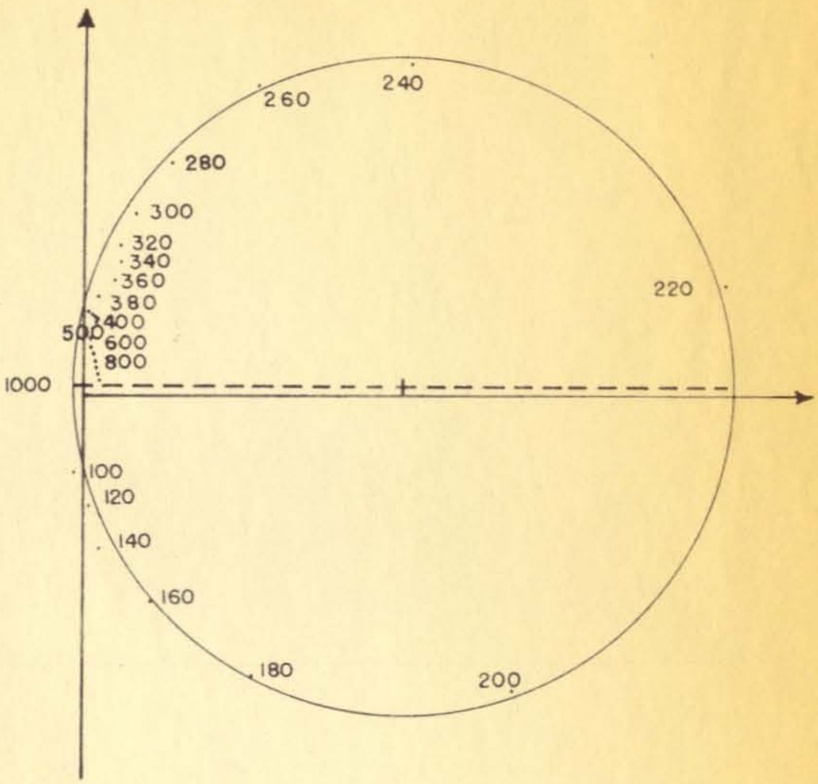
Velocity Response of a 3 Degree of Freedom System so excited as to be predominantly vibrating in its 2nd mode.



Damping: Classical

Figure 7

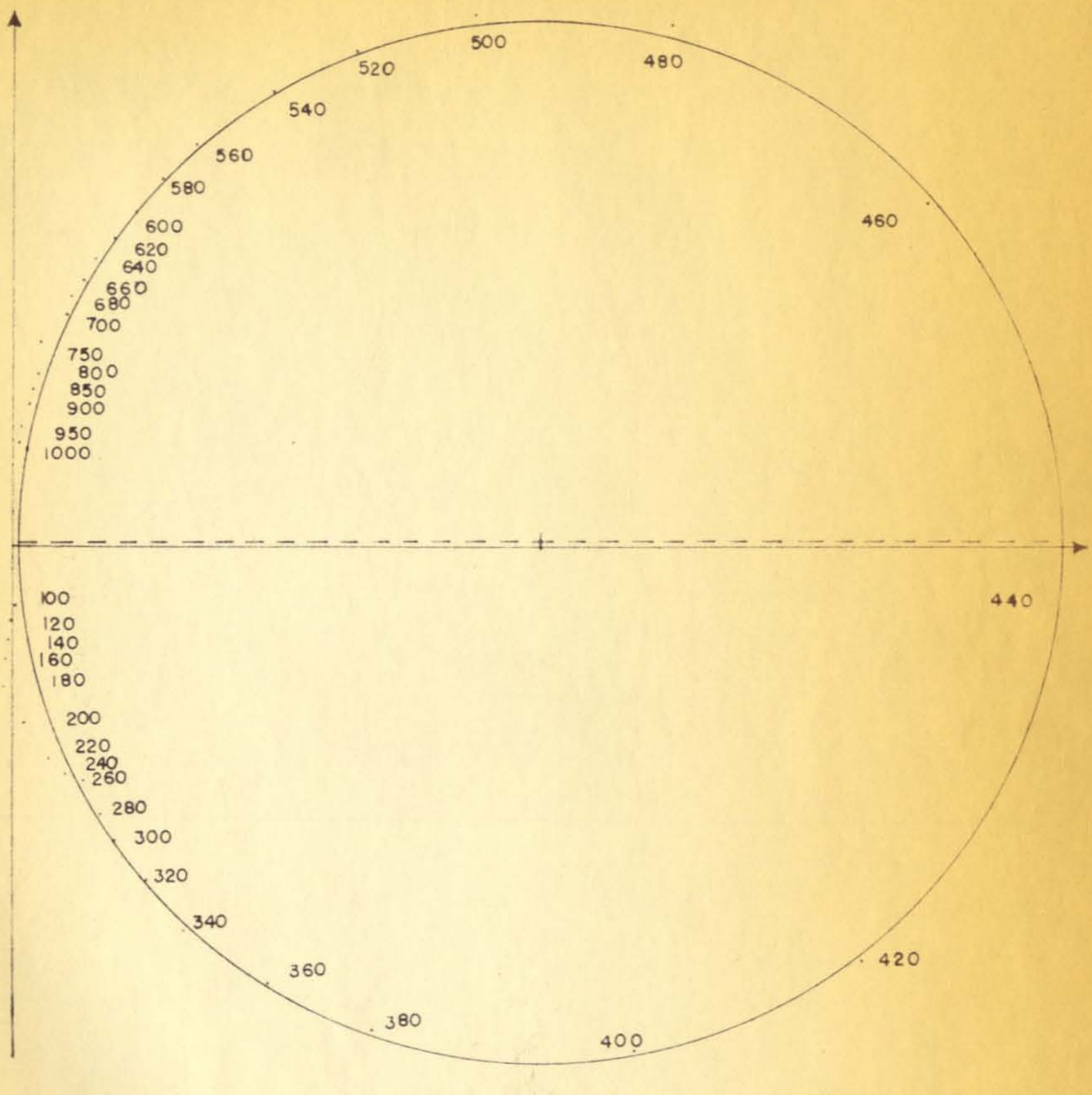
Velocity Response of a 3 Degree of Freedom System so excited as to be predominantly vibrating in its 3rd mode.



Damping: Non-Classical

Figure 8

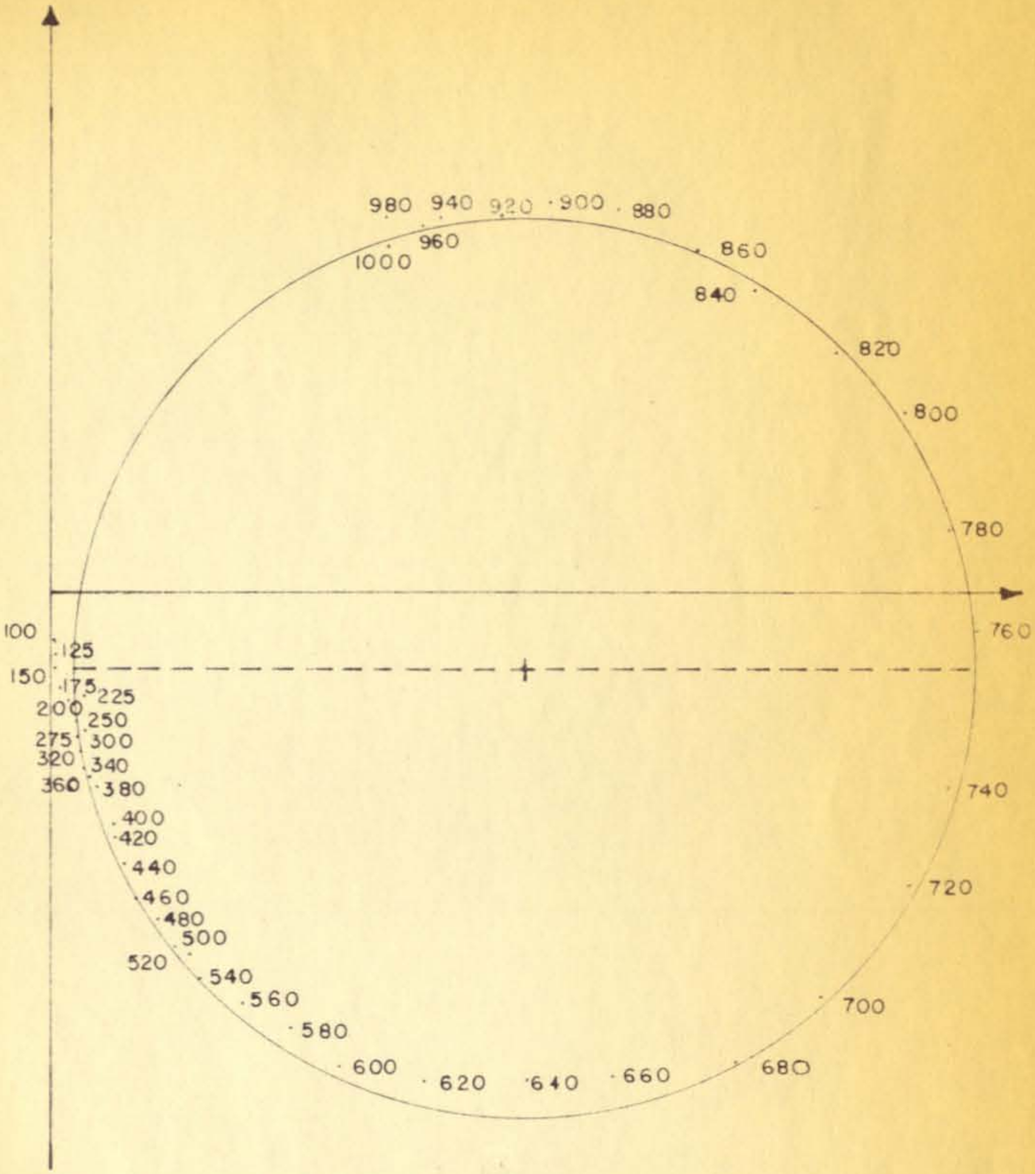
Velocity Response of a 3 Degree of Freedom System so excited as to be predominantly vibrating in its **1st** mode.



Damping: Non-Classical

Figure 9

Velocity Response of a 3 Degree of Freedom System so excited as to be predominately vibrating in its 2nd mode.



Damping: Non-Classical

Figure 10

Velocity Response of a 3 Degree of Freedom System so excited as to be predominantly vibrating in its 3rd mode.

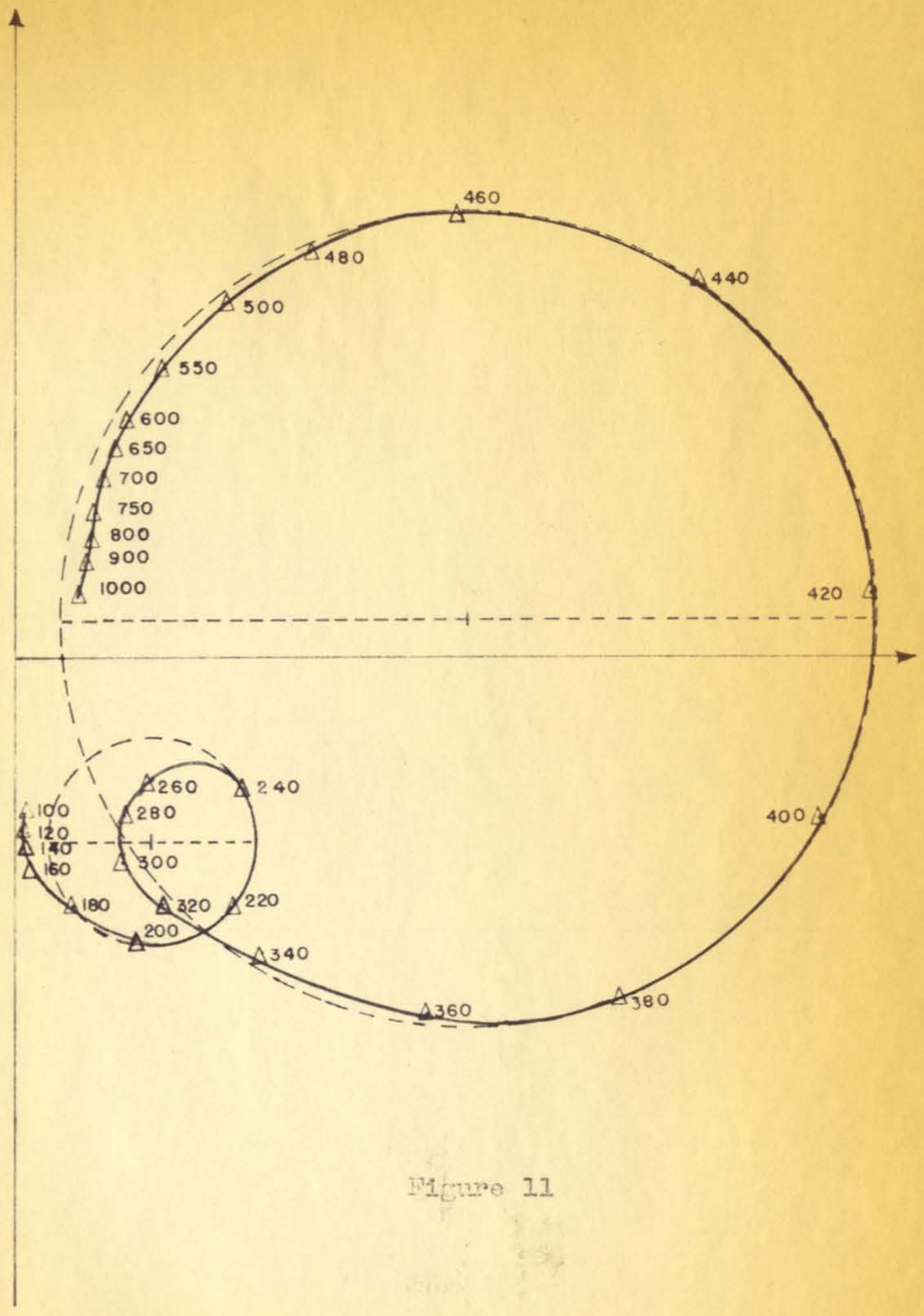
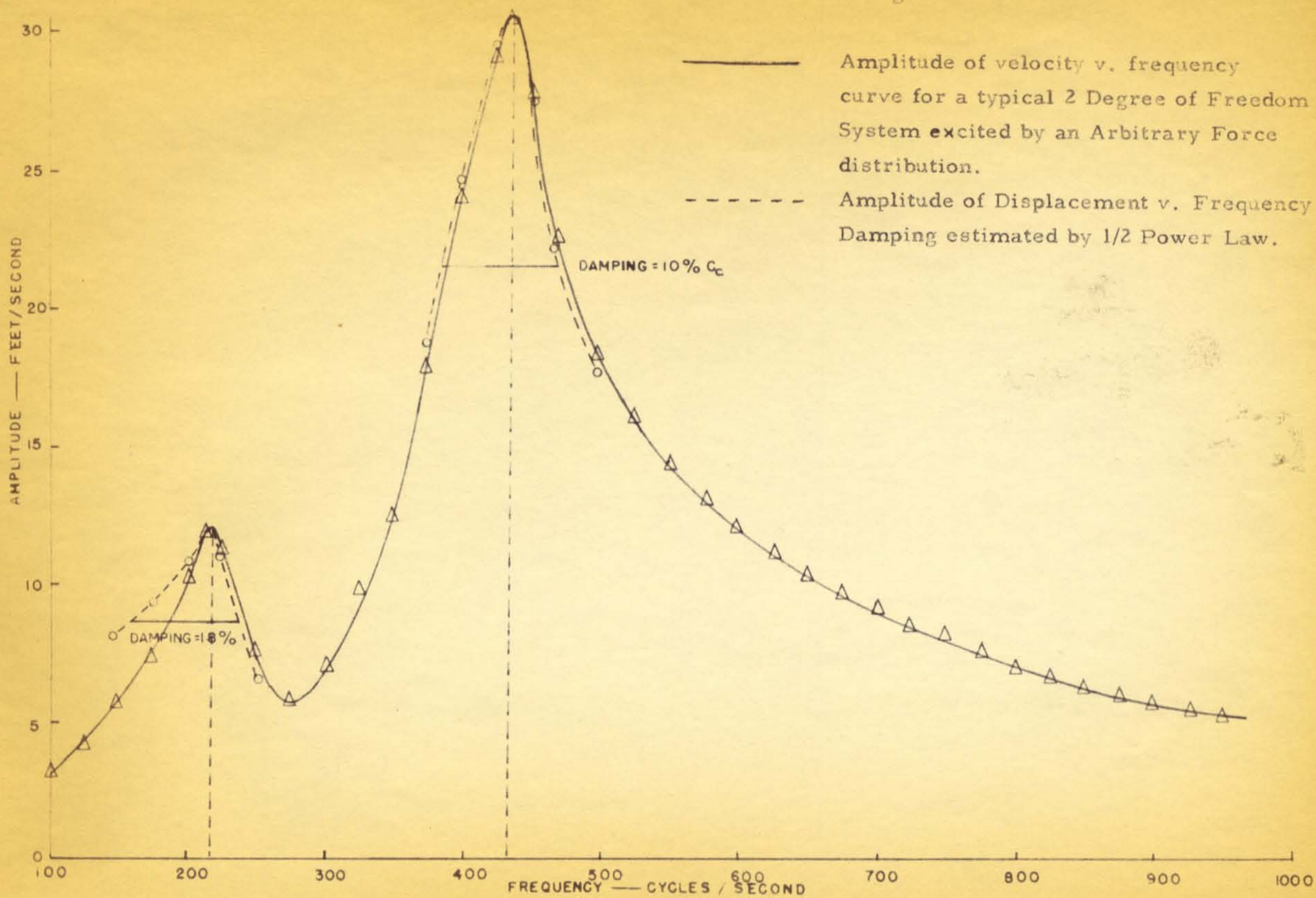


Figure 11

Typical Velocity Response of a 2 Degree of Freedom System excited by an arbitrary force distribution.

Figure 12


Amplitude of velocity v. frequency curve for a typical 2 Degree of Freedom System excited by an Arbitrary Force distribution.
Amplitude of Displacement v. Frequency Damping estimated by 1/2 Power Law.
DAMPING = 10% C_c
DAMPING = 1.8%
AMPLITUDE — FEET/SECOND
FREQUENCY — CYCLES / SECOND
0
5
10
15
20
25
30
100
200
300
400
500
600
700
800
900
1000

Calculation of Damping in Each Mode

Classical System

Percentage Initial Damping

	Exact	From Circles
1st Mode	15.88	14.24
2nd mode	11.25	11.78
3rd mode	9.97	10.13

Non-Classical System

Equivalent Classical Damping

	Exact	From Circles
1st mode	15.88	16.98
2nd mode	11.25	13.15
3rd mode	9.97	11.24

Determination of Natural Frequencies

3 Degree of Freedom System

Classical Damping:

	Exact	From Circle
1st mode	215	216
2nd mode	445	444
3rd mode	752	754

Non-Classical Damping:

	Exact	From Circle
1st mode	215	216
2nd mode	441	442
3rd mode	742	755

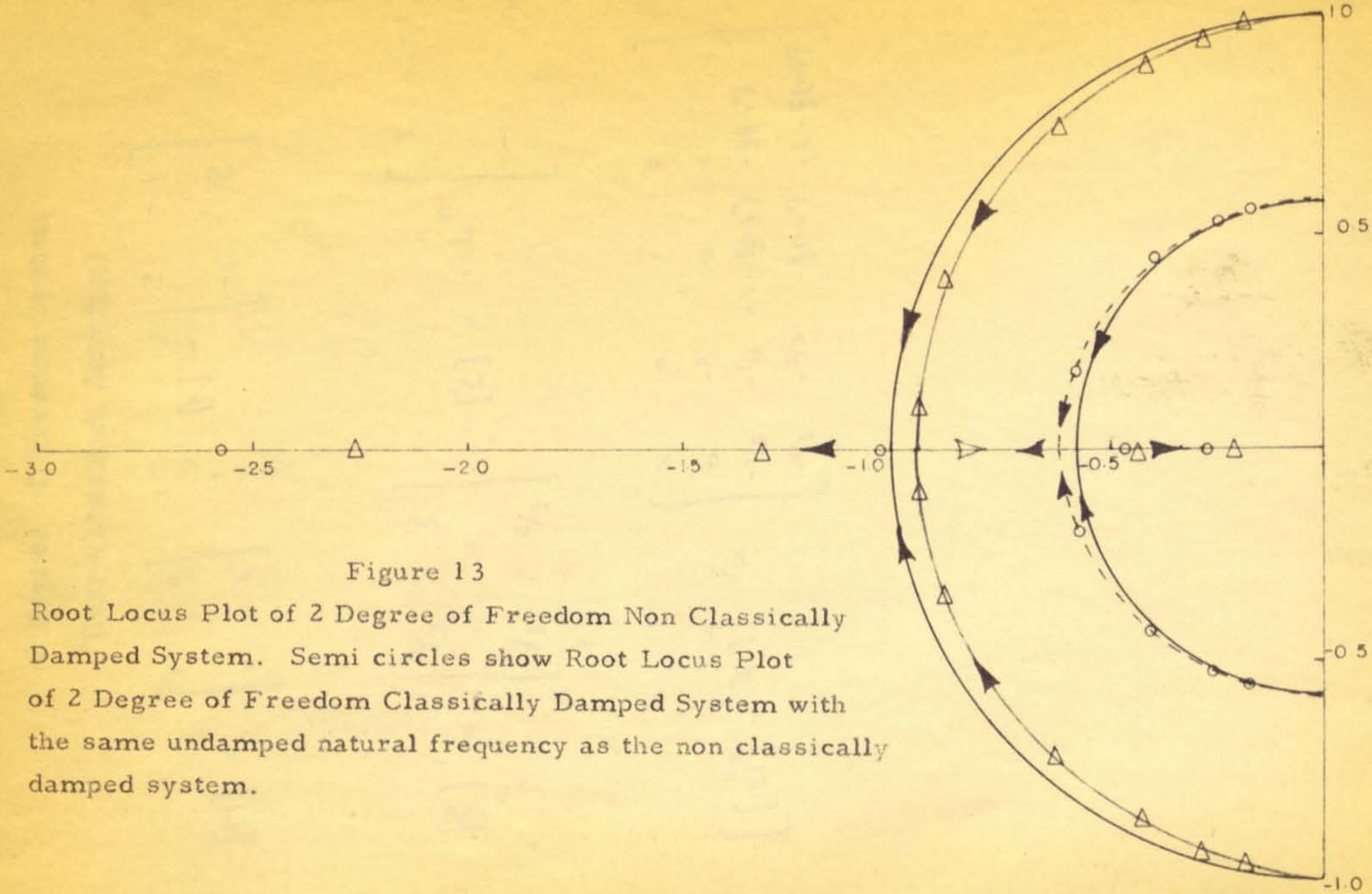


Figure 13

Root Locus Plot of 2 Degree of Freedom Non Classically Damped System. Semi circles show Root Locus Plot of 2 Degree of Freedom Classically Damped System with the same undamped natural frequency as the non classically damped system.

Root Locus Plot

2 Degree of Freedom System

(Non-Classical Damping)

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad [K] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[K]^{-1} = \begin{bmatrix} 2/3 & \frac{1}{3} \\ 1/3 & 2/3 \end{bmatrix} \quad ; \quad [C] = 1 \cdot 4^{n} \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

$$\left[[U] - \lambda I \right] = \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -2/3 & -1/3 & -\lambda - \frac{5}{3}(1 \cdot 4^{n}) & -\frac{1}{3}(1 \cdot 4^{n}) \\ -1/3 & -2/3 & -\frac{2}{3}(1 \cdot 4^{n}) & -\lambda - \frac{7}{3}(1 \cdot 4^{n}) \end{bmatrix}$$

Root Locus Data

n	λ		
0	-.187	±	.9811
	-.170	±	.557
1	-.281	±	.956
	-.256	±	.524
2	-.418	±	.903
	-.398	±	.441
3	-.607	±	.754
	-.578	±	.143
4	-.890		.373
	-.458		
	-1.022		
5	.958		.078
	-.274		
	-2.574		
6	-.421		
	-1.311		
	-.153		
	-4.001		
7	-.213		
	-2.261		
	-.101		
	-6.021		

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