ANALYSIS OF HETEROGENEOUS REACTORS
CONTAINING MODERATING FUEL ELEMENTS

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ABSTRACT

The Feinberg-Galanin method for heterogeneous reactors is formulated by using a two-group model instead of an age kernel. This treatment is then extended to take into account secondary effects such as fast fission and thermalization of neutrons inside a rod which may contain moderator. The use of a single coefficient in a Feinberg-Galanin approach allows one to relate the source and sink strength of the fuel element to the thermal flux only. By defining a set of four coefficients it is possible to connect the strengths of thermal and fast neutron sources and sinks to both thermal and fast fluxes. A method of calculating these four coefficients α_1, β_1, α_2 and β_2 is presented.

The critical fuel mass of a fission electric cell reactor is calculated by using the four coefficient method. A value of approximately 0.77 kg is obtained, compared to a critical mass of 1.3 kg estimated from two-group homogeneous theory.

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I. INTRODUCTION

The theory of heterogeneous reactors was first developed using the homogeneous model by calculating the critical data of an equivalent homogeneous reactor. The actual reactor is divided into cells whose shape is determined by the fuel element distribution. Using the Wigner-Seitz unit cell model, each cell is then replaced by a homogeneous circular cell whose properties are the average properties of the actual cell. Diffusion theory is then used to obtain criticality.

The homogeneous method has several disadvantages:

i) it does not give the fine structure of the flux in a finite core, which makes it difficult to know the fission rate in the fuel.

ii) it is completely irrelevant for a reactor with a small number of elements.

iii) the side elements of the reactor, where the flux varies considerably across the cell, are poorly accounted for.

iv) the actual disposition of the rods in a finite core has no influence on the homogeneous calculations. The same number of fuel rods in a square array or in a hexagonal array gives the same results.

v) it is inaccurate in treating the effects of singularities such as control rods, especially if the symmetry is destroyed by these singularities.

Subsequently, Feinberg (1) and Galanin (2) developed a heterogeneous method applicable to infinite moderator media. This method was extended to finite media of rectangular shape by Meetz (5) and cylindrical shape by Jonsson (6).

Feinberg and Galanin consider each fuel or control rod as a singularity. These singularities are considered as external to the moderator and are treated as localized sources and sinks. The properties of the rods are contained in the thermal constant γ which connects the thermal net current to the thermal flux at the surface of the fuel element.

This allows one to relate the strength of the thermal neutron sink at the slug to the thermal neutron flux at its surface. The singularity is then considered as a sink of thermal neutrons and a source of fast neutrons. The strength of the source is related to the strength of the sink through the coefficient η , the average number of fission neutrons produced per neutron absorbed in the rod. This coefficient may be different for each rod.

The Feinberg-Galanin method is a big improvement over the homogeneous method for reactors with a small number of rods. But it is not completely satisfactory for such reactors as the fission-electric cell reactor, or the vortex tube reactor which have heterogeneous fuel elements containing a significant amount of moderator. In other words, in such a reactor fast neutrons can be produced in a fuel element by fission, but slow neutrons can also be produced by slowing down inside a fuel element. Neither of the above two methods treats this kind of reactor properly; the homogeneous method does not localise the singularity, and the Feinberg-Galanin method neglects the slowing down inside the singularity.

In order to introduce this effect in a heterogeneous reactor calculation, we reconstruct a Feinberg-Galanin method in a two-group model (section II). Then we introduce the additional source and sink effects in the fast and thermal groups (section III).

Hence with the latter method of treating a heterogeneous reactor we consider a fuel element as:

- a source of fast neutrons (fission).
- a source of thermal neutrons (thermalization).
- a sink of fast neutrons (radiative capture and fast fission).
- a sink of thermal neutrons (thermal absorption).

The source and sink terms are related to each other by two constants: η_1 and η_2, the average number of neutrons produced per fast and thermal absorption respectively in the fuel element, and p the probability that a neutron slowing down inside the fuel element reaches thermal energy.

The change introduced here with respect to the Feinberg-Galanin method is that the sink terms are related to both the thermal and fast fluxes at the surface of the rod. These relationships necessitate the establishment of four coefficients which must be determined. Section IV gives an approximative method of obtaining these coefficients for a simple fuel element.

In a numerical example (section V) we consider a different type of fuel element and make the necessary corrections to the previous method of obtaining the coefficients.

We treat here the case of fuel rods only; the extension to control rods is straightforward. In the latter case the coefficients η are equal to zero.

II. FEINBERG-GALANIN METHOD WITH TWO-GROUP MODEL

Consider an infinite moderating medium containing a finite number of fuel elements. These fuel elements are assumed to be cylinders of finite length and parallel to each other forming a core embedded in the moderator. It is assumed that either the distance between these elements is large compared with their transverse dimensions, or that the lattice is sufficiently symmetrical so that the flux near the elements possesses enough symmetry to consider the elements as line sources. To use diffusion theory in the moderator only, we make the further assumption that the distance between two fuel elements is large compared with the diffusion length.

Applying two-group diffusion theory one can write equations for the overall fluxes:

$$-D_1 \nabla^2 \phi_1(\vec{\rho}) + (\Sigma_R + \Sigma_a^{(1)}) \phi_1(\vec{\rho}) = \eta \sum_{k=1}^{N} S_k(\vec{\rho})\, \delta(\vec{\rho} - \vec{\rho}_k) \tag{2.1}$$

$$-D_2 \nabla^2 \phi_2(\vec{\rho}) + \Sigma_a^{(2)} \phi_2(\vec{\rho}) = p\, \Sigma_R \phi_1(\vec{\rho}) - \sum_{k=1}^{N} S_k(\vec{\rho})\, \delta(\vec{\rho} - \vec{\rho}_k) \tag{2.2}$$

where $\phi_1(\vec{\rho})$ and $\phi_2(\vec{\rho})$ refer to the fast and thermal flux respectively in the moderator, Σ_R is the removal cross section of the moderator;

Σ_R can be defined in a two-group theory by(*):

$$\Sigma_R = \frac{D}{\tau_{th}}$$

we have introduced $\Sigma_a^{(1)}$ a fast absorption cross section for the moderator. This may be negligible. N is the total number of fuel elements.

On the right hand sides are the densities of the different neutron sources of the system. We see that each fuel element is considered as a line sink capturing S_k thermal neutrons per unit time and unit length. S_k depends on the position along the fuel element (along the z axis) which is the reason for introducing $S_k(\vec{\rho})$ in the above equations. Note that the Dirac delta functions used in the equations are two dimensional delta functions so that the product $S_k(\vec{\rho})\ \delta(\vec{r}-\vec{r}_k)$ is the sink density of the fuel element k at point $\vec{\rho}$.

The geometry of the fuel elements suggests the use of cylindrical coordinates. We separate the variables in the differential equations. Let:

$$\phi(\vec{\rho}) = \phi(\vec{r})\ \phi(z) \tag{2.3}$$

$$\phi(z) \propto \cos B_z z \tag{2.4}$$

*Ref. 3 - 8.157, p. 459.

$$B_z = \frac{\pi}{2h + 2\Delta_z} \tag{2.5}$$

where Δ_z is the reflector saving on one side along the z axis, along which the reflector needs not to be infinite. $S_k(\vec{\rho})$ which is directly proportional to the flux can be written:

$$S_k(\vec{\rho}) = S_k \cos B_z z$$

where S_k is now a constant to be determined for each rod. We are left with the two dimensional equations:

$$-\nabla^2 \phi_1(\vec{r}) + \kappa_1^2 \phi_1(\vec{r}) = \frac{\eta}{D_1} \sum_{k=1}^{N} S_k \, \delta(\vec{r} - \vec{r}_k) \tag{2.6}$$

$$-\nabla^2 \phi_2(\vec{r}) + \kappa_2^2 \phi_2(\vec{r}) = \frac{p\Sigma_R}{D_2} \phi_1(\vec{r}) - \frac{1}{D_2} \sum_{k=1}^{N} S_k \, \delta(\vec{r} - \vec{r}_k) \tag{2.7}$$

where

$$\kappa_1^2 = \frac{\Sigma_R + \Sigma_a^{(1)}}{D_1} + B_z^2 \qquad \kappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} + B_z^2 \tag{2.8}$$

Solving the fast flux equation we look for a solution of the form:

$$\phi_1(\vec{r}) = \sum_{k=1}^{N} A_k K_0(\kappa_1 |\vec{r} - \vec{r}_k|) \tag{2.9}$$

which is the solution for a superposition of k line sources located at different positions λ_k and satisfies the boundary condition for an infinite moderator medium (vanishing flux at infinity).

One can easily find the coefficients A_k by making the following assumption(*):

The flux at a rod emplacement is the sum of a symmetrical rapidly varying function due to the rod itself and an unsymmetrical slowly varying function due to all the other rods. In computing the derivative about a rod we will neglect the derivative of the slowly varying function when compared with the derivative of the other. The latter function at the rod k is:

$$\varphi_k = A_k \, K_0(\kappa_1 |\bar{r} - \bar{r}_k|)$$

In order to evaluate the coefficients A_k, the finite radius b of the rod must now be considered. The number of fast neutrons leaving the surface of the rod per unit length and time is:

$$-D_1 \frac{\partial \phi_1}{\partial \lambda} 2\pi b = 2\pi b \, D_1 \kappa_1 A_k K_1(\kappa_1 b) = \eta S_k \tag{2,10}$$

where S_k is the number of thermal neutrons absorbed per unit length and time at the center of rod k; then:

$$A_k = \frac{\eta S_k}{2\pi b D_1 \kappa_1 K_1(\kappa_1 b)} \tag{2.11}$$

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*Ref. 3, p. 740.

Using the thermal constant defined by

$$\gamma = \frac{-2\pi b \; J_2(\vec{r}_k)}{\phi_2(\vec{r}_k)} = \frac{S_k}{\phi_2(\vec{r}_k)} \tag{2.12}$$

where $\phi_2(\vec{r}_k)$ and $J_2(\vec{r}_k)$ are the thermal flux and net current (taken as positive when directed outward) at the surface of fuel element k, we obtain for the fast flux:

$$\phi_1(\vec{r}) = \frac{\eta\gamma}{2\pi b D_1 \kappa_1 K_1(\kappa_1 b)} \sum_{k=1}^{N} \phi_2(\vec{r}_k) \, K_0(\kappa_1 |\vec{r} - \vec{r}_k|) \tag{2.13}$$

The thermal equation 1.7 is then:

$$-\nabla^2 \phi_2(\vec{r}) + \kappa_2^2 \, \phi_2(\vec{r}) = m \sum_{k=1}^{N} \phi_2(\vec{r}_k) \, K_0(\kappa_1 |\vec{r} - \vec{r}_k|) - \frac{\gamma}{D_2} \sum_{k=1}^{N} \phi_2(\vec{r}_k) \, \delta(\vec{r} - \vec{r}_k) \tag{2.14}$$

where $$m = p \, \frac{\Sigma_R}{D_2} \, \frac{\eta\gamma}{2\pi b D_1 \kappa_1 K_1(\kappa_1 b)} \tag{2.15}$$

We solve this equation by a Fourier transform technique(*) noting that:

$$\mathcal{F}\left\{K_0 |\kappa \vec{r}|\right\} = \frac{1}{\kappa^2 + s^2}$$

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*Ref. 3, p. 707.

Let $$\psi(\vec{s}) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \phi(\vec{r})\, e^{i\vec{s}.\vec{r}}\, d\vec{r} \tag{2.16}$$

then $$\phi(\vec{r}) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \psi(\vec{s})\, e^{-i\vec{s}.\vec{r}}\, d\vec{s}$$

The transform of equation 2.14 is

$$(s^2 + K^2)\,\psi_2(\vec{s}) = m \sum_{k=1}^{N} \frac{e^{i\vec{s}.\vec{r}_k}\,\phi_1(\vec{r}_k)}{K_1^2 + s^2}$$

$$- \frac{1}{2\pi}\,\frac{\gamma}{D_2} \sum_{k=1}^{N} \phi_2(\vec{r}_k)\, e^{i\vec{s}.\vec{r}_k}$$

$$\therefore\; \psi_2(\vec{s}) = \frac{1}{K_2^2 + s^2}\left(\frac{m}{K_1^2 + s^2} - \frac{\gamma}{2\pi D_2}\right) \sum_{k=1}^{N} \phi_2(\vec{r}_k)\, e^{i\vec{s}.\vec{r}_k}$$

Inversion of $\psi_2(\vec{s})$ is given by:

$$\phi_2(\vec{r}) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{1}{K_2^2 + s^2}\left(\frac{m}{K_1^2 + s^2} - \frac{\gamma}{2\pi D_2}\right) \times$$

$$\times \sum_{k=1}^{N} \phi_2(\vec{r}_k)\, e^{i\vec{s}.\vec{r}_k}\, e^{-i\vec{s}.\vec{r}}\, d\vec{s}$$

Let $$d\vec{s} = s\, ds\, d\theta$$

$$\phi_2(\vec{r}) = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{s}{\kappa_2^2+s^2}\left(\frac{m}{\kappa_1^2+s^2} - \frac{\gamma}{2\pi D_2}\right)\sum_{k=1}^{N}\phi_2(\vec{r}_k)\,e^{-is|\vec{r}-\vec{r}_k|\cos\theta}\,ds\,d\theta$$

$$\phi_2(\vec{r}) = \frac{1}{2\pi}\int_0^{\infty}\frac{s}{\kappa_2^2+s^2}\left(\frac{m}{\kappa_1^2+s^2} - \frac{\gamma}{2\pi D_2}\right)\sum_{k=1}^{N}\phi_2(\vec{r}_k)\,ds\int_0^{2\pi} e^{isz\cos\theta}\,d\theta$$

where $\quad z = |\vec{r} - \vec{r}_k|$

Noting that $\quad \displaystyle\int_0^{2\pi} e^{isz\cos\theta}\,d\theta = 2\pi J_0(sz)$

$$\phi_2(\vec{r}) = \sum_{k=1}^{N}\phi_2(\vec{r}_k)\left\{ m\int_0^{\infty}\frac{s\,J_0(sz)\,ds}{(\kappa_1^2+s^2)(\kappa_2^2+s^2)} - \frac{\gamma}{2\pi D_2}\int_0^{\infty}\frac{s\,J_0(sz)\,ds}{(\kappa_2^2+s^2)}\right\}$$

$$\phi_2(\vec{r}) = \sum_{k=1}^{N}\phi_2(\vec{r}_k)\left\{\frac{m}{\kappa_2^2-\kappa_1^2}\left[\int_0^{\infty}\frac{s\,J_0(sz)\,ds}{\kappa_1^2+s^2} - \int_0^{\infty}\frac{s\,J_0(sz)\,ds}{\kappa_2^2+s^2}\right]\right.$$

$$\left. - \frac{\gamma}{2\pi D_2}\int_0^{\infty}\frac{s\,J_0(sz)\,ds}{\kappa_2^2+s^2}\right\}$$

Noting that $\quad \displaystyle\int_0^{\infty}\frac{s\,J_0(sz)\,ds}{\kappa^2+s^2} = K_0(\kappa z)$

$$\phi_2(\vec{r}) = \sum_{k=1}^{N} \phi_2(\vec{r}_k)\left\{\frac{m}{\kappa_2^2-\kappa_1^2}\left[K_0(\kappa_1|\vec{r}-\vec{r}_k|) - K_0(\kappa_2|\vec{r}-\vec{r}_k|)\right] - \frac{\gamma}{2\pi D_2} K_0(\kappa_2|\vec{r}-\vec{r}_k|)\right\} \tag{2.17}$$

Define

$$H(|\vec{r}-\vec{r}_k|) = \frac{m}{\kappa_2^2-\kappa_1^2} K_0(\kappa_1|\vec{r}-\vec{r}_k|) - \left(\frac{m}{\kappa_2^2-\kappa_1^2} + \frac{\gamma}{2\pi D_2}\right) K_0(\kappa_2|\vec{r}-\vec{r}_k|) \tag{2.18}$$

then

$$\phi_2(\vec{r}) = \sum_{k=1}^{N} \phi_2(\vec{r}_k)\, H(|\vec{r}-\vec{r}_k|) \tag{2.19}$$

The flux at the surface of each rod is given by:

$$\phi_2(\vec{r}_n) = \sum_{k=1}^{N} \phi_2(\vec{r}_k)\, H(|\vec{r}_n-\vec{r}_k|) \tag{2.20}$$

This is a system of N linear homogeneous equations whose unknowns are the $\phi(\vec{r}_k)$'s . For a non-trivial solution of $\phi(\vec{r}_k)$, the determinant of the system must vanish. This is the criticality condition.

III. THE FOUR-COEFFICIENT METHOD

We consider the same assembly as the one defined in section II, but the fuel elements are allowed now to have non-negligible slowing-down properties. We replace each fuel element by:

- a sink of thermal neutrons
- a sink of fast neutrons due to absorption and thermalization
- a source of fast neutrons due to fission
- a source of thermal neutrons due to thermalization inside the rod.

To write the balance equations we need to relate the source and sink strengths which we do with the coefficients η and p.

We use here an overall resonance escape probability as a first approach; it is the same probability as the one used in homogeneous methods. We will indicate in the conclusion how this approximation can be improved.

1) The Neutron-Balance Equations

We write the same equations as in section II but where $S_k^{(1)}$ and $S_k^{(2)}$ the fast and thermal neutron sink terms appear now instead of one thermal sink term:

$$\left(-D_1\nabla^2+\Sigma_R+\Sigma_a^{(1)}\right)\phi_1(\vec{\rho}) = \eta_2\sum_{k=1}^{N} S_k^{(2)}(\vec{\rho})\,\delta(\vec{r}-\vec{r}_k) + (\eta_1-1)\sum_{k=1}^{N} S_k^{(1)}(\vec{\rho})\,\delta(\vec{r}-\vec{r}_k) \qquad (3.1)$$

$$\left(-D_2 \nabla^2 + \Sigma_a^{(2)}\right) \phi_2(\vec{\rho}) = p\, \Sigma_R\, \phi_1(\vec{\rho}) - \sum_{k=1}^{N} S_k^{(2)}(\vec{\rho})\, \delta(\vec{r} - \vec{r}_k) + p \sum_{k=1}^{N} S_k^{(1)}(\vec{\rho})\, \delta(\vec{r} - \vec{r}_k) \tag{3.2}$$

where $S_k^{(1)}$ is the number of neutrons which disappear from fast group per unit length and time at the center of rod k.

$S_k^{(2)}$ is the number of thermal neutrons absorbed per unit length and time at the center of rod k.

η_1 is the average number of fast neutrons produced per fast absorption in rod k (may depend on k).

η_2 is the average number of fast neutrons produced per thermal absorption in rod k.

We note the extra terms in the right hand side of these equations:

(i) $(\eta_1 - 1)\, S_k^{(1)}(\vec{\rho})\, \delta(\vec{r} - \vec{r}_k)$ accounts for fast absorption and fast fission in rod k.

(ii) $p\, S_k^{(1)}(\vec{\rho})\, \delta(\vec{r} - \vec{r}_k)$ accounts for thermalisation in rod k.

In these equations the z dependancy can be removed by letting:

$$\phi(\vec{\rho}) = \phi(\vec{r}) \cos B_z z$$

We then obtain

$$\left(-\nabla^2+\kappa_1^2\right)\phi_1(\vec{r}) = \frac{\eta_2}{D_1}\sum_{k=1}^{N} S_k^{(2)}\delta(\vec{r}-\vec{r}_k) + \frac{\eta_1 - 1}{D_1}\sum_{k=1}^{N} S_k^{(1)}\delta(\vec{r}-\vec{r}_k) \tag{3.3}$$

$$\begin{aligned}\left(-\nabla^2+\kappa_2^2\right)\phi_2(\vec{r}) = {} & \frac{p}{D_2}\Sigma_R\,\phi_1(\vec{r}) - \frac{1}{D_2}\sum_{k=1}^{N} S_k^{(2)}\delta(\vec{r}-\vec{r}_k) \\ & + \frac{p}{D_2}\sum_{k=1}^{N} S_k^{(1)}\delta(\vec{r}-\vec{r}_k)\end{aligned} \tag{3.4}$$

where:

$$\kappa_1^2 = \frac{\Sigma_R+\Sigma_a^{(1)}}{D_1} + B_z^2 \qquad \kappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} + B_z^2 \qquad B_z = \frac{\pi}{2h+2\Delta_z} \tag{3.5}$$

We wish now to express $S_k^{(1)}$ and $S_k^{(2)}$ in terms of the fluxes at the surface of each rod $\phi_1(\vec{r}_k)$ and $\phi_2(\vec{r}_k)$. In order to do this we define the coefficients:

α_1 : probability that a fast neutron entering the fuel element escapes from the fuel element as a fast neutron.

β_1 : probability that a fast neutron born from fission inside the fuel element escapes from the fuel element as a fast neutron.

α_2 : probability that a thermal neutron entering the fuel element escapes from the fuel element.

β_2 : probability that a neutron thermalized inside a fuel element escapes from this fuel element.

These four coefficients depend on the nuclear and geometrical properties of the rod. One way of getting a reasonable approximation for them is given in the next section.

Using these coefficients we can write:

$$S_k^{(1)} = 2\pi b J_1^-(1-\alpha_1) + \eta_2 S_k^{(2)}(1-\beta_1) + \eta_1 S_k^{(1)}(1-\beta_1) \tag{3.6}$$

$$S_k^{(2)} = 2\pi b J_2^-(1-\alpha_2) + p\, S_k^{(1)}(1-\beta_2) \tag{3.7}$$

where we use p the overall resonance escape probability as an approximation like stated before.

J^- is the partial current going inward at the rod surface.

b is the radius of the rod.

The fast and thermal net currents can be expressed:

$$2\pi\bar{b}\, J_1 = 2\pi\bar{b}\left(j_1^+ - j_1^-\right) = \eta_2 S_k^{(2)} \beta_1 + S_k^{(1)} \left(\eta_1 \beta_1 - 1\right) \tag{3.8}$$

$$2\pi\bar{b}\, J_2 \doteq 2\pi\bar{b}\left(j_2^+ - j_2^-\right) = p\, S_k^{(1)} \beta_2 - S_k^{(2)} \tag{3.9}$$

and from diffusion theory we get:

$$j_1^+ + j_1^- = \frac{\phi_1(\vec{r}_k)}{2} \tag{3.10}$$

$$j_2^+ + j_2^- = \frac{\phi_2(\vec{r}_k)}{2} \tag{3.11}$$

where the $j's$ and $\phi's$ denote the partial currents and fluxes at the outside surface of a slug.

Note that due to the assumption of symmetry made at the beginning of section II, the flux at the surface of a rod is independent of the azimuthal angle.

With the six linear relationships 3.6 through 3.11, one can find the S_k's in terms of the ϕ's :

$$S_k^{(1)} = a_1 \phi_1(\vec{r}_k) + b_1 \phi_2(\vec{r}_k) \tag{3.12}$$

$$S_k^{(2)} = a_2 \phi_2(\vec{r}_k) + b_2 \phi_1(\vec{r}_k) \tag{3.13}$$

where:

$$\begin{aligned}
a_1 &= \pi b \; \frac{(1-\alpha_1)(1+\alpha_2)}{M} \\
b_1 &= -\pi b \, \eta_2 \; \frac{(1-\alpha_2)(3\beta_1 - 2 - \alpha_1\beta_1)}{M} \\
a_2 &= \pi b \; \frac{(1-\alpha_2)\left[1+\eta_1(3\beta_1 - 2) + \alpha_1(1-\eta_1\beta_1)\right]}{M} \\
b_2 &= -\pi b \, p \; \frac{(1-\alpha_1)(3\beta_2 - 2 - \alpha_2\beta_2)}{M} \\
M &= \left[1+\eta_1(3\beta_1 - 2) + \alpha_1(1-\eta_1\beta_1)\right](1+\alpha_2) - p\,\eta_2(3\beta_2 - 2 - \alpha_2\beta_2)(3\beta_1 - 2 - \alpha_1\beta_1)
\end{aligned} \tag{3.14}$$

Thus, the differential equations 3.3 and 3.4 become:

$$\begin{aligned}
(-\nabla^2 + \kappa_1^2)\phi_1(\vec{r}) &= \frac{\eta_2}{D_1} \sum_{k=1}^{N} \left[a_2 \phi_2(\vec{r}_k) + b_2 \phi_1(\vec{r}_k)\right] \delta(\vec{r} - \vec{r}_k) \\
&+ \frac{\eta_1 - 1}{D_1} \sum_{k=1}^{N} \left[a_1 \phi_1(\vec{r}_k) + b_1 \phi_2(\vec{r}_k)\right] \delta(\vec{r} - \vec{r}_k)
\end{aligned} \tag{3.15}$$

$$(-\nabla^2+\kappa_2^2)\,\phi_2(\vec{r}) = \frac{p}{D_2}\Sigma_R\,\phi_1(\vec{r}) - \frac{1}{D_2}\sum_{k=1}^{N}\left[a_2\,\phi_2(\vec{r}_k)+b_2\,\phi_1(\vec{r}_k)\right]\delta(\vec{r}-\vec{r}_k)$$
$$+\frac{p}{D_2}\sum_{k=1}^{N}\left[a_1\,\phi_1(\vec{r}_k)+\bar{b}_1\,\phi_2(\vec{r}_k)\right]\,\delta(\vec{r}-\vec{r}_k) \tag{3.16}$$

or

$$(-\nabla^2+\kappa_1^2)\,\phi_1(\vec{r}) = \frac{1}{D_1}\sum_{k=1}^{N}\left\{\phi_1(\vec{r}_k)\left[\eta_2 b_2+(\eta_1-1)a_1\right]+\phi_2(\vec{r}_k)\left[\eta_2 a_2+(\eta_1-1)b_1\right]\right\}\delta(\vec{r}-\vec{r}_k) \tag{3.17}$$

$$(-\nabla^2+\kappa_2^2)\,\phi_2(\vec{r}) = \frac{p}{D_2}\Sigma_R\,\phi_1(\vec{r})$$
$$+\frac{1}{D_2}\sum_{k=1}^{N}\left\{\phi_1(\vec{r}_k)\left[p a_1-\bar{b}_2\right]+\phi_2(\vec{r}_k)\left[p\,\bar{b}_1-a_2\right]\right\}\delta(\vec{r}-\vec{r}_k) \tag{3.18}$$

These can be written:

$$(-\nabla^2+\kappa_1^2)\,\phi_1(\vec{r}) = \sum_{k=1}^{N}(\phi_{1k})\,\delta(\vec{r}-\vec{r}_k) \tag{3.19}$$

$$(-\nabla^2+\kappa_2^2)\,\phi_2(\vec{r}) = \frac{p}{D_2}\Sigma_R\,\phi_1(\vec{r})+\sum_{k=1}^{N}(\phi_{2k})\,\delta(\vec{r}-\vec{r}_k) \tag{3.20}$$

where:

$$(\phi_1 k) = \frac{1}{D_1}\left\{\phi_1(r_k)\left[\eta_2 b_2 + (\eta_1 - 1)a_1\right] + \phi_2(r_k)\left[\eta_2 a_2 + (\eta_1 - 1)b_1\right]\right\} \tag{3.21}$$

$$(\phi_2 k) = \frac{1}{D_2}\left\{\phi_1(r_k)\left[p a_1 - b_2\right] + \phi_2(r_k)\left[p b_1 - a_2\right]\right\} \tag{3.22}$$

2) <u>Solution of the Balance Equations. Criticality Condition</u>

For an infinite moderator a solution of equation 3.19 is:

$$\phi_1(\vec{r}) = \sum_{k=1}^{N} A_k K_0(\kappa_1 |\vec{r} - \vec{r}_k|) \tag{3.23}$$

To compute A_k, we again neglect in the derivative of the flux at a rod k, the derivative of the flux due to the other rods when compared with the derivative of the flux due to the rod k. We compute the current at the boundary of the k element in the same manner as in section II. The result is:

$$A_k = \frac{(\phi_1 k)}{2\pi b \kappa_1 K_1(\kappa_1 b)} = (\phi_1 k) F \tag{3.24}$$

where:

$$F = \frac{1}{2\pi b \kappa_1 K_1(\kappa_1 b)} \tag{3.25}$$

Then

$$\phi_1(\vec{r}) = F \sum_{k=1}^{N} (\phi_{1k})\, K_0(\kappa_1 |\vec{r} - \vec{r}_k|) \tag{3.26}$$

By substitution into equation 3.20 we get:

$$\begin{aligned}(-\nabla^2 + \kappa_2^2)\, \phi_2(\vec{r}) &= \frac{r}{D_2} \Sigma_R F \sum_{k=1}^{N} (\phi_{1k})\, K_0(\kappa_1 |\vec{r} - \vec{r}_k|) \\ &\quad + \sum_{k=1}^{N} (\phi_{2k})\, \delta(\vec{r} - \vec{r}_k)\end{aligned} \tag{3.27}$$

•

As in section II we use a Fourier transform procedure to solve this equation. The two-dimensional Fourier transform of equation 3.27 defined as in 2.16 is:

$$\begin{aligned}S^2 \psi_2(\vec{s}) + \kappa_2^2 \psi_2(\vec{s}) &= \frac{r}{D_2} \Sigma_R F \sum_{k=1}^{N} \frac{(\phi_{1k})\, e^{i \vec{s} \cdot \vec{r}_k}}{\kappa_1^2 + S^2} + \\ &\quad + \frac{1}{2\pi} \sum_{k=1}^{N} (\phi_{2k})\, e^{i \vec{s} \cdot \vec{r}_k}\end{aligned} \tag{3.28}$$

and:

$$\psi_2(\vec{s}) = \frac{1}{\kappa_2^2+s^2}\left\{\frac{p}{D_2}\,\Sigma_R F \sum_{k=1}^{N} \frac{(\phi_{1k})\, e^{i\vec{s}\cdot\vec{r}_k}}{\kappa_1^2+s^2} + \frac{1}{2\pi}\sum_{k=1}^{N} (\phi_{2k})\, e^{i\vec{s}\cdot\vec{r}_k}\right\} \tag{3.29}$$

Taking the inverse Fourier transform the same way we did in section II:

$$\begin{aligned}\phi_2(\vec{r}) = {} & \frac{p}{D_2}\,\frac{\Sigma_R F}{\kappa_2^2-\kappa_1^2}\sum_{k=1}^{N} (\phi_{1k})\Big[K_0(\kappa_1|\vec{r}-\vec{r}_k|) - K_0(\kappa_2|\vec{r}-\vec{r}_k|)\Big] \\ & + \frac{1}{2\pi}\sum_{k=1}^{N} (\phi_{2k})\, K_0(\kappa_2|\vec{r}-\vec{r}_k|)\end{aligned} \tag{3.30}$$

To get the criticality condition, one computes from equations 3.26 and 3.30 the thermal and fast fluxes at the surface of each rod:

$$\begin{aligned}\phi_1(\vec{r}_m) = {} & \frac{F}{D_1}\sum_{k=1}^{N}\Big\{\phi_1(\vec{r}_k)\big[\eta_2 b_2 + (\eta_1-1)a_1\big] + \phi_2(\vec{r}_k)\big[\eta_2 a_2 + (\eta_1-1)b_1\big]\Big\}\ \times \\ & \times K_0(\kappa_1|\vec{r}_m-\vec{r}_k|)\end{aligned} \tag{3.31}$$

$$\begin{aligned}\phi_2(\vec{r}_m) = {} & \frac{p}{D_2}\,\frac{\Sigma_R F}{\kappa_2^2-\kappa_1^2}\,\frac{1}{D_1}\sum_{k=1}^{N}\Big\{\phi_1(\vec{r}_k)\big[\eta_2 b_2 + (\eta_1-1)a_1\big] + \phi_2(\vec{r}_k)\big[\eta_2 a_2 + (\eta_1-1)b_1\big]\Big\}\ \times \\ & \times \Big\{K_0(\kappa_1|\vec{r}_m-\vec{r}_k|) - K_0(\kappa_2|\vec{r}_m-\vec{r}_k|)\Big\}\ + \\ & + \frac{1}{2\pi}\,\frac{1}{D_2}\sum_{k=1}^{N}\Big\{\phi_1(\vec{r}_k)\big[p a_1 - b_2\big] + \phi_2(\vec{r}_k)\big[p b_1 - a_2\big]\Big\} K_0(\kappa_2|\vec{r}_m-\vec{r}_k|)\end{aligned} \tag{3.32}$$

where N is the number of rods.

- P is defined by 3.25

- $a_1 b_1 a_2 b_2$ are defined by 3.14

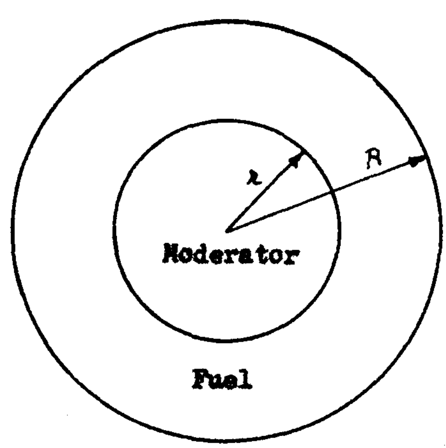
Hence we end up with a system of $2N$ linear homogeneous equations whose $2N$ unknowns are the $\phi_1(\lambda_k)$ and $\phi_2(\lambda_k)$. Non-trivial solutions exist only if $\Delta = 0$, the determinant of the system is zero. The criticality condition is that the $2N$ order determinant Δ must vanish.

Practically, we can take into account symmetries so that several fuel elements have the same surface flux $\phi(\lambda_k)$ and the number of unknowns can be considerably reduced. Most of the time the order of the determinant is smaller than $2N$.

IV. CALCULATION OF THE FOUR COEFFICIENTS

The four coefficients $\alpha_1 \beta_1 \alpha_2 \beta_2$ introduced in section III depend on the nuclear properties and geometrical configuration of the fuel element. Each fuel element if different from the others can have a different set of these coefficients. We consider here a simple model of fuel element and give an approximate method of obtaining these four coefficients.

The fuel element is composed of two concentric cylindrical regions. The inner region contains moderating material $(\Sigma_a \ll \Sigma_s)$ and the outher cylinder contains fuel $(\Sigma_a \sim \Sigma_s)$. The cross section of the fuel element is shown below.



1) Transmission Probabilities

We calculate first a few probabilities which will be useful in obtaining these coefficient. Let us define:

P_1 : probability that a neutron coming from the inner moderator goes through the fuel shell without making any collision.

P_2 : probability that a neutron escapes from the fuel element after a scattering collision in the fuel.

P_3 : probability that a neutron enters the inner moderator after a scattering collision in the fuel.

π_1 : probability that a neutron coming from outside goes through the fuel shell only and escapes from the fuel element without a collision.

π_2 : probability that a neutron coming from outside reaches the inner moderator without a collision in the fuel.

π_3 : probability that a fast neutron entering the inner moderator does not thermalize inside the moderator.

π_4 : probability that a thermal neutron escapes the inner moderator.

These probabilities (except π_3 and π_4) must be defined for both fast and thermal neutrons. A general expression will be derived for each of them in which the appropriate cross sections (fast or thermal) must be used.

In computing these probabilities we make the following approximations:

- the scattering is isotropic in the laboratory system.
- the angular distribution of neutrons impinging on the outside surface of the fuel element is isotropic.
- the collision density $\Sigma\varphi$ in each region of the fuel element is space independent for both thermal and fast neutrons.

a) The probability P_1 : probability that a neutron coming from the inner moderator goes through the fuel shell without making any collision.

We will assume as a first approximation that the angular distribution of the neutrons going into the moderator is uniform. This is reasonable if diffusion theory applies because the net current is zero at the boundary of the moderator if there are no absorptions inside.

It is shown in Appendix B that, subject to certain conditions, the angular distribution of the neutrons which leave the inner moderator is isotropic provided that the angular distribution of the incoming neutrons is also isotropic.

The probability that a neutron coming out of dA (fig. 4.1) through a solid angle $d\Omega$ about θ and φ goes through the fuel shell without colliding is $e^{-\Sigma_F S_0}$ where Σ_F is the total cross section of the fuel. According to Appendix B:

$$J_1^+ \, d\Omega = \frac{1}{\pi} \cos^2\theta \cos\varphi \, d\varphi \, d\theta$$

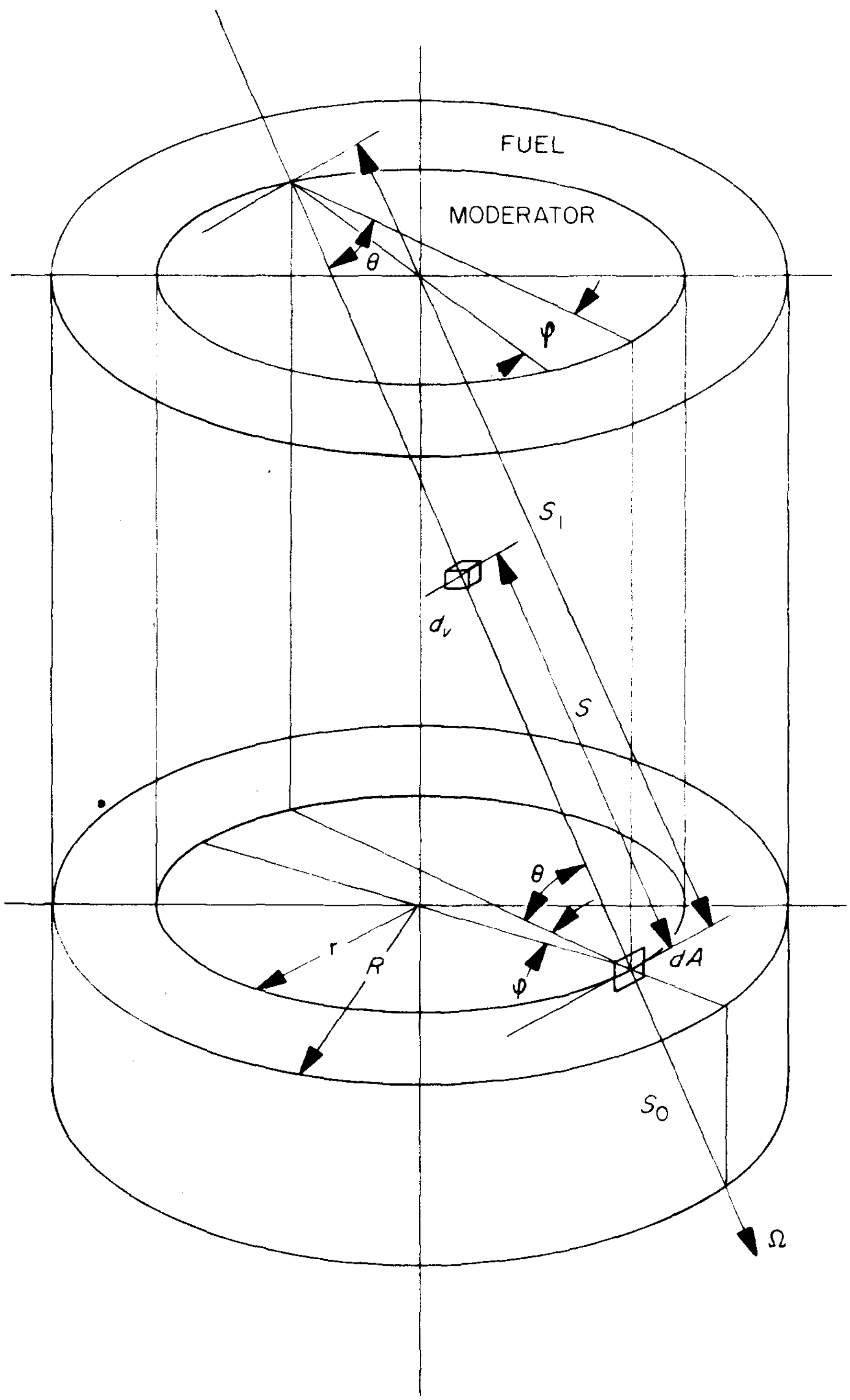


Fig. 4.1

is the probability that a neutron going through dA is in solid angle $d\Omega$ about θ and φ. The product $e^{-\Sigma_F S_0} f_1^{+} d\Omega$ is in the probability that a neutron going through dA is in solid angle $d\Omega$ and goes through the fuel shell without a collision. P_1 is the sum of these elementary probabilities over the solid angle:

$$P_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \, e^{-\Sigma_F S_0} \, d\varphi \, d\theta$$

where $S_0 = \frac{R}{\cos\theta}\left(\sqrt{1-\frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)$ according to (fig. 4.1)

$$P_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \, e^{-\Sigma_F \frac{R}{\cos\theta}\left(\sqrt{1-\frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)} d\varphi \, d\theta \tag{4.1}$$

$$P_1 = \frac{4}{\pi} Z\left(\frac{r}{R}, R, \Sigma_F\right) \tag{4.2}$$

where, writing $x = \frac{r}{R}$ and $y = R$

$$Z(x, y, \Sigma) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \, e^{-\Sigma \frac{R}{\cos\theta}\left(\sqrt{1-x^2\sin^2\varphi} - x\cos\varphi\right)} d\varphi \, d\theta \tag{4.3}$$

b) The probability P_2 : Probability that a neutron escapes from the fuel element after a scattering collision in the fuel. We assume scattering is isotropic in the fuel, and suppose the flux can be considered as a constant inside the fuel.

Within these restrictions, each volume element dv is considered as a unit source. The probability that a neutron coming from dv reaches an area dA on the outside surface of the fuel element is the product of $e^{-\Sigma_F S}$ and of the fraction of solid angle through which dA is seen from dv

$$\frac{dA \cos\theta \cos\varphi}{4\pi S^2}$$

where S is the distance between dA and dv.

To get P_2 we integrate this product over the volume and divide by the source strength which is v :

$$P_2 = \frac{1}{v} \iiiint\limits_{A\,S\,\theta\,\varphi} \frac{dA \cos\theta \cos\varphi}{4\pi S^2} e^{-\Sigma_F S} S^2 dS \cos\theta \, d\varphi \, d\theta \qquad (4.4)$$

we note that there are two kinds of limits of integration, depending on whether φ is larger or smaller than α (fig. 4.2) where $\alpha = \sin^{-1}\frac{r}{R}$.

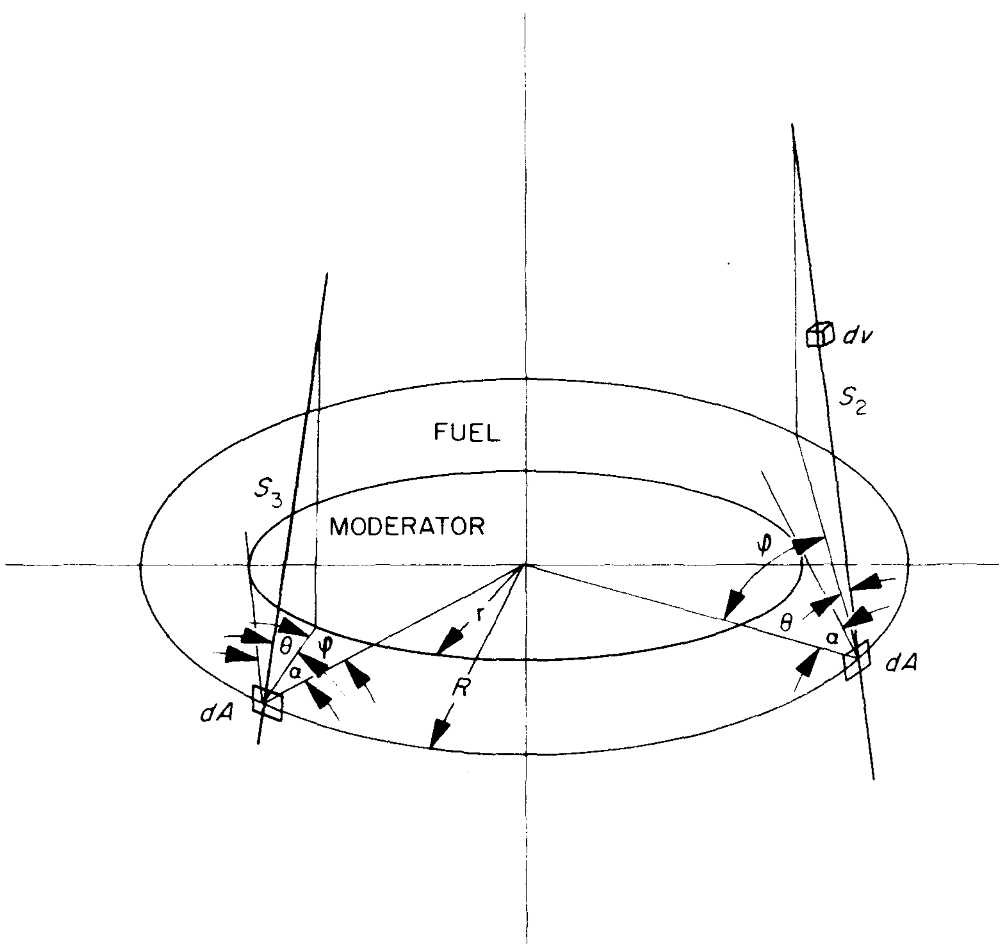

dv
S_2
FUEL
S_3
MODERATOR
φ
θ
α
r
R
dA
θ
φ
α
dA

Fig. 4.2

These two limits are:

$$S_2 = 2R\,\frac{\cos\varphi}{\cos\theta}$$

$$S_3 = \frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2} - \sin^2\varphi}\right)$$

As all the dA's are equivalent, equation 4.4 now becomes:

$$P_2 = \frac{4A}{4\pi v}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}}\int_0^{S_2}\cos^2\theta\cos\varphi\, e^{-\Sigma_F s}\, ds\, d\varphi\, d\theta + \int_0^{\frac{\pi}{2}}\int_0^{\alpha}\int_0^{S_3}\cos^2\theta\cos\varphi\, e^{-\Sigma_F s}\, ds\, d\varphi\, d\theta\right\}$$

$$A = 2\pi R h$$

$$v = \pi(R^2 - r^2)h$$

$$P_2 = \frac{2R}{\Sigma_F\pi(R^2-r^2)}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}}\cos^2\theta\cos\varphi\left(1 - e^{-\Sigma_F 2R\frac{\cos\varphi}{\cos\theta}}\right)d\varphi\, d\theta + \right. \tag{4.5}$$

$$\left. + \int_0^{\frac{\pi}{2}}\int_0^{\alpha}\cos^2\theta\cos\varphi\left(1 - e^{-\Sigma_F\frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2} - \sin^2\varphi}\right)}\right)d\varphi\, d\theta\right\}$$

$$P_2 = \frac{2R}{\Sigma_F\pi(R^2-r^2)}\left\{\frac{\pi}{4} - I\left(\frac{r}{R}, R, \Sigma_F\right) - W\left(\frac{r}{R}, R, \Sigma_F\right)\right\} \tag{4.6}$$

where:

$$I(x,y,\Sigma) = \int_{0}^{\frac{\pi}{2}}\int_{\varphi=\alpha}^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \, e^{-\Sigma 2y \frac{\cos\varphi}{\cos\theta}} \, d\varphi \, d\theta \tag{4.7}$$

$$W(x,y,\Sigma) = \int_{0}^{\frac{\pi}{2}}\int_{\varphi=0}^{\alpha} \cos^2\theta \cos\varphi \, e^{-\Sigma \frac{y}{\cos\theta}\left(\cos\varphi - \sqrt{x^2 - \sin^2\varphi}\right)} \, d\varphi \, d\theta \tag{4.8}$$

c) The probability P_3 : probability that a neutron enters the inner moderator after a scattering collision in the fuel.

We consider now a small volume element dv in the fuel and a small area dA on the surface of the inner moderator and apply the same reasoning as for P_2 . The limit of integration (fig. 4.3) is now:

$$s_4 = \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{\lambda^2}{R^2}\sin^2\varphi} - \frac{\lambda}{R}\cos\varphi\right).$$

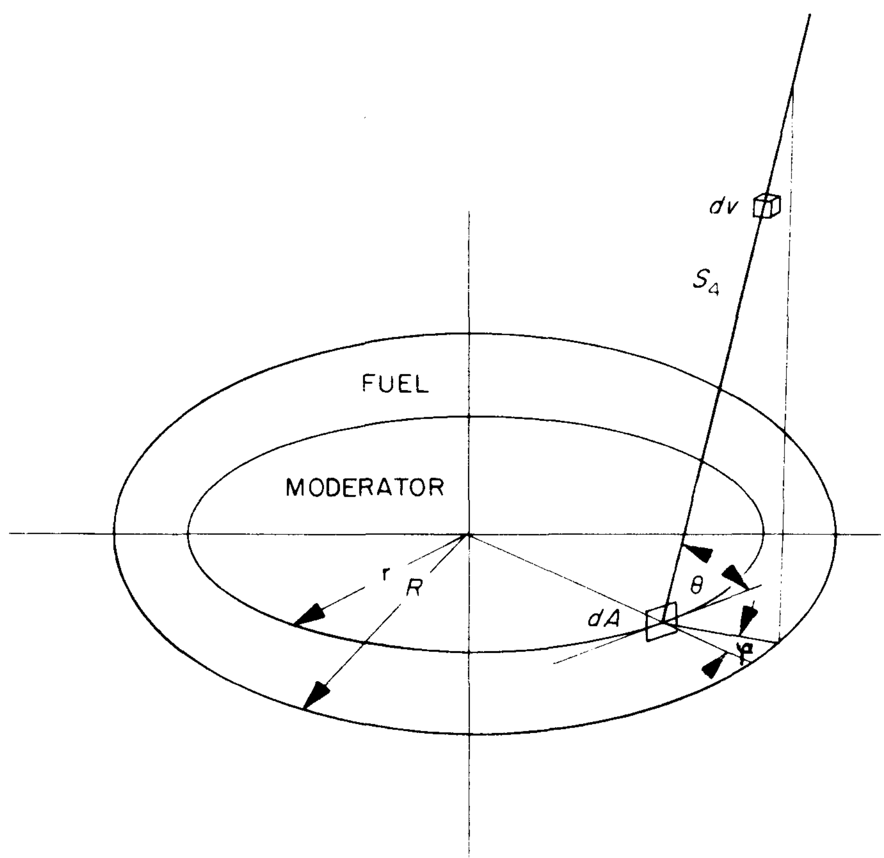


Fig. 4.3

and

$$P_3 = \frac{1}{\sigma} \iiiint\limits_{A\,S\,\theta\,\varphi} \frac{dA \cos\theta \cos\varphi}{4\pi S^2}\, e^{-\Sigma_F S}\, s^2\, ds \cos\theta \, d\varphi \, d\theta \tag{4.9}$$

$$P_3 = \frac{2r}{\Sigma_F \pi (R^2 - r^2)} \times \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \left(1 - e^{-\Sigma_F \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)}\right) d\varphi\, d\theta \tag{4.10}$$

$$P_3 = \frac{2r}{\Sigma_F \pi (R^2 - r^2)} \left\{ \frac{\pi}{4} - Z\left(\frac{r}{R}, R, \Sigma_F\right) \right\} \tag{4.11}$$

where

$Z(x, y, \Sigma)$ is defined by equation 4.3.

One can confirm that the normalization is correct by calculating $P_2 + P_3$ which is equal to 1 without further collisions:

$$\frac{2R}{\pi(R^2 - r^2)} \left\{ \int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} 2R \frac{\cos\varphi}{\cos\theta} \cos^2\theta \cos\varphi \, d\varphi\, d\theta + \int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \cos\varphi \frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2} - \sin^2\varphi}\right) d\varphi\, d\theta \right\}$$

$$+ \frac{2r}{\pi(R^2 - r^2)} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right) d\varphi\, d\theta = 1$$

d) The probability π_1 : probability that a neutron coming from outside goes through the fuel shell only and escapes from the fuel element without a collision.

Let us assume an isotropic angular distribution for the velocities of the neutrons coming into the fuel element

$$j^{-}(\theta,\varphi)\, d\varphi\, d\theta = \cos^2\theta \cos\varphi \, d\varphi\, d\theta \tag{4.12}$$

The probability that a neutron traveling in solid angle $d\Omega$ about θ and φ escapes is $e^{-\Sigma_F S_2}$ provided $\varphi > \alpha$ (see fig. 4.2). Hence:

$$\pi_1 = \frac{\iint j^{-}(\theta,\varphi)\, e^{-\Sigma_F S_2}\, d\varphi\, d\theta}{\iint j^{-}(\theta,\varphi)\, d\varphi\, d\theta}$$

$$\pi_1 = \frac{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta \cos\varphi\, e^{-\Sigma_F S_2}\, d\varphi\, d\theta}{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi\, d\varphi\, d\theta}$$

as

$$S_2 = 2R\,\frac{\cos\varphi}{\cos\theta}$$

$$\therefore \quad \pi_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \, e^{-\Sigma_F 2R \frac{\cos\varphi}{\cos\theta}} \, d\varphi \, d\theta \tag{4.13}$$

$$\pi_1 = \frac{4}{\pi} \, I\left(\frac{\lambda}{R}, R, \Sigma_F\right) \tag{4.14}$$

where $I(X, y, \Sigma)$ is defined by (4.7)

e) The probability π_2 : probability that a neutron coming from outside reaches the inner moderator without a collision in the fuel.

Using the same procedure as for π_1 we get

$$\pi_2 = \frac{\iint j^-(\theta,\varphi) \, e^{-\Sigma_F S_3} \, d\varphi \, d\theta}{\iint j^-(\theta,\varphi) \, d\varphi \, d\theta}$$

where according to (fig. 4.2)

$$S_3 = \frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{\lambda^2}{R^2} - \sin^2\varphi}\right)$$

$$\therefore \quad \pi_2 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \cos\varphi \, e^{-\Sigma_F \frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{\lambda^2}{R^2} - \sin^2\varphi}\right)} \, d\varphi \, d\theta \tag{4.15}$$

$$\pi_2 = \frac{4}{\pi} W\left(\frac{\lambda}{R}, R, \Sigma_F\right) \tag{4.16}$$

where $W(x, y, \Sigma)$ is defined by (4.8).

f) The probability π_3 : probability that a fast neutron entering the inner moderator does not thermalize inside the moderator.

Let us distinguish two cases:

1) The neutrons born inside the slug from fission; these neutrons have a well known average lethargy and one can have an idea of the average number of scattering collisions which will make these neutrons thermal.

Knowing the probability that a neutron escapes at each scattering collision, one can determine the probability that a fast neutron escapes without becoming thermal.

We use the notations and results of Appendix A:

λ : probability that a neutron going through the moderator does not collide

ς : probability that a neutron after a scattering collision in the moderator escapes from the moderator

In this case we can assume $\frac{\Sigma_s}{\Sigma_t} = 1$ because we are considering only fast neutrons.

Calling N_1 the average number of collisions which make a fission neutron thermal we get:

$$\pi_3 = \lambda + (1-\lambda)\varsigma + (1-\lambda)(1-\varsigma)\varsigma + \ldots \qquad \ldots + (1-\lambda)(1-\varsigma)^{N_1-2}\varsigma$$

$$\pi_3 = \lambda + (1-\lambda)\varsigma\,\frac{1-(1-\varsigma)^{N_1-1}}{1-(1-\varsigma)} = \lambda + (1-\lambda)\left[1-(1-\varsigma)^{N_1-1}\right] \qquad (4.17)$$

ii) The fast neutrons which enter the slug from outside. These neutrons belong to the fast group, but actually their lethargy is not well defined and is spread between thermal and minimum lethargies.

In the context of the two group model, one can say that at each scattering collision the average probability that a fast neutron becomes thermal is $\frac{\Sigma_R}{\Sigma_s}$, neglecting fast absorptions in the moderator. Therefore, using the results of Appendix A:

$$\pi_3' = \lambda + (1-\lambda)\left(1-\frac{\Sigma_R}{\Sigma_s}\right)\frac{\varsigma}{1-(1-\varsigma)\left(1-\frac{\Sigma_R}{\Sigma_s}\right)} \qquad (4.18)$$

In these expressions: $\lambda = \frac{4}{\pi}\, I(0, r, \Sigma_m)$

$$\varsigma = \frac{2\ell}{3}\left\{-2 + \left(2\ell + \frac{1}{\ell}\right) I_1(\ell) K_1(\ell) + I_0(\ell) K_1(\ell) - I_1(\ell) K_0(\ell) + 2\ell\, I_0(\ell) K_0(\ell)\right\}$$

where $\ell = \Sigma_m r$

g) The probability π_4 : probability that a thermal neutron escapes the inner moderator.

This probability accounts for the thermal absorption which takes place in the moderator. Most of the time it is very close to 1.

π_4 is equivalent to the transmission coefficient of the moderator slug. Hence according to the results of Appendix A:

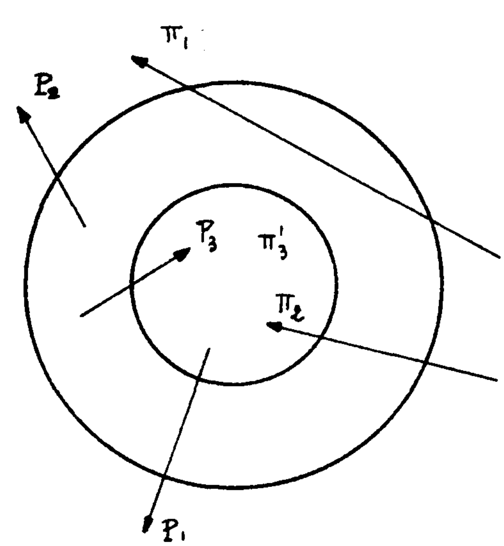
$$\pi_4 = \lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}\frac{\xi}{1-(1-\xi)\frac{\Sigma_s}{\Sigma_t}} \tag{4.19}$$

where $\frac{\Sigma_s}{\Sigma_t}$ of the moderator is used.

2) <u>Expression of the Four Coefficients in Terms of Transmission Probabilities</u>

a) The coefficient α_1 : probability that a fast neutron entering the fuel element escapes as a fast neutron.

Fig. 4.5



Let $\omega = \frac{\Sigma_s}{\Sigma_t}$ in the fuel

S fast neutrons enter the fuel element per unit time and per unit length. X of these neutrons after entering the inner moderator leave it again as fast neutrons per unit length and time. Hence:

$[S(1-\pi_1-\pi_2)+X(1-P_1)]\,\omega$ neutrons make at least one scattering collision in the fuel.

$[S(1-\pi_1-\pi_2)+X(1-P_1)]\,\omega(1-P_2-P_3)\,\omega$ make two scattering collisions in the fuel.

The total number of scattering collisions these neutrons make in the fuel is: (by summation)

$$\frac{[S(1-\pi_1-\pi_2)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}$$

Thus:

$$S\pi_1 + \frac{[S(1-\pi_1-\pi_2)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}\,P_2 + X P_1 \qquad (4.20)$$

neutrons escape

$$S\pi_2 + \frac{[S(1-\pi_1-\pi_2)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}\,P_3 \qquad (4.21)$$

enter the inner moderator

Hence:

$$x = \pi_3' \left\{ S\pi_2 + \frac{S(1-\pi_1-\pi_2) + x(1-P_1)}{1-(1-P_2-P_3)\omega} \; \omega P_3 \right\}$$

and solving for x ,

$$x = S\pi_3' \; \frac{\pi_2 - (1-P_2-P_3)\omega\pi_2 + (1-\pi_1-\pi_2)\omega P_3}{1-(1-P_2-P_3)\omega - (1-P_1)\omega P_3 \pi_3'} \tag{4.22}$$

Substituting equation 4.22 into equation 4.20 we get the number of neutrons escaping:

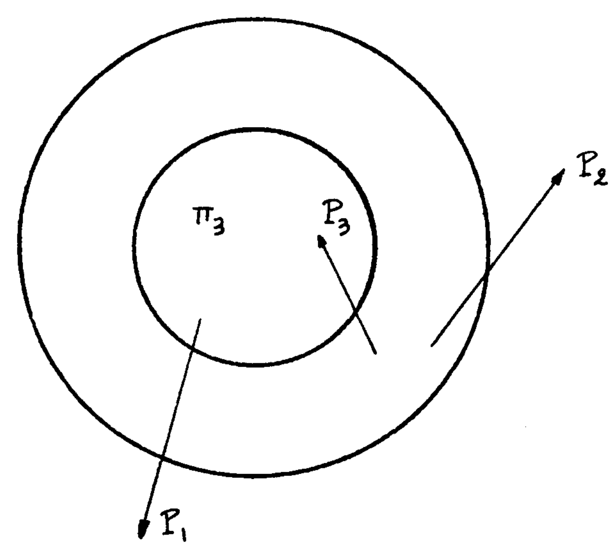
$$S\alpha_1 = S\pi_1 + \frac{S(1-\pi_1-\pi_2)\omega P_2}{1-(1-P_2-P_3)\omega}$$

$$+ S\pi_3' \; \frac{\pi_2 - (1-P_2-P_3)\omega\pi_2 + (1-\pi_1-\pi_2)\omega P_3}{1-(1-P_2-P_3)\omega - (1-P_1)\omega P_3 \pi_3'} \left[\frac{(1-P_1)\omega P_2}{1-(1-P_2-P_3)\omega} + P_1 \right]$$

$$\alpha_1 = \pi_1 + \frac{(1-\pi_1-\pi_2)\omega P_2}{1-(1-P_2-P_3)\omega} + \varepsilon_1 \; \frac{(P_2 + P_1P_2 - P_1)\omega + P_1}{1-(1-P_2-P_3)\omega} \tag{4.23}$$

where $$\varepsilon_1 = \pi_3' \; \frac{\pi_2 - (1-P_2-P_3)\omega\pi_2 + (1-\pi_1-\pi_2)\omega P_3}{1-(1-P_2-P_3)\omega - (1-P_1)\omega P_3 \pi_3'} . \tag{4.24}$$

b) The coefficient β_1 : probability that a fast neutron born from fission inside the fuel element escapes from the fuel element as fast neutrons.



S neutrons are born from fission in the fuel element per unit length and time. X of these after entering the inner moderator leave it again as fast neutrons.

$\left[S(1-P_2-P_3)+X(1-P_1)\right]\omega$ make at least one scattering collision in the fuel.

The total number of scattering collisions these neutrons make in the fuel is:

$$\frac{[S(1-P_2-P_3)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}$$

Thus

$$SP_2 + \frac{[S(1-P_2-P_3)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}\,P_2 + XP_1 \qquad (4.25)$$

of these escape

$$SP_3 + \frac{[S(1-P_2-P_3)+X(1-P_1)]\,\omega}{1-(1-P_2-P_3)\,\omega}\,P_3 \qquad (4.26)$$

enter the inner moderator

Hence:

$$X = \pi_3 \left\{ SP_3 + \frac{[S(1-P_2-P_3)+X(1-P_1)]\,\omega\,P_3}{1-(1-P_2-P_3)\,\omega} \right\}$$

and solving for X,

$$X = \frac{S\,P_3\,\pi_3}{1-(1-P_2-P_3)\,\omega-(1-P_1)\,\omega\,P_3\,\pi_3} \qquad (4.27)$$

Substituting equation 4.27 into equation 4.25

$$\beta_1 = P_2 + \frac{(1-P_2-P_3)\omega P_2}{1-(1-P_2-P_3)\omega} + \frac{P_3 \pi_3}{1-(1-P_2-P_3)\omega-(1-P_1)\omega P_3 \pi_3} \times \frac{P_1+\omega(P_2+P_1P_3-P_1)}{1-(1-P_2-P_3)\omega} \qquad (4.28)$$

c) The coefficient α_2 : probability that a thermal neutron entering the fuel element escapes from the fuel element.

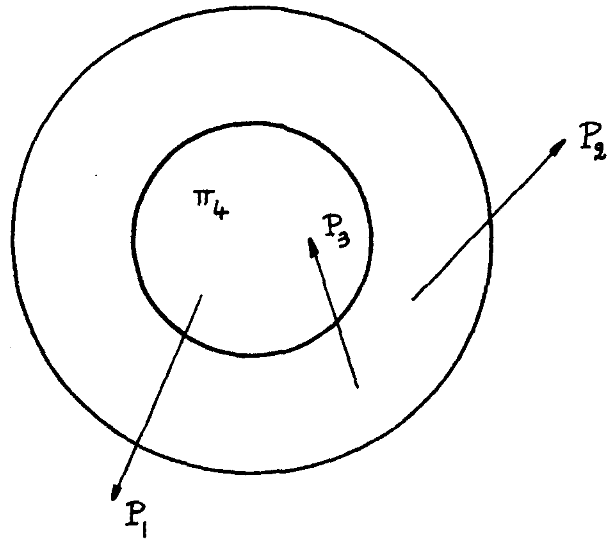
One can use here the same procedure as for α_1 where π_4 replaces π_3' :

$$\alpha_2 = \pi_1 + \frac{(1-\pi_1-\pi_2)\omega P_2}{1-(1-P_2-P_3)\omega} + \varepsilon_2 \frac{(P_2+P_1P_3-P_1)\omega + P_1}{1-(1-P_2-P_3)\omega} \qquad (4.29)$$

where:

$$\varepsilon_2 = \pi_4 \frac{\pi_2-(1-P_2-P_3)\omega\pi_2+(1-\pi_1-\pi_2)\omega P_3}{1-(1-P_2-P_3)\omega-(1-P_1)\omega P_3 \pi_4} \qquad (4.30)$$

d) The coefficient β_2 : probability that a thermal neutron thermalised in the fuel element escapes from the fuel element.



S neutrons are thermalized in the slug per unit length and time. x of these after being in the fuel return into the inner moderator and then leave again. $(S+x)(1-P_1)\,\omega$ neutrons make at least one scattering collision in the fuel.

The total number of scattering collisions these neutrons make in the fuel is:

$$\frac{(S+x)(1-P_1)\,\omega}{1-(1-P_2-P_3)\,\omega}$$

Thus:

$$(S+x)P_1 + \frac{(S+x)(1-P_1)\,\omega}{1-(1-P_2-P_3)\,\omega}\,P_2 \qquad (4.31)$$

neutrons escape from the slug

$$\frac{(S+X)(1-P_1)\omega}{1-(1-P_2-P_3)\omega} \quad P_3 \tag{4.32}$$

re-enter the inner moderator

Hence:

$$X = \pi_4 \frac{(S+X)(1-P_1)\omega P_3}{1-(1-P_2-P_3)\omega}$$

solving for X :

$$X = S \frac{(1-P_1)\omega P_3 \pi_4}{1-(1-P_2-P_3)\omega-(1-P_1)\omega P_3 \pi_4} \tag{4.33}$$

Substituting equation 4.33 into equation 4.31

$$\beta_2 = \left\{ 1 + \frac{(1-P_1)\omega P_3 \pi_4}{1-(1-P_2-P_3)\omega-(1-P_1)\omega P_3 \pi_4} \right\} \left\{ P_1 + \frac{(1-P_1)\omega P_2}{1-(1-P_2-P_3)\omega} \right\}$$

$$\beta_2 = \frac{P_1 + \omega(P_2 + P_1 P_3 - P_1)}{1-(1-P_2-P_3)\omega-(1-P_1)\omega P_3 \pi_4} \tag{4.34}$$

V. NUMERICAL EXAMPLE

We calculate here the critical mass of a fission-electric cell reactor. This reactor contains fuel elements consisting of a beryllium cylinder of 4.0 cm diameter covered with a thin layer of fully enriched uranium (93.5% U^{235}). Between this layer and the outside of the element there is a vacuum gap of 0.5 cm. Fig. 5.1 shows an idealized cell configuration.

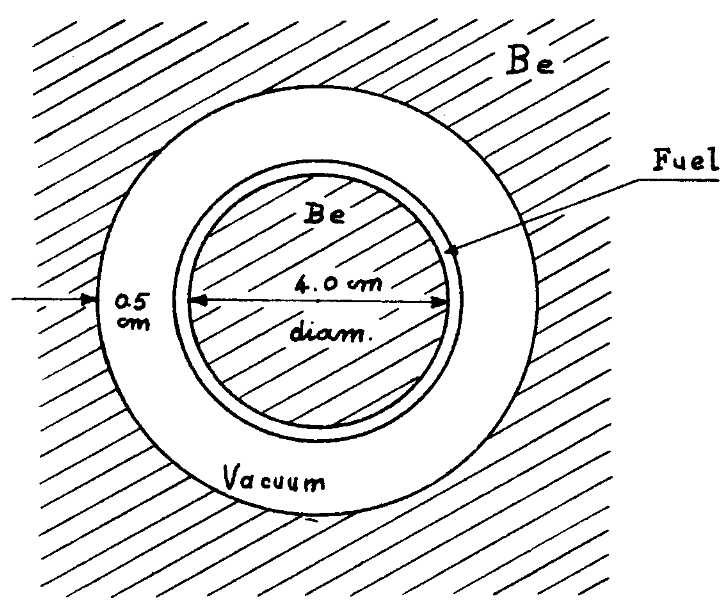


Fig. 5.1

These elements are 100 cm long and 253 of them are contained in an hexagonal array inside a 50 cm radius core. The complete fuel element lattice is shown in (fig. 5.4). The whole core is surrounded by an infinite reflector made of the same material as the core moderator.

Being guided by Appendix C results, we select several fuel loadings and find out at what loading the determinant of equations 3.31 and 3.32 is zero. We give the results here for the following values of the total mass of uranium: 0.20, 0.40, 0.60, 0.70, 0.75, 0.80, 1.00, and 1.30 kg.

First, we notice that our fuel element is different from the one considered in section IV. This will bring a small correction to the transmission probabilities. Then we construct our determinant taking into account the symmetry simplifications. Most of this work is done on a computer and a brief outline is presented below.

1) Theoretical Considerations

The presence of a vacuum gap brings a small correction to the results of section IV.1. We assume here that the angular distribution of the velocity of the neutrons at the outer boundary of the vacuum gap is isotropic. We consider the vacuum gap as a part of the fuel element. With this assumption, the angular distribution of the velocity of the neutrons at the outer boundary of the fuel layer is not isotropic, and we derive now an expression for this distribution.

At the outer boundary of the vacuum gap the distribution is:

$$j(\theta',\varphi')\, d\theta'\, d\varphi' \propto \cos^2\theta' \cos\varphi'\, d\theta'\, d\varphi'$$

where the angles are defined according to (fig. 5.2) and (fig. 5.3). In order to use the results derived in section IV.1 we must express this distribution in terms of the angles θ and φ.

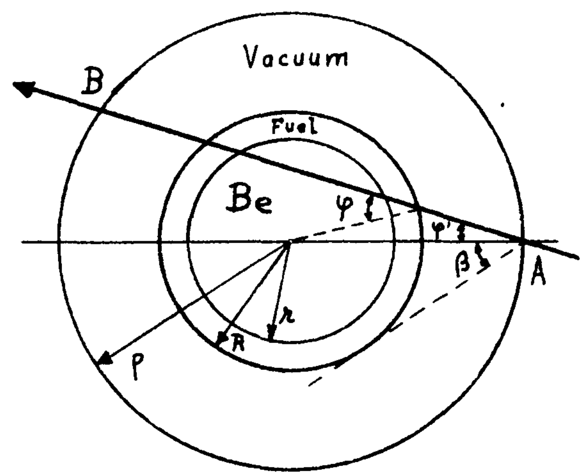


Fig. 5.2

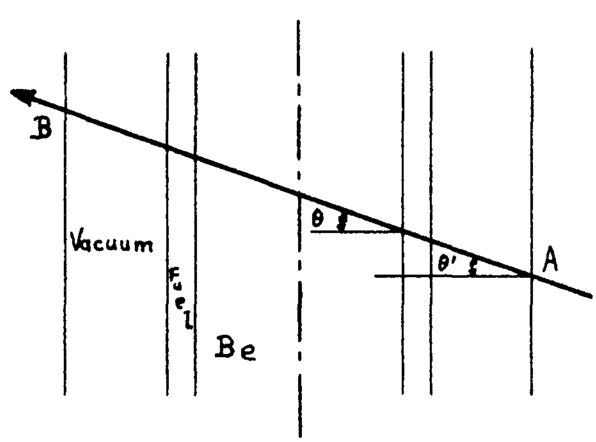


Fig. 5.3

We see immediately from (fig. 5.2) and (fig. 5.3) that:

$$\theta = \theta'$$

$$\sin\varphi = \frac{\rho}{R}\sin\varphi'$$

Hence:

$$j(\theta,\varphi)\,d\varphi\,d\theta = \cos^2\theta\sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\;d\varphi\,d\theta$$

which when normalised becomes:

$$j(\theta,\varphi) = \frac{\cos^2\theta\sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}}{4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos^2\theta\sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\;d\varphi\,d\theta}$$

Integrating the denominator we get:

$$j(\theta,\varphi)\,d\varphi\,d\theta = \frac{1}{E}\,\frac{1}{\pi}\,\cos^2\theta\sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\;d\varphi\,d\theta \qquad (5.1)$$

where E is the complete elliptic integral of the second kind:

$$E = \int_0^{\frac{\pi}{2}}\sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\;d\varphi \qquad (5.2)$$

If we retain the assumption made in section IV.1, only the probabilities involving neutrons coming from outside the fuel element are affected by the change in the distribution due to the vacuum gap. Hence, P_1, P_2 and P_3 remain unchanged:

$$P_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \; e^{-\Sigma_F \frac{R}{\cos\theta}\left(\sqrt{1-\frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)} d\varphi \, d\theta \qquad (5.3)$$

$$P_2 = \frac{2R}{\pi\Sigma_F(R^2-r^2)}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta\cos\varphi\left[1 - e^{-\Sigma_F 2R\frac{\cos\varphi}{\cos\theta}}\right] d\varphi\, d\theta \; + \right. \qquad (5.4)$$

$$\left. + \int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta\cos\varphi\left[1 - e^{-\Sigma_F\frac{R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2}-\sin^2\varphi}\right)}\right] d\varphi\, d\theta\right\}$$

$$P_3 = \frac{2r}{\pi\Sigma_F(R^2-r^2)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta\cos\varphi\left[1 - e^{-\Sigma_F\frac{R}{\cos\theta}\left(\sqrt{1-\frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)}\right] d\varphi\, d\theta \qquad (5.5)$$

However Π_1 and Π_2 now become:

$$\Pi_1 = \frac{1}{E}\,\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_{\alpha}^{\frac{\pi}{2}} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; e^{-\Sigma_F 2R\frac{\cos\varphi}{\cos\theta}}\, d\varphi\, d\theta \tag{5.6}$$

$$\Pi_2 = \frac{1}{E}\,\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; e^{-\Sigma_F \frac{R}{\cos\theta}\left(\cos\varphi-\sqrt{\frac{\lambda^2}{R^2}-\sin^2\varphi}\right)}\, d\varphi\, d\theta \tag{5.7}$$

Due to the fact that the fuel layer is very thin, P_1 is very close to unity. In order to have more accuracy in calculating $1-P_1$ we define the quantity:

$$A_0 = 1-P_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta\cos\varphi\left[1-e^{-\Sigma_F\frac{R}{\cos\theta}\left(\sqrt{1-\frac{\lambda^2}{R^2}\sin^2\varphi}-\frac{\lambda}{R}\cos\varphi\right)}\right] d\varphi\, d\theta \tag{5.8}$$

Similarily $P_2 + P_3$ is very close to unity, and the quantity $1-P_2-P_3$ is very small. So we define two quantities:

- B_0 = probability that a neutron which would have escaped the fuel element after a scattering collision in the fuel, makes another collision in the fuel.

- C_0 = probability that a neutron which would have gone into the inner moderator after a scattering collision in the fuel, makes another collision in the fuel.

We see immediately from sections IV.1.b and IV.1.C that:

$$B_o = \frac{A}{\pi v}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}}\int_0^{S_2} \cos^2\theta \cos\varphi \left(1 - e^{-\Sigma_F s}\right) ds\, d\varphi\, d\theta + \int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}}\int_0^{S_3} \cos^2\theta \cos\varphi \left(1 - e^{-\Sigma_F s}\right) ds\, d\varphi\, d\theta\right\} \quad (5.9)$$

$$C_o = \frac{A}{\pi v}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{S_4} \cos^2\theta \cos\varphi \left(1 - e^{-\Sigma_F s}\right) ds\, d\varphi\, d\theta \quad (5.10)$$

Hence:

$$B_o = \frac{2R}{\pi \Sigma_F (R^2 - r^2)}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \left[\Sigma_F 2R \frac{\cos\varphi}{\cos\theta} - 1 + e^{-\Sigma_F 2R \frac{\cos\varphi}{\cos\theta}}\right] d\varphi\, d\theta + \right.$$
$$\left. + \int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \cos\varphi \left[\frac{\Sigma_F R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2} - \sin^2\varphi}\right) - 1 + e^{-\frac{\Sigma_F R}{\cos\theta}\left(\cos\varphi - \sqrt{\frac{r^2}{R^2} - \sin^2\varphi}\right)}\right] d\varphi\, d\theta\right\} \quad (5.11)$$

$$C_o = \frac{2r}{\pi \Sigma_F (R^2 - r^2)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \left[\frac{\Sigma_F R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right) - 1 + \right.$$
$$\left. + e^{-\Sigma_F \frac{R}{\cos\theta}\left(\sqrt{1 - \frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right)}\right] d\varphi\, d\theta \quad (5.12)$$

Finally, $\pi_1 + \pi_2$ is also close to unity. For the same reason as before we define two new quantities:

$\mathcal{D}_o$ = probability that a neutron, which coming from outside would have gone through the fuel shell without entering the inner moderator, makes a prior collision in the fuel.

F_o = probability that a neutron, which coming from outside would have gone through the fuel shell and entered the inner moderator, makes a prior collision in the fuel.

We see immediately from sections IV.1.d and IV.1-e that, taking into account the change in distribution mentioned above, $\mathcal{D}_o$ and F_o are:

$$\mathcal{D}_o = \frac{1}{E}\,\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\left[1-e^{-\Sigma_F 2R\frac{\cos\varphi}{\cos\theta}}\right] d\varphi\, d\theta \tag{5.13}$$

$$F_o = \frac{1}{E}\,\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\left[1-e^{-\frac{\Sigma_F R}{\cos\theta}\left(\cos\varphi-\sqrt{\frac{\lambda^2}{R^2}-\sin^2\varphi}\right)}\right] d\varphi\, d\theta \tag{5.14}$$

Comparing these expressions (5.8) through (5.14) with expressions (5.3) through (5.7) we can deduce immediately the relationships:

$$P_1 = 1 - A_o \tag{5.15}$$

$$P_2 = \frac{2R}{\pi(R^2-\lambda^2)}\left\{\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta\cos\varphi\, 2R\frac{\cos\varphi}{\cos\theta}\, d\varphi\, d\theta + \right.$$
$$\left. + \int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta\cos\varphi\,\frac{R}{\cos\theta}\left(\cos\varphi-\sqrt{\frac{\lambda^2}{R^2}-\sin^2\varphi}\right) d\varphi\, d\theta\right\} - B_o \tag{5.16}$$

$$P_2 = \frac{2R^2}{\pi(R^2-r^2)}\left(\frac{\pi}{2} - \frac{\pi}{4}\frac{r^2}{R^2} - \frac{\alpha}{2} - \frac{\sin 2\alpha}{4}\right) - B_0 \qquad (5.17)$$

$$P_3 = \frac{2r}{\pi(R^2-r^2)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi \frac{R}{\cos\theta}\left(\sqrt{1-\frac{r^2}{R^2}\sin^2\varphi} - \frac{r}{R}\cos\varphi\right) d\varphi\, d\theta - C_0 \qquad (5.18)$$

$$P_3 = \frac{2R^2}{\pi(R^2-r^2)}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2} - \frac{r^2}{R^2}\frac{\pi}{4}\right) - C_0 \qquad (5.19)$$

$$\pi_1 = \frac{1}{E}\,\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_\alpha^{\frac{\pi}{2}} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; d\varphi\, d\theta - D_0 \qquad (5.20)$$

$$\pi_1 = \frac{G}{E} - D_0$$

$$\pi_2 = \frac{1}{E}\cdot\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\alpha} \cos^2\theta \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; d\varphi\, d\theta - F_0 \qquad (5.21)$$

$$\pi_2 = \frac{E-G}{E} - F_0$$

where

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; d\varphi \qquad (5.22)$$

$$G = \int_\alpha^{\frac{\pi}{2}} \sqrt{1-\frac{R^2}{\rho^2}\sin^2\varphi}\; d\varphi \qquad (5.23)$$

And finally:

$$1 - P_2 - P_3 = B_\circ + C_\circ \tag{5.24}$$

$$1 - \pi_1 - \pi_2 = D_\circ + F_\circ \tag{5.25}$$

2) Data

Geometrical data:

λ = 2.0 cm

R = $\lambda + \tau$

τ = thickness of the fuel layer (see Table 5.2)

ρ = 2.5 cm

ε = scale factor equal numerically to the distance between two rods in cm (see section V.4) = 2.9934

σ = ρ = 2.5 cm

Other data: All the nuclear constants are obtained from references indicated in Appendix C (section II)

$\Sigma_F^{(1)}$ = 0.91017 cm^{-1}

$\Sigma_F^{(2)}$ = 33.389 cm^{-1}

$\Sigma_a^{(2)}$ = 0.001237 cm^{-1}

$\Sigma_R = \sum_{\mu_{ch}} \Sigma_s^{(1)}$ = 0.0087658 cm^{-1}

$$\frac{\Sigma_R}{\Sigma_s^{(1)}} = 0.0102$$

$$\Sigma_s^{(2)} = 0.8662 \text{ cm}^{-1}$$

$$D_1 = \Sigma_R \tau_{tR} = 0.852 \text{ cm}$$

$$D_2 = 0.4156 \text{ cm}$$

$$B_z^2 = 0.0004567 \text{ cm}^{-2}$$

$$\kappa_1^2 = \frac{\Sigma_R}{D_1} + B_z^2 = (0.10365 \text{ cm}^{-1})^2$$

$$\kappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} + B_z^2 = (0.0586 \text{ cm}^{-1})^2$$

$\omega_1 = \frac{\sigma_s}{\sigma_t} = 0.526$ **fast values in the fuel**

$\omega_2 = \frac{\sigma_s}{\sigma_t} = 0.01434$ **thermal values in the fuel**

$\eta_1 = \dfrac{\nu \Sigma_F^{(1)} V_F}{V_F \Sigma_a^F + V_M \Sigma_R}$ **depends on fuel loading**

M kg	η_1
0.20	.006289
0.40	.01253
0.60	.01874
0.70	.02182
0.75	.02336
0.80	.02490
1.00	.03102
1.30	.04011

η_2 = 2.08

λ = $\frac{4}{\pi} I_1(\Sigma_t b)$ = 0.0713 see Appendix A, equation 22

ς = 0.2688 see Appendix A, equation 35

N_1 = average number of collisions to thermalize a fission neutron. We saw in Appendix C that 84.

$$(1-\varsigma)^{83} \ll 1$$

hence

$$\pi_3 = \lambda + (1-\lambda)\left[1 - (1-\varsigma)^{N_1 - 1}\right] \simeq 1$$ see equation 4.17

$$\pi_3' = \lambda + (1-\lambda)\left(1 - \frac{\Sigma_R}{\Sigma_s^{(1)}}\right) \frac{\varsigma}{1-(1-\varsigma)\left(1-\frac{\Sigma_R}{\Sigma_s^{(1)}}\right)}$$ = 0.9659 see equation 4.18

$$\pi_4 = \lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_t} \frac{\varsigma}{1-(1-\varsigma)\frac{\Sigma_s}{\Sigma_t}}$$ = 0.9948 see equation 4.19

$$F = \frac{1}{2\pi b\, \kappa_1 K_1(\kappa_1 b)}$$ = 14.619 see equation 3.25

p = 0.97 see Appendix C, section 3

3) Transmission Probabilities and the Four Coefficients

From calculations programmed on an IBM 7090 computer, we obtain the following numerical values for the integrals defined by equations 5.8 through 5.12. Table 5.2 gives the values for the fast group, and Table 5.3 the corresponding values for the thermal group.

M_{hg}	τ microns	A_o	B_o	C_o	D_o	F_o
0.20	0.336398	.000061177	.000131035	.000103630	.000036230	.000107305
0.40	0.672791	.000122333	.000167924	.000115178	.000072170	.000210011
0.60	1.009179	.000183438	.000224196	.000144812	.000107906	.000310018
0.70	1.177371	.000213938	.000252276	.000161450	.000125695	.000359246
0.75	1.261466	.000229183	.000265575	.000170636	.000134576	.000383683
0.80	1.345561	.000244425	.000282829	.000180428	.000143447	.000408025
1.00	1.681934	.000305412	.000343488	.000216471	.000178880	.000504569
1.30	2.186487	.000396771	.000436008	.000272753	.000231771	.000646887

Table 5.2

M_{hg}	A_o	B_o	C_o	D_o	F_o	E	G
0.20	.0022379	.00218819	.00151551	.00096529	.0039011	1.276337	.00347873
0.40	.0044621	.00411905	.00296710	.00172416	.0075763	1.276325	.00492067
0.60	.0066724	.00591796	.00436780	.00238600	.0111069	1.276313	.00602691
0.70	.0077713	.00678134	.00505059	.00269122	.0128287	1.276307	.00650947
0.75	.0083195	.00720505	.00538789	.00283861	.0136796	1.276304	.00673779
0.80	.0088670	.00762426	.00572272	.00298285	.0145249	1.276301	.00695863
1.00	.0110506	.00926055	.00703921	.00353250	.0178545	1.276289	.00778019
1.30	.0143020	.01160504	.00894695	.00428839	.0226998	1.276627	.00887030

Table 5.3

From these results, using the expressions 4.23, 4.28, 4.29, 4.34 of section IV we obtain the four coefficients α_1, β_1, α_2 and β_2:

$M k_g$	α_1	β_1	α_2	β_2
0.20	.966465	1.0	.987871	.997807
0.40	.966203	1.0	.981372	.995609
0.60	.966037	1.0	.975144	.993426
0.70	.965961	1.0	.972110	.992341
0.75	.965947	1.0	.970611	.991801
0.80	.965933	1.0	.969122	.991261
1.00	.965846	0.999945	.963260	.989107
1.30	.965703	0.999752	.954730	.985899

Table 5.4

These can be transformed into the four coefficients a_1, $\bar{b}_1$, a_2 and $\bar{b}_2$ defined by equations 3.14.

$M k_g$	a_1	$\bar{b}_1$	a_2	$\bar{b}_2$
0.20	.13394	-0.0016998	.047927	-0.0005044
0.40	.13499	-0.0026397	.073850	-0.0006452
0.60	.13566	-0.0035509	.098857	-0.0007692
0.70	.13596	-0.0039994	.111096	-0.0008265
0.75	.13601	-0.0042192	.117158	-0.0008536
0.80	.13607	-0.0044380	.123185	-0.0008801
1.00	.13641	-0.0052925	.147014	-0.0009809
1.30	.13698	-0.0065043	.182026	-0.0011168

Table 5.5

4) Construction and Solution of the Determinant of Equations 3.31 and 3.32

We write these equations here in the following way:

$$\phi_1(\bar{r}_m) = \frac{F}{D_1} \sum_{k=1}^{N} \left\{ \phi_1(\bar{r}_k) A_0' + \phi_2(\bar{r}_k) B_0' \right\} K_0(\kappa_1 \varepsilon |\bar{r}_m - \bar{r}_k|) \tag{5.26}$$

$$\begin{aligned} \phi_2(\bar{r}_m) = {} & \frac{p}{D_2} \frac{\Sigma_R F}{\kappa_2^2 - \kappa_1^2} \frac{1}{D_1} \sum_{k=1}^{N} \left\{ \phi_1(\bar{r}_k) A_0' + \phi_2(\bar{r}_k) B_0' \right\} \left\{ K_0(\kappa_1 \varepsilon |\bar{r}_m - \bar{r}_k|) - K_0(\kappa_2 \varepsilon |\bar{r}_m - \bar{r}_k|) \right\} \\ & + \frac{1}{2\pi} \frac{1}{D_2} \sum_{k=1}^{N} \left\{ \phi_1(\bar{r}_k) C_0' + \phi_2(\bar{r}_k) D_0' \right\} K_0(\kappa_2 \varepsilon |\bar{r}_m - \bar{r}_k|) \end{aligned} \tag{5.27}$$

where we define:

$$\begin{aligned} A_0' &= \eta_2 b_2 + (\eta_1 - 1) a_1 \\ B_0' &= \eta_2 a_2 + (\eta_1 - 1) b_1 \\ C_0' &= p a_1 - b_2 \\ D_0' &= p b_1 - a_2 \end{aligned} \tag{5.28}$$

ε is a scale factor. In (fig. 5.4) the distance between two rods is taken as unity. ε is the ratio between the actual distance in the reactor and this unit. As mentioned in section V.2, $\varepsilon = 2.9934$.

The values of the coefficients defined in equations 5.28 can be expressed now for the different fuel loadings in terms of the results of table 5.5.

M_{k_g}	A'_o	B'_o	C'_o	D'_o
0.20	-0.13414	0.10138	0.13042	-0.04957
0.40	-0.13464	0.15621	0.13159	-0.07641
0.60	-0.13472	0.20910	0.13236	-0.10230
0.70	-0.13471	0.23499	0.13271	-0.11497
0.75	-0.13461	0.24781	0.13278	-0.12125
0.80	-0.13451	0.26055	0.13287	-0.12749
1.00	-0.13422	0.31092	0.13330	-0.15215
1.30	-0.13381	0.38486	0.13399	-0.18833

Table 5.6

We can now construct our determinant.

In order to simplify the determination of the coefficients of $\phi_1(\lambda_{\hat{k}})$ and $\phi_2(\lambda_{\hat{k}})$ we utilize the symmetry of the fuel element locations. From figure 5.4 we see that we need consider only the flux at the elements located in 1/12 of the reactor. These are the elements shown in the shaded area of figure 5.4 and their coordinates (in a triangular coordinate system) are listed in the second column of table 5.7. As indicated by the first column of the table, only 28 different values for each of the thermal and fast fluxes need be considered.

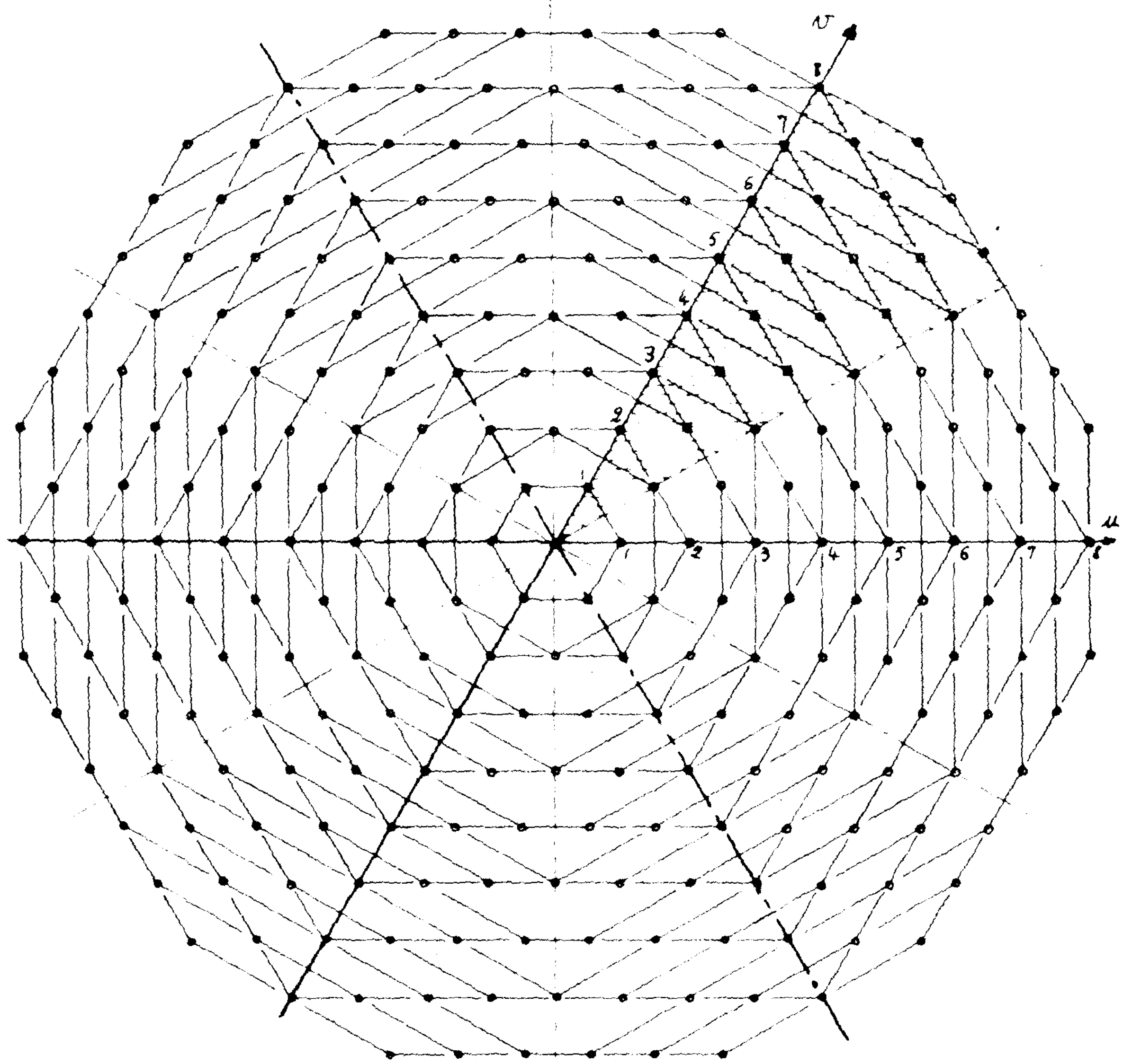


Fig 54

m	fuel elements coordinates (k = 1 to 253)											
1	0 0											
2	0 1		-1 1		-1 0		0-1		1-1		1 0	
3	0 2		-2 2		-2 0		0-2		2-2		2 0	
4	0 3		-3 3		-3 0		0-3		3-3		3 0	
5	0 4		-4 4		-4 0		0-4		4-4		4 0	
6	0 5		-5 5		-5 0		0-5		5-5		5 0	
7	0 6		-6 6		-6 0		0-6		6-6		6 0	
8	0 7		-7 7		-7 0		0-7		7-7		7 0	
9	0 8		-8 8		-8 0		0-8		8-8		8 0	
10	1 1		-1 2		-2 1		-1-1		1-2		2-1	
11	1 2	-1 3	-2 3	-3 2	-3 1	-2-1	-1-2	1-3	2-3	3-2	3-1	2 1
12	1 3	-1 4	-3 4	-4 3	-4 1	-3-1	-1-3	1-4	3-4	4-3	4-1	3 1
13	1 4	-1 5	-4 5	-5 4	-5 1	-4-1	-1-4	1-5	4-5	5-4	5-1	4 1
14	1 5	-1 6	-5 6	-6 5	-6 1	-5-1	-1-5	1-6	5-6	6-5	6-1	5 1
15	1 6	-1 7	-6 7	-7 6	-7 1	-6-1	-1-6	1-7	6-7	7-6	7-1	6 1
16	1 7	-1 8	-7 8	-8 7	-8 1	-7-1	-1-7	1-8	7-8	8-7	8-1	7 1
17	2 2		-2 4		-4 2		-2-2		2-4		4-2	
18	2 3	-2 5	-3 5	-5 3	-5 2	-3-2	-2-3	2-5	3-5	5-3	5-2	3 2
19	2 4	-2 6	-4 6	-6 4	-6 2	-4-2	-2-4	2-6	4-6	6-4	6-2	4 2
20	2 5	-2 7	-5 7	-7 5	-7 2	-5-2	-2-5	2-7	5-7	7-5	7-2	5 2
21	2 6	-2 8	-6 8	-8 6	-8 2	-6-2	-2-6	2-8	6-8	8-6	8-2	6 2
22	2 7	-2 9	-7 9	-9 7	-9 2	-7-2	-2-7	2-9	7-9	9-7	9-2	7 2
23	3 3		-3 6		-6 3		-3-3		3-6		6-3	
24	3 4	-3 7	-4 7	-7 4	-7 3	-4-3	-3-4	3-7	4-7	7-4	7-3	4 3
25	3 5	-3 8	-5 8	-8 5	-8 3	-5-3	-3-5	3-8	5-8	8-5	8-3	5 3
26	3 6	-3 9	-6 9	-9 6	-9 3	-6-3	-3-6	3-9	6-9	9-6	9-3	6 3
27	4 4		-4 8		-8 4		-4-4		4-8		8-4	
28	4 5	-4 9	-5 9	-9 5	-9 4	-5-4	-4-5	4-9	5-9	9-5	9-4	5 4

Table 5.7

Fuel elements coordinates

Each row represents points of equal flux.

By inspection of (fig. 5.4) we determine those elements outside the shaded area which have the same surface flux as each of the 28 elements inside that area. The coordinates of these are arranged in table 5.7 such that each row corresponds to a single flux level, due to symmetry.

Hence, by suitably grouping the coefficients of $\phi_1(\vec{r}_k)$ and $\phi_2(\vec{r}_k)$ in equations 5.26 and 5.27 the number of N terms in each summation can be effectively reduced to 28, resulting in a determinant of order 56.

Each element of the determinant is a summation of the terms corresponding to all the elements listed in the same row of table 5.7.

The term on the left side of each of equations 5.26 and 5.27 is transferred to the right side by adding an extra -1 to the diagonal terms of the determinant. In order to represent the contribution of $\phi(\vec{r}_k)$ to $\phi(\vec{r}_m)$ when $k = m$, we replace $K_0(\kappa\varepsilon|\vec{r}_k - \vec{r}_m|)$ by $K_0(\kappa\varepsilon\bar{b})$ where $\bar{b}$ is the radius of the element. For this case $\vec{r}_k - \vec{r}_m$ would be equal to zero.

The construction and calculation of the determinant are programmed on an IBM 7090 computer. The program input are the parameters whose values are given in section V.2, the four coefficient A'_0, B'_0, C'_0, D'_0 and table 5.7.

- The following results have been obtained:

M kg	Δ
0.20	1.1548×10^{6}
0.40	1.4292×10^{7}
0.60	2.7905×10^{7}
0.70	1.9350×10^{7}
0.75	6.6451×10^{6}
0.80	-1.3570×10^{7}
1.00	-1.8307×10^{8}
1.30	-4.1577×10^{8}

Table 5.8

For $M = 0$ the value of the determinant is $\Delta = 1$ because in this case:

$$A'_0 = B'_0 = C'_0 = D'_0 = 0$$

As the mass is increased, the value of the determinant increases from 1 (for zero mass) to the values indicated in table 5.8. The first zero occurs between the values $M = 0.75$ kg and $M = 0.80$ kg. This first zero corresponds to criticality. (See Fig. 5.5)

Further studies could be undertaken to determine the degree of precision required in such calculations. For instance, if one wants to know what error is made on the determination of the critical mass, one has to make an error analysis on the value of the determinant. This error is due to the uncertainty of the data and to the limit of accuracy used in solving the determinant. This latter uncertainty becomes more important as the size of the determinant increases, because when a high order determinant takes on small values (near the zero of the determinant), the relative error involved in the computing process is considerably increased. A knowledge of the magnitude of the error allows one to determine an upper and lower bound for the critical loading.

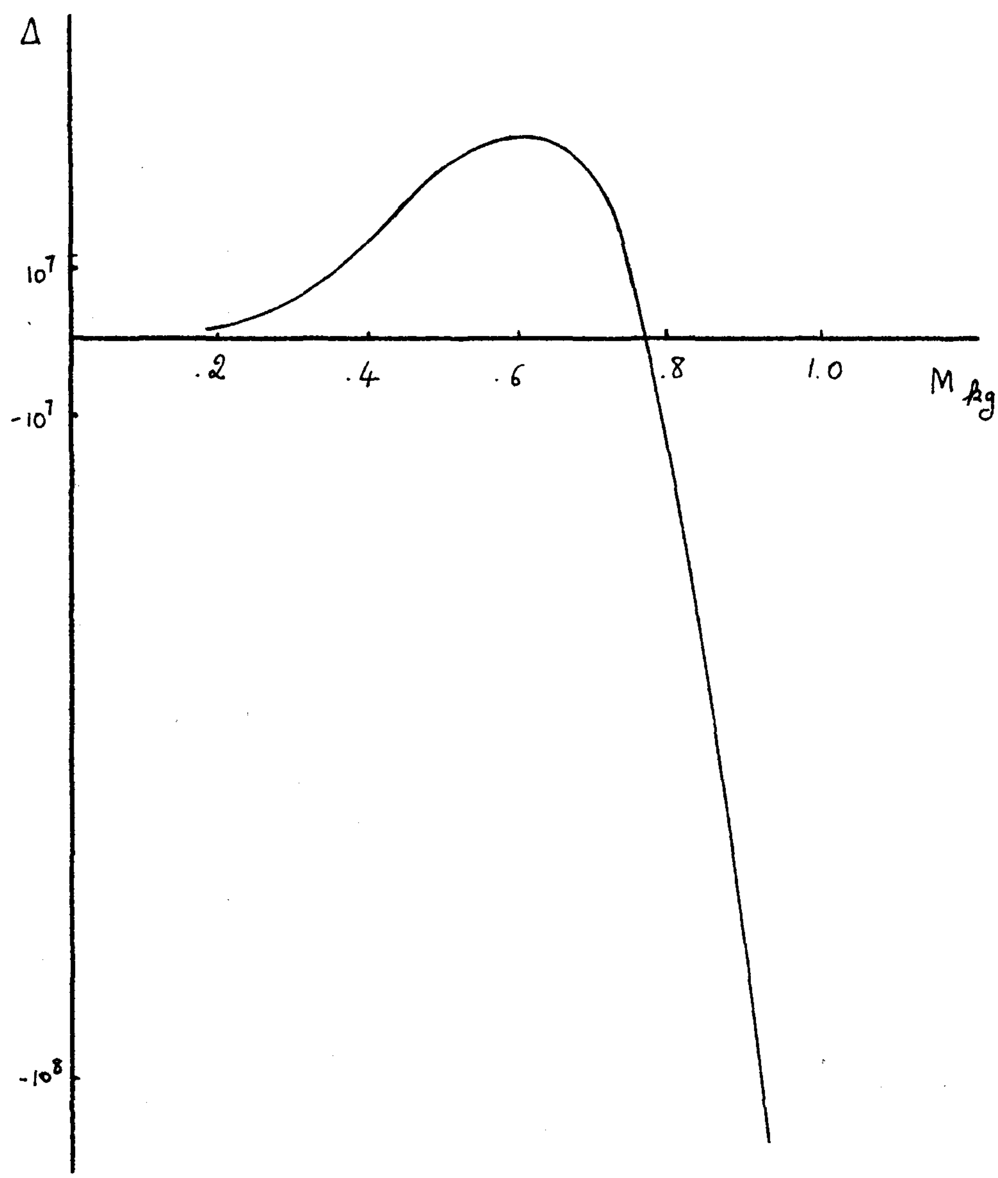


Fig. 5.5

The Value of the Determinant Δ Versus the Fuel Loading

VI. CONCLUSION

Before performing these calculations, one should investigate the validity of the method outlined here. From the critical masses obtained, $M_c = 1.3\ kg$ from homogeneous theory (Appendix C) and $M_c = 0.77\ kg$ from the four coefficient method (section V), one observes a noticeable difference. Actually, the example chosen here is not suitable for the four coefficient method. This is due principally to the fact that the distance between two fuel elements is not large when compared with the diffusion length, and also to the small thickness of the fuel layer. Therefore, it should be expected that for this particular case, homogeneous theory is more appropriate. The four coefficient method is mostly applicable to assemblies having a small number of fuel elements, fuel elements of different kinds, variation in the lattice configuration along the radius, and other features which are not easily handled by homogeneous theory. Nevertheless, the example treated in section V shows that, with the help of large digital computers, the four coefficient method may be applied to reactors containing a significant number of fuel elements.

Furthermore, the critical masses obtained here should not be expected to agree with experimental measurements. As was mentioned in Appendix C, the values of the thermal cross sections used are those evaluated at an energy corresponding to 2,200 m/s, and the averaging over thermal distribution has not been performed here. The thermal absorption cross sections should be decreased by 11%. By performing calculations identical to those carried out to evaluate the change due to modifications in the diffusion length (Appendix C), one can evaluate the importance of this correction. It is found that the critical mass would be decreased by about 22%.

The flux distribution in the moderator can be obtained from equations 3.26 and 3.30. These equations include the terms $(\phi_1 k)$ and $(\phi_2 k)$ defined in 3.21 and 3.22. They depend on $\phi_1(\overline{r}_k)$ and $\phi_2(\overline{r}_k)$, the fluxes at the surface of each fuel element. The latter are solutions of the homogeneous linear equations 3.31 and 3.32 whose determinant has been made zero. Hence the solution is not unique. A value for one $\phi(\overline{r}_k)$ must be arbitrarily chosen. By substituting this value in equations 3.31 and 3.32 one gets a system of linear non-homogeneous equations. One of these equations is redundant because we are left with one more equation than unknowns. This non-homogeneous system can be solved by calculating determinants of order 2N-1 at most. In simple cases the size of the determinant is m-1 where m is the reduced number of unknowns corresponding to the elements located in the shaded area of (fig. 5.4).

Further improvements can be made to the four coefficient method. As was said at the beginning of section III, the resonances in the fuel are taken into account by an overall resonance escape probability. Actually, the resonance absorptions take place mainly in the fuel, and it might be worthwhile to take into account the localization of this phenomenon. This might be done for instance by introducing a third group. One observes that the resonances occur in a relatively narrow range of energy for uranium (between about 2 ev and 200 ev). Using the flux distribution of an intermediate group, it is then possible to determine at what rate these neutrons go into each fuel element. A set of cross sections is to be evaluated, including resonance properties, for this particular group. In other words, one applies the heterogeneous method to the intermediate group and takes into account the fact that resonances are localized in the fuel elements.

This localization is an important fact in heterogeneous reactors, because the increase of the resonance escape probability is the principal improvement over homogeneous reactors.

The introduction of three groups requires the use of nine coefficients instead of the four in the method outlined in section III. These nine coefficients relate the three sink terms to the three fluxes at the surface of each fuel element, and are equivalent to the four coefficients defined in equations 3.12 and 3.13.

The next improvement would be to consider a finite medium. This is necessary for the case of thin reflectors. One might apply the refinements developed in the present investigation to the Jonsson theory (6) for example.

It should not be forgotten that all these extra effects if taken into account give rise to mathematical complications. For instance, the size of the determinant of the criticality condition is doubled in the four coefficient method, and is tripled if three groups are used. Hence, depending on the number of fuel elements to be used, a compromise must be found between the degree of complexity one can afford and the accuracy one desires.

APPENDIX A

THE THERMAL CONSTANT IN CYLINDRICAL GEOMETRY

The thermal constant γ of a slug in a diffusing medium is defined as the ratio of the net flow per unit time of thermal neutrons into the slug to the value of the thermal flux at the surface of the slug.

One way of getting an approximate value of this constant is presented here, making the following assumptions:

- the fuel rod consists of one kind of material
- the angular distribution of neutrons entering the slug is isotropic (a first order correction to this approximation is done here)
- the collision density is constant inside the fuel

The thermal coefficient is related to the transmission coefficient t = fraction of all neutrons incident upon the surface of a lump which pass through the lump without being absorbed:

$$t = \frac{J^+}{J^-}$$

$$\gamma = \frac{-2\pi b J}{\phi} = \pi b \frac{J^- - J^+}{J^- + J^+}$$

$$\gamma = \pi b \frac{1-t}{1+t} \qquad \text{(A.1)}$$

J and ϕ are the net current and the flux at the surface of the rod. j^+ and j^- are the partial currents at the same surface, j^+ being the current going outward. b is the radius of the slug, which is assumed to be circular cylindrical.

We will compute t and deduce γ from this relationship.

1) The Transmission Coefficient t

Among the neutrons entering the slug, some scatter, some are absorbed, and some escape without undergoing collisions. In section A2 we will compute the fraction λ which escape without undergoing any collision. The fraction $(1 - \lambda)$ therefore make at least one collision.

Let $\Sigma_t =$ total macroscopic cross section of the material inside the lump

$\Sigma_s =$ macroscopic scattering cross section of the same material

Thus, among the $(1 - \lambda)$ neutrons which make a collision, $(1 - \lambda)\frac{\Sigma_s}{\Sigma_t}$ make a scattering collision. In section A3 we will compute the fraction S of these latter neutrons which having made a scattering collision, then escape from the slug. For that we will make the two following assumptions:

i) The scattering is isotropic in the laboratory system in the slug material. This assumption is acceptable for heavy nuclei where $\bar{\mu}_0 = \frac{2}{3A}$ The average cosine of the scattering angle becomes small when compared to unity.

ii) The collision density in the slug is independent of position.

Once ς is found, we know that the fraction

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_t}(1-\varsigma) \tag{A.2}$$

of the incoming neutrons make at least two collisions and the fraction

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_t}(1-\varsigma)\frac{\Sigma_s}{\Sigma_t} \tag{A.3}$$

make at least two scattering collisions. By the same procedure, we know that the fraction

$$(1-\lambda)\frac{\Sigma_s}{\Sigma_t}\left[(1-\varsigma)\frac{\Sigma_s}{\Sigma_t}\right]^n \tag{A.4}$$

of the incoming neutrons make at least $(n+1)$ scattering collisions, and that at each scattering generation, a fraction ς of the scattering neutrons escape from the slug.

Hence, the total fraction of incoming neutrons which escape from the slug after any number of collisions is:

$$t = \lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}\varsigma + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}(1-\varsigma)\frac{\Sigma_s}{\Sigma_t}\varsigma + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}(1-\varsigma)^2\left(\frac{\Sigma_s}{\Sigma_t}\right)^2\varsigma + \dots \tag{A.5}$$

$$t = \lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}\varsigma\left[1 + (1-\varsigma)\frac{\Sigma_s}{\Sigma_t} + (1-\varsigma)^2\left(\frac{\Sigma_s}{\Sigma_t}\right)^2 + \dots\right] \tag{A.6}$$

$$= \lambda + (1-\lambda)\frac{\Sigma_s}{\Sigma_t}\varsigma\frac{1}{1-(1-\varsigma)\frac{\Sigma_s}{\Sigma_t}} = \lambda + (1-\lambda)\frac{\varsigma\Sigma_s}{\Sigma_t-(1-\varsigma)\Sigma_s} \tag{A.7}$$

Hence, the thermal constant is, from equation A.1

$$\gamma = \pi \bar{b}\,\frac{(1-\lambda)\left[\Sigma_t-(1-\varsigma)\Sigma_s\right]-(1-\lambda)\varsigma\Sigma_s}{(1+\lambda)\left[\Sigma_t-(1-\varsigma)\Sigma_s\right]+(1-\lambda)\varsigma\Sigma_s} \tag{A.8}$$

$$\gamma = \pi \bar{b}\,\frac{(1-\lambda)(\Sigma_t-\Sigma_s)}{(1+\lambda)(\Sigma_t-\Sigma_s)+2\varsigma\Sigma_s} \tag{A.9}$$

2) The Probability λ

λ is the probability that a neutron entering the slug will escape without making any collision.

This probability is equivalent to the transmission coefficient t of a lump computed in a first flight approximation(*), but where the cross section used is not the absorption cross section but the total cross section.

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*Ref. 3, p. 247.

a) Expression for the Probability λ

We need the angular distribution of the velocities of the thermal neutrons impinging upon the surface of the slug.

A first order correction to the isotropic distribution is given by the diffusion theory(*):

$$f^{-}(\mu,\varphi)\,d\mu\,d\varphi = \frac{\mu\,d\mu\,d\varphi}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)|\cos\beta}{\Sigma_s^{(m)}}\right] \tag{A.10}$$

where μ is the cosine of the angle which the direction of motion of the incident neutron makes with the normal to the surface of the lump.

φ is the azimuth of the neutron direction about the normal.

$\Sigma_s^{(m)}$ is the scattering cross section of the diffusion medium outside the lump.

β is the angle between the direction of motion of the incident neutron and the flux gradient.

In our present case, the flux gradient can be considered as always oriented perpendicular to the surface of the slug, neglecting the gross variation of the flux in the reactor when compared with the local variation at the slug boundary.

Hence, $\cos\beta \simeq \mu$ and the neutron angular distribution is:

$$f^{-}(\mu,\varphi)\,d\mu\,d\varphi = \frac{\mu\,d\mu\,d\varphi}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)|\,\mu}{\Sigma_s^{(m)}}\right] \tag{A.11}$$

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*Ref. 3, p. 171.

where $\nabla\phi(0)$ can be approximated from a diffusion theory calculation. For cylindrical geometry one gets:

$$\frac{\nabla\phi(0)}{\phi(0)} = \frac{\phi'(0)}{\phi(0)} \simeq \frac{\kappa I_1(\kappa b)}{I_0(\kappa b)} \qquad (A.12)$$

where $\kappa^2 = \frac{\Sigma_a^{(F)}}{D^{(F)}}$ (slug material constants). (A.13)

Hence:

$$j^-(\mu,\varphi)d\mu d\varphi = \frac{\mu d\mu d\varphi}{4\pi}\,\phi_0\left[1 + \frac{\kappa I_1(\kappa b)}{I_0(\kappa b)}\,\frac{\mu}{\Sigma_s^{(m)}}\right] \qquad (A.14)$$

$$j^-(\mu,\varphi)d\mu d\varphi = \frac{\mu d\mu d\varphi}{4\pi}\,\phi_0(1+g\mu) \qquad (A.15)$$

where $$g = \frac{\kappa}{\Sigma_s^{(m)}}\,\frac{I_1(\kappa b)}{I_0(\kappa b)} \qquad (A.16)$$

The integral, over all directions into the lump, of the velocity distribution is:

$$\frac{\phi_0}{4\pi}\int_0^1 \mu(1+g\mu)d\mu\int_0^{2\pi} d\varphi = \frac{\phi_0}{4}\left(1+\frac{2g}{3}\right) \qquad (A.17)$$

The probability that a neutron will pass through a distance s of material without making any collision is $e^{-\Sigma_t s}$ where Σ_t is the total cross section of the material.

We can now express the probability λ as:

$$\lambda = \frac{\iint j(\mu,\varphi)\, e^{-\Sigma_t s(\mu,\varphi)}\, d\mu\, d\varphi}{\iint j(\mu,\varphi)\, d\mu\, d\varphi} \qquad (A.18)$$

$$\lambda = \frac{1}{\pi\left(1+\frac{2g}{3}\right)} \int_0^1 \mu(1+g\mu)\, d\mu \int_0^{2\pi} e^{-\Sigma_t s(\mu,\varphi)}\, d\varphi \qquad (A.19)$$

g : defined in A.16

Σ_t : total cross section of the slug material

$s(\mu,\varphi)$: straight line path in the direction μ, φ through the slug

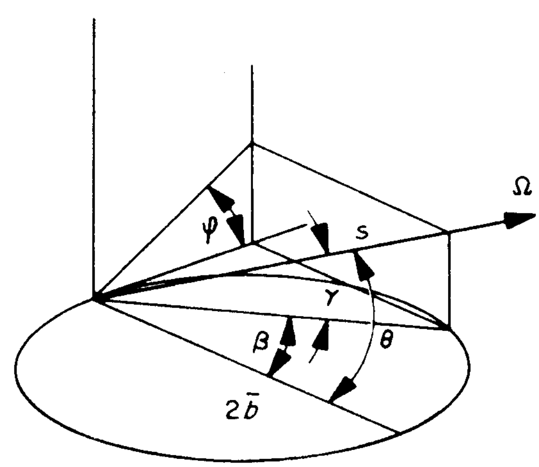


Fig. A.1

b) Calculation of the Probability λ in Cylindrical Geometry from Expression A.19

$$\lambda = \frac{1}{\pi\left(1+\frac{2g}{3}\right)} \int_0^1 \mu(1+g\mu)\,d\mu \int_0^{2\pi} e^{-\Sigma_t S(\mu,\varphi)}\,d\varphi$$

We make the change of variables defined by:

$$\mu = \cos\beta \cos\gamma$$

$$tg\,\varphi = \frac{tg\,\gamma}{\sin\beta}$$

$$S = 2b\,\frac{\cos\beta}{\cos\gamma} \qquad \text{(see Fig. A.1)}$$

The Jacobian of the transformation is

$$J = \cos\gamma$$

Hence:

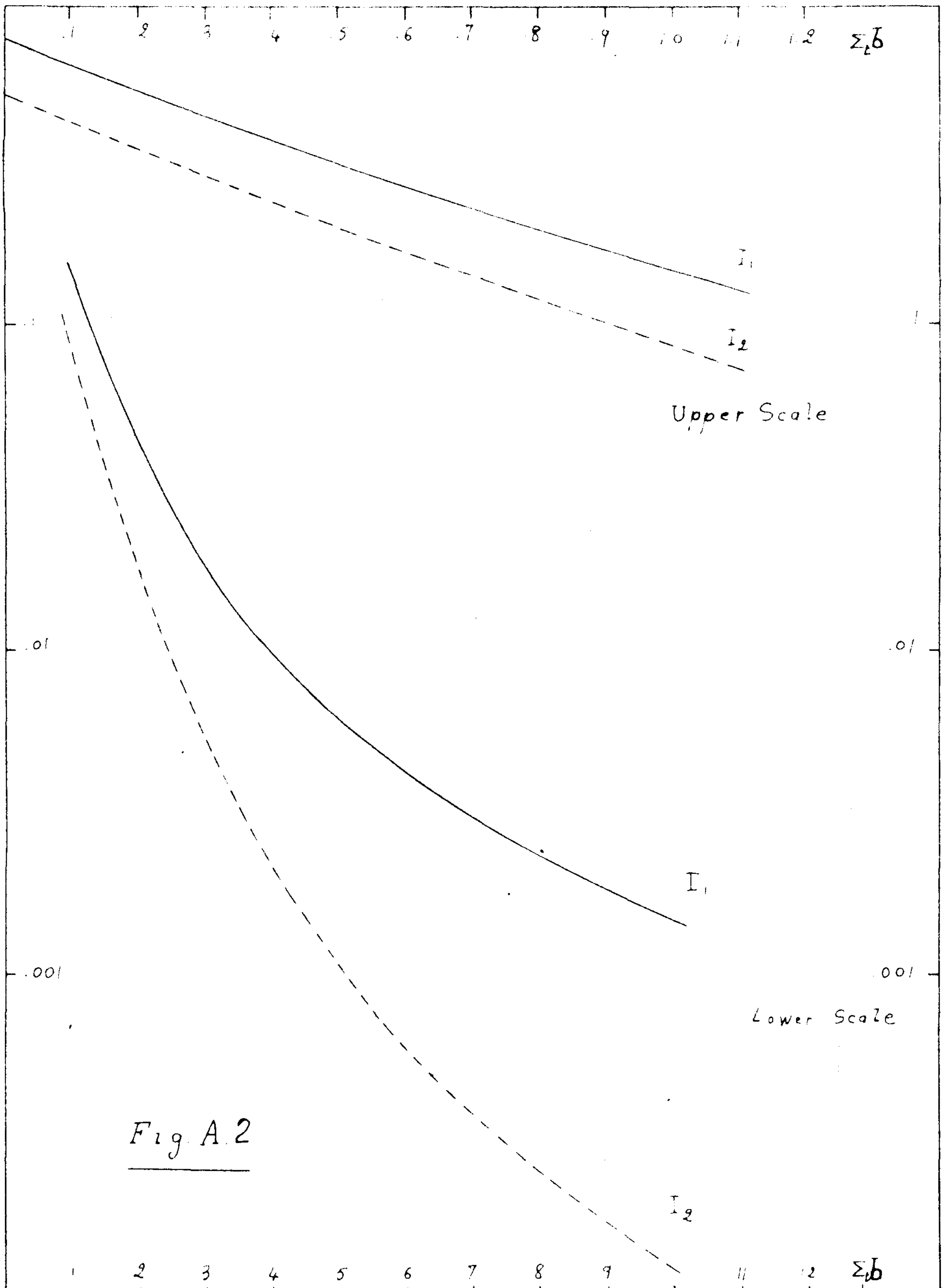
$$\lambda = \frac{4}{\pi\left(1+\frac{2g}{3}\right)} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos\beta\cos\gamma\,(1+g\cos\beta\cos\gamma)\cos\gamma\, e^{-\Sigma_t 2b\frac{\cos\beta}{\cos\gamma}}\,d\beta\, d\gamma \qquad \text{(A.20)}$$

$$\lambda = \frac{4}{\pi(1+\frac{2}{3}g)}\left\{\iint_0^{\frac{\pi}{2}} \cos\beta\cos^2\gamma\, e^{-\Sigma_t \ell \bar{b}\frac{\cos\beta}{\cos\gamma}} d\beta d\gamma + g\iint_0^{\frac{\pi}{2}} \cos^2\beta\cos^3\gamma\, e^{-\Sigma_t \ell \bar{b}\frac{\cos\beta}{\cos\gamma}} d\beta d\gamma\right\} \qquad (A.21)$$

These integrals must be evaluated numerically. Writing

$$\lambda = \frac{4}{\pi(1+\frac{2}{3}g)}\left(I_1 + g\, I_2 \right)$$

the values of I_1 and I_2 are plotted on (Fig. A.2) as functions of $\Sigma_t \bar{b}$.



.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 $\Sigma_t b$
I_1
I_2
1
Upper Scale
.01
.001
I_1
Lower Scale
I_2
1 2 3 4 5 6 7 8 9 11 12 $\Sigma_t b$

Fig. A.2

3) The Probability ς

ς is the probability, that a neutron will escape from the cylindrical slug after a scattering collision.

We made the assumptions that the scattering is homogeneous and isotropic inside the slug. Hence, the system is equivalent to a uniform source material whose shape is a circular cylinder, and ς is the probability that a neutron born in this source will escape.

This problem has already been treated (Ref. 4) and a short demonstration will be given here:

Consider a source of volume v and unit source density bounded externally by area A. The number of particles coming from the volume element dv and going through the area dA is:

$$dv \; \frac{dA \cos\theta}{4\pi\lambda^2} \; e^{-\Sigma_t \lambda} \qquad \text{(A.22)}$$

where λ is the distance between dv and dA and θ is the angle between the vector $\vec{\lambda}$ and the outward normal to A. Σ_t is the total macroscopic cross section.

Integrating over A and v yields the number of particles going out of the volume v ; dividing by the total source strength v yields the probability that a particle escapes without undergoing another collision.

$$\mathcal{S} = \frac{1}{v}\int_v dv \int_A dA\ \frac{\cos\theta}{4\pi r^2}\ e^{-\Sigma_t r} \qquad \text{(A.23)}$$

In cylindrical geometry every dA is equivalent, and

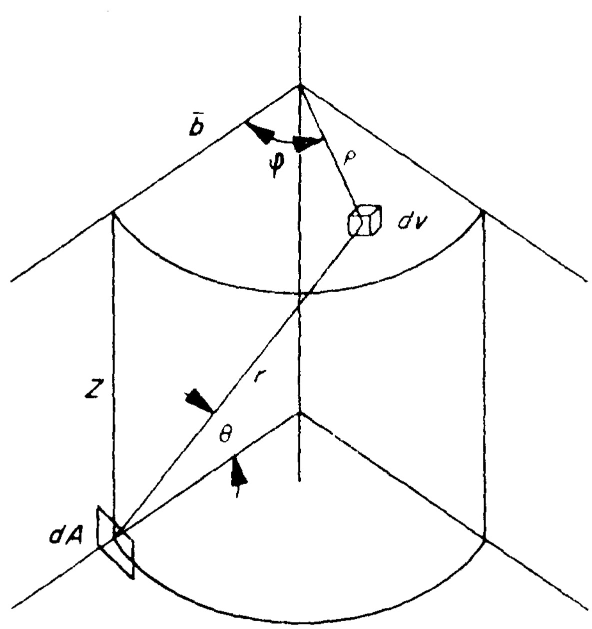


Fig. A.3

$$\mathcal{S} = \frac{A}{v}\int_v dv\ \frac{\cos\theta}{4\pi r^2}\ e^{-\Sigma_t r} \qquad \text{(A.24)}$$

For an infinite cylinder:

$$G = \frac{2}{b}\int_0^b \rho\, d\rho \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz\, \frac{e^{-\Sigma_t r}}{4\pi r^2} \cos\theta \tag{A.25}$$

where

$$r^2 = z^2 + b^2 + \rho^2 - 2 b \rho \cos\varphi$$

$$\cos\theta = \frac{b - \rho\cos\varphi}{r}$$

We carry out the integration over z with the substitution:

$$\Sigma_t^2 r^2 = f^2 + \Sigma_t^2 z^2 = \xi^2 f^2 \tag{A.26}$$

$$G = \frac{2}{b}\int_0^b \rho\, d\rho \int_0^{2\pi} d\varphi \int_{-\infty} (b - \rho\cos\varphi)\, \frac{e^{-\Sigma_t r}}{4\pi r^3}\, dz$$

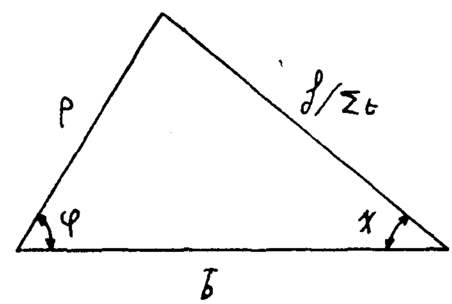
$$G = \frac{2}{4\pi b}\int_0^b \rho\, d\rho \int_0^{2\pi} (b - \rho\cos\varphi)\, d\varphi \int_{-\infty}^{\infty} \frac{e^{-\Sigma_t r}}{r^3}\, dz \tag{A.27}$$

where

$$\int_{-\infty}^{\infty} \frac{e^{-\Sigma_t r}}{r^3}\, dz = \frac{2\Sigma_t^2}{f^2}\int_1^{\infty} \frac{e^{-\xi f}}{\xi^2(\sqrt{\xi^2 - 1})}\, d\xi \tag{A.28}$$

$$\int_{-\infty}^{\infty} \frac{e^{-\Sigma_t r}}{r^3}\, dz = \frac{2\Sigma_t^2}{f}\int_1^{\infty} K_1(f t)\, \frac{dt}{t} \tag{A.29}$$

Projecting (fig. A.3) on a horizontal plane:



$$\cos\chi = \frac{(\bar{b} - \rho\cos\varphi)\,\Sigma_t}{s} \qquad \text{(A.30)}$$

$$\varsigma = \frac{\Sigma_t}{\pi \bar{b}} \int_0^{\bar{b}} \rho\, d\rho \int_1^{\infty} \frac{dt}{t} \int_0^{2\pi} K_1(st) \cos\chi\, d\varphi \qquad \text{(A.31)}$$

Observing that

$$K_1(st)\cos\chi = \sum_{m=-\infty}^{\infty} K_{m+1}(\Sigma_t \bar{b} t)\, I_m(\Sigma_t \rho t) \cos m\varphi \qquad \text{(A.32)}$$

and carrying out the integration over φ **:**

$$\varsigma = \frac{2\Sigma_t}{\bar{b}} \int_0^{\bar{b}} \rho\, d\rho \int_1^{\infty} K_1(\Sigma_t \bar{b} t)\, I_0(\Sigma_t \rho t)\, \frac{dt}{t} \qquad \text{(A.33)}$$

$$S = 2\int_1^{\infty} K_1(\Sigma_t \bar{b}\, t)\ I_1(\Sigma_t \bar{b}\, t)\ \frac{dt}{t^2} \tag{A.34}$$

$$S = \frac{2\Sigma_t b}{3}\Big\{-2 + \Big(2\Sigma_t \bar{b} + \frac{1}{\Sigma_t \bar{b}}\Big) I_1(\Sigma_t \bar{b})\, K_1(\Sigma_t \bar{b})$$

$$+ I_0(\Sigma_t \bar{b})\, K_1(\Sigma_t \bar{b}) - I_1(\Sigma_t \bar{b})\, K_0(\Sigma_t \bar{b}) + 2\Sigma_t \bar{b}\, I_0(\Sigma_t \bar{b})\, K_0(\Sigma_t \bar{b})\Big\} \tag{A.35}$$

APPENDIX B

THE ANGULAR DISTRIBUTION OF NEUTRONS LEAVING A CYLINDRICAL MODERATOR ROD

We consider the cylindrical region, containing moderator, of the fuel element of section IV. We show that provided the angular distribution of the incoming neutrons is isotropic, the angular distribution of the outgoing neutrons is also isotropic, if the following assumptions are made:

- there is no absorption in the moderator
- the flux is independent of position inside the moderator
- scattering is isotropic in the laboratory system

By isotropic distribution, we mean that considering a small area dA and defining the angles θ and φ according to (fig. B.1):

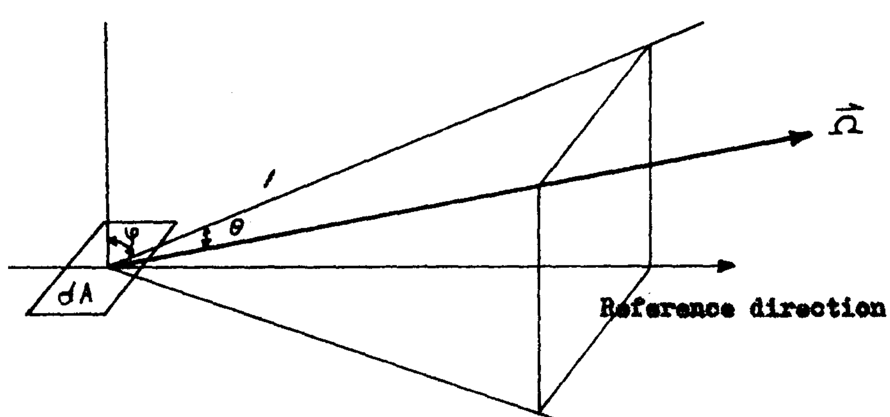


Fig. B.1

the number of neutrons going through dA from one given side to the other in solid angle $d\Omega$ is proportional to

$$dA \cos\theta \cos\varphi \; d\Omega = dA \cos^2\theta \cos\varphi \; d\varphi \, d\theta$$

Let $f^-(\theta,\varphi)\, d\varphi \, d\theta = \cos^2\theta \cos\varphi \, d\varphi \, d\theta$ (B.1)

be the angular distribution of the neutrons going into the moderator. Among these some scatter before going out and have the angular distribution $f_+^{(1)}(\theta,\varphi)\, d\varphi \, d\theta$ when leaving and some go straight through and have the angular distribution $f_+^{(2)}(\theta,\varphi) d\varphi d\theta$ when leaving. Hence, if $f_+^{(1)}$ and $f_+^{(2)}$ are weighted with correct coefficients so that the number of neutrons going in equals the number going out, the angular distribution of the neutrons going out is:

$$f^+(\theta,\varphi)\, d\varphi \, d\theta = f_+^{(1)}(\theta,\varphi) d\varphi d\theta' + f_+^{(2)}(\theta,\varphi)\, d\varphi \, d\theta \tag{B.2}$$

First, we calculate $f_+^{(1)}(\theta,\varphi)\, d\varphi \, d\theta$. Consider a small volume element dv inside the moderator (fig. 4.1). The number of neutrons coming from dv per unit time and going through dA is:

$$e^{-\Sigma_m s} \frac{dA \cos\theta \cos\varphi}{4\pi s^2} \, dv = e^{-\Sigma_m s} \frac{dA \cos\theta \cos\varphi}{4\pi s^2} \, s^2 \cos\theta \, d\varphi \, d\theta \, ds$$

where one neutron per unit volume and time is coming from dv ; $e^{-\Sigma_m s}$ is the probability that a neutron coming from dv reaches dA without colliding; and Σ_m is the scattering cross section of the moderator. Since all the dA are equivalent:

$$j_+^{(1)}(\theta,\varphi)\,d\varphi\,d\theta \propto \cos^2\theta\cos\varphi\,d\varphi\,d\theta\int_0^{s_1} e^{-\Sigma_m s}\,ds$$

$$j_+^{(1)}(\theta,\varphi)\,d\varphi\,d\theta \propto \cos^2\theta\cos\varphi\left(1-e^{-\Sigma_m s_1}\right)d\varphi\,d\theta \tag{B.3}$$

where s and s_1 are defined according to (fig. 4.1).

Now we calculate $j_+^{(2)}(\theta,\varphi)\,d\varphi\,d\theta$. First, we note that the angles θ and φ are the same at the entrance and at the exit for the non-colliding neutrons (fig. 4.1). Hence, the neutrons incoming through the solid angle $d\Omega$ about φ,θ have the probability $e^{-\Sigma_m s_1}$ of reaching the other side of the surface without colliding:

$$j_+^{(2)}(\theta,\varphi)\,d\varphi\,d\theta = e^{-\Sigma_m s_1}\cos^2\theta\cos\varphi\,d\varphi\,d\theta \tag{B.4}$$

which has the correct weight factor corresponding to an incoming current:

$$j_-(\theta,\varphi)\,d\varphi\,d\theta = \cos^2\theta\cos\varphi\,d\varphi\,d\theta$$

To find the weight factor of $j_+^{(1)}(\theta,\varphi)\,d\varphi\,d\theta$ we consider the neutrons which go in and collide. There are:

$$\int dA\int_\theta\int_\varphi\left(1-e^{-\Sigma_m s_1}\right)j_-(\theta,\varphi)\,d\varphi\,d\theta = \int dA\int_\theta\int_\varphi\left(1-e^{-\Sigma_m s_1}\right)\cos^2\theta\cos\varphi\,d\varphi\,d\theta$$

of them.

and there are:

$$\int dA \int_\theta \int_\varphi j_+^{(1)}(\theta,\varphi)\, d\varphi\, d\theta = \int dA \int_\theta \int_\varphi \left(1-e^{-\Sigma_m s_1}\right) \cos^2\theta \cos\varphi\, d\varphi\, d\theta$$

going out. These two expressions must be equal. We see immediately that:

$$j_+^{(1)}(\theta,\varphi) = \left(1-e^{-\Sigma_m s_1}\right) \cos^2\theta \cos\varphi \tag{B.5}$$

The angular distribution of the neutrons going out is then

$$j^+(\theta,\varphi)\, d\varphi\, d\theta = e^{-\Sigma_m s_1} \cos^2\theta \cos\varphi\, d\varphi\, d\theta + \left(1-e^{-\Sigma_m s_1}\right) \cos^2\theta \cos\varphi\, d\varphi\, d\theta$$
$$= \cos^2\theta \cos\varphi\, d\varphi\, d\theta$$

which is an isotropic distribution, the same as the incoming neutron distribution.

In order to consider $j^+(\theta,\varphi) d\varphi\, d\theta$ as a probability, it is conveniently normalised as:

$$j^+(\theta,\varphi)\, d\varphi\, d\theta = \frac{\cos^2\theta \cos\varphi\, d\varphi\, d\theta}{4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^2\theta \cos\varphi\, d\varphi\, d\theta}$$

$$j^+(\theta,\varphi)\, d\varphi\, d\theta = \frac{1}{\pi} \cos^2\theta \cos\varphi\, d\varphi\, d\theta \tag{B.6}$$

APPENDIX C

INITIAL ESTIMATE OF CRITICAL MASS

In order to calculate the critical mass from the four coefficients method (section III), the criticality condition must be satisfied. The calculations may be performed by assuming several fuel loadings and determining the value of the determinant for each.

A first guess can be obtained by homogenizing the reactor and using two-group theory to calculate the critical mass of an equivalent homogeneous reactor. For illustrative purposes, the reactor described in section V will be considered here.

1) Theoretical Introduction

We recall briefly the two-group homogeneous model described in Reference 3, section 8.4. Fast fission has been included.

The diffusion equations in the core are:

$$-D_1 \nabla^2 \phi_1(\vec{\rho}) + \Sigma_R^{(1)} \phi_1(\vec{\rho}) = \nu \Sigma_f^{(2)} \phi_2(\vec{\rho}) + \nu \Sigma_f^{(1)} \phi_1(\vec{\rho})$$

$$-D_2 \nabla^2 \phi_2(\vec{\rho}) + \Sigma_a^{(2)} \phi_2(\vec{\rho}) = p \Sigma_R^{(1)} \phi_1(\vec{\rho}) \qquad \text{(C.1)}$$

In the reflector:

$$-D_3 \nabla^2 \phi_3(\vec{\rho}) + \Sigma_R^{(3)} \phi_3(\vec{\rho}) = 0$$

$$-D_4 \nabla^2 \phi_4(\vec{\rho}) + \Sigma_a^{(4)} \phi_4(\vec{\rho}) = \Sigma_R^{(3)} \phi_3(\vec{\rho})$$

Indices 1 and 2 refer to fast and thermal quantities, respectively, in the core. Indices 3 and 4 refer to fast and thermal quantities, respectively, in the reflector.

ϕ refers to the flux.

D refers to the diffusion constant.

$\Sigma_R^{(1)}$ and $\Sigma_R^{(3)}$ are the removal cross sections in the core and reflector, respectively.

$\Sigma_f^{(1)}$ and $\Sigma_f^{(2)}$ are the fast and thermal fission cross sections, respectively.

p is the resonance escape probability in the core.

Define

$$k_\infty^{(1)} = \frac{\nu \Sigma_f^{(1)}}{\Sigma_R^{(1)}} \qquad k_\infty^{(2)} = \frac{\nu p \Sigma_f^{(2)}}{\Sigma_a^{(2)}}$$

$$\varkappa_1^2 = \frac{\Sigma_R^{(1)}}{D_1} \qquad \varkappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2} \tag{C.2}$$

$$\varkappa_3^2 = \frac{\Sigma_R^{(3)}}{D_3} \qquad \varkappa_4^2 = \frac{\Sigma_a^{(4)}}{D_4}$$

and $$\chi_1^2 = \varkappa_1^2 (1 - k_\infty^{(1)})$$

The solution of equations C.1 is:

$$\phi_1(\vec{\rho}) = A_1 Z(\vec{\rho}) + C_1 W(\vec{\rho})$$

$$\phi_2(\vec{\rho}) = a_1 A_1 Z(\vec{\rho}) + a_2 C_1 W(\vec{\rho}) \qquad (C.3)$$

$$\phi_3(\vec{\rho}) = S_1 U(\vec{\rho})$$

$$\phi_4(\vec{\rho}) = T_1 V(\vec{\rho}) - a_3 S_1 U(\vec{\rho})$$

where Z **,** W**,** U **and** V **are solutions of**

$$\nabla^2 Z + \mu^2 Z = 0$$

$$\nabla^2 W - \lambda^2 W = 0 \qquad (C.4)$$

$$\nabla^2 U - \varkappa_3^2 U = 0$$

$$\nabla^2 V - \varkappa_4^2 V = 0$$

$$a_1 = \frac{p\Sigma_R^{(1)}}{D_2\mu^2 + \Sigma_a^{(2)}}$$

$$a_2 = \frac{p\Sigma_R^{(1)}}{-D_2\lambda^2 + \Sigma_a^{(2)}} \qquad (C.5)$$

$$a_3 = \frac{\Sigma_R^{(3)}}{D_4(\varkappa_3^2 - \varkappa_4^2)}$$

$$2\mu^2 = -(\varkappa_2^2 + \chi_1^2) + \left\{(\varkappa_2^2 + \chi_1^2)^2 + 4\varkappa_1^2\varkappa_2^2\left(k_\infty^{(2)} - 1 + k_\infty^{(1)}\right)\right\}^{1/2} \qquad (C.6)$$

$$-2\lambda^2 = -(\varkappa_2^2 + \chi_1^2) - \left\{(\varkappa_2^2 + \chi_1^2)^2 + 4\varkappa_1^2\varkappa_2^2\left(k_\infty^{(2)} - 1 + k_\infty^{(1)}\right)\right\}^{1/2}$$

In the present case, the geometry is assumed to be a cylinder with an infinite side reflector. The reflected ends are taken into account by introducing the reflector savings.

$$Z(\vec{\rho}) = J_0(B_1 \lambda) \cos B_z z \qquad Z'(\vec{\rho}) = -B_1 J_1(B_1 \lambda) \cos B_z z$$

$$W(\vec{\rho}) = I_0(B_2 \lambda) \cos B_z z \qquad W'(\vec{\rho}) = B_2 I_1(B_2 \lambda) \cos B_z z$$

$$B_1^2 = \mu^2 - B_z^2 \qquad B_2^2 = \lambda^2 + B_z^2 \qquad B_z = \frac{\pi}{2R + 2\Delta z}$$

$$U(\vec{\rho}) = K_0(B_3 \lambda) \cos B_z z \qquad U'(\vec{\rho}) = -B_3 K_1(B_3 \lambda) \cos B_z z \tag{C.7}$$

$$V(\vec{\rho}) = K_0(B_4 \lambda) \cos B_z z \qquad V'(\vec{\rho}) = -B_4 K_1(B_4 \lambda) \cos B_z z$$

$$B_3^2 = \varkappa_3^2 + B_z^2 \qquad B_4^2 = \varkappa_4^2 + B_z^2$$

By applying the boundary conditions to these solutions we get the criticality condition:

$$\Delta = \begin{vmatrix} Z & W & -U & 0 \\ -D_1 Z' & -D_1 W & D_3 U' & 0 \\ a_1 Z & a_2 W & a_3 U & -V \\ -a_1 D_2 Z & -a_2 D_2 W & -a_3 D_4 U & D_4 V \end{vmatrix} = 0$$

where the functions Z, W, U, V and their derivatives are evaluated at the boundary of the core.

Defining:

$$\varsigma \equiv \frac{Z'}{Z} \qquad \omega \equiv \frac{W'}{W} \qquad \psi \equiv \frac{U'}{U} \qquad \beta \equiv \frac{V'}{V} \tag{C.8}$$

the criticality condition becomes: $\varsigma = \frac{N}{D}$

where

$$\frac{N}{D} \equiv \frac{\omega\,\psi(a_3 D_1 D_4 + a_2 D_2 D_3) - D_1 D_4 \omega (a_1 + a_3)\beta + D_3 D_4 \psi (a_1 - a_2)\beta}{D_1 D_2 \omega (a_2 - a_1) - D_1 D_4 \beta (a_2 + a_3) + \psi (a_1 D_2 D_3 + a_3 D_1 D_4)} \tag{C.9}$$

We will calculate ς and the value of the quotient $\frac{N}{D}$ for various values of fuel loading. After trial and error, we select 1.2, 1.3 and 1.4 kg and see where the criticality condition is satisfied.

2) <u>Data</u>

a) Densities (Ref. 3, p. 46)

uranium $d = 18.7$ g/cm^3

beryllium $d = 1.85$ g/cm^3

b) Cross Sections (Ref. 3, p. 46)

The thermal cross sections have been selected at energy corresponding to 2,200 m/s and no correction has been made for averaging over thermal distribution.

The fast cross sections have been obtained by averaging over the lethargy range corresponding to energies $1\,MeV$ and $1\,eV$ the cross sections given in Reference 7.

Thermal Cross Sections in barns

	σ_a	σ_s	σ_f
Beryllium	0.01	7	0
Uranium 235	687	10	580
Uranium 238	2.75	8.2	0

Fast cross sections in barns:

beryllium $\sigma_s^{(1)}$ = 6 barns

uranium $\sigma_f^{(1)}$ = 13.9 barns

The age to thermal is assumed to be the same as in pure moderator. According to Reference 3, p. 415, we choose:

$$\tau_{th} = 97.2\ \mathrm{cm}^2$$

Average number of neutrons produced per fission $\nu = 2.47$.

e) Geometrical data

Radius of the core	R_o =	50 cm
Height of the core	2h =	100 cm
Inside radius of one element	r =	2.0 cm
Outside radius of one element	ρ =	2.5 cm
Total volume of the core	V =	7.854×10^5 cm^3
Volume of one cell	v =	3.1043×10^3 cm^3
Radius of the equivalent cell	a =	3.143 cm
Volume of beryllium in one cell	v' =	2.3975×10^3 cm^3
Equivalent density of beryllium	d' =	1.4287 g/cm^3

3) Resonance Escape Probability p

Due to the fact that the fuel is fully enriched uranium in small concentration, the resonance escape probability is close to unity. Hence, we can use an approximation for the resonance integral:

$$I = \frac{V_F N_F}{\xi \Sigma_s V} \int_0^{u_{th}} \sigma_a(u)\, du$$

By integrating $\sigma_a(u)$ graphically over the lethargy range, we get I = 0.0302 for a fuel mass of 1.3 kg . Therefore, the corresponding value for the resonance escape probability is p = 0.970.

We will keep this value constant for all the values of fuel mass. This is not a good approximation, but satisfactory for the present purpose.

4) Removal Cross Section and Fast Diffusion Coefficients

We use the approximation

$$\Sigma_R = \frac{\xi \Sigma_s}{\mu_{tA}}$$

in order to calculate Σ_R. Then the fast diffusion coefficient may be obtained by the relationship

$$\Sigma_R = \frac{D_1}{\tau_{tA}}$$

The approximation is reasonably good for the low fuel concentrations used here.

$$\xi = 1 + \frac{\alpha \ln \alpha}{1-\alpha} = 1 - \frac{0.44628}{0.5625} = .20662$$

$$\frac{\xi}{\mu_{tA}} = \frac{0.20662}{17.4998} = \frac{1}{84.7} \quad \text{(Assume u = 0 at E = 1 Mev.)}$$

$$\Sigma_s^{(1)} = \frac{1.4287 \times 0.602 \times 6}{9} = 0.573405 \text{ cm}^{-1}$$

$$\Sigma_R^{(1)} = \frac{0.5734}{84.7} = 0.0067698 \text{ cm}^{-1}$$

$$\tau'_{tA} = \tau_{tA} \left(\frac{v}{v'}\right)^2$$

$$= 97.2 \left(\frac{31.04}{23.97}\right)^2 = 162.99 \text{ cm}^2$$

$$D_1 = 162.99 \times .0067698 = 1.034 \text{ cm}$$

5) Beryllium Macroscopic Cross Sections

$$\Sigma_a = \frac{1.4287 \times 0.602 \times 0.01}{9} = 9.5517 \times 10^{-4} \text{ cm}^{-1}$$

$$\Sigma_s = \frac{1.4287 \times 0.602 \times 7}{9} = 0.66862 \text{ cm}^{-1}$$

$$D_2 = \frac{1}{3 \Sigma_s (1 - \bar{\mu}_0)}$$

$$= \frac{1}{3 \times 0.66862 \times 0.9259} = 0.5383 \text{ cm}$$

6) Thermal Cross Sections

Mass of Uranium $M\,kg$	1.2	1.3	1.4
Equivalent Density of $U = \frac{M}{V}\ g/cm^3$	0.0015279	0.0016552	0.0017825
$\Sigma_a^{(2)}$ fuel only cm^{-1}	0.00250652	0.00271535	0.00292419
$\Sigma_s^{(2)}$ fuel only cm^{-1}	0.0000386	0.0000418	0.0000450
$\Sigma_f^{(2)}$ • cm^{-1}	0.00211568	0.00229195	0.00246823
$\Sigma_a^{(2)}$ cm^{-1}	0.00346169	0.00367052	0.00387936
$\Sigma_s^{(2)}$ cm^{-1}	0.66866	0.66866	0.66866
$\Sigma_f^{(1)}$ cm^{-1}	0.0000544	0.0000589	0.0000634

The total absorption cross section has been obtained here by adding the cross section of the fuel and the cross section of the moderator. This can be done because according to Appendix D the disadvantage factor is very close to unity.

7) Calculation of μ^2 and λ^2

$$2\mu^2 = -(\varkappa_2^2 + \chi_1^2) + \theta$$

$$-2\lambda^2 = -(\varkappa_2^2 + \chi_1^2) - \theta$$

$$\theta = \left\{(\varkappa_2^2 + \chi_1^2)^2 + 4\varkappa_1^2\varkappa_2^2\left(k_\infty^{(2)} - 1 + k_\infty^{(1)}\right)\right\}^{1/2}$$

Mass of Uranium kg	1.2	1.3	1.4
$\varkappa_1^2 = \frac{\Sigma_\ell^{(1)}}{D_1}$ cm^{-2}	0.00613540	0.00613540	0.00613540
$\varkappa_2^2 = \frac{\Sigma_a^{(2)}}{D_2}$ cm^{-2}	0.00643078	0.00681872	0.00720668
$k_\infty^{(1)} = \nu \frac{\Sigma_f^{(1)}}{\Sigma_\ell^{(1)}}$	0.019848	0.021489	0.023131
$1 - k_\infty^{(1)}$	0.980152	0.978511	0.976869
$\chi_1^2 = \varkappa_1^2(1 - k_\infty^{(1)})$ cm^{-2}	0.00601362	0.00600355	0.00599348
$k_\infty^{(2)} = \nu p \frac{\Sigma_f^{(2)}}{\Sigma_a^{(2)}}$	1.46430	1.49605	1.52438
$\varkappa_2^2 + \chi_1^2$ cm^{-2}	0.0124444	0.0128223	0.0132001
$(\varkappa_2^2 + \chi_1^2)^2$ cm^{-4}	0.000154863	0.000164411	0.000174242
$\varkappa_1^2 \varkappa_2^2$ cm^{-4}	0.0000394554	0.0000418355	0.0000442158
$k_\infty^{(2)} - 1 + k_\infty^{(1)}$	0.48415	0.51754	0.54751
$4\varkappa_1^2\varkappa_2^2(k_\infty^{(2)} - 1 + k_\infty^{(1)})$ cm^{-4}	0.0000764093	0.0000866062	0.0000968344
θ^2 cm^{-4}	0.000231272	0.000251017	0.000271076
θ cm^{-2}	0.0152077	0.0158435	0.0164643
$\theta - (\varkappa_2^2 + \chi_1^2)$ cm^{-2}	0.0027633	0.0030212	0.0032642
$\theta + (\varkappa_2^2 + \chi_1^2)$ cm^{-2}	0.0276521	0.0286658	0.0296644
μ^2 cm^{-2}	0.0013816	0.0015106	0.0016321
λ^2 cm^{-2}	0.0138260	0.0143330	0.0148322

8) **Calculation of a_1 and a_2**

$$a_1 = \frac{p\,\Sigma_R^{(1)}}{D_2\mu^2 + \Sigma_a^{(2)}} \qquad a_2 = \frac{p\,\Sigma_R^{(1)}}{-D_2\lambda^2 + \Sigma_a^{(2)}}$$

p	0.970		
$\Sigma_R^{(1)}$ cm^{-1}	0.0067698		
D_2 cm	0.5382		
Mass of Uranium kg.	1.2	1.3	1.4
$\Sigma_a^{(2)}$ cm^{-1}	0.00346169	0.00367052	0.00387936
μ^2 cm^{-2}	0.0013816	0.0015106	0.0016321
λ^2 cm^{-2}	0.0138260	0.0143330	0.0148322
$D_2\mu^2 + \Sigma_a^{(2)}$ cm^{-1}	0.00420540	0.00448367	0.00475792
$D_2\lambda^2 - \Sigma_a^{(2)}$ cm^{-1}	0.00398084	0.00404488	0.00410481
a_1	1.56149	1.46458	1.38016
a_2	1.64957	1.62346	1.59975

9) <u>Calculation of a_3</u>

$$a_3 = \frac{\Sigma_R^{(3)}}{D_4\,(\varkappa_3^2 - \varkappa_4^2)}$$

$$\Sigma_R^{(3)} = \frac{\Sigma_s^{(3)}}{84.7} = \frac{1.85 \times 0.602 \times 6}{84.7 \times 9} = 0.0087658 \text{ cm}^{-1}$$

$$D_3 = \Sigma_R^{(3)}\,\tau_{tR} = 97.2 \times 0.0087658 = 0.852 \text{ cm}$$

$$D_4 = \frac{1}{3\,\Sigma_s^{(4)}(1-\bar{\mu}_0)} = 0.4156 \text{ cm}$$

$$\varkappa_3^2 = 0.010288 \text{ cm}^{-2}$$

$$\varkappa_4^2 = 0.0029772 \text{ cm}^{-2}$$

$$\therefore\ a_3 = 2.8848$$

10) <u>Calculation of the Axial Buckling B_z</u>

For an infinite reflector the savings are approximately (Reference 8 - 15.14):

$$\Delta_z \simeq \frac{D_3}{D_4\,\varkappa_4} = 23.5 \text{ cm}$$

hence $2R + 2\Delta_z = \cdot 147$ cm.

$$B_z^2 = \left(\frac{\pi}{147}\right)^2 = 0.00045672 \text{ cm}^{-2}$$

11) <u>Calculation of B_1 and B_2</u>

$$B_1^2 = \mu^2 - B_z^2$$

$$B_2^2 = \lambda^2 + B_z^2$$

Mass of Uranium Kg	1.2	1.3	1.4
μ^2 cm^{-2}	0.0013816	0.0015106	0.0016321
λ^2 cm^{-2}	0.0138260	0.0143330	0.0148322
B_1^2 cm^{-2}	0.0009249	0.0010539	0.0011754
B_2^2 cm^{-2}	0.0142827	0.0147896	0.0152889
B_1 cm^{-1}	0.030412	0.032465	0.034282
B_2 cm^{-1}	0.11951	0.12161	0.12365

12) <u>Calculation of B_3 and B_4</u>

$B_3^2 = 0.010288 + 0.0004567 = 0.0107447\ cm^{-2}$

$B_4^2 = 0.0027772 + 0.0004567 = 0.0033339\ cm^{-2}$

$B_3 = 0.10366\ cm^{-1}$

$B_4 = 0.05860\ cm^{-1}$

13) Calculation of ξ and ω

R_o = 50 cm Radius of the core

Mass of Uranium kg	1.2	1.3	1.4
B_1 cm^{-1}	0.030412	0.032465	0.034282
B_2 cm^{-1}	0.11951	0.12161	0.12365
$B_1 R_o$	1.5206	1.6232	1.7141
$B_2 R_o$	5.9755	6.0805	6.1825
$J_o(B_1 R_o)$	0.5000	0.4422	0.3898
$J_1(B_1 R_o)$	0.5606	0.5721	0.5785
$I_o(B_2 R_o)$	65.75	72.36	79.43
$I_1(B_2 R_o)$	59.96	66.11	72.69
$Z(R_o)$	0.5000	0.4422	0.3898
$-Z'(R_o)$	0.017049	0.018573	0.019832
$W(R_o)$	65.75	72.36	79.43
$W'(R_o)$	7.166	8.039	8.988
$-\xi$	0.034098	0.042001	0.050877
ω	0.10898	0.11109	0.11315

14) <u>Calculation of ψ and β</u>

$$B_3 = 0.10366 \text{ cm}^{-1} \qquad B_3R_o = 5.183$$

$$B_4 = 0.05860 \text{ cm}^{-1} \qquad B_4R_o = 2.930$$

$$K_o(B_3R_o) = 0.003020 \qquad K_o(B_4R_o) = 0.03765$$

$$K_1(B_3R_o) = 0.003298 \qquad K_1(B_4R_o) = 0.04366$$

$$U(R_o) = 0.003020 \qquad V(R_o) = 0.03765$$

$$U'(R_o) = -0.00034187 \qquad V'(R_o) = -0.002558$$

$$\psi = -0.11320 \qquad \beta = -0.067941$$

15) <u>Calculation of N/D</u>

$$\frac{N}{D} = \frac{\omega\psi(a_3D_1D_4 + a_2D_2D_3) - D_1D_4\omega\beta(a_1+a_3) + D_3D_4\psi\beta(a_1 - a_2)}{D_1D_2\omega(a_2 - a_1) - D_1D_4\beta(a_2+a_3) + \psi(a_1D_2D_3 + a_3D_1D_4)}$$

$D_1 = 1.034$ cm

$D_2 = 0.5383$ cm

$D_3 = 0.8520$ cm

$D_4 = 0.4156$ cm

Mass of Uranium kg	1.2	1.3	1.4
$a_3 D_1 D_4$ cm^2	1.3229		
$-a_2 D_2 D_3$ cm^2	0.75655	0.74454	0.73367
$-\omega\psi$	0.012336	0.012575	0.012808
$-\omega\psi(a_3 D_1 D_4 + a_2 D_2 D_3) cm^2$	0.0069865	0.0072728	0.0075468
$-D_1 D_4 \omega\beta$ cm^2	0.00339537	0.00346111	0.00352529
$a_1 + a_3$	4.4463	4.3494	4.2649
$-D_1 D_4 \omega\beta (a_1 + a_3) cm^2$	0.0150968	0.0150537	0.0150350
$D_3 D_4 \psi\beta (a_1 - a_2) cm^2$	0.0087447	0.0084095	0.0081148
N cm^2	0.016855	0.0161904	0.015603
$D_1 D_2 \omega$ cm^2	0.064729	0.065983	0.067206
$-D_1 D_2 \omega (a_2 - a_1) cm^2$	0.20785	0.20375	0.20026
$-D_1 D_4 \beta (a_2 + a_3) cm^2$	0.038484	0.039300	0.040038
$a_1 D_2 D_3$ cm^2	0.71615	0.67171	0.63296
$-\psi(a_1 D_2 D_3 + a_3 D_1 D_4) cm^2$	0.23082	0.22579	0.22140
$-D$ cm^2	0.40019	0.39024	0.38162
$-N/D$	0.042117	0.041488	0.040886

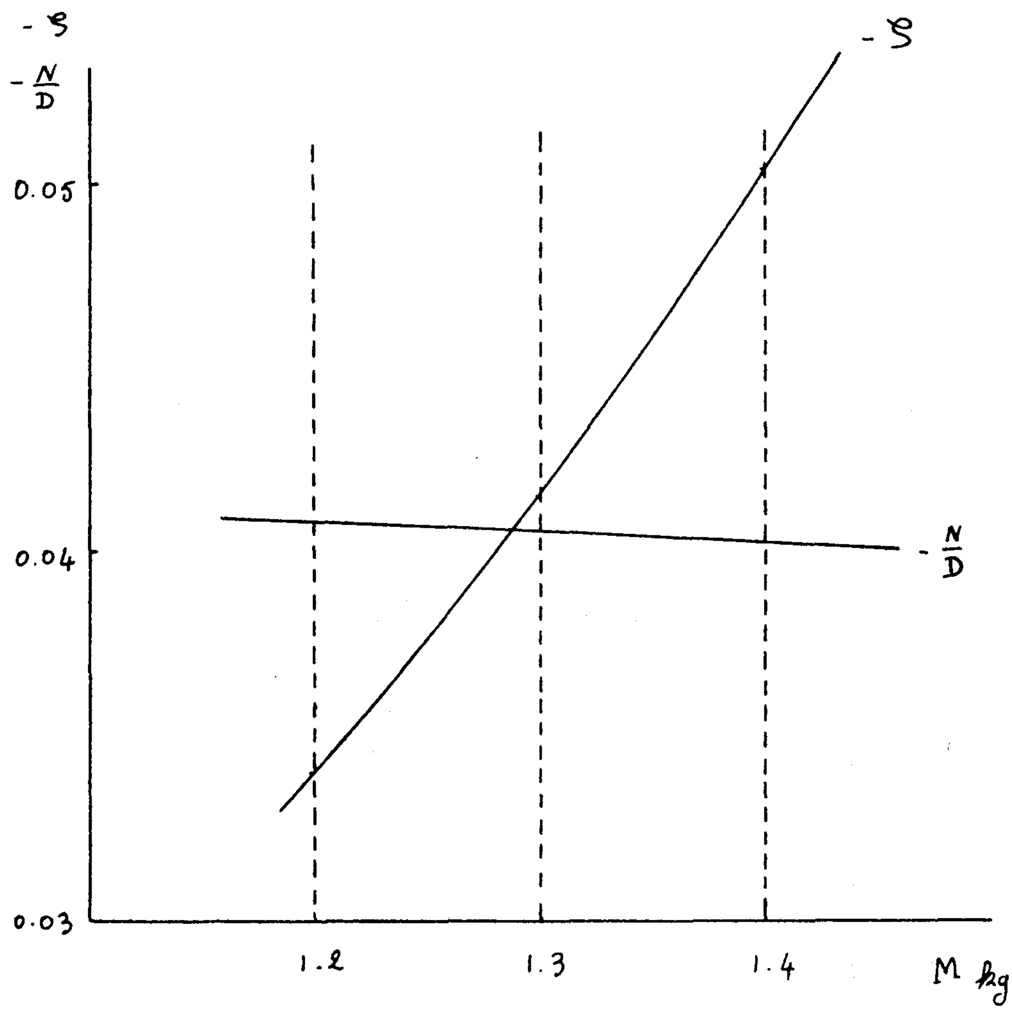


Fig. 4.2

Within the limits of accuracy of the method we can conclude the critical mass is $M_c = 1.3$ kg .

It should be noticed that this homogeneous calculation is not carried out in complete detail . For instance, the presence of holes in the reactor is taken into account by its influence on the average density of the materials. Behrens (9) has proposed a better way of taking such holes into account by correcting the diffusion length. We want to show here, that for our present purpose the importance of this correction is small.

First we notice that a plain density correction gives the following results for the reactor described above:

$$\frac{L}{L_0} = 1.228$$

where L is the diffusion length of the actual medium with holes and L_0 the diffusion length of the medium without holes.

The Behrens correction gives:

$$\frac{L}{L_0} = 1.268$$

the difference between the two corrections is about 3.2%.

We perform now a rapid calculation to see what is the influence on the critical mass of such a difference.

In a two group theory criticality is obtained when:

$$\frac{\eta \varepsilon p f}{(1+L^2B^2)(1+\tau_{th}B^2)} = 1$$

in this expression L and $\sqrt{\tau_{th}}$ are affected by a change of 3.2%, hence, the critical mass will be affected, too. A small change in the fuel loading affects mainly the factor f :

$$\frac{\eta \varepsilon p \dfrac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^M}}{(1+L^2B^2)(1+\tau_{th}B^2)} = 1$$

A straightforward calculation gives:

$$\frac{\Delta M}{M} = \frac{\Delta \Sigma_a^F}{\Sigma_a^F} = \frac{\Delta L}{L}\,\frac{2B^2}{\eta \varepsilon p}\left(L^2 + \tau_{th} + 2L^2\tau_{th}B^2\right)\frac{(\Sigma_a^F + \Sigma_a^M)^2}{\Sigma_a^F \Sigma_a^M}$$

where we have made $\dfrac{\Delta\sqrt{\tau_{th}}}{\sqrt{\tau_{th}}} = \dfrac{\Delta L}{L}$

Using the above numerical results we find that:

$$\frac{\Delta M}{M} = 0.59\ \frac{\Delta L}{L}$$

which gives a difference of 1.9% in the fuel loading deduced above. This correction does not affect the usefulness of our present homogeneous calculation whose goal is to give a first estimate of the critical mass.

APPENDIX D

THE DISADVANTAGE FACTOR

In the reactor described in section V, the fuel is contained in thin shells. Each of these shells gives rise to an equivalent cell as defined in homogeneous theories. We shall show here that the disadvantage factor in such cells is very close to unity.

$$\zeta = \frac{\overline{\phi_M}}{\overline{\phi_F}} \simeq 1$$

We first consider a cell where no vacuum gap exists (see fig. D.1). We use diffusion theory in the moderator(*) and take into account the fuel by means of transmission coefficients. We assume there is a uniform source density S_o of thermal neutrons in the moderator. The flux in the moderator is solution of the equation:

$$-D_2 \nabla^2 \phi(r) + \Sigma_a^M \phi(r) = S_o$$

We choose the origin at the center of the cell. Using the condition that the flux must be finite, the solution is:

$$\phi_I = \frac{S_o}{\Sigma_a^M}\left[1 + A\, I_o(\varkappa r)\right]$$

$$\phi_{II} = \frac{S_o}{\Sigma_a^M}\left[1 + B\, K_o(\varkappa r) + C\, I_o(\varkappa r)\right]$$

*Ref. 3, p. 648.

where regions I and II are defined according to (fig. D.1)

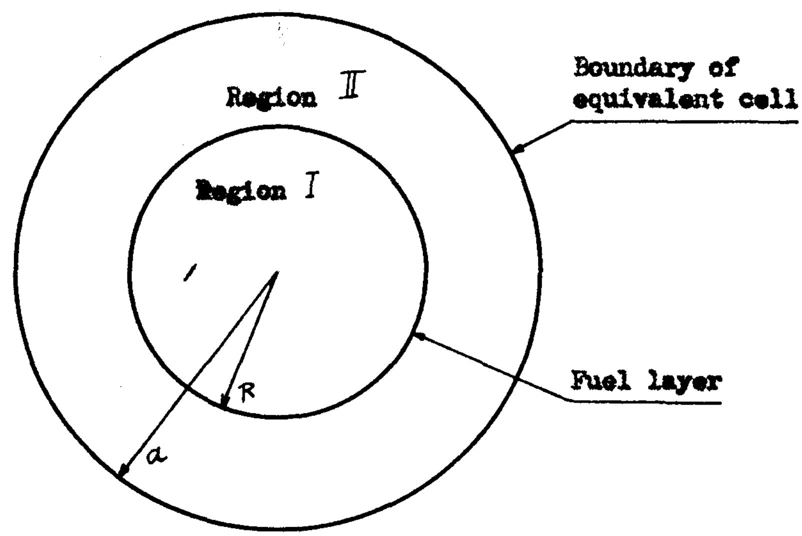


Fig. D.1

We find the constants A , B and C by using the boundary conditions:

1) At the outside boundary of the equivalent cell the net current is zero.

ii) At the fuel shell location, the partial currents are related by:

$$J_I^-(R) = t^- J_{II}^-(R)$$

$$J_{II}^+(R) = t^+ J_I^+(R)$$

where

t^+ = transmission coefficient of the fuel shell from inside to outside.

t^- = transmission coefficient of the fuel shell from outside to inside.

We note that

$t^+ = P_1$ (see equation 4.1)

and

$t^- = \pi_2$ (see equation 4.15)

Applying the first boundary condition gives:

$$B = C \frac{I_1(\mathcal{X} a)}{K_1(\mathcal{X} a)} \equiv C y$$

The second condition gives the coefficients A and C :

$$A = -\frac{(1-t^-)(\gamma K_0 + I_0 + 2D\gamma\varkappa K_1 - 2D\varkappa I_1) + t^-(1-t^+)(\gamma K_0 + I_0 - 2D\gamma\varkappa K_1 + 2D\varkappa I_1)}{\Delta}$$

$$C = -\frac{(1-t^+)(I_0 + 2D\varkappa I_1) + t^+(1-t^-)(I_0 - 2D\varkappa I_1)}{\Delta}$$

where:

$$\Delta = (I_0 + 2D\varkappa I_1)(\gamma K_0 + I_0 + 2D\gamma\varkappa K_1 - 2D\varkappa I_1)$$
$$- t^- t^+ (I_0 - 2D\varkappa I_1)(\gamma K_0 + I_0 - 2D\gamma\varkappa K_1 + 2D\varkappa I_1)$$

and

$$I_0 \equiv I_0(\varkappa R) \qquad I_1 \equiv I_1(\varkappa R)$$
$$K_0 \equiv K_0(\varkappa R) \quad , \quad K_1 \equiv K_1(\varkappa R)$$

This enables us to calculate $\bar{\phi}_M$:

$$\bar{\phi}_M = 2\pi \cdot \frac{\int_0^R \phi_I(r)\, r\, dr + \int_R^a \phi_{II}(r)\, r\, dr}{\pi a^2}$$

$$\bar{\phi}_M = \frac{S_0}{\Sigma_a^M}\left\{ 1 + \frac{2}{\varkappa a^2}\left[A R I_1(\varkappa R) - C\gamma a K_1(\varkappa a) + C a I_1(\varkappa a) + C\gamma R K_1(\varkappa R) - C R I_1(\varkappa R) \right] \right\}$$

To calculate $\overline{\phi}_F$, we assume that, because the fuel shell is very thin, the flux inside the fuel is linear, so that:

$$\overline{\phi}_F \simeq \frac{\phi_I(R) + \phi_{II}(R)}{2}$$

We apply this to the cell which has the following characteristics (see Appendix C, section 2.c):

$$R = 2.0 \text{ cm}$$

$$a = 3.143 \text{ cm}$$

For a thickness of fuel shell of τ = 1.8 microns we obtained the values of P_1 and π_2 from equations 4.1 and 4.15 in the four coefficient calculations. Hence:

$$1 - t^- = 0.012$$

$$1 - t^+ = 0.020$$

Substituting these values in the above equations and using $\varkappa$ = 0.0586 cm^{-1}

$$\Delta = 0.859$$
$$\gamma = 0.0176$$
$$A = -0.0436$$
$$B = -0.000672$$

$$C = -\,0.0382$$

$$\therefore \quad \phi_1(R) = 0.956$$

$$\phi_{II}(R) = 0.960$$

$$\therefore \quad \bar{\phi}_F = \frac{\phi_1(R) + \phi_2(R)}{2} = 0.958$$

also

$$\bar{\phi}_M = 0.969$$

$$\therefore \quad S = \frac{\bar{\phi}_M}{\bar{\phi}_F} = 1.011$$

This small deviation from unity indicates that the flux depression in the fuel shell is very small. Therefore, if we replace an annulus of moderator outside the fuel shell by a vacuum, the flux depression will also be small. The disadvantage factor should then be close to the value calculated above.

Symbols

B_z = longitudinal buckling $\quad B_z = \dfrac{\pi}{2h + 2\Delta_z}$

b = radius of a fuel element

D_1, D_2 = fast and thermal diffusion constants

h = half length of a fuel element

$K_n(x)$ = modified Bessel function of the second kind and of the n^{th} order

M = fuel loading in kg

N = total number of fuel elements

N_1 = average number of collisions to make a fission neutron thermal

p = resonance escape probability

r = radius of the inner moderator cylinder of a fission-electric cell element

R = outside radius of the fuel layer of a fission-electric cell element

$\bar{r}$ = two dimensional vector: space variable in transverse flux equations after removal of the z dependancy

$\bar{\rho}$ = three dimensional vector: space variable in overall flux equations

ρ = outside radius of the vacuum gap of a fission-electric cell element

$S_k^{(1)}, S_k^{(2)}$ = fast and thermal sink strength of the singularity

t = transmission coefficient of a slug

α = $\sin^{-1} \frac{\lambda}{R}$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ = four coefficients connecting sink strengths to fluxes (section III)

γ = thermal constant

Δ_z = reflector savings on one end along the axis

ε = scale factor

κ = transverse component of the inverse of diffusion length

$\varkappa$ = inverse of the diffusion length

η_1 = average number of neutrons produced per fast absorption in the fuel element

η_2 = average number of neutrons produced per thermal absorption in the fuel element

ξ = probability that a neutron after colliding in the fuel shell escapes without making an extra collision

λ = probability that a neutron going through a fuel shell does not collide

ν = average number of neutrons produced per fission

$\Sigma_a^{(1)}, \Sigma_a^{(2)}$ = fast and thermal absorption cross sections of the moderator

Σ_R = removal cross section of the moderator

$\phi_1(\vec{r}) =$ fast flux at point $\vec{r}$

$\phi_2(\vec{r}) =$ thermal flux at point $\vec{r}$

$\phi_1(\vec{r}_s), \phi_2(\vec{r}_s) =$ flux at the surface of fuel element

$\omega =$ ratio of the scattering and total cross section in a given material

$\tau =$ thickness of the fuel layer in a fuel element

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