IRREVERSIBLE THERMODYNAMICS AND VARIATIONAL PRINCIPLES WITH APPLICATIONS TO VISCOELASTICITY

Thesis by

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To my patient wife

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ABSTRACT

A unified theory of the thermo-mechanical behavior of viscoelastic media is developed from studying the thermodynamics of irreversible processes, and includes discussions of the general equations of motion, crack propagation, variational principles, and approximate methods of stress analysis.

The equations of motion in terms of generalized coordinates and forces are derived for systems in the neighborhood of a stable equilibrium state. They represent a modification of Biot's theory in that they contain explicit temperature dependence, and a thermodynamically consistent inclusion of the time-temperature superposition principle for treating media with temperature-dependent viscosity coefficients. The stress-strain-temperature and energy equations for viscoelastic solids follow immediately from the general equations and, along with equilibrium and strain-displacement relations, they form a complete set for the description of the thermo-mechanical behavior of media with temperature-dependent viscosity. In addition, an energy equation for crack propagation is derived and examined briefly for its essential features by applying it to a specific problem.

The thermodynamic equations of motion are then used to deduce new variational principles for generalized coordinates and forces, employing convolution-type functionals. Anticipating various engineering applications, the formulation is phrased alternately in terms of mechanical displacement, stresses, entropy displacement, and temperature in thermally and mechanically linear solids. Some special variational principles are also suggested for applications wherein the nonlinear thermal effects of temperature dependent viscosity and dissipation may be important.

Building upon the basic variational formulation, it is next shown that when these convolution functionals are Laplace-transformed with respect to time, some convenient minimum principles result which can be employed for the approximate calculation of transformed, viscoelastic responses. The characteristic time dependence of exact and approximate solutions is then derived and used in relating error in approximate viscoelastic solutions to error in the associated elastic solutions.

The dissertation is concluded with a study of some approximate methods of viscoelastic analysis. First, the important problem of inverting complicated Laplace transforms to physical time-dependent solutions is resolved by advancing two easily applied, approximate methods of transform inversion. These inversion methods and variational principles are then used in some illustrative, numerical, examples of stress and heat conduction analysis.

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PART I

GENERAL EQUATIONS OF IRREVERSIBLE THERMODYNAMICS WITH APPLICATIONS TO THE THERMO-MECHANICAL BEHAVIOR OF VISCOELASTIC SOLIDS

1.1. Introduction

The thermodynamics of irreversible processes (TIP) has been used by several workers to develop a macroscopic theory of linear irreversible phenomena. Probably the most unified and elegant treatment was presented by Biot (1, 2); we shall not attempt to list many of the other papers dealing with the subject, since a relatively complete list can be found in reference 2. Biot derived the linear equations governing a general, inhomogeneous, thermodynamic system which is in the neighborhood of a stable equilibrium state. In addition, he clearly illustrated their utility when he used them in deriving the stress-strain equations of an anisotropic, isothermal, viscoelastic solid (1); presenting a unified treatment of thermoelastic damping (3); studying the behavior of porous media (4); and deducing variational principles (2, 5). While Biot's emphasis has been on the mechanics of solids, the general equations could also be applied to other irreversible phenomena, such as reacting gases near an equilibrium composition.

One of the primary objectives of this dissertation is to establish the relation between temperature and deformations in viscoelastic media by using TIP. As a motivation for the theoretical development which is given in Part I, let us consider certain aspects of the theories proposed by Biot and some other investigators which relate to the treatment of thermal effects. The initial Biot formulation (1) permitted the thermodynamic system to have a non-uniform temperature distribution, but temperature variations were treated as hidden coordinates (i.e., the temperature at the geometric boundary of the system was maintained at a constant reference temperature). As a consequence, for example, it was not possible to use the results to deduce viscoelastic stress-strain equations with explicit temperature dependence. Later, Biot generalized his theory to admit temperature variations at the boundary (2) and arrived at an analogy between thermal and mechanical variables. In particular, it was shown that a small excess temperature (actual temperature minus the reference temperature) applied to the boundary plays the role of a generalized force, while its conjugate coordinate is entropy displacement (heat flow into the system divided by the reference temperature). However, in addition to the assumption that the excess temperature is small, it is implicit in this analogy that temperature rise due to the second order term in the energy equation (Rayleigh Dissipation Function) must be neglected. Another restrictive assumption which was introduced in his formulation is that the viscous properties of the system are independent of temperature. These assumptions made by Biot are valid within the domain of a completely linear theory; however, they are impractical for some applications. For example, if one were to use this analogy in deducing the coupled thermo-mechanical equations for a viscoelastic solid, important effects would be neglected. For, not only are the viscous properties of a viscoelastic solid usually very sensitive to temperature, but transient deformations maintained over a long enough time may produce a significant temperature rise as a result of dissipation.

Recently, Hunter (6) and Chu (7) have dealt directly with the derivation of the stress-strain-temperature equations and the associated energy equation of isotropic, viscoelastic solids. Even though they have relaxed certain of the above mentioned assumptions on thermal effects, the applicability of their results is restricted since all of the implications of TIP were not utilized. For example, Hunter assumes that the specific heat and thermal expansion coefficient are algebraic factors rather than integral (or differential) operators, while Chu only considers particularly simple stress-strain laws. In another paper Eringen (8) also introduces some explicit thermal effects in a study of more general systems, however he tacitly makes the same assumptions cited above in regard to Hunter's work, and assumes further that viscosity is independent of temperature.

In Part I-A of this dissertation we shall use TIP as a means of deriving equations of motion in terms of generalized coordinates and forces, but proceed with the thermodynamic formulation differently than Biot and Eringen in order to: 1) obtain explicit temperature dependence in the equations, 2) allow for temperature-dependent viscous properties, and 3) include the effect of dissipation on temperature (or heat flow). The results will be arrived at by studying the behavior of a thermodynamic system with a spacewise uniform temperature which is not necessarily constant in time, rather than with an arbitrary temperature distribution as permitted by Biot. It will be seen that the equations of motion reduce to those derived by Biot when thermal effects are omitted.

Following the derivation and solution of the equations governing the behavior of a thermally homogeneous system, we present an exposition of Biot's linear thermodynamic theory (2); this presentation will serve the dual purpose of contrasting the features of his development with ours and providing background information which will be needed in Part II.

The second half of Part I is concerned primarily with using the results obtained in the first half to deduce the stress-strain-temperature equations and energy equation of anisotropic viscoelasticity. This objective is readily accomplished by interpreting a viscoelastic material element as a closed thermodynamic system at a uniform temperature, and then associating mechanical strains and stresses with the generalized coordinates and forces. When these equations are combined with the infinitesimal strain-displacement relations and equilibrium equations, a complete set of field equations is obtained for the coupled thermo-mechanical, small deformation behavior of viscoelastic solids. The set is, in general, nonlinear due to the dissipation function in the energy equation and the assumed temperature dependent viscosity.

It should be emphasized that the relations between stress, strain, and temperature are deduced directly from TIP, without introducing spring-dashpot models in the development. However, it is shown that for most practical cases all moduli, coefficients of expansion, and specific heats are integral (or differential) operators which can be represented by the well-known mechanical models consisting of Maxwell elements in parallel or Voigt elements in series. This analogy was pointed out by Biot (2) for isothermal moduli, but the proof for the specific heats and thermal expansion coefficients is believed to be new. Recent experimental work by Kovacs and others (9, p. 411) substantiates the deduction for thermal expansion of several polymers.

The situation in which this Voigt or Maxwell representation is not thermodynamically admissible occurs only when the temperature is transient and the viscous properties of the material are such that the time-temperature superposition principle (9, p. 209) is not obeyed. In such a case, a mechanical model can still be used to represent the behavior, however an array of springs and dashpots is needed which is more general than the simple combinations of Maxwell and Voigt elements. As a practical matter, however, it is known that the superposition principle applies to most linear viscoelastic metals and nonmetals, and is also predicted by simple molecular models (9, p. 201; 10).

Providing the time-temperature superposition principle is applicable, there is another interesting implication of the thermodynamic analysis. To be specific let us suppose that a relaxation modulus of a given material is found to obey this superposition principle, in which the relation between time and temperature is given by a certain temperature dependent "shift factor." Then the implication is that <u>all</u> other mechanical and thermal properties (moduli, thermal expansion coefficients, heat capacities) which are associated with the same molecular processes also obey this superposition principle and have the same shift factor. It may be noted that experimental work by Kovacs, Marvin, and others (9, p. 223 and p. 414) confirms this deduction for the bulk and shear moduli and the thermal expansion coefficients of some polymers. Birefringence in polymers, of interest to photoelasticians, is another property which is expected to have the same shift factor since, just as stress-strain behavior, it is directly related to molecular configurations (11). For example, it is anticipated that the time-dependent strain and stress optical coefficients reported by Theocaris and Mylonas (12) for one temperature, have the same time-temperature superposition behavior as the corresponding moduli.

To conclude Part I, we indicate how TIP could be used to introduce rate effects into finite deformation and crack propagation problems. Equations are postulated which enable one to make a finite deformation analysis providing the free energy function is known. In addition, an energy equation for viscoelastic solids is proposed which can be used in predicting the propagation of a crack if its direction of travel is known. The equation is examined briefly for its essential features when applied to a special problem.

A. GENERAL THEORY

1.2. Derivation of the Thermodynamic Equations for Systems at a Uniform Temperature

We consider now a thermodynamic system of unit mass which has a prescribed, space-wise constant temperature. Its thermodynamic state is assumed to be defined by n state variables q_i (generalized coordinates), and by either temperature or internal energy. If the system is not spacewise uniform except for temperature, the state is defined by subdividing the system into cells sufficiently small that properties can be considered as uniform in each cell; this can be done as long as it is not necessary to make the size of each cell comparable with atomic dimensions. Extensive properties such as total entropy and internal energy are evaluated by a summation of local values over all cells. The coordinates q_i are very general and can represent such varied quantities as mechanical strains, molecular configurations in a polymer, location of interstitial atoms in a metal, chemical composition of reacting gases, etc. A generalized force Q_i, conjugate to the variable q_i, is defined by the condition that $Q_i \delta q_i$ is an increment of external work per unit volume done on the system.

While the equations of motion which are derived in this section are quite general in that they are applicable to linear systems satisfying the above assumptions, we shall use them later only for deducing the thermo-mechanical equations of a linear, anisotropic, viscoelastic solid. For this specific situation, six of the coordinates q_i $(i = 1, \ldots, 6)$ are associated with the six independent components of the infinitesimal strain tensor, e_{ij}. The remaining $(n-6)$ variables q_i $(i = 7, \ldots, n)$ are used to represent "internal degrees of freedom," and are called hidden coordinates. Hidden coordinates are defined by the condition that their conjugate (externally applied) forces are always zero. The "molecular configuration" in a polymer is an important example of a hidden coordinate. With q_i $(i = 1, \ldots, 6)$ as the strains, e_{ij}, the increment of work done on an element of unit mass in a small change δq_i is (13, p. 82)

$$\rho^{-1} Q_i \delta q_i = \rho^{-1} \sigma_{ij} \delta e_{ij} \tag{1.1}$$

where σ_{ij} is a component of the stress tensor and ρ is the density.

Consequently, we shall later associate the generalized force Q_i $(i = 1, \ldots, 6)$ with the six independent components of the stress tensor.

TIP will now be applied in order to obtain the equations governing the behavior of the system defined above as it passes through non-equilibrium states.[*] First, we must calculate the rate at which entropy is produced as a result of irreversibility in order to apply Onsager's Principle, which will be stated later. This calculation requires that the basic hypothesis of TIP be used (14); namely, the entropy of a system which is sufficiently close to equilibrium can be defined by classical thermodynamic variables.[**] Assuming that the entropy of a system of unit mass, s, is an explicit function of internal energy, u, and the generalized coordinates, q_i, the change in entropy is[***]

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[*] By equilibrium we will mean classical thermodynamic equilibrium, which some authors prefer to call thermostatic equilibrium.

[**] The legitimacy of this hypothesis has been investigated by several workers. For example, Prigogine (15) has used the kinetic theory of gases to show that for transport processes, the domain of validity of this assumption extends throughout the domain of validity of linear phenomenological laws (Fourier's law of heat conduction, Fick's law for mass diffusion, etc.). In the case of chemical reactions, he has shown that the reaction rates must be sufficiently slow so as not to perturb the Maxwell equilibrium distribution of velocities of each component to an appreciable extent; this excludes only reactions with abnormally low energy of activation. Thus, for most processes, the assumption that the entropy can be defined by classical thermodynamic variables is expected to be valid even quite far from equilibrium.

[***] Unless specified otherwise, we shall use the tensor notation that a repeated index is to be summed out, e.g.

$$\left(\frac{\partial s}{\partial q_i}\right)_{u, q_i'} dq_i \equiv \sum_{i=1}^{n} \left(\frac{\partial s}{\partial q_i}\right)_{u, q_i'} dq_i$$

and

$$Q_i dq_i \equiv \sum_{i=1}^{n} Q_i dq_i$$

$$ds = \left(\frac{\partial s}{\partial u}\right)_{q_i} du + \left(\frac{\partial s}{\partial q_i}\right)_{u,\, q_i'} dq_i \qquad (1.2)$$

where q_i' denotes all coordinates $q_1, \ldots, q_n$ with exception of q_i. From the First Law of Thermodynamics the increment of heat, dh, added to the unit mass is

$$dh = du - \rho^{-1} Q_i \, dq_i \qquad (1.3)$$

When $dq_i = 0$ the incremental process is reversible and from the Second Law of Thermodynamics $T\,ds = dh$, so that, using equations 1.2 and 1.3, we have

$$T \left(\frac{\partial s}{\partial u}\right)_{q_i} = 1 \qquad (1.4)$$

where T is the (instantaneous) absolute temperature of the system. This permits the entropy change to be written as

$$T\,ds = du + T\left(\frac{\partial s}{\partial q_i}\right)_{u,\, q_i'} dq_i \qquad (1.5)$$

which is Gibb's equation in generalized form. It is convenient to define the state function $Q_i^{(R)}$ as

$$-\rho^{-1} Q_i^{(R)} \equiv T\left(\frac{\partial s}{\partial q_i}\right)_{u,\, q_i'} \qquad (1.6)$$

and call $Q_i^{(R)}$ a reversible force. With this notation, equation 1.5 becomes

$$T\,ds = du - \rho^{-1} Q_i^{(R)} \, dq_i \qquad (1.7)$$

Imagine now that the system is immersed in a large heat reservoir which has the same temperature as the system. Assuming that the combined system consisting of the reservoir and actual system is insulated, the entropy change of the reservoir is $ds_R = -dh/T$. Consequently, the incremental entropy change of the total system, ds', is

$$ds' = ds + ds_R = ds - \frac{dh}{T} \qquad (1.8a)$$

which is the entropy change due to irreversibility. This entropy change is readily evaluated for the system under consideration by subtraction of equation 1.3 from 1.7, to find

$$ds' = \frac{1}{\rho T}\left(Q_i - Q_i^{(R)}\right) dq_i \qquad (1.8b)$$

which, when divided by the time increment dt, yields the desired expression for the rate at which entropy is produced,

$$\dot{s}' = \frac{1}{\rho T} X_i \dot{q}_i \qquad (1.9a)$$

where

$$X_i \equiv Q_i - Q_i^{(R)} \qquad (1.9b)$$

and the dot denotes differentiation with respect to time. $\dot{s}'$ is termed the entropy production per unit mass and X_i the irreversible component of force applied to the coordinate q_i. This force may be viewed, for example, as an internal force which arises from "viscosity" in the system and resists the motion of q_i.

The important principle of TIP is now introduced, namely Onsager's Principle, which can be stated as follows (14): If the entropy production is written in the form

$$\mathring{s}' = X''_i \dot{q}_i \tag{1.10}$$

and the forces X''_i are proportional to the "fluxes" $\dot{q}_i$, that is,

$$X''_i = b''_{ij} \dot{q}_j \qquad i = 1, \ldots, n \tag{1.11}$$

then the matrix of coefficients b''_{ij} is symmetric. Equation 1.11 can be written in the notation of 1.9 by setting

$$X''_i = \frac{1}{\rho T} X_i \; ; \qquad b''_{ij} = \frac{1}{\rho T} b_{ij} \tag{1.12}$$

so that we obtain the set of equations for the q_i,

$$Q_i = Q_i^{(R)} + b_{ij}(T) q_j ; \quad i = 1, 2, \ldots, n \tag{1.13}$$

with $b_{ij} = b_{ji}$, and the matrix b_{ij} may be a function of temperature.

Another property of the matrix b_{ij} is deduced by applying a corollary of the Second Law of Thermodynamics, which states that for all possible processes (16)

$$ds' \geq 0 \tag{1.14}$$

Substitution of equations 1.13 into 1.9 and using this property of the entropy change, ds', yields

$$\mathring{s}' = \frac{1}{\rho T} b_{ij} \dot{q}_i \dot{q}_j \geq 0 \qquad \text{for } \dot{q}_i \dot{q}_i > 0 \tag{1.15}$$

which implies that b_{ij} is a positive semi-definite matrix. The equality sign is needed in order to allow for reversible processes, e.g. elastic deformation.

Strictly speaking, linearity between fluxes and forces must be introduced as an assumption, rather than as information given by Onsager's Principle. However, this linearity has been found to be generally true for the thermodynamic systems sufficiently close to equilibrium; indeed, most systems exhibit this property quite far from equilibrium.* Of course, the ultimate justification of this assumption and its domain of validity depend on the ability of the resulting equations 1.13 to predict experimental observations.

A more useful form of these equations is obtained by using definition 1.6 to express $Q_i^{(R)}$ as a function of q_i and T. To do so, we introduce the Helmholtz free energy, f, which is defined as

$$f \equiv u - Ts \qquad (1.16)$$

Substitution of f into equation 1.7 yields

$$df = - s\ dt + \rho^{-1} Q_i^{(R)}\ dq_i \qquad (1.17)$$

which implies

$$-s = \left(\frac{\partial f}{\partial T}\right)_{q_i} \qquad (1.18a)$$

$$\rho^{-1} Q_i^{(R)} = \left(\frac{\partial f}{\partial q_i}\right)_{T, q_i'} \qquad (1.18b)$$

Use of identity 1.18b permits equations 1.13 to be written as

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*Chemical reactions and the mechanical behavior of metals appear to be the only important exceptions. However, even in these cases linearity exists when the systems are sufficiently close to equilibrium (19, 10).

$$Q_i = \rho \left(\frac{\partial f}{\partial q_i}\right)_{T, q_i'} + b_{ij}(T)\dot{q}_j \; ; \quad i = 1, 2, \ldots, n \tag{1.19}$$

which is a set of n equations of motion for the q_i, under the action of prescribed forces, Q_i, and temperature, which are not required to be constant in time.

It should be emphasized that the set 1.19 is expected to be valid for many systems which are not even close to equilibrium. However, through a consideration of the free energy we now wish to specialize these equations to a general system which is in the neighborhood of a reference state, defined as the state in which all forces Q_i are zero, the temperature is T_r, and the system is in thermodynamic equilibrium. By expanding the free energy in a Taylor series and neglecting powers higher than second order one obtains

$$f = \left(\frac{\partial f}{\partial T}\right)_r \theta + \left(\frac{\partial f}{\partial q_j}\right)_r q_j + \frac{1}{2}\left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_r q_i q_j$$

$$+ \theta\left(\frac{\partial^2 f}{\partial q_j \partial T}\right)_r q_j + \frac{1}{2}\left(\frac{\partial^2 f}{\partial T^2}\right)_r \theta^2 \tag{1.20}$$

where $\theta \equiv T - T_r$; q_i and f are arbitrarily taken as zero at the reference state, denoted by the subscript r.

Some useful properties of the coefficients in the series 1.20 will now be enumerated. First, it is observed from equations 1.19 that, by definition of the reference state,

$$\left(\frac{\partial f}{\partial q_i}\right)_r = 0; \quad i = 1, 2, \ldots, n \tag{1.21}$$

In addition, definition 1.16 and identity 1.18a show that

$$\left(\frac{\partial u}{\partial T}\right)_{q_i} = T\left(\frac{\partial s}{\partial T}\right)_{q_i} = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_{q_i} \tag{1.22}$$

and

$$\left(\frac{\partial f}{\partial T}\right)_r = -s_r \tag{1.23}$$

in which $(\partial u/\partial T)_{q_i}$ is the specific heat at constant generalized coordinates, c_q, as seen from equation 1.3. Furthermore, within the region of validity of expansion 1.20, the specific heat is a constant given by

$$c_q = -T_r\left(\frac{\partial^2 f}{\partial T^2}\right)_r \tag{1.24}$$

If we also make the definitions,

$$\begin{aligned} \rho_r\left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_r &= a_{ij} = a_{ji} \\ \rho_r\left(\frac{\partial^2 f}{\partial T \partial q_i}\right)_r &= -\beta_i \end{aligned} \tag{1.25}$$

the free energy expansion becomes

$$f = -s_r\theta + \frac{1}{2\rho_r}a_{ij}q_iq_j - \frac{\theta}{\rho_r}\beta_iq_i - \frac{c_q}{2T_r}\theta^2 \tag{1.26}$$

Further information about the free energy can be deduced by first combining the energy equation 1.3, definition 1.16, and 1.8a, to find

$$df = -T\,ds' - s\,dT + \rho^{-1}Q_i dq_i \tag{1.27}$$

Under the assumption that a stable equilibrium state exists at the temperature T_e (which is not necessarily equal to T_r) with all forces zero, the constant temperature behavior of f is found in the neighborhood of this state by integrating 1.27 with $dT = 0$,

$$f - f_e = T_e(s'_e - s') + \int_{q_{ie}}^{q_i} \rho^{-1} Q_i \, dq_i \tag{1.28}$$

where the subscript e refers to the equilibrium state. Consider first the limit case of reversible processes (e.g., motion of an elastic body) so that $s'_e = s'$ and the state function f is equal to the work done in displacing the system from equilibrium. Mechanical stability requires this work to be positive and therefore f is a minimum at the equilibrium point. For the actual case in which processes are irreversible, we can determine the nature of f by applying the classical thermodynamic result (16) that the entropy of an isolated system is a maximum at a stable equilibrium point. If the case is considered in which $Q_i = 0$, the system of interest and its reservoir make up an isolated system; hence at equilibrium s' is a maximum and 1.28 shows that f is a minimum.* Thus, the equilibrium state is distinguished by

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*When the reference state is defined such that the system is under forces which are not all zero, it is necessary to work with the Gibb's free energy, g. For example, in the case of a gas whose reference state is at a pressure p_e, the Gibb's free energy is

$$g = f + p_e v \tag{a}$$

where v is the specific volume. It is seen that this casts equation 1.28 in the form

$$g - g_e = T_e(s'_e - s') + \int_{q_{ie}}^{q_i} Q_i \rho^{-1} dq_i \tag{b}$$

where the force conjugate to q_v (= - v) is $(p - p_e)\rho$. *We see that* g, rather than f, is a minimum at the reference state.

the conditions

$$\left(\frac{\partial f}{\partial q_i}\right)_{T_e} = 0 \qquad i = 1, \ldots, n \tag{1.29}$$

and for stability

$$\left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_{T_e} dq_i dq_j \geq 0 \;;\; dq_i dq_i > 0 \tag{1.30}$$

in which the derivatives in equations 1.29 and 1.30 are evaluated at an equilibrium state that is not necessarily at the reference temperature T_r. It should be observed that f retains this behavior even when forces Q_i are applied, since it is a function of state variables and therefore does not depend explicitly on the external forces. Also, we have included the equality sign in 1.30 in order to allow for the condition of neutral stability (non-unique equilibrium state) with respect to some configurations. It will be seen later that this inclusion leads to steady flow under the action of timewise constant forces.

Substitution of expansion 1.26 into equation 1.29 for the equilibrium condition yields an expression for the equilibrium configuration, q_{ie}, thus

$$a_{ij} q_{je} = \theta \beta_i \tag{1.31}$$

Also, inequality 1.30 requires for stability at the equilibrium state

$$a_{ij} dq_i dq_j \geq 0; \qquad dq_i dq_i > 0 \tag{1.32}$$

which implies that a_{ij} is a positive semi-definite matrix.

Let us now substitute the free energy expansion 1.26 into equations 1.19, thus deriving linear equations of motion for the system when subjected to mechanical and thermal perturbations:

$$a_{ij}q_j + b_{ij}(T)\dot{q}_j = Q_i + \beta_i\theta; \quad i = 1, 2, \ldots, n \qquad (1.33)$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, and these matrices are positive semi-definite. It is important to observe that a_{ij} is constant, but the matrix b_{ij}, which represents the "viscosity" in the system, can be a strong function of temperature, and therefore time-dependent when the temperature is transient. These equations reduce to those derived by Biot (1) if the temperature is fixed at its reference value, $\theta = 0$.

Thus, it is observed that two of the three objectives stated in the introduction have been accomplished; that is 1) the equations of motion 1.33 contain explicit temperature dependence, and 2) the viscous properties are permitted to be temperature dependent. The third objective, which is to bring in the effect of dissipation on temperature (or heat flow), is reached in section 1.4 in a study of the energy equation.

1.3. Solution of the Equations

The basic thermodynamic equations 1.33 will now be solved to obtain q_i as an explicit function of the thermal and mechanical loading. A standard, convenient method of solving these equations involves the determination of a transformation which simultaneously diagonalizes a_{ij} and b_{ij} (i.e., uncouples the equations of motion). It is well-known that this transformation can always be found when the matrices are independent of time (18), which is the case for our equations if either b_{ij} is independent of temperature or the temperature is constant in time. However, it is not actually necessary that the matrix

b_{ij} be constant, but only that each element be proportional to the same function of time; this follows from the fact that a change of time variable can be made which reduces the differential equations 1.33 to ones with constant coefficients. Namely, if

$$b_{ij} = F(T)b'_{ij} \tag{1.34}$$

where b'_{ij} is independent of temperature, we can define a "reduced time" variable t' by

$$dt' = \frac{dt}{F[T(t)]} \tag{1.35a}$$

or

$$t' = \int_0^t \frac{du}{F[T(u)]} \tag{1.35b}$$

which casts equations 1.33 in the form

$$a_{ij}q_j + b'_{ij}\frac{dq_j}{dt'} = Q_i + \beta_i\theta \tag{1.36}$$

In the remainder of Part I it will be assumed that 1.34 is applicable in order that the solutions, which are derived by diagonalization, will be valid under transient temperature conditions. Further, as a matter of convention, we shall assume that $F(T) > 0$.*

The details of solving equations 1.36 will not be given since the results can be written down simply by analogy with Biot's solutions (1). Thus, we deduce that the solution of equations 1.36, in operational notation, is

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*It will be seen later that $F(T)$ is equal to the well-known polymer shift factor, a_T (17). Since the use of a_T reduces the time and temperature into one variable, t', this reduction is often called the time-temperature superposition principle.

$$q_i = S_{ij} Q_j + a_i^o \theta; \qquad i, j = 1, 2, \ldots, n \tag{1.37a}$$

where[*]

$$S_{ij} \equiv \sum_s \frac{C_{ij}^{(s)}}{1 + \tau_s' Fp} + \frac{C_{ij}}{Fp} \tag{1.37b}$$

$$a_i^o \equiv \sum_s \frac{C_{ij}^{(s)} \beta_j}{1 + \tau_s' Fp} \tag{1.37c}$$

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[*] This operational notation is purely formal, and can be defined by means of the solutions of first order differential equations. For example, if in equation 1.37b we set

$$q_i^{(s)} = \frac{C_{ij}^{(s)} Q_j(t)}{1 + \tau_s' Fp} \tag{a}$$

then the first-order differential equation is

$$q_i^{(s)} + \tau_s' F \frac{dq_i^{(s)}}{dt} = C_{ij}^{(s)} Q_j^{(t)} \tag{b}$$

which has the solution

$$q_i^{(s)} = \frac{e^{-\frac{1}{\tau_s'}\left(\int_0^t \frac{du}{F[T(u)]}\right)}}{\tau_s'} \left\{ \int_0^t \left[e^{\frac{1}{\tau_s'}\left(\int_0^v \frac{du}{F[T(u)]}\right)} C_{ij}^{(s)} Q_j(v) \frac{dv}{F[T(v)]} \right] dv + C_s \right\} \tag{c}$$

where the C_s are constants to be determined from initial conditions ($C_s = 0$ if the system is undisturbed at $t = 0$). Thus the operational form (a) is to be interpreted as the integral operator in (c). Similarly,

$$\frac{C_{ij} Q_j(t)}{Fp} \equiv \int_0^t \frac{C_{ij} Q_j(v)\, dv}{F[T(v)]} + C_s \tag{d}$$

A second interpretation can be made in terms of the Laplace (or Fourier) transform. If we denote the reduced-time transform of a function, $y(t')$, as

$$\overline{y}(p') = \int_0^\infty e^{-p't'} y(t') dt' \tag{e}$$

where the symbol p' represents the transform parameter, then S_{ij} and a_i^o (with Fp replaced by p') are transfer functions relating transformed forces, $\overline{Q}_i$, and temperature, $\overline{\theta}$, to transformed generalized coordinates $\overline{q}_i$. In view of the generality and simplicity embodied in the operational notation in 1.37, we shall use it throughout this dissertation.

and

$$Fp \equiv F\frac{d}{dt} = \frac{d}{dt},$$

The summation $\sum_s$ is extended to all distinct, finite, roots, τ'_s, of the vanishing determinant,

$$|a_{ij} - \frac{1}{\tau'} b'_{ij}| = 0 \tag{1.38}$$

Also, since the implied summation in equation 1.37a extends over all non-zero forces, the operator S_{ij} relates each coordinate to the forces applied to (k) observed coordinates, with the number of hidden coordinates being (n-k). However, the operational coefficient of thermal expansion, α^o_i, defined in equation 1.37c contains a summation over all indices $j = 1, 2, \ldots, n$.

Some important properties of the coefficients in equations 1.37 follow from the symmetry and positive semi-definite character of the matrices a_{ij} and b'_{ij}. It can be shown that

$$C^{(s)}_{ij} = C^{(s)}_{ji} \; ; \quad C_{ij} = C_{ji} \tag{1.39}$$

and that these matrices are independent of temperature and are positive semi-definite. In view of 1.39, S_{ij} is symmetric. Furthermore, the constants τ'_s (which we shall call retardation times in analogy with their significance in the stress-strain equations of polymers) are real and non-negative. A zero retardation time occurs when $|b'_{ij}| = 0$, which corresponds to a reversible process (e.g. elastic deformation).

The coefficients C_{ij} correspond to an infinite retardation time, which occurs when $|a_{ij}| = 0$. It is seen from equation 1.37b that the existence of C_{ij} leads to steady-flow (coordinates which increase linearly with time) under the action of constant forces. Also it can be shown from the equilibrium condition 1.31 that such steady-flow terms do not occur in the operational coefficient of expansion, a_i^o.

We would now like to invert solution 1.37 and express the applied forces in terms of observed coordinates and temperature. This inversion is most easily done by returning to the original equations 1.36, which can be represented in matrix form as

$$\left[\begin{array}{ccc|ccc} A_{11} & \cdots & A_{1k} & A_{1,k+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} & A_{k,k+1} & \cdots & A_{k,n} \\ \hline A_{k+1,1} & & & & & \\ \vdots & & & & M & \\ A_{n,1} & & & & & \end{array}\right] \left[\begin{array}{c} q_1 \\ \vdots \\ q_k \\ \hline \xi_{k+1} \\ \vdots \\ \xi_n \end{array}\right] = \left[\begin{array}{c} Q_1+\beta_1\theta \\ \vdots \\ Q_k+\beta_k\theta \\ \hline \beta_{k+1}\theta \\ \vdots \\ \beta_n\theta \end{array}\right] \qquad (1.40)$$

where $A_{ij} = a_{ij} + Fpb_{ij}$ with $Fp \equiv d/dt$, and the coordinates $\xi_{k+1} \ldots \xi_n$ are hidden. The subsystem M is a symmetric square matrix relating the $(n-k)$ hidden coordinates to k observed coordinates and temperature. This submatrix can always be diagonalized, in accordance with the previous discussion, and therefore we can assume that the hidden variables ξ_s, $s = k+1, \ldots, n$, are the normal coordinates associated with the matrix M, which becomes

$$\left[\; M \;\right] = \begin{bmatrix} A_{k+1} & & O \\ & A_s & \\ O & & A_n \end{bmatrix} \qquad (1.41)$$

The elements of M are $A_s = a_s + Fpb^{\dagger}_s$. The non-negative character of free energy and entropy production requires (assuming $F(T) > 0$):

$$a_s \geq 0 \ ; \quad b^{\dagger}_s \geq 0 \tag{1.42}$$

Whenever $a_s = 0$ or $b^{\dagger}_s = 0$, certain coefficients in 1.40 must vanish in order to satisfy this non-negative requirement on free energy at constant temperature and entropy production. This condition on the coefficents can be seen by writing these functions with the diagonal submatrix; from equation 1.26 we have for $\theta = 0$,

$$f = \frac{1}{2\rho_r}\left\{\sum_{ij=1}^{k} a_{ij}q_i q_j + 2\sum_{i=1}^{k}\sum_{s=k+1}^{n} a_{is}q_i\xi_s + \sum_{s=k+1}^{n} a_s\xi_s^2\right\} \geq 0 \tag{1.43}$$

and from equation 1.15,

$$\dot{s}^{\dagger} = \frac{F}{T\rho_r}\left\{\sum_{ij=1}^{k} b^{\dagger}_{ij}\dot{q}_i\dot{q}_j + 2\sum_{i=1}^{k}\sum_{s=k+1}^{n} b^{\dagger}_{is}\dot{q}_i\dot{\xi}_s + \sum_{s=k+1}^{n} b^{\dagger}_s\dot{\xi}_s^2\right\} \geq 0 \tag{1.44}$$

where the summation signs are now used in order to distinguish between summation over hidden and observed coordinates. If $b^{\dagger}_s$ is zero, then all coefficients $b^{\dagger}_{is}$ corresponding to this (s) value must also be zero as a result of this non-negative character of entropy production. Similarly, if an a_s vanishes, then all corresponding coefficients a_{is} must also vanish. In addition, $a_{is} = 0$ implies from the equilibrium condition 1.31 that the corresponding thermal coefficient β_s must be zero.

With these points in mind, we can use 1.40 to solve for the normal coordinates ξ_s in terms of the observed coordinates and temperature; these expressions are then substituted into the first (k) equations in 1.40 to obtain the external forces as explicit functions of observed coordinates and temperature. This procedure leads to the following result:

$$Q_i = \sum_{j=1}^{k} T_{ij} q_j - \beta_i^o \theta \qquad (i = 1, \ldots, k) \tag{1.45a}$$

where

$$T_{ij} = \sum_s \frac{FpD_{ij}^{(s)}}{Fp + \frac{1}{\rho'_s}} + D_{ij} + D'_{ij} Fp \tag{1.45b}$$

$$\beta_i^o = \sum_s \frac{FpB_i^{(s)}}{Fp + \frac{1}{\rho'_s}} + B_i \tag{1.45c}$$

with the definitions

$$D_{ij}^{(s)} = \psi_i^{(s)} \psi_j^{(s)} \tag{1.45d}$$

$$D_{ij} = a_{ij} - \sum_s \frac{a_{is} a_{sj}}{a_s} \tag{1.45e}$$

$$D'_{ij} = b'_{ij} - \sum_s \frac{b'_{is} b'_{sj}}{b'_s} \tag{1.45f}$$

$$\psi_i^{(s)} = \frac{a_{is}}{a_s^{1/2}} - \frac{b'_{is}}{b'_s} a_s^{1/2} \tag{1.45g}$$

$$B_i^{(s)} = \beta_s \left[\frac{a_{is}}{a_s} - \frac{b'_{is}}{b'_s} \right] \tag{1.45h}$$

$$B_i = \beta_i - \sum_s \frac{a_{is} \beta_s}{a_s} \tag{1.45i}$$

$$\rho'_s = \frac{b'_s}{a_s} \tag{1.45j}$$

where s is not summed unless indicated, and we have the symmetry property

$$T_{ij} = T_{ji} \tag{1.46}$$

The matrices composing T_{ij} are positive semi-definite. It is clear from the definition 1.45d that this property is true for $D^{(s)}_{ij}$. The non-negativeness of the matrix D_{ij} is shown by setting $\theta = 0$ and changing q_j very slowly. The work done in this process must be non-negative, i.e.,

$$\int_0^{q_i} Q_i dq_i = \frac{1}{2} D_{ij} q_i q_j \geq 0 \tag{1.47}$$

from which it follows that D_{ij} is non-negative. Similarly, the power input $Q_i \dot{q}_i$ must be non-negative for high rates of change of q_i; therefore D'_{ij} has this same property.

The constant ρ'_s, defined in 1.45j, is called the relaxation time associated with the s^{th} hidden normal coordinate. These constants play a role which is similar to that of the retardation times τ'_s; however, τ'_s and ρ'_s are generally not equal since the latter represents the eigenvalues of the submatrix M while the quantities τ'_s are eigenvalues of the entire $(n \times n)$ matrices.

1.4. The Energy Equation

In this section we first derive the energy equation in terms of temperature and observed generalized coordinates. The coordinates are then replaced by generalized forces by using results from the previous section. It will be seen that by expressing the energy equation as a function of observed variables, we are led to the concept of operational specific heats. We further show that when the energy equation is linearized, the total heat added to the element and the temperature perturbation act essentially as conjugate coordinate and force, respectively.

Let us rewrite the energy equation 1.3 by assuming that the internal energy is a function of (n) generalized coordinates and temperature:

$$dh = c_q dT + \left[\left(\frac{\partial u}{\partial q_i} \right)_{T, q_i'} - \rho^{-1} Q_i \right] dq_i \qquad (1.48)$$

where

$$c_q = \left(\frac{\partial u}{\partial T} \right)_{q_i} \qquad (1.49)$$

Making this same modification of equation 1.7 yields

$$ds = \left(\frac{\partial u}{\partial T} \right)_{q_i} \frac{dT}{T} + \frac{1}{T} \left[\left(\frac{\partial u}{\partial q_i} \right)_{T, q_i'} - \rho^{-1} Q_i^{(R)} \right] dq_i \qquad (1.50)$$

Since the entropy is a state function we can write

$$\frac{1}{T} \frac{\partial^2 u}{\partial q_i \partial T} = \frac{\partial}{\partial T} \left[\frac{1}{T} \left(\frac{\partial u}{\partial q_i} - \rho^{-1} Q_i^{(R)} \right) \right] \qquad (1.51)$$

and obtain,

$$\frac{\partial u}{\partial q_i} - \rho^{-1} Q_i = - T \frac{\partial}{\partial T} (\rho^{-1} Q_i^{(R)}) - \rho^{-1} (Q_i - Q_i^{(R)}) \qquad (1.52)$$

Use of identity 1.18b, expansion 1.26 for the free energy, and equation 1.13 from Onsager's principle casts 1.52 in the form

$$\frac{\partial u}{\partial q_i} - \rho^{-1} Q_i = T \rho_r^{-1} \beta_i - \rho_r^{-1} b_{ij} \dot{q}_j \qquad (1.53)$$

If this equation is substituted into the energy equation 1.48 and we divide by dt and ρ_r^{-1} we find

$$\mathring{H} = C_q \mathring{\theta} + T\beta_i \mathring{q}_i - 2D \tag{1.54}$$

where it was convenient to introduce the Rayleigh Dissipation Function defined as

$$D \equiv \frac{1}{2} b_{ij} \mathring{q}_i \mathring{q}_j \tag{1.55}$$

and capital letters are used to denote quantites per unit volume of the reference state, i.e.,

$$\mathring{H} \equiv \rho_r \mathring{h} \quad ; \quad C_q \equiv \rho_r^{-1} c_q \tag{1.56}$$

It is seen from equation 1.15 that D is proportional to the entropy production per unit volume,

$$D = \frac{1}{2} T \mathring{S}' \tag{1.57}$$

Equation 1.54 can be written in terms of the reduced time, t', by multiplying it by F, thus

$$FpH = C_q Fp\theta + T\beta_i Fpq_i - 2FD \tag{1.58a}$$

or, explicitly

$$\frac{dH}{dt'} = C_q \frac{d\theta}{dt'} + T\beta_i \frac{dq_i}{dt'} - b'_{ij} \frac{dq_i}{dt'} \frac{dq_j}{dt'} \tag{1.58b}$$

Let us now eliminate the hidden coordinates from 1.58 by writing them in terms of observed coordinates and temperature by using the last (n-k) equations of 1.40. With normal coordinates ξ_s for hidden variables we find

$$\frac{1}{F}(\beta_i Fpq_i) = \frac{1}{F}\left\{\sum_{i=1}^{k} \beta_i^o Fpq_i + \left[\sum_s \frac{\beta_s^2/b_s'}{Fp+\frac{1}{\rho_s'}} + \beta^2\right]Fp\theta\right\} \tag{1.59}$$

where β_i^o is the thermal expansion operator defined in equation 1.45c, and

$$\beta^2 \equiv \sum_s \frac{\beta_s^2}{a_s} \tag{1.60}$$

in which the summation is extended over all values of s corresponding to zero relaxation times. Substitution of 1.59 into 1.58a yields

$$FpH = \left[C_q + T\beta^2 + T\sum_s \frac{\beta_s^2/b_s'}{Fp+\frac{1}{\rho_s'}}\right]Fp\theta + T\sum_{i=1}^{k}\beta_i^o Fpq_i - 2FD \tag{1.61}$$

D can also be expressed as a function of observed coordinates and temperature by using equation 1.40. It is interesting to observe that if we neglect the second order dissipation function D and assume $\theta/T_r \ll 1$, the operational coefficient operating on $p\theta$ defines an operational specific heat for fixed observed coordinates, i.e.,

$$C_q^o \equiv \left(\frac{\delta H}{\delta\theta}\right)_{q_i(i=1,\ldots,k)} = C_q + T_r\beta^2 + T_r\sum_s \frac{\beta_s^2/b_s'}{Fp+\frac{1}{\rho_s'}} \tag{1.62}$$

The symbol δ is used since this specific heat is not an exact differential. All coefficients in this operational specific heat are non-negative since $a_s \geq 0$ and $b'_s \geq 0$.

The energy equation can also be written in terms of applied forces in place of observed coordinates. This is easily done by using equations 1.37 in order to replace all of the generalized coordinates appearing in equation 1.58a. Carrying out the substitution, we find

$$FpH = [\, C_q + T\beta_i a_i^o]\, Fp\theta + T \sum_{i=1}^{k} a_i^o FpQ_i - 2FD \tag{1.63}$$

where a_i^o is defined in equation 1.37c. Again, by neglecting the dissipation function and setting $T = T_r$, an operational specific heat at constant applied forces is obtained,

$$C_Q^o \equiv \left(\frac{\delta H}{\delta \theta}\right)_{Q_i} = C_q + T_r \beta_i a_i^o = C_q + T_r \sum_s \frac{\gamma^{(s)}}{1+\tau'_s Fp} \tag{1.64}$$

where $\gamma^{(s)} = \beta_i \beta_j C_{ij}^{(s)}$. C_Q^o has the same form as the specific heat 1.62, and the coefficients are non-negative since $C_{ij}^{(s)}$ is positive semi-definite, i.e.,

$$\gamma^{(s)} = \beta_i \beta_j C_{ij}^{(s)} \geq 0 \tag{1.65}$$

It is of interest to invert equations 1.61 and 1.63 so as to obtain temperature difference, θ, as a function of heat flow and observed variables. This step is easily carried out by analogy with the solutions in section 1.3, if, in equation 1.54 we neglect D and evaluate T at the reference value, T_r. Under the assumption that this linearization is valid, integration yields

$$\theta = \frac{H}{C_q} - \frac{T_r}{C_q} \sum_{i=1}^{n} \beta_i q_i \tag{1.66}$$

Upon substitution of this expression into the general equations 1.33

there results

$$[\,a_{ij} + \frac{T_r}{C_q}\,\beta_i\beta_j]\,q_j + b_{ij}\mathring{q}_j = Q_i + \beta_i\,\frac{H}{C_q}\,, \quad i = 1, 2, \ldots, n \qquad (1.67)$$

The matrix $T_r/C_q[\,\beta_i\beta_j]$ is symmetric and non-negative, so that 1.67 has the same properties as the original equations 1.33. As a consequence, all of the results of section 1.3 are applicable if we replace θ by H/C_q and a_{ij} by $a_{ij} + \frac{T_r}{C_q}\,\beta_i\beta_j$. It may be noted in this regard that by letting $H = 0$, the adiabatic operators are obtained.

Thus, in analogy with equation 1.59 we find

$$\beta_{iH}q_i = \sum_{i=1}^{k} \beta^{o}_{iH}\,q_i + \left[\sum_{s} \frac{\beta^{2}_{sH}/b_{sH}}{Fp + \frac{1}{\rho_{sH}}} + \beta^{2}_{H}\right]\frac{H}{C_q} \qquad (1.68)$$

where the subscript (H) denotes the coefficients associated with the new set of hidden normal coordinates. These coefficients are not the same as in 1.59 since the hidden coordinates must now be chosen such that they are normal with respect to the matrices in equations 1.67. Substitution of equation 1.68 into 1.66 yields

$$\begin{aligned} C_q\theta &= H - T_r\beta_i q_i \\ &= \left[\left(C_q - T_r\sum_{s}\frac{\beta^{2}_{sH}}{a_{sH}}\right) + T_r\sum_{s}\left(\frac{\frac{\beta^{2}_{sH}}{a_{sH}}\,Fp}{Fp + \frac{1}{\rho_{sH}}}\right)\right]\frac{H}{C_q} \\ &\quad - T_r\sum_{i=1}^{k}\beta^{o}_{iH}\;q_i \end{aligned} \qquad (1.69)$$

where the summation $\sum_{s}$ is extended over all values of s $(k+1 \leq s \leq n)$

for which $a_{sH} \neq 0$. The term $(C_q - T_r \sum_s \beta_{sH}^2 / a_{sH})$ is positive since a low rate of heat addition, with $q_i = 0$ $(i = 1, \ldots, k)$, must produce a temperature rise. Therefore all coefficients appearing in the operator on H/C_q are positive. A similar result is obtained when the observed coordinates are replaced by the applied forces.

It is interesting to note that the operator on H/C_q has the same positive property and form as the diagonal operators T_{ii} appearing in equations 1.45 (except for the matrix D'_{ij}). In this regard the heat added to the element plays the role of a "coordinate" whose conjugate "force" is the temperature perturbation θ. This same property exists in connection with the linearized energy equation 1.63 in integrated form,

$$H = \left[C_q + T_r \sum_s \frac{\gamma^{(s)}}{1 + \tau'_s F p} \right] \theta + T_r \sum_{i=1}^{k} a_i^o Q_i \qquad (1.70)$$

Comparison of this expression with 1.37 shows that temperature again appears as a force, with heat added as the conjugate coordinate (except for the matrix C_{ij}). We can therefore write 1.37 in the general form

$$q_i = \sum_{j=1}^{k+1} S_{ij} Q_j \; ; \quad i = 1, 2, \ldots, n+1 \qquad (1.71)$$

and its inverse 1.45 (with θ replaced by H/C_q in accordance with equation 1.67) as

$$Q_i = \sum_{j=1}^{k+1} T_{ij_H} q_j \; ; \qquad i = 1, 2, \ldots, k+1 \qquad (1.72)$$

where temperature is included as one of the forces, and heat added as its conjugate coordinate. From different considerations, Biot (2) observed a similar role of the thermal variables for systems with a non-uniform temperature. However, it must be emphasized that this is not true unless $\theta/T << 1$ and the effect of dissipation on temperature is negligible, i.e., when the energy equation is linearized. In addition, even with this linearization, the analogy is not complete because of the existence of the matrices D'_{ij} and C_{ij} in equations 1.45b and 1.37b, respectively.

1.5. Mechanical Models

Biot (2) has pointed out that the operational equations 1.37 and 1.45 for the isothermal case $(\theta = 0)$ can be represented by mechanical models consisting of Voigt elements in series (Kelvin model, figure 1.1) and Maxwell elements in parallel (Wiechert model, figure 1.2), respectively. We shall now show that by making an intuitive modification of these models, they can also be used to represent non-isothermal behavior. Also, it will be seen that the operational specific heats, for example, have these same mechanical analogs.

Consider first the series arrangement of Voigt elements, each consisting of a spring and dashpot, as shown in figure 1.1. The viscosity of each dashpot is denoted by η_s and the compliance of the corresponding spring is k_s.

It is assumed that each spring extends linearly with temperature in the absence of force so that the force-displacement law for the s^{th} spring is

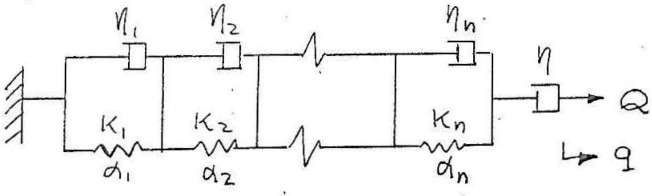

Figure 1.1. Kelvin Model

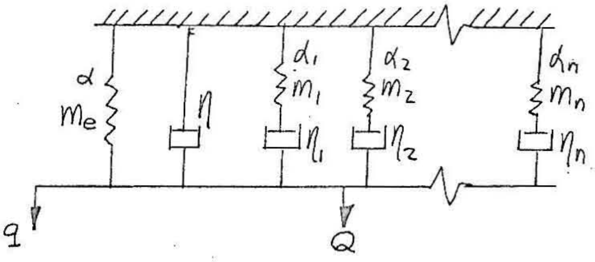

Figure 1.2. Wiechert Model

$$Q_{s_k} = \frac{q_s}{k_s} - \frac{\alpha_s}{k_s}\theta \qquad (1.73)$$

where Q_{s_k} is the force in the s^{th} spring and q_s is the corresponding displacement. Also, the force in the s^{th} dashpot is

$$Q_{s_\eta} = \eta_s' F p q_s \qquad (1.74)$$

where we assume that η_s' is a constant while $F(T)$ is the temperature dependence. Using the assumption that the total force $(Q = Q_{s_k} + Q_{s_\eta})$ in each Voigt element is the same, and that the total displacement of the model is $q = \sum_s q_s$, we derive, operationally, the force-displacement-temperature equation:

$$q = \left[\sum_s \frac{k_s}{1+\tau_s' Fp} + \frac{1}{\eta' Fp} \right] Q + \left[\sum_s \frac{\alpha_s}{1+\tau_s' Fp} \right] \theta \qquad (1.75)$$

where the retardation time has been defined as $\tau_i' = \eta_i' k_i$. This result is identical in form with equation 1.37, which verifies the model representation if we let $q(t)$ be an observed generalized coordinate and $Q(t)$ be a generalized force. The correspondence between coefficients is obvious,

$$C_{ij}^{(s)} = k_s \; ; \quad C_{ij} = \frac{1}{\eta'} \quad ; \qquad C_{ij}^{(s)} \beta_j = \alpha_s \tag{1.76}$$

It is also observed that we can think of the thermal expansion operator, α^o,

$$\alpha^o \equiv \sum_s \frac{\alpha_s}{1+\tau_s' Fp} \tag{1.77}$$

and specific heats 1.62 and 1.64 as being represented by their own Kelvin model (but without a free dashpot η') for which θ is the applied force.

In view of the remarks made previously about the coefficients in equation 1.37, compliances, k_s, and viscosities, η', corresponding to diagonal components, i.e. C_{ii} and $C_{ii}^{(s)}$, are positive. However, this property is not required by thermodynamics for the off-diagonal operators.

The force-displacement-temperature equation associated with the Wiechert model shown in figure 1.2 can be derived in a similar fashion. Denoting the modulus of the s^{th} spring by m_s, the viscosity by $F\eta_s'$, and the coefficient of thermal expansion by α_s, we obtain the following relations:

$$Q_s = m_s[q_{s_m} - \alpha_s\theta] = \eta'_s F p q_{s_\eta} \quad ; \qquad \rho'_s \equiv \frac{\eta'_s}{m_s}$$

$$q = q_{s_m} + q_{s_\eta} \tag{1.78}$$

and

$$Q = \sum_s Q_s$$

where q_{s_m} and q_{s_η} are the displacements in the s^{th} spring and dashpot, respectively. The expressions 1.78 combine to yield

$$Q = \left[m_e + \eta' Fp + \sum_s \frac{m_s Fp}{Fp + \frac{1}{\rho'_s}} \right] q - \left[m_e \alpha + \sum_s \frac{m_s \alpha_s Fp}{Fp + \frac{1}{\rho'_s}} \right] \theta \tag{1.79}$$

This equation has the same form as the general operational expression 1.45; consequently, as before, a formal correspondence between model parameters and thermodynamic derivatives can be made. In addition, thermodynamics requires that the spring moduli be positive only when they are associated with a model representing a diagonal term of the matrix T_{ij}. Furthermore, the operator

$$\beta^o \equiv m_e \alpha + \sum_s \frac{m_s \alpha_s Fp}{Fp + \frac{1}{\rho'_s}} \tag{1.80}$$

can be represented by its own model but without the dashpot (η').

It can be noted in passing that if all expansion coefficients are equal, i.e. $\alpha_s = \alpha$, and $\eta' = 0$, then equation 1.79 becomes

$$Q = \left[m_e + \sum_s \frac{m_s Fp}{Fp + \frac{1}{\rho'_s}} \right] \left[q - \alpha\theta \right] \tag{1.81}$$

With $Q = 0$, the strain is simply proportional to temperature change. The assumption of equal expansion coefficients would appear to be physically reasonable for isotropic viscoelastic solids, for example. However, the behavior resulting from such an assumption contradicts experimental findings, at least for polymers near the glass transition temperature (9).

As a final remark, it is clear from the figures that if the Kelvin and Wiechert models are used to represent the same system, they must exhibit the same basic behavior. For example, if steady flow and instantaneous deformation occur in a viscoelastic solid, then the Kelvin model must have a free spring and dashpot. The operator equation becomes

$$q = \left[\sum_s \frac{k_s}{1+\tau^*_s Fp} + k_o + \frac{1}{\eta^* Fp} \right] Q + \left[\alpha_o + \sum_s \frac{\alpha_s}{1+\tau^*_s Fp} \right] \theta \qquad (1.82)$$

where k_o and α_o correspond to the free spring for which $\tau^*_o = 0$. The equivalent Wiechert model must be such that $m_e = 0$ and $\eta^* = 0$, which casts the inverse operator equation in the form

$$Q = \left[\sum_s \frac{m_s Fp}{Fp + \frac{1}{\rho'_s}} \right] q - \left[\sum_s \frac{m_s \alpha_s Fp}{Fp + \frac{1}{\rho'_s}} \right] \theta \qquad (1.83)$$

1.6. Generalization to Systems with Non-Uniform Temperature

We shall now derive the linearized equations for a closed thermodynamic system whose temperature may now vary from point to point. This section therefore is an exposition of Biot's thermodynamic theory (2).

It is assumed that the system is in the neighborhood of a stable equilibrium point, so that its thermodynamic state is completely defined by n variables q_i. As before, these coordinates are very general, but we shall now include local temperatures (and later heat flow) in this set of variables. In addition, generalized forces Q_i are defined somewhat differently than before, in that $Q_i \delta q_i$ is to represent total work, rather than work per unit volume. The system is to be divided into cells in order to specify its state if it is not uniform. It is clear that such a system is, for example, the thermodynamic model of a viscoelastic solid whose mechanical and thermal properties may be non-uniform.

The derivation follows a pattern which is similar to the one used in section 1.2; namely, entropy production is first evaluated in order to apply Onsager's principle, and then the form of an energy function is determined for small departures of the system from a reference state.

To begin, we assume that the system has its geometric boundary fully covered with one or more heat reservoirs, in which the temperature of the i^{th} reservoir is T_i. Later, it will be necessary to assume that T_i is close to a reference value T_r, and therefore we shall introduce for convenience the temperature difference θ_i defined as $\theta_i = T_i - T_r$. The total system consisting of the actual system and all of the reservoirs is assumed to be insulated. The entropy change of this total adiabatic system during any incremental process is

$$dS'_T = dS_T + \sum_i dS_i \tag{1.84}$$

where dS_i is the entropy increase of the i^{th} reservoir and dS_T that of the system of interest. It should be noted in this regard that the system's entropy, S_T, is the total value which is calculated by summing over all cells. Conservation of energy is expressed as

$$dU_T = Q_j dq_j + \sum_i dh_i \tag{1.85}$$

where U_T is the system's total internal energy and dh_i the increment of heat injected into the system by the i^{th} reservoir. By hypothesis, the reservoirs undergo only reversible processes, hence we can write

$$dh_i = -dS_i(T_r + \theta_i) = -dS_i T_r \left(1 + \frac{\theta_i}{T_r}\right) \quad \text{(i not summed)} \tag{1.86}$$

and solving for $\sum_i dS_i$,

$$T_r \sum_i dS_i = -\frac{dh_i}{\left(1 + \frac{\theta_i}{T_r}\right)} = -\sum_i dh_i + \frac{\theta_i dh_i}{\left(1 + \frac{\theta_i}{T_r}\right) T_r} \tag{1.87}$$

Substituting equation 1.87 into 1.84 and using the energy equation 1.85, we find

$$T_r dS'_T = T_r dS_T - dU_T + Q_j dq_j + \frac{\theta_i dh_i}{\left(1 + \frac{\theta_i}{T_r}\right) T_r} \tag{1.88}$$

The state function V is now defined,

$$V_T \equiv U_T - T_r S_T \tag{1.89}$$

which Biot calls the generalized free energy. Let us assume that $\theta_i/T_r \ll 1$, so that equation 1.88 becomes

$$T_r dS'_T = -dV_T + \sum_j Q_j dq_j + \frac{\theta_i dh_i}{T_r} \tag{1.90}$$

or, per unit time

$$T_r \dot{S}'_T = -\dot{V}_T + Q_j \dot{q}_j + \theta_i \frac{\dot{h}_i}{T_r} \tag{1.91}$$

Biot calls the quantity $\dot{h}_i/T_r$ the entropy flow $\dot{S}_i$, and its time integral $S_i \equiv h_i/T_r$ entropy displacement. This title is somewhat misleading since $\dot{h}_i/T_r$ is the actual entropy flow only when $\theta_i = 0$. However, this name will be retained even with $\theta \neq 0$ for lack of a better term.

The function S'_T is the entropy of an adiabatic system and therefore $\dot{S}'_T$ is the entropy production for which Onsager's principle is applicable. In accordance with the remarks in section 1.2, we must first express the right-hand side of equation 1.91 such that it is in the form

$$T_r \dot{S}'_T = X_j \dot{q}_j \tag{1.92}$$

where $\dot{q}_j$ are thermodynamic state variables. It is clear from equation 1.91 that it is necessary to include the entropy displacement, $S_i = h_i/T_r$, in this collection of variables. Since entropy displacement defines, in effect, the net heat addition, it is necessary to be able to express V as a function of local heat addition rather than local temperature. The restriction under which heat addition is a state variable can be deduced by referring to the energy equation 1.54. It is observed that by neglecting the second-order dissipation function, D, equation 1.54 can be integrated and then used to replace the local temperature dependence of V_T by heat addition.

In the remainder of this section we shall assume that entropy displacement, h_i/T, rather than temperature, is included in the set of generalized coordinates. Furthermore, equation 1.90 shows that θ_i plays the role of an external force conjugate to the variable h_i/T_r, which allows us to incorporate the excess temperature θ_i into the set of generalized forces. With this association, equation 1.91 can be written

$$T_r \mathring{S}'_T = - \mathring{V}_T + Q_j \mathring{q}_j \tag{1.93a}$$

or

$$T_r \mathring{S}'_T = \left(- \frac{\partial V_T}{\partial q_j} + Q_j \right) \mathring{q}_j \tag{1.93b}$$

where the implied summation extends over all mechanical and thermal variables.

Onsager's principle is now used (see equations 1.10, 1.11) by assuming linearity between the "forces" X_i and "fluxes" $\mathring{q}_i$ which appear in the entropy production 1.92, thus

$$Q_i - \frac{\partial V_T}{\partial q_i} = X_i = b_{ij} \mathring{q}_j \tag{1.94}$$

where

$$b_{ij} = b_{ji}$$

from which it follows that

$$T_r \mathring{S}'_T = b_{ij} \mathring{q}_i \mathring{q}_j \equiv 2D_T \geq 0 \tag{1.95}$$

The reference state is defined as in section 1.2; namely, the equilibrium state for which $Q_i = q_i = 0$. For motion in which $Q_i = 0$, we have from equation 1.93a

$$T_r S_T' = - V_T \tag{1.96}$$

Since the entropy of an isolated system, S_T', is a maximum at equilibrium, V_T is a minimum. This minimum character of V_T in the neighborhood of the reference state exists of course even when $Q_i \neq 0$ since V_T is a function of only state variables. Hence, V_T is a quadratic function of q_i when terms higher than second order are neglected in a Taylor series expansion, thus

$$V_T = \frac{1}{2} a_{ij} q_i q_j \tag{1.97}$$

where

$$a_{ij} \equiv \left(\frac{\partial^2 V_T}{\partial q_i \partial q_j} \right)_{q_i = 0} = a_{ji}$$

and a_{ij} is a positive semi-definite matrix.

The linear equations of motion are now easily derived by substituting the free energy 1.97 into equation 1.94, which yields

$$a_{ij} q_j + b_{ij} \dot{q}_j = Q_i \qquad i = 1, 2, \ldots, n \tag{1.98}$$

where $b_{ij} = b_{ji}$, $a_{ij} = a_{ji}$, and both of these matrices are positive semi-definite.

These equations are of the same form as equations 1.33 which were derived for a system at a prescribed, spacewise, uniform temperature. However, the matrices a_{ij} and b_{ij} in 1.98 are not equal to the corresponding matrices in 1.33; this observation follows from the fact that the thermal variables, entropy displacement and temperature perturbation, are now included in the sets of generalized coordinates and forces. Furthermore, if b_{ij} in the set of equations 1.98 is temperature dependent, this set is non-linear unless the temperature is prescribed throughout the thermodynamic system. Also, the simple temperature dependence 1.34 is not realistic if the temperature is not spacewise constant. In view of these complications, it is assumed that b_{ij} in 1.98 is a constant matrix, or, what is equivalent, we retain only the constant, reference value in a Taylor series expansion of $b_{ij}(T)$.

With this latter assumption, all of the results of section 1.3 are applicable for the solution of equations 1.98 after setting $F = 1$, $\beta_i = 0$, and dropping the primes on ρ_s' and τ_s'. In addition, it is observed that the analogy between thermal and mechanical variables, which was deduced in section 1.4 after linearizing the energy equation (see equations 1.71 and 1.72) is consistent with the results of the present section.

Finally, we should add that Biot has indicated that the theory in this section is applicable to more general systems, such as open systems and systems with electrical energy (2). The extension is made by simply choosing the correct generalized "forces" and "coordinates" as was done with temperature and entropy displacement.

B. APPLICATIONS TO VISCOELASTIC SOLIDS

1.7. The Stress-Strain-Temperature Equations

a. Anisotropic continuum. In view of our earlier remarks in section 1.2 on the association of stress and strain with generalized forces and coordinates, the results of section 1.3 can be used to deduce the stress-strain-temperature equations of a general viscoelastic, anisotropic solid. This straightforward procedure of obtaining the most general, thermodynamically admissible, viscoelastic equations is to be contrasted with the work of previously mentioned authors, e.g. Hunter (6), in which TIP was not fully utilized.

Denoting the three orthogonal coordinate axes by x_1, x_2, x_3, we let

$$Q_1 = \sigma_{11},\ Q_2 = \sigma_{22},\ Q_3 = \sigma_{33},\ Q_4 = \sigma_{23},\ Q_5 = \sigma_{31},\ Q_6 = \sigma_{12}$$

$$q_1 = e_{11},\ q_2 = e_{22},\ q_3 = e_{33},\ q_4 = 2e_{23},\ q_5 = 2e_{31},\ q_6 = 2e_{12} \tag{1.99a}$$

where

$$e_{ij} = e_{ji} \quad \text{and} \quad \sigma_{ij} = \sigma_{ji} \tag{1.99b}$$

With these definition, equation 1.45 becomes

$$\sigma_{\mu\nu} = Z^{ij}_{\mu\nu} e_{ij} - \beta^{o}_{\mu\nu}\theta \tag{1.100a}$$

in which the operational moduli are[*]

$$Z^{ij}_{\mu\nu} = \sum_s \frac{FpD^{ij(s)}_{\mu\nu}}{Fp + \frac{1}{\rho'_s}} + D^{ij}_{\mu\nu} + D'^{ij}_{\mu\nu}\, Fp \qquad (1.100b)$$

and the thermal expansion operators are

$$\beta^{o}_{\mu\nu} = \sum_s \frac{Fp\beta^{(s)}_{\mu\nu}}{Fp + \frac{1}{\rho'_s}} + \beta_{\mu\nu} \qquad (1.100c)$$

where $\rho'_s > 0$. The operators satisfy the symmetry property

$$Z^{ij}_{\mu\nu} = Z^{ij}_{\nu\mu} = Z^{ji}_{\mu\nu}$$

$$\beta^{o}_{\mu\nu} = \beta^{o}_{\nu\mu} \qquad (1.101)$$

as consequences of the symmetry of σ_{ij} and e_{ij}. TIP also requires that they satisfy the property

$$Z^{ij}_{\mu\nu} = Z^{\mu\nu}_{ij} \qquad (1.102)$$

The inverse of these operators can be obtained in a similar fashion by using equations 1.37. This allows us to write

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[*] For many viscoelastic materials there are a large number of relaxation times which are closely spaced. This permits the series in equation 1.100b to be approximated by integrals

$$Z^{ij}_{\mu\nu} = \int_0^\infty \frac{FpH^{ij}_{\mu\nu}(\rho)\, d\rho}{(Fp + \frac{1}{\rho})\rho} + D^{ij}_{\mu\nu} + D'^{ij}_{\mu\nu}\; Fp \qquad (a)$$

in which $H^{ij}_{\mu\nu}(\rho)$ is called the relaxation spectrum. Thus, the internal coordinate space is replaced by the mathematical model of a continuum, in the same sense that a macroscopic system is represented by a continuum.

$$e_{\mu\nu} = A^{ij}_{\mu\nu}\,\sigma_{ij} + a^{o}_{\mu\nu}\,\theta \tag{1.103a}$$

where the operational compliances are

$$A^{ij}_{\mu\nu} = \sum_{s} \frac{C^{ij(s)}_{\mu\nu}}{1 + \tau'_{s} Fp} + \frac{C^{ij}_{\mu\nu}}{Fp} + C'^{ij}_{\mu\nu} \tag{1.103b}$$

$$a^{o}_{\mu} = \sum_{s} \frac{a^{(s)}_{\mu\nu}}{1 + \tau'_{s} Fp} + a'_{\mu\nu} \tag{1.103c}$$

where $\tau'_{s} > 0$, and the same symmetry properties exist as shown in equations 1.101 and 1.102. We have removed from the summation in 1.103b and 1.103c those coefficients corresponding to zero retardation times. These coefficients, $C'^{ij}_{\mu\nu}$ and $a'_{\mu\nu}$, provide the solid with "instantaneous elasticity," i.e., instantaneous straining under the action of timewise step change in stress or temperature. They are thermodynamically admissible when we require only that entropy production be non-negative, rather than positive.

The coefficients $C^{ij}_{\mu\nu}$ give rise to strains which change linearly in time under constant applied stress, i.e. "steady flow." These coefficients occur when the free energy change can be zero for some configurational motion.

It should be observed that the operators in equations 1.100 and 1.103 are the most general ones which are permitted by thermodynamics. For example, the operational coefficient of expansion, $a^{o}_{\mu\nu}$, cannot contain a term $a_{\mu\nu}/Fp$ such as appears in the operational compliances; similarly, $\beta^{o}_{\mu\nu}$ is not allowed to have a term $\beta'_{\mu\nu}Fp$. Furthermore, the simple operational form is less general than would be obtained by means of a purely mathematical approach to linear viscoelasticity. In order to illustrate this point, consider the general linear relation between stress-strain and temperature

$$\left[\sum_s a^{ij(s)}_{\mu\nu}\, p^s\right]\sigma_{ij} = \left[\sum_s b^{ij(s)}_{\mu\nu}\, p^s\right] e_{ij} - \left[\sum_s c^{(s)}_{\mu\nu} p^s\right]\theta \qquad (1.104)$$

where

$$p \equiv \frac{d}{dt}\,.$$

Solving for the stress, under the assumption of time independent coefficients, we obtain

$$\sigma_{\mu\nu} = P^{ij}_{\mu\nu}\, e_{ij} - \gamma^{o}_{\mu\nu}\,\theta \qquad (1.105)$$

where the elements of the matrices $P^{ij}_{\mu\nu}$ and $\gamma^{o}_{\mu\nu}$ are to be expanded in partial fractions. However, as pointed out by Biot (2), these partial fractions may be quite different from those in equations 1.100 because:

1. The roots $\frac{1}{\rho_s}$ of $\left[\sum_s a^{ij(s)}_{\mu\nu}\, p^s\right]$ may be complex conjugates. This condition can arise in the thermodynamic results only when hidden variables possess non-random kinetic energy, which we have neglected from the start.
2. There may be fractions of the type $\left(\frac{1}{p + \frac{1}{\rho_s}}\right)^n$ corresponding to multiple roots.
3. The matrix $P^{ij}_{\mu\nu}$ is not necessarily symmetric, i.e.

$$P^{ij}_{\mu\nu} \neq P^{\mu\nu}_{ij}$$

Additional properties of the matrices in the operators $Z^{ij}_{\mu\nu}$ and $A^{ij}_{\mu\nu}$ follow immediately from the properties of the general ones relating Q_i, q_i, and θ. For example, all constant matrices composing $Z^{ij}_{\mu\nu}$ and $A^{ij}_{\mu\nu}$ are positive semi-definite. However, for stability, it is required for all tensors $Y_{ij} \neq 0$ that

$$\left[\sum_s D^{ij(s)}_{\mu\nu} + D^{ij}_{\mu\nu} + D'^{ij}_{\mu\nu} \right] Y_{ij} Y_{\mu\nu} > 0 \tag{1.106a}$$

and

$$\left[\sum_s C^{ij(s)}_{\mu\nu} + C^{ij}_{\mu\nu} + C'^{ij}_{\mu\nu} \right] Y_{ij} Y_{\mu\nu} > 0 \tag{1.106b}$$

which implies that $Z^{ij}_{\mu\nu}$ and $A^{ij}_{\mu\nu}$ are positive definite (rather than semi-definite) when Fp is real and positive. These requirements 1.106 follow from the observation that if it were possible to find $Y_{ij} \neq 0$ such that expression 1.106a or 1.106b vanished, then $|Z^{ij}_{\mu\nu}|$ or $|A^{ij}_{\mu\nu}|$ would vanish for all Fp.

b. Isotropic continuum. The stress-strain equations for a linear isotropic material are obtained from the general relations 1.100a and 1.103a just as in elasticity. That is, we require the relation between stress, strain and temperature to remain invariant under all rotations of the coordinate axes. This implies that there are only two operational moduli or compliances and one thermal expansion operator. Using notation analogous to that used for elastic bodies we can write

$$\sigma_{ij} = 2\mu(p) e_{ij} + \lambda(p) \delta_{ij} \vartheta - \beta^{o} \theta \tag{1.107}$$

where

$$\vartheta \equiv e_{11} + e_{22} + e_{33}\;; \quad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and $\mu(p)$ and $\lambda(p)$ the operational equivalents of the Lamé elastic constants. These operators, as well as the one for thermal expansion, are obtained from equations 1.100 and 1.103. We have

$$\mu(p) = \sum_s \frac{Fp\mu^{(s)}}{Fp + \frac{1}{\rho_s'}} + \mu + \mu' Fp \qquad (1.108)$$

with a similar representation for $\lambda(p)$, and

$$\beta^o = \sum_s \frac{Fp\beta^{(s)}}{Fp + \frac{1}{\rho_s'}} + \beta \qquad (1.109)$$

The coefficients in $\mu(p)$ are positive since $\mu(p)$ corresponds to the diagonal term Z_{23}^{23} (13), however this positiveness is not required for $\lambda(p)$.

It is often convenient to have the operator corresponding to the bulk modulus, which is defined by

$$\frac{\Theta}{3} = K(p)\vartheta - \beta^o\theta \qquad (1.110)$$

where $\Theta \equiv \sigma_1 + \sigma_2 + \sigma_3$, from which it follows that

$$K(p) = \frac{2}{3}\,\mu(p) + \lambda(p) \qquad (1.111)$$

With simple pressurization, Θ and are the only observed conjugate variables, and hence $K(p)$ must be of the form

$$K(p) = \sum_s \frac{FpK^{(s)}}{Fp + \frac{1}{\rho_s'}} + K + K' Fp \qquad (1.112)$$

in which all coefficients are positive. The inverse of equation 1.110 is written

$$= B(p)\,\frac{\Theta}{3} + \alpha^{o}\,\theta \qquad (1.113a)$$

where

$$B(p) = [\,K(p)]^{-1} = \sum_{s} \frac{B^{(s)}}{1+\tau_{s}'Fp} + \frac{B}{Fp} + B' \qquad (1.113b)$$

$$\alpha^{o} = B(p)\,\beta^{o} = \sum_{s} \frac{\alpha^{(s)}}{1+\tau_{s}'Fp} + \alpha' \qquad (1.113c)$$

with all positive coefficients in the operational bulk compliance $B(p)$. A similar result is obtained for the operational tensile modulus and compliance by letting the one-dimensional stress and strain be the observed conjugate variables.

1.8. The Coupled Thermo-Mechanical Field Equations

a. Anisotropic continuum. For a complete description of the thermo-mechanical behavior of viscoelastic solids it is necessary to include the equations of strain-displacement, mechanical equilibrium, and heat conduction along with the stress-strain-temperature and energy equations considered above. We shall assume that strains are small so that the strain-displacement equations can be written as*

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*A comma before a subscript denotes differentiation with respect to the corresponding orthogonal cartesian coördinate, e.g.,

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

and

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.114}$$

where the displacement components (u_1, u_2, u_3) are referred to an orthogonal set of cartesian axes (x_1, x_2, x_3). The equilibrium equations are

$$\sigma_{ij,j} + F_i = 0 \; ; \tag{1.115}$$

where F_i is the body force per unit volume. Also, the experimental law of heat conduction for a general anisotropic body is*

$$K_{ij}\theta,_j = -\overset{\circ}{h}_i \; ; \tag{1.116}$$

where $\overset{\circ}{h}_i$ is the heat flow per unit area in the x_i direction and K_{ij} is the thermal conductivity tensor. TIP can be used to show that K_{ij} is positive definite and symmetric (14). The net amount of heat flow into an infinitesimal element of unit volume is

$$\dot{H} = -\overset{\circ}{h}_{i,i} \tag{1.117}$$

so that from equation 1.116

$$\overset{\circ}{H} = (K_{ij}\theta,_j),_i \tag{1.118}$$

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*We have omitted interaction coefficients which give rise to coupling between the irreversible processes of heat flow and deformation, such as included by Eringen (8). This omission is done on the basis of Curie's symmetry principle which states that macroscopic causes do not have more elements of symmetry than the effects they produce (19); while heat flow is a vector, the thermodynamic variables considered in section 1.2 are assumed to be tensors of rank two (e.g. strains) and scalars (e.g. hidden coordinates).

Under the assumption that the time-temperature superposition principle is valid, we can write the energy equation 1. 61 as

$$F(K_{ij}\theta,_{j}),_{i} = C_e^o Fp\theta + T\beta_{ij}^o Fpe_{ij} - 2FD \qquad (1.119)$$

where C_e^o is the operational specific heat at zero strain defined by equation 1. 62, β_{ij}^o is given by 1. 100c, and D is the dissipation function which can be expressed in terms of stress or strain rates and temperature by means of equation 1. 40. The energy equation can also be expressed in terms of stresses by using equation 1. 63.

Equations 1. 114, 1. 115, 1. 119, and the stress-strain temperature equations 1. 100, together with appropriate boundary conditions, form a complete set for calculation of the sixteen dependent variables σ_{ij}, e_{ij}, u_i, and θ. It should be noted that they are nonlinear if properties are temperature dependent or if the dissipation function is retained.

b. <u>Isotropic continuum.</u> The equations of isotropic viscoelasticity are easily obtained from the preceding ones. The strain-displacement and equilibrium equations 1. 114 and 1. 115 remain the same, of course, while the energy equation 1. 119 becomes

$$F(K\theta,_{j}),_{j} = C_e^o Fp\theta + T\beta^o Fp \; - 2FD \qquad (1.120)$$

and the stress-strain-temperature equations are given by 1. 107.

We would like to conclude this section by considering a few practical points in regard to solving the general set of viscoelastic equations. First, a simplification may be achieved through a change of variables given by

$$x_i' = x_i$$

$$t' = \int_0^t \frac{dV}{F[T(v, x_i)]} \tag{1.121}$$

which was suggested by Morland and Lee (20) for the uncoupled problem. Under this substitution all spacial derivatives transform according to

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i'} + \frac{\partial}{\partial t'} \times \frac{\partial t'}{\partial x_i} \tag{1.122}$$

and the stress-strain-temperature equations and specific heat have operators with constant coefficients. Because of the second term appearing in equation 1.122 this transformation may or may not simplify the calculation, depending on the particular application. For example, if the temperature is transient, but independent of x_i, then the spacial derivatives 1.122 in the new coordinates x_i' are the same as with x_i; in this case the equations are much simpler in the primed variables t' and x_i'. If, however, the temperature is independent of time, but a function of x_i, the original system (t, x_i) is more convenient.

It is often possible to neglect the terms in the energy equation 1.119 (or 1.120) which are due to straining and to take the specific heat as an algebraic factor, rather than an operator.* These approximations

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*The effect of rate-of-temperature-change on the specific heat at constant pressure has been studied experimentally by Davies and Jones (21). Using several polymers and super-cooled liquids, they found that the specific heat is rate-dependent with moderate rates, if the temperature is near the glass transition temperature. Furthermore, for all substances studied, it was observed that for sudden permit the temperature distribution to be obtained from the classical heat conduction equation. The mechanical equations for stresses and displacements are then solved with this result. These latter equations are linear in this case, but they have variable coefficients due to the function F(T) if the temperature is not constant.

The error introduced through the assumption which makes the temperature distribution independent of the straining can be readily estimated for some problems by comparing the limit case of adiabatic deformation with isothermal deformation. The dependence of the energy equation on e_{ij} (except for D) and the operational character of C_e^o results from the free energy derivatives β_i, as can be seen from equation 1.54. These are analogous to the coefficients which occur in elastic bodies in that they represent "reversible" coupling. These coefficients are known to produce only small temperature changes (with small strains) in steel and polymers (6), for example. The dissipation function, however, may produce considerable temperature

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temperature changes, the short time (high rate) value of specific heat was approximately one-half the long time (low rate) value. The theoretical expression for the constant pressure (or force) specific heat, which is given by equation 1.64, is consistent with this latter observation; namely, the positive property of all coefficients predicts that the short time value will always be smaller than the long time response to sudden changes in temperature.

It has also been found that the coefficient of expansion of many isotropic polymers has the same quantitative behavior near the glass transition temperature (9). Such similar behavior is not surprising in view of the close thermodynamic connection which exists between the heat capacity and thermal expansion operators, as exhibited by the relation 1.64 in generalized notation, specifically

$$C_Q^o = C_q + T_r \beta_i a_i^o$$

changes if deformation rates are non-zero for sufficiently long times. But for simple loading conditions, such as uniaxial creep under constant load, a temperature rise due to dissipation of only a few degrees is typical for small-strain, adiabatic deformation of polymers (6).

1.9. Extensions to Large Deformation and Crack Propagation Theory

This section deals with the role of thermodynamics in problems of finite viscoelastic deformation and crack propagation. Since the dissertation is concerned mainly with small deformation behavior, our discussion will be brief. However, it is hoped that the comments suggest a fruitful approach to the solution of such problems.

a. Large deformation theory. The thermodynamics of irreversible processes provides a natural means of incorporating viscous rate effects into the large deformation theory of solids. Indeed, if the imposed strain rates are not too high it is reasonable to assume that the linear rate law 1.11 is applicable, regardless of the magnitude of deformation. Such an assumption was used by Biot (22) when he indicated how the stress-strain equations given in section 1.7 can be used in an incremental formulation of large strain problems.

An alternate approach is to work directly with the equations of motion 1.19, for a unit mass, namely

$$Q_i = \rho \left(\frac{\partial f}{\partial q_i}\right)_{T, q_i'} + b_{ij}\dot{q}_j \; ; \quad i = 1, \ldots, n \qquad (1.123)$$

where the matrix b_{ij} may possibly be a function of temperature and generalized coordinates. If the free energy density, f, and the temperature and coordinate dependence of b_{ij} were known, these equations could be used to deduce the stress-strain-temperature equations of large deformation viscoelasticity.

b. Crack propagation theory. An energy equation will be derived which should be useful in the study of isothermal, viscoelastic crack propagation. First it must be realized that equation 1.123 is not only valid for a homogeneous system of unit mass, but with a modification of the notation can be applied to an arbitrary, inhomogeneous system at spacewise constant temperature.

Let us now multiply equation 1.123 by $\dot{q}_i$, sum over all variables, assume a constant temperature (in time as well as space), and thereby obtain the energy equation,

$$Q_i\mathring{q}_i = \mathring{F}_T + b_{ij}\mathring{q}_i\mathring{q}_j \qquad (1.124a)$$

or, equivalently

$$Q_i\mathring{q}_i = \mathring{F}_T + 2D_T \qquad (1.124b)$$

where $Q_i\mathring{q}_i$ is the total rate of work input to the system, $\mathring{F}_T$ is the rate of change of total Helmholtz free energy (including molecular or atomic bond energy), and D_T is the total dissipation. This equation is applicable, in particular, to an inhomogeneously strained solid with propagating cracks when the kinetic energy is negligible. On the other hand if the fracture takes place at high speed, it would be necessary to include the kinetic energy K_T, giving in this case

$$Q_i\mathring{q}_i = \dot{F}_T + 2D_T + \mathring{K}_T \qquad (1.124c)$$

An illustrative example: We shall now illustrate the usefulness of equation 1.124c by using it to predict slow[*] crack motion in a specific problem. Anticipating a fracture problem currently being investigated by Knauss (23), consider the geometry shown in figure 1.3. An infinitely long, thin, plate of viscoelastic material, containing a semi-infinite crack along the x-axis, is clamped along its upper and lower edges.

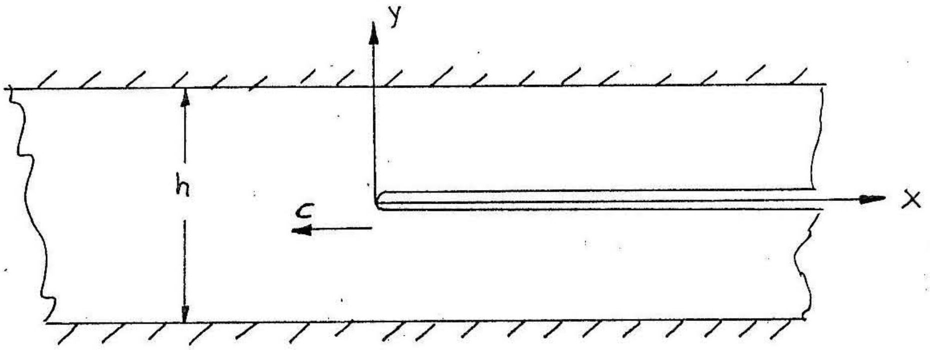

Figure 1.3. Crack Propagation in a Long Sheet

It is desired to find an expression for the steady-state crack velocity, c, which is attained after the grips are pulled apart and then held stationary. It should be noted, however, that this steady-state assumption is made only for simplicity, and is not required by equation 1.124c.

In this stationary state the rate of change of kinetic energy is zero and the rate of change of free energy is

$$\dot{F}_T = E_s c - E_\infty c \qquad (1.125)$$

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[*] We define a slow moving crack as one in which the effect of stress waves on crack velocity is negligible.

where E_∞ is the (constant) elastic strain energy per unit length in the sheet at large, negative values of x, and E_s is the surface (bond) energy per unit crack length (both surfaces of crack).* In addition, since the grips are fixed the rate of energy input is zero. Thus, according to 1.124c, the crack velocity must satisfy the equation

$$0 = -(E_\infty - E_s)c + 2D_T \tag{1.126}$$

which, in this form, is valid for large strains.

Before considering a calculation of D_T, let us assume that the velocity, c, is small, and use 1.126 to determine its value. Since D_T vanishes when the velocity is zero, and is also positive definite, the dissipation function can be approximated at small velocities by

$$D_T \simeq \frac{1}{2} bc^2 \tag{1.127}$$

where

$$b \equiv \left(\frac{\partial^2 D_T}{\partial c^2}\right)_{c=0}$$

which, when substituted into equation 1.126, yields the stable, steady-state velocity**

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*A tacit assumption used in writing equation 1.125, which must be emphasized, is that all bond-breaking is associated with the formation of one crack; hence, in order for this expression to be reasonably accurate, little or no fracture should occur away from the line of the crack. When this assumption does not apply, one can still consider E_s as containing all of the broken-bond energy, however it probably will be velocity dependent.

**It is easy to show that when $E_\infty \geq E_s$ this velocity is stable, while $c = 0$ is not stable. Such a proof is made by examination of equation 1.126; when the right-hand side is positive for small positive velocity perturbations, Δc, about a solution, c, this solution is stable since the maintenance of such a perturbation requires positive external work. On the other hand, a negative right-hand side implies an instability since the velocity could increase without the help of an external energy source.

$$c = \frac{E_{\infty} - E_s}{b} \quad ; \quad E_{\infty} \geq E_s \tag{1.128}$$

as well as the solution $c = 0$ which is unstable when $E_{\infty} > E_s$; the condition $E_{\infty} = E_s$ corresponds to the well-known Griffith criterion for initiation of crack growth in elastic bodies. Thus, if the elastic energy per unit length far ahead of the crack tip remains slightly larger than the surface energy absorbed per unit crack length, the crack will run at a low velocity given approximately by equation 1.128.

When the velocity is not small enough to permit the quadratic approximation to D_T, it becomes necessary to make an explicit calculation of D_T in order to examine the propagation characteristics. We shall not attempt here to make more than an approximate calculation of the dissipation using a simplified model and small strain theory. Consider therefore the crack shape shown in figure 1.4, in which the actual (dotted lines) shape is approximated by the solid, straight lines, and the distance L may be a function of the crack speed c. It will be assumed that the field for $x < 0$ is unaffected by the crack, and has the constant strain components,

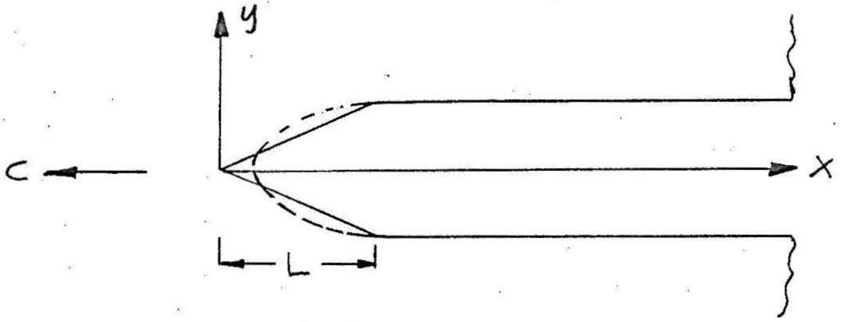

Figure 1.4. Idealized Crack

$$e_y \equiv e_\infty,\ e_x = 0\ ; \quad x < 0 \tag{1.129a}$$

while for $x > 0$ we shall assume that the strain e_y is independent of y, but has the approximate x dependence,

$$\begin{aligned} e_y &= \left(1 - \frac{x}{L}\right) e_\infty\ ; \quad 0 \leq x \leq L \\ e_y &= 0\ ; \quad x \geq L \end{aligned} \tag{1.129b}$$

and

$$e_x = 0\ ; \quad x > 0 \tag{1.129c}$$

In addition, all shear strains will be neglected in the calculation of D_T.

With the strain components given by 1.129a, plane stress, and the further simplifying assumption of incompressibility, the stress-strain law is

$$\sigma_y = \frac{4}{3} m_e e_y \tag{1.130}$$

so that the strain energy per unit sheet length becomes

$$E_\infty = \frac{2}{3} h m_e e_\infty^2 \tag{1.131}$$

where m_e is the equilibrium (long-time) uniaxial tensile modulus, and the sheet thickness is arbitrarily taken as unity.

The dissipation will be calculated using the general Wiechert model, figure 1.2, (but without the free dashpot, η) to represent the uniaxial tensile response of the sheet. The dissipation per unit volume is given by

$$2D = \frac{4}{3} \sum_s \eta_s \dot{e}_{s_\eta}^2 \tag{1.132}$$

The dashpot strain rates, $\dot{e}_{s_\eta}$, can be evaluated most easily by using the steady-state transformation, $\frac{d}{dt} \rightarrow c\,\frac{d}{dx}$, so that

$$2D = \frac{4}{3} c^2 \sum_s \eta_s \left(\frac{de_{s_\eta}}{dx} \right)^2 \tag{1.133}$$

In order to evaluate de_{s_η}/dx in terms of the strain components given by equation 1.129b, the model equations given in section 1.5 are used, but with d/dt replaced by c(d/dx). Carrying out this calculation, and integrating the result over $0 \leq x < \infty$, we finally obtain the total energy dissipation,

$$2D_T = \frac{4}{3} \frac{c^2 h e_\infty^2}{L} \sum_s \eta_s \left[1 - \frac{c\rho_s}{L} \left(1 - e^{-L/c\rho_s}\right) \right] \tag{1.134}$$

in which it is assumed that ρ_s and L are finite and non-zero, but it is not required that L be independent of velocity. First, consider the low speed $(c\rho_s/L \ll 1)$ and higher speed $(c\rho_s/L \gg 1)$ limit cases.

The dissipation at low velocities is

$$2D_T = \frac{4}{3} \frac{c^2 h e_\infty^2}{L} \sum_s \eta_s \quad ; \quad \frac{c\rho_s}{L} \ll 1 \tag{1.135}$$

so that from equation 1.127 we calculate $b = 4he_\infty^2 \sum_s \eta_s / 3L$, and the velocity is given by equations 1.128 and 1.131, thus

$$c = \frac{\frac{2}{3} h m_e e_\infty^2 - E_s}{\frac{4\ he_\infty^2}{3L} \sum_s \eta_s} \quad ; \quad \frac{c\rho_s}{L} \ll 1 \tag{1.136}$$

which is stable when the numerator is positive (see previous footnote).

Another interesting limit case is when $\frac{c\rho_s}{L} \gg 1$, in which case the dissipation is

$$2D_T = \frac{2}{3} che_\infty^2 (m_g - m_e) - \frac{2}{9} hLe_\infty^2 \sum_s \frac{m_s}{\rho_s} \; ; \quad \frac{c\rho_s}{L} >> 1 \qquad (1.137)$$

where we have defined the "glassy" modulus m_g as

$$m_g = \sum_s m_s + m_e \qquad (1.138)$$

Substitution of 1.137 into the energy equation 1.126 yields

$$c = \frac{\frac{2}{9} hLe_\infty^2 \sum_s \frac{m_s}{\rho_s}}{\frac{2}{3} he_\infty^2 (m_g - 2m_e) + E_s} \; ; \quad \frac{c\rho_s}{L} >> 1 \qquad (1.139)$$

which is a stable solution when the denominator does not vanish. It is observed that, regardless of the value of E_s and e_∞, the velocity is never unbounded when $m_g \geq 2m_e$, which is the case for polymers below their glass transition temperature; in fact it is usually true that $m_g \approx 10^3 m_e$ (9).*

For cases other than low $(\frac{c\rho_s}{L} << 1)$ and higher $(\frac{c\rho_s}{L} >> 1)$ speeds, the velocity must be calculated from the equation

$$c \sum_s \eta_s \left[1 - \frac{c\rho_s}{L} (1 - e^{-L/c\rho_s}) \right] = \frac{\frac{2}{3} hm_e e_\infty^2 - E_s}{\frac{4he_\infty^2}{3L}} \qquad (1.140)$$

---

*It should perhaps be noted that for brittle materials $m_g \approx m_e$, and it is theoretically possible to make e_∞ such that the denominator in 1.139 vanishes, and therefore have $c \to \infty$. Furthermore, it is recalled that under our assumptions kinetic energy does not put an upper limit on the speed since it does not enter into the energy equation 1.126 under steady-state conditions; the kinetic energy affects only the crack acceleration. The unboundedness occurs because the assumed dissipation is too small to absorb the strain energy ahead of the crack, and consequently this energy must go into increasing the kinetic energy. Of course with sufficiently high crack speeds in real materials, there will be enough energy transfer through stress wave action to put an upper limit on the speed.

It can be easily shown that for each s,

$$c\eta_s\left[1 - \frac{c\rho_s}{L}\left(1-e^{-L/c\rho_s}\right)\right] \rightarrow \frac{L\eta_s}{2\rho_s} = \frac{Lm_s}{2} \text{ as } c \rightarrow \infty \qquad (1.141)$$

and that this function increases monotonically from zero, with increasing c, to the limiting value $Lm_s/2$. Therefore, the sum in equation 1.140 increases monotonically with c, so that there is at most one solution to equation 1.140 for each set of physical constants and strain. Furthermore, as long as

$$0 < \frac{\frac{2}{3}hm_e e_\infty^2 - E_s}{\frac{4he_\infty^2}{3L}} < \frac{L}{2}\sum_s m_s = \frac{L}{2}(m_g - m_e) \qquad (1.142)$$

there will be one finite, stable, velocity c which satisfies equation 1.140. Criterion 1.142 can be written as

$$m_g - 2m_e > -\frac{3}{2}\,\frac{E_s}{he_\infty^2} \qquad (1.143)$$

with $\frac{2}{3}hm_e e_\infty^2 > E_s$, which is the same condition needed to prevent vanishing of the denominator in equation 1.139, as it should be. As observed earlier, we see from this inequality that when $m_g > 2m_e$ the velocity will never be (mathematically) infinite, regardless of the magnitude of the applied strain e_∞.

The crack propagation model will not be pursued further here, but this example does indicate a simple method for incorporating a dissipation mechanism into the criterion for slow fracture of viscoelastic media.

PART II

VARIATIONAL PRINCIPLES FOR IRREVERSIBLE SYSTEMS WITH APPLICATIONS TO THE THERMO-MECHANICAL BEHAVIOR OF VISCOELASTIC SOLIDS

2.1. Introduction

The important role of minimum principles in mechanics is well established. In the area of static, elastic stress analysis, for example, variational methods have been applied to appropriate energy functionals in order to generate approximate theories and numerical solutions to varying degrees of accuracy. The potential energy (13) is used to obtain approximate displacements, while if only stresses are desired the complementary energy (13) can be used. Hemp (24) has extended these principles to problems in thermoelasticity for the case in which the temperature field is unaffected by deformation. Inertia can be included in the elastic problem by using Hamilton's principle (25).

While much of the interest in variational principles for solids has been with these reversible processes, several notable papers have appeared recently dealing with the thermodynamic development and application of principles for systems with irreversibility. Biot (2, 5) has shown that the general equations of linear irreversible systems, equations 1.98, can be obtained from an operational-variational principle. This basic principle was applied to heat conduction (3), thermoelasticity (3), and isothermal viscoelasticity (5). Rosen (26) utilized a principle of minimum entropy production to derive stationary functions for heat conduction and viscous fluid flow. Besseling (27) obtained variational principles which are directly applicable to the nonlinear plasticity and creep problems of metals. The distinguishing characteristic of all these variational principles for irreversible systems is that the stationary functionals do not contain an integration with respect to time; hence, time and its derivatives are treated more or less as known parameters in the variation. An important implication therefore is that the stationary point of these functionals is not a true minimum, but is minimum only when time derivatives are not varied. This restriction on the admissible variations is to be contrasted with Hamilton's principle for reversible dynamic processes in which both time and space are treated alike.

Besseling believes that this special role of the time variable in a basic variational principle results from the defective treatment given time in classical non-relativistic thermodynamics, while Rosen states that this role probably results from the fact that analyses of irreversible problems are approximate because of the statistical methods used.

One of the purposes of this chapter is to show, however, that the general equations of linear systems are the Euler equations of a variational principle which is analogous to Hamilton's principle in that it consists of a time integral which has definite thermodynamic significance. In order to obtain this variational principle we have introduced a functional which is a convolution integral with respect to time. It is shown that this new concept treats time dependence as an initial value problem in the sense that varied paths have to satisfy only initial conditions.

The above mentioned Euler equations 1.98 are *differential* equations for generalized coordinates which are conjugate to specified generalized forces. It is also of practical importance to establish a functional whose Euler equations solve the inverse problem; namely, integral equations for unknown forces that are conjugate to prescribed coordinates. It will be seen that the coordinate principle represents an extension of the potential energy theorem for displacements in linear elastic bodies to general linear irreversible systems, while the variational principle for generalized forces is an extension of the complementary energy theorem. We shall derive this "complementary" principle directly from the original one for coordinates. The principle for coordinates and the one for forces are called "homogeneous" variational principles.

Following this development, we apply the results to the determination of functionals whose Euler equations are the field equations for the thermal and mechanical behavior of linear solids. It will be seen that all of the functionals can be deduced directly from the basic thermodynamic ones by calculating the appropriate free energy and dissipation functions. That such a unified treatment is possible can be attributed to the fact that linear solids must obey the same equations which are common to all processes satisfying the assumptions of TIP.

In order to illustrate the essential features of applying the basic principles, we consider first the simple heat conduction problem using the analogy between thermal and mechanical variables discussed in Part I. It will be recalled that the excess temperature (difference between actual and reference values) applied to a boundary plays the role of a force whose conjugate coordinate is the "entropy displacement" (total amount of heat which has flowed in through this boundary divided by the reference temperature). Thus, by applying the coordinate principle we arrive at a variational principle for the entropy displacement field, while application of the complementary theorem leads to one in terms of temperature. This temperature principle is similar to the one given by Rosen (26), but derived in a different manner.

Following this simple example, we derive variational principles for the linear thermo-mechanical behavior of viscoelastic solids. By linearity we mean that all thermodynamic variables are related to one another through linear differential or integral equations. Since temperature (or entropy displacement) is one of the variables, it must be sufficiently close to a reference value so that all properties (including viscosity) can be taken as constants with respect to temperature. Furthermore, strains must be small enough to use the linear strain-displacement relations. Linearity imposes another restriction which limits the coupling between temperature and deformation to reversible effects; it is necessary to assume that the influence of viscous dissipation on temperature is negligible.

A principle which is homogeneous in displacements (mechanical and entropy displacements) and one which is homogeneous in stresses (mechanical stresses and temperature) are deduced. Further, variational principles for mixed mechanical and thermal variables are given; namely, one for mechanical displacement and temperature, and one for mechanical stresses and entropy displacement. It will be seen that the stress and mixed variable principles can be derived directly from the original displacement theorem, in accordance with the general theory. In addition, it is shown that a generalization of Reissner's principle (28, 29) for mechanical displacements and stresses to thermo-viscoelasticity is obtained as a direct result of this change of variables, and that Reissner's principle is essentially the complementary theorem in which Lagrange multipliers are used with the equilibrium equations.

Another point of interest is concerned with the relation that exists between the thermo-elastic and thermo-viscoelastic variational principles when the above linearity assumptions are valid. It is shown that Laplace time-transforms of the functionals associated with visco-elastic and elastic principles are identical in form, with the only difference being that the viscoelastic one contains operational quantities in place of elastic constants. Thus, an approximate (or exact) transformed thermo-viscoelastic solution can be obtained from a transformed elastic solution by simply replacing elastic constants with appropriate operators. This is an extension of the correspondence rule deduced by Biot for isothermal, anisotropic viscoelasticity (2), as well as the one stated by Lee for exact solutions with isotropic media (30).

An additional item, which is of importance for the calculation of approximate solutions, is that the two homogeneous principles for mechanical and thermal stresses and displacements are true minimum principles when the Laplace transform parameter is real and positive. However, the two non-homogeneous principles (i.e. the one for mechanical displacements and temperature, and the one for mechanical stresses and entropy displacement) have this minimum property only if the temperature field is prescribed, so that the thermal variables are independent of mechanical stresses and displacements.

Following the discussion of principles for linear behavior, we suggest how these principles can be modified to include temperature dependent properties as well as the influence of viscous dissipation. However, we were not able to deduce full variational principles, but only ones in which certain artificial constraints are required in the variational process. Perhaps further investigation will yield principles in which such constraints are not required. Nevertheless, it is indicated briefly how these modified principles can be used in approximate analyses.

2.2. Basic Homogeneous Variational Principles for Linear Systems

a. Principle for generalized coordinates. We consider the same thermodynamic system defined in section 1.6, whose state is specified by n generalized coordinates q_i. The equations governing the behavior of the system were shown to be

$$a_{ij}q_j + b_{ij}\dot{q}_j = Q_i \qquad (i, j = 1, 2, \ldots, n) \tag{2.1}$$

where

$$a_{ij} = a_{ji} \qquad \text{and} \qquad b_{ij} = b_{ji} \tag{2.2}$$

and both matrices, a_{ij}, and b_{ij}, are assumed to be constant. An equivalent representation of equation 2.1 is the Lagrangian form

$$\frac{\partial V_T}{\partial q_i} + \frac{\partial D_T}{\partial \dot{q}_i} = Q_i \tag{2.3}$$

where it will be recalled that V_T is the generalized free energy defined as

$$V_T \equiv U_T - T_r S_T \tag{2.4}$$

with

U_T = total internal energy
S_T = total entropy
T_r = reference temperature

which is a non-negative quadratic function for small departures from the reference state,

$$V_T = \frac{1}{2} a_{ij} q_i q_j \geq 0 \tag{2.5}$$

Similarly, the dissipation function, D_T, is a non-negative function which is proportional to the total entropy production $\dot{S}_T'$,

$$D_T = \frac{1}{2} T_r \dot{S}_T' \geq 0 \tag{2.6}$$

and is given by

$$D_T = \frac{1}{2} b_{ij} \dot{q}_i \dot{q}_j \geq 0 \tag{2.7}$$

Biot (2) has indicated that equation 2.1 can be obtained from an operational-variational principle in which an operational form of the dissipation function is introduced as

$$D_T' \equiv \frac{1}{2} b_{ij} p q_i q_j \tag{2.8}$$

p is the time derivative, d/dt, which must be treated like a constant parameter in calculating the variation of D_T'. The equations of motion

2.1 are then equivalent to

$$\delta I' = \delta\{V_T + D'_T - Q_i q_i\} = \delta V_T + \delta D'_T - Q_i \delta q_i = 0 \qquad (2.9)$$

for all arbitrary variations of q_i and with all Q_i prescribed.

Use of the time derivative in this fashion is purely formal and does not lead to a variational principle that has a physical significance analogous to Hamilton's principle, in the sense that this latter principle consists of a time integral of certain energy functions. Of course, as pointed out by Biot, this operational notation is compact and therefore is convenient to use in applications because of its simplicity. However, we will now discuss a variational principle which does not require this operational notation, but does contain 2.9 as a special case. This more general one will be seen to have a wider range of application in the approximate solutions of linear problems.

First we introduce the functional $I_q(t)$ given by the convolution time integral,

$$I_q(t) = \int_0^t [V_T^* + D_T^* - Q_\ell(\tau) q_\ell(t-\tau)]\, d\tau \qquad (2.10a)$$

where

$$V_T^* \equiv \frac{1}{2} a_{ij} q_j(\tau) q_i(t-\tau) \qquad (2.10b)$$

$$D_T^* \equiv \frac{1}{2} b_{ij} \frac{dq_j(\tau)}{d\tau} q_i(t-\tau) \qquad (2.10c)$$

Before proceeding with the statement of the theorem, a few remarks will be made in regard to the term $Q_\ell(\tau) q_\ell(t-\tau)$. Equations 2.1 contain all coordinates needed to define the thermodynamic state, including those which are specified in a particular problem. However, only those which are not specified (and whose conjugate force must be given) should be included in the term $Q_\ell(\tau)q_\ell(t-\tau)$, while V_T^* and D_T^* are to contain all coordinates. Also, since the generalized forces Q_j are defined in such a way that $Q_\ell(\tau)\delta q_\ell(\tau)$ is the virtual work of forces external to the system, there may be a large number of coordinates whose conjugate force is identically zero, particularly if the system is an inhomogeneous collection of cells representing the thermodynamic model of a continuum. The theorem is now stated:

> Considering a linear irreversible system as defined above and assumed to be at rest in its reference state $q_i = \overset{\circ}{0}$ for $t = 0$, the actual path followed by the state variables q_i is determined by making $I_q(t)$ stationary with respect to all small variations δq_i of the unspecified coordinates from the actual values. That is, the Euler equations of
>
> $$\delta I_q = \int_0^t \{\delta[V_T^* + D_T^*] - Q_\ell(\tau)\delta q_\ell(t-\tau)\}d\tau = 0 \qquad (2.11)$$
>
> are equations 2.1
>
> $$a_{\ell j}q_j + b_{\ell j}\overset{\circ}{q}_j = Q_\ell \qquad (2.12)$$
>
> where the index ℓ ranges only over those values corresponding to unspecified coordinates while j indicates summation over all n coordinates.

We shall prove this theorem by showing that the stationary condition 2.11 requires that q_j satisfy 2.12. The variation is obtained by letting $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$ in 2.10a with $q_i(0) = \delta q_i(0) = 0$ and dropping terms of second order in $\delta q_i(t)$:

$$\delta I_q = \frac{1}{2}\int_0^t \{a_{ij}[q_i(\tau)\delta q_j(t-\tau) + \delta q_i(\tau)q_j(t-\tau)] + b_{ij}[\frac{d\delta q_i(\tau)}{d\tau} q_j(t-\tau) + \frac{dq_i(\tau)}{d\tau}\delta q_j(t-\tau)]\} d\tau - \int_0^t Q_\ell(\tau)\delta q_\ell(t-\tau)d\tau$$

Using the symmetry of a_{ij} we find

$$a_{ij}\int_0^t \delta q_i(\tau)q_j(t-\tau)d\tau = a_{ij}\int_0^t q_i(\tau)\delta q_j(t-\tau)d\tau$$

Also, by integrating the factor of b_{ij} by parts, using the symmetry of b_{ij}, and applying the initial condition $\delta q_i(0) = q_i(0) = 0$ yields

$$b_{ij}\int_0^t \frac{d\delta q_i(\tau)}{d\tau} q_j(t-\tau)d\tau = b_{ij}\int_0^t \frac{dq_i(\tau)}{d\tau} \delta q_i(t-\tau)d\tau$$

Since $\delta q_i = 0$ for $i \neq \ell$, the variation equation 2.11 becomes

$$\delta I_q = \int_0^t \{a_{\ell j}q_j(\tau) + b_{\ell j}\frac{dq_j(\tau)}{d\tau} - Q_\ell(\tau)\}\delta q_\ell(t-\tau)d\tau = 0 \qquad (2.13)$$

Because $\delta q_\ell(t-\tau)$ is arbitrary, the Euler equations are 2.12, which proves the theorem. Further, it can be easily shown that the second variation of I_q is not positive definite, which implies that the value of I_q in terms of the actual q_i is not necessarily an absolute minimum.

As a converse theorem, it can be stated that the actual path followed by the process is such as to make I_q stationary with respect to all small variations of unprescribed state variables satisfying zero initial conditions.

If the state variables are not initially zero but have some other prescribed value, the variational principle can be extended to this case by making either a shift of the time origin or adding $-\frac{1}{2} b_{ij} q_i(t) q_j(0)$ to the functional I_q.

In addition if some coordinates contribute kinetic energy (in the macroscopic sense), the internal energy will be composed of the usual static terms plus the energy $\frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j$ $(m_{ij} = m_{ji})$; hence, the functional to be used for the variational principle is

$$I_{qT} = \int_0^t \{ T_T^* + V_T^* + D_T^* - Q_\ell(\tau) q_\ell(t-\tau) \} d\tau \tag{2.14}$$

where V_T^* and D_T^* are defined by 2.10b and 2.10c, and

$$T_T^* \equiv \frac{1}{2} m_{ij} \frac{dq_i(t-\tau)}{d(t-\tau)} \frac{dq_j(\tau)}{d\tau}$$

The stationary condition on I_{qT} yields the Euler equations

$$m_{\ell j} \ddot{q}_j + b_{\ell j} \dot{q}_j + a_{\ell j} q_j = Q_\ell \tag{2.15}$$

with the initial conditions $q_i(0) = \dot{q}_i(0) = 0$.

It is important to observe that the convolution variational principle treats time dependence as an initial value problem, as it actually is; whereas in Hamilton's principle q_i must be prescribed at two different end points in time, which corresponds to a boundary value problem in this respect.

To show the correspondence between the functional I_q and Biot's I', equation 2.9, we take the Laplace (or Fourier) transform of I_q in 2.10:

$$\overline{I}_q(p) = \int_0^\infty I_q(t)e^{-pt}dt \;;\; I_q = 0 \text{ for } t \leq 0 \tag{2.16}$$

in which the transformed variable is denoted with a bar. The transform as a function of the complex parameter p is the Laplace transform, while if we let $p \to i\omega$ (ω = frequency) then definition 2.16 becomes the Fourier transform of I_q. Using the product rule of transform theory with equation 2.10, we find

$$\overline{I}_q = \frac{1}{2}[a_{ij}\overline{q}_i(p)\overline{q}_j(p) + b_{ij}p\overline{q}_i(p)\overline{q}_j(p)] - \overline{Q}_\ell(p)\overline{q}_\ell(p) \tag{2.17}$$

Comparing this with I' in 2.9 it is clear that if the variables in 2.9 are interpreted as being transforms of the physical quantities q_i, and the time operator p as the parameter p, then

$$\overline{I}_q(p) = I'(p) \tag{2.18}$$

Furthermore, a property which is of considerable practical importance is that the stationary point of $I_q(t)$ makes $I_q(p)$ an absolute minimum when p is real and positive. This positive definite nature of $\overline{I}_q$ will be discussed in Part III in regard to obtaining approximate solutions.

In deriving the following variational principle for forces, as well as principles for a continuum, it will be helpful to evaluate V^*_T and D^*_T by using a property of bilinear functions. Namely, multiplying 2.1 by $q_i(t-\tau)$ yields

$$a_{ij}q_j(\tau)q_i(t-\tau) + b_{ij}\frac{dq_j(\tau)}{d\tau}q_i(t-\tau) = Q_i(\tau)q_i(t-\tau)$$

so that

$$2[V^*_T + D^*_T] = Q_i(\tau)q_i(t-\tau) \tag{2.19}$$

It is important to note that Q_iq_i in 2.19 consists of <u>all</u> state variables, whether or not q_i is prescribed in a particular problem.

b. <u>Principle for generalized forces.</u> A change of variables in 2.10 will now be made in order to obtain a variational principle whose Euler equations are the inverse of equations 2.1. We shall call this a "complementary" or "force" principle in distinction to the previous one for coordinates or "displacements." The words "force" and "displacement" are used in more general sense than meaning just the mechanical quantities, as will become clear in the section 2.3 on applications. It will be seen that this variable change leads to a "complementary" variational principle in analogy to the well-known complementary energy principle of elasticity. Strictly speaking, the derivation of the complementary principle from the one for coordinates is heuristic and hence must ultimately be rigorously justified on the basis that it leads to the correct Euler equations. However, it will be seen later that the procedure followed in the derivation here is very useful in deducing the appropriate functions for a continuum.

Let us first write

$$Q_iq_i = Q_kq_k + Q_\ell q_\ell \tag{2.20}$$

where $1 \leq i \leq n$, and k ranges over those indices corresponding to specified coordinates, while ℓ indicates summation over the specified forces. Substituting equation 2.20 into I_q, equation 2.10a, yields

$$I_q = \int_0^t [(V_T^* + D_T^* - Q_i(\tau)q_i(t-\tau)) + Q_k(\tau)q_k(t-\tau)]\, d\tau \tag{2.21}$$

I_Q is now defined as the negative of I_q,

$$I_Q \equiv - I_q \equiv \int_0^t \{[\, Q_i(\tau) q_i(t-\tau) - (V_T^* + D_T^*)] - Q_k(\tau) q_k(t-\tau)\} \, d\tau \qquad (2.22)$$

which can also be written as

$$I_Q(t) = \int_0^t \{V_T^* + D_T^* - q_k(\tau) Q_k(t-\tau)\} \, d\tau \qquad (2.23)$$

where identity 2.19 has been used. The function $V_T^* + D_T^*$ in equation 2.23 is to be expressed in terms of Q_i by solving equations 2.1 and then substituting the result, $q_i(Q_i)$, into identity 2.19. In view of the discussion in sections 1.3 and 1.6, the solution to equation 2.1, for all coordinates, is given by equation 1.37 after setting $\tau_s' F \equiv \tau_s =$ constant and dropping the terms $\beta_i \theta$; thus,

$$q_i = \left[\sum_s \frac{C_{ij}^{(s)}}{1+\tau_s p} + \frac{C_{ij}}{p} \right] Q_i \qquad (2.24a)$$

and $V_T^* + D_T^*$ becomes

$$\begin{aligned} V_T^* + D_T^* &= \frac{1}{2} Q_i(t-\tau) \left[\sum_s \frac{C_{ij}^{(s)} Q_j(\tau)}{1+\tau_s p} + \frac{C_{ij} Q_j(\tau)}{p} \right] \\ &\equiv \frac{1}{2} Q_i(t-\tau) \left\{ \sum_s \frac{C_{ij}^{(s)}}{\tau_s} e^{-t/\tau_s} \int_0^t e^{v/\tau_s} Q_j(v) \, dv \right. \\ &\qquad \left. + C_{ij} \int_0^t Q_j(v) \, dv \right\} \end{aligned} \qquad (2.24b)$$

It should be noted that equation 2.24a is valid for all coordinates, q_i, while only those with index k (prescribed coordinates) are needed to determine the unknown forces Q_k. Therefore, on the basis of the above relation between I_Q and I_q we expect the Euler equations of the variation of 2.23 to be

$$q_k = \left[\sum_s \frac{C_{kj}^{(s)}}{1+\tau_s p} + \frac{C_{kj}}{p} \right] Q_j$$

$$\equiv \sum_s \frac{C_{kj}^{(s)}}{\tau_s} e^{-t/\tau_s} \int_0^t e^{v/\tau_s} Q_j(v) dv + C_{kj} \int_0^t Q_j(v)\, dv \qquad (2.25)$$

The complementary force theorem can now be stated:

> Considering a linear system as defined above and assumed to be at rest in its reference state at $t = 0$, the actual path followed by the forces Q_k, which are conjugate to the specified generalized coordinates q_k, is determined by making I_Q stationary with respect to all small variations δQ_k from the actual values; that is, the integral equations 2.25 are the Euler equations of the variation
>
> $$\delta I_Q = \int_0^t \{\delta [V_T^* + D_T^*] - q_k(\tau) \delta Q_k(t-\tau)\} \, d\tau = 0 \qquad (2.26)$$
>
> where I_Q is defined by equation 2.23.

By completing this variation, the Euler equations are found to be equations 2.25.

It is clear that the converse theorem is also true, i.e., the variation 2.26 vanishes as a result of 2.25. In addition, it can be shown that the actual value of I_Q is not an extremal, but is just a stationary point. However, as with $\overline{I}_q$ for p real and positive, $\overline{I}_Q$ is an absolute minimum for the actual path.

It is noted from equation 2.22 that the name "complementary" principle is appropriate since

$$I_Q(t) + I_q(t) = 0 \tag{2.27}$$

for the actual solution.

Another point of interest concerns the physical meaning that can be attached to the convolution functionals 2.10a and 2.23 in contrast to the standard form used for reversible processes, e.g. Hamilton's principle. A basic fact is, of course, that energy is degraded in any natural (irreversible) process. For example, if a viscoelastic body is loaded by mechanical forces, but is not in thermodynamic equilibrium, the motion of the state variables is such that mechanical energy is converted continuously into thermal energy, with an associated entropy increase. However, by using products of forward and backward running coordinates and forces in energy functionals, this irreversibility is removed in the sense that these functionals have the stationary property during the entire process.

2.3. Application of the Basic Homogeneous Principles to Viscoelastic Solids

While the general variational principles developed in section 2.2 apply directly to a system defined by n thermodynamic state variables, it is the purpose of this section to use them to formulate variational principles for the thermal and mechanical behavior of solids whose thermodynamic state is, in general, described by an infinite number of these variables. As mentioned earlier, a differential mass element of a continuum can be interpreted as being a uniform cell whose thermodynamic state is defined by m variables, say; and that this interpretation is valid as long as spacewise changes of variables are small relative to characteristic atomic or molecular distances. The total system is then defined by a set of m variables which vary continuously throughout the body. Consequently, the thermodynamic functions, V_T and D_T, and the external energy supplied to the system are given by volume and surface integrals with integrands consisting of a sum over the m variables of each cell.

It will be seen that this method of deducing continuum principles, in which thermodynamic functions are suggested by the discrete cell analysis, has several advantages. For one, the determination of variational principles is straightforward even though they are not at all obvious by simply examining the field (Euler) equations. In addition, this method provides functionals which are expressed in terms of thermodynamic invariants, and consequently the functionals are independent of the particular coordinate system used. Another advantage is that the Euler equations are guaranteed to be consistent with thermodynamics.

We now proceed to determine both "displacement" and "force" (or stress) principles for anisotropic media; first for pure heat conduction and then for the combined thermo-mechanical behavior of viscoelastic media. It will be assumed that inertia effects due to straining are negligible and that all variables vanish at the time origin. If this is not the case in any particular problem, the appropriate functional can be modified as discussed in section 2.2. We also assume that the geometric boundaries of the solid do not change with time except for small deformations and that all properties are independent of temperature.

a. A simple example--heat conduction.

α) Principle for entropy displacement: The thermodynamic state of a solid which is assumed to experience only thermal energy changes is defined by the absolute temperature, T, at each point. However, it has been shown that the variational principle 2.11 applies if entropy displacement, rather than temperature, is used as the thermodynamic variable. That this can be done follows from the conservation of energy statement

$$\operatorname{div} \vec{\dot{h}} \equiv \dot{h}_{i,i} = -C\,\dot{\theta} \tag{2.28}$$

where

$\vec{\dot{h}} = (\dot{h}_1, \dot{h}_2, \dot{h}_3)$ = heat flux vector

C = specific heat, per unit volume (assumed independent of temperature)

The relation between entropy displacement, $\vec{S} = (S_1, S_2, S_3)$, and temperature is then obtained by integration,

$$S_{i,i} = -\frac{C\theta}{T_r} \tag{2.29}$$

where the entropy displacement is defined as

$$S_i \equiv \frac{1}{T_r}\int_0^t \dot{h}_i\,dt = \frac{h_i}{T_r} \tag{2.30}$$

It is clear from equation 2.29 that S_i completely specifies the temperature, and hence the thermodynamic state of the body. It is interesting to observe that the temperature is only a function of div $\vec{S}$ and therefore is independent of curl $\vec{S}$. This property is analogous to the role of mechanical displacement, $\vec{u}$, since the rotation of material elements, curl $\vec{u}$, has no effect on the thermodynamic state of each element.

Let us now evaluate the terms V_T^* and D_T^* in I_q, equation 2.10, by using identity 2.19. Biot (2) calculated these functions from their thermodynamic definitions; however, for our purposes it is easier to use the identity

$$- \int_A \theta n_i S_i \, dA = Q_i q_i = 2[V_T^* + D_T^*] \qquad (2.31)$$

where n_i is the <u>outer</u> unit normal to the exterior surface A, and the variables t and τ are implied. In writing equation 2.31 we have used the thermo-mechanical analogy discussed in Part I, section 6. In this, the boundary temperature θ acts as a generalized force in the sense that

$$- \int_A \theta n_i \delta S_i \, dA = Q_i \delta q_i \qquad (2.32)$$

is virtual work (or energy addition) done on the body; the negative sign accounts for the definition that n_i is an <u>outer</u> normal. Since V_T^* and D_T^* are volume integrals of density functions, they can be determined by applying the divergence theorem to the surface integral in 2.31. This provides the identity

$$\int_A \theta n_i S_i dA = \int_B [\theta S_{i,i} + \theta_{,i} S_i] \, dB \qquad (2.33)$$

where B denotes volume integration.

From equation 2.29 we have

$$\theta S_{i,i} = -\frac{T_r}{C}(S_{i,i})^2 \qquad (2.34)$$

In order to express the second term in the volume integral in 2.33 as a function of entropy displacement, we must utilize the law of heat conduction given in Part I, equation 1.116,

$$K_{ij}\theta,_j = -\mathring{h}_i \equiv -T_r\dot{S}_i \; ; \quad K_{ij} = K_{ji} \qquad (2.35)$$

which can be inverted to find

$$\theta,_i = -T_r\lambda_{ij}\mathring{S}_j \qquad (2.36)$$

where λ_{ij} is the thermal resistivity matrix defined as

$$\lambda_{ij} = [K_{ij}]^{-1} \qquad (2.37)$$

which is assumed to be independent of θ. Using equation 2.36 we now have

$$\theta,_i S_i = -T_r\lambda_{ij}\mathring{S}_i S_j \qquad (2.38)$$

Substitution of equations 2.38 and 2.34 into identity 2.33 and using relation 2.31 yields

$$V_T^* + D_T^* = \int_B \left\{ \frac{T_r}{2C} S_{i,i}(\tau)S_{j,j}(t-\tau) + \frac{T_r}{2}\lambda_{ij}\mathring{S}_i(\tau)S_j(t-\tau)\right\} dB \qquad (2.39)$$

On comparing this with equations 2.5, 2.7, and 2.10, the generalized free energy density, V (per unit volume) is identified as

$$V = \frac{T_r}{2C}(S_{i,i})^2 \qquad (2.40)$$

which can also be expressed as a function of temperature by means of the energy equation 2.29,

$$V = \frac{C\theta^2}{2T_r} \qquad (2.41)$$

The dissipation density, D, is observed to be

$$D = \frac{T_r}{2}\lambda_{ij}\dot{S}_i\dot{S}_j \qquad (2.42)$$

or, in terms of temperature,

$$D = \frac{K_{ij}}{2T_r}\theta,_i\theta,_j \qquad (2.43)$$

The functional I_s for entropy displacement, corresponding to I_q in the general theory, can now be written by substituting equation 2.39 into equation 2.10a,

$$I_s = \int_0^t\int_B \{ \frac{T_r}{2C} S_{i,i}(\tau)S_{j,j}(t-\tau) + \frac{T_r}{2}\lambda_{ij}\dot{S}_i(\tau)S_j(t-\tau)\} \, dB\, d\tau$$
$$+ \int_0^t\int_{A_\theta} \Theta(\tau)n_iS_i(t-\tau)dA\, d\tau \qquad (2.44)$$

where A_θ is the portion of the surface on which $\theta = \Theta$ is prescribed, and S_in_i must satisfy the boundary conditions on A_s where heat flow is prescribed. This latter requirement arises from the fact that q_i in the general theory must satisfy constraints, which are the heat flow boundary conditions. In order to simplify the notation, we shall use the usual notation for convolution integrals,

$$f^*g \equiv \int_0^t f(\tau)g(t-\tau)\, d\tau \qquad (2.45)$$

where f and g are functions of time. Two useful properties of the product f^*g are

$$f^*g = g^*f \;; \quad (f+g)^*h = f^*h + g^*h \qquad (2.46)$$

In this simplified notation, the functional 2.44 becomes

$$I_s = \int_B \{ \frac{T_r}{2C} S^*_{j,j} S_{i,i} + \frac{T_r}{2} \lambda_{ij} \dot{S}_i{}^* S_j \} \, dB$$

$$+ \int_{A_\theta} \Theta^* n_i S_i \, dA \qquad (2.47)$$

The variational principle for entropy displacement can be stated as

$$\delta I_s = 0 \qquad (2.48)$$

for all arbitrary variations of S_i consistent with its boundary conditions. It can be shown that the Euler equations of 2.48 are the heat conduction equations

$$[\frac{1}{C} S_{i,i}]_{,j} = \lambda_{ji} \mathring{S}_i \qquad \text{in } B \qquad (2.49)$$

and the natural boundary condition is

$$S_{i,i} = - \frac{C\Theta}{T_r} \qquad \text{on } A_\theta \qquad (2.50)$$

That these are indeed the experimentally correct equations for entropy displacement is verified by substitution of equation 2.29 into the law 2.36.*

The Euler equations of 2.48 can be obtained in the same form as the general equations 2.1 if we first write

$$S_i = S_{ij}(x_i) q_j(t) \qquad (2.51)$$

---

*In passing it may be recalled that even though the principle 2.48 was derived from thermodynamics under the assumption of $\theta/T_r << 1$, it may have a wider range of applicability; this follows from the fact that the temperature range for which it is sufficiently accurate rests ultimately on the accuracy of its Euler equations in satisfying experimentally observed behavior.

where the quantities $S_{ij}(x_i)$ are assumed functions of x_i, consistent with the boundary conditions on S_i, and $q_j(t)$ are unspecified functions of time. Substitution of 2.51 into the variational equations 2.48 yields differential equations for the q_j which are identical in form with the basic relations 2.1.

β) Principle for temperature: In numerical applications, once the entropy displacement is found from equation 2.48, the temperature distribution can be calculated from the energy equation 2.29. This procedure is similar to the one followed with the potential energy theorem of elasticity, wherein displacements are first calculated from the variational principle, and then stress-strain equations are used to find stresses. However, as in the elasticity problem, in many cases it may be more desirable to obtain temperature ("stress") directly from a variational principle. This can be done by using the complementary function 2.23 for heat conduction, in which θ corresponds to the generalized force Q_i. Equation 2.41 provides us immediately with V_T^* as a function of temperature. The function D_T^* is found by combining equations 2.38 and 2.35,

$$D_T^* = \int_B D^* dB = -\frac{1}{2}\int_B \theta,_i S_i = \frac{1}{2T_r}\int_B \theta,_i K_{ij}\frac{\theta,_j}{p} dB \tag{2.52}$$

The functional for temperature, I_θ, can now be written by making the appropriate substitution in 2.23, thus

$$I_\theta = \frac{1}{2}\int_B \left\{\frac{C}{T_r}\theta^*\theta + \frac{1}{T_r}\theta^*,_i\left[K_{ij}\frac{\theta,_j}{p}\right]\right\} dB + \int_{A_S} \theta^* n_i S_i dA \tag{2.53}$$

where A_S is the portion of the boundary where heat flow is prescribed. It will probably be more convenient to use the time derivative of I_θ in applications,

$$T_r \overset{\circ}{I}_\theta = \frac{1}{2}\int_B \{C\dot{\theta}^* \theta + \theta^*_{,i}[K_{ij}\theta_{,j}]\}\, dB + \int_{A_S} \theta^* n_i \overset{\circ}{h}_i\, dA \qquad (2.54)$$

The temperature distribution is found from the variational equation[*]

$$\delta \dot{I}_\theta = 0 \qquad (2.55)$$

for all variations of θ which satisfy the boundary conditions on A_θ. It can be easily verified that the Euler equation of 2.55 is

$$C\dot{\theta} = [K_{ij}\theta_{,j}]_{,i} \qquad \text{in } B \qquad (2.56)$$

and the boundary condition is

$$n_i K_{ij}\theta_{,j} = -\overset{\circ}{h}_i n_i \qquad \text{on } A_S \qquad (2.57)$$

Since these are also obtained by combining equations 2.29 and 2.35, the variational principle is valid under the same conditions as is the entropy displacement principle. It may be noted in passing that equation 2.56 for an isotropic, homogeneous solid takes the well-known form

$$\frac{C}{K}\dot{\theta} = \nabla^2\theta \qquad (2.58)$$

---

[*] This variational principle is similar to Rosen's (26); however, he does not include a time integral so that $\dot{\theta}$ must be held constant in making the variation.

Let us now show that identity 2.27 is valid for the application at hand. Let $\theta(x_i, t)$ be the actual temperature distribution, and S_i the corresponding entropy displacement for a body with n_iS_i prescribed on A_S and θ prescribed on A_θ. Adding functionals 2.47 and 2.53,

$$I_s + I_\theta = 2\int_0^t [V_T^* + D_T^*]\, d\tau + \int_A \theta^* n_i S_i dA \qquad (2.59)$$

Since $(V_T^* + D_T^*)$ satisfies equation 2.31 we arrive at the simple result that for the "exact" solution,

$$I_s(t) + I_\theta(t) = 0 \qquad (2.60)$$

The same complementary property can be expected in other applications as a consequence of the fact that the underlying thermodynamic equations of motion are the same for all linear systems. In fact, this simple example of heat transfer illustrates well most of the essential features found in other applications.

b. <u>Thermo-viscoelasticity.</u> In this section variational principles for the linearized equations of thermo-viscoelasticity will be derived. It should be recalled that the thermodynamic linearity assumption requires material properties to be independent of temperature and the effect of dissipation on temperature to be neglected. Even though these assumptions are often too restrictive for thermal stress problems, several important sub-cases can be obtained from the general linear analysis. For example thermoelasticity, viscoelasticity (without temperature), and heat conduction appear as special cases.

a) Principle for mechanical and entropy displacements: We proceed as in the previous example by evaluating $V_T^* + D_T^*$ in terms of displacement with the aid of equation 2.19. For the case of both mechanical and thermal perturbations we have

$$Q_i q_i = \int_A T_i u_i dA + \int_B F_i u_i dB - \int_A \theta n_i S_i dA \qquad (2.61a)$$

where T_i is the surface force per unit area, F_i the prescribed body force per unit volume, u_i the mechanical displacement, and the variables t and τ are implied. The entropy displacement integral is written in terms of density functions by utilizing equation 2.33. The integrals representing mechanical energy can also be expressed as volume integrals by means of the divergence theorem. We find (13)

$$\int_A T_i u_i dA + \int_B F_i u_i dB = \int_B [\sigma_{ij,j} + F_i] u_i dB + \int_B \sigma_{ij} e_{ij} dB \qquad (2.61b)$$

where e_{ij} is the strain tensor,

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (2.62)$$

and σ_{ij} the stress tensor. Since V_T^* and D_T^* are defined as functions of state variables, they must be evaluated under the condition of mechanical equilibrium; this requires (neglecting inertia),

$$\sigma_{ij,j} + F_i = 0 \qquad \text{in } B \qquad (2.63)$$

Substitution of equation 2.61a into equation 2.19, after making use of equations 2.33, 2.61b, and 2.63 yields

$$2(V_T^* + D_T^*) = \int_B \{\sigma_{ij} e_{ij} - \theta S_{i,i} - \theta_{,i} S_i\} dB \qquad (2.64)$$

The last term in equation 2.64 can be expressed immediately as a function of entropy displacement by using the heat conduction law, 2.36, to obtain equation 2.38; the assumption that the heat conduction equation is unaffected by deformation was discussed in section 1.8. Also, the operational stress-strain-temperature equation 1.100a, is used to obtain

$$\sigma_{ij} e_{ij} = [Z^{\mu\nu}_{ij} \; e_{\mu\nu}] e_{ij} - [\beta^{o}_{ij} \theta] e_{ij} \qquad (2.65)$$

In the previous example it was possible to evaluate the second term in equation 2.64 , $\theta S_{i,i}$, through the use of the energy equation 2.29. However, we must now utilize the linearized version of equation 1.119, which can be written as

$$H = - T_r S_{i,i} = C^{o}_{e} \theta + T_r \beta^{o}_{ij} e_{ij} \qquad (2.66)$$

or

$$\theta = - T_r [C^{o}_{e}]^{-1} S_{i,i} - T_r [C^{o}_{e}]^{-1} \beta^{o}_{ij} e_{ij} \qquad (2.67)$$

where, in equation 1.119, we have set $\rho'_s = \rho_s$ = constant, $D = 0$, and integrated with respect to time. The inverse of the specific heat operator, $[C^{o}_{e}]^{-1}$, is of the form

$$[C^{o}_{e}]^{-1} = d_e + \sum_{s} \frac{d^{(s)}_{e} p}{p + \frac{1}{\rho_{SH}}} \qquad (2.68)$$

in which d_e and $d^{(s)}_{e}$ are positive constants; this property is obtained by comparing equation 2.67 and the earlier equation 1.69. It is also seen from equation 1.69 that $[C^{o}_{e}]^{-1} \beta^{o}_{ij}$ has this same form, but the coefficients are not necessarily positive.

Let us now substitute equations 2.65, 2.67, and 2.38 into identity 2.64 to find,

$$2\int_0^t (V_T^* + D_T^*)d\tau = \int_B \{ [Z_{ij}^{\mu\nu} e_{\mu\nu}] * e_{ij} + \frac{\theta}{T_r} * [C_e^o \theta] + T_r \lambda_{ij} \dot{S}_i * S_j \} dB \tag{2.69}$$

in which the convolution notation 2.45 is indicated and we have used the fact that

$$[\beta_{ij}^o \theta] * e_{ij} = \theta * [\beta_{ij}^o e_{ij}] \tag{2.70}$$

This identity can easily be verified by operating on it with the Laplace transform. Elimination of the temperature from equation 2.69 and substituting the result into equation (2.10a) yields the functional to be used in the variational principle:

$$\begin{aligned} I_{us} = \frac{1}{2}\int_B \{ & [Z_{ij}^{\mu\nu} e_{\mu\nu}] * e_{ij} + T_r [(C_e^o)^{-1} (S_{i,i} + \beta_{ij}^o e_{ij})] * [S_{i,i} + \beta_{ij}^o e_{ij}] \\ & + T_r \lambda_{ij} \dot{S}_i * S_j \} dB - \int_{A_T} T_i * u_i dA - \int_B F_i * u_i dB \\ & + \int_{A_\theta} \Theta * n_i S_i dA \end{aligned} \tag{2.71}$$

where the force vector T_i is prescribed on A_T and temperature Θ is prescribed on A_θ. Also, the strains must be expressed as functions of displacement by using equation 2.62.

The equations for mechanical and entropy displacements are determined by the stationary condition

$$\delta I_{us} = 0 \tag{2.72}$$

for all arbitrary variations of u_i and S_i compatible with their boundary conditions. By carrying out the variation we find that the

Euler equations of 2.72 are the three equilibrium equations;

$$[Z^{ij}_{\mu\nu} + T_r\beta^o_{\mu\nu}\beta^o_{ij}\,(C^o_e)^{-1}]\,u_{i,j\nu} + T_r(C^o_e)^{-1}\beta^0_{\mu\nu}S_{i,i\nu} + F_\mu = 0 \qquad \text{in B} \tag{2.73}$$

and the three heat conduction equations

$$C^o_e\lambda_{ij}\dot{S}_i - S_{i,ij} - \beta^o_{\mu\nu}u_{\mu,\nu j} = 0 \qquad \text{in B} \tag{2.74}$$

This same set of equations can be obtained directly from the stress-strain-temperature equations, energy equation 2.67, and heat conduction equation 2.36; on the basis of this remark as well as the fact that the natural boundary conditions of the variation 2.72 are

$$n_j\sigma_{ij}(u_\nu, S_\mu) \equiv \{[Z^{\mu\nu}_{ij} + T_r\beta^o_{\mu\nu}\beta^o_{ij}\,(C^o_e)^{-1}]\,e_{\mu\nu} + T_r(C^o_e)^{-1}\beta_{ij}S_{\mu,\mu}\}n_j = T_i \qquad \text{on } A_T \tag{2.75}$$

$$S_{i,i} = -\frac{C^o_e\Theta}{T_r} \qquad \text{on } A_\theta \tag{2.76}$$

we conclude that the variational principle is valid within the region of validity of the field equations 2.73 and 2.74.

Some special cases of the variational principle 2.72 will now be considered. For viscoelasticity without thermal effects, set $\theta = 0$ in equation 2.64 and use the result to write

$$I_u = \frac{1}{2}\int_B \sigma^*_{ij}e_{ij}dB - \int_B F^*_i u_i dB - \int_{A_T} T^*_i u_i dA \tag{2.77}$$

in which the stress tensor is to be expressed as a function of the strains through the operator equation appropriate for the desired degree of anisotropy. In order to obtain the functional for the case in which there are strains due to temperature, but the *temperature* field is prescribed, it is necessary to first write equation 2.64 in terms of strains and temperature. This situation will be considered later in connection with "non-homogeneous" variational principles.

The functional for adiabatic deformation is obtained immediately from equation 2.71 by setting $S_i = S_{i,i} = 0$. Furthermore, the variational principle for thermoelasticity is deduced from 2.71 by simply replacing the operators $Z^{ij}_{\mu\nu}$, C^o_e and β^o_{ij} with their corresponding elastic constants. It is also clear that the functional for pure heat conduction results by equating the displacement field to zero.

In view of the remarks on the general principles in section 2.2, the variational principle 2.72 can be formulated in terms of the Laplace transforms of displacement. This is done by transforming I_{us} with the help of the convolution theorem, thus

$$\overline{I}_{us} = \frac{1}{2}\int_B \{Z^{\mu\nu}_{ij}\overline{e}_{\mu\nu}\overline{e}_{ij} + T_r(C^o_e)^{-1}(\overline{S}_{i,i} + \beta^o_{ij}\overline{e}_{ij})^2 + T_r\lambda_{ij}p\overline{S}_i\overline{S}_j\}\,dB$$

$$-\int_B \overline{F}_i\overline{u}_i\,dB + \int_{A_\theta}\overline{\Theta}n_i\overline{S}_i\,dA - \int_{A_T}\overline{T}_i\overline{u}_i\,dA \qquad (2.78)$$

where all time derivatives in the operational coefficients are replaced by the transform parameter p. It can be shown that the stationary point of $\overline{I}_{us}$ is actually an absolute minimum for p real and positive. Furthermore, it is clear that the Euler equations and natural boundary conditions of

$$\delta\overline{I}_{us} = 0 \qquad (2.79)$$

are the same equations one obtains by transforming equations 2.73, 2.74, 2.75 and 2.76.

Thus, it is seen that the thermo-viscoelastic variational principle in terms of transformed variables is formally identical with the one for thermo-elasticity. This leads to a correspondence rule in which an approximate (or exact) transformed viscoelastic solution can be obtained directly from a transformed "associated" elastic solution as calculated from a variational principle; it is only necessary to replace all material constants by the appropriate operators, e.g., $Z^{ij}_{\mu\nu}(p)$. This same rule extends to all principles which we shall derive from the basic thermodynamic ones. The correspondence rule stated here is a generalization of the one shown by Biot for isothermal viscoelasticity (2).

Before we discuss the complementary principle, let us examine briefly an alternate method of calculating $V^*_T + D^*_T$, i.e., use of the thermodynamic definitions for the generalized free energy and entropy production directly. This will show more clearly the close correspondence between the basic form of the variational principle given in section 2.2 and the one derived in this section.

First, the generalized free energy density, V, (per unit volume) is obtained by relating it to the Helmholtz free energy density, F. We have the definitions

$$V \equiv U - T_r S \tag{2.80a}$$

$$F \equiv U - (\theta + T_r)S = V - \theta S \tag{2.80b}$$

Reference to equations 1.18a and 1.26 shows that

$$V = F - \theta \left(\frac{\partial F}{\partial T} \right)_{q_i} = \frac{1}{2} \left[a_{ij} q_i q_j + \frac{C_q \theta^2}{T_r} \right] \tag{2.81}$$

where C_q is the specific heat (per unit volume). It is implicit in the expression that the local thermodynamic state is defined by excess temperature, θ, and the n coordinates q_i, six of which $(q_1, q_2, \ldots, q_6)$ are the mechanical strains e_{ij}. Therefore, the total free energy is

$$V_T = \frac{1}{2}\int_B \{a_{ij}q_iq_j + \frac{C_q}{T_r}\theta^2\}\, dB \tag{2.82}$$

Since the variational principle must be written in terms of entropy displacement, rather than temperature, it is necessary to use the linearized energy equation 1.66. Elimination of temperature difference, θ, provides us with the appropriate form of the generalized free energy,

$$V_T = \frac{1}{2}\int_B \{a_{ij}q_iq_j + \frac{T_r}{C_q}(S_{i,i} + \beta_iq_i)^2\}\, dB \tag{2.83}$$

The entropy production per unit volume separates into non-thermal and thermal components.* The thermal component leads to the dissipation per unit volume given by equation 2.42

$$D_t \equiv \frac{T_r}{2}\lambda_{ij}\dot{S}_i\dot{S}_j \tag{2.84}$$

The non-thermal contribution to dissipation is obtained from equation 1.55,

$$D_m \equiv \frac{1}{2}b_{ij}\dot{q}_i\dot{q}_j \tag{2.85}$$

Thus, we can write the total dissipation function as

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*This follows from the assumption that the thermal contribution to entropy production is of a different degree of symmetry than the other causes, as discussed in section 1.8.

$$D_T = \frac{1}{2}\int_B \{b_{ij}\dot{q}_i\dot{q}_j + T_r\lambda_{ij}\dot{S}_i\dot{S}_j\}\, dB \tag{2.86}$$

It is now possible to substitute V_T^* and D_T^*, corresponding to the functions given by equations 2. 83 and 2. 86, directly into the variational principle 2. 11. We thereby obtain a variational principle for hidden coordinates as well as the observed variables of mechanical and entropy displacement,

$$\delta I_{qs} = 0 \tag{2.87}$$

with the definition

$$\begin{aligned} I_{qs} = \frac{1}{2}\int_B \{(a_{ij}q_j + b_{ij}\dot{q}_j)^* q_i + \frac{T_r}{C_q}(S_{i,i} + \beta_i q_i)^*(S_{i,i} + \beta_i q_i) \\ + T_r\lambda_{ij}\dot{S}_i^* S_j\}\, dB \\ - \int_{A_T} T_i^* u_i\, dA - \int_B F_i^* u_i\, dB + \int_{A_\theta} \Theta^* n_i S_i\, dA \end{aligned} \tag{2.88}$$

where the strains e_{ij} $[= q_i \ (i = 1, 2, \ldots, 6)]$ are to be expressed as functions of the mechanical displacements using equation 2. 62. It can be shown that the Euler equations of 2. 87 are not only the mechanical equilibrium and heat conduction equations, but also (n-6) equations for hidden coordinates as given by equation 1. 40. If, however, we initially eliminate the hidden coordinates from equation 2. 88 by solving the (n-6) equations, then the functional 2. 88 becomes identical with the first one we derived, equation 2. 71.

β) Principle for mechanical stresses and temperature: The complementary principle for stresses and temperature will now be deduced from the basic functional 2. 22. While it was possible to use the simpler form 2. 23 in deriving the heat conduction principle, it cannot be used now. This stems from the fact that when we introduce equation 2.19, $[2(V_T^* + D_T^*) = Q_i(\tau)q_i(t-\tau)]$, it is tacitly assumed that the internal stress field is in equilibrium. The additional term which must be added if this is not the case is

$$\int_B [\sigma_{ij,j} + F_i] u_i \, dB$$

as seen by referring to equation 2.61b. Thus we have

$$Q_i q_i - (V_T^* + D_T^*) = \int_B [\sigma_{ij,j} + F_i] u_i dB + (V_T^* + D_T^*) \quad (2.89)$$

where $V_T^* + D_T^*$ is to be written as a function of stress and temperature with the help of equation 2.64.

For the mechanical component, use is made of the stress-strain-temperature equation 1.103a derived in Part I, so that

$$\sigma_{ij} e_{ij} = \sigma_{ij}[A_{ij}^{\mu\nu} \sigma_{\mu\nu}] + \sigma_{ij}[a_{ij}^{o} \theta] \quad (2.90)$$

In order to express the first thermal integrand in equation 2.64 in terms of temperature we use the energy equation 1.63. After linearizing and integrating with respect to time we find,

$$S_{i,i} = -\frac{C_\sigma^o}{T_r} \theta - a_{ij}^o \sigma_{ij} \quad (2.91)$$

where C_σ^o is the constant-stress specific heat operator whose form is given by equation 1.64. Therefore

$$-\theta S_{i,i} = \frac{\theta}{T_r} [C_\sigma^o \theta] + \theta [a_{ij}^o \sigma_{ij}] \quad (2.92)$$

The second thermal term is found immediately by using the heat conduction law 2.35,

$$-\theta,_i S_i = \frac{\theta,_i}{T_r}\,[\,K_{ij}\frac{\theta,_j}{p}\,] \tag{2.93}$$

Now that $V_T^* + D_T^*$ is known as a function of temperature and stress, we can substitute equation 2.89 into equation 2.22 to obtain the complementary functional $I_{\sigma\theta}$,

$$\begin{aligned} I_{\sigma\theta} = \int_B [\,\sigma_{ij,j} + F_i]^{*} u_i \, dB + \frac{1}{2}\int_B \{\sigma_{ij}^{*}[\,A_{ij}^{\mu\nu}\sigma_{\mu\nu}] + 2\theta^{*}[\,a_{ij}^{o}\sigma_{ij}] \\ + \frac{\theta^{*}}{T_r}[\,C_{\sigma}^{o}\theta] + \frac{\theta^{*},_i}{T_r}[\,K_{ij}\frac{\theta,_j}{p}\,]\}dB \\ - \int_{A_u} U_i^{*}\sigma_{ij}n_j\,dA + \int_{A_S}\theta^{*}n_iS_i\,dA \end{aligned} \tag{2.94}$$

where displacements U_i are prescribed on A_u and the entropy-displacement (i. e., heat flow) is given on A_s. The mechanical displacement field, u_i, can be eliminated from this functional by requiring that all admissible states of stress in the variational principle satisfy the equilibrium equations. This is accomplished by expressing the stresses as derivatives of certain stress functions (31). However, an equivalent procedure is to interpret continuous functions u_i as Lagrange multipliers associated with side condition,

$$\sigma_{ij,j} + F_i = 0 \qquad \text{in } B \tag{2.95}$$

This latter method will be used since it leads to an easier proof.

As the complementary principle we state that actual stress and temperature distributions are determined by the variational equation

$$\delta I_{\sigma\theta} = 0 \qquad (2.96)$$

where all admissible stresses satisfy the stress boundary conditions,

$$\sigma_{ij} n_j = T_i \qquad \text{on} \quad A_T \qquad (2.97)$$

and all admissible temperature fields satisfy the boundary conditions on A_θ. Performing the variation with stresses and displacements varied independently we arrive at the Euler equations, which are the six compatibility conditions

$$A_{ij}^{\mu\nu} \sigma_{\mu\nu} + a_{ij}^{o} \theta = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{in} \quad B \qquad (2.98)$$

the three equilibrium conditions,

$$\sigma_{ij,j} + F_i = 0 \qquad \text{in} \quad B \qquad (2.99)$$

and the conservation of energy statement,

$$\left[K_{ij} \frac{\theta_{,j}}{p}\right]_{,i} = C_\sigma^o \theta + T_r a_{ij}^o \sigma_{ij} \qquad \text{in} \quad B \qquad (2.100)$$

In addition, the natural boundary conditions of the variation are the conditions on the Lagrange multiplier u_i,

$$u_i = U_i \qquad \text{on} \quad A_u \qquad (2.101)$$

and the entropy displacement boundary condition,

$$n_i K_{ij} \frac{\theta_{,j}}{p} = - T_r S_i n_i \qquad \text{on} \quad A_S \qquad (2.102)$$

Equations 2.98 - 2.102 are a complete set for calculation of the stresses and temperature field. It is clear that these are the correct equations; the compatibility condition 2.98 relates stress-dependent strains to displacement-dependent strains, while equation 2.100 is simply the linearized energy equation 2.91 combined with heat conduction law 2.35.

It is interesting that if the equilibrium equations are not identically satisfied in the variation, application of Lagrange multipliers with the equilibrium equations gives a principle for displacements as well as stresses and temperature. In fact, it will now be shown that this principle is an extension of Reissner's principle for elastic stresses and displacements (28) to thermo-viscoelasticity, except for the constraint on boundary values. The functional 2.94 is cast in Reissner's form by using identity 2.61b in order to replace the first integral in this functional. We find

$$I_{\sigma u} = \frac{1}{2}\int_B \{-\sigma^*_{ij}(u_{i,j} + u_{j,i}) + \sigma^*_{ij}[A^{\mu\nu}_{ij}\sigma_{\mu\nu}]\}\,dB + \int_{A_T} T^*_i u_i\,dA + \int_B F^*_i u_i\,dB \qquad (2.103)$$

where the thermal terms have been omitted for simplicity. As before, T_i is the prescribed stress vector on A_T and F_i is the prescribed body force. Reissner's principle for viscoelasticity can be stated as follows:

> Among all displacements which satisfy the displacement boundary conditions and among all stress states, the actual stresses and displacements are determined by the variational

equation

$$\delta I_{\sigma u} = 0 \qquad (2.104)$$

in which the stresses and displacements are varied independently.

As Euler equations we find the six stress-displacement compatibility conditions,

$$A_{ij}^{\mu\nu}\sigma_{\mu\nu} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{in} \quad B \qquad (2.105)$$

and the three equilibrium equations

$$\sigma_{ij,j} + F_i = 0 \qquad \text{in} \quad B \qquad (2.106)$$

The natural boundary conditions are on the stresses,

$$\sigma_{ij}n_j = T_i \qquad \text{on} \quad A_T \qquad (2.107)$$

These field equations and boundary conditions are sufficient for calculation of the six stresses and three displacement components. It should be noted that the admissible displacements in 2.103 must satisfy their boundary conditions, while the stresses are not restricted. However, just the converse is true in the functional 2.94. Nevertheless, from a practical standpoint, in using these principles to generate approximate solutions it probably will be desirable to choose stresses and displacements which satisfy their respective boundary conditions. In this case, the two principles are identical.

In concluding this section on the complementary principle, it should be remarked that it has properties which are similar to those of the displacement principle 2.72; i.e., concerning special limit cases, the correspondence rule, and an absolute minimum point.

As an example of an important limit case, consider the situation occurring most frequently in practice in which the temperature change due to stresses is negligible, so that the temperature is prescribed variable in the functional 2.94 (i.e., it is calculated prior to making the stress analysis by using the equation for heat conduction). In this case, the pure thermal terms in 2.94 do not affect the variation and can therefore be omitted. Thus, the functional becomes

$$I_{\sigma\theta} = \int_B [\sigma_{ij,j} + F_i] \overset{*}{} u_i \, dB + \frac{1}{2}\int_B \{\sigma_{ij} \overset{*}{} [A_{ij}^{\mu\nu}\sigma_{\mu\nu}] + 2\theta \overset{*}{} [\overset{o}{a}_{ij}\sigma_{ij}]\} \, dB - \int_{A_u} U_i \overset{*}{} \sigma_{ij} n_j \, dA \tag{2.108}$$

in which θ is to be held constant when calculating the variation.

In regard to the minimum property, it can be shown that the exact stresses and temperature make $\overline{I}_{\sigma\theta}(p)$, with p real and positive, an absolute minimum with respect to all stresses satisfying the equilibrium equations 2.95 and stress boundary condition 2.97, and all temperature fields satisfying the boundary conditions on temperature.* It should be added that in proving this minimum property, one must use the thermodynamic relations between $A_{ij}^{\mu\nu}$, $\overset{o}{a}_{ij}$, and $\overset{o}{C}_{\sigma}$ which are given in generalized coordinate notation in Part I.

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*This minimum character does not exist if the varied states of stress do not identically satisfy the equilibrium equations and stress boundary conditions. Thus, the transform of the functional in Reissner's principle, equation 2.103, is just stationary on the positive real p-axis.

2.4. Determination of Non-Homogeneous Variational Principles from the Homogeneous Principles for Linear Systems

So far in the applications we have utilized two basic variational theorems which are homogeneous in displacement and stress variables; i.e. equation 2.11 for mechanical and thermal displacements, and equation 2.26 for mechanical and thermal stresses. In addition, it was shown that the stress theorem could be derived directly from the original one for displacements by making a change of variables. Furthermore, in section 2.3b this change of variables led to a non-homogeneous principle for mechanical displacements and stresses by retaining the equilibrium equations with Lagrange multipliers. An analogous principle for temperature and entropy displacement could have also been deduced in the heat conduction example; in this case the heat conduction law 2.36 would appear with a Lagrange multiplier.

Thus, it seems reasonable that a principle could be derived for the calculation of all functions, u_i, σ_{ij}, θ, and S_i. Indeed, it can be shown that all of the above mentioned principles are special cases of this more general one.

We shall not here derive this general principle, but examine briefly two additional variational principles which are also special cases. It will be seen that they are of practical importance and represent companions to the homogeneous thermo-viscoelastic principles in section 2.3b. The first one considered is for mechanical displacements and temperature and the second one is for mechanical stresses and entropy displacement.

a. Principle for mechanical displacements and temperature. The functional is derived most easily by means of a heuristic argument using generalized variables. Let $Q_j q_j$ be divided into mechanical variables $q_r Q_r$ and thermal variables $q_s Q_s$. In analogy to the derivation of equation 2. 22 we obtain the functional

$$I_{Qq} = \int_0^t \{[V_T^* + D_T^* - Q_s q_s] + Q_k q_k - Q_i q_i\} d\tau \qquad (2.109)$$

where the index k indicates summation over prescribed thermal coordinates, and index i indicates summation over prescribed mechanical forces. The term in the square brackets is to be written as a function of mechanical coordinates and thermal forces.

We now identify the thermal energy $Q_s q_s$ with its continuum representation

$$Q_s q_s \equiv - \int_A \theta n_i S_i \, dA \qquad (2.110)$$

and first use identity 2. 33 as well as 2. 64 in determining the displacement and temperature dependence of $(V_T^* + D_T^* - q_s Q_s)$; the procedure for eliminating the mechanical stresses and entropy displacement is similar to what we have followed with the previous cases. When the results of this calculation are substituted into equation 2. 109 the functional for displacement and temperature is obtained,

$$I_{u\theta} = \frac{1}{2} \int_B \{[Z_{ij}^{\mu\nu} e_{\mu\nu}]^* e_{ij} - 2\theta^* [\beta_{\mu\nu}^o e_{\mu\nu}] - \frac{\theta^*}{T_r} [C_e^o \theta]$$

$$- \frac{\theta,_i^*}{T_r} [K_{ij} \frac{\theta,_j}{p}]\} dB - \int_{A_T} T_i^* u_i \, dA - \int_B F_i^* u_i \, dB - \int_{A_S} \theta^* n_i S_i \, dA$$

$$(2.111)$$

where T_i is prescribed on A_T and S_i is specified on A_S. The displacement and temperature fields are determined by the stationary condition

$$\delta I_{u\theta} = 0 \tag{2.112}$$

in which all admissible displacements and temperatures must agree with their respective boundary values on A_u and A_θ. The Euler equations are the three equilibrium equations,

$$Z^{\mu\nu}_{ij} u_{\mu,\nu j} - \beta^{o}_{ij}\theta,_{j} + F_i = 0 \qquad \text{in } B \tag{2.113}$$

and the energy equation

$$\left[K_{ij} \frac{\theta,_j}{p} \right],_i = C^{o}_{e}\theta + T_r \beta^{o}_{ij} e_{ij} \qquad \text{in } B \tag{2.114}$$

with the natural boundary conditions

$$[Z^{\mu\nu}_{ij} u_{\mu,\nu} - \beta^{o}_{ij}\theta] n_j = T_i \qquad \text{on } A_T \tag{2.115}$$

$$n_i \left[K_{ij} \frac{\theta,_j}{p} \right] = - T_r S_i n_i \qquad \text{on } A_S \tag{2.116}$$

It is easily verified that these equations are the complete, correct set for the description of the displacement and temperature distributions.

An important limit case of 2.112 is when the temperature field is prescribed, and therefore is assumed independent of displacements. The variational principle for this condition is obtained by omitting the pure thermal terms in equation 2.111 and holding the temperature fixed in the variation 2.112. With the temperature prescribed, it can be shown that among all displacements satisfying the displacement boundary conditions, those which satisfy the equilibrium equations make $\overline{I}_{u\theta}$ an absolute minimum when p in on the positive real axis. However, if the temperature is a function of the displacement, $\overline{I}_{u\theta}$ is only stationary.

b. <u>Principle for mechanical stresses and entropy displacement.</u> The functional for the mixed variables of mechanical stress and entropy displacement is derived in a similar manner. We find that the field equations and natural boundary conditions for σ_{ij} and S_i are obtained from the variational equation

$$\delta I_{\sigma S} = 0 \tag{2.117a}$$

where

$$\begin{aligned} I_{\sigma S} = \int_B [\sigma_{ij,j} + F_i]^* u_i dB + \frac{1}{2}\int_B \{[A_{ij}^{\mu\nu}\sigma_{\mu\nu}]^*\sigma_{ij} \\ - T_r[(C_\sigma^o)^{-1}(S_{i,i} + \alpha_{ij}^o\sigma_{ij})]^*[S_{i,i} + \alpha_{ij}^o\sigma_{ij}] - T_r\dot{S}_i^*[\lambda_{ij}S_j]\} dB \\ - \int_{A_u} U_i^* \sigma_{ij} n_j dA - \int_{A_\theta} \Theta^* n_i S_i dA \end{aligned} \tag{2.117b}$$

in which U_i is given on A_u and Θ is specified on A_θ. The admissible stresses and entropy displacements in 2.117 must satisfy the boundary conditions on A_T and A_S, respectively. Furthermore, as with the complementary principle, the displacements appearing in the first integral are Lagrange multipliers which drop out if all varied states of stress are chosen such that the equilibrium equations are identically satisfied. We find that the Euler equations are the six compatibility conditions,

$$[A_{ij}^{\mu\nu} - T_r(C_\sigma^o)^{-1} a_{\mu\nu}^o a_{ij}^o]\sigma_{\mu\nu} - T_r(C_\sigma^o)^{-1}\alpha_{ij}^o S_{\mu,\mu} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.118}$$

and the heat conduction equation

$$T_r \lambda_{ij} \dot{S}_j = [\, T_r (C^o_\sigma)^{-1} (S_{\mu,\mu} + a^o_{\mu\nu} \sigma_{\mu\nu})]_{,i} \qquad (2.119)$$

with the natural boundary conditions

$$u_i = U_i \qquad \text{on} \qquad A_u \qquad (2.120)$$

$$T_r (C^o_\sigma)^{-1} (S_{\mu,\mu} + \alpha^o_{\mu\nu} \sigma_{\mu\nu}) = - \Theta \quad \text{on} \quad A_\theta \qquad (2.121)$$

which form the correct, complete, set for the calculation of stress and entropy displacement fields.

When the entropy displacement is prescribed, the exact stress state makes the functional $\overline{I}_{\sigma s}(p)$ an absolute minimum (with p real and positive) with respect to varied stress states which satisfy equilibrium and stress coundary bonditions. As with the previous non-homogeneous principle, this minimum property does not exist in the general coupled thermo-mechanical problem. Furthermore, it is clear that the adiabatic limit case is obtained by setting $S_i \equiv 0$ in the functional 2.117.

2.5. Comments on Special Variational Principles when Dissipation and Temperature Dependent Viscosity are Considered

It has been emphasized that the previous variational principles are valid only for thermodynamically linear systems, and that this assumption requires the dissipation function in the energy equation and the temperature dependence of material properties to be neglected. There does not appear, at this time, to be a variational principle which can be used for these nonlinear effects in the general thermo-mechanical problem, without imposing artificial constraints on admissible variations. However, by slightly modifying the functionals in the linear variational principles and imposing certain constraints, some useful results can be obtained. As illustrations of the method, let us briefly consider two cases.

First, we shall indicate the necessary modifications of the mechanical displacement-temperature principle presented in section 2.4a in order to include dissipation, but retain all other assumptions. The only Euler equation which is changed is the energy equation 2.114. It is seen from equation 1.119 that the energy equation with dissipation is

$$[K_{ij}\frac{\theta_{,j}}{p}]_{,i} = C_e^o\theta + T_r\beta_{ij}^o e_{ij} - 2\frac{D_m}{p} \qquad (2.122)$$

where D_m is the mechanical dissipation per unit volume, equation 2.85, which is to be expressed as a function of temperature and strains. Thus, if the integral

$$2\int_B \left(\frac{D_m}{p}\right)^* \theta \, dB \qquad (2.123)$$

is added to the functional $I_{u\theta}$, and $\frac{D_m}{p}$ is held constant in the variation, the energy equation 2.122 will be obtained as one of the Euler equations. It is observed that this procedure treats $2\frac{D_m}{p}$ as if it were a known thermal energy source.

Now, consider the case in which the temperature field is prescribed and the viscosity is temperature dependent, but with the time-temperature superposition principle applicable. If the temperature is constant in time, the two functionals , 2.108 for stresses and

2.111 for displacements, can be used directly without modification.

On the other hand, when the temperature varies with both time and space, these original principles must be modified in order to obtain the correct Euler equations. The necessary modifications are determined by examination of the effect of the temperature on the Euler equations and boundary conditions. If the variable transformation 1.121 is used to change from real time to reduced time, the time dependence of material properties is removed; however new terms with time-dependent coefficients are introduced through the spacial derivatives, according to equation 1.122. Consequently, the Euler equations and boundary conditions, as functions of reduced time, will contain terms which are identical with those for a problem without temperature dependent viscosity, as well as terms which arise from the spacial variation of temperature. Thus, variational principles for mechanical variables can be obtained immediately from the previous ones, 2.108 and 2.111, if the necessary terms are added to the functionals (as was done with the dissipation function considered above) and these terms are treated as known quantities in the variation; of course, these modified functionals depend on reduced time, rather than physical time.

Another approach is to use full operational-variational principles, such as suggested by Biot (3). These can be employed for the most general case of coupled thermo-mechanical behavior. The appropriate functionals are obtained by simply removing the convolution integrals from the functionals in the previous sections, and adding the dissipation function, if desired. Variation of these functionals will lead to the correct Euler equations and boundary conditions if all time derivatives, p, dissipation, and all temperature dependent properties are treated as known quantities. Once the variation is carried out and the Euler equations are obtained, these fictitiously constrained quantities are to take on their actual significance.

Finally, as a practical point, it should be added that even though the modified principles suggested in this section appear to be quite artificial, it is possible to use them to calculate approximate solutions. This follows from the fact that the stationary condition makes the approximate Euler equations orthogonal to certain weighting functions; thus, the use of these principles is closely related to the well-known methods of Galerkin and Kantorovich (13). Consider, for example, the problem of calculating an approximate solution in which the solution is assumed in the form of a series of prescribed spacial functions with arbitrary, time dependent coefficients. By substituting the series into the appropriate functional and carrying out the variation with respect to the coefficients, a set of integro-differential equations for the coefficients is obtained. The ease with which these resulting equations can be solved relies, of course, on the particular problem, and the practicality of such a procedure will depend upon further study.

PART III

MATHEMATICAL PROPERTIES OF SOLUTIONS OBTAINED FROM LAPLACE TRANSFORMED VARIATIONAL PRINCIPLES

3.1. Introduction

There are several ways in which the preceding variational principles for viscoelastic media can be used to obtain approximate solutions. For example, the methods of Ritz and Kantorovich (13), commonly used with the minimum principles of elasticity, could be applied to the convolution principles; time dependence would be treated just the same as spacial dependence. However, for many cases, the simplest procedure will be to take the variation of a Laplace transformed functional and thereby obtain an approximate or exact transformed solution; the time dependence is then found by inversion of the transform. Of course, the final step of inverting the transforms may prove to be extremely difficult if standard, exact methods are used. Therefore, it will often be expedient to use approximate inversion techniques; this subject is discussed later in Part IV. Discussion of the convolution principles, in relation to their usefulness for the calculation of approximate variational solutions, is also deferred to Part IV, where two examples are given to illustrate the essential features.

In this Part, transformed variational principles will be studied. First, we shall deduce the general form of time dependence of exact and approximate solutions which are obtained from certain transformed principles; all prescribed loads and displacements (thermal and mechanical) are assumed to be step functions of time applied at $t = 0$.

This analysis is then used in a discussion of the relation between errors in approximate and exact viscoelastic and elastic solutions. The conclusions reached in Part III apply to responses calculated from those principles whose transform on the positive, real p-axis attains an absolute minimum for the exact solution; namely, the homogeneous principles 2.72 and 2.96 for both mechanical and thermal variables, and the non-homogeneous principles 2.112 and 2.117 with the thermal variables prescribed throughout the body. The remaining principles given in Part II do not fall within the scope of the present discussion.

While all results established in this Part apply directly to bodies subjected to step inputs in time, they can be also used to obtain the behavior for arbitrary time-dependent displacement and load prescriptions; one needs only to employ the Duhamel-superposition integral. Furthermore, for simplicity, but without loss of generality, we have omitted all thermal variables in the actual calculations.

Before summarizing the results, let us discuss briefly the motivation for our study of time dependence. A clue to the fact that definite statements can be made about the time dependence of solutions is provided by the general thermodynamic equations 1.98

$$a_{ij}q_i + b_{ij}\dot{q}_j = Q_i \tag{3.1}$$

With zero initial conditions, the transformed solution to these equations is given by equation 2.24a

$$\overline{q}_i = \Big[\sum_s \frac{C_{ij}^{(s)}}{1 + \tau_s p} + \frac{C_{ij}}{p} \Big] \overline{Q}_j \tag{3. 2}$$

If, for example, all forces are prescribed as step functions of time, i. e.

$$Q_i = \begin{cases} 0 \quad , t < 0 \\ \tilde{Q}_i \quad , t > 0 \end{cases} \tag{3.3}$$

where the $\tilde{Q}_i$ are constants, the time dependence of q_i can be written immediately as

$$q_i = \sum_s C_{ij}^{(s)} (1 - e^{-t/\tau_s}) \tilde{Q}_j + C_{ij} \tilde{Q}_j t \tag{3. 4a}$$

or, if steady flow does not occur

$$q_i = \sum_s C_{ij}^{(s)} (1 - e^{-t/\tau_s}) \tilde{Q}_j . \tag{3. 4b}$$

Thus, as was initially observed by Biot (5), apart from the steady flow term $(C_{ij} Q_j t)$ the time dependence of all coordinates is given by a series of decaying exponentials. By interpreting the observed coordinates as mechanical displacements in viscoelastic media, it is expected that the actual displacements would have the same time dependence if all applied loads are step functions of time.

In section 3. 2 it is shown that approximate displacements likewise have this property. Also, it is proved that this behavior extends to mixed boundary conditions and to exact and approximate stresses obtained from the above mentioned variational principles.

We should mention that a rigorous proof is made for only those responses which are represented by a finite sum of spacial functions with time dependent coefficients. However, on the basis of certain physically reasonable arguments, it is postulated that this exponential behavior applies to all viscoelastic responses.

After establishing the time dependence of solutions, we use the results to examine the error between approximate and exact viscoelastic solutions. It is shown that uniform convergence for certain ranges of elastic constants, of a sequence of approximate, associated, elastic solutions to the exact solution, implies that the corresponding sequence of approximate viscoelastic solutions converges (in the mean) to the exact function; this argument rests on the assumption that certain infinite series and improper integrals are absolutely convergent or else that the transient responses are quadratically integrable over $0 \leq t < \infty$.

Knowledge that the time dependence of stress and displacement solutions is of the form 3.4 has additional practical importance. For example, it is known a priori that all of the singularities of Laplace transformed solutions are on the non-positive real axis. Thus, in order to invert a transformed solution exactly, it is only necessary to study behavior on this axis rather than the entire complex p-plane. Also, the form 3.4 lends itself readily to an approximate inversion method which will be discussed in Part IV.

3.2. Time Dependence of Solutions for Step-Displacement and Step-Load Inputs

a. Displacement response. In this section we shall use the transformed displacement variational principle to calculate time dependence of displacements which are given by the series

$$u_i = f_i^{(\alpha)}(x_i)q_\alpha(t) + \tilde{U}_i(x_i)H(t) \tag{3.5}$$

In this expression summation over α (= 1, 2, . . . , N) is implied, $f_i^{(\alpha)}(x_i)$ are functions of the coordinates x_i and vanish on the portion of the boundary A_u where displacements are prescribed, q_α are time-dependent generalized coordinates, and U_i ($\equiv \tilde{U}_i H$) is the prescribed displacement vector for which $H(t)$ is the Heaviside step function,

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{3.6}$$

It is not required that these displacements satisfy the equilibrium equations or stress boundary conditions.

Let us now apply the Laplace transformed variational principle 2.79 in order to calculate $\overline{q}_\alpha(p)$ when the $f_i^{(\alpha)}$ are given functions. For simplicity thermal effects are neglected and zero initial conditions are assumed. The appropriate displacement functional is

$$\overline{I}_u \equiv \frac{1}{2}\int_B \{Z_{ij}^{\mu\nu}\overline{e}_{ij}\overline{e}_{\mu\nu}\}dB - \int_B \overline{F}_i\overline{u}_i dB - \int_{A_T} \overline{T}_i\overline{u}_i dA \tag{3.7}$$

where $\overline{T}_i$ and $\overline{F}_i$ are prescribed forces given by

$$\overline{T}_i = \frac{\widetilde{T}_i(x_i)}{p} \quad ; \quad \overline{F}_i = \frac{\widetilde{F}_i(x_i)}{p} \tag{3.8a}$$

and the transformed displacement vector is

$$\overline{u}_i = f_i^{(\alpha)}\overline{q}_\alpha + \frac{\widetilde{U}_i}{p} \tag{3.8b}$$

Also, from 1.100b

$$Z^{ij}_{\mu\nu} = \sum_s \frac{pD^{ij(s)}_{\mu\nu}}{p+\frac{1}{\rho_s}} + D^{ij}_{\mu\nu} + pD'^{ij}_{\mu\nu} \tag{3.9a}$$

where each matrix in 3.9a is positive semi-definite, i.e. for all e_{ij}

$$D^{ij(s)}_{\mu\nu} e_{\mu\nu} e_{ij} \geq 0; \quad D^{ij}_{\mu\nu} e_{\mu\nu} e_{ij} \geq 0; \quad D'^{ij}_{\mu\nu} e_{\mu\nu} e_{ij} \geq 0 \tag{3.9b}$$

but the matrix made up of the sum of those in 3.9a is positive definite,

$$\Big[\sum_s D^{ij(s)}_{\mu\nu} + D^{ij}_{\mu\nu} + D'^{ij}_{\mu\nu}\Big] e_{\mu\nu} e_{ij} > 0; \quad e_{ij}e_{ij} > 0 \tag{3.9c}$$

We now write $\overline{I}_u$ as a function of $\overline{q}_\alpha$ by using the transformed strain-displacement relations

$$\overline{e}_{ij} = \frac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i}) \tag{3.10}$$

to find

$$\overline{I}_u = \frac{1}{2}\int_B Z^{ij}_{\mu\nu}\Big[e^{(\alpha)}_{ij}\overline{q}_\alpha + \frac{\widetilde{E}_{ij}}{p}\Big]\Big[e^{(\beta)}_{\mu\nu}\overline{q}_\beta + \frac{\widetilde{E}_{\mu\nu}}{p}\Big]dB$$
$$- \int_B \frac{\widetilde{F}_i}{p}\Big(f_i^{(\alpha)}\overline{q}_\alpha + \frac{\widetilde{U}_i}{p}\Big)dB - \int_{A_T} \frac{\widetilde{T}_i}{p}\Big(f_i^{(\alpha)}\overline{q}_\alpha + \frac{\widetilde{U}_i}{p}\Big)dA \tag{3.11}$$

where

$$e^{(\alpha)}_{ij} \equiv \frac{1}{2}\Big[f^{(\alpha)}_{i,j} + f^{(\alpha)}_{j,i}\Big] \tag{3.12}$$

$$\widetilde{E}_{ij} \equiv \frac{1}{2}\Big[\widetilde{U}_{i,j} + \widetilde{U}_{j,i}\Big] \tag{3.13}$$

The generalized coordinates can be found by minimizing $\overline{I}_u$ with respect to each $\overline{q}_\alpha$. This leads to the following N linear algebraic equations,

$$C_{\alpha\beta}\overline{q}_\alpha = \frac{1}{p^2}\widetilde{Q}_\beta \qquad (3.14)$$

in which the following definitions are employed,

$$C_{\alpha\beta} \equiv \int_B \{\sum_s \frac{D^{ij(s)}_{\mu\nu}}{p + \frac{1}{\rho_s}} + \frac{D^{ij}_{\mu\nu}}{p} + D'^{ij}_{\mu\nu}\} e^{(\alpha)}_{\mu\nu} e^{(\beta)}_{ij} dB \qquad (3.15a)$$

$$\widetilde{Q}_\beta \equiv \int_{A_T} \widetilde{T}_i f_i^{(\beta)} dA + \int_B \widetilde{F}_i f_i^{(\beta)} dB - \int_B \{\sum_s \frac{pD^{ij(s)}_{\mu\nu}}{p + \frac{1}{\rho_s}} + D^{ij}_{\mu\nu} + D'^{ij}_{\mu\nu} p\} \widetilde{E}_{\mu\nu} e^{(\beta)}_{ij} dB \qquad (3.15b)$$

It is noted that the singularities of $\widetilde{Q}_\beta$ are simple poles (or branch cuts if $\rho_s = \rho_s(x_i)$) on the negative real p-axis, but if the boundary conditions are all on stress, $\widetilde{Q}_\beta$ is independent of p. Also, from the symmetry of $Z^{ij}_{\mu\nu}$ we have $C_{\alpha\beta} = C_{\beta\alpha}$. The operational moduli in equations 3.15 have been left in the volume integrals since properties may be functions of x_i. We shall now establish the dependence of $\overline{q}_\alpha$ on p, and thereby obtain the time dependence of displacements.

First, the following theorem will be proved:

Theorem I - The singularities of $\overline{q}_\alpha$ occur only on the non-positive real p-axis.

The proof will be made by showing that the determinant of $C_{\alpha\beta}$ ($|C_{\alpha\beta}|$) does not vanish when p is complex or real and positive. Let $p = u + iv$

and substitute this into 3.15 to find

$$C_{\alpha\beta} = R_{\alpha\beta} - ivI_{\alpha\beta} \tag{3.16}$$

where

$$R_{\alpha\beta} \equiv \int_B \{ \sum_s \frac{(u + \frac{1}{\rho_s})D^{ij(s)}_{\mu\nu}}{(u+\frac{1}{\rho_s})^2 + v^2} + \frac{uD^{ij}_{\mu\nu}}{u^2+v^2} + D'^{ij}_{\mu\nu} \} e^{(\alpha)}_{ij} e^{(\beta)}_{\mu\nu}\, dB \tag{3.17a}$$

$$I_{\alpha\beta} \equiv \int_B \{ \sum_s \frac{D^{ij(s)}_{\mu\nu}}{(u+\frac{1}{\rho_s})^2 + v^2} + \frac{D^{ij}_{\mu\nu}}{u^2+v^2} \} e^{(\alpha)}_{ij} e^{(\beta)}_{\mu\nu}\, dB \tag{3.17b}$$

and

$$I_{\alpha\beta} = I_{\beta\alpha}\, ; \qquad R_{\alpha\beta} = R_{\beta\alpha} \tag{3.17c}$$

It is noted by reference to equations 3.9 that $R_{\alpha\beta}$ is positive definite when $u > 0$, but it is indefinite when $u < 0$; also $I_{\alpha\beta}$ is positive semi-definite for all u and v.

Let us assume that $|C_{\alpha\beta}| = 0$ and determine the permissible values of u and v. This means that a non-trivial (real or complex) solution y_α can be found such that

$$(R_{\alpha\beta} - ivI_{\alpha\beta})y_\alpha = 0 \tag{3.18}$$

If the complex conjugate of y_α is denoted by $\overline{y}_\alpha$, multiplying equation 3.18 by $\overline{y}_\beta$ and summing yields

$$R_{\alpha\beta}y_\alpha\overline{y}_\beta = ivI_{\alpha\beta}y_\alpha\overline{y}_\beta \tag{3.19}$$

Since $R_{\alpha\beta}$ and $I_{\alpha\beta}$ are real symmetric matrices, $R_{\alpha\beta}y_\alpha\overline{y}_\beta$ and $I_{\alpha\beta}y_\alpha\overline{y}_\beta$ are real numbers; in addition, the latter one is non-negative.

First, assume that $I_{\alpha\beta}y_\alpha\overline{y}_\beta > 0$. But equation 3.19 cannot be satisfied unless v is zero since the left-hand side is real while the right-hand side is imaginary. Now, suppose $I_{\alpha\beta}y_\alpha\overline{y}_\beta = 0$ which can be seen from equation 3.17a to imply that

$$R_{\alpha\beta}y_\alpha\overline{y}_\beta = \int_B D'^{ij}_{\mu\nu}\, e^{(\alpha)}_{ij}\, e^{(\beta)}_{\mu\nu}\, dB\, y_\alpha\overline{y}_\beta \tag{3.20}$$

But equation 3.9c requires that this be a (non-zero) positive number and therefore equation 3.19 cannot be satisfied. Thus, the determinant of $C_{\alpha\beta}$ cannot vanish unless p is real.

It only remains to show that there are no zeros of $|C_{\alpha\beta}|$ on the positive real axis. That this is indeed the case follows immediately from the fact that $R_{\alpha\beta}$ is positive definite when $u > 0$. Theorem I is therefore proved for the most general stress-strain relations which are thermodynamically admissible.

Further information about the singularities of $\overline{q}_\alpha$ will now be obtained. However, in the following discussion we shall assume that the relaxation times, ρ_s, are independent of x_i. This assumption permits $C_{\alpha\beta}$ (defined by equation 3.15a) to be written as

$$C_{\alpha\beta} = \sum_s{}' \frac{F^{(s)}_{\alpha\beta}}{p + \frac{1}{\rho_s}} + \frac{F_{\alpha\beta}}{p} + F'_{\alpha\beta} \tag{3.21}$$

with the definitions

$$F^{(s)}_{\alpha\beta} \equiv \int_B \{D^{ij(s)}_{\mu\nu}\, e^{(\alpha)}_{\mu\nu}\, e^{(\beta)}_{ij}\}\, dB \tag{3.22a}$$

$$F_{\alpha\beta} \equiv \int_B \{D^{ij}_{\mu\nu}\, e^{(\alpha)}_{\mu\nu}\, e^{(\beta)}_{ij}\}\, dB \tag{3.22b}$$

$$F'_{\alpha\beta} \equiv \int_B \{D'^{ij}_{\mu\nu} e^{(\alpha)}_{\mu\nu} e^{(\beta)}_{ij}\} dB \qquad (3.22c)$$

and the equation for $\overline{q}_\alpha$ becomes

$$\left[\sum_s \frac{pF^{(s)}_{\alpha\beta}}{p+\frac{1}{\rho_s}} + F_{\alpha\beta} + pF'_{\alpha\beta}\right]\overline{q}_\alpha = \frac{1}{p}\tilde{Q}_\beta \qquad (3.23)$$

The following theorem will be proved:

Theorem II - When the relaxation times, ρ_s, are constant the singularities of $\overline{q}_\alpha$ are simple poles except at the origin where a double pole may occur.

For the present let us assume that $\tilde{Q}_\beta$ is constant. Also let $\alpha = 1, 2, \ldots, N$ and $s = 1, 2, \ldots, M$, so that we can write equation 3.23 as

$$G_{\alpha\beta}\overline{q}_\alpha = \frac{1}{p}\tilde{Q}_\beta \prod_{s=1}^{M} (p+\frac{1}{\rho_s}) \qquad (3.24)$$

in which each element of $G_{\alpha\beta}$ is at most a polynomial of order $N(M+1)$. Theorem I implies

$$|G_{\alpha\beta}| = G \prod_{r=1}^{R} (p+\frac{1}{\gamma_r})^{m_r} \qquad (3.25)$$

where G and γ_r are real positive constants, m_r is the multiplicity of the r^{th} root $(p = -\frac{1}{\gamma_r})$, and $\sum_{r=1}^{R} m_r \leq N(M+1)$. It follows that $\overline{q}_\alpha$ can be expressed as a ratio of polynomials in p given by

$$\overline{q}_{\alpha} = \frac{a_{\alpha\beta}^{(j)}\, p^{j}}{\prod_{r=1}^{R} \left(p + \frac{1}{\gamma_r}\right)^{m_r}} \left(\frac{\tilde{Q}_{\beta}}{p}\right) \tag{3.26}$$

The ratio multiplying $\tilde{Q}_{\beta}/p$ can be written as a sum of partial fractions if the order of the numerator is lower than that of the denominator; if they are of equal order then $\overline{q}_{\alpha}$ will contain an additional constant term multiplying $\tilde{Q}_{\beta}/p$. That either of these conditions is always satisfied can be shown to follow from equation 3.23 by letting $p \to \infty$. If $|D'^{ij}_{\mu\nu}| > 0$, then from equation 3.22c we have $|F'_{\alpha\beta}| > 0$, and equation 3.23 shows that $\overline{q}_{\alpha}$ must behave like $1/p^2$ as $p \to \infty$; hence

$$\frac{a_{\alpha\beta}^{j} p^{j}}{\prod_{r=1}^{R} \left(p + \frac{1}{\gamma_r}\right)^{m_r}} \sim \frac{1}{p} \quad \text{as } p \to \infty \tag{3.27a}$$

If, however, $|F'_{\alpha\beta}| = 0$ (which can only occur if $|D'^{ij}_{\mu\nu}| = 0$) then

$$\frac{a_{\alpha\beta}^{j} p^{j}}{\prod_{r=1}^{R} \left(p + \frac{1}{\gamma_r}\right)^{m_r}} \sim \text{constant as } p \to \infty \tag{3.27b}$$

Consequently, it is always possible to write $\overline{q}_{\alpha}$ as the partial fraction series

$$\overline{q}_{\alpha} = \frac{\tilde{Q}_{\beta}}{p} \left[\sum_{r=1}^{R} \sum_{j=1}^{m_r} \frac{S_{\alpha\beta}^{(j)}}{\left(p + \frac{1}{\gamma_r}\right)^{j}} + S_{\alpha\beta} \right] \tag{3.28}$$

where $S_{\alpha\beta}^{(j)}$ and $S_{\alpha\beta}$ are real symmetric matrices.

The order of the poles of $\overline{q}_\alpha$ can be determined by examining the behavior of $\overline{q}_\alpha$ in conjunction with equation 3.23 as p approaches the roots of $|G_{\alpha\beta}|$. Consider, then, $p = \epsilon - \frac{1}{\gamma_t}$ and $|\epsilon| << 1$; with p close to $-\gamma_t$ only the term in equation 3.28 which behaves like ϵ^{-m_t} need be retained, thus

$$\overline{q}_\alpha \simeq \frac{\tilde{Q}_\beta S^{(t)}_{\alpha\beta}}{p\epsilon^{m_t}} \equiv \frac{g^{(t)}_\alpha}{p\epsilon^{m_t}} \tag{3.29}$$

Multiplying equation 3.23 by $\overline{q}_\beta$ and summing over β, and then substituting equation 3.29 for $\overline{q}_\alpha$ yields

$$\sum_s \frac{F^{(s)}}{p + \frac{1}{\rho_s}} + \frac{F}{p} + F' = \frac{1}{p} \tilde{Q}_\beta g^{(t)}_\beta \epsilon^{m_t} \tag{3.30}$$

where $p = \epsilon - \frac{1}{\gamma_t}$ and

$$F^{(s)} \equiv F^{(s)}_{\alpha\beta}\, g^{(t)}_\alpha g^{(t)}_\beta \geq 0 \tag{3.31a}$$

$$F \equiv F_{\alpha\beta} g^{(t)}_\alpha g^{(t)}_\beta \geq 0 \tag{3.31b}$$

$$F' \equiv F'_{\alpha\beta} g^{(t)}_\alpha g^{(t)}_\beta \geq 0 \tag{3.31c}$$

The right-hand side of equation 3.30 has a zero of order m_t, whose value must agree with the left-hand side. However, the latter can have only simple zeros at all poles of $\overline{q}_\alpha$ since its first derivative can never vanish in the finite p plane, which is

$$-\left[\sum_s \frac{F^{(s)}}{(p + \frac{1}{\rho_s})^2} + \frac{F}{p^2}\right] < 0 \tag{3.32}$$

Thus $m_t = 1$ for all finite and infinite values of γ_t. It is to be noted that this conclusion applies even when a root γ_t is equal to one of the relaxation times ρ_s. Reference to equation 3.30 shows that this equality can occur only if $F^{(s)} = 0$ (this requires $|D^{ij(s)}_{\mu\nu}| = 0$). Also, equation 3.23 indicates that the determinant $|G_{\alpha\beta}|$ has a zero at the origin if and only if $|F_{\alpha\beta}| = 0$ (which requires $|D^{ij}_{\mu\nu}| = 0$). If this latter condition exists $\bar{q}_\alpha$ has a double pole at the origin.

Theorem II is thus proved for the case in which $\tilde{Q}_\beta$ is constant. Furthermore, the previous considerations lead to the following time dependence of the generalized coordinates:

1. When $|D^{ij}_{\mu\nu}| > 0$, $|D'^{ij}_{\mu\nu}| > 0$:

$$q_\alpha = \tilde{Q}_\beta \sum_{r=1}^{N(M+1)} \gamma_r S^{(r)}_{\alpha\beta}(1 - e^{-t/\gamma_r}) \tag{3.33a}$$

2. When $|D^{ij}_{\mu\nu}| > 0$, $|F'_{\alpha\beta}| = 0$:

$$q_\alpha = \tilde{Q}_\beta \left[\sum_{r=1}^{N(M+1)} \gamma_r S^{(r)}_{\alpha\beta}(1-e^{-t/\gamma_r}) + S_{\alpha\beta} \right] \tag{3.33b}$$

3. When $|F_{\alpha\beta}| = |F'_{\alpha\beta}| = 0$

$$q_\alpha = \tilde{Q}_\beta \left[\sum_{r=1}^{N(M+1)-1} \gamma_r S^{(r)}_{\alpha\beta}(1-e^{-t/\gamma_r}) + S_{\alpha\beta} + S'_{\alpha\beta} t \right] \tag{3.33c}$$

The restriction that the $\tilde{Q}_\beta$ are constant can be easily removed. When $\tilde{Q}_\beta$ are of the general form given by equation 3.15b, with ρ_s constant, then $\bar{q}_\alpha$ contains simple poles at $-\frac{1}{\rho_s}$ as well as at $-\frac{1}{\gamma_r}$.* $q_\alpha(t)$ therefore has time dependence similar to that shown

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*This can be shown to be true even when $(-1/\rho_s)$ is a zero of $|G_{\alpha\beta}|$ by using the fact that $F^{(s)}$ vanishes in such a case.

in equations 3.33, except there will be additional exponentials with time constants ρ_s. Also the correspondence between the vanishing of a given determinant and the time dependence indicated in cases 1, 2, and 3 above will not necessarily be the same. For example, if $\tilde{T}_i$ and $\tilde{F}_i$ in equation 3.15b are zero and $D^{ij}_{\mu\nu} = 0$, then $\bar{q}_\alpha$ will contain at most a simple pole at the origin; hence it will not have the term proportional to time which is shown in equation 3.33c.

b. Stress response. Stresses which are derived from the transformed complementary principle can be shown to have time dependence similar to that of the displacements discussed above. We consider stresses that are given by

$$\sigma_{ij} = f^{(\alpha)}_{ij}(x_i)Q_\alpha(t) + \tilde{\sigma}_{ij}(x_i)H(t) \tag{3.34}$$

in which α is to be summed out ($\alpha = 1, 2, \ldots, N$), $f^{(\alpha)}_{ij}$ ($= f^{(\alpha)}_{ji}$) are given functions of the coordinates x_i which vanish on A_T where stresses are prescribed, Q_α are time dependent functions which we shall call generalized stresses, and on A_T the vector $\tilde{\sigma}_{ij}n_jH$ is equal to the prescribed surface force, T_i. It is further assumed that for each α, the $f^{(\alpha)}_{ij}$ satisfy the equilibrium equations

$$f^{(\alpha)}_{ij,j} = 0 \tag{3.35}$$

and the stresses $\tilde{\sigma}_{ij}H(t)$ satisfy the equilibrium equations with prescribed body forces $\tilde{F}_iH(t)$, hence

$$\tilde{\sigma}_{ij,j} + \tilde{F}_i = 0 \tag{3.36}$$

It is not required that the stresses σ_{ij} satisfy compatibility or the boundary conditions on displacement.

The Laplace transform of the complementary functional, equation 2.94, is (with temperature neglected for simplicity)

$$\overline{I}_\sigma \equiv \frac{1}{2}\int_B \{A^{\mu\nu}_{ij}\overline{\sigma}_{ij}\overline{\sigma}_{\mu\nu}\}\,dB - \int_{A_u}\overline{U}_i\overline{\sigma}_{ij}n_j\,dA \qquad (3.37)$$

where $\overline{U}_i$ is the transformed, prescribed surface displacement

$$\overline{U}_i = \frac{\widetilde{U}_i(x_i)}{p} \qquad (3.38)$$

and the operational compliance matrix $A^{\mu\nu}_{ij}$ is given by equation 1.103a

$$A^{ij}_{\mu\nu} = \sum_s \frac{C^{ij(s)}_{\mu\nu}}{1+\tau_s p} + \frac{C^{ij}_{\mu\nu}}{p} + C'^{ij}_{\mu\nu} \qquad (3.39)$$

and each matrix composing $A^{ij}_{\mu\nu}$ satisfies the same properties as those composing $Z^{\mu\nu}_{ij}$ given by equations 3.9b and 3.9c.

The generalized stresses are obtained just as the generalized coordinates were in the previous discussion. Namely, the transformed stresses,

$$\overline{\sigma}_{ij} = f^{(\alpha)}_{ij}\overline{Q}_\alpha + \frac{\widetilde{\sigma}_{ij}}{p} \qquad (3.40)$$

are substituted into $\overline{I}_\sigma$ and then the stationary condition $\delta\overline{I}_\sigma = 0$ provides us with a set of N algebraic equations for the $\overline{Q}_\alpha$, which is

$$B_{\alpha\beta}\overline{Q}_\alpha = \frac{1}{p}\widetilde{q}_\beta \qquad (3.41)$$

where we define

$$B_{\alpha\beta} \equiv \int_B \left\{ \sum_s \frac{C^{ij(s)}_{\mu\nu}}{1+\tau_s p} + \frac{C^{ij}_{\mu\nu}}{p} + C'^{ij}_{\mu\nu} \right\} f^{(\alpha)}_{\mu\nu} f^{(\beta)}_{ij}\,dB \qquad (3.42a)$$

$$\tilde{q}_\beta \equiv \int_{A_u} \tilde{U}_i f_{ij}^{(\beta)} n_j dA - \int_B \left\{ \sum_s \frac{C_{\mu\nu}^{ij(s)}}{1+\tau_s p} + \frac{C_{\mu\nu}^{ij}}{p} + C_{\mu\nu}^{\prime ij} \right\} f_{ij}^{(\beta)} \tilde{\sigma}_{\mu\nu} dB \qquad (3.42b)$$

The similarity between the present set of equations 3.41 and equations 3.14 which occur with the displacement principle is evident. However, an important difference is in the factors $\frac{1}{p}$ and $\frac{1}{p^2}$ on the right-hand side which multiply the prescribed quantities q_β and $\tilde{Q}_\beta$; it is seen that $\overline{Q}_\beta$ cannot have a double pole at the origin.* Thus, in analogy with the previous theorems I and II we can state two companion theorems:

<u>Theorem III</u> - The singularities of $\overline{Q}_a$ occur only on the non-positive real p-axis.

<u>Theorem IV</u> - When the retardation times, τ_s, are constant the singularities of $\overline{Q}_a$ are simple poles.

The time dependence of the generalized stresses is therefore

$$Q_a(t) = \sum_r T_a^{(r)} e^{-t/\lambda_r} + T_a \qquad (3.43)$$

where $T_a^{(r)}$ and T_a are constants and the λ_r are positive constants. Since a double pole at the origin does not occur there is no term proportional to time.

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*There is another point which should be mentioned concerning the behavior of $\overline{Q}_\beta$ at $p = \infty$. It is noted from equations 3.41 and 3.42 that if $\tilde{U}_i \neq 0$ and $|C_{\mu\nu}^{\prime ij}| = 0$, then $\overline{Q}_\beta$ is a non-zero constant at $p = \infty$. Since this leads to an infinite (delta function) stress at $t = 0$ we shall rule this out in all of the following work by assuming $|C_{\mu\nu}^{\prime ij}| > 0$. However, the analysis could be extended, if desired, to include this singular behavior.

c. *Generalizations.* Strictly speaking, the theorems in sections 3. 2a and 3. 2b apply to a restricted class of stresses and displacements. To reiterate, theorems I and III are rigorously valid for only those approximate and exact solutions which can be expressed as finite sums of terms, each of which is a simple product of a space dependent function and a time dependent function. Except for this restriction, they are valid for the most general stress-strain relations consistent with thermodynamics; it should be noted that the integral representation (see first footnote in section 1. 7a) in which $\sum_s$ is replaced by $\int_0^\infty$ can be used also. However, theorems II and IV require, in addition to the restriction cited above, that the finite sum $\sum_s$ representation of the stress-strain relations be used. Furthermore, the relaxation and retardation times must be constant; this requirement can actually be modified in that these time constants need be constant only over finite regions of a body, since the total volume integrals in equations 3. 15 and 3. 42 then separate into integrals each having constant values of τ_s or ρ_s.

When the above mentioned conditions are satisfied, we showed that the time dependence of all solutions is given by a series of decaying exponentials, with possibly a term proportional to time. In this section we shall discuss the time dependence of generalized coordinates and stresses when the restrictions are removed. The most general behavior which is expected will be first postulated and then the statements will be justified, to some extent, on physical grounds and by showing that the correct result is obtained for special cases.

Consider now the following postulates for stresses and displacements in media which are subjected to loads and displacements that are step functions of time. Exact solutions as well as approximate solutions obtained from the transformed variational principles are applicable.

<u>Postulate I</u> - Thermodynamically admissible displacements and stresses in bounded media can be represented by finite or uniformly convergent infinite series as given by equations 3.5 and 3.34, respectively.

This postulate appears to be reasonable since one expects that a thermodynamically admissible stress or displacement field can always be approximated arbitrarily well by a finite number of spacial functions (Fourier series, for example), each with a time dependent coefficient. Also, it is expected that when the body is unbounded, the discrete sum is replaced by an integral (e.g. Fourier integral).

<u>Postulate II</u> - The most general time dependence of the generalized coordinates is

$$q_a = \int_0^\infty S_a(\tau) e^{-t/\tau} d\tau + S_a + S_a' t \tag{3.44a}$$

and the generalized stresses is

$$Q_a = \int_0^\infty T_a(\tau) e^{-t/\tau} d\tau + T_a \tag{3.44b}$$

where $S_a(\tau)$ and $T_a(\tau)$ are spectral distributions of the variable τ which may consist entirely or partly of Dirac-delta functions.

It is clear that the finite degree of freedom cases considered in sections 3. 2a and 3. 2b are obtained immediately by setting

$$S_{\alpha}(\tau) \equiv \sum_{r} S_{\alpha}^{(r)}\delta(\tau-\gamma_{r}) \tag{3. 45a}$$

$$T_{\alpha}(\tau) \equiv \sum_{r} T_{\alpha}^{(r)}\,\delta(\tau-\lambda_{r}) \tag{3. 45b}$$

where $\delta(\tau)$ is the Dirac delta function,

$$\begin{aligned} &\delta(\tau) = 0, \quad \tau \neq 0 \;; \\ &\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1 \end{aligned} \tag{3. 46}$$

Thus, the forms 3. 44a and 3. 44b for the generalized variables appear to be a natural extension of the earlier results to problems for which finite sums are replaced by integrals and infinite sums.

As additional justification of the postulate, let us examine the case in which certain orthogonality properties exist; it will be sufficient to consider just the stress principle. Suppose that the spacial functions $f_{ij}^{(\alpha)}$ in equation 3. 34 satisfy the orthogonality property (see equations 3. 41 and 3. 42)

$$\int_{B} \{ A_{\mu\nu}^{ij}\, f_{\mu\nu}^{(\alpha)} f_{ij}^{(\beta)} \} dB \;= B_{\beta}\delta_{\alpha\beta} \equiv \; B_{\alpha\beta} \quad (\beta \text{ not summed}) \tag{3. 47a}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta

$$\delta_{\alpha\beta} \equiv \begin{cases} 1\,, & \alpha = \beta \\ 0\,, & \alpha \neq \beta \end{cases} \tag{3. 47b}$$

and the finite sum ($\sum_s$) in $A^{ij}_{\mu\nu}$ is to be replaced by an infinite integral ($\int_0^\infty$). With this orthogonality property the set of equations 3.41 reduces to the uncoupled set

$$B_\beta \overline{Q}_\beta = \frac{1}{p}\,\overline{q}_\beta \qquad (\beta \text{ not summed}) \tag{3.48}$$

The time dependence of $\overline{Q}_\beta$ is obtained immediately from Stieltjes transform theory (32) (assuming certain reasonable convergence properties of the improper integrals) and it is found to be the same as given by equation 3.44b.

With the postulates appearing to be valid, we now turn to an error analysis which makes direct use of them.

3.3. Relation Between the Error in Approximate Viscoelastic and Elastic Solutions

It will be assumed in the following calculations that all infinite series and improper integrals are absolutely convergent in order to perform the necessary operations, such as interchanging order of integrations. Strictly speaking, such an assumption should be verified for each particular problem but we shall not pursue this point any further. Use of Postulates I and II in connection with the series 3.5 and 3.34 yields the most general time dependence of both approximate and exact displacements and stresses,

$$u_i = \int_0^\infty f_i^{(\alpha)} S_\alpha(\tau) e^{-t/\tau} d\tau + (S_\alpha f_i^{(\alpha)} + \tilde{U}_i) + (f_i^{(\alpha)} S'_\alpha) t \tag{3.49a}$$

$$\sigma_{ij} = \int_0^\infty f_{ij}^{(\alpha)} T_\alpha(\tau) e^{-t/\tau} d\tau + (T_\alpha f_{ij}^{(\alpha)} + \tilde{\sigma}_{ij}) \tag{3.49b}$$

where, by assumption

$$u_i = \sigma_{ij} = 0 \quad \text{for} \quad t < 0 \tag{3.49c}$$

For the present let us omit the steady-flow term in u_i, and concern ourselves with a representative exact solution

$$\psi_e \equiv \int_0^{\infty} \varphi_e(\tau) e^{-t/\tau} d\tau + \psi_e(\infty) \tag{3.50a}$$

and approximate solution

$$\psi_a \equiv \int_0^{\infty} \varphi_a(\tau) e^{-t/\tau} d\tau + \psi_a(\infty) \tag{3.50b}$$

where ψ, $\psi(\infty)$, and $\varphi(\tau)$ are implied functions of x_i which represent the corresponding functions in u_i or σ_{ij}; i.e., ψ is to be interpreted as either a stress or displacement in order to simplify the notation. It will also be convenient to define the transient component of the solutions as $\Delta\psi \equiv \psi - \psi(\infty)$. Let us now integrate the square error, $\epsilon_{\tau}^2 = (\Delta\psi_e - \Delta\psi_a)^2$, over all positive time to find,

$$\epsilon^2 = \int_0^{\infty} [\varphi_e(\tau) - \varphi_a(\tau)] [\Delta\overline{\psi}_e(\tfrac{1}{\tau}) - \Delta\overline{\psi}_a(\tfrac{1}{\tau})] \, d\tau \tag{3.51}$$

where $\Delta\overline{\psi}(\frac{1}{\tau})$ denotes the transient component transformed with respect to the transform parameter $\frac{1}{\tau} = p$, and it is assumed that $\epsilon^2 < \infty$. Now with p on the positive real axis, $p[\overline{\psi}(p)]$ $(= p\Delta\overline{\psi} + \psi(\infty))$ is identical with an elastic solution whose elastic moduli or compliances are numerically equal to $Z_{\mu\nu}^{ij}$ or $A_{\mu\nu}^{ij}$. Furthermore, $\psi(\infty)$ is an elastic solution with moduli which are equal to $Z_{\mu\nu}^{ij}$ evaluated at $p = 0$. Therefore, an important implication of equation 3.51 is that if $\overline{\psi}_a(\frac{1}{\tau})$ is one solution out of a sequence of approximate elastic solutions which is known to converge uniformly to the exact solution (with respect to elastic constants whose values range over the values taken by $Z^{ij}_{\mu\nu}$ or $A^{ij}_{\mu\nu}$ on the real interval $0 \leqslant p < \infty$), then the corresponding sequence of time-dependent solutions converges (in the mean) to the exact solution. An equivalent statement is that the integrated square error in the time-dependent solution is "small" when the error on the real p axis is sufficiently "small" for $0 \leqslant p < \infty$.

The foregoing conclusion can be readily modified to allow for the presence of a steady-flow term in the displacements, equation 3.49a. It is only necessary to redefine ψ as

$$\psi \equiv \int_0^\infty \varphi(\tau)e^{-t/\tau}d\tau + \psi(\infty) + \psi' t \qquad (3.52a)$$

and $\Delta\psi$ as

$$\Delta\psi \equiv \psi - \psi(\infty) - \psi' t \qquad (3.52b)$$

With the exception of ψ_a', the convergence of ψ_a to ψ_e is established by the previous results. However, the sequence of values for ψ_a' will also converge to the correct value if a sequence of associated elastic solutions with compliances equal to $C^{ij}_{\mu\nu}$ (see equation 3.39) has this behavior. This follows from the fact that the long-time ($p \rightarrow 0$) value of $\dot{\psi}$ is the same as an elastic solution with compliances equal to $(pA^{ij}_{\mu\nu})_{p=0}$.

The problem of relating error in time dependence to error on the positive real p-axis can be approached in another way. Erdelyi (33, 34) has derived Laplace inversion formulas from which it is possible to expand $\Delta\psi(t)$ in an infinite series of orthogonal functions, whose coefficients contain $\Delta\overline{\psi}(p)$ evaluated at a discrete number of positive, real values of p. The only assumption needed is that $\Delta\psi(t)$ be quadratically integrable, i. e.

$$\int_0^\infty [\Delta\psi(t)]^2 \, dt < \infty \tag{3.53}$$

which is satisfied by $\Delta\psi_e$ and $\Delta\psi_a$ if the integrals in equations 3. 50 are absolutely convergent. These inversion formulas lead to the same conclusion deduced from equation 3. 51; namely, a "small" error on the positive real p-axis implies a "small" integrated square error in time.

It is clear that the error estimate, equation 3. 51, cannot actually be used in practice since it requires knowledge of an exact solution. Rather, it only indicates that a sufficiently close approximation to an associated elastic solution leads to a good approximate viscoelastic solution. It is felt that a stronger statement of what is meant by "sufficiently close" must come from a study of numerical examples. Illustrations along these lines will be discussed in Part IV.

PART IV

APPROXIMATE METHODS OF TRANSFORM INVERSION AND NUMERICAL APPLICATIONS OF VARIATIONAL PRINCIPLES

4.1. Introduction

Some numerical examples are given in this Part which illustrate the use of variational methods in obtaining approximate viscoelastic solutions. It has already been mentioned in Part II how a transformed viscoelastic solution can be easily derived from an associated elastic problem by applying the correspondence rule. However, as the resulting transform is often very difficult to invert, particularly when realistic material properties are used, we shall first proceed to develop two methods of approximate inversion, each of which can often be used as a check on the other, followed by specific illustrations of the variational process.

The first one considered in section 4.2 is a modification of Alfrey's rule (9), and the second one is a collocation method which is based on the characteristic time dependence deduced in Part III. With both techniques it is necessary to know only an associated elastic solution numerically for certain ranges of elastic constants and numerical values of the operational properties (e.g. moduli) for real, positive values of p. It will be seen that a time dependent solution can be calculated with very little effort once these numerical quantities have been determined. Examples in which the results of these approximate inversion methods are compared to exact inversions are presented in an expanded version elsewhere (35).

In addition to the usefulness of the first method for inversion, for those problems in which it is applicable it also shows that if the associated elastic solution is approximate, the "quality" of the viscoelastic response is essentially the same as that of the elastic problem.

Section 4.3 gives two numerical examples using transformed variational principles. In the first we derive the approximate transformed displacements in a thin plate with mixed boundary conditions. The transform is deduced using an elastic solution obtained from the potential energy principle. This example serves two purposes. For one, the transform, though a very complicated function of an operational Poisson's ratio, is easily inverted to illustrate the simplicity of the approximate methods given in section 4.2. Secondly, the type and location of all singularities of the transformed solution are examined. The findings are not used to calculate the exact inversion (of the approximate transform) numerically, but to show that the analytical form of time dependence agrees with that predicted from Postulates I and II in Part III.

As the next example, we use the complementary energy principle to calculate approximate stresses in a long, symmetrically loaded cylinder. Two different approximate solutions are obtained and both the elastic and viscoelastic responses are compared with the exact stresses. This comparison serves to illustrate the relation between elastic and viscoelastic errors which was discussed in Part III.

To conclude our discussion on numerical applications of variational principles, it is suggested in section 4.4 how convolution principles can be employed when it is not convenient or possible to use the Laplace transform method. This latter situation arises when assumed solutions are nonlinear functions of generalized coordinates and is illustrated by a one-dimensional heat conduction problem.

4. 2. Approximate Methods of Laplace Transform Inversion

a. Direct method. Of the two inversion techniques to be discussed, the simplest one is called the "direct method". It will be shown to yield good results when the derivative of the time dependent solution with respect to logarithmic time, log t, is a slowly varying function of log t. A modification of this method is also suggested in order to handle functions for which the derivative of their logarithm has this slowly varying property.

The problem which we pose is to find an approximate representation of a viscoelastic response, $\psi(t)$, from the integral equation

$$\overline{\psi}(p) = \int_0^\infty \psi(t)e^{-pt}dt \tag{4. 1}$$

where $\overline{\psi}(p)$ is the Laplace transform of $\psi(t)$ and is known at least numerically for all real, non-negative values of the transform parameter, p. Let us represent $p\overline{\psi}(p)$ as a function of log p and define

$$\hat{f}(u) \equiv p\overline{\psi}(p) \; ; \quad u \equiv \log p \tag{4. 2}$$

$$f(v) \equiv \psi(t) \; ; \quad v \equiv \log t \tag{4. 3}$$

$$w = u + v \tag{4. 4}$$

which renders 4.1 in the form

$$\hat{f}(u) = \ln 10 \int_{-\infty}^{\infty} f(w-u) 10^{w} e^{-10^{w}} \, dw \qquad (4.5)$$

where $\ln \equiv \log_e$ and $\log \equiv \log_{10}$.

The weighting function, $10^{w} e^{-10^{w}}$, is drawn in figure 4.1[*] which shows that it is practically a delta function if $f(v)$ changes slowly enough. This behavior implies that an approximation to $f(v)$ can be obtained directly in terms of $\hat{f}(u)$ by replacing $(\ln 10) 10^{w} e^{-10^{w}}$ with $\delta(w - w_o)$, i.e. a Dirac delta function located at the point w_o,[**] which will yield an approximate inversion formula

$$f(w_o - u) \simeq \hat{f}(u) \qquad (4.6)$$

The point w_o is somewhat arbitrary in view of the spread of the weighting function, which is about two decades. However we shall now calculate the "best" value to use when f is closely approximated by a straight line in the two-decade interval, $|w - w_o| \lesssim 1$. To do so, we first expand $f(v)$ in a Taylor series about the point v_o $(\equiv w_o - u)$

$$f(v) = f(v_o) + f'(v_o)(v - v_o) + \frac{1}{2} f''(v_o)(v - v_o)^2 + \ldots \qquad (4.7)$$

where primes denote differentiation with respect to v. Substitution of this expression into 4.5 yields

$$\hat{f}(u) \simeq f(v_o) + (\ln 10) f'(v_o) \int_{-\infty}^{\infty} (w - w_o) 10^{w} e^{-10^{w}} \, dw \qquad (4.8)$$

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[*] All figures for part IV follow the text.

[**] It should be recalled that a delta function is defined such that

$$\delta(w - w_o) = 0 \text{ if } w \neq w_o; \quad \int_{-\infty}^{\infty} \delta(w - w_o) \, dw = 1$$

where only the constant and linear term in 4.7 have been retained. It is seen that the approximate inversion formula 4.6 is obtained if the integral in 4.8 vanishes. This condition locates w_o at the centroid of the area under the curve $10^w e^{-10^w}$, which is

$$w_o = \int_0^\infty (\log t) e^{-t} dt = \frac{-C}{\ln 10} \tag{4.9a}$$

where C is Euler's constant

$$C = -\int_0^\infty (\ln t) e^{-t} dt \simeq 0.58 \tag{4.9b}$$

When this value for w_o is substituted into 4.6 and the resulting expression is written in terms of the original functions $\psi(t)$ and $p\overline{\psi}(p)$ by using 4.2 and 4.3, we find*

$$\psi(t) \simeq \left[p\overline{\psi}(p)\right]_{p = e^{-C}/t} \tag{4.10a}$$

where $e^{-C} \simeq 0.56$. Due to the skewed form of the weighting function it has been found that a somewhat better formula is

$$\psi(t) \simeq \left[p\overline{\psi}(p)\right]_{p = 0.5/t} \tag{4.10b}$$

It is important to recognize that 4.10 was derived using an assumption about the exact inversion $\psi(t)$, rather than $p\overline{\psi}(p)$. Since in practice this solution will not be known, one must assume that if

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*The fact that the weighting function is essentially zero except for a two decade interval suggests that this inversion will be good at those times in which the solution (as a function of log t) is linear, or nearly linear, for at least two decades. If there is a strong curvature, but it is somewhat removed from this linear region, this curvature should not produce a significant error in the linear portion.

$d^2(p\overline{\psi}(p))/d(\log p)^2$ is small for $p \geqslant 0$ then $\psi(t)$ behaves similarly.

In the event that $\log p\overline{\psi}(p)$ is essentially linear for several decades of p when plotted against $\log p$, so that $p\overline{\psi}(p) \simeq Ap^m$, it can be easily verified for $m < 1$ that an approximate inversion formula is

$$\psi(t) \simeq \left[p\overline{\psi}(p)\right]_{p=\alpha/t} \tag{4.11}$$

where

$$\alpha = \left[\Gamma(1-m)\right]^{-1/m}; \quad \Gamma(1-m) \equiv \int_0^\infty x^{-m} e^{-x} dx$$

While this further approximation is valid for many practical situations, it should be recognized that the approximation is poor when m is close to $+1$.

Finally, we should remark that there does not appear to be an easy method of quantitatively estimating the error involved in this direct method. However, there are some qualitative techniques which can be used. As one, the approximate solution could be transformed numerically or analytically, and then compared to the original transform for $p \geqslant 0$. If the approximate solution is physically acceptable, and its transform is relatively "close" to the original one, it is reasonable to assume that the error in time dependence is small. Another check on the direct method can be made by comparing the inversion to the result of the method which will be discussed next.

b. Collocation method. We would now like to discuss a second technique which is not as simple as the direct method, but it has other advantages. For one, it is not restricted to functions whose derivative is slowly varying with respect to logarithmic time. In addition, the time dependence is given by a simple series of exponentials which can be used readily in the Duhamel integral for the calculation of responses to prescribed loads and displacements that are not step functions. Thirdly, the accuracy of the inversion can be improved by adding more terms to the series.

In Chapter III it was argued that with prescribed quantities as step functions of time, the transient component of both exact and approximate solutions obtained from minimum principles can be expressed in terms of an integral

$$\Delta\psi = \int_0^\infty \varphi(\tau) e^{-t/\tau} \, d\tau \qquad (4.12)$$

Assuming that this integral is absolutely convergent, a Dirichlet series

$$\Delta\psi_D = \sum_{i=1}^{n} S_i e^{-t/\gamma_i} \qquad (4.13)$$

can be used as an approximation to the solution $\Delta\psi(t)$.* The present method makes use of this series for which the γ_i are prescribed positive constants, and the S_i are unspecified coefficients to be calculated by minimizing the total square error between $\Delta\psi$ and $\Delta\psi_D$.

The total square error is

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*Actually, a necessary and sufficient condition on $\Delta\psi(t)$, in order that it can be expanded in a Dirichlet series, is that it be quadratically integrable (33), i. e.

$$\int_0^\infty (\Delta\psi)^2 dt < \infty$$

which is less restrictive than requiring that it be given by the absolutely convergent integral 4.12.

$$E^2 = \int_0^\infty \left(\Delta\psi - \Delta\psi_D \right)^2 dt \qquad (4.14)$$

with the minimization yielding

$$-\frac{1}{2}\frac{\partial E^2}{\partial S_i} = 0 = \int_0^\infty (\Delta\psi - \Delta\psi_D) e^{-t/\gamma_i} dt; \; i = 1, 2, \ldots, n \qquad (4.15a)$$

so that n relations are obtained between the Laplace transforms of $\Delta\psi$ and $\Delta\psi_D$ evaluated at $1/\gamma_i$,

$$\Delta\overline{\psi}_D(1/\gamma_i) = \Delta\overline{\psi}(1/\gamma_i); \; i = 1, 2, \ldots, n \qquad (4.15b)$$

A more convenient form is obtained by multiplying these by γ_i^{-1}, which yields

$$\left[p\Delta\overline{\psi}_D(p) \right]_{p=1/\gamma_i} = \left[p\Delta\overline{\psi}(p) \right]_{p=1/\gamma_i} ; \; i = 1, 2, \ldots, n \qquad (4.16a)$$

or explicitly

$$\sum_{j=1}^{n} \frac{S_j}{1 + \frac{\gamma_i}{\gamma_j}} = \left[p\Delta\overline{\psi}(p) \right]_{p=1/\gamma_i} ; \; i = 1, 2, \ldots, n \qquad (4.16b)$$

These equations are sufficient for calculating the coefficients S_j and hence dependence of the transient component $\Delta\psi(t)$. To obtain the total solution, equation 3.52a, the constants ψ' and $\psi(\infty)$ are evaluated exactly by examining the behavior of $p\overline{\psi}(p)$ and $p^2\overline{\psi}(p)$ as p tends to zero.

Thus, we see that the total square error is minimized by collocating the p-multiplied transform of the Dirichlet series 4.13 and an associated elastic solution, $p\Delta\overline{\psi}(p)$, at n points $p = 1/\gamma_i$. With this elastic solution given numerically or graphically for $0 \leq p < \infty$, suitable values of γ_i can be prescribed simply by inspection.*

With regard to the error involved in this approximate method, it is of interest to calculate the total square error, E^2, by using 4.14,

$$E^2 = \int_0^\infty \left\{ \Delta\psi - \Delta\psi_D \right\} \left\{ \int_0^\infty \varphi(\tau) e^{-t/\tau} d\tau - \sum_{i=1}^{n} S_i e^{-t/\gamma_i} \right\} dt \qquad (4.17)$$

which becomes

$$E^2 = \int_0^\infty \varphi(\tau) \left\{ \Delta\overline{\psi}(\frac{1}{\tau}) - \Delta\overline{\psi}_D(\frac{1}{\tau}) \right\} d\tau - \sum_{i=1}^{n} S_i \left\{ \Delta\overline{\psi}(\frac{1}{\gamma_i}) - \Delta\overline{\psi}_D(\frac{1}{\gamma_i}) \right\} \qquad (4.18)$$

It is important to recognize that the summation in 4.18 does not necessarily vanish unless <u>exact</u> values of $\Delta\overline{\psi}(\frac{1}{\gamma_i})$ are collocated with $\Delta\overline{\psi}_D(\frac{1}{\gamma_i})$. Since, in practice, $\Delta\overline{\psi}(\frac{1}{\gamma_i})$ can be evaluated only within certain numerical accuracy, this summation is generally not zero. Consequently, if an approximate inversion has been obtained with n terms, and it is desired to reduce the square error by using additional terms, it will be necessary in some cases to evaluate the transforms $\Delta\overline{\psi}(\frac{1}{\gamma_i})$ and $\Delta\overline{\psi}_D(\frac{1}{\gamma_i})$ with increased accuracy. When the transforms are calculated with enough numerical accuracy, equation 4.18 shows that if they are "sufficiently close" for $0 \leq p < \infty$, the total square error of the approximate time dependence is "small."

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*Of course, it is possible to determine the parameters γ_i by minimizing the square error 4.14 with respect to each one. However, this procedure leads to a nonlinear set of equations in which the slopes of $\Delta\overline{\psi}_D$ and $\Delta\overline{\psi}$ are collocated at the (initially) unknown points $1/\gamma_i$. Thus, it generally will be desirable to choose enough values of γ_i so that the slope condition is (closely) satisfied using only equation 4.16.

4.3. Numerical Applications of the Transformed Variational Principles

a. Displacement principle-Kantorovich method. We shall now solve, approximately, a two-dimensional mixed boundary-value problem. The method of Kantorovich (13) will be used in conjunction with the potential energy principle to first calculate approximate elastic displacements u and v. The elastic constants will then be replaced by the appropriate viscoelastic operators and the boundary pressure by a transformed pressure. On the basis of the correspondence rule this procedure yields the approximate, transformed viscoelastic solution. The inversion of this solution will be accomplished by using the approximate methods in section 4.2.

Consider now the thin plate shown in figure 4.2, which is loaded by a uniform tensile stress, σ_o, on the edges $x = \pm a$, and clamped along the edges $y = \pm b$. It is assumed to be in plane stress and composed of homogeneous, isotropic, material.

α) Elastic solution: The elastic potential energy V_E, is

$$V_E = \int_B W\,dB - \int_{A_T} T_i u_i\,dA \tag{4.19}$$

where W is the strain energy density and all other symbols have been defined in earlier sections. The stationary condition, $\delta V_E = 0$, is used to determine approximate displacements which are taken in the form

$$u = b(1 - \eta^2)f(\rho) \tag{4.20a}$$

$$v = b(1 - \eta^2)\eta g(\rho) \tag{4.20b}$$

where we define the dimensionless variables

$$\eta \equiv \frac{y}{b}, \quad \rho \equiv \frac{x}{a} \tag{4.20c}$$

It is seen that u and v both satisfy the displacement boundary conditions (as required by the variational principle) and have physically reasonable dependence on y. $f(\rho)$ and $g(\rho)$ are unspecified functions which are calculated from the Euler equations and natural boundary conditions of $\delta V_E = 0$. These equations for f and g could be obtained by substituting equations 4.20 directly into equation 4.19 and then taking the variation. However, the same set results from the simpler procedure of substituting them into the general stationary condition (13)

$$\delta V_E = 0 = -\int_B (\sigma_{ij,j}) \delta u_i dB + \int_{A_T} (\sigma_{ij} n_j - T_i) \delta u_i dA \qquad (4.21)$$

in which the stresses are to be expressed in terms of the displacement by using the stress-strain equations,

$$\sigma_{ij} = \frac{2\mu\nu}{1-2\nu} \delta_{ij} \vartheta + 2\mu e_{ij} \qquad (4.22)$$

For the problem at hand, the variations δu_i are,

$$\delta u = b(1 - \eta^2)\delta f \, ; \quad \delta v = b(1 - \eta^2)\eta \delta g \qquad (4.23)$$

the equilibrium equations, $\sigma_{ij,j}$, are

$$\frac{\partial \sigma_x}{\partial \rho} + \lambda \frac{\partial \tau_{xy}}{\partial \eta} = 0 \, ; \quad \lambda \frac{\partial \sigma_y}{\partial \eta} + \frac{\partial \tau_{xy}}{\partial \rho} = 0 \qquad (4.24)$$

where λ is the aspect ratio, $\lambda \equiv \frac{a}{b}$, and the applied surface force, T_i, is $T_x = \sigma_o$. Substitution of equations 4.23 and 4.24 into the stationary condition 4.21, and noting that δf and δg are arbitrary, yields the Euler equations

$$\int_0^1 \left(\frac{\partial \sigma_x}{\partial \rho} + \lambda \frac{\partial \tau_{xy}}{\partial \eta} \right)(1 - \eta^2) \, d\eta = 0 \qquad (4.25a)$$

$$\int_0^1 \left(\lambda \frac{\partial \sigma_y}{\partial \eta} + \frac{\partial \tau_{xy}}{\partial \rho} \right)(\eta - \eta^3)\, d\eta = 0 \qquad (4.25b)$$

and natural boundary conditions

$$\int_0^1 \left(\sigma_x \Big|_{\rho=1} - \sigma_o\right)(1 - \eta^2)\,d\eta = 0 \qquad (4.26a)$$

$$\int_0^1 \tau_{xy} \Big|_{\rho=1} (\eta - \eta^3)\,d\eta = 0 \qquad (4.26b)$$

By expressing the stresses in terms of displacements, u and v, and performing the indicated integrations, we obtain two differential equations corresponding to equations 4.25

$$\left(\frac{8}{15\lambda^2}\right) f'' - \frac{2}{3}(1 - \nu) f + \frac{2}{15\lambda}(1 + \nu) g' = 0 \qquad (4.27a)$$

$$\frac{(1-\nu)}{\lambda^2}\, \frac{4}{105}\, g'' - \frac{4}{5}\, g - \frac{2}{15\lambda}(1 + \nu) f' = 0 \qquad (4.27b)$$

where primes denote differentiation with respect to ρ, and the boundary conditions at $\rho = 1$,

$$\left(2 + \nu\, \frac{3 - 34\nu + 3\nu^2}{42(1+\nu)}\right) f' - \frac{8\nu(1-\nu)}{42\lambda^2(1+\nu)}\, f''' = \frac{5}{4}\, \frac{\sigma_o}{\mu}\, \lambda(1-\nu) \qquad (4.28a)$$

$$f - \frac{3 - 34\nu + 3\nu^2}{147\lambda^2(1+\nu)}\, f'' + \frac{8(1-\nu)}{147\lambda^4(1+\nu)}\, f^{iv} = 0 \qquad (4.28b)$$

It is also required, on the basis of the symmetry in this problem, that f is an odd function of ρ while g is even.

The solution of these equations leads to the displacements,

$$\frac{u}{\left(\frac{\sigma_o b}{\mu}\right)} = (1-\nu)\beta_1(1 - \eta^2)\left\{ - \sinh \lambda a_2 \rho + \varphi_1 \sinh \lambda a_1 \rho \right\} \qquad (4.29)$$

$$\frac{v}{\left(\frac{\sigma_o b}{\mu}\right)} = (1-\nu)\beta_1\eta(1-\eta^2)\Big\{(a_3+a_4a_2^2)a_2\cosh\lambda a_2\rho - (a_3+a_4a_1^2)a_1\varphi_1\cosh\lambda a_1\rho\Big\} \tag{4.30}$$

where

$$a_1 = \frac{1}{4}\sqrt{\psi+\sqrt{\psi^2-6720}}\;;\quad a_2 = \frac{1}{4}\sqrt{\psi-\sqrt{\psi^2-6720}}$$

$$a_3 = \frac{-3+34\nu-3\nu^2}{42(1+\nu)}\;;\quad a_4 = \frac{4(1-\nu)}{21(1+\nu)}$$

$$\psi = \frac{171-34\nu+3\nu^2}{1-\nu}\;;\quad \varphi_1 = \frac{\sinh\lambda a_2}{\sinh\lambda a_1}\left[\frac{7+2a_2^2(a_3+a_4a_2^2)}{7+2a_1^2(a_3+a_4a_1^2)}\right] \tag{4.31}$$

$$\beta_1 = \frac{1.25}{\left[2-\nu(a_3+a_4a_1^2)\right]\left[\varphi_1a_1\cosh\lambda a_1\right]-\left[2-\nu(a_3+a_4a_2^2)\right]\left[a_2\cosh\lambda a_2\right]}$$

and the involved dependence upon the Poisson's ratio is especially to be noted. As a typical result, $u/(\sigma_o b/\mu)$ is plotted in figure 4.3 against ν for $0.3 \leq \nu \leq 0.5$ and $\eta = 0$, $\rho = 1$, $\lambda = 1$.

β) Viscoelastic solution: To illustrate calculation of the viscoelastic solution, it is sufficient to consider just the u-displacement with $\eta = 0$, $\rho = 1$, $\lambda = 1$. The transformed response is found immediately by the correspondence rule. With the stress σ_o as a step function of time, its transform is σ_o/p and we have

$$\overline{u}(p) = \sigma_o b\,\frac{J(p)}{p}\,\hat{u}(p) \tag{4.32}$$

where it is convenient to introduce the operational shear compliance, $J(p) \equiv 1/\mu(p)$, and the dimensionless function $\hat{u}(p)$, which depends on p through only Poisson's ratio according to equation 4.29, viz.,

$$\hat{u}(p) \equiv (1-\nu)\beta_1\Big\{-\sinh a_2 + \varphi_1\sinh a_1\Big\} \tag{4.33}$$

in which we have set $\lambda = \rho = 1, \ \eta = 0$.

Since $\hat{u}$ is numerically equal to the elastic displacement shown in figure 4.3, for engineering purposes it could be taken as a constant and evaluated as some average value of Poisson's ratio. Then, recognizing that $J(p)/p$ is the transform of the creep compliance in shear, $J_c(t)$, which can be measured experimentally, the displacement would be simply

$$u(t) \simeq \sigma_o b \hat{u} J_c(t) \tag{4.34}$$

Furthermore, for typical polymers, $J_c(\infty)/J_c(0) \simeq 1000$ so that in this case the error due to taking $\hat{u}$ as a constant would be very small relative to the total variation of $u(t)$.

However, since one of our objectives is to illustrate the simplicity of the approximate inversion methods by inverting a transform which is a very involved function of material properties, we shall consider here the p-dependence of $\hat{u}$. In order not to mask this dependence by the large variation of $J(p)$, $\hat{u}$ will be inverted separately. Also, it will be sufficient for our purposes to calculate just the time dependence due to the variation of $\hat{u}$ because the total solution can always be obtained by using the inversion rule for transform products, which provides the result

$$\frac{u(t)}{\sigma_o b} = \int_0^t \frac{dJ_c(\tau)}{d\tau} \, u_\nu(t-\tau) \, d\tau \tag{4.35a}$$

where we have defined

$$\bar{u}_\nu(p) = \frac{\hat{u}(p)}{p} \tag{4.35b}$$

which is the function that will be inverted by the approximate methods.

These methods require that $\nu(p)$ be calculated for $0 \leq p < \infty$. We shall assume that the bulk modulus is constant and evaluate $\nu(p)$ from the well-known relation for isotropic bodies (13)

$$\nu(p) = \frac{3K - 2\mu(p)}{2(3K+\mu(p))} = \frac{3KJ(p)-2}{2(3KJ(p)+1)} \tag{4.36}$$

in which $J(p)$ is to be calculated from experimental shear data obtained on glass-filled polyisobutylene under constant frequency sinusoidal loading (36). Such a test provides the frequency dependence of the so-called complex shear compliance, which is equal to the ratio of the Fourier transformed strain to the transformed stress. The real part of the complex compliance, $J'(\omega)$, is plotted in figure 4.4.

One possible way of determining the operational compliance, $J(p)$, is to express $J(p)$ as an integral of $J'(\omega)$. The integral is derived by first writing the creep compliance as a Fourier sine transform of $J'(\omega)$ (32) and then taking the p-multiplied Laplace transform of the creep compliance. Omitting the details of this calculation we find

$$J(p) = J_G + \frac{2p}{\pi} \int_0^{\infty} \frac{J'(\omega) - J_G}{p^2 + \omega^2} d\omega; \quad \text{Real } p \geq 0 \tag{4.37}$$

where $J_G \equiv J'(\infty)$, which was numerically integrated for p real and positive, and the result is shown in figure 4.4.* It should be noted that this integration is quite laborious because $J'(\omega)$ varies over many decades of frequency. In fact, it was found that an alternate model-

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*The similarity between the curves occurs because the integrand in equation 4.37, when expressed in terms of log ω, contains a weighting function which is similar to the one in the Laplace transform, figure 4.1.

fitting scheme (37) provided the same results with much less effort.

Substitution of J(p) from figure 4.4 into equation 4.36 yields the operational Poisson's ratio which is shown in figure 4.5, with K chosen such that $\nu(p = \infty) = 0.3$.

The transformed displacement $p\bar{u}_\nu(p)$ is now readily obtained for p real and positive by combining the curves in figures 4.3 and 4.5 and recognizing the fact that

$$p\bar{u}_\nu(p) = \frac{u}{p_o b/\mu} \tag{4.38}$$

The direct method of inversion yields immediately

$$u_\nu(t) \simeq \left[p\bar{u}_\nu(p)\right]_{p = 0.5/t} \tag{4.39}$$

whose transient component, $\Delta u_\nu \equiv u_\nu - u_\nu(\infty)$, is plotted in figure 4.6.

An alternate approximation will now be calculated by the collocation method. An approximate solution is assumed in the form

$$u_{\nu_D}(t) = S_o + \sum_{i=1}^{n} S_i e^{-t/\gamma_i} \tag{4.40}$$

where, by definition

$$\Delta u_{\nu_D} \equiv \sum_{i=1}^{n} S_i e^{-t/\gamma_i}$$

and

$$S_o = \left[p\bar{u}_\nu\right]_{p=0} = 0.285$$

The coefficients, S_i, are calculated from the system of equations 4.16b, with $p\Delta\bar{u}_\nu$ replacing $p\Delta\bar{\psi}(p)$. Examination of $p\Delta\bar{u}_\nu$ in figure 4.6 indicates that a five-term series (n=5) will provide a good approximation with

$$\gamma_i = 10^{-(4+i)} \qquad i = 1, 2, \ldots, 5 \tag{4.41}$$

Substitution of these time constants and values for $\left[p\Delta\bar{u}_\nu\right]_{p=1/\gamma_i}$ into 4.16b yields the following system of equations:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} & \frac{1}{10001} \\ \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} & \frac{1}{1001} \\ \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} & \frac{1}{101} \\ \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} & \frac{1}{11} \\ \frac{1}{1.0001} & \frac{1}{1.001} & \frac{1}{1.01} & \frac{1}{1.1} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_1 x10^2 \\ S_2 x10^2 \\ S_3 x10^2 \\ S_4 x10^2 \\ S_5 x10^2 \end{bmatrix} = \begin{bmatrix} +0.20 \\ +1.20 \\ +3.30 \\ +6.25 \\ +8.20 \end{bmatrix} \tag{4.42}$$

Because of the relative magnitude of the matrix elements, this system can be easily solved by iteration to yield the time dependence

$$\begin{aligned} \Delta u_{\nu_D} &= +10^{-2}\left[0.31e^{-10^5 t} + 1.24e^{-10^6 t} + 2.72e^{-10^7 t} \right. \\ &\qquad \left. + 4.69e^{-10^8 t} - 0.60e^{-10^9 t}\right] \\ &= u_{\nu_D}(t) - 0.285 \end{aligned} \tag{4.43}$$

which is plotted in figure 4.6. It was found by adding two additional terms to the series 4.40, with $\gamma = 10^{-8.5}$ and $10^{-9.5}$, only the short time behavior of Δu_{ν_D} was affected slightly as shown in the same figure. In view of this, as well as the reasonably good agreement with the direct method, it can be assumed that the Dirichlet series 4.43 represents a good approximation to the exact inversion.

γ) Behavior of the transformed displacement: Let us now determine the form of time dependence of the u-displacement by examining the singularities of its transform, and thereby determine whether or not it agrees with the behavior 3.50 deduced from Postulates I and II. In order to simplify the analysis, we shall consider the semi-infinite plate shown in figure 4.7.

For this geometry, the transformed displacement is found from equation 4.29 and written as

$$\overline{u}/\sigma_o b = \frac{J(p)}{p} \hat{u}(p) \tag{4.44}$$

with $\hat{u}(p)$ given by

$$\hat{u}(p) = (1-\nu)\beta_2(1-\eta^2)\left\{-e^{-\alpha_2 \frac{x}{b}} + \varphi_2 e^{-\alpha_1 \frac{x}{b}}\right\} \tag{4.45}$$

where we have defined

$$\begin{aligned} \varphi_2 &= \frac{7+2\alpha_2^2(\alpha_3+\alpha_4\alpha_2^2)}{7+2\alpha_1^2(\alpha_3+\alpha_4\alpha_1^2)} \\ \beta_2 &= \frac{1.25}{\left[2-\nu(\alpha_3+\alpha_4\alpha_1^2)\right]\varphi_2\alpha_1-\left[2-\nu(\alpha_3+\alpha_4\alpha_2^2)\right]\alpha_2} \end{aligned} \tag{4.46}$$

and all α_i $(i = 1, \ldots, 4)$ are given in equation 4.31.

First, the singularities of $\hat{u}(p)$ which are seen to depend on p only through Poisson's ratio, will be found. The square roots appearing in α_1 and α_2 give branch points at

$$\nu_o = -13.9,\ -2.13,\ 36.3,\ 2.33,\ 1,\ \pm\infty \tag{4.47}$$

Also, after a considerable amount of algebra, it is found that $\hat{u}(\nu)$ diverges only at the values

$$\nu_o = 2.33,\ -2.76,\ -1 \tag{4.48}$$

which are simple poles in the ν-plane.

All of these singularities will now be shown to lie on the negative real axis in the p-plane by using the relation 4.36 and analytical representations of K(p) and $\mu(p)$ as derived from thermodynamics. Substitution of equations 1.108 and 1.112 into equation 4.36 yields

$$\sum_{s'} \frac{3K^{(s')}(1-2\nu)-2\mu^{(s')}(1+\nu)}{(p+\frac{1}{\rho_{s'}})} + 3K'(1-2\nu)-2\mu'(1+\nu) = 0 \qquad (4.49)$$

where $\rho_{s'}$ can be a positive real number or infinite and all coefficients $K^{(s')}$, $\mu^{(s')}$, K' and μ' are positive. This equation must hold for all values of ν, and in particular the real values given by equations 4.47 and 4.48. Setting $p = r + is$ and equating the imaginary component of 4.49 to zero gives the restriction on s,

$$0 = s\sum_{s'} \frac{3K^{(s')}(1-2\nu_o) - 2\mu^{(s')}(1+\nu_o)}{(r+\frac{1}{\rho_{s'}})^2 + s^2} \qquad (4.50)$$

Since all singular points occur for $\nu \leq -1$ and $\nu > \frac{1}{2}$ it is clear that s must vanish at these points. Also, by setting the real part of 4.49 to zero we find that r must be negative at the singularities of $\hat{u}(p)$. Thus, all poles and branch points of $\hat{u}(p)$ occur on the negative real p-axis.

In order to determine the behavior of $\hat{u}(p)$ in the neighborhood of each singularity (except $\nu = \pm\infty$) we calculate $d\nu/dp$ from equation 4.49 to find

$$\frac{d\nu}{dp} = \frac{1}{2} \cdot \frac{\displaystyle\sum_{s'} \frac{3K^{(s')}(1-2\nu) - 2\mu^{(s')}(1+\nu)}{(p+\frac{1}{\rho_{s'}})^2}}{\displaystyle\sum_{s'} \frac{3K^{(s')} + \mu^{(s')}}{(p+\frac{1}{\rho_{s'}})} + 3K' + \mu'} \qquad (4.51)$$

which does not vanish at any of the values, ν_o, in equations 4.47 and 4.48. Thus, in the neighborhood of each of these values (except $\nu = \pm\infty$)

$$\nu = \nu_o + \left(\frac{d\nu}{dp}\right)_{\nu_o} (p - p_o) + \text{higher order terms} \tag{4.52}$$

so that at each ν_o the behavior of $\hat{u}$ with respect to p is the same as with respect to ν; hence, all poles of $\hat{u}(p)$ are simple, and the character of the branch points of $\hat{u}(\nu(p))$ is the same as for $\hat{u}(\nu)$.

As indicated in equation 4.47, it is found that $\hat{u}$ has branch points at $\nu = \pm\infty$; furthermore, it can be shown that

$$\hat{u} \sim \frac{1}{\sqrt{\nu}} \quad \text{as} \quad \nu \rightarrow \pm\infty \tag{4.53}$$

To examine this in the p-plane, we solve equation 4.49 for ν, which yields

$$\nu = \frac{\displaystyle\sum_{s'} \frac{3K^{(s')} - 2\mu^{(s')}}{p + \frac{1}{\rho_{s'}}} + 3K' - 2\mu'}{2\left[\displaystyle\sum_{s'} \frac{3K^{(s')} + \mu^{(s')}}{p + \frac{1}{\rho_{s'}}} + 3K' + \mu'\right]} \tag{4.54}$$

It is seen that the poles of $\nu(p)$ occur only when the denominator vanishes, and that these are simple poles since the derivative of the denominator does not vanish in the finite p-plane. Thus, at each of the singularities of ν, p_s say,

$$\hat{u} \sim \sqrt{p - p_s} \quad \text{as} \quad p \rightarrow p_s \tag{4.55}$$

In order to complete the examination of $\overline{u}$, equation 4.44, we have only to determine whether or not the poles of J(p) fall on the poles of $\hat{u}(p)$. The form of $J(p)$ is obtained from equation 1.103b,

$$J(p) = \sum_s \frac{J^{(s)}}{1+\tau_s p} + \frac{J}{p} \tag{4.56}$$

where $J^{(s)}$, J, and τ_s are non-negative. We shall also make use of the inverse of the bulk modulus, $B(p) = K(p)^{-1}$, as given by equation 1.113b,

$$B(p) \equiv \frac{1}{K(p)} = \sum_s \frac{B^{(s)}}{1+\tau_s p} + \frac{B}{p} \tag{4.57}$$

where $B^{(s)}$ and B are non-negative. Substitution of equations 4.56 and 4.57 into Poisson's ratio, equation 4.35, yields

$$\begin{aligned} \nu &= \frac{3J - 2B}{2(3J + B)} \\ &= \frac{\displaystyle\sum_s \frac{3J^{(s)} - 2B^{(s)}}{1 + \tau_s p} + \frac{3J - 2B}{p}}{2\left[\displaystyle\sum_s \frac{3J^{(s)} + B^{(s)}}{1 + \tau_s p} + \frac{3J + B}{p}\right]} \end{aligned} \tag{4.58}$$

Now if $p = -1/\tau_r$, which is a pole of $J(p)$ if $J^{(r)} \neq 0$, then Poisson's ratio must satisfy

$$\nu = \frac{1}{2}\,\frac{3J^{(r)} - 2B^{(r)}}{3J^{(r)} + B^{(r)}} \quad \text{or} \quad 3J^{(r)}(1-2\nu) = 2B^{(r)}(1+\nu) \tag{4.59}$$

Here, at each pole of $J(p)$, Poisson's ratio is restricted to the interval $-1 < \nu \leq \frac{1}{2}$ which is free of the singularities of $\hat{u}(p)$.

To summarize, it has been shown that all singularities of $\overline{u}(p)$ are on the non-negative real p-axis and that all poles are simple except at the origin where a double pole occurs if J in equation 4.56 does not vanish. With this information, the time dependence can be calculated by means of the Bromwich-Wagner Inversion Integral (38) given by

$$u(t) = \frac{1}{2\pi i}\int_{BR_1} e^{pt}\bar{u}(p)dp$$

$$\equiv \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{pt}\bar{u}(p)\,dp \qquad (4.60)$$

where $\delta > 0$. Since it is clear from the above examination that $\bar{u}(p)$ behaves like $1/p$ or $1/p^2$ as $p \to \infty$, the Bromwich contour, indicated by BR_1 in the inversion integral, can be deformed to the contour, BR_2, shown in figure 4.8 (38). Writing the integral 4.60 along BR_2 leads to the time dependent solution $u(t)$:

$$u(t) = \int_0^{\infty} e^{-tu}S(u)du + \sum_i S_i e^{-u_i t} + S' t \qquad (4.61)$$

where this integral results from branch cuts along the negative real axis and the series represents contributions from the poles. Furthermore, it can be shown that the integral is absolutely convergent for $0 \leq t < \infty$. Thus, the time dependence of $u(t)$ is in agreement with the general behavior predicted by Postulates I and II.

b. <u>Stress principle - Ritz method.</u> As another illustration of the correspondence rule, we shall apply the Ritz method, in conjunction with the complementary energy principle, to compute two approximate stress distributions in an elastic cylinder and then use the results to obtain viscoelastic stresses. In addition, the approximations will be compared to the exact solution.

a) Elastic solution: Consider an isotropic, homogeneous, cylinder in plane strain with the outer boundary rigidly supported and the inner boundary loaded by a pressure, p_i, applied stepwise at $t = 0$, as shown in figure 4.9. The exact elastic stresses are easily found to be

radial: $$\sigma_{re} = - p_i + A_e \left(\lambda^2 - \frac{b^2}{r^2}\right) \tag{4.62a}$$

tangential: $$\sigma_{\theta e} = - p_i + A_e \left(\lambda^2 + \frac{b^2}{r^2}\right) \tag{4.62b}$$

where

$$\lambda = \frac{b}{a} \quad ; \quad A_e = \frac{1 - 2\nu}{1 + \lambda^2 (1 - 2\nu)} \; p_i$$

and the shear stress, $\tau_{r\theta}$, is identically zero because of symmetry.

The approximate stresses are not required to satisfy compatibility, but they must fulfill the equilibrium condition,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{4.63}$$

and the stress boundary condition, $\sigma_r = - p_i$ at $r = a$. Two sets of stresses satisfying these conditions are:

$$\sigma_{r\alpha} = - p_i + \alpha \left(\lambda \frac{r}{b} - 1\right) \tag{4.64a}$$

$$\sigma_{\theta\alpha} = - p_i + \alpha \left(2\lambda \frac{r}{b} - 1\right) \tag{4.64b}$$

$$\sigma_{r\beta} = - p_i + \beta \left(1 - \frac{1}{\lambda} \frac{b}{r}\right) \tag{4.65a}$$

$$\sigma_{\theta\beta} = - p_i + \beta \tag{4.65b}$$

with $\tau_{r\theta} = 0$ in both sets. The coefficients, α and β, are determined by minimizing the complementary energy V^* with respect to each one. For a cylinder in plane strain, the complementary energy is

$$V^* = \int_a^b W(2\pi r)\, dr$$

$$= \frac{2\pi(1-\nu^2)}{E} \int_a^b \left\{ \frac{1}{2}\left(\sigma_r^2 + \sigma_\theta^2\right) - \frac{\nu}{1-\nu}\,\sigma_r \sigma_\theta \right\} r\, dr \qquad (4.66)$$

where W is the strain energy density.

Substitution of stresses 4.64 into V^*, carrying out the integration, and minimizing, i.e.

$$\frac{\partial V^*}{\partial \alpha} = 0$$

yields

$$\alpha = \frac{\left(\frac{1-2\nu}{1-\nu}\right)(\lambda - 1)\, p_i}{-\frac{1}{4\lambda^2} + \left(\frac{5}{4} - \frac{\nu}{1-\nu}\right)\lambda^2 - \left(\frac{1-2\nu}{1-\nu}\right)(2\lambda - 1)} \qquad (4.67a)$$

Similarly, for approximation 4.65

$$\beta = \frac{\left(\frac{1-2\nu}{1-\nu}\right)(\lambda-1)\lambda p_i}{\ln \lambda + \left(\frac{1-2\nu}{1-\nu}\right)(\lambda-1)^2} \qquad (4.67b)$$

Substituting these expressions for α and β into the respective stresses, we obtain the elastic stresses which are, along with the exact solution, shown in figure 4.10 for $\nu = 0.45$ and $\lambda = 2$.

β) Viscoelastic solution: We turn now to calculation of the viscoelastic stresses. On the basis of the correspondence rule, the time dependent stresses are obtained by replacing Poisson's ratio, stresses, and internal pressure in equations 4.64, 4.65, and 4.67 by transformed quantities and then inverting.

For specification of an operational Poisson's ratio, a constant bulk modulus, K, is assumed along with an operational shear modulus given by

$$\mu(p) = \frac{\mu_m p}{p + \frac{1}{\tau}} \tag{4.68}$$

which corresponds to a Maxwell model. Substituting K and $\mu(p)$ into equation 4.36, Poisson's ratio becomes

$$2\nu(p) = \frac{\left(3 - \frac{2\mu_m}{K}\right)\tau p + 3}{\left(3 + \frac{\mu_m}{K}\right)\tau p + 3} \tag{4.69}$$

where we shall take $\mu_m/K = 0.6$ in order that $\nu\,(p = \infty) = 0.25$. The pressure is taken to be a step function, of magnitude p_i, applied at $t = 0$ so that p_i appearing in the elastic solutions is to be replaced by p_i/p. Making these substitutions for ν and p_i and then inverting yields the approximate time-dependent stresses:

$$\sigma_{r\alpha}/p_i = -1 + 0.416\left(2\,\frac{r}{b} - 1\right)e^{-0.325\,t/\tau} \tag{4.70a}$$

$$\sigma_{\theta\alpha}/p_i = -1 + 0.416\left(4\,\frac{r}{b} - 1\right)e^{-0.325t/\tau} \tag{4.70b}$$

$$\sigma_{r\beta}/p_i = -1 + 0.980\left(1 - \frac{1}{2}\,\frac{b}{r}\right)e^{-0.284t/\tau} \tag{4.71a}$$

$$\sigma_{\theta\beta}/p_i = -1 + 0.980\; e^{-0.284t/\tau} \tag{4.71b}$$

Furthermore, it is easy to verify that the exact stresses are given by

$$\sigma_{re}/p_i = -1 + 0.167\left(4 - \frac{b^2}{r^2}\right)e^{-0.278t/\tau} \tag{4.72a}$$

$$\sigma_{\theta e}/p_i = -1 + 0.167\left(4 + \frac{b^2}{r^2}\right)e^{-0.278t/\tau} \tag{4.72b}$$

All of these stresses are plotted in figure 4.11 for $r/b = 0.7$.

It is seen from the equations that the time dependence is in complete agreement with Postulates I and II. Furthermore, the graph shows that the quality of approximation (β) is better than that of (α) for both the viscoelastic and elastic solutions, which is not unexpected in view of the remarks made in section 3.3.

4.4. Numerical Applications of Convolution Variational Principles

The ease with which approximate viscoelastic solutions were calculated in the previous two sections suggests that for most linear problems it is not essential to use the convolution principles. However, we shall now discuss two classes of problems for which it may be either more convenient or else necessary to use them instead of the transformed principles and related approximate (or exact) inversion methods.

a. Comments on the dynamic problem. In wave propagation and vibration problems the time dependence of a response may be a rapidly changing and oscillating function of logarithmic time.* In such a case, the direct method is not expected to yield good results. The collocation method can still be used, in principle, if the transient response is quadratically integrable. However, many exponential terms may be needed to obtain a satisfactory solution, which will possibly require an extremely accurate evaluation of the transforms (see equation 4.18). Thus, while the approximate transform may be

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*This behavior is actually more of an exception than a rule for viscoelastic wave propagation problems. In fact Arenz (39) is currently applying the direct and collocation methods to one- and two-dimensional problems with considerable success.

relatively easy to obtain, inversion may be impractical to perform.

In view of this possible complication, a simpler procedure might be to assume approximate solutions which consist of prescribed functions of time along with some arbitrary parameters. These parameters could be calculated from the condition that the appropriate convolution functional be stationary.

Let us briefly illustrate this with the displacement principle, equation 2.111. With temperature neglected for simplicity, but with kinetic energy included (see equation 2.14) we have

$$I_u = \frac{1}{2}\int_0^{t_1}\int_B \left\{ \left[Z_{ij}^{\mu\nu} e_{\mu\nu}(\tau) \right] e_{ij}(t_1-\tau) + \rho_r \frac{du_i(t_1-\tau)}{d(t_1-\tau)} \frac{du_i(\tau)}{d\tau} \right\} dB\, d\tau$$

$$- \int_0^{t_1}\int_B F_i(\tau)u_i(t_1-\tau)dBd\tau - \int_0^{t_1}\int_{A_T} T_i(\tau)u_i(t_1-\tau)dA\, d\tau \tag{4.73}$$

where t_1 is written for the upper limit on the time integral to emphasize the fact that we must restrict ourselves to a definite interval of time, $0 \leq t \leq t_1$, in the approximate method. Assume displacements represented by the series

$$u_i = C_\alpha f_i^{(\alpha)}(x_i, t) + \tilde{U}_i \tag{4.74}$$

in which $f_i^{(\alpha)}$ are specified functions of x_i and t that vanish on A_u, the $\tilde{U}_i$ are prescribed boundary displacements, and the C_α are arbitrary constants which are to be determined from the stationary condition

$$\delta I_u = 0 \tag{4.75}$$

for all arbitrary variations δC_a. By carrying out the integrations in equation 4.73 and the variations, a system of linear algebraic equations is obtained for determination of the constants C_a. It can be shown that the matrix of coefficients which multiply C_a is symmetric; this is a consequence of the symmetry of $Z_{ij}^{\mu\nu}$ and the property of the convolution integral that

$$(Of)^{*}g = f^{*}(Og) \qquad (4.76)$$

where O is an operator (e.g. an element of $Z_{ij}^{\mu\nu}$), and f and g are time dependent functions. It is important to reiterate that I_u is made stationary for just the interval $0 \leq t \leq t_1$, and hence the resulting solution will not be valid for times larger than t_1.

b. Heat conduction example. There is another class of problems in which the transformed principles (or correspondence rule) cannot be used. Namely, that class for which the assumed solution is a nonlinear function of the unspecified generalized coordinates. This application will be illustrated with a one-dimensional heat conduction problem.

Consider the isotropic, homogeneous, semi-infinite solid shown in figure 4.12. At $t = 0$, the face $x = 0$ is suddenly brought to a constant temperature θ_o. It is assumed that the temperature, θ, is zero for $t < 0$.

For comparison purposes, let us first give the exact solution which is derived by solving the heat conduction equation

$$\frac{C}{K}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} \tag{4.77}$$

with

$$\theta = 0 \quad \text{for } t < 0,\ x \geq 0$$

$$\theta = \theta_o \quad \text{for } x = 0,\ t > 0$$

$$\theta \rightarrow 0 \quad \text{as } x \rightarrow \infty,\ t > 0$$

where C is the specific heat per unit volume and K is the heat conduction coefficient. The solution is well-known and is given by

$$\frac{\theta_e}{\theta_o} = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{\frac{Kt}{C}}}^{\infty} e^{-\xi^2} d\xi \tag{4.78}$$

For the approximate solution a simple form is assumed which satisfies the temperature boundary conditions at $x = 0$ and $x = \infty$:

$$\frac{\theta_a}{\theta_o} = e^{-x/q(t)} \tag{4.79}$$

where $q(t)$ is a generalized coordinate that is to be determined by using the variational principle, equation 2.55. For the particular case considered here, the variational principle is

$$T_r \delta \dot{I}_\theta = 0 = \delta\left[\int_0^\infty \left(C \frac{\partial\theta^*}{\partial t}\theta + K\frac{\partial\theta^*}{\partial x}\frac{\partial\theta}{\partial x}\right) dx\right] \tag{4.80a}$$

which is equivalent to

$$\int_0^{t_1}\int_0^\infty \left(C\frac{\partial\theta(\tau)}{\partial\tau} - K\frac{\partial^2\theta(\tau)}{\partial x^2}\right)\delta\theta(t_1-\tau)dx\,d\tau = 0 \tag{4.80b}$$

Substitution of the temperature 4.79 and the variation,

$$\delta\theta_a = \frac{\theta_o x}{q^2} e^{-x/q}\delta q \tag{4.81}$$

into equation 4.80b and recognizing that δq is arbitrary yields*

$$\int_0^{\infty} \left[\frac{dq(\tau)}{d\tau} x^2 - \frac{K}{C} x \right] e^{-x \left(\frac{1}{q(\tau)} + \frac{1}{q(t_1-\tau)} \right)} dx = 0 \tag{4.82}$$

Performing the integration over x the Euler equation for $q(\tau)$ is obtained, thus

$$\frac{2C}{K} \frac{dq(\tau)}{d\tau} = \frac{1}{q(\tau)} + \frac{1}{q(t_1 - \tau)} \tag{4.83}$$

in which t_1 is to be treated as a constant parameter. This is solved by first noting that it can also be written as

$$\frac{2C}{K} \frac{dq(t_1-\tau)}{d(t_1-\tau)} = \frac{1}{q(\tau)} + \frac{1}{q(t_1-\tau)} \tag{4.84}$$

Taking the ratio of 4.83 and 4.84 we find

$$-dq(t_1 - \tau) = dq(\tau) \tag{4.85a}$$

which integrates to

$$q(t_1 - \tau) = q(t_1) - q(\tau) \tag{4.85b}$$

When this is substituted into the differential equation 4.83 and the result integrated we find

$$\hat{q}^2 \left(1 - \frac{2}{3}\hat{q}\right) = \frac{K}{C} \frac{t_1 \hat{\tau}}{q^2(t_1)} \tag{4.86a}$$

in which the non-dimensional variables are defined,

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*It is interesting to observe that the weighting function,

$$e^{-x \left(\frac{1}{q(\tau)} + \frac{1}{q(t_1-\tau)} \right)}$$

is a symmetric function with respect to the point $\tau = t_1/2$ as the time variable τ ranges over the interval $0 \leq \tau \leq t_1$.

$$\hat{q} \equiv \frac{q(\tau)}{q(t_1)} \quad ; \quad \hat{\tau} = \frac{\tau}{t_1} \tag{4.86b}$$

so that $0 \leq \hat{\tau} \leq 1$. Setting $\hat{\tau} = \hat{q} = 1$ in equation 4.86 yields the value for $q(t_1)$:

$$q(t_1) = \sqrt{\frac{3Kt_1}{C}} \tag{4.87}$$

Using this value for $q(t_1)$ in equation 4.86a provides an expression directly relating $\hat{q}$ and $\hat{\tau}$,

$$\hat{q}^2 \left(1 - \frac{2}{3}\hat{q}\right) = \frac{1}{3}\hat{\tau} \tag{4.88}$$

which is plotted in figure 4.13. The values for $\hat{q}$ can now be used in the solution if the temperature 4.79 is written as

$$\frac{\theta_a}{\theta_o} = e^{-\hat{x}/\hat{q}(\hat{\tau})} \tag{4.89a}$$

where

$$\hat{x} \equiv \frac{x}{q(t_1)} \tag{4.89b}$$

This approximate solution is plotted in figure 4.14 along with the exact temperature, which, in terms of the variables $\hat{x}$ and $\hat{\tau}$ is

$$\frac{\theta_e}{\theta_o} = \frac{2}{\sqrt{\pi}} \int_{0.866(\hat{x}/\sqrt{\hat{\tau}})}^{\infty} e^{-\xi^2} d\xi \tag{4.90}$$

It is of interest to compare the approximate temperature given above with one calculated using $\delta\theta(\tau)$ as a weighting function in equation 4.80b in place of $\delta\theta(t_1 - \tau)$. Such a procedure is analogous to the way in which a spacial weighting function is employed in Galerkin's method (13). Using equation 4.79 and $\delta\theta(\tau)$ in equation 4.80b we find

$$\int_0^{\infty} \left(\frac{dq}{d\tau} x^2 - \frac{K}{C} x \right) e^{-2 \frac{x}{q}} dx = 0 \tag{4.91}$$

which yields the differential equation

$$q \frac{dq}{d\tau} = \frac{K}{C} \tag{4.92}$$

This is easily solved to find*

$$q = \sqrt{\frac{2K\tau}{C}} \tag{4.93}$$

so that the approximate temperature is

$$\frac{\theta_\beta}{\theta_o} = e^{-1.225 \, (\hat{x}/\sqrt{\hat{\tau}})} \tag{4.94}$$

which is also shown in figure 4.14.

It is observed that for $\hat{\tau} \lessapprox 0.8$, θ_α is closer than θ_β to the exact solution, except at small values of $\hat{x}$. Because of this, θ_α appears to be the best approximation in the sense that

$$\int_0^1 \int_0^{\infty} (\theta_\alpha - \theta_e)^2 d\hat{x}\, d\hat{\tau} < \int_0^1 \int_0^{\infty} (\theta_\beta - \theta_e)^2 d\hat{x}\, d\hat{\tau} \tag{4.95}$$

It should be added, in this regard, that the stationary condition on the convolution functional, equation 4.80, minimizes the transform, $\overline{I}_\theta$, with respect to $\overline{q}$ for p real and positive. In contrast, condition 4.91 is not associated with the minimization of a functional.

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*It is of interest to mention that the same result is obtained by applying the direct method of inversion to the exact transformed solution of equation 4.77.

Although further study on the use of convolution variational principles is needed, this section does suggest that they are useful tools for approximate analysis.

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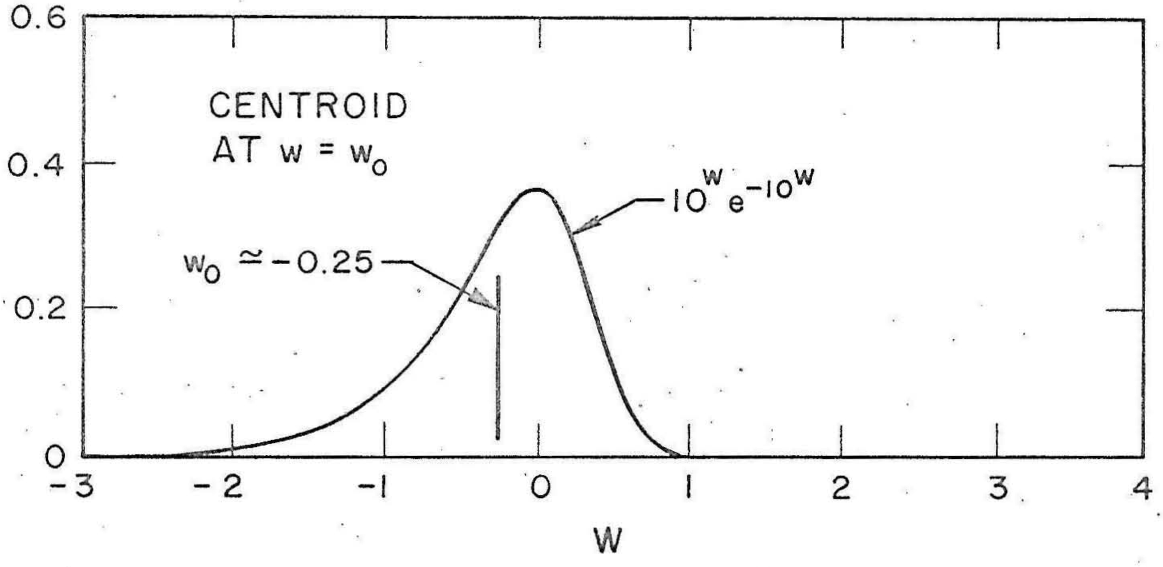

Figure 4.1. Weighting Function in Laplace Transform with Logarithmic Time Scale

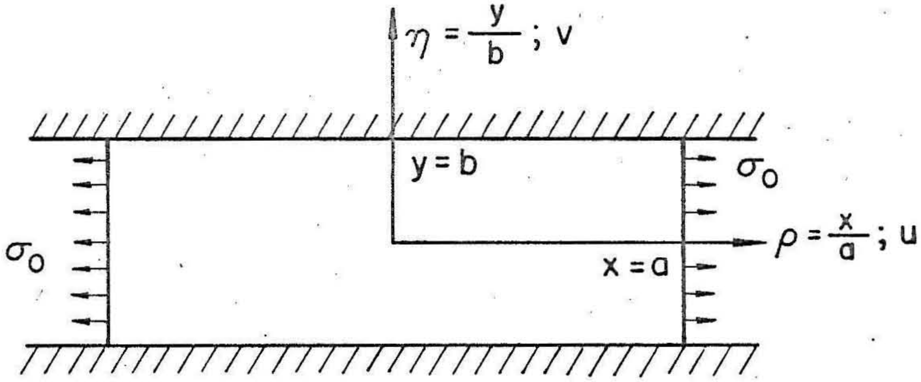

Figure 4.2. Clamped Plate with Uniform Tensile Stress on the Ends

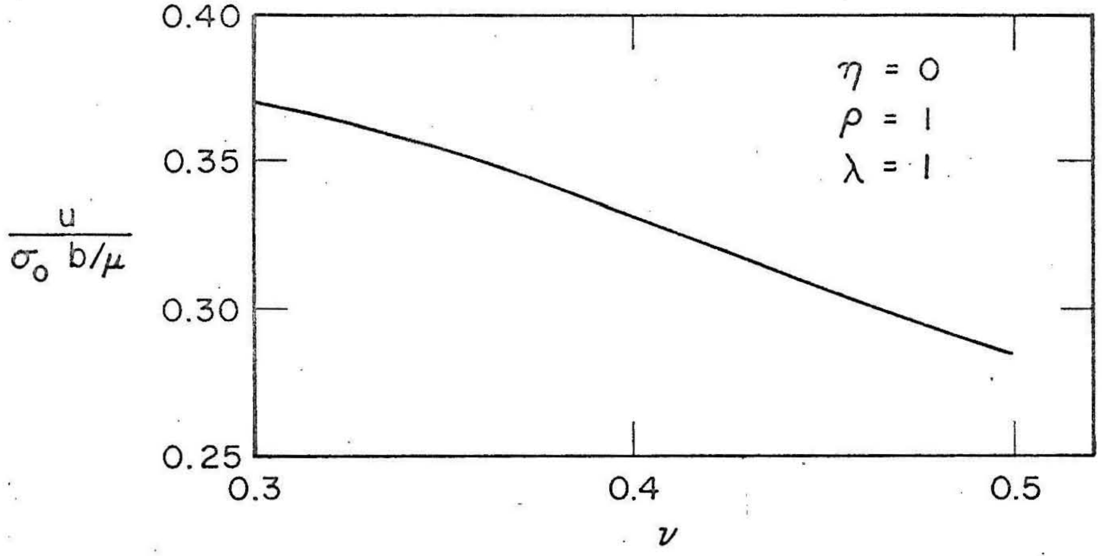

Figure 4.3. Dependence of Displacement on Poisson's Ratio

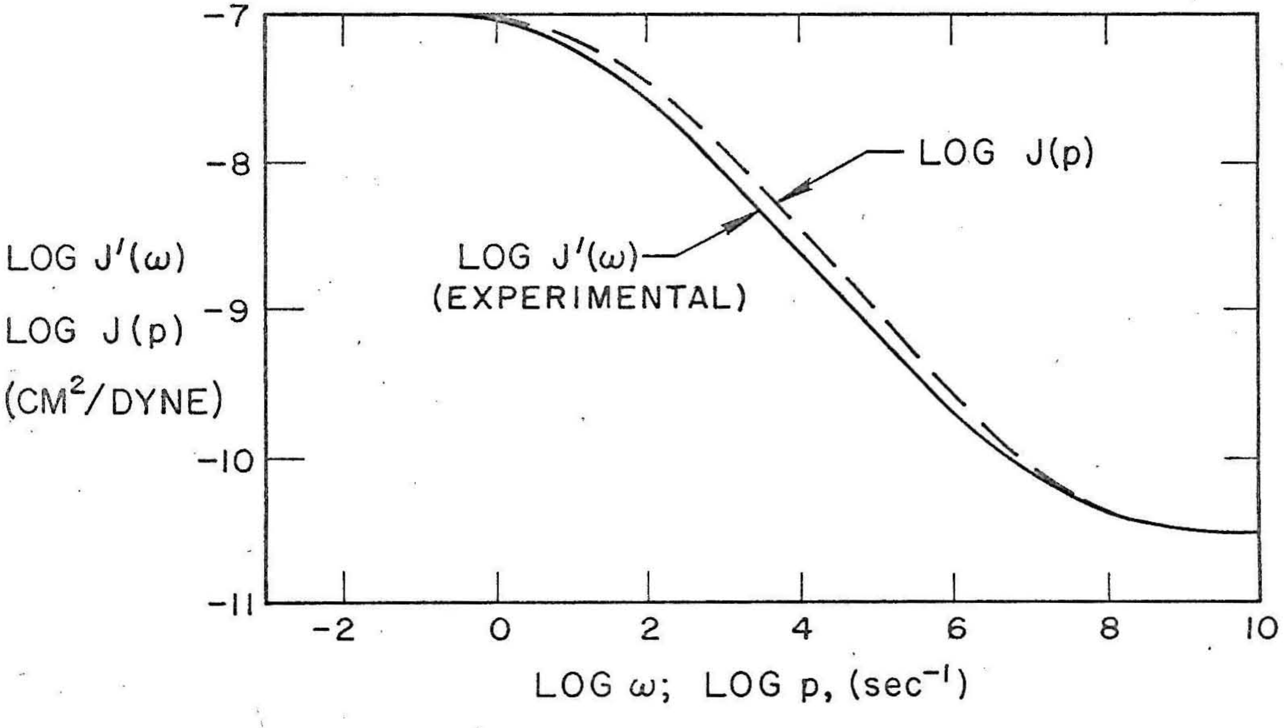

Figure 4.4. Real Part of Complex Shear Compliance and Operational Compliance for Glass-Filled Polyisobutylene at 12.5°C

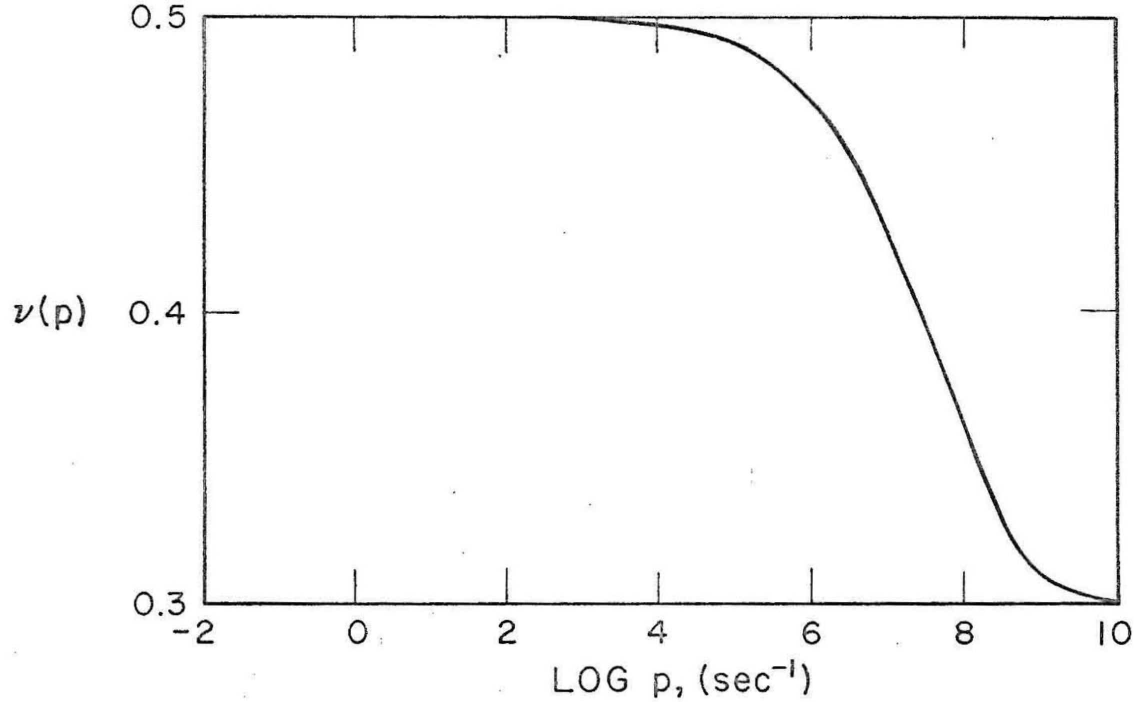

Figure 4.5. Operational Poisson's Ratio

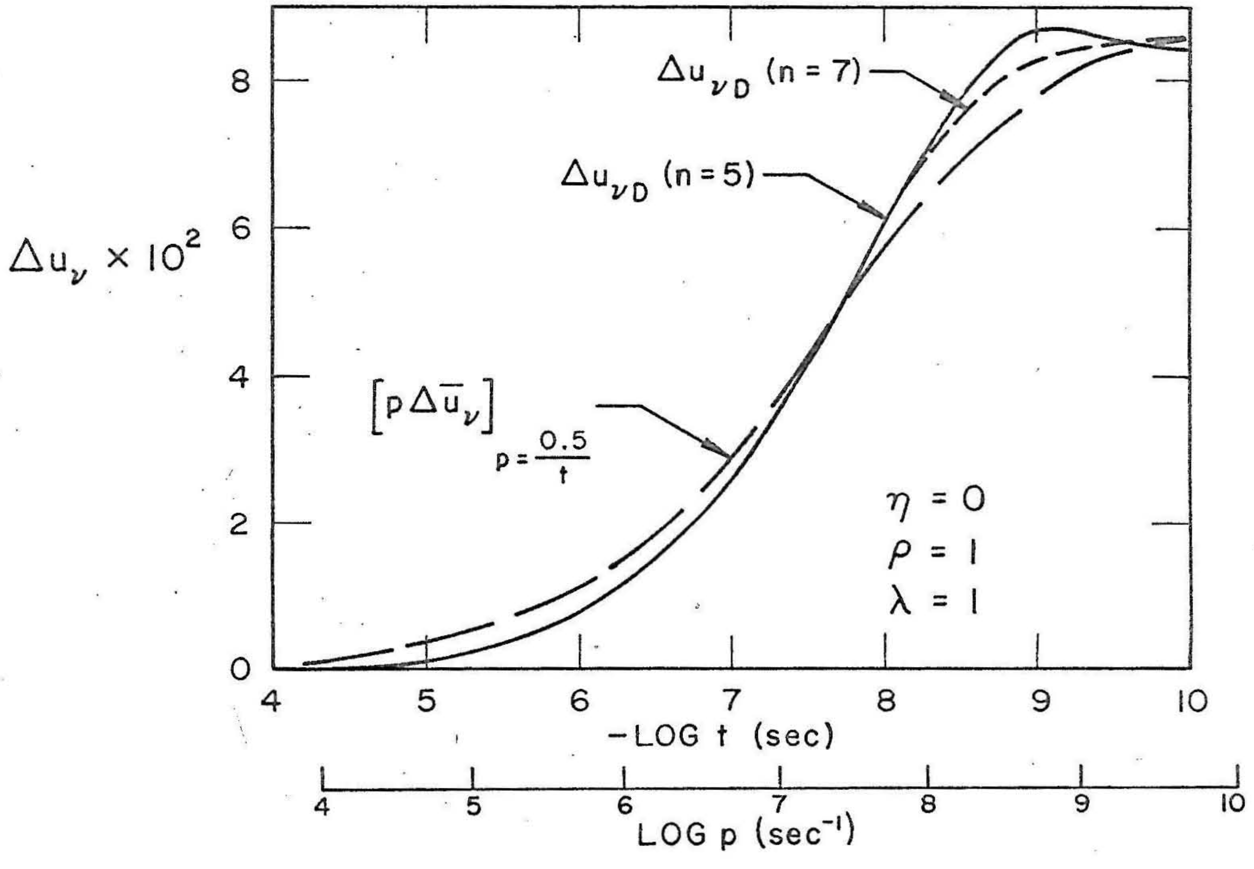

Figure 4.6. Time Dependence of Displacement Due to Change in Poisson's Ratio

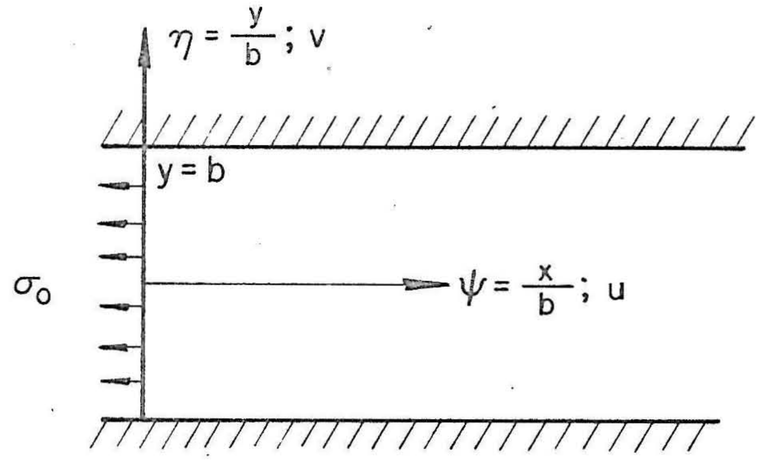

Figure 4.7. Semi-Infinite Clamped Plate with Uniform Tensile Stress on the End

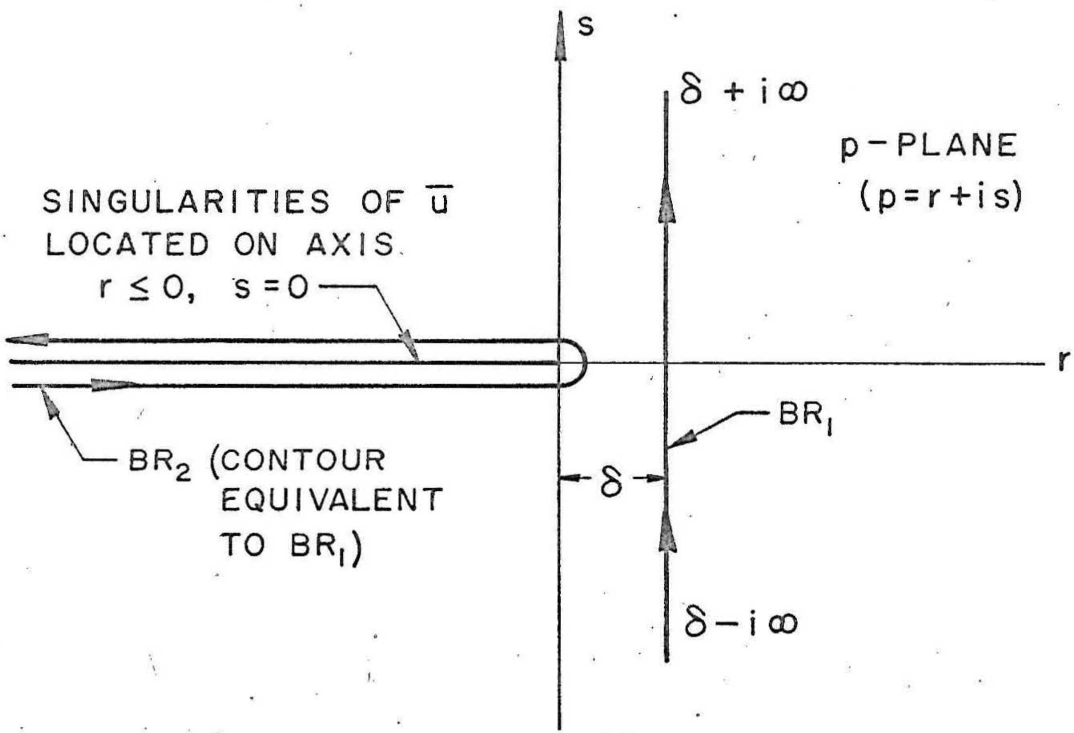

Figure 4.8. Bromwich Contour, BR_1, and Equivalent Contour, BR_2

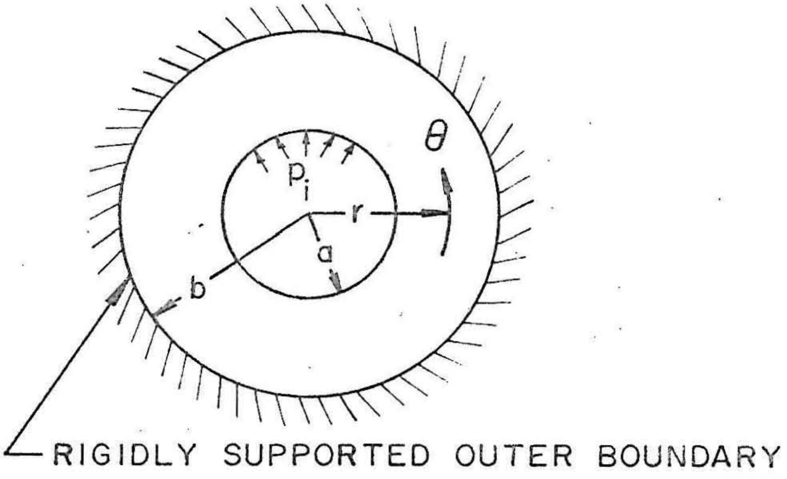

Figure 4.9. Internally Pressurized Cylinder

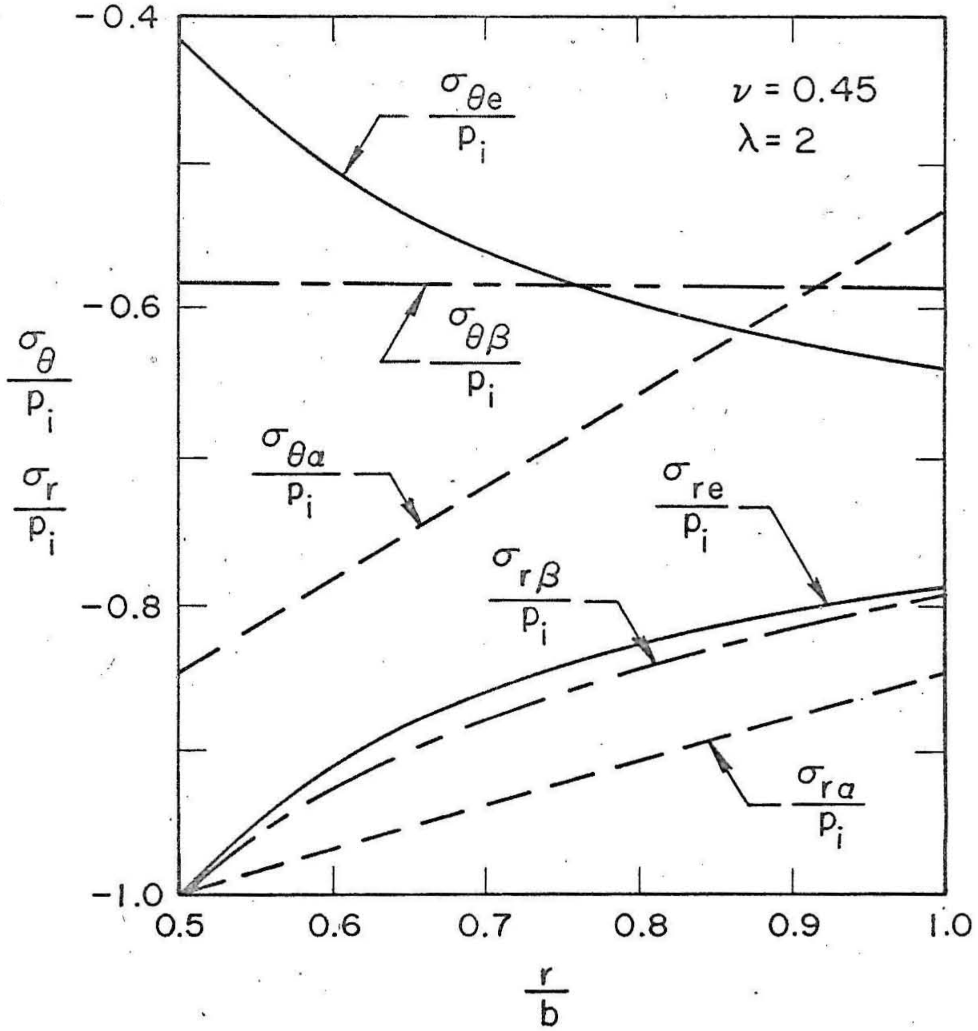

Figure 4.10. Radial Dependence of Elastic Stresses in Cylinder

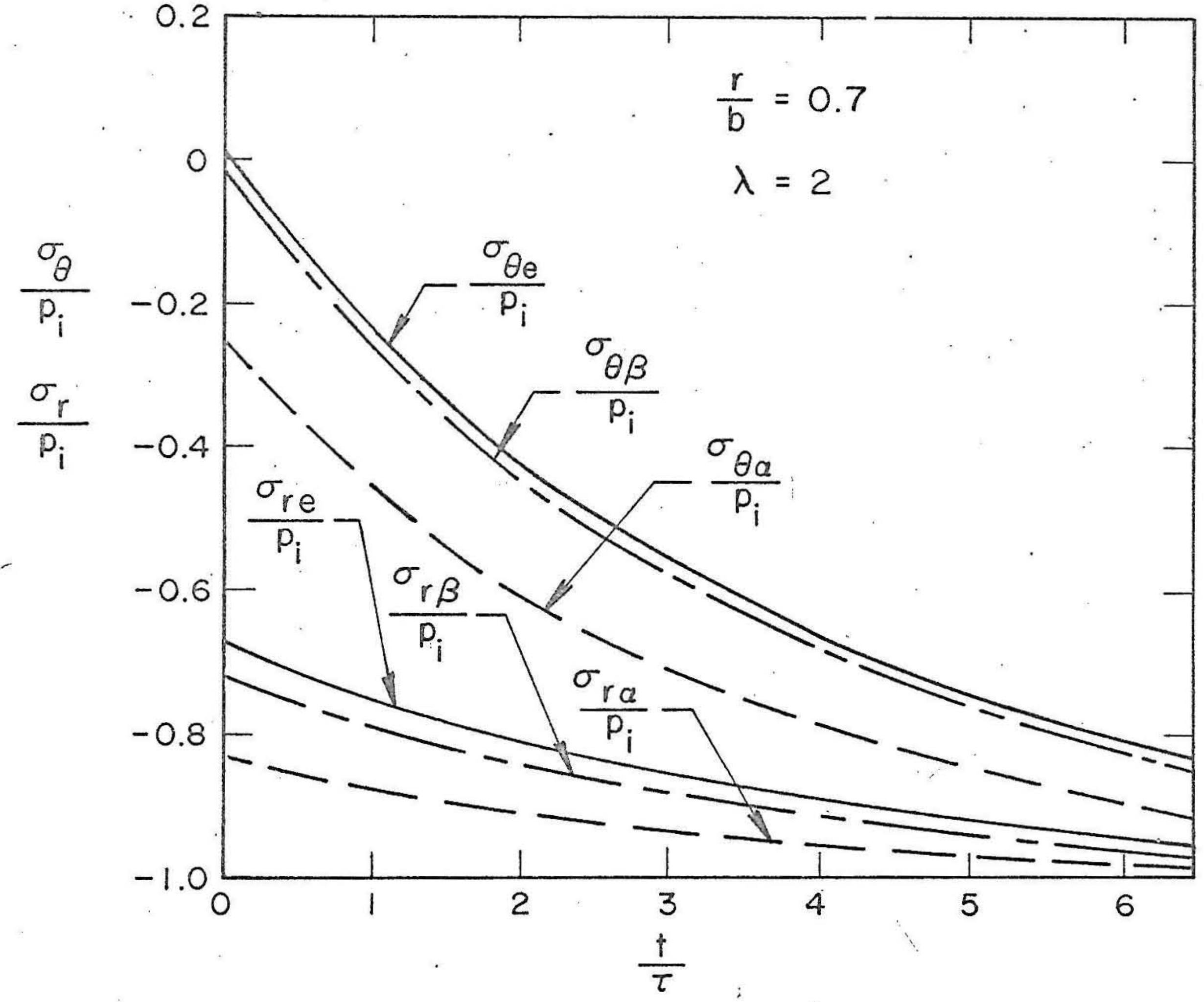

Figure 4.11. Time Dependence of Viscoelastic Stresses in Cylinder

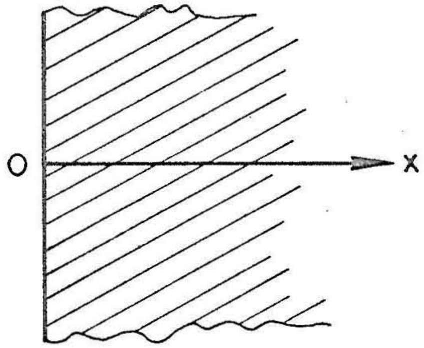

Figure 4.12. Semi-Infinite Solid for Thermal Analysis

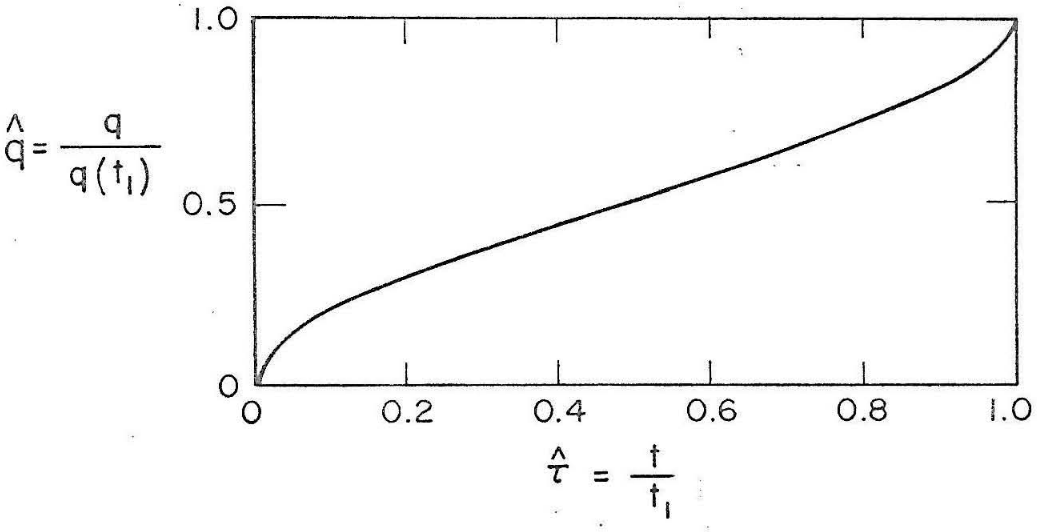

Figure 4.13. Time Dependence of Generalized Coordinate

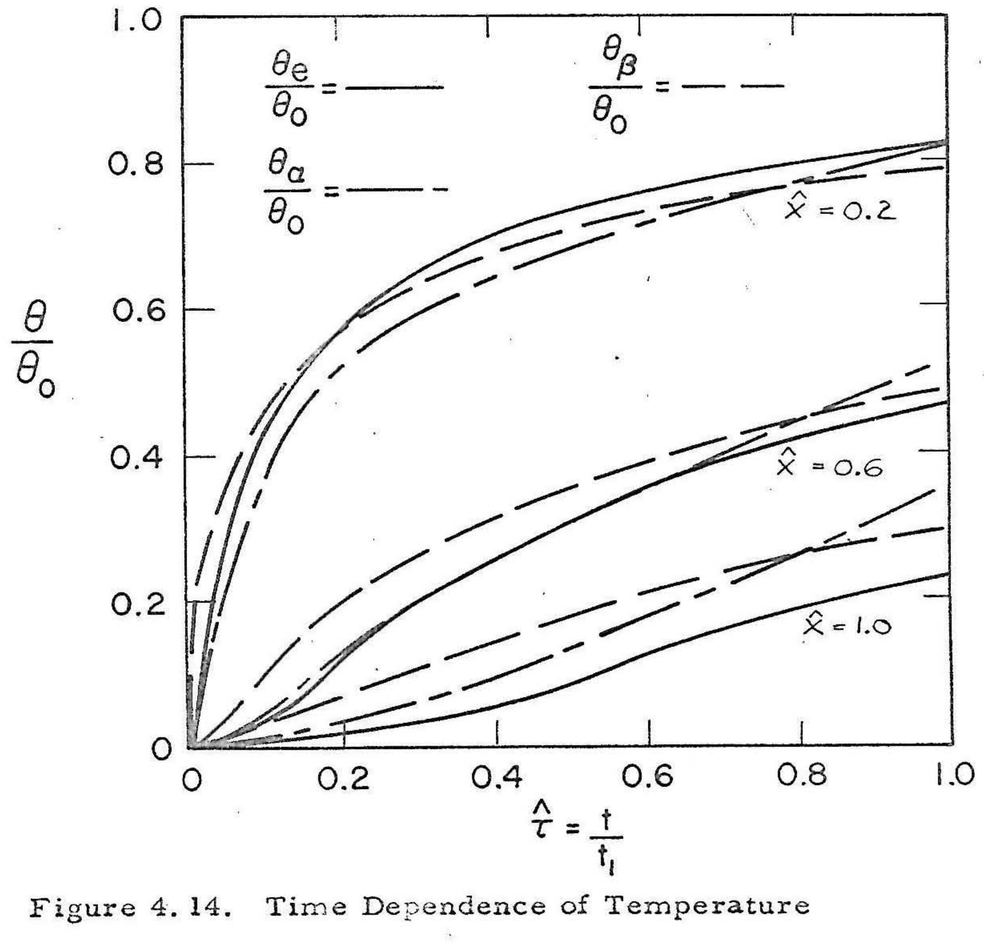

Figure 4.14. Time Dependence of Temperature