STUDIES ON THERMAL STRESSES IN ELASTIC SOLIDS

Thesis by

Che-Min Cheng

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Contents

Summary

Some results, following Gibbs and Murnaghan, on the general thermodynamical properties of a continuous and isotropic medium are reviewed in Part I. These discussions lead to the formulation of various thermodynamic functions for a thermo-elastic solid in small strain. The expression for the free energy is useful, in particular, for approximate solutions of thermal stress problems involving either steady or transient heating. Also in Part I, a rather general condition is established under which the inertia effect due to transient thermal expansions may be neglected. Conditions under which temperature distributions may be calculated independently of stresses and strains are also given. Attention is given to the order of approximations involved in such simplifications.

The general results in Part I are applied to two problems in Part II and Part III. The problem of thermal shock, a type of failure due to sudden heating or cooling, is studied in Part II. The analytic results obtained there are compared with the experimental results on thermal shock carried out by N.A.C.A. investigators on circular ceramic and ceramal dics. The correlation between theory and experiment is considered satisfactory.

Thermal stresses in thin cylindrical shells and plates are formulated and discussed in Part III. It is assumed that the temperature varies only across the thickness, and the

Young's modulus may be an arbitrary function of temperature. A convention regarding the choice of the reference surface is introduced, by means of which the present theory becomes comparable to the ordinary theory of plates and shells. Methods based on similarity considerations are devised such that the resulting stresses and strains in a shell or plate caused by temperature gradient and external loads can be predicted by experimenting with a similar specimen at a uniform temperature. These considerations are motivated by the necessity to overcome the difficulties both in analytic calculations and experimental measurements of stresses and strains at elevated temperatures, especially when transient heating and complicated loads are involved. Such a situation arises, for example, in the combustion chamber of a rocket engine, where stresses produced by supporting seats are often too complicated to compute by purely analytical methods.

Part I

Some General Remarks on the Thermodynamic Properties of an Elastic Solid under Thermal Stresses

I. Some General Results of Strain Analysis

The coordinates of each point of a continuous medium in a reference state will be denoted by x^i. In this coordinate system h_{ij} shall be used to denote the metric tensor. In any other state of strain, a new coordinate system y^α will be employed. $g_{\alpha\beta}$ will be used to denote the metric tensor in this coordinate system. Then in general

$$y^\alpha = y^\alpha(x^1, x^2, x^3) \,, \qquad x^i = x^i(y^1, y^2, y^3) \,. \tag{1.1}$$

These functions will be assumed to have continuous derivatives. Then*

$$dy^\alpha = a^\alpha_i \, dx^i \,, \qquad dx^i = b^i_\alpha \, dy^\alpha \,, \tag{1.2}$$

where

$$a^\alpha_i = \frac{\partial y^\alpha}{\partial x^i} \,, \qquad b^i_\alpha = \frac{\partial x^i}{\partial y^\alpha} \,, \tag{1.3}$$

$\| a^\alpha_i \|$ and $\| b^i_\alpha \|$ being reciprocal matrices. It can be shown by means of coordinate transformations that the a^α_i are contravariant vectors with respect to y^α, and covariant vectors with respect to x^i. On the other hand the b^i_α are covariant vectors in y^α, and contravariant vectors in x^i. The italic and Greek indices therefore indicate the tensorial characteristics of the various quantities. Let ds_0 be the length of an infinitesimal element dx^i at x^i in the reference state, and let ds be its length in the deformed state. Then (Ref. 1)

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* Summation convention with respect to dummy indices will be followed throughout the text.

$$ds^2 - ds_0^2 = g_{\alpha\beta}dy^\alpha dy^\beta - h_{ij}dx^i dx^j \qquad (1.4)$$

gives the change in length of this infinitesimal element. The last equation may be written, by means of equations (1.2) and (1.3) in either of the following two ways:

$$ds^2 - ds_0^2 = (g_{\alpha\beta} - h_{ij}b^i_\alpha b^j_\beta)dy^\alpha dy^\beta, \qquad (1.5)$$

$$ds^2 - ds_0^2 = (g_{\alpha\beta}a^\alpha_i a^\beta_j - h_{ij})dx^i dx^j. \qquad (1.6)$$

$h_{ij}b^i_\alpha b^j_\beta$ is clearly a scalar with respect to x^i, and a covariant tensor (symmetric) with respect to y^α. The strain tensors $\epsilon_{\alpha\beta}$ are then defined as

$$\epsilon_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - h_{ij}b^i_\alpha b^j_\beta). \qquad (1.7)$$

$\epsilon_{\alpha\beta}$ are thus invariants with respect to coordinate transformations of x^i. Similarly, the strain tensor with respect to x^i can be defined as

$$e_{ij} = \frac{1}{2}(g_{\alpha\beta}a^\alpha_i a^\beta_j - h_{ij}). \qquad (1.8)$$

Therefore

$$ds^2 - ds_0^2 = 2\epsilon_{\alpha\beta}dy^\alpha dy^\beta, \qquad (1.9)$$

$$ds^2 - ds_0^2 = 2e_{ij}dx^i dx^j. \qquad (1.10)$$

Moreover

$$\epsilon_{\alpha\beta} = e_{ij}b^i_\alpha b^j_\beta,$$
$$e_{ij} = \epsilon_{\alpha\beta}a^\alpha_i a^\beta_j, \qquad (1.11)$$

and

$$g_{\alpha\beta} = (2e_{ij} + h_{ij})b^i_\alpha b^j_\beta. \qquad (1.12)$$

Strain Invariants

Since $\epsilon_{\alpha\beta}$ is a symmetric tensor of the second rank, three independent invariants may be formed. They shall be denoted by I_1, I_2 and I_3. Thus

$$I_1 = \epsilon_\alpha^\alpha ,$$
$$I_2 = \frac{1}{2!} \epsilon_\beta^\alpha \epsilon_\alpha^\beta , \qquad (1.13)$$
$$I_3 = \frac{1}{3!} \epsilon_\beta^\alpha \epsilon_\gamma^\beta \epsilon_\alpha^\gamma .$$

Similarly, three invariants may be formed from the e_j^i,

$$I_{01} = e_i^i ,$$
$$I_{02} = \frac{1}{2!} e_j^i e_i^j , \qquad (1.14)$$
$$I_{03} = \frac{1}{3!} e_j^i e_k^j e_i^k .$$

Principal Directions of Strain:

In the neighborhood of y^α, there exist three directions along which $(ds^2 - ds_0^2)/ds^2$ attains maximum or minimum values. These directions are called the principal directions of strain, in the y^α coordinates. Let λ^α be a unit vector at y^α. Then $\lambda^\alpha = \frac{dy^\alpha}{ds}$. The condition that $ds^2 - ds_0^2/ds^2$ attain a stationary value is, according to equation (1.9),

$$\delta (\epsilon_{\alpha\beta} \lambda^\alpha \lambda^\beta) = 0 ,$$

with the restriction that

$$g_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1 .$$

Let Λ be the Lagrange multiplier. Then

$$(\epsilon_{\alpha\beta} \lambda^\alpha - \Lambda g_{\alpha\beta} \lambda^\alpha) \delta \lambda^\beta = 0$$

which reduces to the following three equations, since $\delta\lambda^\alpha$ are arbitrary;

$$(\epsilon_{\alpha\beta} - \Lambda g_{\alpha\beta}) \lambda^\alpha = 0 . \qquad (1.15)$$

These three homogeneous equations may be solved for λ^α provided that their determinant vanishes identically, i.e.

$$| \epsilon_{\alpha\beta} - \Lambda g_{\alpha\beta} | = 0 . \qquad (1.16)$$

In general three distinct roots exist. It is easy to show that none of these roots may be complex, for if Λ is a complex root, then its conjugate $\bar{\Lambda}$ must also be a root. Substituting Λ and $\bar{\Lambda}$ into equations (1.15), two sets of λ^{α} may be determined. Moreover, if λ^{α} corresponds to Λ, the conjugate of λ^{α}, namely $\bar{\lambda}^{\alpha}$, must be the solution corresponding to $\bar{\Lambda}$. Hence, by equations (1.15),

$$(t_{\alpha\beta} - \Lambda g_{\alpha\beta})\,\lambda^{\alpha} = 0 \;,$$

$$(t_{\alpha\beta} - \bar{\Lambda} g_{\alpha\beta})\,\bar{\lambda}^{\alpha} = 0 \;.$$

Contracting the first set of these equations by $\bar{\lambda}^{\beta}$, the second set by λ^{β}, and subtracting the resulting two equations one obtains,

$$\bar{\Lambda} g_{\alpha\beta}\lambda^{\alpha}\bar{\lambda}^{\beta} - \Lambda g_{\alpha\beta}\bar{\lambda}^{\alpha}\lambda^{\beta} = t_{\alpha\beta}\lambda^{\alpha}\bar{\lambda}^{\beta} - t_{\alpha\beta}\bar{\lambda}^{\alpha}\lambda^{\beta}.$$

which, on account of the symmetry of $g_{\alpha\beta}$ and $t_{\alpha\beta}$, reduces to

$$g_{\alpha\beta}\lambda^{\alpha}\bar{\lambda}^{\beta}(\bar{\Lambda} - \Lambda) = 0 \;.$$

Now, if one puts* $\lambda^{\alpha} = a^{\alpha} + i\,b^{\alpha}$, then

$$g_{\alpha\beta}\lambda^{\alpha}\bar{\lambda}^{\beta} = g_{\alpha\beta}a^{\alpha}a^{\beta} + g_{\alpha\beta}b^{\alpha}b^{\beta},$$

which is not equal to zero. Consequently $\bar{\Lambda} - \Lambda = 0$. It follows that Λ cannot be complex.

When equation (1.16) possesses distinct roots, $\Lambda_{(1)}$, $\Lambda_{(2)}$, $\Lambda_{(3)}$, it can be shown by exactly the same method used in the last paragraph that

$$g_{\alpha\beta}\lambda^{\alpha}_{(\gamma)}\lambda^{\beta}_{(\delta)} = 0 \;, \quad \gamma \neq \delta \;;$$

$$g_{\alpha\beta}\lambda^{\alpha}_{(\gamma)}\lambda^{\beta}_{(\delta)} = 1 \;, \quad \gamma = \delta \;,$$

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* $i = \sqrt{-1}$

where $\lambda^{\alpha}_{(\gamma)}$ is the unit vector corresponding to $\Lambda_{(\gamma)}$. Consequently, the three principal directions of strain are mutually perpendicular. When equation (1.16) has multiple roots, there is some indeterminancy in the three sets of principal directions. However, three mutually perpendicular principal directions can always be constructed. Equation (1.16) may be written alternatively as

$$| \epsilon^{\alpha}_{\beta} - \Lambda \delta^{\alpha}_{\beta} | = 0 .$$

Expanding the determinant and making use of the strain invariants defined previously, one obtains

$$\Lambda^3 - \Lambda^2 I_1 + \tfrac{1}{2}(I_1^2 - I_2)\Lambda + (I_1 I_2 - 2 I_3 - \tfrac{I_1^3}{6}) = 0 . \qquad (1.17)$$

Hence

$$\begin{aligned} J_1 &= \Lambda_{(1)} + \Lambda_{(2)} + \Lambda_{(3)} = I_1 , \\ J_2 &= \Lambda_{(1)}\Lambda_{(2)} + \Lambda_{(2)}\Lambda_{(3)} + \Lambda_{(3)}\Lambda_{(1)} = \tfrac{1}{2}(I_1^2 - I_2) , \\ J_3 &= \Lambda_{(1)}\Lambda_{(2)}\Lambda_{(3)} = -(I_1 I_2 - 2 I_3 - \tfrac{I_1^3}{6}) . \end{aligned} \qquad (1.18)$$

J_1, J_2, J_3 are, of course, also strain invariants. The same analysis may be carried out with respect to the x^i coordinate system. Using the subscript o to differentiate the present case from the one just discussed, one obtains

$$\begin{aligned} & | e^i_j - \Lambda_o \delta^i_j | = 0 , \\ & (e_{ij} - \Lambda_o h_{ij}) \lambda^i_o = 0 , \\ & \Lambda_o^3 - \Lambda_o^2 I_{o1} + \tfrac{1}{2}(I_{o1}^2 - I_{o2})\Lambda_o + (I_{o1} I_{o2} - 2 I_{o3} - \tfrac{I_{o1}^3}{6}) = 0 , \\ & J_{o1} = \Lambda_{o(1)} + \Lambda_{o(2)} + \Lambda_{o(3)} = I_{o1} , \\ & J_{o2} = \Lambda_{o(1)}\Lambda_{o(2)} + \Lambda_{o(2)}\Lambda_{o(3)} + \Lambda_{o(3)}\Lambda_{o(1)} = \tfrac{1}{2}(I_{o1}^2 - I_{o2}) , \\ & J_{o3} = \Lambda_{o(1)}\Lambda_{o(2)}\Lambda_{o(3)} = -(I_{o1} I_{o2} - 2 I_{o3} - \tfrac{I_{o1}^3}{6}) , \end{aligned} \qquad (1.19)$$

The above analysis shows that one may use any three strain invariants and the three principal directions of strain to specify the state of strain at any point. Depending on whether the Eulerian or Lagrangian point of view is assumed, either set of strain invariants and principal directions may be used.

II. Thermodynamical Functions and Stress-Strain Relations

It will be assumed that the thermodynamic state of the continuous medium is uniquely determined by its state of strain, and its temperature. Hence all the thermodynamic functions of such a medium must be expressible in terms of $\epsilon_{\alpha\beta}$ (or e_{ij}) and the temperature, but nothing else. As stated in the previous section, in place of $\epsilon_{\alpha\beta}$ (or e_{ij}) any three independent strain invariants and the corresponding three principal directions may be used. If in addition a material is isotropic, then the thermodynamic functions cannot have preferred directional properties. In this case, all of the thermodynamic functions can only depend on the strain invariants, if ψ is the free energy (Ref. 2) per unit mass, then in general,

$$\psi = \psi(I_1, I_2, I_3, T) , \qquad (2.1)$$

or

$$\psi = \psi(I_{01}, I_{02}, I_{03}, T).$$

Other thermodynamical functions can be similarly expressed.

The stress-strain relation can be derived from the free energy ψ for arbitrary strains and temperature. If S denotes the entropy per unit mass, and w denotes the work done per unit volume of the material, then, according to thermodynamic principles,

$$\delta\psi = -S\delta T + \frac{1}{\rho}\delta w \qquad (2.2)$$

where ρ is the density. When δw is expressed in terms of stresses and strains, then the above equation can be shown to give rise to the stress-strain relation.

To find the expression for δw, consider now the variations in the expressions for strains caused by small variations δy^{α}. In this variation the x^{i} are kept constant. Therefore $\delta h_{ij} = 0$, $\delta dx^{i} = 0$. Moreover, all variations in tensorial quantities must be calculated covariantly. That is, if $T^{\alpha\cdots\beta}_{\gamma\cdots\sigma}$ is a tensor, then $\delta T^{\alpha\cdots\beta}_{\gamma\cdots\sigma}$ is defined as

$$\delta T^{\alpha\cdots\beta}_{\gamma\cdots\sigma} = \frac{D T^{\alpha\cdots\beta}_{\gamma\cdots\sigma}}{D y^{\delta}} \delta y^{\delta}$$

where $\frac{D}{Dy^{\delta}}$ denotes covariant differentiation. From this it follows that $\delta g_{\alpha\beta} = 0$. With this understanding it can be readily shown that

$$\delta ds^2 = \left(\frac{D\delta y_{\alpha}}{Dy^{\beta}} + \frac{D\delta y_{\beta}}{Dy^{\alpha}}\right) dy^{\alpha} dy^{\beta} . \qquad (2.3)$$

But $\delta(ds_0^2) = 0$. Thus, by equation (1.10), one obtains

$$\delta e_{ij} = \frac{1}{2}\left(\frac{D\delta y_{\alpha}}{Dy^{\beta}} + \frac{D\delta y_{\beta}}{Dy^{\alpha}}\right) a^{\alpha}_{i} a^{\beta}_{j} . \qquad (2.4)$$

Similarly, by equations (1.9) and (2.3) one obtains

$$\delta \epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{D\delta y_{\alpha}}{Dy^{\beta}} + \frac{D\delta y_{\beta}}{Dy^{\alpha}}\right) - \epsilon_{\delta\beta}\frac{D\delta y^{\delta}}{Dy^{\alpha}} - \epsilon_{\alpha\delta}\frac{D\delta y^{\delta}}{Dy^{\beta}} . \qquad (2.5)$$

The results in equations (2.4) and (2.5) are useful in deriving the general equations of stress-strain relations. If $\frac{D\delta y_{\alpha}}{Dy^{\beta}} + \frac{D\delta y_{\beta}}{Dy^{\alpha}} = 0$ then $\delta ds^2 = 0$, showing that the virtual displacements are rigid body motions. In particular $\frac{D\delta y_\alpha}{Dy^\beta} = 0$ is the condition for rigid body translation.

Virtual Work

The stresses in a deformed medium may be expressed as a symmetric tensor of the second order. It shall be denoted by $\tau^{\alpha\beta}$. Its covariant and mixed forms can be obtained by contraction with the metric tensor $g_{\alpha\beta}$ in the usual manner. If the material is in static equilibrium, and if no body forces exist, the total virtual work $\delta\overline{W}$ due to surface tractions is clearly given by

$$\delta\overline{W} = \int \delta w \, dv = \int \tau^{\alpha\beta}\delta y_\alpha \, ds_\beta \quad , \quad ds_\beta = \sqrt{g}\, e_{(\alpha\beta\gamma)}\, dy^\beta dy^\gamma$$

where ds_β is the elementary surface, $g = |g_{ij}|$, and $e_{(\alpha\beta\gamma)}$ is the permutation symbol. The volume and surface integrals are to be extended throughout the volume and surface of the material. By the divergence theorem,

$$\delta\overline{W} = \int \frac{D\tau^{\alpha\beta}}{Dy^\beta}\delta y_\alpha \, dv + \int \tau^{\alpha\beta}\frac{D\delta y_\alpha}{Dy^\beta} dv$$

Since the material is in equilibrium, $\delta\overline{W} = 0$ for any rigid body motion. In particular $\delta\overline{W} = 0$ when $\frac{D\delta y_\alpha}{Dy^\beta} = 0$ by equation (2.5). Consequently,

$$\frac{D\tau^{\alpha\beta}}{Dy^\beta} = 0 \, ,$$

and, hence, in general

$$\delta w = \tau^{\alpha\beta}\frac{D\delta y_\alpha}{Dy^\beta} = \frac{1}{2}\tau^{\alpha\beta}\left(\frac{D\delta y_\alpha}{Dy^\beta} + \frac{D\delta y_\beta}{Dy^\alpha}\right).$$

Since $\tau^{\alpha\beta}$ is symmetric, by means of equations (2.4) one obtains

$$\delta w = \tau^{\alpha\beta} b^i_\alpha b^j_\beta \delta e_{ij} \qquad (2.6)$$

Substituting this into equation (2.2), one obtains

$$\delta\psi = -S\,\delta T + \frac{1}{\rho}\tau^{\alpha\beta} b^i_\alpha b^j_\beta \,\delta e_{ij} , \qquad (2.7)$$

or

$$\frac{\partial\psi}{\partial e_{ij}} = \frac{1}{\rho}\tau^{\alpha\beta} b^i_\alpha b^j_\beta .$$

Because ψ is a function of T and the strain invariants only, the above equation gives the relation between stresses and strains at various temperatures. Hence it may be regarded as the stress-strain relation. To find the stress-strain relations in terms of $\epsilon_{\alpha\beta}$, consider the variation of ψ with respect to $\epsilon_{\alpha\beta}$. Clearly, when T is held constant, $\delta\psi = \frac{1}{\rho}\delta w$. Thus

$$\frac{\partial\psi}{\partial\epsilon_{\alpha\beta}}\delta\epsilon_{\alpha\beta} = \frac{1}{\rho}\tau^{\alpha\beta}\frac{D\delta y_\alpha}{Dy^\beta} .$$

Making use of equation (2.5) and noting that $\frac{\partial\psi}{\partial\epsilon_{\alpha\beta}}$ is symmetric in α,β one obtains from the last equation

$$\frac{\partial\psi}{\partial\epsilon_{\alpha\beta}}\left(\frac{D\delta y_\alpha}{Dy^\beta} - 2\epsilon^\delta_\beta\frac{D\delta y_\delta}{Dy^\alpha}\right) = \frac{1}{\rho}\tau^{\alpha\beta}\frac{D\delta y_\alpha}{Dy^\beta} .$$

Since the derivatives $\frac{D\delta y_\alpha}{Dy^\beta}$ are arbitrary, the last equation gives

$$\tau^{\alpha\beta} = \rho\left(\frac{\partial\psi}{\partial\epsilon_{\alpha\beta}} - 2\epsilon^\alpha_\delta\frac{\partial\psi}{\partial\epsilon_{\delta\beta}}\right) \qquad (2.8)$$

This equation then relates $\epsilon_{\alpha\beta}$ and $\tau^{\alpha\beta}$, with temperature as a parameter. It clearly expresses the stress-strain relation. Thus if ψ is known the stress-strain relation can be found by direct differentiation. Moreover, since $\frac{\partial\psi}{\partial T} = -S$, and $\psi = U - TS$, the entropy S and internal energy U can also be evaluated very easily. In isothermal processes $\delta\psi = \frac{1}{\rho}\delta w$;

thus ψ plays the same role as the strain energy function in ordinary elasticity. Many attempts have been made in the past to find a general expression for ψ . However, on account of the fact that when the strains are large most materials exhibit irreversible phenomena such as yielding, these attempts have not been successful, except in a few isolated cases. (Ref. 3)

III. Equations of Motion

The equations of motion of a continuous medium are most conveniently expressed in terms of the Eulerian coordinates y^α . When the medium is in motion, equations (1.1) must be written as

$$y^\alpha = y^\alpha(x^1, x^2, x^3, t)$$
$$x^i = x^i(y^1, y^2, y^3, t) \qquad (3.1)$$

The Lagrange coordinates x^i are of course independent of the time t*. The velocity vector is clearly $v^\alpha = \frac{\partial y^\alpha}{\partial t}$ by the first set of equations (3.1). Replacing x^i by y^α in $\frac{\partial y^\alpha}{\partial t}$, one obtains the expression for the velocity field in y^α,

$$v^\alpha = v^\alpha(y^1, y^2, y^3, t) \ . \qquad (3.2)$$

Newton's law of motion for the continuous media can therefore be expressed as

$$\int \tau^{\alpha\beta} \lambda_\alpha \, dS_\beta + \int F^\alpha \rho \lambda_\alpha \, dV = \frac{D}{Dt} \int \rho v^\alpha \lambda_\alpha \, dV$$

where λ_α is an arbitrary parallel vector field. $\frac{D}{Dt}$ denotes the substantial derivative $\frac{\partial}{\partial t} + v^\alpha \frac{D}{Dy^\alpha}$. $\frac{D}{Dt}\int \rho v^\alpha \lambda_\alpha \, dV = \int \rho \frac{Dv^\alpha}{Dt} \lambda_\alpha \, dV$, since ρdV is constant following the path of motion, and $\frac{D\lambda_\alpha}{Dt} = 0$.

* The total derivative of x^i with respect to t is zero.

Consequently by the divergence theorem, the last equation becomes,

$$\frac{D\tau^{\alpha\beta}}{Dy^{\beta}} + \rho F^{\alpha} = \rho \frac{Dv^{\alpha}}{Dt} \qquad (3.3)$$

IV. Continuity Condition :

In terms of the Eulerian coordinates, the law of conservation of mass may be written as,

$$\frac{D}{Dt}\int \rho \, dV = 0 \; .$$

But $\frac{D}{Dt}\int dV = \int v^{\alpha} dS_{\alpha} = \int \frac{Dv^{\alpha}}{Dy^{\alpha}} dV$. Hence

$$\frac{D}{Dt}\int \rho \, dV = \int \frac{D\rho}{Dt} dV + \int \rho \frac{Dv^{\alpha}}{Dy^{\alpha}} dV = 0$$

for any arbitrary bulk of the medium. Therefore the continuity equation becomes,

$$\frac{D\rho}{Dt} + \rho \frac{Dv^{\alpha}}{Dy^{\alpha}} = 0 \; . \qquad (3.4)$$

V. The Energy Equation:

If it is assumed that the medium is in quasi-equilibrium, then the thermodynamic functions are well determined at all times. The equation of energy balance may then be written as

$$\int \tau^{\alpha\beta} v_{\alpha} \, dS_{\beta} + \int F^{\alpha} \rho v_{\alpha} \, dV + \frac{D}{Dt}\int \rho Q \, dV = \frac{D}{Dt}\int \rho U \, dV + \frac{D}{Dt}\int \tfrac{1}{2}\rho v^{\alpha} v_{\alpha} \, dV \, ,$$

where the first two integrals on the left hand side of the equation denote the rate of work done on the medium, the third term denotes the rate of heat in-put, Q being the heat in-put per unit mass. The two terms on the right-hand side denote respectively the rates of increase of internal energy and kinetic energy. By means of the divergence theorem and the equations of equilibrium, the last equation reduces immediately to

$$\rho \frac{DQ}{Dt} = \rho \frac{DU}{Dt} - \tau^{\alpha\beta} \frac{Dv_\alpha}{Dy^\beta} \tag{5.1}$$

But $\tau^{\alpha\beta} \frac{Dv_\alpha}{Dy^\beta}$ is the rate of increase of strain work. (Section II). Since $\delta U = \frac{1}{\rho}\delta w + T ds$, the last equation becomes

$$\frac{DQ}{Dt} = T \frac{DS}{Dt} \tag{5.2}$$

Hence, the fundamental relation between entropy and heat input is reestablished. This result could have been written down immediately since only quasi-equilibrium processes are considered.

VI. Passage to the Lagrangian Coordinate System when the Displacements and Strains are Very Small

When the strains and displacements are small, the Lagrangian coordinate system is the most convenient one to employ. In this case one puts

$$\begin{aligned} y^i &= x^i + u^i , \\ u^i &= u^i(x^1, x^2, x^3, t) , \end{aligned} \tag{6.1}$$

and there is no longer any need to use Greek indices. Then the metric tensor g_{ij} may be expressed as

$$g_{ij} = h_{ij}(x^1+u^1, x^2+u^2, x^3+u^3) . \tag{6.2}$$

By (6.1)

$$a^i_j = \delta^i_j + \frac{\partial u^i}{\partial x^j} \tag{6.3}$$

Where the differential coefficients $\frac{\partial u^i}{\partial x^j}$ are by assumption small quantities. Their products and terms such as $u^k \frac{\partial u^i}{\partial x^j}$ are accordingly neglected. By means of (1.8) e_{ij} can be evaluated by expanding g_{ij} in a Taylor series and neglecting products of small terms u^i, $\frac{\partial u^i}{\partial x^j}$. One obtains

$$e_{ij} = \frac{1}{2}\left(\frac{Du_i}{Dx^j} + \frac{Du_j}{Dx^i}\right) . \qquad (6.4)$$

By equation (1.11), it can be verified that

$$e_{ij} = \epsilon_{ij} + \epsilon_{\ell i}\frac{\partial u^\ell}{\partial x^j} + \epsilon_{j\ell}\frac{\partial u^\ell}{\partial x^i} \qquad (6.5)$$

Hence, to the present order of approximation $e_{ij} = \epsilon_{ij}$, and there is no longer any need to distinguish the two sets of strain tensors. Similarly, one obtains

$$\rho = \rho_0(1-\theta) \quad , \quad \theta = I_1 = I_{01} \qquad (6.6)$$

To the same order of approximation as in (6.5), the stress-strain relation may be written as

$$\frac{\partial(\rho\psi)}{\partial e_{ij}} = \tau^{ij} . \qquad (6.7)$$

VII. Thermodynamic Functions and Stress-Strain Relations for an Elastic, Isotropic Material

If the strains are small, most solids are elastic in any isothermal change of state. Since the change in volume is small, all thermodynamic functions may be based on specific values per unit volume, as in (6.7). For an isotropic, elastic material in small strain, the most general form for the free energy per unit volume is then

$$\psi = a + bI_1 + cI_2 + dI_1^2 \qquad (7.1)$$

where a, b, c, d may be functions of temperature but not of strains. It is clear that the above expression corresponds to a power series expansion in the e^i_j, where the third and higher order terms have been neglected. Since only small strains are considered, by equation (6.7)

$$\tau^i_j = b\delta^i_j + ce^i_j + 2d\delta^i_j e^k_k . \qquad (7.2)$$

The coefficients a, b, c can be evaluated as follows:

a) when the material is allowed to expand freely as the temperature is raised, one must have $ds^2 = (1+\beta)\, ds_o^2$, where is the linear strain due to thermal expansion, and is assumed to be small so that its square may be neglected. It follows then that $e^i_j = \beta \delta^i_j$. Moreover, $\tau^i_j = 0$. Consequently equation (7.2) yields one relation between b, c, d at any temperature,

$$b + \beta(c + 6d) = 0 . \tag{7.3a}$$

b) If the material is under pure tension F in the direction of the unit vector λ^i_T , then $\tau^i_j \lambda_{Ti} \lambda^j_T = F$ and $\tau^i_j \lambda_{oi} \lambda^j_o = 0$, λ^i_o being any unit vector normal to λ^i_T . If E is the Young's modulus and ν the Poissons ratio, then by definition,

$$E = \frac{\tau^i_j \lambda_{Ti} \lambda^j_T}{e^i_j \lambda_{Ti} \lambda^j_T - \beta} , \quad -\nu = \frac{e^i_j \lambda_{oi} \lambda^j_o - \beta}{e^i_j \lambda_{Ti} \lambda^j_T - \beta} . \tag{7.4}$$

Now

$$F = b + c e^i_j \lambda^j_T \lambda_{Ti} + 2d e^k_k ,$$

$$0 = b + c e^i_j \lambda^j_o \lambda_{oi} + 2d e^k_k .$$

Hence

$$E = \frac{c\,(e^i_j \lambda^j_T \lambda_{Ti} - e^i_j \lambda^j_o \lambda_{oi})}{e^i_j \lambda^j_T \lambda_{Ti} - \beta} .$$

As $F \to 0$, $e^i_j \to \beta \delta^i_j$, and one obtains

$$c = \frac{E}{1+\nu} . \tag{7.3b}$$

By a similar method, one obtains

$$b = -\beta \left\{ \frac{E}{1+\nu} + \frac{3E\nu}{(1+\nu)(1-2\nu)} \right\} . \tag{7.3c}$$

Equations (7.3a,b,c) then enable one to express b, c, d in terms of the empirical material constants E, ν, β which in general are functions of temperature. Introducing the Lamé constant λ , and the rigidity modulus μ , b, c, d may be most conveniently expressed as

$$c = 2\mu \quad , \; d = \tfrac{1}{2}\lambda \quad , \; b = -\beta(2\mu+3\lambda) \; .$$

Consequently,

$$\psi = a - \beta(2\mu+3\lambda)\, I_1 + 2\mu I_2 + \tfrac{1}{2}\lambda I_1^2 \; , \qquad (7.4)$$

$$\tau^i_j = -\beta(2\mu+3\lambda)\,\delta^i_j + 2\mu e^i_j + \lambda e^\kappa_\kappa \delta^i_j \, . \qquad (7.5)$$

It may be remarked that in evaluating a, b, c by the above procedure, the reference state is assumed free from stress and strain.

In the expression for ψ the function a(T) remains to be determined. Before doing this, it is convenient first to give the expression for the specific entropy and internal energy per unit volume. Since $S = -\frac{\partial\psi}{\partial T}$, one obtains

$$S = \frac{da}{dT} - \left\{ \tfrac{1}{2}(e^\kappa_\kappa)^2 \frac{d\lambda}{dT} + e^i_\kappa e^\kappa_i \frac{d\mu}{dT} \right\} + e^\kappa_\kappa \frac{d}{dT}\{\beta(2\mu+3\lambda)\}. \qquad (7.6)$$

Similarly, because $\psi = U - ST$, where U is the specific internal energy per unit volume,

$$U = T^2\left\{ -\frac{d}{dT}\left(\frac{a}{T}\right) - \tfrac{1}{2}(e^\kappa_\kappa)^2 \frac{d}{dT}\left(\frac{\lambda}{T}\right) - e^\kappa_i e^i_\kappa \frac{d}{dT}\left(\frac{\mu}{T}\right) + e^\kappa_\kappa \frac{d}{dT} \frac{\beta(2\mu+3\lambda)}{T} \right\}. \qquad (7.7)$$

If C_v is defined as the specific heat per unit volume when strains are kept at zero, then, since $c_v = \frac{dQ}{dT}\Big|_{e^i_j=0} = T\frac{dS}{dT}\Big|_{e^i_j=0}$, it follows from (7.6) that

$$C_v = -T\frac{d^2a}{dT^2}$$

Consequently, if C_v is determined, a(T) can be obtained by quadrature, i.e.,

$$a(T) = -\int^T dT' \int^{T'} \frac{C_v}{T''}\, dT'' + C_1 T + C_2$$

If one puts $C_1, C_2 = 0$ at the reference state where both stresses and strains vanish, then

$$\psi = -\int^T dT' \int^{T'} \frac{C_v}{T''} dT'' - \beta(2\mu+3\lambda) + \mu e^\kappa_i e^i_\kappa + \tfrac{1}{2}\lambda(e^\kappa_\kappa)^2 \quad , \qquad (7.8)$$

$$S = \int^T \frac{C_v}{T'}\, dT' + e^\kappa_\kappa \frac{d}{dT}\{\beta(2\mu+3\lambda)\} - \left\{\tfrac{1}{2}(e^\kappa_\kappa)^2 \frac{d\lambda}{dT} + e^\kappa_i e^i_\kappa \frac{d\mu}{dT}\right\} , \qquad (7.9)$$

$$U = -\int^T dT' \int^{T'} \frac{C_v}{T''} dT'' + T\int^T \frac{C_v}{T'} dT' - T^2\left\{\tfrac{1}{2}(e^\kappa_\kappa)^2 \frac{d}{dT}\left(\frac{\lambda}{T}\right) + e^i_\kappa e^\kappa_i \frac{d}{dT}\left(\frac{\mu}{T}\right) - e^\kappa_\kappa \frac{d}{dT} \frac{\beta(2\mu+3\lambda)}{T} \right\} . \qquad (7.10)$$

Then

$$dS = \frac{\partial S}{\partial T} dT + \frac{\partial S}{\partial I_1} dI_1 + \frac{\partial S}{\partial I_2} dI_2 ,$$

where

$$\frac{\partial S}{\partial I_1} = - I_1 \frac{d\lambda}{dT} + \frac{d}{dT}\{\beta(2\mu+3\lambda)\} ,$$

$$\frac{\partial S}{\partial I_2} = - 2 \frac{d\mu}{dT}$$

$$\frac{\partial S}{\partial T} = \frac{c_v}{T} - \left\{ \frac{1}{2} I_1^2 \frac{d^2\lambda}{dT^2} + 2 I_2 \frac{d^2\mu}{dT^2} \right\} + I_2 \frac{d^2}{dT^2}\{\beta(2\mu+3\lambda)\} .$$

For most solids in small strain

$$- \frac{T}{c_v}\left[\frac{1}{2} I_1^2 \frac{d^2\lambda}{dT^2} + 2 I_2 \frac{d^2\mu}{dT^2} - I_2 \frac{d^2}{dT^2}\{\beta(2\mu+3\lambda)\} \right]$$

is an extremely small number. Hence one may simply put

$$\frac{\partial S}{\partial T} = \frac{c_v}{T} .$$

With this approximation, the expression for dS becomes

$$dS = \frac{c_v}{T} dT + \left\{ - I_1 \frac{d\lambda}{dT} + \frac{d}{dT}[\beta(2\mu+3\lambda)] \right\} de_k^k - 2 e_i^j \frac{d\mu}{dT} de_j^i \qquad (7.11)$$

For any adiabatic process, the heating and cooling effect due to strain can be easily examined by means of the above equation. For example, if the material is expanded in an adiabatic process such that $e_j^i = \delta_j^i \frac{I_1}{3}$, then

$$\left.\frac{dT}{dI_1}\right|_{e_j^i = \delta_j^i \frac{I_1}{3}} = - \frac{T}{c_v}\left\{ - I_1 \frac{d\lambda}{dT} + \frac{d}{dT}[\beta(2\mu+3\lambda)] - \frac{2}{3}\frac{d\mu}{dT} I_1 \right\} \qquad (7.12)$$

VIII. Equations of Motion and the Energy Equation for Small Displacements

Let the X^i represent a rectangular Cartesian coordinate system. Then if L is a characteristic length of the solid

x^i , y^i and u^i may be made dimensionless through division by L. Let be a characteristic time interval of the system. Define

$$x^{i*} = \frac{x^i}{L} \quad , \quad u^{i*} = \frac{u^i}{L} \quad , \quad \tau = \frac{t}{t_0} \; .$$

By small displacement, it is meant that if x^i is measured in the scale L, t in the scale t_0, u^{i*} are very small quantities, in comparison with unity. This implies that $\frac{\partial u^{i*}}{\partial x^{j*}}$, $\frac{\partial u^{i*}}{\partial \tau} = v^{i*}$ $\frac{\partial v^{i*}}{\partial x^j}$, $\frac{\partial^2 u^{i*}}{\partial \tau^2}$ are all small, and are of the same order as u^{i*} . Consequently $\theta = I_1$ is of the same order as u^{i*} . The acceleration term $\rho \frac{Dv^i}{Dt}$ may now be simplified as $u^{i\prime}$ follows:

In Cartesian coordinates

$$\rho \frac{Dv^i}{Dt} = \rho_0 (1+\theta) \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial y^j} \right)$$

$$= \rho_0 (1+\theta) \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^\ell} \frac{\partial x^\ell}{\partial y^j} \right)$$

$$= \frac{\rho_0 L}{t_0^2} \left\{ \frac{\partial^2 u^{i*}}{\partial \tau^2} + u^{i*} \frac{\partial v^{i*}}{\partial x^{\ell*}} \frac{\partial x^{\ell*}}{\partial y^{j*}} \right\} .$$

Clearly $\frac{\partial x^{\ell*}}{\partial y^{i*}}$ is of order 1 or smaller since $dy^i = dx^j(\delta^i_j + \frac{\partial u^i}{\partial x^j})$. Thus the only important term contributing to acceleration is

$$\frac{L}{t_0^2} \rho_0 \frac{\partial^2 u^*}{\partial \tau^2} = \rho_0 \frac{\partial^2 u^i}{\partial t^2} \; .$$

Similarly, the resultant force by stresses, $\frac{\partial \tau^{ij}}{\partial y^j}$ for small displacement and strain, can be approximated by $\frac{\partial \tau^{ij}}{\partial x^j}$, and the contribution of body force $\rho F^i(y^1, y^2, y^3, t)$ · by $\rho_0 F^i(x^1, x^2, x^3, t)$ provided $\frac{\partial F^i}{\partial x^j}$ are not of greater order than F^i. Thus for small displacements, the equations of motion become

$$\frac{\partial \tau^{ij}}{\partial x^j} + \rho_0 F^i = \rho_0 \frac{\partial^2 u^i}{\partial t^2}$$

In curvilinear coordinates they clearly are

$$\frac{D\tau^{ij}}{Dx^j} + \rho_0 F^i = \rho_0 \frac{\partial^2 u^i}{\partial t^2} \; . \tag{8.1}$$

With the same order of approximation (replacing $\frac{D}{Dt}$ by $\frac{\partial}{\partial t}$) the energy equation becomes

$$\frac{\partial Q}{\partial t} = T\frac{\partial S}{\partial t} \qquad (8.2)$$

where Q, S may be considered as the specific heat input per unit volume, since the change in ρ is of higher order than is considered here. If the empirical relation

$$\frac{\partial Q}{\partial t} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^i} k h^{ij} \frac{\partial}{\partial x^j} T$$

is used, where k is the coefficient of heat conduction, equation (8.2) becomes

$$\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^i} k h^{ij} \frac{\partial}{\partial x^j} T = T\frac{\partial S}{\partial t} . \qquad (8.3)$$

Substituting (7.11) in (8.3), one obtains, for an isotropic and elastic medium

$$\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^i} k h^{ij} \frac{\partial}{\partial x^j} T = c_v\frac{\partial T}{\partial t} + T\left\{-I_1\frac{d\lambda}{dT} + \frac{d}{dT}[\beta(2\mu+3\lambda)]\right\}\frac{\partial e^k_k}{\partial t} - 2Te^i_j\frac{\partial e^j_i}{\partial t}\frac{d\mu}{dT} \cdot \qquad (8.4)$$

VIII. Thermal Stress-Problem:

When stresses and strains are due to uneven thermal expansion both the energy equation and the equilibrium equation may be considerably simplified. Consider first the energy equation. Let the material be heated by an external agent at temperature T_0 . Then it is physically clear that e^i_j will be of the order αT_0 , where α is a very small number of the order of the linear thermal expansion coefficient. Hence, one may put

$$T = T_0 T^* \ , \quad \lambda = \lambda_0 \lambda^* \ , \quad \mu = \lambda_0 \mu^*$$

$$\beta = \beta^* \alpha T_0 \ , \quad k = k_0 k^* \ ,$$

Similarly put

$$e^i_j = \alpha T_0 e^{*i}_j$$

where the starred quantities are so chosen that their magnitudes are of order unity. Let a Cartesian coordinate system be used such that L is the characteristic linear dimension of the system. Then put

$$x^{*i} = \frac{x^i}{L} \quad , \quad \tau = \frac{t}{t_0} \quad ,$$

where t_0 is a characteristic time of the system. Then equation (8.4) becomes

$$\frac{\partial}{\partial x^{*i}} k^* \frac{\partial}{\partial x^{*i}} T = \frac{c_v L^2}{k_0 t_0} \left\{ \frac{\partial T^*}{\partial t} + \frac{\lambda_0 \alpha^2 T_0}{c_v} \left[\frac{\partial e^{*k}_k}{\partial \tau} \left\{ -e^{*k}_k \frac{d\lambda^*}{dT^*} - \frac{d}{dT^*}[\beta^*(2\mu^* + 3\lambda^*)] \right\} - 2T^* e^{*i}_j \frac{d\mu^*}{dT^*} \frac{\partial e^{*i}_j}{\partial \tau} \right] \right\}.$$

The parameter $\frac{\lambda_0 \alpha^2 T_0}{c_v}$ is in general a small quantity in comparison with unity*. Hence, regarding the starred quantities as of order unity, the terms multiplied by the small dimensionless parameter may be neglected. Hence the energy equation reduces to the ordinary heat conduction equation, i.e.,

$$\frac{1}{\sqrt{h}} \frac{\partial}{\partial x^i} k h^{ij} \frac{\partial}{\partial x^j} T = c_v \frac{\partial T}{\partial t} \quad . \qquad (8.5)$$

The calculation of temperature is thus independent of stress

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* For steel $E = 30 \times 10^6$ lb/in^2, $\nu = \frac{1}{3}$, $\alpha = 6.7 \times 10^{-6}$ in/in$^\circ$F, $c_v = 360$ in-lb/in^3 $^\circ$F, this parameter, $\lambda\alpha^2 T_0/c_v = 2.9 \times 10^{-6} T_0$. If T_0 is of the order 1000° F, $\lambda\alpha^2 T_0/c_v \sim 2.9 \times 10^{-3}$

or strain. From the above equation it also appears that the characteristic time governing the heat equation should be taken as

$$t_0 = \frac{c_v L^2}{k_0} \tag{8.6}$$

Reduction of the equilibrium equations can be discussed along the same line, when stresses and strains are produced primarily by uneven temperature distributions. Clearly one may put

$$\tau^{ij} = \lambda_0 \alpha T_0 \tau^{*ij} \quad , \quad u^i = \alpha T_0 L u^{*i}$$

where τ^{*ij} , u^{*i} are of order 1. Introducing these dimensionless quantities into the equation of equilibrium (8.1) one obtains*, in Cartesian coordinates,

$$\frac{\partial \tau^{ij}}{\partial x^{*}_{j}} = \frac{\rho_0 k^2}{\lambda_0 c_v^2 L^2} \frac{\partial^2 u^{*i}}{\partial \tau^2}$$

It will be recognized that $L\left(\frac{\rho_0}{\lambda_0}\right)^{\frac{1}{2}}$ is the characteristic time of wave motion in the solid. This is in general much smaller than the characteristic time for heat conduction. In a time interval of the order $\frac{c_v L^2}{k}$ either a large number of waves have occurred, or the wave motion has already disappeared. In either case no net effect due to motion is felt. Mathematically this means that since $\frac{\partial \tau^{*ij}}{\partial x^{*}_{j}}$ and $\frac{\partial^2 u^{*i}}{\partial \tau^2}$ may be reasonably regarded as of order unity, then if $\frac{\rho_0 k^2}{\lambda_0 c_v^2 L^2}$ is very small

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* F^i is put equal to zero for the sake of argument. If F^i is a static force, the conclusion in this section is still valid.

the inertia term on the right-hand side of the last equation may be dropped, i.e.

$$\frac{\partial \tau^{*ij}}{\partial x^{*j}} = 0 ,$$

or, more generally,

$$\frac{D\tau^{ij}}{Dx^{j}} = 0 . \tag{8.7}$$

Strains produced in solids by thermal expansion are small. Hence, for the problem of thermal stresses, the theory of small strain is valid. The analysis made in the last section shows, furthermore, that the temperature distribution can be calculated independently of stress and strain, by means of the ordinary heat conduction equation, (8.5). Then the stresses, strains, and displacements can be determined using the static equilibrium equations (8.7), the stress-strain relations, equations (7.5), and the relations between strains and displacements, equations (6.4). Aside from the calculation of temperature, which must be determined first, the thermal stress problem is identical with the ordinary elasticity problem, except that in the latter case λ and μ are usually regarded as constant. For the problem of thermal stress, E , μ , α must be experimentally determined as functions of temperature.

IX. The Minimum Energy Principle as Applied to Thermal Stress Problems

From the thermodynamic point of view, if the solid is in thermodynamic equilibrium, thermally as well as mechanically, its state of equilibrium may be determined by purely thermodynamical considerations. For instance, one requires, for the mechanical and thermodynamical equilibriumof an isolated system, that the total entropy of the system be a maximum. The cases of adiabatic and isothermal changes of state have been fully discussed by Gibbs(Ref. 4). However, for the study of thermal stresses where temperature gradient exists in a conducting medium, thermodynamical arguments become invalid, because thermal equilibrium is not realized as a result of the conduction of heat. Hence, if one wishes to formulate a variational problem for thermal stresses, one must rely on mechanical considerations only, namely the concept of virtual work. For the discussion of thermal stresses it has been shown that the following equations of equilibrium derived for small strains are adequate,

$$\frac{D\tau^{ij}}{DX^{j}} + F^{i}\rho = 0 ,$$

when the given body forces F^{i} and the given surface tractions τ^{ij} are static. Let u_i be the displacements satisfying both the equilibrium equations and boundary conditions for a given temperature distribution. It is assumed that the displacements u_i are specified on S' of the entire surfa ce S" and the surface tractions are specified on the rest of S". Let δu_i be arbitrary and small variations in u_i , compatible with the given constraints. Then δu_i are clearly zero on S'. Contracting the equilibrium equations by δu_i and integrating throughout the volume occupied by the solid, one obtains

$$\int \tau^{ij} \frac{D\delta u_i}{Dx^j} dv - \int F^i \rho \delta u_i dv - \int_{S'} \tau^{ij} \delta u_i ds_j = 0 \quad .$$

Now, $\tau^{ij} \frac{D\delta u_i}{Dx^j} = \delta w = \delta \psi \big|_T$. Consequently, if the temperature T is kept constant in the variation, the last condition is equivalent to

$$\delta \left\{ \int \psi \, dv - \int F^i \rho u_i dv - \int \tau^{ij} u_i ds_j \right\} = 0 \quad , \qquad (9.1)$$

since F^i, τ^{ij} are given, and δdv, δds_j are quantities whose variations are of higher order than those of others in the theory of small strains. (9.1) is then the necessary and sufficient conditions for the mechanical equilibrium of the system. If the expression in the bracket of the last equations is denoted by Φ , then Φ reaches a stationary value at the equilibrium state (mechanical but not necessarily thermal). If the expression for the free energy ψ derived previously is used, it can be shown that Φ actually reaches a minimum value. For

$$\int \delta \psi \big|_T dv = \int \tau^{ij} \frac{D\delta u_i}{Dx^j} dv + \int R \, dv \quad ,$$

where

$$R = \mu (\delta e^i_j)(\delta e^j_i) + \tfrac{1}{2} \lambda (\delta e^i_i)^2$$

Hence,

$$\delta \Phi = \int R \, dv \quad . \qquad (9.2)$$

But $R > 0$, unless $\delta e^i_j = 0$. Therefore Φ is an absolute minimum when mechanical equilibrium is established. This result may be stated in words as follows: of all displacements satisfying given boundary conditions, those satisfying the equilibrium conditions make Φ a minimum, for each temperature distribution. It may be recalled that within the present order of approximation, for each problem the temperature distribution may be determined from the heat conduction equation.

The counterpart of this result when the temperature is uniform everywhere is well known. The present result is an extension of the minimum energy principle in ordinary elasticity. It may be used as a basis for approximate solutions of thermal stress problems.

Part I

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Part II

Resistance to Thermal Shock

RESISTANCE TO THERMAL SHOCK

Introduction

Resistance to thermal shock is the strength of the material against failure during rapid heating or cooling. The current interest in high temperature designs using ceramic and ceramal materials, which are relatively weak in thermal shock in comparison with metals, necessitates a closer examination of this composite property. To help the search for better materials, it is important to determine the physical and mechanical properties of the material which contribute to high thermal shock resistance. W. G. Lidman and A. R. Bobrowsky (Ref. 1) argued that the resistance to thermal shock is proportional to the value of $\frac{\sigma_0 k}{\alpha E}$, where k is the coefficient of heat conduction of the material, σ_0 the ultimate strength, α the coefficient of linear thermal expansion, and E Young's modulus of elasticity. It is the purpose of this part to give a more complete theory of resistance to thermal shock, together with a comparison with experimental data.

The failure of material is the result of high stress. In rapid heating or cooling, the stress is generated by the non-uniform temperature and hence non-uniform expansion. Then, the problem of resistance to thermal shock is really the problem of computing the thermal stress under such conditions. In order to correlate the theory with experiment, the particular case analyzed is that of a plate heated or cooled rapidly. For this simple shape, the computation is quite easy. In fact, the problem of thermal stress in a plate is already known (cf. Ref. 2), while the temperature distributions in the plate have also been calculated before (cf. Ref. 3). The specific task of this discussion is then to bring these elements together and properly interpret the results for a clear understanding of the problem of thermal shock.

Thermal Stress in a Flat Plate

Consider a flat plate with a uniform thickness much smaller than its lateral dimensions. Then if cooling and heating of the plate take place through heat transfer at the surfaces, the temperature of the material is different at different distances to the plate surface, but for the greater part of the plate the temperature is independent of the location in the plane of the plate. If the median plane of the plate is taken as the x-y plane (Fig. 1), then the temperature T is a function of z only. Let σ_x, σ_y, σ_z, and e_x, e_y, e_z be the normal stresses and normal strains in the x, y and z directions respectively. The shearing stresses and shearing strains obviously vanish in this problem. The relations between the stresses and strains are

$$\begin{aligned} e_x &= \frac{1}{E}\{\sigma_x - \nu(\sigma_y + \sigma_z)\} + e_T, \\ e_y &= \frac{1}{E}\{\sigma_y - \nu(\sigma_z + \sigma_x)\} + e_T, \qquad (1) \\ e_z &= \frac{1}{E}\{\sigma_z - \nu(\sigma_x + \sigma_y)\} + e_T, \end{aligned}$$

where e_T is the thermal strain due to thermal expansion, and E, ν are Young's modulus and Poisson's ratio. E and ν may be functions of temperature. Due to the lack of shearing stresses and applied surface forces, σ_z is identically zero by the equilibrium of forces in the z-direction.

Now treat the case where the temperature is symmetrical with respect to the median plane of the plate. Then the only deformation the plate can undergo is a uniform lateral expansion. Therefore $e_x = e_y$ and both are constants. Then $\sigma_x = \sigma_y$. Assume that the plate is not restrained at the edges, then net tensile force at the edges must vanish or when b is the half-thickness of the plate $\int_{-b}^{b} \sigma_x dz = 0$. By using these relations, it can be easily deduced that,

$$e_x = e_y = \frac{\int_{-b}^{b} \frac{E e_T}{1-\nu} dz}{\int_{-b}^{b} \frac{E}{1-\nu} dz} \qquad (2)$$

$$e_z = \frac{-\int_{-b}^{b} \frac{E e_T}{\nu} dz}{\int_{-b}^{b} \frac{E}{\nu} dz}$$

and, hence,

$$\sigma_x = \sigma_y = \frac{E}{1-\nu} \left\{ \frac{\int_{-b}^{b} \frac{E e_T}{1-\nu} dz}{\int_{-b}^{b} \frac{E}{1-\nu} dz} - e_T \right\} \qquad (3)$$

These equations show that given temperature distribution in the plate, the stress and strain at any point can be easily calculated, provided E, ν, e_T are known functions of temperature. If E, ν are constants and take $e_T = \alpha T$ where T is the change in temperature from the initially uniform state and α the constant coefficient of thermal expansion, Equation (3) reduces to

$$\sigma_x = \sigma_y = \frac{E\alpha}{1-\nu}\left\{\frac{1}{2b}\int_{-b}^{b} T\,dz - T\right\} . \tag{4}$$

By introducing the dimensionless parameters

$$\xi = \frac{z}{b} \quad , \quad T^* = \frac{T}{T_o} \quad , \quad \bar{\bar{T}}^* = \frac{1}{2}\int_{-1}^{1} T^* d\xi \ ,$$

$$\sigma^* = \frac{\sigma_x (1-\nu)}{\alpha E T_o} \quad ,$$

where T_o is any convenient reference temperature, Equation (4) simplifies to

$$\sigma^* = \bar{\bar{T}}^* - T^* . \tag{5}$$

Hence the reduced stress is equal to the difference between the average reduced temperature and the local reduced temperature.

For the more general case, where the temperature in the plate is not symmetric with respect to median plane but still independent of x and y, we need only make the change

$$e_x = e_y = e + \beta z \tag{6}$$

where e and β are two constants so far undetermined. While the conditions $\sigma_x = \sigma_y$ and $\sigma_z = 0$, remain true. That a linear variation of e_x across the thickness is correct follows from the symmetry consideration imposed by our assumptions on the uniformity of temperature with respect to x and y, and the absence of external constraints. In addition to conditions of vanishing average tensile stress across the thickness of the plate, there is the further condition of vanishing bending moment, due to the assumption of free edges of the plate.

The stress-strain relations are now

$$e_x = e_y = \frac{1-\nu}{E}\sigma_x + e_T \ ,$$
$$e_z = -\frac{2\nu}{E}\sigma_x + e_T \ . \qquad (7)$$

In view of later development in this paper, it is convenient to take the thickness to be b instead of 2b as in the previous case. Then the force and moment conditions become

$$\int_{-b/2}^{b/2} \sigma_x \, dz = 0 \ ,$$
$$\int_{-b/2}^{b/2} \sigma_x z \, dz = 0 \ . \qquad (8)$$

By integrating now the first of Equations (7) with respect to z, and, by using means of (8), the following equations for the two unknowns e and β result,

$$e\int_{-b/2}^{b/2} \frac{E}{1-\nu} dz + \beta \int_{-b/2}^{b/2} \frac{Ez}{1-\nu} dz = \int_{-b/2}^{b/2} \frac{E e_T}{1-\nu} dz \ ,$$
$$e\int_{-b/2}^{b/2} \frac{Ez}{1-\nu} dz + \beta \int_{-b/2}^{b/2} \frac{Ez^2}{1-\nu} dz = \int_{-b/2}^{b/2} \frac{E e_T z}{1-\nu} dz \ . \qquad (9)$$

Here again, e, β and, hence, $e_x, e_y, \sigma_x, \sigma_y$ and e_z are determined when the temperature distribution is known, and when E, ν, e_T are known functions of temperature. If $e = \alpha T$, and if E, ν, α are constant, the following equation for the reduced stress is readily obtained,

$$\sigma^* = \bar{T}^* + 12\xi \int_{-1/2}^{1/2} T^* \xi \, d\xi - T^*, \tag{10}$$

where, as before,

$$\xi = \frac{z}{b}, \quad T^* = \frac{T}{T_0}, \quad \bar{T}^* = \int_{-1/2}^{1/2} T^* \, d\xi,$$

$$\sigma^* = \frac{\sigma_x (1-\nu)}{\alpha E T_0}$$

Because for this case the thickness of the plate is b, $-\frac{b}{2} \le z \le \frac{b}{2}$. Therefore when T is symmetric with respect to z, $\int_{-1/2}^{1/2} T^* \xi \, d\xi = 0$; then Equation (10) reduces to Equation (5).

To actually compute the thermal stress in a plate one needs to know the temperature distribution. For this purpose, consider two special cases: As the first case, take a plate heated or cooled uniformly on both sides by a constant heat source. As the second case, consider a plate uniformly heated or cooled on one side, and insulated on the other. As far as temperature distribution is considered, only the second case needs to be computed, since at the median plane of the plate in the first case there is no heat transfer due to the symmetry of the temperature distribution and the median plane can be considered as insulated. This is the reason that the thickness of the symmetric case is taken to be 2b and the thickness for the second case is taken to be b.

Transient Temperature Distribution:

Proceed now to the calculation of the temperature distribution in an infinite plate with one side uniformly heated or cooled and the other side insulated. Take b as the constant thickness, and x and y-axis of a Cartesian coordinate system in the middle plane. For small strain, i.e., when the shape and size of the plate are only slightly changed by thermal strain, the ordinary heat conduction equation may be used. If k is the coefficient of heat conduction, h the surface conductivity at $z = -b/2$, c the specific heat per unit volume, and if k, h, c are constants, the heat conduction equation is,

$$k \frac{\partial^2 T}{\partial z^2} = h(T - T_0) .$$

The boundary conditions are,

$$k \frac{\partial T}{\partial z} = h(T - T_0) , \quad z = -b/2 ;$$

$$\frac{\partial T}{\partial z} = 0 , \quad z = b/2 ,$$

where T_0 = const. is now taken as the temperature of the heat source. Initially we consider the plate to be at uniform temperature T = 0. The above equations may be written in dimensionless form if we introduce,

$$T^* = \frac{T}{T_0} \quad , \quad \xi = \frac{z}{b} \; ,$$

$$f = \frac{kt}{cb^2} \quad , \quad R = \frac{k}{hb} \; . \qquad (11)$$

R is the so-called resistance ratio, and f the Fourier number.

Thus

$$\frac{\partial^2 T^*}{\partial \xi^2} = \frac{\partial T^*}{\partial t}$$

and $\quad R \dfrac{\partial T^*}{\partial \xi} = 1 - T^* \quad$ at $\quad \xi = -1/2 \; , \qquad (12)$

and $\quad \dfrac{\partial T^*}{\partial \xi} = 0 \quad$ at $\quad \xi = 1/2 \; ,$

$\quad T^* = 0 \quad$ at $\quad f = 0 \; .$

The solution to the above equation is well known (cf. Ref. 3). They are written down here for reference:

$$T^* = 1 - \sum_{k=0}^{\infty} \frac{2e^{-\eta_k^2 f}}{\eta_k} \frac{\cos \eta_k (\frac{1}{2} - \xi)}{(1 + R + R^2 \eta_k^2) \sin \eta_k}$$

$$\bar{T}^* = 1 - \sum_{k=0}^{\infty} \frac{2e^{-\eta_k^2 f}}{\eta_k^2} \frac{1}{(1 + R + R^2 \eta_k^2)} \qquad (13)$$

$$\int_{-1/2}^{1/2} T^* \xi \, d\xi = - \sum_{k=0}^{\infty} \frac{2e^{-\eta_k^2 f}}{\eta_k^2} \frac{\frac{1}{\eta_k} - \frac{1}{\eta_k}\cos \eta_k - \frac{1}{2}\sin \eta_k}{(1 + R + R^2 \eta_k^2) \sin \eta_k}$$

where η_k are roots of the transcendental equation

$$R \eta \tan \eta = 1 \qquad (14)$$

For small values of f, Equations (13) may be expressed asymptotically as (cf. Ref. 4) ,

$$T^* = \operatorname{erfc}\left(\frac{\frac{1}{2}+\xi}{2\sqrt{\tau}}\right) + \operatorname{erfc}\left(\frac{\frac{3}{2}-\xi}{2\sqrt{\tau}}\right) - e^{\frac{1}{R}(\frac{1}{2}+\xi)+\frac{\tau}{R^2}}\operatorname{erfc}\left\{\frac{\frac{1}{2}+\xi}{2\sqrt{\tau}}+\frac{\sqrt{\tau}}{R}\right\}$$
$$- e^{\frac{1}{R}(\frac{3}{2}-\xi)+\frac{\tau}{R^2}}\operatorname{erfc}\left(\frac{\frac{3}{2}-\xi}{2\sqrt{\tau}}+\frac{\sqrt{\tau}}{R}\right) + O(\tau^{\frac{1}{2}}e^{-\frac{1}{4\tau}}) \quad ,$$

$$\bar{T}^* = \frac{2\tau^{\frac{1}{2}}}{\sqrt{\pi}} - R + R\,e^{\frac{\tau}{R^2}}\operatorname{erfc}\left(\frac{\sqrt{\tau}}{R}\right) + O(\tau^{\frac{1}{2}}e^{-\frac{1}{4\tau}}) \quad , \qquad (15)$$

$$\int_{-1/2}^{1/2} T^*\xi\,d\xi = -\frac{1}{2}\bar{T}^* + R + \tau - \frac{2R^2\tau^{\frac{1}{2}}}{\sqrt{\pi}} - R^2 e^{\frac{\tau}{R^2}}\operatorname{erfc}\left(\frac{\sqrt{\tau}}{R}\right) + O(\tau^{\frac{1}{2}}e^{-\frac{1}{4\tau}}) \quad ,$$

where $\operatorname{erfc} u$ is the complimentary error function, defined as

$$\operatorname{erfc} u = 1 - \operatorname{erf} u = \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-s^2}ds^2 \ .$$

Computation of Thermal Stresses

For the plate with both sides heated it follows from Equations (5), (13) and (15) that

$$\sigma^* = \sum_{k=0}^{\infty} \frac{2e^{-\eta_k^2\tau}}{\eta_k}\frac{1}{(1+R+R^2\eta_k^2)}\left(\frac{\cos\eta_k\xi}{\sin\eta_k} - \frac{1}{\eta_k}\right) \quad , \qquad (16)$$

$$\sigma^* = \frac{2\tau^{\frac{1}{2}}}{\sqrt{\pi}} - R + R\,e^{\frac{\tau}{R^2}}\operatorname{erfc}\left(\frac{\sqrt{\tau}}{R}\right) - \operatorname{erfc}\left(\frac{1+\xi}{\sqrt{\tau}}\right) - \operatorname{erfc}\left(\frac{1-\xi}{\sqrt{\tau}}\right)$$
$$+ e^{\frac{1+\xi}{R}+\frac{\tau}{R^2}}\operatorname{erfc}\left\{\frac{1+\xi}{2\sqrt{\tau}}+\frac{\sqrt{\tau}}{R}\right\} + e^{\frac{1-\xi}{R}+\frac{\tau}{R^2}}\operatorname{erfc}\left\{\frac{1-\xi}{2\sqrt{\tau}}+\frac{\sqrt{\tau}}{R}\right\} \qquad (17)$$
$$+ O(\tau^{\frac{1}{2}}e^{-\frac{1}{4\tau}}) \quad ,$$

where now $-1 \le \xi \le 1$. It is to be noted that, since

$$\sigma^* = \bar{T}^* - T^* \ ,$$

T^* being symmetric and monotonic to either side of the z-axis, maximum stress must occur at $\xi = \pm 1$ and $\xi = 0$. In the case of heating, i.e., $T \geq 0$, σ^* at $\xi = \pm 1$ will be a compression and that at $\xi = 0$ a tension. The converse is true during cooling. Consequently in the following numerical computation, stresses are computed only at $\xi = -1$ and $\xi = 0$.

Maximum tensile and compressive stresses in the plate are computed according to Equations (16) and (17), for various values of R. The result is plotted in Figs. 2 and 3. In the following paragraphs the approximate formulas for maximum stresses at $\xi = \pm 1$ and $\xi = 0$ for very small and very large values of R are given.

By Equations (16) and (17), at $\xi = \pm 1$,

$$\sigma^* = \sum_{k=0}^{\infty} \frac{2e^{-\eta_k^2 f}}{1 + R + R^2\eta_k^2}\left(R - \frac{1}{\eta_k^2}\right) , \tag{18}$$

$$\sigma^* = \frac{2f^{\frac{1}{2}}}{\sqrt{\pi}} + (1+R)\left\{e^{\frac{f}{R^2}} \operatorname{erfc}\left(\frac{\sqrt{f}}{R} - 1\right)\right\} + O\left(f^{\frac{1}{2}} e^{-\frac{1}{4f}}\right) . \tag{19}$$

When $R = 0$, σ^* obviously reaches its maximum at $f = 0$. Hence it is expected that for very small R, Equation (19) should be adequate for computational purposes. To determine maximum σ^*, set

$$\frac{\partial \sigma^*}{\partial f} = 0 .$$

Equation (19) then gives

$$\frac{1}{1+R} = \sqrt{\pi}\, e^{\frac{f}{R^2}}\, \frac{f^{\frac{1}{2}}}{R}\, \mathrm{erfc}\left(\frac{f^{\frac{1}{2}}}{R}\right) . \tag{20}$$

Hence, given $\sqrt{f}/R$, one can compute R and hence determine f. The applicability of Equation (20) is of course limited to small values of f, since it is an asymptotic solution. For four decimal point accuracy, f is limited to values below 0.1. Corresponding to this value of f, $R \sim 0.32$. It can be seen from the structure of Equation (20) that as R decreases to zero, $\sqrt{f}/R$ must increase to infinity, although f itself decreases. In fact,

$$\mathrm{erfc}(\sqrt{f}/R) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{f}{R^2}}\, \frac{R}{\sqrt{f}}\left[1 - \frac{1}{2}\left(\frac{R}{\sqrt{f}}\right)^2 + \frac{3}{4}\left(\frac{R}{\sqrt{f}}\right)^4 + O\left\{\left(\frac{R}{\sqrt{f}}\right)^6\right\}\right] .$$

To this order of approximation ,

$$f = \frac{3R^2}{1-\left(1-\frac{12R}{1+R}\right)^{1/2}} . \tag{21}$$

It can be shown that when $R < 0.005$, the above formula is adequate for four place accuracy in f. When f is known, it is but a simple matter to calculate σ^*. To see the influence of R on σ^*_{max}, compute the latter and, to the first order of approximation in R, obtain

$$\sigma^*_{max} = -1 + \sqrt{\frac{8}{\pi}}\, R^{1/2} \tag{22}$$

On the other hand when $R \rightarrow \infty$, f also increases as R increases. Hence one may use Equation (18), the first two terms of which give accurate answers up to four significant figures, when f is larger than 0.10. After a similar analysis as outlined previously, one obtains, for maximum σ^* at $\xi = \pm 1$,

$$\sigma^*_{max} = -\frac{2(\frac{1}{\eta_0^2} - R)}{1 + R + R^2\eta_0^2} e^{-\eta_0^2 f}\left(1 - \frac{\eta_0^2}{\eta_1^2}\right) ,$$

where η_0, η_1 are the first two roots of $R\eta \tan \eta = 1$, and

$$f = \frac{1}{\eta_1^2 - \eta_0^2} \log\left(\frac{1 + R + R^2\eta_0^2}{1 + R + R^2\eta_1^2} \; \frac{R\eta_1^2 - 1}{1 - R\eta_0^2}\right) .$$

Now, for large values of R,

$$\eta_0 = R^{-1/2} - \frac{1}{6} R^{-3/2} + O(R^{-5/2}) , \quad \eta_1 = \pi + \frac{1}{\pi} R^{-1} - \frac{1}{\pi^3} R^{-2} + O(R^{-3}) . \qquad (23)$$

By making use of these results, one obtains to the first order of approximation,

$$\sigma^*_{max} = \frac{1}{3R} , \quad R \to \infty . \qquad (24)$$

This result may be used when $R > 10$, to yield answers of two significant figures.

At the point $\xi = 0$, preliminary calculations show that when $R = 0$, $f = 0.102$, for maximum σ^*. Hence the Fourier series solution (16) may be used for small as well as large values of R, when numerical solutions are not carried beyond four significant figures. While Equations (23) correspond to large values of R, one must use, for small values of R,

$$\eta_0 = \frac{\pi}{2}(1+R) + O(R^2) ,$$

$$\eta_1 = \frac{3\pi}{2}(1+R) + O(R^2) . \qquad (25)$$

Accordingly, for σ^*_{max} at $\xi = 0$, the following relations hold,

$$\sigma^*_{max} \sim \frac{1}{6R} \quad , \quad R \to \infty \quad ; \tag{26}$$

$$\sigma^*_{max} \sim \frac{32}{9\pi^2} \frac{(\pi-2)^{9/8}}{(3\pi+2)^{1/8}} \left(1 - \frac{6\pi+5}{\pi-2} R\right), \quad R \to 0. \tag{27}$$

Comparison between Equations (22), (24), (26), (27) shows that the ratio of maximum stresses on the surface to that at the center is about 2 to 1 when $R \to \infty$, and is about 10 to 3 when $R \to 0$.

For the plate with one side heated and the other side insulated, the maximum stresses are computed for $\xi = -1/2$ and $\xi = 0$. The results are plotted on Figs. 2 and 3. It should be noted that in the present case the maximum stresses at $\xi = -1/2$ or $\xi = 0$ may not be the absolute maximum stresses in the whole plate, for given R. However, this deviation is not considered to be important, and the computed maximum stresses are representative.

For comparison with the previous case, the following result is given for σ^*_{max} at $\xi = -1/2$, when R is very small,

$$\sigma^*_{max} = -1 + \sqrt{\frac{32}{\pi}}\, R^{1/2} \tag{28}$$

Due to the difference in the definition in the thickness of the plate, for the same plate, R in this case is only one-half of the R in the previous case. Eq. (28) then shows that the thermal stress at the surface is smaller when only one side of a plate is heated, as is to be expected, since bending of the plate constitutes a relief of the thermal stress.

Application to the Evaluation of Thermal Shock Resistivity of Certain Ceramic and Ceramal Materials

An experimental method (Cf. Ref. 1 and 5), currently used by N.A.C.A. for the evaluation of the thermal shock properties of ceramic and ceramal materials is conducted in the following way: A specimen 2 inches in diameter and 1/4 inch in thickness is subjected to repeated cycles of heating and cooling until fracture occurs. Heating is performed in a furnace of controlled temperature, and cooling in an air stream at about room temperature. The specimen was first subjected to 25 temperature cycles with a furnace temperature of 1800°F. If it survived this treatment, 25 cycles were repeated in succession with furnace temperature at 2000°F, 2200°F, and 2400°F, excepting when failure occurred. The results obtained at the N.A.C.A. Lewis Flight Propulsion Laboratory are tabulated in Table I.

Test results showed that all specimens failed by tension during cooling, with the exception of stabilized zirconium dioxide which failed by shear. Hence only tensile strengths are given in Table I. A correction on the value of tensile strength for stabilized zirconium has already been made in the table.

W. G. Lidman and A. R. Bobrowsky reported (Cf. Ref. 1) that the increase of thermal shock strength was accompanied by an increase in the ratio $\frac{\sigma_o k}{\alpha E}$, where σ_o is the fracture tensile strength of the material under consideration. This correlation was qualitative and, in so far as the parameter $\frac{\sigma_o k}{\alpha E}$ is not dimensionless has only limited applications. However, the previous results on the calculation of maximum stress in a plate can be used to determine a more satisfactory correlation between the thermal shock test results and the physical parameters of the material.

The deciding factor whether a brittle material fails or does not fail is clearly the ratio of the failure strength to the actual stress. If the ratio exceeds one, it will fail, and vise versa. Moreover, general fatigue tests show that when a specimen is repeatedly stressed through cycles, the number of cycles that the specimen can survive bears a statistical relation to this ratio. This relation, however, may vary from material to material and becomes less definite when the stress cycles are operated close to the failure strength. Applied to the phenomena of thermal stresses, the above considerations should still be true, if all changes in physical and mechanical properties due to change of temperature are properly taken care of. Hence one still can take this ratio as the criterion for thermal shock evaluations, and write

$$S = \frac{\sigma_0}{\sigma} \tag{29}$$

where σ now stands for the maximum thermal stress actually reached in the specimen. When S is less than unity, we can predict with certainty that the specimen will fail. S can be called the thermal shock resistivity.

To find the maximum stress in the circular plate tested by N.A.C.A., the results of the previous section for a plate heated on both sides can be applied with the following assumptions:

(1) The plate is infinite in extent, and constant in thickness,

(2) The physical parameter E, ν, α, k, c, h are constant,

(3) The plate is heated (or cooled) uniformly on both sides by the same heat source (or sink). The time duration is such that at the end of each temperature cycle, a uniform temperature is re-established in the plate,

(4) Small strain.

Applied to our present problem, assumptions (4) and (3) are justified by the very nature of the experiment. Assumption (2) is made on account of the fact that little information is available on the variation of E, ν, α, k, c, and h with respect to temperature. Assumption (1) seems warranted in view of the uncertainty introduced by assumption (2), in addition to the fact that the test specimen was reasonably thin.

The theoretical results in the previous section computed for a plate heated or cooled on both sides may be expressed as ,

$$\sigma = \frac{\alpha E T_0}{1-\nu} \sigma^*_{max} \tag{30}$$

where σ is the maximum stress at $Z = \pm b$. These points are chosen simply because at these points, maximum stresses occur both during cooling and heating. By the definition of S, Eq. (29),

$$S = \frac{(1-\nu)\sigma_0}{\alpha E T_0} \frac{1}{\sigma^*_{max}} \tag{31}$$

S depends only on R and another dimensionless parameter $\frac{(1-\nu)\sigma_0}{\alpha E T_0}$. The dependence of S on R is not, in general, linear, as shown in Figure 2. However, as $\sigma^*_{max} \sim \frac{1}{3R}$, when R becomes sufficiently large, say R > 10, it is seen that S is, then, proportional to R or $\frac{k}{hb}$. Thus, when plates of the same thickness are tested in identical ways, we have,

$$S \propto \frac{(1-\nu)\sigma_0 k}{\alpha E}$$

for large values of R. This is the relation used by Lidman and Bobrowsky. As noted before, the above relation is sufficiently accurate for two significant figure computations, when R > 10.

To compare the test results, given in Table I, with our thermal shock resistivity, take h to be the same for all the specimens, since they were tested under identical conditions. The value h = 50 B.t.u./F° ft.^{2} hr gives a satisfactory correlation. We also put $\nu = 1/3$, and take 60°F as the room temperature. In the following table, the thermal shock resistivity theoretically evaluated is compared with the test results.

In Table II, except for ceramal, only data given in Table I have been used, together with Figs. 2,3, and Eq. (31). For ceramal, at 2000°F, σ_0 is taken to be 22,000 lb/in.^{2}, which is the mean value of the two tensile strengths at 1800°F and 2200°F. Where two different values for k are given, two values for S are evaluated. The reason that the tensile strength at high temperature is used instead of the tensile strength at low temperature becomes clear, if one recalls that none of the specimens failed by compression during the test, and the maximum stress occurs always on the surface.

The assertion that whether the specimen will or will not survive any thermal cycle will depend on the value of S, is satisfactorily confirmed by the correlation between experimental results and computations based on our simplified theory. The values of S for titanium carbide at 2000°F, 2200°F, and 2400°F are probably too high, since in the calculation the ultimate tensile strength at 1800°F is used instead of its exact values at these high temperatures, owing to the lack of data. The same comment goes for S at 2000°F, calculated for beryllium oxide.

Concluding Remarks

On the strength of this correlation with experiments presented in the previous section, one may conclude that for the type of experiment on thermal shocks, described above, the question of immediate failure upon application of a temperature gradient can be satisfactorily answered by calculating the thermal shock resistivity S according to the present simple theory. When S is smaller than unity, one can be quite sure that the specimen will suffer immediate failure.

The question as to how many cycles a specimen can undergo, before failure eventually occurs, remains open. The N.A.C.A. test procedure described above can not be used as the basis of such an analysis, since the thermal cycles were stopped as soon as 25 cycles were completed without failure. Yet, from the practical point of view this question is of greater interest than that of immediate failure. Ceramic and ceramal materials are used generally for machine parts that have only a limited life. Hence in the study of the relationship between the number of cycles N and thermal shock resistivity, the primary interest is in low or mediumly high values of N, in contrast to the usual fatigue tests on metals. Perhaps an empirical relation between S and N of the following type,

$$N = e^{m(S-1)} \quad , \quad S > 1 \; ,$$

may be constructed, where m is a positive number to be determined experimentally. . This sort of relationship, of course, can be used only in a statistical sense. A great deal of deviation is to be expected when S is close to unity.

It may also be pointed out that the thermal shock resistance is by no means an intrinsic property of a material. Thermal shock resistance depends rather strongly on the manner, in which heat is supplied, and the form of the specimen, as seen from the influence of b and h on S, according to the simple theory described above. This fact limits greatly the usefulness of thermal shock experiments carried out on special specimens and under special conditions to no more than an evaluation of the comparative merits of various materials against thermal shock. Such tests cannot predict the performance of any particular design subjected to a specified heating or cooling condition. For any specific design, the thermal stress induced in the material by the non-uniform and unsteady heating and cooling must be first determined. Then with the known strength of the material under repeated stress, the performance of the design can be ascertained. This is the rational theory of thermal shock resistance. It is the primary purpose of this paper to demonstrate what fundamental physical parameters are essential to the construction of a rational theory, by which the thermal shock phenomenon may be analyzed for any given material, of any shape and under any service condition.

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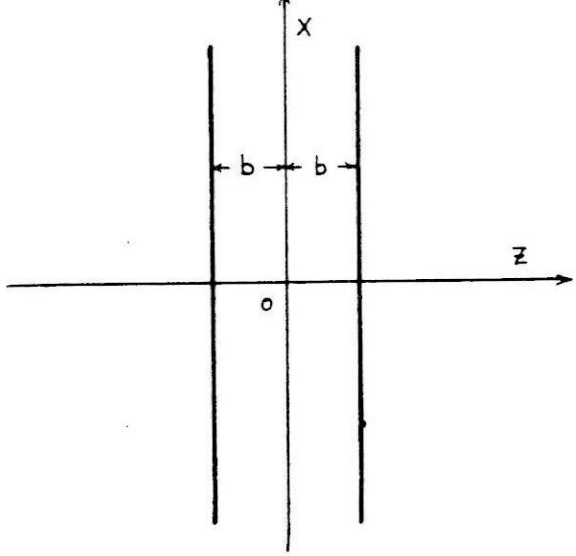
TABLE I

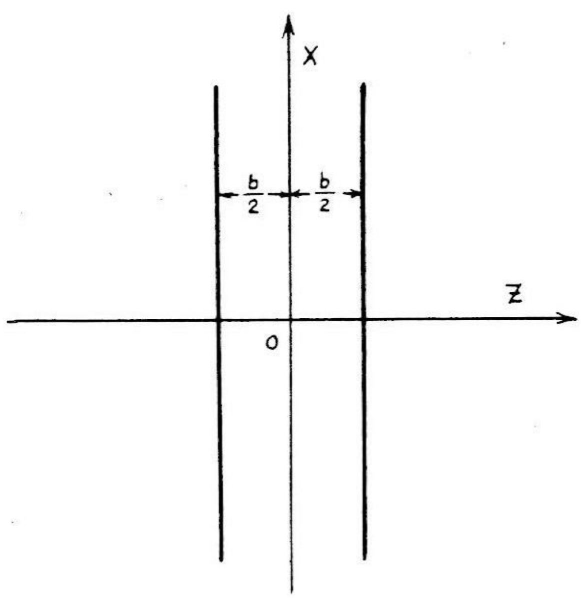
Material	Physical Properties				Thermal Shock Evaluation (Cycles Completed)			
	α(60°-1100°F) in/in.° F	K B.t.u./(hr)(ft²)(°F/in)	E lb./in.²	Tensile Strength lb/in²	1800°F	2000°F	2200°F	2400°F
Ceramal 80% TiC 20% CO	5.0 x 10^{-6}	247	5.5 x 10^7	22,000 (1800°F) 11,000 (2200°F)	25	25	25	25
Titanium Carbide (TiC)	4.1 x 10^{-6}	240	5.5 x 10^7	17,200 (1800°F)	25	25	25	21
Beryllium Oxide (BeO)	5.28 x 10^{-6}	223	4.28 x 10^7	6,200 (1800°F)	25	3	—	—
Zircon (Zr Si O_4)	2.24 x 10^{-6}	11.6 14.5(1832°F)	2.4 x 10^7	5,700 (1800°F)	1	—	—	—
Magnesium Oxide (MgO)	6.94 x 10^{-6}	17.7(2192°F) 40.8(2,012°F)	1.24 x 10^7	3,100 (1800°F)	½	—	—	—
94% Zirconium Dioxide 6% Calcium Oxide (CaO)	4.95 x 10^{-6}	6.4(2192°F)	2.5 x 10^7	6,750	—	—	—	—

TABLE II

Material	Thermal Shock Evaluation	Furnace Temperature Room Temperature (60°F)			
		1800°F	2000°F	2200°F	2400°F
Ceramal	Exp.	25	25	25	25
80% TiC					
20% CO	S	5.5	3.3	1.5	
Titanium	Exp.	25	25	25	21
Carbide	S	3.4	3.0	2.7	2.5
Beryllium	Exp.	25	3	--	--
Oxide	S	1.7	1.05	--	--
Zircon	Exp.	1	--	--	--
	S	0.45	--	--	--
		0.54			
		½			
Magnesium	Exp.	0.14	--	--	--
Oxide	S	0.28	--	--	--
94% Zirconium	Exp.	0			
Dioxide 6%	S	0.10	--	--	--
Calcium Oxide					



a. Both Sides Uniformly Heated



b. Uniformly Heated on Side z=-b/2
and Insulated on Side z=b/2

Fig. 1

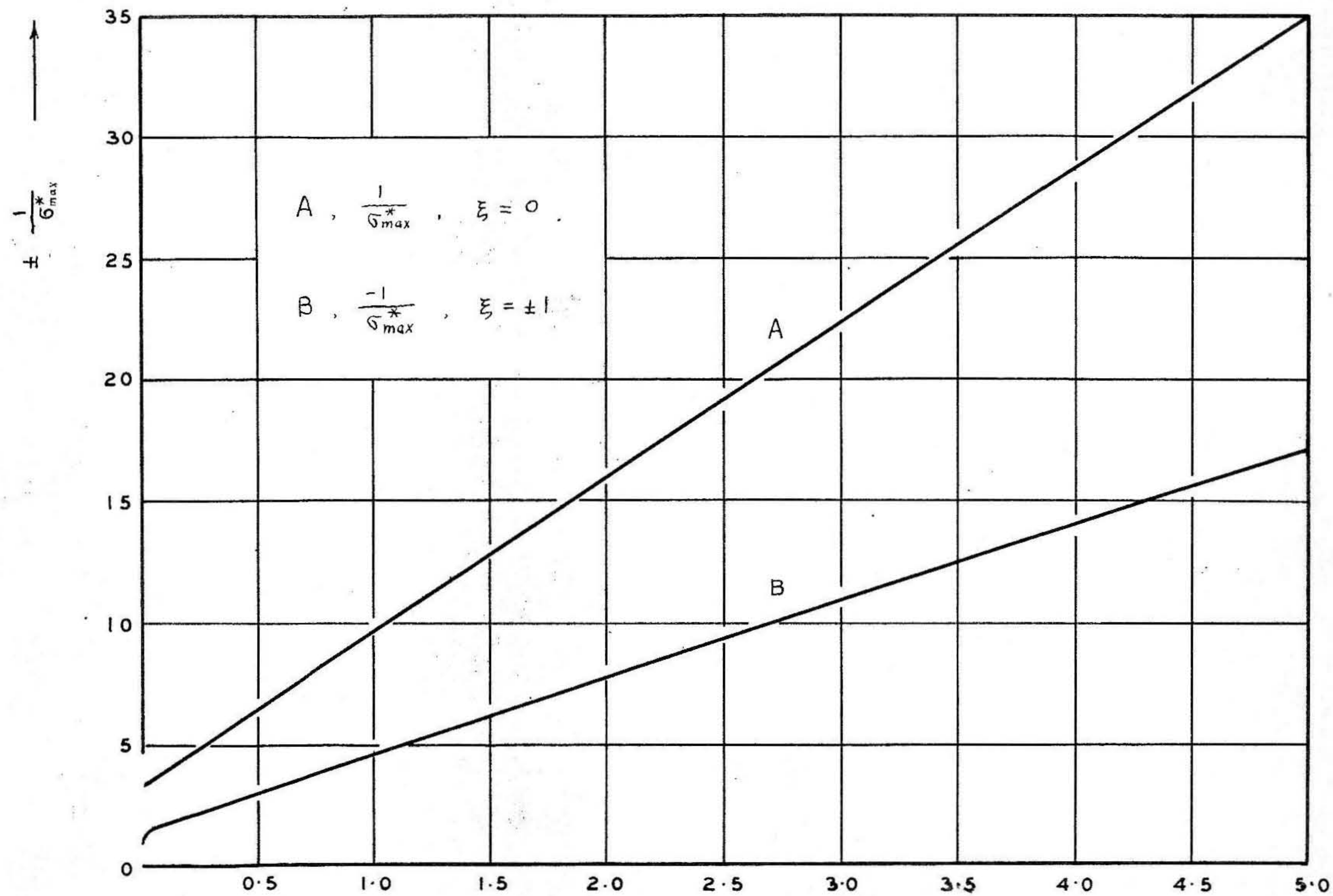


Fig. 2 Maximum Thermal Stress in a Flat Plate with Both Sides Uniformly Heated

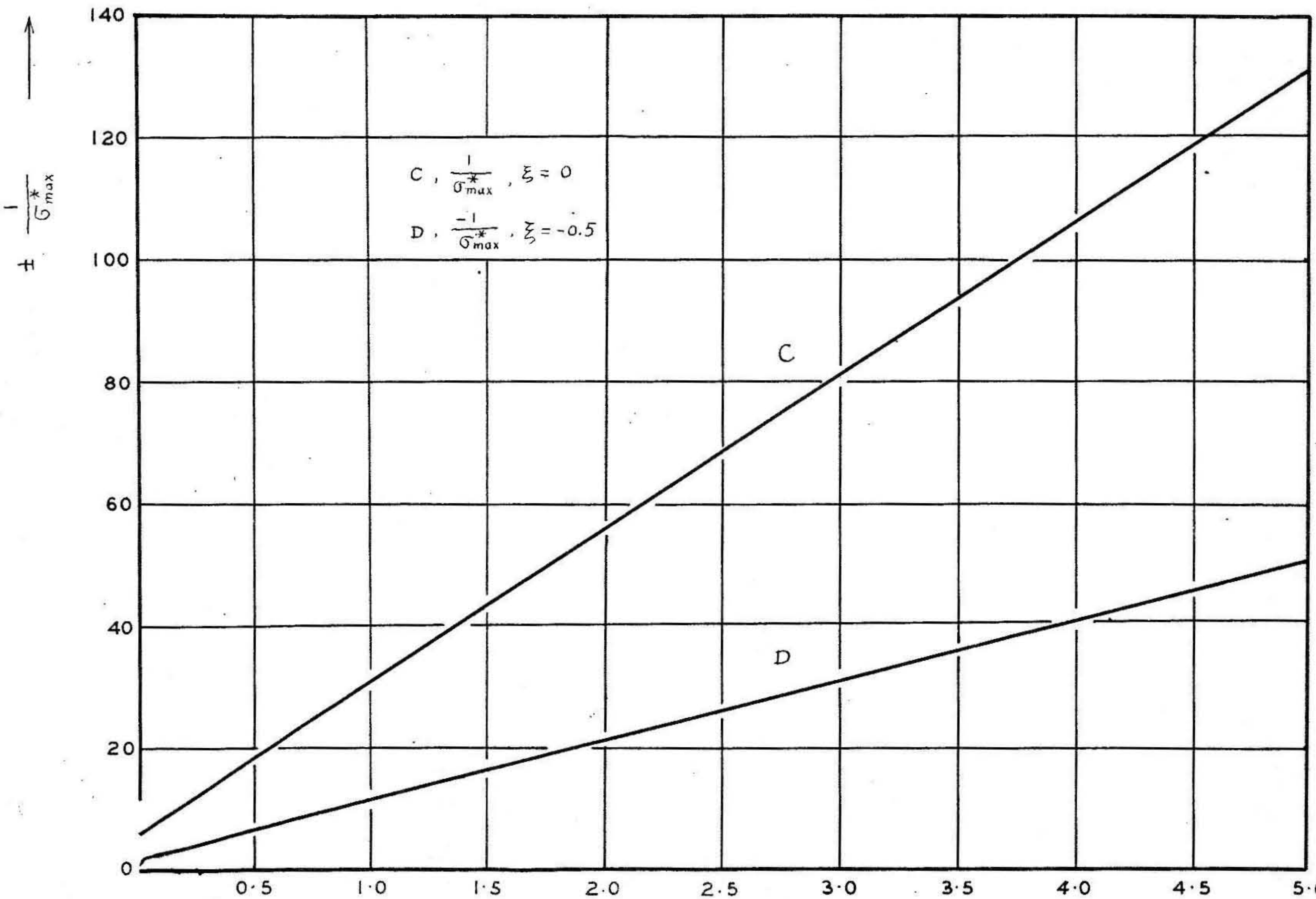


Fig. 3 Thermal Stresses in a Flat Plate Uniformly Heated at z=-b/2 and Insulated at z=b/2 R →

Part III

Thermal Stresses in Thin-Walled Cylindrical Shells and Plates

I. Thermal Stresses in a Thin Cylindrical Shell:

Relations between Sectional Forces and Moments and Strains in the Reference Surface

If a thin cylinder is slightly deformed the geometry of the deformed cylinder may be described by the strains and changes of curvature of a cylindrical reference surface embedded in the shell. In the usual theory of thin shells where the temperature is uniform, the reference surface is conveniently taken as the middle surface. When temperature gradients are present, however, it is not always expedient to make this choice, as will be seen presently. Accordingly the position of the middle surface shall, for the moment, remain unspecified.

When the displacements in a thin shell are sufficiently small, elements which are normal to the reference surface prior to deformation may be assumed to remain normal. On account of the thinness of the shell it may also be assumed that the stresses normal to the shell are zero. External loads normal to the surface are then sustained by bending moments and shearing forces. With this understanding the strain throughout the shell may be expressed in terms of displacements in the reference surface.

Let x, y, z be a local Cartesian coordinate system with origin on the cylindrical reference surface such that x points in the axial direction of the cylindrical shell, y is in the circumferential direction, and z is normal to the reference surface and points toward the center. Also, let u, v, w be displacements along the directions x, y, and z. Then the strains in the reference plane may be expressed as (Cf. Ref. 1)

$$e_x^{(0)} = \frac{\partial u}{\partial x} \quad , \quad e_y^{(0)} = \frac{1}{R}\frac{\partial v}{\partial \theta} - \frac{w}{R} \quad , \quad \gamma_{xy}^{(0)} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} \, , \qquad (1.1)$$

where $e_x^{(0)}$, $e_y^{(0)}$ are the normal strains, $\gamma_{xy}^{(0)}$ the shear strain, $d\theta$ the angle extended by a circumferential element, and R is the radius of the reference surface. The above relations are true for small strains and displacements. The corresponding strain components in any other surface are

$$e_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \quad , \quad e_y = \frac{1}{R}\frac{\partial v}{\partial \theta} - \frac{w}{R} - \frac{z}{R^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right) ,$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} - z\frac{z}{R^2}\left(\frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial v}{\partial x}\right) . \qquad (1.2)$$

By Hooke's Law, since $\sigma_z = 0$ by assumption,

$$\sigma_x = \frac{E}{1-\nu^2}\left\{(e_x + \nu e_y) - (1+\nu) e_T\right\} ,$$
$$\sigma_y = \frac{E}{1-\nu^2}\left\{(e_y + \nu e_x) - (1+\nu) e_T\right\} , \qquad (1.3)$$
$$\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} ,$$

where E is the Young's Modulus, ν the Poisson's ratio, and e_T the thermal expansion strain. The sectional forces and moments defined by

$$N_x = \int \sigma_x \, dz \quad , \quad N_y = \int \sigma_y \, dz \quad , \quad N_{xy} = \int \tau_{xy} \, dz \qquad (1.4)$$
$$M_x = -\int \sigma_x z \, dz \quad , \quad M_y = -\int \sigma_y z \, dz \quad , \quad M_{xy} = -\int \tau_{xy} z \, dz \, ,$$

may be expressed in terms of u, v, and w (Fig. 1). All integrations with respect to z from here on shall be understood to extend throughout the thickness of the shell.

Before carrying out the integration, an additional assumption will be made. It shall be assumed that ν , the Poisson's ratio is constant and independent of temperature. This assumption is justifiable since ν generally does not differ significantly from 1/3. Then, from equations (1.2), (1.3) and (1.4) one obtains,

$$N_x = D^{(0)}(e_x^{(0)} + \nu e_y^{(0)}) - D^{(1)}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{1}{R^2}\frac{\partial v}{\partial \theta} + \nu \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right) - N_T ,$$

$$-M_x = D^{(1)}(e_x^{(0)} + \nu e_y^{(0)}) - D^{(2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{1}{R^2}\frac{\partial v}{\partial \theta} + \nu \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right) - M_T ,$$

where,

$$D^{(n)} = \frac{1}{1-\nu^2}\int E z^n dz , \quad n = 0, 1, 2,$$

$$N_T = \frac{1}{1-\nu}\int E e_T dz ,$$

$$M_T = \frac{1}{1-\nu}\int E e_T z dz \qquad (1.5)$$

Similar expressions for the other sectional moments and forces may be obtained. N_T and M_T are the forces and moments required to keep all strains identically zero, as is seen from the expressions for N_x and M_x. In fact, when all strains are zero $M_x = M_y = M_T$ and $N_x = N_y = -N_T$. $D^{(0)}$ is equal to $\frac{Eb}{1-\nu^2}$ and $D^{(2)}$ is equal to $\frac{Eb^3}{12(1-\nu^2)}$, when E is constant. In accordance with the terminology used in the theory of plates and shells, $D^{(2)}$ shall be called the sectional modulus. $D^{(1)}$ is taken to be equal to zero if E is constant and if the middle surface is taken as the reference surface.

Since the position of the reference surface can be freely chosen, within the approximations of small displacement, one can make $D^{(1)}$ equal to zero by properly choosing the reference surface. This simple device not only makes the formal analysis of thermal stresses in plates and shells simpler and more comparable to that of the ordinary theory of plates and shells, but also is physically significant. Let x', y', z' be another Cartesian coordinate system with origin on the middle surface but otherwise similarly oriented as the x, y, z coordinate system. Let the origin of the x, y, z coordinate be located at z_o'. z_o' can now be determined by the condition $D^{(1)} = 0$. For

$$D^{(1)} = \frac{1}{1-\nu^2}\int E z\, dz = \frac{1}{1-\nu^2}\left(\int_{-b/2}^{b/2} E z' dz' - z_o \int_{-b/2}^{b/2} E\, dz'\right)$$

Hence,

$$z_o' = \frac{\int_{-b/2}^{b/2} E z' dz'}{\int_{-b/2}^{b/2} E\, dz'} \tag{1.6}$$

When the temperature distribution is known across the thickness, both $\int_{-b/2}^{b/2} E\, dz'$ and $\int_{-b/2}^{b/2} E z' dz'$ can be computed. Therefore z_o' is a well determined quantity for each temperature distribution. Consequently, this reference surface can be chosen for each problem. However, it must be remembered that this choice of reference surface is possible only when the temperature distribution is uniform along axial and circumferential directions. If the temperature distribution is not so, then a surface on which $D^{(1)} = 0$ will not be cylindrical and therefore can not serve the purpose of a reference surface.

Suppose that the temperature decreases as z or z_o' increases. Then, since E generally decreases with increase of temperature, z_o' will be positive. That is, the reference surface is shifted from the hotter side toward the cooler side. Physically this can be interpreted to mean a decrease in the effective thickness of the shell, since the hotter side now takes up a smaller share of the load than it otherwise would, if E did not decrease with increase of temperature.

With this agreement on the choice of reference surface the expressions for sectional moments and forces are greatly simplified. At this stage, it is convenient to express all equations in dimensionless forms by introducing the following dimensionless parameters: Let L be a characteristic length of the cylindrical shell, and define

$$
\begin{aligned}
&\varphi = \frac{u}{L}, \quad \psi = \frac{v}{L}, \quad \omega = \frac{w}{L}, \\
&\xi = \frac{x}{L}, \quad n = \frac{L}{R}, \\
&\mathcal{N}_\xi = \frac{N_x}{D^{(0)}}, \quad \mathcal{N}_\eta = \frac{N_y}{D^{(0)}}, \quad \mathcal{N}_{\xi\eta} = \frac{N_{xy}}{D^{(0)}}, \quad \mathcal{N}_T = \frac{N_T}{D^{(0)}}, \\
&\mathcal{M}_\xi = \frac{M_x L}{D^{(2)}}, \quad \mathcal{M}_\eta = \frac{M_y L}{D^{(2)}}, \quad \mathcal{M}_{\xi\eta} = \frac{M_{xy} L}{D^{(2)}}, \quad \mathcal{M}_T = \frac{M_T L}{D^{(2)}}, \\
&e_\xi = e_x, \quad e_\eta = e_y, \quad \gamma_{\xi\eta} = \gamma_{xy} .
\end{aligned}
\tag{1.7}
$$

Then, the reduced sectional moments and forces are related to φ, ψ, ω by the following equations:

$$
\begin{aligned}
\mathcal{N}_\xi &= \frac{\partial \varphi}{\partial \xi} - \nu n \omega + n \nu \frac{\partial \psi}{\partial \theta} - \mathcal{N}_T , \\
\mathcal{N}_\eta &= \nu \frac{\partial \varphi}{\partial \xi} - n \omega + n \frac{\partial \psi}{\partial \theta} - \mathcal{N}_T , \\
\mathcal{N}_{\xi\eta} &= \frac{1}{2}(1-\nu)\left(\frac{\partial \psi}{\partial \xi} + n \frac{\partial \varphi}{\partial \theta} \right) , \\
\mathcal{M}_\xi &= \frac{\partial^2 \omega}{\partial \xi^2} + n^2 \nu \frac{\partial^2 \omega}{\partial \theta^2} + n^2 \nu \frac{\partial \psi}{\partial \theta} + \mathcal{M}_T , \\
\mathcal{M}_\eta &= \nu \frac{\partial^2 \omega}{\partial \xi^2} + n^2 \frac{\partial^2 \omega}{\partial \theta^2} + n^2 \frac{\partial \psi}{\partial \theta} + \mathcal{M}_T , \\
\mathcal{M}_{\xi\eta} &= n(1-\nu)\left(\frac{\partial^2 \omega}{\partial \xi \partial \theta} + \frac{\partial \psi}{\partial \xi} \right).
\end{aligned}
\tag{1.8}
$$

Equations of Equilibrium

The usual equations of equilibrium for shells when expressed in terms of sectional forces and moments are derived simply from a consideration of the geometry of the deformed middle surface. Since the geometry of deformation of any other surface parallel to the middle surface is similar to that of the middle plane for small displacements, the equilibrium equations are also similar. If the temperature distribution is uniform in the axial direction and circumferential directions, the reference surface chosen by putting $D^{(1)} = 0$ is actually parallel to the middle surface. Hence, when this is the case, the equations of equilibrium can be written down immediately. There are only five equations of equilibrium-- three for equilibrium of forces, only two for the equilibrium of moments, since all moments are in the plane of the reference surface according to the approximate theory (Cf. Ref. 1).

$$\begin{aligned}
&\frac{1}{n}\frac{\partial N_\xi}{\partial \xi} + \frac{\partial N_{\xi\eta}}{\partial \theta} - \frac{1}{n} Q_\xi \frac{\partial^2 \omega}{\partial \xi^2} - \frac{1}{n} N_{\xi\eta}\frac{\partial^2 \psi}{\partial \xi^2} - Q_\eta\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial^2 \omega}{\partial \xi \partial \theta}\right) - N_\eta\left(\frac{\partial^2 \psi}{\partial \xi \partial \theta} - \frac{\partial \omega}{\partial \xi}\right) = 0 \quad ,\\
&\frac{\partial N_\eta}{\partial \theta} + \frac{1}{n}\frac{\partial N_{\xi\eta}}{\partial \xi} + \frac{1}{n} N_\xi \frac{\partial^2 \psi}{\partial \xi^2} - Q_\xi\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial^2 \omega}{\partial \xi \partial \theta}\right) + N_{\xi\eta}\left(\frac{\partial^2 \psi}{\partial \xi \partial \theta} - \frac{\partial \omega}{\partial \xi}\right)\\
&\qquad\qquad + Q_\eta\left(1 + n\frac{\partial \psi}{\partial \theta} + n\frac{\partial^2 \omega}{\partial \theta^2}\right) = 0 \quad ,\\
&\frac{1}{n}\frac{\partial Q_\xi}{\partial \xi} + \frac{\partial Q_\eta}{\partial \theta} + 2N_{\xi\eta}\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial^2 \omega}{\partial \xi \partial \theta}\right) + \frac{1}{n} N_\xi \frac{\partial^2 \omega}{\partial \xi^2} + N_\eta\left(1 + n\frac{\partial \psi}{\partial \theta} + n\frac{\partial^2 \omega}{\partial \theta^2}\right) - \Gamma = 0 \quad ,\\
&\frac{1}{n}\frac{\partial M_{\xi\eta}}{\partial \xi} + \frac{\partial M_\eta}{\partial \theta} + \frac{1}{n} M_\xi \frac{\partial^2 \psi}{\partial \xi^2} + M_{\xi\eta}\left(\frac{\partial^2 \psi}{\partial \xi \partial \theta} - \frac{\partial \omega}{\partial \xi}\right) + \frac{1}{n}\beta Q_\eta = 0 \quad ,\\
&-\frac{\partial M_{\xi\eta}}{\partial \theta} - \frac{1}{n}\frac{\partial M_\xi}{\partial \xi} + \frac{1}{n} M_{\xi\eta}\frac{\partial^2 \psi}{\partial \xi^2} + M_\eta\left(\frac{\partial^2 \psi}{\partial \xi \partial \theta} - \frac{\partial \omega}{\partial \xi}\right) - \frac{1}{n}\beta Q_\xi = 0 \quad ,
\end{aligned} \tag{1.9}$$

where

$$Q_\xi = \frac{Q_x}{D^{(0)}} \quad , \quad Q_\eta = \frac{Q_y}{D^{(0)}} \quad ,$$

$$\Gamma = \frac{pR}{D^{(0)}} \quad , \quad \beta = L^2 \frac{D^{(0)}}{D^{(2)}} \quad , \tag{1.10}$$

and Q_x, Q_y are the sectional shearing forces, p the normal pressure (opposite to z). Equation (1.8) and (1.9) are now the fundamental equations to be solved for N_ξ, N_η, $N_{\xi\eta}$, M_ξ, M_η, $M_{\xi\eta}$, Q_ξ, Q_η, φ, ψ, and ω.

Some Simple Solutions of Thermal Stresses in Thin Cylindrical Shells

Consider now a thin cylinder with non-uniform temperature distribution only across the thickness. If left free, the wall of this cylinder will deform in an axially symmetrical manner, but remains straight except at points very close to the ends. This can be shown readily. Since the problem is axially symmetrical, $\psi = 0$, and all quantities must depend only on ξ . Equations (1.8) and (1.9) are reduced now to

$$\mathcal{N}_\xi = \frac{d\varphi}{d\xi} - \nu n\omega - \mathcal{N}_T , \qquad (a)$$

$$\mathcal{N}_\eta = \nu \frac{d\varphi}{d\xi} - n\omega - \mathcal{N}_T , \qquad (b) \quad (1.11)$$

$$\mathcal{M}_\xi = \frac{d^2\omega}{d\xi^2} + \mathcal{M}_T , \qquad (c)$$

$$\mathcal{M}_\eta = \nu \frac{d^2\omega}{d\xi^2} + \mathcal{M}_T , \qquad (d)$$

and

$$\frac{1}{n}\frac{d\mathcal{N}_\xi}{d\xi} + \mathcal{N}_\eta \frac{d\omega}{d\xi} - \frac{1}{n} Q_\xi \frac{d^2\omega}{d\xi^2} = 0 , \qquad (e)$$

$$\frac{1}{n}\frac{dQ_\xi}{d\xi} + \frac{1}{n}\mathcal{N}_\xi \frac{d^2\omega}{d\xi^2} + \mathcal{N}_\eta = 0 , \qquad (f) \quad (1.12)$$

$$\frac{1}{n}\frac{d\mathcal{M}_\xi}{d\xi} + \mathcal{M}_\eta \frac{d\omega}{d\xi} + \frac{1}{n}\beta Q_\xi = 0 . \qquad (g)$$

$\mathcal{N}_{\xi\eta}$, $\mathcal{M}_{\xi\eta}$, Q_η are clearly zero by symmetry. Furthermore $\mathcal{N}_\xi = 0$, since there is no load acting on the cylinder. By neglecting $\mathcal{M}_\eta \frac{d\omega}{d\xi}$ in (g) as small, the following equation for ω is obtained from (f) and (g) ,

$$\frac{d^4\omega}{d\xi^4} + \beta(1-\nu^2) n^2 \omega = - n\beta(1-\nu)\mathcal{N}_T .$$

For a long cylinder, with free end at $\xi = 0$, the boundary conditions are $\mathcal{M}_\xi = 0$, $Q_\xi = 0$, and $\mathcal{N}_\xi = 0$. The solution for $\xi \geq 0$ is,

$$\omega = -\frac{\mathcal{N}_T}{n(1+\nu)} + \frac{\mathcal{M}_T}{2\delta^2} e^{-\delta\xi}(\sin\delta\xi - \cos\delta\xi) ,$$

where

$$4\delta^4 = n^2\beta(1-\nu^2)$$

Hence,

$$\mathcal{M}_\xi = \mathcal{M}_T \{ 1 - e^{-\delta\xi}(\cos\delta\xi + \sin\delta\xi) \} ,$$

$$\mathcal{M}_\eta = \mathcal{M}_T \{ 1 - \nu e^{-\delta\xi}(\cos\delta\xi + \sin\delta\xi) \} ,$$

$$Q_\xi = -2\mathcal{M}_T \frac{\delta}{\beta} e^{-\delta\xi} \sin\delta\xi , \qquad (1.13)$$

$$\mathcal{N}_\eta = 2\mathcal{M}_T e^{-\delta\xi} \frac{\delta^2}{n\beta} (\cos\delta\xi - \sin\delta\xi) .$$

δ is a positive number of the order $\frac{L}{\sqrt{Rb}}$. Therefore, these results show that if the cylinder is long and thin, *i.e.* $\frac{L}{\sqrt{Rb}} >> 1$ the thermal bending moment causes only a local disturbance in the deformation of the shell as a whole. For most parts of the shell, the deflection ω , and also φ are caused entirely by $\mathcal{N}_T$. This result may be interpreted to mean that in the calculation for the general deformation of a shell, the effect of $\mathcal{M}_T$ is only of secondary importance.*

Another simple result is the following. To keep the heated cylinder perfectly straight, such that

$$\varphi = k_1 \xi \quad , \quad \psi = 0 \quad , \quad \omega = -\frac{k_2}{n} \tag{1.14}$$

where k_1 , k_2 are respectively the axial and normal strains in the reference surface, the following end conditions are required

$$\begin{aligned} \mathcal{N}_\xi &= k_1 + \nu k_2 - \mathcal{N}_T \\ \mathcal{M}_\xi &= \mathcal{M}_T \end{aligned} \tag{1.15}$$

together with a normal load

$$\Gamma = \nu k_1 + k_2 - \mathcal{N}_T \tag{1.16}$$

Under this condition, then, throughout the shell

$$\begin{aligned} \mathcal{N}_\xi &= k_1 + \nu k_2 - \mathcal{N}_T \ , \\ \mathcal{N}_\eta &= \nu k_1 + k_2 - \mathcal{N}_T \ , \\ \mathcal{M}_\xi &= \mathcal{M}_\eta = \mathcal{M}_T \end{aligned} \tag{1.17}$$

whereas, Q_ξ , Q_η , $\mathcal{N}_{\xi\eta}$, $\mathcal{M}_{\xi\eta} = 0$. The interesting fact about this is that to arrest the bending of the cylindrical wall, the required bending moment is always $\mathcal{M}_T$, independent of the presence of $\mathcal{N}_\xi$ and Γ (stability considerations are excluded).

A Perturbation Method for Linearizing the Fundamental Equations

For a very long and thin cylinder where thermal stresses

---

* Instead of neglecting $\mathcal{M}_\eta \frac{d\omega}{d\xi}$ entirely, one may keep $\mathcal{M}_T \frac{d\omega}{d\xi}$. It can then be shown that the above conclusion remains true if L/R is sufficiently large.

caused by a radial temperature gradient are present, the displacements are essentially uniform both in the axial direction (φ = const.) and in the radial direction (ω = const.), as shown in the last section. If in addition a uniform normal load and a uniform axial load are also present, the long cylinder clearly will remain straight, provided the axial load does not reach critical conditions. However, when loads other than those just mentioned act on the cylinder, the cylinder generally will be distorted from its straight position. The magnitude of such distortions depends on the relative magnitudes of the disturbing loads and the stresses already present due to the combined effect of thermal stress, the uniform normal load and the axial force. If the disturbing loads are sufficiently small, these small distortions can be studied as small deviations from the straight state. Thus a perturbation procedure can be set up by which the fundamental equations for the cylinder may be linearized. As a result, the displacements, stresses, and strains are mainly caused by thermal expansion, the uniform normal load and the uniform axial load, whereas only small fractions of them are due to the additional external loads, which shall be called the perturbation loads. Likewise, the normal loads and axial load shall be called the primary loads.

For a cylinder of finite length there is no apriori reason to assume that the end disturbances can be studied as small perturbations. This depends entirely on the boundary conditions prescribed at the ends. However, if the prescribed end conditions are such that they differ only slightly from the straight cylinder under the primary loads, then the same perturbations procedure applies also at the ends.

It is shown in the last section that the state of strain due to the primary loads $\Gamma^{(0)}$ and $N_\xi^{(0)}$ can be completely characterized by k_1 and k_2. (Equations (1.11), (1.15), (1.16), (1.17)). Accordingly, the various quantities associated with k_1 and k_2 shall be denoted by a superscript (0), and all perturbation quantities caused by the disturbing loads shall be denoted by the same letters with superscript (1). The perturbation quantities are small by assumption so that their products may be neglected in comparison with the primary quantities. Hence one can put,

$$\begin{aligned}
&\varphi = \varphi^{(0)} + \varphi^{(1)}, \quad \psi = \psi^{(1)}, \quad \omega = \omega^{(0)} + \omega^{(1)}, \\
&N_\xi = N_\xi^{(0)} + N_\xi^{(1)}, \quad N_\eta = N_\eta^{(0)} + N_\eta^{(1)}, \quad N_{\xi\eta} = N_{\xi\eta}^{(1)}, \\
&M_\xi = M_\xi^{(0)} + M_\xi^{(1)}, \quad M_\eta = M_\eta^{(0)} + M_\eta^{(1)}, \quad M_{\xi\eta} = M_{\xi\eta}^{(1)}, \\
&Q_\xi = Q_\xi^{(1)}, \quad Q_\eta = Q_\eta^{(1)}, \quad \Gamma = \Gamma^{(0)} + \Gamma^{(1)},
\end{aligned} \tag{1.17a}$$

such that

$$\begin{aligned}
&\nu k_1 + k_2 - N_T = \Gamma^{(0)}, \\
&k_1 + \nu k_2 - N_T = N_\xi^{(0)}, \\
&N_\eta^{(0)} = \Gamma^{(0)}, \quad M_\xi^{(0)} = M_\eta^{(0)} = M_T, \\
&\varphi^{(0)} = k_1 \xi, \quad \psi^{(0)} = -\frac{k_2}{n},
\end{aligned} \tag{1.17b}$$

following the results in equations (1.14), (1.15), (1.16) and (1.17). Substituting these into equation (1.8) and (1.9), and neglecting products of perturbation quantities, which are assumed small, the following equations can be obtained

$$
\begin{aligned}
N_\xi^{(1)} &= \frac{\partial \varphi^{(1)}}{\partial \xi} - \nu n \omega^{(1)} + n\nu \frac{\partial \psi^{(1)}}{\partial \theta} , \\
N_\eta^{(1)} &= \nu \frac{\partial \varphi^{(1)}}{\partial \xi} - n\omega^{(1)} + n \frac{\partial \psi^{(1)}}{\partial \theta} , \\
N_{\xi\eta}^{(1)} &= \frac{1}{2}(1-\nu)\left(\frac{\partial \psi^{(1)}}{\partial \xi} + n \frac{\partial \varphi^{(1)}}{\partial \theta}\right), \\
M_\xi^{(1)} &= \frac{\partial^2 \omega^{(1)}}{\partial \xi^2} + n^2\nu \frac{\partial^2 \omega^{(1)}}{\partial \theta^2} + \nu n^2 \frac{\partial \psi^{(1)}}{\partial \theta} , \\
M_\eta^{(1)} &= \nu \frac{\partial^2 \omega^{(1)}}{\partial \xi^2} + n^2 \frac{\partial^2 \omega^{(1)}}{\partial \theta^2} + n^2 \frac{\partial \psi^{(1)}}{\partial \theta} , \\
M_{\xi\eta}^{(1)} &= (1-\nu)\, n \left(\frac{\partial^2 \omega^{(1)}}{\partial \xi \partial \theta} + \frac{\partial \psi^{(1)}}{\partial \xi}\right) ,
\end{aligned}
\qquad (1.18)
$$

and

$$
\begin{aligned}
&\frac{1}{n}\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \nu \frac{\partial^2 \psi^{(1)}}{\partial \xi \partial \theta} - \nu \frac{\partial \omega^{(1)}}{\partial \xi} + \frac{1}{2}(1-\nu)\left(\frac{\partial^2 \psi^{(1)}}{\partial \theta \partial \xi} + n \frac{\partial^2 \varphi^{(1)}}{\partial \theta^2}\right) = N_\eta^{(0)}\left(\frac{\partial^2 \psi^{(1)}}{\partial \xi \partial \theta} - \frac{\partial \omega^{(1)}}{\partial \xi}\right), \\
&\nu \frac{\partial^2 \varphi^{(1)}}{\partial \xi \partial \theta} - n \frac{\partial \omega^{(1)}}{\partial \theta} + n \frac{\partial^2 \psi^{(1)}}{\partial \theta^2} + \frac{1}{2n}(1-\nu)\left(\frac{\partial^2 \psi^{(1)}}{\partial \xi^2} + n \frac{\partial^2 \varphi^{(1)}}{\partial \xi \partial \theta}\right) \\
&\quad + \frac{n}{\beta}\left\{(1-\nu)\left(\frac{\partial^3 \omega^{(1)}}{\partial \xi^2 \partial \theta} + \frac{\partial^2 \psi^{(1)}}{\partial \xi^2}\right) + \nu \frac{\partial^3 \omega^{(1)}}{\partial \xi^2 \partial \theta} + n^2 \frac{\partial^3 \omega^{(1)}}{\partial \theta^3} + n^2 \frac{\partial^2 \psi^{(1)}}{\partial \theta^2}\right\} \\
&\quad = -\left(\frac{1}{n} N_\xi^{(0)} + \frac{1}{\beta} M_\xi^{(0)}\right)\frac{\partial^2 \psi^{(1)}}{\partial \xi^2} , \qquad (1.19) \\
&2(1-\nu)\, n \left(\frac{\partial^4 \omega^{(1)}}{\partial \xi^2 \partial \theta^2} + \frac{\partial^3 \psi^{(1)}}{\partial \xi^2 \partial \theta}\right) + \frac{1}{n}\left(\frac{\partial^4 \omega^{(1)}}{\partial \xi^4} + n^2\nu \frac{\partial^4 \omega^{(1)}}{\partial \xi^2 \partial \theta^2} + n^2\nu \frac{\partial^3 \psi^{(1)}}{\partial \xi^2 \partial \theta}\right) \\
&\quad + n\left(\nu \frac{\partial^4 \omega^{(1)}}{\partial \xi^2 \partial \theta^2} + n^2 \frac{\partial^4 \omega^{(1)}}{\partial \theta^4} + n^2 \frac{\partial^3 \psi}{\partial \theta^3}\right) - \beta\left(\nu \frac{\partial \varphi^{(1)}}{\partial \xi} - n\omega^{(1)} + n \frac{\partial \psi^{(1)}}{\partial \theta}\right) + \beta \Gamma^{(1)} \\
&\quad = \left(\frac{1}{n}\beta N_\xi^{(0)} - M_\eta^{(0)}\right)\frac{\partial^2 \omega^{(1)}}{\partial \xi^2} + n\beta N_\eta^{(0)}\left(\frac{\partial \psi^{(1)}}{\partial \theta} + \frac{\partial^2 \omega^{(1)}}{\partial \theta^2}\right) .
\end{aligned}
$$

One remarkable result shown by equations (1.19) is that $M_\xi^{(0)}$ and $M_\eta^{(0)}$ only appear in combination with $\frac{1}{n}\beta N_\xi^{(0)}$. Translated into physical quantities,

$$
\frac{M_\xi^{(0)}}{\frac{1}{n}\beta N_\xi^{(0)}} = \frac{M_\eta^{(0)}}{\frac{1}{n}\beta N_\xi^{(0)}} = \frac{M_T}{R\ N_x^{(0)}} = \frac{\int e_T E z\, dz}{R\, N_x^{(0)}(1-\nu)} .
$$

For transient heating from an initially even temperature, $e_T E$ is either all negative or all positive across the plate. Hence, by the mean value theorem

$$
\left| \int \frac{e_T E z\, dz}{1-\nu} \right| = \epsilon b \left| \int \frac{E e_T\, dz}{1-\nu} \right| = \epsilon b N_T
$$

where $\epsilon < 1$. Generally, for sudden heating, ϵ is much smaller than unity. Now if $|N_x^{(0)}|$ is larger than $\frac{b}{R}|N_T|$ then the terms involving $M_\xi^{(0)}$ and $M_\eta^{(0)}$ may be dropped.

A Possible Method for Determining Experimentally the Combined Stresses and Strains Under the Influence of Temperature and External Loads

If one neglects $\epsilon \frac{b}{R}|N_T|$ in comparison with $N_x^{(0)}$ on account of the smallness of the thickness b, equations (1.19) become:

$$
\begin{aligned}
&\frac{1}{n}\frac{\partial^2\varphi^{(1)}}{\partial\xi^2} + \nu\frac{\partial^2\psi^{(1)}}{\partial\xi\partial\theta} - \nu\frac{\partial\omega^{(1)}}{\partial\xi} + \frac{1}{2}(1-\nu)\left(\frac{\partial^2\psi^{(1)}}{\partial\theta\partial\xi} + n\frac{\partial^2\varphi^{(1)}}{\partial\theta^2}\right) = N_\eta^{(0)}\left(\frac{\partial^2\psi^{(1)}}{\partial\xi\partial\theta} - \frac{\partial\omega^{(1)}}{\partial\xi}\right) , \\
&\nu\frac{\partial^2\varphi^{(1)}}{\partial\xi\partial\theta} - n\frac{\partial\omega^{(1)}}{\partial\theta} + n\frac{\partial^2\psi^{(1)}}{\partial\theta^2} + \frac{1}{2n}(1-\nu)\left(\frac{\partial^2\psi^{(1)}}{\partial\xi^2} + n\frac{\partial^2\varphi^{(1)}}{\partial\xi\partial\theta}\right) \\
&\qquad + \frac{n}{\beta}\left\{(1-\nu)\left(\frac{\partial^3\omega^{(1)}}{\partial\xi^2\partial\theta} + \frac{\partial^2\psi^{(1)}}{\partial\xi^2}\right) + \nu\frac{\partial^3\omega^{(1)}}{\partial\xi^2\partial\theta} + n^2\frac{\partial^3\omega^{(1)}}{\partial\theta^3} + n^2\frac{\partial^2\psi^{(1)}}{\partial\theta^2}\right\} \\
&\qquad = -\frac{1}{n}N_\xi^{(0)}\frac{\partial^2\psi^{(1)}}{\partial\xi^2} , \qquad\qquad (1.20) \\
&2(1-\nu)n\left(\frac{\partial^4\omega^{(1)}}{\partial\xi^2\partial\theta^2} + \frac{\partial^3\psi^{(1)}}{\partial\xi^2\partial\theta}\right) + \frac{1}{n}\left(\frac{\partial^4\omega^{(1)}}{\partial\xi^4} + n^2\nu\frac{\partial^4\omega^{(1)}}{\partial\xi^2\partial\theta^2} + n^2\nu\frac{\partial^3\psi^{(1)}}{\partial\xi^2\partial\theta}\right) \\
&\qquad + n\left(\nu\frac{\partial^4\omega^{(1)}}{\partial\xi^2\partial\theta^2} + n^2\frac{\partial^4\omega^{(1)}}{\partial\theta^4} + n^2\frac{\partial^3\psi}{\partial\theta^3}\right) - \beta\left(\nu\frac{\partial\varphi^{(1)}}{\partial\xi} - n\omega^{(1)} + n\frac{\partial\psi^{(1)}}{\partial\theta}\right) + \beta\Gamma^{(1)} \\
&\qquad = \frac{\beta}{n}N_\xi^{(0)}\frac{\partial^2\omega^{(1)}}{\partial\xi^2} + n\beta N_\eta^{(0)}\left(\frac{\partial\psi^{(1)}}{\partial\theta} + \frac{\partial^2\omega^{(1)}}{\partial\theta^2}\right) .
\end{aligned}
$$

For a cylinder with uniform temperature and no initial strain, $e_T = 0$ and, hence, $N_T, M_T = 0$. All the previous equations remain true by putting $N_T, M_T = 0$, and by identifying the reference surface with the middle surface. To differentiate the cold cylinder problem from the previous one, all quantities having to do with the cold shell will be distinguished by a bar (-), e.g. $\bar{N}_x$, $\bar{N}_\xi$, etc. Consider now a similar perturbation procedure. The equations for $\bar{\varphi}^{(1)}$, $\bar{\psi}^{(1)}$ and $\bar{\omega}^{(1)}$ will be exactly the same as equation (1.20). Hence, the perturbation displacements obey the same equations. Moreover, equations relating $\bar{N}_\xi^{(1)}$, $\bar{N}_\eta^{(1)}$, $\bar{N}_{\xi\eta}^{(1)}$, $\bar{M}_\xi^{(1)}$, $\bar{M}_\eta^{(1)}$, $\bar{M}_{\xi\eta}^{(1)}$ and derivatives of $\bar{\varphi}^{(1)}$, $\bar{\psi}^{(1)}$ and $\bar{\omega}^{(1)}$ are identical to equations (2.8). Thus, within the present order of approximation, there is complete similarity between the perturbation displacements, sectional forces and moments of a heated cylindrical shell and those of an unheated cylindrical shell.

The dimensionless parameters in equations (1.20) are n and β , where n describes the geometry, and β describes the elastic property of the shell at various temperatures. ν is assumed to be constant for the materials under consideration.

Now if a cold shell is chosen that makes $\bar{\beta} = \beta$, and n = n and if it is loaded by an end force $\bar{N}_\xi^{(0)} = N_\xi^{(0)}$, a normal load $\bar{\Gamma}^{(0)} = \Gamma^{(0)}$, and if in addition $\bar{\Gamma}^{(1)} = \epsilon\Gamma^{(1)}$, then similarity between the two sets of linearized equations requires that

$$(\bar{\varphi}^{(1)}, \bar{\psi}^{(1)}, \bar{\omega}^{(1)}) = \epsilon (\varphi^{(1)}, \psi^{(1)}, \omega^{(1)})$$

$$(\bar{N}_\xi^{(1)}, \bar{N}_\eta^{(1)}, \bar{N}_{\xi\eta}^{(1)}, \bar{M}_\xi^{(1)}, \bar{M}_\eta^{(1)}, \bar{M}_{\xi\eta}^{(1)}, \bar{Q}_\xi^{(1)}, \bar{Q}_\eta^{(1)}) = \epsilon (N_\xi^{(1)}, N_\eta^{(1)}, N_{\xi\eta}^{(1)}, M_\xi^{(1)}, M_\eta^{(1)}, M_{\xi\eta}^{(1)}, Q_\xi^{(1)}, Q_\eta^{(1)})$$

everywhere in the two shells, provided these conditions are met on the boundaries. In general, $N_\xi^{(0)}$, $\Gamma^{(0)}$, $\Gamma^{(1)}$ can be estimated for a given problem. Hence $\bar{N}_\xi^{(0)}$, $\bar{\Gamma}^{(0)}$ are known and $\bar{\Gamma}^{(1)}$ may be suitably chosen. Using the boundary conditions for the original shell problem, the boundary conditions for the perturbation quantities can be calculated, once $N_\xi^{(0)}$, $\Gamma^{(0)}$ have been estimated. Consequently, the boundary conditions for the perturbation equations of the cold cylinder may be calculated. These and the boundary conditions satisfied by $\bar{N}_\xi^{(0)}$ and $\bar{\Gamma}^{(0)}$ then add up to form the complete boundary conditions for the cold shell. Hence, if the displacements, sectional forces and moments for the cold shell under this

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It can be shown that for an axially symmetric problem, it is not necessary to assume that $\frac{1}{n}\beta N_\xi^{(0)} \gg M_T$ in order to establish the similarity relations, because M_T now occurs only in one combination with $\frac{1}{n}\beta N_\xi^{(0)}$. It is then possible to make this composite quantity equivalent to $\bar{N}_\xi^{(0)}$.

set of boundary conditions are known, one may trace the above procedure backwards and calculate, in a very simple manner, the displacements, sectional forces and moments of the original problem. Theoretically, this is not very interesting, for it is just as easy or just as difficult to solve a cold cylindrical shell problem as it is to solve a heated one. Experimentally, however, there is a real advantage in so far as the difficulties for measurements of stress and strain at high and transient temperatures can be avoided by doing an experiment on a cold specimen.

Before entering into a more detailed discussion of the practical usefulness of this suggested scheme, the procedure outlined in the preceding paragraph will be formulated first:

(1) For a given cylindrical shell, with known boundary conditions (B_0) and known temperature distribution across the thickness, first evaluate n, β . The radius of the middle surface may be used to evaluate n.

(2) Make an estimate of $N_\xi^{(0)}$ and $\Gamma^{(0)}$, and then determine $\Gamma^{(1)}$.

(3) Modify the original boundary conditions by deducting the contributions due to $N_\xi^{(0)}$, M_T , $\Gamma^{(0)}$. For example, if the edge moment is initially M_ξ , it should be reduced by the amount M_T . Denote the new set of boundary conditions by (B_1).

(4) Choose a thin cylindrical shell such that

$$\bar{\beta} = \beta ,$$
$$\bar{n} = n .$$

(5) Apply to this cylinder at corresponding points normal loads such that $\bar{\Gamma}^{(1)} = \epsilon \Gamma^{(1)}$. Specifically, if ϖ , T , F , denote

distributed, line, and point loads, then

$$\bar{L}\,\bar{W}/\bar{D}^{(0)} = \epsilon\, L\, \overline{W} / D^{(0)} \quad ,$$
$$\bar{T}/\bar{D}^{(0)} = \epsilon\, T / D^{(0)} \quad ,$$
$$\bar{F}/\bar{L}\,\bar{D}^{(0)} = \epsilon\, F / L\, D^{(0)} \quad .$$

(6) Now add the boundary conditions contributed by

$$\bar{N}_{\xi}^{(0)} = N_{\xi}^{(0)} \quad ,$$
$$\bar{\Gamma}^{(0)} = \Gamma^{(0)} \quad ,$$

to the set of boundary conditions $\epsilon(B_1)$. The notation means that all boundary conditions (B_1) are to be magnified by the factor ϵ. The new boundary conditions resulting from the above addition are the boundary conditions to be applied on the comparison or cold shell.

(7) If the local strains are measured on both surfaces of the comparison cylinder, then the local strains at corresponding points can be computed in the following manner:

$$e_x = \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \quad .$$

Now, using the notations in (1.17a) and (1.17b)

$$e_x = \kappa_1 + \left(\frac{\partial \varphi^{(1)}}{\partial \xi} - \frac{z}{L}\frac{\partial \omega^{(1)}}{\partial \xi}\right) \quad .$$

But

$$\varphi^{(1)} = \epsilon^{-1}\,\bar{\varphi}^{(1)} = \epsilon^{-1}\left(\bar{\varphi} - \bar{\varphi}^{(0)}\right) \quad ,$$
$$\omega^{(1)} = \epsilon^{-1}\,\bar{\omega}^{(1)} = \epsilon^{-1}\left(\bar{\omega} - \bar{\omega}^{(0)}\right) ,$$

where

$$\frac{\partial \bar{\varphi}^{(0)}}{\partial \xi} = \bar{\kappa}_1 \quad , \quad \bar{\omega}^{(0)} = -\frac{\bar{\kappa}_2}{\bar{n}} = -\frac{\bar{\kappa}_2}{n} \quad ,$$

κ_1, κ_2 being the axial and circumferential strains. Moreover, since $N_{\xi}^{(0)} = \bar{N}_{\xi}^{(0)}$, $\Gamma^{(0)} = \bar{\Gamma}^{(0)}$, by equations (1.17b)

$$\bar{\kappa}_1 = \kappa_1 - \frac{N_T}{D^{(0)}(1+\nu)} \quad , \quad \bar{\kappa}_2 = \kappa_2 - \frac{N_T}{D^{(0)}(1+\nu)} \quad .$$

By means of these equations,

$$e_x = \kappa_1\left(1 - \frac{1}{\epsilon}\right) + \frac{1}{\epsilon}\,\frac{N_T}{D^{(0)}(1+\nu)} + \left(\frac{\partial \bar{\varphi}}{\partial \xi} - \frac{z}{L}\frac{\partial^2 \bar{\omega}}{\partial \xi^2}\right)\frac{1}{\epsilon}$$

is obtained. Now if $\bar{e}_{x[1]}$ and $\bar{e}_{x[2]}$ denote the strains on the inner and outer surfaces of the cold cylinder ($\bar{z} = \frac{\bar{b}}{2}$ and $\bar{z} = -\frac{\bar{b}}{2}$), then by equations (1.2)

$$\frac{\partial \bar{\varphi}}{\partial \xi} = \frac{1}{2}\left(\bar{e}_{x[1]} + \bar{e}_{x[2]} \right) ,$$
$$\frac{\partial^2 \bar{\omega}}{\partial \xi^2} = \frac{\bar{L}}{\bar{b}}\left(\bar{e}_{x[2]} - \bar{e}_{x[1]} \right) .$$

Hence

$$e_x = k_1\left(1-\frac{1}{\epsilon}\right) + \frac{1}{\epsilon}\frac{N_T}{D^{(0)}(1+\nu)} + \frac{1}{\epsilon}\left\{\frac{1}{2}\left(\bar{e}_{x[1]} + \bar{e}_{x[2]}\right) - \frac{\bar{z}}{\bar{b}}\frac{\bar{L}}{L}\left(\bar{e}_{x[2]} - \bar{e}_{x[1]}\right)\right\} .$$

Similarly

$$e_y = k_2\left(1-\frac{1}{\epsilon}\right) + \frac{1}{\epsilon}\frac{N_T}{D^{(0)}(1+\nu)} + \frac{1}{\epsilon}\left\{\frac{1}{2}\left(\bar{e}_{y[1]} + \bar{e}_{y[2]}\right) - \frac{\bar{z}}{L}\frac{\bar{L}}{\bar{b}}\left(\bar{e}_{y[2]} - \bar{e}_{y[1]}\right)\right\} ,$$
$$\gamma_{xy} = \frac{1}{\epsilon}\left\{\frac{1}{2}\left(\bar{\gamma}_{xy[1]} + \bar{\gamma}_{xy[2]}\right) - \frac{\bar{z}}{L}\frac{\bar{L}}{\bar{b}}\left(\bar{\gamma}_{xy[2]} - \gamma_{xy[1]}\right)\right\} . \qquad (1.22)$$

where $\bar{z}$ is referred to the reference surface (not the middle surface) in the heated shell.

For $\epsilon = 1$, equations (1.22) become

$$e_x = k$$
$$e_y = \frac{N_T}{D^{(0)}(1+\nu)} + \frac{1}{2}\left(\bar{e}_{y[1]} + \bar{e}_{y[1]}\right) - \frac{\bar{z}}{L}\frac{\bar{L}}{\bar{b}}\left(\bar{e}_{y[2]} - \bar{e}_{y[1]}\right) \qquad (1.23)$$
$$\gamma_{xy} = \frac{1}{2}\left(\bar{\gamma}_{xy[1]} + \bar{\gamma}_{xy[2]}\right) - \frac{\bar{z}}{L}\frac{\bar{L}}{\bar{b}}\left(\bar{\gamma}_{xy[2]} - \bar{\gamma}_{xy[1]}\right)$$

The terms $\frac{1}{2}\left(\bar{e}_{x[1]} + \bar{e}_{x[2]}\right)$, $\frac{1}{2}\left(\bar{e}_{y[1]} + \bar{e}_{y[2]}\right)$, $\frac{1}{2}\left(\bar{\gamma}_{xy[1]} + \bar{\gamma}_{xy[2]}\right)$ clearly express the strain in the middle plane, and $\frac{1}{\bar{b}}\left\{\bar{e}_{x[2]} - \bar{e}_{x[1]}\right\}$ $\frac{1}{\bar{b}}\left\{\bar{e}_{y[2]} - \bar{e}_{y[1]}\right\}$, $\frac{1}{\bar{b}}\left\{\bar{\gamma}_{xy[2]} - \bar{\gamma}_{xy[1]}\right\}$ are the three curvatures. When $\bar{e}_x$, $\bar{e}_y$, $\bar{\gamma}_{xy}$ are measured on the surfaces of comparison cylinders, then e_x , e_y , γ_{xy} may be directly calculated at any point. The stresses can then be computed according to (1.14).

Consider now the following example. A long cylindrical shell of mean radius 5 in. and wall thickness 1/4 in. is made of the high temperature alloy, Timken 16-25-6. (Nominal composition Cr. 16%, Ni 25%, Mo 6%). (Ref. 2) In the elastic range, the Young's Modulus is plotted as a function of temperature in Fig. 2. Assuming that the inner surface is suddenly heated to a temperature T_0 and the outer surface is insulated, all the pertinent data concerning the heated shell may be calculated in the following manner. It is assumed that T_0 is 1000°F, $R' = \frac{k}{hb} = 0$ and $f = \frac{kt}{cb^2} = 0.09$, where k is the thermal conductivity, h the surface conductivity, b the thickness of the plate, t time, and c the specific heat per unit volume. α may be taken as 9 x 10^{-6} in/in, and ν as 0.286. Under this particular set of conditions, z_0', $D^{(0)}$, $D^{(2)}$, N_T and M_T can be easily calculated if the temperature distribution is first computed as in a flat plate because the thickness to radius ratio is small enough. The temperature distribution is plotted in Fig. 3. Using this curve and the curve E/E_0 at various temperatures, z_0', $D^{(0)}$, $D^{(2)}$, N_T, M_T are computed graphically. (Fig. 4). The results are:

For $R' = \frac{k}{hb} = 0$, $f = 0.09$, $\alpha = 9 \times 10^{-6}$ in/in °F, $R = 5$ in, $b = 0.25$ in,

$$z_0' = -0.0386\,b \quad , (\;0\;) \quad ,$$
$$D^{(0)} = 0.908\,\frac{E_0 b}{1-\nu^2} , \left(\frac{E_0 b}{1-\nu^2} \right) \quad ,$$
$$D^{(2)} = 0.0462\,\frac{E_0 b^3}{1-\nu^2} \quad , \left(\frac{1}{12}\,\frac{E_0 b^3}{1-\nu^2} \right) ,$$
$$N_T = 0.272\,\frac{E_0 \alpha T_0 b}{1-\nu} \quad , \left(0.341\,\frac{E_0 \alpha T_0 b}{1-\nu} \right) ,$$
$$M_T = 0.155\,\frac{E_0 \alpha T_0 b^2}{1-\nu^2} \quad , \left(0.0798\,\frac{E_0 \alpha T_0 b^2}{1-\nu} \right) ,$$

where E_0 is the Young's Modulus at room temperature and is equal to 28.2 x 10^6 lg/in.^{2}. For E = constant the equivalent values for z_0', $D^{(0)}$, $D^{(2)}$, N_T and M_T are written out in the brackets as shown in the above tabulation. Substituting the numerical values for b, T_0, E_0, α and ν one obtains

$$N_T = 36{,}200 \text{ lb/in} \quad , \quad M_T = 5{,}160 \text{ lb}.$$

In order to make use of the linearized approximation for comparison with a cold cylinder, the condition is that

$$\left| \frac{M_T}{R N_x^{(0)}} \right| << 1$$

By assuming that $|M_T / R N_x^{(0)}| \leq 0.05$, one obtains

$$N_x^{(0)} = \pm 20{,}600 \text{ lb/in}^2 .$$

If this axial force is produced by internal pressure p, then

$$p \geq 8{,}280 \text{ lb/in}^2 .$$

That this pressure should be so large is due to the fact that R' is assumed to be zero. In practical applications R' is never equal to zero. As a result, M_T is generally much smaller than in this particular example.

To choose a comparison cylinder for this problem, one must make

$$\bar{\beta} = \beta$$
$$\bar{n} = n$$

where $\beta = L^2 \frac{D^{(0)}}{D^{(2)}}$, $n = \frac{L}{R}$. If $L = \bar{L}$, $R = \bar{R}$, then a little computation shows that

$$\beta = 19.7 \frac{L^2}{b^2} .$$

Consequently,

$$\frac{19.7}{b^2} = \frac{12}{\bar{b}^2} .$$

Then

$$\bar{b} = 0.782 \, b ,$$

a result which is independent of the Young's Modulus of the comparison cylinder. It also shows that the effective thickness is decreased, as is to be expected.

Two points must now be discussed regarding the practical applicability of the above scheme of comparison. It has been pointed out that the linearization procedure may become invalid when the edge conditions are unfavorable. Then the above scheme of comparison, strictly speaking, does not apply. However, this point may be disposed of by saying that the error involved is probably significant only in the immediate neighborhood of the boundary, and the information derived from a comparison cylinder may still be useful from the engineering point of view.

The second point also has to do with the boundary conditions. It is seen from the previous discussion that if the heated cylinder has a hinged edge, a definite moment, a definite radial displacement and a definite axial load must be applied at the corresponding end of the comparison cylinder. To apply an axial load is experimentally simple, but it is difficult to produce a given radial displacement and especially difficult to apply a definite moment on the edge. It appears to the author that this point must remain a serious difficulty with regard to the usefulness of the present theory.

II Thermal Stresses in a Thin Flat Plate:

The formulation of the general thermal stress problem in a thin flat plate can be done in the same manner as that of a shell. Again, the temperature distribution is assumed to vary only across the thickness, the the reference surface is taken in such a way that $D^{(1)} = \int E z \, dz = 0$. To illustrate the use of the free energy $\bar{\Psi}$ discussed in Part I, the general equations of equilibrium shall be derived in this section by the energy method.

Let x, y, z be a Cartesian coordinate system such that x, y are in the reference plane and z is normal to them. Then for small strains, σ_z, γ_{xz}, γ_{yz} may be assumed to be zero. Consequently, the stress-strain relations and free energy $\bar{\Psi}$ may be written as*

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + e_T ,$$
$$e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + e_T , \qquad (2.1)$$
$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} ,$$

and $$\bar{\Psi} - a + \frac{1}{2}\frac{E}{1-\nu}e_T^2 = \frac{1}{2}\frac{E}{1-\nu^2}\left\{(e_x+e_y)^2 - 2(1-\nu)\left(e_xe_y - \frac{1}{4}\gamma_{xy}^2\right)\right\} - \frac{e_T E(e_x+e_y)}{1-\nu} .$$

The strain expression may be written in two parts, one corresponding to extension in the reference plane and the other corresponding to bending about the reference plane. Thus one may write

$$e_x = e_x' + z e_x'' , \quad e_y = e_y' + z e_y'' , \quad \gamma_{xy} = \gamma_{xy}' + z\gamma_{xy}'' .$$

It is well known that if extension in the reference plane due to bending is included

$$e_x' = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad e_y' = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy}' = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} . \qquad (2.2)$$

Similarly, $$e_x'' = -\frac{\partial^2 w}{\partial x^2} , \quad e_y'' = -\frac{\partial^2 w}{\partial y^2} , \quad \gamma_{xy}'' = -2\frac{\partial^2 w}{\partial x\partial y} ,$$

where u, v, w are the displacements in the x, y, z directions respectively. With these expressions for the strains, recalling that $\int E z\, dz = 0$, and that ν = constant, one obtains

$$\int\left(\bar{\Psi} - a + \frac{1}{2}\frac{E}{1-\nu}e_T^2\right)dz = w' + w'' ,$$

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* See Part I

where $$\omega' = \frac{1}{2} D^{(0)} \{ (e_x' + e_y')^2 - 2(1-\nu)(e_x' e_y' - \frac{1}{4}\gamma_{xy}'^2) \} - (e_x' + e_y') N_T \quad ,$$
$$\omega'' = \frac{1}{2} D^{(2)} \{ (e_x'' + e_y'')^2 - 2(1-\nu)(e_x'' e_y'' - \frac{1}{4}\gamma_{xy}''^2) \} - (e_x'' + e_y'') M_T \; .$$

$D^{(0)}$, $D^{(2)}$, N_T , M_T have been identified previously. Let p be the normal load on the plate (regarded as negative in the positive direction). Then the virtual work done by p is $-\delta\iint p w \, dx\, dy$. The work done at the ends gives rise only to boundary terms, which do not affect the form of the equilibrium equations. Hence, the equilibrium equations may be obtained by setting*

$$\delta \iint (\omega' + \omega'' + p w) \, dx \, dy = 0 \qquad (2.1)$$

for arbitrary variations in the u, v, w, while T is kept constant at each point. Now,

$$\delta \iint p w \, dx\, dy = \iint p \delta w \, dx \, dy \; ,$$
$$\delta \iint \omega'' dx\, dy = \delta \iint \frac{D^{(2)}}{2} \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] dx\, dy + \delta \iint M_T \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) dx\, dy$$
$$= \iint D^{(2)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w \, \delta w \, dx \, dy \; .$$

In the last expression, the boundary terms have not been written out, since they will eventually cancel out on account of boundary conditions. To obtain $\delta\iint \omega' dx\, dy$, it is convenient to make use of N_x , N_y , N_{xy} , defined previously as the sectional normal forces and shearing forces. It can be shown that, aside from boundary terms,

$$-\delta \iint \omega' dx\, dy = \iint \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u \, dx\, dy + \iint \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v \, dx\, dy$$
$$+ \iint \left\{ N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \frac{\partial w}{\partial x} + \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \frac{\partial w}{\partial y} \right\} \delta w \, dx\, dy \, .$$

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* Part I, (9.2).

By (2.3) the equilibrium equations are therefore,

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 , \\ &\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 , \\ &D^{(2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = -p + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} . \end{aligned} \tag{2.2}$$

These are the same equations as one would obtain for a plate at constant temperature, confirming the fact that formally the equations of equilibrium cannot change. The only effect of temperature is on the form of the stress-strain relations. One now introduces Airy's stress function F such that

$$N_x = \frac{\partial^2 F}{\partial y^2} , \quad N_y = \frac{\partial^2 F}{\partial x^2} , \quad N_{xy} = - \frac{\partial^2 F}{\partial x \partial y} .$$

Recalling that N_x, N_y, N_{xy} are not independent but are connected by the compatibility relation,

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 ,$$

obtained from (2.2) by eliminating u, v, one obtains the following two fundamental equations,

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F = (1-\nu^2) D^{(0)} \left\{ \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right\} , \\ &D^{(2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = -p + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} . \end{aligned} \tag{2.4}$$

N_x, N_y, N_{xy}, M_x, M_y, M_{xy} are of course given by,

$$\begin{aligned} N_x &= D^{(0)}\left\{ \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right\} - N_T , \\ N_y &= D^{(0)}\left\{ \nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right\} - N_T , \\ N_{xy} &= \frac{1}{2} D^{(0)} (1-\nu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right) , \\ M_x &= M_T + D^{(2)} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) , \\ M_y &= M_T + D^{(2)} \left(\nu \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) , \\ M_{xy} &= (1-\nu) D^{(2)} \frac{\partial^2 w}{\partial x \partial y} . \end{aligned} \tag{2.5}$$

Introducing the following non-dimensional quantities,

$$\varphi = \frac{u}{L}, \quad \psi = \frac{v}{L}, \quad \omega = \frac{w}{L}, \quad \xi = \frac{x}{L}, \quad \eta = \frac{y}{L}$$

$$\mathcal{N}_\xi = \frac{N_x}{D^{(0)}}, \quad \mathcal{N}_\eta = \frac{N_y}{D^{(0)}}, \quad \mathcal{N}_{\xi\eta} = \frac{N_{xy}}{D^{(0)}},$$

$$\mathcal{M}_\xi = \frac{LM_x}{D^{(2)}}, \quad \mathcal{M}_\eta = \frac{LM_y}{D^{(2)}}, \quad \mathcal{M}_{\xi\eta} = \frac{LM_{xy}}{D^{(2)}},$$

$$\Gamma = \frac{pL}{D^{(0)}}, \quad \mathcal{M}_T = \frac{M_T L}{D^{(2)}}, \quad \mathcal{N}_T = \frac{N_T}{D^{(0)}},$$

$$F^* = \frac{F}{D^{(0)}L^2}, \quad \beta^{-1} = \frac{D^{(2)}}{L^2 D^{(0)}}, \quad e_\xi = e_x, \quad e_\eta = e_y, \quad \gamma_{\xi\eta} = \gamma_{xy},$$

The fundamental equations may be written as

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2 F^* = (1-\nu^2)\left\{\left(\frac{\partial^2\omega}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2\omega}{\partial\xi^2}\frac{\partial^2\omega}{\partial\eta^2}\right\},$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2 \omega = \beta\left(-\Gamma + \frac{\partial^2F^*}{\partial\xi^2}\frac{\partial^2\omega}{\partial\eta^2} - \frac{\partial^2F^*}{\partial\xi\partial\eta}\frac{\partial^2\omega}{\partial\xi\partial\eta} + \frac{\partial^2F^*}{\partial\eta^2}\frac{\partial^2\omega}{\partial\xi^2}\right). \qquad (2.6)$$

where

$$\mathcal{N}_\xi = \frac{\partial^2F^*}{\partial\eta^2}, \quad \mathcal{N}_\eta = \frac{\partial^2F^*}{\partial\xi^2}, \quad \mathcal{N}_{\xi\eta} = -\frac{\partial^2F^*}{\partial\xi\partial\eta}, \qquad (2.7)$$

and

$$\begin{aligned}
\mathcal{N}_\xi &= \frac{\partial\varphi}{\partial\xi} + \frac{1}{2}\left(\frac{\partial\omega}{\partial\xi}\right)^2 + \nu\frac{\partial\psi}{\partial\eta} + \frac{\nu}{2}\left(\frac{\partial\omega}{\partial\eta}\right)^2 - \mathcal{N}_T, \\
\mathcal{N}_\eta &= \nu\frac{\partial\varphi}{\partial\xi} + \frac{\nu}{2}\left(\frac{\partial\omega}{\partial\xi}\right)^2 + \frac{\partial\psi}{\partial\eta} + \frac{1}{2}\left(\frac{\partial\omega}{\partial\eta}\right)^2 - \mathcal{N}_T, \\
\mathcal{N}_{\xi\eta} &= \frac{1}{2}(1-\nu)\left(\frac{\partial\psi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta} + \frac{\partial\omega}{\partial\eta}\frac{\partial\omega}{\partial\xi}\right), \\
\mathcal{M}_\xi &= \mathcal{M}_T + \frac{\partial^2\omega}{\partial\xi^2} + \nu\frac{\partial^2\omega}{\partial\eta^2}, \\
\mathcal{M}_\eta &= \mathcal{M}_T + \nu\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}, \\
\mathcal{M}_{\xi\eta} &= (1-\nu)\frac{\partial^2\omega}{\partial\xi\partial\eta}.
\end{aligned} \qquad (2.8)$$

The problem of a flat plate having a thermal gradient of the present type is therefore reduced to the solution of equation (2.6). As remarked previously, these equations are formally the same as if no thermal gradient is present. The real difference, however, lies in the fact that equations (2.8) are no longer homogeneous.

To use the energy method for approximate solutions, (Ref. 3), an appropriate expression for ω which satisfies the boundary condition involving ω may be assumed with undetermined constants. Then by means of the first of equations (2.6), F may be solved. From (2.7) and (2.8) the strain components may now be determined. Using this information $\iint \omega' dx\,dy$, $\iint \omega'' dx\,dy$ can be calculated. The principle of minimum thermodynamic potential then enables one to obtain equations connecting the undetermined coefficients. The difficulty of the plate problem is in the solution of the bi-harmonic equations, such that all the appropriate boundary conditions are satisfied. The situation is more complicated when a temperature gradient exists, because it results in more complicated boundary conditions as shown by equations (2.8).

In the present discussion no attempt shall be made to solve the problem analytically. The objective shall be to discover, if possible, some similarity rule by means of which the stresses and strains in a heated plate may be predicted from tests made on cold specimens. One could proceed, in a similar manner as in the case of thin cylindrical shells, by a perturbation procedure. However, as in the case of a shell the edge conditions for the comparison plate may be difficult to secure experimentally. On the other hand, there is a much simpler method which can be used for plates if one is willing to use a curved plate of constant curvature. It is clear that if the heated plate is free from external constraints and loads, it will bend into a spherical shape, together with lateral extensions, when a temperature gradient exists and is in the direction normal to the plate. Under such conditions the sectional forces and moments must all vanish, although local stresses may exist. In the approximate theory of plates only the sectional forces and moments are important. Consequently, a heated plate, after free expansion, may be regarded as a curved plate of constant curvature, i.e., as if no local stress exists, so far as the calculation of deformation and strains is concerned. From this point of view, to obtain a comparison plate one need only choose a curved plate of appropriate curvature, somewhat over-sized to allow for the lateral expansions. To determine the displacements of free expansion, it should first be noticed that M_T is generally small. Therefore extension of the reference plane due to w may be neglected. This means that the non-linear terms, $(\frac{\partial w}{\partial \xi})^2$, $(\frac{\partial w}{\partial \eta})^2$, $(\frac{\partial w}{\partial \eta})(\frac{\partial w}{\partial \xi})$, may be dropped from the expressions (2.8). Hence, putting all the sectional moments and forces to zero and integrating, one obtains:

$$\omega = -\frac{M_T}{2}(\xi^2+\eta^2) \ , \quad \varphi = \frac{N_T}{1+\nu}\xi \ , \quad \psi = \frac{N_T}{1+\nu}\eta \ , \qquad (2.9)$$

the displacement at $\xi = \eta = 0$ being taken as zero. The curvature of the deformed plate is clearly given by $-\frac{M_T}{D^{(2)}}$. Put now $\varphi = \varphi^{(0)} + \varphi^{(1)}$, $\psi = \psi^{(0)} + \psi^{(1)}$, $\omega = \omega^{(0)} + \omega^{(1)}$, where $\varphi^{(0)}$, $\psi^{(0)}$, $\omega^{(0)}$ are given in equations (2.9), and substitute them into (2.6). The resultant equations are

$$\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^2 \omega^{(1)} = \beta\left\{-\Gamma + \frac{\partial^2 F^*}{\partial \eta^2}\frac{\partial^2(\omega^{(1)}+\omega^{(0)})}{\partial \xi^2} - 2\frac{\partial^2 F^*}{\partial \xi \partial \eta}\frac{\partial^2(\omega^{(1)}+\omega^{(0)})}{\partial \xi \partial \eta} + \frac{\partial^2 F^*}{\partial \xi^2}\frac{\partial^2(\omega^{(1)}+\omega^{(0)})}{\partial \eta^2}\right\},$$

$$\begin{aligned}\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^2 F^* &= (1-\nu^2)\left\{\left(\frac{\partial^2 \omega^{(1)}}{\partial \xi \partial \eta}\right)^2 - \left(\frac{\partial^2 \omega^{(1)}}{\partial \xi^2}\right)\left(\frac{\partial^2 \omega^{(1)}}{\partial \eta^2}\right) + M_T\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\omega^{(1)} - M_T^2\right\} \\ &\cong (1-\nu^2)\left\{\left(\frac{\partial^2 \omega^{(1)}}{\partial \xi \partial \eta}\right)^2 - \left(\frac{\partial^2 \omega^{(1)}}{\partial \xi^2}\right)\left(\frac{\partial^2 \omega^{(1)}}{\partial \eta^2}\right) + M_T\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\omega^{(1)}\right\} ,\end{aligned} \qquad (2.10)$$

since $\mathcal{M}_T$ is assumed to be small. The relations connecting $\mathcal{N}_\xi$, $\mathcal{N}_\eta$, $\mathcal{N}_{\xi\eta}$, $\mathcal{M}_\xi$, $\mathcal{M}_\eta$, $\mathcal{M}_{\xi\eta}$ and $\omega^{(1)}$, $\varphi^{(1)}$, $\psi^{(1)}$ remain the same as (2.8) if $\mathcal{N}_T$, $\mathcal{M}_T$ in these expressions are put to zero,* i.e.,

$$\begin{aligned}
\mathcal{N}_\xi &= \frac{\partial \varphi^{(1)}}{\partial \xi} + \frac{1}{2}\left(\frac{\partial \omega^{(1)}}{\partial \xi}\right)^2 + \nu \frac{\partial \psi^{(1)}}{\partial \eta} + \frac{\nu}{2}\left(\frac{\partial \omega^{(1)}}{\partial \eta}\right)^2 , \\
\mathcal{N}_\eta &= \frac{\partial \psi^{(1)}}{\partial \eta} + \frac{1}{2}\left(\frac{\partial \omega^{(1)}}{\partial \eta}\right)^2 + \nu \frac{\partial \varphi^{(1)}}{\partial \xi} + \frac{\nu}{2}\left(\frac{\partial \omega^{(1)}}{\partial \xi}\right)^2 , \\
\mathcal{N}_{\xi\eta} &= \frac{1}{2}(1-\nu)\left(\frac{\partial \psi^{(1)}}{\partial \xi} + \frac{\partial \varphi^{(1)}}{\partial \eta} + \frac{\partial \omega^{(1)}}{\partial \xi}\frac{\partial \omega^{(1)}}{\partial \eta}\right) , \\
\mathcal{M}_\xi &= \frac{\partial^2 \omega^{(1)}}{\partial \xi^2} + \nu \frac{\partial^2 \omega^{(1)}}{\partial \eta^2} , \\
\mathcal{M}_\eta &= \nu \frac{\partial^2 \omega^{(1)}}{\partial \xi^2} + \frac{\partial^2 \omega^{(1)}}{\partial \eta^2} , \\
\mathcal{M}_{\xi\eta} &= (1-\nu)\frac{\partial^2 \omega^{(1)}}{\partial \xi \partial \eta} .
\end{aligned} \tag{2.11}$$

The set of equations (2.10), (2.11) will be recognized to be the same equations for a slightly curved plate. Hence, one may immediately conclude that the problem of a heated plate of the present type may be compared with that of a curved plate such that $\bar{\Gamma} = \Gamma$, $\bar{\beta} = \beta$, and $-\frac{M_T L}{D^{(0)}} = \frac{\bar{L}}{\bar{R}}$, where the barred quantities are referred to the comparison cylinder, $\frac{1}{\bar{R}}$ being the curvature of the comparison cylinder. The boundary conditions for the comparison plate can be easily established by taking $\bar{\omega} = \omega - \omega^{(0)}$, $\bar{\varphi} = \varphi - \varphi^{(0)}$, $\bar{\psi} = \psi - \psi^{(0)}$, together with similar relations between their derivatives.

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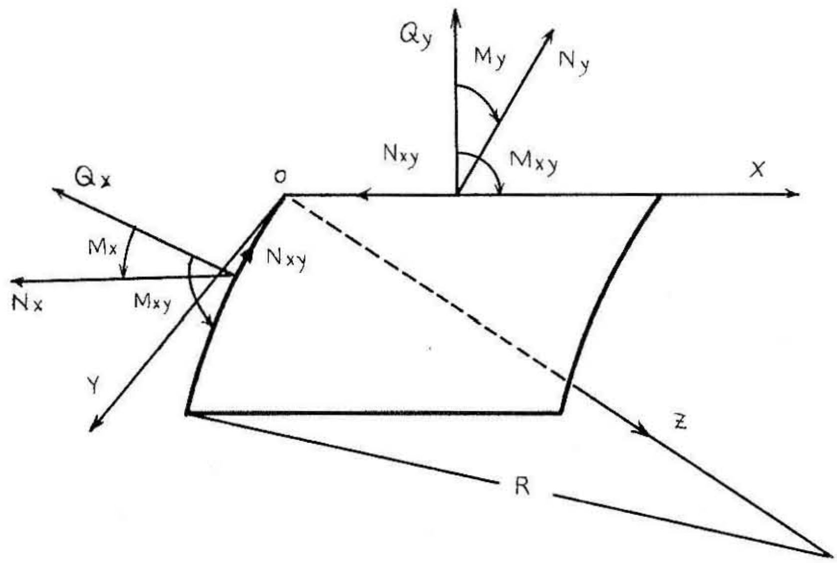
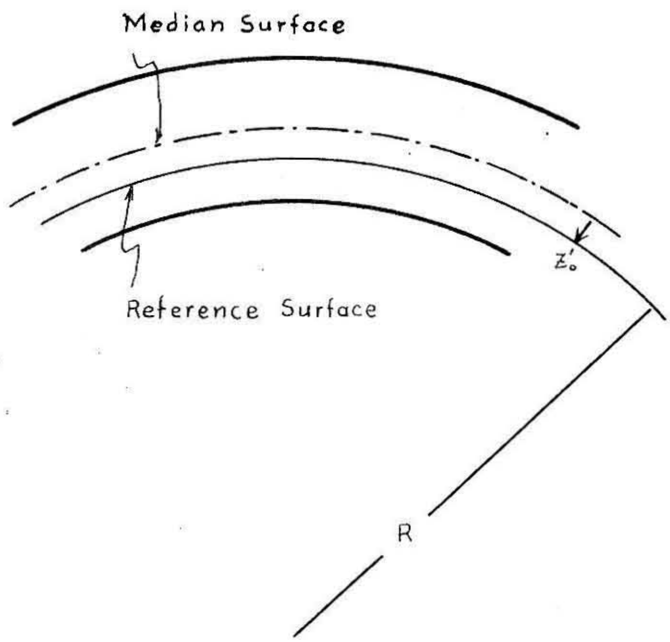
* Stretching of reference plane due to $\mathcal{M}_T$ being neglected.



a. Forces and Moments on an Element of a Cylindrical Shell



b. The Median Surface and the Reference Surface of a Cylindrical Shell

Fig. 1

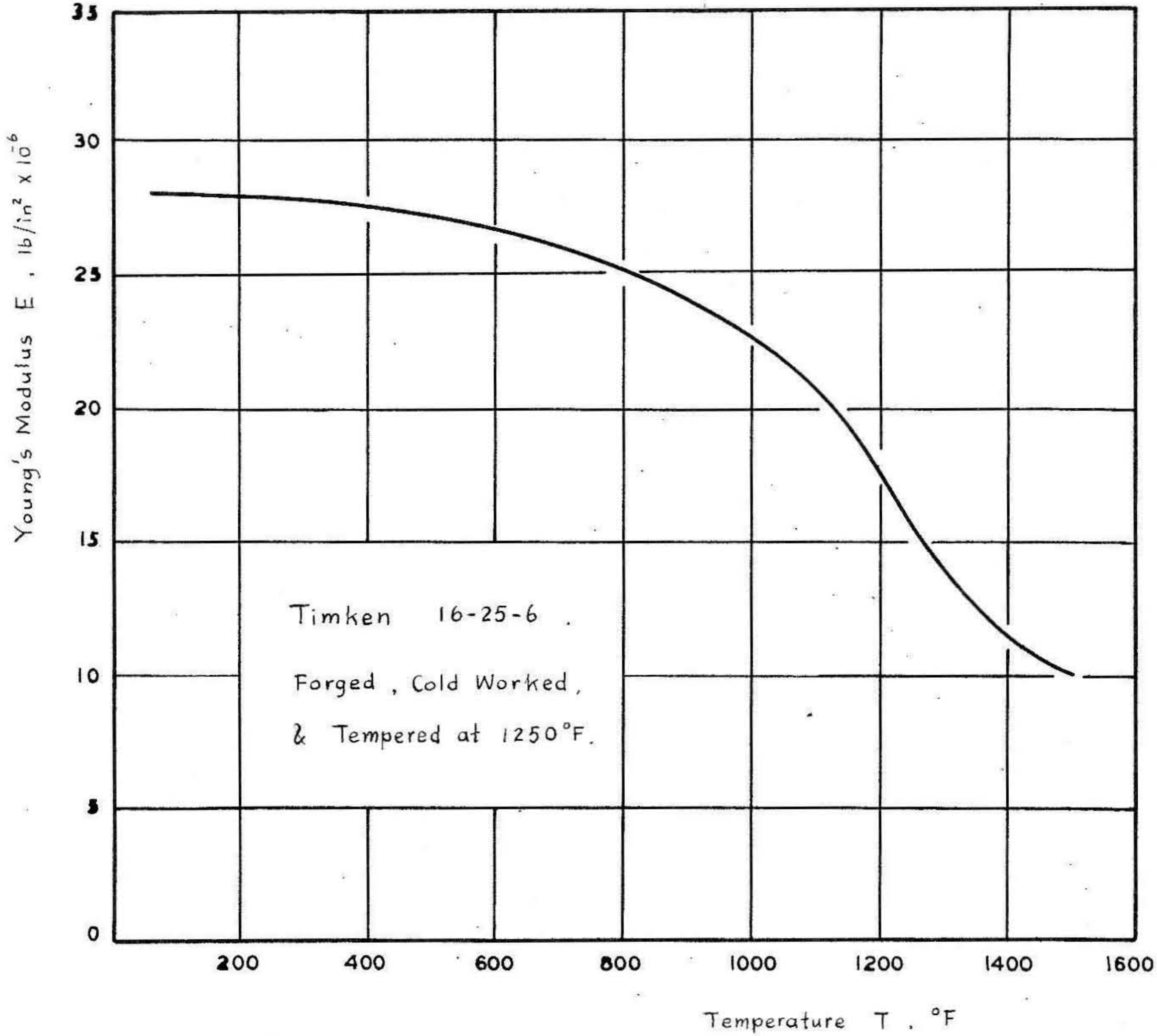

Young's Modulus E , lb/in² x 10⁻⁶
35
30
25
20
15
10
5
0
200
400
600
800
1000
1200
1400
1600
Temperature T , °F
Timken 16-25-6 .
Forged , Cold Worked ,
& Tempered at 1250°F.

Fig. 2

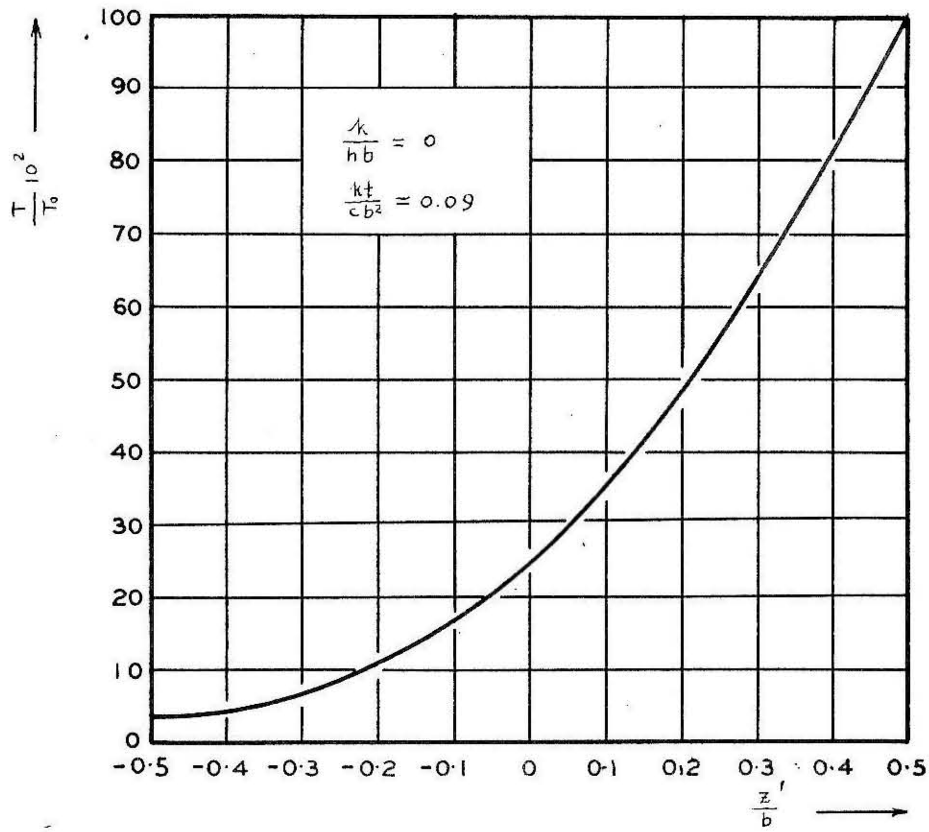
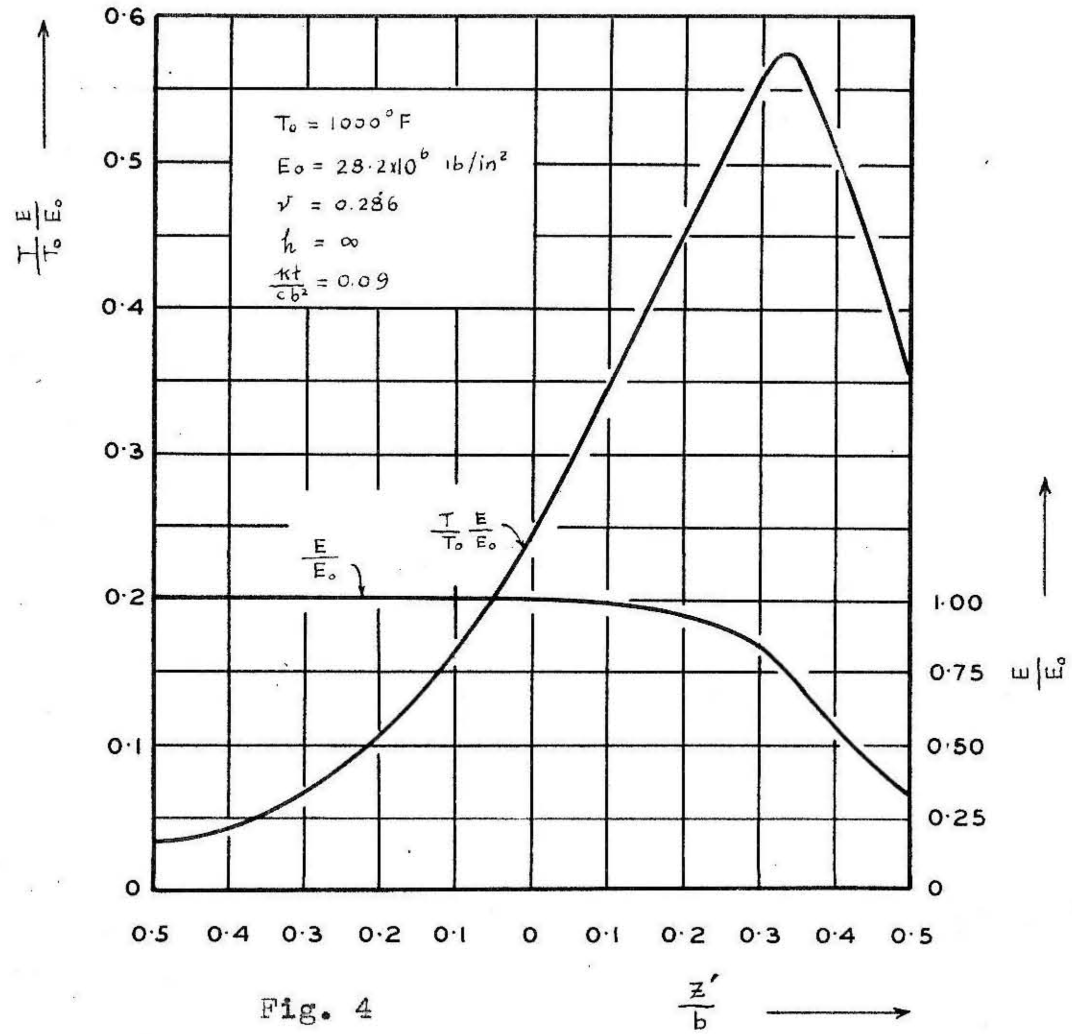


Fig.3 Temperature Distribution in the Direction Normal to the Wall



$T_o = 1000°F$
$E_o = 28.2 \times 10^6$ lb/in²
$\nu = 0.286$
$h = \infty$
$\frac{\kappa t}{c b^2} = 0.09$
$\frac{T}{T_o}\frac{E}{E_o}$
$\frac{E}{E_o}$
0·6
0·5
0·4
0·3
0·2
0·1
0
1·00
0·75
0·50
0·25
0
0·5 0·4 0·3 0·2 0·1 0 0·1 0·2 0·3 0·4 0·5
$\frac{z'}{b}$

Fig. 4

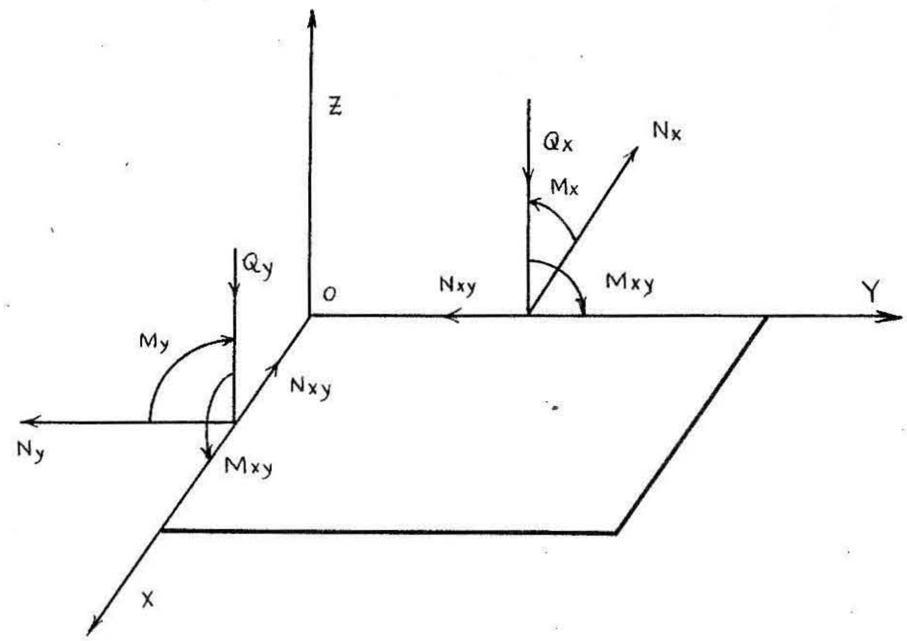


Fig. 5 Forces and Moments on an Element of a Plate

Part III

References

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2. F. H. Clark, "Metals at High Temperatures", (1950) pp. 163-172.

3. Th. von Kármán and H. S. Tsien, "The Buckling of Thin Cylindrical Shells under Axial Compression", J. Ae. S. (1941), Vol. 8, p. 303.