THE QUANTUM ELECTRODYNAMICS OF A MEDIUM

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ABSTRACT

By using the Feynman diagram technique, a unified analysis is given of the Quantum Electrodynamics of a Medium. We consider both an atomic medium and an electron gas. The photon propagator in a medium is calculated by summing the most highly divergent diagrams in each term of the perturbation series expansion of the photon propagator. An explicit form for the interaction amplitude of two arbitrary currents in a medium is given. From this amplitude a complete complex dielectric function is defined (at the pole of a photon propagator). Furthermore, we have examined the photon propagator for its poles in order to obtain dispersion relations which yield the energy-momentum relation for free motion of the system. We have considered, in detail, an atomic system and an electron gas. In both cases explicit dispersion relations are found over a wide range of energy and momentum variables. Effects of finite temperatures are discussed. Also we have obtained the energy loss of fast incident charged particles passing through an atomic medium from the self energy of the incident particle in the medium. The energy loss so obtained consists of three parts: loss due to excitation of atoms, loss due to ionization of atoms, and a Cerenkov loss. General features of the energy loss are discussed. We also give a number of expressions for the loss for various incident particles.

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I. INTRODUCTION

The purpose of this paper is to give a unified treatment of the Quantum Electrodynamics of a Medium. By the Quantum Electrodynamics of a Medium we mean the quantum mechanics of a system of charged particles (medium) interacting with the electromagnetic field (i.e., scalar longitudinal and transverse photons). For the most part we will be dealing with an atomic medium (i.e., a medium consisting of atoms interacting with the electromagnetic field).

There are a number of physical problems of interest that arise in connection with the interacting system (medium plus photon field). For example; the propagation of photons through the medium (index of refraction), the normal modes of the interacting system, the energy loss of fast charged particles passing through the medium, etc. are typical problems. In the past these problems have been studied mainly from a classical point of view. For example, the energy loss problem was first treated by N. Bohr in 1915 using classical techniques.* Many people have since extended Bohr's classical treatment.

It was not until 1956 that Tidman (1) gave a non-phenomenological quantum mechanical treatment of the energy loss that included the contribution to the loss from large impact parameters (or equivalently small momentum transfers). It is in this region of large impact parameters where it is necessary to include the passive effects of the medium on the energy loss. Tidman uses the hamiltonian approach, that is, the

*A complete history of this problem may be found in the paper by Tidman.

approach described in Heitler's book (2). This approach necessitates the use of a number of approximations (e.g., that the Coulomb interaction in the medium does not differ from that in vacuum). Since Tidman's treatment of the atoms of the medium is non-relativistic, his results are applicable only to small momentum transfers. Furthermore, Tidman makes the approximation that the dielectric function (the square of the index of refraction) is real. None of these approximations are necessary in the method used here, the Feynman diagram technique. By this method the problem may be treated in a completely four-dimensional fashion, and it follows that the results are valid for all values of the momentum transfer. Another consequence of our four-dimensional formalism is the general validity and usefulness of the current conservation law (e.g., this law enables us to derive the modification due to the medium of the coulomb interaction). Finally we are able to obtain the complete complex dielectric function for both large and small momentum transfers. The relation between the imaginary part of the dielectric function and the energy loss of fast charged particles is derived.

In addition we have applied these methods to a consideration of the normal modes of the system consisting of the atomic medium together with the electromagnetic field. By a simple extension of these methods, we are able to derive the complete relativistic dispersion relation for the electron gas, a relation which reduces in the non-relativistic approximation to that of Bohm and Pines (3).

We feel that the elegance and generality of the Feynman diagram technique is particularly suited to a complete description of all coherent physical processes of interest in a medium.

II. PHOTON PROPAGATION IN A MEDIUM AND THE INDEX OF REFRACTION

A. Qualitative Features of the Photon Propagator in a Medium

In the following we are using gaussian units with $\hbar = c = 1$; then $e^2 \approx 1/137$. Four-vectors will be denoted by small letters [e.g., $k = (\omega, \vec{K})$]. The dot product of two four a, b is taken as $a \cdot b = a_t b_t - \vec{a} \cdot \vec{b}$. Also, the notation $\not{a} = a_t \gamma_t - \vec{a} \cdot \vec{\gamma}$ is used.

In this section we will discuss the photon propagator in a medium. The following assumptions are made in this paper. The medium is assumed to be nonconductive and to consist of N identical, infinitely heavy, nonpolar, nonmagnetic, randomly situated atoms per cubic centimeter. We take N to be small enough so that we may neglect any direct interaction between atoms of the medium. We assume that the intrinsic properties of the atoms of the medium are known, that is, the energy eigenfunctions ψ_n, energy eigenvalues E_n, and line widths γ_n are assumed known. Also, for simplicity, we will consider one electron atoms. We take the temperature of the medium to be so low that in the ground state of the medium each atom is in its ground state. In Chapter IV we will discuss the effects of a finite temperature on the photon propagator.

In order to obtain the photon propagator in a medium we proceed as follows. Consider two current sources in the medium. One current, $J(x_1)$, is at an arbitrary space-time point 1; the other, $J'(x_2)$, at an arbitrary space-time point 2.

We will be working exclusively in momentum space so without loss in generality the currents may be taken to vary as $j_\mu e^{-ik \cdot x_1}$.

Let us ask for the probability amplitude that the current at point 1 emits a photon, the photon propagating through the medium from point 1 to point 2 subsequently being absorbed by the current at point 2. Let us call this amplitude $\mathcal{A}$. We write $\mathcal{A}$ in the following manner

$$\mathcal{A} = \iint d^4x_1 d^4x_2 J_\mu(x_1) P_{\mu\nu}(x_1, x_2) J'_\nu(x_2) .$$

Writing $J_\mu(x_1) = j_\mu e^{ik\cdot x_1}$ (emits) and $J'_\nu(x_2) = j'_\nu e^{-ik'\cdot x_2}$ (absorbs) we have

$$\mathcal{A} = j_\mu \left[\iint d^4x_1 d^4x_2 P_{\mu\nu}(x_1, x_2) e^{i(k\cdot x_1 - k'\cdot x_2)} \right] j'_\nu$$

$$= j_\mu P_{\mu\nu}(k, k') j'_\nu . \qquad \text{(II-1)}$$

Now, due to the homogeneity of the medium $P_{\mu\nu}(x_1, x_2)$ must be a function only of the difference $|x_1 - x_2|$. Therefore

$$P_{\mu\nu}(k, k') = (2\pi)^4 \delta^4(k - k') \pi_{\mu\nu}(k)$$

For the vacuum $\pi_{\mu\nu}$ is just $-4\pi/k^2 \delta_{\mu\nu}$.* With a medium present $\pi_{\mu\nu}$ will differ from the vacuum case. It will contain the effect of the medium on the propagation properties of the photon. In the rest of this chapter we will concentrate on calculating the "propagator" $\pi_{\mu\nu}$.

Physically we would expect that if the spatial separation between points 1 and 2 is larger than the mean separation between the atoms of the medium, which we take to be of the order of the Bohr radius, the

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*$\delta_{\mu\nu}$ = +1 if $\mu = \nu = 4$, = -1 if $\mu = \nu = 1, 2, 3$ and zero otherwise.

effect of the medium will be to modify the propagation properties of the photon from the case of propagation in vacuum. The atoms of the medium will passively scatter the photons; for example, there is a finite probability that the following process will occur: A photon of momentum k excites an atom of the medium to a virtual excited state, the excited atom then decays re-emitting a photon of momentum k. On the other hand, for a spatial separation that is smaller than the separation between atoms of the medium we expect that the atoms will have little effect on the photon propagator for vacuum.

B. Calculation of the Photon Propagator in a Medium

The coupling between electrically charged particles and the electromagnetic field is characterized by the dimensionless constant $e^2 \approx 1/137$. Because this constant is much less than one the usual method of computing amplitudes in quantum electrodynamics is to use perturbation theory (expanding the amplitude in a power series in e^2). This is the method that is used here. In order to calculate the contributions to the amplitudes in perturbation theory we use the Feynman diagram technique. The utility in using Feynman diagrams is that the contribution from each term in the perturbation expansion can be written down by inspection. Figure 1 shows some of the lower order diagrams associated with $\pi_{\mu\nu}$. Diagram 1a represents the propagation of a photon (of momentum k) from 1 to 2 without interacting with the medium (i.e., the photon propagates from 1 to 2 as a free particle). Diagram 1b represents a photon propagating from 1 to 3 as a free particle. At 3 the

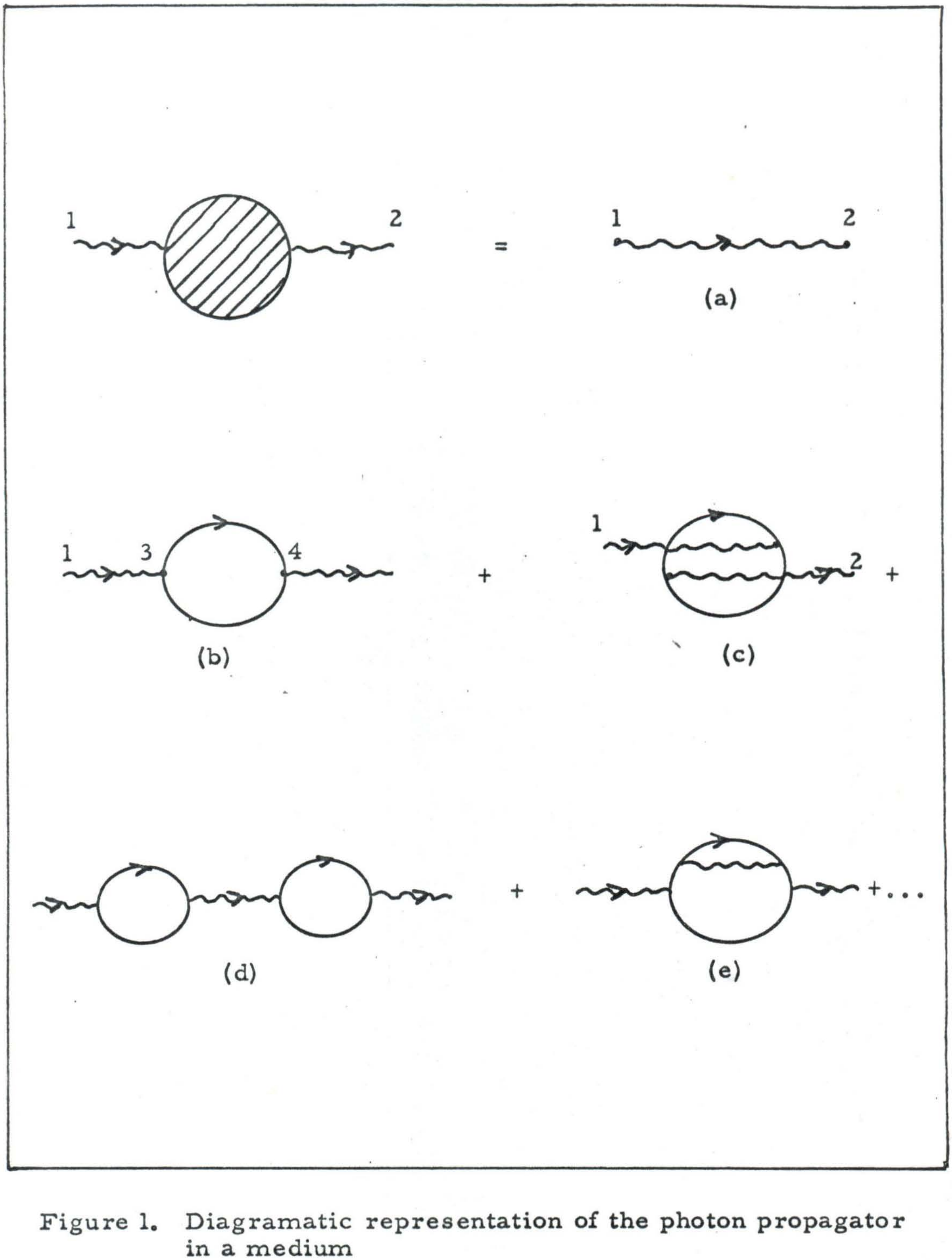

Figure 1. Diagramatic representation of the photon propagator in a medium

photon interacts with an atom of the medium causing the atom to make a transition to an excited state. From 3 to 4 we have an atom in a virtual excited state. We represent this by 

. At 4 the atom de-excites, emitting a photon (of momentum k) which propagates freely from 4 to 2. Diagram 1c differs from diagram 1b only in that the photon that arrives at 2 is emitted before the photon coming from 1 is absorbed. Diagram 1a, 1b, 1c represent the complete 2nd order (in e) contribution to the photon propagator in a medium. Diagrams 1d, 1e are part of the fourth order expansions for $\pi_{\mu\nu}$.

To second order $P_{\mu\nu}$ is given by,

$$P_{\mu\nu} = -\frac{4\pi}{k^2 + i\epsilon}\left[(2\pi)^4\delta^4(k-k')\delta_{\mu\nu} + P^{(2)}_{\mu\nu}\right]^{*} \qquad \text{(II-2)}$$

where $P^{(2)}_{\mu\nu}$ is of the form $(2\pi)^4\delta^4(k-k')\beta^{(2)}_{\mu\nu}$. In II-2, $P^{(2)}_{\mu\nu}$ is $-4\pi/(k')^2$ times the amplitude per unit volume that a photon of polarization μ, momentum k excites an atom of the medium to an axcited state with the atom subsequently emitting a photon of momentum k', polarization ν plus $-4\pi/(k')^2$ times the amplitude per unit volume that a photon of polarization ν, momentum k' is emitted by an atom of the medium, the atom being raised to a virtual excited state with the atom subsequently de-exciting by absorbing a photon of polarization μ, momentum k.

Diagrams 1a, 1b, 1c are the complete contribution to the photon propagator to second order. In the fourth order there are two kinds of diagrams that contribute: iterations of second order diagrams (see for

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*To avoid complexity in notation in the following we will drop the $i\epsilon$ on the Feynman propagator.

example figure 1d) and new types of diagrams that cannot be obtained from iterating second order diagrams (figure 1e shows a typical diagram of this type). Let us call a diagram as proper if it cannot be reduced to two simpler diagrams by cutting a single photon line. Thus, in figure 1, (b) and (c) are proper second order diagrams and (e) is an example of a proper fourth order diagram while (d) is not a proper diagram. In the n'th order of perturbation theory we get contributions from n'th order proper diagrams plus contributions from iterations of lower order proper diagrams.

Call $\beta^{(n)}_{\mu\nu}$ the sum of all the proper diagrams of the n'th order. Then the total amplitude for the medium to absorb a photon of momentum k, polarization μ and the medium subsequently re-emitting a photon of momentum k polarization ν is given by [omitting the factor $(2\pi)^4\delta^4(k-k')$]

$$P_{\mu\nu} \sim \pi_{\mu\nu} = \frac{-4\pi}{k^2}\left(\delta_{\mu\nu} + \beta^{(2)}_{\mu\nu} + \beta^{(4)}_{\mu\nu} + \beta^{(2)}_{\mu\sigma}\beta^{(2)}_{\sigma\nu} + \beta^{(6)}_{\mu\nu} + \beta^{(2)}_{\mu\sigma}\beta^{(2)}_{\sigma\rho}\beta^{(2)}_{\rho\nu} + \dots\right)^{*} \qquad \text{(II-3a)}$$

$$= \frac{-4\pi}{k^2}\left\{\left(\delta - \beta^{(2)} - \beta^{(4)} - \dots\right)_{\mu\nu}\right\}^{-1} \qquad \text{(II-3b)}$$

That is, π is proportional to the inverse of the matrix $(\delta-\beta^{(2)}-\beta^{(4)}-\dots)$ which just involves proper diagrams. What we are doing in calculating $\pi_{\mu\nu}$ is essentially calculating the self energy of a virtual photon in a medium due to virtual interactions with particles of the medium. The net result of these virtual interactions will be to shift the pole of the

*We are using the Feynman summation convention for repeated indices.

photon propagator at $\omega = K$ (vacuum case) to somewhere else. That is, the infinite series II-3a is a perturbation expansion calculation of this shift of the pole. Since e^2 is so small we expect that the location of the pole will be close to $\omega = K$ and also that the nature of the pole will be the same as the vacuum case (e.g., a simple pole). We note that each successive term in II-3a is getting more and more divergent near $\omega = K$. Now β^2 contains a factor e^2/k^2, $\beta^{(4)}$ a factor $e^2(e^2/k^2)$ etc. So, for example, in the fourth order we get two terms, $\beta^{(4)}$ and $(\beta^{(2)})^2$. Both of these terms are of order e^4 but near $\omega = K$, $(\beta^{(2)})^2$, the iterated second order term, dominates the $\beta^{(4)}$ term because of the extra factor $1/k^2$. It is easily seen that this will be true to all orders (i.e., the iterated second order term will be dominant near $\omega = K$). The expression II-3b is telling us that the location of the new pole will be determined where the matrix $k^2(\delta-\beta^{(2)}-\beta^{(4)}-\ldots)^{-1}$ is singular. Physically, at the pole, we obtain the relation (dispersion relation) between ω (energy) and K (momentum) for real processes in a medium. As an approximation to II-3b we take the expression

$$\pi_{\mu\nu} = -\frac{4\pi}{k^2}\{(\delta-\beta^{(2)})_{\mu\nu}\}^{-1} \qquad \text{(II-4)}$$

This expression is the sum of the most highly divergent terms of the perturbation expansion II-3a in each order. We shall see that we do get a pole which is simple and close to $\omega = K$.

C. GENERAL EXPRESSION FOR $\beta_{\mu\nu}$

Applying the Feynman rules (4) to diagrams 1b and 1c we get

$$P^{(2)}_{\mu\nu} = -\frac{i4\pi e^2}{k^2}\sum_i \int d^4x_3\, d^4x_4\, e^{ik\cdot x_4}\overline{\Psi}^i_o(\vec{x}_4, t_4)\gamma_\mu K^A_+(x_4, x_3)\gamma_\nu \Psi^i_o(\vec{x}_3, t_3)$$

$$\times\, e^{-ik'\cdot x_3} + \text{terms with } k \rightleftarrows -k' \quad * \qquad \text{(II-5)}$$

where

$$K^A_+(x_4, x_3) = \begin{cases} \displaystyle\sum_{\text{Pos. } E_n} \varphi_n(\vec{x}_4)\overline{\varphi_n}(\vec{x}_3)e^{-iE_n(t_4-t_3)} & \text{for } t_4 > t_3 \\ \displaystyle -\sum_{\text{Neg. } E_n} \varphi_n(\vec{x}_4)\overline{\varphi_n}(\vec{x}_3)e^{-iE_n(t_4-t_3)} & \text{for } t_4 < t_3 \end{cases} \qquad \text{(II-6)}$$

$\sum_i$ signifies the sum over all the atoms in a unit volume, $\Psi^i_n(x) = \varphi^i_n(\vec{x})e^{-iE_nt}$ are the stationary solutions for the ith atomic electron (i.e., the Ψ_n's are the solution of $(i\nabla - e\not{A}^{ExT} - m)\Psi = 0$, normalized to $\int \varphi^*\varphi\, d^3\vec{x} = 1$, where A^{ExT} is the external field acting on the electron i.e., the coulomb field of the nucleus), the quantities γ_μ $(\mu = x, y, z, t)$ are the familiar gamma matrices which satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$, and $\overline{\Psi}$ is the adjoint of Ψ $(= \Psi^+\gamma_t)$. The first term of II-5 corresponds to diagram 1b and the second term to diagram 1c.

After inserting II-6 into II-5 and combining terms we get:

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*Henceforth we drop the superscript (2).

$$P_{\mu\nu} = \frac{i4\pi e^2}{(k')^2} \sum_i \int_{-\infty}^{\infty} dt_3 \int_{-\infty}^{\infty} \sum_{\text{Pos. } E_n} e^{-it_3(E_o+\omega)} e^{it_4(E_o+\omega)}$$

$$\times e^{-iE_n(t_4-t_3)} \left(\int d^3x_1 \bar{\varphi}_o \gamma_\mu e^{-i\vec{K}\cdot\vec{x}_1} \varphi_n\right)_i \left(\int d^3x_2 \bar{\varphi}_n \gamma_\nu e^{-i\vec{K}'\cdot\vec{x}_2} \varphi_o\right)_i$$

$$+ \text{ terms with } k \rightleftarrows -k' \text{ for } t_4 > t_3$$

$$= -\frac{i4\pi e^2}{(k')^2} \cdots - \sum_{\text{Neg. } E_n} \cdots \text{ for } t_4 < t_3 \qquad \text{(II-7)}$$

Consider the atomic matrix elements in II-7. For bound states the wave functions φ_n are non-zero for values of $|\vec{x}|$ up to the order of the Bohr radius a_o. So for small K ($Ka_o \ll 1$) we can expand the exponential factor $e^{i\vec{K}\cdot\vec{x}}$ as series of powers of $\vec{K}$: $e^{i\vec{K}\cdot\vec{x}} \approx 1 + i\vec{K}\cdot\vec{x} = 1 + iKz$; we choose a coordinate system with the z axis along the vector $\vec{K}$. Also in this region of K we make the nonrelativistic approximation in the treatment of the atomic electrons. Physically in this region of K the atom receives a momentum which is small compared with the original intrinsic momentum (me^2) of the atomic electrons. On the other hand in the opposite limit, that of large K ($Ka_o \gg 1$), it is evident physically that we can regard the atomic electrons as free, and at rest (atom receives a momentum which is large compared to me^2). This can also be seen from the atomic matrix elements in II-7. For large K, the integrand contains a rapidly oscillating factor $e^{iK\cdot x}$, and the integral is almost zero if φ_n does not contain a similar factor. Such a function φ_n corresponds to an ionized atom with the momentum (space part) of the emitted electron given by the law of conservation of momentum between the incident photon ($\vec{K}$) and the original momentum of the electron (which is much smaller than $\vec{K}$).

D. CALCULATION OF $\beta_{\mu\nu}$ IN THE CASE OF SMALL K

First let us consider the case of small K ($Ka_o \ll 1$). Now

$$K_+^A(x_4, x_3) = \frac{1}{2\pi i} \sum_{\text{all } n} \int_{-\infty}^{\infty} \frac{d\Omega e^{i(t_4-t_3)\Omega}}{E_n(1-i\epsilon)+\Omega} \varphi_n(\vec{x}_4)\bar{\varphi}_n(\vec{x}_3) \qquad \text{(II-8)}$$

where ϵ is to approach zero through positive values after the Ω integration has been done. Upon inserting II-8 into II-5 and rearranging terms we get,*

$$P_{\mu\nu} = -\frac{i4\pi e^2}{(k')^2} \sum_i \int_{-\infty}^{\infty} d\Omega\, dt_1 dt_2 e^{-it_1(E_o+\omega+\Omega)} e^{it_2(E_o+\omega+\Omega)} \frac{1}{2\pi i}$$

$$\times \sum_{\text{all } n} \left(\int d^3\vec{x}_2 \bar{\varphi}_o e^{-i\vec{K}'\cdot\vec{x}_2} \gamma_\nu \varphi_n\right)_i \left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\mu e^{i\vec{K}\cdot\vec{x}_1} \varphi_o\right)_i$$

$$\times \frac{1}{E_n(1-i\epsilon)+\Omega} + \text{terms with } k \rightleftarrows -k'$$

Performing the integrals over t_1 and t_2 we get,

$$P_{\mu\nu} = -\frac{4\pi e^2}{(k')^2} (2\pi)\delta(\omega-\omega') \sum_{i,n} \left\{ \int_{-\infty}^{\infty} d\Omega\, \delta(\Omega+E_o+\omega) \left(\int d^3\vec{x}_2 \bar{\varphi}_o e^{-i\vec{K}'\cdot\vec{x}_2} \gamma_\nu \varphi_n\right)_i \right.$$

$$\times \left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\mu e^{i\vec{K}\cdot\vec{x}_1} \varphi_o\right)_i \frac{1}{E_n(1-i\epsilon)+\Omega} + \int_{-\infty}^{\infty} d\Omega\, \delta(\Omega+E_o-\omega)$$

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*In the following the dummy integration variables will be denoted by the subscripts 1 and 2.

$$\times \left(\int d^3\vec{x}_2 \bar{\varphi}_o e^{i\vec{K}\cdot\vec{x}_2} \gamma_\mu \varphi_n\right)_i \left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\nu e^{-i\vec{K}'\cdot\vec{x}_1} \varphi_o\right)_i \frac{1}{E_n(1-i\epsilon)+\Omega} \Big\}$$

$$= -\frac{4\pi e^2}{(k')^2} (2\pi)\delta(\omega-\omega') \sum_{i,n} \Big\{ \left(\int d^3\vec{x}_2 \bar{\varphi}_o e^{-i\vec{K}'\cdot\vec{x}_2} \gamma_\nu \varphi_n\right)_i$$

$$\times \left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\mu e^{i\vec{K}\cdot\vec{x}_1} \varphi_o\right)_i \frac{1}{E_n(1-i\epsilon)-\omega-E_o}$$

$$+ \left(\int d^3\vec{x}_2 \varphi_o e^{i\vec{K}\cdot\vec{x}_2} \gamma_\mu \varphi_n\right)_i \left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\nu e^{-i\vec{K}'\cdot\vec{x}_1} \varphi_o\right)_i$$

$$\times \frac{1}{E_n(1-i\epsilon)+\omega-E_o} \Big\} \qquad \text{(II-9)}$$

To proceed further, we first examine the contribution from positive energy states. In II-9 for a given atom (i) the $\varphi_n^i \equiv \varphi_n(\vec{x}-\vec{a}_i)$ where $\vec{a}_i$ is the position of the i'th nucleus. So if we write, say,

$$\left(\int d^3\vec{x}_1 \bar{\varphi}_n \gamma_\mu e^{i\vec{K}\cdot\vec{x}_1} \varphi_o\right)_i$$

as

$$\left(\int d^3(\vec{x}_1-\vec{a}_i) \bar{\varphi}_n \gamma_\mu e^{i\vec{K}\cdot(\vec{x}_1-\vec{a}_i)} \varphi_o\right)_i e^{-i\vec{K}\cdot\vec{a}_i}$$

and similarly for the other terms then this form has the integration variable centered around the i'th nucleus. Since all of the atoms are identical the atomic matrix elements for different i are equal; the only dependence on i in II-9 will appear in a factor $e^{-i\vec{a}_i\cdot(\vec{K}-\vec{K}')}$. Then

$$\sum_i e^{-\vec{a}_i\cdot(\vec{K}-\vec{K}\,)} \to N \int d^3\vec{a} e^{-i\vec{a}\cdot(\vec{K}-\vec{K}')}$$

$$= N(2\pi)^3 \delta^3(\vec{K}-\vec{K}')$$

Putting everything together, we finally get that

$$P_{\mu\nu} = -\frac{4\pi e^2 N}{k^2}(2\pi)^4\delta^4(k-k)\sum_{\text{all } n}\left\{\frac{<0|e^{-i\vec{K}\cdot\vec{x}}\gamma_\nu|n><n|\gamma_\mu e^{i\vec{K}\cdot\vec{x}}|0>}{E_n(1-i\epsilon)-E_o-\omega}\right.$$

$$\left. + \frac{<0|e^{i\vec{K}\cdot\vec{x}}\gamma_\mu|n><n|\gamma_\nu e^{-i\vec{K}\cdot\vec{x}}|0>}{E_n(1-i\epsilon)-E_o+\omega}\right\} \qquad \text{(II-10a)}$$

$$= (2\pi)^4\delta^4(k-k')\beta_{\mu\nu} \qquad \text{(II-10b)}$$

where $<n|\gamma_\mu e^{i\vec{K}\cdot\vec{x}}|0> = \int d^3\vec{x}\varphi_n\gamma_\mu e^{i\vec{K}\cdot\vec{x}}\varphi_o$. Consider the atomic matrix element

$$a_\mu = <n|\gamma_\mu e^{i\vec{K}\cdot\vec{x}}|0>^{*} \qquad \text{(II-11)}$$

where $\mu = (t, x, y, z)$.

In the nonrelativistic approximation a_μ becomes:

$$a_t \cong <n|\gamma_t(1+iKz)|0>$$

$$= (1+iKz)_{no}$$

$$= iKz_{no}$$

where $z_{no} = \int \varphi_n^+ z\varphi_o d^3\vec{x}$ (the matrix element of 1 between n and o vanish because these states are orthogonal to each other). Also

$$a_x \cong <n|\gamma_x|0>$$

$$= \int d^3\vec{x}\varphi_n^+ a_x\varphi_o$$

$$= \dot{x}_{no},$$

---

* $|0>$ represents the ground state of the atom.

(since in the nonrelativistic approximation $\vec{\alpha}$ is replaced by $\dot{\vec{x}}$)

$$= i(E_n - E_o)x_{no}$$

$$\equiv i\omega_{no}x_{no}$$

similarly

$$a_y = i\omega_{no}y_{no} \quad \text{and} \quad a_z = i\omega_{no}z_{no}$$

Also for a given value of E_n there is a sum over the angular momentum states of the atom.* As is well known in the dipole approximation $\Delta\ell = \pm 1$. Since the ground state of the atom is an s state the excited states must be p states. So for a given E_n there are three angular momentum states $(m_\ell = 0, \pm 1)$. Now it is easy to see that within the dipole approximation that the only non-zero terms of $\beta_{\mu\nu}$ are: β_{11}, β_{22}, β_{33}, β_{34}, β_{43}, and β_{44}, where the indices 4 corresponds to t, 3 to z (the direction of $\vec{K}$), 1 and 2 to x_+ and x_- respectively $\left(\frac{-x-iy}{\sqrt{2}}, \frac{x-iy}{\sqrt{2}}\right)$. The reason for this is the following: consider, for example, β_{13}. Physically we are asking for the amplitude, in the dipole approximation, that an atom absorbs a right-handed polarized photon and emits a longitudinal photon. Now for a given value of E_n

$$\beta_{\mu\nu} \sim a_\mu^+(+)a_\nu(+) + a_\mu^+(0)a_\nu(0) + a_\mu^+(-)a_\nu(-)$$

where the ± 1 and 0 correspond to excited states with $m_\ell = \pm 1$ and 0

*We disregard spin here because the interaction hamiltonian, j· A, does not involve spin variables, hence the spin operator commutes with the hamiltonian implying that the spin is a conserved quantity.

respectively.* So

$$\beta_{13} \sim a_1^+(+)a_3(+) + a_1^+(0)a_3(0) + a_1^+(-)a_3(-) = 0$$

since, for example, $a_3(+) = a_1(0) = a_1(-) = 0$.** Similarly, all off diagonal elements except β_{34} and β_{43} are zero. Now consider the diagonal terms

$$\beta_{11} \sim (\,|a_1(+)|^2 + |a_1(0)|^2 + |a_1(-)|^2) = |a_1(+)|^2 \equiv \omega_{no}^2\,|z_{no}|^2 \;***$$

Also

$$\beta_{22} \sim |a_2(-)|^2 = |a_1(+)|^2 = \omega_{no}^2\,|z_{no}|^2$$

$$\beta_{33} \sim |a_3(0)|^2 = |a_1(+)|^2 = \omega_{no}^2\,|z_{no}|^2$$

and

$$\beta_{44} \sim |a_4(0)|^2 = K^2|z_{no}|^2$$

also

$$\beta_{34} = \beta_{43} \sim K\omega_{no}|z_{no}|^2$$

Therefore from II-10 and II-12 we obtain that the contributions from the sum over positive energy states is given by

$$\beta_{11} = \beta_{22} = \beta_{33} = \frac{8\pi Ne^2}{k^2}\sum_{n+}\omega_{no}^2\,|z_{no}|^2\left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right) \qquad \text{(II-13a)}$$

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*$\varphi_n(\pm, 0)$ contain a factor of $e^{\pm i\phi}$ and $e^{i0\phi}$ respectively.

**(See (2-11)).

***Henceforth $|z_{no}|^2$ denotes the dipole matrix element summed over the orbital angular momentum states.

$$\beta_{44} = \frac{8\pi N e^2}{k^2} \sum_{n+} K^2 |z_{no}|^2 \left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right), \quad \beta_{34} = \beta_{43} = \frac{\omega}{K}\beta_{44} \qquad \text{(II-13b)}$$

Now in the dipole approximation we calculate the contribution from negative energy states. For a negative energy state $E_n \sim -(m +$ Order Rydberg). Now $E_o \sim m$ + Order Rydberg so $E_n - E_o \sim -2m +$ O(Ryd.). So for ω much less than m we get, from II-9, that the contribution to say β_{11} ($= \beta_{22} = \beta_{33}$) becomes (neglecting terms of order Ryd./m)

$$\beta_{11} \sim \sum_{n-} \langle 0|\gamma_1|n^-\rangle\langle n^-|\gamma_1|0\rangle \left(-\frac{1}{2m} - \frac{1}{2m}\right)$$

$$= -\frac{1}{m} \sum_{\text{all } n} \langle 0|\gamma_1 \rho\, |n\rangle\langle n|\gamma_1|0\rangle$$

$$= -\frac{1}{m} \langle 0|\gamma_1 \rho\, \gamma_1|0\rangle \qquad \text{(II-14)}$$

where ρ is the projection operator for negative energy states (i.e., $\rho\,|n^+\rangle = 0$, $\rho\,|n^-\rangle = |n^-\rangle$. Neglecting terms of the order Ryd./m ρ can be taken to be:

$$\rho = \frac{\not{p} - m}{2m} \sim \frac{1}{2}(\gamma_t - 1) \qquad \text{(II-15)}$$

Substituting II-15 into II-14 we obtain

$$\beta_{44} \sim -\frac{1}{m} \langle 0|\gamma_4 \rho\, \gamma_4|0\rangle = -\frac{1}{m}\langle 0|\gamma_t \frac{(\gamma_t + 1)}{2}|0\rangle = -\frac{1}{m} \qquad \text{(II-16)}$$

II-16 can be recognized as the contribution from the A^2 term in the nonrelativistic hamiltonian.

The contribution to β_{44} from negative energy states is

$$\beta_{44} \sim -\frac{1}{m} <0|\gamma_4 \not{\mathcal{P}} \gamma_4|0> = -\frac{1}{m} <0|\gamma_t \frac{(\gamma_t-1)}{2} \gamma_t|0> = 0$$

Also, the contributions to all off diagonal elements are zero. For example

$$\beta_{12} \sim -\left[\frac{1}{2m} <0|\gamma_1 \frac{(\gamma_t-1)}{2} \gamma_2|0> + <0|\gamma_2 \frac{(\gamma_t-1)}{2} \gamma_1|0>\right]$$

$$= \frac{1}{2m} <0|(\gamma_1\gamma_2 + \gamma_2\gamma_1)|0>$$

$$= 0$$

Also

$$\beta_{34} \sim -\frac{1}{2m}\left[<0|\gamma_3 \frac{(\gamma_t-1)}{2} \gamma_4|0> + <0|\gamma_4 \frac{(\gamma_t-1)}{2} \gamma_3|0>\right]$$

$$= \frac{1}{2m}\left[<0|\gamma_3\gamma_4 \frac{(\gamma_t-1)}{2} |0> + <0| \frac{(\gamma_t-1)}{2} \gamma_4\gamma_3|0>\right]$$

$$= 0$$

since $\gamma_t|0> = |0>$.

Summing II-16 over all of the atoms per unit volume gives a factor N. That is (replacing all the factors) the non-zero contribution from negative energy states per unit volume is,

$$\beta_{11} = \beta_{22} = \beta_{33} = \frac{4\pi e^2}{k^2}\left(-\frac{N}{m}\right) \qquad \text{(II-16')}$$

From II-13 and II-16' we get that the contribution to β from both positive and negative energy states per unit volume is,

$$\beta_{11} = \beta_{22} = \beta_{33} = \frac{4\pi Ne^2}{k^2}\left\{\sum_{n+}\left[\omega_{no}^2 |z_{no}|^2 \left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right)\right] - \frac{1}{m}\right\} \quad \text{(II-17a)}$$

$$\beta_{34} = \frac{\omega}{K}\beta_{44}; \; \beta_{44} = \frac{4\pi Ne^2}{k^2} \sum_{n+} K^2 |z_{no}|^2 \left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right)^{*} \quad \text{(II-17b)}$$

We remark that the expression II-17 could have been written down by inspection. Consider, say β_{11}. This is the amplitude for transverse waves propagating exciting-de-exciting atoms. Treating the atomic electrons as nonrelativistic the amplitude for diagrams 1b, 1c can be obtained as follows: Amplitude for propagation from 1 to 3, $-4\pi/k^2$, amplitude for a transverse photon to excite atom to nth state, $ie\omega_{no} z_{no}$.** The amplitude for the system to propagate in the intermediate state 3-4 is $1/(E_{in} - E_{int})$, where E_{in} is the initial energy of the system (just prior to 3) and E_{int} is the energy of the system in the intermediate state. For both daigrams $E_{in} = E_o + \omega$. For 1b, $E_{int} = E_n$ and for 1c, $E_{int} = E_n + 2\omega$. Amplitude for the photon to be emitted by the atom $-ie\omega_{no} z_{no}^{+}$. Amplitude to propagate from 4 to 2, $- 4\pi/k^2$.

Also in the nonrelativistic approximation there is a contribution, for transverse waves only, from the $e^2/m\,(\vec{A} \cdot \vec{A})$ term in the hamiltonian. This term contributes, per atom, an amount e^2/m. Putting everything together and holding back one of the factors of $-4\pi/k^2$ we get

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*Henceforth we write $\sum_{n+}$ as $\sum_{n}$.

**We imply that the sum over angular momentum states has been performed.

$$\beta_{11} = -\frac{4\pi e^2 N}{k^2}\left\{\sum_n \left[(-i\omega_{no} z^+_{no})\left(\frac{1}{\omega+E_o-E_n} + \frac{1}{\omega+E_o-2\omega-E_n}\right)(i\omega_{no} z_{no})\right] + \frac{1}{m}\right\}$$

$$= \frac{4\pi Ne^2}{k^2}\left\{\sum_n \left[\omega^2_{no}|z_{no}|^2\left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right)\right] - \frac{1}{m}\right\}$$

This is just II-17a. (To get the contribution per unit volume we just multiplied by N since the atoms do not interact with each other.) To get β_{44} we note that for coulomb photons there is no contribution from the $\vec{A}\cdot\vec{A}$ term hence there is no factor of $1/m$, and the atomic matrix elements are $\sim Kz_{no}$.

$$\beta_{44} = -\frac{4\pi e^2 N}{k^2}\sum_n (-iKz^+_{no})\left(\frac{1}{\omega+E_o-E_n} + \frac{1}{\omega+E_o-2\omega-E_n}\right)(iKz_{no})$$

$$= \frac{4\pi Ne^2}{k^2}\sum_n K^2|z_{no}|^2\left(\frac{1}{\omega_{no}-\omega} + \frac{1}{\omega_{no}+\omega}\right)$$

which is just II-17b. The expressions II-17 may be simplified by making use of the Thomas Reiche sum rule (5) which says that

$$\sum_n f_{no} = \sum_n 2m\omega_{no}|z_{no}|^2 = 1^*$$

or

$$\frac{1}{m} = 2\sum_n \omega_{no}|z_{no}|^2 \qquad \text{(II-18)}$$

Substituting II-18 into II-17a we obtain

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*The quantities $f_{no} = 2m\omega_{no}|z_{no}|^2$ are called the oscillator strengths.

$$\beta_{11} = \frac{4\pi Ne^2}{k^2} \sum_n |z_{no}|^2 \omega_{no} \left(\frac{\omega_{no}}{\omega_{no}-\omega} + \frac{\omega_{no}}{\omega_{no}+\omega} - 2 \right) \tag{II-19}$$

$$= 8\pi Ne^2 \frac{\omega^2}{k^2} \sum_n \omega_{no} |z_{no}|^2 \frac{1}{\omega_{no}^2 - \omega^2} \tag{II-20a}$$

and from II-13

$$\beta_{44} = 8\pi Ne^2 \frac{K^2}{k^2} \sum_n \omega_{no} |z_{no}|^2 \frac{1}{\omega_{no}^2 - \omega^2} \tag{II-20b}$$

The expressions II-20a, b may be simplified by expressing them in terms of the oscillator strengths f_{no} and the plasma frequency $\omega_p = (4\pi Ne^2/m)^{1/2}$. In terms of these parameters we get that*

$$\beta_{11} = \omega_p^2 \frac{\omega^2}{k^2} \sum_n \frac{f_n}{\omega_n^2 - \omega^2} \tag{II-21a}$$

and

$$\beta_{44} = \omega_p^2 \frac{K^2}{k^2} \sum_n \frac{f_n}{\omega_n^2 - \omega^2} \tag{II-21b}$$

It appears from II-21 that for $\omega = \omega_n$, β is infinite. The reason for this is that up to now we have assumed that the energy eigenvalues are discrete. Actually each eigenvalue is not discrete but is spread out about some mean energy E_n with a half width γ_n. Physically, each excited state has a finite probability of decaying which implies an uncertainty in the energy. In Appendix A it is

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*Henceforth we omit the subscript o.

shown that when the finite lifetime of the excited states is taken into account the expressions for β become

$$\beta_{11} = \beta_{22} = \beta_{33} = \omega_p^2 \frac{\omega^2}{k^2} \sum \frac{f_n \left(\frac{\omega_{\gamma_n}}{\omega_n}\right)}{\omega_{\gamma_n}^2 - \omega^2}$$

$$\beta_{44} = \omega_p^2 \frac{K^2}{k^2} \sum_n \frac{f_n \left(\frac{\omega_{\gamma_n}}{\omega_n}\right)}{\omega_{\gamma_n}^2 - \omega^2} \qquad \text{(II-22)*}$$

where $\omega_{\gamma_n} = \omega_n - i\frac{\gamma_n}{2}$.

E. Current-Current Interactions in a Medium in the Case of Small K

We are now in a position to return to the expression II-1 for the amplitude for the emission and absorption of photons by currents in a medium. Here we carry out the implicit summation implied in II-1 to obtain an explicit form for the interaction of currents in a medium. First of all let's consider the form of the interaction in vacuum. In this case II-1 takes the form

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*It may be noted that II-22 may be obtained from II-21 a, b by substituting ω_{γ_n} for ω_n. A semi-plausible justification for this is that the amplitude for a meta-stable state contains the factors $e^{-(\gamma/2)t}e^{-iEt} = e^{-i(E-i\frac{\gamma}{2})t}$. Thus the energy of a meta-stable state can be regarded as complex with the imaginary part being $-\gamma/2$. Also we note that γ is the total line breadth of the excited states. However in the following we will consider the idealized case where we take γ to be the spontaneous line breadth. This would be the case for isolated atoms at zero temperature.

$$\mathcal{A} = -4\pi j_\mu \frac{\delta_{\mu\nu}}{k^2} j'_\nu$$

$$= -4\pi j_\mu \frac{1}{k^2} j'_\mu \qquad \text{(II-23)}$$

where j, j' are the two currents involved. Now since all currents are conserved (i. e., $j_{\mu,\mu} = 0$ which in momentum space reads $k_\mu j_\mu = 0$), II-23 can be simplified as follows. If instead of choosing the space directions x, y, z one direction parallel to K (photon momentum) and two directions transverse to $\vec{K}$ are taken the matrix element $\mathcal{A}$ can be written (suppressing the factor -4π)

$$\mathcal{A} = j_\mu \frac{1}{k^2} j'_\mu$$

$$= j_4 \frac{1}{k^2} j'_4 - j_3 \frac{1}{k^2} j'_3 - \sum_{2\text{ tr. direc.}} j_{tr} \frac{1}{k^2} j'_{tr.} \qquad \text{(II-24)}$$

where j_3 is the component of j parallel to K and $j_{tr.}$ represents the component of j in either of the transverse directions. The fourth component of the current four vector, j_4, is the charge density. By the conservation of current j_3 can be expressed in terms of j_4 (or vice versa) as follows: From $k_\mu j_\mu = 0$ it follows that $\omega j_4 - K j_3 = 0$ or

$$j_3 = \frac{\omega}{K} j_4 \qquad \text{(II-25)}$$

Inserting II-25 into II-24 and combining terms we get

$$\mathcal{A} = -4\pi\left[j_4 j_4'(1-\frac{\omega^2}{K^2})\frac{1}{k^2} - \sum_{2\text{ tr. direc.}} j_{tr.}\frac{1}{k^2} j_{tr.}'\right]$$

$$= 4\pi\left[\frac{j_4 j_4'}{K^2} + \sum_{2\text{ tr. direc.}} j_{tr.}\frac{1}{\omega^2 - K^2} j_{tr.}'\right] \qquad \text{(II-26)}$$

Now $1/K^2$ represents a coulomb field in momentum space and j_4 is the charge density so the first term of II-26 represents an instantaneous coulomb interaction (since it is independent of ω) while the second term contains the delayed interaction through transverse waves.

In a medium we have that

$$\mathcal{A} = j_\mu \pi_{\mu\nu} j_\mu' \qquad \text{(II-27)}$$

where $\pi_{\mu\nu}$ has been discussed in Section C, D, and E. To second order $\pi_{\mu\nu}$ is given by

$$\pi_{\mu\nu} = -\frac{4\pi}{k^2}(\delta_{\mu\nu} + \beta_{\mu\nu}), \text{ so that } \mathcal{A} \text{ becomes}$$

$$\mathcal{A} = -\frac{4\pi}{k^2} j_\mu(\delta_{\mu\nu} + \beta_{\mu\nu}) j_\nu' . \qquad \text{(II-28)}$$

Carrying out the indicated summations implied in II-28 we have (temporarily suppressing the factor $-4\pi/k^2$)

$$\mathcal{A} \sim j_4 j_4'' - j_3 j_3' - \sum_{2\text{ tr. direc.}} j_{tr.} j_{tr.}' + j_4 \beta_{44} j_4' - j_4 \beta_{43} j_3'$$

$$- j_3 \beta_{34} j_4' + j_3 \beta_{33} j_3' + \sum_{2\text{ tr. direc.}} j_{tr.} \beta_{tr.\,tr.} j_{tr.}' \qquad \text{(II-29)}$$

In arriving at II-29 we have made use of the fact that all off diagonal matrix elements of β are zero except β_{43} and β_{34} ($= \beta_{43}$).

Consider the term of II-29 that depends on the transverse directions; this is given by

$$\frac{4\pi}{k^2} \sum_{2\text{ tr. direc.}} j_{tr.} j'_{tr.} (1 - \beta_{tr.,tr.})$$

To get the contribution from all orders we note that since the transverse part of β is diagonal that the transverse part of the inverse of $(\delta - \beta)$ is just given by the reciprocal of its transverse diagonal elements. That is to all orders the transverse part of $\mathcal{A}(\mathcal{A}_\tau)$ is given by

$$\mathcal{A}_\tau = \frac{4\pi}{k^2} \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{1 + \beta_{tr.,tr.}} \quad \text{or from II-22}$$

$$= 4\pi \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{k^2 \left(1 + \omega_p^2 \frac{\omega^2}{k^2} \sum_n \frac{f_n \omega_{\gamma_n} / \omega_n}{\omega_{\gamma_n}^2 - \omega^2}\right)}$$

$$= 4\pi \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{\left(k^2 + \omega_p^2 \omega^2 \sum_n \frac{f_n \omega_{\gamma_n} / \omega_n}{\omega_{\gamma_n}^2 - \omega^2}\right)}$$

or since $k^2 = \omega^2 - K^2$

$$\mathcal{A}_\tau = 4\pi \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{\omega^2 \eta - K^2} \qquad \text{(II-30)}$$

where

$$\eta = 1 + \frac{k^2}{\omega^2}\beta_{tr.,tr.} = 1 + \omega_p^2 \sum_n \frac{f_n \omega_{\gamma_n}/\omega_n}{\omega_{\gamma_n}^2 - \omega^2} \qquad \text{(II-31)}$$

On comparing II-30 to the corresponding term of II-26 we see that the effect of the medium on the emission and absoprtion of transverse polarized photons is contained in the function η. Note that η is independent of K.

Now we consider the scalar-longitudinal terms of $\mathcal{A}$ ($\mathcal{A}_c$). From II-29 these are

$$\mathcal{A}_c = j_4 j_4' - j_3 j_3' + j_4 \beta_{44} j_4' - j_4 \beta_{43} j_3' - j_3 \beta_{34} j_4' + j_3 \beta_{33} j_3' \qquad \text{(II-32)}$$

To simplify this expression we note that since all currents involved here are conserved (j, j' and the atomic electron currents) the longitudinal components of currents can be related to the scalar component via II-25 (remember that 3 is the longitudinal direction, direction of K and 4 is the scalar or time direction). Upon using II-25, II-32 becomes

$$\begin{aligned}\mathcal{A}_c &= j_4 j_4' \left\{1 - \frac{\omega^2}{K^2} + \beta_{44}\left(1 - \frac{2\omega^2}{K^2} + \frac{\omega^4}{K^4}\right)\right\} \\ &= j_4 j_4'\left(1 - \frac{\omega^2}{K^2}\right)\left\{1 + \left(1 - \frac{\omega^2}{K^2}\right)\beta_{44}\right\} \\ &= -\frac{k^2}{K^2} j_4 j_4' \left(1 - \frac{k^2}{K^2}\beta_{44}\right) \qquad \text{(II-33)}\end{aligned}$$

To get the contribution to all orders we note that in every order we can use the same trick to eliminate longitudinal currents in favor of scalar ones. With this fact it is easy to see that to all orders II-33 becomes

$$\mathcal{A}_c = -\frac{k^2}{K^2} j_4 j_4' \left\{1 - \frac{k^2}{K^2}\beta_{44} + \left(\frac{k^2}{K^2}\right)^2 (\beta_{44})^2 + \ldots\right\}$$

$$= -\frac{k^2}{K^2} j_4 j_4' \frac{1}{1 + \frac{k^2}{K^2}\beta_{44}} \qquad \text{(II-34)}$$

upon substituting for β_{44} from II-22 and replacing the factor $-4\pi/k^2$, II-34 becomes:

$$\mathcal{A}_c = j_4 \frac{4\pi}{K^2\eta} j_4' \qquad \text{(II-35)}$$

where $\eta = 1 + \frac{k^2}{K^2}\beta_{44}$ and is given explicitly in II-31. On comparing II-35 to the corresponding term of II-26 we see again that the effect of the medium on the propagation of scalar photons is contained in η.*

On combining II-30 and II-35 we obtain the desired result:

$$j_\mu \pi_{\mu\nu} j_\nu' = 4\pi\left[j_4 \frac{1}{K^2\eta} j_4' + \sum_{2\text{ tr. direc.}} j_{tr.} \frac{1}{\omega^2\eta - K^2} j_{tr.}' \right] \qquad \text{(II-36)}$$

This is the amplitude to propagate any type of polarized photons through the medium (scalar, longitudinal and transverse) for values of K such that $Ka_o \ll 1$.

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*The elimination of longitudinal photons in favor of scalar ones is by no means necessary. If we chose we could have done the contrary that is, by the inverse of II-25 we replace j_4's by j_3's. If this is carried out the analogue of II-35 is $j_3 \frac{4\pi}{\omega^2\eta} j_3'$ (the amplitude to propagate the longitudinally polarized photons).

F. Index of Refraction of a Medium

In order to make a connection of the photon propagator with an index of refraction we consider Maxwell's equations for a medium which is characterized by a phenomenological index of refraction n (and dielectric function $\epsilon = n^2$). As is well known, Maxwells equations can be written as

$$\nabla^2 A_{tr.} - \frac{n^2}{c^2} \frac{\partial^2 A_{tr.}}{\partial t^2} = 4\pi J_{tr.} \qquad \text{(II-37a)}$$

and

$$\nabla^2 A_4 = - \frac{4\pi J_4}{\epsilon}^{*} \qquad \text{(II-37b)}$$

In momentum space II-37a, and II-37b become

$$(K^2 - \omega^2 n^2) a_{tr.} = 4\pi j_{tr.}$$

and

$$K^2 a_4 = \frac{4\pi}{\epsilon} j_4$$

or

$$a_{tr.} = \frac{4\pi j_{tr.}}{\omega^2 n^2 - K^2} \qquad \text{(II-38a)}$$

and

$$a_4 = \frac{4\pi j_4}{K^2 \epsilon} \qquad \text{(II-38b)}$$

Now $a_{tr.}$ and a_4 are the potentials produced in a medium from the current j. The coupling with another current j' is $j' \cdot a$ or

$$4\pi \left[j_4 \frac{1}{K^2 \epsilon} j_4' + \sum_{2\ \text{tr. direc.}} j_{tr.} \frac{1}{\omega^2 n^2 - K^2} j_{tr.}' \right]$$

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*Note the gauge used for II-37 is $\vec{\nabla} \cdot \vec{A} = 0$ which implies in general that $A_3 = 0$.

Comparing II-39 to II-36 we see that the function η given by II-31 can be regarded as the dielectric function of the medium. Another way of looking at the situation is to use the fact that at the location of the pole of the propagator one obtains the relation between energy and momentum (ω, K) of the free waves. Now in a medium the phase velocity of the waves is given by $\omega/K = 1/n$ where n is the index of refraction, i. e., the amplitude for the propagation of a wave in a medium is proportional to $e^{-i(\omega t - Kx)} = e^{-i\omega(t-nx)}$. Now from II-36 the pole of the transverse part of the propagator occurs at $\omega^2\eta - K^2 = 0$ or $\omega^2/K^2 = 1/\eta$; that is η can be interpreted as the square of the index of refraction. On further examining II-36 we see that there is also the possibility of another pole from the coulomb term;that is another pole occurs when $\eta = 0$.* Since η is not a function of K, we see that at the pole $d\omega/dK = 0$, i. e., the group velocity is identically zero. That is the free waves due to the coulomb term does not represent a propagating disturbance but a purely oscillating one. We will return to quantitative nature of this pole later on.

We are not the first to give a full quantum theory of the index of refraction. Tidman (1) has given a quantum theory of the index based on an atomic model similar to the one used here. However the method that he uses to obtain the index is quite different from ours. We obtained the index from the photon propagator in a medium (at the pole). Tidman considers only transverse waves and uses ordinary perturbation theory (as opposed to the Feynman diagram method) to calculate what

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*There is also a pole at $K^2 = 0$.

he calls the polarization energy of the medium (the self energy of the system (medium plus radiation) due to the real photons). He then makes a canonical transformation of the radiation field variables. The index is obtained by defining the parameter in this transformation in such a way as to account for the polarization energy. The real part of the index that he obtains is the same as ours. The imaginary part is different. However Tidman's imaginary part can be made equal to ours if we retain the terms that he drops. That is, replace ω_{no} by $\omega_{no} - (i\gamma_n/2)$ everywhere in Tidman's equation 5.7.

G. Discussion of the Index of Refraction

Let us return to II-31 for η. We have

$$\eta = 1 + \omega_p^2 \sum_n \frac{f_n\left(1 - \frac{i\gamma_n}{\omega_{no}}\right)}{(\omega_{no} - \frac{i\gamma_n}{2} - \omega)(\omega_{no} - \frac{i\gamma_n}{2} + \omega)} \qquad \text{(II-40)}$$

or since $\gamma_n \ll \omega_{no}$

$$\eta \approx 1 + \omega_p^2 \sum_n \frac{f_n}{\omega_{no}^2 - \omega^2 - i\gamma_n\omega_{no}} \qquad \text{(II-41)}$$

We see from II-40 that η (as well as $\omega^2\eta - K^2$) has poles in both the lower and upper half ω plane. That is our η is not causal. On the other hand the index computed classically is causal. For example if we treat the atomic election as bound to an origin by a damped harmonic potential of characteristic frequencies ω_n and damping constants γ_n the index would be

$$\eta_c = 1 + \omega_p^2 \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\gamma_n\omega}$$

which has poles only in the lower half plane. In our quantum treatment a causal propagator does not appear. The propagator that we obtain is similar to vacuum Feynman propagators (i. e., poles in upper and lower ω plane) where the mass of the particle is given an infinitesimal negative imaginary part. As long as we are dealing with positive frequencies the predictions of causal propagators and Feynman propagators are the same.

We note that because of the spherical symmetry of the atoms (after orbital angular momentum states have been summed over) that $\beta_{33} = (\omega^2/K^2)\beta_{44}$. That is, $\beta_{tr.,tr.} = (\omega^2/K^2)\beta_{44}$ etc. Here we have calculated $\beta_{tr.,tr.}$ and β_{44} separately and we see from II-22 that indeed these results are attained.

In deriving the index we have assumed that the density of the medium is low enough so that we may neglect any direct interaction between the atoms. That is,we have assumed that the field that acts on a particular atom is just the applied external photon field. Actually the external field may induce a non-zero moment in the atoms; the field produced by these atomic moments may be non-zero at the atomic sites. The field acting on the atoms then is a sum of this induced field and the applied external field. We expect that as the density of the medium increases the effect of this induced field becomes increasingly more important. This effect was first treated by Lorentz (e. g., see the review article by W. Brown (6)). As yet we have not been able to incorporate this effect within the framework of the present theory with a quantum mechanical analysis. However, we believe that the formulae that are later developed, giving the energy loss of fast charged particles in terms of the index, are valid in dense media, where the index that we have calculated is incorrect. That is, in a dense medium, by using a more exact index (e. g., computed classically or determined experimentally) in the energy loss expressions, we believe that one obtains the correct result for the energy loss.

H. Calculation of $\beta_{\mu\nu}$ in the Case of Large K

We now return to expression II-8 and consider the case of large K ($Ka_o >> 1$). In this region the atomic matrix elements a_μ (see II-11) are almost zero ($e^{i\vec{K}\cdot\vec{X}}$ is rapidly oscillating) unless $|n>$ contains a compensating exponential. This corresponds to an ionized atom. Since for $K >> \frac{1}{a_o}$ implies that the momentum of the atomic electron is much larger than the intrinsic momentum of the atomic electron in its ground state (which is of the order $me^2 = \frac{1}{a_o}$) we can consider the initial state of the atomic electron to be free and at rest. Therefore, $K_+^A(x_4, x_3)$ reduces to free particle propagator in vacuum. It is easy to see that equation II-5 reduces to

$$P_{\mu\nu} = -\frac{4\pi e^2}{k^2}(2\pi)^4\delta^4(k-k')\sum_i \bar{u}_i\left(\gamma_\mu \frac{1}{\not{p}_{io} + \not{k} - m}\gamma_\nu\right)u_i$$

$$+ \text{ terms with } k \rightarrow -k \qquad \text{(II-41)}$$

where u_i is the ith electron spinor of momentum p_o, $p_o = (m, 0, 0, 0)$, and $\bar{u}u = 1$. We have

$$P_{\mu\nu} = -\frac{4\pi e^2 N}{k^2}(2\pi)^4\delta^4(k-k')\left\{\bar{u}\left(\gamma_\mu \frac{1}{\not{p}_o + \not{k} - m}\gamma_\nu\right)u\right.$$

$$\left. + \text{terms with } k \rightarrow -k\right\} \qquad \text{(II-41)}$$

$$= (2\pi)^4\delta^4(k-k')\beta_{\mu\nu}$$

On averaging the initial spin state and summing over the spin states of the intermediate state we get, from II-41 and II-42,

$$\beta_{\mu\nu} = -\frac{4\pi e^2 N}{k^2}\left(\frac{1}{2}\right)\left(\frac{1}{2m}\right)\left\{\frac{Sp\ [(\not{p}_o + m)\gamma_\mu(\not{p}_o + \not{k} + m)\gamma_\nu]}{(\not{p}_o + \not{k})^2 - m^2}\right.$$

$$\left. + \text{terms with } k \rightarrow -k\right\}$$

$$= -\frac{\omega_p^2}{k^2}\left\{\frac{2p_{o\mu}p_{o\nu} + k_\mu p_{o\nu} + k_\nu p_{o\mu} - \delta_{\mu\nu}(p_o \cdot k)}{k^2 + 2p_o \cdot k}\right.$$

$$\left. + \frac{2p_{o\mu}p_{o\nu} - k_\mu p_{o\nu} - k_\nu p_{o\mu} + \delta_{\mu\nu}(p_o \cdot k)}{k^2 - 2p_o \cdot k}\right\} \qquad \text{(II-43)}$$

where

$$\omega_p^2 = \frac{4\pi N e^2}{m}$$

and

$$p_o = m\delta_{\mu 4}$$

Explicitly the elements of β are

$$\beta_{11} = \beta_{22} = \beta_{33} = \frac{4m^2\omega_p^2}{(k^2 + 2m\omega)(k^2 - 2m\omega)}\left(\frac{\omega^2}{k^2}\right) \qquad \text{(II-44a)}$$

$$\beta_{44} = \frac{K^2}{\omega^2}\beta_{33} = \frac{4m^2\omega_p^2}{(k^2+2m\omega)(k^2-2m\omega)}\left(\frac{K^2}{k^2}\right) \tag{II-44b}$$

$$\beta_{34} = \beta_{43} = \frac{\omega}{K}\beta_{44}, \text{ all others are zero.} \tag{II-44c}$$

I. Current-Current Interaction in the Case of Large K

Now we are in a position to calculate II-1 for the case of large K. We have

$$\mathcal{A} = j_\mu \pi_{\mu\nu} j'_\nu \tag{II-1}$$

where

$$\pi_{\mu\nu} = -\frac{4\pi}{k^2}\left[(\delta-\beta)_{\mu\nu}\right]^{-1} \tag{II-4}$$

Carrying out the summation implied in II-1 and using β as given in II-44 we obtain, after going through the same procedure that led to II-36,*

$$\mathcal{A} = 4\pi\left[\frac{j_4 j'_4}{K^2(1+\frac{k^2}{K^2}\beta_{44}} + \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{k^2(1+\beta_{tr.\,tr.})}\right]$$

$$= 4\pi\left[\frac{j_4 j'_4}{K^2\eta} + \sum_{2\text{ tr. direc.}} \frac{j_{tr.} j'_{tr.}}{(\omega^2\eta - K^2)}\right] \tag{II-45}$$

where

$$\eta(\omega, K) = 1 + \frac{4m^2\omega_p^2}{(k^2+2m\omega)(k^2-2m\omega)} \tag{II-46}$$

In this case we see that η is a function of both ω and K but not a function of direction which is to be expected since we are dealing with

*Here only the elements of β are different but in both cases $k_\mu \beta_{\mu\nu} = 0$.

an isotropic medium. This is to be contrasted to the case $Ka_o \ll 1$ where η is a function of ω only.

J. Real Processes in a Medium; Poles of the Photon Propagator

As is well known, real processes correspond to poles in the formulae for virtual processes. In this section we will examine the photon propagator for its poles. Poles arise from both the coulomb term and the transverse term. First let us examine poles from the coulomb term.

Poles will arise in the coulomb term at the zeros of η.* For the case of $Ka_o \ll 1$ we have from II-31 that the poles are determined from

$$1 + \omega_p^2 \sum_n \frac{f_n \left(\omega_{\gamma_n}/\omega_n\right)}{\omega_{\gamma_n}^2 - \omega^2} = 0 \tag{II-47}$$

For the moment let us consider the case of only one excited state; in this case II-47 becomes

$$1 + \frac{\omega_p^2}{\left(\omega_1 - \frac{i\gamma}{2}\right)^2 - \omega^2} = 0 \tag{II-48}$$

where we have approximated $\omega_\gamma/\omega = 1 - i\gamma/2\omega_1$ in the numerator of

*The pole at $K = 0$ corresponds to the interaction of two charge densities at very large (infinite) mutual separation. Therefore, in general, $\eta(\omega, K=0)$ gives the effective force between the two charges (i.e., if we were to go to the coordinate space representation of the coulomb interaction and take the limit $|x| \to \infty$ the only component of $\eta(\omega, K)$ that contributes is $\eta(\omega, K = 0)$. In the following we will be interested in the poles arising at $\eta = 0$.

II-47 by 1. The solution of II-48 is

$$\omega^2 = \left(\omega_1 - \frac{i\gamma}{2}\right)^2 + \omega_p^2$$

for the case at hand $\omega_p^2 << \omega_1^2$ * so we have (also $\gamma << \omega_1$).

$$\omega_1 \cong \omega_1 + \frac{\omega_p^2}{2\omega_1} - \frac{i\gamma}{2}$$

or

$$\omega_{Re} = \omega_1 + \frac{\omega_p^2}{2\omega_1} \qquad \text{(II-49)}$$

$$\omega_{Im} = -\frac{\gamma}{2}$$

We see that the medium will oscillate at a frequency given by $\omega_{Re} = \omega_1[1 + (\omega_p^2/2\omega_1^2)]$. Therefore, there should be an absorption at ω_{Re} in say the energy loss of a fast charged particle passing through a thin film of this hypothetical medium of one state atoms. Note that ω does not depend on K so the group velocity $d\omega/dK = 0$. Hence this mode corresponds to a pure oscillatory mode. The imaginary part of ω, $-\gamma/2$, tells us that the life time of this mode is $1/\gamma$. This is reasonable since the only decay mechanism in our theory is the line breadth of the excited states. That is, after time $1/\gamma$ essentially all the atoms will be in the ground state.

For the case of many excited states we proceed as follows. From the case of one excited state we saw that the pole is essentially

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*For $N \sim 10^{20}$ per cm^3 $(\omega_p/\omega_1 \sim 10^{-3}$. Note, for solids, ω_p is of the Ryd., and in some cases larger than the Ryd. In these cases the pole is dominated by the ω_p term (the plasmon).

at the frequency of the excited state. Now for frequencies near ω_n one term in the sum in II-47 will dominate. That is, for frequencies near say ω_n II-47 becomes

$$1 + \frac{\omega_p^2 f_n}{\omega_{\gamma_n}^2 - \omega^2} + r_n = 0 \tag{II-50}$$

where

$$r_n = \sum_m{}' \frac{\omega_p^2 f_m}{\omega_{\gamma_m}^2 - \omega^2} \quad \text{(the prime means to omit the term for } m = n)$$

$$\cong - \frac{\omega_p^2}{\omega^2} \sum_{m=1}^{n-1} f_m + \omega_p^2 \sum_{m=n+1} \frac{f_m}{\omega_{\gamma_m}^2}$$

$$\ll 1$$

Calling $1 + r_n = R_n$ the solution of II-50 is

$$\omega^2 = \omega_{\gamma_n}^2 + \left(\frac{f_n}{R_n}\right)\omega_p^2 \tag{II-51}$$

Note that since $\omega_p^2 \ll \omega_{\gamma_n}^2$, R_n can be taken to be real. That is the pole frequencies are at

$$\omega_{Re} = \omega_n + \left(\frac{f_n}{R_n}\right) \frac{\omega_p^2}{2\omega_n} \qquad (n = 1, 2, 3, \ldots)$$

and (II-52)

$$\omega_{Im} \sim - \frac{\gamma_n}{2} \qquad (n = 1, 2, 3, \ldots)$$

So, for many excited states we get a pole for each state the location of which is slightly modified, as seen from II-52, due to the effects of the non-resonant states. So much for the case of small K.

For the case of large K we have, from II-46, that the poles are determined from*

$$1 + \frac{4m^2\omega_p^2}{(k^2 + 2m\omega)(k^2 - 2m\omega)} = 0$$

or

$$1 + \frac{4m^2\omega_p^2}{(\omega^2 + 2m\omega - K^2)(\omega^2 - 2m\omega - K^2)} = 0 \qquad \text{(II-53)}$$

We wish solutions of II-53 for $Ka_o >> 1$ or $K >> me^2$. We remark that if we had considered the hypothetical problem of photon propagation through a medium consisting of free electrons initially at rest** (with a positive smeared out background charge to give charge neutrality) we would have arrived at II-53 as the determining equation for the dispersion formula for all values of K. So let us solve II-53 for all K. For small values of K, in particular $K^2 << \omega_p^2$, we can compare our results to the results of Bohm and Pines (3) who have

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*We have omitted the $i\epsilon$ from the Feynman propagators. In the following the poles will be on the real axis. That is, there will be no damping of the waves. The delta functions arising from the $i\epsilon$'s imply an infinitely narrow width to the excitations (i. e., an infinitely long life time). This is to be expected here since we have assumed a collisionless medium. That is, once we excite an excitation (collective or single particle), there is no mechanism for the excitation to transfer energy into other modes. For example, if a single electron, at rest, picks up energy ω and momentum K from the photon field $((\omega+m)^2 = K^2+m^2)$, it will always have energy $(m+\omega)$ and momentum K since it does not collide with any other electrons of the medium. That is, it will always remain a free particle. These considerations are correct only at zero temperature. At finite temperatures because of the continuous distribution of P, the delta functions will give a finite imaginary part. Damping at finite temperatures is discussed in Chapter IV.

**Hypothetical because the Pauli exclusion principle will not allow more than one electron in a quantum state, which implies we cannot have a system of electrons all at rest.

found the dispersion formula for longitudinal waves in an electron gas in the non-relativistic approximation.

First we note that II-53 is even in ω. So, in the following, we will consider solutions only for positive ω. In the non-relativistic approximation, $\omega << m$, II-53 becomes

$$1 - \frac{4m^2\omega_p^2}{4m^2\omega^2 - K^4} = 0$$

or

$$1 - \frac{\omega_p^2}{\omega^2 - \frac{K^4}{4m^2}} = 0 \qquad \text{(II-54)}$$

with a solution,

$$\omega^2 = \omega_p^2 + \frac{K^4}{4m^2} \qquad \text{(II-55)}$$

For $K^2 << m\omega_p$ we get

$$\omega = \omega_p\left(1 + \frac{K^4}{8m^2\omega_p^2} + \ldots\right) \qquad \text{(II-56)}$$

Equations II-54 and II-56 are the Bohm-Pines results for $\vec{P}_i = 0$ ($\vec{P}_i$ are the momenta of the electrons); see equations 57 and 67 of reference 3. The restriction $\vec{P}_i = 0$ is not a serious limitation to us. For if we had not made the assumption of $\vec{P}_i = 0$, we would have had, instead of II-43, that

$$\beta_{\mu\nu} = - \frac{\omega_p^2}{Nk^2} \sum_{i=1}^{N} \left(\frac{m}{E_i}\right) \left\{ \frac{2p_{i\mu}p_{i\nu} + k_\mu p_{i\nu} + k_\nu p_{i\mu} - \delta_{\mu\nu}(p_i \cdot k)}{k^2 + 2p_i \cdot k} \right.$$

$$\left. + \text{ terms with } k \rightarrow -k \right\} \qquad \text{(II-43')}$$

leading to, instead of II-46,

$$\eta = 1 + \frac{k^2}{K^2}\beta_{44}$$

$$= 1 + \frac{\omega_p^2}{N}\sum_i \frac{4E_i^2\left[1 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right](m/E_i)}{(k^2+2E_i\omega)(k^2-2E_i\omega)+8E_i\omega(\vec{P}_i\cdot K) - 4(\vec{P}_i\cdot\vec{K})^2}$$

$$= 1 - \frac{\omega_p^2}{N}\sum_i \frac{\left[1 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right](m/E_i)}{(\omega - \vec{v}_i\cdot\vec{K})^2 - \left(\frac{k^2}{2E_i}\right)^2} \qquad \text{(II-57)}$$

leading to a dispersion relation $(\eta = 0)$

$$1 = \frac{\omega_p^2}{N}\sum_i \frac{\left[1 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right](m/E_i)}{(\omega - \vec{v}_i\cdot\vec{K})^2 - \left(\frac{k^2}{2E_i}\right)^2} \qquad \text{(II-58a)}$$

or

$$\omega^2 = \frac{\omega_p^2}{N}\sum_i \frac{\left[1 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right](m/E_i)}{\left(1 - \frac{\vec{v}_i\cdot K}{\omega}\right)^2 - \left(\frac{k^2}{2E_i\omega}\right)^2} \qquad \text{(II-58b)}$$

Equation II-58b is the relativistic analogue of the Bohm-Pines dispersion relation (see equation 57 of reference 3). Now, for sufficiently small K, we may expand the denominator of II-58b in powers of $\frac{\vec{v}_i\cdot\vec{K}}{\omega}$ and $\left(\frac{k^2}{2E_i\omega}\right)^2$, and get a solution for $\omega(K)$. Doing this, we obtain

$$\omega^2 \cong \frac{\omega_p^2}{N}\sum_i \left[1 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right](m/E_i)\left\{1+2\frac{\vec{v}_i\cdot\vec{K}}{\omega} + 3\left(\frac{\vec{v}_i\cdot\vec{K}}{\omega}\right)^2 + \left(\frac{k^2}{2E_i\omega}\right)^2\right\}$$

In the non-relativistic approximation $(E_i = m(1 + \frac{v_i^2}{2}),\ v_i << 1,$ $\omega << m)$ we obtain

$$\omega^2 \cong \frac{\omega_p^2}{N} \sum_i \left\{ 1 + 2\left(\frac{\vec{v}_i \cdot \vec{K}}{\omega_p}\right) + 3\left(\frac{\vec{v}_i \cdot \vec{K}}{\omega_p}\right)^2 + \left(\frac{\omega_p^2 - K^2}{2m\omega_p}\right)^2 - \left(\frac{\vec{v}_i \cdot \vec{K}}{K}\right)^2 - \frac{1}{2} v_i^2 \right\}$$

(where we have set $\omega = \omega_p$ in second, third, and fourth terms). Assuming an isotropic distribution of $\vec{v}_i$ we get, after averaging over directions $(\overline{\cos\theta} = 0,\ \overline{\cos^2\theta} = \frac{1}{3})$, that

$$\omega^2 = \omega_p^2 \left\{ 1 + \frac{K^2}{\omega_p^2} \langle v^2 \rangle + \frac{K^4}{4m^2\omega_p^2} - \frac{5}{6} \langle v^2 \rangle - \frac{K^2}{2m^2} + \frac{\omega_p^2}{4m^2} \right\} \quad \text{(II-58c)}$$

where

$$\langle v^2 \rangle = \frac{1}{N} \sum_i v_i^2$$

Now, if we disregard the last three terms of II-58c for the moment, we have the dispersion formula given by Bohm and Pines (see equation 66 of reference 3). Clemmow and Wilson (7) have considered a relativistic electron gas by using the relativistic Boltzmann equation. They have worked out the lowest order corrections to the Bohm-Pines formula. They get

$$\omega^2 = \omega_p^2 \left\{ 1 + \frac{K^2}{\omega_p^2} \langle v^2 \rangle - \frac{5}{6} \langle v^2 \rangle \right\}$$

(see equation 39 of reference 7).

If we define $\overline{\omega}_p^2 = \omega_p^2 (1 - \frac{5}{6} \langle v^2 \rangle + \frac{\omega_p^2}{4m^2})$ (i.e., an effective plasma frequency), we get

$$\omega^2 = \bar{\omega}_p^2 \left\{ 1 + \frac{K^2}{\omega_p^2} <v^2> + \frac{K^4}{4m^2\omega_p^2} - \frac{K^2}{2m^2} \right\}.$$

which, to the same approximation, is

$$\omega^2 = \bar{\omega}_p^2 \left\{ 1 + \frac{K^2}{\bar{\omega}_p^2} <v^2> + \frac{K^4}{4m^2\bar{\omega}_p^2} - \frac{K^2}{2m^2} \right\} \qquad \text{(II-58e)}$$

Now if $<v^2> \gg (\omega_p/m)^2$ the last term in II-58e is negligible compared to the second term. If we assume a Maxwellian distribution for $<v^2>$ ($<v^2> = \frac{3\kappa T}{m}$ (where κ = Boltzmann's constant = 8.62×10^{-5} e.V./°K)), we get $<v^2> = \frac{3\kappa T}{m} \gg (\omega_p/m)^2$, or $\kappa T \gg (\omega_p/m)^2 m/3$. For $N \sim 10^{20}$ we have $\kappa T \gg 10^{-8}$ e.V. or $T \gg 10^{-4}$ degrees Kelvin ($<v^2> \gg 10^{-13}$). So, for just about all cases of interest, the fourth term of II-58e is small.* In fact, Bohm and Pines point out that the third term is small compared to the second term. Note that at $K=0$, we get $\omega^2 = \bar{\omega}_p^2 \cong \omega_p^2(1 - \frac{5}{6}<v^2>)$ which is a shift in the plasma frequenty. This shift at say room temperature is $\frac{5}{6}<v^2> \cong 10^{-7}$, i.e. very small. We note that we have not included the effects of higher order proper diagrams and that therefore the correction terms are only of academic interest. We have made no attempt to estimate these higher order terms.

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*For $N \sim 10^{19} - 10^{20}$, $(\omega_p/m)^2 \sim 10^{-15}$; for Maxwell-Boltzman statistics to apply, $T \gtrsim 100°$ K; for $T = 100°$ K, $<v^2> \sim 10^{-11}$; for $T = 1500°$ K, $<v^2> \sim 10^{-7}$. So we can take $\bar{\omega}_p^2 = \omega_p^2(1 - \frac{5}{6}<v^2>)$. Below 100 K Fermi statistics applies, and, in this case, for $N \sim 10^{19}$, $<v^2> \sim 10^{-8}$, so $(\omega_p/m)^2$ is still negligible. Consider some real plasmas: Solar corona $N \sim 10^6$, $(\omega_p/m)^2 \sim 10^{-27}$, $T \sim 10^6$, $<v^2> \sim 10^{-3}$; thermonuclear plasma $N \sim 10^6$, $(\omega_p/m)^2 \sim 10^{-17}$, $T \sim 10^8$, $<v^2> \sim 10^{-1}$.

Returning now to the case at hand, large K and $\vec{P}_i = 0$, we now continue the discussion of the solutions of II-53. From II-55 we get that for $K^2 >> m\omega_p$,

$$\omega = \frac{K^2}{2m} \left(1 + \frac{2m^2\omega_p^2}{K^4} + \ldots\right) \qquad \text{(II-59)}$$

II-59 is the dispersion formula for non-relativistic values of K ($K << m$). For larger values of ω we get from II-53 on dropping the ω_p term, that

$$k^4 \cong 4m^2\omega^2 \qquad (\omega \gtrsim m)$$

or

$$(\omega \pm m)^2 = K^2 + m^2$$

$$\omega = \pm m \pm (K^2 + m^2)^{1/2} \qquad \text{(II-60)}$$

It is interesting to note that a new solution exists for small K; namely $\omega \cong 2m(1 + \frac{K^2}{2m^2})$. For K=0 this gives $\omega = 2m$. This root is connected with the phenomenon of pair production. Namely, a virtual longitudinal photon is able to make a pair, electron-positron, if it has a frequency at least equal to 2m. The other positive root of II-60, which is the extension of II-59 for relativistic values of K, is

$$\omega = -m + (K^2 + m^2)^{1/2} \qquad \text{(II-61)}$$

which, for $K >> m$, is

$$\omega = K - m$$

Before going on to the transverse case, we include a brief discussion on the order of magnitude of the various quantities involved.

For our case of the atomic system, the quantities which characterize the atoms are the Rydberg (Ryd) and the Bohr radius a_o. The medium is characterized by the plasma frequency ω_p. First of all,

$$\left(\frac{\omega_p}{Ryd}\right)^2 = 16\pi N a_o^3$$

$$\cong 2\pi N \times 10^{-24}$$

So, for $N \ll 10^{24}$, $\omega_p^2 \ll (Ryd)^2$. This is our case, e.g., most gases at S.T.P. For metals ω_p can be of the order of a Rydberg. Second, the case of large K, $Ka_o \gg 1$ implies $K \gg 1/a_o = me^2$. For $K_o = me^2$ we see that in the case considered here, $\omega_p^2 \ll K_o^2$.* Summing up, small K's are of the order of the Rydberg and large K's are of the order of a few percent of the electron mass.

In summary then, the dispersion formula for the case of our atomic system for longitudinal photons consists of two branches: one branch starts out, for $Ka_o \ll 1$, as $\omega_n + (\omega_p^2/2\omega_n)$, $n = 1, 2, \ldots$. As K increases these lines gradually merge into $K^2/2m$ which asymptotically tends to $K - m$ for $K \gg m$. The other branch starts out at $2m$ as $2m + \frac{K^2}{2m}$ tending to $K + m$ for $K \gg m$. These dispersion curves are plotted in figure 2.

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* Numerically for $N \sim 10^{20}/cm^3$, $\omega_p \sim 10^{-2}$ e.V.; Ryd $\cong$ 13.7 e.V.; $K_o \sim \frac{1}{2} \times 10^4$ e.V. In the usual cgs units, $K \sim \frac{1}{\text{length}}$. Here, however, since we are using natural units ($\hbar = c = 1$), $\frac{1}{\text{length}}$ is equivalent to energy.

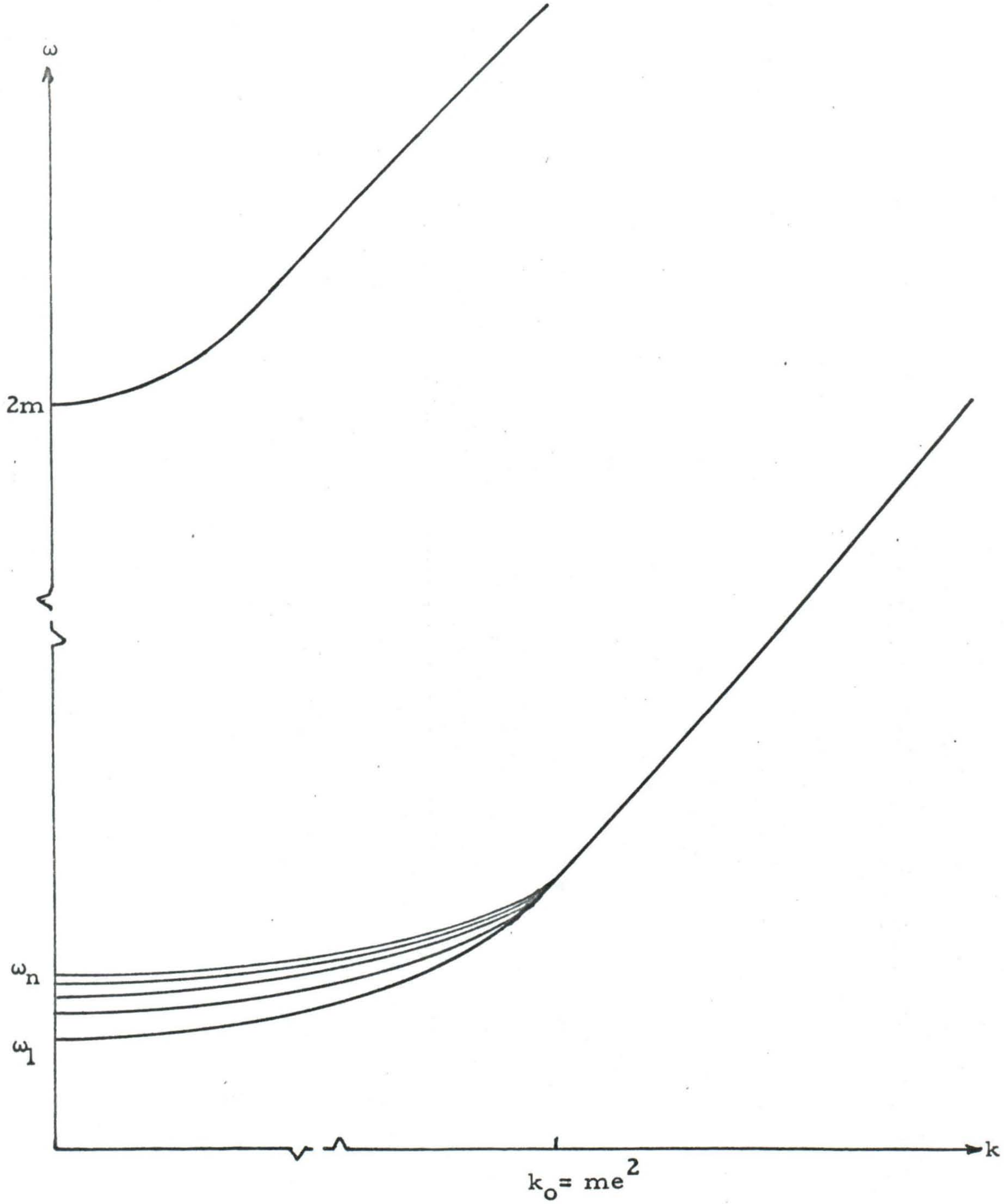

Figure 2. Longitudinal dispersion curves for the atomic system

Now we go on to discuss the poles arising in the transverse term. Poles arise from the transverse term when $\omega^2\eta - K^2 = 0$. Again we will discuss separately the cases of large and small K. First consider the case of small K. From II-31 we have

$$\omega^2\left(1 + \omega_p^2 \sum_n \frac{f_n}{\omega_{\gamma_n}^2 - \omega^2}\right) - K^2 = 0^* \tag{II-62}$$

Again we first consider only one excited state in order to get an idea of what is going on. In this case II-62 becomes

$$\omega^2\left(1 + \frac{\omega_p^2}{\omega_\gamma^2 - \omega^2}\right) - K^2 = 0$$

where $\omega_\gamma = \omega_1 - \frac{i\gamma}{2}$.

Breaking this up into real and imaginary parts yields

$$\omega^2\left(1 - \frac{\omega_p^2\,[(\omega^2 - \bar{\omega}^2) - i\gamma\omega_1]}{(\omega^2 - \bar{\omega}^2)^2 + \gamma^2\omega_1^2}\right) - K^2 = 0 \tag{II-63}$$

where

$$\bar{\omega}^2 = \omega_1^2 - \frac{\gamma^2}{4}$$

To simplify writing, let $x = \frac{\omega^2}{\omega_p^2}$, $\Omega = \frac{\bar{\omega}^2}{\omega_p^2}$, $\Gamma = \frac{\gamma\omega_1}{\omega_p^2}$, and $y = \frac{K^2}{\omega_p^2}$; II-63 becomes

$$x\left(1 - \frac{(x - \Omega) - i\Gamma}{(x - \Omega)^2 + \Gamma^2}\right) - y = 0 \tag{II-64}$$

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* Again we have approximated $\omega_{\gamma_n}/\omega_n$ by 1.

For gases, $\Omega \sim 10^3 - 10^4$, taking $\gamma \sim 10^{-6}\,\omega_1$ and $\omega_1 \sim$ Ryd., $\Gamma \sim 10^{-2} - 10^{-3}$. The real part of II-63 is,

$$x\left(1 - \frac{(x-\Omega)}{(x-\Omega)^2 + \Gamma^2}\right) = y_{Re} \qquad \text{(II-65)}$$

Equation II-65 is plotted in figure 3a. We see that, near the origin, $y_{Re} \cong x$ with a slope $\frac{dx}{dy_{Re}} \cong 1 - \frac{1}{2\Omega} \cong 1$. At $x = \Omega$, $y = \Omega$; while at $x = \Omega \pm \Gamma$, $y_{Re} \cong (1 \pm \frac{1}{\Gamma})\,\Omega \sim 10^6$ (resonance); as $x \to \infty$, $y_{Re} \to x$.

The imaginary part of II-63 is

$$\frac{x}{(x-\Omega)^2 + \Gamma^2} = y_{Im} \qquad \text{(II-66)}$$

Equation II-66 is plotted in figure 3b. We see that y_{Im} is non-zero only for $x \sim \Omega$; for $x = \Omega$, $y_{Im} = \frac{\Omega}{\Gamma} \sim 10^5 - 10^7$.

The alteration of the results for one excited state to the case of many excited states is essentially that, instead of having one resonance at $x = \omega_1^2/\omega_p^2$, there will be many resonances located at ω_1^2/ω_p^2, ω_2^2/ω_p^2, ... The effect of the non-resonant states r_n (see II-50) will in general be very small compared to the resonant term. Specifically at $x = \Omega_n$, $y_{Re} = \Omega_n(1 + r_n)$, $r_n \ll 1$, instead of $y_{Re} = x$, while at $x = \Omega_n \pm \Gamma_n$, r_n can be ignored compared with Ω_n/Γ_n. The imaginary part for many states gives a series of spikes at $x = \Omega_n$, $n = 1, 2, \ldots$.

For the atomic system, we are interested in values of K up to the order of $K_o = 1/a_o = me^2 \sim 10^3$ Ryd. This is far beyond the

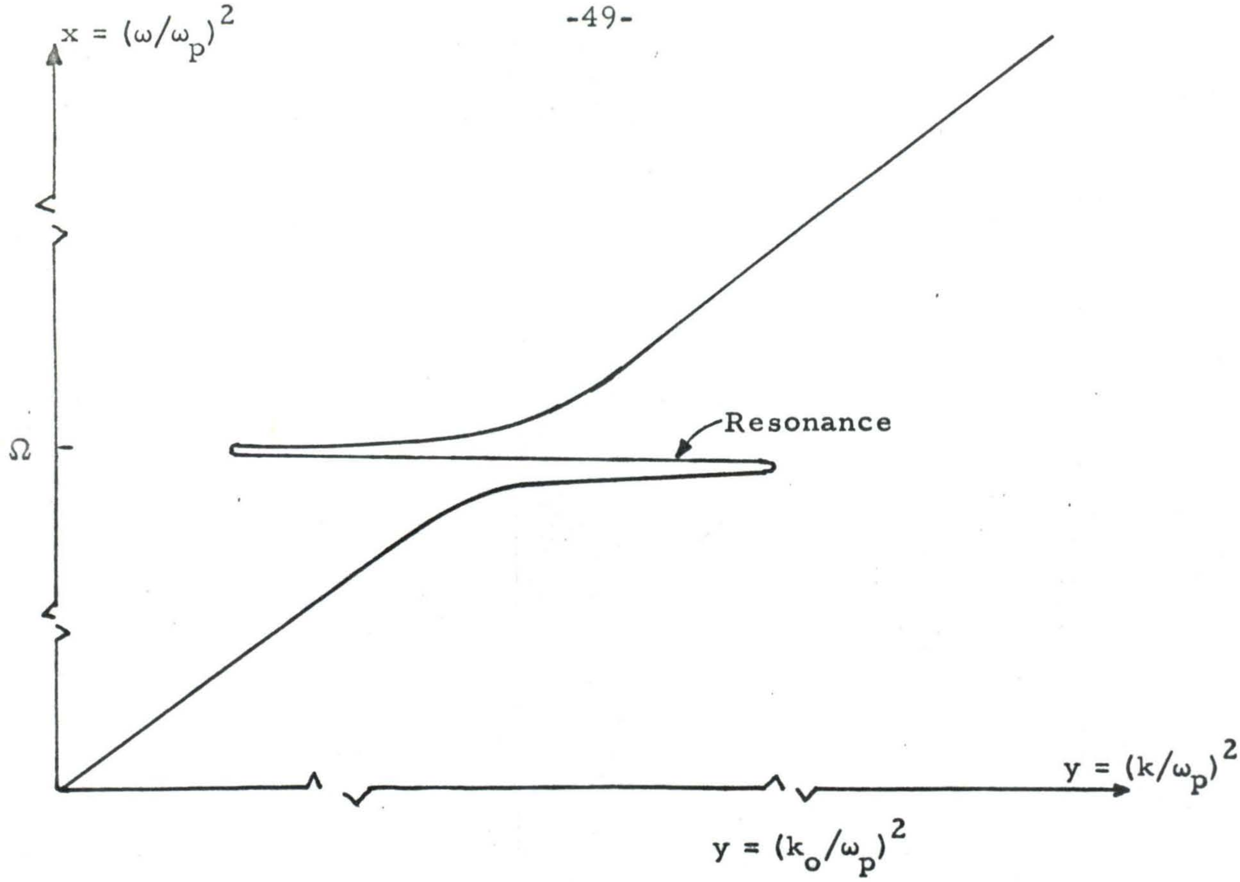

Figure 3a. Transverse dispersion curve for the atomic system; real part

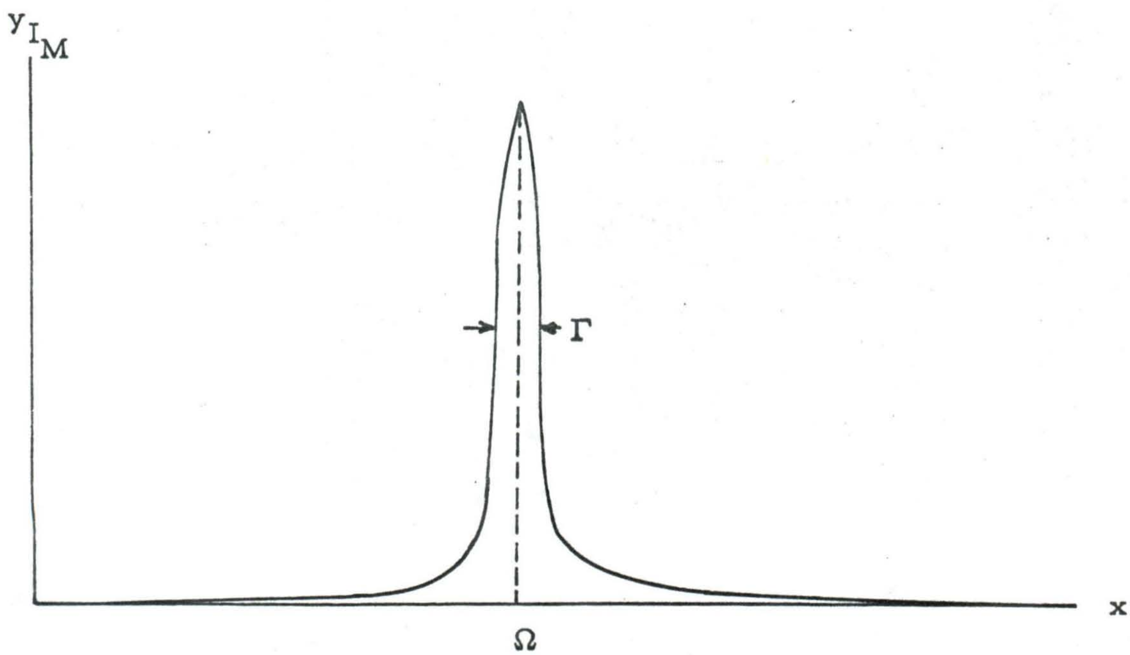

Figure 3b. Transverse dispersion curve for the atomic system; imaginary part

resonance region (see figure 3a).

Now we go to the region of large K ($Ka_o >> 1$). From II-45 we have

$$\omega^2\left(1 + \frac{4m^2\omega_p^2}{(k^2 + 2m\omega)(k^2 - 2m\omega)}\right) - K^2 = 0$$

or

$$k^6 - 4m^2\omega^2(k^2 - \omega_p^2) = 0 \qquad \text{(II-67)}$$

Again, we will find solutions of II-67 for all K just as was done in the longitudinal case. For our atomic system we want the solution for $K >> K_o = me^2$.

To solve II-67 we first try to find solutions for $k^2 >> \omega_p^2$; II-67 becomes

$$k^2 - 4m^2\omega^2k^2 = 0$$

or

$$k^2 = \pm 2m\omega$$

and

$$k^2 = 0$$

For $2m\omega$ to be much greater than ω_p^2 implies that $\omega >> \omega_p(\frac{\omega_p}{m}) \sim 10^{-7}\,\omega_p$ (for gases at S. T. P.). So for $\omega >> 10^{-7}\,\omega_p$, the solutions of $k^2 = \pm 2m\omega$ are

$$\omega = \pm m \pm (m^2 + K^2)^{1/2} ;$$

the two positive solutions are

$$\omega_1 = m + (m^2 + K^2)^{1/2}$$

$$= 2m + \frac{K^2}{2m} \quad \text{for } (K \ll m) \qquad \text{(II-68a)}$$

$$= K + m \quad \text{for } (K \gg m)$$

and

$$\omega_2 = -m + (m^2 + K^2)^{1/2}$$

$$= \frac{K^2}{2m} \quad \text{for } (K \ll m) \qquad \text{(II-68b)}$$

$$= K - m \quad \text{for } (K \gg m)$$

Again we see from ω_1 that at $K = 0$, $\omega = 2m$, i.e., a transverse photon is able to make an electron position pair if its frequency is at least 2m. Now, $\omega_2 = K^2/2m + \ldots$ is a solution as long as $\omega \gg \omega_p(\omega_p/m) = 10^{-7}\ \omega_p$ which implies that $K^2 \gg \omega_p^2$. So the conditions on ω_2 are

$$\omega_2 = \frac{K^2}{2m} \quad \text{for } (\omega_p \ll K \ll m) .$$

Now, the solution $k^2 = 0$ certainly does not satisfy $k^2 \ll \omega_p^2$. However, to find the third solution try $k^2 = \delta$, where $\delta \sim \omega_p^2$. We get, from II-67, that $\delta^3 = 4m^2\omega^2(\delta - \omega_p^2)$, which has the approximate solution $\delta = \omega_p^2$ for $\omega \gg \omega_p(\omega_p/m) = 10^{-7}\omega_p$. We have $k^2 = \omega_p^2$ or

$$\omega^2 = K^2 + \omega_p^2 \qquad \text{(II-69)}$$

For $K = 0$, $\omega = \omega_p$ ($\gg 10^{-7}\ \omega_p$) so II-69 is valid for all K. II-69 is recognized as the dispersion formula for transverse waves in a plasma for $\vec{P}_i = 0$. (8). For $K \gg \omega_p$, we have $\omega^2 = K^2$, which is the dispersion formula for free light. Now we fix up ω_2 for $K << \omega_p$. For $K << \omega_p$, $k^2 \approx \omega^2$, II-67 becomes

$$\omega^6 - 4m^2\omega^6(\omega^2 - \omega_p^2) = 0$$

with solutions

$$\omega^2 = 0$$

$$\omega^2 = \omega_p^2$$

and

$$\omega^2 = (2m)^2$$

$\omega = \omega_p$ is recognized as the limiting form of II-69 for $K^2 << \omega_p^2$; $\omega = 2m$ is recognized as ω_1 for $K = 0$. Therefore, $\omega = 0$ is the limiting form of ω_2 for small K. The three solutions are plotted in figure 4. Again we state that the portion of the K axis that we are interested in is for $K >> me^2$.

We remark that the assumption $\vec{P}_i = 0$ is not a serious limitation. We could, just as we did in the coulomb case, rederive everything with $\vec{P}_i \neq 0$. From II-43' we find that the transverse dielectric function ($\eta_{tr.}$) is

$$\eta_{tr.} = 1 + \frac{k^2}{\omega^2}\beta_{tr.,tr.}$$

$$= 1 - \frac{\omega_p^2}{N\omega^2}\sum_i \frac{m}{E_i}\left\{\frac{\left(1 - \frac{\vec{v}_i\cdot\vec{K}}{\omega}\right)^2 - \left[v_i^2 - \left(\frac{\vec{v}_i\cdot\vec{K}}{K}\right)^2\right]\frac{k^2}{2\omega^2}}{\left(1 - \frac{\vec{v}_i\cdot\vec{K}}{\omega}\right)^2 - \left(\frac{k^2}{2E_i\omega}\right)^2}\right\} \quad \text{(II-70)}$$

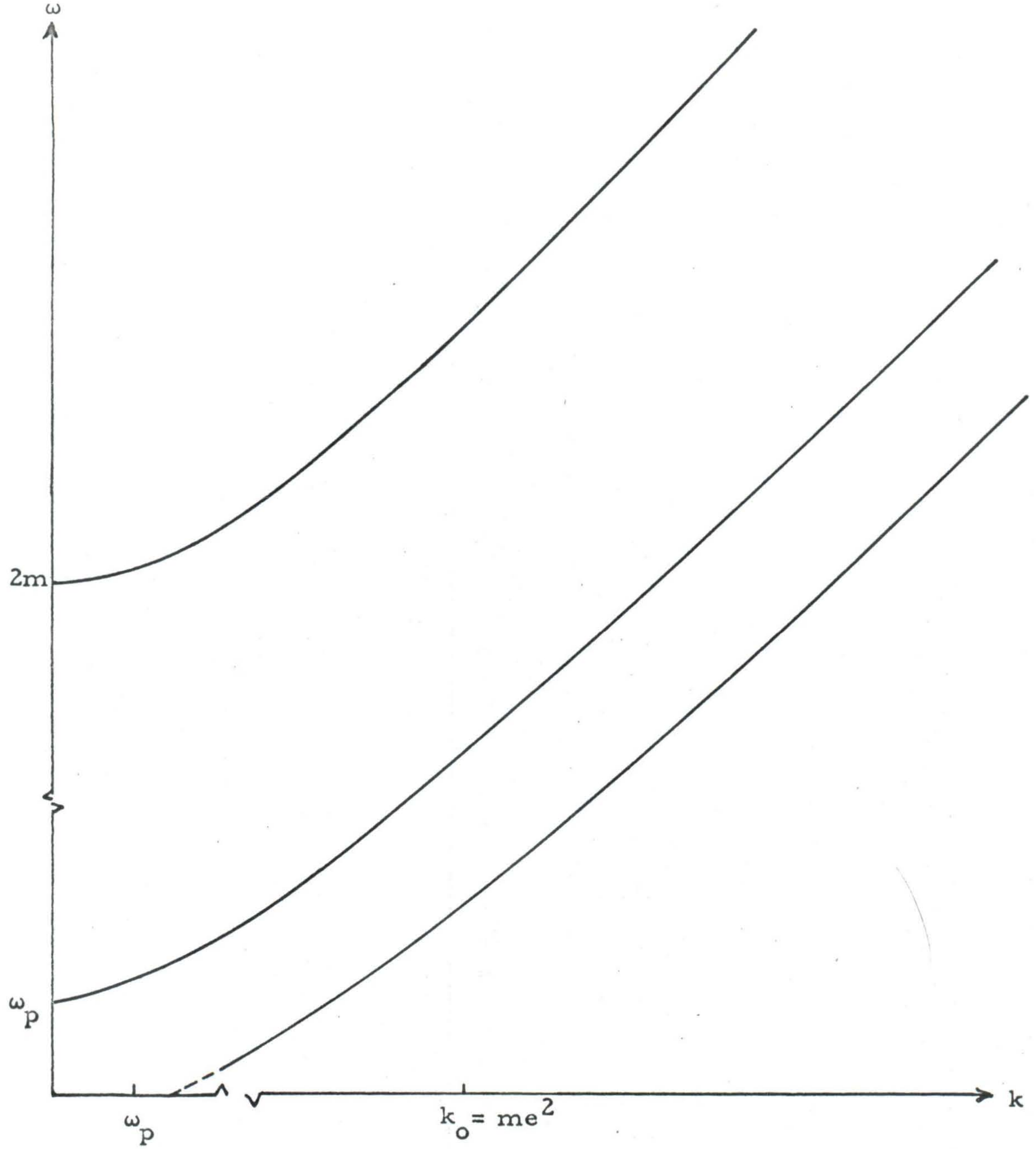

Figure 4. Transverse dispersion curves for the electron gas

We note that for $P_i \neq 0$, the transverse and coulomb dielectric function are not equal. From II-70 we find, in the non-relativistic approximation, that the correction to the dispersion formula $\omega^2 = K^2 + \omega_p^2$, for small K, is

$$\omega^2 = K^2 + \bar{\omega}_p^2 \left[1 + \frac{K^4}{4m^2 \bar{\omega}_p^2} + \frac{K^2}{3\bar{\omega}_p^2} \langle v^2 \rangle - \frac{K^2}{2m^2} \right]^* \qquad \text{(II-71)}$$

where

$$\bar{\omega}_p^2 = \omega_p^2 \left[1 - \frac{5}{6} \langle v^2 \rangle + \frac{\omega_p^2}{4m^2} \right]$$

Except for II-71 we note that, in both the longitudinal and transverse cases, the poles where ω_p (collective term) can be neglected are identical (compare II-59 - II-61 with II-68 a, b). These poles correspond to single particle excitations by longitudinal and transverse photons respectively; they should be equal, since a particle which is initially at rest and picks up energy ω and momentum K from either a longitudinal or a transverse photon must satisfy $(\omega + m)^2 = K^2 + m^2$, which is just the determining relation for these poles.

II-71 is the dispersion formula for light propagating through the medium, the ω_p terms being the collective effect of the medium on this propagation, and the K dependent terms the single particle effects. For $\omega_p = 0$ we get $\omega = K$, i.e., free light. For $\vec{P_i} \neq 0$

*The second and fourth terms in the braces are, for small K, a higher order correction to the ω_p^2 term and do not have anything to do with the motion of the electrons. We have included them here for completeness.

we have seen that $\eta_{tr.} \neq \eta_{coul.}$. This is to be expected. For, consider a photon in the lab system [described by a vector potential a_μ = constant $e_\mu e^{ik\cdot x}$ (e_μ is a unit vector)]. Now, in the rest frame of an electron $a'_{tr.} \neq a'_{long.}$ (or, equivalently $E'_{tr.} \neq E'_{long.}$), and hence the recoil of the electron to a transverse photon is different from that to a longitudinal (or coulomb) photon (i. e., the transverse and coulomb index are not equal). Finally we note that, when $K = 0$, the results for the longitudinal and transverse cases are identical. This is to be expected since when $K = 0$ (no direction is specified), one cannot tell the difference between longitudinal and transverse oscillations.

III. ENERGY LOSS OF RELATIVISTIC CHARGED PARTICLES IN A MEDIUM

A. Relation to Loss

Up until now we have been considering only the propagation of photons through matter. That is, we have discussed only the interaction of the electromagnetic field with the medium. Now we consider the interaction of a charged particle, as opposed to the atomic electrons, with the medium and the electromagnetic field. Specifically we will be interested in the energy loss of a relativistic charged particle of mass M passing through the medium. In order to get the energy loss per unit path length, $\frac{dE}{dx}$, we will first calculate the probability per second that the incident charged particle, of energy E (momentum $(E^2 - M^2)^{1/2}$), makes a transition to a state of energy $(E - \omega)$ in range $d\omega$ (momentum $(E-\omega)^2 - M^2)^{1/2}$), that is, the particle loses energy ω. Calling this quantity $d\Gamma_{\omega\to\omega+d\omega}$ the energy loss per unit path length $\frac{dE}{dx}$ is given by

$$\frac{dE}{dx} = \int_0^\infty \frac{\omega}{v}\, d\Gamma_{\omega\to\omega+d\omega}$$

$$= \int_0^\infty \frac{\omega}{v} \left(\frac{d(\Gamma_{\omega\to\omega+d\omega}}{d\omega} \right) d\omega \qquad \text{(III-1)}$$

where v is the speed of the incident particle, the integral in III-1 extends only over positive frequencies since the particle can loose only positive energy (medium absorbs only positive frequencies). So the problem is to calculate the decay rate for a relativistic charged particle. In order to do so we proceed as follows: Consider the self energy of a particle in the medium, that is, the emission of a virtual photon by the particle with the particle subsequently reabsorbing the photon. In a medium the photon has an amplitude to interact with the atoms of the medium. From time dependent perturbation theory it is known that the only effect the above process has on the wave function representing the particle is to cause a change in its phase proportional to the time interval T over which the perturbation is applied. The resulting wave function is proportional to

$$e^{-iEt}e^{-i\Delta Et}$$

where ΔE is the energy shift due to the virtual photons. In general* ΔE may have a real and an imaginary part. The real part represents the correction to the energy eigenvalue due to the emission and re-absorption of photons (i.e., the real part represents the change in mass of the particle due to virtual photons). The imaginary part of the self energy represents the loss in amplitude required by the fact that the probability that the particle remains in a state of energy E (momentum $P = (E^2 - M^2)^{1/2}$) decreases with time. That is, the imaginary part of ΔE (Im ΔE) is minus one-half the total decay rate out of the initial state. This can be seen as follows: when the total decay rate is Γ_T (i.e., probability of decay proportional to $e^{-\Gamma_T t}$), the amplitude of remaining in the original state contains a

---

*In vacuum the imaginary part of ΔE is zero in accordance with the fact that a free particle in vacuum cannot emit a photon.

factor $e^{-\Gamma_{T/2}t}$, and the time dependent wave function has a factor $e^{-\Gamma_{T/2}t} e^{-i\mathcal{E}t} = e^{-i(\mathcal{E}-i\Gamma_{T/2})t}$.* Thus $-\Gamma_T/2$ is the imaginary part of the energy. Thus the ultimate problem is to compute the self energy of a particle in a medium and then to take its imaginary part. This method of computing decay rates has for example been used by DuBois (9) to calculate plasmon damping in an electron gas.

B. Self Energy and Decay Rate of a Particle in a Medium

In order to calculate the self energy we will again use perturbation theory via Feynman diagrams. Consider the diagrams in figure 5. Diagrams a-d..., represent the exchange of one virtual photon and its interaction with the medium. Diagrams e-f... represent the exchange of two or more photons. Diagram α represents the sum of all one photon exchange diagrams. Diagram β represents the sum of all two photon exchange diagrams. The effect of the medium is that the photon propagator is modified. Physically the particle interacts with the bare electromagnetic field (coupling ~ $e\gamma_\mu$) emitting a photon; the photon propagates through the medium being absorbed and re-emitted by atoms of the medium (propagator $\pi_{\mu\nu}$); finally the particle interacts with the bare electromagnetic field (coupling ~ $e\gamma_\nu$) absorbing the photon. Here we will neglect two, three, etc. photon exchange processes since they are certainly smaller than a one photon

*Here $\mathcal{E}$ is the total real part of the energy (i.e., $\mathcal{E}$ includes the effect of virtual photons etc.)

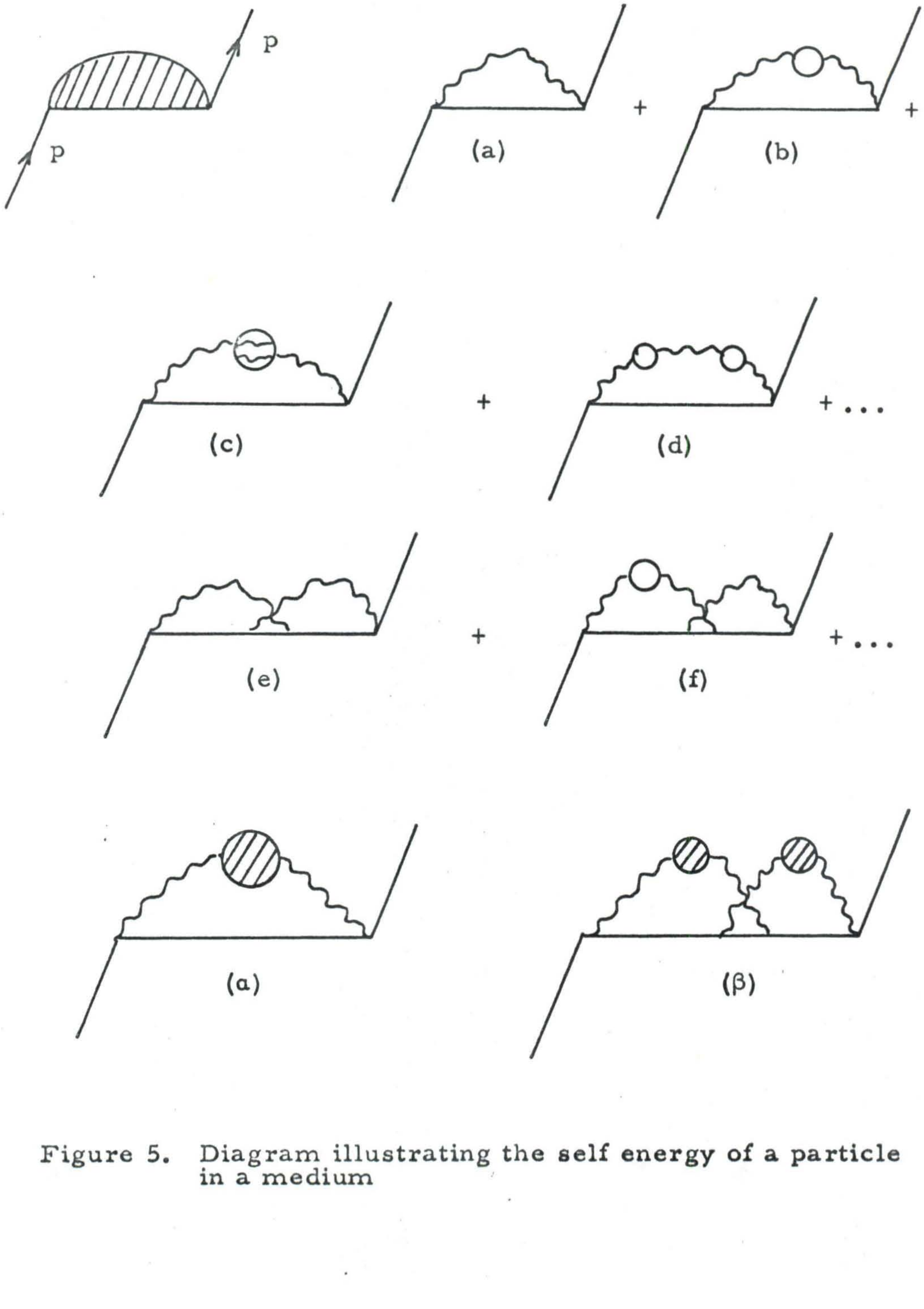

Figure 5. Diagram illustrating the self energy of a particle in a medium

exchange process because of the weak coupling constant e^2. It should be noted that the "bubbles" in figure 5 represent the propagation of atoms in excited states and not propagation of electron-position pairs.

We now write down the amplitude that corresponds to figure 6. Calling this amplitude $\mathcal{M}$, we have

$$\mathcal{M} = ie^2 \int \frac{d^4k}{(2\pi)^4} \overline{U}_p \left(\gamma_\nu \frac{1}{\not{p}-\not{k}-M} \gamma_\mu\right) U_p \pi_{\mu\nu} \qquad \text{(III-2)}$$

where U_p = the free particle spinor of four momentum p ($p = (E, \vec{P})$) normalized to $\overline{U}_p U = 2m$

$\pi_{\mu\nu}$ = photon propagator in a medium which is given in II-4

and

p = particle four momentum which here is taken as relativistic.

With the normalization used here the relation between $\mathcal{M}$ and ΔE is given by

$$\Delta E = \frac{\mathcal{M}}{2E} \qquad \text{(III-3)}$$

We are interested in $\Gamma_T = -2 \text{ Im } \Delta$ which from III-2 and III-3 is

$$\Gamma_T = \frac{-e^2}{E} \text{Re} \int \frac{d^4k}{(2\pi)^4} \overline{U}_p \left(\gamma_\nu \frac{1}{\not{p}-\not{k}-M} \gamma_\mu\right) U_p \pi_{\mu\nu} \qquad \text{(III-4)}$$

where Re denotes real part and $d^4k = dk_4 d^3\vec{K}$. The limits on the integral are from $-\infty$ to $+\infty$. Now if in III-4 we perform the integration over K we are left with an integral to do over k_4. Call the integrand of this integral $F(k_4)$. Then

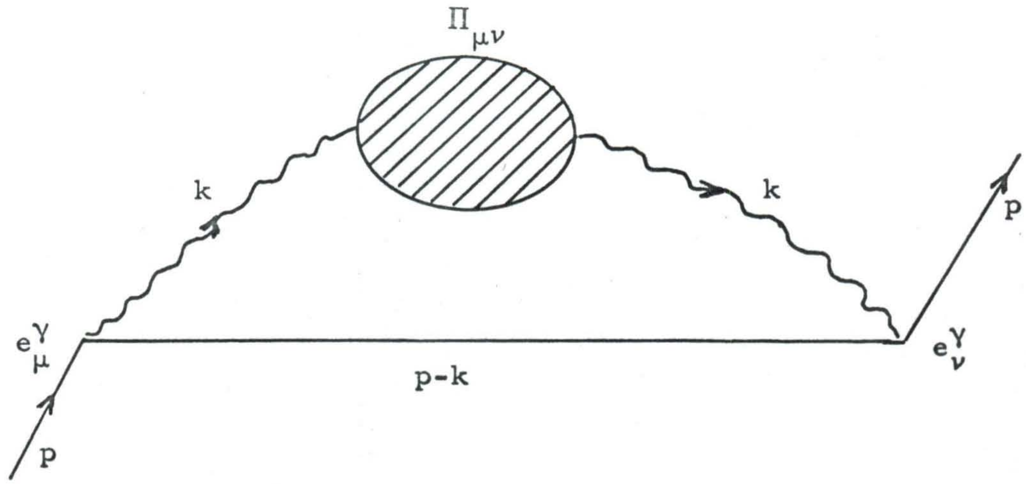

Figure 6. Diagram used to calculate the self energy of a particle in a medium

$$\Gamma_T = \int_{-\infty}^{\infty} F(k_4)\, dk_4$$

$$= \int_{0}^{\infty} [F(k_4) + F(-k_4)]\, dk_4 \qquad \text{(III-5)}$$

where in III-5 we have expressed Γ_T as an integral over positive values of the fourth component of k. We now can associate the integrand of III-5 as being the differential decay rate $\frac{d\Gamma_{\omega \to \omega+d\omega}}{d\omega}$.*

That is

$$\frac{d\Gamma}{d\omega} = F(\omega) + F(-\omega) \qquad \text{(III-6)}$$

where

$$F(\omega) = \frac{-e^2}{(2\pi)^4 E} \operatorname{Re} \int d^3\vec{K}\, \overline{U}_p (\gamma_\nu \frac{1}{\not{p}-\not{k}-M} \gamma_\mu) U_p \pi_{\mu\nu} \qquad \text{(III-7)}$$

We write $F(\omega)$ as follows

$$F(\omega) = \frac{-e^2}{(2\pi)^4 E} \operatorname{Re} \int d^3\vec{K}\, X_{\mu\nu} \pi_{\mu\nu} \qquad \text{(III-8)}$$

where

$$X_{\mu\nu} = \overline{U}_p (\gamma_\nu \frac{1}{\not{p}-\not{k}-M} \gamma_\mu)\, U_p\, . \qquad \text{(III-9)}$$

Next we carry out the implied summations in III-9. To second order

$$\pi_{\mu\nu} = -\frac{4\pi}{k^2} (\delta_{\mu\nu} + \beta_{\mu\nu})\, .$$

Now proceeding in a manner that is exactly the same as the one that led from II-28 to II-36, we find that to all orders

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*Henceforth we call the integration variable in III-5 ω. Also we drop the subscript $\omega \to \omega+d\omega$ on $d\Gamma$.

$$f \equiv X_{\mu\nu}\pi_{\mu\nu} = 4\pi\left\{\frac{X_{44}}{K^2\eta} + \sum_{2\text{ tr. direc.}} \frac{X_{\text{tr., tr.}}}{\omega^2\eta - K^2}\right\} \qquad \text{(III-10)}$$

where

$$\eta = 1 + \frac{k^2}{K^2}\beta_{44} \qquad \text{(III-11)}$$

and β_{44} is given in II-22 for $Ka_o << 1$ and in II-44 for $Ka_o >> 1$. In arriving at III-10 we have, in dealing with the scalar longitudinal terms, replaced γ_3 by $(\omega/K)\gamma_4$ etc. (4). Now

$$F(\omega) = \frac{-e^2}{(2\pi)^4 E} \operatorname{Re} \int d^3\vec{K} f$$

$$= \frac{-e^2}{(2\pi)^4 E} \operatorname{Re}\left\{ \int_{K<K_o} d^3\vec{K} f + \int_{K>K_o} d^3\vec{K} f\right\} \qquad \text{(III-12)}$$

which we write as

$$F(\omega) = \frac{-e^2}{(2\pi)^4 E} \operatorname{Re}\,(I_1 + I_2) \qquad \text{(III-13)}$$

where

$$I_1 = \int_{K<K_o} d^3\vec{K}\, f_1$$

$$I_2 = \int_{K>K_o} d^3\vec{K}\, f_2 \qquad \text{(III-14)}$$

and

$$K_o = \frac{1}{a_o}$$

For f_1 we will use the expression for η in the range $Ka_o << 1$ and for f_2 the expression for η in the range $Ka_o >> 1$. That is, in f_1

we use β_{44} as given in II-22, and in f_2 we use β_{44} as given in II-44. Strictly speaking, II-22 and II-44 should not be used near K_o. However, as is usually the case, after performing the integrals the dependence on K_o in I_1 and I_2 is proportional to ln (K_o) and ln $(\frac{1}{K_o})$ respectively. The final answer $(\sim I_1 + I_2)$ is then independent of K_o. It is convenient to rewrite I_1 as

$$I_1 = \int_{\text{all } K} d^3\vec{K} f_1 - \int_{K>K_o} d^3\vec{K} f_1 \qquad \text{(III-15)}$$

III-12 becomes

$$F(\omega) = \frac{-e^2}{(2\pi)^4 E} \operatorname{Re}\left\{ \int_{\text{all } K} d^3\vec{K} f_1 + \int_{K>K_o} d^3\vec{K}(f_2 - f_1) \right\} \qquad \text{(III-16)}$$

$$= F_R(\omega) + F_S(\omega) \qquad \text{(III-17)}$$

where

$$\left.\begin{aligned} F_R &= \frac{-e^2}{(2\pi)^4 E} \operatorname{Re} \int_{\text{all } K} d^3\vec{K} f_1 \\ \text{and} \quad F_S &= \frac{-e^2}{(2\pi)^4 E} \operatorname{Re} \int_{K>K_o} d^3\vec{K}(f_2 - f_1) \end{aligned}\right\} \qquad \text{(III-18)}$$

To proceed further we need an explicit form for $X_{\mu\nu}$. $X_{\mu\nu}$ is given in III-9. Rationalizing the denominator in III-9 and averaging over the initial spins and summing over the final spins of the particle we get

$$X_{\mu\nu} = \frac{1}{(p-k)^2 - M^2} \left(\frac{1}{2}\right) \operatorname{Sp}\left\{(\not p + M)\gamma_\nu(\not p - \not k + M)\gamma_\mu\right\}$$

$$= \frac{2}{k^2 - 2p \cdot k} \left\{ 2p_\mu p_\nu - k_\mu p_\nu - k_\mu p_\nu + \delta_{\mu\nu}(p \cdot k) \right\} \qquad \text{(III-19)}$$

From III-19 we find that

$$X_{44} = \frac{2}{k^2 - 2p \cdot k} \left\{ 2E^2 - 2E\omega + p \cdot k \right\}$$

$$= \frac{4E^2}{k^2 - 2p \cdot k} \left\{ 1 - \frac{\omega}{2E} - \frac{\vec{P} \cdot \vec{K}}{2E^2} \right\} \qquad \text{(III-20a)}$$

and

$$X_{tr.,tr.} = \frac{2}{k^2 - 2p \cdot k} \left\{ 2P^2_{tr.} - p \cdot k \right\}$$

$$= \frac{4}{k^2 - 2p \cdot k} \left\{ P^2_{tr.} - \frac{E\omega - \vec{P} \cdot \vec{K}}{2} \right\} \qquad \text{(III-20b)}$$

where $P_{tr.}$ is a component of $\vec{P}$ transverse to $\vec{K}$. Now in all of the cases considered here η is independent of direction so upon substituting III-20a and III-20b into III-10 we find that:

$$f = \frac{16\pi}{k^2 - 2p \cdot k} \left\{ \frac{E^2(1 - \omega/2E - \vec{P} \cdot \vec{K}/2E^2)}{K^2\eta} \right.$$

$$\left. + \frac{P_\perp^2 - E\omega + \vec{P} \cdot \vec{K}}{\omega^2\eta - K^2} \right\} \qquad \text{(III-21)}$$

where

$$P_\perp^2 = \sum_{2\ \text{tr. direc.}} P^2_{tr.} = P^2(1 - \cos^2\theta) \ ;$$

θ is the angle between $\vec{P}$ and $\vec{K}$.

Equation III-21 becomes:

$$f = \frac{16\pi E^2}{k^2 - 2p\cdot k}\left\{\frac{1 - \frac{\omega}{2E} - \frac{\vec{P}\cdot\vec{K}}{2E^2}}{K^2\eta} + \frac{v^2(1-\cos^2\theta) - \frac{\omega}{E} + \frac{\vec{P}\cdot\vec{K}}{E^2}}{\omega^2\eta - K^2}\right\} \qquad \text{(III-22)}$$

In the next two sections we evaluate the integrals F_R and F_S respectively.

C. On the Evaluation of the Integral $F_R(\omega)$

From III-22 and III-18 we have that

$$F_R = \frac{-e^2 E}{\pi^3}\text{Re}\int_{\text{all }K} d^3\vec{K}\ \frac{1}{k^2 - 2p\cdot k}\left\{\frac{1 - \frac{\omega}{2E} - \frac{\vec{P}\cdot\vec{K}}{2E^2}}{K^2\eta_1}\right.^{*}$$

$$\left. + \frac{v^2(1-\cos^2\theta) - \frac{\omega}{E} + \frac{\vec{P}\cdot\vec{K}}{E^2}}{\omega^2\eta_1 - K^2}\right\} \qquad \text{(III-23)}$$

where η_1 is given in II-31. Now

$$F_R = -\frac{e^2 E}{\pi^3}\text{ Re }(C + T) \qquad \text{(III-24)}$$

where

$$C = C_1 + C_2 + C_3 \qquad \text{(III-25)}$$

$$C_1 = \int d^3\vec{K}\ \frac{1}{(k^2 - 2p\cdot k)K^2\eta_1} \qquad \text{(III-26a)}$$

$$C_2 = -\frac{\omega}{2E}\int d^3\vec{K}\ \frac{1}{(k^2 - 2p\cdot k)K^2\eta_1} \qquad \text{(III-26b)}$$

---

*Henceforth we omit the "all K" on the K integral.

$$C_3 = -\frac{1}{2E^2}\int d^3\vec{K}\,\frac{\vec{P}\cdot\vec{K}}{(k^2-2p\cdot k)K^2\eta_1} \tag{III-26c}$$

and

$$T = T_1 + T_2 + T_3 \tag{III-27}$$

where

$$T_1 = v^2\int d^3\vec{K}\,\frac{(1-\cos^2\theta)}{(k^2-2p\cdot k)(\omega^2\eta_1 - K^2)} \tag{III-28a}$$

$$T_2 = -\frac{\omega}{E}\int d^3\vec{K}\,\frac{1}{(k^2-2p\cdot k)(\omega^2\eta_1 - K^2)} \tag{III-28b}$$

$$T_3 = \frac{1}{E^2}\int d^3\vec{K}\,\frac{\vec{P}\cdot\vec{K}}{(k^2-2p\cdot k)(\omega^2\eta_1 - K^2)} \tag{III-28c}$$

These integrals are evaluated in Appendix B. The results are:*

$$C_1 = \alpha\ln\left(\frac{1-\frac{a}{2}+(1-a)^{1/2}}{-a/2}\right) \tag{III-29a}$$

$$C_2 = -\frac{\omega}{2E}C_1 \tag{III-29b}$$

$$C_3 = -\alpha(v^2/2)\left[(1-a)^{1/2} + \frac{a}{2}\ln\left(\frac{1-\frac{a}{2}+(1-a)^{1/2}}{-a/2}\right)\right] \tag{III-29c}$$

$$T_1 = \alpha\left(\frac{v^2P^2}{\omega^2}\right)\left[-\frac{\delta}{4}(1-a)^{1/2} + \frac{\epsilon}{2}\delta^{1/2} + \left(\delta-\frac{\epsilon^2}{4}\right)\ln\left(\frac{1-\frac{\epsilon}{2}+(1-\epsilon+\delta)^{1/2}}{-\left(\frac{\epsilon}{2}-\delta^{1/2}\right)}\right)\right.$$
$$\left. -\frac{a^2}{4}\ln\left(\frac{1-\frac{a}{2}+(1-a)^{1/2}}{-a/2}\right)\right] \tag{III-29d}$$

---

*In the remainder of this section we drop the subscript 1 on η.

$$T_2 = \alpha(\frac{\omega}{E}\eta)\ln\left(\frac{1-\frac{\epsilon}{2}+(1-\epsilon+\delta)^{1/2}}{-(\frac{\epsilon}{2}-\delta^{1/2})}\right) \qquad \text{(III-29e)}$$

$$T_3 = \alpha(-\eta v^2)\left[(1-\epsilon+\delta)^{1/2}-\delta^{1/2}+\frac{\epsilon}{2}\ln\left(\frac{1-\frac{\epsilon}{2}+(1-\epsilon+\delta)^{1/2}}{-(\frac{\epsilon}{2}-\delta^{1/2})}\right)\right] \qquad \text{(III-29f)}$$

where

$$\alpha = i\pi^2/P\eta$$

$$a = 2\omega/Pv(1-\omega/2E)$$

$$\delta = (\omega^2/P^2)\eta \qquad \text{(III-30)}$$

$$\epsilon = a + \delta$$

It is to be noted that we want the real part of the resulting expression.

Next we substitute the expression III-30 into III-28 and III-29. We then expand the resulting expressions as a power series in ω/E. Now, as we shall see, the values of ω that contribute to the decay rate from the F_R terms are of the order of the Rydberg. So we only need the lowest order terms in the resulting expansion of III-28 and III-29 as a series in ω/E. For, taking the incident particle to have a mass the order of the proton mass and taking its energy to be of the order of its rest mass then $\omega/E \sim 10^{-7}$, which is very small. So, excluding the unlikely possibility of a very large coefficient of ω/E etc.,* we drop these higher order terms. We have

$$F_R = -\frac{e^2E}{\pi^3}\text{Re}(C+T) = -\frac{e^2E}{\pi^3}\text{Re}(C_1+C_2+C_3+T_1+T_2+T_3) =$$

*This has been verified by direct calculation of the coefficient of the ω/E term.

$$= -\frac{e^2}{\pi v} \mathrm{Re}\left[\left(\frac{i}{\eta}\right) \left\{ (v^2\eta-1) \ln\left(\frac{-2}{\frac{\omega}{Pv} - \left(\frac{\omega^2\eta}{P^2}\right)^{1/2}} \right) \right.\right.$$

$$\left.\left. + \frac{\omega}{Pv}\left(\frac{\omega^2\eta}{P^2}\right)^{1/2}\left(-\frac{v^2P^2}{\omega^2}\right) - \frac{v^2}{2}\right\} + O\left(\frac{\omega}{E}\right) + \ldots \right] \qquad \text{(III-31)}$$

In obtaining III-31, we have dropped terms that do not contribute a real part. We wish the contribution to the decay rate that arises from F_R. Calling this contribution $(d\Gamma/d\omega)_R$, we have from III-6

$$\left(\frac{d\Gamma}{d\omega}\right)_R = F_R(\omega) + F_R(-\omega)$$

Noting that η is an even function of ω, we obtain

$$\left(\frac{d\Gamma}{d\omega}\right)_R = -\frac{e^2}{\pi v} \mathrm{Re}\left[\left(\frac{i}{\eta}\right) \left\{ (v^2\eta-1) \left(\ln\left(\frac{-2}{\frac{\omega}{Pv} - \left(\frac{\omega^2\eta}{P^2}\right)^{1/2}} \right) \right.\right.\right.$$

$$\left.\left.\left. + \ln\left(\frac{-2}{\frac{-\omega}{Pv} - \left(\frac{\omega^2\eta}{P^2}\right)^{1/2}} \right) \right) - v^2 \right\} \right]$$

$$= -\frac{e^2}{\pi v} \mathrm{Re}\left[\left(\frac{i}{\eta}\right) \left\{ (v^2\eta-1) \left(\ln\left(-\frac{(2Pv/\omega)^2}{1-v^2\eta}\right) - v^2 \right) \right\} \right]$$

$$= -\frac{e^2}{\pi v}\left[(\mathrm{Im}\,\eta^{-1}) \left\{ \ln \frac{(2Pv/\omega)^2}{|1-v^2\eta|} - v^2 \right\} - \mathrm{Re}(v^2-\eta^{-1})(\pi-\theta) \right]$$

$$\text{(III-32)}$$

where

$$|1-\eta v^2| = +\left[(1-v^2\,\mathrm{Re}\,\eta)^2 + (v^2\,\mathrm{Im}\,\eta)^2 \right]^{1/2} \qquad \text{(III-33)}$$

$$\theta = \tan^{-1}\left(\frac{-v^2 \operatorname{Im}\eta}{1-v^2 \operatorname{Re}\eta}\right)$$

$$= \pi - \varphi \qquad \text{for } 1-v^2 \operatorname{Re}\eta > 0$$

$$= \varphi \qquad \text{for } 1-v^2 \operatorname{Re}\eta < 0 \tag{III-34}$$

and

$$\varphi = \tan^{-1}\left|\frac{v^2 \operatorname{Im}\eta}{1-v^2 \operatorname{Re}\eta}\right| \tag{III-35}$$

The contribution to the energy loss from $(d\Gamma/d\omega)_R$ is obtained by multiplying equation III-32 by ω/v and integrating over positive frequencies. Calling this contribution $(dE/dx)_R$, we have

$$\left(\frac{dE}{dx}\right)_R = -\frac{e^2}{\pi v^2}\int_0^{\infty} \omega d\omega \ \operatorname{Im}\eta^{-1}\left[\ln\left(\frac{(2Pv/\omega)^2}{|1-v^2\eta|}\right) - v^2\right]$$

$$- \operatorname{Re}(v^2 - \eta^{-1})(\pi-\theta) \tag{III-36}$$

We leave III-36 as it stands for the moment, going on in Section D to evaluate the contribution to the decay rate that arises from F_S. In Section E we will combine the results of this section and Section D to get the complete energy loss formula.

D. On the Evaluation of the Integral $F_S(\omega)$

From III-18 we have

$$F_S = -\frac{e^2}{(2\pi)^4 E} \operatorname{Re} \int_{K>K_o} d^3\vec{K}\,(f_2 - f_1)$$

where

$$f_1 = X_{\mu\nu}\pi_{1\mu\nu}$$

and (III-37)

$$f_2 = X_{\mu\nu}\pi_{2\mu\nu}$$

In the following a quantity with the subscript 1 or 2 affixed to it means that we are to use the expressions for the photon propagator in the regions $Ka_o \ll 1$ and $Ka_o \gg 1$ respectively. In III-22 we have given an explicit form for f.* We have used this explicit form in the last section to calculate F_R. This explicit form (for f_1) was useful because η_1 does not depend on K. That is, on performing the K integration, η_1 does not depend on the integration variables and so can be treated as a constant. The K integration was done by the Feynman parameterization technique. This technique is useful in calculating F_R since the integration variables go over all K space. Now, F_S has a restriction on the range of integration; namely, $K > K_o$. Also, F_S contains η_2 (through the f_2 term), which is a function of both ω and K (see II-46) contrasted to η_1, which is a function of ω only. In fact, η_2 is a rather complicated function of ω and K which makes the explicit form for f_2 quite unmanageable. So, in order to perform the integration of the f_2 term, we proceed as follows: We note that since $K > K_o$ the correction to the photon propagator is very small so we will use the second order expansion of f_2.

*By explicit form we mean an explicit form for the inverse of the matrix $(\delta - \beta)$.

Specifically, we shall use the following expression for f_2 in the calculation of F_S:

$$f_2 = X_{\mu\nu}\,\pi_{2\mu\nu}$$

$$\cong -\frac{4\pi}{k^2} X_{\mu\nu}(\delta_{\mu\nu} + \beta_{2\mu\nu}) \qquad \text{(III-38)}$$

where, from II-43,

$$\beta_{2\mu\nu} = -\frac{\omega_p^2}{k^2}\left\{\frac{2p_{o\mu}p_{o\nu} + k_\mu p_{o\nu} + k_\nu p_{o\mu} - \delta_{\mu\nu}(p_o \cdot k)}{k^2 + 2p_o \cdot k + i\epsilon} + \text{terms } k \to -k\right\}$$

(III-39)

where $p_{o\mu} = m\delta_{\mu\nu}$.

We note that what we are really after is the contribution to the decay rate that comes from F_S. Calling this contribution Γ_S, we have from III-6 that

$$\Gamma_S \equiv \int_0^\infty \left(\frac{d\Gamma}{d\omega}\right) d\omega$$

$$= \int_0^\infty (F_S(\omega) + F_S(-\omega))\, d\omega$$

$$= \int_{-\infty}^\infty F_S(\omega)\, d\omega \qquad \text{(III-40)}$$

Upon substituting III-38 and III-39 into III-40 via III-37 we find (suppressing the factor $-\frac{e^2}{(2\pi)^4 E}$ Re) that

$$\Gamma_S = (1) + (2) \qquad \text{(III-41)}$$

where

$$(1) = \int_0^\infty d\omega \left[\int_{K>K_o} d^3\vec{K} \left(-\frac{4\pi\delta_{\mu\nu}}{k^2} X_{\mu\nu} - \pi_{1\mu\nu} X_{\mu\nu} \right) + \text{terms with } \omega \to -\omega \right]$$

(III-42)

and

$$(2) = -\int_0^\infty d\omega \int_{K>K_o} d^3\vec{K} \left[\frac{4\pi}{k^2} \beta_{2\mu\nu} X_{\mu\nu} + \text{terms with } \omega \to -\omega \right] \quad \text{(III-43)}$$

We first consider (1). Now

$$-\frac{\delta_{\mu\nu} X_{\mu\nu}}{k^2} = -\frac{1}{k^2} \left[X_{44} - X_{33} - \sum_{2 \text{ tr. direc.}} X_{tr.,tr.} \right]$$

$$= \frac{X_{44}}{K^2} + \sum_{2 \text{ tr. direc.}} \frac{X_{tr.,tr.}}{k^2} \qquad \text{(III-44)}$$

where we have used $X_{33} = \frac{\omega^2}{K^2} X_{44}$ in obtaining III-44 and X is given in III-20. Also $\pi_{1\mu\nu} X_{\mu\nu} \equiv f_1$, and is given in III-22 with $\eta = \eta_1$. Substituting in III-42, we find:

$$(1) = 4\pi \int_0^\infty d\omega \Bigg\{ \int_{K>K_o} d^3\vec{K} \frac{X_{44}}{K^2} \left(1 - \frac{1}{\eta_1}\right)$$

$$+ \sum_{2 \text{ tr. direc.}} X_{tr.,tr.} \left[\frac{1}{\omega^2 - K^2} - \frac{1}{\omega^2 \eta_1 - K^2} \right]$$

$$+ \text{terms with } \omega \to -\omega \Bigg\} \qquad \text{(III-45)}$$

First we consider the transverse terms in (1). We note that for $|\omega| >>$ Order Ryd., $\eta_1 \sim 1 - (\omega_p/\omega)^2 \approx 1$. So for $|\omega| >>$ Order Ryd. the two terms in parenthesis cancel. Also, for $|\omega| <$ Order Ryd. we may neglect the ω^2 and $\omega^2\eta_1$ terms with respect to the K^2 terms since $K > K_o$. For, taking η_1 at a resonance ($\eta_1 \sim 10^3$)* we have $\omega^2\eta_1/K^2 < \omega^2\eta_1/K_o^2 \sim 10^{-3}$ for $\omega \sim$ Ryd. That is, the two terms again cancel. So we get no contribution to (1) from the transverse terms.

Now we consider the contribution of the coulomb term to (1). From III-20a (replacing the $i\epsilon$ on the Feynman propagator) we have

$$X_{44} = \frac{4E^2}{k^2 - 2p\cdot k + i\epsilon}\left[1 - \frac{\omega}{2E} - \frac{(\vec{P}\cdot\vec{K})}{2E^2}\right]$$

$$= 4E^2\left[1 - \frac{\omega}{2E} - \frac{(\vec{P}\cdot\vec{K})}{2E^2}\right]\left[\frac{P.V.}{k^2 - 2p\cdot k} - i\pi\delta(k^2 - 2p\cdot k)\right] \quad \text{(III-20a)}$$

where P. V. denotes principal value. Uponssubstituting X_{44} into III-45, we get two terms. One term is from the P. V. term in III-20a, and the other term is from the delta function in III-20a. Remembering that we want the real part of (1) we proceed as follows. First, consider the contribution from the P. V. term. This contribution is proportional to

$$P.V.\int_0^\infty d\omega\left\{\int_{K>K_o}\frac{d^3\vec{K}}{K^2}\left(1 - \text{Re}\,\frac{1}{\eta_1}\right)\left[\frac{1 - \frac{\omega}{2E} - \frac{(\vec{P}\cdot\vec{K})}{2E^2}}{k^2 - 2p\cdot k}\right.\right.$$

$$\left.\left. + \text{ terms with } \omega \rightarrow -\omega\right]\right\} \quad \text{(III-46)}$$

In order to evaluate the contribution from this term we note that in the integral over ω, $1 - \text{Re}\,\frac{1}{\eta_1}$ is non-zero only in the immediate neighborhood of the resonance frequencies of η_1. Specifically, near

*We have taken $N \sim 10^{19} - 10^{20}$ and $\gamma/\text{Ryd} \sim 10^{-7}$.

a resonance frequency $\bar{\omega}_n$, we get from II-31 that

$$1 - \mathrm{Re}\,\frac{1}{\eta_1} \cong \frac{\omega_p^2(\omega^2 - \bar{\omega}_n^2)}{(\omega^2 - \bar{\omega}_n^2)^2 + \gamma_n^2\omega_n^2} \qquad \text{(III-47)}$$

where in III-47 we have omitted the small contribution from non-resonant states, and $\bar{\omega}_n^2 \cong \omega_n^2 + \omega_p^2$. That is, in the integral over ω, we get a sum of integrals of the form III-47 located at the frequencies $\bar{\omega}_1, \bar{\omega}_2, \ldots$. Now, near each resonant frequency, the terms in the square brackets of III-46 are slowly varying functions of ω. Hence, we may, as a first approximation substitute $\bar{\omega}_n$ for ω near each resonance in the terms in the square brackets. Doing this, the only ω dependence is in the $1 - \mathrm{Re}\,\frac{1}{\eta_1}$ term. Now, it is easy to see that the P. V. term will not contribute to the energy loss. This is so because the contribution to the energy loss from the P. V. term is obtained by inserting into the integrand a term proportional to ω (see III-1). The resulting integral, near each resonance, is of the form

$$\int_a^b \frac{(\omega^2 - \bar{\omega}_n^2)\,\omega\,d\omega}{(\omega^2 - \bar{\omega}_n^2)^2 + \gamma_n^2\omega_n^2} \sim \int \frac{x\,dx}{x^2 + \gamma_n^2\omega_n^2} = 0$$

where the resonant frequency $\bar{\omega}_n$ is included in the interval a-b. Since there is a negligible contribution to the integral for frequencies outside the immediate neighborhood of $\bar{\omega}_n$, the limits may be selected to symmetrical about $\bar{\omega}_n$ and the integral vanishes.*

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*By expanding equation III-46 in a Taylor series about $\bar{\omega}_n$ we find that the next term is of order γ/K_0 ($\sim 10^{-9}$) times the other terms that we are keeping.

Now we consider the contribution to (1) from the delta function term of III-20a. Substituting this term into III-45 and replacing all the numerical factors, we find

$$(1) = \frac{e^2}{(2\pi)^4 E} \mathrm{Re}\left\{ \int_0^\infty d\omega \left[\int_{K>K_o} \frac{d^3\vec{K}}{K^2} (16\pi E^2)(1 - \frac{\omega}{2E} - \frac{(\vec{P}\cdot\vec{K})}{2E^2}) \right.\right.$$

$$\left.\left. \times (- i\pi\delta(k^2 - 2p\cdot k)(1 - \frac{1}{\eta_1})) + \text{terms with } \omega \to -\omega \right]\right\}$$

$$= -\frac{e^2 E}{\pi^2} \int_0^\infty \mathrm{Im}\,(\eta_1^{-1}) d\omega \left[\int_{K>K_o} d^3\vec{K}\,\delta(k^2 - 2p\cdot k) \right.$$

$$\left. \times (1 - \frac{\omega}{2E} - \frac{(\vec{P}\cdot\vec{K})}{2E^2}) + \text{terms with } \omega \to -\omega \right] \qquad \text{(III-48)}$$

Consider the terms in the square brackets in III-48. Call this term (a); we have

$$(a) = 2\pi \int_{K_o}^\infty dK \int_1^{-1} d(\cos\theta)\,\delta(k^2 - 2E\omega + 2PK\cos\theta)(1 - \frac{\omega}{2E}$$

$$- \frac{PK\cos\theta}{2E^2}) + \text{terms } \omega \to -\omega$$

$$= \frac{\pi}{P} \int_{K_o}^\infty \frac{dK}{K} \int_1^{-1} d(\cos\theta)\,\delta(\cos\theta - \frac{\omega}{Kv} + \frac{k^2}{2PK})(1 - \frac{\omega}{2E}$$

$$- \frac{PK\cos\theta}{2E^2}) + \text{terms with } \omega \to -\omega \qquad \text{(III-49)}$$

We note that because of the delta function there are no contributions to the integral from momentum transfers $K \gtrsim 2P$. In obtaining this result, we have made use of the fact that because of the factor $\mathrm{Im}(\eta_1^{-1})$

the values of ω that contribute will be of the order of a Ryd. (i.e., $\text{Im}\left(\frac{1}{\eta_1}\right)$ is a sharp spike at each resonant frequency which are of the order of Ryd.). With this upper limit 2P we have

$$(a) = \frac{\pi}{P}\int_{K_o}^{2P} \frac{dK}{K}\left[1 - \frac{\omega}{2E} - \frac{1}{2E^2}\left(E\omega - \frac{k^2}{2}\right)\right] + \text{terms with } \omega \rightarrow -\omega$$

$$\cong \frac{\pi}{P}\int_{K_o}^{2P} \frac{dK}{K}\left[1 - \frac{\omega}{E} + \frac{1}{4E^2}(\omega^2 - K^2)\right] + \text{terms with } \omega \rightarrow -\omega$$

(III-50)

The integral in III-50 is elementary; the result is

$$(a) = \frac{\pi}{P}\left[\left(1 - \frac{\omega}{E} + \frac{\omega^2}{4E^2}\right)\ln\left(\frac{2P}{K_o}\right) - \frac{v^2}{2} + \frac{K_o^2}{8E^2}\right] + \text{terms with } \omega \rightarrow -\omega$$

$$= \frac{\pi}{P}\left[\ln\left(\frac{2P}{K_o}\right) - \frac{v^2}{2}\right] + \text{terms with } \omega \rightarrow -\omega \qquad \text{(III-51)}$$

In arriving at III-51 we have made use of the facts that $K_o^2 = (me^2)^2 << E^2 = \left(\frac{M^2}{1-v^2}\right)$*, and that the values of ω that contribute are of the order of the Ryd. Substituting (a) into III-48 we find that

$$(1) = \frac{e^2}{\pi v}\int_0^\infty \text{Im}\left(\frac{1}{\eta_1}\right) d\omega \left[\ln\left(\frac{2P}{K_o}\right) - \frac{v^2}{2}\right] + \text{terms with } \omega \rightarrow -\omega$$

$$= \frac{e^2}{\pi v}\int_0^\infty \text{Im}\left(\frac{1}{\eta_1}\right) d\omega \left[\ln\left(\frac{2P}{K_o}\right)^2 - v^2\right] \qquad \text{(III-52)}$$

In arriving at III-52, we made use of the fact that η_1 is an even function of ω.

Next we calculate the contribution to Γ_s that arises from (2). From III-43 we have (replacing the factors previously withheld)

*For example, taking the incident particle to be a proton and with $E = 10$ M; $(K_o/E)^2 \sim 10^{-12}$.

$$(2) = \frac{e^2}{(2\pi)^4 E} \operatorname{Re} \int d^4k \left(\frac{4\pi}{k^2} \beta_{2\mu\nu} X_{\mu\nu} \right)$$

Upon substituting for β and X from III-39 and III-19 respectively, we find

$$(2) = -\frac{e^2 \omega_p^2}{2\pi^3 E} \operatorname{Re} \int_{K > K_o} d^4k \left(\frac{1}{k^2}\right)\left(\frac{1}{k^2}\right) \frac{2p_\mu p_\nu - k_\mu p_\nu - k_\nu p_\mu + \delta_{\mu\nu}(p\cdot k)}{k^2 - 2p\cdot k + i\epsilon}$$

$$\times \left[\frac{2p_{o\mu}p_{o\nu} + k_\mu p_{o\nu} + k_\nu p_{o\mu} - \delta_{\mu\nu}(p_o \cdot k)'}{k^2 + 2p_o\cdot k + i\epsilon} + \text{terms with } k \rightarrow -k \right] \tag{III-53}$$

In order to evaluate III-53 we note that since in III-53 we are dealing with the ordinary Feynman propagators we may use the well known rule (10) to evaluate the real part of the integral in III-53. The rule says that to obtain the imaginary part of a given Feynman diagram* one imagines that we cut the Feynman diagram in all possible ways. For each cut the contribution to the imaginary part is obtained by replacing each Feynman propagator, $1/(p^2 - M^2)$, that is cut through by $(2\pi)\delta(p^2 - M^2)$; include an overall factor of $-1/2$ and integrate only over positive frequencies.

The Feynman diagram that corresponds to the first term in (2) is shown in figure 7.** We immediately see that the only cut that will give a non-zero contribution to (2) is the one that cuts the particle line and the electron line. For, if we cut the particle line and a photon line, we get zero. Physically, this is because this cut cor-

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*We have factored out a factor i from the expression for the self-energy integral (see III-2 and III-4). Hence we want the real part here.

**We shall see that the second term in (2) gives no contribution.

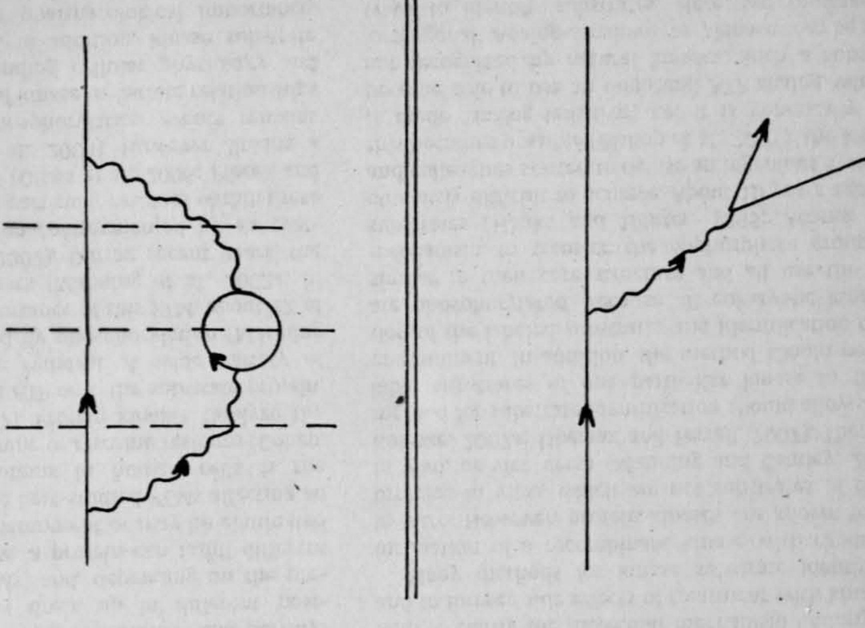

Figure 7. The process of diagram cutting to obtain decay rates from the self energy

responds to a process where a free electron decays into a state of lower energy by emitting a free photon. But this process cannot occur since energy and momentum cannot be conserved simultaneously. Mathematically we get zero for this cut since we get a factor $\delta(k^2)$ which makes the integral vanish. Hence III-53 becomes

$$(2) = \frac{e^2\omega_p^2}{\pi E} \int_{K>K_o} \theta(\omega) \frac{d^4k}{k^4} \Big[\delta(k^2-2p\cdot k)\delta(k^2+2p_o\cdot k)$$

$$\times (2p_\mu p_\nu - k_\mu p_\nu - k_\nu p_\mu + \delta_{\mu\nu} p\cdot k)(2p_{o\mu}p_{o\nu} + k_\mu p_{o\nu} + k_\nu p_{o\mu} - \delta_{\mu\nu} p_o\cdot k)$$

$$+ \text{terms with } k \rightarrow -k\Big] \qquad \text{(III-54)}$$

where $\theta(\omega) = 1$ for $\omega > 0$; $= 0$ for $\omega < 0$. Combining terms we get

$$(2) = \frac{e^2\omega_p^2}{\pi E} \int_{K>K_o} \theta(\omega) \frac{d^4k}{k^4} \Big[\delta(2k\cdot(p+p_o))\delta(k^2+2p_o\cdot k)(4E^2m^2 - 4m^2E\omega$$

$$- 2M^2m\omega + 2m^2\omega^2 - 2m^3\omega) + \text{terms with } k \rightarrow -k\Big] \quad \text{(III-55)}$$

In arriving at III-55 frequent use has been made of the delta functions. First of all we note that the term with $k \rightarrow -k$ is zero. This is so because this term contains factors of $\delta(k^2-2p\cdot k)$ and $\delta(k^2-2p_o\cdot k)$. Physically this term corresponds to the square of the amplitude of a real process where a particle of energy E loses positive energy ω and the atomic electron of energy m (at rest) loses positive energy ω.* Clearly this is impossible. Mathematically we get zero because the

*Note the $\theta(\omega)$ factor insures that only positive ω contribute.

delta functions imply that $\cos\theta$ must be imaginary.

Now let us evaluate the integrals in III-55. We have

$$(2) = \frac{4e^2m^2\omega_p^2E}{\pi}\int_0^\infty d\omega\int_{K_o}^\infty \frac{K^2dK}{K^4}\int_1^{-1}d(\cos\theta)\int_0^{2\pi}d\varphi\,\delta(k^2+2p_o\cdot k)$$

$$\times\,\delta(2k\cdot(p+p_o))(1-\frac{\omega}{E}-\frac{M^2\omega}{2mE^2}+\frac{\omega^2}{2E^2}-\frac{m\omega}{2E^2}) \qquad \text{(III-56)}$$

The first delta function implies that $k^2 = \omega^2 - K^2 = -2m\omega$. It also implies that since $K \geq K_o$ and $K_o \ll m$ that

$$\omega \geq \omega_o \cong K_o^2/2m \qquad \text{(III-57)}$$

The second delta function implies an upper limit to ω. This is because $|\cos\theta| \leq 1$. Now, from the second delta function, we have that $\cos\theta = \omega/Kv - m\omega/PK$. From this, and noting that K is related to ω through the first delta function, it follows that

$$\omega \leq \omega_M = \frac{2mP^2}{m^2+M^2+2mE} \qquad \text{(III-58)}$$

Doing the integrals over φ, K and θ via the delta functions, III-56 becomes

$$(2) = \left(\frac{2\pi Ne^4}{mv}\right)\int_{\omega_o}^{\omega_M}\frac{d\omega}{\omega^2}\left[1-\left(\frac{1}{E}+\frac{M^2}{2mE^2}+\frac{m}{2E^2}\right)\omega+\frac{\omega^2}{2E^2}\right]$$

$$\text{(III-59)}$$

In order to give a physical interpretation of III-59 we proceed as follows. We note that III-59 is the contribution to the decay rate that comes from momentum transfers greater than K_o. We have pointed out before that for momentum transfers greater than K_o the atomic electrons can be treated as being free and at rest. Hence in a collision with the external particle the atomic electron picks up energy and momentum given by the law of conservation of four momentum applied to a particle at rest. That is, we must have $(m+\omega)^2 = K^2 + m^2$, but this is just what the second delta function in III-55 is telling us. The incident free particle of energy E interacts with the atomic electrons losing energy ω after which it is a free particle of energy $E - \omega$ and momentum given by the law of conservation of four momentum; i.e., $(E - \omega)^2 = K^2 + M^2$, but this is just what the first delta function in III-55 is telling us. We also note that since $K > K_o$, or, equivalently, the impact parameters involved are less than the mean separation between the atoms, that the contribution to the decay rate for $K > K_o$ can also be obtained as the decay rate that results from the direct collision between the incident particle and the atomic electrons (assumed to be free and at rest). That is, the incident particle and the electron interact via a virtual photon (amplitude $\sim 1/k^2$). This process is depicted in figure 8. If one applies the Feynman rules to this diagram to obtain the decay rate (i.e., $\Gamma = \frac{2\pi}{(\Pi\, 2E)_{in}(\Pi\, 2E)_{out}} |M|^2$ x density of states) we arrive at III-59.

We can now substitute III-52 and III-59 into III-41 to get Γ_s. However, we really want the contribution to the energy loss that arises from these terms; call this contribution $(\frac{dE}{dx})_S$, we have, from III-1

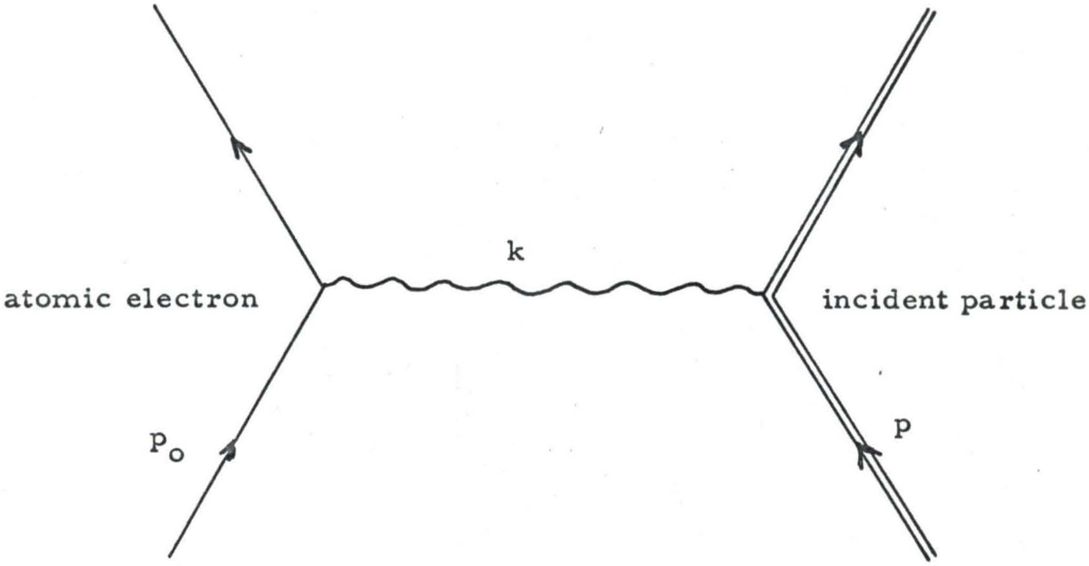

Figure 8. Feynman diagram illustrating the direct collision between the incident particle and the atomic electron

$$\left(\frac{dE}{dx}\right)_S = \int_{\omega > 0} \frac{\omega}{v} \left(\frac{d\Gamma}{d\omega}\right)_S d\omega \qquad \text{(III-60)}$$

Upon substituting III-52 and III-59 into III-40 and then substituting III-40 into III-60 we find

$$\left(\frac{dE}{dx}\right)_S = \frac{2\pi Ne^4}{mv^2} \int_{\omega_o}^{\omega_M} \frac{d\omega}{\omega} \left[1 - \left(\frac{1}{E} + \frac{M^2}{2mE^2} + \frac{m}{2E^2}\right)\omega + \frac{\omega^2}{2E^2}\right]$$

$$+ \frac{e^2}{\pi v}\int_0^{\infty} \text{Im}\left(\frac{1}{\eta_1}\right) d\omega \left[\ln\left(\frac{2P}{K_o}\right)^2 - v^2\right] \qquad \text{(III-61)}$$

In the next section we will combine the results of this and the last section to obtain the total energy loss per length of a charged particle passing through the medium.

E. General Expressions for the Energy Loss

In this section we combine the results of Section C and Section D to obtain general expressions for the energy loss per unit length of a fast charged particle in a medium.

Combining the contributions to the loss from F_R and F_S (equations III-36 and III-61) we obtain

$$\frac{dE}{dx} = \left(\frac{dE}{dx}\right)_R + \left(\frac{dE}{dx}\right)_S$$

$$= -\frac{e^2}{\pi v^2}\int_0^{\infty} \omega\, d\omega \left[\text{Im}\,\eta^{-1} \ln\left(\frac{(K_o v/\omega)^2}{|1 - v^2\eta|}\right) - \text{Re}(v^2 - \eta^{-1})(\pi - \theta)\right]$$

$$+ \frac{2\pi Ne^4}{mv^2}\int_{\omega_o}^{\omega_M} \frac{d\omega}{\omega}\left[1 - \omega\left(\frac{1}{E} + \frac{M^2}{2mE^2} + \frac{m}{2E^2}\right) + \frac{\omega^2}{2E^2}\right] \qquad \text{(III-62)}$$

where η is given in II-31. Call the first integral in III-62 A and the second B. For the moment let us concentrate on B. Upon performing the ω integral we find

$$B = \frac{2\pi Ne^4}{mv^2}\left[\ln\left(\frac{\omega_M}{\omega_o}\right) - \frac{\omega_M}{E} + \frac{\omega_M M^2}{2mE^2} + \frac{m\omega_M}{2E^2} + \frac{\omega_M^2}{4E^2}\right]^* \qquad \text{(III-63)}$$

where, from III-57, $\omega_o = K_o^2/2m$, and ω_M, the maximum energy transfer, is given in III-58. In obtaining III-63 we have neglected terms of the order ω_o/E compared with ω_M/E, etc. Eq. III-63 is the contribution to the energy loss from momentum transfers greater than K_o ($= \sqrt{2m\omega_o}$). From now on let us take the case $M >> m$. For very large momenta ($P >> M^2/m$) ω_M becomes

$$\omega_M \cong E \qquad \text{for } P >> M^2/m^{**} \qquad \text{(III-64a)}$$

On the other hand, if $M >> m$, and if the condition $P << M^2$ m is satisfied, then ω_M becomes

$$\omega_M = \frac{2mv^2}{1 - v^2} \qquad \text{for } P << M^2/m^{***} \qquad \text{(III-64b)}$$

As an example, let us take the case given in III-64a. In this case III-63 becomes

$$B = \frac{2\pi Ne^4}{mv^2}\left[\ln\left(\frac{2mE}{K_o^2}\right) - \frac{3}{4}\right] \qquad \text{(III-65)}$$

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*For most cases of interest the third and fourth terms in the braces are negligible.

**For muons this condition states $P >> 200\ M_\mu$.

***For muons this condition states $P << 200\ M_\mu$.

where we have used $\omega_o = K_o^2/2m$ in obtaining III-65.

Another case of interest is where one asks for the energy loss where the atomic electron picks up energy ω less than some value ω', where $\omega' << M$. This case is of interest experimentally (for example (11)). In this case the upper limit on the integral in B is ω' (instead of ω_M) and we find (dropping terms of order ω'/E)

$$B = \frac{2\pi Ne^4}{mv^2} \ln \left(\frac{2m\omega'}{K_o^2} \right) \tag{III-66}$$

Now let us concentrate on the first term in III-62. Call this term A; we have

$$A = -\frac{e^2}{\pi v^2} \int_0^\infty \omega \, d\omega \quad \mathrm{Im}\, \eta^{-1} \ln \left(\frac{(K_o v/\omega^2)}{|1 - v^2 \eta|} \right)$$

$$- \mathrm{Re}\, (v^2 - \eta^{-1})(\pi - \theta) \tag{III-67}$$

This expression is exact in the sense that we have not made any assumption about the relative size of $\mathrm{Re}\,\eta$ and $\mathrm{Im}\,\eta$ in deriving it. The only property of η that we have used is that it is only a function of ω. Now, as we shall see, the frequencies that contribute to the integral giving A are given at the pole of η. In our case of gases, $\mathrm{Im}\,\eta$ is small (order $\gamma\, \mathrm{Ryd}/\omega_p^2 \sim 10^{-3}$, for $\gamma \sim 10^{-7}$ Ryd), so we can expand the arctan that gives θ. From equations III-33 through III-35 we find

$$\mathrm{Re}(v^2 - \eta^{-1})(\pi - \theta) = v^2\, \mathrm{Im}\, \eta^{-1} \left(\mathrm{Re}\, \eta + \frac{v^2 (\mathrm{Im}\, \eta_1)^2}{1 - v^2 \mathrm{Re}\, \eta}\right) \quad \text{for } 1 - v^2 \mathrm{Re}\, \eta > 0$$

$$\cong v^2 \mathrm{Im}\, \eta^{-1} (\mathrm{Re}\, \eta) \quad \text{for } 1 - v^2 \mathrm{Re}\, \eta > 0 \tag{III-68a}$$

and

$$\mathrm{Re}(v^2 - \eta^{-1})(\pi-\theta) = v^2 \mathrm{Im}\,\eta^{-1}(\mathrm{Re}\,\eta + \frac{v^2(\mathrm{Im}\,\eta)^2}{1-v^2\mathrm{Re}\,\eta}) + \pi v^2(1 - \frac{1}{v^2}\mathrm{Re}\,\eta^{-1})$$

$$\text{for } 1 - v^2 \mathrm{Re}\,\eta < 0$$

$$\cong v^2 \mathrm{Im}\,\eta^{-1}(\mathrm{Re}\,\eta) + \pi v^2(1 - \frac{1}{v^2}\mathrm{Re}\,\eta^{-1})$$

$$\text{for } 1 - v^2 \mathrm{Re}\,\eta < 0 \qquad \text{(III-68b)}$$

We see from III-68 that for $1 - v^2 \mathrm{Re}\,\eta < 0$ (i.e., for the velocity of the particle greater than the phase velocity of light in the medium) we obtain an extra term to the energy loss, namely $\pi v^2(1 - \frac{1}{v^2}\mathrm{Re}\,\eta^{-1})$. This is the well known Cerenkov radiation term (12). Substituting III-68 into III-67 we obtain

$$A = \frac{-e^2}{\pi v^2}\int_0^\infty \omega\, d\omega\, \mathrm{Im}\,\eta^{-1}\left[\ln\left(\frac{(K_o v\ \omega)^2}{|1-v^2\eta|}\right) - v^2 \mathrm{Re}\,\eta\right]$$

$$+ \frac{e^2}{\pi v^2}\int' \omega\, d\omega \left(v^2 - \frac{\mathrm{Re}\,\eta}{|\eta|^2}\right) \qquad \text{(III-69)}$$

where the prime on the last integral means that we integrate only over those frequencies such that $1 - v^2 \mathrm{Re}\,\eta < 0$. The first term in III-69 is the contribution to the energy loss due to the excitation of atoms of the medium for momentum transfers less than K_o (or equivalently; from impact parameters greater than $1/K_o$). The second term in III-69 is the Cerenkov contribution to the loss.

Focusing our attention on the first term in III-69, we find from II-31 that near each resonance

$$\text{Im}\,\eta^{-1} = \frac{\text{Im}\,\eta}{|\eta|^2} = -\frac{\gamma_n \omega_n \omega_p^2}{(\omega^2 - \bar{\omega}_n^2)^2 + \gamma_n^2 \omega_n^2}$$

where

$$\bar{\omega}_n^2 = \omega_n^2 + \left(\frac{f_n}{R_n}\right) \omega_p^2 \qquad \text{(III-70)}$$

and

$$R_n = 1 + r_n$$

$$\approx 1 \text{ (see equation II-50)}$$

From III-69 and III-70 we see that the absorptions which give rise to the energy loss are shifted from the atomic frequencies. This shift, which is proportional to ω_p^2, is one of the effects of the medium on the energy loss. Physically, the loss is divided between the individual excitations of the atoms and a collective loss proportional to ω_p^2 (i.e., proportional to N). That is, part of the energy transfer goes into the excitation of collective oscillations in the medium. In the case of gases, $\omega_p^2 \sim 10^{-4}\,\omega_n^2$, and the collective loss is small compared with the individual excitation loss.

We wish to perform the integral in the first term of III-69. We note that away from the immediate neighborhood of $\bar{\omega}_n$, $\text{Im}\,\eta^{-1}$ is practically zero while at $\omega = \omega_n$, $\text{Im}\,\eta^{-1}$ has a sharp maximum. Also, near each resonance, the terms in braces in III-69 are slowly varying functions of ω so we may substitute $\bar{\omega}_n$ for ω in these terms. The resulting integral, near the n'th resonance, becomes

$$\int_a^b \omega\, d\omega\, \text{Im}\,\eta^{-1} = -\,\omega_p^2 \frac{\pi}{2} f_n \qquad \text{(III-71)}$$

where the interval a-b contains the resonance frequency and may be extended to $\pm\infty$ since $\mathrm{Im}\,\eta^{-1}$ vanishes away from the resonance.

Incorporating these results, we find

$$A = \frac{2\pi Ne^4}{mv^2}\sum_n f_n\left[\ln\left(\frac{(K_o\frac{v}{\omega_n})^2}{|1-v^2\eta(\overline{\omega}_n)|}\right) - v^2 \mathrm{Re}\,\eta(\overline{\omega}_n)\right]$$

$$+ \text{ Cerenkov term} \qquad \text{(III-72)}$$

where

$$|1-v^2\eta(\overline{\omega}_n)| = +[(1-v^2\mathrm{Re}\,\eta(\overline{\omega}_n))^2 + (v^2\mathrm{Im}\,\eta(\overline{\omega}_n))^2]^{1/2} \qquad \text{(III-73)}$$

Now, by combining the expressions for A and B, we obtain an explicit expression for the energy loss. For the total loss ($\omega \leq \omega_m$) we find

$$\left(\frac{dE}{dX}\right)_T = \frac{2\pi Ne^4}{mv^2}\left[\sum_n f_n\left[\ln\left(\frac{2m\,\omega_M(v/\overline{\omega}_n)^2}{|1-v^2\eta(\overline{\omega}_n)|}\right) - v^2\mathrm{Re}\,\eta(\omega_n)\right]\right.$$

$$\left. - \frac{\omega_M}{E} + \frac{\omega_M^2}{4E^2}\right] + \text{ Cerenkov term}. \qquad \text{(III-74)}$$

For ultra relativistic incident particle energies III-74 reduces to

$$\left(\frac{dE}{dX}\right)_T = \frac{2\pi Ne^4}{m}\left[\sum_n f_n\left[\ln\left(\frac{2mE/\overline{\omega}_n^2}{|1-\eta(\overline{\omega}_n)|}\right) - \mathrm{Re}\,\eta(\overline{\omega}_n)\right] - \frac{3}{4}\right]$$

$$+ \text{ Cerenkov term}. \qquad \text{(III-75)}$$

For the energy loss with energy transfer less than some energy ω' we find

$$\left(\frac{dE}{dX}\right)_{<\omega'} = \frac{2\pi Ne^4}{mv^2}\sum_n f_n\left[\ln\left(\frac{2m\omega'/\overline{\omega}_n^2}{|1-v^2\eta(\overline{\omega}_n)|}\right) - v^2 \mathrm{Re}\,\eta(\overline{\omega}_n)\right]. \quad \text{(III-76)}$$

From equation III-76 we see that for the case of a rare gas, where Cerenkov radiation is not possible except at extremely high incident energies, that the energy loss of energy transfers less than ω' remains finite. In older theories (13) we would have, instead of $1 - v^2\eta$, $1 - v^2$ in the ln term. That is, in these older theories the energy loss due to excitation and ionization for energy transfers less than ω' diverges for $v \to 1$. Here we obtain a plateau in the energy loss. The physical reason for this plateau is that the field at large distances from the particles trajectory remains finite as opposed to varying as $(1 - v^2)^{-1}$ as in vacuum. The field at large distances can be thought of as being modified due to the passive scatterings of the photon field by the atoms of the medium. These rescatterings cause a destructive interference limiting the range of action.

From III-75 we see that the contribution to the total loss due to excitation-ionization by an ultrarelativistic particle is proportional to $\ln(E)$ which diverges as $E \to \infty$. This divergence is due to collisions involving large momentum transfers (i.e., small impact parameters). Now as the incident particle becomes more and more relativistic it can transfer more and more energy to the atomic electron. At extreme relativistic energies the particle can transfer all its kinetic energy to an atomic electron ($\omega_M = E$).

Budini (14) has, by semi-classical methods, calculated the energy loss of relativistic particles in a medium. Specifically he calculates the energy loss due to collisions occurring at distances greater than some impact parameter ρ. Budini makes a classical calculation of the Poynting vector at a distance ρ from the path of the incident particle. From the Poynting vector he obtains the photon spectrum by dividing by $\hbar\omega$ (i.e., the number of photons of energy ω emerging from a cylinder of radius ρ about the path of the incident particle). To get the loss Budini then multiplies the photon spectrum by the atomic photo absorption cross section and integrates over all positive frequencies. Budini then combines this result with the loss for smaller impact parameters (which he obtains from the Bethe-Bloch theory (13)) to get the energy loss for energy transfers less than some value ω'. Budini obtains an expression that is the same as equation III-76. He then goes on to show that the theoretical results agree well with experiment. Budini also discusses the contribution to the energy loss due to Cerenkov radiation. The interested reader is referred to Budini's paper for more details.

Tidman (1) has given a non-phenomenological quantum mechanical treatment of the energy loss due to excitations occurring at distances greater than $1/K_o$ from the path of the incident particle. His method is quite different from ours. He uses the hamiltonian formalism to compute the decay rate to the second order (in e^2) in perturbation theory. Tidman uses an unmodified coulomb hamiltonian (i.e., $\sim 1/K^2$). For the transverse interaction Tidman performs a canonical transformation on the usual transverse interaction term to obtain a modified interaction term. This modified term describes the interaction of currents with photons whose energy-momentum relation is $\omega = K/n$, where n is the index of refraction. In Tidman's theory n is real. Tidman also assumes that the characteristic absorptions occur at the atomic frequencies. Tidman obtains a loss proportional to

$$\left[\ln (K_o\omega/v)^2 + \frac{1}{n(\omega_n)^4} \ln \frac{1}{(1 - v^2 n(\omega_n)^2)} - v^2/n(\omega_n)^2\right]$$

In order to make a comparison with Tidman's work we consider the case where $\text{Im}\,\eta$ is a series of infinitely narrow lines which are located at the atomic frequencies (i.e., a series of delta functions). In this case III-69 yields an energy loss proportional to

$$\left[\frac{1}{n(\omega_n)^4}\left[\ln (K_o\omega_n/v)^2 + \ln \frac{1}{(1 - v^2 n(\omega_n)^2)}\right] - v^2/n(\omega_n)^2\right]$$

This result was obtained with the term $|\eta|$ in the denominator of III-69 replaced by $n^2 \equiv \text{Re}\,\eta$. The difference between our result and Tidman's is in the coefficient of the coulomb term (i.e., the $\ln (K_o\omega_n/v)^2$ term). The reason for this is that Tidman uses the direct coulomb interaction $(1/K^2)$ as compared to our modified coulomb interaction $(1/K^2\eta)$.

The advantage in obtaining the energy loss from the self energy is that we are able to handle in a straightforward manner a continuous distribution for $\text{Im}\,\eta$. The loss so obtained contains all the contributions to the loss (which in this case is the excitation ionization loss and the Cerenkov loss). In particular we have seen that the excitation loss is a combination of individual atomic excitations (single particle effects) and a coherent collective atomic excitation loss.

In deriving the energy loss we have assumed that the incident particle has spin 1/2, is distinguishable from the atomic electrons (no exchange effects), and has no anomalous moment. That is, the energy loss expression is ideal for the muon as the incident particle. The effects due to other incident particles will modify only the high momentum transfer part of the energy loss expression, that is, the expression for B (equation III-63) will be different for different incident particles. We have pointed out that the expression for B is identical with the energy loss as computed from the direct collision between the incident particle and the atomic electrons (see the discussion following equation III-59).

Rossi (15) has tabulated the differential cross section for various incident particles and electrons. We find that the modifications of our result for various incident particles are:

1) <u>ELECTRONS:</u> When the energy of the primary electron is large compared with its rest mass the 2nd square brackets in III-62 are replaced by

$$\left[\frac{1}{(E - \omega)^2} \left(1 - \omega/E + (\omega/E)^2\right)^2 \right] \qquad \text{(III-77)}$$

and $M \equiv m$. The modification of the total energy loss is that the $- 3/4$ in equation III-75 is replaced by $(9/8 - \ln 4)$.

2) POSITRONS: When the energy of the positron is large compared with its rest mass the square brackets in III-62 are replaced by

$$\left[(1 - \omega/E + (\omega/E)^2\right]^2 \qquad \text{(III-78)}$$

and $M \equiv m$. The modification of the total loss is that the $-3/4$ in equation III-75 is replaced by $-11/12$.

3) PARTICLES OF MASS M AND SPIN O: The square brackets in equation III-62 are replaced by

$$\left[1 - v^2(\omega/\omega_M)\right] \qquad \text{(III-79)}$$

where ω_M, the maximum energy transfer, is given in equation III-58. The modification of the total loss is that the terms outside of the inner square bracket in equation III-74 are replaced by $-v^2$. For ultrarelativistic incident energies the $-3/4$ in equation III-75 is replaced by -1.

4) PROTONS: Pauli, in his review article (16), has given the cross section for the scattering of electrons by particles of spin 1/2, and magnetic moment $\mu \neq 1$ (the proton's moment is 2.79). We find, that the modification of our results due to the anomalous magnetic moment of the proton is that, for incident energies much greater than the order of $10^5\ M_P$ the energy loss is dominated by the contribution of the anomalous moment, and is given by

$$\left(\frac{dE}{dX}\right)_T = \frac{2\pi N e^4}{m}\left[\frac{(\mu-1)^2}{4}\ (m/M)(E/M)\right] \qquad \text{(III-80)}$$

That is, the energy loss depends linearly on the incident energy.

IV. SOME MISCELLANEOUS TOPICS

A. Effect of Finite Temperatures on the Photon Propagator for Small K

In Chapter II, Section D, we calculated the photon propagator for the atomic system for $Ka_o << 1$ where we assumed that the system was at zero temperature (e.g., all of the atoms were assumed to be in the ground state). In this section we will remove the restriction of zero temperature and calculate the photon propagator. We will consider the case of small K ($Ka_o << 1$). The case of large K ($Ka_o >> 1$) was treated previously (see Chapter II, Section I).

In the case of finite temperature, the initial state of the atom may be any one of the states of the atom (ground state plus excited states). We assume that the probability that the atom is in the m'th state is given by Maxwell Boltzmann statistics. That is, we take that the normalized probability P_m for the atom to be in the m'th state is

$$P_m = \frac{1}{Q} e^{-E_m/\kappa T} \tag{IV-1}$$

where

$$Q = \sum_n e^{-E_n/\kappa T} \tag{IV-2}$$

Now, as we have seen in Chapter II, the photon propagator in the medium may be broken down into two parts: the coulomb propagator $\mathcal{A}_c$ and the transverse propagator $\mathcal{A}_T$. We have (see II-34 and II-31)

$$\mathcal{A}_c = \frac{4\pi}{K^2\eta} = \frac{4\pi}{K^2(1 + \frac{k^2}{K^2}\beta_{44})} \tag{IV-3a}$$

and

$$\mathcal{A}_T = \frac{4\pi}{\omega^2\eta - K^2} = \frac{4\pi}{\omega^2(1 + \frac{k^2}{\omega^2}\beta_{tr.,tr.}) - K^2} \qquad \text{(IV-3b)}$$

We remark that, because of the spherical symmetry of the atoms (after orbital angular momentum states have been summed over), $\beta_{tr.,tr} = \beta_{33}$, and because of current conservation, $\beta_{33} = \omega^2/K^2\,\beta_{44}$. That is, $\beta_{tr.,tr.} = \omega^2/K^2\,\beta_{44}$. Therefore, the photon propagator is characterized by one quantity which we take to be β_{44}.

In the case of finite temperature, β_{44} is given by:

$$\beta_{44} = -\frac{4\pi N}{k^2}\sum_n\Big\{\sum_m\Big[(-iez_{nm}K)\Big(\frac{1}{(\omega+E_m) - E_n - i\epsilon} + \frac{1}{(\omega+E_m) - (E_n+2\omega) - i\epsilon}\Big)(iez_{nm}K)\Big]P_m\Big\} \qquad \text{(IV-4)}$$

This is because β_{44} is $-4\pi/k^2$ times the amplitude per unit volume that a coulomb photon interacts with an atom in an arbitrary initial state m causing the atom to make a transition to an arbitrary intermediate state n; the atom propagating in state n subsequently making a transition back to state m by emitting a coulomb photon. Now, P_m is the probability that the atom is initially in state m; $iez_{nm}K$ is the amplitude, in the dipole approximation, that the atom interacts with the coulomb field making a transition from state m to n. The amplitude to propagate in the intermediate state is $\frac{1}{E_{initial} - E_{int.} - i\epsilon}$; there are two possibilities, $E_{initial} = \omega + E_n$, $E_{int.} = E_n$ (see figure 1b), and

$E_{initial} = \omega + E_n$, $E_{int.} = E_n + 2\omega$ (see figure 1c). The amplitude for the atom to make a transition from state m to state n by emitting a coulomb photon is $-iez^{+}_{nm}$. We sum over all initial states m and all intermediate states n to include all possible initial and intermediate states. Since we are assuming the atoms to be independent, the amplitude per unit volume is N times the amplitude per atom, hence the factor N.

Substituting P_m from IV-1 and combining terms, IV-4 becomes

$$\beta_{44} = \frac{\omega_p^2 K^2}{k^2} \sum_m \frac{e^{-E_m/\kappa T}}{Q} \sum_n \frac{f_{nm}}{\omega_{nm}^2 - \omega^2}$$

where

$$\omega_{nm} = E_n - E_m \qquad \text{(IV-5)}$$

$$f_{nm} = 2m\omega_{nm}|z_{nm}|^2 \;*$$

and

$$Q = \sum_n e^{-E_n/\kappa T}$$

In arriving at IV-5 we have made use of the Thomas-Reiche sum rule which states that $\sum_n f_{nm} = 1$. So far we have assumed that the energy eigenvalues of the atom are discrete. Actually all of the energy levels except the ground state are spread out due to the finite lifetimes of the excited states. In order to take the finite lifetime into account we proceed in exactly the same way as is done in Appendix A with the only difference being that here, since the initial and intermediate states

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*Note: $f_{nm} \equiv 0$ and $f_{nm} = -f_{mn}$.

are in general not the ground state (i.e., they both have an energy spread), we replace E_m and E_n by $\mathcal{E}_m$ and $\mathcal{E}_n$ respectively and average all $\mathcal{E}_m$ and $\mathcal{E}_n$ with weights $G(\mathcal{E}_m)$ and $G(\mathcal{E}_n)$ respectively. If we choose the form A-4 for the function G and then perform the resulting integrals in a similar manner as is done in Appendix A the result is:

$$\beta_{44} = \frac{\omega_p^2 K^2}{k^2} \sum_m \frac{e^{-E_m/\kappa T}}{Q} \left(\sum_n \frac{f_{nm}\Omega_{nm}/\omega_{nm}}{\Omega_{nm}^2 - \omega^2} \right)$$

where

$$\Omega_{nm} = \omega_{nm} - i\gamma_{nm}/2 \qquad \text{(IV-6)}$$

and

$$\gamma_{nm} = \gamma_n + \gamma_m {}^{*}$$

Since

$$\gamma_n << \omega_n$$

we have

$$\beta_{44} \cong \frac{\omega_p^2 K^2}{k^2} \sum_m \frac{e^{-E_m/\kappa T}}{Q} \left(\sum_n \frac{f_{nm}}{\Omega_{nm}^2 - \omega^2} \right)$$

which may be written as

$$\beta_{44} = \frac{\omega_p^2 K^2}{Qk^2} \sum_{\substack{n,m \\ n>m}} \left(e^{-E_m/\kappa T} - e^{-E_n/\kappa T} \right) \left(\frac{f_{nm}}{\Omega_{nm}^2 - \omega^2} \right) \qquad \text{(IV-7)}$$

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*We note that IV-6 cannot be obtained from IV-4 by replacing E_n by $E_n - i\gamma_n/2$, etc. If these replacements are made one obtains the difference rather than the sum of the line breadths.

Now, from IV-3 and IV-7 we find

$$\eta = 1 + \frac{k^2}{K^2}\beta_{44}$$

$$= 1 + \frac{\omega_p^2}{Q}\sum_{\substack{n,m \\ n>m}}\left(e^{-E_m/\kappa T} - e^{-E_n/\kappa T}\right)\frac{f_{nm}}{\Omega_{nm}^2 - \omega^2} \qquad \text{(IV-8)}$$

The sum sign in IV-8 means to sum over the discrete energy levels and an integral over the continuous levels. In general, IV-8 cannot be reduced to a simpler form.

Now, as an example, we consider a simplified case. Consider a two state system. Calling $E_o = 0$, $E_1 = \omega$, and the half life of the excited state γ, we find (calling $\omega_\gamma = \omega_1 - i\gamma/2$)

$$Q = 1 + e^{-\omega_1/\kappa T}$$

and

$$\eta = 1 + \left(\frac{1 - e^{-\omega_1/\kappa T}}{1 + e^{-\omega_1/\kappa T}}\right)\frac{\omega_p^2}{\omega_\gamma^2 - \omega^2}$$

$$= 1 + \frac{\omega_p^2}{\omega_\gamma^2 - \omega^2}\tanh\frac{\omega_1}{2\kappa T} \qquad \text{(IV-9)}$$

We see from IV-9 that, as $T \to 0$, $\eta \to 1 + \frac{\omega_p^2}{\omega_\gamma^2 - \omega^2}$, and as $T \to \infty$, $\eta \to 1$.* The fact that $\eta \to 1$ as $T \to \infty$ can be understood physically as follows. For very high temperatures each state has equal probability of being occupied. Now, being that excited states are occupied, they will decay incoherently to lower states. This incoherence will destroy the coherent effect of a photon exciting the atom with its subsequently

*Also from the general formula IV-8 for η it can be seen that as $T \to \infty$, $\eta \to 1$.

de-exciting. From IV-8 we get that for $\kappa T >> E_n$

$$\eta = 1 + \frac{\omega_p^2}{\kappa T \Omega} \sum_{\substack{n,m \\ n>m}} \frac{\omega_{nm} f_{nm}}{\Omega_{nm}^2 - \omega^2} \qquad \text{(IV-10)}$$

That is, for high temperatures (compared with the Rydberg*) $(\eta - 1)$ is inversely proportional to the temperature.

For the two state system we find that the modification of the coulomb pole due to finite temperatures is that the real part of the pole is located at

$$\omega^2 = \omega_1^2 + \omega_p^2 \tanh(\omega_1/\kappa T) \, . \qquad \text{(IV-11)}$$

We see from IV-11 that for $\kappa T >> \omega_1$, $\omega = \omega_1$. The reason for this is connected with the fact that at high temperatures any coherent collective effects are washed out by the incoherence due to the high temperature (i.e., at high temperatures the atoms are being excited and de-excited randomly).

Finally, we note that the effect of a finite temperature on the energy loss of a fast charged particle passing through the medium is that for η in equation III-72 we use equation IV-8 instead of equation II-31.

B. Damping

In this section we indicate briefly how damping arises in the case of finite temperatures. The system considered here is an electron gas (with a positive background charge to give overall charge neutrality). We consider only the longitudinal case since the method is easily applied to the transverse case.

*This corresponds to $T >> 10^5\ ^{\circ}K$.

What we want to calculate is the absorption probability of free waves in an electron gas. The dispersion formula that determines the energy-momentum relation of the waves arise at the zeros of the dielectric function. In Chapter II, Section I, we have discussed the real part (dispersive part) of the dispersion formula. Now, we consider the imaginary part (absorptive part) of the dispersion relation.

As we have seen, poles of the coulomb propagator arise when

$$\eta = 1 + \frac{k^2}{K^2} \beta_{44} = 0 \tag{IV-12}$$

where

$$\beta_{\mu\nu} = -\frac{\omega_p^2}{Nk^2} \sum_i \left(\frac{m}{E_i}\right) \left[\frac{2p_{i\mu}p_{i\nu} + k_\mu p_{i\nu} + k_\nu p_{i\mu} - \delta_{\mu\nu}(p_i \cdot k)}{k^2 + 2p_i \cdot k + i\epsilon} + \text{terms with } k \to -k \right] \tag{IV-13}$$

In IV-13, we have included the $i\epsilon$'s that arise from the prescription that the mass of the particle has an infinitesimal negative imaginary part. From IV-13 we see that for zero temperature ($p_{i\mu} = m\delta_{\mu 4}$), that Im β is proportional to delta functions ($1/x \pm i\epsilon = \text{P.V.}/x \mp i\pi\delta(x)$). For finite temperatures, where there is a continuous distribution of p_i, we shall see that we obtain a finite imaginary part. Both terms in the brackets of IV-13 contribute to the imaginary part of β_{44}. Specifically,

$$\text{Im}\, \beta_{44} = \frac{\pi\omega_p^2}{Nk^2} \sum_i \left(\frac{m}{E_i}\right) \left[(2E_i^2 + 2E_i\omega + k^2/2)\delta(k^2 + 2p_i \cdot k) + \text{terms with } k \to -k \right] \tag{IV-14}$$

For a continuous distribution of p_i,

$$\frac{1}{N}\sum_i \rightarrow \int \frac{d^3\vec{P}}{(2\pi)^3} f(P) \qquad \text{(IV-15)}$$

where f(P) is the distributive function* for P (Maxwell-Boltzmann, Fermi-Dirac, or what have you). Substituting IV-15 into IV-14 and then substituting IV-14 into IV-12, we find

$$\text{Im}\ \eta = \frac{\pi\omega_p^2}{K^2}\int \frac{d^3\vec{P}}{(2\pi)^3} f(P)\left(\frac{m}{E}\right)\Big[(2E^2 + 2E\omega + k^2/2)\delta(k^2 + 2p\cdot k) + \text{terms with } k \rightarrow -k\Big] \qquad \text{(IV-16)}$$

where

$$E = (P^2 + m^2)^{1/2} .$$

We now discuss the interpretation of the two terms in the square brackets of IV-16. The first term is proportional to the square of the amplitude (probability) that a wave of momentum k is absorbed by an electron of momentum p, the electrons then having momentum p+k (absorption of a wave). This process is illustrated in figure 9a. The second term is proportional to the square of the amplitude that a wave of momentum k stimulates an electron of momentum p to emit a wave of momentum k, the electrons having momentum p-k (stimulated emission of a wave). This process is illustrated in figure 9b. In both cases the initial states are the same (a wave of momentum k). However, the final states are different (an extra electron of momentum p+k; two waves of momentum k, and an extra electron of momentum p - k). In both cases, energy and momentum can be conserved.**

*We normalize f such that $\int f(P)/(2\pi)^3 d^3\vec{P} = 1$.

**This would not be true at zero temperature; in this case stimulated emission is not possible.

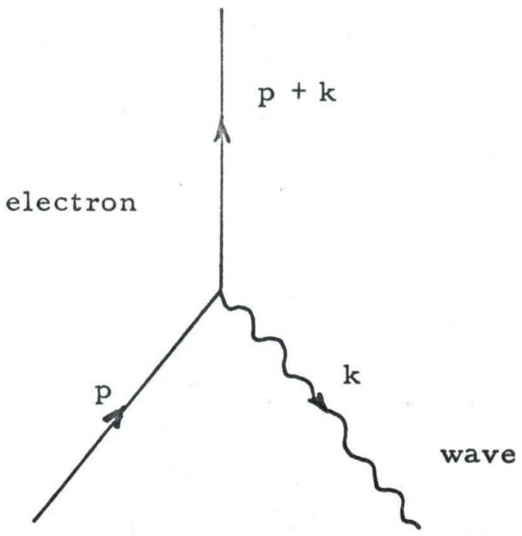

Figure 9a. Absorption of a Wave

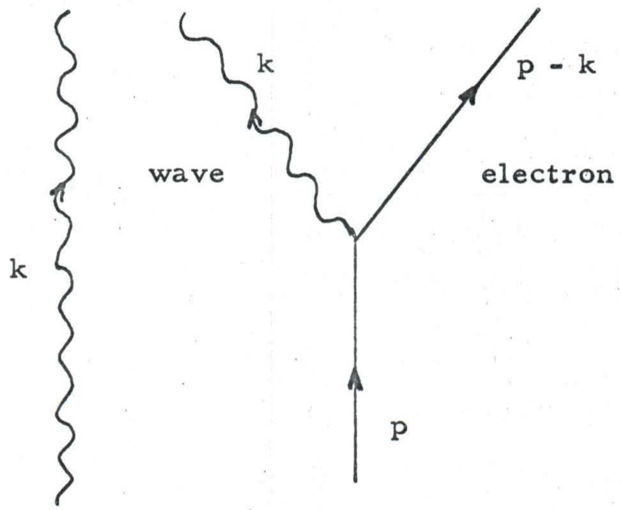

Figure 9b. Stimulated Emission of a Wave

That is, the initial state (consisting of a free wave) may go into two possible final states, as is illustrated in figures 9a, b. From IV-16, we see that the probabilities for these two processes are to be added together. That is, $\mathrm{Im}\ \eta$ is proportional to the total probability that the initial state changes with time. Now, what we want is the probability for the effective absorption (absorption of energy) of a wave. The probability of effective absorption is the difference between the probability of absorption and the probability of stimulated emission. That is, the effective absorption is obtained by taking the difference of the two terms in the brackets of IV-16 (rather than their sum).

Call equation IV-16 with the plus sign between the two terms replaced by a minus, $\mathrm{Im}\ \eta_{\mathrm{eff.}}$ We will use $\mathrm{Im}\ \eta_{\mathrm{eff.}}$ in the dispersion formula $(\eta = \mathrm{Re}\ \eta + i\ \mathrm{Im}\ \eta_{\mathrm{eff.}} = 0)$ to find the effective damping of waves. We have

$$\mathrm{Im}\ \eta_{\mathrm{eff.}} = \frac{\pi\omega_p^2}{K^2} \int \frac{d^3\vec{P}}{(2\pi)^3}\ f(P) \left(\frac{m}{E}\right) \Big[(2E^2 + 2E\omega + k^2/2)\,\delta(k^2 + 2p\cdot k) - \text{terms with } k \to -k \Big] \qquad \text{(IV-17)}$$

We want to find the damping of free waves of frequency $\omega_{Re}(K)$* or, equivalently, the imaginary part of the energy (frequency) near ω_{Re}. By expanding η $(= \mathrm{Re}\ \eta + i\ \mathrm{Im}\ \eta_{\mathrm{eff.}})$ about ω_{Re} we find

$$\omega_{Im} = \left. \frac{\mathrm{Im}\ \eta_{\mathrm{eff.}}}{\frac{d}{d\omega}(\mathrm{Re}\ \eta)} \right|_{\omega = \omega_{Re}} \qquad \text{(IV-18)}$$

---

*$\omega_{Re}(K)$ is determined from the equation $\mathrm{Re}\ \eta = 0$.

We note that equation IV-17 is general in the sense that it holds for arbitrary values of the four momentum and for arbitrary distribution functions.

Now, let us consider, as a special case, the Maxwell-Boltzmann distribution function (17). Also let us consider the two extreme limits, the non-relativistic limit, and the ultra-relativistic limit.

First we consider the non-relativistic limit. In this case we have

$$f(P) = \frac{(2\pi)^3}{(2\pi m \kappa T)^{3/2}} e^{-P^2/2m\kappa T}$$

$$\kappa T <<< m \tag{IV-19}$$

and

$$E = m$$

By equating the real part of η to zero we have found (see equation II-58)

$$\begin{aligned} \omega_{Re}^2(K) &= \omega_p^2 + K^2 < v^2 > \\ &= \omega_p^2 \qquad (\text{for } K^2 < v^2 > << \omega_p^2) \end{aligned} \tag{IV-20}$$

This is the relation between ω and K for free waves in the non-relativistic limit. From IV-17 and IV-29 we find

$$\text{Im } \eta_{eff.} = \frac{\pi\omega_p^2}{K^2} \int \frac{d^3\vec{P}}{(2\pi)^3} f(P)(2m^2) \left[\delta(k^2 + 2p\cdot k) - \delta(k^2 - 2p\cdot k) \right] \tag{IV-21}$$

Now,

$$k^2 \pm 2p \cdot k = k^2 \pm 2m(\omega - Kv \cos\theta)$$

Since $|\cos\theta| \leq 1$ and noting that we want $\text{Im } \eta_{eff.}$ for $\omega = \omega_{Re} \approx \omega_P << m$, we find (using IV-19)

$$\text{Im}\ \eta_{\text{eff.}} \cong -\frac{\omega_p^2}{K^3}\left(\frac{2\pi m^3}{\kappa T}\right)^{1/2} e^{-(m\omega_p^2)/(2\kappa T K^2)} \sinh\left(\frac{\omega_p}{2\kappa T}\right)$$

$$= -\left(\frac{\pi}{2}\right)^{1/2} \frac{1}{K^3}\left(\frac{\omega_p^2 m}{\kappa T}\right)^{3/2} e^{-(m\omega_p^2)/(2\kappa T K^2)}$$

$$(\text{for}\ \omega_p << \kappa T) \qquad \text{(IV-22)}$$

From IV-18 and IV-20, we find, for $\omega_p << \kappa T$, that

$$\omega_{\text{Im}} = -\omega_p\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{m\omega_p^2}{\kappa T K^2}\right)^{3/2} e^{-(m\omega_p^2)/(2\kappa T K^2)} \qquad \text{(IV-23)}$$

This is called Landau Damping (18). So much for the non-relativistic case.

Now we consider the ultra-relativistic limit. In this case, $\kappa T >> m$, $E = P >> m$, and

$$f(P)/(2\pi)^3 = 1/8\pi(\kappa T)^3 e^{-P/\kappa T} \qquad \text{(IV-24)}$$

For completeness, we first discuss briefly the real part of η in this limit. From II-57 we find

$$\text{Re}\ \eta = 1 - a \qquad \text{(IV-25)}$$

where

$$a = \frac{m\omega_p^2}{4(\kappa T)^3\omega^2}\int_m^\infty P\,dP\, e^{-P/\kappa T} H(P) \qquad \text{(IV-26)}$$

$$H(P) = \int_{-1}^{+1} \frac{(1-x^2)\,dx}{(1-ax)^2 - b^2} \qquad \text{(IV-27)}$$

$$a = K/\omega$$

and (IV-28)

$$b = k^2/2P\omega$$

The integral in IV-27 must be taken in the sense of the principle value. We have set the lower limit of the P integral in IV-26 arbitrarily at the order of the rest mass m. The integral in IV-27 is elementary. The result is

$$H = -\frac{1}{2}a^2\left[\frac{1-a^2}{b} + b\right]F - \frac{G}{a^2} - \frac{2}{a} \qquad \text{(IV-29)}$$

where

$$F = \ln\left|\frac{1 + b/(1-a)}{1 - b/(1-a)}\right| + \ln\left|\frac{1 - b/(1+a)}{1 + b/(1+a)}\right| \qquad \text{(IV-30)}$$

and

$$G = \ln\left|\frac{b^2 - (1-a)^2}{b^2 - (1+a)^2}\right| \qquad \text{(IV-31)}$$

Now, let us just consider the case where ω and K are much less than P ($b \ll 1$, and $1 - a \ll b$. In this case we find (from equations IV-26 - IV-31)

$$a \cong -m\omega_p^2/(\kappa T)K^2\left[1 + \omega/2K \ln\left|\frac{\omega - K}{\omega + K}\right| + \frac{1}{12}(K/\kappa T)^2 \ln(m/\kappa T)\right] \qquad \text{(IV-32)}$$

Next, we consider the imaginary part. From IV-17 and IV-24 we find

$$\operatorname{Im}\eta_{\text{eff.}} = \frac{\pi\omega_p^2 m}{4(\kappa T)^3 K^2}\int_m^\infty P\,dP\,e^{-P/\kappa T}\left[(2P^2 + 2P\omega + k^2/2)\ (k^2 + 2p\cdot k)\right.$$

$$\left. - \text{terms with } k \rightarrow -k\right] \qquad \text{(IV-33)}$$

For ω and K small compared with P, we find

$$\text{Im}\,\eta_{\text{eff.}} = \frac{\dot{\omega}\pi m\omega_p^2}{4(\kappa T)K^3} \qquad \text{for } v_{ph} = \omega/K < 1$$

$$= 0 \qquad \text{for } v_{ph} = \omega/K > 1 \qquad \text{(IV-34)}$$

The reason there is no damping for $v_{ph} > 1$ is, that since the electrons cannot have a speed greater than one a wave travelling with a phase velocity greater than one cannot transfer energy (on the average) to the electrons.

Silin (19) has considered the coulomb dielectric function of the ultra-relativistic plasma using classical techniques. The real part of Silin's dielectric function is identical with the first two terms of equation IV-32, the third term being a quantum mechanical correction [i.e., the third term is $\sim (\hbar Kc/\kappa T)^2 \ln (mc^2/\kappa T)$]. The imaginary part of Silin's dielectric function is identical with $\text{Im}\,\eta_{\text{eff.}}$ as given in equation IV-36.

We note that the results obtained here are, strictly speaking, of an academic nature only. At very high temperatures effects such as radiation of electromagnetic waves by the electrons will considerably modify the distribution function.

V. SUMMARY AND CONCLUSIONS

We have calculated the photon propagator in a medium by using the four-dimensional Feynman diagram method. All four directions of photon polarization (scalar, longitudinal, and transverse) are treated on an equal footing. By invoking the current conservation law, longitudinal photons are eliminated. We have calculated an explicit form for the interaction of two arbitrary currents in a medium. From this interaction we are able to define a coulomb and transverse dielectric function (which are identical in most cases considered here). By examining the photon propagator for its poles we have been able to obtain dispersion relations which yield the energy-momentum relation of the free motion of the system. We have discussed both the real (dispersive) and imaginary (absorptive) parts of the dispersion relations. For example, we have obtained corrections to the work of Bohm and Pines in the non-relativistic limit, and to the work of Silin in the ultra-relativistic limit.

By calculating the self energy of a fast incident particle we have been able to obtain the total transition probability out of the particles initial state and hence, the energy loss. The energy loss was found to be composed of three parts; the excitation loss, the ionization loss, and the Cerenkov loss. In particular, we have derived the dependence of the excitation loss on the dielectric function.

We have seen that the Feynman diagram technique is particularilarily suited to give a unified treatment of a large number of physical processes that take place in a medium. Although our approach has been quantum mechanical, many of the results that we have obtained are derivable classically (e. g., the atomic index of refraction is essentially the classical result). The advantage of our method is, that by a single scheme (via the photon propagator) we have been able to treat a large number of problems, which previously were each treated by a different approach. That is, the Feynman diagram approach is a way (undoubtedly there are others) by which one can give a unified picture of the Quantum Electrodynamics of a Medium.

Finally, we are left with some problems of interest which have not been treated here. In our opinion, the major problem is the contribution of higher order proper diagrams to the photon propagator. We expect that these higher order terms become increasingly more important as the density of the medium increases. That is, we really would like to apply the theory developed here to a medium other than gases (e. g., to glass), and until we have an idea of the effects of these higher terms we are, so to speak, left in the dark. Another problem that seems amenable to our method is the properties of a non-isotropic system (e. g., an atomic medium in the presence of an external magnetic field).

APPENDIX A

In this appendix we will modify the theory presented in Chapter 2 to take into account the finite lifetime of the excited states of the atoms. To do this we imagine that each excited state is really made up of a large number (infinite) of discrete states of energy $\mathcal{E}_n$ ($\mathcal{E}_n$ being real) centered around E_n with weight $G(\mathcal{E}_n)$. We now recalculate expressions II-20a and II-20b by first calculating for an energy level $\mathcal{E}_n$ (discrete and real) and then averaging over $\mathcal{E}_n$ [weight $G(\mathcal{E}_n)$]. We choose a $G(\mathcal{E}_n)$ such that its expectation value is E_n and half width is γ_n, the line width.* That is, a $G(\mathcal{E}_n)$ is chosen such that its effects describe a broadened energy level. In what follows, to avoid unnecessary complexity in notation, we consider only one excited state above the ground state.

First let us consider the transverse part of β. For a single excited state II-17a reads (suppressing an irrelevant multiplicative factor)

$$\beta_{11} \sim \omega_1^2 |z_1|^2 \left(\frac{1}{\omega_1 - \omega} + \frac{1}{\omega_1 + \omega}\right) - \frac{1}{m} \tag{A-1}$$

where $\omega_1 \equiv E_1(1 - i\epsilon) - E_o$ (see II-10).

Also, to simplify the notation let us put $E_o = 0$, that is, we are measuring energy relative to the ground state. A-1 becomes

$$\beta_{11} \sim \left\{ \omega_1^2 |z_1|^2 \left(\frac{1}{\omega_1 - \omega - i\epsilon} + \frac{1}{\omega_1 + \omega - i\epsilon}\right) - \frac{1}{m} \right\} \tag{A-2}$$

---

* γ_n is the total line width of the excited state.

Now to take into account the effects of the finite lifetime we desire to replace ω_1 by $\mathcal{E}$ and average overall $\mathcal{E}$ with weight $G(\mathcal{E})$. Specifically, A-2 becomes

$$\beta_{11} \sim \frac{\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E}) \frac{\mathcal{E}^2 |z_\epsilon|^2}{\mathcal{E} - \omega - i\epsilon} + \frac{\mathcal{E}^2 |z_\epsilon|^2}{\mathcal{E} + \omega - i\epsilon} - \frac{1}{m}}{\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E})} \qquad \text{(A-3)}$$

A function which permits the pertinent integrals to be done is given by

$$G(\mathcal{E}) = \frac{\gamma}{2\pi} \frac{1}{(\mathcal{E} - \omega_1)^2 + \frac{\gamma^2}{4}} \qquad \text{(A-4)}$$

where $\gamma \ll \omega_1$. The factor of $\frac{\gamma}{2\pi}$ in A-4 makes $G(\mathcal{E})$ have the property $\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E}) = 1$. It is readily verified that this choice for $G(\mathcal{E})$ satisfies the properties discussed in the preceding paragraph. Substituting A-4 into A-3 we get

$$\beta_{11} \sim \int_{-\infty}^{\infty} \frac{\gamma}{2\pi} \frac{d\mathcal{E}}{(\mathcal{E} - \omega_1)^2 + \frac{\gamma^2}{4}} \left\{ \frac{\mathcal{E}^2 |z|^2}{\mathcal{E} - \omega - i\epsilon} + \frac{\mathcal{E}^2 |z|^2}{\mathcal{E} + \omega - i\epsilon} - \frac{1}{m} \right\} \qquad \text{(A-5)}$$

where we take $|z|$ to be independent of $\mathcal{E}$ since $G(\mathcal{E})$ is large only in the immediate neighborhood of ω_1. Now for one excited state the sum rule, II-18, becomes $2m\omega_1 |z_1|^2 = 1$. For many discrete states spread about ω_1 with weight $G(\mathcal{E})$, the sum rule becomes

$$\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E}) 2m\mathcal{E} |z|^2 = 1$$

or

$$\frac{1}{m} = \int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E})\, 2\mathcal{E}\, |z|^2 \qquad \text{(A-6)}$$

Substituting A-6 into A-5 and using A-4 we get

$$\beta_{11} \sim \int_{-\infty}^{\infty} d\mathcal{E}\, \frac{\frac{\gamma}{2\pi}}{(\mathcal{E} - {}_1)^2 + \frac{\gamma^2}{4}} \left\{ \frac{\mathcal{E}^2}{\mathcal{E} - \omega - i\epsilon} + \frac{\mathcal{E}^2}{\mathcal{E} + \omega - i\epsilon} - 2\mathcal{E} \right\} |z|^2$$

$$= \int_{-\infty}^{\infty} d\mathcal{E}\, \frac{\frac{\gamma}{2\pi}|z|^2}{(\mathcal{E} - \omega_1)^2 + \frac{\gamma^2}{4}} \left\{ \frac{\mathcal{E}^2}{\mathcal{E} - \omega - i\epsilon} + \frac{\mathcal{E}^2}{\mathcal{E} + \omega - i\epsilon} \right\} - 2\omega_1 |z|^2 \qquad \text{(A-7)}$$

since $\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E})\, \mathcal{E} = \omega_1$. To evaluate the integral in A-7 we use contour integration. Considering just the integral of A-7 we have

$$\beta_{11} \sim \int_{-\infty}^{\infty} d\mathcal{E}\, \frac{\frac{\gamma}{2\pi}}{(\mathcal{E} - \omega_1 + \frac{i\gamma}{2})(\mathcal{E} - \omega_1 - \frac{i\gamma}{2})} \quad \frac{\mathcal{E}^2}{\mathcal{E} - \omega - i\epsilon} + \frac{\mathcal{E}^2}{\mathcal{E} + \omega - i\epsilon} \qquad \text{(A-8)}$$

Now the terms in the braces and the factor $(\mathcal{E} - \omega_1 - \frac{i\gamma}{2})$ has poles in the UHP $\mathcal{E}$ plane. The only pole in the LHP $\mathcal{E}$ plane comes from the factor $(\mathcal{E} - \omega_1 + \frac{i\gamma}{2})$. So on applying Cauchy's theorem we choose to close the contour in the LHP. Hence, by Cauchy's theorem

$$\beta_{11} = -2\pi i \,(\text{residue at } \mathcal{E} = \omega_1 - \frac{i\gamma}{2}) - \int_c \qquad \text{(A-9)}$$

where $\int_c$ is the integral of the integrand of A-8 evaluated along the infinite semicircle in the LHP. Now

$$-2\pi i(\text{residue at } \mathcal{E}=\omega_1-\frac{i\gamma}{2}) = -2\pi i\left(\frac{\gamma}{2\pi}\right)\left(\frac{1}{-i\gamma}\right)\left(\omega_o-\frac{i\gamma}{2}\right)^2$$

$$\times\left\{\frac{1}{\omega_1-\omega-\frac{i\gamma}{2}}+\frac{1}{\omega_1+\omega-\frac{i\gamma}{2}}\right\}^*$$

$$=\left(\omega_1-\frac{i\gamma}{2}\right)^2\left\{\frac{1}{\omega_1-\omega-\frac{i\gamma}{2}}+\frac{1}{\omega_1+\omega-\frac{i\gamma}{2}}\right\} \qquad \text{(A-10)}$$

Next

$$\int_c = \lim_{R\to\infty}\int_0^{-\pi}\frac{\left(\frac{\gamma}{2\pi}\right)iRe^{i\theta}d\theta}{(Re^{i\theta}-\omega_o)+\frac{\gamma^2}{4}}\left\{\frac{R^2e^{2i\theta}}{Re^{i\theta}-\omega-i\epsilon}+\frac{R^2e^{2i\theta}}{Re^{i\theta}+\omega-i\epsilon}\right\}$$

$$=\lim_{R\to\infty}\int_0^{-\pi}\frac{\left(\frac{\gamma}{2\pi}\right)iR^3e^{3i\theta}(2)\,d\theta}{R^3e^{3i\theta}}$$

$$= i\gamma \qquad \text{(A-11)}$$

Upon substituting A-10 and A-11 into A-9 we get that (replacing the term retained in A-7)

$$\beta_{11}\sim|z|^2\left[\left(\omega_1-\frac{i\gamma}{2}\right)^2\left\{\frac{1}{\omega_1-\omega-\frac{i\gamma}{2}}+\frac{1}{\omega_1+\omega-\frac{i\gamma}{2}}\right\}+i\gamma-2\omega_1\right]$$

$$=|z|^2\left(\omega_1-\frac{i\gamma}{2}\right)\left[\frac{\omega_1-\frac{i\gamma}{2}}{\omega_1-\omega-\frac{i\gamma}{2}}+\frac{\omega_1-\frac{i\gamma}{2}}{\omega_1+\omega-\frac{i\gamma}{2}}-2\right]$$

$$=2\omega^2|z|^2\left(\omega_1-\frac{i\gamma}{2}\right)\left[\frac{1}{\left(\omega_1-\frac{i\gamma}{2}\right)^2-\omega^2}\right] \qquad \text{(A-12)}$$

Now consider the coulomb term β_{44}: For a single state, et cetera, II-17b reads

---

* ϵ has now been set equal to zero.

$$\beta_{44} \sim K^2 |z|^2 \left(\frac{1}{\omega_1 - \omega - i\epsilon} + \frac{1}{\omega_1 + \omega - i\epsilon} \right)$$

$$\rightarrow \frac{K^2 |z|^2 \int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E}) \left\{ \frac{1}{\mathcal{E} - \omega - i\epsilon} + \frac{1}{\mathcal{E} + \omega - i\epsilon} \right\}}{\int_{-\infty}^{\infty} d\mathcal{E}\, G(\mathcal{E})}$$

Proceeding in a manner similar to the one used on β_{11} gives that the result of averaging over many states for β_{44} is

$$\beta_{44} \sim K^2 |z|^2 \left(\frac{1}{\omega_1 - \omega - \frac{i\gamma}{2}} + \frac{1}{\omega_1 + \omega - \frac{i\gamma}{2}} \right)$$

$$= K^2 |z|^2 \frac{1}{\left(\omega_1 - \frac{i\gamma}{2}\right)^2 - \omega^2} \qquad \text{(A-13)}$$

The obvious extension to many excited states is that in A-12 and A-13 $\omega_1 \rightarrow \omega_n$, $\gamma \rightarrow \gamma_n$ and $\sum_n$. The final result is II-22.

The problem of natural line shape has been discussed by Low (20), where he applies the covariant methods of Feynman and Dyson to the problem of line shape. He discusses both the line shift and line shape.* Specifically, Low applies his techniques to calculate the elastic scattering rear resonance of photons by a one electron atom in its ground state. He considers only effects which essentially come from diagram 1b. His overall result is that ω_n is replaced by $\omega_n - \frac{i\gamma_n}{2}$. By using Low's method and including diagram 1c we get the same result as the averaging procedure used here. It should be noted that Low gives a method

*Here we have assumed that E_n is the exact energy (real) eigenvalue which includes radiative effects, et cetera.

to calculate the line width where we have assumed that it is known.

We note that these results were obtained with the particular function G given in A-4 which was chosen because it enabled the averaging process to be done analytically. Of course we expect the results are general and do not depend upon the particular function chosen. However, we must admit that we were unable to do the integrals analytically with such a seemingly appropriate weighting function as the gaussian.

APPENDIX B

In this appendix we give a method for the evaluation of the integrals in equations III-26 and III-27. Here we will explicitly evaluate the most complicated of those integrals. The other integrals are done in exactly the same way as we will show for the integral T_1. From III-28a we have

$$T_1 = (v^2) \int_{\text{all } K} d^3\vec{K} \; \frac{(1 - \cos^2\theta)}{(k^2 - 2p \cdot k)(\omega^2\eta - K^2)} \tag{B-1}$$

To evaluate this integral we use the Feynman parameterization technique. The following relations will be useful:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + (1-x)b]^2} \tag{B-2}$$

and

$$\frac{1}{a^2 b} = \int_0^1 \frac{2y\, dy}{[ay + (1-y)b]^3} \tag{B-3}$$

Now

$$\cos^2\theta = \left(\frac{\vec{P} \cdot \vec{K}}{PK}\right)^2 = \frac{P_\alpha P_\beta K_\alpha K_\beta}{P^2 K^2} \tag{B-4}$$

where $\alpha, \beta = 1, 2, 3$.

Equation B-1 becomes (dropping the all K under the integral sign

$$T_1 = \frac{v^2}{P^2} \int d^3\vec{K} \; \frac{P^2K^2 - K_\alpha K_\beta P_\alpha P_\beta}{(K^2 - 2\vec{P}\cdot\vec{K} + A)(K^2 - \omega^2\eta)K^2} \tag{B-5}$$

where $A = 2E\omega - \omega^2$. Using B-2 we find

$$T_1 = \frac{v^2}{P^2} \int_0^1 dx \int d^3\vec{K} \; \frac{(P^2K^2 - K_\alpha K_\beta P_\alpha P_\beta)}{(K^2 - 2\vec{P}\cdot\vec{K} + A)(K^2 - \omega^2\eta x)^2} \tag{B-6}$$

Now we apply B-3 to get

$$T_1 = \frac{v^2}{P^2} \int_0^1 2y\,dy \int_0^1 dx \int d^3\vec{K} \, \frac{(P^2K^2 - K_\alpha K_\beta P_\alpha P_\beta)}{[K^2 - 2\vec{P}\cdot\vec{K}(1-y) - \omega^2\eta xy + (1-y)A]^3}$$

$$= \frac{v^2}{P^2} \int_0^1 2y\,dy \int_0^1 dx \int d^3\vec{K} \frac{P^2K^2 - K_\alpha K_\beta P_\alpha P_\beta}{[(\vec{K} - (1-y)\vec{P})^2 + A(1-y) - \omega^2\eta xy - (1-y)^2P^2]^3} \tag{B-7}$$

Letting $Q_\alpha = K_\alpha - (1-y)P_\alpha$ we find

$$T_1 = 2v^2 \int_0^1 y\,dy \int_0^1 dx \int \frac{d^3\vec{Q}\left(Q^2 - \frac{Q_\alpha Q_\beta P_\alpha P_\beta}{P^2}\right)}{(Q^2 + \Delta)^3} \tag{B-8}$$

where $\Delta = A(1-y) - \omega^2\eta xy - (1-y)^2P^2$. (B-9)

In arriving at B-8 we have omitted terms in the numerator linear in Q since these terms vanish because they give an odd integral and we are integrating over all Q. Also, because of the spherical symmetry $Q_\alpha Q_\beta$ is equivalent to $\frac{Q^2}{3}\delta_{\alpha\beta}$. Equation B-8 becomes

$$T_1 = \frac{4}{3} v^2 \int_0^1 y\,dy \int_0^1 dx \int \frac{Q^2 d^3\vec{Q}}{(Q^2 + \Delta)^3} \tag{B-10}$$

Now, by elementary integration, it is easily verified that

$$\int \frac{Q^2 d^3\vec{Q}}{(Q^2 + \Delta)^3} = \frac{3}{4}\pi^2 \Delta^{-1/2} .$$

Equation B-10 becomes

$$T_1 = \pi^2 v^2 \int_0^1 y\,dy \int_0^1 dx \frac{1}{(A(1-y)-\omega^2\eta xy-(1-y)^2P^2)^{1/2}} \qquad \text{(B-11)}$$

$$= -\frac{i\pi^2 v^2}{P}\int_0^1 y\,dy \int_0^1 \frac{dx}{[(1-y)^2-\frac{A}{P^2}(1-y)+\frac{\omega^2}{P^2}\;xy]^{1/2}}$$

$$= \frac{2i\pi^2 P v^2}{\omega^2\eta}\int_0^1 dt \left[\left(t^2-t\left(a+\frac{\omega^2}{P^2}\right)+\frac{\omega^2\eta}{P^2}\right)^{1/2}-(t^2-at)^{1/2}\right] \qquad \text{(B-12)}$$

where

$$a = \frac{2\omega}{Pv}\left(1-\frac{\omega}{2E}\right).$$

Completing the square under the radical sign in B-12 gives

$$T_1 = \frac{2i\pi^2 P v^2}{\omega^2\eta}\int_{-\epsilon/2}^{1-\epsilon/2} dx\left(x^2+\delta-\frac{\epsilon^2}{4}\right)^{1/2}-\int_{-a/2}^{1-a/2}\left(x^2-\frac{a^2}{4}\right)^{1/2}dx \qquad \text{(B-13)}$$

where

$$\delta = \left(\frac{\omega^2}{P^2}\right)\eta$$

and (B-14)

$$\epsilon = a+\delta$$

The integral in B-13 is elementary. The result is

$$T_1 = \alpha\left(v^2\frac{P^2}{\omega^2}\right)\left[-\frac{\delta}{4}(1-a)^{1/2}+\frac{\epsilon}{2}\delta^{1/2}+\left(\delta-\frac{\epsilon^2}{4}\right)\ln\left(\frac{1-\frac{\epsilon}{2}+(1-\epsilon+\delta)^{1/2}}{-(\frac{\epsilon}{2}-\delta)^{1/2}}\right)\right.$$

$$\left.-\frac{a^2}{4}\ln\left(\frac{1-\frac{a}{2}+(1-a)^{1/2}}{-\frac{a}{2}}\right)\right] \qquad \text{(B-15)}$$

where $\alpha = i\frac{\pi^2}{P\eta}$. (B-16)

The other integrals are not as complicated as T_1 since they only involve one parameterization while T_1 involved two parameterizations. These integrals were evaluated using the same method as used on T_1 with the results given in III-29.

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