EXACT TRANSIENT SOLUTION OF SOME PROBLEMS OF ELASTIC WAVE PROPAGATION

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1960

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ABSTRACT

Exact solutions are obtained for three problems of progressive elastic wave propagation in bounded media: (1) SH wave propagation from an impulsive point source in an infinite plate; (2) torsional waves in a solid cylinder; (3) radiation from an impulsive source of compressional and of shear waves in an infinite solid plate held between smooth rigid surfaces. The Laplace transform method is used.

Problems (1) and (3) are shown to be closely related. For these problems the solution is expressed both as an infinite series of normal modes and an infinite series of multiple reflections, and it is shown that the two representations of the solution are related by Poisson's summation formula. Solutions are obtained for both a delta-function and a unit function input.

Problem (2) is solved as an infinite series of normal modes for an impulsive shear stress source distributed over a normal section of the cylinder. The case of a point source on the axis of the cylinder is examined in detail.

Problem (3) involves mixed boundary conditions. A relation between the solution of this problem and wave propagation in a free plate is discussed.

ACKNOWLEDGEMENTS

Grateful acknowledgement is made to my research director, C. H. Dix, and to Frank Press, for their constant encouragement and help. I have also had the benefit of stimulating discussions of this work with R. D. Mindlin and W. B. Kamb, and have received invaluable assistance from R. L. Shreve and G. P. Mayhew.

Professor J. C. Jaeger very kindly made funds available for numerical computations on SILLIAC, the electronic computer at the University of Sydney. Acknowledgement is made to L. E. Howard, who helped the author to learn the SILLIAC programming code, and to Mrs. A. Tingley, who typed the manuscript.

CHAPTER I
INTRODUCTION

1.1. Introduction. - This work deals with the exact solution of several problems of progressive elastic wave propagation in bounded media: impulsive radiation from a point source of 1) SH waves in a horizontal plate; 2) torsional waves in a circular cylinder; and 3) compressional and shear waves in a plate held between smooth frictionless rigid surfaces.

The common characteristic of these problems is that only one type of wave - compressional or shear - is present in each problem. This circumstance obviates the complexities encountered in problems in which conversion of wave type takes place at the boundaries. In this simpler type of problem, the double integral in terms of which the solution is expressed is simple enough that it can be evaluated exactly for any range of distance from the source. Sato (1954) demonstrated the equivalence of SH waves in the solid medium and sound waves in fluids, so the solutions of the first two problems mentioned above are also solutions of the equivalent fluid problems.

1.2. SH wave propagation in a free plate. - The problem of the vibration of a solid plate was first considered in detail by Poisson in 1828 (see, for example, Love, 1892, p. 496), and various special aspects of the general problem have been treated by many workers since that time. Because of the complexity of the phenomena of wave propagation in a medium with more than one plane boundary, these studies have been made using one of two approximation techniques: The "normal mode" method has been to use a steady-state harmonic source to derive a formal solution in terms of an integral over the wave number. The wave number is then regarded as a complex variable, and the integral is expressed in terms of integrals around the branch cuts of the integrand and a series of residues. For large distances from the source, the branch line integrals decay with distance more rapidly than the residues, so the solution is given approximately by the residues alone. The equation giving the location of the poles of the integrand furnishes a relation between the phase velocity of the harmonic solution and its wave length, and provides information about the form of the solution at large distances. The response to an impulsive source is written as a Fourier integral involving the residues; this integral can be evaluated approximately for large distances by the stationary phase method (see, for example, Ewing, Jardetzky, and Press, 1957; and Folk, Fox, Shook, and Curtis, 1958).

The "ray-theory" method, which is typified by the work of Mencher (1953) and Spencer (1956), involves setting up the problem of an impulsive source in a plate or series of layers by means of the Laplace transform, and then to expand the integrand in an infinite series. The solution is desired only on the epicentral line, so that the series is quite easily inverted term by term, and each term of the series may be identified with a different multiple reflection of the incident spherical wave.

This method gives no information about the distortion at a distant receiver of the incident wave as a result of its propagation in a bounded medium, unless the response is approximated by summing a finite number of the multiple reflections. This work has been carried out for one case, that of a layered liquid half-space, by Pekeris and Longman (1958).

The case of SH waves from a point source in a solid free plate is simple enough that an exact solution, in the form of an infinite series of normal modes, can be found without making any approximations. The normal mode solution is valid close to the source as well as far away.

The multiple reflection expansion of the solution is also easily obtained. It is found that close to the source a few terms of the multiple reflection representation of the solution are adequate to describe the motion at the receiver, but many terms of the normal mode series must be added to arrive at the same result. The opposite is the case far away from the source, where the multiple reflection representation becomes only slowly convergent.

A simple analytical relationship between the two exact representations of the solution exists: Poisson's summation formula transforms the multiple reflection series into the normal mode series.

In Chapter 3 we use the saddle-point method to carry out an approximation to the solution for large distances from the source, and show that the higher modes behave exactly the same near the source as far away from it. The approximation shows that the exact solution satisfies the classical group velocity dispersion equation for this problem.

1.3. <u>Wave propagation in a plate with mixed boundary condition.</u> - By mixed boundary conditions we mean that instead of requiring stress or displacement to vanish, we seek a solution in which one component of stress and one component of displacement vanish at the faces of the plate. A point source of compressional and of shear waves is considered in turn. Since the particular set of mixed boundary conditions used here ensures that no conversion of wave type takes place at the faces of the plate, the form of the solution for both problems is the same as that of the SH wave problem. For the shear wave source, however, the plate vibrates antisymmetrically, so that the zero order term present in the other problems is missing.

1.4. Torsional waves in a solid cylinder. - The problem of determining the natural periods of torsional vibration of a solid homogeneous isotropic circular cylinder seems first to have been studied by Pochhammer (1876) and independently by Chree (1889). In both these classic papers, the authors were interested in the much more difficult problems of flexural and longitudinal vibrations of cylinders, and neither author carried the torsional vibration problem farther than deriving equation 5.8 of this work; neither author presented a thorough discussion of that equation. Pochhammer did point out that the zero'th eigenvalue, $k_o = 0$, corresponds to a mode of vibration in which each transverse section of the cylinder rotates as a whole about its center, without angular distortion.

Interest in torsional vibrations then turned to a great number of engineering problems. For over fifty years after Pochhammer's paper was published, almost the only references to torsional waves in the literature are graphical or numerical solution of the lowest eigenfrequency of composite cylinders, etc. Rayleigh, in his Theory of Sound (1877) only shows that the lowest mode torsional wave travels with the shear wave velocity. Love (1892) simply recapitulates Pochhammer on the subject, as does the Handbuch der Physik (1928).

In 1935 B. Sen and Y. Nomura independently considered the vibrations of an infinitely long cylinder driven by a sinusoidally varying torque applied to the radial surface of the cylinder. The main interest of both authors was in the resonances between the harmonic driving force and the modes of vibration of the cylinder, rather than in torsional wave propagation.

More recently, Morse and Feshbach (1953, p. 1844) considered the same problem in more general terms, and discussed the torsional impedance seen by the driving mechanism. Kolsky (1953, p. 65) derived the phase and group velocity equations for free torsional waves. Torsional waves are only mentioned by Ewing, Jardetzky, and Press (1957, p. 311).

In Chapter 5 we derive the response to an impulsive shear stress distributed over a cross section of the cylinder instead of on the radial surface. The general solution found here holds for an arbitrary distribution of stress but only one particular stress distribution is discussed in detail.

CHAPTER 2

SH WAVE PROPAGATION IN A HORIZONTAL FREE PLATE

2.1. Introduction. - We consider a solid, homogeneous, isotropic, perfectly elastic plate of infinite extent, of thickness H, bounded on both sides by vacuum.

Cylindrical coordinates (fig. 1) r, θ, z will be used, with the origin in one face and the z-axis normal to the faces of the plate, so that the plate lies between the planes $z = 0$ and $z = H$.

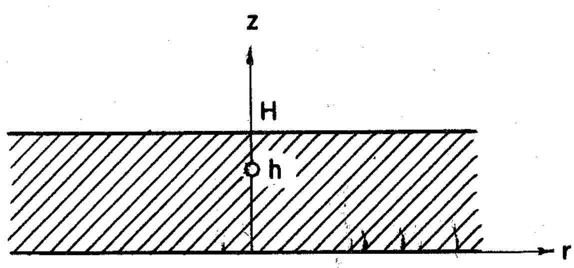


Figure 1: Point source in solid plate.

A point source of horizontally polarized shear waves is located on the z-axis a distance $h < H$ from the origin. The excitation by the source being given, we will find the displacement at any point in the plate. The particular excitation function we use will be described in section 2.3.

2.2 Equations of motion and boundary conditions. - In order to deal only with SH waves, we consider only the case in which the displacement is axially symmetric and contains no radial or vertical components. This requires that the displacement $u(r, \theta, z)$ satisfy

$$(2.1) \qquad \underset{\sim}{u} = \underset{\sim}{e}_\theta u_\theta$$

where $\underset{\sim}{e}_\theta$ is the unit vector in the θ-direction and u_θ is the

θ-component of $\underset{\sim}{u}$.

In Appendix A we show that in this special case, the displacement is given by a single equation

$$(2.2) \quad \nabla^2 u_\theta - \frac{u_\theta}{r^2} = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}$$

where $\beta = (\mu/\rho)^{1/2}$ is the shear wave velocity, μ the modulus of rigidity and ρ the density of the solid.

Equation 2. 2 is not the wave equation, but if we derive $\underset{\sim}{u}$ from a vector potential $\underset{\sim}{A}$,

$$(2.3) \quad u_\theta = - \frac{\partial A_z}{\partial r},$$

then A_z satisfies the wave equation

$$(2.4) \quad \nabla^2 A_z = \frac{1}{\beta^2}\frac{\partial^2 A_z}{\partial t^2}$$

(see Appendix A).

From now on we drop the subscripts on u_θ and A_z, since all other components of $\underset{\sim}{A}$ and $\underset{\sim}{u}$ vanish.

The boundary conditions are that the surfaces $z = 0$ and $z = H$ be free from stress. In Appendix A we show that τ_{rz} and τ_{zz} are identically zero, and

$$(2.5) \quad \tau_{z\theta} = -\mu \frac{\partial^2 A}{\partial r \partial z} = \mu \frac{\partial u}{\partial z} .$$

Thus the boundary conditions are

$$(2.6) \quad \tau_{z\theta} = -\mu \frac{\partial^2 A}{\partial r \partial z} = 0 \quad \text{at } z = 0 \text{ and } z = H \text{ for all } r \text{ and } t.$$

2.3. The source. - We consider the source to be a spherical cavity of small radius R_o, containing some apparatus which exerts on the face of the cavity shear stress $\tau_{r\theta}$, displacement u_θ, torque N_z, or any other physical quantity which will serve to radiate SH waves into the plate. In what follows we will always consider R_o to be so small compared to the distance between the source and the observation point that the transient wave from the near side of the cavity arrives at sensibly the same time as that from the far side.

Although the solution we derive is strictly valid only up to the time the first reflected wave reaches the source -- all the reflected waves will be diffracted by the source -- we consider R_o to be so small that the diffraction effects may be ignored.

The wave radiated by the source must satisfy the equations of motion for an axially-symmetric point source in an infinite medium, equation 2.4. We select the following solution of 2.4 to be the representation of the source:

$$(2.7) \quad A_s = \frac{1}{R} f(t - \frac{R}{\beta}) ,$$

where

$$(2.8) \quad R = [(z - h)^2 + r^2]^{1/2}$$

and $f(t - \frac{R}{\beta})$ is a function of R and t which for $R = R_o$ gives the distribution of potential A over the face of the source cavity.

We now tentatively choose the source function

$$(2.9)\quad A_s(R_o, t) = -\frac{A_o}{R_o}\, 1(t - \frac{R_o}{\beta}) = \begin{cases} 0 & t < \frac{R_o}{\beta} \\ -\frac{A_o}{R_o} & t > \frac{R_o}{\beta} \end{cases}$$

where A_o is independent of R and t, and $1(t - \frac{R_o}{\beta})$ is the unit function.

A_o is shown in Appendix G to be related to the total torque T exerted by the source on the surrounding medium by

$$A_o = \frac{T}{8\pi\mu}.$$

Blake (1952) showed that for a source of radial stress the constant $A_o^{(rad)}$ is

$$A_o^{(rad)} = R_o^3 P_o / \rho a^2,$$

where P_o is the amplitude of the radial stress and a is the compressional wave velocity of the solid medium.

The wave radiated from the source is, from 2.9,

$$(2.10)\quad A_s(R, t) = -\frac{A_o}{R}\, 1(t - \frac{R}{\beta}).$$

But at least the first two derivatives of $A_s(R, t)$ must exist for it to satisfy the equation of motion, and this unit function has no derivative at $t = R/\beta$.

This difficulty may be circumvented in several ways. One can regard the differential equations as the limiting case of the corresponding difference equations for infinitely small mesh size. There are no continuity restrictions on the solutions of the difference equation, so we may use the unit function as an input. The solution of the difference equations approaches the solution of the differential equations as the mesh size approaches zero (Evans, 1918), and because there is no trouble at any state of the limiting process, there is no trouble in approaching the mathematically discontinuous step-function as closely as we please.

This is tantamount to regarding the unit function as the limit of a sequence of functions which are each continuous and infinitely differentiable. Lighthill (1958) calls the limit of such a sequence a "generalised function," and shows (p. 18-20) that, subject only to some restrictions on how the sequence of regular functions is to be defined (which are discussed below), the limiting process can be interchanged with the processes (among others) of differentiation, Fourier transformation, and inverse Fourier transformation.

Therefore, using a properly defined sequence of regular functions (which we may suppose to be zero with at least two of its derivatives at $t = 0$) to represent the unit function, we may carry through solution of the problem for the unit function itself, knowing that the sequence of solutions approaches that solution as the sequence of regular source functions approaches the unit function. There are some mathematical details involved in this approach, which will not concern us here.

We stipulate, then, that the response to the step input will always eventually be used in a convolution integral to obtain the response to a source function $F(t - \frac{R}{\beta})$ which really is twice differentiable (see, for example, Cagniard, 1939, pages 11-15):

(2.11) $$A_c(r, z, t) = \int_0^t F'(\tau) A(r, z, t-\tau)\, d\tau \, .$$

It is this response, $A_c(r, z, t)$, which must be regarded as the solution to the real physical problem. The response to the unit step-function source is interesting because it is mathematically tractable, and because it can be used as a building block to synthesize the solution to the response of any real physical source.

Regarding the step function as a generalised function in Lighthill's sense makes it possible to write 2.11 as

$$(2.12) \qquad A_c(r, z, t) = \int_0^t F(\tau)A'(r, z, t-\tau)d\tau ,$$

where $A'(r, z, t)$ is now the response to a δ-function input, an input which is the time derivative of 2.9. It will be seen later that the solutions to all the problems considered in this thesis are very much simpler for a δ-function input than for a step-function input. The response to both a step-function and a δ-function will be given in each case.

Limitations on the available computer time made it impossible to include numerical examples of the response to a continuous source function.

2.4. Laplace transform of the problem. - We now make a Laplace transform of the whole problem. We define $\overline{A}(r, z, p)$ as the Laplace transform of $A(r, z, t)$ by

$$(2.13) \qquad \overline{A}(r, z, p) = \int_0^\infty e^{-pt}A(r, z, t)dt = \mathcal{L}\{A(r, z, t)\}$$

We will use

$$(2.14)\quad \begin{cases} \mathcal{L}\left[\dfrac{\partial^n A(r,z,t)}{\partial r^k \partial z^{n-k}}\right] = \dfrac{\partial^n \overline{A}(r,z,p)}{\partial r^k \partial z^{n-k}} \\ \mathcal{L}\left[\dfrac{\partial^2 A(r,z,t)}{\partial t^2}\right] = p^2\overline{A}(r,z,p) - pA(r,z,0) - \left[\dfrac{\partial A(r,z,t)}{\partial t}\right]_{t=0} \end{cases}$$

(Hildebrand, 1948, Chapter 2). The last two terms on the right side of the second of equations 2.14 may be set equal to zero, because of the way we defined the sequence of functions leading to the unit function. Thus 2.4 transforms to

$$(2.15)\quad \nabla^2\overline{A} = \frac{p^2}{\beta^2}\overline{A},$$

and the boundary conditions 2.6 transform to

$$(2.16)\quad -\overline{\tau}_{z\theta} = \mu\frac{\partial^2\overline{A}}{\partial r\partial z} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = H.$$

The step-function source term 2.10 becomes

$$(2.17)\quad \mathcal{L}\left\{A_s(r,z,t)\right\} = \mathcal{L}\left\{-\frac{A_o}{R}\,1(t-\frac{R}{\beta})\right\} = \frac{A_o}{pR}\exp(-\frac{pR}{\beta}),$$

while the delta-function source term is

$$\mathcal{L}\left\{A_s(r,z,t)\right\} = \mathcal{L}\left\{-\frac{A_o}{R}\,\delta(t-\frac{R}{\beta})\right\} = \frac{A_o}{R}\exp(-\frac{pR}{\beta}).$$

2.5. Formal solution. - A formal solution of 2.15 which is finite at $r = 0$ is easily found to be

$$(2.19)\quad \overline{A}(r,z,p) = \int_0^\infty J_o(kr)[B_1(k)e^{-\ell(z-h)} + B_2(k)e^{\ell(z-h)}]\,dk,$$

where

$$(2.20) \qquad \ell = (k^2 + \frac{p^2}{\beta^2})^{1/2} .$$

The δ-function source term may be written in its integral representation

$$(2.21) \qquad \overline{A}_s(r, z, p) = \frac{A_o}{R} \exp(-\frac{pR}{\beta}) = A_o \int_0^\infty \frac{k}{\ell} J_o(kr) e^{-\ell |z-h|} dk$$

(Ewing, Jardetzky, and Press, 1957, p. 13), so that the complete solution is

$$(2.22) \qquad \overline{A}(r, z, p) = \int_0^\infty J_o(kr)[B_1(k) e^{-\ell(z-h)} + B_2(k) e^{\ell(z-h)} + A_o \frac{k}{\ell} e^{-\ell |z-h|}] dk,$$

where $B_1(k)$ and $B_2(k)$ are to be determined from the boundary conditions 2.16.

Substituting 2.22 into 2.16, we obtain two simultaneous integral equations for $B_1(k)$ and $B_2(k)$:

$$(2.23) \qquad \begin{cases} \mu \int_0^\infty J_o(kr)[A_o k e^{-\ell h} - B_1(k) e^{\ell h} + B_2(k) e^{-\ell h}] dk = 0 \\ \mu \int_0^\infty J_o(kr)[-A_o k e^{-\ell(H-h)} - B_1(k) e^{-\ell(H-h)} + B_2(k) e^{\ell(H-h)}] dk = 0 \end{cases}$$

A sufficient condition for equations 2.23 to be satisfied is:

$$(2.26)\quad \begin{cases} B_1 e^{\ell h} - B_2 e^{-\ell h} = A_o \dfrac{k}{\ell} e^{-\ell h} \\[2ex] B_1 e^{-\ell(H-h)} - B_2 e^{\ell(H-h)} = -A_o \dfrac{k}{\ell} e^{-\ell(H-h)}, \end{cases}$$

from which B_1 and B_2 are given by

$$(2.27)\quad \begin{cases} B_1 = A_o \dfrac{k}{\ell} e^{-\ell h} \dfrac{\cosh \ell(H-h)}{\sinh \ell H} \\[2ex] B_2 = A_o \dfrac{k}{\ell} e^{-\ell h} \dfrac{\cosh \ell h}{\sinh \ell H} \quad . \end{cases}$$

We could substitute 2.27 back into 2.23, but some simplification can be made. Because of the absolute value sign in 2.22, $\overline{A}(r, z, p)$ has different expressions for $z \gtrless h$. We can use the first of equations 2.26 to eliminate B_2 in the expression for $\overline{A}(r, z, p)$ in the range $0 < z < h$, and the second of equations 2.26 to eliminate B_1 in the range $h < z < H$. The formal solution is finally

$$(2.28)\quad \begin{cases} \overline{A}(r, z, p)\Big]_{z<h} = 2A_o \displaystyle\int_0^{\infty} J_o(kr) \frac{k \cosh \ell z \cosh \ell (H-h)}{\ell \sinh \ell H} \, dk \\[2ex] \overline{A}(r, z, p)\Big]_{z>h} = 2A_o \displaystyle\int_0^{\infty} J_o(kr) \frac{k \cosh \ell h \cosh \ell (H-z)}{\ell \sinh \ell H} \, dk \, . \end{cases}$$

We notice as a check that the first and second equations 2.28 are obtainable from each other by interchanging z and h.

We must eventually show that neither integrand in equations 2.28 becomes infinite on the path of integration.

<u>2.6. Evaluation of the integrals $\overline{A}(r, z, p)$</u>. - The integrals 2.28 are

both of the form

(2. 29) $$\overline{A}(r, z, p) = 2A_o \int_0^{\infty} \frac{k \cosh a\ell \cosh b\ell}{\ell \sinh \ell H} J_o(kr)dk$$

if we define

(2. 30)

	a	b
$0 < z < h$	z	H - h
$h < z < H$	h	H - z

Therefore we will evaluate 2. 29 and then substitute for a and b according as $z \gtrless h$.

In order to carry out the integration in 2. 29, it is convenient to consider k as a complex variable and the integral as a contour integral in the complex k-plane.

We define a complex variable

(2. 31) $$\zeta = k + i\eta$$

where k and η are real, and we choose the sign of

$$\ell = [\zeta^2 + \frac{p^2}{\beta^2}]^{1/2}$$

to be positive when ζ and p are real and positive.

$J_o(\zeta r)$ is not finite for large ζ unless $\eta = 0$. In order to close the path $(0, \infty)$ we substitute for $J_o(\zeta r)$

(2. 32) $$2J_o(\zeta r) = H_o^{(1)}(\zeta r) + H_o^{(2)}(\zeta r)$$

(Watson, 1952, p. 74), so that 2. 29 splits into two integrals which we can treat separately:

$$\overline{A}(r, z, p) = A_o \int_0^\infty I_1(\zeta) d\zeta + \int_0^\infty I_2(\zeta) d\zeta \tag{2.33}$$

where

$$\begin{cases} I_1(\zeta) = G(\zeta) H_o^{(1)}(\zeta r) \\ I_2(\zeta) = G(\zeta) H_o^{(2)}(\zeta r) \\ G(\zeta) = \dfrac{\zeta \cosh a\ell \cosh b\ell}{\ell \sinh \ell H} = \dfrac{N(\zeta)}{D(\zeta)} . \end{cases} \tag{2.34}$$

We can now close the contour for I_1 around the lower half-plane and that for I_2 around the upper half-plane, since

$$\lim_{|\zeta| \to \infty} H_o^{(1),(2)}(\zeta) = 0 \quad \text{if} \quad \text{Im}(\zeta) \gtrless 0$$

(Watson, 1952, p. 198).

We define the following contours in the ζ-plane:

For I_1:

1. (C_1): ϵ to R along the real axis;
2. (R_1): R to $-R + i\epsilon$ along an arc of radius R in the upper half-plane;
3. (D_1): $-R + i\epsilon$ to $+i\epsilon$ along the negative real axis and slightly above it;
4. (E_1): $+i\epsilon$ to ϵ along an arc of radius ϵ in the upper half-plane.

For I_2:

1. (C_2): ϵ to R along the real axis;
2. (R_2): R to $-R - i\epsilon$ along an arc of radius R in the lower half-plane;

3. (D_2): $- R - i\epsilon$ to $- i\epsilon$ along the negative real axis and slightly below it;

4. (E_2): $- i\epsilon$ to ϵ along an arc of radius ϵ in the lower half-plane.

The care in keeping away from the negative real axis and the origin is made necessary by the fact that the Hankel functions have logarithmic singularities at the origin, and the negative real axis is a convenient place to put the requisite branch cut.

These paths are shown in fig. 2 (next page). We will eventually take the limit of these integrals as $\epsilon \to 0$ and $R \to \infty$.

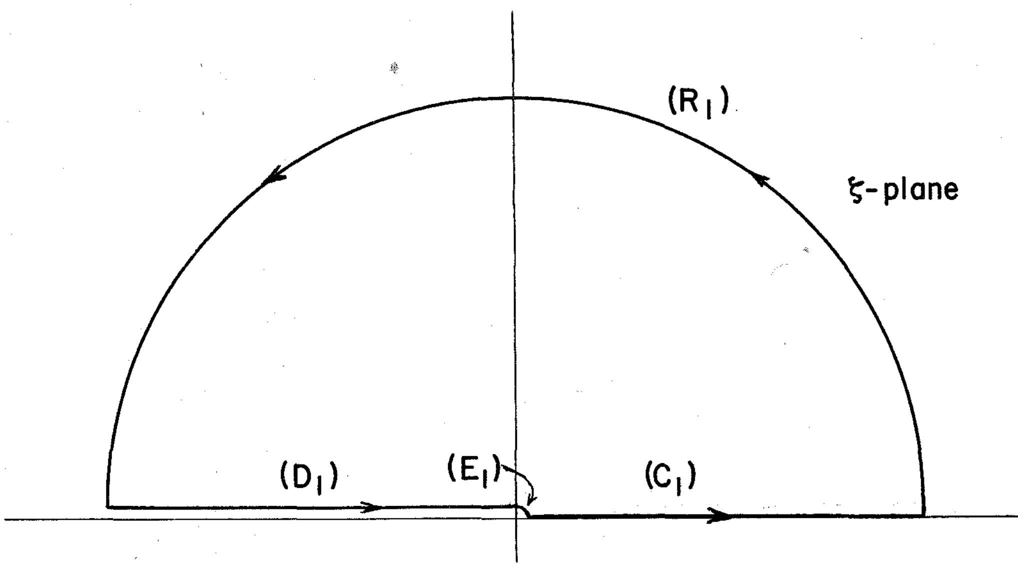
We see immediately that the contributions of (E_1) and (E_2) must cancel each other out, for integrating $J_o(\zeta r)$ around the origin in a circle of vanishing radius contributes nothing.

It is also clear that no branch cuts in the ζ-plane, other than the one down the negative real axis, are necessary to keep I_1 and I_2 single valued, for $G(\zeta)$ is an even function of ℓ, and although the points $\zeta = \pm \frac{ip}{\beta}$ are branch points of ℓ, they are not branch points of $G(\zeta)$.

By Cauchys theorem,

$$(2.35) \quad \begin{cases} \int_{(C_1)} + \int_{(R_1)} + \int_{(D_1)} + \int_{(E_1)} = 2\pi i \sum \text{Res}\ (I_1) \\ \\ \int_{(C_2)} + \int_{(R_2)} + \int_{(D_2)} + \int_{(E_2)} = 2\pi i \sum \text{Res}\ (I_2) \end{cases}$$

where $\sum \text{Res}\ (I_1)$ denotes the sum of all the residues of I_1 in the upper half-plane, and $\sum \text{Res}\ (I_2)$ denotes the sum of all the residues of I_2 in



a. Path of integration for I_1.

b. Path of integration for I_2.

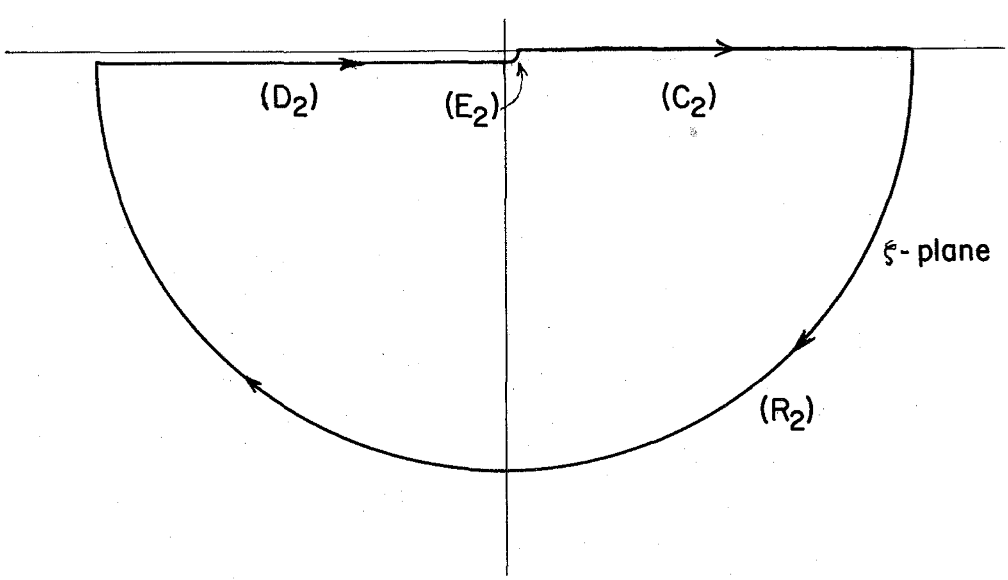


Figure 2. Paths of integration for I_1 and I_2.

the lower half-plane.

We will show first that the integrals over (R_1) and (R_2) vanish as $R \to \infty$; then that the sum of the integrals over (E_1) and (E_2) vanish as $\epsilon \to 0$; and last that the sum of the integrals over (D_1) and (D_2) is equal to the sum of the integrals over (C_1) and (C_2). This last step leaves us with nothing but the residues, since from 2.33,

$$\overline{A}(r, z, p) = \int_{(C_1)} + \int_{(C_2)} .$$

The integrals over R_1 and R_2: we assume that R is very large. When $|\zeta| = R \to \infty$,

$$\cosh a\ell \to \cosh a\zeta \to \frac{1}{2} e^{a\zeta}$$

$$\cosh b\ell \to \frac{1}{2} e^{b\zeta}$$

$$H_o^{(1)}(\zeta r) \to \left[\frac{2}{\pi r\zeta}\right]^{1/2} e^{i(\zeta r - \pi/4)}$$

$$H_o^{(2)}(\zeta r) \to \left[\frac{2}{\pi r\zeta}\right]^{1/2} e^{-i(\zeta r - \pi/4)}$$

(Watson, 1952, p. 198). On (R_1), $\mathrm{Im}(\zeta) > 0$, and

$$I_1(\zeta) < \left| e^{\zeta(a+b-H)} \left[\frac{2}{\pi\zeta r}\right]^{1/2} e^{-\eta r} \right| .$$

Reference to 2.30 shows that

$$a + b - H = \begin{cases} z - h \text{ when } h > z \\ h - z \text{ when } z > h \end{cases}$$

so that the coefficient of ζ in the exponential is always negative. Thus on (R_1), I_1 is always less than

$$|\zeta^{-1/2}e^{-M\zeta}| \, ,$$

where M is real and positive, and as $|\zeta| = R \to \infty$, I_1 vanishes and the integral over (R_1) is zero.

For (R_2), exactly the same reasoning shows that since $\mathrm{Im}(\zeta) < 0$ on (R_2), the integral over (R_2) vanishes as $R \to \infty$. We notice, however, that if there are poles of the integrands all the way out to infinity, the limit must be taken in such a way that (R_1) and (R_2) never pass directly through one of the poles.

<u>The integrals over (E_1) and (E_2):</u> when $|\zeta|$ is small,

$$(2.36) \quad \begin{cases} H_o^{(1)}(\zeta r) \to 1 + iY_o(\zeta r) \to 1 + \frac{2i}{\pi}\log(\zeta r) \\ H_o^{(2)}(\zeta r) \to 1 - iY_o(\zeta r) \to 1 - \frac{2i}{\pi}\log(\zeta r) \, . \end{cases}$$

(Hildebrand, 1948, p. 161). We may conveniently take

$$\zeta = \epsilon\, e^{i\theta}$$

since on (E_1) and (E_2) only the phase of ζ is changing.

For small ϵ,

$$G(\zeta) \propto \epsilon \quad .$$

From (2.36), and using the definition of the logarithm of a complex variable,

$$
(2.37) \quad \begin{cases} H_o^{(1)}(\zeta r) \to +\frac{2i}{\pi}\log(\epsilon r) - \frac{2\theta}{\pi} + 1 \\ H_o^{(2)}(\zeta r) \to -\frac{2i}{\pi}\log(\epsilon r) + \frac{2\theta}{\pi} + 1 . \end{cases}
$$

Integrating the first term on the right of both of equations 2.37 around the appropriate small arc and adding, we find that the contributions of these two terms cancel each other out identically, as we expected. Integrating the last two terms on the right around the proper small arcs and adding, we find that this contribution to the whole solution vanishes as $\epsilon \to 0$.

<u>The integrals on (D_1) and (D_2):</u> we had

$$
\int_{(D_1)} = \int_{-\infty+i\epsilon}^{-\epsilon+i\epsilon} G(\zeta)H_o^{(1)}(\zeta r)d\zeta .
$$

Now let

$$
\xi = e^{-i\pi}\zeta ,
$$

rotating the path through the upper half plane to a position just below the real axis:

$$
\int_{(D_1)} = -\int_{\infty-i\epsilon}^{\epsilon-i\epsilon} G(e^{i\pi}\xi)H_o^{(1)}(re^{i\pi}\xi)d\xi .
$$

But

$$
H_o^{(1)}(e^{i\pi}\xi r) = -H_o^{(2)}(\xi r)
$$

(Magnus and Oberhettinger, 1954, p. 17), and

$$
(2.38) \qquad G(e^{i\pi}\xi) = -G(\xi) .
$$

So

$$\int_{(D_1)} = \int_{\epsilon - i\epsilon}^{\infty - i\epsilon} G(\xi) H_o^{(2)}(\xi r) d\xi .$$

Similarly,

$$\int_{(D_2)} = \int_{-\infty - i\epsilon}^{-\epsilon - i\epsilon} G(\xi) H_o^{(2)}(\xi r) d\xi ,$$

and on putting

$$\xi = e^{i\pi} \zeta$$

to rotate the path through the lower half plane to a position just above the real axis, we have

$$\int_{(D_2)} = - \int_{\infty + i\epsilon}^{\epsilon + i\epsilon} G(e^{-i\pi}\xi) H_o^{(2)}(e^{-i\pi} r\xi) d\xi .$$

Using 2.38 and

$$H_o^{(2)}(e^{-i\pi}\xi r) = -H_o^{(1)}(\xi r)$$

(Magnus and Oberhettinger, 1954, p. 17), we have

$$\int_{(D_2)} = \int_{\epsilon + i\epsilon}^{\infty + i\epsilon} G(\xi) H_o^{(2)}(\xi r) d\xi$$

and looking back at the definition of our contours, we have, as $\epsilon \to 0$,

$$\int_{(D_1)} + \int_{(D_2)} = \int_{(C_1)} + \int_{(C_2)} .$$

Then by Cauchy's theorem, we have finally

(2.39) $$\overline{A}(r,z,p) = A_o\int_{(C_1)} I_1(\zeta)d\zeta + A_o\int_{(C_2)} I_2(\zeta)d\zeta$$

$$= \pi i A_o^{*}[\sum \text{Res}(I_1) + \sum \text{Res}\ (I_2)]\ .$$

<u>The residues of I_1 and I_2</u>: $I_1(\zeta)$ was

(2.40) $$I_1(\zeta) = H_o^{(1)}(\zeta r)\frac{\zeta \cosh a\ell \cosh b\ell}{\ell \sinh H\ell} = H_o^{(1)}(\zeta r)\frac{N(\zeta)}{D(\zeta)}\ .$$

Poles in the upper half-plane are located where

(2.41) $$D(\zeta) = \ell \sinh H\ell = 0,$$

since neither $H_o^{(1)}(\zeta r)$ nor $N(\zeta)$ has poles for finite ζ in the upper half-plane.

An obvious set of poles is

(2.42) $$\ell = \pm \frac{in\pi}{H} \qquad n = 1,\ 2,\ 3,\ \ldots$$

or

(2.42a) $$\zeta = \zeta_n = iZ_n = i[\frac{p^2}{\beta^2} + (\frac{n\pi}{H})^2]^{1/2}\ ,$$

which are located on the curves $\text{Re}(\ell) = 0$. We show in Appendix B that these are the only zeros of $D(\zeta)$ in the whole ζ-plane.

Since $H_o^{(2)}$ has no poles in the lower half-plane, 2.42a gives the poles of I_2 as well as those of I_1.

At this point we must be careful to remember that we must eventually integrate over the Bromwich contour in the p-plane, and consider the possibility of some of the poles of I_1 being outside the upper half-plane for some unfavorable values of p. To pin down the poles more precisely, we let $p = s + i\sigma$ be complex, and write the conditions for $\mathrm{Re}(\ell) = 0$ and $D(\ell) = 0$:

$$\ell^2 = \zeta^2 + \frac{p^2}{\beta^2} = k^2 - \eta^2 + \frac{s^2 - \sigma^2}{\beta^2} + 2i\left[k\eta + \frac{s\sigma}{\beta^2}\right] = -\left(\frac{n\pi}{H}\right)^2,$$

or

$$(2.43)\quad \begin{cases} k^2 - \eta^2 = \dfrac{\sigma^2}{\beta^2} - \left[\dfrac{s^2}{\beta^2} + \left(\dfrac{n\pi}{H}\right)^2\right] \\[2ex] k\eta = -\dfrac{s\sigma}{\beta^2} \quad . \end{cases}$$

Thus the poles lie at the intersections of two sets of hyperbolae: one set with the axes of the ζ-plane as asymptotes, the other with the lines $\zeta = e^{\pm i\pi/4}$ as asymptotes. Close examination of 2.43 shows that (provided only $s > 0$) although the location of the poles does indeed depend on the position of p in the p-plane, no pole which is in the upper half of the ζ-plane for any one value of p can ever be found in the lower half of the ζ-plane for any other value of p. Use of this fact will be necessary later on.

All the poles of I_1 are simple: all the poles but the one given by $n = 0$ are clearly simple, and for that one, $\ell = 0$, so that

$$D(\zeta) = \ell \sinh \ell H = \ell\left[\ell H + \frac{(\ell H)^3}{3!} + \ldots\right]$$

behaves near $\ell = 0$ as

$$\ell^2 H = \left(\zeta + \frac{ip}{\beta}\right)\left(\zeta - \frac{ip}{\beta}\right)H ,$$

and thus this is a simple pole too.

The residue at $\ell = 0$ is

(2.44) $$\mathrm{Res}\,(I_1)\Big|_{\zeta=ip/\beta} = \lim_{\zeta\to ip/\beta} \left[(\zeta - \frac{ip}{\beta}) I_1(\zeta) \right] = \frac{1}{2H} H_o^{(1)}\left(\frac{ipr}{\beta}\right).$$

The residue at $\zeta = iZ_n$ is

(2.45) $$\mathrm{Res}\,(I_1)\Big|_{\zeta=iZ_n} = \left[\frac{N(\zeta)}{\frac{d}{d\zeta} D(\zeta)} \right]_{\zeta=iZ_n}$$

$$= \frac{(-1)^n}{H} \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} H_o^{(1)}(irZ_n).$$

Using

$$H_o^{(1)}(iu) = \frac{2}{\pi i} K_o(u)$$

(Hildebrand, 1948, p. 161), 2.44 and 2.45 sum to

(2.46) $$\sum \mathrm{Res}(I_1) = \frac{1}{\pi i H}\left[K_o\left(\frac{pr}{\beta}\right) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} K_o(rZ_n) \right].$$

<u>The residues of I_2:</u> The poles in the lower half-plane are clearly one-by-one the complex conjugates of the poles in the upper half-plane. In Appendix C we show that this implies that the residues of I_2 are the complex conjugates of the residues of I_1. Using the fact that poles of I_1 never become poles of I_2 for any given value of p, we have

(2.47) $$\overline{A}(r,z,p) = \pi i A_o\left[\sum \mathrm{Res}\,(I_1) + \sum \mathrm{Res}\,(I_2) \right]$$

$$= \pi i A_o \left[\sum \mathrm{Res}\,(I_1) + \overline{\sum \mathrm{Res}\,(I_1)} \right] = 2\pi i A_o \mathrm{Re}\left[\sum \mathrm{Res}\, I_1 \right]$$

$$(2.48) \qquad \overline{A}(r,z,p) = \frac{2A_o}{H} \mathrm{Re}\left[K_o\left(\frac{pr}{\beta}\right) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H}\cos\frac{n\pi b}{H} K_o(rZ_n)\right]$$

where

$$Z_n = \left[\frac{p^2}{\beta^2} + \left(\frac{n\pi}{H}\right)^2\right]^{1/2}.$$

<u>2.7. Inverse Laplace transform of A(r, z, p).</u> - From Bateman Manuscript Project (1954), vol. 1, p. 277 and p. 283,

$$(2.49) \qquad \begin{cases} \mathcal{L}^{-1}\left\{K_o\left(\frac{pr}{\beta}\right)\right\} = \left(t^2 - \frac{r^2}{\beta^2}\right)^{-1/2} 1\left(t - \frac{r}{\beta}\right) \\ \mathcal{L}^{-1}\left\{K_o\left(\frac{r}{\beta}[p^2 + c_n^2]^{1/2}\right)\right\} = \left(\frac{\pi c_n}{2}\right)^{1/2} \left(t^2 - \frac{r^2}{\beta^2}\right)^{-1/4} \cdot \end{cases}$$

$$J_{-1/2}\left(c_n\left[t^2 - \frac{r^2}{\beta^2}\right]^{1/2}\right) 1\left(t - \frac{r}{\beta}\right),$$

where we have put $c_n = \frac{n\pi\beta}{H}$. But

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x,$$

so the second of equations 2.49 is

$$\mathcal{L}^{-1}\left\{K_o\left(\frac{r}{\beta}[p^2 + c_n^2]^{1/2}\right)\right\} = \left(t^2 - \frac{r^2}{\beta^2}\right)^{-1/2} \cdot \cos\left(c_n\left[t^2 - \frac{r^2}{\beta^2}\right]^{1/2}\right) 1\left(t - \frac{r}{\beta}\right).$$

Thus the inverse transform of 2.48 is

$$(2.50) \qquad A(r,z,t) = \frac{4A_o}{\gamma H}\sum_{n=0}^{\infty} (-1)^n \epsilon_n \cos\frac{n\pi a}{H}\cos\frac{n\pi b}{H}\cos\frac{n\pi\beta\gamma}{H} \; 1\left(t - \frac{r}{\beta}\right),$$

where ϵ_n is the Neumann factor ($\epsilon_n = 1$ for $n = 0$, $\epsilon_n = 2$ for $n \geqslant 1$),

(2.51) $$\gamma = \left(t^2 - \frac{r^2}{\beta^2}\right)^{1/2}$$

and a and b are defined by 2.30:

(2.30)

	a	b
$0 < z < h$	z	$H - h$
$h < z < H$	h	$H - z$

The displacement $u(r, z, t)$ is

(2.52) $$u(r, z, t) = -\frac{\partial A}{\partial r} = \frac{1}{\beta} A(r, z, t)\delta\left(t - \frac{r}{\beta}\right)$$
$$- \frac{4A_o r}{\beta^2 \gamma^2 H} \sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} \cdot$$
$$\left[\frac{1}{\gamma^2}\cos\frac{n\pi\beta\gamma}{H} + \frac{n\pi\beta}{H}\sin\frac{n\pi\beta\gamma}{H}\right] 1\left(t - \frac{r}{\beta}\right).$$

The series in 2.50 does not converge. This might have been anticipated in a problem whose input is a function as pathological as the δ-function. However, even before the rigorous justification of the discontinuous functions by Schwarz, Temple, and Lighthill, it would have been possible to make sense of 2.50 by taking the Cesaro sum (C, 1) of the series in 2.50. See, for example, Franklin (1940), p. 486, or Knopp (1951), Chapter 13. The sum (C, 1) of the series in 2.50 is exactly the same as the sum found below by a more satisfactory method.

We first condense the three cosine terms in 2.50 into a single cosine:

(2.53) $$\cos\frac{n\pi b}{H}\cos\frac{n\pi\beta\gamma}{H} = \frac{1}{4}\sum_{m=1}^{4}\cos\frac{n\pi E_m}{H}$$

where

(2.54) $$E_m = \begin{cases} \beta\gamma - a - b, & m = 1 \\ \beta\gamma + a - b, & m = 2 \\ \beta\gamma - a + b, & m = 3 \\ \beta\gamma + a + b, & m = 4 . \end{cases}$$

2.50 is now

(2.55) $$A(r, z, t) = \frac{A_o}{\gamma H} 1(t - \frac{r}{\beta}) \sum_{m=1}^{4}\sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos\frac{n\pi E_m}{H}.$$

In section 2.9 the solution is represented as an infinite sequence of multiple reflections. We postpone further comment on the normal mode representation of the solution 2.55 until section 2.10, where it will be shown that the multiple reflection expansion and 2.55 are identical, and are transformable one into the other by the use of Poisson's summation formula.

Since we will be able to show that the two representations of the solution are identical, we return to an examination of the normal mode solution 2.50.

To concentrate attention on the radial propagation, we put both source and receiver on the bottom face of the plate, and examine the first few terms of 2.50:

(2.56)

n	A
0	$\frac{4A_o}{\gamma H} 1(t - \frac{r}{\beta})$
1	$- \frac{4A_o}{\gamma H} \cos(\pi\beta\gamma) 1(t - \frac{r}{\beta})$
2	$\frac{4A_o}{\gamma H} \cos(2\pi\beta\gamma) 1(t - \frac{r}{\beta})$

Defining the dimensionless parameters τ, κ, and Γ as

(2.57)
$$\begin{cases} \tau = \frac{t\beta}{H} \\ \kappa \equiv r/H \\ \Gamma = (\tau^2 - \kappa^2)^{1/2} \end{cases}$$

2.50 becomes

(2.58)
$$A(r, z, t) = \frac{4A_o\beta}{H^2\Gamma} \sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} \cos(n\pi\Gamma)\, 1(\tau - \kappa),$$

and 2.56 is

(2.59)

n	A
0	$\frac{4A_o\beta}{H^2\Gamma} 1(\tau - \kappa)$
1	$- \frac{4A_o\beta}{H^2\Gamma} \cos(\pi\Gamma) 1(\tau - \kappa)$
2	$\frac{4A_o\beta}{H^2\Gamma} \cos(2\pi\Gamma) 1(\tau - \kappa)$

These three terms are plotted against τ in fig. 3 (next page) for $\kappa = H = \beta = A_o = 1$. Each term begins with a singularity at $\tau = 1$, and for a given κ, decays with time as $\tau^{-1/2}$, as we expect for two-dimensionally guided waves. From 2. 50 we see that for large t at a given r, the oscillatory terms become damped sinusoids of period

(2. 60) $$T_o = \frac{2H}{n\beta} .$$

The relative amplitude of the modes depends entirely on the choice of z and h.

The relation between 2. 50 and the normal mode solution for large r is developed in the next chapter.

2. 8. Solution for step-function input. - The solution for the step-function input is obtained from 2.50 by integrating with respect to time:

$$A_1(r, z, t) = \int_0^t A(r, z, t')dt'$$

or

(2. 61) $$A_1(r, z, t) = \frac{2A_o}{H}\left[\cosh^{-1}\left(\frac{t\beta}{r}\right) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H} \cdot \cos\frac{n\pi b}{H}\int_{r/\beta}^{t} \frac{\cos(n\pi\beta\gamma'/H)}{\gamma'} dt' \; 1(t-\frac{r}{\beta})\right]$$

$$= \frac{2A_o}{H}\left[\cosh^{-1}\left(\frac{t\beta}{r}\right) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H} \cdot \cos\frac{n\pi b}{H}\int_0^{\gamma} \frac{\cos(c_n\eta)}{(\eta^2 + r^2/\beta^2)^{1/2}} d\eta \; 1(t-\frac{r}{\beta})\right] .$$

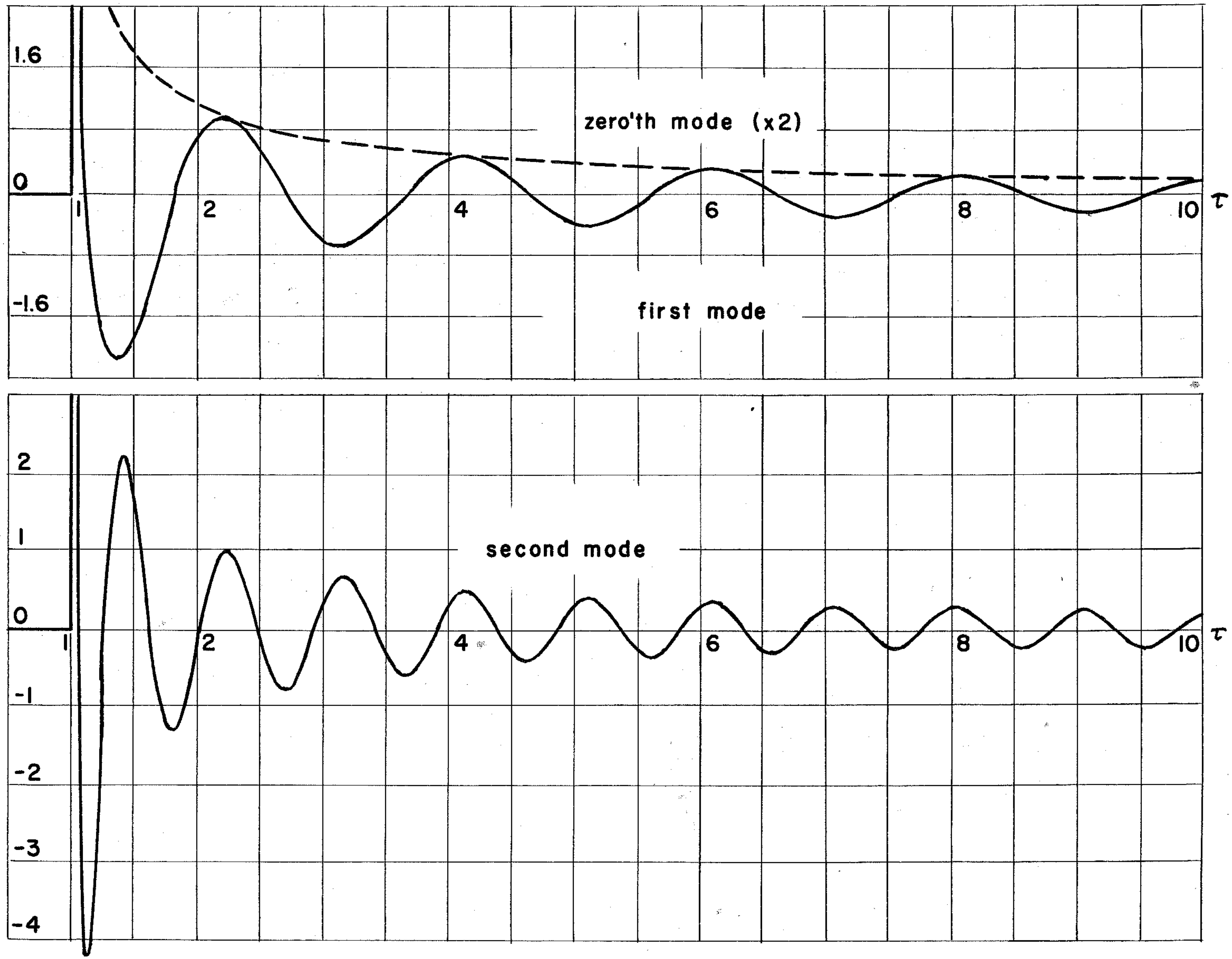


Figure 3: The first three terms of $A(r, z, t)$

The displacement is

$$(2.62)\qquad u_1(r,z,t) = \frac{1}{\beta} A_1(r,z,t)\,\delta(t-\frac{r}{\beta}) - \frac{2A_o r}{H\beta^2}\left\{\frac{\beta^2}{r^2} + \frac{1}{\gamma(t+\gamma)} + \sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H}\cos\frac{n\pi b}{H}\cdot \left[\int_0^{\gamma} \frac{\cos c_n\eta}{\left[\eta^2+\frac{r^2}{\beta^2}\right]^{3/2}}\,d\eta + \frac{\cos c_n\gamma}{t\gamma}\right]\right\} 1(t-\frac{r}{\beta}) .$$

Although $A_1(r,z,t)$ becomes indefinitely large for large t, $u_1(r,z,t)$ remains finite for all $t > 0$. In fact, as $t \to \infty$, the residual deformation in the plate is given by

$$(2.63)\qquad \lim_{t\to\infty} u_1(r,z,t) = -\frac{2A_o r}{H\beta^2}\left[\frac{\beta^2}{r^2} + 2\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi a}{H}\cdot \cos\frac{n\pi b}{H}\,\beta c_n K_1(\frac{c_n r}{\beta})\right]$$

(Bateman Manuscript Project, 1954, vol. 1, p. 11).

Using the dimensionless numbers, τ, κ, and Γ, 2.62 is

$$(2.64)\qquad u_1(\kappa,z,\tau) = \frac{1}{\beta} A_1(\kappa,z,\tau)\,\delta(\tau-\kappa) - \frac{2\kappa A_o}{H^2}\cdot \left\{\frac{1}{\kappa^2} + \frac{1}{\Gamma(\tau+\Gamma)} + 2\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi a}{H}\cos\frac{n\pi b}{H}\cdot \left[\int_0^{\Gamma}\frac{\cos(n\pi\xi)}{(\xi^2+\kappa^2)^{3/2}}\,d\xi + \frac{\cos n\pi\Gamma}{\tau\Gamma}\right]\right\} .$$

With $z = h = 0$, the first few terms of 2.64, excluding the δ-function term, are given by 2.65:

(2.65)

n	u_1
0	$-\frac{2\kappa A_o}{H^2}\left[\frac{1}{\kappa^2}+\frac{1}{\Gamma(\tau+\Gamma)}\right]1(\tau-\kappa)$
1	$\frac{4\kappa A_o}{H^2}\left[\int_0^\Gamma \frac{\cos \pi\eta}{(\eta^2+\kappa^2)^{3/2}}\, d\eta + \frac{\cos \pi\Gamma}{\tau\Gamma}\right]1(\tau-\kappa)$
2	$\frac{4\kappa A_o}{H^2}\left[\int_0^\Gamma \frac{\cos 2\pi\eta}{(\eta^2+\kappa^2)^{3/2}}\, d\eta + \frac{\cos 2\pi\Gamma}{\tau\Gamma}\right]1(\tau-\kappa)$

The integrals in 2.64 are not expressible in terms of finite combinations of elementary functions (Ritt, 1948, Chapter 3), so numerical integration is necessary. The method of integration used for all the integrals in this thesis is described in Appendix H. The compilations were carried out on SILLIAC, the electronic computer at the University of Sydney, Australia.

The first three terms of $u_1(r, z, t)$, as given in 2.65 were computed for $\kappa = H = A_o = r = 1$. The results are shown in fig. 4 (next page).

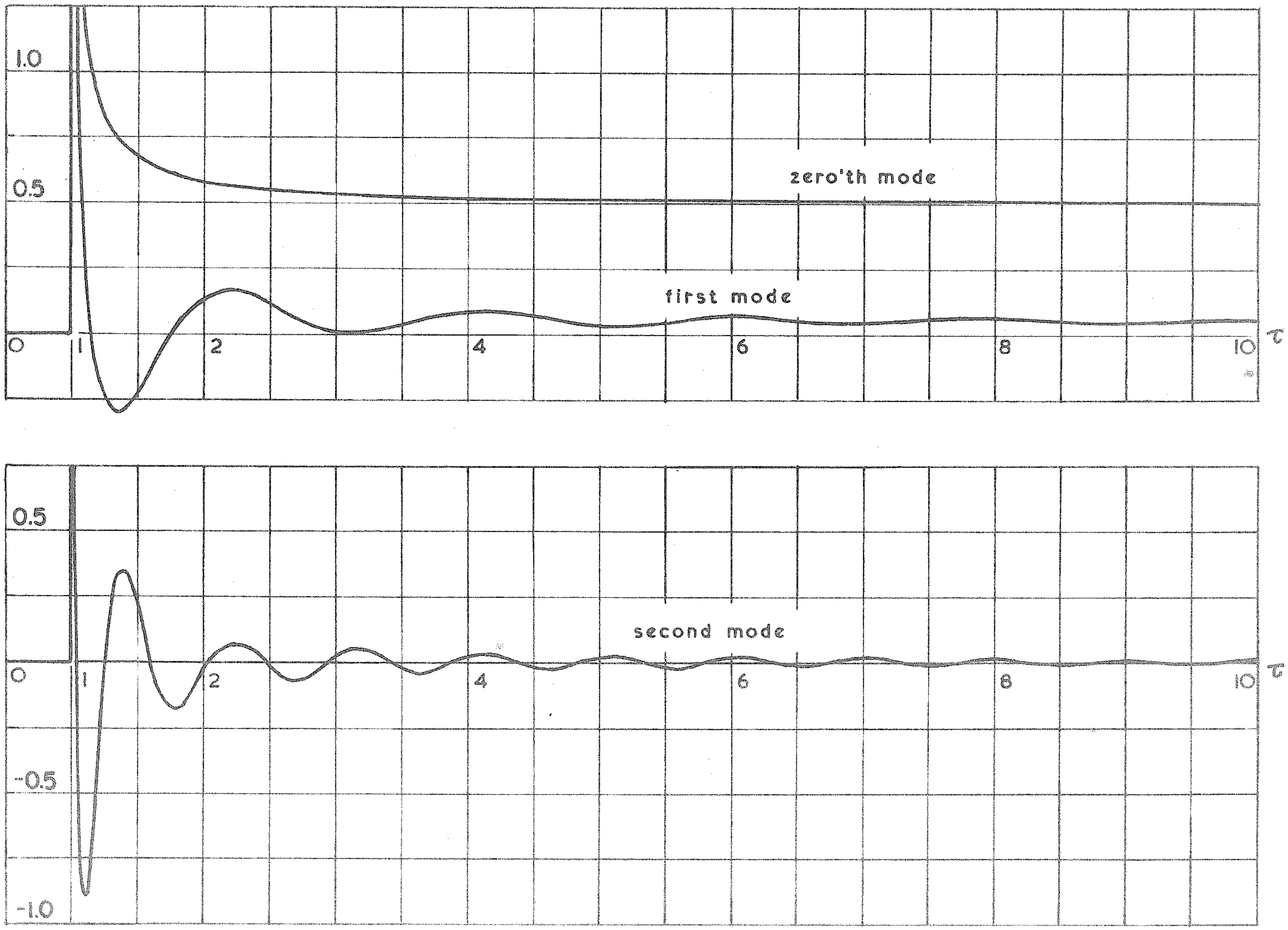


Figure 4: The first three terms of $u_1(r, z, t)$

2.9. Expansion of A(r, z, t) in an infinite series of multiple reflections. -

We may write 2.29

(2.66) $$\overline{A_o}(r,z,t) = A_o\int_0^\infty \frac{k}{\ell} e^{-\ell H} J_o(kr)(1-e^{-2\ell H})^{-1} \cdot$$

$$[e^{\ell(a+b)} + e^{-\ell(a+b)} + e^{\ell(a-b)} + e^{-\ell(a-b)}]\, dk.$$

We know that

(2.67) $$(1 - e^{-2\ell H})^{-1} = \sum_{n=0}^{\infty} e^{-2nH\ell}$$

provided that $e^{-2\ell H} < 1$, i.e., if $Re(\ell) > 0$, which is always true in our case. Substituting 2.67 into 2.66, we have

(2.68) $$\overline{A}(r,z,t) = A_o\int_0^\infty \sum_{m=1}^{4}\sum_{n=0}^{\infty} \frac{k}{\ell} J_o(kr) \exp(-B_{mn}\ell)dk,$$

where we have defined

(2.69) $$B_{mn} = \begin{cases} a + b + (2n+1)H, & m = 1 \\ -a + b + (2n+1)H, & m = 2 \\ a - b + (2n+1)H, & m = 3 \\ -a - b + (2n+1)H, & m = 4. \end{cases}$$

The infinite series is uniformly convergent, so we may invert the order of summation and integration. We know from 2.8 and 2.21 that

$$\int_0^\infty \frac{k}{\ell} J_o(kr) \exp(-B_{mn}\ell)dk = \frac{1}{R_{mn}} \exp(-\frac{p}{\beta} R_{mn})$$

where

$$R_{mn} = [\, r^2 + B_{mn}^2 \,]^{1/2} \quad ,$$

provided only that B_{mn} is positive, as it is in our case. Then 2.68 is

$$(2.70) \qquad \overline{A}(r,z,p) = A_o \sum_{m=1}^{4} \sum_{n=0}^{\infty} \frac{1}{R_{mn}} \exp\left(-\frac{p}{\beta} R_{mn}\right)$$

and the inverse transform of this is just

$$(2.71) \qquad A(r,z,t) = A_o \sum_{m=1}^{4} \sum_{n=0}^{\infty} \frac{1}{R_{mn}} \delta\left(t - \frac{R_{mn}}{\beta}\right) .$$

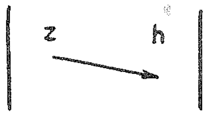
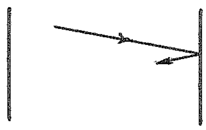
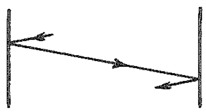
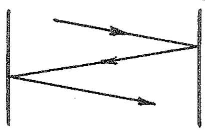
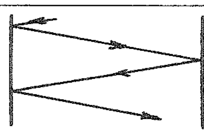
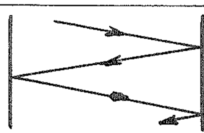
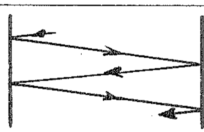
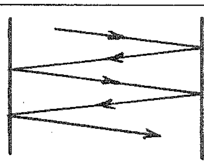
It is easy to show that the double series gives all possible multiple reflections of the incident wave from the faces of the plate. We write out the case $0 < z < h$; the case $h < z < H$ may be obtained by interchanging z and h wherever they appear in what follows.

For $0 < z < h$

$$(2.72) \qquad \begin{cases} B_{1n} = -z + h + 2nH \\ B_{2n} = z + h + 2nH \\ B_{3n} = -z - h + 2(n+1)H \\ B_{4n} = z - h + 2(n+1)H \end{cases}$$

and the corresponding waves for the first few n are shown in the table on the following page.

<u>2.10. Relation between the multiple reflection expansion and the normal mode representation of the solution.</u> - Having solved the problem two different ways, each of which is exact, we must now show that the two solutions are in fact identical, as the existence of a uniqueness theorem

Order	Path	z-component of path length	Exponent
0'th		z - h	B_{10}
1st		z + h	B_{20}
		-z - h + 2H	B_{30}
2d		z - h + 2H	B_{40}
		-z + h + 2H	B_{11}
3d		z + h + 2H	B_{21}
		-z - h + 4H	B_{31}
4th		z - h + 4H	B_{41}
		-z + h + 4H	B_{12}

for elastic wave problems (Love, 1892, p. 176) demands.

The two representations are:

$$(2.71) \qquad A(r,z,t) = A_o \sum_{m=1}^{4} \sum_{n=0}^{\infty} \frac{1}{R_{mn}} \delta(t - \frac{R_{mn}}{\beta})$$

where

$$(2.69) \qquad B_{mn} = \begin{cases} a + b + (2n+1)H, & m = 1 \\ -a + b + (2n+1)H, & m = 2 \\ a - b + (2n+1)H, & m = 3 \\ -a - b + (2n+1)H, & m = 4. \end{cases}$$

and

$$(2.55) \qquad A(r,z,t) = \frac{A_o}{\gamma H} 1(t - \frac{r}{\beta}) \sum_{m=1}^{4} \sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos \frac{n\pi E_m}{H}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \geq 1 \end{cases} , \qquad \gamma = [t^2 - \frac{r^2}{\beta^2}]^{1/2} ,$$

and

$$(2.54) \qquad E_m = \begin{cases} \beta\gamma - a - b, & m = 1 \\ \beta\gamma + a - b, & m = 2 \\ \beta\gamma - a + b, & m = 3 \\ \beta\gamma + a + b, & m = 4. \end{cases}$$

We will demonstrate the identity of the two solutions only for the first term $m = 1$, since the other terms are equivalent by exactly the same reasoning. For this term, 2.71 is

(2.73) $$A(r, z, t) = A_o \sum_{n=0}^{\infty} \frac{1}{R_n} \delta(t - \frac{R_n}{\beta})$$

where

(2.74) $$R_n = [r^2 + (a + b + H + 2nH)^2]^{1/2} .$$

The function A is identically zero except at values of t such that $t = R_n/\beta$. We therefore rearrange the argument of the delta-function to make the series into an equally spaced row of delta-functions as follows: $t - R_n/\beta = 0$ implies

$$\beta^2 t^2 - r^2 - (a + b + H + 2nH)^2 = 0$$

or

(2.75) $$\beta\gamma - a - b - H - 2nH = 0.$$

Changing the argument of the delta-function from $t - R_n/\beta$ to

$$f(t) = \beta\gamma - a - b - H - 2nH$$

requires multiplication by $df/dt = \beta t/\gamma$ (Friedrichs, 1955, p. 29). Hence 2.73 is

(2.76) $$A(r, z, t) = \frac{A_o}{\gamma} \sum_{-\infty}^{\infty} \delta(\beta\gamma - a - b - H - 2nH)\, 1(t - \frac{r}{\beta}) .$$

This transformation has been carried out on the understanding that $\gamma = (t^2 - r^2/\beta^2)^{1/2}$ is real: since we know that $A(r, z, t)$ is identically zero until at least $t = r/\beta$, we have inserted the unit function $1(t-r/\beta)$

in 2.76. This also allows us to extend the lower limit summation from 0 to $-\infty$. But by Lighthill (1958), p. 67,

$$(2.77)\qquad \sum_{-\infty}^{\infty} \delta(x-2nH) = \frac{1}{2H}\sum_{-\infty}^{\infty} e^{in\pi x/H} = \frac{1}{H}\sum_{n=0}^{\infty} \epsilon_n \cos\frac{n\pi x}{H}$$

Thus 2.76 is

$$(2.78)\qquad A(r,z,t) = \frac{A_o}{\gamma H}\sum_{n=0}^{\infty} \epsilon_n \cos\frac{n\pi}{H}(b\gamma-a-b-H)$$

$$= \frac{A_o}{\gamma H}\sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos\frac{n\pi}{H}(\beta\gamma - a - b)\ ,$$

which is the first term of 2.55, since from 2.54,

$$E_1 = \beta\gamma - a - b.$$

Equation 2.77 is the Poisson summation formula for the delta-function. For functions which Lighthill (1958) calls "good" ($F(x)$ is good if it is infinitely differentiable for all x and is $O(x^{-M})$ for all numbers M as $x \to \infty$) the Poisson formula is

$$(2.79)\qquad c\sum_{-\infty=n}^{\infty} F(nc) = \sum_{m=-\infty}^{\infty} \overline{F}(m/c)\ .$$

(Lighthill, 1958, p. 69) where $\overline{F}$ is the Fourier transform of F.

Since the convolution of a delta-function (such as one term of the multiple reflection representation of the solution of this problem for a delta-function source) with a good function is a good function, we may expect that for a physically realizable source the normal mode representing the solution will still be obtainable from the multiple reflection representation by the use of 2.79.

It is interesting to conjecture whether this relationship holds in more complex problems, such as a compressional wave source in a plate or cylinder. It would not be true in a problem of wave propagation in a system including one or two semi-infinite layers, because in this case the solution consists of more than just the residues of an integral: the normal mode representation is defined as just the sum of the residues. The normal mode representation is important even in such problems because the contributions of the residues fall off with distance less rapidly than the branch line integrals, and thus become predominant for a receiver far away from the source.

If relation 2.79 were found to hold for the simpler case of problems in which no semi-infinite layer is involved, this relation might well provide information about the minimum source-receiver distance at which a few terms of the normal mode series may be expected to represent the solution, to a given accuracy.

In the present problem, fig. 3 shows that the first mode alone, or even the first few modes, are not sufficient to give a fair representation of the solution as close as $r = H$. Looking at the phase relationship between the first two modes, we see that the negative half-cycles of the quasi-sinusoidal first mode will tend to be cancelled out, and the positive half-cycles reinforced, by the addition of the higher modes. By adding more and more modes we eventually get large narrow spikes located approximately at the maxima of the first mode -- which is what the multiple reflection expansion predicts. The exact position of the spikes depends directly on h and z in the multiple reflection representation, and in the normal mode representation this dependence is in the factors $\cos \frac{n\pi a}{H} \cos \frac{n\pi b}{H}$ in equation 2.50.

We must go to a physically realizable source function in order to investigate the question of the minimum radial source-receiver distance at which the first few terms of the normal mode expansion is an adequate representation of the solution. Clearly the first few normal modes will be a very poor representation at any distance for a delta-function input.

The procedure would be to pass both representations of the solution through a low-pass filter whose cutoff frequency was low enough to isolate effectively, say, the lowest mode. This would amount to convoluting the solutions for a delta-function source, with a physically realizable source function whose Fourier transform is very small above a certain (sufficiently low) frequency.

Unfortunately, it does not appear to be possible to carry this integration out, even for an ideal low-pass filter. It is clear, however, that here too the general principle holds true that for a receiver close to the source, in a given time interval, far fewer terms of the multiple reflection representation than of the normal mode representation are needed to give a good picture of the solution (and vice versa for a far-away receiver).

<u>2.11. Approximation of the solution at large distances from the source.</u> - We derive in this section an approximate solution which is valid when

$r >> H$. We had in 2.48

$$\overline{A}(r,z,p) = \frac{2A_o}{H} \mathrm{Re}\left\{K_o(pr) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} K(\frac{r}{\beta} Z_n)\right\}, \tag{2.80}$$

where

$$Z_n = [p^2 + c_n^2]^{1/2}, \quad c_n = \frac{n\pi\beta}{H}.$$

For large values of the argument, the modified Bessel function is

$$K_o(x) \approx \left[\frac{2}{\pi x}\right]^{1/2} e^{-x}$$

(Hildebrand, 1948, p. 162), so that for large r, 2.80 is approximately

$$\overline{A}(r,z,p) \approx \mathrm{Re}\left\{E_o(\frac{\beta}{rp})^{1/2} \exp(-\frac{pr}{\beta}) + \sum_{n=1}^{\infty} E_n(\frac{\beta}{rZ_n})^{1/2} \exp(-\frac{rZ_n}{\beta})\right\} \tag{2.81}$$

where we have defined

$$\left\{\begin{aligned} E_o &= \frac{2A_o}{H}\sqrt{\frac{2}{\pi}} \\ E_n &= \frac{4A_o}{H}\sqrt{\frac{2}{\pi}}(-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H}. \end{aligned}\right. \tag{2.82}$$

The inverse Laplace transform of 2.81 is

$$A(r,z,t) = E_o \mathrm{Re}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sqrt{\frac{\beta}{rp}} \exp[p(t-\frac{r}{\beta})]\,dp \tag{2.83}$$

$$+ \mathrm{Re}\frac{1}{2\pi i}\sum_{n=1}^{\infty} E_n\int_{c-i\infty}^{c+i\infty}\sqrt{\frac{\beta}{rZ_n}} \exp[r(\frac{pt}{r} - \frac{Z_n}{\beta})]\,dp,$$

where the order of summation and integration can be inverted, since both the integral and the series are uniformly convergent.

The first integral is

(2.84) $$(\pi t)^{-1/2} 1(t - \frac{r}{\beta})$$

(Magnus and Oberhettinger, 1954, p. 127). Since we have already taken r to be large, the second integral is in a form suitable for a saddle-point approximation.

A saddle point occurs where $\frac{d}{dp}(\frac{pt}{r} - \frac{Z_n}{\beta}) = 0$, or

(2.85) $$\frac{t}{r} = \frac{p}{\beta}[p^2 + c_n^2]^{-1/2}.$$

2.85 implies that the maximum contribution appears to have travelled with a velocity

(2.86) $$U = \frac{r}{t} = \beta[1 + \frac{p^2}{c_n^2}]^{1/2},$$

which is the group velocity in problems in which the solution is formed from the harmonic steady state by Fourier synthesis. We postpone further discussion of 2.86 until the next chapter.

The saddle point is located at p_o in the complex p-plane. Solving 2.85 for p_o,

(2.87) $$p_o = \begin{cases} tc_n[\frac{r^2}{\beta^2} - t^2]^{-1/2} & t < \frac{r}{\beta} \\ -itc_n[t^2 - \frac{r^2}{\beta^2}]^{-1/2} & t > \frac{r}{\beta} \end{cases}$$

As t increases from zero, the saddle point thus moves out from the origin of the p-plane along the real axis, approaching infinity as $t \to \frac{r}{\beta}$. Just after $t = \frac{r}{\beta}$, the saddle point is at $-i\infty$. It moves up the negative imaginary axis, approaching $-ic_n$ as $t \to \infty$.

To find the path of steepest descent through the saddle point, we expand $f(p) = \frac{pt}{r} - \frac{Z_n}{\beta}$ in a Taylor series around p_o and require that

$$(2.88) \qquad f(p) - f(p_o) = \frac{1}{2}(p - p_o)^2 f''(p_o) + \ldots$$

be real and negative. In Appendix D we show that the path is along the real axis when $t < \frac{r}{\beta}$, and makes an angle $\phi = \frac{\pi}{4}$ with the real axis when $t > \frac{r}{\beta}$. Then using the formula for the saddle point approximation,

$$\int F(p)e^{rf(p)}dp \approx \frac{e^{i\phi}\sqrt{2\pi}}{[\,|rf''(p_o)|\,]^{1/2}}\, e^{rf(p_o)}F(p_o)$$

(Ewing, Jardetzky, and Press, 1957, p. 367), the series in 2.83 is

$$(2.89) \qquad \sum_{n=1}^{\infty} E_n \sqrt{2\pi}\,\left[\frac{r^2}{\beta^2} - t^2\right]^{-1/2} \exp\left[-c_n\left(\frac{r^2}{\beta^2} - t^2\right)^{1/2}\right] \qquad t < \frac{r}{\beta}$$

and

$$(2.90) \qquad \sum_{n=1}^{\infty} E_n \sqrt{2\pi}\,\left[t^2 - \frac{r^2}{\beta^2}\right]^{-1/2} \cos\left(c_n\left[t^2 - \frac{r^2}{\beta^2}\right]^{1/2}\right) \qquad t > \frac{r}{\beta}.$$

Equation 2.90 is identical to the higher order terms in the exact solution 2.50. The apparent forerunner 2.89 has been introduced by the approximation method and has no physical reality: the greater r is, the more closely 2.86 approximates the solution before $t \doteq r/\beta$. But it is clear that as r becomes large, 2.86 is very small except at $t = r/\beta$, and in the limit as $r \to \infty$, it is zero even at $t = r/\beta$.

CHAPTER 3

NORMAL MODE THEORY FOR SH WAVES IN A PLATE

3.1. Introduction. - In the general problem of wave propagation in wave guides the residues at the poles of the integrands in terms of which the solution is expressed are called normal modes (Pekeris, 1948, p. 12). These residues decay with distance from the source less rapidly than the branch line integrals, so that far away from the source the normal modes are the predominant part of the solution.

The normal mode solutions are characterised by a standing-wave distribution of amplitude in the direction normal to the guide boundaries, in which the number of nodes is given by the mode number. It has been shown that in layered media the normal mode solutions arise physically from constructively-interfering multiple reflections from the guide boundaries (see Ewing, Jardetzky, and Press, 1957, pp. 140 and 156, and references). The period equation for the wave guide, which relates the frequency or wavelength of the harmonic wave to its phase velocity in the guide direction, may thus be written directly from the physical condition of constructive interference of multiply-reflected waves.

Mathematically, the period equation is identical to the equation which locates the poles of the integrand: $D = 0$, where D is the denominator of the integrand in the formal solution.

In this chapter we set up the problem of SH wave propagation in a plate with a harmonic source, and solve the problem approximately for large distances from the source. We will show first that the period equation can be simply derived from the principle of constructive interference.

3. 2. Constructive interference of multiply-reflected SH waves in a plate. - Consider one crest of a harmonic shear wave train far enough away from the source that the wave surface may be considered plane. The condition for constructive interference at a point A (fig. 5) is for the wave crest just arriving at A to be in phase with the wave crest which has travelled the additional distance ABC. This condition may be written

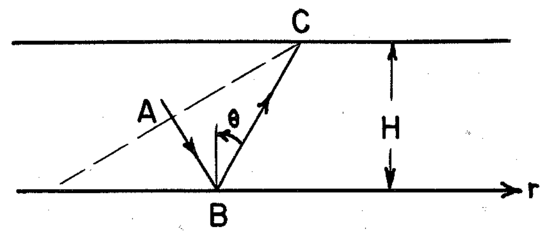


Figure 5: Constructive interference of successive multiple reflections

(3.1) $$\overline{ABC} = n\lambda'$$

where n is an integer and λ' is the wavelength along the ray ABC. There is no phase shift at the boundaries of the plate for SH waves.

The wave phase velocity along the ray is β, the shear wave velocity. The wavelength and phase velocity in the horizontal direction are given by

(3. 2) $$\begin{cases} \lambda = \dfrac{\lambda'}{\sin\theta} \\ c = \dfrac{\beta}{\sin\theta} \end{cases}$$

where θ is the angle between the wave normal and the normal to the faces of the plate. We can show that

(3. 3) $$\overline{ABC} = 2H\cos\theta\ .$$

Using 3. 3 and 3. 2, 3.1 becomes

$$(3.4) \qquad 2H \cot\theta = 2H\left(\frac{c^2}{\beta^2} - 1\right)^{1/2} = n\lambda .$$

In terms of the wave number $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$, 3.4 is

$$(3.5) \qquad kH\left(\frac{c^2}{\beta^2} - 1\right)^{1/2} = n\pi .$$

Solving for the phase velocity, 3.5 becomes

$$(3.6) \qquad \frac{c}{\beta} = \left(1 - \frac{c_n^2}{\omega^2}\right)^{-1/2} = \left[1 - \left(\frac{n\beta}{2H}\right)^2 f^{-2}\right]^{-1/2} ,$$

where $c_n = \frac{n\pi\beta}{H}$ and $f = \frac{\omega}{2\pi}$, the frequency.

The group velocity $U = \frac{d\omega}{dk}$ is

$$(3.7) \qquad \frac{U}{\beta} = \left(1 - \frac{c_n^2}{\omega^2}\right)^{1/2} = \frac{1}{c/\beta} .$$

(See, for example, Ewing, Jardetzky, and Press, 1957, pp. 293-295). 3.6 and 3.7 are shown in fig. 6 plotted against the dimensionless parameter $\xi = \frac{Hf}{\beta}$. The integer n is the mode number. We see that each mode has a cutoff frequency

$$(3.8) \qquad f_o = \frac{1}{T_o} = \frac{n\beta}{2H}$$

at which the group velocity is zero and the phase velocity infinite. Infinite phase velocity implies normal incidence at the faces of the plate, and we see that the cutoff period T_o is the time required for n round trips across the plate at normal incidence.

Equation 3.7 describes the variation of period of the signal with time, but contains no information about how the amplitude varies. 3.7

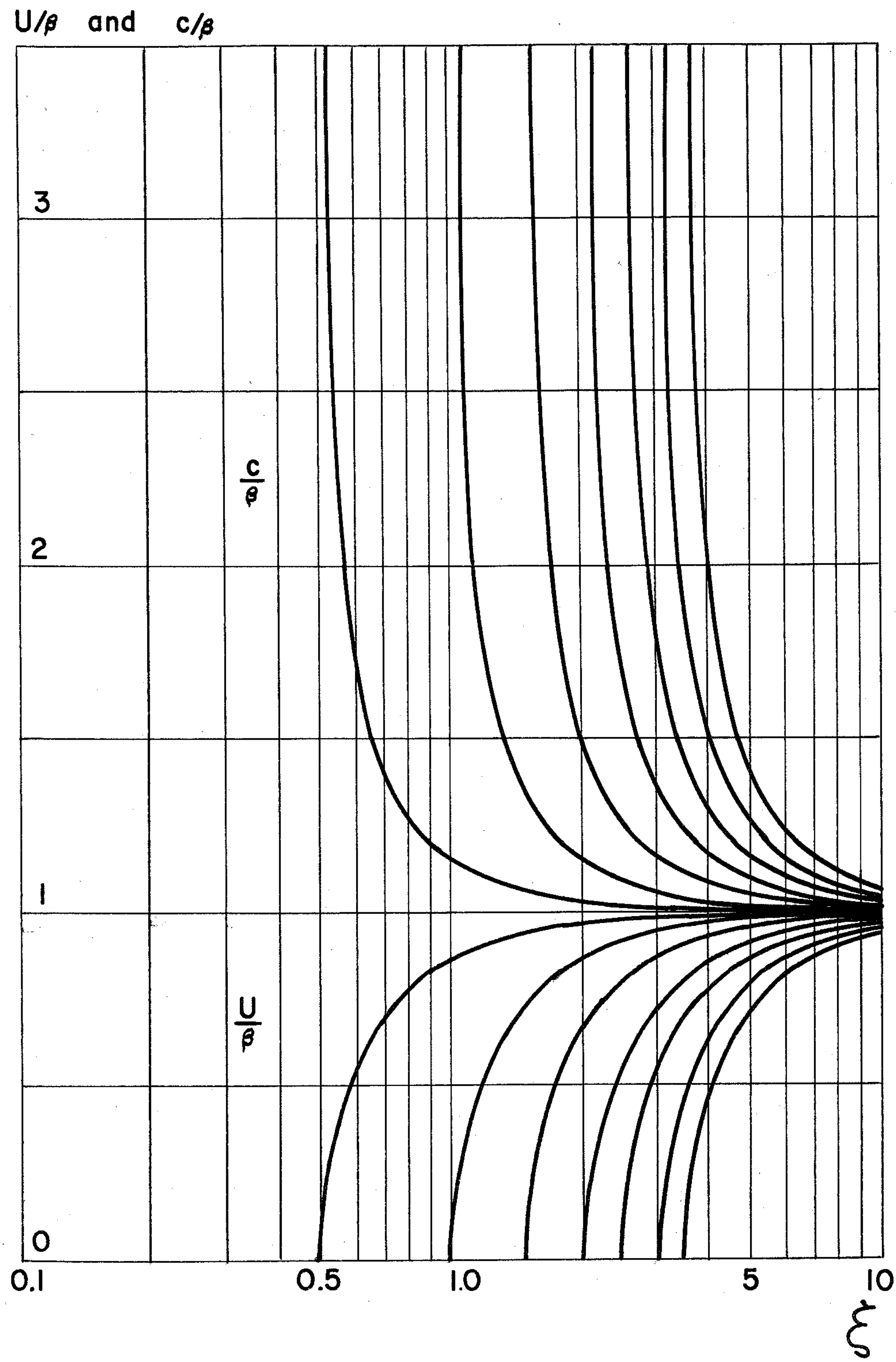


Figure 6: Phase and group velocity curves for the first seven modes of SH wave propagation in a solid plate.

implies that very far away from a source which radiates all frequencies, the front of the received signal will appear to have travelled with the shear wave velocity β. The signal will begin with an infinitely high frequency component, and the frequency of the signal will decrease, rapidly at first and then more and more slowly, until after a long time the signal is approximately a sinusoid of frequency f_o.

This is exactly how the exact solution of Chapter 2 behaves, not only far away from the source, but near the source as well. The saddle point approximation of section 2.11 established 3.7 analytically as the velocity with which the maximum contribution to the solution travels, far away from the source. It has been shown (Broer, 1951), that in all wave propagation problems in which no energy is dissipated in heat, the group velocity is the velocity of energy propagation. However, the whole concept of group and phase velocity is based on plane harmonic waves. Close to the source, where the wave fronts are not even approximately plane, we cannot expect even to be able to define a group velocity.

3.3. Approximate solution of the problem of SH waves in a plate by normal mode theory. - We start with equations 2.3-2.6:

(2.3) $$u = -\frac{\partial A}{\partial r},$$

(2.4) $$\nabla^2 A = \frac{1}{\beta^2}\frac{\partial^2 A}{\partial t^2}$$

(2.5) $$\tau_{z\theta} = -\mu\frac{\partial^2 A}{\partial z \partial r} \quad \text{at } z = 0 \text{ and } z = H,$$

but instead of making a Laplace transform of the problem, we assume that the source emits a steady cosine wave of period $\frac{2\pi}{\omega}$, and has been doing so for an infinite time, so that all transients have died out and we are dealing with a steady state problem.

We use the exponential notation for the harmonic wave emitted by the source, with the agreement that only the real part of the final solution is to be used. Writing

$$A = \mathrm{Re}\,\varphi(r,z)e^{i\omega t}, \tag{3.9}$$

we have to solve

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} + \frac{\omega^2}{\beta^2}\varphi = 0 \tag{3.10}$$

subject to the boundary conditions

$$-\mu\frac{\partial^2\varphi}{\partial r\partial z} = 0 \quad \text{at} \quad z = 0 \text{ and } z = H\,. \tag{3.11}$$

Using the notation

$$k_\beta = \frac{\omega}{\beta}, \quad \nu = (k^2 - k_\beta^2)^{1/2},$$

a solution to 3.10 which is finite at $r = 0$ is

$$\varphi(r,z) = \int_0^\infty J_o(kr)[\,f(k)e^{\nu(z-h)} + g(k)e^{-\nu(z-h)}]\,dk\,. \tag{3.12}$$

Taking a point source of A at $z = h$, $r = 0$,

$$\begin{aligned} A^{(o)} &= \mathrm{Re}A_o e^{i\omega t}\frac{1}{R}\exp(-ik_\beta R) \\ &= \mathrm{Re}A_o e^{i\omega t}\int_0^\infty J_o(kr)e^{-\nu|z-h|}\frac{k}{\nu}\,dk\,, \end{aligned}$$

where

$$R = [(z-h)^2 + r^2]^{1/2},$$

we have the formal solution

(3.13) $$A(r,z,t) = \mathrm{Re} A_o e^{i\omega t} \int_0^\infty J_o(kr)[f(k)e^{\nu(z-h)} + g(k)e^{-\nu(z-h)} + \frac{k}{\nu} e^{-\nu|z-h|}]\,dk .$$

Substituting 3.13 into 3.11, solving for f(k) and g(k), and simplifying exactly as in section 2.5, we find

(3.14) $$A(r,z,t) = \mathrm{Re}\, 2A_o e^{i\omega t} \int_0^\infty J_o(kr) \frac{k \cosh \nu a \cosh \nu b}{\nu \sinh \nu H}\,dk$$

where a and b are as defined in 2.30.

Following the development of section 2.6 exactly, we split the Bessel function into two Hankel functions, and integrate the two separate integrands around contours in the upper and lower halves of the ζ-plane. The poles of the integrands lie on curves $\mathrm{Re}(\nu) = 0$ at

(3.15) $$\nu = \pm \frac{in\pi}{H},$$

which is identical to 3.5, as we expect.

The analysis of the integrals is the same as in the last chapter, except that we have replaced p by $i\omega$. Evaluating the residues of the integrands, we find

(3.16) $$A(r,z,t) = \frac{2A_o\pi}{H} \mathrm{Re}\left\{ e^{i\omega t}\left[H_o^{(2)}\left(\frac{\omega r}{\beta}\right) + 2\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} H_o^{(2)}\left(\frac{rY_n}{\beta}\right)\right]\right\}$$

where

$$Y_n = (\omega^2 - c_n^2)^{1/2}, \qquad c_n = \frac{n\pi\beta}{H}.$$

We now suppose that r is very large, so that we can replace the Hankel functions by their asymptotic approximations

$$H_o^{(2)}(x) \approx (2/\pi x)^{1/2} \exp\left[-i\left(x - \frac{\pi}{4}\right)\right]$$

(Hildebrand, 1948, p. 162). Then 3.16 is

$$(3.17) \qquad A(r,z,t) \approx \frac{2A_o(2\pi\beta)^{1/2}}{H(r)^{1/2}} \operatorname{Re}\left\{\exp\left(\frac{3i\pi}{4}\right)\left[\frac{1}{(\omega)^{1/2}}\exp\left[i\omega\left(t - \frac{r}{\beta}\right)\right] + 2\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi a}{H}\cos\frac{n\pi b}{H}\frac{1}{(Y_n)^{1/2}}\exp\left[i(\omega t - rY_n)\right]\right]\right\}$$

This is the steady state response. We can generalize this result for an arbitrary time variation at the source, S(t). If the Fourier transform $g(\omega)$ of S(t) exists, the response to S(t) is given by

$$A_s(r,z,t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g(\omega)A(r,z,t;\omega)\, d\omega .$$

In our case, then,

$$(3.18) \qquad A_s(r,z,t) = \frac{2A_o(\beta)^{1/2}}{H(r)^{1/2}} \operatorname{Re}\left\{\exp\left(\frac{3i\pi}{4}\right)\int_{-\infty}^{\infty}(\omega)^{-1/2}\exp\left[i\omega\left(t - \frac{r}{\beta}\right)\right]\cdot g(\omega)d\omega\right\}$$
$$+ \frac{4A_o(\beta)^{1/2}}{H(r)^{1/2}} \operatorname{Re}\left\{\exp\left(\frac{3i\pi}{4}\right)\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi a}{H}\cdot \cos\frac{n\pi b}{H}\int_{-\infty}^{\infty}(Y_n)^{-1/2}\exp\left[i(\omega t - rY_n)\right]g(\omega)d\omega\right\}$$

(see, for example, Ewing, Jardetzky, and Press, 1957, p. 143). The first integral in 3.18 is clearly zero for $t < \frac{r}{\beta}$. For $t > \frac{r}{\beta}$, it reproduces the source function with distortion introduced by the factor $1/(\omega)^{1/2}$. It is not useful to carry out the integration for individual $g(\omega)$, but for $g(\omega) = 1$, this integral may be shown to be equal to 2.74. See Lapwood (1949, p. 66) for a discussion of similar integrals.

We may make a stationary phase approximation to the second integral in 3.18. Rewriting the integral in the common notation for this approximation (see Ewing, Jardetzky, and Press, 1957, Appendix A), we have

$$I = \int F(\omega) e^{irf(\omega)} d\omega$$

where r is large, and

$$F(\omega) = Y_n^{-1/2} g(\omega)$$

$$f(\omega) = \frac{\omega t}{r} - \frac{Y_n}{\beta}$$

$$Y_n = (\omega^2 - c_n^2)^{1/2}.$$

The maximum contribution to the integral comes from $f'(\omega) = 0$, or

(3.19) $$\frac{t}{r} = \frac{\omega/\beta}{(\omega^2 - c_n^2)^{1/2}}.$$

The velocity with which the maximum contribution to the integrand appears to travel is therefore

$$\frac{r}{t} = (1 - c_n^2/\omega^2)^{1/2} \tag{3.20}$$

But this velocity is just the group velocity U (Jeffreys and Jeffreys, 1950, p. 512), so that 3.20 is identical to 3.7.

The harmonic steady state and the Laplace transform methods are closely related to one another: it can be shown that a solution in terms of the harmonic steady state may be obtained from the Laplace transform formulation simply by letting the positive real variable p move along a path in the first or fourth quadrants of the p-plane to a point $i\omega$ on the imaginary axis (Cagniard, 1939, p. 140). Substituting $i\omega$ for p in equation 2.83 makes it identical to equation 3.19.

An approximate evaluation of the higher mode solutions has already been carried out in section 2.10, so we need not carry through the stationary-phase approximation of the series term in 3.18. We notice, however, that this term decays with r as $1/(r)^{1/2}$ before integration, and (since the integration brings down another $1/(r)^{1/2}$) $1/r$ after integration. This is what we expect of two-dimensionally guided waves.

CHAPTER 4

WAVE PROPAGATION IN A PLATE WITH MIXED BOUNDARY CONDITIONS

4.1. Introduction - In this chapter we consider separately compressional and shear wave propagation in a solid plate held between smooth frictionless rigid walls. The boundary conditions corresponding to this physical condition are

$$(4.1)\quad \begin{cases} \tau_{rz} = 0 \\ u_z = 0 \end{cases} \quad \text{at } z = 0, \text{ and } z = H, \text{ for all } r \text{ and } t.$$

4.2. The compressional wave source. - The plate is as shown in fig. 1. We now have to deal with the components of displacement u_r, u_z which have been zero in the previous problem. We use the notation of Ewing, Jardetzky, and Press (1957, Chapter 1). The displacement components are derived from potentials φ, W defined by

$$(4.2)\quad \begin{cases} u_r = \dfrac{\partial\varphi}{\partial r} - \dfrac{\partial W}{\partial z} \\ u_\theta = 0 \\ u_z = \dfrac{\partial\varphi}{\partial z} + \dfrac{\partial W}{\partial r} + \dfrac{W}{r} \end{cases}$$

and φ and W satisfy

$$(4.3)\quad \begin{cases} \nabla^2\varphi = \dfrac{1}{\alpha^2}\dfrac{\partial^2\varphi}{\partial t^2} \\ \nabla^2 W - \dfrac{W}{r^2} = \dfrac{1}{\beta^2}\dfrac{\partial^2 W}{\partial t^2} \end{cases}$$

where α and β are the compressional and shear wave velocities, respectively.

We assume as before that a point source is located at $(0, h<H)$, but the source now radiates compressional waves. For simplicity, we assume that the source emits a δ-function of potential φ. The generalization to a source of pressure or volume may be carried out straightforwardly (Dix, 1954), but it is not useful for our purpose here to carry out the generalization.

The source function, then, is

$$\varphi_o = \frac{\Phi_o}{R} \delta(t - \frac{R}{\alpha}) \tag{4.4}$$

where

$$R = [(z-h)^2 + r^2]^{1/2}$$

and Φ_o is a constant whose dimensions are $(\text{length})^3$.

As in Chapter 2, we make a Laplace transform of the entire problem. Drawing on the relations of Chapter 2, we find that solutions of 4.3 are

$$\begin{cases} \overline{\varphi} = \int_0^\infty J_o(kr)[A(k)e^{\ell z} + B(k)e^{-\ell z} + \Phi_o \frac{k}{\ell} e^{-\ell|z-h|}] dk \\ \overline{W} = \int_0^\infty J_1(kr) [C(k)e^{\nu z} + D(k)e^{-\nu z}] dk \end{cases} \tag{4.5}$$

where

$$\ell = (k^2 + \frac{p^2}{\alpha^2})^{1/2}, \quad \nu = (k^2 + \frac{p^2}{\beta^2})^{1/2} . \tag{4.6}$$

The boundary conditions are

(4.7)
$$\begin{cases} \overline{\tau}_{rz} = \mu\left[\frac{1}{\beta^2}\frac{\partial^2\overline{W}}{\partial t^2} + 2\frac{\partial^2\overline{\varphi}}{\partial r\partial z} - 2\frac{\partial^2\overline{W}}{\partial z^2}\right] = 0 \\ \overline{u}_z = \frac{\partial\overline{\varphi}}{\partial z} + \frac{\partial\overline{W}}{\partial r} + \frac{\overline{W}}{r} = 0 \end{cases}$$

at $z = 0$ and $z = H$, for all r and some real positive p.

Substituting 4.5 into 4.7, we find that A, B, C, D are determined by

(4.8)
$$\begin{cases} 2\ell A - 2\ell B - 2k^2C - 2k^2D = -2\Phi_o ke^{-\ell h} \\ 2\ell Ae^{\ell H} - 2\ell Be^{-\ell H} - 2k^2Ce^{\nu H} - 2k^2De^{-\nu H} = 2\Phi_o ke^{-\ell(H-h)} \\ 2\ell A - 2\ell B - (2k^2 + \frac{p^2}{\beta^2})C - (2k^2 + \frac{p^2}{\beta^2})D = -2\Phi_o ke^{-\ell H} \\ 2\ell Ae^{\ell H} - 2\ell Be^{\ell H} - (2k^2 + \frac{p^2}{\beta^2})Ce^{\nu H} - (2k^2 + \frac{p^2}{\beta^2})De^{-\nu H} \\ \qquad = 2\Phi_o ke^{-\ell(H-h)} . \end{cases}$$

Solving 4.8, we find that $C = D = 0$, and

(4.9)
$$\begin{cases} A = \Phi_o \frac{k}{\ell} e^{-\ell H} \frac{\cosh \ell h}{\sinh \ell H} \\ B = \Phi_o \frac{k}{\ell} \frac{\cosh \ell(H-h)}{\sinh \ell H} . \end{cases}$$

Substituting 4.9 into 4.5 and using 4.8 to simplify the result, we find

$$
(4.10)\quad \begin{cases} \overline{W} = 0 \\ \overline{\varphi} = 2\Phi_o \displaystyle\int_0^{\infty} \frac{k \cosh \ell a \cosh \ell b}{l \sinh \ell H} J_o(kr)\, dk \end{cases}
$$

where a and b are defined in 2.30. Thus $\overline{\varphi}$ has exactly the same form as $\overline{A}$ has in the SH wave problem, and we have immediately from 2.50:

$$
(4.11)\quad \begin{cases} \varphi(r, z, t) = \dfrac{4\Phi_o}{H} \left(t^2 - \dfrac{r^2}{a^2}\right)^{-1/2} \displaystyle\sum_{n=0}^{\infty} \epsilon_n (-1)^n \cos\frac{n\pi a}{H} \cos\frac{n\pi b}{H} \cdot \\ \qquad \cos\left[c_n^{\prime}\left(t^2 - \dfrac{r^2}{a^2}\right)^{1/2}\right] 1\left(t - \dfrac{r}{a}\right) \\ W(r, z, t) = 0 \end{cases}
$$

where we have put $c_n^{\prime} = \dfrac{n\pi a}{H}$.

4.3. <u>The shear wave source.</u> - Now we consider the same problem as was treated in the last section, except that the source radiates shear waves instead of compressional waves.

We define a potential ψ,

$$
(4.12)\quad W = -\frac{\partial \psi}{\partial r},
$$

so that

$$
(4.13)\quad \begin{cases} u_r = \dfrac{\partial \varphi}{\partial r} + \dfrac{\partial^2 \psi}{\partial r \partial z} \\ u_\theta = 0 \\ u_z = \dfrac{\partial \varphi}{\partial z} - \dfrac{\partial^2 \psi}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} = \dfrac{\partial \varphi}{\partial z} + \dfrac{\partial^2 \psi}{\partial z^2} - \dfrac{1}{\beta^2}\dfrac{\partial^2 \psi}{\partial t^2} \end{cases}
$$

and now φ and ψ satisfy

$$(4.14)\quad \begin{cases} \nabla^2\varphi = \dfrac{1}{\alpha^2}\dfrac{\partial^2\varphi}{\partial t^2} \\ \nabla^2\psi = \dfrac{1}{\beta^2}\dfrac{\partial^2\psi}{\partial t^2}. \end{cases}$$

The boundary conditions are

$$(4.15)\quad \begin{cases} \tau_{rz} = \mu\left[2\dfrac{\partial^2\varphi}{\partial r\partial z} + 2\dfrac{\partial^3\psi}{\partial r\partial z^2} - \dfrac{1}{\beta^2}\dfrac{\partial^3\psi}{\partial r\partial t^2}\right] = 0 \\ u_z = \dfrac{\partial\varphi}{\partial z} + \dfrac{\partial^2\psi}{\partial z^2} - \dfrac{1}{\beta^2}\dfrac{\partial^2\psi}{\partial t^2} = 0 \end{cases} \quad \text{at } z = 0 \text{ and } z = H, \text{ for all } r \text{ and } t.$$

Now we take the source function to be

$$(4.16)\qquad \psi_o = \frac{\Psi_o}{R}\,\delta(t - \frac{R}{\beta})\,.$$

The solutions of the Laplace-transformed equations 4.14 may be written

$$(4.17)\quad \begin{cases} \overline{\varphi} = \displaystyle\int_0^\infty J_o(kr)[\,A(k)e^{\ell z} + B(k)e^{-\ell z}]\,dk \\ \overline{\psi} = \displaystyle\int_0^\infty J_o(kr)[\,C(k)e^{\nu z} + D(k)e^{-\nu z} + \Psi_o\frac{k}{\nu}\,e^{-\ell\,|z-h|}]\,dk \end{cases}$$

where ℓ and ν are as defined in 4.6, and A, B, C, D must be determined from the boundary conditions 4.15. Substituting 4.17 into 4.15, we find that $A = B = 0$, and

$$(4.18)\qquad C = -\Psi_o\frac{k}{\nu}\,e^{-\nu H}\frac{\sinh \nu h}{\sinh \nu H}\,,\qquad D = -\Psi_o\frac{k}{\nu}\,\frac{\sinh \nu(H-h)}{\sinh \nu H}\,.$$

Substituting 4.18 into 4.17 and simplifying as we have done before, we find

$$
(4.19)\qquad \begin{cases} \overline{\varphi} = 0 \\[2ex] \overline{\psi} = 2\displaystyle\int_0^\infty \Psi_o \frac{k \sinh \nu a \sinh \nu b}{\nu \sinh \nu H} J_o(kr)\, dk \end{cases}
$$

where a and b are as defined in 2.30. $\overline{\psi}$ has exactly the same form as $\overline{\varphi}$ had in the last section, except that there are hyperbolic sines instead of hyperbolic cosines in the numerator.

The evaluation of the integral in 4.19 is similar to that carried out in section 2.6, except for the expression for the residues. We start with the evaluation of the residues.

$\underline{\nu = 0}$: As $\nu \to 0$, the integrand approaches

$$
\frac{kab}{H} H_o^{(1),(2)}\left(\frac{ipr}{\beta}\right) ,
$$

so that $\nu = 0$ is not a pole.

$\underline{\nu = \pm \frac{in\pi}{H}}$: At these points, $\zeta = \zeta_n = iZ_n = i\left[\frac{p^2}{\beta^2} + C_n^2\right]^{1/2}$, where $\underline{C_n = \frac{n\pi}{H}}$. The residue of I_1 is

$$
(4.20)\qquad \mathrm{Res}(I_1)\Big|_{\zeta=iZ_n} = \left[\frac{N(\zeta)}{\frac{d}{d\zeta} D(\zeta)}\right]_{\zeta=iZ_n}
$$

$$
= \frac{1}{H}(-1)^{n+1} \sin\frac{n\pi a}{H} \sin\frac{n\pi b}{H} H_o^{(1)}(iZ_n r) .
$$

We have, then,

$$\overline{\psi}(r,z,p) = \frac{4\Psi_o}{H} \sum_{n=1}^{\infty} (-1)^{n+1} \sin\frac{n\pi a}{H} \sin\frac{n\pi b}{H} K_o(rZ_n) \;, \tag{4.21}$$

which becomes after inversion

$$\left\{ \begin{aligned} &\psi(r,z,t) = \frac{4\Psi_o}{H}\left(t^2 - \frac{r^2}{\beta^2}\right)^{-1/2} \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \\ &\sin\frac{n\pi a}{H} \sin\frac{n\pi b}{H} \cos\left(c_n\left[t^2 - \frac{r^2}{\beta^2}\right]^{1/2}\right) 1\left(t - \frac{r}{\beta}\right) \\ &\varphi = 0. \end{aligned} \right. \tag{4.22}$$

ψ thus has the same form as φ had in the last section, except that 1) there is no zero-order term; and 2) the displacement is antisymmetric in z about the middle surface of the plate, instead of symmetric.

4.4. Discussion. - It is important to remember that although there is a remarkable unity in these problems of wave propagation, the displacement vector $\underset{\sim}{u}$ is considerably different in each case. It is a straightforward but algebraically complicated process to compute the response to a source of radial pressure, say, instead of the potential φ_o used in 4.4, or to a torque source instead of the potential ψ_o in 4.16.

The mixed boundary condition problem has a physical significance which was discussed by Mindlin (1955). The very much more difficult problem of the plate in a vacuum may be approached by relaxation of the boundary condition that the normal displacement vanish at the faces of the plate: assume that linearly elastic springs are uniformly inserted between the plate and the rigid bodies on either side of the plate. Then the boundary conditions are:

$$(4.23)\quad \begin{cases} \tau_{rz} = 0 & \text{at } z = 0 \text{ and } z = H \\ \tau_{zz} = \begin{cases} -\mathscr{k} u_z & \text{at } z = H \\ \mathscr{k} u_z & \text{at } z = 0 . \end{cases} \end{cases}$$

when the springs are rigid ($\mathscr{k}$ infinitely large), 4.23 is the same as 4.1. When the spring constant approaches zero, the problem becomes that of a plate in a vacuum.

Mindlin (1955) first established two sets of dispersion curves for shear and compressional wave propagation in the rigidly held plate. These may be derived from the equations locating the poles of the integrands of the last two sections:

$$\ell = \pm \frac{in\pi}{H} \quad \text{or} \quad kH\left(\frac{c^2}{\alpha^2} - 1\right)^{1/2} = n\pi$$

$$\nu = \pm \frac{im\pi}{H} \quad \text{or} \quad kH\left(\frac{c^2}{\beta^2} - 1\right)^{1/2} = m\pi$$

(it is now necessary to distinguish between the integers n and m). Mindlin then showed that the network of curves (n, m even) govern modes in which the plate vibrates symmetrically about its middle surface, while the network (n, m odd) governs the antisymmetric modes.

The problem of the free plate may also be discussed in terms of symmetric and antisymmetric modes, so it is to be expected that in the relaxation process the curves (n, m even) will be closely connected with the symmetrical free plate dispersion curves, and the (n, m odd) curves with the antisymmetrical free plate dispersion curves.

Considering only the antisymmetric modes for the moment, Mindlin showed that as k is decreased from infinity, the dispersion curves for the higher modes of the relaxed problem still pass through the lattice of points of intersection of the curves (n, m odd), but move away from the lattice of points of intersection of the curves (n, m even). In the limit as $k \rightarrow 0$, the dispersion curves for the higher antisymmetric modes still pass through the lattice of points of intersection of (n, m odd) and have moved down to pass through the points of intersection of the curves (n, m even) as shown in fig. 7 (Mindlin's fig. 2.101, 1955) on the next page.

An exactly similar result holds for the symmetric modes, the dispersion curves always passing through the points of intersection of (n, m even) as k decreases, and approaching the points of intersection of (n, m odd) as $k \rightarrow 0$.

For the lower modes, certain other complications arise.

This "terrace-like structure" of the higher modes of vibration of the free plate is shown in fig. 8. page 66 (Mindlin's fig. 2.112, 1955). The relaxation of equations 4. 24 to the free plate problem has also been further developed by Mindlin (1957) and Tolstoy and Usdin (1957).

It may prove that knowledge of the exact solution of the two mixed boundary condition problems can be used to approximate the solution to the free plate problem.

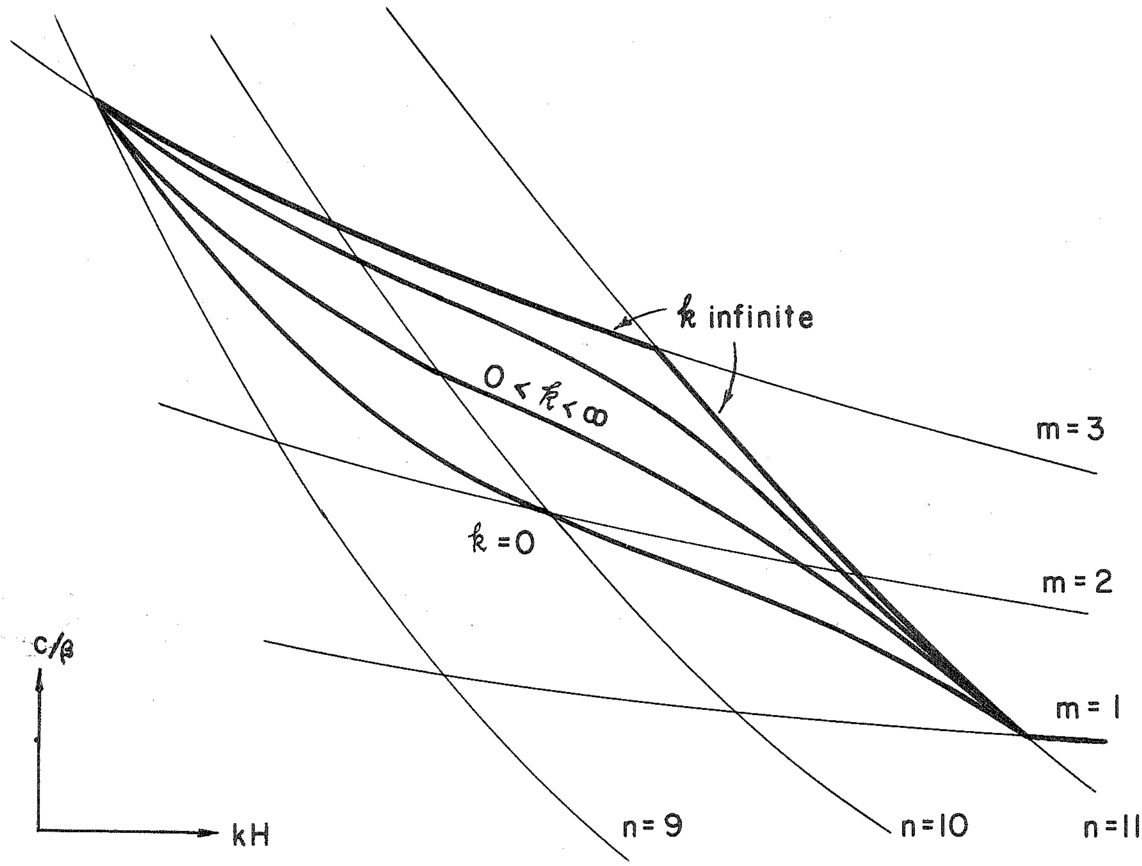


Figure 7: Effect on the dispersion curves of the higher symmetric modes, of relaxing the spring constant (after Mindlin).

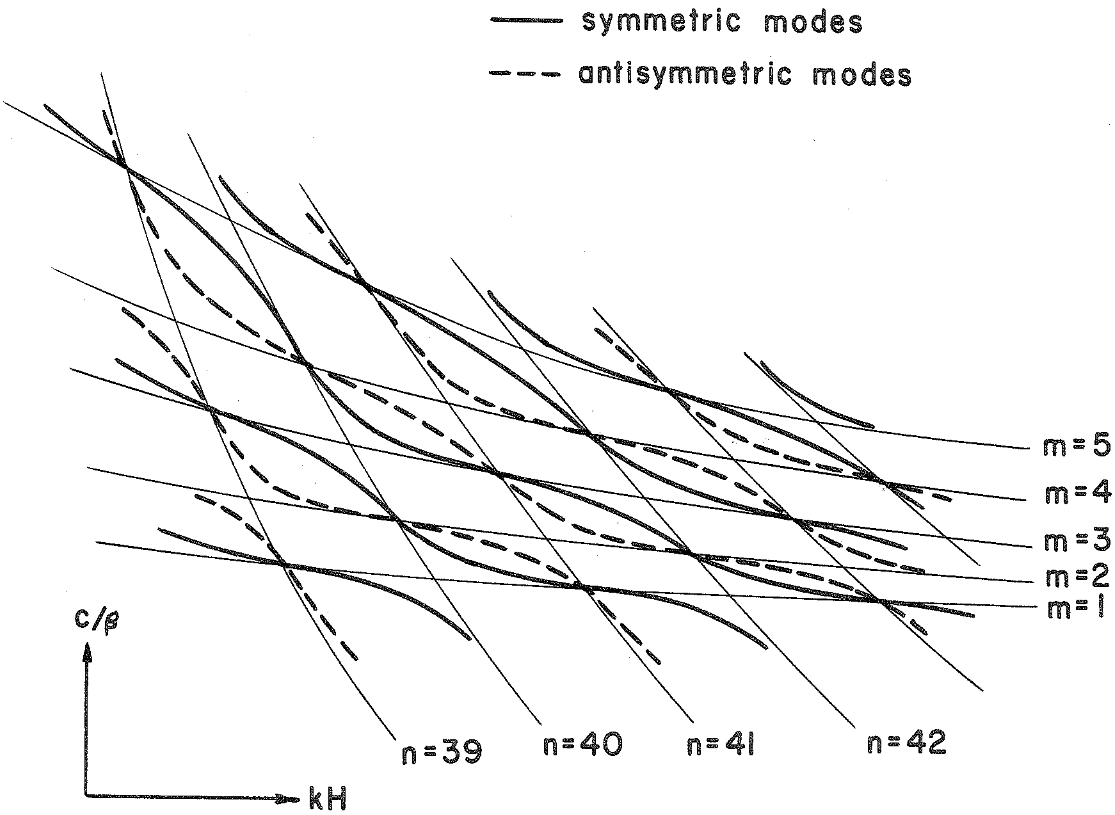


Figure 8: Terrace-like structure of the dispersion curves for a free plate (after Mindlin).

CHAPTER 5

TORSIONAL WAVES IN A SOLID CYLINDER

5.1. Introduction. - We consider an infinitely long, solid, homogeneous, isotropic, perfectly elastic cylinder of finite radius a, surrounded by a vacuum. It is convenient to take the axis of the cylinder along the z-axis of a system of cylindrical coordinates r, θ, z.

The problem is: given a function of time and the radial distance which is integrable with respect to r and is twice differentiable with respect to both r and t, but is otherwise arbitrary; let this function $f(r, t)$ describe the distribution of shear stress $\tau_{z\theta}$ across a plane normal section of the cylinder. We choose $f(r, t)$ such that it is identically zero before $t = 0$, so that the problem is one of progressive wave propagation. We will then determine the displacement u_θ at any point in the cylinder at any time t.

For brevity we will consider the cylinder to be semi-infinite, i.e., to extend from the source plane (for which we choose the plane $z = 0$) to infinity only in the positive z direction. The end of the cylinder is not stress-free, since the stress is defined to be $f(r, t)$ there. In the more general case of a doubly-infinite cylinder, the waves would spread in both directions from the source plane. However, there would be no interaction between the two halves of the cylinder, and the solution in one half of the cylinder would be obtainable from that in the other half merely by changing the sign of the coordinate z, wherever it appeared.

5.2. Statement of the problem. - We will solve the equation of motion for the θ-component of displacement (see Appendix A),

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial^2 t} \tag{5.1}$$

subject to the boundary conditions that the stress vanishes at the radial surface and that the shear stress at $z = 0$ is given by $f(r, t)$:

$$\left\{\begin{array}{ll} \tau_{r\theta} = \mu \dfrac{\partial u_\theta}{\partial z} = f(r, t) & \text{at } z = 0 \\ \\ \tau_{r\theta} = \mu \dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right) = 0 & \text{at } r = a, \end{array}\right. \tag{5.2}$$

where $f(r, t)$ is subject to the restrictions mentioned in the last section.

As in the previous chapters, we make a Laplace transform of the whole problem, obtaining

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{p^2}{\beta^2}\bar{u} - \frac{\bar{u}}{r^2} = 0 \tag{5.3}$$

$$\left\{\begin{array}{ll} \bar{\tau}_{r\theta} = \dfrac{\partial}{\partial r}\left(\dfrac{\bar{u}}{r}\right) = 0 & \text{at } r = a \\ \\ \bar{\tau}_{r\theta} = \mu \dfrac{\partial \bar{u}}{\partial z} = \bar{f}(r, p) & \text{at } z = 0 \end{array}\right. \tag{5.4}$$

where the bar denotes the Laplace-transformed function, and we have dropped the subscript on $\bar{u}_\theta$.

A solution of 5.3 which remains finite as $z \to \infty$ and $r \to 0$ is

$$\bar{u}(r, z, p) = A_o r \exp\left(-\frac{pz}{\beta}\right) + \sum_{n=1}^{\infty} A_n \exp(-\ell_n z) J_1(k_n r), \tag{5.5}$$

where

$$(5.6) \qquad \ell_n = (k_n^2 + \frac{p^2}{\beta^2})^{1/2}$$

We must determine the k_n and A_n from the boundary conditions 5.4. We have at $r = a$,

$$(5.7) \qquad \overline{\tau}_{r\theta} = -\mu \sum_{n=1}^{\infty} A_n \exp(-\ell_n z) \frac{k_n}{a} J_2(k_n a) = 0$$

since

$$J_2(x) + J_o(x) = \frac{2}{x} J_1(x)$$

(Hildebrand, 1948, p. 163). One way for 5.7 to be satisfied for all z is for all the k_n to be solutions of

$$(5.8) \qquad J_2(k_n a) = 0.$$

We take this as the determining condition for the k_n, so that $k_n a$ from now on designates the nth solution of 5.8 .

It can be shown that 5.8 is just the period equation for Love waves propagating circumferentially around a cylinder (Sezawa, 1927).

In order to find the A_n, we must expand $\overline{f}(r, p)$ in a series of eigenfunctions of the problem and match coefficients term by term. In Appendix E we show that

$$(5.10) \qquad \overline{f}(r, p) = rb_o(p) + \sum_{n=1}^{\infty} b_n(p) J_1(k_n r) ,$$

where

$$(5.11)\quad \begin{cases} b_o(p) = \dfrac{4}{a^4}\displaystyle\int_0^a \overline{f}(r, p)\, r^2 dr \\ \\ b_n(p) = \dfrac{2}{a^2 J_1^2(k_n a)} \displaystyle\int_0^a f(r, p) J_1(k_n r)\, r\, dr\ . \end{cases}$$

At $z = 0$,

$$(5.12)\qquad \overline{\tau}_{z\theta} = \mu \frac{\partial \overline{u}}{\partial z} = -\mu A_o \frac{pr}{\beta} - \mu \sum_{n=1}^{\infty} \ell_n A_n J_1(k_n r) = \overline{f}(r, p)\ .$$

Equating coefficients in 5.10 and 5.12, we find

$$(5.13)\quad \begin{cases} A_o = -\dfrac{\beta b_o(p)}{\mu p} \\ \\ A_n = -\dfrac{b_n(p)}{\mu \ell_n} \end{cases}$$

so that the transformed solution may be written

$$(5.14)\qquad \overline{u}(r, z, p) = -\frac{1}{\mu}\left[b_o(p) \frac{\beta r}{p} \exp\left(-\frac{pz}{\beta}\right) + \sum_{n=1}^{\infty} \frac{b_n(p)}{\ell_n} \exp(-\ell_n z) J_1(k_n r) \right]$$

or using 5.11,

$$(5.15)\qquad \overline{u}(r, z, p) = -\frac{1}{\mu}\left[\frac{4}{a^4} \frac{r}{p} \exp\left(-\frac{pz}{\beta}\right) \int_0^a \overline{f}(\xi, p)\xi\, d\xi + \sum_{n=1}^{\infty} \frac{2}{a^2 \ell_n} \exp(-\ell_n z) \frac{J_1(k_n r)}{J_1^2(k_n a)} \int_0^a \overline{f}(\xi, p) J_1(k_n \xi)\xi\, d\xi \right] .$$

In order to carry out the inverse transformation we must be more specific about the function $f(r, t)$.

<u>5.3. The source function $f(r, t)$.</u> - We consider only functions which are separable in their r and t dependence:

$$f(r, t) = g(r)h(t) .$$

We now take the time variation $h(t)$ to be the δ-function. Then we have

$$\bar{u}(r, z, p) = -\frac{4}{\mu a^4}\left[\beta \frac{r}{p} \exp\left(-\frac{pz}{\beta}\right) \int_0^a g(\xi)\xi \, d\xi \right. \tag{5.16}$$

$$\left. + \sum_{n=1}^{\infty} \frac{a^2}{2\ell_n} \exp(-\ell_n z) \frac{J_1(k_n r)}{J_1^2(k_n a)} \int_0^a g(\xi) J_1(k_n \xi)\xi \, d\xi \right] .$$

The integral in the series is difficult to evaluate except for a few special distributions $g(r)$. One simple distribution for which the integral is known is

$$g(r) = \begin{cases} \frac{2Q}{\pi b^4} r & r < b \\ 0 & a < r < b \end{cases} \tag{5.17}$$

where the coefficient of r is chosen so that the total torque exerted on the cylinder is Q (see Appendix F).

Using 5.17 the integrals in 5.16 become

$$(5.18)\quad \begin{cases} \displaystyle\int_0^a g(r)r^2 dr = \frac{2Q}{\pi b^4}\int_0^b r^3 dr = \frac{1}{2\pi}Q \\ \displaystyle\int_0^a g(r)J_1(k_n r)r\,dr = \frac{2Q}{\pi b^4}\int_0^b J_1(k_n r)r^2 dr \\ \displaystyle\qquad = \frac{2Q}{\pi b^2 k_n} J_2(k_n b) \end{cases}$$

(Watson, 1952, p. 132). We can also represent a point source of torque on the axis of the cylinder by taking the limit of 5.18 as $b \to 0$:

$$(5.19)\quad \begin{cases} \displaystyle\lim_{b\to 0}\int_0^a g(r)r\ dr = \frac{1}{2\pi}Q \\ \displaystyle\lim_{b\to 0}\int_0^a g(r)J_1(k_n r)r\,dr = \frac{Qk_n}{4\pi} \end{cases}$$

For simplicity we will use the point source 5.17 and 5.19, but it should be remembered that nothing in the following analysis depends on the source distribution function $g(r)$, and the factors in 5.19 may be replaced by the more general case, 5.16.

With 5.17 and 5.19, 5.16 becomes

$$(5.20)\quad \overline{u}(r, z, p) = -\frac{2Q}{\mu\pi a^4}\left[\beta\frac{r}{p}\exp\left(-\frac{pz}{\beta}\right) + \frac{a^2}{4}\sum_{n=1}^{\infty}\frac{k_n}{\ell_n}\exp(-\ell_n z)\frac{J_1(k_n r)}{J_1^2(k_n a)}\right].$$

5.4. Inverse Laplace transform of the solution. - We notice first that for the term linear in r, we have a very simple solution even for an arbitrary time function h(t):

$$\mathcal{L}^{-1}\left\{\frac{\overline{h}(p)}{p} \exp\left(-\frac{pz}{\beta}\right)\right\} = \int_0^t h(\tau)d\tau \ 1\left(t - \frac{z}{\beta}\right) . \tag{5.21}$$

5.21 is exactly the solution we expect from elementary reasoning if the normal sections of the cylinder are constrained to rotate as a whole. The series in 5.20 therefore describes that part of the solution which the elementary approach to the torsional wave problem ignores.

The inverse Laplace transforms of the functions of p in 5.20 are known:

$$(5.22)\quad \begin{cases} \mathcal{L}^{-1}\left\{\frac{1}{p} \exp\left(-\frac{pz}{\beta}\right)\right\} = 1\left(t - \frac{z}{\beta}\right) \\ \mathcal{L}^{-1}\left\{\frac{1}{\ell_n} \exp(-\ell_n z)\right\} = \mathcal{L}^{-1}\left\{\frac{\beta \exp\left(-\frac{z}{\beta}(p^2 + k_n^2\beta^2)^{1/2}\right)}{(p^2 + k_n^2\beta^2)^{1/2}}\right\} \end{cases}$$

$$= \beta J_o(k_n\beta\gamma)1\left(t - \frac{z}{\beta}\right)$$

(Magnus and Oberhettinger, 1954, p. 133), with the notation

$$\gamma = (t^2 - z^2/\beta^2)^{1/2} .$$

Thus the inverse transform of 5.20 is

(5.23) $$u(r,z,t) = -\frac{2Q\beta}{\mu\pi a^4}\left[r + \frac{a^2}{4\beta}\sum_{n=1}^{\infty}\frac{k_n J_1(k_n r)}{J_1^2(k_n a)} \quad J_o(k_n\beta\gamma)\right] 1(t - \frac{z}{\beta}) .$$

The inverse transform of the more general 5.16 is

(5.24) $$u(r,z,t) = -\frac{4}{\mu a^4}\left[r\int_0^a g(\xi)\xi^2 d\xi + \frac{a^2}{2}\sum_{n=1}^{\infty}\frac{k_n J_1(k_n r)}{J_1^2(k_n a)} \quad J_o(k_n\beta\gamma)\int_0^a g(\xi)J_1(k_n\xi)\xi d\xi\right] 1(t-\frac{z}{\beta}) .$$

5.5. Discussion. - The solution 5.23 or 5.24 consists of two parts: one non-dispersive term (the elementary torsional wave) which is the time integral of the input stress and which depends linearly on the radius; and the series of higher modes of dispersed waves. The relative amplitude of the different modes depends entirely on the initial stress distribution. 5.24 shows that any given mode may be excited to the exclusion of all others by selecting $g(r) = J_1(k_j r)$, and similarly any given mode may be rejected by proper selection of $g(r)$.

We may rewrite 5.23 in terms of the dimensionless parameters

(5.25) $$\tau = \frac{\beta t}{a} , \quad \kappa = \frac{z}{a} , \quad \Gamma = (\tau^2 - \kappa^2)^{1/2}$$

(5.26) $$u(r,\kappa,\tau) = -\frac{2Q}{\mu\pi a^4}\left[r + \frac{a^2}{4\beta}\sum_{n=1}^{\infty}\frac{k_n J_1(k_n r)}{J_1^2(k_n a)} J_o(k_n a\Gamma)\right] 1(\tau - \kappa) .$$

$J_o(k_n a\Gamma)$ is plotted against τ in fig. 9 (next page) for $\kappa = n = 1$. We see that the higher modes have the same general time variation as the higher modes of SH waves in the plate. In fact, if we make a saddle-

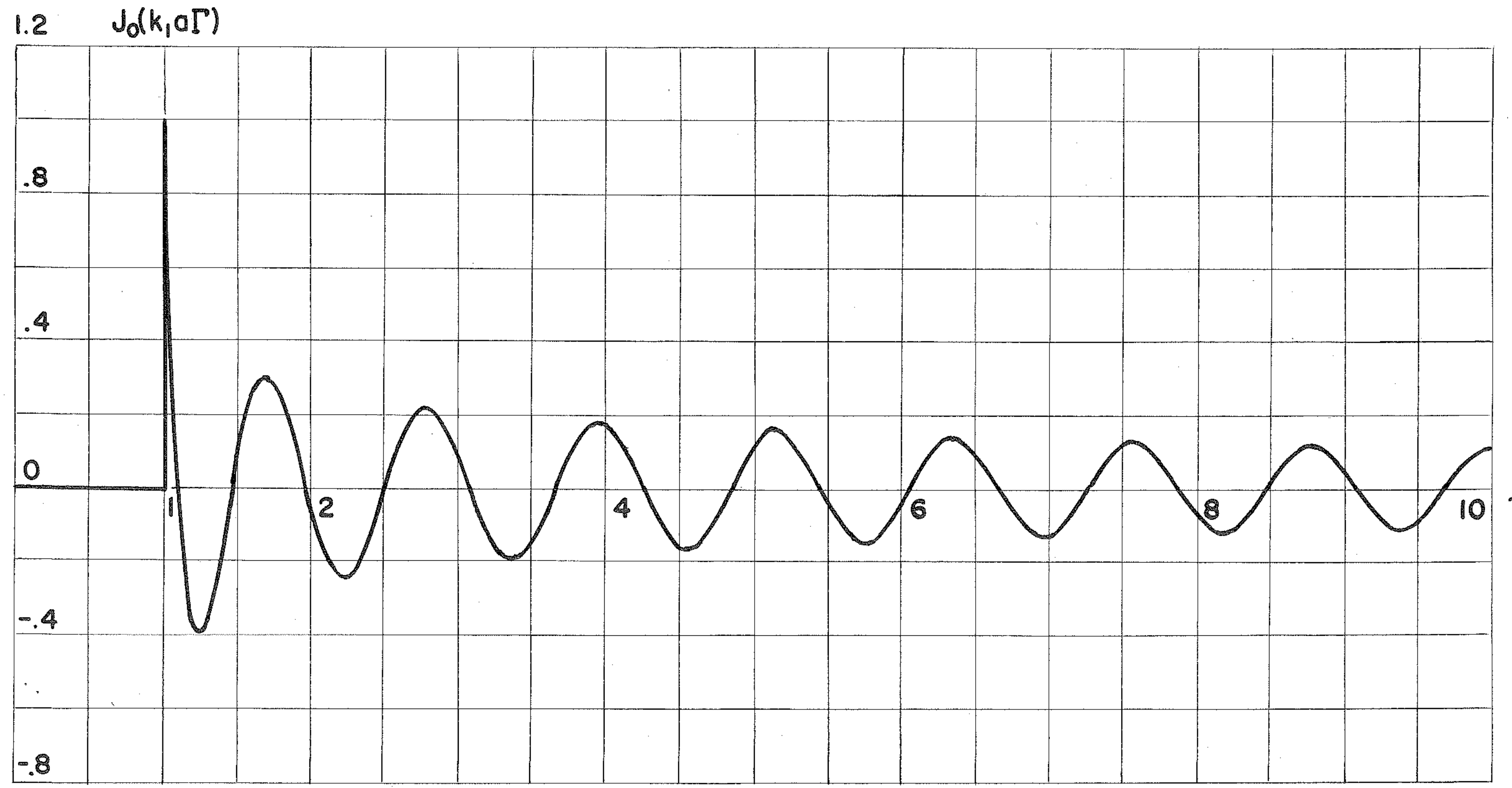


Figure 9: $J_o(k_1 a\Gamma)$

point approximation to 5.20 for large z, we find that the saddle-point is located where

$$(5.27) \qquad \frac{z}{t} = U = \left[1 - \frac{k_n^2 \beta^2}{\omega^2}\right]^{1/2}$$

where we have substituted $i\omega$ for p. This is the same dispersion curve as we found in Chapters 2 and 3, equations 2.86 and 3.20. See also Kolsky (1953) for an elementary derivation of 5.27.

Notice that 5.26 shows that at $z = 0$, at the source plane, the response begins at $t = 0$ and continues for all time with a time dependence given by $J_o(k_n at)$. It is not clear whether this infinitely long transient response is a result of the multiple reflections from the surface of the cylinder at the plane $z = 0$, or of backward propagation of energy from the complex of multiply-reflected waves as it propagates down the cylinder. We expect such backward transmission to take place, and also that the input signal will be distorted upon reflection.

This is a completely different situation from that considered in Chapter 2. There the boundaries were normal to the axis of torque of the source, and there can never be any motion at the line $r = 0$ in that case. Here the boundary is parallel to the axis of torque of the source, yet curved so that the displacement on the boundary is always tangential to the boundary. It does not seem possible to predict whether the interaction of a spherical wave front with such a boundary will produce a surface wave (for the case of such a torque source below a plane boundary, a surface wave is produced: Pinney, 1954).

Solution of this problem in the usual way, starting with a point source on the axis of the cylinder and evaluating an integral in terms of residues and branch line integrals is extremely difficult, as the attempt to solve almost the same problem by T. W. Spencer (unpublished portion of Ph. D. thesis, California Institute of Technology, 1956) showed. This difficulty is what forced us to assume a solution in the form of a series of eigenvalues in the first place. This series has turned out to be the normal mode representation of the solution (standing waves in the r direction, propagation in the z direction) and hence is presumably a series of residues of the integral representation for the solution.

The attempt has not been made to invert 5.26 by Poisson's summation formula and study the result to determine whether the inverted series represents a sum of multiple reflections.

The function $J_o(\beta c\gamma)$ is well-known as the Green's function for the one-dimensional Klein-Gordon equation (Morse and Feshbach, 1953, p. 1343; van der Pol, 1950, p. 331)

$$\frac{\partial^2\psi}{\partial z^2} - c^2\psi = \frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2} \tag{5.28}$$

which governs problems such as the vibrating string with additional stiffness forces along the string, or constant-temperature wave motion in deep water (Carslaw and Jaeger, 1948, p. 183). The Green's function for the telegraph equation is very similar (Morse and Feshbach, 1953, p. 867). So far as the author is aware, however, the function has not been obtained as the solution of an elastic wave problem, nor has the relation between the dispersion curve and the transient solution been discussed.

5.6. Response to a step-function input. - Integration of 5.23 with respect to time gives the step-function response

(5.29) $$u_1(r, z, t) = -\frac{2Q}{\mu\pi a^4}\left[rU(t - \frac{r}{\beta}) + \frac{a^2}{4\beta}\sum_{n=1}^{\infty}\frac{k_n J_1(k_n r)}{J_1^2(k_n(a)}\cdot\int_{\frac{z}{\beta}}^{t} J_o(k_n\beta\gamma)dt\right]1(t - \frac{z}{\beta})$$

where $U(t - z/\beta)$ is the unit ramp-function. The oscillatory part of $u_1(r, z, t)$ is now given by the integral in 5.29. With the dimensionless parameters τ, κ, and Γ the integral is

(5.30) $$\frac{a}{\beta}\int_{\kappa}^{\tau} J_o(k_n a\Gamma)d\tau = \frac{a}{\beta}\int_0^{\Gamma}\frac{xJ_o(k_n ax)}{(x^2+\kappa^2)^{1/2}}dx .$$

This integral was evaluated for $n = 1$ ($k_1 a = 5.1356223$), $a=\beta=\kappa=1$, on SILLIAC, using the numerical method described in Appendix H. The result is shown in fig. 10 (next page).

We see that for large τ at a given κ, 5.30 as well as 5.26 becomes a damped sinusoid of period

$$T = \frac{2\pi}{k_n a} .$$

As $\tau \rightarrow \infty$, the residual displacement in the rod increases without limit as a result of the ramp-function in 5.29. The contribution of the

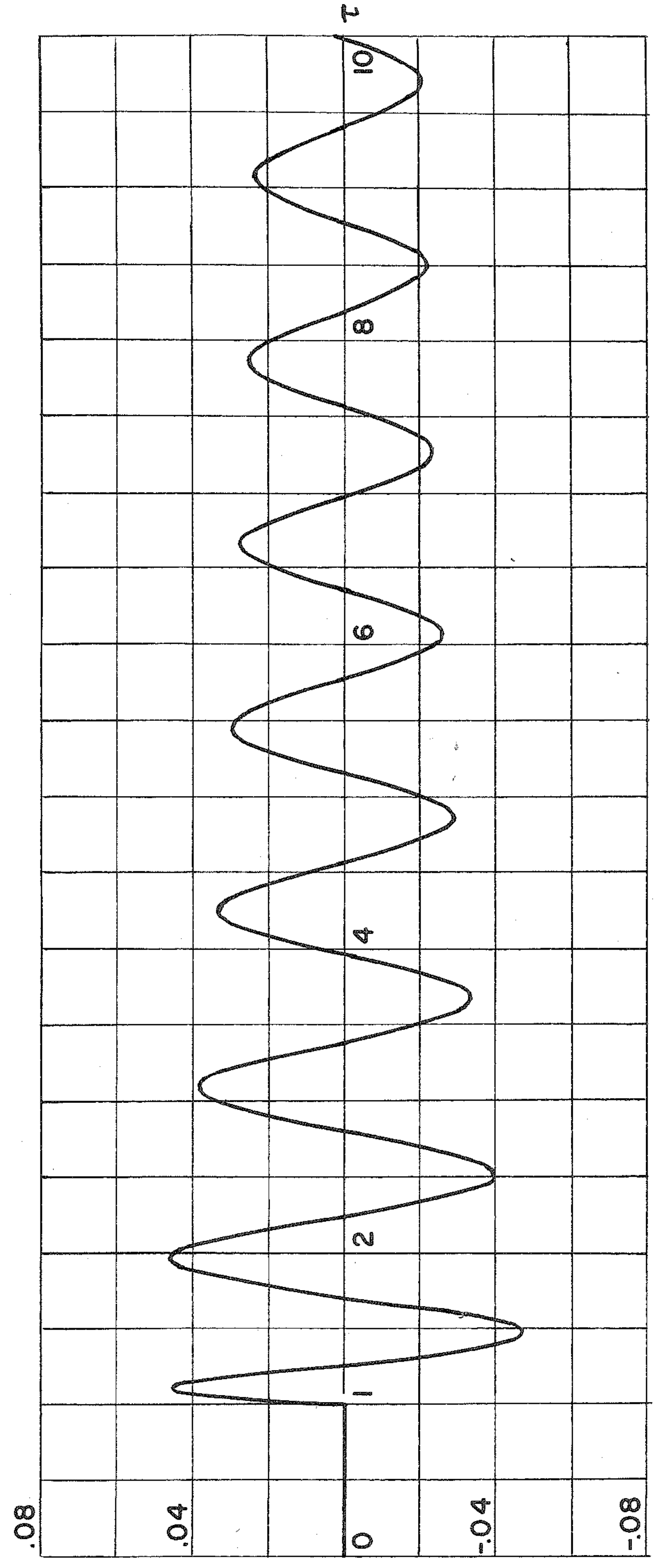


Figure 10: $\frac{a}{\beta} \int_{\kappa}^{\tau} J_o(k_1 a \Gamma)\, d\tau$

higher order modes to the residual displacement is

$$\frac{a^2}{4\beta^2} \sum_{n=1}^{\infty} \frac{J_1(k_n r)}{J_1^2(k_n a)} \exp(-\kappa k_n a)$$

(Watson, 1952, p. 434).

5.7. The cylinder of finite length. - In this section we examine the response of a cylinder of finite length to an impulsive torsional source at one end, and consider the question whether reflection of the torsional wave at the free end may cause distortion of the pulse shape.

We add to 5.4 the boundary condition that

(5.31) $$\tau_{z\theta} = 0 \quad \text{at} \quad z = H,$$

taking H to be the length of the cylinder. The solution 5.5 becomes

(5.32) $$\bar{u}(r, z, p) = A_o r \exp\left(-\frac{pz}{\beta}\right) + B_o r \exp\left(\frac{pz}{\beta}\right) + \sum_{n=1}^{\infty} \left[A_n \exp(-\ell_n z) + B_n \exp(\ell_n z)\right] J_1(k_n r) .$$

Inserting 5.32 into the boundary conditions 5.4 and 5.25 and using 5.10, we find that the A's and B's satisfy

(5.33) $$\begin{cases} A_o - B_o = -\dfrac{\beta b_o}{\mu p} \\ A_n - B_n = -\dfrac{\beta b_n}{\mu \ell_n} \\ A_o \exp\left(-\dfrac{pH}{\beta}\right) - B_o \exp\left(\dfrac{pH}{\beta}\right) = 0 \\ A_n \exp(-\ell_n H) - B_n \exp(\ell_n H) = 0 \end{cases}$$

from which

$$(5.34)\quad \begin{cases} A_o = -\dfrac{\beta b_o \exp(\frac{pH}{\beta})}{2\mu p \sinh(\frac{pH}{\beta})} \ ; \quad B_o = -\dfrac{\beta b_o \exp(-\frac{pH}{\beta})}{2\mu p \sinh(\frac{pH}{\beta})} \\ \\ A_n = -\dfrac{b_n \exp(\ell_n H)}{2\mu\ell_n \sinh(\ell_n H)} \ ; \quad B_n = -\dfrac{b_n \exp(-\ell_n H)}{2\mu\ell_n \sinh(\ell_n H)} \ . \end{cases}$$

Thus 5.32 is, after some rearrangement,

$$(5.35)\qquad \overline{u}(r,z,p) = -\frac{1}{\mu}\left[\frac{b_o \cosh[\frac{p}{\beta}(H-z)]}{p \sinh(\frac{pH}{\beta})}\, r + \sum_{n=1}^{\infty} \frac{b_n \cosh[\ell_n(H-z)]}{\ell_n \sinh(\ell_n H)}\, J_1(k_n r)\right] .$$

For convenience, we specialize to the point source described by equations 5.17 and 5.19, so that 5.35 becomes

$$(5.36)\qquad \overline{u}(r,z,p) = -\frac{2Q}{\mu\pi a^4}\left[\frac{\cosh[\frac{p}{\beta}(H-z)]}{\frac{p}{\beta}\sinh(\frac{pH}{\beta})}\, r + \frac{a^2}{4\beta}\sum_{n=1}^{\infty} k_n \frac{\cosh[\ell_n(H-z)]}{\ell_n \sinh(\ell_n H)}\, \frac{J_1(k_n r)}{J_1^2(k_n a)}\right] .$$

In order to carry out the inversion of 5.36 we expand the functions of p in 5.36 in an infinite series: substituting

$$\frac{1}{\sinh(Hx)} = \sum_{j=0}^{\infty} \exp[-(2j+1)Hx]$$

(where $x = \frac{p}{\beta}$ in the first term of 5. 37, and $x = \ell_n$ in the remaining terms), which is valid since $Hx > 0$ (cf. 2. 67), we have

$$\text{(5.37)} \qquad \frac{\cosh[x(H-z)]}{x \sinh(xH)} = \sum_{j=0}^{\infty} \left\{ \exp[-x(2jH+z)] + \exp[-x(2\{j+1\}H-z)] \right\}$$

so if we define

$$\text{(5.38)} \qquad \begin{cases} L_j = 2jH + z \\ M_j = 2(j+1)H - z, \end{cases}$$

5. 36 becomes

$$\text{(5.39)} \qquad \overline{u}(r, z, p) = -\frac{2Q}{\mu\pi a^4} \left\{ \sum_{j=0}^{\infty} \frac{\beta r}{p} \exp\left(-\frac{pL_j}{\beta}\right) + \sum_{j=0}^{\infty} \frac{\beta r}{p} \exp\left(-\frac{pM_j}{\beta}\right) + \sum_{n=1}^{\infty} \frac{k_n a^2}{4} \left[\sum_{j=0}^{\infty} \frac{1}{\ell_n} \exp(-\ell_n L_j) + \sum_{j=0}^{\infty} \frac{1}{\ell_n} \exp(-\ell_n M_j) \right] \frac{J_1(k_n r)}{J_1^2(k_n a)} \right\} .$$

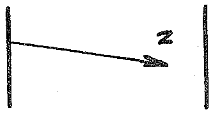
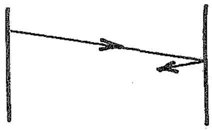
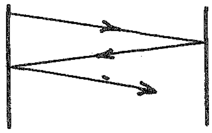
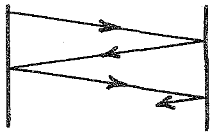
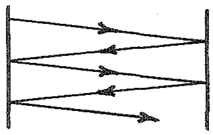
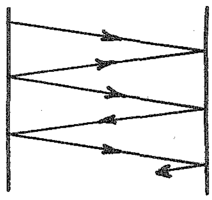
But 5. 39 is identical to 5. 20 if we identify L_j and M_j in turn with z in 5. 20. We can write down the inverse Laplace transform of 5. 39 immediately from 5. 23:

(5.40)
$$u(r,z,t) = -\frac{2Q\beta}{\mu\pi a^4}\sum_{j=0}^{\infty}\Bigg\{ r\,[\,1(t-\frac{L_j}{\beta}) + 1(t-\frac{M_j}{\beta})]$$

$$+ \frac{a^2}{4\beta}\sum_{n=1}^{\infty}\frac{k_n J_1(k_n r)}{J_1^2(k_n a)}\Bigg[J_o(k_n\beta[\,t^2 - \frac{L_j^2}{\beta^2}\,]^{1/2})1(t-\frac{L_j}{\beta})$$

$$+ J_o(k_n\beta\,[\,t^2 - \frac{M_j^2}{\beta^2}\,]^{1/2})1(t-\frac{M_j}{\beta})\Bigg]\Bigg\} .$$

The physical meaning of 5.40 is clear. Each j-term corresponds to a different multiple reflection of the incident set of waves from the ends of the cylinder. The table on the next page gives the first few multiple reflections and the associated coefficients. There is no distortion of the incident waves by the reflection at the ends of the cylinder, and each multiple reflection has exactly the same form as the incident waves 5.23.

Order	Path	z-component of path length	Coefficient
0		z	L_o
1		2H - z	M_o
2		2H + z	L_1
3		4H - z	M_1
4		4H + z	L_2
5		6H - z	M_2

APPENDIX A

DERIVATION OF ELASTIC EQUATION OF MOTION FOR SH WAVES

In cylindrical coordinates, r , θ , z , the general elastic equations of motion are

$$\text{(A. 1)} \qquad \begin{aligned} &\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \rho F_r = \rho \frac{\partial^2 u_r}{\partial t^2} \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \rho F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + \rho F_z = \rho \frac{\partial^2 u_z}{\partial t^2} \end{aligned}$$

(Love, 1892, p. 90), where (u_r, u_θ, u_z) is the vector displacement, (F_r, F_θ, F_z) is the body force, and the τ_{ij} are the components of stress. Hooke's law in cylindrical coordinates is

$$\text{(A. 2)} \qquad \left\{ \begin{aligned} &\tau_{ii} = \lambda \Delta + 2\mu\, \epsilon_{ii} && i = r, \theta, z \\ \\ &\tau_{ij} = \frac{1}{2}\mu\, \epsilon_{ij} && i, j = r, \theta, z \end{aligned} \right.$$

where ϵ_{ij} are the components of strain, Δ is the dilatation, and λ and μ are Lame's constants. Equation A. 2 follows from Hooke's law in rectangular coordinates (Love, 1892, p. 102) and the relations between strain and displacement (Love, 1892, p. 101), by transforming to cylindrical coordinates.

The components of strain are

$$\text{(A. 3)}\quad \begin{cases} \epsilon_{rr} = \dfrac{\partial u_r}{\partial r} & \epsilon_{r\theta} = \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} \\ \epsilon_{\theta\theta} = \dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \epsilon_{z\theta} = \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} + \dfrac{\partial u_\theta}{\partial z} \\ \epsilon_{zz} = \dfrac{\partial u_z}{\partial z} & \epsilon_{rz} = \dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r} \end{cases}$$

and the dilatation is

$$\text{(A. 4)}\qquad \Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r}$$

(Love, 1892, p. 56).

We seek solutions to equation A. 1 which have the following properties:

$$\text{(A. 5)}\quad \begin{cases} 1.\ \dfrac{\partial \underset{\sim}{u}}{\partial \theta} = 0 \ \text{(independence of } \theta \text{)} \\ 2.\ u_r = u_z = 0 . \end{cases}$$

Under these conditions equations A. 3 and A. 4 reduce to

$$\text{(A. 6)}\quad \begin{cases} \epsilon_{rr} = \epsilon_{zz} = \epsilon_{\theta\theta} = \epsilon_{rz} = \Delta = 0 \\ \epsilon_{r\theta} = \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \\ \epsilon_{\theta z} = \dfrac{\partial u_\theta}{\partial z} \end{cases}$$

so that equation A. 2 becomes

$$\text{(A. 7)}\quad \begin{cases} \tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{rz} = 0 \\ \tau_{r\theta} = \mu\left[\dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right] \\ \tau_{z\theta} = \mu\,\dfrac{\partial u_\theta}{\partial z} \end{cases}$$

We assume that body forces are absent. Equation A. 1 reduces to the single equation

$$\text{(A. 8)} \qquad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2}$$

Substituting equation A. 7 into A. 8, and putting $\beta^2 = \mu/\rho$, we find

$$\text{(A. 9)} \qquad \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \nabla^2 u_\theta - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}$$

It is more convenient to work with the wave equation than with an equation like A. 9 . We now find a potential from which u_θ may be derived, which satisfies the wave equation.

By Helmholtz' theorem, any vector field which is finite, continuous, and which vanishes at infinity can be represented as the sum of the gradient of a scalar and the curl of a vector whose divergence is zero (Morse and Feshbach, 1953, p. 52):

$$\underset{\sim}{u} = \nabla\phi + \nabla \times \underset{\sim}{A} , \qquad \nabla \cdot \underset{\sim}{A} = 0 .$$

In cylindrical coordinates

$$\text{(A. 10)} \qquad \begin{cases} \phi = \underset{\sim}{e}_r \dfrac{\partial\phi}{\partial r} + \underset{\sim}{e}_\theta \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} + \underset{\sim}{e}_z \dfrac{\partial\phi}{\partial z} \\[2ex] \nabla \times \underset{\sim}{A} = \underset{\sim}{e}_r \left[\dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z}\right] + \underset{\sim}{e}_\theta \left[\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}\right] \\[2ex] \qquad + \underset{\sim}{e}_z \left[\dfrac{1}{r}\dfrac{\partial(rA_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta}\right] \end{cases}$$

(Magnus and Oberhettinger, 1954, p. 145). By the two conditions (equation A. 5 , ϕ and the components of $\underset{\sim}{A}$ must satisfy the following equations:

$$\text{(A. 11)} \quad \begin{cases} \dfrac{\partial\phi}{\partial r} + \dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z} = u_r = 0 \\[2ex] \dfrac{\partial\phi}{\partial z} + \dfrac{1}{r}A_\theta + \dfrac{\partial A_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta} = u_z = 0 \\[2ex] \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} + \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} = u_\theta \;. \end{cases}$$

A solution of equations A. 11 which is independent of θ is

$$\text{(A. 12)} \quad \begin{cases} \phi = A_r = A_\theta = 0 \\[2ex] u_\theta = \dfrac{\partial A_z}{\partial r} \end{cases}$$

We know that $\underset{\sim}{A}$ satisfies the wave equation

$$\text{(A. 13)} \qquad -\nabla\times\nabla\times\underset{\sim}{A} = \frac{1}{\beta^2}\frac{\partial^2\underset{\sim}{A}}{\partial t^2}$$

(Morse and Feshbach, 1953, p. 143), so that the equation of motion for A_z is

$$\text{(A. 14)} \qquad \nabla^2 A_z = \frac{1}{\beta^2}\frac{\partial^2 A_z}{\partial t^2}$$

Equation A. 9 is obtained by differentiating equation A. 14 with respect to r .

APPENDIX B

THE ROOTS OF $\ell \sinh \ell H = 0$

We write

(B.1) $$\ell = \left[\zeta^2 + \frac{p^2}{\beta^2}\right]^{1/2} = x + iy \ .$$

Then $D(\zeta)$ becomes

(B.2) $$\begin{aligned} \ell \sinh \ell H &= \frac{1}{2}(x+iy)\left[e^{Hx+iHy} - e^{-Hx-iHy}\right] \\ &= \frac{1}{2}(x+iy)\left[\cos(hy)\sinh(Hx) \right. \\ &\qquad \left. + i\sin(Hy)\cosh(Hx)\right] \\ &= \left[x\cos(Hy)\sinh(Hx) - y\sin(Hy)\cosh(hx)\right] \\ &\qquad + i\left[y\cos(Hy)\sinh(Hx) + x\sin(Hy)\cosh(Hx)\right] \ . \end{aligned}$$

For $D(\zeta)$ to vanish, the real and imaginary partsof equation B.2 must vanish separately:

(B.3) $$\begin{aligned} \mathrm{Re}\left[\ell \sinh(\ell H)\right] &= x\cos(Hy)\sinh(Hx) - y\sin(Hy)\cosh(Hx) \\ &\equiv L_1 - L_2 = 0 \end{aligned}$$

(B.4) $$\begin{aligned} \mathrm{Im}\left[\ell \sinh(\ell H)\right] &= y\cos(Hy)\sinh(Hx) + x\sin(Hy)\cosh(Hx) \\ &\equiv L_3 + L_4 = 0 \end{aligned}$$

where we have put

$$\text{(B.5)} \quad \begin{cases} L_1 = x\cos(Hy)\sinh(Hx) \\ L_2 = y\sin(Hy)\cosh(Hx) \\ L_3 = y\cos(Hy)\sinh(Hx) \\ L_4 = x\sin(Hy)\cosh(Hx) \end{cases}$$

There are only three possible ways for equation B. 3 to be satisfied:

$$\text{(B.6)} \quad \begin{cases} \text{a. } L_1 = L_2 = 0 \\ \text{b. } L_1 > 0 \text{ and } L_2 < 0 \\ \text{c. } L_1 < 0 \text{ and } L_2 > 0 . \end{cases}$$

Similarly, for equation B. 4 to be satisfied,

$$\text{(B.7)} \quad \begin{cases} \text{a. } L_3 = 0 \text{ and } L_4 = 0 \\ \text{b. } L_3 > 0 \text{ and } L_4 < 0 \\ \text{c. } L_3 < 0 \text{ and } L_4 > 0 \end{cases}$$

Denote y in the nth quadrant by y_n, and y in the nth or mth quadrant by $y_{n,m}$. Then from the table on the next page, we see that the conditions for B. 6 and B. 7 to be satisfied simultaneously are

1. If $x = 0$, $y = \pm \frac{n\pi}{H}$;

2. If $x < 0$, then equation B. 6 is satisfied only for $y_{1,3}$ - but equation B. 7 can be satisfied only for $y_{2,4}$;

3. If $x > 0$, then equation B. 6 is satisfied only for $y_{1,3}$ - but again equation B. 7 can be satisfied only for $y_{2,4}$.

Therefore, the only roots of $D(\zeta)$ are

$$\mathrm{Re}\,\ell = 0; \quad \mathrm{Im}\,\ell = \pm \frac{n\pi}{H} .$$

	L_1		L_2	Range of x and y for (B. 6) to be satisfied
$L_1 = 0$	$x = 0$ any y $y = \pm \frac{(n+\frac{1}{2})\pi}{H}$ any x	$L_2 = 0$	$y = 0$ any x $y = \pm \frac{n\pi}{H}$ any x	$x = y = 0$ $x = 0$ and $y = \pm \frac{n\pi}{H}$
$L_1 > 0$	$y_{,1,4}$ any y	$L_2 > 0$	$y_{1,2}$ any x $y < 0$ any x	$y_{\pm 1}$ any x
$L_1 < 0$	$y_{2,3}$ any x	$L_2 < 0$	$y_{3,4}$ any x	$y_{\pm 3}$ any x
	L_3		L_4	Range of x and y for (B. 7) to be satisfied
$L_3 = 0$	$y = 0$ any x $x = 0$ any y $y = \pm \frac{(n+\frac{1}{2})\pi}{H}$ any x	$L_4 = 0$	$x = 0$ any y $y = \pm \frac{n\pi}{H}$ $x = y = 0$	$x = 0$ any y
$L_3 > 0$	$y_{1,4}$ $x > 0$ $y_{2,3}$ $x < 0$	$L_4 < 0$	$y_{3,4}$ $x > 0$ $y_{1,2}$ $x < 0$	$y_{\pm 4}$ $x > 0$ $y_{\pm 2}$ $x < 0$
$L_3 < 0$	$y < 0$ and $x > 0$ $y_{1,4}$ $x < 0$ $y_{2,3}$ $x > 0$	$L_4 > 0$	$y_{1,2}$ $x > 0$ $y_{3,4}$ $x < 0$	$y_{\pm 2}$ $x > 0$ $y_{\pm 4}$ $x < 0$

APPENDIX C

SYMMETRY OF $H_o^{(1)}(\zeta r)$, $H_o^{(2)}(\zeta r)$, AND $G(\zeta)$ IN THE ζ-PLANE

We investigate here the changes in $G(\zeta)$ and the Hankel functions when ζ is changed to its complex conjugate $\overline{\zeta}$.

First, define x and y as the real and imaginary parts of ℓ :

$$\ell = \left[\zeta^2 + \frac{p^2}{\beta^2}\right]^{1/2} = x + iy .$$

Thus,

$$\ell^2 = \zeta^2 + \frac{p^2}{\beta^2} = k^2 - \eta^2 + 2ik + \frac{p^2}{\beta^2} = x^2 + 2ixy - y^2 ,$$

since we defined $\zeta = k + i\eta$.

Now, if we change ζ to $\overline{\zeta} = k - i\eta$,

$$\ell^2 = x^2 - y^2 + 2ixy = k^2 - \eta^2 - 2ik + \frac{p^2}{\beta^2}$$

This corresponds to a change of sign of y , so

(C. 1) $$\ell(\overline{\zeta}) = \overline{\ell(\zeta)}$$

Next,

$$\cosh(\ell) = \cosh(x+iy) = \cosh(x)\cos(y) + i(\sinh(x)\sin(y)$$

$$\cosh(\overline{\ell}) = \cosh(x-iy) = \cosh(x)\cos(y) - i\sinh(x)\sin(y)$$

so

$$\cosh(\overline{\ell}) = \overline{\cosh(\ell)} \quad \text{and} \quad \sinh(\overline{\ell}) = \overline{\sinh(\ell)} .$$

Finally, $H_o^{(2)}(\overline{\zeta r}) = \overline{H_o^{(1)}(\zeta r)}$

(Watson, 1952, p. 168, equations (5) and (6)).

We had

$$G(\xi) = \frac{\cosh(\ell a)\cosh(\ell b)}{\ell \sinh(\ell H)}$$

so that

$$G(\overline{\xi}\,) = \frac{\overline{\xi \cosh(\ell a)}\;\overline{\cosh(\ell b)}}{\overline{\ell \sinh(\ell H)}} = \overline{G(\xi)}$$

From equation 2.29,

$$I_1(\xi) = G(\xi)\, H_o^{(1)}(\xi r)$$

$$I_2(\xi) = G(\xi)\, H_o^{(2)}(\xi r)$$

so

(C. 2) $$I_2(\overline{\xi}) = G(\overline{\xi})\, H_o^{(2)}(\overline{\xi} r) = \overline{G(\xi)}\;\overline{H_o^{(1)}(\xi r)} = \overline{I_1(\xi)}$$

Thus changing ξ to its complex conjugate changes I_2 to the complex conjugate of I_1.

It also follows from the above that

(C. 3) $$\left[\frac{N(\overline{\xi})\, H_o^{(2)}(\overline{\xi} r)}{\frac{d}{d\xi}\, D(\overline{\xi})}\right] = \overline{\left[\frac{N(\xi) H_o^{(1)}(\xi r)}{\frac{d}{d\xi}\, D(\xi)}\right]} \quad .$$

APPENDIX D

PATHS OF STEEPEST DESCENT FOR APPROXIMATE INTEGRATION OF (2.80)

On a path of steepest descent of a function f(p) , Im(f) = 0. At the saddle point p_o of f(p) , Re(f) is a maximum. Then the equation determining the path in the neighborhood of the saddle point is

(D.1) $$f(p) - f(P_o) = \tfrac{1}{2}(p-p_o)^2 f''(p_o) + \dots = \text{real, negative.}$$

Defining $p = p_o + s + i\sigma$, equation D.1 becomes for $t < \frac{r}{\beta}$

$$-(s^2 - \sigma^2 + 2is\sigma) = \text{real, negative}$$

or

(D.2) $$\begin{cases} s\sigma = 0 \\ s^2 - \sigma^2 > 0 \end{cases}$$

Hence the path is as shown in Figure 11, and the angle between the s-axis and the path is zero.

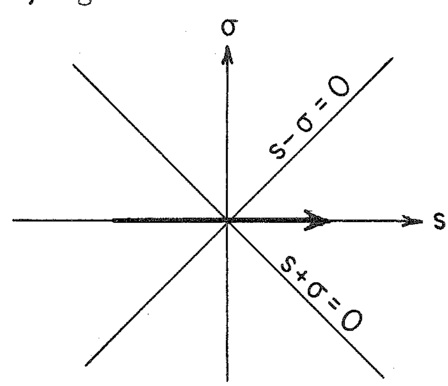


Figure 11: Path of steepest descent for $t < r/\beta$.

For $t > \frac{r}{\beta}$, equation D.1 is

$$i(s^2 - \sigma^2 + 2is\sigma) = \text{real, negative}$$

or

(D.3) $$\begin{cases} s^2 - \sigma^2 = (s+\sigma)(s-\sigma) = 0 \\ s\sigma > 0 \end{cases}$$

Hence the path is as shown in Figure 12, and the angle ϕ between the s-axis and the path is $\frac{\pi}{4}$.

As a check, we notice that in both cases $f''(p_o)e^{2i\phi}$ is real and negative (Jeffreys and Jeffreys, 1950, p. 504).

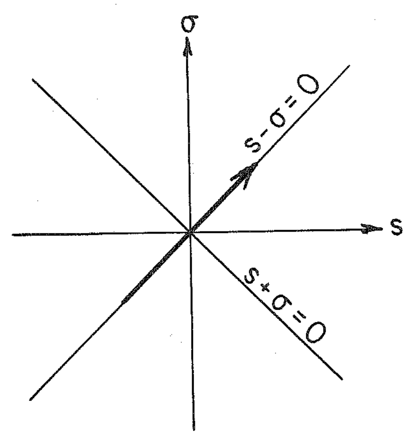


Figure 12: Path of steepest descent for $t > r/\beta$.

APPENDIX E

EXPANSION OF $\bar{f}(r, p)$ IN A SERIES OF EIGENFUNCTIONS OF THE TORSIONAL WAVE PROBLEM

The eigenfunctions w_j are

$$\text{(E. 1)} \quad \begin{cases} w_o = c_o r & j = 0 \\ w_j = c_j J_1(k_j r) & j = 1, 2, 3, \ldots \end{cases}$$

where the c_j are normalization factors to be determined so that the w_j are an orthonormal set. (The set of eigenfunctions is generally given as the $J_1(k_j r)$ alone, but this is not a complete set. See Courant and Hilbert, 1953, p. 424.)

We can easily show that the w_j are orthogonal with respect to the weighting function r: if w_o is involved,

$$\text{(E. 2)} \quad \int_o^a w_o w_j \, r \, dr = \int_o^a c_o c_j \, r^2 J_1(k_j r) \, dr = c_o c_j \frac{a^2}{k_j^2} J_2(k_j a)$$

(Watson, 1952, p. 132). But $J_2(k_j a)$ vanishes by the boundary condition (2.8). If w_o is not involved, for two eigenfunctions such that $i \neq j$,

$$\text{(E. 3)} \quad \int_o^a w_i w_j \, r \, dr = \int_o^a c_i c_j \, r \, J_1(k_i r) \, J_1(k_j r) \, dr$$
$$= (k_j^2 - k_i^2) \left[k_i a \, J_2(k_i a) \, J_1(k_j a) - k_j a \, J_2(k_j a) \, J_1(k_i a) \right]$$

which vanishes since $k_j^2 \neq k_i^2$ and $J_2(k_i a) = J_2(k_j a) = 0$.

We determine the c_j by the normality condition

$$\int_0^a w_j^2 \, r \, dr = 1 \, .$$

For $j = 0$, we find

(E. 4) $$c_o = \frac{2}{a^2} \, ,$$

and c_j is found from

$$\int_0^a c_j^2 \, J_1^2(k_j r) \, r \, dr = \frac{1}{2} c_j^2 \, a^2 \left[J_1^2(k_j a) - J_o(k_j a) \, J_2(k_j a) \right] = 1$$

(Margenau and Murphy, 1943, p. 117) to be

(E. 5) $$c_j = \frac{\sqrt{2}}{a \, J_1(k_j a)}$$

We try to represent a function $\overline{f}(r, p)$ as a series of eigen-functions (E. 1):

(E. 6) $$\overline{f}(r, p) = \sum_{n=0}^{\infty} B_n \, w_n(r, p) = B_o c_o \, r + \sum_{n=1}^{\infty} B_n c_n \, J_1(k_n r)$$

where c_o and c_n are given by equations E. 4 and E. 5, and the B_n must be determined for each particular $\overline{f}(r, p)$ in the usual way: if equation E. 6 is multiplied by $r \, J_1(k_n r)$ and integrated with respect to r from 0 to a, the term in B_o vanishes - the integral is the same as in equation E. 2 - and we find

(E. 7) $$B_n = \frac{\sqrt{2}}{a \, J_1(k_n a)} \int_0^a \overline{f}(r, p) \, J_1(k_n r) \, r \, dr \, .$$

If we multiply equation E. 6 by $r^2 dr$ and integrate over r from 0 to a, every term of the series vanishes - the integral in each term is the same as the integral (E. 2) - and we find

(E. 8) $$B_o = \frac{2}{a^2} \int_0^a \bar{f}(r,p)\, r^2\, dr \;.$$

Thus, equation E. 6 is

(E. 9) $$\bar{f}(r,p) = \frac{4}{a^4}\, r \int_0^a \bar{f}(r,p)\, r^2\, dr + \sum_{n=1}^{\infty} \frac{2}{a^2} \frac{J_1(k_n r)}{J_1^{\,2}(k_n a)} \int_0^a \bar{f}(r,p)\, J_1(k_n r)\, r\, dr \;.$$

or, for brevity,

(E. 10) $$\bar{f}(r,p) = b_o\, r + \sum_{n=1}^{\infty} b_n\, J_1(k_n r)$$

where

(E. 11) $$\begin{cases} b_o = \dfrac{4}{a^4} \displaystyle\int_0^a \bar{f}(r,p)\, r^2\, dr \\[2ex] b_n = \dfrac{2}{a^2\, J_1^{\,2}(k_n a)} \displaystyle\int_0^a \bar{f}(r,p)\, J_1(k_n r)\, r\, dr \;. \end{cases}$$

APPENDIX F

CALCULATION OF TORQUE EXERTED ON THE CYLINDER BY THE SOURCE

We take

$$\tau_{z\theta} = f(r,t) = \delta(t)\, g(r) = \begin{cases} 0 & b < r < a \\ \dfrac{2r}{\pi b^4}\, Q\, \delta(t) & 0 < r < b \end{cases} \qquad \text{(F. 1)}$$

where $\delta(t)$ is the Dirac δ-function, and Q is to be shown to be equal to the total torque exerted by the source.

The force exerted by the source is

$$dF = f(r,t)\, dA = 2\pi f(r,t)\, r\, dr\, .$$

The torque due to the couple consisting of this force and the force complementary to it is

$$dN = r\, dF = 2\pi r^2 f(r,t)\, dr\, ,$$

so that the total torque is

$$N = 2\pi \int_o^a r^2 f(r,t)\, dr = \frac{4}{b^4} Q \int_o^b r^3\, dr\, \delta(t) = Q\, \delta(t) \qquad \text{(F. 2)}$$

which was to be shown.

APPENDIX G

DERIVATION OF THE CONSTANT IN EQUATION 2.9

In this Appendix we determine the relation between the total torque injected into the solid medium by the point source and the displacement field in the solid.

We assume that the point source is equivalent to a small cylinder of radius a and height $2b$ (fig. 13) over the surface of which we may prescribe time-varying torques which will produce the same effect in the solid as the actual point source, provided that the distance of observation is sufficiently large.

The torque exerted by the cylinder on the surrounding medium is the sum of the torque exerted by the two ends and the torque exerted by the curved face of the cylinder (Pinney, 1954). With obvious notation, the total torque is

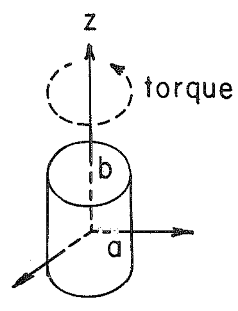


Figure 13: The source cylinder

$$\text{(G.1)} \quad T = -4\pi \int_0^a r^2 \tau_{z\theta} dr \bigg|_{z=b} - 4\pi a^2 \int_0^b \tau_{r\theta} dz \bigg|_{r=a} .$$

From A.7 and A.12,

$$\text{(G.2)} \quad \begin{cases} \tau_{z\theta} = -\mu \dfrac{\partial^2 A}{\partial r \partial z} \\ \tau_{r\theta} = -\mu\left(\dfrac{\partial^2 A}{\partial r^2} - \dfrac{A}{r}\right) . \end{cases}$$

If f(t) denotes the time variation at the source, we may take

(G. 3) $A_s = f(t)/R$

(G. 4) $R = (z^2 + r^2)^{1/2}$,

so that G. 2 is approximately, as R becomes small,

(G. 5)
$$\begin{cases} \tau_{z\theta} = -3\mu r z f(t) R^{-5} + 0(R^{-4}) \\ \\ \tau_{r\theta} = -3\mu r^2 f(t) R^{-5} + 0(R^{-4}). \end{cases}$$

Substituting G. 5 into G. 1, we have

(G. 6) $$T = 12\pi\mu f(t) \; b\int_0^a r^3(r^2+b^2)^{-5/2} dr + a^4\int_0^b (z^2+a^2)^{-5/2} dz \quad .$$

Changing the variables of integration in G. 6, we have

$$T = 12\pi\mu f(t) \int_{ctn^{-1} a/b}^{\pi/2} \cos^3\theta \, d\theta + \int_0^{\tan^{-1} b/a} \cos^3\theta \, d\theta$$

or

$$T = 12\pi\mu f(t) \int_0^{\pi/2} \cos^3\theta \, d\theta = 8\pi\mu f(t) \; .$$

Thus

(G. 8) $f(t) = T/8\pi\mu$

and we may take the constant A_o in 2. 9 as

(G. 9) $A_o = T/8\pi\mu$.

Jeffreys (1931), in a little-known paper, considered the SH radiation from a spherical source in an infinite medium; the constant in front of his solution can be shown to be the same as G. 9. Jeffreys also considered radiation from a spherical pressure source in an infinite medium, for the case $\lambda = \mu$, arriving at the same result as Blake (1952), and considerably antedating Blake and the other workers to whom Blake refers.

APPENDIX H

METHOD OF NUMERICAL INTEGRATION

In order to evaluate integrals of the form

(H.1) $$I = \int_A^B f(x) \cos ax \, dx \, ,$$

where a may be large, Filon's method (Tranter, 1951, chapter 5) has advantages over other methods of numerical integration. This method enables us to use an interval no smaller than is necessary to integrate

$$\int_A^B f(x) \, dx$$

to the required accuracy.

In Filon's method we divide the interval (A, B) into an even number of intervals of width h :

(H.2) $$x = hs \, , \qquad s = 0, 1, 2, \ldots 2n-1, 2n.$$

A parabola is fitted to the function f(x) cos ax at three consecutive values of s , and the integration is carried out explicitly from $x = (s-1)h$ to $x = (s+1)h$. Summing over all the intervals, the result is

(H.3) $$I = h \left\{ \alpha \left[f(B) \sin aB - f(A) \sin aA \right] + \beta C^e + \gamma C^o \right\}$$

where C^e denotes the sum of all the even ordinates of f(x) cos ax , less half the end ordinates, C^o denotes the sum of all the odd ordinates of f(x) cos ax, and

$$(H.4)\quad \begin{cases} \alpha = \theta^{-3}(\theta^2 + \theta \sin\theta \cos\theta - 2\sin^2\theta) \\ \beta = 2\,\theta^{-3}(\theta \quad 1 + \cos^2\theta \quad - 2 \sin\theta \cos\theta) \\ \gamma = 4\,\theta^{-3}(\sin\theta - \theta\cos\theta) \\ e = ah\,. \end{cases}$$

When $a \longrightarrow 0$, equation H. 3 reduces to Simpson's rule.

In order to reduce the amount of machine time required to carry out the integrations in Chapters 2 and 5, a modification of Filon's method was devised: instead of a parabola, a fifth-order curve was fitted to the three consecutive ordinates, counting the two end points as double points. The result on summing up the contributions from each panel is a formula which is considerably more accurate than equation H. 2.

The fifth-order curve is obtained by truncating the Taylor series for $f(x)$ about $x = x_s$, where $x_s = A + hs$:

$$(H.5)\qquad f(x) \approx A_1 + A_1(x-x_s) + A_2(x-x_s)^2 + \ldots + A_5(x-x_s)^5 \;.$$

The A's are determined in terms of f_s, f'_s, f_{s+1}, f'_{s+1}, f_{s-1}, f'_{s-1}; where $f_s = f(x_s)$, $f'_s = \frac{d}{dx} f(x_s)$, etc.

Substituting equation H. 5 into equation H. 1, and integrating by parts five times, we obtain (with $\theta = ah$)

$$(H.6)\qquad I_s = \frac{h}{\theta}\sin ax \left\{ \left[(f_{s+1} - f_{s-1}) - \frac{h^2}{\theta^2}(f''_{s+1} - f''_{s-1}) + \frac{h^4}{\theta^4}(f''''_{s+1} - f''''_{s-1}) \right] \cos\theta - \frac{h}{\theta}\left[(f'_{s+1} + f'_{s-1}) - \frac{h^2}{\theta^2}(f'''_{s+1} - f'''_{s-1}) + 2\frac{h^4}{\theta^4} f'''''_{s} \right] \sin\theta \right\}$$

$$+ \frac{h}{\theta}\cos ax \left\{ \left[(f_{s+1} + f_{s-1}) - \frac{h^2}{\theta^2}(f''_{s+1} + f''_{s-1}) + \frac{h^4}{\theta^4}(f''''_{s+1} + f''''_{s-1}) \right] \sin\theta + \frac{h}{\theta}\left[(f'_{s+1} - f'_{s-1}) - \frac{h^2}{\theta^2}(f'''_{s+1} - f'''_{s-1}) \right] \cos\theta \right\} .$$

The higher derivatives occurring in equation H.6 can be expressed in terms of the A's in equation H.5, and thence in terms of the six numbers which are assumed known: f_s, f'_s, f_{s+1}, f'_{s+1}, f_{s-1}, f'_{s-1}. Making these substitutions, collecting terms in $(s+1)$, s, and $(s-1)$, and writing

$$\cos ax = \cos(ax_{s+1} - \theta)$$
$$\sin ax = \sin(ax_{s+1} - \theta)$$

and

$$\cos ax = \cos(ax_{s-1} + \theta)$$
$$\sin ax = \sin(ax_{s-1} + \theta) ,$$

we find an equation for I_s. Summing over all the panels leads to the final form

$$\text{(H. 7)} \quad I = \int_A^B f(x)\cos ax\,dx = h\Big\{\Phi\left[f(B)\sin aB - f(A)\sin aA\right]$$

$$+ \Xi\, h\left[f'(B)\cos aB - f'(A)\cos aA\right] + \Lambda C^{e}_{\cos}$$

$$+ h\,\Psi\, C'^{e}_{\sin} + T\, C^{o}_{\cos} + h\,\Theta\, C'^{o}_{\sin}\Big\}$$

where

(H. 8)

$C^{e}_{\cos}$ is the sum of all the even ordinates of $f(x)\cos ax$, less half the end ordinates;

$C'^{e}_{\sin}$ is the sum of all the even ordinates of $f'(x)\sin ax$, less half the end ordinates;

$C^{o}_{\cos}$ is the sum of all the odd ordinates of $f(x)\cos ax$;

$C'^{o}_{\sin}$ is the sum of all the odd ordinates of $f'(x)\sin ax$;

and the Greek capital letters are parameters depending only on h and a :

$$
(\text{H. }9)\quad
\begin{cases}
\Theta = \theta^{-6}\left[16\theta(15-\theta^2)\cos\theta + 48(2\theta^2-5)\sin\theta\right] \\
\Upsilon = \theta^{-6}\left[16\theta(3-\theta^2)\sin\theta - 48\,\theta^2\cos\theta\right] \\
\Xi = \theta^{-6}\left[2\theta(\theta^2-24)\sin\theta\cos\theta + 15(\theta^2-4)\cos^2\theta \right. \\
\qquad\qquad \left. + \theta^4 - 27\theta^2 + 60\right] \\
\Psi = 2\theta^{-6}\left[\theta(12-5\theta^2) + 15(\theta^2-4)\sin\theta\cos\theta \right. \\
\qquad\qquad \left. + 2\theta(24-\theta^2)\cos^2\theta\right] \\
\Lambda = 2\theta^{-6}\left[\theta(156-7\theta^2)\sin\theta\cos\theta + 3(60-17\theta^2)\cos^2\theta \right. \\
\qquad\qquad \left. - 15(12-5\theta^2)\right] \\
\Phi = \theta^{-6}\left[\theta(\theta^4+8\theta^2-24) + \theta(7\theta^2-156)\cos^2\theta \right. \\
\qquad\qquad \left. + 3(60-17\theta^2)\sin\theta\cos\theta\right] .
\end{cases}
$$

In exactly the same way, it can be shown that

$$
(\text{H. }10)\quad \int_A^B f(x)\sin ax\,dx = h\Big\{\Phi\left[f(A)\cos aA - f(B)\cos aB\right]
$$
$$
+ h\,\Xi\left[f'(B)\sin aB - f'(A)\sin aA\right] + \Lambda C^{e}_{\sin}
$$
$$
- h\,\Psi C'^{e}_{\cos} + \Upsilon C^{o}_{\sin} - h\,\Theta\, C'^{o}_{\cos}\Big\} ,
$$

with an obvious change of notation in the C's .

Clearly equations H. 9 are useless for evaluating the parameters for small θ . Upon expanding the parameters in powers of θ , we find

$$
(H. 11)\quad \begin{cases}
\Theta = -\frac{16}{105}\theta + \frac{8}{945}\theta^3 - \frac{2}{10395}\theta^5 + \frac{1}{405405}\theta^7 - \frac{1}{48648600}\theta^9 \\
\Upsilon = \frac{16}{15} - \frac{8}{105}\theta^2 + \frac{2}{945}\theta^4 - \frac{1}{31185}\theta^6 + \frac{1}{3243240}\theta^8 \\
\Xi = -\frac{1}{15} + \frac{2}{105}\theta^2 - \frac{1}{315}\theta^4 + \frac{2}{7425}\theta^6 - \frac{62}{4729725}\theta^8 \\
\Psi = -\frac{8}{105}\theta + \frac{16}{945}\theta^3 - \frac{104}{51975}\theta^5 + \frac{256}{2027025}\theta^7 - \frac{16}{3274425}\theta^9 \\
\Lambda = \frac{14}{15} - \frac{16}{105}\theta^2 + \frac{22}{945}\theta^4 - \frac{608}{311850}\theta^6 + \frac{268}{2837835}\theta^8 \\
\Phi = \frac{19}{105}\theta - \frac{2}{63}\theta^3 + \frac{1}{275}\theta^5 - \frac{2}{8775}\theta^7 + \frac{34}{3869775}\theta^9 \quad .
\end{cases}
$$

The expansions, H. 11, give better accuracy than equations H. 9 when θ is less than about 0.9.

In the limit as $a \rightarrow 0$, equations H. 11 show that equation H. 7 reduces to the modified Simpson's rule described by Lanczos (1957, p. 417):

$$
(H. 12)\qquad I = \frac{h}{15}\left\{14\, C^{e}_{cos} + 16\, C^{o}_{cos} - h\left[f'(B) - f'(A)\right]\right\} \quad .
$$

It can also be shown that equation H. 10 reduces to equation H. 12 for $a \rightarrow \frac{\pi}{2}$.

The ratio of the error of the modified Simpson's rule to the error of the standard Simpson's rule is

$$
(H. 13)\qquad \frac{2h^2}{105}\,\frac{f''''''(\vartheta)}{f''''(\vartheta)}
$$

(Lanczos, 1957, p. 417), where ϑ is some point in the interval (A, B). We may expect to obtain a similar increase in accuracy over Filon's method with the present modification.

To check the accuracy of the modification [and the expansions, H. 11], the integral

$$\int_{0.5}^{1.5} e^{x} \cos(\pi x)\, dx$$

was evaluated to nine decimal places, using the electronic computer at the University of Sydney. The correct value of the integral is -1.7718441, to nine places. The errors, for several different intervals, were:

Interval	Number of points	Error, Filon's method	Error, modification
0.1	11	-.00000141	$< 10^{-9}$
0.25	5	-.00070660	-.00000016
0.5	3	-.00051522	-.00008785

This represents a satisfactory increase in accuracy, and enables us to use a larger value of h in the computations.

The increased accuracy is paid for by the necessity of evaluating not only the function at each computation point, but its derivative as well - a heavy price in many cases. However, the f(x) in the integrals evaluated here has a simple form, the derivative of which is simple to calculate.

For the integral in Chapter 5, which contains a Bessel function, an approximation to the Bessel functions was used:

$$J_o(x) = \sum_{n=o}^{7} (x/4)^{2n} B_n \qquad 0 < x < 4$$

$$J_o(x) = 2x^{-\frac{1}{2}} \sum_{n=0}^{5} (4/x)^{2n} P_n \cos(x-\pi/4)$$

$$- \sum_{n=0}^{5} (4/x)^{2n+1} Q_n \sin(x-\pi/4) \qquad x > 4$$

(H. 14)

$$J_1(x) = \sum_{n=0}^{7} (x/4)^{2n+1} B'_n \qquad 0 < x < 4$$

$$J_1(x) = 2x^{-\frac{1}{2}} \sum_{n=0}^{5} (4/x)^{2n} P'_n \cos(x-3\pi/4)$$

$$- \sum_{n=0}^{5} Q'_n (4/x)^{2n+1} \sin(x-3\pi/4) \qquad x > 4$$

(Hitchcock, 1958,), where the B's , P's , and Q's are numerical coefficients which decrease in magnitude quite rapidly with increasing n , so that terminating the series at n = 5 or n = 7 results in an error less than $5 \cdot 10^{-9}$ in each case. Equations H. 14 were derived by approximating the Taylor series and the asymptotic series for $J_o(x)$ and $J_1(x)$ by series of Tchebyshev polynomials.

In evaluating the integral of Chapter 5, the range of integration was split into two parts, corresponding to $x \gtrless 4$ in equations H. 14. For $x < 4$ the integration was carried out explicitly by the modified Simpson's rule, equation H. 12. For $x > 4$ the numerical method described in this appendix was used.

APPENDIX I

ADDITIONAL REFERENCES

In this appendix are listed references which bear on the problems treated here, but which are not referred to in the text. Generally, only references not mentioned in Chapter 6 of Ewing, Jardetzky, and Press (1957) are given here.

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