Some constructions, related to noncommutative tori; Fredholm modules and the Beilinson–Bloch regulator

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2015

(Defended May 6, 2015)

Acknowledgments

I would like to thank my advisor, Prof. Matilde Marcolli, for her guidance and expert advice throughout all of my work at Caltech. I wish to express my sincere thanks to Ilya Kachkovskiy for multiple discussions throughout my PhD at Caltech, and for hints on the preparation of the text of this thesis. I am grateful to Eugene Ha for multiple discussions of the topic of Chapter 3 and helping to correct inaccuracies in the draft version of it, and to Jan Jitse for multiple discussions of the topic of Chapter 1. I place on record, my sincere thank you to my wife, Valeria Kasatkina, for the continuous encouragement, love, and emotional support.

Abstract

A noncommutative 2-torus is one of the main toy models of noncommutative geometry, and a noncommutative n-torus is a straightforward generalization of it. In 1980, Pimsner and Voiculescu in [17] described a 6-term exact sequence, which allows for the computation of the K-theory of non-commutative tori. It follows that both even and odd K-groups of n-dimensional noncommutative tori are free abelian groups on 2^{n-1} generators. In 1981, the Powers–Rieffel projector was described [19], which, together with the class of identity, generates the even K-theory of non-commutative 2-tori. In 1984, Elliott [10] computed trace and Chern character on these K-groups. According to Rieffel [20], the odd K-theory of a noncommutative n-torus coincides with the group of connected components of the elements of the algebra. In particular, generators of K-theory can be chosen to be invertible elements of the algebra. In Chapter 1, we derive an explicit formula for the first non-trivial generator of the odd K-theory of noncommutative tori. This gives the full set of generators for the odd K-theory of noncommutative 3-tori and 4-tori.

In Chapter 2, we apply the graded-commutative framework of differential geometry to the polynomial subalgebra of the noncommutative torus algebra. We use the framework of differential geometry described in [27], [14], [25], [26]. In order to apply this framework to noncommutative torus, the notion of the graded-commutative algebra has to be generalized: the "signs" should be allowed to take values in $U(1)$, rather than just $\{-1, 1\}$. Such generalization is well-known (see, e.g., [8] in the context of linear algebra). We reformulate relevant results of [27], [14], [25], [26] using this extended notion of sign. We show how this framework can be used to construct differential operators, differential forms, and jet spaces on noncommutative tori. Then,

we compare the constructed differential forms to the ones, obtained from the spectral triple of the noncommutative torus. Sections 2.1–2.3 recall the basic notions from [27], [14], [25], [26], with the signs $(-1)^{\bullet\cdot\bullet}$ replaced with $\lambda(\bullet, \bullet)$. In Section 2.4, we apply these notions to the polynomial subalgebra of the noncommutative torus algebra. This polynomial subalgebra is similar to a free graded-commutative algebra. We show that, when restricted to the polynomial subalgebra, Connes construction of differential forms gives the same answer as the one obtained from the graded-commutative differential geometry. One may try to extend these notions to the smooth noncommutative torus algebra, but this was not done in this work.

A reconstruction of the Beilinson–Bloch regulator (for curves) via Fredholm modules was given by Eugene Ha in [12]. However, the proof in [12] contains a critical gap; in Chapter 3, we close this gap. More specifically, we do this by obtaining some technical results, and by proving Property 4 of Section 3.7 (see Theorem 3.9.4), which implies that such reformulation is, indeed, possible. The main motivation for this reformulation is the longer-term goal of finding possible analogs of K_2 and of the regulators for noncommutative spaces. This work should be seen as a necessary preliminary step for that purpose.

For the convenience of the reader, we also give a short description of the results from [12], as well as some background material on central extensions and Connes–Karoubi character.

Contents

Introduction

For any positive integer n, the noncommutative n-tori C^*-algebras $A_\theta^{(n)}$ are a family of C^*-algebras which generalize the algebra $C(\mathbb{T}^n, \mathbb{C})$ of continuous functions on n-torus. The algebra $A_\theta^{(n)}$ is defined as the universal C^*-algebra generated by n unitary generators $U_1, U_2, \ldots, U_n$ subject to relations $U_l U_j = e^{2\pi i \theta_{lj}} U_j U_l$. Here, θ is an $n \times n$ antisymmetric matrix with elements in $\mathbb{R}$. When $\theta = 0$, we have $A_\theta^{(n)} \simeq C(\mathbb{T}^n, \mathbb{C})$. The algebra $A_\theta^{(n)}$ only depends on fractional parts of the elements of the matrix θ. One can define a noncommutative analogue of the (normalized) integral $\int : C(\mathbb{T}^n, \mathbb{C}) \to \mathbb{C}$. This is a specific map $\tau \colon A_\theta^{(n)} \to \mathbb{C}$, satisfying $\tau(1) = 1$, and $\tau(ab) = \tau(ba)$. This map can be extended to the map $\tau \colon M_m\big(A_\theta^{(n)}\big) \to \mathbb{C}$ with $\tau(a) = \tau(\operatorname{Tr}(a))$. Here, $M_m(A)$ is the algebra of $m \times m$ matrices over an algebra A. The family of noncommutative tori is the most widely studied class of noncommutative spaces: see, e.g., [9], [19], [11].

This work consists of 3 independent chapters, related to the notion of noncommutative torus. We will now describe these chapters.

Chapter 1

K-theory of C^*-algebras associates two abelian groups, $K_0(A)$ and $K_1(A)$, to every C^*-algebra A. These can be seen as invariants of the algebra A. These groups may be used to distinguish one algebra from another. The group $K_0(A)$ is defined in terms of equivalence classes $[p]_0$ of orthogonal projections $p \in \bigcup_{m=1}^{\infty} M_m(A)$. The group $K_1(A)$ consists of classes $[a]_1$ of invertible elements $a \in \bigcup_{m=1}^{\infty} \mathrm{GL}_m(A)$, where $\mathrm{GL}_m(A) = \operatorname{Inv}(M_m(A))$ is the group of invertible $m \times m$ matrices with elements in A. These functors, K_0 and K_1, are analogous to corresponding functors of the topological

K-theory: when $A = C(X, \mathbb{C})$ is the algebra of continuous functions on a compact Hausdorff space X, we have isomorphisms $K_0(A) \simeq K^0(X)$ and $K_1(A) \simeq K^1(X) \simeq K^0(SX)$, where $SX \simeq [0,1] \times X/(0,x) \sim (0,y), (1,x) \sim (1,y)$ is the suspension of X.

For the case of noncommutative tori, these groups were computed by Pimsner and Voiculescu in [17]: $K_0(A_\theta^{(n)}) \simeq K_1(A_\theta^{(n)}) \simeq \mathbb{Z}^{2^{n-1}}$.

In [6], [5] (see also the translation [4]), Connes introduced the map $\mathrm{Ch}\colon K_0(A_\theta^{(n)}) \oplus K_1(A_\theta^{(n)}) \to \mathbb{R} \otimes \Lambda G$, where $G \simeq \mathbb{Z}^n$. Its 0th component $\mathrm{Ch}^0\colon K_0(A_\theta^{(n)}) \to \mathbb{R}$ is a noncommutative analogue of dimension of a vector bundle: in general $\mathrm{Ch}^0([p]_0) = \tau(p)$, and for $\theta = 0$ one has $\mathrm{Ch}^0([p]_0) = \mathrm{Tr}(p(x))$ for every $x \in \mathbb{T}^n$, when p is interpreted as a projector-valued function in $C\left(\mathbb{T}^n, M_m(A_\theta^{(n)})\right)$. The map Ch is called Chern character or Chern–Connes character.

Unlike the commutative case, $\mathrm{Ch}^0([p]_0) = \tau(p)$ is not always an integer. In particular, for $\theta_{12} \in (0,1)$ Rieffel and Powers [19] constructed a projector $P_{\theta_{12}} \in A_\theta^{(2)}$ such that $\tau(P_{\theta_{12}}) = \theta_{12}$, and the classes $[1]_0, [P_{\theta_{12}}]_0$ generate $K_0(A_\theta^{(2)})$.

Elliott [10] described an isomorphism $K_0(A_\theta^{(n)}) \oplus K_1(A_\theta^{(n)}) \simeq \Lambda G$, where $G \simeq \mathbb{Z}^n$. Under this isomorphism $\Lambda^{\mathrm{even}}(G)$ corresponds to $K_0(A_\theta^{(n)})$, and $\Lambda^{\mathrm{odd}}(G)$ corresponds to $K_1(A_\theta^{(n)})$. Elliott also described an explicit formula for the Chern–Connes character under this isomorphism. Under the Elliott's isomorphism $1 \in \Lambda^0 G$ corresponds to $[1]_0 \in K_0(A_\theta^{(n)})$, $e_l \in \Lambda^1 G$ corresponds to $[U_l]_1 \in K_1(A_\theta^{(n)})$, $e_l \wedge e_j \in \Lambda^2 G$ corresponds to $[P_{\theta_{lj}}]_0 \in K_0(A_\theta^{(n)})$, and $e_l \wedge e_j \wedge e_k \in \Lambda^3 G$ corresponds to $[a_{ljk}]_1 \in K_1(A_\theta^{(n)})$, where unitary $a_{ljk} \in A_\theta^{(n)}$ is explicitly described by Chapter 1 of this work.

Chapter 1 of this work is devoted to finding an explicit formula for a unitary $a \in A_\theta^{(3)}$ such that classes $[U_1]_1, [U_2]_1, [U_3]_1, [a]_1$ generate $K_1(A_\theta^{(3)}) \simeq \mathbb{Z}^4$. In Section 1.2, we find one formula for such unitary a. This formula is written in terms of Araki expansionals [1]. The approach we use relies on using the 6-term exact sequence described by Pimsner and Voiculescu in [17], and is similar to the one used by Rieffel in [20, proof of 8.2]. In Section 1.3, we use an ansatz, similar to the one used by Rieffel and Powers for describing the projector $P_{\theta_{12}}$. This method gives a much simpler formula for such unitary a. Note that unitaries, produced by Sections 1.2 and 1.3, may differ from each other. They, however, generate the same class in K-theory.

As a longer-term goal, one would like to have some explicit understanding of all the generators of the odd K-theory of all the noncommutative tori.

Chapter 2

The noncommutative torus algebra $A_\theta^{(n)}$ has subalgebras $\mathcal{A}_\theta^{(n)}$ and $\mathrm{A}_\theta^{\mathrm{(n),poly}}$. The algebra $\mathcal{A}_\theta^{(n)}$ is analogous to the algebra $C^\infty(\mathbb{T}^n, \mathbb{C})$ of smooth functions in the commutative case. The algebra $\mathrm{A}_\theta^{\mathrm{(n),poly}}$ consists of polynomials in $U_1, \dots, U_n$ with complex coefficients.

Given an abelian group Γ with a bilinear antisymmetric map $(-1)^{\bullet \cdot \bullet}\colon \Gamma \times \Gamma \to \{-1, 1\}$, we say that A is a graded algebra if $A = \bigoplus_{g \in \Gamma} A_g$, and algebra multiplication satisfies $A_g A_h \subset A_{g+h}$. An element $a \in A$ is said to be homogeneous if $a \in A_g$ for some $g \in \Gamma$. In this case we say that g is the grading degree of a, and write $\widetilde{a} = \deg a = g$. A graded algebra A is said to be graded-commutative if $ab = (-1)^{\widetilde{a} \cdot \widetilde{b}} ba$ for all homogeneous $a, b \in A$.

Given a graded-commutative algebra A, one can define the jet bundle $J^k(A)$, the algebra of differential forms $\Lambda(A)$, and other objects of differential geometry. The corresponding framework was developed in [27], [14], [25], [26]. The goal of Chapter 2 is to apply this framework to the algebra $\mathrm{A}_\theta^{\mathrm{(n),poly}} = \bigoplus_{I \in \mathbb{Z}^n} \mathbb{C} U^I$. In order to do this, we should generalize this differential geometry framework to allow more general signs, which is done by replacing the sign function $(-1)^{\bullet \cdot \bullet}$ with $\lambda\colon \Gamma \times \Gamma \to U(1)$, and applying the Koszul sign rule where appropriate. Such generalization of the notion of graded-commutative algebra has been considered in [8]. It is relatively straightforward to extend the mentioned framework to the new sign function. In Sections 2.1–2.3 we do this for the notions of the jet bundle, derivations, multiderivations, and differential forms. The main result of Chapter 2 is the explicit description of the main constructions from this framework of differential geometry in the case of the polynomial algebra $\mathrm{A}_\theta^{\mathrm{(n),poly}}$ of the noncommutative torus. These are the module of derivations $D\big(\mathrm{A}_\theta^{\mathrm{(n),poly}}\big)$, the jet bundle $J^k\big(\mathrm{A}_\theta^{\mathrm{(n),poly}}\big)$, and the algebra $\Lambda\big(\mathrm{A}_\theta^{\mathrm{(n),poly}}\big)$ of differential forms.

There is a well-known algebra of differential forms $\Omega_D\left(\mathcal{A}_\theta^{(n)}\right)$ on $\mathcal{A}_\theta^{(n)}$, constructed by Connes. In Section 2.5, we get $\Omega_D\big(\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}\big)$ by applying this Connes framework to $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$, and compare it to the algebra $\Lambda\big(\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}\big)$. It turns out that $\Omega_D\big(\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}\big) \simeq \Lambda\big(\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}\big)$.

As a longer-term goal, one would like to extend these results from the polynomial subalgebra $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ to the smooth subalgebra $\mathcal{A}_\theta^{(n)}$. One of the goals of developing notions, such as jet bundles in the context of noncommutative tori, is to obtain other examples of natural geometric differential operators on noncommutative tori, by mimicking analogous constructions in ordinary differential geometry.

Chapter 3

In algebraic K-theory and algebraic geometry, there is a notion of regulator. The word “regulator” is used for homomorphisms from (algebraic) K-groups to cohomology groups (see, e.g., [23]). Such maps may be seen as algebraic analogues of the Chern character from the topological K-theory. In Chapter 3, we consider the Beilinson–Bloch regulator [2]. While it is defined in a more general setting, we will consider it only in the case of $K_2(X)$, where X is a closed Riemann surface. In this case, the Beilinson–Bloch regulator is a specific homomorphism $K_2(X) \to H^1(X, \mathbb{C}^*)$.

The main goal of a long-term project, started by Eugene Ha, is to generalize the Beilinson–Bloch regulator to the case of noncommutative spaces. Now, we list the main obstacles towards this goal.

Notion of a space. In the context of the Beilinson–Bloch regulator and $K_2(X)$, space X is understood as a variety. In partular, for every finite subset $S \subset X$, the variety X has a ring $\mathcal{O}(X \setminus S)$ of meromorphic functions with no poles outside of S. One of the main notions of a “space” in noncommutative geometry is the notion of a spectral triple. It is an analogue of a smooth Riemannian manifold endowed with a $\mathrm{Spin}^{\mathbb{C}}$ structure. An even spectral triple $(\mathcal{A}, H, D, \gamma)$ consists of

- a Hilbert space H;
- a self-adjoint unitary $\gamma \in B(H)$;

- an algebra $\mathcal{A} \subset B(H)$, such that all $a \in \mathcal{A}$ satisfy $a\gamma = \gamma a$;
- a (potentially unbounded) operator D, satisfying $D\gamma = -\gamma D$, s.t. $[D, a]$ is bounded for every $a \in \mathcal{A}$.

Here, $B(H)$ is the algebra of bounded operators on a Hilbert space H. Noncommutative spectral triple doesn't provide a notion of meromorphic functions. So, the first obstacle is the need to understand a suitable noncommutative analogue of the notion of a meromorphic function.

Algebraic K-theory. In noncommutative geometry, one uses the notion of K-theory of C^*-algebras and some generalizations of it. Although one can define $K_n(A)$ for $n > 1$, Bott periodicity implies that $K_{n+2}(A) \simeq K_n(A)$. This is different from algebraic K-theory, where $K_2(R)$ is rarely isomorphic to $K_0(R)$. In order to generalize the Beilinson–Bloch regulator, one needs to devise a suitable noncommutative analogue of algebraic K-theory.

The definition of the Beilinson–Bloch regulator. Beilinon–Bloch regulator is defined in terms of values and integrals of meromorphic functions on X. The notions of a point and a loop are not defined in noncommutative case. Therefore one needs to reformulate the definition of the Beilinson–Bloch regulator to avoid the use of these notions.

The unfinished manuscript of Eugene Ha [12] partially addresses this obstacle by proposing an alternative definition of the Beilinson–Bloch regulator using the Connes–Karoubi character on the universal 2-summable Fredholm module. If in the definition of an even spectral triple above we impose 2 additional conditions, namely that D is bounded, and $[D, a]$ is a Hilbert–Schmidt operator for every $a \in \mathcal{A}$, we get a 2-summable Fredholm module. Although typically the operator D is unbounded, Connes [6] first introduces the notion of a character for n-summable Fredholm modules, and then extends it to deal with a spectral triple, satisfying certain additional conditions, but having an unbounded operator D. If the Hilbert space H is separable, any 2-summable Fredholm module can be "embedded" into a universal 2-summable Fredholm module $\mathcal{M}^1$. Connes and Karoubi [7] described a homomorphism $\tau_2^{\mathrm{CK}}\colon K_2(\mathcal{M}^1) \to \mathbb{C}^*$. Eugene Ha conjectured a possible alternative definition of the Beilinson–Bloch regulator in his manuscript [12]. Given a loop $\gamma\colon S^1 \to X \setminus S$, one has an embedding $\rho_\gamma\colon \mathcal{O}(X \setminus S)$. The main step in the definition of the Beilinson–Bloch regulator is defining the maps $r_S\colon K_2(\mathcal{O}(X \setminus S)) \to H^1(X \setminus S, \mathbb{C}^*)$. Eugene Ha suggested doing this using the formula

$$\langle r_S(u), [\gamma] \rangle = \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_*\right)(u). \tag{1}$$

After one writes this formula, it remains to prove that it describes a well-defined regulator, and this regulator coincides with the original Beilinson–Bloch regulator. A possible strategy for proving this fact was outlined in [12]. We provide a complete proof in Chapter 3, which is somewhat different from the approach suggested in [12], though it follows the same main strategy.

The main result of Chapter 3 is that equality (1) holds, where the map r_S in its left hand side comes from the original Beilinson–Bloch regulator. The proof is based on the computation of the right hand side of (1) on Steinberg symbols, and uses multiple results from [18]. As explained in Section 3.7, since the original Beilinson–Bloch regulator is well defined, it follows that the equality (1) can be used as an alternative definition of it.

Chapter 1

4th generators of the odd K-theory of 3-dimensional noncommutative tori

1.1 Introduction

In this section, we describe the notation, definitions, and main known results used in this work and related to it.

1.1.1 Noncommutative tori

Let n be a positive integer, and let θ be an $n \times n$ antisymmetric matrix with elements in $\mathbb{R}$. Let $A_\theta^{(n)}$ be the universal unital C^*-algebra generated by unitaries $U_1, \dots, U_n$ subject to relations $U_l U_j = e^{2\pi i \theta_{lj}} U_j U_l$, where $l, j = 1, \dots, n$; then, this algebra is called a noncommutative torus algebra. Now we introduce the following multi-index notation. Let α be an element of $\mathbb{Z}^n$. By definition, put

$$U^\alpha = U_1^{\alpha_1} U_2^{\alpha_2} \dots U_n^{\alpha_n} \in A_\theta^{(n)}, \qquad |\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_n| . \tag{1.1}$$

An element $a \in A_\theta^{(n)}$ is said to be smooth if

$$a = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha U^\alpha, \tag{1.2}$$

where $a_\alpha \in \mathbb{C}$, and $\forall N \in \mathbb{N}\ \exists C_N \in \mathbb{R}\ \forall \alpha \in \mathbb{Z}^n\ \ |a_\alpha| < C_N\,(1+|\alpha|)^{-N}$. By definition, put

$$\mathcal{A}_\theta^{(n)} = \left\{ a \in A_\theta^{(n)} \;\middle|\; a \text{ is smooth} \right\}. \tag{1.3}$$

One might want to write an arbitrary element $a \in A_\theta^{(n)}$ in the form 1.2, with the only condition on the coefficients that the series converge. Unfortunately, that's not always possible: e.g., if $a = f(U_1)$ for some continuous function f, then the series above is just the Fourier series for f (with uniform convergence), and there are continuous functions for which the Fourier series doesn't converge uniformly (see, e.g., [13, remark after proof of Theorem 2.1]). It is possible to resolve this issue by constructing a space, analogous to the Hilbert space of square-integrable functions on S^1 in case $n = 1$, but we don't intend to do that.

1.1.2 Derivations

Definition 1.1.1. Suppose A is an algebra. A map $\xi \colon A \to A$ is called a derivation of A if the following conditions hold:

1. ξ is $\mathbb{C}$-linear;
2. for any a, b such that $a, b \in A$, we have

$$\xi(ab) = \xi(a)b + a\xi(b). \tag{1.4}$$

By definition, put

$$\delta_l \colon \mathcal{A}_\theta^{(n)} \to \mathcal{A}_\theta^{(n)} \colon \sum_{\alpha \in \mathbb{Z}^n} a_\alpha U^\alpha \mapsto \sum_{\alpha \in \mathbb{Z}^n} \alpha_l a_\alpha U^\alpha. \tag{1.5}$$

For any integer $l \in \{1, \ldots, n\}$ the map δ_l is a derivation of $\mathcal{A}_\theta^{(n)}$.

1.1.3 Even K-theory

For the convenience of the reader, we briefly revisit the definition of the K-theory of a C^*-algebra. For every C^*-algebra A, there are two groups: $K_0(A)$ and $K_1(A)$. One may define $K_n(A)$ for arbitrary nonnegative integer n, but due to the Bott periodicity, they only depend on whether n is even or odd. Thus, $K_0(A)$ and $K_1(A)$ are sometimes called even and odd K-theory respectively. The $K_0(A)$ is defined as follows.

Definition 1.1.2. Let A be a C^*-algebra and $p \in A$. If $p = p^2$, we say that p is idempotent. If, in addition, p is self-adjoint, i.e. $p = p^*$, we say that p is a projector. If p, q are 2 projectors in A, we say that they are Murray–von Neumann equivalent, and write $p \sim q$ if and only if there is an element $v \in A$ such that $p = v^*v$ and $q = vv^*$.

Let $M_n(A)$ be the algebra of $n \times n$ matrices over A. We denote the set of projectors in $M_n(A)$ with $\mathrm{Proj}_n(A)$. If $p \in \mathrm{Proj}_n(A)$, $q \in \mathrm{Proj}_m(A)$, define $p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in \mathrm{Proj}_{n+m}(A)$. By identifying p with $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we embed $\mathrm{Proj}_n(A)$ into $\mathrm{Proj}_m(A)$ for $m \geq n$. Define the semigroup $KD_0(A)$ by

$$KD_0(A) = \varinjlim_n \mathrm{Proj}_n(A)/\sim_0, \tag{1.6}$$

where $p \sim_0 q$ if and only if there exists n s.t. p and q are Murray–von Neumann equivalent as elements of $\mathrm{Proj}_n(A)$. While operation $\oplus$, defined above, doesn't respect the inclusions, the class $[p \oplus q]$ is a well-defined element of $KD_0(A)$ for any $p, q \in \bigcup_n \mathrm{Proj}_n(A)$. This operation induces a commutative and associative operation $+$ on $KD_0(A)$, allowing us to view $KD_0(A)$ as a monoid with unit $[0]$.

Then, by definition, the group $K_0(A)$ is the Grothendieck group of the monoid $KD_0(A)$.

The definition of the group $K_0(A)$ is stable under minor changes; e.g., one can consider all idempotents instead of projectors, and replace Murray–von Neumann equivalence with homotopy, and still construct the same group $K_0(A)$.

1.1.4 Odd K-theory and "unstable" odd K-theory

Here, we define the K-group $K_1(A)$ for a C^*-algebra A, and give some additional notation along the way to explain an important result of Rieffel.

Definition 1.1.3. Let A be a C^*-algebra. Let $\operatorname{Inv}(A)$ denote the group of its invertible elements, and let $\mathrm{GL}_k(A) = \operatorname{Inv}(M_k(A))$ be the group of invertible $k \times k$ matrices over A (in particular, $\mathrm{GL}_1(A) = \operatorname{Inv}(A)$). Let $\mathrm{GL}(A)$ be the injective limit of $\mathrm{GL}_n(A)$ with respect to inclusions $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) : \mathrm{GL}_n(A) \to \mathrm{GL}_{n+1}(A)$. Let $\operatorname{Inv}^0(A)$, $\mathrm{GL}^0_n(A)$, and $\mathrm{GL}^0(A)$ denote the connected components of the identity in these groups, so that

$$\pi_0\left(\operatorname{Inv}(A)\right) = \operatorname{Inv}(A)/\operatorname{Inv}^0(A), \qquad \pi_0\left(\mathrm{GL}_n(A)\right) = \mathrm{GL}_n(A)/\mathrm{GL}^0_n(A), \tag{1.7}$$

$$\pi_0\left(\mathrm{GL}(A)\right) = \mathrm{GL}(A)/\mathrm{GL}^0(A). \tag{1.8}$$

By definition, $K_1(A) = \pi_0(\mathrm{GL}(A)) = \mathrm{GL}(A)/\mathrm{GL}^0(A)$.

As in the case of K_0, the definition of K_1 is robust under minor changes. For example, one could use unitaries instead of generic invertible operators. Also, one can first go to the group of connected components, and then take the direct limit: $K_1(A) \simeq \varinjlim_n \mathrm{GL}_n(A)/\mathrm{GL}^0_n(A)$.

A natural question is whether the sequence $\mathrm{GL}_n(A)/\mathrm{GL}^0_n(A)$ stabilizes, so that the natural maps

$$\mathrm{GL}_n(A)/\mathrm{GL}^0_n(A) \to \mathrm{GL}_{n+1}(A)/\mathrm{GL}^0_{n+1}(A), \quad \mathrm{GL}_n(A)/\mathrm{GL}^0_n(A) \to K_1(A) \tag{1.9}$$

are bijective starting from some n. In general, the answer is no, and these maps are neither injective, nor surjective. However, for the case of noncommutative tori, according to Rieffel [20, Theorems 8.1 and 8.3], we have

Theorem 1.1.4. *Let $\theta \in M_n(\mathbb{R})$ be an anti-symmetric matrix with at least one component being irrational. Then, for $A = A^{(n)}_\theta$, all maps in (1.9) for all $n \in \mathbb{N}$ are group isomorphisms.*

We want to construct the 4th generator $[a]_1$ of $K_1\big(A_\theta^{(3)}\big)$. In the view of this theorem, we know that such a exists in $\mathrm{Inv}\big(A_\theta^{(3)}\big)$ (at least, when θ has some irrationality). It remains to construct it explicitly. Our first construction, described in 1.2, uses some of the ideas from the proof of [20, Proposition 8.2], used by Rieffel to show the above result.

1.1.5 Trace

Suppose τ is a trace on a C^*-algebra A (that is, bounded positive linear map $A \to \mathbb{C}$, satisfying $\tau(ab) = \tau(ba)$), and $a \in M_k(A)$; then by definition, put

$$\tau(a) = \sum_{l=1}^{n} \tau(a_{ll}) = \tau(\mathrm{Tr}(a)). \tag{1.10}$$

Lemma 1.1.5. *Let A be a C^*-algebra, let τ be a trace on A, and let p, q be projectors in $M_k(A)$. Suppose that p is Murray–von Neumann equivalent to q, i.e., $p \sim q$; then, $\tau(p) = \tau(q)$.*

Proof. The claim follows from the definition of Murray–von Neumann equivalence and the trace property. □

Suppose that $[p]_0 \in K_0(A)$, and let $\tau([p]_0) = \tau(p)$. From Lemma 1.1.5 it follows that the value $\tau([p]_0)$ is well defined.

There exists a natural trace τ on the C^*-algebra $A_\theta^{(n)}$ such that if $a \in \mathcal{A}_\theta^{(n)}$ is an element of the form (1.2), then $\tau(a) = a_0$. In the rest of the paper, τ will denote this natural trace on $A_\theta^{(n)}$ and its extension to $M_n\big(A_\theta^{(n)}\big)$ and $K_0\big(A_\theta^{(n)}\big)$.

1.1.6 Chern character

The map $\tau\colon K_0\big(A_\theta^{(n)}\big) \to \mathbb{C}$ is useful, because it allows us to distinguish between different classes of projectors. There is an analogous map, serving the same purpose for the odd K-theory. This map is $\mathrm{Ch}^1\colon K_1\big(A_\theta^{(n)}\big) \to \mathbb{C}^n$, defined on the classes $[a]$ of smooth invertible elements $a \in \mathrm{Inv}\left(\mathcal{A}_\theta^{(n)}\right)$ with

$$\mathrm{Ch}_k^1([a]) = \tau\left(a^{-1}\delta_k a\right), \qquad k \in \{1, \ldots, n\}. \tag{1.11}$$

Substituting $a = U_j$ we get

$$\mathrm{Ch}_k^1([U_j]) = \delta_{kj}. \tag{1.12}$$

This map is a component of the Chern character. In the context of noncommutative geometry, the Chern character was introduced by Connes; see [5] (see also the English translation [4]) and [6].

1.1.7 Elliott's paper

Elliott's 1984 paper [10] considers noncommutative tori algebras $A_\theta^{(n)}$ as obtained from a pair (G, θ), where $\mathbb{Z}^n \simeq G \subset \mathrm{Inv}\big(A_\theta^{(n)}\big)/\mathbb{C}^*$ is the abelian group, generated by classes of U_k, $k = 1, \ldots, n$. Let U_g for $g \in G$ be U_I, where g represents the class of U_I. Then, map $\theta\colon G \wedge G \to \mathbb{R}$ is such that $U_g U_h = e^{2\pi i \theta(g \wedge h)} U_h U_g$. The data (G, θ) is indeed sufficient to reconstruct the algebra. In order to do this, one has to choose $\alpha\colon G^2 \to \mathbb{T}$ such that

$$\alpha(g, h)\alpha(h, g) = e^{2\pi i \theta(g \wedge h)}, \tag{1.13}$$

set

$$A_\theta^{(n)} = A_{G,\theta} = C^*\langle \{U_g\}_{g \in G} \mid U_g U_h = \alpha(g, h) U_{g+h} \rangle, \tag{1.14}$$

and prove that the algebra doesn't depend on the choice of α satisfying (1.13). Here, $C^*\langle\text{generators} \mid \text{relations}\rangle$ is the universal C^*-algebra, defined by the given generators and relations. Main results from the Elliott's paper can be summarized in the following theorem.

Theorem 1.1.6. *There is a natural isomorphism $K_*\big(A_\theta^{(n)}\big) \simeq \Lambda G$. Under this isomorphism:*

1. $K_0\big(A_\theta^{(n)}\big) \simeq \Lambda^{even} G$, $K_1\big(A_\theta^{(n)}\big) \simeq \Lambda^{odd} G$;

2. $\tau = \mathrm{Ch}^0 \colon K_0\big(A_\theta^{(n)}\big) \to \mathbb{R}$ *corresponds to* $\exp_\wedge(\theta)$*;*

3. $\mathrm{Ch}^1 \colon K_1\big(A_\theta^{(n)}\big) \to G \otimes_{\mathbb{Z}} \mathbb{R}$ *corresponds to* $1_G \wedge \exp_\wedge(\theta)$*;*

4. $\mathrm{Ch} \colon K_*\big(A_\theta^{(n)}\big) \to \Lambda G \otimes_{\mathbb{Z}} \mathbb{R}$ *corresponds to* $\exp_\wedge(1 \wedge \theta)$*.*

In particular, it follows from Elliott's paper that

$$\mathrm{Ch}^1\left(K_1\big(A_\theta^{(3)}\big)\right) = \mathbb{Z}^3 + \mathbb{Z}(\theta_{23}, -\theta_{13}, \theta_{12}). \tag{1.15}$$

This fact also follows from our construction (see Subsection 1.2.7).

1.1.8 Pimsner–Voiculescu 6-term exact sequence

The main tool in the K-theory of noncommutative tori is the Pimsner–Voiculescu 6-term exact sequence. Let $A_\theta^{(n)}$ be a noncommutative torus algebra, and let $A_\theta^{(n-1)}$ denote its C^*-subalgebra, generated by $U_1, \dots, U_{n-1}$. Note that only $(n-1)\times(n-1)$ submatrix of θ is actually used as parameters for this subalgebra, so we are slightly abusing the notation. Let $i \colon A_\theta^{(n-1)} \to A_\theta^{(n)}$ be the standard embedding. It follows from Pimsner–Voiculescu 6-term exact sequence that there are the following 2 short exact sequences, relating K-theories of $A_\theta^{(n-1)}$ and $A_\theta^{(n)}$:

$$0 \longrightarrow K_1\big(A_\theta^{(n-1)}\big) \xrightarrow{K_1(i)} K_1\big(A_\theta^{(n)}\big) \xrightarrow{\delta_1^{\mathrm{PV}}} K_0\big(A_\theta^{(n-1)}\big) \longrightarrow 0, \tag{1.16}$$

$$0 \longrightarrow K_0\big(A_\theta^{(n-1)}\big) \xrightarrow{K_0(i)} K_0\big(A_\theta^{(n)}\big) \xrightarrow{\delta_0^{\mathrm{PV}}} K_1\big(A_\theta^{(n-1)}\big) \longrightarrow 0, \tag{1.17}$$

Using induction, it follows that

$$K_0\big(A_\theta^{(n)}\big) \simeq \mathbb{Z}^{2^{n-1}}, \qquad K_1\big(A_\theta^{(n)}\big) \simeq \mathbb{Z}^{2^{n-1}}. \tag{1.18}$$

1.1.9 Rieffel projector

The generators of $K_1(A_\theta^{(3)})$ can be chosen to be $[U_1]_1$, $[U_2]_1$, $[U_3]_1$, and $[a]_1$ with $a \in \mathrm{Inv}(A_\theta^{(3)})$. The first 3 generators are given by "one-letter" formulas, but describing the last one explicitly requires some work. A similar situation exists with $K_0(A_\theta^{(2)})$: it is generated by $[1]_0$ and $[P_\theta]_0$, where P_θ is the Rieffel–Powers projector, introduced in [19] (see also its discussion in the textbook [9]). To keep our introduction self-contained, and to introduce the notation, we provide, following [19] and [9], the description of the Rieffel projector here. We also note that from the construction below, it follows that P_θ can be chosen to lie in $\mathcal{A}_\theta^{(2)}$. Computations below are done in $A_\theta^{(2)}$. The antisymmetric 2×2 matrix θ is determined by one number θ_{12}, so for simplicity, we will write θ instead of θ_{12}. The algebra $A_\theta^{(2)}$ depends only on the class $\theta + \mathbb{Z}$ of θ in $\mathbb{R}/\mathbb{Z}$, and the Rieffel projector is defined for $\theta \in \mathbb{R} \setminus \mathbb{Z}$, so we will assume $\theta \in (0,1)$. We set $U = U_1$, $V = U_2$, so $UV = e^{2\pi i\theta} VU$. Let's search in A_θ for a projector of the form

$$P_\theta = h(U)V^* + f(U) + g(U)V. \tag{1.19}$$

Here, $f, g, h\colon S^1 \to \mathbb{C}$ are functions from the unit circle to the set of complex numbers. By definition, P_θ, given by (1.19), is a projector if and only if it satisfies the equalities $P_\theta = P_\theta^*$ and $P_\theta^2 - P_\theta = 0$. Conjugating (1.19), we get

$$P_\theta^* = \bar{g}(e^{2\pi i\theta} U)V^* + \bar{f}(U) + \bar{h}(e^{-2\pi i\theta} U)V. \tag{1.20}$$

Therefore, the projector P_θ, given by (1.19), is self-adjoint if and only if

$$f(z) = \bar{f}(z), \tag{1.21}$$

$$h(z) = \bar{g}(e^{2\pi i\theta} z). \tag{1.22}$$

Thus, we can take a note that f should be real-valued, and replace h with its expression in (1.19). We get

$$P_\theta = \bar{g}(e^{2\pi i\theta} U)V^* + f(U) + g(U)V. \tag{1.23}$$

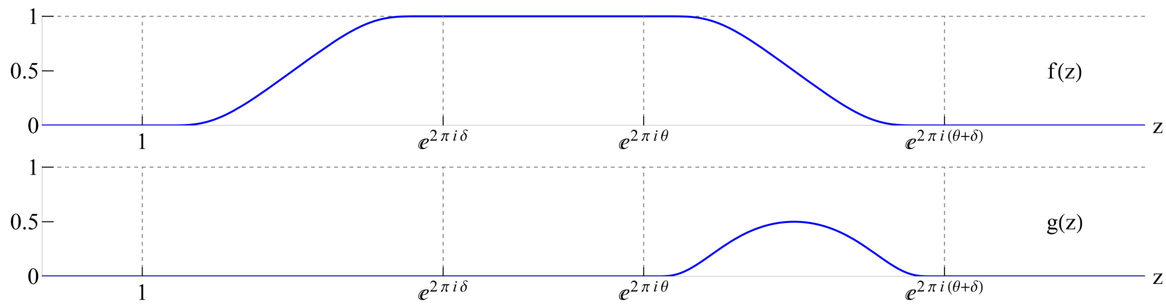


Figure 1.1: $f(z)$ and $g(z)$.

We note that $P_\theta^2 - P_\theta$ is automatically self-adjoint. Therefore, it is enough to require that terms with non-positive powers of V vanish. We get

$$P_\theta^2 - P_\theta = \bar{g}(e^{2\pi i\theta}U)\bar{g}(e^{4\pi i\theta}U)(V^*)^2 + \bar{g}(e^{2\pi i\theta}U)\left(f(e^{2\pi i\theta}U) + f(U) - 1\right)V^* + \left(\bar{g}(e^{2\pi i\theta}U)g(e^{2\pi i\theta}U) + g(U)\bar{g}(U) - f(U)(1 - f(U))\right) + (\dots)V + (\dots)V^2 \quad (1.24)$$

Fix any $\delta \in (0, \min(\theta, 1-\theta))$. As shown on Figure 1.1, we ask f to be a function $S^1 \to [0,1]$, s.t. $f(e^{2\pi i t})$ is 0 for $t \in [-1+\theta+\delta, 0]$, 1 for $t \in [\delta, \theta]$, and it changes continuously from 0 to 1 and vice-versa on the rest of the circle. Moreover, we require that $f(e^{2\pi i(t+\theta)}) = 1 - f(e^{2\pi i t})$ for $t \in [0, \delta]$. Then, we set $g(e^{2\pi i t}) = \sqrt{f(e^{2\pi i t})(1 - f(e^{2\pi i t}))}$ for $t \in [\theta, \theta+\delta]$, and 0 otherwise. For such choice of f and g coefficients of $(V^*)^2$, V^* and 1 vanish in the expression (1.24) for $P_\theta^2 - P_\theta$. Coefficients of V and V^2 will then vanish automatically, since $P_\theta^* = P_\theta$.

We compute $\delta_U P_\theta$ and $\delta_V P_\theta$. We recall that derivations $\delta_U = \delta_1$ and $\delta_V = \delta_2$ of A_θ are defined by

$$\begin{aligned} \delta_U U &= U, & \delta_U V &= 0, \\ \delta_V U &= 0, & \delta_V V &= V. \end{aligned} \quad (1.25)$$

Let $\dot{g}$ denote the derivative of g along the circle, so that

$$\dot{g}\left(e^{2\pi i\varphi}\right) = \frac{1}{2\pi i}\frac{\partial}{\partial\varphi}g\left(e^{2\pi i\varphi}\right), \qquad \dot{g}(z) = z\frac{\partial g(z)}{\partial z}, \qquad \delta_U g(U) = \dot{g}(U). \quad (1.26)$$

Then,

$$\begin{aligned} \delta_U P_\theta &= \dot{\bar{g}}(e^{2\pi i\theta}U)V^* + \dot{f}(U) + \dot{g}(U)V, \\ \delta_V P_\theta &= -\bar{g}(e^{2\pi i\theta}U)V^* + g(U)V. \end{aligned} \tag{1.27}$$

Later, in Subsection 1.2.5, we will investigate a "twisted" version of this projector, which still gives the same class in K-theory.

1.2 The 4th generator: semi-explicit formula

In this part, we construct the 4th generator of $K_1\big(A_\theta^{(3)}\big)$, using the Pimsner–Voiculescu short exact sequence. A similar approach was used by Rieffel in [20, proof of 8.2].

1.2.1 Pimsner–Voiculescu lemma

In order to construct the 4th generator, we will use the Pimsner–Voiculescu exact sequence (1.16):

$$0 \longrightarrow K_1\big(A_\theta^{(n-1)}\big) \xrightarrow{K_1(i)} K_1\big(A_\theta^{(n)}\big) \xrightarrow{\delta_1^{\mathrm{PV}}} K_0\big(A_\theta^{(n-1)}\big) \longrightarrow 0. \tag{1.28}$$

The map δ_1^{PV} comes from the index map in the 6-term exact sequence of K-theory, so we call δ_1^{PV} an index map. Since $K_1\big(A_\theta^{(n)}\big) \simeq \mathbb{Z}^{2^{n-1}}$, it has 2^{n-1} generators. From this sequence (1.28), we see that in order to list all generators of $K_1\big(A_\theta^{(n)}\big)$, it is enough to list all 2^{n-2} generators of $K_1\big(A_\theta^{(n-1)}\big)$ and any preimages for each of 2^{n-2} generators of $K_0\big(A_\theta^{(n-1)}\big)$. If we proceed inductively to compute explicit formulas for generators of $K_0\big(A_\theta^{(n)}\big)$ and $K_0\big(A_\theta^{(n)}\big)$, then on the n-th inductive step, we will know 2^{n-2} generators of $K_1\big(A_\theta^{(n)}\big)$: these are images from $K_1\big(A_\theta^{(n-1)}\big)$. It is harder to find explicitly preimages of projectors under the index map. We will soon see that $[U_n]_1 \in -\left(\delta_1^{\mathrm{PV}}\right)^{-1}([1]_0)$ (it directly follows from [17]). This gives n out of 2^{n-1} generators, which solves the problem of finding all generators for $n = 1, 2$. The rest of this section is devoted to the construction of a preimage of the Rieffel projector, i.e., of a, satisfying $[a]_1 \in \left(\delta_1^{\mathrm{PV}}\right)^{-1}[P_{\theta_{12}}]_0$. This will give $n + \binom{n}{3}$ out of 2^{n-1} generators, so it will give all generators in the case $n = 3, 4$. Now we give the key results from

[17], allowing us to work with the index map.

Lemma 1.2.1. *Let* F *be a projector in* $M_m\big(A_\theta^{(n-1)}\big)$, *and let*

$$a = (1 - F) + FxU_n^* F \in M_m\big(A_\theta^{(n)}\big), \tag{1.29}$$

where $x \in M^m\big(A_\theta^{(n-1)}\big)$ *is such that* a *is unitary. Then,* $\delta^{PV}[a]_1 = [F]_0$. *Moreover,* $K_1\big(A_\theta^{(n)}\big)$ *is generated by classes* $[a]_1$ *of unitary elements of the form* (1.29).

Note that the condition "a given by (1.29) is unitary" is a nontrivial condition on x and F.

Proof. The last statement, saying that classes of unitary elements of the form (1.29) generate $K_1\big(A_\theta^{(n)}\big)$, is the lemma 1.2 of [17]. The first fact is stated in the proof of Lemma 2.3. To prove it, one needs to trace the definition of the index map δ_1^{PV}, and use the definition of the index map from the 6-term exact sequence of K-theory. □

1.2.2 Adapting the Pimsner–Voiculescu lemma

Now we will adapt the lemma above to fit our specific needs.

Lemma 1.2.2. *Let* F *be an projector, and* x *be a unitary element in* $M_m\left(\mathcal{A}_\theta^{(n-1)}\right)$. *Assume that they satisfy* $U_n^* F U_n = x^* F x$, *and let* a *be the element of the form* (1.29), *constructed from these* F *and* x.

1. *a is unitary;*

2. $\delta_1^{PV}[a]_1 = [F]_0$;

3. $\mathrm{Ch}_j^1(a) = \tau(\delta_j(x)x^* F)$ *for* $j = 1, \ldots, n-1$;

4. $\mathrm{Ch}_n^1(a) = -\tau(F)$.

Note that in the lemma we use $\mathcal{A}$ instead of A: we require both x and F to be smooth.

Proof. 1. By substituting (1.29), we get that $a^*a = (1-F) + FF_1F$ and $aa^* = (1-F) + FF_2F$, where

$$F_1 = U_n x^* F x U_n^* = U_n U_n^* F U_n U_n^* = F, \tag{1.30}$$

$$F_2 = x U_n^* F U_n x^* = x x^* F x x^* = F, \tag{1.31}$$

so, indeed, $a^*a = aa^*$.

2. Follows from the previous lemma.

3. In the computations below, we will use the following properties of the trace τ: $\tau(yz) = \tau(zy)$ and $\tau(\delta_j(y)) = 0$ for any $y, z \in M_m\left(\mathcal{A}_\theta^{(n)}\right)$ and any $j \in \{1, \ldots, n\}$. From these, it follows that $\tau(F^l \partial_j F) = \frac{1}{l+1}\tau(\partial_j(F^{l+1})) = 0$. For $j \in \{1, \ldots, n-1\}$, substituting (1.29) into (1.11), and using $a^{-1} = a^*$, we get

$$\begin{aligned}
\mathrm{Ch}_j^1(a) = \tau\left(-(1-F)\delta_j(F) + FU_n x^* F \delta_j(F x U_n^* F)\right) = \\
\tau\Big(-(1-F)\delta_j(F) + FU_n x^* F \delta_j(F) x U_n^* F + \\
FU_n x^* F \delta_j(x) U_n^* F + FU_n x^* F x U_n^* \delta_j(F)\Big) = \\
\tau\Big(\big(-(1-F) + xU_n^* F U_n x^* F + FU_n x^* F x U_n^*\big)\delta_j(F) + U_n^* F U_n x^* F \delta_j(x)\Big).
\end{aligned} \tag{1.32}$$

Now, we use $U_n^* F U_n = x^* F x$ to get

$$\begin{aligned}
\mathrm{Ch}_j^1(a) = \tau\Big(\big(-(1-F) + F + F\big)\delta_j(F) + x^* F x x^* F \delta_j(x)\Big) = \\
\tau\Big((-1+3F)\delta_j(F) + x^* F \delta_j(x)\Big) = \tau\big(\delta_j(x) x^* F\big)
\end{aligned} \tag{1.33}$$

as desired.

4. Similarly,

$$\mathrm{Ch}_n^1(a) = \tau\big(FU_n x^* F \delta_n(F x U_n^* F)\big) = -\tau\big(FU_n x^* F x U_n^* F\big) = -\tau(F). \tag{1.34}$$

□

1.2.3 Homotopy and unitary equivalence

Let A be a C^*-algebra, and $\mathcal{A} \subset A$ — its Fréchet subalgebra, containing the unit of A. Let $\{F_t\}_{t\in[0,1]}$ be a family of projectors in $\mathcal{A}$, such that the function $t \mapsto F_t \colon [0,1] \to \mathcal{A}$ has a continuous derivative. The goal of this subsection is to informally construct a unitary $x \in \mathcal{A}$, s.t. $F_1 = x^* F_0 x$. In the next subsection, we will formalize the answer using Araki expansionals (see [1]), and prove that it indeed satisfies the desired properties and solves the equation $F_1 = x^* F_0 x$. Our construction is inspired by the following result from the K-theory of C^*-algebras from [3, Proposition 4.3.2].

Lemma 1.2.3. *If e, f are projectors with $\|e - f\| < 1$, then $z^{-1} e z = f$ with $z = (2e-1)(2f-1)/2 + 1/2$, $\|z - 1\| \leq \|e - f\|$.*

Naively, one may want to use this lemma with $e = F_0$, $f = F_1$ and take $x = z$, where z is given by lemma above. There are two problems with that. First, we don't know, whether $\|F_1 - F_0\| < 1$ or not, and, second, z given by the formula above may fail to be unitary. Instead, we will try to construct the family $\{x_t\}_{t\in[0,1]}$ s.t.

$$F_t = x_t^* F_0 x_t, \tag{1.35}$$

and then take $x = x_1$. Assuming $x_t^{-1} = x_t^*$, we get

$$F_{t+\varepsilon} = (x_t^{-1} x_{t+\varepsilon})^{-1} F_t x_t^{-1} x_{t+\varepsilon}. \tag{1.36}$$

Then, we try to use Lemma 1.2.3 with $e = F_t$ and $f = F_{t+\varepsilon}$ to get an expression for $x_t^{-1} x_{t+\varepsilon}$, assuming ε is small. We get

$$x_t^{-1} x_{t+\varepsilon} = 1 + (2F_t - 1)\partial_t F_t \varepsilon + O(\varepsilon^2). \tag{1.37}$$

Using the limit $\varepsilon \to 0$, (1.37) gives

$$\partial_t x_t = x_t (2F_t - 1)\partial_t F_t. \tag{1.38}$$

Together with the initial condition $x_0 = 1$, that should describe x_t.

1.2.4 Araki expansionals

The Araki expansional [1] is an object, which is supposed to solve the equations of the form (1.38). Here, we slightly adjust the definition from [1] to fit our purposes.

Definition 1.2.4. Let A be a Banach algebra, and let $f\colon s \mapsto f(s)\colon [\alpha,\beta] \to A$ be a norm-continuous function. Define

$$\operatorname{Exp}_r\left(\int_\alpha^\beta;\ f(s)ds\right) = \sum_{n=0}^{\infty} \int_\alpha^\beta dt_1 \int_\alpha^{t_1} dt_2 \dots \int_\alpha^{t_{n-1}} dt_n f(t_n)\dots f(t_1). \tag{1.39}$$

If $f\colon [\alpha,\beta] \to A$ is piece-wise continuous, s.t. it is continuous on $[t_l, t_{l+1}]$ for $\alpha = t_0 \le t_1 \le \cdots \le t_L = \beta$, we define

$$\operatorname{Exp}_r\left(\int_\alpha^\beta;\ f(s)ds\right) = \operatorname{Exp}_r\left(\int_{t_0}^{t_1};\ f(s)ds\right)\cdots\operatorname{Exp}_r\left(\int_{t_{L-1}}^{t_L};\ f(s)ds\right). \tag{1.40}$$

These two definitions agree with each other; i.e., if f is continuous on $[\alpha,\beta]$, the second definition gives the same result as the first one. The integrals are well defined, and the sum in (1.39) converges absolutely. Moreover, (1.39) makes sense and gives the same result for a piece-wise continuous function, as (1.40).

We are interested in the case when A is a C^*-algebra, and f takes values in its Fréchet subalgebra. In this case, $\operatorname{Exp}_r\left(\int_\alpha^\beta;\ f(s)ds\right)$ is defined to be an element of A, and we want it to be element of $\mathcal{A}$. We have the following lemma.

Lemma 1.2.5. *Let A be a C^*-algebra and $\mathcal{A}$ be its Fréchet subalgebra, for which the inclusion $\mathcal{A} \hookrightarrow A$ is continuous. Let $f\colon [\alpha,\beta] \to \mathcal{A}$ be a piece-wise continuous function (with respect to the Fréchet topology on $\mathcal{A}$). Then,* $\operatorname{Exp}_r\left(\int_\alpha^\beta;\ f(s)ds\right) \in \mathcal{A}$.

Proof. From the definition (1.40), we see that it is enough to show the lemma in the case when f is continuous. In this case, the integrals in (1.39) are defined in $\mathcal{A}$, and since the inclusion $\mathcal{A} \hookrightarrow A$ is continuous, have the same value in $\mathcal{A}$, as they have in A. Similarly, the sum in (1.39) is absolutely convergent in any of the seminorms on

$\mathcal{A}$ and, by the same argument, converges in $\mathcal{A}$ to the same value, as it does in A. Thus, the result of (1.39) belongs to $\mathcal{A}$. □

We also need the following lemma:

Lemma 1.2.6 ([1, Proposition 2]). *If $f\colon [\alpha,\beta] \to A$ is a piece-wise continuous function, then for $t \in (\alpha,\beta)$, we have*

$$\partial_t \operatorname{Exp}_r\left(\int_\alpha^t;\ f(s)ds\right) = \operatorname{Exp}_r\left(\int_\alpha^t;\ f(s)ds\right) f(t). \tag{1.41}$$

Comparing (1.38) and (1.41), we propose the following solution for x_t:

$$x_t = \operatorname{Exp}_r\left(\int_0^t;\ (2F_s - 1)\partial_s F_s ds\right). \tag{1.42}$$

It remains to show that x_t satisfies the desired properties.

Lemma 1.2.7. *Let A be a C^*-algebra and $\mathcal{A}$ be its Fréchet subalgebra, for which the inclusion $\mathcal{A} \hookrightarrow A$ is continuous, and $1_A \in \mathcal{A}$; let $F_t \in \mathcal{A}$ be projectors for $t \in [0,1]$ s.t. the function $t \mapsto F_t\colon [0,1] \to \mathcal{A}$ is continuous and piece-wise continuously differentiable; and let x_t be given by* (1.42)*. Then, $\{x_t\}_{t\in[0,1]}$ is a continuously differentiable family of self-adjoint operators in $\mathcal{A}$ with $x_0 = 1$, satisfying* (1.38) *(for all t, where F_t is differentiable) and* (1.35) *(for all $t \in [0,1]$).*

Proof. It is enough to prove the lemma for the case of continuously differentiable $t \mapsto F_t$: otherwise, we can apply the lemma to each segment.

First, note that x_t is differentiable with respect to t, and satisfies (1.38), which follows from (1.42) using [1, prop. 2]. From (1.38), it follows that the derivative $\partial_t x_t$ is continuous. Equality $x_0 = 1$ follows from the definition of Exp_r.

We then prove that x_t is self-adjoint for $t \in [0,1]$. Let $y_t = (2F_t - 1)\partial_t F_t$. We claim that $y_t^* = -y_t$. Indeed,

$$(y_t^* + y_t)/2 = (\partial_t F_t)(2F_t - 1)/2 + (2F_t - 1)\partial_t F_t/2 =$$
$$(\partial_t F_t)F_t + F_t\partial_t F_t - \partial_t F_t = \partial_t(F_t^2 - F_t) = \partial_t(0) = 0. \tag{1.43}$$

Let $a_t = x_t^* x_t - 1$. We want to know that $a_0 = 0$, and

$$\partial_t a_t = (\partial_t x_t^*) x_t + x_t^* \partial_t x_t = y_t^* x_t^* x_t + x_t^* x_t y_t = y_t^* a_t + a_t y_t. \tag{1.44}$$

$a_t = 0$ is a solution of this equation, so it remains to give some argument about uniqueness of the solution of (1.44). We don't know a suitable reference for that, so we show directly that (1.44) implies $a_t = 0$. Integrating, we get

$$\|a_t\| \leq \int_0^t \|y_s^* a_s + a_s y_s\| ds \leq \int_0^t 2\|y_s\|\|a_s\| ds. \tag{1.45}$$

We know that F_t, x_t and $\partial_t F_t$ are continuous. Therefore, $\|a_t\|$ and $\|y_t\|$ are continuous on $[0,1]$, so there are C_1, C_2 s.t. $\|a_t\| \leq C_1$, $\|y_t\| \leq C_2$ for all $t \in [0,1]$. Thus, iterating (1.45), we get

$$\|a_t\| \leq C_1 (2C_2 t)^k / k!. \tag{1.46}$$

Thus, $a_t = 0$, since the r.h.s. of (1.46) goes to 0 as $k \to \infty$.

Now we show that x_t satisfies (1.35). Similarly to the above, we let $b_t = x_t^* F_0 x_t - F_t$. We compute

$$y_t^* F_t + F_t y_t = (\partial_t F_t)(2F_t - 1)F_t + F_t(2F_t - 1)(\partial_t F_t) = (\partial_t F_t)F_t + F_t(\partial_t F_t) = \partial_t(F_t^2) = \partial_t F_t. \tag{1.47}$$

Therefore,

$$\partial_t b_t = y_t^* F_0 + F_0 y_t - \partial_t F_t = 0. \tag{1.48}$$

Integrating, and using $b_0 = 0$ we get $b_t = 0$ as desired.

Finally, we have to prove that $x_t \in \mathcal{A}$. To prove this, we note that the expression inside the integral in (1.42) is, by assumption, a continuous function $[0,1] \to \mathcal{A}$, and Araki exponential of a continuous function with values in a Fréchet subalgebra of a C^*-algebra lies in that Fréchet subalgebra (because all the integrals and series in the definition of Exp_r converge in any of the semi-norms; see, e.g., formula (2.2) of [1], defining Exp_r). □

Note that we can rescale the construction above and starting with a path $\{F_t\}_{t\in[T_0,T_1]}$, get $\{x_t\}_{t\in[T_0,T_1]}$, given by

$$x_t = \mathrm{Exp}_r\left(\int_{T_0}^{t};\ (2F_s-1)\partial_s F_s ds\right), \tag{1.49}$$

and satisfying

$$F_t = x_t^* F_{T_0} x_t, \tag{1.50}$$

with the same technical conditions, as in Lemma 1.2.7.

1.2.5 Twisted Rieffel projector

Our next goal is to construct unitary $x \in \mathcal{A}_\theta^{(n-1)}$, satisfying the conditions of the lemma 1.2.2 with $F = P_{\theta_{lj}}$, i.e. unitary x, satisfying $U_n^* P_{\theta_{lj}} U_n = x^* P_{\theta_{lj}} x$. All the computations will be done in the subalgebra of $\mathcal{A}_\theta^{(n)}$, generated by U_l, U_j, and U_n, which is isomorphic to $\mathcal{A}_{\theta'}^{(3)}$ for θ' being the appropriate submatrix of θ. Therefore, we will assume $n = 3, l = 1, j = 2$. We denote $U = U_1$, $V = U_2$, and $W = U_3$. Whenever the arbitrary n have to be restored, we will have to replace $U = U_1$ with U_l, $V = U_2$ with U_j, and $W = U_3$ with U_n. We will now join the Rieffel projector $P_{\theta_{12}}$ with its "twisted" version $W^* P_{\theta_{12}} W$ by a piece-wise continuously differentiable path, and obtain x_t and x using (1.49) and Lemma 1.2.7.

Let's denote the torus action on $A_\theta^{(2)}$ by β, so that for $s, t \in \mathbb{R}$ the automorphism $\beta_{s,t}$ of the algebra $A_\theta^{(2)}$ satisfies

$$\beta_{s,t}(U) = e^{2\pi i s} U, \qquad\qquad \beta_{s,t}(V) = e^{2\pi i t} V. \tag{1.51}$$

Then, for all $a \in A_\theta^{(2)} \subset A_\theta^{(3)}$, we have

$$\beta_{\theta_{13},\theta_{23}} a = W^* a W. \tag{1.52}$$

Note that $\beta_{s,t}$ only depends on the classes of s and t modulo $\mathbb{Z}$, i.e. on the point of the torus $\mathbb{R}^2/\mathbb{Z}^2$, hence the name "torus action." It remains to join $\mathrm{id} = \beta_{0,0}$ with $\beta_{\theta_{13},\theta_{23}}$

piece-wise smoothly, apply these automorphisms to $P_{\theta_{12}}$, and apply Lemma 1.2.7 to the obtained path in the space of projectors. This construction works for any path, but we will choose a specific one to obtain a specific unitary x. Note that we can change θ_{13} and θ_{23} by any integer without changing the algebra. We will use that to assume that both θ_{13} and θ_{23} are nonnegative, and choose the path $\{F_t\}_{t\in[0,T]}$, where

$$T = \theta_{13} + \theta_{23}, \quad F_t = \beta_{t,0} P_{\theta_{12}} \quad \text{for } t \in [0, \theta_{13}], \quad F_{\theta_{13}+t} = \beta_{\theta_{13},t} P_{\theta_{12}} \quad \text{for } t \in [0, \theta_{23}]. \tag{1.53}$$

Since the formula (1.49), we are planning to use for $x = x_T$, contains $\partial_t F_t$, we will compute it. First, note that for $a \in \mathcal{A}_\theta^{(2)}$ we have

$$\partial_s(\beta_{s,t} a) = 2\pi i \delta_U(\beta_{s,t} a) = 2\pi i \beta_{s,t}(\delta_U a), \quad \partial_t(\beta_{s,t} a) = 2\pi i \delta_V(\beta_{s,t} a) = 2\pi i \beta_{s,t}(\delta_V a). \tag{1.54}$$

Therefore, using (1.27) and the definition of β, we get

$$\partial_s(\beta_{s,t} P_{\theta_{12}}) = 2\pi i \left(e^{-2\pi i t} \dot{\bar{g}}(e^{2\pi i(\theta_{12}+s)} U) V^* + \dot{f}(e^{2\pi i s} U) + e^{2\pi i t} \dot{g}(e^{2\pi i s} U) V \right), \tag{1.55}$$

$$\partial_t(\beta_{s,t} P_{\theta_{12}}) = 2\pi i \left(-e^{-2\pi i t} \bar{g}(e^{2\pi i(\theta_{12}+s)} U) V^* + e^{2\pi i t} g(e^{2\pi i s} U) V \right). \tag{1.56}$$

Also, applying $\beta_{s,t}$ to the expression (1.23) for $P_{\theta_{12}}$, we get

$$\beta_{s,t} P_{\theta_{12}} = e^{-2\pi i t} \bar{g}(e^{2\pi i(\theta_1 2+s)} U) V^* + f(e^{2\pi i s} U) + e^{2\pi i t} g(e^{2\pi i s} U) V. \tag{1.57}$$

Given these expressions, the following expression for x, obtained from Lemma 1.2.7, is somewhat explicit:

$$x = \operatorname{Exp}_r \left(\int_0^{\theta_{13}} ; \ (2\beta_{s,0} P_{\theta_{12}} - 1) \partial_s(\beta_{s,0} P_{\theta_{12}}) ds \right) \operatorname{Exp}_r \left(\int_0^{\theta_{23}} ; \ (2\beta_{\theta_{13},t} P_{\theta_{12}} - 1) \partial_t(\beta_{\theta_{13},t} P_{\theta_{12}}) dt \right). \tag{1.58}$$

Alternatively, choosing a straight-line path, we get another solution $\widetilde{x}$ of $W^* P_{\theta_{12}} W =$

$x^*P_{\theta_{12}}x$:

$$\widetilde{x} = \mathrm{Exp}_r\left(\int_0^1;\ (2\beta_{s\theta_{13},s\theta_{23}}P_{\theta_{12}} - 1)\partial_s(\beta_{s\theta_{13},s\theta_{23}}P_{\theta_{12}})ds\right). \tag{1.59}$$

Note that one can continuously deform one solution into another (by deforming the path), where deformation doesn't leave the space of solutions of $W^*P_{\theta_{12}}W = x^*P_{\theta_{12}}x$.

1.2.6 The 4th generator

Now, according to Lemma 1.2.2 and the construction above, the 4th generator of $K_1(A_\theta^{(3)})$ can be written as $[a]_1$, where

$$a = (1 - P_{\theta_{12}}) + P_{\theta_{12}}xW^*P_{\theta_{12}}, \tag{1.60}$$

where x is given by (1.58). Later, in Section 1.3, we will give a much more simple and explicit expression. Now, we will compute the value of $\mathrm{Ch}^1([a]_1)$ for this generator.

1.2.7 Chern character

From Lemma 1.2.2, we know that

$$\mathrm{Ch}^1_3([a]_1) = -\tau(P_{\theta_{12}}) = -\theta_{12}, \tag{1.61}$$

and for $j = 1, 2$,

$$\mathrm{Ch}^1_j([a]_1) = \tau(\delta_j(x)x^*P_{\theta_{12}}). \tag{1.62}$$

Since replacing x with $\widetilde{x}$ in (1.60) doesn't change the homotopy class of a, it doesn't change the value of the Chern character. Thus,

$$\mathrm{Ch}^1_j([a]_1) = \tau(\delta_j(\widetilde{x})\widetilde{x}^*P_{\theta_{12}}). \tag{1.63}$$

Consider $\widetilde{x}_t$, which is given by the same formula (1.59), except for the upper integration, which is replaced with t. Since $\widetilde{x}_0 = 1$, we have $\tau(\delta_j(\widetilde{x}_0)\widetilde{x}_0^*P_{\theta_{12}}) = 0$. We let

$\widetilde{F}_t = \beta_{t\theta_{13},t\theta_{23}} P_{\theta_{12}}$, and $\widetilde{y}_t = (2\widetilde{F}_t - 1)\partial_t \widetilde{F}_t$, so that $\partial \widetilde{x}_t = \widetilde{x}_t \widetilde{y}_t$ and $\widetilde{y}_t^* = -\widetilde{y}_t$. We get

$$\begin{aligned}\mathrm{Ch}_j^1([a]_1) = \int_0^1 \partial_t \tau(\delta_j(\widetilde{x}_t)\widetilde{x}_t^* P_{\theta_{12}})dt = \int_0^1 \tau\Big(\big(\delta_j(\widetilde{x}_t\widetilde{y}_t) + \delta_j(\widetilde{x}_t)\widetilde{y}_t^*\big)\widetilde{x}_t^* P_{\theta_{12}}\Big)dt = \\ \int_0^1 \tau\big(\widetilde{x}_t \delta_j(\widetilde{y}_t)\widetilde{x}_t^* P_{\theta_{12}}\big)dt = \int_0^1 \tau\left(\delta_j(\widetilde{y}_t)\widetilde{F}_t\right) dt = \\ -\int_0^1 \tau\left(\widetilde{y}_t \delta_j \widetilde{F}_t\right) dt = -\int_0^1 \tau\left((2\widetilde{F}_t - 1)\partial_t \widetilde{F}_t \delta_j \widetilde{F}_t\right) dt. \quad (1.64)\end{aligned}$$

From (1.54), we know that $\partial_t \widetilde{F}_t = 2\pi i \left(\theta_{13}\delta_U \widetilde{F}_t + \theta_{23}\delta_V \widetilde{F}_t\right)$. Let

$$I_{lj} = -2\pi i \int_0^1 \tau\left((2\widetilde{F}_t - 1)\partial_l \widetilde{F}_t \delta_j \widetilde{F}_t\right) dt. \quad (1.65)$$

Then,

$$\mathrm{Ch}_j^1([a]_1) = \theta_{13} I_{1j} + \theta_{23} I_{2j}, \quad (1.66)$$

so it remains to evaluate I_{lj}. Note that derivations δ_U and δ_V are invariant under automorphisms $\beta_{s,t}$. Therefore, the expression in I_{lj} doesn't depend on t, and

$$I_{lj} = -2\pi i \tau\left((2P_{\theta_{12}} - 1)\partial_l P_{\theta_{12}} \delta_j P_{\theta_{12}}\right). \quad (1.67)$$

We have

$$\begin{aligned}I_{lj} = -2\pi i \tau\left((2P_{\theta_{12}} - 1)\partial_l P_{\theta_{12}} \delta_j P_{\theta_{12}}\right) = 2\pi i \tau\left(\partial_l P_{\theta_{12}}(2P_{\theta_{12}} - 1)\delta_j P_{\theta_{12}}\right) = \\ 2\pi i \tau\left((2P_{\theta_{12}} - 1)\delta_j P_{\theta_{12}} \partial_l P_{\theta_{12}}\right) = -I_{jl}. \quad (1.68)\end{aligned}$$

In particular, $I_{jj} = 0$. So, it remains to evaluate $I_{12} = -I_{21}$. We could use the explicit expression for $P_{\theta_{12}}$, substitute it into the definition of I_{12}, and evaluate the trace. That would involve evaluating 4 integrals of products of functions f and g from the definition of $P_{\theta_{12}}$. There is a simpler way to do so. From (1.67), we know that I_{12} is a smooth function of θ_{12} for $\theta_{12} \in (0,1)$ (and doesn't depend on anything else). So, from now on, we will denote this function with $I_{12}\colon (0,1) \to \mathbb{C}$, and write

$I_{12}(\theta_{12})$ instead of I_{12}. From the above, we know that

$$\mathrm{Ch}^1([a]_1) = (-\theta_{23}I_{12}(\theta_{12}), \theta_{13}I_{12}(\theta_{12}), -\theta_{12}). \tag{1.69}$$

Lemma 1.2.8. $I_{12}(\theta) = 1$ *for all* $\theta \in (0, 1)$.

Proof. Let's consider a noncommutative torus with $\theta_{12} = -\theta_{13} = \theta_{23} = \theta$. Then, noncommutative 3-torus algebra $A_\theta^{(3)}$ has automorphisms, induced by cyclic permutations of U, V, W. Therefore, the image of Ch^1 in $\mathbb{R}^3$ is symmetric with respect to cyclic permutations of the components. On the other hand, it is equal to $\mathbb{Z}^3 + \mathbb{Z}(-I_{12}(\theta)\theta, -I_{12}(\theta)\theta, -\theta)$.

Let's consider the case of irrational $\theta \in (0, 1)$, and consider the image of Ch^1 modulo $\mathbb{Z}^3$. This image is generated by $e_1 = (-I_{12}(\theta)\theta, -I_{12}(\theta)\theta, -\theta)$. Let $e_2 = (-\theta, -I_{12}(\theta)\theta, -I_{12}(\theta)\theta)$ and $e_3 = (-I_{12}(\theta)\theta, -\theta, -I_{12}(\theta)\theta)$ be cyclic permutations of e_1. Due to the symmetry of the algebra with respect to cyclic permutations of θ_{ij}, we get $e_2 \equiv \pm e_1 \pmod{\mathbb{Z}^3}$. But then, $e_3 \equiv \pm e_2$ and $e_1 \equiv \pm e_3$ with the same sign, so $e_1 \equiv \pm e_3 \equiv e_2 \equiv \pm e_1$, and therefore, the sign is $+$. Then, from $e_2 \equiv e_1$, we get $e_2 - e_1 = (n_1, n_2, n_3)$. The first coordinate of this equation gives $n_1 = I_{12}(\theta)\theta - \theta$.

Now, consider function $n(\theta) = I_{12}(\theta)\theta - \theta$ defined for all $\theta \in (0, 1)$. Since $I_{12}(\theta)$ is smooth, this function is continuous. From the above, we know that at all irrational points it takes integer values. This is only possible when $n(\theta) = n_1$ is a universal integer constant. So,

$$I_{12}(\theta) = 1 + n_1/\theta. \tag{1.70}$$

To show that $n_1 = 0$, consider a non-commutative 3-torus with arbitrary $\theta_{12}, \theta_{13}, \theta_{23} \in (0, 1)$. For such $A_\theta^{(3)}$, we have

$$\mathrm{Ch}^1\left(K_1\big(A_\theta^{(3)}\big)\right) = \mathbb{Z}^3 + \mathbb{Z}(-\theta_{23}I_{12}(\theta_{12}), \theta_{13}I_{12}(\theta_{12}), -\theta_{12}). \tag{1.71}$$

By a similar permutation argument, this image should coincide with

$$\mathbb{Z}^3 + \mathbb{Z}(-\theta_{23}, \theta_{13}I_{12}(\theta_{23}), -\theta_{12}I_{12}(\theta_{23})). \tag{1.72}$$

Comparing the first components, we get

$$-\theta_{23}s + m = -\theta_{23}I_{12}(\theta_{12}) \tag{1.73}$$

for some $m \in \mathbb{Z}$, $s \in \{-1, 1\}$ (m and s might depend on θ_{ij}). Substituting I_{12} from (1.70), and simplifying, we get

$$(1-s)\theta_{23} + m + n_1\theta_{23}/\theta_{12} = 0. \tag{1.74}$$

Choosing θ_{23} and θ_{12} s.t. 1, θ_{23} and θ_{23}/θ_{12} are linearly independent over $\mathbb{Q}$, we get $n_1 = 0$. $\square$

Thus, the values of the first component of the Chern character on the 4-th generator, described above, are

$$\mathrm{Ch}^1([a]_1) = (-\theta_{23}, \theta_{13}, -\theta_{12}). \tag{1.75}$$

1.3 The 4th generator: explicit formula

1.3.1 Introduction

As we noted earlier, we can only work with the $n = 3$, i.e., with noncommutative 3-torus. As earlier, we let $U = U_1$, $V = U_2$, $W = U_3$.

For this construction, we assume $\theta_{12}, \theta_{13} \notin \mathbb{Z}$. This condition is a bit stronger, than the condition from the previous section, where we required only θ_{12} to be non-integer. However, if only θ_{12} is non-integer, we can choose generators $\widetilde{U} = U$, $\widetilde{V} = UV$, $\widetilde{W} = W$, and perform the construction below. So, when we account for possibility of a "change of variables" before performing the construction of a generator, both semi-explicit construction above, and the explicit construction below, only require one of $\theta_{12}, \theta_{13}, \theta_{23}$ to be non-integer.

Given $\theta_{12}, \theta_{13} \notin \mathbb{Z}$, we construct an (explicit formula for an) element $a \in \mathcal{A}_\theta^{(3)}$, satisfying (1.75), and such that elements $[U]_1, [V]_1, [W]_1, [a]_1$ generates $K_1\big(A_\theta^{(3)}\big)$ as a free abelian group (see Theorem 1.3.3).

Since the definition of the algebra $A_\theta^{(n)}$ is invariant under changing elements of the matrix θ by integers, and, as we agreed above, θ_{12}, θ_{13} are not integers, in the construction below, we assume that $\theta_{12}, \theta_{13} \in (0, 1)$.

1.3.2 Ansatz

By analogy with the construction of the Rieffel projector [19] (its nice description given in [21, Exercise 5.8]), we will try to search for a unitary a of the form

$$a = f_0(U)W^* + f_1(U)VW^* + f_2(U)V + f_3(U) \tag{1.76}$$

for some continuous functions $f_0, f_1, f_2, f_3 \colon \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} = \{z \in \mathbb{C} \colon\ |z| = 1\}$. For such element a to be unitary, it has to satisfy $aa^* = 1$ and $a^*a = 1$. The following lemma shows that it is enough to check only one of these 2 conditions.

Lemma 1.3.1. *Let A be a unital C^*-algebra with a faithful trace, $a \in A$ with $aa^* = 1$. Then, $a^*a = 1$.*

Proof. From $aa^* = 1$, we know by C^*-algebra theory that a^*a is a projector, too. Let $p = 1 - a^*a$. Let τ be a faithful trace on A. We have

$$\tau(p) = \tau(1 - a^*a) = \tau(1) - \tau(a^*a) = \tau(1) - \tau(aa^*) = \tau(1) - \tau(1) = 0.$$

Since p is a projector, $p \geq 0$. Then, since τ is faithful, from $\tau(p) = 0$ we know that $p = 0$. Thus, $1 = a^*a$. $\square$

Let's look at the equation $aa^* = 1$. We can write aa^* in a polynomial-like form

$$aa^* = \sum_{l,j=-1}^{1} c_{l,j} V^l W^j \tag{1.77}$$

with $c_{l,j}$ being functions of U. Thus, the equality $aa^* = 1$ is equivalent to a system of 9 equations on functions of U. Not all of them are independent: one can see that the equation for $c_{l,j}$ is equivalent to the equation for $c_{-l,-j}$. Since the spectrum of U is $\mathbb{T}$, we can replace U with a variable z, and require the equations to be satisfied for any $z \in \mathbb{T}$. We are left with the following system of 5 equations on functions f_0, f_1, f_2, f_3:

$$\sum_{l=0}^{3} |f_l(z)|^2 = 1, \tag{1.78}$$

$$f_3(z)\bar{f}_0\left(e^{-2\pi i\theta_{13}}z\right) + e^{2\pi i\theta_{23}} f_2(z)\bar{f}_1\left(e^{-2\pi i\theta_{13}}z\right) = 0, \tag{1.79}$$

$$f_1(z)\bar{f}_0\left(e^{-2\pi i\theta_{12}}z\right) + f_2(z)\bar{f}_3\left(e^{-2\pi i\theta_{12}}z\right) = 0, \tag{1.80}$$

$$f_1(z)\bar{f}_3\left(e^{2\pi i(\theta_{13}-\theta_{12})}z\right) = 0, \tag{1.81}$$

$$f_2(z)\bar{f}_0\left(e^{-2\pi i(\theta_{13}+\theta_{12})}z\right) = 0. \tag{1.82}$$

By construction, if functions f_l satisfy these equations for any $z \in \mathbb{T}$, then $aa^* = 1$ and, thus, a is unitary.

1.3.3 Smooth solution

We will now find a smooth solution of the system (1.78)–(1.82). We will construct functions $f_l\colon \mathbb{T} \to \mathbb{C}$ from functions $f_{l,2}\colon \mathbb{R} \to \mathbb{C}$ via

$$f_l(z) = \sum_{t\in\mathbb{R}\colon\, e^{2\pi i t}=z} f_{l,2}(t). \tag{1.83}$$

Let's choose $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$ s.t. $x_3 - \theta_{13} = x_1$, $x_2 - \theta_{12} = x_0$. For example, that can be done with

$$x_1 = \theta_{12}(1-\theta_{13}), \qquad x_2 = \theta_{12}, \qquad x_3 = 1-(1-\theta_{12})(1-\theta_{13}). \tag{1.84}$$

As a zeroth approximation to the functions $f_{l,2}$, we take indicator-like functions $f_{l,0}\colon \mathbb{R} \to \mathbb{C}$, defined by

$$f_{l,0}(t) = \begin{cases} 1 & \text{if } x_l \le t < x_{l+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.85}$$

If we use $f_{l,0}$ instead of $f_{l,2}$, equations (1.78–1.82) will be satisfied, but these are not continuous, and, thus, can't be applied to elements of a C^*-algebra. To smoothen them, we choose $\varepsilon < \min_{l=0,\dots,3}(x_{l+1} - x_l)/4$ and a non-negative smooth function ω with support in $[-\varepsilon, \varepsilon]$ and integral equal to 1, denote with $\star$ the convolution operation, and define

$$f_{l,1} = \sqrt{\omega \star (f_{l,0})^2}. \tag{1.86}$$

Unfortunately, that will break equations (1.79) and (1.80), for z replaced with $e^{2\pi i t}$: equation (1.79) for $t \in (x_3 - \varepsilon, x_3 + \varepsilon)$ and (1.80) — for $t \in (x_4 - \varepsilon, x_4 + \varepsilon)$. To be more precise, in each of the equations two products will be exactly the same, but the coefficients will not add up to 0: $1 + e^{2\pi i \theta_{23}} \neq 0$ and $1 + 1 \neq 0$. To resolve this, we tweak $f_{3,1}$ by setting

$$f_{3,2}(t) = e^{2\pi i \varphi(t)} f_{3,1}(t) \tag{1.87}$$

with any smooth function $\varphi\colon \mathbb{R} \to \mathbb{R}$, satisfying

$$e^{2\pi i \varphi(t)} = -e^{2\pi i \theta_{23}} \quad \text{for } t < x_3 + \varepsilon, \tag{1.88}$$

$$e^{2\pi i \varphi(t)} = -1 \qquad \text{for } t > x_4 - \varepsilon. \tag{1.89}$$

That is, $\varphi(t)$ has predefined values (up to adding an integer) outside $(x_3 + \varepsilon, x_4 - \varepsilon)$ and has to smoothly interpolate in-between. It follows that $\varphi(x_4) - \varphi(x_3) = -\theta_{23} + n_\varphi$ for some integer n_φ. Moreover, by making an appropriate choice of φ we can make n_φ to be equal to any given integer. In order to fix a specific generator a, we will later fix $n_\varphi = 1$. For $l = 0, 1, 2$ let

$$f_{l,2} = f_{l,1}. \tag{1.90}$$

By construction, functions $f_{l,2}$ satisfy the following properties.

1. For any $t \in [-\varepsilon, \varepsilon]$, we have

$$|f_{l,2}(x_l - t)| = |f_{j,2}(x_j - t)| \quad \text{and} \quad |f_{l,2}(x_{l+1} + t)| = |f_{j,2}(x_{j+1} + t)| . \tag{1.91}$$

2. $f_{l,2}(t) = 0$ for $t \notin (x_l - \varepsilon, x_{l+1} + \varepsilon)$, $|f_{l,2}(t)| = 1$ for $t \in [x_l + \varepsilon, x_{l+1} - \varepsilon]$.

3. For $l \neq 3$, we have $f_{l,2}(t) \in \mathbb{R}_+ = \{r \in \mathbb{R} \colon r \geq 0\}$ for all $t \in \mathbb{R}$; $f_{3,2}(t)\overline{s_{\text{left}}} \in \mathbb{R}_+$ for $t < x_3 + \varepsilon$, $f_{3,2}(t)\overline{s_{\text{right}}} \in \mathbb{R}_+$ for $t > x_4 - \varepsilon$, where $s_{\text{left}} = -e^{2\pi i \theta_{23}}$, $s_{\text{right}} = -1$.

4. Functions $t \mapsto |f_{l,2}(t)|^2$ sum up to 1:

$$\sum_{l=0}^{3} \sum_{k \in \mathbb{Z}} |f_{l,2}(t+k)|^2 = 1. \tag{1.92}$$

Here, the last property follows from linearity of the convolution with ω and equality $\omega \star 1 = 1$.

Lemma 1.3.2. *Let $f_{l,2}$ for $l = 0, 1, 2, 3$ be any functions $\mathbb{R} \to \mathbb{C}$, satisfying properties 1–4 above, and let functions f_l be defined by* (1.83) *from these $f_{l,2}$. Then, functions f_l satisfy equations (1.78–1.82).*

Proof. Equation (1.78) follows from (1.92).

To show (1.79), notice that $f_3(z)$ may only be non-zero for $z = e^{2\pi i t}$ with $t \in (x_3 - \varepsilon, x_4 + \varepsilon)$, while $f_0(e^{-2\pi i \theta_{13}} z)$ may only be nonzero for $z = e^{2\pi i t}$ with $t \in (x_0 + \theta_{13} - \varepsilon, x_3 + \varepsilon)$ (here we've used that $x_1 + \theta_{13} = x_3$). Also notice that $x_0 + \theta_{13} - \varepsilon < x_3 - \varepsilon < x_3 + \varepsilon < x_4 + \varepsilon$, and, since $\theta_{13} = x_3 - x_1 > x_3 - x_2 > 4\varepsilon$, we have $(x_4 + \varepsilon) - (x_0 + \theta_{13} - \varepsilon) < 1$. Therefore, we don't deal with two different values of t giving the same value of $z = e^{2\pi i t}$, and the first term in (1.79) may only be nonzero for $z = e^{2\pi i t}$ with $t \in (x_3 - \varepsilon, x_3 + \varepsilon)$. A similar argument shows that the same is true for the second term of (1.79). Then, for $t = x_3 + t_1 \in (x_3 - \varepsilon, x_3 + \varepsilon)$ and $z = e^{2\pi i t}$, using (1.83), we get

$$\left| f_3(z) \bar{f}_0(e^{-2\pi i \theta_{13}} z) \right| = |f_{3,2}(t)| \cdot |f_{0,2}(t - \theta_{13})| = |f_{3,2}(x_3 + t_1)| \cdot |f_{0,2}(x_1 + t_1)|,$$

$$\left| f_2(z) \bar{f}_1(e^{-2\pi i \theta_{13}} z) \right| = |f_{2,2}(t)| \cdot |f_{1,2}(t - \theta_{13})| = |f_{2,2}(x_3 + t_1)| \cdot |f_{1,2}(x_1 + t_1)|.$$

However, by Property 1 above, we get $|f_{3,2}(x_3 + t_1)| = |f_{1,2}(x_1 + t_1)|$ and $|f_{0,2}(x_1 + t_1)| = |f_{2,2}(x_3 + t_1)|$. Thus, the terms have the same absolute value, but by Property 3, they have opposite signs. So, they cancel each other.

Equation (1.80) is satisfied by the similar reason.

Let us now show that (1.81) is satisfied too. Let

$$a_1 = x_1 - \varepsilon, \qquad b_1 = x_2 + \varepsilon,$$
$$a_2 = x_3 - \theta_{13} + \theta_{12} - \varepsilon = x_1 + \theta_{12} - \varepsilon, \quad b_2 = x_4 - \theta_{13} + \theta_{12} + \varepsilon = 1 + x_2 - \theta_{13} + \varepsilon.$$

Then, the first term in (1.81), $f_1(z)$, may only be non-zero for $z = e^{2\pi it}$ with $t \in (a_1, b_1)$, while the second term $f_3(e^{2\pi i(\theta_{13}-\theta_{12})}z)$ — with $t \in (a_2, b_2)$. Notice that $a_2 - b_1 = x_1 - x_0 - 2\varepsilon > 0$ and $a_1 + 1 - b_2 = x_3 - x_2 - 2\varepsilon > 0$. Thus, $a_1 < b_1 < a_2 < b_2 < a_1 + 1$. So, the product in (1.81) is 0 everywhere.

A similar computation shows that (1.82) is satisfied, too. □

1.3.4 Non-smooth solution

Although we will be working with a smooth solution, described above, one can deform it to get a similar solution, given by simpler formulas, but lacking smoothness. We assume $x_0, \ldots, x_4$, ε and n_φ are chosen as above. In this solution, functions f_l for $l = 0, 1, 2$ are given by

$$f_l(z) = \begin{cases} \sqrt{1/2 + (t - x_l)/(2\varepsilon)} & \text{if } z = e^{2\pi it} \text{ with } t \in (x_l - \varepsilon, x_l + \varepsilon), \\ 1 & \text{if } z = e^{2\pi it} \text{ with } t \in [x_l + \varepsilon, x_{l+1} - \varepsilon], \\ \sqrt{1/2 - (t - x_{l+1})/(2\varepsilon)} & \text{if } z = e^{2\pi it} \text{ with } t \in (x_{l+1} - \varepsilon, x_{l+1} + \varepsilon), \\ 0 & \text{otherwise.} \end{cases} \tag{1.93}$$

and f_3 is given by

$$f_3(z) = \begin{cases} -\exp(2\pi i\theta_{23})\sqrt{1/2 + (t - x_3)/(2\varepsilon)} \\ \qquad\qquad\qquad\qquad \text{if } z = e^{2\pi it} \text{ with } t \in (x_3 - \varepsilon, x_3 + \varepsilon), \\ -\exp\big(2\pi i(\theta_{23} - n_\varphi)(x_4 - \varepsilon - t)/(x_4 - x_3 - 2\varepsilon)\big) \\ \qquad\qquad\qquad\qquad \text{if } z = e^{2\pi it} \text{ with } t \in [x_3 + \varepsilon, x_4 - \varepsilon], \\ -\sqrt{1/2 - (t - x_4)/(2\varepsilon)} \\ \qquad\qquad\qquad\qquad \text{if } z = e^{2\pi it} \text{ with } t \in (x_4 - \varepsilon, x_4 + \varepsilon), \\ 0 \\ \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{1.94}$$

1.3.5 Chern character

In this section, we compute the value $\mathrm{Ch}^1([a]_1)$ for the element a constructed above. Note that since unitaries, constructed in 1.3.3 and 1.3.4, are homotopic to each other, they give the same class $[a]_1$, and, thus, the same value $\mathrm{Ch}^1([a]_1)$. In the computation below, we will use the smooth version.

Using the trace property of τ and unitarity of a, we get

$$\mathrm{Ch}^1_j([a]_1) = \tau(a^{-1}\delta_j a) = \tau((\delta_j a)a^*). \tag{1.95}$$

Substituting a from (1.76),

$$\mathrm{Ch}^1_2([a]_1) = \mathrm{Ch}^1_V([a]_1) = \tau((f_1(U)VW^* + f_2(U)V)\cdot \\ (f_0(U)W^* + f_1(U)VW^* + f_2(U)V + f_3(U))^*) = \tau\left(|f_1(U)|^2 + |f_2(U)|^2\right) = \\ \int_0^1 \left(\left|f_1(e^{2\pi it})\right|^2 + \left|f_2(e^{2\pi it})\right|^2\right) dt = (x_2 - x_1) + (x_3 - x_2) = x_3 - x_1 = \theta_{13}. \tag{1.96}$$

Similarly,

$$\mathrm{Ch}^1_3([a]_1) = \mathrm{Ch}^1_W([a]_1) =$$
$$-\tau((f_0(U)W^* + f_1(U)VW^*)(f_0(U)W^* + f_1(U)VW^* + f_2(U)V + f_3(U))^*) =$$
$$\tau\left(|f_0(U)|^2 + |f_1(U)|^2\right) = (x_1 - x_0) + (x_2 - x_1) = x_2 - x_0 = \theta_{12}. \quad (1.97)$$

Finally,

$$\mathrm{Ch}^1_1([a]_1) = \mathrm{Ch}^1_U([a]_1) = \tau\Big((\delta_U f_0(U))W^* + (\delta_U f_1(U))VW^* +$$
$$(\delta_U f_2(U))V + (\delta_U f_3(U))\big)\big(f_0(U)W^* + f_1(U)VW^* + f_2(U)V + f_3(U)\big)^*\Big) =$$
$$\tau\left(\sum_{l=0}^{3}\big(\delta_U f_l(U)\big)\bar{f}_l(U)\right) = \frac{1}{2\pi i}\sum_{l=0}^{3}\int_{\mathbb{R}}\big(\partial_t f_{l,2}(t)\big)\bar{f}_{l,2}(t)dt =$$
$$\frac{1}{2\pi i}\sum_{l=0}^{3}\int_{\mathbb{R}}\frac{1}{2}\Big(\big(\partial_t f_{l,2}(t)\big)\bar{f}_{l,2}(t) - \big(\partial_t \bar{f}_{l,2}(t)\big)f_{l,2}(t)\Big)dt. \quad (1.98)$$

One can notice that for $l \neq 3$ expression inside the integral is zero, and for $l = 3$, it may only be nonzero for $t \in (x_3 + \varepsilon, x_4 - \varepsilon)$, where it is equal to $2\pi i \partial_t \varphi(t)$ (see (1.87)). Thus,

$$\mathrm{Ch}^1_1([a]_1) = \mathrm{Ch}^1_U([a]_1) = \varphi(x_4) - \varphi(x_3) = -\theta_{23} + n_\varphi. \quad (1.99)$$

Thus, we have obtained,

$$\mathrm{Ch}^1([a]_1) = (-\theta_{23} + n_\varphi, \theta_{13}, -\theta_{12}). \quad (1.100)$$

From now on, let's fix a with $n_\varphi = 0$, so that

$$\mathrm{Ch}^1([a]_1) = (-\theta_{23}, \theta_{13}, -\theta_{12}). \quad (1.101)$$

Note that if at least one of θ_{12}, θ_{13} and θ_{23} is irrational, it follows from (1.101) that $[U]_1, [V]_1, [W]_1, [a]_1$ generate $K_1\big(A^{(3)}_\theta\big)$ as a free abelian group.

1.3.6 The 4th generator and Pimsner–Voiculescu short exact sequence

The main goal of this section is the following theorem.

Theorem 1.3.3. *Suppose $\theta \notin M_3(\mathbb{Z})$. Let a be given by* (1.76) *with functions f_l satisfying conditions of Lemma 1.3.2. Then,*

1. *$\delta_1^{PV}([a]_1) = [P_{\theta_{12}}]_0$, where $P_{\theta_{12}}$ is the Rieffel projector;*
2. *$K_1\big(A_\theta^{(3)}\big)$ is generated by $[U]_1$, $[V]_1$, $[W]_1$, and $[a]_1$.*

Note that if θ has at least one irrational element, then Theorem 1.3.3 follows from the previous section.

Proof. Construction of Pimsner–Voiculescu short exact sequence can be summarized by the following commutative diagram.

$$
\begin{array}{ccccccccccc}
K_1(J) & \xrightarrow{i_*} & K_1(\mathcal{T}) & \xrightarrow{p_*} & K_1(\mathcal{T}/J) & \xrightarrow{\partial_J} & K_0(J) & \xrightarrow{i_*} & K_0(\mathcal{T}) & \xrightarrow{p_*} & K_0(\mathcal{T}/J) \\
\widetilde{\psi}_* \uparrow \simeq & & \| & & \widetilde{\pi}_* \downarrow \simeq & & \widetilde{\psi}_* \uparrow \simeq & & \| & & \widetilde{\pi}_* \downarrow \simeq \\
K_1(A \otimes K) & \xrightarrow{\psi_*} & K_1(\mathcal{T}) & \xrightarrow{\pi_*} & K_1(A \times_\alpha \mathbb{Z}) & \xrightarrow{\partial_{\mathcal{T}}} & K_0(A \otimes K) & \xrightarrow{\psi_*} & K_0(\mathcal{T}) & \xrightarrow{\pi_*} & K_0(A \times_\alpha \mathbb{Z}) \\
\uparrow \simeq & & d_* \uparrow \simeq & & \| & & \uparrow \simeq & & d_* \uparrow \simeq & & \| \\
K_1(A) & \xrightarrow{\mathrm{id}_* - \alpha(-1)_*} & K_1(A) & \xrightarrow{i_*} & K_1(A \times_\alpha \mathbb{Z}) & \xrightarrow{\delta_1^{\mathrm{PV}}} & K_0(A) & \xrightarrow{\mathrm{id}_* - \alpha(-1)_*} & K_0(A) & \xrightarrow{i_*} & K_0(A \times_\alpha \mathbb{Z})
\end{array}
\tag{1.102}
$$

Here, the first two rows are 6-term exact sequences, associated with short exact sequences of C^*-algebras. The last row represents the Pimsner–Voiculescu 6-term exact sequence. All rows are isomorphic to each other. We use the diagram (1.102) for $A = A_\theta^{(2)}$, i.e., the C^*-subalgebra in $A_\theta^{(3)}$, generated by U and V. Then, $A \times_\alpha \mathbb{Z}$ is $A_\theta^{(3)}$. We write a in the form

$$a = a_1 W^* + a_2, \tag{1.103}$$

where

$$a_1 = f_0(U) + f_1(U)V, \qquad a_2 = f_2(U)V + f_3(U). \tag{1.104}$$

Let $\widetilde{a} \in \mathcal{T}$ be given by

$$\widetilde{a} = a_1 W^* \otimes S^* + a_2 \otimes 1, \tag{1.105}$$

so that $\pi(\widetilde{a}) = a$.

Note that for any $b \in A_\theta^{(2)}$ we have

$$(W \otimes S) \cdot (b \otimes 1) = (WbW^* \otimes 1) \cdot (W \otimes S), \quad (W \otimes S)^* \cdot (W \otimes S) = 1 \otimes 1, \tag{1.106}$$

$$(W \otimes S) \cdot (W \otimes S)^* = 1 \otimes (1 - P). \tag{1.107}$$

That is, all commutation relations for $W \otimes S$ are the same, as for W, except (1.107), where an additional $P \otimes 1$ is subtracted. If we have a computation (involving only expanding products of expressions, polynomial in W and W^* with coefficients in $A_\theta^{(2)}$) in $A_\theta^{(3)}$, then we have the counterpart computation in $\mathcal{T}$, which is the same up to replacement $W \to W \otimes S$, except when we multiply $W \otimes S$ by $(W \otimes S)^*$ (in this order). In this last case, we should subtract $P \otimes 1$ from the result.

We know that $aa^* = 1$. Therefore, $\widetilde{a}\widetilde{a}^* = 1$, so $\widetilde{a}^*$ is an isometry. We know that $a^*a = 1$. Therefore,

$$\widetilde{a}^*\widetilde{a} = 1 - (a_1 W^*)^*(a_1 W^*) \otimes P = 1 - W a_1^* a_1 W^* \otimes P. \tag{1.108}$$

Note that in particular from (1.108), it follows that $W a_1^* a_1 W^*$, $a_1^* a_1$ and $a_1 a_1^*$ are projectors. From (1.108), we have

$$\partial_{\mathcal{T}}([a]_1) = [\widetilde{a}\widetilde{a}^*]_0 - [\widetilde{a}^*\widetilde{a}]_0 = [1]_0 - [1 - W a_1^* a_1 W^* \otimes P]_0 = [W a_1^* a_1 W^* \otimes P]_0. \tag{1.109}$$

Thus,

$$\delta_1^{\mathrm{PV}}([a]_1) = [W a_1^* a_1 W^*]_0 = [a_1^* a_1]_0 = [a_1 a_1^*]_0. \tag{1.110}$$

Now,

$$a_1 a_1^* = (f_0(U) + f_1(U)V)(f_0(U) + f_1(U)V)^* = V^* \bar{g}_1(U) + g_0(U) + g_1(U)V, \tag{1.111}$$

where

$$g_1(U) = \bar{f}_0\left(e^{-2\pi i\theta_{12}}U\right) f_1(U), \qquad g_0(U) = |f_0(U)|^2 + |f_1(U)|^2 . \tag{1.112}$$

It is easy to see that $a_1 a_1^*$ is the Rieffel projector.

The second claim of the theorem follows from the first and the exactness of the sequence (1.102) in $K_1(A \times_\alpha \mathbb{Z})$. $\square$

Chapter 2

Differential calculus on graded-commutative algebras, associated with noncommutative tori

2.1 Graded algebras, modules and linear differential operators

We use notation from [27], [14], [25], [26] for the constructions of differential geometry. We also use the notation for the sign rule from [8]. Since our main goal is the application of the construction to the algebra of noncommutative torus, we restrict all definitions to the case of unital $\mathbb{C}$-algebras, and use signs in $U(1) = \{z \in \mathbb{C}\colon\ |z| = 1\}$. In general, one may use k-algebras for any field k, and any signs in $k^* = k \setminus \{0\}$.

Definition 2.1.1. A grading group is an abelian group Γ endowed with a bilinear antisymmetric map $\lambda\colon \Gamma \times \Gamma \to U(1)$. We say that $\mathbb{C}$-algebra A is Γ-graded commutative if and only if it is represented as a direct sum $A = \bigoplus_{g\in\Gamma} A_g$ with $A_f A_g \subset A_{f+g}$ for $f, g \in G$, and $ab = \lambda(f, g)ba$ for $a \in A_f, b \in A_g$. If $a \in A_f$, we say that a is a homogeneous element of A of degree f, and denote its degree f with $\widetilde{a} = \deg a$, so that the commutation relation above can be written as

$$ab = \lambda(\widetilde{a}, \widetilde{b})ba. \tag{2.1}$$

The Koszul sign rule allows us, given a definition involving commutative algebras, to construct a definition involving graded-commutative ones. To do that, one has to take each equality of that definition, and choose a "standard" order of terms in products. Then, for every product-like expression, having a different order of terms, one has to multiply it by a "sign," corresponding to the permutation we have performed. For example, if we have variables a, b, c, d, and choose the standard order to be $abcd$, then the expression $-3dbca$ have to be replaced with $-3\lambda(\widetilde{a}+\widetilde{b}+\widetilde{c},\widetilde{d})\lambda(\widetilde{a},\widetilde{b}+\widetilde{c})dbca$.

Definition 2.1.2. Let A be a Γ-graded commutative algebra. We say that M is a (graded) A-module if and only if M is an A-module, endowed with a grading $M = \bigoplus_{g\in\Gamma} M_g$, s.t. $A_f M_g \subset M_{f+g}$.

In this work, all algebras are, by default, assumed to be graded-commutative and unital, all modules are assumed to be graded. For any (Γ, λ) as above, we can interpret $\mathbb{C}$ as (Γ, λ)-commutative algebra, by defining $\mathbb{C}_0 = \mathbb{C}$ and $\mathbb{C}_g = \{0\}$ for $g \in \Gamma \setminus \{0\}$.

Definition 2.1.3. Let A be an algebra, and let P and Q be A-modules. Then, we define $\mathcal{H}om(P,Q) = \mathcal{H}om_A(P,Q)$ to be the module of graded A-linear maps from P to Q. That is,

$$\mathcal{H}om_A(P,Q) = \bigoplus_{g\in\Gamma} \mathcal{H}om_{A,g}(P,Q), \tag{2.2}$$

$$\begin{aligned}\mathcal{H}om_{A,g}(P,Q) = \{\varphi\colon P \to Q :\ \ & \varphi(P_f) \subset Q_{g+f};\\ & \forall p_1, p_2 \in P\ \varphi(p_1+p_2) = \varphi(p_1) + \varphi(p_2);\\ & \forall p \in P\ \ \forall \text{ homogeneous } a \in A\ \ \varphi(ap) = \lambda(g,\widetilde{a})a\varphi(p)\}.\end{aligned} \tag{2.3}$$

The A-module structure is given by $(a\varphi)(p) = a\varphi(p)$.

Let again A be an algebra, and P and Q be A-modules. Note that P and Q can be re-interpreted as $\mathbb{C}$-modules, and, thus, the definition above defines a $\mathbb{C}$-module $\mathcal{H}om_{\mathbb{C}}(P,Q)$. It has two A-module structures: $(a\cdot\varphi)(p) = a\varphi(p)$ and $(a\cdot_R\varphi)(p) = \lambda(\widetilde{a},\widetilde{\varphi})\varphi(ap)$. We reserve the name $\mathcal{H}om_{\mathbb{C}}(P,Q)$ for the module with the first module structure $(\cdot)$ and denote the same set, endowed with the second module structure $(\cdot_R)$, by $\mathcal{H}om^{+}_{\mathbb{C}}(P,Q)$.

Note that $\mathcal{H}om_A(P,Q)$ can be interpreted as a submodule of both $\mathcal{H}om_{\mathbb{C}}(P,Q)$ and $\mathcal{H}om^+_{\mathbb{C}}(P,Q)$:

$$\mathcal{H}om_A(P,Q) = \{\varphi \in \mathcal{H}om_{\mathbb{C}}(P,Q) : \forall a \in A \;\; a \cdot \varphi = a \cdot_R \varphi\}. \tag{2.4}$$

Definition 2.1.4. Let Δ, ∇ be homogeneous elements of $\mathcal{H}om_{\mathbb{C}}(P,P)$. We define their graded commutator to be

$$[\Delta, \nabla] = \Delta \circ \nabla - \lambda(\widetilde{\Delta}, \widetilde{\nabla}) \nabla \circ \Delta. \tag{2.5}$$

The definition is extended to non-homogeneous elements by $\mathbb{C}$-linearity.

It satisfies the following properties:

1. bilinearity:

$$[\Delta, \nabla_1 + \nabla_2] = [\Delta, \nabla_1] + [\Delta, \nabla_2], \quad [\Delta_1 + \Delta_2, \nabla] = [\Delta_1, \nabla] + [\Delta_2, \nabla], \tag{2.6}$$

$$c[\Delta, \nabla] = [c\Delta, \nabla] = [\Delta, c\nabla] \text{ for } c \in \mathbb{C}; \tag{2.7}$$

2. (graded) anti-symmetry:

$$[\nabla, \Delta] = -\lambda(\widetilde{\nabla}, \widetilde{\Delta})[\Delta, \nabla]; \tag{2.8}$$

3. (graded) version of Jacobi identity:

$$[\nabla, [\Delta, \Box]] + \lambda(\widetilde{\nabla} + \widetilde{\Delta}, \widetilde{\Box})[\Box, [\nabla, \Delta]] + \lambda(\widetilde{\nabla}, \widetilde{\Delta} + \widetilde{\Box})[\Delta, [\Box, \nabla]] = 0. \tag{2.9}$$

Alternatively, the Jacobi identity can be written as

$$[\nabla, [\Delta, \Box]] = [[\nabla, \Delta], \Box] + \lambda(\widetilde{\nabla}, \widetilde{\Delta})[\Delta, [\nabla, \Box]]. \tag{2.10}$$

Definition 2.1.5. Let A be an algebra, let P, Q be A-modules, and let $a \in A$. Note that a can be interpreted as an operator "multiplication by a" $P \to P$ and $Q \to Q$. Define $\delta_a\colon \mathcal{H}om_{\mathbb{C}}(P,Q) \to \mathcal{H}om_{\mathbb{C}}(P,Q)$ with $\delta_a\Delta = [a,\Delta] = a\circ\Delta - \lambda(\widetilde{a},\widetilde{\Delta})\Delta\circ a$. For $a \in A^{\otimes n}$, we define $\delta_a^n\colon \mathcal{H}om_{\mathbb{C}}(P,Q) \to \mathcal{H}om_{\mathbb{C}}(P,Q)$ with $\delta^n_{a_1\otimes\cdots\otimes a_n} = \delta_{a_1}\circ\cdots\circ\delta_{a_n}$. We define the module of linear differential operators from P to Q of order $\le k$ with

$$\mathrm{Diff}_k(P,Q) = \{\Delta \in \mathcal{H}om_{\mathbb{C}}(P,Q)\colon \forall a \in A^{\otimes(k+1)}\ \delta_a^{k+1}\Delta = 0\}. \tag{2.11}$$

Define the module $\mathrm{Diff}_k^+(P,Q)$ by replacing $\mathcal{H}om_{\mathbb{C}}(P,Q)$ in (2.11) with $\mathcal{H}om^+_{\mathbb{C}}(P,Q)$, so that $\mathrm{Diff}_k^+(P,Q)$ has the same elements, as $\mathrm{Diff}_k(P,Q)$, but uses $\cdot_R$ for its A-module structure.

Note that $\mathrm{Diff}_{-1}(P,Q) = 0$, $\mathrm{Diff}_0(P,Q) = \mathcal{H}om_A(P,Q)$, $\mathrm{Diff}_0(A,Q) = Q$, and

$$\mathrm{Diff}_{k+1}(P,Q) = \{\Delta \in \mathcal{H}om_{\mathbb{C}}(P,Q)\colon \forall a \in A\ \delta_a\Delta \in \mathrm{Diff}_k(P,Q)\}. \tag{2.12}$$

Moreover, (2.12) will remain true, if we replace "$\forall a$" with "$\forall$ homogeneous a".

Lemma 2.1.6. *If $\Delta \in \mathrm{Diff}_k(P,Q)$, $\nabla \in \mathrm{Diff}_l(Q,R)$, then $\nabla\circ\Delta \in \mathrm{Diff}_{k+l}(P,R)$.*

Proof. It is enough to prove the lemma for homogeneous Δ and ∇. We use induction in $k+l$. If $k=-1$ or $l=-1$, then $\nabla\circ\Delta = 0 \in \mathrm{Diff}_{k+l}(P,R)$. Now, assume $k,l \ge 0$, and let a be a homogeneous element of A. Using the definition of δ_a, we get

$$\delta_a(\nabla\circ\Delta) = \delta_a(\nabla)\circ\Delta + \lambda(\widetilde{a},\widetilde{\nabla})\nabla\circ\delta_a(\Delta). \tag{2.13}$$

By induction hypothesis, the right hand side belongs to $\mathrm{Diff}_{k+l-1}(P,R)$, so $\nabla\circ\Delta \in \mathrm{Diff}_{k+l}(P,R)$ by (2.12). ☐

Lemma 2.1.7. *If $\Delta \in \mathrm{Diff}_k(A,A)$, $\nabla \in \mathrm{Diff}_l(A,A)$, then $[\nabla,\Delta] \in \mathrm{Diff}_{k+l-1}(A,A)$.*

Proof. We prove the statement by induction in $k+l$. If $l=0$, map ∇ has to coincide with multiplication by some $a \in A$. We get

$$[\nabla,\Delta] = [a,\Delta] = \delta_a\Delta \in \mathrm{Diff}_{k-1}(A,A). \tag{2.14}$$

The same argument proves the lemma for $k = 0$. Now, assume $k, l > 0$, and notice that the inductive step follows from the Jacobi identity:

$$\delta_a([\nabla, \Delta]) = [a, [\nabla, \Delta]] = [[a, \nabla], \Delta] + \lambda(\widetilde{a}, \widetilde{\nabla})[\nabla, [a, \Delta]] = [\delta_a \nabla, \Delta] + \lambda(\widetilde{a}, \widetilde{\nabla})[\nabla, \delta_a \Delta]. \tag{2.15}$$

□

2.2 Jet Spaces

Definition 2.2.1. Let A be an algebra, and let P be an A-module. Define a module $J^k(P)$ together with a k-th order differential operator $j_k \colon P \to J^k(P)$ to be a representing object of $\mathrm{Diff}_k(P, \bullet)$. Thus, if $J^k(P)$ exists, $\mathrm{Diff}_k(P, \bullet) \simeq \mathcal{H}om_A(J^k(P), \bullet)$ and j_k is the preimage of the map $id_{J^k(P)} \in \mathcal{H}om_A(J^k(P), J^k(P))$. We say $j_k \colon P \to J^k(P)$ is the universal differential operator of order $\leq k$, acting on the module P.

$$\begin{array}{ccc} J^k(P) & & \\ {\scriptstyle j_k}\uparrow & \searrow^{\exists ! \varphi_\nabla} & \\ P & \xrightarrow{\ \nabla\ } & Q \end{array} \tag{2.16}$$

As illustrated by the diagram above, the definition of $(j_k, J^k(P))$ is equivalent to the following statement: for any A-module Q and any $\nabla \in \mathrm{Diff}_k(P, Q)$, there is a unique $\varphi_\nabla \in \mathcal{H}om_A(J^k(P), Q)$, satisfying the equality $\varphi_\nabla \circ j_k = \nabla$.

Lemma 2.2.2. *The universal differential operator $j_k \colon P \to J^k(P)$ exists for any algebra A and any A-module P.*

Proof. Consider the module $\widetilde{J}^k(P) = A \otimes_{\mathbb{C}} P$, with the multiplication defined by $a(b \otimes p) = (ab) \otimes p$, and consider a $\mathbb{C}$-linear map $\widetilde{j}_k \colon P \to \widetilde{J}^k \colon p \mapsto 1 \otimes p$. Let I be the submodule of $\widetilde{J}^k$, generated by $\{\delta_a(\widetilde{j}_k)(p) : a \in A^{\otimes(k+1)}, p \in P\}$. Then, let $J^k(P) = \widetilde{J}^k / I$ with j_k induced by $\widetilde{j}_k$. Note that submodule I introduces only those restrictions, which are satisfied by any element $\nabla \in \mathrm{Diff}_k(P, Q)$. Therefore, $(j_k, J^k(P))$ represents $\mathrm{Diff}_k(P, \bullet)$. □

Lemma 2.2.3. *Functor* $\mathrm{Diff}_k^+(\bullet, R)$ *is representable for any* A*-module* R.

Proof. Let Q be an A-module, and consider the map

$$\mathrm{Diff}_k^+(Q,R) \to \mathcal{H}om_A(Q, \mathrm{Diff}_k^+(A,R))\colon \nabla \mapsto \varphi_\nabla, \tag{2.17}$$

where φ_∇ is given by

$$\varphi_\nabla(q)(a) = \lambda(\widetilde{q}, \widetilde{a})\nabla(aq). \tag{2.18}$$

The following computations check that $a \mapsto \varphi_\nabla(q)(a)$ is a differential operator, and that $\nabla \mapsto \varphi_\nabla$ is a homomorphism:

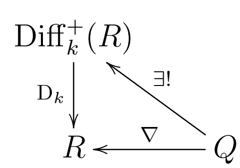
$$(\delta_b(\varphi_\nabla(q)))(a) = b(\varphi_\nabla(q)(a)) - \lambda(\widetilde{b}, \widetilde{\nabla} + \widetilde{q})\varphi_\nabla(q)(ba) = \\ \lambda(\widetilde{q}, \widetilde{a})b(\nabla(aq)) - \lambda(\widetilde{b}, \widetilde{\nabla})\lambda(\widetilde{q}, \widetilde{a})\nabla(baq) = \lambda(\widetilde{q}, \widetilde{a})(\delta_b\nabla)(aq) = \varphi_{\delta_b\nabla}(q)(a), \tag{2.19}$$

$$\varphi(b \cdot_R \nabla)(q)(a) = \lambda(\widetilde{q}, \widetilde{a})(b \cdot_R \nabla)(aq) = \lambda(\widetilde{q}, \widetilde{a})\lambda(\widetilde{b}, \widetilde{\nabla})\nabla(baq) = \\ \lambda(\widetilde{b}, \widetilde{\nabla} + \widetilde{q})\varphi_\nabla(q)(ba) = (b \cdot_R (\varphi_\nabla(q)))(a) = (b\varphi_\nabla)(q)(a). \tag{2.20}$$

We have constructed a well-defined natural transformation of functors, given by the family of maps $\nabla \mapsto \varphi_\nabla$. Its inverse is given by $\nabla \mapsto \nabla_\varphi$, where

$$\nabla_\varphi(q) = \varphi(q)(1). \tag{2.21}$$

□



(2.22)

Definition 2.2.4. As illustrated by diagram (2.22), we denote the representing object of $\mathrm{Diff}_k^+(\bullet, R)$, given by the lemma above, with $(\mathrm{D}_k, \mathrm{Diff}_k^+(R))$, so that

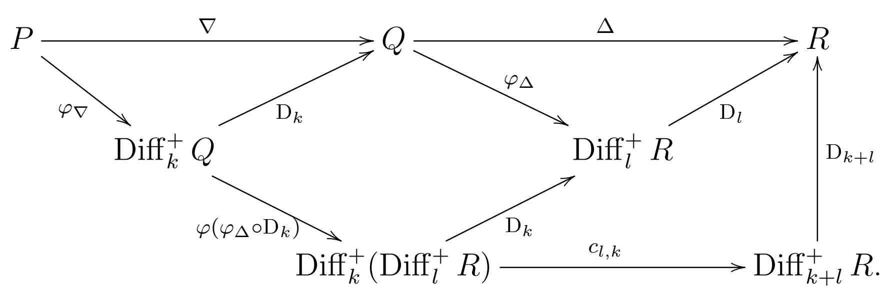
$$\mathrm{Diff}_k^+(R) = \mathrm{Diff}_k^+(A, R), \quad \mathrm{D}_k \in \mathrm{Diff}_k(\mathrm{Diff}_k^+(R), R), \quad \mathrm{D}_k(\nabla) = \nabla(1), \quad \deg(\mathrm{D}_k) = 0. \tag{2.23}$$

Definition 2.2.5. Composition $\mathrm{D}_l \circ \mathrm{D}_k \colon \mathrm{Diff}_k^+(\mathrm{Diff}_l^+ R) \to R$ is a differential operator of order $\leq k + l$. Therefore, by the universal property (2.22), we get a homomorphism $\mathrm{Diff}_k^+(\mathrm{Diff}_l^+ R) \to \mathrm{Diff}_{k+l}^+ R$. By definition, let's call this homomorphism "universal composition" $c_{l,k}$.

Universal composition is associative in a certain sense (see [14, page 22]), and generates a natural transformation of functors $\mathrm{Diff}_k^+(\mathrm{Diff}_l^+ \bullet) \to \mathrm{Diff}_{k+l}^+$:

$$\begin{CD} \mathrm{Diff}_k^+(\mathrm{Diff}_l^+ R) @>{\mathrm{D}_k}>> \mathrm{Diff}_l^+ R \\ @V{c_{l,k}}VV @VV{\mathrm{D}_l}V \\ \mathrm{Diff}_{k+l}^+ R @>{\mathrm{D}_{k+l}}>> R \end{CD} \quad . \tag{2.24}$$

Given any pair of differential operators $\nabla \in \mathrm{Diff}_k^+(P, Q), \Delta \in \mathrm{Diff}_l^+(Q, R)$, we have the following commutative diagram:



(2.25)

Here, maps, generated by the universal property (2.22), are marked with $\varphi_\bullet$ or $\varphi(\bullet)$. One also has

$$c_{l,k} \circ \varphi(\varphi_\Delta \circ \mathrm{D}_k) \circ \varphi_\nabla = \varphi(\Delta \circ \nabla). \tag{2.26}$$

Finally, we introduce the standard bijections i^{+-} and i^{-+}, changing the module structure:

$$i_k^{+-}: \mathrm{Diff}_k^{+}(P,Q) \to \mathrm{Diff}_k(P,Q), \qquad i_k^{-+}\colon \mathrm{Diff}_k^{+}(P,Q) \to \mathrm{Diff}_k(P,Q). \tag{2.27}$$

We have the following observation.

Lemma 2.2.6. *i_k^{+-} and i_k^{-+} are differential operators of order $\leq k$. Operator $D_k^- = D_k \circ i^{-+}$ is a homomorphism.*

Proof. The first statement can be shown by inductively applying

$$(\delta_a i_k^{+-})(\Delta) = i_k^{+-}(\delta_a \Delta), \qquad (\delta_a i_k^{-+})(\Delta) = -i_k^{-+}(\delta_a \Delta). \tag{2.28}$$

The second one can be obtained by unrolling the definitions above, similarly to computations in Lemma 2.2.3. □

2.3 Differential forms

In this section, we define differential forms over a graded-commutative algebras. Essentially, we repeat the definitions from [25] and [26], and add necessary signs $\lambda(\bullet, \bullet)$.

2.3.1 Definitions

Definition 2.3.1. Let A be an algebra, and P be an A-module. We let

$$D(P) = D_A(P) = \bigoplus_{g \in \Gamma} (D_A(P))_g \subset \mathcal{H}om_{\mathbb{C}}(A, P), \tag{2.29}$$

$$(D_A(P))_g = \left\{ \xi \in \mathcal{H}om_{\mathbb{C}}(A,P) \colon \xi(ab) = \lambda(\widetilde{\xi}, \widetilde{a}) a \xi(b) + \lambda(\widetilde{\xi} + \widetilde{a}, \widetilde{b}) b \xi(a) \right\}. \tag{2.30}$$

We say that $D_A(A)$ is the module of derivations of algebra A, and $D_A(P)$ is the module of P-valued derivations of A. The condition in the right hand side of (2.30) is called the (graded) Leibniz's rule.

One can check that $D_A(P) = \{\xi \in \mathrm{Diff}_1(A,P)\colon \xi(1) = 0\} = \ker(\mathrm{D}_1^-)$. Similarly to Lemma 2.2.2, one can check that the functor D_A is representable. We denote its representing object with $(d, \Lambda^1(A))$, and call it "the module of differential 1-forms." This definition can be extended to define higher differential forms. To do this, we will define D_k, and let $(d, \Lambda^k(A))$ to be the corresponding representing objects. Then, the algebra of differential forms will be $\Lambda(A) = \bigoplus_{k=0}^{\infty} \Lambda^k(A)$. If A is graded by (Γ, λ), then $\Lambda(A)$ is a $(\Gamma \oplus \mathbb{Z}, \lambda_1)$-graded-commutative algebra, where $\lambda_1(g+n, h+m) = \lambda(g,h)(-1)^{nm}$. By slightly abusing the notation, we will use letter λ instead of λ_1.

Definition 2.3.2. Let S be any subset of A-module Q. By definition, let $D(S \subset Q)$ be the set of those derivations $\xi\colon A \to Q$, for which $\xi(A) \subset S$. Similarly, let $\mathrm{Diff}_k^+(S \subset Q)$ be $\{\Delta \in \mathrm{Diff}_k^+(Q)\colon \Delta(A) \subset S\}$. We inductively define functors D_k and $\mathcal{P}_k^+$ with

$$D_0(Q) = Q, \qquad \mathcal{P}_0^+(Q) = Q,$$
$$D_{k+1}(Q) = D(D_k(Q) \subset \mathcal{P}_k^+(Q)), \qquad \mathcal{P}_{k+1}^+(Q) = \mathrm{Diff}_1^+(D_k(Q) \subset \mathcal{P}_k^+(Q)).$$

The module structures on $D_0(Q)$ and $\mathcal{P}_0^+(Q)$ are inherited from Q. For $k \geq 1$ the module structure on $D_k(Q)$ is given by

$$(a\xi)(b) = a(\xi(b)), \tag{2.31}$$

where the multiplication on the right-hand side is the one from $D_{k-1}(Q)$. The module structure on $\mathcal{P}_k^+(Q)$ is inherited from $\mathrm{Diff}^+(\mathcal{P}_{k-1}^+(Q))$. We define the modules $\mathcal{P}_k(Q)$. These modules coincide with $\mathcal{P}_k^+(Q)$ as abelian groups, and have the module structure, defined by (2.31).

One can check that these modules are well defined. The following lemma performs the least trivial of these checks.

Lemma 2.3.3. *Multiplication in $D_k(Q)$ is well defined.*

Proof. We prove the lemma by induction in k. There is nothing to check for $D_0(Q) = Q$, so consider the multiplication in $D_{k+1}(Q)$ for $k \geq 0$ and let $\xi \in D_{k+1}(Q)$. We have to check that $a\xi \in D_{k+1}(Q)$. Note that $(a\xi)(b) = a(\xi(b)) \in D_k(Q)$. It remains to check that $a\xi$ satisfies the Leibniz's rule. For $k = 0$, i.e. $\xi \in D_1(Q)$, the statement follows from the fact that $D_0(Q) = Q = \mathcal{P}_0^+(Q)$, so $D_1(Q) = D(Q)$. For $k \geq 1$, we have

$$(a\xi)(bc) = a \cdot (\xi(bc)) = a \cdot \left(\lambda(\widetilde{\xi}, \widetilde{b}) b \cdot_R (\xi c) + \lambda(\widetilde{\xi} + \widetilde{b}, \widetilde{c}) c \cdot_R (\xi b)\right). \qquad (2.32)$$

In the last expression, $\cdot$ is the multiplication in $D_k(Q)$, and $\cdot_R$ is the multiplication in $\mathcal{P}_k^+(Q)$. As follows from their definitions, these operations commute with each other up to the standard sign. That is, for any $\nabla \in \mathrm{Hom}_{\mathbb{C}}(P, Q)$ and any homogeneous $a, b \in A$, one has

$$a \cdot (b \cdot_R \nabla) = \lambda(\widetilde{a}, \widetilde{b}) b \cdot_R (a \cdot \nabla). \qquad (2.33)$$

Applying this to (2.32), we get

$$(a\xi)(bc) = \lambda(\widetilde{\xi} + \widetilde{a}, \widetilde{b}) b \cdot_R (a \cdot (\xi c)) + \lambda(\widetilde{\xi} + \widetilde{b} + \widetilde{a}, \widetilde{c}) c \cdot_R (a \cdot (\xi b)) =$$
$$\lambda(\widetilde{\xi} + \widetilde{a}, \widetilde{b}) b \cdot_R ((a\xi) c) + \lambda(\widetilde{\xi} + \widetilde{b} + \widetilde{a}, \widetilde{c}) c \cdot_R ((a\xi) b) \qquad (2.34)$$

as desired. □

We can iterate the construction above and give the following definition.

Definition 2.3.4. Define, inductively in k, functors $D_k(D_l \subset \mathcal{P}_l^+)$ and $\mathcal{P}_k^+(D_l \subset \mathcal{P}_l^+)$ with

$$D_0(D_l \subset \mathcal{P}_l^+) = D_l, \qquad \mathcal{P}_0^+(D_l \subset \mathcal{P}_l^+) = \mathcal{P}_l^+,$$

$$D_{k+1}(D_l \subset \mathcal{P}_l^+) = D\left(D_k(D_l \subset \mathcal{P}_l^+) \subset \mathcal{P}_k^+(D_l \subset \mathcal{P}_l^+)\right),$$

$$\mathcal{P}_{k+1}^+(D_l \subset \mathcal{P}_l^+) = \mathrm{Diff}_1^+\left(D_k(D_l \subset \mathcal{P}_l^+) \subset \mathcal{P}_k^+(D_l \subset \mathcal{P}_l^+)\right).$$

Module structures and functors $\mathcal{P}_k(D_l \subset \mathcal{P}_l^+)$ are, then, defined in the same way, as above.

This definition doesn't add anything new, as noted in the following lemma.

Lemma 2.3.5.

$$D_k(D_l \subset \mathcal{P}_l^+) \simeq D_{k+l}, \qquad \mathcal{P}_k^+(D_l \subset \mathcal{P}_l^+) \simeq \mathcal{P}_{k+l}^+, \qquad \mathcal{P}_k(D_l \subset \mathcal{P}_l^+) \simeq \mathcal{P}_{k+l}.$$

Proof. The statement of the lemma follows from the definitions above by induction in k. □

Corollary 2.3.5.1. *There are natural inclusions*

$$\alpha_k \colon D_{k+1}(Q) \hookrightarrow D_k(\mathrm{Diff}_1^+(Q)), \qquad \alpha_{k,l} \colon D_{k+l}(Q) \hookrightarrow D_k(D_l(Q)),$$

where α_k is a differential operator of order ≤ 1 and $\alpha_{k,l}$ is a homomorphism. Grading degrees of both inclusions are equal to 0.

Proof. For $l = 0$, the second inclusion is trivial, so we assume that $l \geq 1$. Due to Lemma 2.3.5, it is enough to find the inclusions

$$\widetilde{\alpha}_k \colon D_k(D(Q) \subset \mathrm{Diff}_1^+(Q)) \hookrightarrow D_k(\mathrm{Diff}_1^+(Q)), \quad \widetilde{\alpha}_{k,l} \colon D_k(D_l(Q) \subset \mathcal{P}_l^+(Q)) \hookrightarrow D_k(D_l(Q)).$$

For $k = 0$, we take $\widetilde{\alpha}_0 = i_1^{-+}|_{D(Q)}$ (see Lemma 2.2.6), and $\widetilde{\alpha}_{0,l} = \mathrm{id}_{D_l(Q)}$. Then, define inductively $\widetilde{\alpha}_k$ and $\widetilde{\alpha}_{k,l}$ by

$$\widetilde{\alpha}_{k+1}\xi = \widetilde{\alpha}_k \circ \xi, \qquad \widetilde{\alpha}_{k+1,l}\xi = \widetilde{\alpha}_{k,l} \circ \xi. \tag{2.35}$$

Note that $\widetilde{\alpha}_k$ is a differential operator of order ≤ 1, because

$$((\delta_a \delta_c \widetilde{\alpha}_{k+1})\xi)(b) = (\delta_a \delta_c \widetilde{\alpha}_k)(\xi(b)) = 0. \tag{2.36}$$

where the last equality follows from induction hypothesis. It remains to check that maps $\widetilde{\alpha}_{k+1,l}$ are well defined; i.e., $\widetilde{\alpha}_{k+1,l}\xi \in D_{k+1}(D_l(Q))$. We have

$$((\widetilde{\alpha}_{k+1,l}\xi)(ab))(c) = (\widetilde{\alpha}_{k,l}(\xi(ab)))\,(c) = (\widetilde{\alpha}_{k,l}(\xi(ab)))\,(c) - \lambda(\widetilde{\xi}+\widetilde{a}+\widetilde{b},\widetilde{c})c\,(\widetilde{\alpha}_{k,l}(\xi(ab)))\,(1) =$$
$$\left(\widetilde{\alpha}_{k,l}(\xi(a)\circ b + \lambda(\widetilde{a},\widetilde{b})\xi(b)\circ a)\right)(c) - \lambda(\widetilde{\xi}+\widetilde{a}+\widetilde{b},\widetilde{c})c\left(\widetilde{\alpha}_{k,l}(\xi(a)\circ b + \lambda(\widetilde{a},\widetilde{b})\xi(b)\circ a)\right)(1) =$$
$$\widetilde{\alpha}_{k,l}(\xi(a))(bc) + \lambda(\widetilde{a},\widetilde{b})\widetilde{\alpha}_{k,l}(\xi(b))(ac) - \lambda(\widetilde{\xi}+\widetilde{a}+\widetilde{b},\widetilde{c})c\widetilde{\alpha}_{k,l}(\xi(a))(b) -$$
$$\lambda(\widetilde{\xi}+\widetilde{a}+\widetilde{b},\widetilde{c})\lambda(\widetilde{a},\widetilde{b})c\widetilde{\alpha}_{k,l}(\xi(b))(a). \quad (2.37)$$

Here, $\circ a$ is the composition with a multiplication operator by a. Note that these multiplications are defined differently for $k = 0$, $l \geq 1$ and $k \geq 1$. However, the computation (2.37) works in both cases. By induction hypothesis, $\widetilde{\alpha}_{k,l}(\xi(a))$ and $\widetilde{\alpha}_{k,l}(\xi(b))$ satisfy the Leibniz's rule, so the above equality simplifies to

$$((\widetilde{\alpha}_{k+1,l}\xi)(ab))(c) = \lambda(\widetilde{\xi}+\widetilde{a},\widetilde{b})b\widetilde{\alpha}_{k,l}(\xi(a))(c) + \lambda(\widetilde{\xi},\widetilde{a})a\widetilde{\alpha}_{k,l}(\xi(b))(c) =$$
$$\left(\lambda(\widetilde{\xi}+\widetilde{a},\widetilde{b})b(\widetilde{\alpha}_{k+1,l}\xi)(a) + \lambda(\widetilde{\xi},\widetilde{a})a(\widetilde{\alpha}_{k+1,l}\xi)(b)\right)(c)$$

as desired.

An alternative approach to prove this corollary can be found in [26, page 251, Corollary 1]. □

Definition 2.3.6. By definition, let $(d_k, \Lambda^k(A)) = (d_k, \Lambda^k)$ be the representing object of the functor D_k. Let $d\colon \Lambda^k \to \Lambda^{k+1}$ denote the image of the universal operator $d_{k+1} \in D_{k+1}\Lambda^{k+1}$ under

$$d_{k+1} \in D_{k+1}\Lambda^{k+1} \overset{\alpha_k}{\hookrightarrow} D_k(\mathrm{Diff}_1^+ \Lambda^{k+1}) \simeq \mathcal{H}om_A(\Lambda^k, \mathrm{Diff}_1^+ \Lambda^{k+1}) \simeq \mathrm{Diff}_1^+(\Lambda^k, \Lambda^{k+1}). \quad (2.38)$$

The wedge product is the image of d_{k+l} under

$$d_{k+l} \in D_{k+l}(\Lambda^{k+l}) \overset{\alpha_{k,l}}{\hookrightarrow} D_k(D_l(\Lambda^{k+l})) \simeq \mathcal{H}om_A(\Lambda^k, \mathcal{H}om_A(\Lambda^l, \Lambda^{k+l})) \simeq$$
$$\mathcal{H}om_A(\Lambda^k \otimes \Lambda^l, \Lambda^{k+l}). \quad (2.39)$$

We introduce $\Lambda = \bigoplus_{l=0}^{\infty} \Lambda^l$ with $\wedge$ as the multiplication and $\Gamma \oplus \mathbb{Z}$ as the grading group. An element $a \in \Lambda^l$ of grading degree $g \in \Gamma$ is now assigned a new degree $(g, l) \in \Gamma \oplus \mathbb{Z}$. Let $d\colon \Lambda \to \Lambda$ be the operator, constructed out of operators $d\colon \Lambda^k \to \Lambda^{k+1}$.

Later, in Theorem 2.3.9, we will prove that $d^2 = 0$.

In the following lemma, for any A-module Q and any nonnegative integer k, let ψ be the natural isomorphism $D_k(Q) \to \mathcal{H}om_A(\Lambda^k, Q)$. For example, for $Q = \Lambda^k$ we have $\psi(d_k) = \mathrm{id}_{\Lambda^k}$.

Lemma 2.3.7. *For any module R, we have the following commutative diagram:*

$$\begin{array}{ccccc} D_{k+1}R & & \xrightarrow{\alpha_k} & & D_k(\mathrm{Diff}_1^+ R) \\ \downarrow \psi & & & & \downarrow \psi \\ \mathcal{H}om_A(\Lambda^{k+1}, R) & \xrightarrow{f \mapsto f \circ d} & \mathrm{Diff}_1^+(\Lambda^k, R) & \underset{D_1 \circ g \leftarrow g}{\overset{\Delta \mapsto \varphi(\Delta)}{\rightleftarrows}} & \mathcal{H}om_A(\Lambda^k, \mathrm{Diff}_1^+ R). \end{array} \tag{2.40}$$

Proof. We first notice that it is enough to show the commutativity of (2.40) for $R = \Lambda^{k+1}$ on element $d_{k+1} \in D_{k+1}\Lambda^{k+1}$. Indeed, if $\xi \in D_{k+1}R$, then $\psi(\xi) \in \mathcal{H}om_A(\Lambda^{k+1}, R)$. Since ψ, α_k and other arrows on the diagram are natural transformations of functors, we can obtain (using $\psi(\xi)$) join each node of the diagram (2.40) to the corresponding node the same diagram, but with R replaced with Λ^{k+1}, thus obtaining a "cube" diagram. For example, for the vertical arrow ψ on the left of (2.40), we get a commutative diagram

$$\begin{array}{ccc} D_{k+1}\Lambda^{k+1} & \xrightarrow{D_{k+1}(\psi(\xi))} & D_{k+1}R \\ \downarrow \psi & & \downarrow \psi \\ \mathcal{H}om_A(\Lambda^{k+1}, \Lambda^{k+1}) & \xrightarrow{h \mapsto \psi(\xi) \circ h} & \mathcal{H}om_A(\Lambda^{k+1}, R). \end{array} \tag{2.41}$$

From the commutativity of this diagram (and bijectivity of ψ), we get that $\xi = D_{k+1}(\psi(\xi))(d_{k+1})$. In this way, the commutativity of the diagram (2.40) on ξ, indeed, follows from the commutativity on d_{k+1} of the same diagram with $R = D_{k+1}\Lambda^{k+1}$.

That commutativity on $d_{k+1} \in D_{k+1}\Lambda^{k+1}$ is the equality $\psi(\alpha_k(d_{k+1})) = \varphi(\psi(d_{k+1}) \circ d)$, and it follows from the definition of d and the equality $\psi(d_{k+1}) = \mathrm{id}_{\Lambda^{k+1}}$. $\square$

2.3.2 Explicit description of D_l

Take $\xi \in D_l(Q)$ and $a_1, \dots, a_l \in A$. Define

$$\xi(a_1, \dots, a_l) := \xi(a_1)(a_2) \dots (a_l) \in Q. \tag{2.42}$$

Then, the following lemma holds ([14, page 28]).

Lemma 2.3.8. *Let ξ be a homogeneous map $A^n \to Q$, $\mathbb{C}$-linear in each argument. Then, $\xi \in D_l(Q)$ (in the sense of (2.42)) if and only if it satisfies the following properties.*

1. [skew-symmetry] *For each $m = 1, \dots, l-1$ and homogeneous a_m, a_{m+1}, we have*

$$\xi(a_1, \dots, a_m, a_{m+1}, \dots, a_l) = -\lambda(\widetilde{a}_m, \widetilde{a}_{m+1})\xi(a_1, \dots, a_{m+1}, a_m, \dots, a_l).$$

2. [multiderivation] *For each $m = 1, \dots, l$ and fixed homogeneous a_j ($j = 1, \dots, l$; $j \neq m$) map $a_m \mapsto \lambda(\sum_{j>m} \widetilde{a}_j, \widetilde{a}_m)\xi(a_1, \dots, a_l) \in Q$ is a derivation of grading degree* $\deg(\xi) + \sum_{j \neq m} \deg(a_j)$.

The A-module structure on $D_l(Q)$ is given by $(a\xi)(a_1, \dots, a_l) = a(\xi(a_1, \dots, a_l))$.

Proof. For $\xi \in D_l(Q)$ with $l \geq 2$, we get $0 = \xi(ab)(1) = \xi(a)(b) + \lambda(\widetilde{a}, \widetilde{b})\xi(b)(a)$. Thus, ξ is "skew-symmetric" in the first 2 arguments. Applying this fact to $\xi(a_1) \dots (a_{m-1}) \in D_{l-m+1}(Q)$, we get the (graded) skew-symmetry.

The multi-derivation property for $m = 1$ follows from the definition of $\alpha_{1,l-1} \colon D_l(Q) \to D(D_{l-1}(Q))$. For $m > 1$, it follows from the skew-symmetry.

The description of the module structure follows by induction from the definition of the module structure of $D_l(Q)$:

$$(a\xi)(a_1) \dots (a_l) = (a\xi(a_1))(a_2) \dots (a_l) = a(\xi(a_1)(a_2) \dots (a_l)).$$

Finally, assume that ξ satisfies Properties 1 and 2. Using induction in l, we will prove that $\xi \in D_l(Q)$. Cases $l = 0$ and $l = 1$ are trivial, so assume $l \geq 2$, and let's show that $\xi \in D_l(Q) = D(D_{l-1}(Q) \subset \mathcal{P}^+_{l-1}(Q))$. If $a_1 \in A$, then $\xi(a_1) \in D_{l-1}(Q)$ by induction hypothesis. It remains to show that $\xi\colon A \to \mathcal{P}^+_{l-1}(Q)$ is a derivation, i.e. that

$$\xi(ab) = \lambda(\widetilde{\xi}, \widetilde{a})a \cdot_R \xi(b) + \lambda(\widetilde{\xi} + \widetilde{a}, \widetilde{b})b \cdot_R \xi(a).$$

This equality is similar to the Leibniz's rule that we have from the multi-derivation property, but has the usual multiplication replaced with $\cdot_R$. To show that this modified Leibniz's rule is also satisfied, we apply both sides to a general $c \in A$:

$$\xi(ab)(c) = \lambda(\widetilde{a}, \widetilde{b})\xi(b)(ac) + \xi(a)(bc).$$

This equality can be checked by expanding both sides using the multi-derivation property, and, then, cancelling terms using (graded) skew-symmetry. □

2.3.3 Properties

Theorem 2.3.9. Λ *and* d *satisfy the following properties:*

1. $d^2 = 0$*;*
2. Λ *is a* $(\Gamma + \mathbb{Z})$*-graded algebra,* $A = \Lambda^0$ *is its subalgebra;*
3. $d \in D_\Lambda(\Lambda)$ *with* $\deg(d) = (0, 1)$.

Proof. Fix an integer $k \geq 0$ and consider operators $d_{(k)}\colon \Lambda^k \to \Lambda^{k+1}$ and $d_{(k+1)}\colon \Lambda^{k+1} \to \Lambda^{k+2}$. Here, we temporarily introduced a lower index (k) to distinguish operators $d_{(k)}$ from one another, and put the index in brackets to distinguish $d_{(k)}$ from $d_k \in D_k(\Lambda^k)$. We need to prove that $d_{(k+1)} \circ d_{(k)} = 0$ or, equivalently, $\varphi(d_{(k+1)} \circ d_{(k)}) = 0$. As in (2.26), this can be rewritten as

$$c_{1,1} \circ \varphi(\varphi(d_{(k+1)}) \circ \mathrm{D}_1) \circ \varphi(d_{(k)}) = 0. \tag{2.43}$$

We have

$$d_{k+2} \in D_{k+2}\Lambda^{k+2} \overset{\alpha_{k+1}}{\hookrightarrow} D_{k+1}(\mathrm{Diff}_1^+ \Lambda^{k+2}) \overset{\alpha_k}{\hookrightarrow} D_k((\mathrm{Diff}_1^+)^2\Lambda^{k+2}) \xrightarrow{D_k(c_{1,1})} D_k(\mathrm{Diff}_2^+ \Lambda^{k+2}).$$

Let ∂^2 be the image of d_{k+2} in $D_k(\mathrm{Diff}_2^+ \Lambda^{k+2})$. We observe that $\psi(\partial^2) = c_{1,1} \circ \psi(\alpha_k(\alpha_{k+1}(d_{k+2})))$. Using Lemma 2.3.7 and definition of d, we rewrite this to

$$\psi(\partial^2) = c_{1,1} \circ \varphi(\psi(\alpha_{k+1}(d_{k+2})) \circ d_{(k)}) = c_{1,1} \circ \varphi(\varphi(d_{(k+1)})) \circ d_{(k)}).$$

To check that this coincides with the lhs of (2.43), we notice that left composition with D_1 is the inverse of φ and, thus, invertible. We compute

$$\mathrm{D}_1 \circ \varphi(\varphi(d_{(k+1)})) \circ d_{(k)}) = \varphi(d_{(k+1)})) \circ d_{(k)},$$

$$\mathrm{D}_1 \circ \varphi(\varphi(d_{(k+1)}) \circ \mathrm{D}_1) \circ \varphi(d_{(k)}) = \varphi(d_{(k+1)}) \circ \mathrm{D}_1 \circ \varphi(d_{(k)}) = \varphi(d_{(k+1)}) \circ d_{(k)}.$$

Therefore, it is enough to prove that $\partial^2 = 0$. Notice that the composition

$$D(\mathrm{Diff}_1^+ \Lambda^{k+2}) \overset{\alpha_0}{\hookrightarrow} (\mathrm{Diff}_1^+)^2\Lambda^{k+2} \xrightarrow{\mathrm{D}_1} \mathrm{Diff}_1^+(\Lambda^{k+2})$$

is 0. Thus, (from definition of $c_{1,1}$) we get $c_{1,1} \circ \alpha_0 = 0$. Then, from recursive definition of α_k we get $D_k(c_{1,1}) \circ \alpha_k = 0$ and, hence, $\partial^2 = 0$. Therefore, $d^2 = 0$.

Since D_0 is the identity functor, it is represented by A; thus, $\Lambda^0 = A$. Next, we need to prove that wedge product in Λ^0 coincides with the original multiplication. This follows from the more general fact that the wedge product $\Lambda^0 \otimes \Lambda^l \to \Lambda^l$ coincides with the standard A-module structure of Λ^l. This fact, in turn, follows from "triviality" of most maps in (2.39) for $k = 0$:

$$\mathrm{id}_{\Lambda_l} \in \mathcal{H}om_A(\Lambda_l, \Lambda_l) \simeq D_l(\Lambda^l) = D_0(D_l(\Lambda^l)) \simeq \mathcal{H}om_A(A, \mathcal{H}om_A(\Lambda^l, \Lambda^l)) \simeq \mathcal{H}om_A(A \otimes \Lambda^l, \Lambda^l). \quad (2.44)$$

The associativity of the wedge product in Λ follows from the definition and corresponding property of $\alpha_{\bullet,\bullet}$, i.e.,

$$\alpha_{k,l} \circ \alpha_{k+l,m} = D_k(\alpha_{l,m}) \circ \alpha_{k,l+m},$$

which follows from the explicit description of D_k (Lemma 2.3.8). □

2.4 Noncommutative torus

2.4.1 Introduction

In this section, we apply the above constructions to the polynomial subalgebra of n-dimensional noncommutative torus, $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$. For such algebra, the grading group is $\mathbb{Z}^n$. Given an antisymmetric $n \times n$ matrix θ with entries in $\mathbb{R}$, the sign function is given by

$$\lambda(I,J) = e^{2\pi i I_l \theta_{lj} J_j}. \tag{2.45}$$

Note that the only additional restriction on λ, imposed by (2.45), is $\lambda(I,I) = 1$ for all $I \in \mathbb{Z}^n$: in general antisymmetry and bilinearity conditions on λ allow for $\lambda(I,I) \in \{-1,1\}$. The algebra $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ is then a free graded-commutative algebra, generated by invertible elements $U^1, \ldots, U^n$ with $\deg U^l = e_l = (0,\ldots,0,1,0,\ldots,0)$ with 1 on the l-th place. We don't use any other structures on this algebra. This definition makes each component $(\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}})_I$ be a one dimensional $\mathbb{C}$-vector space $\mathbb{C}U^I$, where, by definition, $U^I = \prod_{l=1}^n (U^l)^{I_l}$. Note that we use the upper indices for U^l to conform with the standard notation, used in the approach to differential geometry we use.

In order to reduce the clutter, we use the following Einstein notation: indices denoted by j,k,l,m go from 1 to n, indices I,J,K,L go over $\mathbb{Z}^n$. If in a product one of the letters above appears twice as an index, as in $a_I U^I$, then the summation over this index is understood. However, appearances of indices inside the arguments of $\lambda(\bullet,\bullet)$ do not count toward the "appears twice" threshold. When this rule doesn't work well, we will write the summation sign explicitly. All summations are assumed to have only a finite number of non-zero terms; e.g., when we write $a_I U^I$, only a finite number of values a_I are allowed to be nonzero. We write $l+I$, $-l+I$, $I+l$, and $I-l$ to mean that 1 is added or subtracted from the l-th component of I.

We write $\cdot$ in the argument of λ to indicate the sign, needed in

$$U^I U^J = \lambda(I \cdot J) U^{I+J}. \tag{2.46}$$

We allow ourselves to use multiple $\cdot$-s in the argument of λ, and mix small and big indices, with the obvious meaning, as illustrated by the following formula:

$$U^l U^I (U^k)^{-1} U^J = \lambda(l \cdot I \cdot (-k) \cdot J) U^{l+I-k+J} \tag{2.47}$$

2.4.2 Derivations

Using Leibniz's rule, we note that to describe derivation $\xi \in D(A)$, it is enough to give a list of its values on generators $U^1, \ldots, U^n$ of algebra A. It's easy to check that derivations $\frac{\partial}{\partial U^l}$, returning 1 on U^l and 0 on U^j when $j \neq l$, exist. Therefore, $D(A)$ is a free module, generated by $\frac{\partial}{\partial U^l}$, so general ξ can be written as

$$\xi = \xi^l_I U^I \frac{\partial}{\partial U^l}$$

and its action on a generic element $a \in A$ is given by

$$\xi a = \left(\xi^l_I U^I \frac{\partial}{\partial U^l}\right) \left(a_J U^J\right) = \sum_{l,I,J} \lambda(I \cdot (-l) \cdot J) \xi^l_I a_J J_l U^{I-l+J}.$$

If $\xi \in D(Q)$, then coefficients $\xi^l_I U^I$ are to be replaced with generic elements $\xi^l \in Q$, and $D(Q)$ is isomorphic to $Q^{\oplus n}$ — direct sum of n copies of Q. Note that ξ^l are to be understood as operators "multiplication by ξ^l," given in the homogeneous case by $\xi^l a = \lambda(\xi^l, a) a \xi^l$.

2.4.3 Differential operators

The basic building blocks for differential operators are compositions of derivations $\frac{\partial}{\partial U^l}$. In order to simplify the description of them, we enhance our multi-index notation in the following way. By definition, we let

$$|I| = I_1 + \cdots + I_n, \qquad I! = I_1! \cdots I_n!, \tag{2.48}$$

$$I \geq 0 \Leftrightarrow I_1 \geq 0 \ \& \ \cdots \ \& \ I_n \geq 0, \qquad I \geq J \Leftrightarrow I - J \geq 0, \tag{2.49}$$

$$\frac{\partial^{|I|}}{\partial U^I} = \left(\frac{\partial}{\partial U^n}\right)^{I_n} \circ \cdots \circ \left(\frac{\partial}{\partial U^1}\right)^{I_1}. \tag{2.50}$$

Note that $I!$ in (2.48) and the operator in (2.50) are defined only for $I \geq 0$. Note that we use the reverse order, when I is in "lower" position. We have

$$\frac{\partial^{|I|}}{\partial U^I} U^J = \frac{J!}{(J-I)!} \lambda((-I) \cdot (J-I)) U^{J-I}. \tag{2.51}$$

Strictly speaking, this formula is valid when $J \geq 0$. To get the formula for J with negative components, one should replace the coefficient $J!/(J-I)!$ with its analytic continuation to avoid the undefined expression of the form ∞/∞.

Define δ^l to be δ_{U^l}, $\delta^I = (\delta^1)^{I_1} \circ \cdots \circ (\delta^n)^{I_n}$.

Lemma 2.4.1. *Let P and Q be graded $\mathrm{A}_\theta^{(\mathrm{n}),poly}$-modules. The following descriptions of $\mathrm{Diff}_k(P,Q)$ are valid:*

1. $\mathrm{Diff}_k(P,Q) = \{\Delta \in \mathcal{H}om_{\mathbb{C}}(P,Q) \colon \delta^l \Delta \in \mathrm{Diff}_{k-1}(P,Q)$ *for* $l = 1, \ldots, n\}$*;*

2. $\mathrm{Diff}_k(P,Q) = \{\Delta \in \mathcal{H}om_{\mathbb{C}}(P,Q) \colon \forall I \geq 0 \ |I| = k+1 \Rightarrow \delta^I \Delta = 0\}$.

Proof. Using induction in k, we note that the second description follows from the first, i.e., from the fact that $\Delta \in \mathcal{H}om_{\mathbb{C}}(P,Q)$ is a differential operator of order $\leq k$ if and only if

$$\delta^l \Delta \in \mathrm{Diff}_{k-1}(P,Q) \text{ for } l = 1, \ldots, n. \tag{2.52}$$

From the definition of $\mathrm{Diff}_k(P,Q)$, we know that Δ is a differential operator of order $\leq k$ if and only if

$$\delta_a \Delta \in \mathrm{Diff}_{k-1}(P,Q) \text{ for all } a \in A \tag{2.53}$$

So, it is enough to prove that (2.52) implies (2.53). From definition of δ_a, we know that

$$\delta_{\alpha a+\beta b} = \alpha\delta_a + \beta\delta_b \text{ for } \alpha,\beta \in \mathbb{C},\ a,b \in A. \tag{2.54}$$

Thus, it is enough to check (2.53) for $a = U^I$. It follows from the Jacobi identity (2.9) that

$$\delta_{ab}\Delta = a \circ (\delta_b \Delta) + \lambda(b,\Delta)(\delta_a\Delta) \circ b. \tag{2.55}$$

Substituting $a = U^I$, $b = U^l$ we see that conditions (2.53) for $a = U^I$ and $a = U^{I+l}$ are equivalent. Since any multi-index $I \in \mathbb{Z}^n$ can be obtained from 0 by finite number of additions and subtractions of 1 to/from its components, (2.53) is indeed satisfied for $a = U^I$. □

The following lemma shows that $\mathrm{Diff}_k(A,A)$ is a free A-module, $\mathrm{Diff}_k(A,Q) \simeq Q^{\oplus N}$, where $N = \binom{k+n}{n}$.

Lemma 2.4.2. *Any differential operator $\Delta \in \mathrm{Diff}_k(A,Q)$ can be uniquely written as*

$$\Delta = \sum_{|I|\leq k} q_I \frac{\partial^{|I|}}{\partial U^I}. \tag{2.56}$$

For homogeneous Δ, coefficients $q_I \in Q$ are given by

$$q_I = \lambda(\Delta, I)\frac{(-1)^{|I|}}{I!}(\delta^I\Delta)(1). \tag{2.57}$$

Proof. We prove formula (2.56) with coefficients (2.57) for homogeneous $\Delta \in \mathrm{Diff}_k(A,Q)$ by induction in k. If $k=-1$, then both sides of the equation are equal to 0. So, assume $k \geq 0$. Note that by applying both sides of (2.56) to 1, we get $\Delta(1) = q_0$, where q_0 is given by (2.57). Thus, it is enough to show that the difference of lhs and rhs of (2.56) is a homomorphism. To do this, according to Lemma 2.4.1, it is enough to apply δ^l to both sides and prove the resulting equality

$$\delta^l\Delta = \sum_{|I|\le k} \lambda(\Delta, I)\lambda(l, \Delta + I)\frac{(-1)^{|I|}}{I!}(\delta^I\Delta)(1)\ \delta^l\left(\frac{\partial}{\partial U^I}\right).$$

Introducing $J = I - l$, we compute

$$\sum_{|I|\le k} \lambda(\Delta, I)\lambda(l, \Delta + I)\frac{(-1)^{|I|}}{I!}(\delta^I\Delta)(1)\ \delta^l\left(\frac{\partial^{|I|}}{\partial U^I}\right) =$$
$$\sum_{|I|\le k} \lambda(\Delta, I)\lambda(l, \Delta + I)\frac{(-1)^{|I|}}{I!}(\delta^I\Delta)(1)I_l e^{-2\pi l\theta(l\cdot I)}(-1)\frac{\partial^{|I|-1}}{\partial U^{I-l}} =$$
$$\sum_{|J|\le k-1} \lambda(\Delta, J + l)\lambda(l, \Delta + J + l)\frac{(-1)^{|J|}}{J!}(\delta^{J+l}\Delta)(1)e^{-2\pi l\theta(l\cdot(J+l))}\frac{\partial^{|J|}}{\partial U^J} =$$
$$\sum_{|J|\le k-1} \lambda(\Delta, J)\frac{(-1)^{|J|}}{J!}(\delta^J\delta^l\Delta)(1)\frac{\partial^{|J|}}{\partial U^J} = \delta^l\Delta.$$

Here, the last equality follows from the induction hypothesis applied to $\delta^l\Delta$.

Thus, existence of coefficients in (2.56) is shown for homogeneous Δ. For non-homogeneous Δ, it follows from $\mathbb{C}$-linearity of (2.56).

To show uniqueness, assume to the contrary that (2.56) is satisfied with $\Delta = 0$ and some q_I being different from 0. Fix J with minimal $|J|$ s.t. $q_J \neq 0$ and apply both sides of (2.56) to U^J. Then, the left-hand side will be 0, but the right-hand side will be nonzero. Contradiction. □

2.4.4 Jet bundle

It follows from Lemma 2.4.2 that $J^k(A)$ is a free module, generated by vectors

$$e^I = \frac{(-1)^{|I|}}{I!}(\delta^I j_k)(1) \text{ for } |I| \le k. \tag{2.58}$$

For $a \in A$, we can express $j_k(a)$ in terms of these basis elements using the formula equation

$$j_k(a) = \sum_{|I| \le k} \left(\frac{\partial^{|I|}}{\partial U^I} a \right) \frac{(-1)^{|I|}}{I!} (\delta^I j_k)(1). \tag{2.59}$$

Alternatively, we could use basis $j_k(U^I)$ for $|I| \le k$.

2.4.5 Multi-derivations

To describe module $D_k(Q)$, we first introduce multi-derivations $\left(\frac{\partial}{\partial U}\right)_I \in D_k(A)$. They are defined for multi-index $I \in \{0,1\}^n$ with $|I| = k$ by the following inductive procedure: $\left(\frac{\partial}{\partial U}\right)_0 = 1 \in A = D_0(A)$. For I with $|I| = k \ge 0$ if l stays after all 1s in I, i.e., $\forall j \in \{1, \ldots, n\}\ I_j = 1 \Rightarrow l > j$, then $\left(\frac{\partial}{\partial U}\right)_{l+I}$ is given by

$$\left(\frac{\partial}{\partial U}\right)_{I+l} (a_0, a_1, \ldots, a_k) =$$
$$\sum_{l=0}^{k} \lambda(I, a_l) \lambda(a_0 + a_1 + \cdots + a_{l-1}, a_l) \left(\frac{\partial}{\partial U^l} a_l \right) \left(\frac{\partial}{\partial U}\right)_I (a_0, a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_k).$$

One can show by induction in k and the description of derivations, given in Subsection 2.4.2, that the module $D_k(Q)$ can be written as

$$D_k(Q) = \left\{ \sum_{I \in \{0,1\}^n :\ |I| = k} q^I \left(\frac{\partial}{\partial U}\right)_I : q^I \in Q \right\}.$$

2.4.6 Differential forms

Differential forms are given by

$$\sum_I a_I dU^I,$$

where $I \in \{0,1\}^n$. If I has 1 in coordinates $l_1 < l_2 < \cdots < l_k$, then

$$dU^I = dU^{l_1} \wedge \cdots \wedge dU^{l_k}.$$

Differential d is determined by $d^2 = 0$ and the Leibniz's rule in $D(\Lambda A)$.

2.4.7 De Rham cohomologies

Since d is a graded derivation of $\Lambda(A)$ of degree $(0,1)$, we have a chain complex for each grading $I \in \Gamma$. If $I \neq 0$, say $I_l \neq 0$, then we can construct chain homotopy between identity and zero maps with:

$$\mathrm{id}_{(\Lambda(A))_I} = h \circ d + d \circ h$$

for $h = \frac{1}{I_l} U^l i_{\frac{\partial}{\partial U^l}}$. Here, $i_{\frac{\partial}{\partial U^l}}$ is the insertion of vector field $\frac{\partial}{\partial U^l}$, i.e., a derivation of $\Lambda(A)$ of degree $(-\deg U^l, -1)$, generated by relations $i_{\frac{\partial}{\partial U^l}}(dU^j) = \delta_l^j$.

If $I = 0$, then d acts as 0 on $(\Lambda A)_I$.

Therefore, cohomologies of ΛA coincide with $H^k(\Lambda A) = (\Lambda A)_{(0,k)}$. In other words, cohomologies are generated by products of $(U^l)^{-1} dU^l$, so $\dim(H^k) = \binom{n}{k}$, as in the case of the commutative torus.

2.5 Comparison with differential forms, coming from the Dirac operator

In [6], Connes has introduced a flavor of differential calculus, built from an algebra A together with a Fredholm module structure, and has shown that one can replace the requirement for a Fredholm module structure with a representation $\rho\colon A \to B(H)$ of algebra A on a Hilbert space H, together with a (potentially unbounded) operator D, interpreted as a Dirac operator, and satisfying certain properties.

To make long story short, we describe only the ingredients, required to make the comparison of differential forms. These ingredients are taken from [16] and [11]. Where possible, we use the notation from the previous section. Note that most nontrivial objects of noncommutative tori require at least smooth algebra, and don't appear in the polynomial algebra we are considering. This leads to quite trivial results of the comparison below. Also note that the constructions below should be applied to the smooth subalgebra of the noncommutative tori algebra. We apply the same constructions to the polynomial subalgebra to be able to do a comparison with the constructions above.

2.5.1 Trace, representation, and the Dirac operator

We use the multi-index notation from the previous section. Trace τ is defined by

$$\tau(a_I U^I) = a_0. \tag{2.60}$$

Involution is given by

$$(a_I U^I)^* = a_I^* (U_I)^{-1} = \lambda(I \cdot I) a_I^* U^{-I}. \tag{2.61}$$

Here, a_I^* is the conjugate of the complex number a_I. The Hilbert space H_τ is the completion of $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ with respect to the norm $\| \bullet \|_\tau$, defined with

$$\|a\|_\tau^2 = \tau(a^* a) = \sum_I \|a_I\|^2. \tag{2.62}$$

Image of $a_I U^I \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ in H_τ is denoted with $a_I U^I \xi$. In particular, ξ is the image of $1 \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$. Algebra $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ acts on the Hilbert space H_τ with $a(b\xi) = (ab)\xi$. The Hilbert space H is defined to be $H_\tau \otimes \mathbb{C}^{\lfloor n/2 \rfloor}$, where $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ acts only on the first component. The second component is used to represent the Clifford algebra $\mathbb{C}\mathrm{l}_n$ using $2^{\lfloor n/2 \rfloor} \times 2^{\lfloor n/2 \rfloor}$ matrices $\gamma_j = \gamma_j^{(n)}$. These matrices are defined as follows. For $n = 1$, let $\gamma_1^{(1)} = 1$. For odd $n \geq 3$, let

$$\gamma_j^{(n)} = \begin{pmatrix} 0 & \gamma_j^{(n-2)} \\ \gamma_j^{(n-2)} & 0 \end{pmatrix}, \qquad \gamma_{n-1}^{(n)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_n^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.63}$$

For even n, let $\gamma_j^{(n)} = \gamma_j^{(n+1)}$ $(j = 1, \ldots, n)$. These matrices satisfy $\gamma_l \gamma_j + \gamma_j \gamma_l = 2\delta_{lj}$ and $\gamma_j^* = \gamma_j$.

We define operators $U^l \frac{\partial}{\partial U^l}$ (no summation), $l = 1, \ldots, n$ on Hilbert space H by closing

$$U^l \frac{\partial}{\partial U^l}(a\xi \otimes v) = \left(U^l \frac{\partial}{\partial U^l} a \right) \xi \otimes v. \tag{2.64}$$

We let the (unbounded) Dirac operator D, acting on H, to be defined by

$$D = 2\pi \sum_j \gamma_j U^j \frac{\partial}{\partial U^j}. \tag{2.65}$$

2.5.2 Comparison of differential forms

From now on, we abandon the Einstein summation convention, since it's no longer convenient. Differential forms, using the Dirac operator, are defined as follows. Let the bimodule Ω^1 be the set of "universal differential operators," i.e., formal sums

$$\sum_l a_l \delta b_l \qquad \text{(where } a_l, b_l \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}\text{)} \tag{2.66}$$

subject to relations

$$\delta(ab) = a\delta(b) + \delta(a)b, \quad \delta(\alpha a + \beta b) = \alpha \delta a + \beta \delta b \quad \text{(where } a, b \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}},\ \alpha, \beta \in \mathbb{C}\text{)}. \tag{2.67}$$

Then, the bimodule Ω^k of k-forms is defined with

$$\Omega^k = \underbrace{\Omega^1 \otimes_{\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}} \cdots \otimes_{\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}} \Omega^1}_{n \text{ terms}}. \tag{2.68}$$

Using the Leibniz's rule (2.66), we can write any element $a \in \Omega^k$ as

$$a_0 \delta a_1 \cdots \delta a_k = a_0 \delta a_1 \otimes \delta a_2 \cdots \otimes \delta a_k. \tag{2.69}$$

These can be "represented" as operators on H with

$$\pi(a_0 \delta a_1 \cdots \delta a_k) = a_0 [D, a_1] \cdots [D, a_k]. \tag{2.70}$$

We let J_0^k be the kernel of this map, let $J_0 = \bigoplus_k J_0^k$, and let $J = J_0 + \delta(J_0) = \bigoplus_k (J_0^k + \delta(J_0^{k-1}))$. Then, the Connes bimodule of k-forms is defined with

$$\Omega_D^k = \Omega^k / (J_0^k + \delta(J_0^{k-1})). \tag{2.71}$$

We follow the computation, done in [16, Section 6.2], to compute Ω_D^l for the algebra $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$.

- 0-forms.

$$J_0^0 = \{0\}, \qquad \Omega_D^0 \simeq \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}} \simeq \Lambda^0, \tag{2.72}$$

- 1-forms.

  Consider the universal 1-forms ν_l and the corresponding Connes differential 1-forms ν_l^D $(l = 1, \dots, n)$, given by

$$\nu_l = (U^l)^{-1} \delta U^l, \qquad \nu_l^D = \pi(\nu_l)(U^l)^{-1} dU^l = 2\pi\gamma_l. \tag{2.73}$$

  Note that from the explicit expression $2\pi\gamma_l$, we see that ν_l^D commutes with all 0-forms, and ν_l satisfies

$$\pi(\nu_l \nu_j + \nu_j \nu_l) = 2\delta_{jl}. \tag{2.74}$$

  Using the Leibniz's rule (2.66), we can write any 1-form as $\sum_{l=1}^{n} \sum_{k=1}^{N_l} a_{k,l} \nu_l^D b_{k,l}$. Since ν_l^D commutes with all 0-forms, and ν_l^D are linearly independent from each other, we see that each one form can be uniquely written as

$$\sum_{l=1}^{n} a_l \nu_l^D. \tag{2.75}$$

  The corresponding component of the ideal $J_0 + \delta(J_0)$ is $J_0^1 = J_0^1 + \delta(J_0^0)$. As follows from the computation above, it is generated by $a\nu_l - \nu_l a$, $a \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$.

- 2-forms and k-forms.

  Let's denote the ideal of Ω, generated by $J_0^1 + \delta(J_0^1)$, with $\widetilde{J}$. It is generated by

1-forms $a\nu_l - \nu_l a$ (where $a \in \mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$) and 2-forms $\nu_k\nu_l + \nu_l\nu_k$. In particular, $\nu_l^D\nu_l^D = 0$. Relations, imposed by $\widetilde{J}$ make $\Omega/\widetilde{J}$ isomorphic to Λ from Section 2.4. Thus, to show that $\Omega_D \simeq \Lambda$, it remains to show that $J = \widetilde{J}$. It would follow, if we show that for any fixed $k \in \{0, 1, \ldots, n\}$ the products $\prod_{j=1}^k \gamma_{l_j}$ are all linearly independent for $1 \leq l_1 \leq \cdots \leq l_k \leq n$ (because for every $\alpha \in \Omega^k$, operator $\pi(\alpha)$ can be written as a linear combination with coefficients in $\mathrm{A}_\theta^{(\mathrm{n}),\mathrm{poly}}$ of the above matrices up to $\pi(\widetilde{J}^k)$). These products are indeed independent. For even n, that follows from the fact that the representation of $\mathbb{C}\mathrm{l}_n$ is faithful. For odd n, the kernel is of the form $(1 + c\omega)\,\mathbb{C}\mathrm{l}_n$, where ω is an odd element of $\mathbb{C}\mathrm{l}_n$ and $c \in \mathbb{C}$. Thus, for odd n, this kernel doesn't contain any homogeneous elements, and linear independence holds, too.

Thus, for the polynomial algebra, the Connes construction of differential forms gives the same answer, as the one coming from graded-commutative differential geometry.

Chapter 3

Fredholm modules and the Beilinson–Bloch regulator

3.1 The general strategy for constructing the Beilinson–Bloch regulator

Let X be a compact Riemann surface. The goal of the paper is to produce an alternative construction of the Beilinson–Bloch regulator $r\colon K_2(X) \to H^1(X, \mathbb{C}^*)$ using the framework of Fredholm modules (see [2] for the original definition). Using the construction, explained in [12], this map can be reconstructed from the corresponding map r_ξ on the field of fractions $F(X) = \mathcal{O}_\xi$, where ξ is the generic point of X. Originally, this construction is due to Beilinson [2]. The field of fractions can be written as a direct limit of rings of functions on $X \setminus S$, where S goes through (increasing) finite subsets of X:

$$F(X) = \varinjlim\nolimits_S \mathcal{O}(X \setminus S). \tag{3.1}$$

Thus, r_ξ, in its turn, can be reconstructed from the maps r_S:

$$r_S\colon K_2(\mathcal{O}(X \setminus S)) \to H^1(X \setminus S, \mathbb{C}^*), \quad r_\xi = \varinjlim\nolimits_S r_S\colon\ K_2(F(X)) \to \varinjlim_S H^1(X \setminus S, \mathbb{C}^*). \tag{3.2}$$

Thus, in the rest of this work, we will mostly concentrate on constructing the maps r_S. We will now recall the definition of $K_2(R)$ for a ring R, and some related facts and definitions, used in this work.

3.2 Central extensions and K-theory of rings

Here, we summarize necessary facts, related to the group $K_2(R)$ for a unitary ring R. In this subsection, we omit most of the proofs, which can be found, e.g., in [22, Chapters 2 and 4].

Definition 3.2.1. For R a unital (not necessarily commutative) ring, we make the following definitions:

- $\mathrm{Inv}(R)$: the group of invertible elements of R;
- $M_n(R)$: the ring of $n \times n$ matrices over R;
- $\mathrm{GL}_n(R) = \mathrm{Inv}(M_n(R))$;
- $\mathrm{GL}(R)$: the injective limit $\varinjlim \mathrm{GL}_n(R)$, where $n \times n$ matrix a is identified with $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$;
- $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$; this group is called "the group of elementary matrices."

There is an alternative definition of $E(R)$:

Definition 3.2.2. Let e_{ij} be the matrix with 1 in the cell (i, j) and 0 in all other cells; then, $E(R)$ is defined to be the subgroup of $\mathrm{GL}(R)$, generated by matrices $e_{ij}(r) \overset{\mathrm{def}}{=} 1 + e_{ij} r$ for $r \in R$.

Lemma 3.2.3. *Two definitions of $E(R)$ above define the same subgroup of* $\mathrm{GL}(R)$. *Moreover, we have $E(R) = [E(R), E(R)]$.*

Proof. See [22, Prop. 2.1.4]. □

Here, are the main definitions and facts about central extensions.

Definition 3.2.4. Let G be a group, and A be an abelian group, both written multiplicatively. The exact sequence

$$1 \to A \to E \xrightarrow{\pi} G \to 1 \tag{3.3}$$

is called a central extension of G by A if the of A is in the center of E. We often refer to a central extension of the form (3.3) as (E, π) or, simply, π. Central extensions of a group G form a category, in which morphisms are given as follows. Let (E_1, π_1), (E_2, π_2) be two central extensions of G. A morphism from (E_1, π_1) to (E_2, π_2) is a map $\varphi\colon E_1 \to E_2$, making the right square on the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \longrightarrow & E_1 & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & A_2 & \longrightarrow & E_2 & \xrightarrow{\pi_2} & G & \longrightarrow & 1. \end{array} \tag{3.4}$$

Here, the map $i\colon A_1 \to A_2$ is the unique map, making the left square commutative: up to identifying $\ker \pi_k$ with A_k, map i is the restriction of φ to a map $\ker \pi_1 \to \ker \pi_2$. Universal central extension of G is the initial object of the category of central extensions of G.

Central extensions of a group G by a fixed abelian group A also form a category. In that category, morphisms are required to induce (as in (3.4)) the identity map on A.

Note that the last category (with fixed A) is rather trivial, since all its morphisms are isomorphisms.

Definition 3.2.5. Group G is called perfect, if $[G, G] = G$.

Note that according to 3.2.3, for any ring R, group $E(R)$ is perfect.

Definition 3.2.6. Let $A \xrightarrow{i} E \xrightarrow{\pi} G$ be a central extension of the group G. Let $a_1, a_2 \in G$, then $[\pi^{-1}a_1, \pi^{-1}a_2]$ is defined as follows: take any $e_j \in \pi^{-1}(a_j)$ for $j = 1, 2$; then $[\pi^{-1}a_1, \pi^{-1}a_2] = [e_1, e_2] \in E$. If $[a_1, a_2] = 1$ in G, then $i^{-1}[\pi^{-1}(a_1), \pi^{-1}(a_2)]$ is the only preimage of $[e_1, e_2]$.

These are well defined. To be more precise, the following lemma holds.

Lemma 3.2.7.

1. *In the notation of the Definition 3.2.6, the value of* $[\pi^{-1}a_1, \pi^{-1}a_2]$ *doesn't depend on the choice of* e_1, e_2*; therefore,* $[\pi^{-1}a_1, \pi^{-1}a_2]$ *is well defined for all* $a_1, a_2 \in G$*;*

2. $[\pi^{-1}a_1, \pi^{-1}a_2] \in i(A)$ *if and only if* $[a_1, a_2] = 1$*; therefore,* $i^{-1}[\pi^{-1}a_1, \pi^{-1}a_2]$ *is well defined for all* $a_1, a_2 \in G$*, satisfying* $[a_1, a_2] = 1$*.*

We will use the properties of central extensions, summarized by the following theorem.

Theorem 3.2.8.

1. *A group* G *has a universal central extension if and only if* G *is perfect.*

2. *If* (S, p) *is the universal central extension of* G*, then* S *is perfect.*

3. *If* (E, π) *is a central extension of* G*, and* E *is perfect, then* G *is perfect, and* E *is generated by the elements of the form* $[\pi^{-1}a, \pi^{-1}b]$ *for some* $a, b \in G$*.*

4. *Let* $\varphi\colon (E_1, \pi_1) \to (E_2, \pi_2)$ *be a morphism of central extensions. If* E_2 *is perfect, then* φ *is surjective.*

5. *Homomorphic images of perfect groups are perfect. In particular, let* $\varphi\colon (E_1, \pi_1) \to (E_2, \pi_2)$ *be a morphism of central extensions. If* E_1 *is perfect and* φ *is surjective, then* E_2 *is perfect. If* E_1 *is perfect but* φ *is not necessarily surjective, we still have* $\varphi(E_1) = [E_2, E_2]$*.*

6. *If* (E_1, π_1)*,* (E_2, p_2) *are central extensions of* G *and* E_1 *is perfect, then there is at most one morphism* $\varphi\colon (E_1, \pi_1) \to (E_2, p_2)$*. If it exists, it is given by*

$$\varphi([\pi_1^{-1}a, \pi_1^{-1}b]) = [\pi_2^{-1}a, \pi_2^{-1}b]. \tag{3.5}$$

   If this formula gives a well-defined group homomorphism $\varphi\colon E_1 \to E_2$*, it is the morphism of central extensions.*

7. *If A is an abelian group, the set of isomorphism classes of central extensions of G by A can be naturally turned into an abelian group* $\mathrm{Ext}(G,A)$*. If $A_0 \to S \xrightarrow{p} G$ is the universal central extension of G, then* $\mathrm{Ext}(G,A) \simeq \mathrm{Hom}(A_0,A)$*. This isomorphism can be described as follows: take an extension $A \to E \xrightarrow{\pi} G$ and let φ be the unique morphism $(S,p) \to (E,\pi)$. Then, we have the corresponding diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_0 & \longrightarrow & S & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1. \end{array} \tag{3.6}$$

The left vertical arrow f on this diagram gives the desired element of $\mathrm{Hom}(A_0,A)$.

Proof. These facts follow from definitions and theorems from [22, Chapter 4]. Items 1 and 2 are stated in the [22, theorem 4.1.3]. To show Item 3, note that by the definition of the perfect group, $E = [E,E]$, so E is generated by elements of the form $e = [e_1, e_2]$. Such e can be written as $e = [\pi^{-1}\pi e_1, \pi^{-1}\pi e_2]$. Therefore, in Item 4, E_2 is generated by elements of the form $e = [\pi_2^{-1}a_1, \pi_2^{-1}a_2]$ for $a_1, a_2 \in G$. Take any preimages $e_j \in \pi_1^{-1}(a_j)$. Then, $e = [\varphi(e_1), \varphi(e_2)] = \varphi([e_1, e_2])$. Similarly, in Item 5, E_1 is generated by elements $e = [e_1, e_2]$, so E_2 is generated by $\varphi(e) = [\varphi(e_1), \varphi(e_2)]$. To prove the last sentence of 5, note that for $e = [e_1, e_2] \in E_1$ we have $\varphi(e) = [\varphi(e_1), \varphi(e_2)]$. On the other hand, if $e = [e_1, e_2] \in E_2$, then $e = \varphi([\pi_1^{-1}\pi_2 e_1, \pi_1^{-1}\pi_2 e_2])$. In Item 6 for perfect E_1, Formula (3.5) determines (possibly ambiguously) the values of φ on commutators, generating E_1, and follows from commutativity of the diagram (3.4) in the definition of morphism of central extension. If such φ is a well-defined homomorphism, then for $e = [\pi_1^{-1}a, \pi_1^{-1}b]$, we have $\pi_1(e) = [a,b] = \pi_2(\varphi(e))$. So, the diagram (3.4) is commutative.

The first sentence in Item 7 is stated in [22, Theorem 4.1.16]. By comparing the proof of [22, Theorem 4.1.16] with the description of the map in Item 7, one can see that the map $\mathrm{Ext}(G,A) \to \mathrm{Hom}(A_0,A)$ is a homomorphism of abelian groups. To see that this map is an isomorphism, consider a map $f\colon A_0 \to A$. This gives the diagram of the form (3.6) without E and 3 arrows, connecting E with other groups on the diagram. Group E can be constructed by taking the abelian pushout of the upper left triangle $A \xleftarrow{f} A_0 \to S$: if we denote the map $A_0 \to S$ with i_0, then $E = A \times S/\{(f(a_0), i_0(a_0)^{-1})\colon a_0 \in A_0\}$ with $\pi([(a,s)]) = p(s)$. Finally, pushout is the only choice for E, since if we had some other E_1, then by the universal property of pushout we will have a map $E \to E_1$, which would necessarily be an isomorphism. □

Definition 3.2.9. Let R be a ring. Let

$$1 \to K_2(R) \xrightarrow{i} \mathrm{St}(R) \xrightarrow{\pi} E(R) \to 1 \tag{3.7}$$

be a universal central extension of the group $E(R)$. This exact sequence is well defined up to an isomorphism of central extensions, and for our later purposes any (fixed) representative of this isomorphism class will suffice.

Note, that there is an alternative but equivalent way to define $\mathrm{St}(R)$ in terms of generators and relations, and $K_2(R)$ as a certain subgroup of $\mathrm{St}(R)$ (see [22]).

When $a, b \in E(R)$ commute, $i^{-1}[\pi^{-1}a, \pi^{-1}b]$ is a well-defined element of $K_2(R)$ (see Definition 3.2.6 and Lemma 3.2.7 above). This observation can be used to construct some elements in $K_2(R)$. Often, the following specialization of this construction is used.

Definition 3.2.10. Let $f, g \in \mathrm{Inv}(R)$ be invertible commuting elements of R. Then, the Steinberg symbol $\{f, g\} \in K_2(R)$ is defined by

$$\{f, g\} = i^{-1}[\pi^{-1}\,\mathrm{diag}(f, f^{-1}, 1), \pi^{-1}\,\mathrm{diag}(g, 1, g^{-1})]. \tag{3.8}$$

3.3 The universal 2-summable Fredholm module

This section, and the following one, are concerned with bounded operators on a Hilbert space. In this context, we will use both additive and multiplicative commutators of operators on $B(\mathcal{H})$ (the second one can applied to the invertible operators only). To distinguish them, we let

$$[a, b]_0 = ab - ba, \qquad\qquad [a, b]_1 = aba^{-1}b^{-1} \tag{3.9}$$

Following the Eugene Ha manuscript [12], we give the following definitions, which are originally due to Connes and Karoubi (see [7]).

Definition 3.3.1. Fix two separable Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and define operator $F\colon \mathcal{H} \to \mathcal{H}$ with $F(x_+ + x_-) = x_+ - x_-$ for $x_+ \in \mathcal{H}_+$, $x_- \in \mathcal{H}_-$. The algebra

$$\mathcal{M}^1 = \{a \in B(\mathcal{H})\colon [F, a]_0 \in \mathcal{L}^2(\mathcal{H})\}, \tag{3.10}$$

is called the universal 2-summable Fredholm module. Here, $\mathcal{L}^2(\mathcal{H})$ is the ideal of Hilbert–Schmidt operators.

The purpose of this section is to give an explicit description of the ring $E(\mathcal{M}^1)$ of elementary matrices over $\mathcal{M}^1$. The first step is to describe the group $\mathrm{Inv}(\mathcal{M}^1)$ and its connected component of the identity $\mathrm{Inv}^0(\mathcal{M}^1)$. In order to do this, we follow [18, Section 6.2]. First, note that if $a \in B(\mathcal{H})$ is a bounded operator on $\mathcal{H}$, then in the view of decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, one can interpret it as a matrix

$$a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix}, \tag{3.11}$$

where a_{lj} is a bounded operator $\mathcal{H}_j \to \mathcal{H}_l$ for $l, j \in \{+, -\}$. From the definition of $\mathcal{M}^1$, such a belongs to $\mathcal{M}^1$ if and only if a_{+-} and a_{-+} are Hilbert–Schmidt.

Lemma 3.3.2. *If $a \in \mathcal{M}^1$ is invertible in $B(\mathcal{H})$, then it is invertible in $\mathcal{M}^1$.*

Proof. If $a^{-1} \in B(\mathcal{H})$, then $[F, a^{-1}]_0 = -a^{-1}[F, a]_0 a^{-1} \in \mathcal{L}^2(\mathcal{H})$, so $a^{-1} \in \mathcal{M}^1$. $\square$

The following lemma is analogous to [18, p. 81].

Lemma 3.3.3. *Let x be a bounded operator on $\mathcal{H}_+$. Then, there exists*

$$a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix} \in \mathrm{Inv}(\mathcal{M}^1)$$

with $a_{++} = x$ if and only if x is Fredholm.

Proof. Suppose that x is a Fredholm operator. By definition of Fredholm operators, that, in particular, means that $\operatorname{ran} x$ is closed. Let p be the (orthogonal) projector on $\ker x$ and q be the (orthogonal) projector on $(\operatorname{ran} x)^\perp = \ker x^*$. Let $u\colon \mathcal{H}_+ \to \mathcal{H}_-$ be an isomorphism of Hilbert spaces.

$$a = \begin{pmatrix} x & qu^* \\ up & ux^*u^* \end{pmatrix}.$$

By construction, $a \in \mathcal{M}^1$ and a is a bijection. Therefore, a is invertible by the bounded inverse theorem. Its inverse lies in $\mathcal{M}^1$ by 3.3.3.

On the other hand, let $ab = ba = 1$ for some $a, b \in \mathcal{M}^1$. Then, operators a, b can be written as

$$a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix}, \qquad b = \begin{pmatrix} b_{++} & b_{+-} \\ b_{-+} & b_{--} \end{pmatrix}.$$

So, $a_{++}b_{++} = 1 - a_{+-}b_{-+}$, $b_{++}a_{++} = 1 - b_{+-}a_{-+}$. Thus, a_{++} is invertible up to a compact, and, hence, Fredholm. □

Kuiper's paper [15] proves that all homotopy groups of $\mathrm{Inv}(B(\mathcal{H}))$ vanish. We will only need the following special case of that.

Lemma 3.3.4. *If $\mathcal{H}$ is a Hilbert space, then $\mathrm{Inv}(B(\mathcal{H}))$ is connected.*

Lemma 3.3.5. *Let $\mathcal{H} = l^2(\mathbb{N}_0)$ be the Hilbert space of sequences $\mathbb{N}_0 = \mathbb{N} \cup \{0\} \to \mathbb{C}$, and let S be the standard shift operator on $\mathcal{H}$. If $a \in B(\mathcal{H})$ is a Fredholm operator, then $a = b_1 S^n S^{*m} b_2$ for some $b_1, b_2 \in \mathrm{Inv}(B(\mathcal{H}))$, $n = \dim(\operatorname{ran} a)^\perp$, $m = \dim\ker a$.*

Proof. Let $a_1\colon (\ker a)^\perp \to \operatorname{ran} a$ be the restriction of a. a_1 is invertible by the Bounded inverse theorem. The intuitive idea of the proof is that a acts like a_1, except that it kills $\ker a$ and maps nothing to $(\operatorname{ran} a)^\perp$. So, by using b_1 and b_2, we can identify $\ker a$ with $l^2(\{1,\dots,m\})$ and $(\operatorname{ran} a)^\perp$ with $l^2(\{1,\dots,m\})$ in such a way that a in this new representation coincides with S^nS^{*m}. Below are technical details, describing how to do this.

Let $\mathbb{N}_k = \{l \in \mathbb{Z}\colon l \geq k\}$. Using $\dim(\ker a)^\perp = \operatorname{card}(\mathbb{N}) = \dim \operatorname{ran} a$, take any isomorphisms $c_1\colon l^2(\mathbb{N}_n) \to \operatorname{ran} a$ and $\widetilde{c}_2\colon (\ker a)^\perp \to l^2(\mathbb{N}_m)$. Let $s\colon l^2(\mathbb{N}_m) \to l^2(\mathbb{N}_n)$ be the restriction of S^nS^{*m}. Note that $c_1 s\widetilde{c}_2$ and a_1 are invertible operators $(\ker a)^\perp \to \operatorname{ran} a$, and define $c_2 = \widetilde{c}_2(c_1 s\widetilde{c}_2)^{-1}a_1$, so that $c_1 s c_2 = a_1$. Then, $a = b_1 S^n S^{*m} b_2$, where

$$b_1\colon \mathcal{H} = l^2(\{0,\dots,n-1\}) \oplus l^2(\mathbb{N}_n) \to (\operatorname{ran} a)^\perp \oplus \operatorname{ran} a = \mathcal{H}, \tag{3.12}$$

$$b_2\colon \mathcal{H} = \ker a \oplus (\ker a)^\perp \to l^2(\{0,\dots,m-1\}) \oplus l^2(\mathbb{N}_m) = \mathcal{H}, \tag{3.13}$$

and operators b_1 and b_2 act as c_1 and c_2 on the second components of the decompositions above, and as any invertible operator on the first ones. □

Let $\operatorname{Inv}^0(\mathcal{M}^1)$ be the connected component of the identity in $\operatorname{Inv}(\mathcal{M}^1)$. The following lemma is a special case of [18, Prop. 6.2.4]

Lemma 3.3.6. *Let $a \in \operatorname{Inv}(\mathcal{M}^1)$. Then, $a \in \operatorname{Inv}^0(\mathcal{M}^1)$ if and only if* $\operatorname{Index}(a_{++}) = 0$.

Proof. Note that $a \mapsto \operatorname{Index}(a_{++})$ is a continuous function from $\operatorname{Inv}(\mathcal{M}^1)$ to $\mathbb{Z}$, and that $\operatorname{Index}(1) = 0$. Therefore, for any $a \in \operatorname{Inv}^0(\mathcal{M}^1)$, we have $\operatorname{Index}(a_{++}) = 0$. It remains to show that the set $I = \{a \in \operatorname{Inv}(\mathcal{M}^1) \mid \operatorname{Index}(a_{++}) = 0\}$ is connected.

To do this, we take any $a \in I$ and multiply it by elements of $\operatorname{Inv}^0(\mathcal{M}^1)$ until we get an element of $\operatorname{Inv}^0(\mathcal{M}^1)$. This will ensure that $c_1 a c_2 = c_3$ for some $c_1, c_2, c_3 \in \operatorname{Inv}^0(\mathcal{M}^1)$, and, thus, $a = c_1^{-1} c_3 c_2^{-1} \in \operatorname{Inv}^0(\mathcal{M}^1)$. Consider the operator $\widetilde{a}$, which is the same as a but with removed anti-diagonal components:

$$\widetilde{a} = \begin{pmatrix} a_{++} & 0 \\ 0 & a_{--} \end{pmatrix}. \tag{3.14}$$

Since a is invertible, and $\widetilde{a} - a$ is compact, we have $\operatorname{Index} \widetilde{a} = \operatorname{Index} a = 0$. On the other hand, $\operatorname{Index} \widetilde{a} = \operatorname{Index}(a_{++}) + \operatorname{Index}(a_{--})$, so, since $\operatorname{Index} a_{++} = 0$, we have $\operatorname{Index} a_{--} = 0$. Without loss of generality, we can identify both $\mathcal{H}_+$ and $\mathcal{H}_-$ with $l^2(\mathbb{N}_0)$. By applying 3.3.5, we can find diagonal c_1, c_2 s.t. $a_1 = c_1 a c_2$ is of the form

$$a_1 = \begin{pmatrix} S^{n_+} S^{*n_+} & a_{+-} \\ a_{-+} & S^{n_-} S^{*n_-} \end{pmatrix} \tag{3.15}$$

for some nonnegative integers n_+, n_-. Since the set of invertible operators is open, there is a ball in $\mathcal{M}^1$ with the center in a_1, s.t. $a_1^{-1} a_2 \in \operatorname{Inv}^0(\mathcal{M}^1)$ for all a_2 in that ball. Take

$$a_2 = \begin{pmatrix} S^{n_+} S^{*n_+} & (1 - S^n S^{*n}) a_{+-} (1 - S^n S^{*n}) \\ (1 - S^n S^{*n}) a_{-+} (1 - S^n S^{*n}) & S^{n_-} S^{*n_-} \end{pmatrix}. \tag{3.16}$$

Since a_{-+} and a_{+-} are compact, $\lim_{n\to\infty}(1 - S^n S^{*n}) a_{+-} (1 - S^n S^{*n}) = a_{+-}$ (and similarly for a_{-+}), so for large enough integer n, operator a_2 will lie in the ball described above. Fix any such $n \geq \max(n_-, n_+)$.

Note that a_2 of the form (3.16) acts nontrivially only on $2n$-dimensional subspace $l^2(\{0, \ldots, n-1\}) \oplus l^2(\{0, \ldots, n-1\})$ of $\mathcal{H}$, and as identity operator on the orthogonal complement. Also note that all invertible operators, satisfying this property, lie in $\mathcal{M}^1$: indeed, their anti-diagonal components are finite-dimensional and, hence, Hilbert–Schmidt. The space of such operators is isomorphic to $\mathrm{GL}_{2n}(\mathbb{C})$, which is connected. So, $a_2 \in \operatorname{Inv}^0(\mathcal{M}^1)$. □

Before we start proving the main lemma of this subsection, we describe one class of bounded operators, expressible as a multiplicative commutator $[\bullet, \bullet]_1$ in $B(\mathcal{H})$.

Lemma 3.3.7. *Let* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ *be a Hilbert space with* $\dim \mathcal{H}_1 \geq \max(\dim \mathcal{H}_0, \aleph_0)$. *Then, for any invertible bounded operator* a *on* $\mathcal{H}_0$, *there are bounded operators* c *and* d *on* $\mathcal{H}$ *s.t.* $[c,d]_1 = cdc^{-1}d^{-1} = a \oplus 1$. *Here,* $a \oplus 1$ *is the operator, mapping* $x_0 + x_1$ *to* $ax_0 + x_1$ *for any* $x_j \in \mathcal{H}_j$, $j = 0, 1$.

Proof. It is enough to consider $\mathcal{H} = \mathcal{H}' = l^2(\mathbb{Z}, \mathcal{H}_0) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}'_n$, where the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is given by $\mathcal{H}_0 = \mathcal{H}'_0$, $\mathcal{H}_1 = \bigoplus_{n\neq 0} \mathcal{H}'_n$. Indeed, for general $\mathcal{H}$ by our construction $\mathcal{H}'_0$ is isomorphic to $\mathcal{H}_0$ and $\bigoplus_{n\neq 0} \mathcal{H}'_n$ can be embedded in $\mathcal{H}_1$ because $\dim \mathcal{H}_1 \geq \max(\dim \mathcal{H}_0, \aleph_0) = \dim \bigoplus_{n\neq 0} \mathcal{H}'_n$. So, if we construct operators c and d, acting on $\mathcal{H}'$ and satisfying the conditions of the lemma, then under these isomorphisms they correspond to operators, acting on some part of $\mathcal{H}$. To get the lemma for $\mathcal{H}$, continue c and d to the whole $\mathcal{H}$ by making them act as identity on the orthogonal complement of this part.

Every element of $\mathcal{H}'$ can be written as a sequence $x = \{x_j\}_{j\in\mathbb{Z}} = \sum_{j\in\mathbb{Z}} e_j x_j$, where e_j is the standard isomorphism $\mathcal{H}_0 \to \mathcal{H}'_j$, and $x_j \in \mathcal{H}_0$. We define c and d with

$$ce_n = e_{n+1}, \qquad de_n = \begin{cases} e_n a^{-1} & \text{if } n \geq 0, \\ e_n & \text{otherwise.} \end{cases} \tag{3.17}$$

Then, by direct computation we get

$$[c,d]_1 e_n = \begin{cases} e_n a & \text{if } n = 0, \\ e_n & \text{otherwise.} \end{cases} \tag{3.18}$$

as desired. □

Let $\mathrm{GL}^0(\mathcal{M}^1)$ be the connected component of the identity in

$$\mathrm{GL}(\mathcal{M}^1) = \varinjlim \mathrm{GL}_n(\mathcal{M}^1).$$

Note that $\mathrm{GL}_n(\mathcal{M}^1)$ can be interpreted as the group of invertible operators a on $\mathcal{H}^n = \mathcal{H}^n_+ \oplus \mathcal{H}^n_-$, satisfying $[F \otimes 1_n, a]_0 \in \mathcal{L}^2(\mathcal{H}^n)$. Thus, by choosing an isomorphism between $\mathcal{H}^n_\pm$ and $\mathcal{H}_\pm$, we can identify $\mathrm{GL}_n(\mathcal{M}^1)$ with $\mathrm{Inv}(\mathcal{M}^1) = \mathrm{GL}_1(\mathcal{M}^1)$. Therefore, we can apply to $\mathrm{GL}_n(\mathcal{M}^1)$ all the lemmas above.

Lemma 3.3.8. $E(\mathcal{M}^1) = \mathrm{GL}^0(\mathcal{M}^1) = \{a \in \mathrm{GL}(\mathcal{M}^1)\colon \operatorname{Index}(a_{++}) = 0\}$.

Proof. We start from the second equality. By continuity of the function $a \mapsto \operatorname{Index}(a_{++})$, so $\mathrm{GL}^0(\mathcal{M}^1) \subset \{a \in \mathrm{GL}(\mathcal{M}^1)\colon \operatorname{Index}(a_{++}) = 0\}$. To prove the converse inclusion note, take $a \in \mathrm{GL}(\mathcal{M}^1)$ with $\operatorname{Index}(a_{++}) = 0$. Note that $a \in \mathrm{GL}_n(\mathcal{M}^1)$ for some n. By using the isomorphism, identifying $\mathrm{GL}_n(\mathcal{M}^1)$ with $\operatorname{Inv}(\mathcal{M}^1)$, and applying Lemma 3.3.6, we see that a belongs to the connected component of the identity in $\mathrm{GL}_n(\mathcal{M}^1)$ and, thus, $a \in \mathrm{GL}^0(\mathcal{M}^1)$.

We know that $E(\mathcal{M}^1)$ is generated by matrices $e_{ij}(a)$ for $a \in \mathcal{M}^1$. $e_{ij}(ta) \in \mathrm{GL}(\mathcal{M}^1)$ for $t \in [0,1]$, and, thus, $e_{ij}(a)$ lie in the connected component of $e_{ij}(0) = 1$ of $\mathrm{GL}(\mathcal{M}^1)$. Thus, $E(\mathcal{M}^1) \subset \mathrm{GL}^0(\mathcal{M}^1)$.

To prove the converse inclusion, note that for any $\varepsilon > 0$ group $\mathrm{GL}^0(\mathcal{M}^1)$ is generated by operators $a \in \mathrm{GL}^0(\mathcal{M}^1)$ with $\|a - 1\| < \varepsilon$. So, we take $\varepsilon = 1$ and $a \in \mathrm{GL}^0(\mathcal{M}^1)$ with $\|a-1\| < 1$. By definition of $\mathrm{GL}(\mathcal{M}^1)$, we have $a \in \mathrm{GL}_{n-1}(\mathcal{M}^1)$ for n large enough. We interpret a as 2×2 matrix acting on $\mathcal{H}^n = \mathcal{H}^n_+ \oplus \mathcal{H}^n_-$ (and acting trivially on the last copy of $\mathcal{H}$), and note that a_{++} is an invertible operator $\mathcal{H}^n_+ \to \mathcal{H}^n_+$. Therefore, we apply the LDU decomposition to such a:

$$a = \begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix} = LDU, \text{ where} \tag{3.19}$$

$$L = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \tag{3.20}$$

$$x = a_{++}^{-1} a_{-+}, \qquad s = a_{++}, \qquad t = a_{--} - a_{-+} a_{++}^{-1} a_{+-}, \qquad y = a_{++}^{-1} a_{+-}. \tag{3.21}$$

Note that s and t act trivially (as identity operators) on the last $\mathcal{H}_\pm$ in $\mathcal{H}^n_\pm$, because we took $a \in \mathrm{GL}_{n-1}(\mathcal{M}^1)$ and interpreted it as an element of $\mathrm{GL}_n(\mathcal{M}^1)$. This allows us to represent s and t as multiplicative commutators by applying Lemma 3.3.7. So, $D \in [\mathrm{GL}(\mathcal{M}^1), \mathrm{GL}(\mathcal{M}^1)]_1 = E(\mathcal{M}^1)$. Noting that L^* is the same as U up to replacing y with x^*, we see that it is enough to show $U \in E(\mathcal{M}^1)$. This is indeed true: one can check by direct computation that

$$U = \left[\begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right]_1 \in [\mathrm{GL}(\mathcal{M}^1), \mathrm{GL}(\mathcal{M}^1)]_1 = E(\mathcal{M}^1). \tag{3.22}$$

□

Note that $\mathrm{GL}(\mathcal{M}^1) = \varinjlim \mathrm{GL}_n(\mathcal{M}^1)$, and $\mathrm{GL}_n(\mathcal{M}^1)$ can be interpreted as a subset of the algebra $B(\mathcal{H}^n)$ of bounded operators on $\mathcal{H}^n$. Therefore, elements of $\mathrm{GL}(\mathcal{M}^1)$ and, in particular, of $E(\mathcal{M}^1) = \mathrm{GL}^0(\mathcal{M}^1)$, can be interpreted as bounded operators on $l^2(\mathbb{N}_0, \mathcal{H})$. Using this interpretation, we write $\mathrm{GL}(\mathcal{M}^1) = \bigcup_n \mathrm{GL}_n(\mathcal{M}^1)$. We summarize the descriptions of $\mathrm{GL}(\mathcal{M}^1)$ and $E(\mathcal{M}^1)$ from this point of view.

Lemma 3.3.9. *Let $a \in B(l^2(\mathbb{N}_0, \mathcal{H}))$. Then,*

1. *$a \in \mathrm{GL}(\mathcal{M}^1)$ if and only if all of the following conditions hold:*
   (a) *a is invertible,*
   (b) *$[a, 1_{l^2(\mathbb{N}_0)} \otimes F] \in \mathcal{L}^2(l^2(\mathbb{N}_0, \mathcal{H}))$ (or, equivalently, a_{+-} and a_{-+} are Hilbert–Schmidt),*
   (c) *there is n s.t. $ax = a^*x = x$ for any $x \in l^2(\mathbb{N}_0 \cap [n, \infty), \mathcal{H})$;*
2. *if all these conditions hold, then a_{++} and a_{--} are Fredholm with $\mathrm{Index}(a_{--}) = -\mathrm{Index}(a_{++})$;*
3. *if $a \in \mathrm{GL}(\mathcal{M}^1)$, $n \in \mathbb{N}$, and the condition (c) holds for this n, then $a \in \mathrm{GL}_n(\mathcal{M}^1)$;*
4. *$a \in E(\mathcal{M}^1) = \mathrm{GL}^0(\mathcal{M}^1)$ if and only if $a \in \mathrm{GL}(\mathcal{M}^1)$ and $\mathrm{Index}(a_{++}) = 0$.*

Proof. This directly follows from definitions, other lemmas and discussions in this subsection. □

3.4 Connes–Karoubi character

The general construction of the Connes–Karoubi characters τ_n^{CK} is written in [7]. Here, we give the explicit description of the map τ_2^{CK}. In this section, we define the Connes–Karoubi character $K_2(\mathcal{M}^1) \to \mathbb{C}^*$. According to Part 7 of Theorem 3.2.8, such a map can be defined by describing the corresponding central extension

$$1 \to \mathbb{C}^* \to \Gamma \to E(\mathcal{M}^1) \to 1. \tag{3.23}$$

In the rest of this subsection, we describe this extension and its properties.

For a Hilbert space $\mathcal{H}$, let $\mathcal{L}^p(\mathcal{H})$ be the ideal of bounded operators $a \in B(\mathcal{H})$, satisfying $|a|^p \in \mathcal{L}^1(\mathcal{H})$, where $\mathcal{L}^1(\mathcal{H})$ is the ideal of trace-class operators. Thus, $\mathcal{L}^p(\mathcal{H})$ coincides with the set of compact operators, whose sequences of singular values belong to $l^p(\dim(\mathcal{H}))$; $\mathcal{L}^2(\mathcal{H})$ is the ideal of Hilbert–Schmidt operators. Following the Eugene Ha paper, we define $\mathcal{E}$ to be the fibre product

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathrm{GL}\big(B(\mathcal{H}_+)\big) \\ \downarrow & & \downarrow{\scriptstyle s \mapsto [s]_{\mathcal{L}^1}} \\ \mathrm{GL}^0(\mathcal{M}^1) & \xrightarrow{a \mapsto [a_{++}]_{\mathcal{L}^1}} & \mathrm{GL}^0\big(B(\mathcal{H}_+)/\mathcal{L}^1(\mathcal{H}_+)\big)\,. \end{array} \tag{3.24}$$

Note that bottom and right arrows on this diagram are surjective. In particular, surjectivity of the bottom one is the statement of Lemma 3.3.3. Using the interpretation of $\mathrm{GL}(\mathcal{H})$ as a subgroup in $\mathrm{Inv}(B(l^2(\mathbb{N}_0, \mathcal{H})))$, introduced in Lemma 3.3.9, we can describe $\mathcal{E}$ explicitly as

$$\mathcal{E} = \big\{(a,s)\colon a \in \mathrm{GL}^0(\mathcal{H}),\ s \in \mathrm{GL}(\mathcal{H}_+),\ \ s \equiv a_{++} \pmod{\mathcal{L}^1}\big\}\,. \tag{3.25}$$

For any Hilbert space $\mathcal{H}$, let $\mathcal{T}(\mathcal{H})$ be the group of operators on $\mathcal{H}$ with (nonzero) determinant (i.e., $\mathcal{T}(\mathcal{H}) = (1+\mathcal{L}^1(\mathcal{H})) \cap \mathrm{Inv}(B(\mathcal{H}))$), let $\mathcal{T} = \mathcal{T}(l^2(\mathbb{N}_0, \mathcal{H}_+)) \cap \mathrm{GL}(\mathcal{H}_+)$, and let $\mathcal{T}_1 = \{s \in \mathcal{T}\colon \det s = 1\}$. Note that $(a, s_1), (a, s_2) \in \mathcal{E}$ (with the same a), then $s_2 s_1^{-1} \in \mathcal{T}$. The Connes–Karoubi character $\tau_2^{\mathrm{CK}}\colon K_2(\mathcal{M}^1) \to \mathbb{C}^*$ is defined (in the sense of 3.2.8) by the central extension

$$1 \to \mathbb{C}^* \to \mathcal{E}/\mathcal{T}_1 \to \mathrm{GL}^0(\mathcal{M}^1) \to 1. \tag{3.26}$$

Here,

$$\mathcal{E}/\mathcal{T}_1 = \left\{(a, [s]) \colon (a, s) \in \mathcal{E},\ [s_1] = [s_2] \iff s_2 s_1^{-1} \in \mathcal{T}_1\right\}, \tag{3.27}$$

and the map $\mathbb{C}^* \to \mathcal{E}\mathcal{T}_1$ sends $\lambda \in \mathbb{C}^*$ to $(1, [s_\lambda])$ with $\det(s_\lambda) = \lambda$. Note that the equivalence class $[s_\lambda]$ is determined by λ. Indeed, if $\det(s_{\lambda,j}) = \lambda$ for $j = 1, 2$, then $\det(s_{\lambda,2} s_{\lambda,1}^{-1}) = \lambda\lambda^{-1} = 1$, and $[s_{\lambda,1}] = [s_{\lambda,2}]$. We will later need the following fact.

Lemma 3.4.1. *The group $\mathcal{E}/\mathcal{T}_1$ is perfect.*

Proof. By definition, we have to show that $\mathcal{E}/\mathcal{T}_1 \subset [\mathcal{E}/\mathcal{T}_1, \mathcal{E}/\mathcal{T}_1]$. Since $\mathrm{GL}^0(\mathcal{M}^1) = E(\mathcal{M}^1)$ is perfect (see Lemma 3.2.3), we know that the group $[\mathcal{E}/\mathcal{T}_1, \mathcal{E}/\mathcal{T}_1]$ contains preimage of every $g \in \mathrm{GL}^0(\mathcal{M}^1)$. It remains to prove that it contains the image of $\mathbb{C}^*$. Let $\lambda \in \mathbb{C}^*$ and $(1, [s_\lambda])$ be its image in $\mathcal{E}/\mathcal{T}_1$. In order to apply Lemma 3.3.7, choose s_λ to act as multiplication by λ on one of the nonzero elements of $\mathcal{H}_+$, and as identity on its orthogonal complement. Then, by Lemma 3.3.7, $s_\lambda = [c, d]_1$ for some invertible $c, d \in \mathrm{Inv}(\mathcal{H}_+)$. Thus, $(1, [s_\lambda]) = [(\widetilde{c}, [c]), (\widetilde{d}, [d])]_1 \in [\mathcal{E}/\mathcal{T}_1, \mathcal{E}/\mathcal{T}_1]$, where $\widetilde{c} = c \oplus \mathrm{id}_{\mathcal{H}_-}$ and $\widetilde{d} = d \oplus \mathrm{id}_{\mathcal{H}_-}$. □

3.5 Fredholm structure on loops

Let S be a finite subset of X, and $\gamma \colon S^1 \to X \setminus S$ be a parametrized smooth loop. In this section, we give a definition of a map $\rho_\gamma \colon \mathcal{O}(X \setminus S) \to \mathcal{M}^1$. This definition, and the proof of its correctness, come from the original manuscript [12].

Let $\mathcal{H}$ be the Hilbert space of square-integrable functions on S^1: $\mathcal{H} = L^2(S^1, d\theta)$. Interpreting S^1 as $\mathbb{R}/(2\pi\mathbb{Z})$, let $z \in \mathcal{H}$ be the function $\theta \mapsto e^{i\theta}$, $\mathcal{H}_+$ and $\mathcal{H}_-$ be the (closed) subspaces of $\mathcal{H}$, generated by z^n for $n \geq 0$ and $n < 0$ respectively, so that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. From now on, we interpret $\mathcal{M}^1$ to be defined using these $\mathcal{H}$, $\mathcal{H}_+$ and $\mathcal{H}_-$. Here, variable θ takes values in $[0, 2\pi)$. For a continuous function $h \colon S^1 \to \mathbb{C}$, we have a multiplication operator $M_h\colon \mathcal{H} \to \mathcal{H}$, defined by $(M_h\xi)(\theta) = h(\theta)\xi(\theta)$. We define

$$\rho_\gamma\colon \mathcal{O}(X \setminus S) \to \mathcal{M}^1\colon f \mapsto M_{f\circ\gamma}. \tag{3.28}$$

Note that *a priori*, we only know that $M_{f\circ\gamma}$ is a bounded operator on $\mathcal{H}$. For ρ to be well defined, we need to know that $M_{f\circ\gamma} \in \mathcal{M}^1$. This is, indeed, true, and was shown in [12, Lemma 1], which is originally [18, Prop. 6.3.1]. Letting $g = f \circ \gamma$, we see that this statement follows from the statement $[F, M_g] \in \mathcal{L}^2$. Later, however, we will need a stronger statement $[F, M_g] \in \mathcal{L}^1$. Therefore, we repeat and enhance the proof of [18, Prop. 6.3.1] to get that stronger statement.

Lemma 3.5.1. *For any smooth function $g\colon S^1 \to \mathbb{C}$ one has $[F, M_g] \in \mathcal{L}^1$. The Hilbert–Schmidt norm $\|[F, M_g]\|_2$ satisfies*

$$\|[F, M_g]\|_2 \le c \sup_\theta |g'(\theta)| = c\|g'\|_\infty. \tag{3.29}$$

for some universal constant $c > 0$.

Proof. In the realization of the universal 2-summable Fredholm module above, operator

$$F\colon \mathcal{H} \to \mathcal{H}\colon x_+ + x_- \mapsto x_+ - x_- \tag{3.30}$$

can be written as

$$(F\xi)(\theta) = P.V. \int_0^{2\pi} K(\theta_1 - \theta)\xi(\theta_1)\frac{d\theta_1}{2\pi}, \tag{3.31}$$

where kernel K is given by

$$K(\theta) = 1 - i\cot(\theta/2). \tag{3.32}$$

From (3.31) we compute the kernel K_1 of $[F, M_g]$:

$$K_1(\theta, \theta_1) = K(\theta_1 - \theta)\left(g(\theta_1) - g(\theta)\right). \tag{3.33}$$

Note that this is smooth everywhere, except may be the diagonal $\theta_1 = \theta$. For θ_1 in a neighborhood of θ, we have

$$K_1(\theta, \theta_1) = ((\theta_1 - \theta) \cdot K(\theta_1 - \theta)) \frac{g(\theta_1) - g(\theta)}{\theta_1 - \theta} \tag{3.34}$$

with both terms being smooth, so K_1 is smooth everywhere on $S^1 \times S^1$. In particular, this allows to get rid of "P.V.." According to [24, Prop. IV.3.5], every operator with kernel in $C^2(S^1 \times S^1)$ is in $\mathcal{L}^1$. By applying this statement to the kernel K_1, we get $[F, M_g] \in \mathcal{L}^1$.

The second statement of the lemma follows from (3.34) and the fact that the Hilbert–Schmidt norm of the operator, given by the kernel K_1, is equal to

$$\left(\int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \left| K_1(\theta_1, \theta) \right|^2 \right)^{1/2} \leq \sup_{\theta, \theta_1} \left| K_1(\theta_1, \theta) \right|. \tag{3.35}$$

□

In what follows, we will be interested in the composition $\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_* \colon K_2(\mathcal{O}(X \setminus S)) \to \mathbb{C}^*$.

3.6 Reparameterization of loops

The goal of this section is to prove that the map $\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_*$ only depends on the oriented path, and not on its parametrization.Let γ be a smooth loop $S^1 \to X \setminus S$, and $\varphi \colon S^1 \to S^1$ be a smooth orientation-preserving map. We are going to prove that $\rho_\gamma = \rho_{\gamma \circ \varphi}$. Following [12] and [18, Section 6.8], we introduce the unitary U_φ by

$$\left(U_\varphi \xi \right)(\theta) = \xi \left(\varphi^{-1}(\theta) \right) \left((\varphi^{-1})'(\theta) \right)^{1/2}. \tag{3.36}$$

Its adjoint $U_\varphi^* = U_\varphi^{-1}$ is then given by

$$\left(U_\varphi^* \xi \right)(\theta) = \xi \left(\varphi(\theta) \right) \left((\varphi)'(\theta) \right)^{1/2}. \tag{3.37}$$

For any invertible operator V, we let

$$\mathrm{Ad}_V : B(\mathcal{H}) \to B(\mathcal{H}) : W \mapsto VWV^{-1}. \tag{3.38}$$

By a direct computation, one can check that

$$\mathrm{Ad}_{U_\varphi} M_{f\circ\gamma\circ\varphi} = M_{f\circ\gamma}. \tag{3.39}$$

In order to be able to use this equality, we need to check that Ad_{U_φ} can be re-interpreted as a map $\mathcal{M}^1 \to \mathcal{M}^1$. The following lemma is, essentially, [18, Prop. 6.8.2].

Lemma 3.6.1.

1. *$U_\varphi \in \mathcal{M}^1$, moreover $[F, U_\varphi] \in \mathcal{L}^1$;*
2. *if $a \in \mathcal{M}^1$, then $\mathrm{Ad}_{U_\varphi}(a) \in \mathcal{M}^1$.*

Proof. The second statement of the lemma follows from the first one. We will show the first one, following a similar strategy to the one used in the proof of 3.5.1. By Lemma 3.5.1, multiplication by a smooth function g satisfies $[F, M_g] \in \mathcal{L}^1$. So, it remains to prove that change of variable operation satisfies the same property. Let us denote this operation with V, so that

$$(V\xi)(\theta) = \xi(\varphi^{-1}(\theta)). \tag{3.40}$$

Using the same notation, as in the proof of Lemma 3.5.1, we compute the kernel K_2 of $[F, V]$:

$$K_2(\theta, \theta_1) = K(\varphi(\theta_1) - \theta)\varphi'(\theta_1) - K(\theta_1 - \varphi^{-1}(\theta)). \tag{3.41}$$

This can only be non-smooth in the neighborhood of the "modified" diagonal $\varphi(\theta_1) = \theta$. We will prove that it is smooth there, too. To do that, we choose a contractible neighbourhood of a point on that modified diagonal, and study the behavior of K_2

there. We let

$$K_3(\theta_2,\theta_1) = K_2(\varphi(\theta_1+\theta_2),\theta_1) = K(\varphi(\theta_1)-\varphi(\theta_1+\theta_2))\varphi'(\theta_1) - K(-\theta_2). \quad (3.42)$$

Since the function $(\theta_2,\theta_1)\mapsto(\varphi(\theta_1+\theta_2),\theta_1)$ is a smooth dif and only ifeomorphism in the preimage of the chosen neighbourhood, it is enough to prove that K_3 is smooth. By Taylor expansion of φ with Peano reminder, for some smooth function φ_2, we have

$$\varphi(\theta_1+\theta_2) = \varphi(\theta_1) + \theta_2\varphi'(\theta_1) + \theta_2^2\,\varphi_2(\theta_2,\theta_1), \quad (3.43)$$

Note that $K(\theta)-(-2i/\theta)$ is smooth, and (3.42) depends linearly on K, so it is enough to prove the smoothness of the expression in the right-hand side of (3.42) with $K(\bullet)$ replaced with $1/(\bullet)$. It is equal to

$$(\varphi(\theta_1)-\varphi(\theta_1+\theta_2))^{-1}\varphi'(\theta_1) - (-\theta_2)^{-1} = \frac{1}{\theta_2}\left(1-\frac{\varphi'(\theta_1)}{\varphi'(\theta_1)+\theta_2\,\varphi_2(\theta_2,\theta_1)}\right). \quad (3.44)$$

Since the expression in the large brackets is smooth near $\theta_2=0$, and equal to 0 for $\theta_2=0$, expression (3.44) is smooth. The statement of the lemma now follows from [24, Prop. IV.3.5]. $\square$

Note that the desired equality $\tau_2^{\mathrm{CK}}\circ(\rho_\gamma)_* = \tau_2^{\mathrm{CK}}\circ(\rho_{\gamma\circ\varphi})_*$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} K_2(\mathcal{O}(X\setminus S)) & \xrightarrow{(\rho_{\gamma\circ\varphi})_*} & K_2(\mathcal{M}^1) \\ \downarrow{\scriptstyle(\rho_\gamma)_*} & & \downarrow{\scriptstyle\tau_2^{\mathrm{CK}}} \\ K_2(\mathcal{M}^1) & \xrightarrow{\tau_2^{\mathrm{CK}}} & \mathbb{C}^* \end{array} \quad . \quad (3.45)$$

Equality (3.39) and lemma (3.6.1) imply that the following diagram, representing the upper left triangle of (3.45), is well defined and commutative.

$$\begin{array}{ccc} \mathcal{O}(X\setminus S) & \xrightarrow{\rho_{\gamma\circ\varphi}} & \mathcal{M}^1 \\ \downarrow{\scriptstyle\rho_\gamma} & \swarrow{\scriptstyle\mathrm{Ad}_{U_\varphi}} & \\ \mathcal{M}^1 & & \end{array} \quad (3.46)$$

Thus, to prove the commutativity of (3.45) it remains to show the commutativity of the following diagram:

$$\begin{array}{ccc} & & K_2(\mathcal{M}^1) \\ & \swarrow^{(\mathrm{Ad}_{U_\varphi})_*} & \downarrow{\scriptstyle \tau_2^{\mathrm{CK}}} \\ K_2(\mathcal{M}^1) & \xrightarrow{\tau_2^{\mathrm{CK}}} & \mathbb{C}^* \end{array} \quad . \tag{3.47}$$

Lemma 3.6.2. *Diagram* (3.47) *is commutative.*

Proof. Using the definition of (3.26), we see that it is enough to prove the existence of a group isomorphism $\psi\colon \mathcal{E}/\mathcal{T}_1 \to \mathcal{E}/\mathcal{T}_1$, making the diagram

$$\begin{CD} 1 @>>> \mathbb{C}^* @>i>> \mathcal{E}/\mathcal{T}_1 @>\pi>> \mathrm{GL}^0(\mathcal{M}^1) @>>> 1 \\ @. @| @VV\psi V @VV\mathrm{Ad}_{U_\varphi}V @. \\ 1 @>>> \mathbb{C}^* @>i>> \mathcal{E}/\mathcal{T}_1 @>\pi>> \mathrm{GL}^0(\mathcal{M}^1) @>>> 1 \end{CD} \tag{3.48}$$

commutative. In fact, any homomorphism ψ will suffice, since by 5-lemma it will automatically be bijective. In order to construct ψ, we will use Item 6 of Theorem 3.2.8 with $\pi_1 = \mathrm{Ad}_{U_\varphi} \circ \pi$, $\pi_2 = \pi$. The group $\mathcal{E}/\mathcal{T}_1$ is perfect according to Lemma 3.4.1. Therefore, by Item 6 of 3.2.8, it remains to check that the right-hand side of

$$\psi([x_1, x_2]) = \left[\pi^{-1} \mathrm{Ad}_{U_\varphi} \pi x_1, \pi^{-1} \mathrm{Ad}_{U_\varphi} \pi x_2\right] \tag{3.49}$$

depends only on $[x_1, x_2]$, and not on x_1, x_2, and that the map ψ thus obtained is a homomorphism. To prove that, take an arbitrary $x = (a, [s]_{\mathcal{T}_1}) \in \mathcal{E}/\mathcal{T}_1$, and, using Lemma 3.4.1, find $x_l = (a_l, [s_l]_{\mathcal{T}_1})$ for $l = 1, 2$ s.t. $x = [x_1, x_2]$. In components, that means that $a = [a_1, a_2]_1$ and $s \equiv [s_1, s_2]_1 \pmod{\mathcal{T}_1}$, i.e., $s^{-1}[s_1, s_2]_1 \in \mathcal{T}_1$. Then, let $b_l \in \mathrm{Ad}_{U_\varphi} a_l$, choose any $t_l \in (b_{l++} + \mathcal{L}^1) \cap \mathrm{GL}(\mathcal{H}_+)$, and let $b = [b_1, b_2]_1$, $t = [t_1, t_2]_1$. Right-hand side of (3.49) is, then, $(b, [t]_{\mathcal{T}_1})$. We have $b = \mathrm{Ad}_{U_\varphi} a$, and, thus, b is determined by x. Since anti-diagonal components of U_φ are in $\mathcal{L}^1$, we have

$$t_l \overset{\mathcal{L}^1}{\equiv} b_{l++} \overset{\mathcal{L}^1}{\equiv} U_{\varphi_{++}} a_{l++} U_{\varphi}{}^*_{++} \overset{\mathcal{L}^1}{\equiv} U_{\varphi_{++}} s_l U_{\varphi}{}^*_{++} \overset{\mathcal{L}^1}{\equiv} \left(U_\varphi (s_l \oplus \mathrm{id}_{\mathcal{H}_-}) U_\varphi{}^{-1}\right)_{++}. \tag{3.50}$$

We now want to use the fact that once b_1, b_2 are fixed, t is determined uniquely (Lemma 3.2.7). Therefore, we can use our freedom of choice of $t_l \in (b_{l++} + \mathcal{L}^1) \cap \mathrm{GL}(\mathcal{H}_+)$ to simplify the computations. Unfortunately, we can't take t_l to be one of the expressions in (3.50), since these are not necessarily invertible. As a workaround, we will use the fact that anti-diagonal components of U_φ belong to $\mathcal{L}^1$ (see Lemma 3.6.1). Let $U_{\varphi,1}$ be U_φ with anti-diagonal components replaced with 0. Then, $U_\varphi \equiv U_{\varphi,1} \pmod{\mathcal{L}^1}$. Since we can continuously deform φ to the identity homeomorphism, thus continuously deforming $U_{\varphi,1}$ to the identity operator, the components $U_{\varphi,1++}$ and $U_{\varphi,1--}$ have Fredholm index 0. Thus, we can make these components invertible by adding finite-dimensional operators to them. Let $\widetilde{U}$ be the operator with these modified (invertible) components. We have $U_\varphi \equiv \widetilde{U} \pmod{\mathcal{L}^1}$ and $\widetilde{U}_{-+} = 0$, $\widetilde{U}_{+-} = 0$. Using this and (3.50), we have

$$b_{l++} \overset{\mathcal{L}^1}{\equiv} \left(U_\varphi(s_l \oplus \mathrm{id}_{\mathcal{H}_-}){U_\varphi}^{-1}\right)_{++} \overset{\mathcal{L}^1}{\equiv} \left(\widetilde{U}(s_l \oplus \mathrm{id}_{\mathcal{H}_-})\widetilde{U}^{-1}\right)_{++} \overset{\mathcal{L}^1}{\equiv}= \widetilde{U}_{++} s_l \widetilde{U}_{++}^{-1}. \tag{3.51}$$

Therefore, as discussed above, we can take

$$t_{l++} = \widetilde{U}_{++} s_l \widetilde{U}_{++}^{-1}. \tag{3.52}$$

Then, $t = [t_{1++}, t_{2++}]_1 = \widetilde{U}_{++}[s_1, s_2]_1 \widetilde{U}_{++}^{-1} = \widetilde{U}_{++} s \widetilde{U}_{++}^{-1}$, so the class $[t]_{\mathcal{T}_1}$ is indeed determined by x, and doesn't depend on x_1, x_2.

Finally, note that ψ is a homomorphism, because, as we have shown above, it is given by

$$\psi((a, [s]_{\mathcal{T}_1})) = \left(\mathrm{Ad}_{U_\varphi} a, \left[\mathrm{Ad}_{\widetilde{U}_{++}} s\right]_{\mathcal{T}_1}\right). \tag{3.53}$$

□

3.7 Moving towards the definition of the Beilinson–Bloch regulator

In this section, following [12], we will discuss the possibility to define the maps r_S and r_ξ using τ_2^{CK}. We will denote our maps with $\widetilde{r}_S$ and $\widetilde{r}_\xi$, and preserve r without tilde for the maps in the original definition of the Beilinson–Bloch regulator. Since r_S is a map $K_2(\mathcal{O}(X \setminus S)) \to H^1(X \setminus S, \mathbb{C}^*) \simeq \mathrm{Hom}(\pi_1(X \setminus S), \mathbb{C}^*)$, in order to define it, it is enough to define the pairing of $r_S(u)$ with $[\gamma]$ for $u \in K_2(\mathcal{O}(X \setminus S))$, $[\gamma] \in \pi_1(X \setminus S)$. Following [12], we would like to let

$$\langle \widetilde{r}_S(u), [\gamma] \rangle = \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_* \right)(u). \tag{3.54}$$

In order to discuss the possibility of using this definition, we denote its right-hand side with $\widetilde{R}_S(u, \gamma)$:

$$\widetilde{R}_S(u, \gamma) = \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_* \right)(u) \in \mathbb{C}^*. \tag{3.55}$$

Note that, by definition, the map $u \mapsto \widetilde{R}_S(u, \gamma)$ is a homomorphism. We would like to prove the following.

1. $\widetilde{R}_S(u, \gamma)$ depends only on the class $[\gamma] \in \pi_1(X \setminus S, x_0)$ and not on the loop γ itself.

2. The map $[\gamma] \mapsto \widetilde{R}_S(u, \gamma)$ is a homomorphism $\pi_1(X \setminus S, x_0) \to \mathbb{C}^*$, i.e., whenever $[\gamma] = [\gamma_1][\gamma_2]$ one has $\widetilde{R}_S(u, \gamma) = \widetilde{R}_S(u, \gamma_1)\widetilde{R}_S(u, \gamma_2)$.

3. Maps $\widetilde{R}_S(u, \gamma_1)$ are compatible with restrictions, i.e., if $S_1 \subset S_2 \subset X$, $\mathrm{res}_{2,1} \colon \mathcal{O}(X \setminus S_1) \to \mathcal{O}(X \setminus S_2)$ is the corresponding restriction map, $x_0 \in X \setminus S$, $u \in K_2(\mathcal{O}(X \setminus S_1))$, and $\gamma \colon (S^1, 1) \to (X \setminus S_2, x_0)$, one has $\widetilde{R}_{S_1}(u, \gamma) = \widetilde{R}_{S_2}((\mathrm{res}_{2,1})_*(u), \gamma)$.

4. $\widetilde{R}_S(u, \gamma) = \langle r_S(u), [\gamma] \rangle$, where r_S is the original Beilinson–Bloch regulator.

Properties 1,2 would then imply that (3.54) defines well-defined homomorphisms

$$\widetilde{r}_S \colon K_2(\mathcal{O}(X \setminus S)) \to H^1(X \setminus S, \mathbb{C}^*);$$

Property 3 would imply that maps $\widetilde{r}_S$ are compatible with restrictions, so we can define $\widetilde{r}_\xi$ as in (3.2); Property 4 would imply that $\widetilde{r}_S = r_S$ and $\widetilde{r}_\xi = r_\xi$. These are not independent: Property 4 would imply 1–3, since the original Beilinson–Bloch generator satisfies 1–3.

In the original manuscript, Properties 1–3 are summarized as [12, Lemma 3]. Unfortunately, we were not able to fill-in all the technical details necessary to finish its proof, so we will only be able to show Property 3 above (see Lemma 3.7.1). After that, in the next section, we will show in Lemma 3.8.1 that Property 4 holds on the Steinberg symbols (see Definition 3.2.10):

4'. Property 4 above holds for u in the subgroup of $K_2(\mathcal{O}(X \setminus S))$, generated by Steinberg symbols. In other words, for any $f, g \in \mathrm{Inv}(\mathcal{O}(X \setminus S))$ we have

$$\widetilde{R}_S(\{f,g\},\gamma) = \langle r_S(\{f,g\}), [\gamma]\rangle \,. \tag{3.56}$$

Note that since both $u \mapsto \widetilde{R}_S(u,\gamma)$ and $u \mapsto \langle r_S(u,[\gamma]\rangle$ are group homomorphisms, it is equivalent to ask Property 4 to hold on Steinberg symbols, and on the subgroup, generated by Steinberg symbols.

Lemma 3.7.1. *Let $S_1, S_2 \subset X$ be finite subsets of X, s.t. $S_1 \subset S_2$, let $\mathrm{res}_{2,1} \colon \mathcal{O}(X \setminus S_1) \to \mathcal{O}(X \setminus S_2)$ be the corresponding restriction map, and let $\gamma \colon S^1 \to X \setminus S_2$ be a smooth path. Then,*

$$\rho_\gamma = \rho_\gamma \circ \mathrm{res}_{2,1} \tag{3.57}$$

as maps $\mathcal{O}(X \setminus S_1) \to \mathcal{M}^1$. In particular, for any $u \in K_2(\mathcal{O}(X \setminus S_1))$ one has $\widetilde{R}_{S_1}(u,\gamma) = \widetilde{R}_{S_2}((\mathrm{res}_{2,1})_* u, \gamma)$, i.e., Property 3 holds.

Note that in (3.57) on the left-hand side ρ_γ is interpreted as a map $\mathcal{O}(X \setminus S_1) \to \mathcal{M}^1$, and on the right — as a map $\mathcal{O}(X \setminus S_2) \to \mathcal{M}^1$.

Proof. In the equality (3.57), ρ_γ on the left maps a function $f \in \mathcal{O}(X \setminus S_1)$ to an operator $M_{f\circ\gamma}$. The restriction map on the right maps f to the same function, restricted to $X \setminus S_2$. This restricted function takes the same values on the loop γ, so we get the same operator $M_{f\circ\gamma}$ by applying ρ_γ to it. Thus, equality (3.57) holds.

The second statement then follows directly from the definition of R_S:

$$R_{S_1}(u,\gamma) = \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_*\right)(u) = \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma \circ \mathrm{res}_{2,1})_*\right)(u) = \\ \left(\tau_2^{\mathrm{CK}} \circ (\rho_\gamma)_*\right)((\mathrm{res}_{2,1})_* u) = R_{S_2}((\mathrm{res}_{2,1})_* u, \gamma). \quad (3.58)$$

□

Lemma 3.7.2. *Properties 4', 3 and 1 above imply Property 4.*

Proof. Assume that maps R_S satisfy 4', 3, 1, let $u_1 \in K_2(\mathcal{O}(X \setminus S_1))$, and let's try to prove 4 for $u = u_1$ and $S = S_1$. Let $\mathrm{res}_1 \colon \mathcal{O}(X \setminus S_j) \to F(X)$ and $\mathrm{res}_{j,k} \colon \mathcal{O}(X \setminus S_k) \to \mathcal{O}(X \setminus S_j)$ be the restriction maps (the finite sets S_2, S_3 will be chosen later). It is well-known that K_2 of a field is generated by Steinberg symbols (see e.g. [22, Theorem 4.3.3]). Therefore, $(\mathrm{res}_1)_*(u_1) = \prod_{j=1}^n \{f_j, g_j\}$ for some rational functions $f_j, g_j \in F(X) \setminus \{0\}$. Let S_2 be the union of S_1 with the set of zeros and poles of all f_j, g_j ($j = 1, \ldots, n$). Then, $\widetilde{u}_2 = \prod_{j=1}^n \{f_j, g_j\}$ is a well-defined element of $K_2(\mathcal{O}(X \setminus S))$. Let $u_2 = (\mathrm{res}_{2,1})_*(u_1)$. Then, $(\mathrm{res}_2)_*(u_2) = (\mathrm{res}_1)_*(u_1) = \prod_{j=1}^n \{f_j, g_j\} = (\mathrm{res}_2)_*(\widetilde{u}_2)$. Thus, there exists a finite set $S_3 \supset S_2$ s.t. $(\mathrm{res}_{3,2})_*(u_2) = (\mathrm{res}_{3,2})_*(\widetilde{u}_2)$. Now, choose arbitrary $\gamma \colon S^1 \to X \setminus S_1$, and let $\widetilde{\gamma}$ be its deformation, which avoids points of S_3. We have

$$\widetilde{R}_{S_1}(u_1,\gamma) = \widetilde{R}_{S_1}(u_1,\widetilde{\gamma}) = \widetilde{R}_{S_3}(u_3,\widetilde{\gamma}) = \\ \widetilde{R}_{S_3}\left(\prod_{j=1}^n \{f_j, g_j\},\ \widetilde{\gamma}\right) = \prod_{j=1}^n \widetilde{R}_{S_3}\left(\{f_j, g_j\}, \widetilde{\gamma}\right) = \prod_{j=1}^n \langle r_{S_3}(\{f_j, g_j\}), [\widetilde{\gamma}]\rangle = \\ \left\langle r_{S_3}\left(\prod_{j=1}^n \{f_j, g_j\}\right),\ [\widetilde{\gamma}]\right\rangle = \langle r_{S_3}(u_3), [\widetilde{\gamma}]\rangle = \langle r_{S_1}(u_1), [\widetilde{\gamma}]\rangle = \langle r_{S_1}(u_1), [\gamma]\rangle. \quad (3.59)$$

□

Since we didn't prove Property 1, we will try to improve the statement above to avoid using it. Note that the proof above shows that, given maps R_S satisfy $4'$

and 3, for any S_1 and $u \in K_2(\mathcal{O}(X \setminus S_1))$, there is a finite set $S_3 \supset S_1$ s.t. 4 holds for any loop γ, which doesn't intersect with S_3. Since such S_3 is finite, "almost all" paths satisfy this condition. Thus, we may hope to replace Property 1 in the list of requirements of the lemma with some continuity condition on the map $\gamma \mapsto R_S(u, \gamma)$. We will do this in Section 3.9.

3.8 Computation of the Beilinson–Bloch regulator on Steinberg symbols

3.8.1 Notation and general observations

The goal of Section 3.8 is to prove Property 4' of $\widetilde{R}_S$ from Section 3.7. From the original definition of the Beilinson–Bloch regulator [2] we know that the right-hand side of (3.56) can be written as

$$\langle r_S(\{f, g\}), [\gamma] \rangle = \exp\left(\frac{1}{2\pi i}\left(\int_\gamma \ln f \, d\ln g - \ln g(x_0)\int_\gamma d\ln f\right)\right). \tag{3.60}$$

Here, x_0 is a point on the path γ, chosen as the starting point of the integrations. Branches of $\ln f$ and $\ln g$ are chosen at this starting point, and analitically continued along the path. One can check that the right-hand side doesn't depend on the choice of x_0 and branches of logarithms. For the definition of the left-hand side of (3.56), see (3.55). Thus, both sides can be written in terms of restrictions of f and g on the circle. To be more precise, let

$$\rho\colon C^\infty(S^1) \to \mathcal{M}^1\colon h \mapsto M_h, \qquad \gamma^*\colon \mathcal{O}(X \setminus S) \to C^\infty(S^1)\colon f \mapsto f \circ \gamma. \tag{3.61}$$

We let $\widetilde{R}$ be the map, analogous to $\widetilde{R}_S$, but acting on $K_2\left(C^\infty(S^1)\right)$. More precisely, we let

$$\widetilde{R}(u) = \tau_2^{\mathrm{CK}} \circ \rho_*\colon K_2\left(C^\infty(S^1)\right) \to \mathbb{C}^*. \tag{3.62}$$

Then, $\widetilde{R}_S(u,\gamma) = \widetilde{R}((\gamma^*)_*(u))$, and, in particular, $\widetilde{R}_S(\{f,g\},\gamma) = \widetilde{R}(\{f\circ\gamma, g\circ\gamma\})$. To deal similarly with (3.60), for $\widetilde{f},\widetilde{g} \in \mathrm{Inv}(C^\infty(S^1))$ we let

$$R(\widetilde{f},\widetilde{g}) = \exp\left(\frac{1}{2\pi i}\left(\int_{S^1} \ln\widetilde{f} d\ln\widetilde{g} - \ln\widetilde{g}(1)\int_{S^1} d\ln\widetilde{f}\right)\right) \tag{3.63}$$

with integrals starting at $1 \in S^1$. Using this notation, it remains to prove that for any $\widetilde{f},\widetilde{g} \in C^\infty(S^1)$ we have

$$\widetilde{R}(\{\widetilde{f},\widetilde{g}\}) = R(\widetilde{f},\widetilde{g}). \tag{3.64}$$

In order to do this, we write $\widetilde{f} = z^n e^f$, $\widetilde{g} = z^m e^g$, where z is the identity function on the circle, n and m are winding numbers of $\widetilde{f}$ and $\widetilde{g}$ respectively, f and g are smooth functions on the circle. We then express both sides of (3.64) in terms of Fourier coefficients of f and g, and compare the results.

3.8.2 Algorithm

Let

$$1 \longrightarrow K_2(C^\infty(S^1)) \xrightarrow{i} \mathrm{St}(C^\infty(S^1)) \xrightarrow{\pi} E(C^\infty(S^1)) \longrightarrow 1 \tag{3.65}$$

be the universal central extension of $E(C^\infty(S^1))$. Note that $E(C^\infty(S^1)) = \mathrm{SL}(C^\infty(S^1)) \simeq \bigcup_n C^\infty(S^1,\mathrm{SL}_n(\mathbb{C}))$ (where $\mathrm{SL}_n(\mathbb{C})$ is interpreted as a subset in $\mathrm{SL}_{n+1}(\mathbb{C})$ using the embedding $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right)$). Given two matrix-valued functions $a,b \in C^\infty(S^1,\mathrm{SL}_n(\mathbb{C}))$ satisfying $[a,b]_1 = 1$, we have $i^{-1}[\pi^{-1}a,\pi^{-1}b] \in K_2(C^\infty(S^1))$. We observe that we can unroll the definitions, used to define $\widetilde{R}$, into the following "algorithm," which, given a,b as above, allows us to "compute" $\widetilde{R}(i^{-1}[\pi^{-1}a,\pi^{-1}b])$:

0. take $a,b \in C^\infty(S^1,\mathrm{SL}_n(\mathbb{C}))$;

1. compute corresponding Toeplitz operators T_a, $T_b\colon \mathcal{H}_+^n \to \mathcal{H}_+^n$;

2. find invertible $s_a \equiv T_a \pmod{\mathcal{L}^1}$, $s_b \equiv T_b \pmod{\mathcal{L}^1}$;

3. return $\widetilde{R}(i^{-1}[\pi^{-1}a, \pi^{-1}b]) = \det([s_a, s_b]_1)$.

The result doesn't depend on the choice of s_a, s_b. Here, the Toeplitz operator T_a of a matrix-valued function $a \in C^\infty(S^1, M_n(\mathbb{C}))$ is defined as follows. Let $P_{+,n} \colon \mathcal{H}^n \to \mathcal{H}^n_+$ be the standard projector. Then, $T_a = P_{+,n} M_a P^*_{+,n}$. We recall that, as we agreed above, $\mathcal{H} = L^2(S^1)$ is the space of square-integrable functions, and $\mathcal{H}_+$ is its closed subspace, generated by z^l for $l \geq 0$. In order to use this algorithm to compute $\widetilde{R}(\{\widetilde{f}, \widetilde{g}\})$ one has to apply this algorithm with

$$a = \operatorname{diag}(\widetilde{f}, \widetilde{f}^{-1}, 1), \qquad b = \operatorname{diag}(\widetilde{g}, 1, \widetilde{g}^{-1}). \tag{3.66}$$

3.8.3 Plan

Here, we outline the plan of the computations. Given $\widetilde{f} = z^n e^f$ and $\widetilde{g} = z^m e^g$, we want to compute $\widetilde{R}(\{\widetilde{f}, \widetilde{g}\})$. Using the properties of Steinberg symbols, we have

$$\begin{aligned}\widetilde{R}(\{\widetilde{f}, \widetilde{g}\}) = \widetilde{R}(\{z^n e^f, z^m e^g\}) = \widetilde{R}\left(\{z, z\}^{nm}\{e^f, z\}^m\{e^g, z\}^{-n}\{e^f, e^g\}\right) = \\ \widetilde{R}\left(\{z, z\}\right)^{nm} \widetilde{R}\left(\{e^f, z\}\right)^m \widetilde{R}\left(\{e^g, z\}\right)^{-n} \widetilde{R}\left(\{e^f, e^g\}\right).\end{aligned} \tag{3.67}$$

Since the second and the third term are of the same form, our computation splits into 4 parts: Term 1, Terms 2 and 3, Term 4 and putting the terms together. Each of these steps is done in the corresponding subsection. The last subsection is devoted to writing $R(\widetilde{f}, \widetilde{g})$ in a similar form (i.e., expressing it using the winding numbers n, m and Fourier coefficients of f, g, where $\widetilde{f} = z^n e^f$, $\widetilde{g} = z^m e^g$), and comparing the results. We use the following Fourier transformation $\hat{f}$ of f:

$$f = \sum_{n=-\infty}^{\infty} \hat{f}_n z^n. \tag{3.68}$$

Term 1, i.e., $\widetilde{R}\left(\{z, z\}\right)$, will be computed explicitly using the algorithm from

Subsection 3.8.2. We express Term 2 as

$$\widetilde{R}\left(\{e^f, z\}\right) = \exp\left(\left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{R}\left(\{e^{tf}, z\}\right)\right), \tag{3.69}$$

so we will compute this derivative. Similarly, for Term 4, we use

$$\widetilde{R}\left(\{e^f, e^g\}\right) = \exp\left(\left.\frac{\partial^2}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} \widetilde{R}\left(\{e^{t_1 f}, e^{t_2 g}\}\right)\right). \tag{3.70}$$

3.8.4 Term 1

Here, we compute $\widetilde{R}(\{\lambda z, \mu z\})$. We will only need it for $\lambda = \mu = 1$, but allowing arbitrary nonzero complex coefficients doesn't make the computation more complex and allows us to check it. We have $a = \operatorname{diag}(\lambda z, \lambda^{-1}\bar{z}, 1)$, $b = \operatorname{diag}(\mu z, 1, \mu^{-1}\bar{z})$. Shift operator notation: $T_z = S$, $S^*S = 1$, $SS^* = 1 - P$, $PS = S^*P = 0$. We get

$$T_a = \operatorname{diag}(\lambda S, \lambda^{-1}S^*, 1), \qquad T_b = \operatorname{diag}(\mu S, 1, \mu^{-1}S^*), \tag{3.71}$$

$$s_a = \begin{pmatrix} \lambda S & P & 0 \\ 0 & \lambda^{-1}S^* & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_b = \begin{pmatrix} \mu S & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & \mu^{-1}S^* \end{pmatrix}. \tag{3.72}$$

$[s_a, s_b]_1 =$

$$\begin{pmatrix} \lambda S & P & 0 \\ 0 & \lambda^{-1}S^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu S & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & \mu^{-1}S^* \end{pmatrix} \begin{pmatrix} \lambda^{-1}S^* & 0 & 0 \\ P & \lambda S & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu^{-1}S^* & 0 & 0 \\ 0 & 1 & 0 \\ P & 0 & \mu S \end{pmatrix} =$$

$$\begin{pmatrix} \lambda\mu S^2 & P & \lambda SP \\ 0 & \lambda^{-1}S^* & 0 \\ 0 & 0 & \mu^{-1}S^* \end{pmatrix} \begin{pmatrix} \lambda^{-1}\mu^{-1}S^{*2} & 0 & 0 \\ \mu^{-1}PS^* & \lambda S & 0 \\ P & 0 & \mu S \end{pmatrix} =$$

$$\begin{pmatrix} S^2S^{*2} + \mu^{-1}PS^* + \lambda SP & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.73)$$

$$\det([s_a, s_b]_1) = \det(S^2S^{*2} + \mu^{-1}PS^* + \lambda SP) = \det \begin{pmatrix} 0 & \mu^{-1} \\ \lambda & 0 \end{pmatrix} = -\lambda\mu^{-1}. \quad (3.74)$$

Thus, $\widetilde{R}(\{\lambda z, \mu z\}) = -\lambda\mu^{-1}$. In particular, $\widetilde{R}(\{z, z\}) = -1$, $\widetilde{R}(\{\alpha, z\}) = \alpha$.

3.8.5 Terms 2,3

In this subsection, we will compute $\frac{\partial}{\partial t}\big|_{t=0} \widetilde{R}(\{e^{tf}, z\})$ and, thus, $\widetilde{R}(\{e^f, z\})$. To describe $\widetilde{R}(\{e^f, z\})$, we apply the algorithm from Subsection 3.8.2 to $a = \mathrm{diag}(e^f, e^{-f}, 1)$, $b = \mathrm{diag}(z, 1, \bar{z})$. We get

$$T_a = \mathrm{diag}(T_{e^f}, T_{e^{-f}}, 1), \qquad T_b = \mathrm{diag}(S, 1, S^*), \quad (3.75)$$

$$s_a = T_a = \begin{pmatrix} T_{e^f} & 0 & 0 \\ 0 & T_{e^{-f}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_b = \begin{pmatrix} S & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & S^* \end{pmatrix}, \quad (3.76)$$

$$[s_a^{-1}, s_b^{-1}]_1 = \begin{pmatrix} T_{e^f}{}^{-1} & 0 & 0 \\ 0 & T_{e^{-f}}{}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S^* & 0 & 0 \\ 0 & 1 & 0 \\ P & 0 & S \end{pmatrix} \begin{pmatrix} T_{e^f} & 0 & 0 \\ 0 & T_{e^{-f}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & S^* \end{pmatrix} =$$
$$\begin{pmatrix} T_{e^f}{}^{-1} S^* & 0 & 0 \\ 0 & T_{e^{-f}}{}^{-1} & 0 \\ P & 0 & S \end{pmatrix} \begin{pmatrix} T_{e^f} S & 0 & T_{e^f} P \\ 0 & T_{e^{-f}} & 0 \\ 0 & 0 & S^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & T_{e^f}{}^{-1} S^* T_{e^f} P \\ 0 & 1 & 0 \\ P T_{e^f} S & 0 & P T_{e^f} P + (1-P) \end{pmatrix}. \tag{3.77}$$

In the last equality, we used that $S^* T_{e^f} S = T_{e^f}$. We compute

$$\det([s_a, s_b]_1) = \det([s_a^{-1}, s_b^{-1}]_1) = 1 + \mathrm{Tr}([s_a^{-1}, s_b^{-1}]_1 - 1) + O(\|f\|^2), \tag{3.78}$$

$$\mathrm{Tr}([s_a^{-1}, s_b^{-1}]_1 - 1) = (1, T_{e^f} 1)_{\mathcal{H}_+} - 1 = (1, e^f)_{\mathcal{H}} - 1 = (1, f)_{\mathcal{H}} + O(\|f\|^2). \tag{3.79}$$

Therefore,

$$\det([s_a, s_b]_1) = 1 + (1, f)_{\mathcal{H}} + O(\|f\|^2), \tag{3.80}$$

so

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \widetilde{R}(\{e^{tf}, z\}) = (1, f)_{\mathcal{H}}, \qquad \widetilde{R}(\{e^f, z\}) = \exp((1, f)_{\mathcal{H}}) = \exp(\hat{f}_0). \tag{3.81}$$

3.8.6 Term 4

In this subsection, we will compute $\left.\frac{\partial^2}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} \widetilde{R}(\{e^{t_1 f}, e^{t_2 g}\})$ and, thus, $\widetilde{R}(\{e^f, e^g\})$. Here, to describe $\widetilde{R}(\{e^f, e^g\})$, we apply the algorithm from Subsection 3.8.2 to $a = \mathrm{diag}(e^f, e^{-f}, 1)$, $a = \mathrm{diag}(e^g, 1, e^{-g})$.

We have

$$s_a = T_a = \mathrm{diag}(T_{e^{t_1 f}}, T_{e^{-t_1 f}}, 1), \qquad s_b = T_b = \mathrm{diag}(T_{e^{t_2 g}}, 1, T_{e^{-t_2 g}}), \tag{3.82}$$

and

$$[s_a^{-1}, s_b^{-1}]_1 = \operatorname{diag}(T_{e^{t_1 f}}{}^{-1} T_{e^{t_2 g}}{}^{-1} T_{e^{t_1 f}} T_{e^{t_2 g}}, 1, 1). \tag{3.83}$$

We compute

$$\frac{\partial^2}{\partial t_1 \partial t_2}\bigg|_{t_1=t_2=0} \widetilde{R}(\{e^{t_1 f}, e^{t_2 g}\}) = \frac{\partial^2}{\partial t_1 \partial t_2} \det\left(T_{e^{t_1 f}}{}^{-1} T_{e^{t_2 g}}{}^{-1} T_{e^{t_1 f}} T_{e^{t_2 g}}\right)\bigg|_{t_1=t_2=0} = \frac{\partial^2}{\partial t_1 \partial t_2} \det\left(1 + T_{e^{t_1 f}}{}^{-1} T_{e^{t_2 g}}{}^{-1}(T_{e^{t_1 f}} T_{e^{t_2 g}} - T_{e^{t_2 g}} T_{e^{t_1 f}})\right)\bigg|_{t_1=t_2=0} = \lim_{t_1, t_2 \to 0} \frac{1}{t_1 t_2} \operatorname{Tr}\left(T_{e^{t_1 f}}{}^{-1} T_{e^{t_2 g}}{}^{-1}(T_{e^{t_1 f}} T_{e^{t_2 g}} - T_{e^{t_2 g}} T_{e^{t_1 f}})\right) = \operatorname{Tr}\left(T_f T_g - T_g T_f\right). \tag{3.84}$$

We will now use the Fourier transforms $\hat{f}, \hat{g}$ of f, g, and their holomorphic and anti-holomorphic parts f_+, g_+ and f_-, g_-. For f, these are given by

$$f = \sum_{l=-\infty}^{\infty} \hat{f}_l z^l = f_- + f_+, \qquad f_+ = \sum_{l=0}^{\infty} \hat{f}_l z^l, \qquad f_- = \sum_{l=-\infty}^{0} \hat{f}_l z^l. \tag{3.85}$$

Using the identities $T_{f_-} T_g = T_{f_- g}$, $T_f T_{g_+} = T_{f g_+}$, and the similar ones with f and g interchanged, we get

$$\frac{\partial^2}{\partial t_1 \partial t_2}\bigg|_{t_1=t_2=0} \widetilde{R}(\{e^{t_1 f}, e^{t_2 g}\}) = \operatorname{Tr}\left(T_{f_-} T_{g_+} - T_{g_+} T_{f_-}\right) - \operatorname{Tr}\left(T_{g_-} T_{f_+} - T_{f_+} T_{g_-}\right). \tag{3.86}$$

One can check that $k, l \geq 0$, we have

$$\operatorname{Tr}\left(T_{z^{-l}} T_{z^k} - T_{z^k} T_{z^{-l}}\right) = \operatorname{Tr}\left(S^{*l} S^k - S^k S^{*l}\right) = \delta_{kl} \operatorname{Tr}(1 - S^l S^{*l}) = l\, \delta_{kl}. \tag{3.87}$$

Therefore, we have

$$\frac{\partial^2}{\partial t_1 \partial t_2}\bigg|_{t_1=t_2=0} \widetilde{R}(\{e^{t_1 f}, e^{t_2 g}\}) = \sum_{l=0}^{\infty} l(\hat{f}_{-l} \hat{g}_l - \hat{g}_{-l} \hat{f}_l) = \sum_{l=-\infty}^{\infty} l \hat{f}_{-l} \hat{g}_l = \frac{1}{2\pi i} \int_{S^1} f(z)(z g'(z)) \frac{dz}{z}. \tag{3.88}$$

3.8.7 Combining terms together

In this subsection we give the answer for values of $\widetilde{R}$ on the Steinberg symbols of $K_2(C^\infty(S^1))$. Let $\widetilde{f}, \widetilde{g}$ be functions from $\mathrm{Inv}(C^\infty(S^1))$, and let us compute $\widetilde{R}(\widetilde{f}, \widetilde{g})$. We write

$$\widetilde{f} = z^n e^f, \qquad \widetilde{g} = z^m e^g, \tag{3.89}$$

where n, m are the winding numbers of f, g. From the above computations, we have

$$\widetilde{R}(\{\widetilde{f}, \widetilde{g}\}) = \Omega(\{z^n e^f, z^m e^g\}) = (-1)^{nm} \exp\left(m\hat{f}_0 - n\hat{g}_0 + \sum_{l=-\infty}^{\infty} l\hat{f}_{-l}\hat{g}_l\right). \tag{3.90}$$

3.8.8 Comparison with the Beilinson–Bloch regulator

We recall that the goal of this section is to show $\widetilde{R}(\{\widetilde{f}, \widetilde{g}\}) = R(\widetilde{f}, \widetilde{g})$, where the right-hand side is given by (3.63). Using equalities $\widetilde{f} = z^n e^f$ and $\widetilde{g} = z^m e^g$, we rewrite $R(\widetilde{f}, \widetilde{g})$ in terms of winding numbers n, m and Fourier coefficients of f and g. Assuming that integration in (3.63) starts at $z = 1$ with $\ln z = 0$, we get

$$R(\widetilde{f}, \widetilde{g}) = \exp\left(\frac{1}{2\pi i}\int_{S^1} (f + n\ln z)\, dg + m\frac{1}{2\pi i}\int_{S^1} (f + n\ln z)\, \frac{dz}{z} - ng(1)\right). \tag{3.91}$$

We compute these integrals:

$$\frac{1}{2\pi i}\int_{S^1} \ln z\, \frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} i\theta d\theta = i\frac{1}{2\pi}(2\pi)^2/2 = \pi i, \tag{3.92}$$

$$\frac{1}{2\pi i}\int_{S^1} f\, \frac{dz}{z} = \hat{f}_0, \tag{3.93}$$

$$\frac{1}{2\pi i}\int_{S^1} \ln z\, dg = g(1) - \int_{S^1} g\, \frac{dz}{z} = g(1) - \hat{g}_0, \tag{3.94}$$

$$\frac{1}{2\pi i}\int_{S^1} f\, dg = \frac{1}{2\pi i}\int_{S^1} f(zg')\frac{dz}{z} = \sum_{l=-\infty}^{\infty} l\hat{f}_{-l}\hat{g}_l. \tag{3.95}$$

Collecting the terms together, we get

$$R(f,g) = \exp\left(\sum_{l=-\infty}^{\infty} l\hat{f}_{-l}\hat{g}_l - n\hat{g}_0 + m\hat{f}_0 + nm\pi i\right) =$$

$$(-1)^{nm}\exp\left(m\hat{f}_0 - n\hat{g}_0 + \sum_{l=-\infty}^{\infty} l\hat{f}_{-l}\hat{g}_l\right) = \widetilde{R}(\{\widetilde{f},\widetilde{g}\}). \quad (3.96)$$

The computations of this section give us the following lemma.

Lemma 3.8.1. *Property 4' of the map $\widetilde{R}_S$ from Section 3.7 holds.*

3.9 Continuity argument and correctness of the definition of the Beilinson–Bloch regulator

We recall our main goals: the first one is to show that (3.54) gives a well-defined map $\widetilde{r}_S\colon K_2(\mathcal{O}(X \setminus S)) \to H^1(X \setminus S, \mathbb{C}^*)$, allowing us to define $\widetilde{r}_\xi$; the second one is to show $\widetilde{r}_S = r_S$, i.e., the regulator we are defining coincides with the original Beilinson–Bloch regulator. Both statements would follow if we show Properties 1–4 of the map $\widetilde{R}_S$, introduced in Section 3.7. We observed that it is enough to show Property 4, as 1,2,3 would follow from it, because they are satisfied by the original Beilinson–Bloch regulator. So far, we've shown Property 3 (Lemma 3.7.1) and 4' (Lemma 3.8.1). We have also shown that combination of 1, 3, and 4' would imply Property 4 in Lemma 3.7.2. We can't use that directly since we didn't get Property 1. Our plan now is to show a property weaker than Property 1, so that the proof of Lemma 3.7.2 would give us the desired Property 4, and, thus, 1,2,3. As we've already alluded in the comment following Lemma 3.7.2, this property is some sort of continuity of $\widetilde{R}_S(u,\gamma)$ in γ. To prove this continuity, we will again look at the exact sequence

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{i} \mathcal{E}/\mathcal{T}_1 \xrightarrow{\pi} \mathrm{GL}^0(\mathcal{M}^1) \longrightarrow 1\,. \quad (3.97)$$

Following [18, Section 6.6], we introduce the topology on $\mathcal{E}$ given by

$$\operatorname{dist}_{\mathcal{E}}((a,s_a),(b,s_b))^2 = \|a-b\|^2 + \|[F,a-b]_0\|_2^2 + \|(s_a-a_{++})-(s_b-b_{++})\|_1^2. \quad (3.98)$$

Here, $\|\bullet\|_p$ is $\mathcal{L}^p$-norm:

$$\|a\|_p = \left(\sum_{j=0}^{\infty} \lambda_j^{p/2}\right)^{1/p}, \quad (3.99)$$

where λ_j are eigenvalues of a^*a. The group operations of $\mathcal{E}$ (product and inversion) are continuous: one can check that by a direct computation. The topology on $\mathcal{E}/\mathcal{T}_1$ is then induced from $\mathcal{E}$. One can check that the projection $\mathcal{E} \to \mathcal{E}/\mathcal{T}_1$ is open.

Topology in $\mathrm{GL}_n(\mathcal{M}^1)$ is given by

$$\operatorname{dist}_{\mathcal{M}^1}(a,b)^2 = \|a-b\|_{\mathcal{M}^1}^2 = \|a-b\|^2 + \|[F,a-b]_0\|_2^2. \quad (3.100)$$

Lemma 3.9.1. *Map* $(\mathrm{GL}_n^0(\mathcal{M}^1))^2 \to \mathcal{E}/\mathcal{T}_1 \colon (a,b) \mapsto [\pi^{-1}a, \pi^{-1}b]$ *is continuous.*

Proof. Since $[\pi^{-1}a,\pi^{-1}b] = [\pi^{-1}b,\pi^{-1}a]^{-1}$, it is enough to prove the continuity in the first argument.

Take $a, b \in \mathrm{GL}_n^0(\mathcal{M}^1)$, and $\varepsilon > 0$. Fix invertible s_a and s_b, satisfying $s_a - a_{++}, s_b - b_{++} \in \mathcal{L}^1(\mathcal{H}_+^n)$. It is enough to prove that there exists $\delta > 0$, such that if $\|a - \widetilde{a}\|_{\mathcal{M}^1} < \delta$ there exists invertible $\widetilde{s}_a$, satisfying $\widetilde{s}_a - \widetilde{a}_{++} \in \mathcal{L}^1$, such that

$$\operatorname{dist}_{\mathcal{E}}((c,s_c),(\widetilde{c},\widetilde{s}_c)) < \varepsilon, \quad (3.101)$$

where $c = [a,b]_1$, $\widetilde{c} = [\widetilde{a},b]_1$, $s_c = [s_a,s_b]_1$, $\widetilde{s}_c = [\widetilde{s}_a,s_b]_1$. We will constrain our choices of δ by $\delta < \|s_a^{-1}\|^{-1}/2$, and choose $\widetilde{s}_a = s_a - a_{++} + \widetilde{a}_{++}$. Then, $\operatorname{dist}((a,s_a),(\widetilde{a},\widetilde{s}_a)) < \delta$, so, by continuity of inversion and multiplication in the group $\mathcal{E}$, there indeed exists $\delta > 0$ satisfying (3.101). □

Lemma 3.9.2. *The standard topology on* $\mathbb{C}^*$ *coincides with the one induced by the inclusion* $i \colon \mathbb{C}^* \to \mathcal{E}/\mathcal{T}_1$.

Proof. Take any $\lambda \in \mathbb{C}^*$. For $s_\lambda \in \mathcal{T}$ with $\det s_\lambda = \lambda$ and any $r > 0$ let

$$B_r^{\mathbb{C}^*}(\lambda) = \{\mu \in \mathbb{C}^* \colon\ |\lambda - \mu| < r\}, \tag{3.102}$$

$$B_r^i(\lambda, s_\lambda) = \{\mu \in \mathbb{C}^* \colon \exists s_\mu \in \mathcal{T} \colon\ \det s_\mu = \mu, \|s_\lambda - s_\mu\|_1 < r\}. \tag{3.103}$$

By taking $s_\lambda = 1 + (\lambda - 1)P$, where P is a one-dimensional projection, we see that $B_r^{\mathbb{C}^*}(\lambda) \subset B_r^i(\lambda, s_\lambda)$. Therefore, the map $i \colon \mathbb{C}^* \to \mathcal{E}/\mathcal{T}_1$ is continuous. On the other hand, det is continuous with respect to the $\| \bullet \|_1$. Therefore, for $r > 0$ (and fixed s_λ) there exists $\delta > 0$ s.t. $\|s_\lambda - s_\mu\|_1 < \delta$ implies $|\det s_\mu - \det s_\lambda| < r$. Thus, for such δ we have $B_\delta^i(\lambda, s_\lambda) \subset B_r^{\mathbb{C}^*}(\lambda)$. □

By combining the discussion and two lemmas above, we get the following observation.

Lemma 3.9.3. *Consider the map*

$$\{(a_j, b_j)\}_{j=1}^N \mapsto \tau_2^{CK}(i^{-1} \prod_{j=1}^N [\pi^{-1} a_j, \pi^{-1} b_j]) \in \mathbb{C}^*, \tag{3.104}$$

acting on collections of $2N$ *operators* $a_j, b_j \in \mathrm{GL}_n^0(\mathcal{M}^1)$*, satisfying*

$$\prod_{j=1}^N [a_j, b_j]_1 = 1. \tag{3.105}$$

This map is continuous with respect to the topology induced from $(M_n(\mathcal{M}^1))^{2n}$.

Theorem 3.9.4. *Property 4 holds.*

Proof. Fix the set S. Choose $u \in K_2(\mathcal{O}(X \setminus S))$, and, using the fact that the Steinberg group is perfect, write it as $u = i^{-1} \prod_{j=1}^N [\pi^{-1} f_j, \pi^{-1} g_j]$ for some $f_j, g_j \in E_n(\mathcal{O}(X \setminus S)) \subset \mathrm{GL}_n(\mathcal{O}(X \setminus S))$ satisfying

$$\prod_{j=1}^N [f_j, g_j]_1 = 1. \tag{3.106}$$

According to the observation, made after Lemma 3.7.2, Property 4 holds for paths γ which avoid a certain set $S_1 \supset S$ (which depends on u). So, let γ be a path $S^1 \to X \setminus S$, which intersects with some points of S_1. Because of Lemma 3.5.1, $M_{f\circ\gamma}$ depends continuously on γ, where topology in the space of paths γ is given by $\mathrm{dist}(\gamma_1, \gamma_2) = \sup_{t\in S^1} \mathrm{dist}(\gamma_1(t), \gamma_2(t)) + \sup_{t\in S^1} \mathrm{dist}(\gamma_1'(t), \gamma_2'(t))$, where some finite covering of X is used to define distances between points and tangent vectors. In this metric, every ball near γ contains paths which avoid the finite set S_1. Thus, the theorem follows from Lemma 3.9.3. $\square$

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