INVESTIGATIONS IN THE THEORY OF ELECTROMAGNETIC SCATTERING FROM EXPANDING DIELECTRIC OBSTACLES

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ABSTRACT

An equation for the reflection which results when an expanding dielectric slab scatters normally incident plane electromagnetic waves is derived using the invariant imbedding concept. The equation is solved approximately and the character of the solution is investigated. Also, an equation for the radiation transmitted through such a slab is similarly obtained. An alternative formulation of the slab problem is presented which is applicable to the analogous problem in spherical geometry. The form of an equation for the modal reflections from a nonrelativistically expanding sphere is obtained and some salient features of the solution are described. In all cases the material is assumed to be a nondispersive, nonmagnetic dielectric whose rest frame properties are slowly varying.

CONTENTS

I. INTRODUCTION

The behavior of electromagnetic radiation in environments containing moving media with time varying constitutive parameters is a subject of considerable interest, both academic and practical. A common formulation of the problem is based on Maxwell's equations with time varying constitutive parameters that describe the medium through which the waves propagate. These equations must be solved subject to appropriate boundary conditions. Of particular interest is the problem of scattering of electromagnetic radiation by a localized object whose character is varying with time. The above approach, however, leads to an unnecessarily extensive calculation. That is, one must calculate the electromagnetic fields everywhere inside and outside the scatterer subject to certain conditions of continuity at the boundary. Fortunately, this is unnecessary because by making use of the concept of invariant imbedding one may circumvent calculation of the fields inside the scatterer and need only consider the external fields which are usually the ones of greatest interest in problems of this type.

Invariant imbedding found its genesis in the now quite well-known paper by V. A. Ambarzumian on the scattering of light by a foggy medium [1]. In that paper he introduced an invariance principle to obtain the scattering. Similar invariance principles were later applied to problems of theoretical astrophysics involving radiative transfer in stellar atmospheres [2]. Subsequently, a method now known as invariant imbedding and based on these invariance principles was applied to a staggering variety of problems [3]. Papas has applied the concept to the problem of reflection of plane electromagnetic waves from a nonuniform slab of dielectric material [4]. More recently, Latham [5] extended this to cylindrical and spherical scatterers and Kritikos, Lee and Papas [6] obtained the scattering of plane waves by nonuniform jet streams by means of invariant imbedding.

The type of scattering problem to be considered here involves a nondispersive, nonmagnetic dielectric scatterer which is expanding or contracting (negatively expanding) with time. The expansion need not be uniform. It is assumed, however, that the evolution of the scatterer is given beforehand and is unmodified by the presence of the electromagnetic radiation. The formulation is quasistatic and is carried out to first order in the velocity of the medium.

2. EXPANDING SLABS

Before considering the general slab problem, two preliminary problems will be discussed to provide some insight with regard to the effects of moving media on the propagation of electromagnetic waves. The magnetic permeability of the medium will always be taken to be that of free space, n will always denote the index of refraction of the medium in the rest frame of that medium, and the velocity of the medium will always be referred to the laboratory frame of reference. In addition, it will be assumed that the index of refraction may be specified independent of the density of the medium.

A. Scattering from a Shock Front

The first situation to be analyzed is depicted in Figure 1. A TEM wave is normally incident on a plane shock front in a gas. The velocities of the gas on both sides of the shock are uniform and normal to the shock front. Also, the indices of refraction n_a and n_b are uniform. The densities are assumed to be such as to satisfy a relativistic continuity equation for the gas [7]. The frame of the shock is considered to be the laboratory frame.

To analyze this situation one assumes plane wave solutions in the rest frames of the media on both sides of the shock front. Thus we have

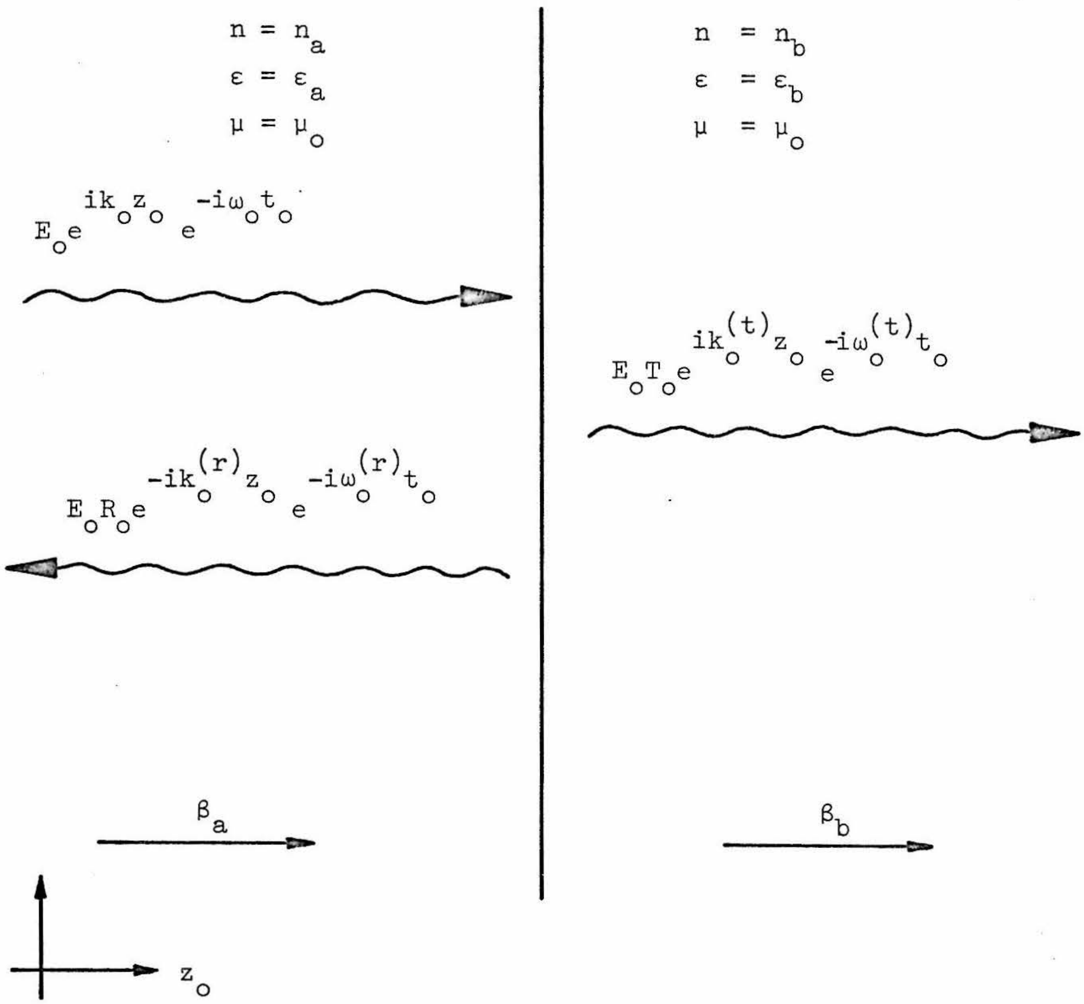


Figure 1. Scattering from a shock front

Viewed in the rest frame of medium a :

$$\vec{E}_1 = \hat{e}_{x_1} E_1 e^{ik_1 z_1} e^{-i\omega_1 t_1}$$

$$\vec{E}_1^{(r)} = \hat{e}_{x_1} E_1 R_1 e^{-ik_1^{(r)} z_1} e^{-i\omega_1^{(r)} t_1} \qquad \text{(2A-1a)}$$

$$\vec{E}_1^{(t)} = \hat{e}_{x_1} E_1 T_1 e^{ik_1^{(t)} z_1} e^{-i\omega_1^{(t)} t_1}$$

medium b :

$$\vec{E}_2 = \hat{e}_{x_2} E_2 e^{ik_2 z_2} e^{-i\omega_2 t_2}$$

$$\vec{E}_2^{(r)} = \hat{e}_{x_2} E_2 R_2 e^{-ik_2^{(r)} z_2} e^{-i\omega_2^{(r)} t_2} \qquad \text{(2A-1b)}$$

$$\vec{E}_2^{(t)} = \hat{e}_{x_2} E_2 T_2 e^{ik_2^{(t)} z_2} e^{-i\omega_2^{(t)} t_2}$$

The subscripts 1 and 2 in these expressions denote the frame in which the quantity is viewed, 1 denoting the rest frame of medium a, and 2 denoting the rest frame of medium b . The associated H fields may be found in the rest frames of the media in which the waves are propagating by making use of the rest frame characteristic impedances of the media; that is,

$$\left.\begin{aligned} \vec{H}_1 &= \hat{e}_{y_1} E_1 \sqrt{\frac{\varepsilon_1}{\mu_o}}\, e^{ik_1 z_1} e^{-i\omega_1 t_1} \\ \vec{H}_1^{(r)} &= -\hat{e}_{y_1} E_1 R_1 \sqrt{\frac{\varepsilon_1}{\mu_o}}\, e^{-ik_1^{(r)} z_1} e^{-i\omega_1^{(r)} t_1} \end{aligned}\right\} \qquad \text{(2A-2a)}$$

$$\vec{H}_2^{(t)} = \hat{e}_{y_2} E_2 T_2 \sqrt{\frac{\varepsilon_2}{\mu_o}}\, e^{ik_2^{(t)} z_2} e^{-i\omega_2^{(t)} t_2} \qquad \text{(2A-2b)}$$

where ε_a and ε_b are the rest frame dielectric permittivities of medium a and medium b respectively. We also note that

$$\left.\begin{aligned} k_1 &= \omega_1 \sqrt{\varepsilon_a \mu_o} = \frac{n_a \omega_1}{c} \\ k_1^{(r)} &= \omega_1^{(r)} \sqrt{\varepsilon_a \mu_o} = \frac{n_a \omega_1^{(r)}}{c} \end{aligned}\right\} \qquad \text{(2A-3a)}$$

$$k_2^{(t)} = \omega_2^{(t)} \sqrt{\varepsilon_b \mu_o} = \frac{n_b \omega_2^{(t)}}{c} \qquad \text{(2A-3b)}$$

Transforming the fields (2A-1) and (2A-2) to the frame of the shock (the laboratory frame) yields

Incident Wave

$$\vec{E}_o = \hat{e}_{x_o} E_o \, e^{ik_o z_o} \, e^{-i\omega_o t_o}$$

$$\vec{H}_o = \hat{e}_{y_o} E_o \sqrt{\frac{\varepsilon_1}{\mu_o}} \, e^{ik_o z_o} \, e^{-i\omega_o t_o} \qquad \text{(2A-4a)}$$

Reflected Wave

$$\vec{E}_o^{(r)} = \hat{e}_{x_o} E_o R_o \, e^{-ik_o^{(r)} z_o} \, e^{-i\omega_o^{(r)} t_o}$$

$$\vec{H}_o^{(r)} = -\hat{e}_{y_o} E_o \sqrt{\frac{\varepsilon_1}{\mu_o}} R_o \, e^{-ik_o^{(r)} z_o} \, e^{-i\omega_o^{(r)} t_o} \qquad \text{(2A-4b)}$$

Transmitted Wave

$$\vec{E}_o^{(t)} = \hat{e}_{x_o} E_o T_o \, e^{ik_o^{(t)} z_o} \, e^{-i\omega_o^{(t)} t_o}$$

$$\vec{H}_o^{(t)} = \hat{e}_{y_o} E_o \sqrt{\frac{\varepsilon_2}{\mu_o}} T_o e^{ik_o^{(t)} z_o} \, e^{-i\omega_o^{(t)} t_o} \qquad \text{(2A-4c)}$$

where

$$E_o = \gamma(1 + n_a\beta_a) E_1 \quad , \quad E_o R_o = \gamma(1 - n_a\beta_a) E_1 R_1$$

$$E_o T_o = \gamma(1 + n_b\beta_b) E_2 T_2 \qquad \text{(2A-5a)}$$

$$k_o = \gamma(1 + \frac{\beta_a}{n_a})\, k_1 \qquad , \qquad k_o^{(r)} = \gamma(1 - \frac{\beta_a}{n_a})\, k_o^{(r)}$$

$$k_o^{(t)} = \gamma(1 + \frac{\beta_b}{n_b}) k_2^{(t)} \qquad (2A\text{-}5b)$$

$$\omega_o = \gamma(1 + n_a\beta_a)\, \omega_1 \qquad , \qquad \omega_o^{(r)} = \gamma(1 - n_a\beta_a)\, \omega_1^{(r)}$$

$$\omega_o^{(t)} = \gamma(1 + n_b\beta_b)\, \omega_2^{(t)} \qquad (2A\text{-}5c)$$

$$z_o = \gamma(z_1 + \beta_a c t_1) \qquad\qquad z_o = \gamma(z_2 + \beta_b c t_2)$$

$$t_o = \gamma(t_1 + \frac{\beta_a}{c} z_1) \qquad\qquad t_o = \gamma(t_2 + \frac{\beta_b}{c} z_2) \qquad (2A\text{-}5d)$$

and $\beta_j = v_j/c$ where v_j is the velocity of medium j in the laboratory frame. Now, working in the frame of the boundary, Maxwell's two curl equations are integrated around a closed loop centered on the boundary and the area of the loop is reduced to zero in the usual way to obtain the boundary conditions on E and H . It is found that tangential E and H must be continuous across the boundary. Applying these boundary conditions to (2A-4) at the shock in the laboratory frame results in

$$R_o = \frac{\sqrt{\frac{\varepsilon_a}{\mu_o}} - \sqrt{\frac{\varepsilon_b}{\mu_o}}}{\sqrt{\frac{\varepsilon_a}{\mu_o}} + \sqrt{\frac{\varepsilon_b}{\mu_o}}} = \frac{n_a - n_b}{n_a + n_b} \qquad (2A\text{-}6a)$$

$$T_o = \frac{\sqrt{\frac{\varepsilon_a}{\mu_o}}}{\sqrt{\frac{\varepsilon_a}{\mu_o}}+\sqrt{\frac{\varepsilon_b}{\mu_o}}} = \frac{2n_a}{n_a + n_b} \tag{2A-6b}$$

$$\omega_o = \omega_o^{(r)} = \omega_o^{(t)} \tag{2A-6c}$$

where R_o is the ratio of the electric field of the reflected wave at the boundary to the electric field of the incident wave at the boundary in the laboratory frame, and T_o is the ratio of the electric field of the transmitted wave at the boundary to the electric field of the incident wave at the boundary also in the laboratory frame. That is, R_o and T_o are the laboratory frame reflection and transmission coefficients of the shock front. Note that expressions (2A-6a) and (2A-6b) are independent of the velocities and are, in fact, the usual results for the scattering at the interface between the same two media at rest! Consequently, $R_o = 0$ and $T_o = 1$ if $n_a = n_b$.

It is remarked in passing that by dividing the k's given in (2A-5b) by the corresponding ω's given by (2A-5c) the well known formulas for the effective index of a moving medium can be obtained; that is,

$$\frac{k_o c}{\omega_o} = n_{eff} = \frac{n_a + \beta_a}{1 + n_a\beta_a}$$

$$\frac{k_o^{(r)} c}{\omega_o^{(r)}} = n_{eff}^{(r)} = \frac{n_a - \beta_a}{1 - n_a\beta_a} \tag{2A-7a}$$

$$\frac{k_o^{(t)} c}{\omega_o^{(t)}} = n_{eff}^{(t)} = \frac{n_b + \beta_b}{1 + n_b \beta_b} \tag{2A-7b}$$

The absence of reflection when $n_1 = n_2$ may be seen to be physically reasonable as follows. Consider the application of a Lorentz transformation to the wave impedance of a plane wave propagating through a medium of index n. In the rest frame of the medium the wave impedance is Z_z and in a frame moving with velocity $\vec{v}$ it is Z_z' where

$$Z_z' = \frac{E_x'}{H_y'} = \frac{\gamma(\vec{E} + \vec{v} \times \vec{B})_x + (1 - \gamma)\,\frac{\vec{E} \cdot \vec{v}}{v^2}\, v_x}{\gamma(\vec{H} - \vec{v} \times \vec{D})_y + (1 - \gamma)\,\frac{\vec{H} \cdot \vec{v}}{v^2}\, v_y}$$

$$= \frac{E_x'}{H_y'} \left[\frac{1 - (1 - \frac{1}{\gamma})\,\frac{\beta_x^2}{\beta^2} - n\beta_z}{1 - (1 - \frac{1}{\gamma})\,\frac{\beta_y^2}{\beta^2} - n\beta_z} \right] \tag{2A-8}$$

or

$$Z_z' = Z_z \left[\frac{1 - (1 - \frac{1}{\gamma})\,\frac{\beta_x^2}{\beta^2} - n\beta_z}{1 - (1 - \frac{1}{\gamma})\,\frac{\beta_y^2}{\beta^2} - n\beta_z} \right]$$

where β_x, β_y, and β_z represent the x,y and z components of the velocity. Note that if the velocity is in the z direction, the expression in brackets becomes unity and we find that for this situation the wave impedance is a Lorentz invariant. Since it is the wave impedance which is relevant in satisfying boundary conditions, and since it is unmodified by a Lorentz transformation, the wave of Figure 1 may propagate through the shock without changing its wave impedance and hence without producing a reflection. Note also that if β_x and β_y are not zero, the reflection is not zero but is none the less second order in β .

B. Scattering from an Expanding Slab (Special Case)

Consider an expanding slab scattering normally incident plane waves. The slab is backed by a perfect absorber as shown in Figure 2. Suppose that the index of refraction on the right of the fixed boundary is somehow maintained at some fixed value n_b (always measured in the rest frame of the material) for all time. Based on experience from the shock problem one might conjecture that the reflection from this object will be independent of the velocity function $\beta(z)$. One would expect that, since the effective (laboratory frame) index of refraction of the material on the right varies with z , the wavelength of the solution on the right will vary in a nearly similar manner with z . Recalling that the effective index is

$$n_{eff} = \frac{n_b + \beta}{1 + n_b \beta} \tag{2B-1}$$

(see (2A-7)) we conjecture that a possible approximate solution on the right might be something like

$$\vec{E} = \hat{e}_x E_o T_o \, e^{i \frac{\omega}{c} \int_o^z \frac{n_b + \beta}{1 + n_b \beta} dz} \, e^{-i\omega t} \tag{2B-2}$$

Substitution into Maxwell's equations using the constitutive relations for a moving medium,

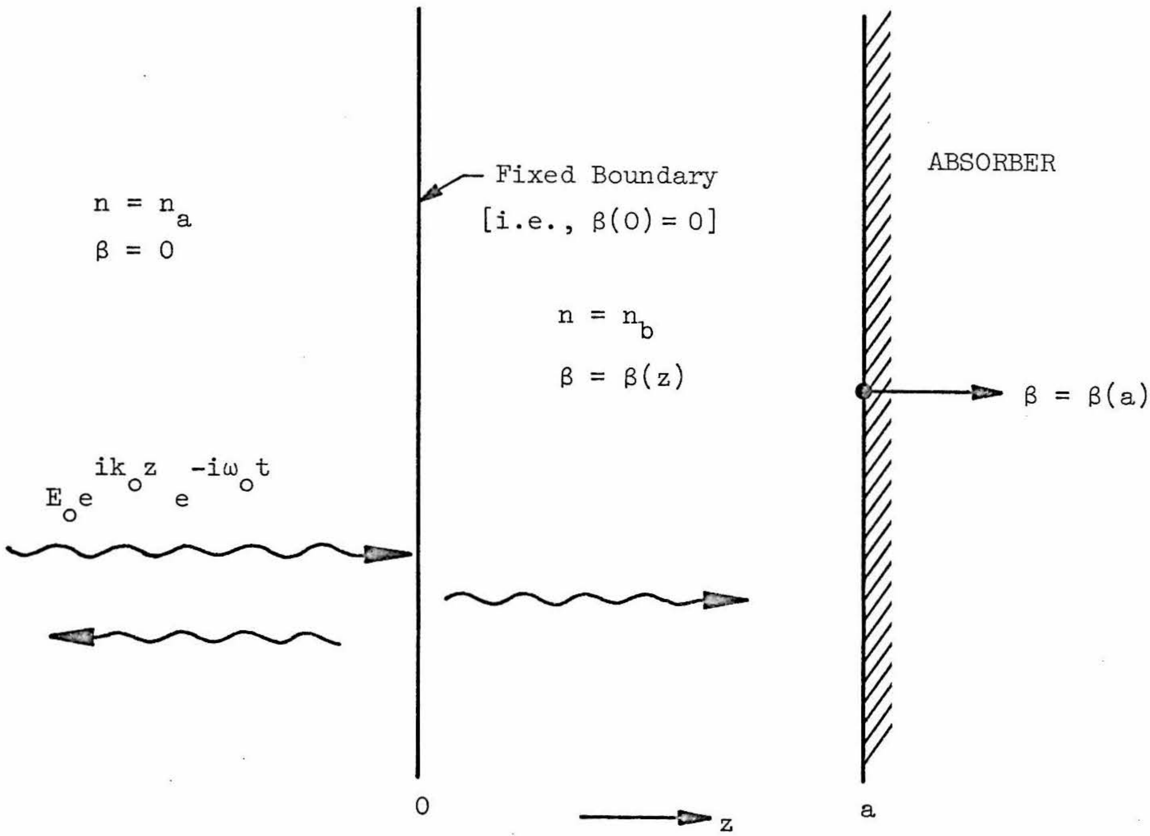


Figure 2. Scattering from an expanding slab

$$\vec{D} + \frac{1}{c^2}\vec{v} \times \vec{H} = n_2^2(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{B} - \frac{1}{c^2}\vec{v} \times \vec{E} = \mu_o(\vec{H} - \vec{v} \times \vec{D}) \qquad (2B\text{-}3)$$

shows (2B-2) to be the exact solution. Now, the expression (2B-2) and its derivative with respect to z take on the same values at $z = 0$ as do the expression for an ordinary plane wave in a medium of index n at rest and its derivative, respectively. This means that the simultaneous equations for R and T obtained by making use of the boundary conditions at $z = 0$ are identical to those obtained in the preceding example (the shock front) and it is seen that here also

$$R = \frac{n_a - n_b}{n_a + n_b} \qquad (2B\text{-}4a)$$

$$T = \frac{2n_a}{n_a + n_b} \qquad (2B\text{-}4b)$$

Thus if n_a and n_b are constants R and T are independent of the velocity of the medium in the slab.

Recall that in this example n_b was taken to be independent of time where, in general, it will not be time independent. When n_b depends on time (2B-2) represents a solution in a quasistatic sense. That is, it is a good approximation to the solution if

$$\left| E \frac{\partial(n_b^2)}{\partial t} \right| << \left| n_b^2 \frac{\partial E}{\partial t} \right| \quad \text{where} \quad E \sim e^{-i\omega t} \qquad (2B\text{-}5)$$

C. The Reflection Function Invariant Imbedding Equation for an Expanding Slab

Consider the expanding slab of dielectric fluid shown in Figure 3. The slab is stratified in that the velocity of the fluid and its rest frame index of refraction are functions of time t and position z only. The velocity is purely z directed. The laboratory frame of reference has been chosen such that the boundary at a is stationary. A unit amplitude, linearly polarized, monochromatic, plane wave of frequency ω_o is normally incident on the slab from the left. Since waves will be reflected from index gradients moving with various velocities within the slab and since a moving reflector results in a Doppler shift, we wish to allow for a reflected wave having a frequency spectrum of nonzero width. We therefore define the reflection function R to be a spectral density function which gives the frequency spectrum of the reflected wave. It is defined to be a function of both the reflected frequency and the incident frequency.

Let us assume that

$$\left|\frac{1}{R}\frac{\partial R}{\partial t}\right| << \omega_m \qquad \text{for} \qquad a \leq z \leq b \qquad (2C\text{-}1a)$$

where ω_m is the lowest frequency for which R is significantly greater than zero. Having made this assumption we may make use of the concept of a time varying spectrum. (See Appendix A.) We also assume that

$$\frac{1}{n}\frac{\partial n}{\partial t} << \frac{c}{n_M}\frac{1}{(b-a)} \qquad \text{for} \quad a \leq z \leq b$$

$$\frac{1}{\beta}\frac{\partial \beta}{\partial t} << \frac{c}{n_M}\frac{1}{(b-a)} \qquad \text{for} \quad a \leq z \leq b \qquad (2C\text{-}1b)$$

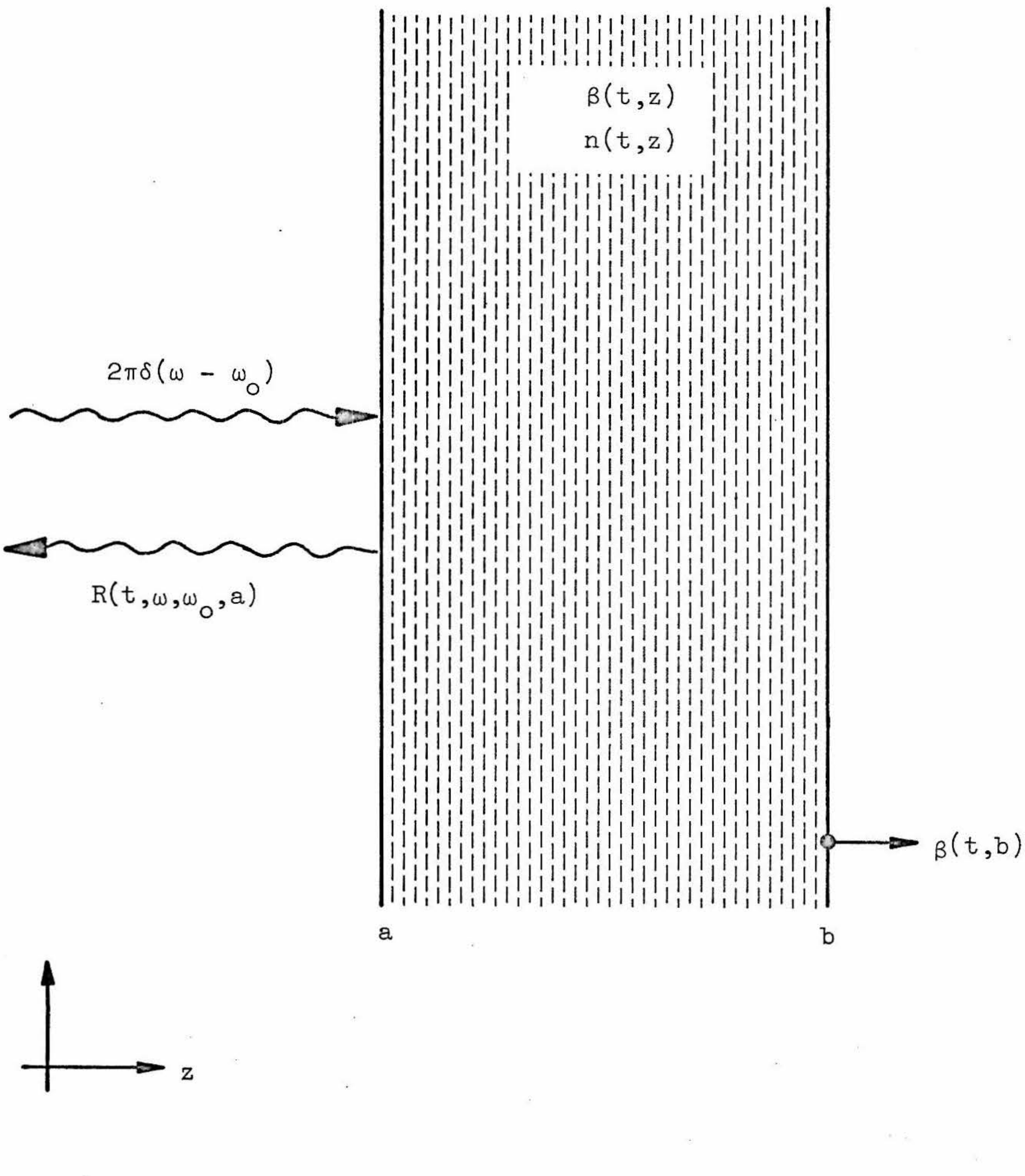


Figure 3. Reflection from the general slab

where n_M is the largest index in the slab and (b-a) is the thickness of the slab. That is, we assume that the properties of the slab do not vary significantly during a time on the order of the time required for the incident wave to completely penetrate it (the quasistatic approximation).

The invariant imbedding formulation of this problem consists of assuming that the reflection from that portion of the slab which lies to the right of a given plane is known and of calculating the change in the reflection due to the addition of a thin layer of material at this plane. This procedure yields a difference equation which, in the limit of vanishing added layer thickness, becomes a differential equation for the reflection function $R(t, \omega_{out}, \omega_{in}, z)$. This equation is to be integrated from the right boundary of the slab where the reflection is known to the left boundary where it is to be found.

Before formulating this problem, let us define a Lorentz frame comoving with the fluid at position z and time t. We define this frame in such a way that its position coordinate ζ is equal to z when its time coordinate τ is equal to t. Thus there will be an infinite number of such frames at any time t each corresponding to a particular choice of position z.

Figure 4 shows the configuration to be used in deriving the invariant imbedding equation for the reflection function. It is assumed that the reflection function at $\zeta + \Delta\zeta$ is known in a Lorentz frame moving with the fluid at $\zeta + \Delta\zeta$, when in this frame the space to the left of $\zeta + \Delta\zeta$ is filled with a homogeneous stationary fluid of index $n(\tau, \zeta+\Delta\zeta)$. A thin slab of fluid of index $n(\tau, \zeta+\Delta\zeta)$ having a

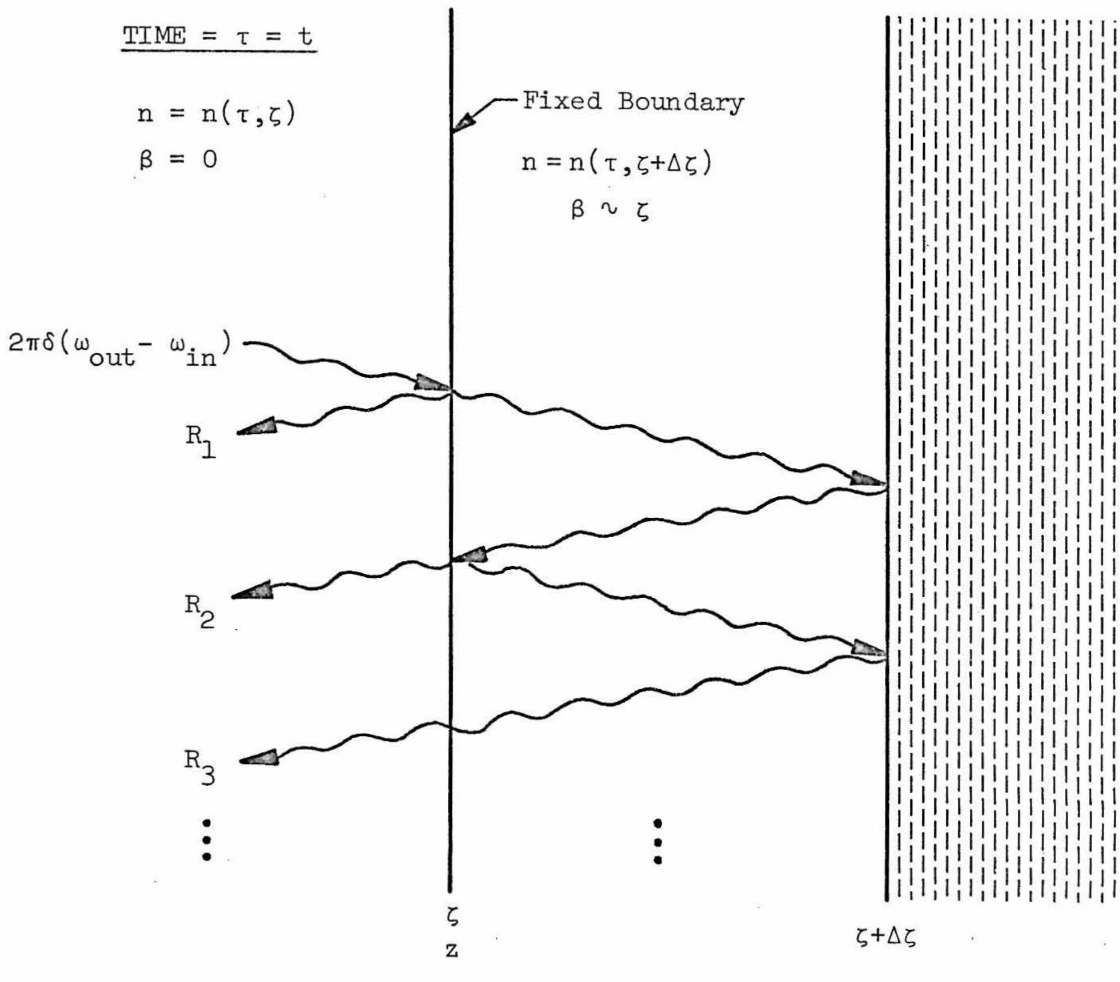


Figure 4. Configuration for derivation of the invariant imbedding equation for the reflection function

constant velocity gradient throughout its thickness is added at $\zeta + \Delta\zeta$ and extends back to ζ. We must calculate the reflection function at ζ in a frame moving with the fluid at ζ, under the assumption that in this frame the region to the left of ζ is homogeneously filled with a stationary fluid of index $n(\tau,\zeta)$. That is, for this calculation a Lorentz frame moving with the fluid at position z and time t in the laboratory frame (or correspondingly, position ζ and time τ in the comoving frame) will be used. Figure 4 shows the situation as seen in this comoving frame. Since it is our intention ultimately to take the limit as $\Delta\zeta$ approaches zero, and since when this is done only terms first order in $\Delta\zeta$ will contribute, all subsequent calculations will be done only to first order in $\Delta\zeta$.

A unit amplitude linearly polarized monochromatic plane wave of frequency ω_{in} is assumed to be normally incident on the composite slab of Figure 4. R_1 is the spectral density function of the wave reflected from the outside of the fixed boundary. R_2 is the spectral density function of the wave transmitted across the fixed boundary and through the added layer, reflected from the slab to the right of $\zeta + \Delta\zeta$ and transmitted back through the added layer and across the fixed boundary. R_3 is the spectral density function of the wave transmitted across the fixed boundary and through the added layer, reflected from the slab to the right of $\zeta + \Delta\zeta$, transmitted back through the added layer, reflected from the inside of the fixed boundary, transmitted through the added layer, reflected from the slab to the right of $\zeta + \Delta\zeta$ and transmitted back through the added layer and across the fixed boundary. Higher order R's are defined similarly. The frequency variable in the spectra given by the R's is ω_{out} .

Equation (2B-4a) leads immediately to the result for R_1 ; that is,

$$R_1 = \frac{n(\tau,\zeta) - n(\tau,\zeta+\Delta\zeta)}{n(\tau,\zeta) + n(\tau,\zeta+\Delta\zeta)} \; 2\pi\delta(\omega_{out} - \omega_{in}) = -\frac{1}{2n}\frac{\partial n}{\partial \zeta}\, 2\pi\delta(\omega_{out} - \omega_{in})\Delta\zeta \tag{2C-2}$$

Similarly,

$$R_2 = \left[\frac{2n(\tau,\zeta)}{n(\tau,\zeta) + n(\tau,\ \zeta+\Delta\zeta)}\right] e^{i\frac{n\omega'}{c}\Delta\zeta} \left[\left(\frac{1 - n\Delta\beta}{1 + n\Delta\beta}\right) R(\tau,\omega'',\omega',\zeta+\Delta\zeta)\right] e^{i\frac{n\omega''}{c}\Delta\zeta}\left[\frac{2n(\tau,\ \zeta+\Delta\zeta)}{n(\tau,\zeta) + n(\tau,\ \zeta+\Delta\zeta)}\right] \tag{2C-3a}$$

where

$$\omega' = \omega_{in}\,(1 - n\Delta\beta)$$

$$\omega'' = \omega_{out}(1 + n\Delta\beta)$$

$$\Delta\beta = \frac{\partial\beta}{\partial\zeta}\,\Delta\zeta$$

The factors $(1 - n\Delta\beta)$ and $(1 + n\Delta\beta)$ arise because $R(\tau,\omega_{out},\omega_{in},\zeta+\Delta\zeta)$ is defined in a Lorentz frame comoving with the fluid at $\zeta+\Delta\zeta$, while we are working in a frame comoving with the fluid at ζ . Thus, to first order in $\Delta\zeta$,

$$R_2 = R + \left[\frac{\partial R}{\partial \zeta} + \omega_{out}\, n \frac{\partial \beta}{\partial \zeta} \frac{\partial R}{\partial \omega_{out}} - \omega_{in}\, n \frac{\partial \beta}{\partial \zeta} \frac{\partial R}{\partial \omega_{in}} + \frac{in}{c}(\omega_{out} + \omega_{in})\, R - 2n \frac{\partial \beta}{\partial \zeta} R \right] \Delta\zeta \qquad (2C\text{-}3b)$$

Similarly,

$$R_3 = \int_{-\infty}^{\infty} R(\tau,\bar{\omega},\omega_{in},\zeta)\, \frac{1}{2n} \frac{\partial n}{\partial \zeta}\, R(\tau,\omega_{out},\bar{\omega},\zeta)\, \frac{d\bar{\omega}}{2\pi}\, \Delta\zeta \qquad (2C\text{-}4)$$

R_j for $j > 3$ is of second or higher order in $\Delta\zeta$ and is therefore negligible in this calculation. Thus we have

$$R = R_1 + R_2 + R_3 \qquad (2C\text{-}5)$$

Substitution of (2C-2), (2C-3b), and (2C-4) into (2C-5) gives

$$\frac{\partial R}{\partial \zeta} + \omega_{out}\, n \frac{\partial \beta}{\partial \zeta} \frac{\partial R}{\partial \omega_{out}} - \omega_{in}\, n \frac{\partial \beta}{\partial \zeta} \frac{\partial R}{\partial \omega_{in}}$$

$$= \frac{1}{2n} \frac{\partial n}{\partial \zeta} \left\{ 2\pi\delta(\omega_{out} - \omega_{in}) - \int_{-\infty}^{\infty} R(\tau,\bar{\omega},\omega_{in},\zeta)\, R(\tau,\omega_{out},\bar{\omega},\zeta) \frac{d\bar{\omega}}{2\pi} \right\}$$

$$- \frac{in}{c}(\omega_{out} + \omega_{in})\, R + 2n \frac{\partial \beta}{\partial \zeta} R \qquad (2C\text{-}6)$$

We now transform from the local comoving frame coordinates (τ,ζ) to the laboratory frame coordinates (t,z). Under our quasistatic assumption and to first order in β this merely amounts to direct replacement of τ with t and ζ with z. The final equation is then

$$\frac{\partial R}{\partial z} + \omega_{out}\, n \frac{\partial \beta}{\partial z} \frac{\partial R}{\partial \omega_{out}} - \omega_{in}\, n \frac{\partial \beta}{\partial z} \frac{\partial R}{\partial \omega_{in}}$$

$$= \frac{1}{2n} \frac{\partial n}{\partial z} \left\{ 2\pi\delta(\omega_{out} - \omega_{in}) - \int_{-\infty}^{\infty} R(t,\bar{\omega},\omega_{in},z)\, R(t,\omega_{out},\bar{\omega},z) \frac{d\bar{\omega}}{2\pi} \right\}$$

$$- \frac{in}{c} (\omega_{out} + \omega_{in})\, R + 2n \frac{\partial \beta}{\partial z} R \qquad (2C-7)$$

This is the quasistatic invariant imbedding equation for the reflection function to first order in β . Notice that we have transformed only the coordinates in terms of which the functions are expressed and have left R, ω_{out}, and ω_{in} in the reference frame comoving with the fluid at z . The quantities R, ω_{out} and ω_{in} have been left as they were because we have in mind that $\beta(a) = 0$ so that at $z = a$ (see Figure 3) the laboratory frame and the comoving frame coincide, thus automatically putting R, ω_{out}, and ω_{in} in the laboratory frame at the final point of the integration.

The properties of (2C-7) are most easily discussed by applying the method of characteristics to it. This will be done in the next subsection. For the moment it is remarked in passing that under certain circumstances R may be written as a function of the difference between its frequency arguments and (2C-7) may then be simplified somewhat by Fourier transformation (See Appendix B.) It is also remarked that a corresponding equation may be derived for the transmission function of the slab (See Appendix C.)

D. Approximate Solution of the Invariant Imbedding Equation

We propose to solve (2C-7) by the method of characteristics [9]. Choosing s to be the parameter designating position along the characteristic curves, we obtain the following set of four ordinary differential equations

$$\frac{dz}{ds} = 1 \tag{2D-1a}$$

$$\frac{d\omega_{out}}{ds} = \omega_{out}\, n \frac{\partial \beta}{\partial z} \tag{2D-1b}$$

$$\frac{d\omega_{in}}{ds} = -\omega_{in}\, n \frac{\partial \beta}{\partial z} \tag{2D-1c}$$

$$\frac{dR}{ds} = \frac{1}{2n}\frac{\partial n}{\partial z}\left\{2\pi\delta(\omega_{out} - \omega_{in}) - \int_{-\infty}^{\infty} R(t,\bar{\omega},\omega_{in},z)\, R(t,\omega_{out},\bar{\omega},z)\frac{d\bar{\omega}}{2\pi}\right\}$$
$$- \frac{in}{c}(\omega_{out} + \omega_{in})\, R + 2n\frac{\partial \beta}{\partial z} R \tag{2D-1d}$$

Equation (2D-1a) indicates that s may be taken equal to z . It is known that both ω_{out} and ω_{in} will be equal when z is equal to b (see Figures 3 and 4). Also, equations (2D-1b) and (2D-1c) indicate that they will differ from this initial value by an amount which is first order in β . Therefore, consistent with calculation to first order in β, ω_{out} and ω_{in} on the right sides of equations (2D-1b) and (2D-1c) may be replaced by ω_o (the value of ω_{in} at $z = a$) . We now have the following set of three ordinary differential equations

$$\frac{d\omega_{out}}{dz} = \omega_o\, n \frac{\partial \beta}{\partial z} \tag{2D-2a}$$

$$\frac{d\omega_{in}}{dz} = -\omega_o \, n \, \frac{\partial \beta}{\partial z} \tag{2D-2b}$$

$$\frac{dR}{dz} = \frac{1}{2n}\frac{\partial n}{\partial z}\left\{2\pi\delta(\omega_{out} - \omega_{in}) - \int_{-\infty}^{\infty} R(t,\overline{\omega},\omega_{in},z)\, R(t,\omega_{out},\overline{\omega},z)\frac{d\overline{\omega}}{2\pi}\right\}$$

$$- \frac{in}{c}(\omega_{out} + \omega_{in})\, R + 2n\,\frac{\partial \beta}{\partial z}\, R \tag{2D-2c}$$

Equations (2D-2a) and (2D-2b) are integrated to yield

$$\omega_{out} = \omega_{out}(b) + \omega_o \int_b^z n\,\frac{\partial \beta}{\partial z'}\, dz' \tag{2D-3}$$

$$\omega_{in} = \omega_{in}(b) - \omega_o \int_b^z n\,\frac{\partial \beta}{\partial z'}\, dz' \tag{2D-4}$$

where $\omega_{out}(b)$ is the value of ω_{out} at $z = b$ on the characteristic and $\omega_{in}(b)$ is the value of ω_{in} at $z = b$ on the characteristic. Substituting (2D-3) and (2D-4) into (2D-2c), we obtain

$$\frac{dR}{dz} = \frac{1}{2n}\frac{\partial n}{\partial z}\, 2\pi\delta[\omega_{out}(b) - \omega_{in}(b) + 2\omega_o \int_b^z n\,\frac{\partial \beta}{\partial z'}\, dz']$$

$$- \frac{in}{c}[\omega_{out}(b) + \omega_{in}(b)]\, R + 2n\,\frac{\partial \beta}{\partial z}\, R \tag{2D-5}$$

where it has been assumed that the slab under consideration is sufficiently tenuous that the nonlinear term may be neglected. Equation (2D-5) is a first order linear ordinary differential equation and may be easily solved. The solution evaluated at the left boundary of the slab $(z = a)$ is

$$R = \int_b^a \frac{1}{2n}\frac{\partial n}{\partial z} 2\pi\delta[\omega_{out}(b) - \omega_{in}(b) + 2\omega_o \int_b^z n \frac{\partial \beta}{\partial z'} dz']$$

$$\times\, e^{-\int_a^z 2n \frac{\partial \beta}{\partial z'} dz'} \, e^{\frac{i}{c}[\omega_{out}(b) + \omega_{in}(b)] \int_a^z n\, dz'} dz$$

$$+ R[t,\omega_{out}(b),\omega_{in}(b),b]\, e^{-\int_a^b 2n \frac{\partial \beta}{\partial z'} dz'} \, e^{\frac{i}{c}[\omega_{out}(b) + \omega_{in}(b)] \int_a^b n\, dz'} \tag{2D-6}$$

Solving (2D-3) and (2D-4) at $z = a$ for $\omega_{out}(b)$ and $\omega_{in}(b)$ as functions of $\omega_{out}, \omega_{in}$ and a, we obtain

$$\omega_{out}(b) = \omega_{out} - \omega_o \int_b^a n \frac{\partial \beta}{\partial z'} dz' \tag{2D-7a}$$

$$\omega_{in}(b) = \omega_{in} + \omega_o \int_b^a n \frac{\partial \beta}{\partial z'} dz' \tag{2D-7b}$$

But, since we are at $z = a$ we know that ω_{in} is ω_o and we will call ω_{out} just ω. Substitution in equation (2D-6) results in

$$R = \int_b^a \frac{1}{2n}\frac{\partial n}{\partial z} 2\pi\delta[\omega - \omega_o + 2\omega_o \int_a^z n \frac{\partial \beta}{\partial z'} dz']$$

$$\times\, e^{-\int_a^z 2n \frac{\partial \beta}{\partial z'} dz'} \, e^{\frac{i}{c}(\omega+\omega_o) \int_a^z n\, dz'} dz$$

$$+ R[t,\omega_{out}(b),\omega_{in}(b),b]\, e^{-\int_a^b 2n \frac{\partial \beta}{\partial z'} dz'} \, e^{\frac{i}{c}(\omega+\omega_o) \int_a^b n\, dz'} \tag{2D-8}$$

The integration from b to a in the first term of this equation may now be carried out and we have

$$R(t,\omega,\omega_o,a) = \frac{2\pi}{\omega_o} \left. \frac{\frac{1}{2n}\frac{\partial n}{\partial z}}{2n\frac{\partial \beta}{\partial z}} \right|_{z=z(\omega)} e^{-\int_a^{z(\omega)} 2n \frac{\partial \beta}{\partial z'} dz'} \; e^{\frac{i}{c}(\omega+\omega_o)\int_a^{z(a)} n\, dz'}$$

$$\times \; W[\omega;\omega(b),\omega(a)] + R[t,\omega_{out}(b),\omega_{in}(b),b] \; e^{-\int_a^b 2n \frac{\partial \beta}{\partial z'} dz'}$$

$$\times \; e^{\frac{i}{c}(\omega+\omega_o)\int_a^b n\, dz'} \qquad \text{(2D-9)}$$

where

$$\omega(z) = \omega_o \left[1 - 2 \int_a^z n \frac{\partial \beta}{\partial z} dz \right] \qquad \text{(2D-10)}$$

and it has been assumed that (2D-10) is one to one over the entire slab. The W function is defined in Figure 5 and it indicates that the first term of the solution (2D-9) is band limited. If (2D-10) is not one to one, the slab must be divided into subslabs over which it is one to one and the integration performed in segments, the result of each integration being the initial condition for the next. In our case, i.e., a slab over which the transformation (2D-10) is one to one, $R(t,\omega_{out},\omega_{in},b)$ will be either zero (if there is no index discontinuity at $z = b$) or proportional to $\delta(\omega_{out} - \omega_{in})$ (if there is an index discontinuity at $z = b$) assuming that n is a constant for $z > b$. Equations (2D-9) and (2D-10) indicate that this delta function becomes $\delta(\omega-\omega(b))$ when it appears in the solution at $z = a$. Its amplitude and phase are also modified by the presence of the exponential factors in the second term of (2D-9). The contribution of the interior of the slab to the spectrum of the reflected wave is given by the first term in (2D-9). Because of the presence of the W function, this contribution

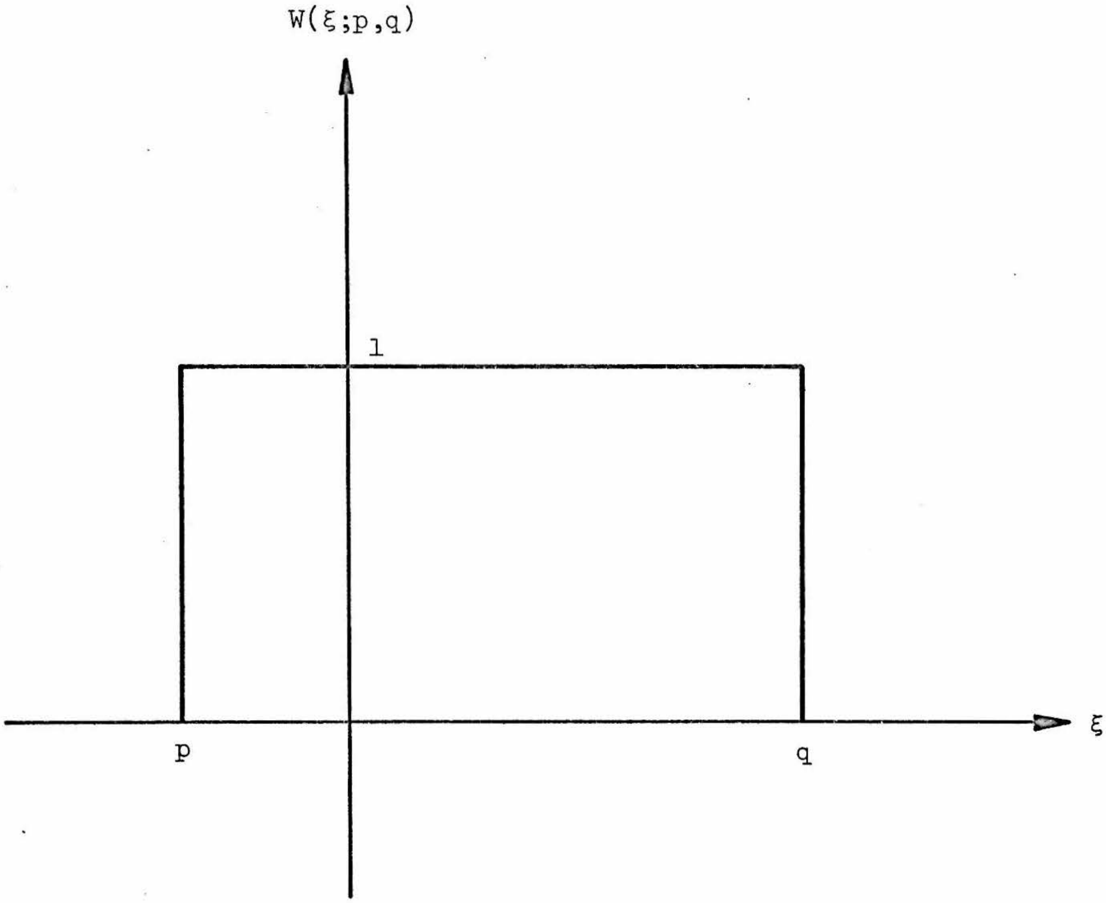


Figure 5. Definition of the W function

is band-limited and extends from $\omega(b)$ to ω_o where $\omega(b)$ is given by (2D-10). The reflection from an index gradient at a given position within the slab has an amplitude proportional to the gradient of the index and has a frequency given by (2D-10) with the position of the index gradient substituted for z. It should be emphasized that ω is the output frequency at $z = a$, i.e., $\omega_{out}(a) = \omega$; therefore, $\omega(b)$ is an output frequency <u>at $z = a$</u> and b is merely the value of z which must be substituted into the variable transformation (2D-10) to obtain this output frequency.

Recall that (2D-9) was derived on the assumption that the slab was "sufficiently tenuous" that the nonlinear term in (2D-2c) could be neglected in obtaining (2D-5). We may now state, somewhat more precisely, that we have found the solution to first order in $(\frac{1}{2n}\frac{\partial n}{\partial z})$ and to first order in $(\frac{n(z_i-\varepsilon) - n(z_i+\varepsilon)}{n(z_i-\varepsilon) + n(z_i+\varepsilon)})$ at any index discontinuity. By this we mean that we have neglected terms containing factors of

$$(\frac{1}{\partial n}\frac{\partial n}{\partial z})^2 \quad \text{or} \quad (\frac{n(z_1-\varepsilon) - n(z_i+\varepsilon)}{n(z_i-\varepsilon) + n(z_i+\varepsilon)})(\frac{n(z_j-\varepsilon) - n(z_j+\varepsilon)}{n(z_j-\varepsilon) + n(z_j+\varepsilon)})$$

which is justified if the slab is "sufficiently tenuous".

It is interesting to note what happens to our solution if $\frac{\partial\beta}{\partial z}$ becomes zero over a finite range of z or at a single value of z. Considering the first term of formula (2D-8) we see that if $\frac{\partial\beta}{\partial z}$ is zero over a finite range, say from z_1 to z_2, where $a \leq z_1 < z_2 \leq b$, the delta function becomes independent of z and may be taken out of the integral from z_1 to z_2 (the integrals from a to z_1 and from z_2 to b remaining as they were). This results in a delta function in the spectrum whose amplitude is given by the integral of the remaining integrand from z_1 to z_2 and whose frequency is given by

$$\omega = \omega_o [1 - 2 \int_a^{z_1} n \frac{\partial\beta}{\partial z'} dz'] \qquad (2D-11)$$

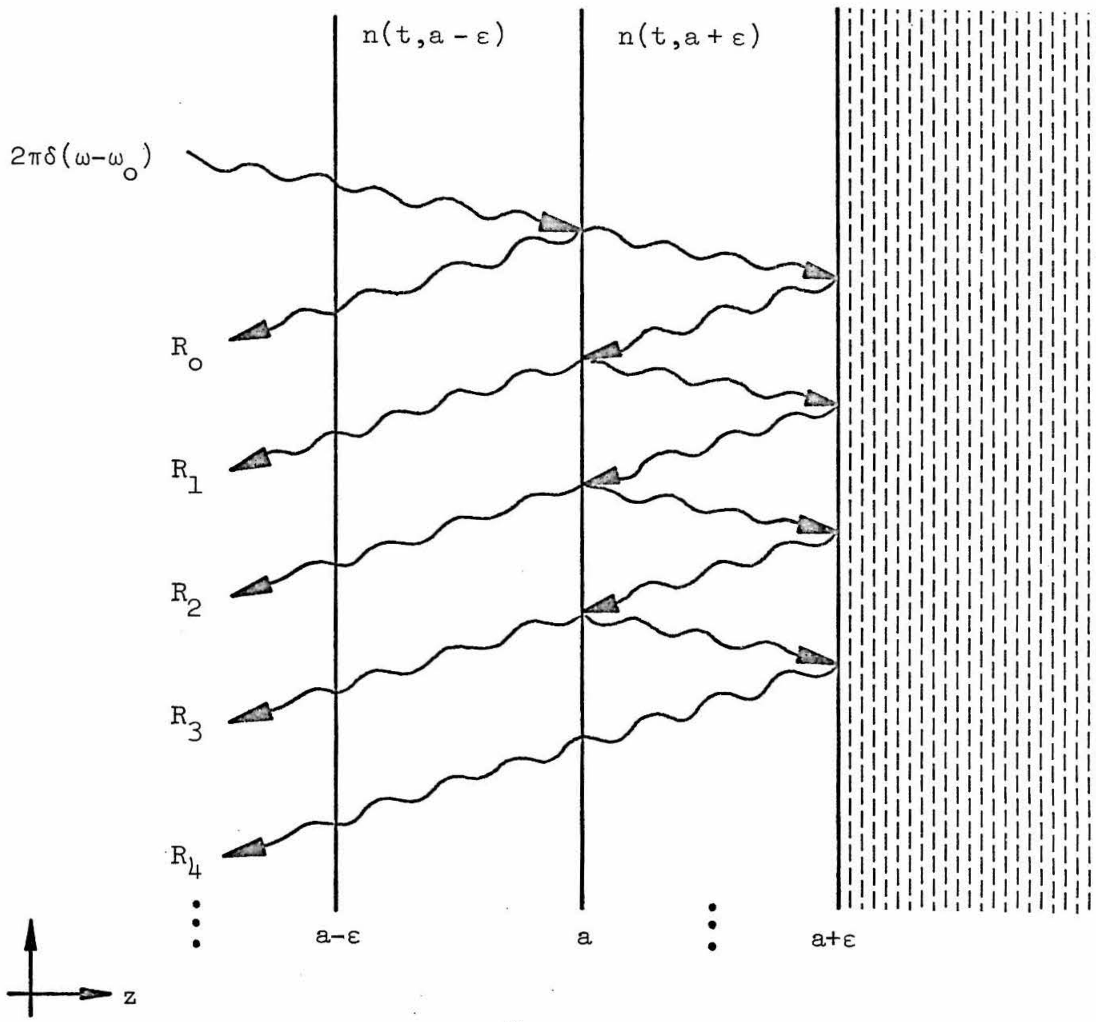
If $z_1 = z_2$ (i.e., $\frac{\partial\beta}{\partial z}$ is zero at a single point) the delta function component in the spectrum is seen to have zero amplitude.

At this point the possibility of an index discontinuity at $z = a$ has not been accounted for. (Discontinuity at $z = b$ is accounted for by the initial function $R(t,\omega_{out},\omega_{in},b)$ in equation (2D-9)). Figure 6 shows the situation at the boundary $z = a$. Here our incident wave falls on the discontinuity at $z = a$ and multiple reflections occur as they did in Figure 4, but in this case the reflection coefficients at the discontinuity for waves incident from the right and from the left are not infinitesimal as they were in Figure 4.

In treating the boundary at $z = a$ it is convenient to have a means of transforming a time varying spectrum into a static spectrum. A formula is now stated and verified which provides a means of performing this transformation. Let $r(t)$ and $s(t)$ be real valued functions of time and let $f[r(t),s(t)]$ be such that

$$F[\omega,s(t)] = \int_{-\infty}^{\infty} f[r(t'),s(t)] \, e^{i\omega t'} \, dt' \quad \text{exists.}$$

Then,



$$R(\omega,\omega_o,a-\varepsilon) = \sum_{i=0}^{\infty} R_i$$

Figure 6. Discontinuity in n at z = a

$$\tilde{G}(\omega) \equiv \int_{-\infty}^{\infty} f[r(t),s(t)]\ e^{i\omega t}\ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F[\omega-\omega',s(t)]\ e^{i\omega' t}\ d\omega' dt \tag{2D-12}$$

This may be demonstrated as follows:

$$\tilde{G}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[r(t'),s(t)]\ \delta(t'-t)\ e^{i\omega t'}\ dt'dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[r(t'),s(t)]\ \frac{1}{2\pi}\ e^{-i\omega'(t'-t)}\ e^{i\omega t'}\ d\omega' dt' dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F[\omega-\omega',s(t)]\ e^{i\omega' t}\ d\omega' dt \quad ,\ \text{q.e.d.}$$

The solution (2D-9) is a function of the form $F[\omega,s(t)]$. Applying (2D-12) we can obtain $\tilde{G}(\omega)$ which we claim to be the solution expressed as a pure (time independent) spectrum function. This claim is only approximately true. The degree of validity of the claim depends upon just how closely $f[r(t),s(t)]$ approximates the true solution of the problem, $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\ e^{-i\omega t}\ d\omega$. This, in turn depends on the validity of the following statement.

$$\left(\left|\frac{\partial f}{\partial s}\right| \left|\frac{ds}{dt}\right|\right)_{max} << \left(\left|\frac{\partial f}{\partial r}\right| \left|\frac{dr}{dt}\right|\right)_{max} \tag{2D-13}$$

or, assuming that $\left|\frac{\partial f}{\partial s}\right|$ and $\left|\frac{\partial f}{\partial r}\right|$ are of the same order of magnitude, (and small)

$$\left|\frac{ds}{dt}\right|_{max} << \left|\frac{dr}{dt}\right|_{max}$$

That is, $r(t)$ must be a "rapidly varying" function of time and $s(t)$

must be a "slowly varying" function of time. Condition (2D-13) is essentially a restatement of equation (2C-1) (See Appendix A).

Returning to Figure 6 and assuming that R has been transformed to a time independent function of frequency, it is easily seen that the limit of the expression for R immediately to the left of the boundary as $\varepsilon \to 0$ is

$$\lim_{\varepsilon \to 0} R(\omega,\omega_o,a-\varepsilon) = \lim_{\varepsilon \to 0} \Bigg[\vec{r}_o(\omega-\omega_o)$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{t}_o(\omega'-\omega_o)\, R(\omega'',\omega',a+\varepsilon)\, \overleftarrow{t}_o(\omega-\omega'')\, \frac{d\omega''}{2\pi}\frac{d\omega'}{2\pi}$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{t}_o(\omega'-\omega_o)\, R(\omega'',\omega',a+\varepsilon)\, \overleftarrow{r}_o(\omega'''-\omega'')\, R(\omega'''',\omega''',a+\varepsilon)$$

$$\times \quad \overleftarrow{t}_o(\omega-\omega'''')\, \frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}\frac{d\omega'''}{2\pi}\frac{d\omega''''}{2\pi} + \cdots \Bigg] \qquad (2D\text{-}14)$$

where $\overleftarrow{r}_o(\omega)$, $\vec{r}_o(\omega)$, $\overleftarrow{t}_o(\omega)$, $\vec{t}_o(\omega)$ are Fourier transforms of $\hat{\overleftarrow{r}}_o(x)$, $\hat{\vec{r}}_o(x)$, $\hat{\vec{t}}_o(x)$, $\hat{\overleftarrow{t}}_o(x)$ and

$$\hat{\overleftarrow{r}}_o(x) = \frac{n(x,a+\varepsilon) - n(x,a-\varepsilon)}{n(x,a+\varepsilon) + n(x,a-\varepsilon)} = -\hat{\vec{r}}_o(x)$$

$$\hat{\overleftarrow{t}}_o(x) = \frac{2n(x,a+\varepsilon)}{n(x,a+\varepsilon) + n(x,a-\varepsilon)}$$

$$\hat{\vec{t}}_o(x) = \frac{2n(x,a-\varepsilon)}{n(x,a+\varepsilon) + n(x,a-\varepsilon)}, \quad \text{and} \quad x = ct\ .$$

The arrows indicate the direction of the incident wave for each reflection or transmission. Notice now that to first order in β the solution

(2D-9) may be written as a function of $\omega-\omega_o$, the ω_o's not associated with ω in this way being constants. This allows us to write (2D-14) in the form

$$\hat{R}(y,a-\varepsilon) = \hat{r}_o(y) + \frac{\hat{\overleftarrow{t}}_o(y)\ \hat{\overrightarrow{t}}_o(y)}{\hat{\overleftarrow{r}}_o(y)} \sum_{j=1}^{\infty} [\hat{\overleftarrow{r}}_o(y)\ \hat{R}(y,a+\varepsilon)]^j$$

$$= \hat{\overrightarrow{r}}_o(y) + \hat{\overrightarrow{t}}_o(y)\ \hat{\overleftarrow{t}}_o(y) \frac{\hat{R}(y,a+\varepsilon)}{1 - \hat{\overleftarrow{r}}_o(y)\ \hat{R}(y,a+\varepsilon)} \qquad \text{(2D-15)}$$

where

$$\hat{F}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \omega_o)\ e^{-i\left(\frac{\omega-\omega_o}{c}\right) y}\ d\omega$$

and

$$F = R,\ \overleftarrow{r}_o,\ \overrightarrow{r}_o,\ \overleftarrow{t}_o,\ \overrightarrow{t}_o \ .$$

Thus

$$R(\omega-\omega_o,a-\varepsilon) = \overrightarrow{r}_o(\omega - \omega_o) + \int_{-\infty}^{\infty} \hat{\overrightarrow{t}}_o(y)\ \hat{\overleftarrow{t}}_o(y) \frac{\hat{R}(y,a+\varepsilon)}{1 - \hat{\overleftarrow{r}}_o(y)\ \hat{R}(y,a+\varepsilon)} \times e^{i\left(\frac{\omega - \omega_o}{c}\right)y}\ dy \qquad \text{(2D-16)}$$

This will not ordinarily be bandlimited. Consistent with dropping the nonlinear term in obtaining (2D-5) (assuming that the slab is sufficiently tenuous to justify doing so) we need only have accounted for the first two reflections in Figure 6; i.e., R_o and R_1, the others being higher than first order in $R(\omega,\omega_o,a+\varepsilon)$. Equation (2D-16), however, would hold even if the slab were not tenuous but, in that case, the function R to be used in (2D-16) would be much more difficult to obtain, as we would no longer be justified in dropping the nonlinear term in (2D-2c). In such cases and in cases where more accuracy is required, one would have to solve equations (2D-1) (or some more appropriate approximation thereto) for the specific cases of interest and this solution would probably be best done numerically using a computer. One might consider improving the accuracy analytically by using a perturbation approach to including the nonlinear term and in this regard we remark that, since each iteration would double the spectral width of the solution, the solution to be substituted into equation (2D-16) would not ordinarily be bandlimited.

Note, also, that consistent with our quasistatic approach to the problem, we could have left R in its time-varying form and left the r_o's and t_o's untransformed. However, the method presented is more generally applicable. That is, given that R has somehow been obtained to better than full quasistatic accuracy, (2D-16) would maintain this accuracy provided a condition like (2D-13) were satisfied.

E. Comments on Brillouin Scattering

Consider an idealized situation where a slab of fluid has in it a plane standing acoustic wave with variation in the $\pm z$ direction, and where the surrounding space is filled with the same fluid having the equilibrium density of the fluid in the slab [10]. A plane electromagnetic wave is normally incident on the slab and we wish to study the properties of the reflected electromagnetic wave. This situation is depicted in Figure 7. We assume that the acoustic wave is switched on at $t = 0$ so that the initial reflection is zero. The data necessary for solution of the equation for the reflection function are

$$n = 1 + \alpha\rho = 1 + \alpha\rho_o[1 - a \sin k_s z \sin \omega_s t]$$

$$\beta = a(1 - \cos k_s z) \cos \omega_s t \qquad \text{(2E-1)}$$

where c_s = the speed of sound = ω_s/k_s

ω_s = frequency of the sound

k_s = magnitude of the acoustic wave vector

ρ_o = equilibrium density of the fluid

a = amplitude of the sound wave

Substituting these expressions into the variable transformation (2D-10), we find that it is badly multivalued. The slab must therefore be divided into subslabs over which the transformation is one to one; i.e., subslabs having thickness equal to half the wavelength of the acoustic wave. The expressions (2E-1) are substituted into (2D-9) within each subslab making use of the variable transformation (2D-10) to express the result in terms of ω. The result would be a sum of N terms, each being a function of time and frequency where N is the number of subslabs.

The bandwidth of the reflection is easily obtained from (2D-9) and (2D-10). For a subslab we find that to first order in a

$$\omega(z) = \omega_o + 2\omega_o a(1 + \alpha\rho_o)(\cos k_s z - \cos k_s z_1)\cos \omega_s t \qquad \text{(2E-2)}$$

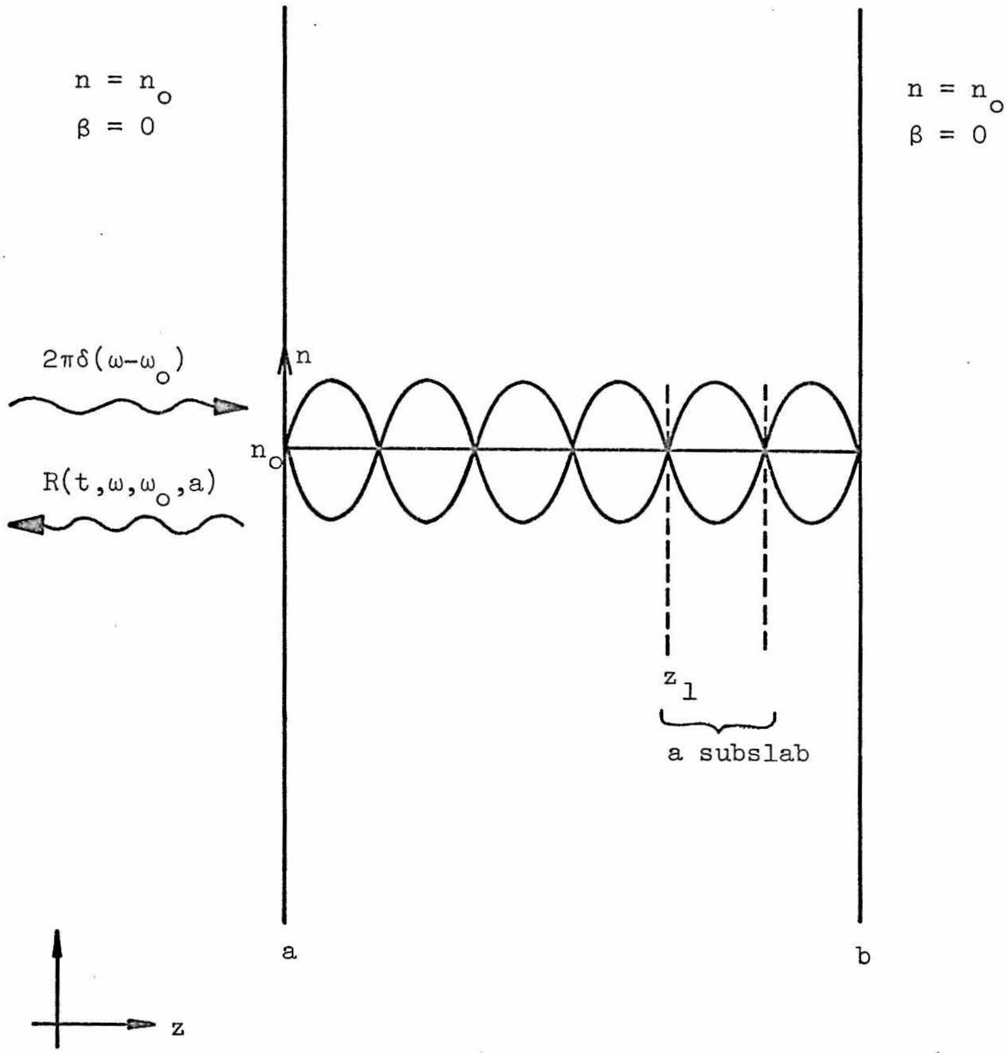


Figure 7. Brillouin scattering

Thus the maximum overall bandwidth is

$$\Delta\omega = 4\omega_o\, a(1 + \alpha\rho_o) \tag{2E-3}$$

and the spectrum will be centered at ω_o .

Another easily noted property of the solution R is that it is periodic in time with period equal to that of the acoustic wave. It may, therefore, be expanded in a Fourier series in time. That is,

$$R(t,\omega,\omega_o,a) = \sum_{n=-\infty}^{\infty} R_n(\omega,\omega_o)\, e^{-in\omega_s t} \tag{2E-4}$$

Applying (2D-12) yields

$$R(\omega,\omega_o,a) = \sum_{n=-\infty}^{\infty} R_n(\omega - n\omega_s,\omega_o) \tag{2E-5}$$

The usual Brillouin scattering result is obtained from this by neglecting all but the terms corresponding to $n=1$ and $n=-1$ and letting $R_n(\omega,\omega_o) \sim \delta(\omega-\omega_o)$.

F. An Alternative Formulation

The invariant imbedding formulation presented previously depends for its relative simplicity upon the translational form invariance of the basic wave functions used, i.e., a plane wave translated in space still looks like a plane wave. Spherical waves do not have this invariance. This indicates that it would be of some advantage to develop a formulation of the plane problem which does not depend on this invariance in anticipation of extension of the theory to spherical scatterers. It is this alternative formulation which will be developed below.

Figure 8 depicts a plane interface between two homogeneous dielectrics of indices n_a and n_b. A monochromatic, linearly polarized, plane electromagnetic wave of frequency ω_o is assumed to be normally incident on the interface as shown. We wish to determine the reflected and transmitted waves. To calculate the reflection and transmission properties of this interface one would ordinarily assume solutions of the following form:

Incident Wave

$$\vec{E}_I = \hat{e}_x E_o \, e^{i\frac{n_a\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t}$$
$$\vec{H}_I = \hat{e}_y E_o \sqrt{\frac{\varepsilon_a}{\mu_o}} \, e^{i\frac{n_a\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t} \qquad z \leq z_o \qquad (2F\text{-}1a)$$

Reflected Wave

$$\vec{E}_R = \hat{e}_x E_o R \, e^{-i\frac{n_a\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t}$$
$$\vec{H}_R = -\hat{e}_y E_o R \sqrt{\frac{\varepsilon_a}{\mu_o}} \, e^{-i\frac{n_a\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t} \qquad z \leq z_o \qquad (2F\text{-}1b)$$

Transmitted Wave

$$\vec{E}_T = \hat{e}_x E_o T \, e^{i\frac{n_b\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t}$$
$$\vec{H}_T = \hat{e}_y E_o T \sqrt{\frac{\varepsilon_b}{\mu_o}} \, e^{i\frac{n_b\omega_o}{c}(z-z_o)} \, e^{-i\omega_o t} \qquad z \geq z_o \qquad (2F\text{-}1c)$$

Requiring continuity of tangential E and H at $z = z_o$ yields the familiar result

$$R = \frac{n_a - n_b}{n_a + n_b} \qquad (2F\text{-}2a)$$

$$T = \frac{2n_a}{n_a + n_b} \qquad (2F\text{-}2b)$$

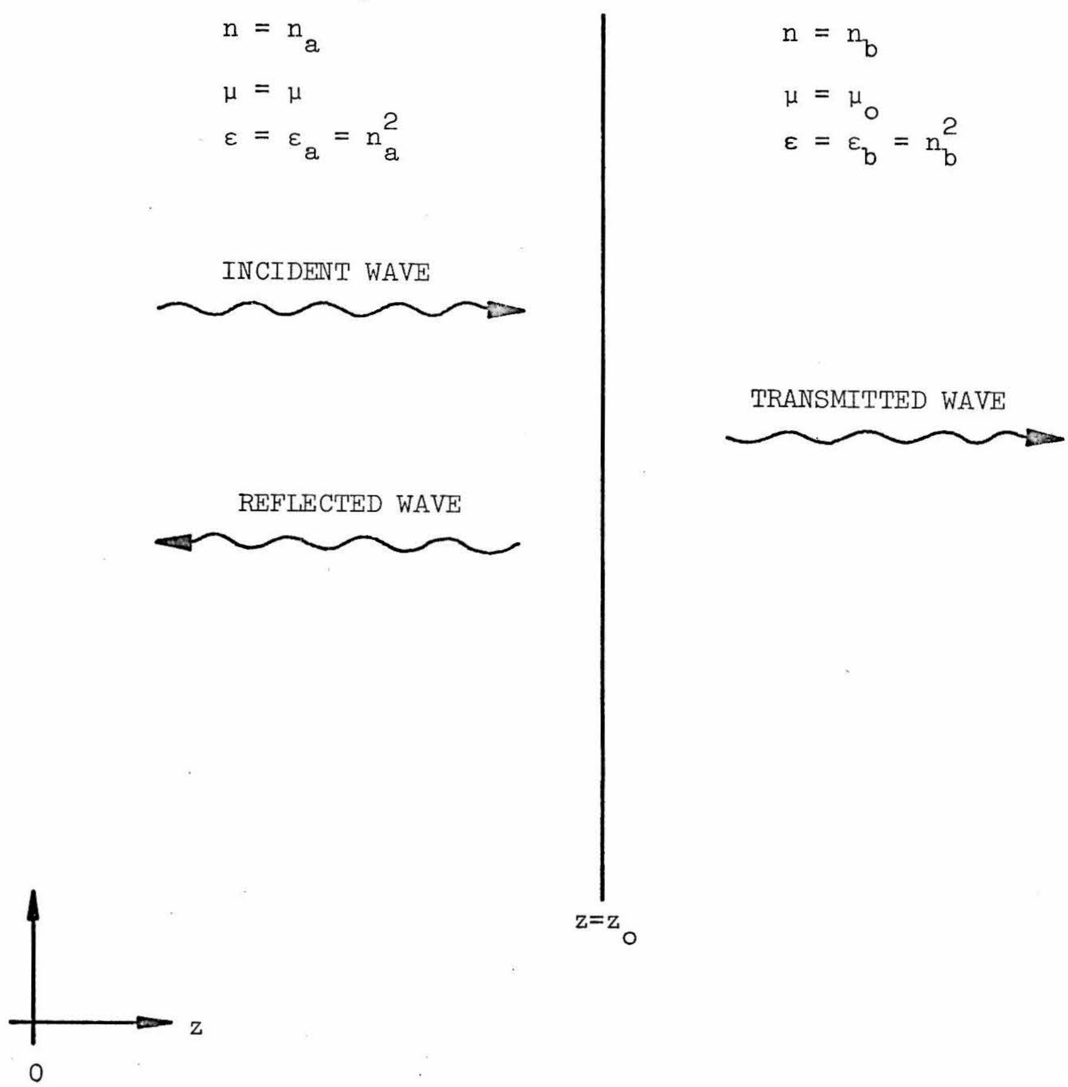


Figure 8. Scattering from a plane dielectric interface

Suppose now that we assume, rather than (2F-1), solutions of the form

Incident Wave

$$\vec{E}_I = \hat{e}_x E_o \, e^{i \frac{n_a \omega_o}{c} z} \, e^{-i\omega_o t}$$

$$\vec{H}_I = \hat{e}_y E_o \sqrt{\frac{\varepsilon_a}{\mu_o}} \, e^{i \frac{n_a \omega_o}{c} z} \, e^{-i\omega_o t} \qquad z \leq z_o \qquad (2F\text{-}3a)$$

Reflected Wave

$$\vec{E}_R = \hat{e}_x E_o R \, e^{-i \frac{n_a \omega_o}{c} z} \, e^{-i\omega_o t}$$

$$\vec{H}_R = -\hat{e}_y E_o R \sqrt{\frac{\varepsilon_a}{\mu_o}} \, e^{-i \frac{n_a \omega_o}{c} z} \, e^{-i\omega_o t} \qquad z \leq z_o \qquad (2F\text{-}3b)$$

Transmitted Wave

$$\vec{E}_T = \hat{e}_x E_o T \, e^{i \frac{n_b \omega_o}{c} z} \, e^{-i\omega_o t}$$

$$\vec{H}_T = \hat{e}_y E_o T \sqrt{\frac{\varepsilon_b}{\mu_o}} \, e^{i \frac{n_b \omega_o}{c}} \, e^{-i\omega_o t} \qquad z \geq z_o \qquad (2F\text{-}3c)$$

that is, solutions whose phase is zero at the origin of coordinates. Requiring continuity of tangential E and H at $z = z_o$ now yields the, perhaps not so familiar, result

$$R = \left(\frac{n_a - n_b}{n_a + n_b}\right) \frac{e^{i \frac{n_a \omega_o}{c} z_o}}{e^{-i \frac{n_a \omega_o}{c} z_o}} \qquad (2F\text{-}4a)$$

$$T = \left(\frac{2n_a}{n_a + n_b}\right) \frac{e^{i \frac{n_a \omega_o}{c} z_o}}{e^{i \frac{n_b \omega_o}{c} z_o}} \qquad (2F\text{-}4b)$$

This, of course, contains the same information as (2F-2), but here we have taken the origin of coordinates as a reference for phase rather than $z = z_o$ as is usually done.

Figure 9 shows a plane interface normal to the z direction. The material to the left of the interface is homogeneous dielectric, having index n_a and moving to the right with velocity $\beta(z_o)$. The medium to the right of the interface is homogeneous and has index n_b but it moves to the right with velocity $\beta(z)$, a function of z. The interface itself moves to the right with velocity $\beta(z_o)$. Again, a monochromatic, linearly polarized, plane electromagnetic wave is assumed to be normally incident from the left and we wish to determine the transmitted and reflected waves. Let us calculate the reflection and transmission properties of this interface in the laboratory frame, taking the origin of laboratory coordinates as a reference for phase and using the effective index (2B-1); that is

$$n^+_{eff_a} = \frac{n_a + \beta(z_o)}{1 + n_a\beta(z_o)} \qquad \text{for the incident wave}$$

$$n^+_{eff_b} = \frac{n_b + \beta(z)}{1 + n_b\beta(z)} \qquad \text{for the transmitted wave}$$

$$n^-_{eff_a} = \frac{n_a - \beta(z_o)}{1 - n_a\beta(z_o)} \qquad \text{for the reflected wave}$$

Assuming solutions of the form (2B-2) on both sides of the boundary and matching at the moving boundary* the reflected and transmitted waves

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*As far as the magnitudes are concerned, this is most easily done in the frame of the boundary. The phases may be handled directly in the laboratory frame.

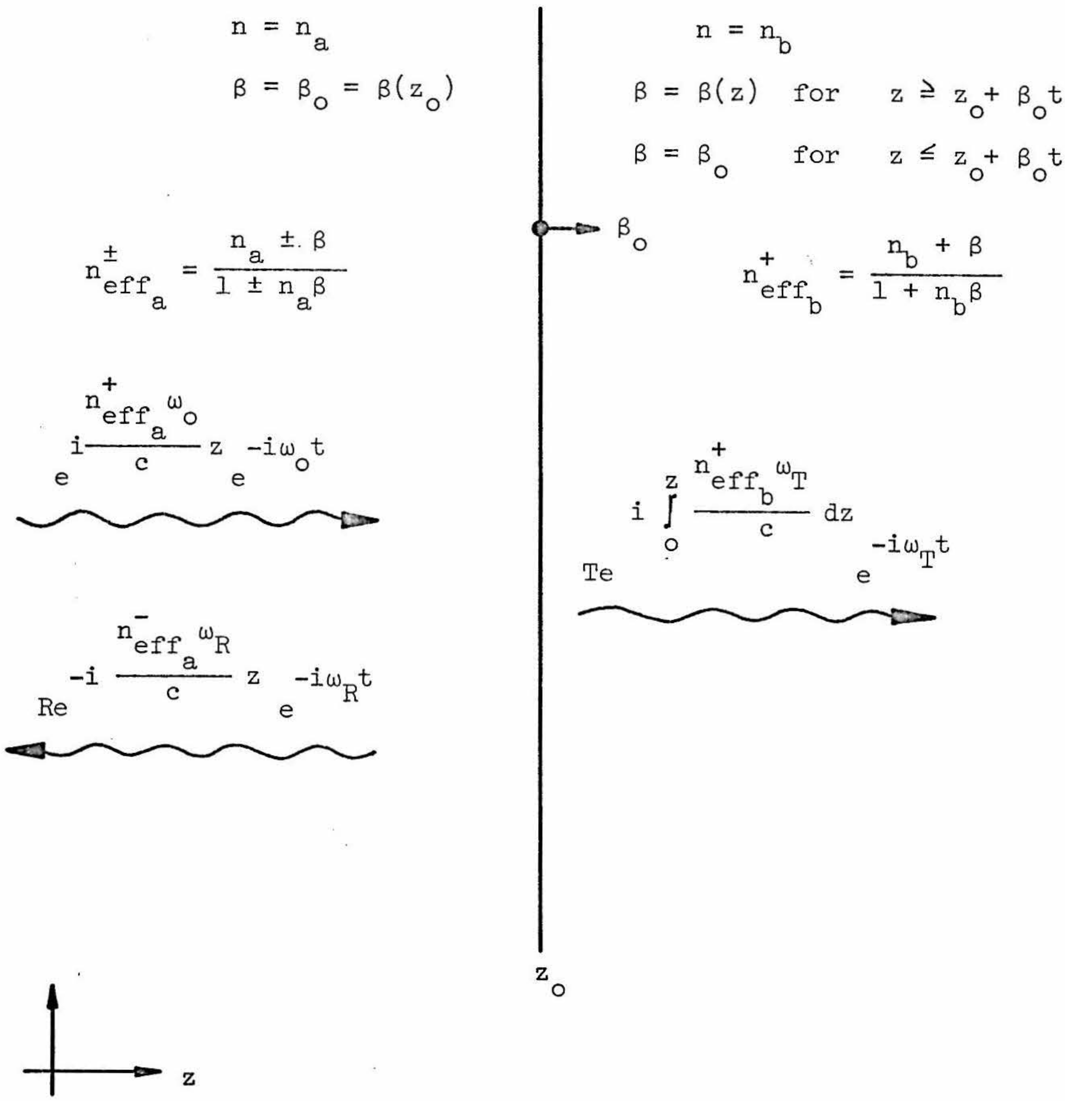

Figure 9. Scattering at a moving plane interface

expressed in the time domain are seen to be

$$\mathrm{Re}^{-i\frac{n^-_{\mathrm{eff}_a}\omega_R}{c}z}\, e^{-i\omega_R t} =$$

$$\left(\frac{1-n_a\beta_o}{1+n_a\beta_o}\right)\left(\frac{n_a-n_b}{n_a+n_b}\right)\left[\frac{e^{i\frac{n^+_{\mathrm{eff}_a}\omega_o}{c}(z_o+\beta_o t)}}{e^{-i\frac{n^-_{\mathrm{eff}_a}\omega_R}{c}(z_o+\beta_o t)}}\right] e^{-i\frac{n^-_{\mathrm{eff}_a}\omega_R}{c}z}\, e^{-i\omega_o t} \tag{2F-5a}$$

$$\mathrm{Te}^{i\int_o^z \frac{n^+_{\mathrm{eff}_b}\omega_T}{c}dz}\, e^{-i\omega_T t} =$$

$$\left(\frac{1+n_a\beta_o}{1+n_b\beta_o}\right)\left(\frac{2n_a}{n_a+n_b}\right)\left[\frac{e^{i\frac{n^+_{\mathrm{eff}_a}\omega_o}{c}(z_o+\beta_o t)}}{e^{i\int_o^{(z_o+\beta_o t)}\left(\frac{n^+_{\mathrm{eff}_b}\omega_T}{c}\right)dz}}\right] e^{i\int_o^z \frac{n^+_{\mathrm{eff}_b}\omega_T}{c}dz}\, e^{-i\omega_o t} \tag{2F-5b}$$

where $\beta_o = \beta(z_o)$.

The factors of $(1 \pm n\beta_o)$ arise because the fields must be made continuous across the index discontinuity in the frame of the discontinuity while we have calculated R and T in the laboratory frame [8] . Now transforming to the frequency domain gives

$$R2\pi\delta(\omega-\omega_R) = \left(\frac{1-n_a\beta_o}{1+n_a\beta_o}\right)\left(\frac{n_a-n_b}{n_a+n_b}\right)\frac{e^{i\frac{n^+_{eff_a}\omega_o}{c}z_o}}{e^{-i\frac{n^-_{eff_a}\omega_R}{c}z_o}}\,2\pi\delta(\omega-\omega_R) \qquad \text{(2F-6a)}$$

$$T2\pi\delta(\omega-\omega_T) = \left(\frac{1+n_a\beta_o}{1+n_b\beta_o}\right)\left(\frac{2n_a}{n_a+n_b}\right)\frac{e^{i\frac{n^+_{eff_a}\omega_o}{c}z_o}}{e^{\frac{i}{c}\int_o^{z_o} n^+_{eff_b}\,\omega_T dz}}\,2\pi\delta(\omega-\omega_T) \qquad \text{(2F-6b)}$$

where
$$\omega_R = \left(\frac{1-n_a\beta_o}{1+n_a\beta_o}\right)\omega_o$$

$$\omega_T = \left(\frac{1+n_b\beta_o}{1+n_a\beta_o}\right)\omega_o$$

The scheme for application of the invariant imbedding concept to the slab of Figure 3, using the above technique, is shown in Figure 10. It is assumed that the reflection function for that portion of the slab to the right of $z+\Delta z$ is known. A thin layer of fluid having index $n(t,z+ z)$ and having a constant velocity gradient throughout its thickness is annexed at $z+ z$. The material to the left of the added layer is homogeneous and has zero velocity gradient throughout. A monochromatic, linearly polarized, plane electromagnetic wave of frequency ω_1 is normally incident on the composite slab from the left and we must calculate the spectrum function of the reflected wave. The frequency variable in the spectra given by the R's in Figure 10 is ω_2, these R's being the results of multiple reflections--similar to those

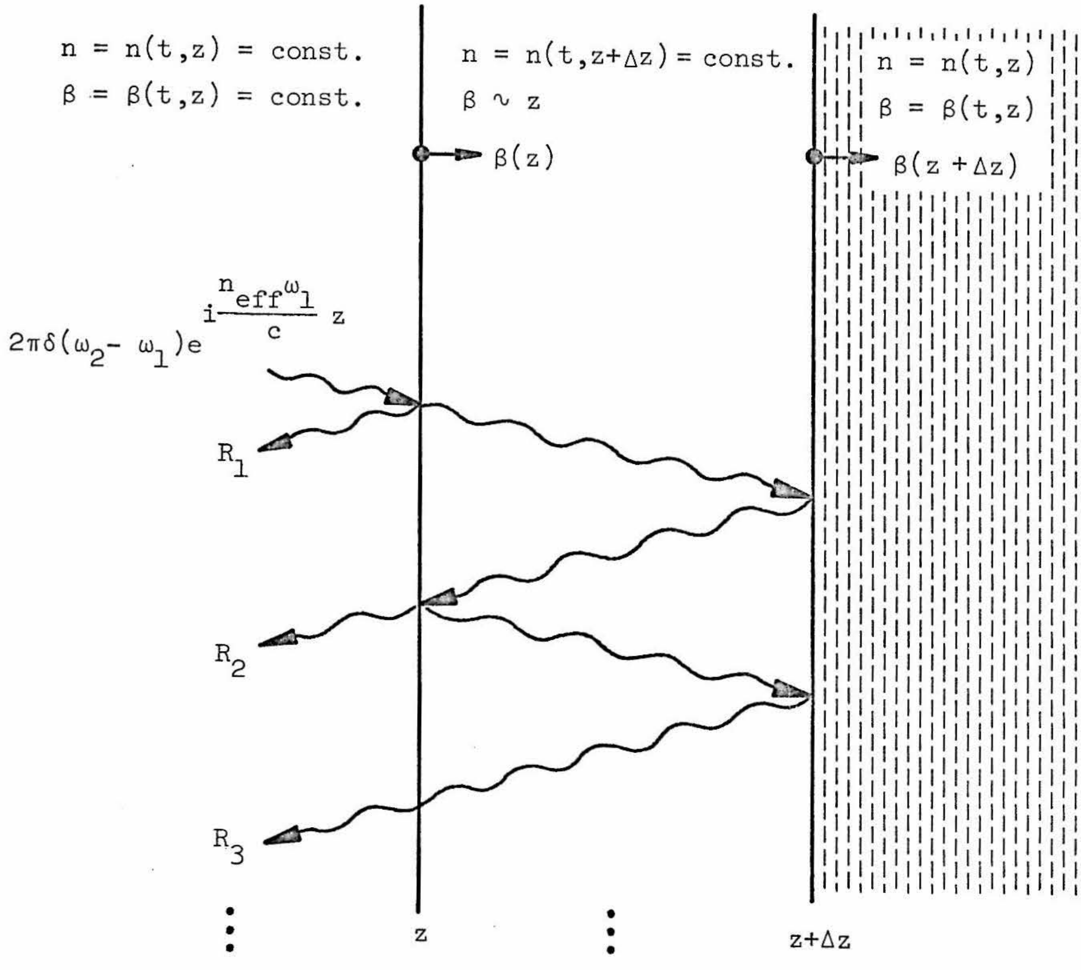

Figure 10. Scheme for laboratory frame derivation of the invariant imbedding equation for the reflection function

described in connection with Figure 4. Evaluating the three significant reflections using (2F-6) and summing them as before, we obtain the invariant imbedding equation for the reflection function expressed in the laboratory frame with a fixed phase reference, i.e., $z = 0$. The result to first order in β is

$$\frac{\partial R}{\partial z} - \omega_2 \beta \frac{\partial n}{\partial z} \frac{\partial R}{\partial \omega_2} + \omega_1 \beta \frac{\partial n}{\partial z} \frac{\partial R}{\partial \omega_1}$$

$$= \frac{1}{2n} \frac{\partial n}{\partial z} \Bigg\{ \left(\frac{1 - n\beta}{1 + n\beta}\right) \delta\left(\frac{\omega_2}{1 - n\beta} - \frac{\omega_1}{1 + n\beta}\right) e^{2i \frac{n}{c} \left(\frac{\omega_2}{1 - n\beta}\right) z}$$

$$- \left(\frac{1 + n\beta}{1 - n\beta}\right) \int_{-\infty}^{\infty} R(t, \bar{\omega}, \omega_1 z)\, e^{-\frac{2in}{c}\left(\frac{\bar{\omega}}{1 - n\beta}\right) z} R(t, \omega_2, \bar{\bar{\omega}}, z) \frac{d\bar{\omega}}{2\pi} \Bigg\}$$

$$-2\beta \frac{\partial n}{\partial z} R + \frac{i}{c} \frac{\partial n}{\partial z} \left(\frac{\omega_1}{1 + n\beta} + \frac{\omega_2}{1 - n\beta}\right) z\, R \qquad \text{(2F-7)}$$

where $\bar{\bar{\omega}} = \bar{\omega}\left(\frac{1 + n\beta}{1 - n\beta}\right)$

This equation may be compared with (2C-7) by means of the variable transformation

$$R_F(t, \omega_2, \omega_1, z) = \left(\frac{1 - n\beta}{1 + n\beta}\right) e^{\frac{i}{c}(\omega_{out} + \omega_{in})\, nz} R_C(t, \omega_{out}, \omega_{in}, z) \qquad \text{(2F-8)}$$

where R_C is the solution to (2C-7), R_F is the solution to (2F-7) and

$$\omega_{out} = \frac{\omega_2}{1 - n\beta}$$

$$\omega_{in} = \frac{\omega_1}{1 + n\beta}$$

It is this formulation which will be used in studying properties of the scattering resulting when a plane wave is incident on a spherically symmetric scatterer.

3. EXPANDING SPHERES

The techniques developed and the experience gained in dealing with the slab case are now applied in an approximate quasistatic analysis of the character of the scattering which results when a plane electromagnetic wave is incident on an expanding dielectric sphere. The plane wave may be written as an infinite sum of spherical waves so the following analysis will concentrate on the scattering of spherical waves. The results are then superposed appropriately to obtain the plane wave scattering. First, a preliminary problem will be considered. Then using the time varying spectrum concept introduced in Section 2, Part C, and assuming a radially stratified sphere, we proceed with the derivation of the form of the invariant imbedding equations. This form is then used to ascertain information regarding the character of the scattered wave. In particular, we study the differences and similarities in the solutions to this problem and that treated by Lam [11], i.e., scattering from an expanding conducting sphere.

A. Eigenfunctions in a Uniform Expanding Sphere

In order to determine the modal reflection functions for an expanding sphere we must first find the eigenfunctions in a uniform radially and spherically symmetrically expanding medium, i.e., the eigenfunctions corresponding to the spherical Hankel functions of argument kr used in the static case. Since we have assumed complete spherical symmetry, the eigenfunctions will be associated with the same angular dependence as were the corresponding Hankel functions, i.e., the spherical harmonics Y_{lm} . Viewed in the laboratory frame (the frame of the center of expansion) the constitutive relations to first order in the velocity of the medium are

$$D = \varepsilon E + \frac{1}{c^2}(n^2 - 1)(v \times H) \qquad (3A\text{-}1a)$$

$$B = \mu_o H - \frac{1}{c^2}(n^2 - 1)(v \times E) \qquad (3A\text{-}1b)$$

The relevant Maxwell's equations are

$$\nabla \times H = -i\omega D \qquad (3A\text{-}2a)$$

$$\nabla \times E = i\omega B \qquad (3A\text{-}2b)$$

the divergence equations being satisfied automatically, since the region is source free. Substitution of (3A-1) and (3A-2a) into (3A-2b) leads to

$$\nabla \times \nabla \times E - k^2 E = -\frac{i\omega}{c^2}(n^2 - 1)\,[v \times (\nabla \times E) + \nabla \times (v \times E)] \qquad (3A\text{-}3)$$

Use of the vector identities

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A \qquad (3A\text{-}4a)$$

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B \qquad (3A\text{-}4b)$$

$$\nabla(A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A) \qquad (3A\text{-}4c)$$

and equations (3A-1) yield

$$\nabla^2 E + k^2 E = -ik(n^2 - 1)[2(v \cdot \nabla)E - (\nabla \cdot v)\,E] \qquad (3A\text{-}5)$$

where $v = \beta c\,\hat{e}_r$

We now assume that the radial dependence of each component of the solution of this equation is of the form

$$E_\ell = \psi_\ell\left[\int_0^r k_{eff}\, dr\right] \tag{3A-6}$$

where $\nabla^2\psi_\ell + k^2\psi_\ell = 0$

(3A-6) is substituted into (3A-5) resulting in a differential equation for k_{eff} , i.e.,

$$\frac{dk_{eff}}{dr} = \frac{\psi_\ell}{\psi_\ell'}\left[k_{eff}^2 - k^2 - \frac{2ik}{c}(n^2-1)\,\frac{\nabla\cdot v}{2|v|}\right] - 2ik\, k_{eff}\,(n^2-1) \tag{3A-7}$$

We assume that the solution of this equation is of the form

$$k_{eff} = k[1 + q_\ell(r)] \qquad \text{where} \qquad q_\ell(r) = O(\beta) \tag{3A-8}$$

Substituting (3A-8) into (3A-7) gives

$$\frac{dq_\ell}{dr} = 2\left[\frac{\psi_\ell}{\psi_\ell'}\, k - ik\beta(n^2-1)\right]q_\ell - 2ik\beta(n^2-1)\left[1 + \frac{1}{k}\,\frac{\nabla\cdot v}{2|v|}\left(\frac{\psi_\ell}{\psi_\ell'}\right)\right] \tag{3A-9}$$

This is a first order linear differential equation and may be solved to yield (to first order in β)

$$k_{eff} = k\left\{1 - e^{2k\int_0^r \frac{\psi_\ell}{\psi_\ell'}\,dr}\int_0^r 2ik\beta(n^2-1)\left[1 + \frac{\nabla\cdot v}{2k|v|}\,\frac{\psi_\ell}{\psi_\ell'}\right] e^{-2k\int_0^r \frac{\psi_\ell}{\psi_\ell'}\,dr}\, dr\right\} \tag{3A-10}$$

The desired eigenfunctions to first order in β are then, merely spherical Hankel functions of argument $\int_{o}^{r} k_{eff}\, dr$. (It may be shown that H satisfies (3A-3) also.)

B. The Invariant Imbedding Formulation for an Expanding Sphere

The geometry of the sphere problem is shown in Figure 11, where it has been assumed that a spherical E type wave is incident on an inhomogeneous dielectric sphere having index $n(t,r)$ and radially expanding with velocity $\beta(t,r)$. The time dependence is again assumed to be slow; see conditions (2C-1). The invariant imbedding formulation of the problem consists of assuming that the reflection from that portion of the sphere inside of a given radius is known, and of calculating the change in the reflection due to addition of a thin shell of fluid at this radius. This results in a difference equation which, in the limit of vanishing added shell thickness, becomes a differential equation for the reflection function. The resulting equation is to be integrated from the center of the sphere where the reflection is known to the surface where it is to be found. This formulation is similar to one carried out by Latham[5] for a static inhomogeneous dielectric sphere but here, of course, the effect of radial expansion is taken into account.

Preliminary to finding the form of the differential equation, expressions analogous to (2F-5) must be found; that is, we must determine the reflection and transmission properties of a radially moving spherical interface between two radially moving dielectric media. We first define the spherical multipole fields as follows. The spherical components of an electric type multipole field of degree ℓ and order

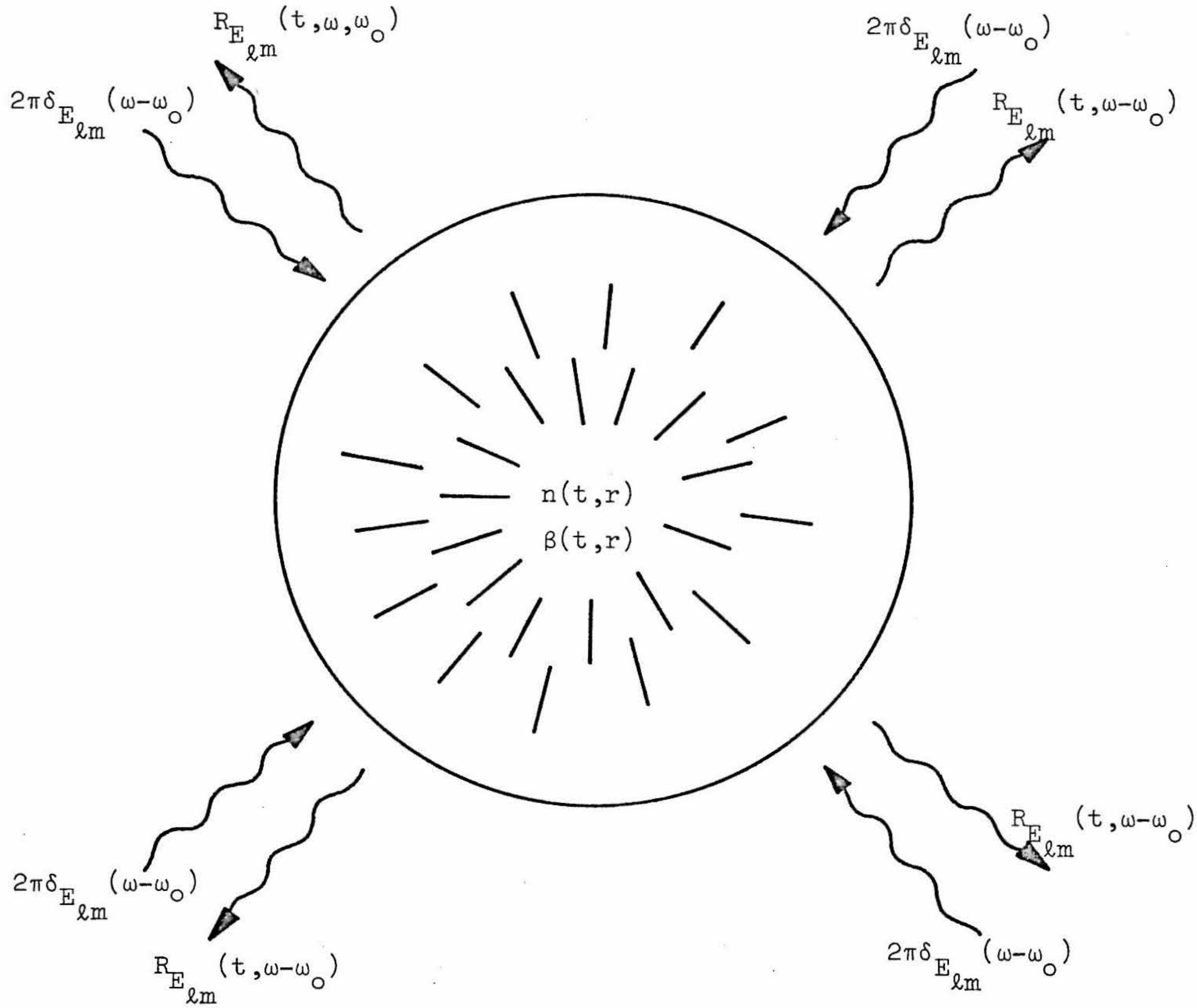

Figure 11. Geometry of the general sphere

m are (after Papas [12])

$$(E_{\ell m})_r = \frac{\ell(\ell+1)}{r} h_\ell(kr)\ Y_{\ell m} \tag{3B-1a}$$

$$(E_{\ell m})_\theta = \frac{1}{r}\frac{d}{dr}[r\ h_\ell(kr)]\ \frac{\partial}{\partial\theta} Y_{\ell m} \tag{3B-1b}$$

$$(E_{\ell m})_\phi = \frac{im}{r \sin\theta}\ \frac{d}{dr}[r\ h_\ell(kr)]\ Y_{\ell m} \tag{3B-1c}$$

$$(H_{\ell m})_r = 0 \tag{3B-1d}$$

$$(H_{\ell m})_\theta = \frac{m\omega\varepsilon}{\sin\theta} h_\ell(kr)\ Y_{\ell m} \tag{3B-1e}$$

$$(H_{\ell m})_\phi = i\omega\varepsilon\ h_\ell(kr)\ \frac{\partial}{\partial\theta} Y_{\ell m} \tag{3B-1f}$$

Similarly the spherical components of a magnetic multipole field of degree ℓ and order m are (again after Papas [12])

$$(H_{\ell m})_r = \frac{\ell(\ell+1)}{r} h_\ell(kr)\ Y_{\ell m} \tag{3B-2a}$$

$$(H_{\ell m})_\theta = \frac{1}{r}\frac{d}{dr}[r\ h_\ell(kr)]\ \frac{\partial}{\partial\theta} Y_{\ell m} \tag{3B-2b}$$

$$(H_{\ell m})_\phi = \frac{im}{r \sin\theta}\frac{d}{dr}[r\ h_\ell(kr)]\ Y_{\ell m} \tag{3B-2c}$$

$$(E_{\ell m})_r = 0 \tag{3B-2d}$$

$$(E_{\ell m})_\theta = -\frac{m\omega\mu_o}{\sin\theta} h_\ell(kr)\ Y_{\ell m} \tag{3B-2e}$$

$$(E_{\ell m})_\phi = -\ i\omega\mu_o\ h_\ell(kr)\ \frac{\partial}{\partial\theta} Y_{\ell m} \tag{3B-2f}$$

The corresponding multipole fields in a region of spherically expanding fluid are obtained by substituting $\int_0^r k_{eff}\, dr$ for kr in the arguments of these expressions.* The spherical harmonics $Y_{\ell m}$ remain as they were.

Now consider an E type wave of degree ℓ (for example) to be incident from without on a spherical boundary at r_o which is moving outward with velocity β_o . The medium inside the boundary has index n_a and moves radially with velocity $\beta(r)$. The medium outside the boundary is homogeneous and has index n_b . It moves radially with velocity $\beta_o = \beta(r_o)$. (See Figure 12). If $\psi_\ell^{(j)}[n_{eff}^{(j)},\omega,r]$ is defined to be $\frac{1}{x}\frac{d}{dx}[x\, h_\ell^{(j)}(x)]\Big|_{x = \int_0^r k_{eff}\, dr}$, then the desired expressions, i.e., those which correspond to (2F-5), are of the form

$$R = P(r_o)\,\frac{\psi_\ell^{(2)}[n_{eff_b}^{(2)},\omega_o,r_o + \beta c\Delta t]}{\psi_\ell^{(1)}[n_{eff_b}^{(1)},\omega_r,r_o + \beta c\Delta t]} \tag{3B-3a}$$

$$T = Q(r_o)\,\frac{\psi_\ell^{(2)}[n_{eff_b}^{(2)},\omega_o,r_o + \beta c\Delta t]}{\psi_\ell^{(2)}[n_{eff_a}^{(2)},\omega_t,r_o + \beta c\Delta t]} \tag{3B-3b}$$

where the superscript on n_{eff} indicates whether it is to be calculated for incoming or outgoing waves ((2) for incoming; (1) for outgoing) and $n_{eff} = ck_{eff}/\omega$. We now proceed to determine the frequency of oscillation of R and T ; that is, we determine the frequencies of the reflected and transmitted waves. The phases of the P and Q functions in (3B-3) vary slowly with r and, therefore, the P and

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*These corresponding expressions must first be written in terms of k and kr only (rather than r and kr).

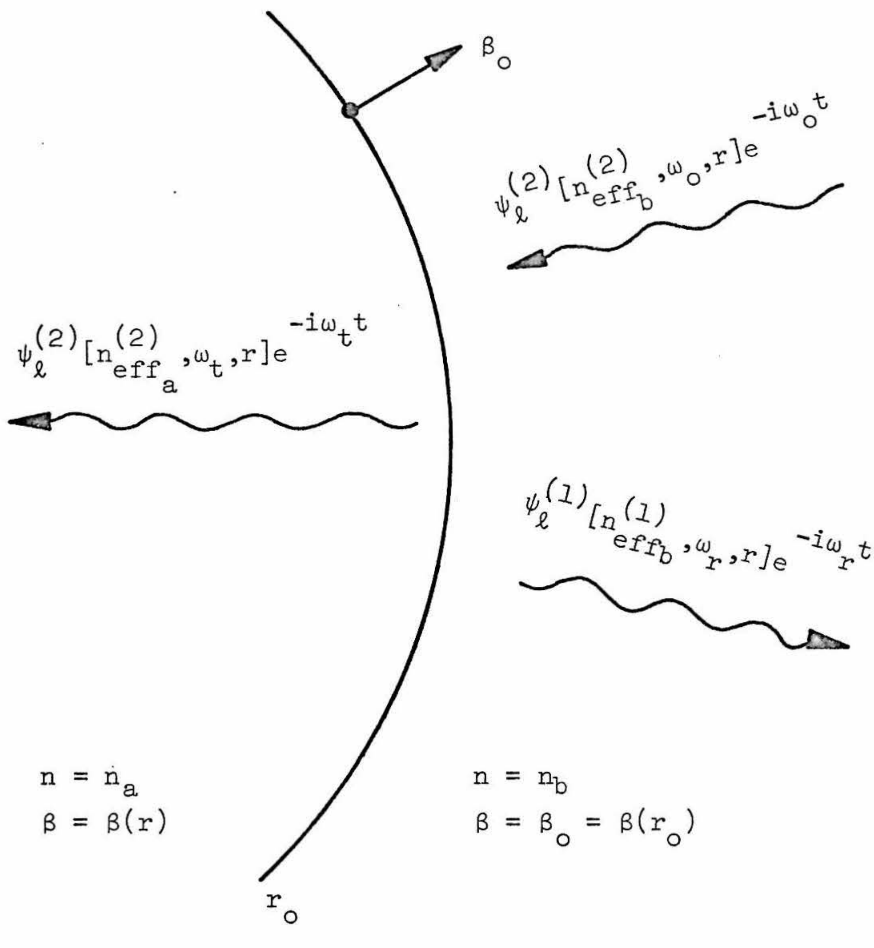

Figure 12. Scattering at a moving spherical interface

Q functions will not substantially affect the following (see Appendix D). Expressions (3B-3) may be rewritten as

$$R\psi_\ell^{(1)}[n^{(1)}_{\text{eff}_b},\omega_r,r]\, e^{-i\omega_o\Delta t} =$$

$$P(r_o)\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r_o]}{\psi_\ell^{(1)}[n^{(1)}_{\text{eff}_b},\omega_r,r_o]}\; e^{\left.\text{Re}\,\frac{\partial}{\partial r}\,\ell n\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r]}{\psi_\ell^{(1)}[n^{(1)}_{\text{eff}_b},\omega_r,r]}\right|_{r=r_o}\beta\delta\Delta t}$$

$$\times\; e^{\left. i\,\text{Im}\,\frac{\partial}{\partial r}\,\ell n\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r]}{\psi_\ell^{(1)}[n^{(1)}_{\text{eff}_b},\omega_r,r]}\right|_{r=r_o}\beta_o c\Delta t}$$

$$\times\; \psi_\ell^{(1)}[n^{(1)}_{\text{eff}_b},\omega_r,r]\, e^{-i\omega_o\Delta t} \tag{3B-4a}$$

$$T\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_a},\omega_t,r]\, e^{-i\omega_o\Delta t} =$$

$$Q(r_o)\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r_o]}{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_a},\omega_t,r_o]}\; e^{\left.\text{Re}\,\frac{\partial}{\partial r}\,\ell n\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r]}{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_a},\omega_t,r]}\right|_{r=r_o}\beta_o c\Delta t}$$

$$\times\; e^{\left. i\,\text{Im}\,\frac{\partial}{\partial r}\,\ell n\,\frac{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_b},\omega_o,r]}{\psi_\ell^{(2)}[n^{(2)}_{\text{eff}_a},\omega_t,r]}\right|_{r=r_o}\beta_o c\Delta t}$$

$$\times\; \psi_\ell^{(2)}[n^{(2)}_{\text{eff}_a},\omega_t,r]\, e^{-i\omega_o\Delta t} \tag{3B-4b}$$

However, we know that the form of R and T must be

$$R = (\cdots)\ \psi_\ell^{(1)}[n^{(1)}_{eff_b}, \omega_r, r]\ e^{-i\omega_r \Delta t} \tag{3B-5a}$$

$$T = (\cdots)\ \psi_\ell^{(2)}[n^{(2)}_{eff_a}, \omega_t, r]\ e^{-i\omega_t \Delta t} \tag{3B-5b}$$

Equating the imaginary part of the exponents in the corresponding expressions in (3B-4) and (3B-5) yields the following equations for the transmitted and reflected frequencies.

$$\omega_r - \beta_o c\ \mathrm{Im}\ \frac{\partial}{\partial r}\{\ell n\ \psi^{(1)}[n^{(1)}_{eff_b}, \omega_r, r]\}_{r=r_o} = \omega_o - \beta_o c\ \mathrm{Im}\ \frac{\partial}{\partial r}\ \{\ell n \psi_\ell^{(2)}[n^{(2)}_{eff_b}, \omega_o, r]\}_{r=r_o} \tag{3B-6a}$$

$$\omega_t - \beta_o c\ \mathrm{Im}\ \frac{\partial}{\partial r}\ \{\ell n\ \psi_\ell^{(2)}[n^{(2)}_{eff_a}, \omega_t, r]\}_{r=r_o} = \omega_o - \beta_o c\ \mathrm{Im}\ \frac{\partial}{\partial r}\ \{\ell n\ \psi_\ell^{(2)}[n^{(2)}_{eff_b}, \omega_o, r]\}_{r=r_o} \tag{3B-6b}$$

Solving these equations to first order in β_o we obtain

$$\omega_r = \omega_o - \beta_o c\,\mathrm{Im}\ \frac{\partial}{\partial r}\ \left\{\ell n\ \frac{\psi_\ell^{(2)}[n_a, \omega_o, r]}{\psi_\ell^{(1)}[n_b, \omega_o, r]}\ \right\}_{r=r_o} \tag{3B-7a}$$

$$\omega_t = \omega_o - \beta_o c\ \mathrm{Im}\ \frac{\partial}{\partial r}\ \left\{\ell n\ \frac{\psi_\ell^{(2)}[n_b, \omega_o, r]}{\psi_\ell^{(2)}[n_a, \omega_o, r]}\right\}_{r=r_o} \tag{3B-7b}$$

These are the frequencies of the waves reflected from and transmitted through the moving boundary of Figure 12. In the invariant imbedding formulation of the sphere problem there arises a situation where $n_a = n(r-\Delta r)$ and $n_b = n(r)$. Under these circumstances

$$\omega_r = \omega_o - \beta_o c \ \mathrm{Im} \frac{\partial}{\partial r} \{\ell n \frac{\psi_\ell^{(2)}}{\psi_\ell^{(1)}} \} \tag{3B-8a}$$

$$\omega_t = \omega_o + \omega_o \beta_o \frac{\partial n}{\partial r} \mathrm{Im} \frac{\partial}{\partial r} \{r \frac{\psi_\ell^{(2)'}}{\psi_\ell^{(2)}} \} \Delta r \tag{3B-8b}$$

where $\psi_\ell^{(2)'} = \frac{1}{r} \frac{\partial}{\partial r} r \ h_\ell^{(2)'} \ (\frac{n\omega r}{c})$

and the prime indicates differentiation of the Hankel function with respect to its argument.

We now have sufficient information to obtain the differential operator and the form of the invariant imbedding equation. In analogy with (2F-7), using (3B-8), we obtain (to first order in β)

$$\frac{\partial R_\ell}{\partial r} - \omega_o \beta \frac{\partial n}{\partial r} \mathrm{Im} \frac{\partial}{\partial r} \left\{ r \frac{\psi_\ell^{(1)'}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right\} \frac{\partial R_\ell}{\partial \omega_2}$$

$$- \omega_o \beta \frac{\partial n}{\partial r} \mathrm{Im} \frac{\partial}{\partial r} \left\{ r \frac{\psi_\ell^{(2)'}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})} \right\} \frac{\partial R_\ell}{\partial \omega_1}$$

$$= A\delta \left\{ \omega_2 - \omega_1 + \beta c \ \mathrm{Im} \frac{\partial}{\partial r} \left| \ell n \frac{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right| \right\} + BR_\ell + C[O(R_\ell^2)] \tag{3B-9}$$

where

$$\psi_\ell^{(j)} = \frac{1}{r}\frac{\partial}{\partial r}[r\, h_\ell^{(j)}(kr)] \qquad \text{for an E wave}$$

$$\psi_\ell^{(j)} = h_\ell^{(j)}(kr) \qquad \text{for an H wave}$$

This equation is to be integrated from the center of the sphere where $R_\ell(t,\omega_2,\omega_1,0) = 2\pi\delta(\omega_2 - \omega_1)$ out to the surface of the sphere where R_ℓ is to be found*. Since the initial value of R_ℓ is of order unity (a unit delta function) we are not justified in dropping the nonlinear term in (3B-9) even if the sphere is very tenuous. However, the equation may be linearized as follows. First, we write the nonlinear term in more explicit form:

$$C[O(R^2)] \sim \int_{-\infty}^{\infty} R_\ell(\bar{\omega},\omega_1)\, Af(\bar{\bar{\omega}})\, R_\ell(\omega_2,\bar{\bar{\omega}})\, \frac{d\bar{\omega}}{2\pi} \tag{3B-10}$$

where $$\bar{\bar{\omega}} = \bar{\omega} - \beta c \,\mathrm{Im}\, \frac{\partial}{\partial r} \left| \ell n \frac{\psi_\ell^{(1)}(\frac{n\bar{\omega}r}{c})}{\psi_\ell^{(2)}(\frac{n\bar{\omega}r}{c})} \right|$$

We then rewrite this term in the form

$$C[O(R_\ell^2)] \sim \int_{-\infty}^{\infty} \{[R_\ell(\bar{\omega},\omega_1) - 2\pi\delta(\bar{\omega}-\omega_1)] + 2\pi\delta(\bar{\omega}-\omega_1)\}\, Af(\bar{\omega})$$

$$\times \{[R_\ell(\omega_2,\bar{\bar{\omega}}) - 2\pi\delta(\omega_2-\bar{\bar{\omega}})] + 2\pi\delta(\omega_2-\bar{\bar{\omega}})\}\, \frac{d\bar{\omega}}{2\pi} \tag{3B-11}$$

where the quantity in square brackets is small (of order A). Writing

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*Actually, the integration may be started at any radius at which R is known.

this term to first order in A, then, results in

$$C[O(R_\ell^2)] \sim Af(\omega_1)\ \delta\left\{\omega_2 - \omega_1 + \beta c\ \mathrm{Im}\ \frac{\partial}{\partial r}\left|\ell n\ \frac{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}\right|\right\} \tag{3B-12}$$

Equation (3B-9) may now be written in the following linearized form (for a tenuous scatterer).

$$\frac{\partial R_\ell}{\partial r} - \omega_o\beta\ \frac{\partial n}{\partial r}\ \mathrm{Im}\ \frac{\partial}{\partial r}\left\{r\ \frac{\psi_\ell^{(1)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})}\right\}\frac{\partial R_\ell}{\partial \omega_2}$$

$$-\omega_o\beta\ \frac{\partial n}{\partial r}\ \mathrm{Im}\ \frac{\partial}{\partial r}\left\{r\ \frac{\psi_\ell^{(2)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}\right\}\frac{\partial R_\ell}{\partial \omega_1}$$

$$= A_1\delta\left\{\omega_2 - \omega_1 + \beta c\ \mathrm{Im}\ \frac{\partial}{\partial r}\left|\ell n\ \frac{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}\right|\right\}$$

$$+ A_2\delta\left\{\omega_2 - \omega_1 + \beta c\ \mathrm{Im}\ \frac{\partial}{\partial r}\left|\ell n\ \frac{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}\right|\right\} + BR_\ell \tag{3B-13}$$

C. Salient Features of the Solution

In view of the fact that the continuous part of the spectrum of the reflected wave will be extremely complicated due to overlap of the various contributions from multiple reflections and due to the complicated nature of the eigenfunctions, and in view of the fact that there is no convenient way to sum the continuous spectrum contributions from the various spherical harmonics in the incident plane wave, we submit that the most informative features of the reflection are the delta function components. That is, it is these discrete components which give the most easily discernable information about the scatterer. Let us, then, study the properties of some of these components and the dependence of the properties on the character of the scatterer. To make this dependence evident we handle (3B-13) by the method of characteristics. This results in the following set of four ordinary differential equations:

$$\frac{dr}{ds} = 1 \tag{3C-1a}$$

$$\frac{d\omega_2}{ds} = -\omega_o \beta \frac{\partial n}{\partial r} \operatorname{Im} \frac{\partial}{\partial r} \left\{ r \frac{\psi_\ell^{(1)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right\} \tag{3C-1b}$$

$$\frac{d\omega_1}{ds} = -\omega_o \beta \frac{\partial n}{\partial r} \operatorname{Im} \frac{\partial}{\partial r} \left\{ r \frac{\psi_\ell^{(2)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})} \right\} \tag{3C-1c}$$

$$\frac{dR}{ds} = A_1\delta \left\{\omega_2 - \omega_1 + \beta c \ \mathrm{Im} \frac{\partial}{\partial r} \left[\ell n \frac{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right] \right\}$$

$$+ A_2\delta \left\{\omega_2 - \omega_1 + \beta c \ \mathrm{Im} \frac{\partial}{\partial r} \left[\ell n \frac{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})} \right] \right\} + BR_\ell \qquad (3C\text{-}1d)$$

Equation (3C-1a) indicates that we may take s equal to r. We thus reduce the problem to solution of the following three ordinary differential equations:

$$\frac{d\omega_2}{dr} = -\omega_o \Delta^{(2)} \qquad (3C\text{-}2a)$$

$$\frac{d\omega_1}{dr} = \omega_o \Delta^{(1)} \qquad (3C\text{-}2b)$$

$$\frac{dR}{dr} = A_1\delta(\omega_2 - \omega_1 - \Delta_1) + A_2\delta(\omega_2 - \omega_1 + \Delta_2) + BR_\ell \qquad (3C\text{-}2c)$$

where

$$\Delta^{(1)} = -\beta \frac{\partial n}{\partial r} \mathrm{Im} \frac{\partial}{\partial r} \left[r \frac{\psi_\ell^{(2)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})} \right]$$

$$\Delta^{(2)} = \beta \frac{\partial n}{\partial r} \mathrm{Im} \frac{\partial}{\partial r} \left[r \frac{\psi_\ell^{(1)\prime}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right]$$

$$\Delta_1 = \beta c \ \mathrm{Im} \frac{\partial}{\partial r} \left[\ell n \frac{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})} \right]$$

$$\Delta_2 = \beta c \ \mathrm{Im} \frac{\partial}{\partial r} \left[\ln \frac{\psi_\ell^{(1)}(\frac{n\omega_o r}{c})}{\psi_\ell^{(2)}(\frac{n\omega_o r}{c})} \right]$$

Equations (3C-2a) and (3C-2b) are now integrated to yield

$$\omega_2 = \omega_2(0) - \omega_o \int_0^r \Delta^{(2)} \, dr \tag{3C-3}$$

$$\omega_1 = \omega_1(0) + \omega_o \int_0^r \Delta^{(1)} \, dr \tag{3C-4}$$

where $\omega_1(0)$ and $\omega_2(0)$ correspond to $r = 0$ on the characteristic. Substituting (3C-3) and (3C-4) into (3C-2c) gives

$$\frac{dR_\ell}{dr} = A_1\delta \left\{ \omega_2(0) - \omega_1(0) - \omega_o \int_0^r (\Delta^{(2)} + \Delta^{(1)})dr - \Delta_1 \right\}$$
$$+ A_2\delta \left\{ \omega_2(0) - \omega_1(0) - \omega_o \int_0^r (\Delta^{(2)} + \Delta^{(1)})dr + \Delta_2 \right\} + BR_\ell \tag{3C-5}$$

or simply,

$$\frac{dR}{dr} = A_1\delta_1 + A_2\delta_2 + BR_\ell \tag{3C-6}$$

The solution of (3C-6) evaluated at the surface of the sphere ($r=r_s$) is

$$R_\ell = \left\{ 2\pi\delta[\omega_2(0) - \omega_1(0)] + \int_0^{r_s} (A_1\delta_1 + A_2\delta_2) \, e^{-\int_0^r B dr} dr \right\} e^{\int_0^r B dr} \tag{3C-7}$$

Solving (3C-3) and (3C-4) at $r = r_s$ for $\omega_1(0)$ and $\omega_2(0)$ in terms of ω_1, ω_2, and r_s we have

$$\omega_2(0) = \omega + \omega_o \int_o^{r_s} \Delta^{(2)} dr \tag{3C-8a}$$

$$\omega_1(0) = \omega_o - \omega_o \int_o^{r_s} \Delta^{(1)} dr \tag{3C-8b}$$

Now the solution (3C-7) may be rewritten in the form

$$R_\ell = 2\pi\delta \quad \omega - \omega_o + \omega_o \int_o^{r_s} (\Delta^{(2)} + \Delta^{(1)}) dr \quad e^{\int_o^{r_s} B\, dr}$$
$$+ \frac{A_1(\hat{r}_1)}{J_1(\hat{r}_1)} e^{\int_{\hat{r}_1}^{r_s} B\, dr} + \frac{A_2(\hat{r}_2)}{J_2(\hat{r}_2)} e^{\int_{\hat{r}_2}^{r_s} B\, dr} \tag{3C-9}$$

where J_1 and J_2 are the appropriate Jacobians of the variable transformations defined by the corresponding delta functions in the integrand of (3C-7) and $\hat{r}_1$ and $\hat{r}_2$ are such that

$$\omega - \omega_o + \omega_o \int_{r_j}^{r_s} (\Delta^{(2)} + \Delta^{(1)}) dr + (-1)^j \Delta_j = 0$$

$$\text{with} \quad j = 1,2$$
$$\text{and} \quad 0 \leq \hat{r}_j \leq r_s$$

This immediately determines the position and width of the spectrum of the reflected wave; that is

$$[\omega_o-(-1)^j \Delta_j(r_s)](-1)^j \leq (-1)^j\omega = (-1)^j \omega_o[1-\int_0^{r_s} (\Delta^{(1)}+ \Delta^{(2)}dr] \qquad (3C\text{-}10)$$

where, when $j = 1$, (3C-10) gives the contribution from δ_1 and when $j = 2$, it gives the contribution from δ_2 and β is assumed positive. The solution (3C-9) also contains a delta function at

$$\omega_\delta = \omega_o\left[1 - \int_0^{r_s} (\Delta^{(2)}+ \Delta^{(1)})\ dr\right] \qquad (3C\text{-}11)$$

Expressions (3C-10) and (3C-11) give the salient features of the modal reflection spectrum represented by (3C-9). That is, they describe the important properties of the spherical wave reflected from a radially expanding sphere of nonconducting fluid containing no discontinuities in index of refraction. The incident wave in this case is a monochromatic spherical wave of degree 1 and order m of either E or H type having frequency ω_o . The reflected wave will be of the same type, the same degree, and the same order but will have a frequency spectrum of width given by (3C-10) and will have a discrete frequency component at ω_δ given by (3C-11).

At this point the possibility of a sharp discontinuity in n at the surface $r = r_s$ has not been accounted for. This possibility may be handled in a manner similar to that used in handling the discontinuity in n at $z = a$ in the slab of Figure 6. However, in the present formulation the boundary is moving and consequently associated with each transmission and reflection there will be a frequency shift given by (3B-7). This treatment will deal only with those reflections corresponding to R_o and R_1 and the delta function component in the reflection corresponding to R_2 in Figure 6. In particular, we are interested in determining the frequencies of the delta functions and the spectral width and position of the term corresponding to R_1.

The frequency of the delta function corresponding to R_o is easily seen to be (see (3B-7a))

$$R_o: \qquad \omega_\delta = \omega_o - \beta(r_s)c \ \mathrm{Im} \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(2)}[n(r_s+\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s+\varepsilon),\omega_o,r]} \right\}_{r=r_s} \tag{3C-12a}$$

Using (3B-7b) with $\psi_\ell^{(2)}$ replaced by $\psi_\ell^{(1)}$ and (3C-11) we find that the delta function in the spectrum corresponding to R_1 is at

$$R_1: \qquad \omega_\delta = \omega_o \left[1 - \int_o^{r_s} (\Delta^{(2)} + \Delta^{(1)}) dr \right] - \beta(r_s)c \ \mathrm{Im} \frac{\partial}{\partial r}$$

$$\times \left\{ \ell n \frac{\psi_\ell^{(2)}[n(r_s+\varepsilon),\omega_o,r]}{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o r]} \right\}_{r=r_s}$$

$$- \beta(r_s)c \ \mathrm{Im} \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s+\varepsilon),\omega_o,r]} \right\}_{r=r_s} \tag{3C-12b}$$

Similarly, the frequency of the delta function in the spectrum corresponding to R_2 is

$$R_2: \qquad \omega_\delta = \omega_o \left[1 - 2 \int_o^{r_s} (\Delta^{(2)} + \Delta^{(1)}) dr \right] - \beta(r_s) c \ \mathrm{Im} \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(2)}[n(r_s+\varepsilon),\omega_o,r]}{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o,r]} \right\}_{r=r_s}$$

$$- \beta(r_s) c \ \mathrm{Im} \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o,r]} \right\}_{r=r_s}$$

$$- \beta(r_s) c \ \mathrm{Im} \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s+\varepsilon),\omega_o,r]} \right\}_{r=r_s} \qquad (3C\text{-}12c)$$

The various frequency shift terms in (3C-12) are easily associated with physical phenomena and were, in fact, obtained by such association. The shift in (3C-12a) is merely the Doppler shift due to reflection from the moving boundary at $r=r_s$. The first term in (3C-12b) accounts for the shift during propagation from the surface at $r=r_s$ to the center of the sphere and back out again. The second and third terms in (3C-12b) account for the shift in crossing the boundary at $r=r_s$ on the way in and on the way out, respectively. Similarly, the first term of (3C-12c) accounts for the shift during propagation from the surface in to the center, back out to the surface, back in to the center, and back out to the surface again. The third term accounts for the shift on reflection from the inner side of the surface at $r=r_s$ and the second and fourth terms again account for the shifts on transmission in through and out through the boundary at $r=r_s$.

These terms take on somewhat more familiar forms in the far zone, i.e., large r_s. That is,

$$R_o: \qquad \omega_\delta \simeq \omega_o[1 + 2n(r_s+\varepsilon)\ \beta(r_s)] \tag{3C-13a}$$

$$R_1: \qquad \omega_\delta \simeq \omega_o \left[1 - \int_o^{r_s} (\Delta^{(2)} + \Delta^{(1)})dr\right] \{1 - 2\beta(r_s)[n(r_s-\varepsilon) - n(r_s+\varepsilon)]\} \tag{3C-13b}$$

$$R_2: \qquad \omega_\delta \simeq \omega_o \left[1 - 2\int_o^{r_s} (\Delta^{(2)} + \Delta^{(1)})dr\right] \{1 - 2\beta(r_s)\ n(r_s-\varepsilon)\}$$
$$\times \{1 - 2\beta(r_s)[n(r_s-\varepsilon) - n(r_s+\varepsilon)]\} \tag{3C-13c}$$

The spectral width and position of the spectrum corresponding to R_1 is also easily found using (3C-10) together with the appropriate boundary effect terms; that is

$$(-1)^j \left\langle \omega_o - (-1)^j\ \Delta_j(r_s) - \beta(r_s)c\ \mathrm{Im}\ \frac{\partial}{\partial r} \left\{ \ell n\ \frac{\psi_\ell^{(2)}[n(r_s+\varepsilon),\omega_o,r]}{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o,r]} \right\}_{r=r_s} \right.$$
$$\left. - \beta(r_s)c\ \mathrm{Im}\ \frac{\partial}{\partial r} \left\{ \ell n\ \frac{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s+\varepsilon),\omega_o,r]} \right\}_{r=r_s} \right\rangle \leq (-1)^j\ \omega$$

$$\leq \left\langle \omega_o \left[1 - \int_o^{r_s} (\Delta^{(1)} + \Delta^{(2)})dr\right] - \beta(r_s)c\ \mathrm{Im}\ \frac{\partial}{\partial r} \left\{ \ell n \frac{\psi_\ell^{(2)}[n(r_s+\varepsilon),\omega_o,r]}{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o,r]} \right\}_{r=r_s} \right.$$
$$\left. - \beta(r_s)c\ \mathrm{Im}\ \frac{\partial}{\partial r} \left\{ \ell n\ \frac{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s+\varepsilon),\omega_o,r]} \right\}_{r=r_s} \right\rangle (-1)^j \tag{3C-14}$$

The spectrum of the reflection resulting from a typical incident spherical mode is shown in Figure 13. Assuming that the surface $r=r_s$ is moving outward, the delta function corresponding to R_o is shifted up in frequency, the one corresponding to R_2 is shifted down, and the position of the one corresponding to R_1 depends in a complicated way on the detailed variation of the velocity and index with radius. The bandwidth and position of the continuous part of the spectrum is given by (3C-14) and this continuous part is seen to lie entirely between the delta functions corresponding to R_o and R_2. The overall modal spectrum is narrow, since its width is linear in $\beta(r_s)$ (to first order in β).

This completes the study of the individual modes. We now proceed to study how these modal reflections combine to produce the reflection due to an incident plane wave.

Consider a plane wave propagating in the positive z direction incident on a sphere in a spherical coordinate system having the z axis as its polar axis. Let the E field of the incident wave be in the x direction. We wish to expand this incident wave into spherical waves of the form given by (3B-1) and (3B-2). Let $X^{(j)}_{\ell mE}$ be the E field of a spherical E wave and let $X^{(j)}_{\ell mH}$ be the E field of a spherical H wave. j=1 implies an outgoing wave and j=2 implies an incoming wave. The desired expansion is

$$E_o \hat{e}_x e^{ikz} = \sum_{\ell m} A^{(1)}_{\ell m} X^{(1)}_{\ell mE} + \sum_{\ell m} A^{(2)}_{\ell m} X^{(2)}_{\ell mE}$$

$$+ \sum_{\ell m} B^{(1)}_{\ell m} X^{(1)}_{\ell mH} + \sum_{\ell m} B^{(2)}_{\ell m} X^{(2)}_{\ell mH} \qquad (3C-15)$$

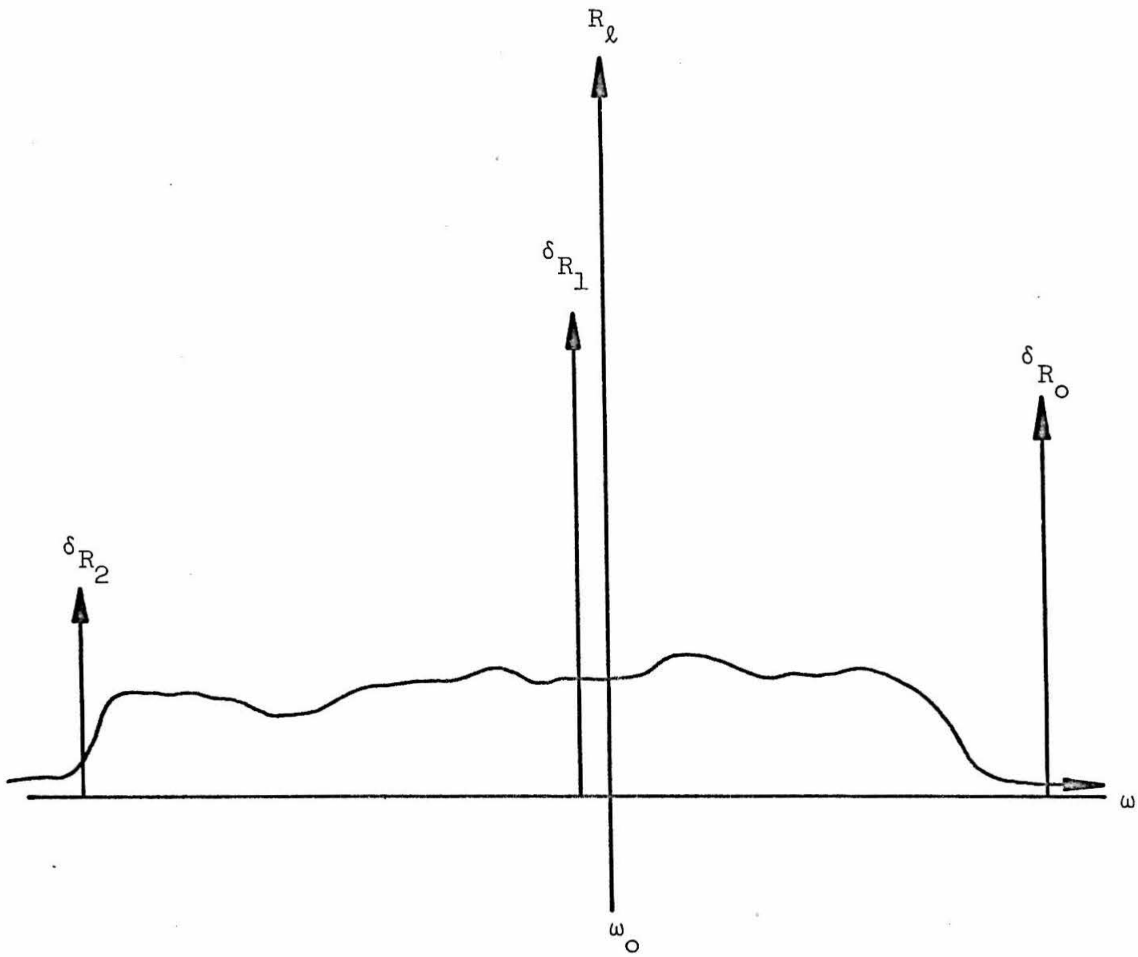

Figure 13. A typical modal reflection spectrum

where

$$A^{(1)}_{\ell,\pm 1} = \pm(i)^{\ell}\sqrt{4\pi}\sqrt{\left(\frac{2\ell+1}{\ell(\ell+1)}\right)}\frac{c}{2\omega_o}E_o = A^{(2)}_{\ell,\pm 1}$$

$$B^{(1)}_{\ell,\pm 1} = (i)^{\ell}\sqrt{4\pi}\sqrt{\left(\frac{2\ell+1}{\ell(\ell+1)}\right)}\frac{1}{2\mu_o\omega_o}E_o = B^{(2)}_{\ell,\pm 1}$$

$$A = B = 0 \qquad \text{for} \qquad m \neq \pm 1$$

The total reflection function will then be

$$E = \sum_{\substack{\ell \\ m=\pm 1}} A^{(2)}_{\ell m}\left[R_{\ell mE} - 2\pi\,\delta(\omega-\omega_o)\right] X^{(1)}_{\ell mE}$$

$$+ \sum_{\substack{\ell \\ m=\pm 1}} B^{(2)}_{\ell m}\left[R_{\ell mH} - 2\pi\,\delta(\omega-\omega_o)\right] X^{(1)}_{\ell mH} \qquad \text{(3C-16)}$$

We are now faced with the task of summing these series which are very slowly convergent under ordinary circumstances. The technique to be applied is as follows. We first make a Watson transformation transforming the sum to a contour integral in the complex l plane [18]. We then evaluate the integral approximately by the method of steepest descent [14]. This technique will be applied to the three discrete components in the solution of Figure 13. It will be found that the three sums have a very satisfying physical interpretation.

Consider first the sum of the terms having frequencies given by (3C-12a). This sum may be written in the form

$$\vec{E}_o = \sum_{\substack{\ell \\ m=\pm 1}} i^{\ell} \sqrt{\left(\frac{2\ell+1}{\ell(\ell+1)}\right)} \begin{pmatrix}\text{slowly varying function} \\ \text{of } r \text{ and } \ell\end{pmatrix}_{r=r_s}$$

$$\times \frac{\psi_{\ell E}^{(2)}[n(r_s+\varepsilon),\omega_o,r_s+\beta\Delta t]}{\psi_{\ell E}^{(1)}[n(r_s+\varepsilon),\omega_o,r_s+\beta\Delta t]} X_{\ell m E}^{(1)} e^{-i\omega_o\Delta t}$$

$$+ \sum_{\substack{\ell \\ m=\pm 1}} i^{\ell} \sqrt{\left(\frac{2\ell+1}{\ell(\ell+1)}\right)} \begin{pmatrix}\text{slowly varying function} \\ \text{of } r \text{ and } \ell\end{pmatrix}_{r=r_s}$$

$$\times \frac{\psi_{\ell H}^{(2)}[n(r_s+\varepsilon),\omega_o,r_s+\beta\Delta t]}{\psi_{\ell H}^{(1)}[n(r_s+\varepsilon),\omega_o,r_s+\beta\Delta t]} X_{\ell m H}^{(1)} e^{-i\omega_o\Delta t} \qquad (3C\text{-}17)$$

The similarity of these series with those treated by Hönl, Maue, and Westpfahl [15] and by Lam [11] is now evident and, using their results, we may write the sum for large r_s and very large r by inspection to within a multiplicative constant. That is,

$$\vec{E}_o \sim \frac{e^{ikr}}{r} e^{-2ikr_s \sin\frac{\theta}{2}} e^{-2ik(r_s)(\sin\frac{\theta}{2})\Delta t} e^{-i\omega_o\Delta t} [(\cos\phi)\hat{e}_\theta + (\sin\phi)\hat{e}_\phi] \qquad (3C\text{-}18)$$

and again we have an angle dependent frequency as did Lam. That is, (3C-18) may be written

$$\vec{E}_o \sim \frac{e^{ikr}}{r} e^{-2ikr_s \sin\frac{\theta}{2}} e^{-i\omega(\theta)\Delta t} [(\cos\phi)\hat{e}_\theta + (\sin\phi)\hat{e}_\phi] \qquad (3C\text{-}19)$$

where

$$\omega(\theta) = \omega_o[1 + 2n(r_s+\varepsilon)\,\beta(r_s)\,\cos(\tfrac{\pi-\theta}{2})]$$

which is consistent with geometrical optics.

Now consider the sum of the terms having frequencies given by (3C-12b). We note that these frequencies depend on ℓ in a rather complicated way due to the integral term in (3C-12b). This implies that the sum we seek will have an angle dependent frequency whose angular dependence is determined in an extremely complicated way by the detailed variation of n and β within the sphere. The theory is quite capable of determining this angular dependence; however, in this presentation we will make a simplifying assumption in order to demonstrate the physical interpretation of the sums without unnecessary complication. We assume that the sphere is homogeneous, which implies that the integral term in (3C-12b) is zero. We also assume, as we did in dealing with (3C-17), that r_s is large. We may now write (3C-12b) as follows:

$$R_1: \qquad \omega_\delta = \omega_o \{1 + 2\beta(r_s)[n(r_s+\varepsilon) - n(r_s-\varepsilon)]\} \tag{3C-20}$$

This is independent of ℓ which implies that the sum R_1 is zero. Had we not made the above assumption, the sum would have been first order in $\beta(r_s)$. The other two sums with which we are dealing--those corresponding to R_o and R_2--are zero order in $\beta(r_s)$ and therefore dominate the spectrum.

Finally, consider the sum of those terms with frequencies given by (3C-12c). Using our assumptions on n, β, and r_s we may write

$$R_2: \qquad \omega_\delta = \bar{\omega} + \beta(r_s)c \ \mathrm{Im}\frac{\partial}{\partial r}\left\{\ell n\,\frac{\psi_\ell^{(2)}[n(r_s-\varepsilon),\omega_o,r]}{\psi_\ell^{(1)}[n(r_s-\varepsilon),\omega_o,r]}\right\}_{r=r_s} \tag{3C-21}$$

where

$$\overline{\omega} = \omega_o \{1 + 2\beta(r_s)[n(r_s+\varepsilon) - n(r_s-\varepsilon)]\}$$

This is identical with (3C-12a) except for a change in the sign of the second term and the replacement of $n(r_s+\varepsilon)$ with $n(r_s-\varepsilon)$ and ω_o with $\overline{\omega}$. In analogy with (3C-19), then, the sum is

$$\vec{E}_2 \sim \frac{e^{ikr}}{r} e^{2ikr_s \sin\frac{\theta}{2}} e^{-i\omega(\theta)\Delta t}[(\cos \phi)\hat{e}_\theta + (\sin \phi)\hat{e}_\phi] \quad (3C\text{-}22)$$

where

$$\omega(\theta) = \overline{\omega}[1 - 2n(r_s-\varepsilon)\, \beta(r_s) \cos(\frac{\pi - \theta}{2})]$$

This corresponds to reflection from the inside of the surface of the sphere and agrees with geometrical optics.

The physical interpretation of R_o and the delta function contributions to R_1 and R_2 are shown in Figure 14.

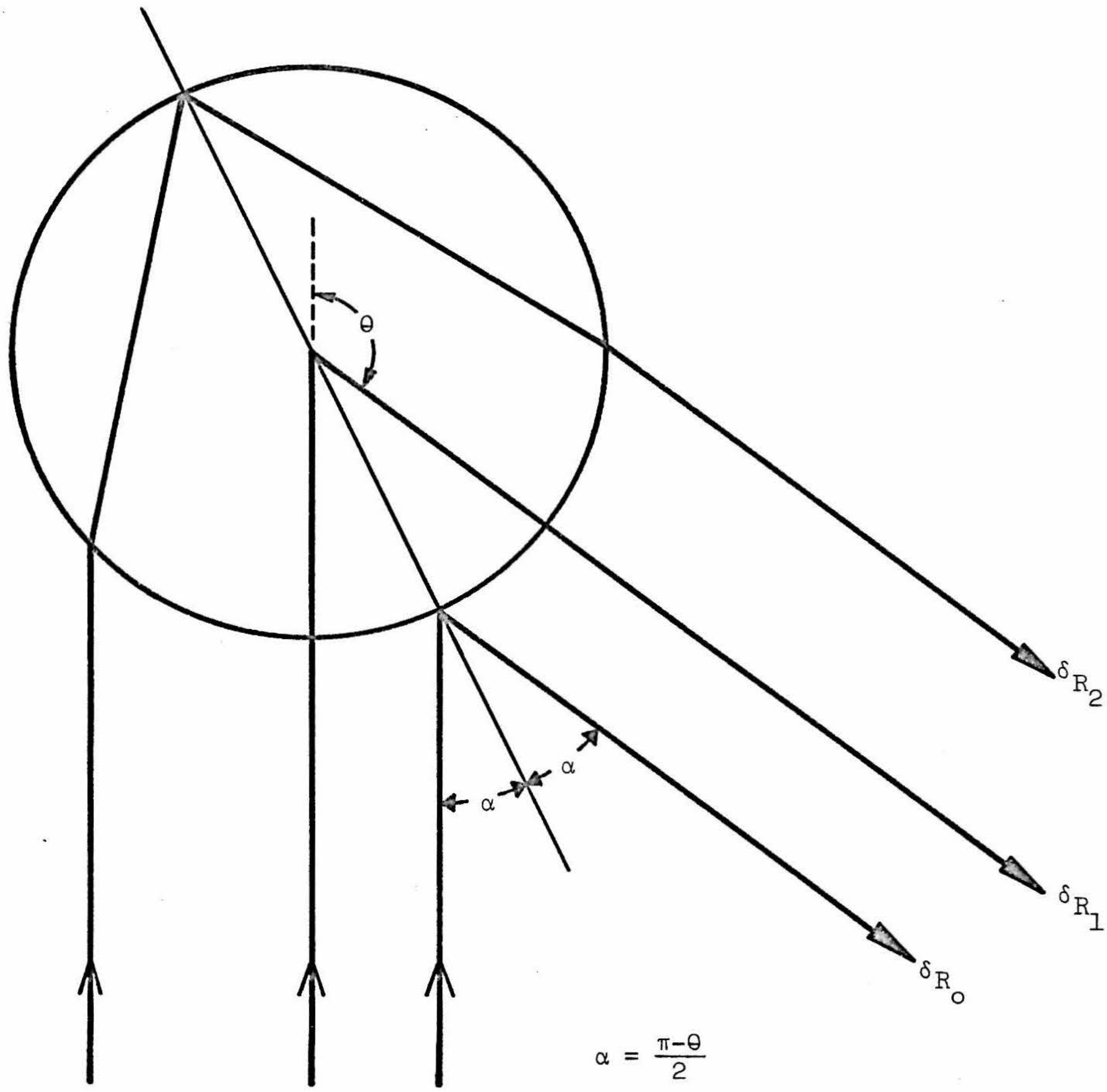

Figure 14. Physical interpretation of discrete spectral components

4. CONCLUDING REMARKS

This thesis presented a method of calculating the scattering which results when a plane electromagnetic wave falls on an expanding nonconducting obstacle. In particular, it dealt with scattering from expanding dielectric slabs and spheres. The problem was formulated by means of the invariant imbedding concept thus circumventing calculation of the fields inside the scatterer.

In dealing with the slab it was found that motion of the medium parallel to the direction of propagation of the wave is insufficient to cause a reflection in that an associated index gradient is necessary. This was found to be due to the fact that the wave impedance of a plane wave is invariant under Lorentz transformation parallel to the direction of propagation of the wave. The equations resulting from the invariant imbedding formulation of the problem were solved approximately to obtain explicit formulas for the spectra of the reflected waves. As an example, the width of the spectral lines produced in Brillouin scattering was computed. It was found that the formulation which was most convenient in the slab geometry was not easily extendable to the spherical geometry, so a second formulation of the slab problem was presented--a formulation more suitable for use in the sphere problem.

The invariant imbedding formulation was then applied to determine some of the properties of the scattered wave resulting when a plane electromagnetic wave falls on an expanding spherical scatterer. It was found that the salient features of the scattered spectrum and those features which give the most easily discernable information about the scatterer are the discrete components (delta functions). The frequencies of two of these components were calculated quasi-statically and to first order in the surface velocity for a large homogeneous expanding sphere, and it was shown that the frequencies have a physical interpretation which agrees with results from geometrical optics.

It is expected that the techniques presented here will prove to be of value in analyzing the data obtained in radar studies of the ionosphere and disturbances in the atmosphere. The spherical analysis may be of particular value in studying the dynamics of explosions by radar as there the motion is primarily radial. The presentation has been aimed at calculating the scattering given the evolution of the scatterer; however, the results were presented in a manner that gives some understanding of the inverse problem.

APPENDIX A

In this appendix the approximations involved in the time varying spectrum concept are discussed further.

In this treatment we are dealing with two essentially distinct time scales. The scale on which the fields oscillate, a scale largely determined by the frequency of the incident wave, and the scale on which the parameters of the scatterer vary, a scale independent of the frequency of the incident wave. Thus, it is reasonable to assume that the solution will vary on two time scales and that it may be written in the form

$$f[r(t),s(t)] \tag{A-1}$$

If we wish to describe this solution as a time varying spectrum function, we proceed as follows. First, expand $f[r(t),s(t)]$ in a Taylor series about a fixed point t_o ; that is,

$$f[r(t),s(t)] = f[r(t_o),s(t_o)] + \left\{\frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial s}\frac{ds}{dt}\right\}(t-t_o) + \cdots \tag{A-2}$$

Now, assume that the function $s(t)$ is associated with the parameter variation time scale and that it, therefore, varies much more slowly than does $r(t)$. Derivatives of $s(t)$ may then be neglected in favor of derivatives of $r(t)$. That is, f may be written approximately as

$$f[r(t),s(t)] \simeq f[r(t_o),s(t_o)] + \frac{\partial f}{\partial r}\frac{dr}{dt}(t-t_o) + \cdots$$

$$= f[r(t),s(t_o)] \qquad \text{(A-3)}$$

Fourier transformation of (A-3) results in

$$F[\omega,s(t_o)] = \int_{-\infty}^{\infty} f[r(t),s(t_o)]\, e^{-i\omega t}\, dt \qquad \text{(A-4)}$$

which is a "time varying spectrum". It has been stated that the condition for validity of this concept is

$$\left|\frac{ds}{dt}\right|_{max} << \left|\frac{dr}{dt}\right|_{max} \qquad \text{(A-5)}$$

Technically this is not quite correct because it admits the cases where $r(t)$ varies slowly everywhere but in a few isolated regions where it changes rapidly. (Since in our case $r(t)$ is more or less uniformly oscillatory, the above situation does not arise.) To account for this we state that (A-5) must hold on every interval of time longer than a few cycles of input signal--say five cycles.

By a property of Fourier transforms we find that

$$i\omega F[\omega,s(t_o)] = \int_{-\infty}^{\infty} \frac{\partial f}{\partial r}\frac{dr}{dt}\, e^{-i\omega t}\, dt \qquad \text{(A-6)}$$

That is,

$$\frac{\dfrac{\partial F}{\partial t_o}}{i\omega F} = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{\partial f}{\partial s}\frac{ds}{dt_o}\, e^{-i\omega t}\, dt}{\displaystyle\int_{\infty}^{\infty} \frac{\partial f}{\partial r}\frac{dr}{dt}\, e^{-i\omega t}\, dt} \qquad \text{(A-7)}$$

Since $s(t)$ is slowly varying compared to $r(t)$, the expression on the right side of (A-7) is very small and we have

$$\left| \frac{1}{F} \frac{\partial F}{\partial t_o} \right| << \omega \tag{A-8}$$

In this presentation the problem was solved for each fixed value of the parameters of the scatterer and each of the solutions thus obtained was called the solution at the time when the parameters of the scatterer took on the values used to obtain it (necessitating conditions (2C-1b)). This is the fully quasistatic approach, a first approximation. The corresponding second approximation is known as the WKB approximation [14] and these two approximations are the first two terms of a series which could, in principle, be carried to any desired degree of accuracy [16]. The exact solution is that which was called $g(t)$.

Thus we see that (2C-1a), (2C-1b), and (2D-13) all have essentially the same meaning.

APPENDIX B

In this appendix equation (2C-7) is simplified somewhat by Fourier transformation.

We note that the solution (2D-9) can be written as a function of $\omega-\omega_o$ and we redefine $R(t,\omega_{out},\omega_{in},z)$ to be $R(t,\omega_{out}-\omega_{in},z)$. Equation (2C-7) then becomes

$$\frac{\partial R}{\partial z} + (\omega_{out}+\omega_{in})\, n \frac{\partial \beta}{\partial z} \frac{\partial R}{\partial \omega_{out}}$$

$$= \frac{1}{2n}\frac{\partial n}{\partial z}\left\{2\pi\delta(\omega_{out}-\omega_{in}) - \int_{-\infty}^{\infty} R(t,\bar{\omega}-\omega_{in},z)\, R(t,\omega_{out}-\bar{\omega},z)\, \frac{d\bar{\omega}}{2\pi}\right.$$

$$- \frac{in}{c}(\omega_{out}+\omega_{in})\, R + 2n\frac{\partial \beta}{\partial z} R \qquad \text{(B-1)}$$

Fourier transformation with respect to ω_{out} yields

$$\frac{\partial \hat{R}}{\partial z} - n\left(1 + y\frac{\partial \beta}{\partial z}\right)\frac{\partial \hat{R}}{\partial y} = (1-\hat{R}^2)\frac{1}{2n}\frac{\partial n}{\partial z}$$

$$+ \left[3n\frac{\partial \beta}{\partial z} - \frac{2i\omega_{in}}{c}\, n\left(1 + y\frac{\partial \beta}{\partial z}\right)\right]\hat{R} \qquad \text{(B-2)}$$

where

$$\hat{R}(t,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(t,\omega_{out}-\omega_{in},z)\, e^{-i\left(\frac{\omega_{out}-\omega_{in}}{c}\right)y}\, d\omega_{out}$$

This is the simplified invariant imbedding equation for the reflection function to first order in β.

We now apply this equation to the slab of Figure 3. Handling (B-2) by the method of characteristics and assuming that the slab under consideration is sufficiently tenuous to permit neglecting the nonlinear term, we obtain the following two ordinary differential equations:

$$\frac{dy}{dz} = -n(1 + y \frac{\partial\beta}{\partial z}) \tag{B-3a}$$

$$\frac{d\hat{R}}{dz} = \frac{1}{2n}\frac{\partial n}{\partial z} + [3n \frac{\partial\beta}{\partial z} - \frac{2i\omega_{in}}{c} n(1 + y \frac{\partial\beta}{\partial z})] \hat{R} \tag{B-3b}$$

Solving (B-3a) yields

$$y(z) = \left[- \int_b^z n e^{\int_b^z n \frac{\partial\beta}{\partial z'} dz'} dz + y(b) \right] e^{-\int_b^z n \frac{\partial\beta}{\partial z'} dz'} \tag{B-4}$$

where $y(b)$ is the value of y corresponding to b on the characteristic. Substituting (B-4) into (B-3b) gives

$$\frac{d\hat{R}}{dz} = \frac{1}{2n}\frac{\partial n}{\partial z} + \{3n \frac{\partial\beta}{\partial z} - \frac{2i\omega_{in}}{c} n[1 + y(z) \frac{\partial\beta}{\partial z}]\} \hat{R} \tag{B-5}$$

Equation (B-5) is a first order linear differential equation which can be easily solved. The solution evaluated at the left boundary of the slab $z = a$ is

$$\hat{R}(t,y,a) = \int_b^a \frac{1}{2n}\frac{\partial n}{\partial z} e^{-\int_a^z I dz} dz + \hat{R}[t,y(b),b]\, e^{\int_b^a I\, dz} \tag{B-6}$$

where

$$I = 3n\frac{\partial\beta}{\partial z} - \frac{2i\omega_o}{c}\, n\left\{1 + y(b)\frac{\partial\beta}{\partial z}\, e^{-\int_b^z n \frac{\partial\beta}{\partial z'} dz'} - \frac{\partial\beta}{\partial z}\int_b^z n\, e^{\int_b^z n \frac{\partial\beta}{\partial z'} dz'} dz\; e^{-\int_b^z n \frac{\partial\beta}{\partial z'} dz'}\right\}$$

Solving (B-4) at $z = a$ for $y(b)$ as a function of y and a yields

$$y(b) = \left[y + \int_b^a n\, e^{\int_a^z n \frac{\partial\beta}{\partial z'} dz'} dz\right] e^{\int_b^a n \frac{\partial\beta}{\partial z'} dz'} \tag{B-7}$$

Substituting in (B-6) we obtain

$$\hat{R}(t,y,a) = \int_b^a \frac{1}{2n}\frac{\partial n}{\partial z}\, e^{-\int_a^z 3n \frac{\partial\beta}{\partial z'} dz'}\; e^{\frac{2i\omega_o}{c}\left[1 - e^{-\int_a^z n \frac{\partial\beta}{\partial z} dz'}\right] y}$$

$$e^{\frac{2i\omega_o}{c}\int_a^z n\, e^{\int_a^z n \frac{\partial\beta}{\partial z'} dz'} dz\; e^{-\int_a^z n \frac{\partial\beta}{\partial z'} dz'}} + \hat{R}[t,y(b),b]\; e^{\int_b^a I\, dz} \tag{B-8}$$

We now make the following change of variable:

$$\omega(z) = \omega_o - 2\omega_o\left[1 - e^{-\int_a^z n \frac{\partial\beta}{\partial z'} dz'}\right] \tag{B-9}$$

This necessitates division of the slab into regions in which this transformation is one to one. For simplicity in this discussion, let us consider a slab in which the transformation is one to one over its entire thickness. Proceeding with the change of variable and making use of the W function defined in Figure 5, we obtain

$$\hat{R}(t,y,a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2\pi}{\omega_o}\right) \left. \frac{\frac{-1}{2n}\frac{\partial n}{\partial z}}{2n \frac{\partial \beta}{\partial z}} \right|_{z=z(\omega)} e^{-\int_a^{z(\omega)} 2n \frac{\partial \beta}{\partial z'} dz'}$$

$$\times\ e^{\frac{i(\omega+\omega_o)}{c} \int_a^{z(\omega)} n\, e^{\int_a^z n \frac{\partial \beta}{\partial z'}} dz}\ e^{-\frac{i}{c}(\omega-\omega_o)y}\ W[\omega;\omega(b),\omega(a)]d\omega$$

$$+ \hat{R}[t,y(b),b]\ e^{-\int_a^b 3n \frac{\partial \beta}{\partial z'}}\ e^{-\frac{i}{c}[\omega(b)-\omega_o]y}$$

$$\times\ e^{\frac{i}{c}[\omega(b) + \omega_o] \int_a^b n\, e^{\int_a^z n \frac{\partial \beta}{\partial z'} dz'} dz} \qquad \text{(B-10)}$$

Comparison with the definition of $\hat{R}$ in (B-2) shows

$$R(t,\omega-\omega_o,a) = - \frac{2\pi}{\omega_o} \left. \frac{\frac{1}{2n}\frac{\partial n}{\partial z}}{2n \frac{\partial \beta}{\partial z}} \right|_{z=z(\omega)} e^{-\int_a^{z(\omega)} 2n \frac{\partial \beta}{\partial z'} dz'}$$

$$\times\ e^{\frac{i}{c}(\omega+\omega_o) \int_a^{z(\omega)} n\, e^{\int_a^z n \frac{\partial \beta}{\partial z'} dz'} dz}\ W(\omega;\omega(b),\omega(a)]$$

$$+ R[t,(\omega-\omega(b))e^{\int_a^b n \frac{\partial \beta}{\partial z'}},b]\ e^{-\int_a^b 2n \frac{\partial \beta}{\partial z'} dz'}$$

$$\times\ e^{\frac{i}{c}[2\omega(b)-\omega + \omega_o] \int_a^b n\, e^{\int_a^z n \frac{\partial \beta}{\partial z'} dz'} dz}$$

which is identical (to first order in β) with (2D-9) except for discrepancies in the phase of the two terms, discrepancies which are first order in β. These discrepancies represent the error in assuming that the solution depends only on the difference between the input and output frequencies.

Thus the condition under which this approach is applicable is that accuracy to first order in β in the phase of the solution be unnecessary.

APPENDIX C

In this appendix the invariant imbedding equation for the transmission function of an expanding slab is derived.

Figure 15 depicts the transmission problem. The laboratory frame here is associated with the right boundary of the slab rather than with the left boundary as in the derivation of the equation for the reflection function. It is assumed that the transmission through that portion of the slab to the left of $\zeta-\Delta\zeta$ is known in a Lorentz frame moving with the fluid at $\zeta-\Delta\zeta$ when in this frame the space to the right of $\zeta-\Delta\zeta$ is filled with a homogeneous stationary fluid of index $n(\tau,\zeta-\Delta\zeta)$. A thin slab of fluid of index $n(\tau,\zeta-\Delta\zeta)$ having a constant velocity gradient throughout its thickness is added at $\zeta-\Delta\zeta$ and extends to ζ. We must calculate the transmission function at ζ (i.e., the transmission through the composite slab) in a frame moving with the fluid at ζ under the assumption that in this frame the region to the right of ζ is homogeneously filled with a stationary fluid of index $n(\tau,\zeta)$. That is, for this calculation a Lorentz frame moving with the fluid at position z at time t (or correspondingly position ζ and time τ in the comoving frame) will be used. Figure 16 shows the situation as seen in this comoving frame. Again, it is our intention to take the limit as $\Delta\zeta$ approaches zero so again calculations will be done only to first order in $\Delta\zeta$.

A unit amplitude, linearly polarized, monochromatic, plane wave of frequency ω_o in the frame of the boundary at $z = a$ is assumed to

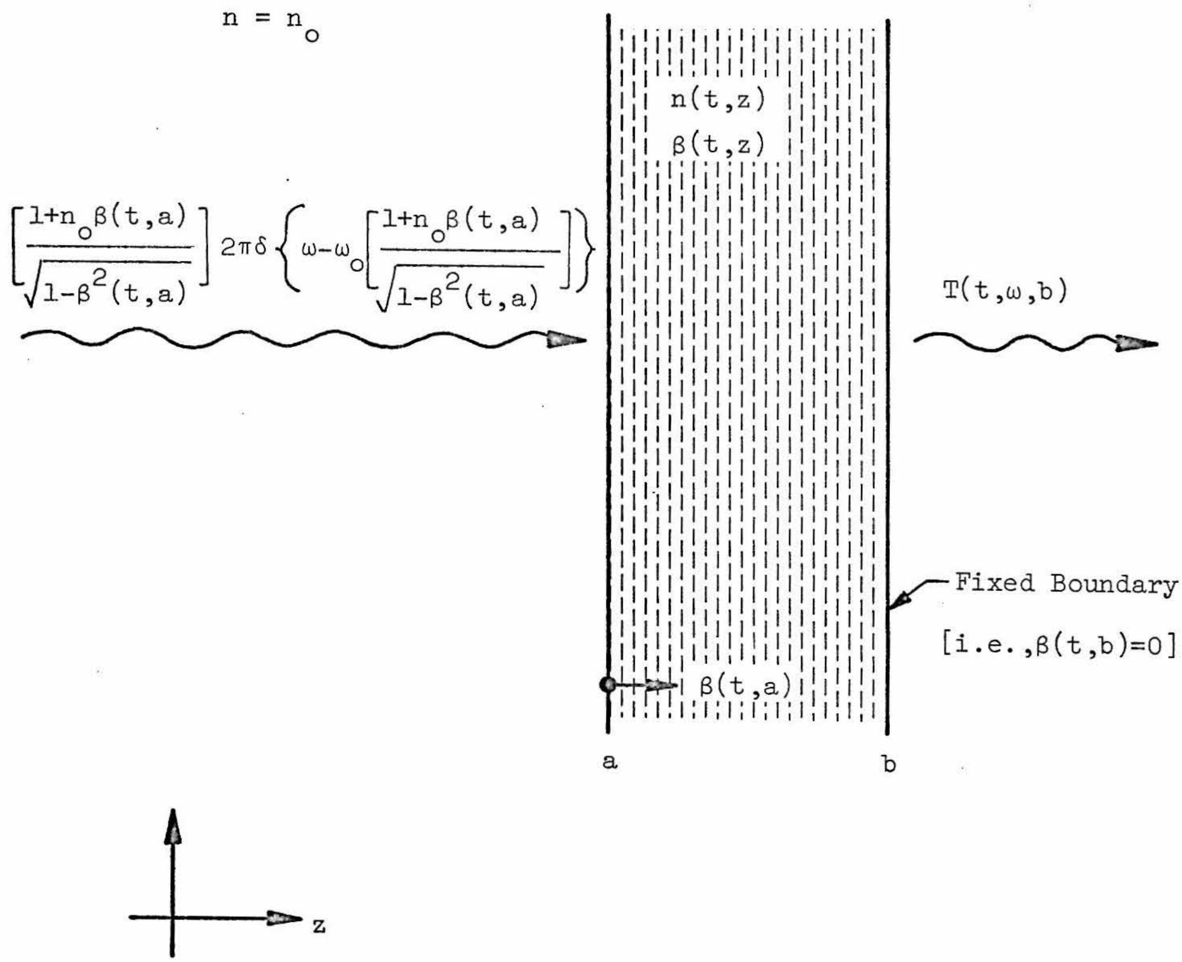

Figure 15. Transmission through the general slab

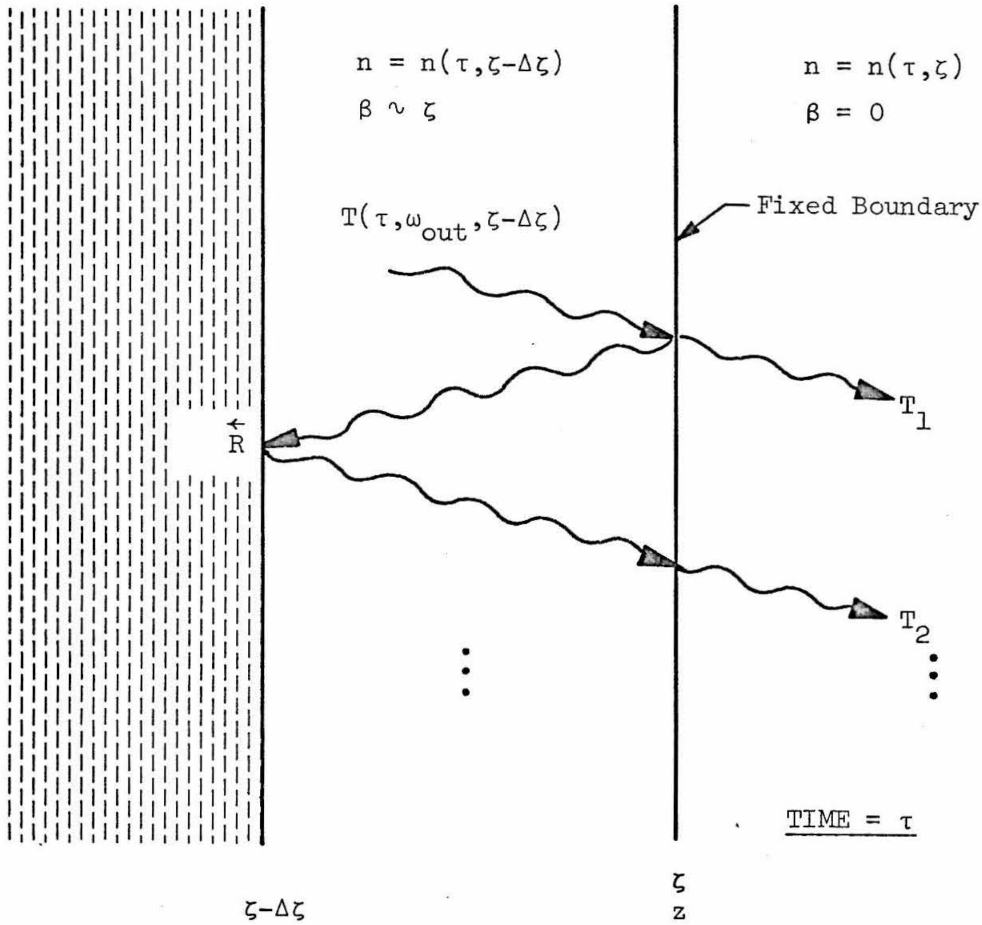

Figure 16. Configuration for derivation of the invariant imbedding equation for the transmission function

be normally incident from the left on the composite slab of Figure 16. It is transmitted through the slab to the left of $\zeta-\Delta\zeta$ and when it appears at $\zeta-\Delta\zeta$ it is described in the local comoving frame by the spectral density function $T(t,\omega_{out},\zeta-\Delta\zeta)$. It is then transmitted through the added layer and across the boundary at ζ to become T_1. Part of it is reflected by the boundary at ζ, transmitted back through the added layer, reflected from the slab to the left of $\zeta-\Delta\zeta$ (whose reflection function has been calculated to be $\vec{R}(\tau,\omega_{out},\omega_{in},\zeta-\Delta\zeta)$), transmitted back through the added layer and across the boundary at to become T_2. Higher order T's are defined similarly.

Equation (2B-4b) leads to

$$T_1 = T(\tau,\omega',\zeta-\Delta\zeta)e^{i\frac{\omega' n}{c}\Delta\zeta}\left[\frac{2n(\tau,\zeta-\Delta\zeta)}{n(\tau,\zeta)+n(\tau,\zeta-\Delta\zeta)}\right](1-n\Delta\beta) \qquad \text{(C-1a)}$$

where

$$\omega' = (1+n\Delta\beta)\omega_{out} \qquad \text{and} \qquad \Delta\beta = \frac{\partial\beta}{\partial\zeta}\Delta\zeta$$

Thus to first order in $\Delta\zeta$ we have

$$T_1 = T = \left[\frac{\partial T}{\partial\zeta} - n\omega_{out}\frac{\partial\beta}{\partial\zeta}\frac{\partial T}{\partial\omega_{out}} + \frac{T}{2n}\frac{\partial n}{\partial\zeta}\frac{i\omega_{out}n}{c}T + n\frac{\partial\beta}{\partial z}T\right]\Delta\zeta \qquad \text{(C-1b)}$$

Again the factors $(1-n\Delta\beta)$ and $(1+n\Delta\beta)$ arise because T_1 and T are defined in different Lorentz frames. In a similar manner we obtain

$$T_2 = -\frac{1}{2n}\frac{\partial n}{\partial\zeta}\int_{-\infty}^{\infty} T(\tau,\bar{\omega},\zeta)\,\overleftarrow{R}(\tau,\omega_{out},\bar{\omega},\zeta)\,\frac{d\bar{\omega}}{2\pi}\,\Delta\zeta \qquad \text{(C-2)}$$

In the above expression $\overleftarrow{R}(t,\omega_{out},\omega_{in},\zeta-\Delta\zeta)$ is the reflection function for that portion of the slab to the left of the plane $\zeta-\Delta\zeta$ for waves incident from the right and it is defined in a Lorentz frame comoving with the fluid at $\zeta-\Delta\zeta$. T_j for $j > 2$ is of second or higher order in $\Delta\zeta$ and is therefore negligible in this calculation. Hence

$$T = T_1 + T_2 \tag{C-3}$$

Substitution of (C-1b) and (C-2) into (C-3) yields

$$\frac{\partial T}{\partial z} - \omega_{out}\, n \frac{\partial \beta}{\partial z} \frac{\partial T}{\partial \omega_{out}} = [\frac{i\omega_{out}\eta}{c} - n\frac{\partial \beta}{\partial z} - \frac{1}{2n}\frac{\partial n}{\partial z}]\, T$$

$$- \frac{1}{2n}\frac{\partial n}{\partial z}\int_{-\infty}^{\infty} T(t,\bar{\omega},z)\; R(t,\omega_{out},\bar{\omega},z)\, \frac{d\bar{\omega}}{2\pi} \tag{C-4}$$

where we have transformed as before from local comoving coordinates to laboratory frame coordinates. This is the invariant imbedding equation for the transmission function to first order in β . Assuming that $\overleftarrow{R}$ is a known function, this equation is a first order linear partial differential equation. It is to be integrated from the left boundary of the slab where the transmission is known to the right boundary where it is to be found.

APPENDIX D

In this appendix the relevant properties of the functions P and Q in equations (3B-3) are discussed.

First, we note that P and Q must reduce to the corresponding expressions R_{11} and R_{12} given by van der Pol and Bremmer [17] when $\beta \to 0$. Hence, we may write to first order in

$$P(r + \beta c\Delta t) = R_{11}(r + \beta c\Delta t) + \beta P_1(r) \tag{D-1a}$$

$$Q(r + \beta c\Delta t) = R_{12}(r + \beta c\Delta t) + \beta Q_1(r) \tag{D-1b}$$

The second term in each of these expressions is time independent and so to study the time dependence of R and T in (3B-3) we need only study the time dependence of R_{11} and R_{12} with the argument $r + \beta ct$. Now the van der Pol and Bremmer expressions involve only the logarithmic derivatives of the Hankel functions. These logarithmic derivatives vary much more slowly than do the Hankel functions themselves, except near the origin for small values of ℓ. However, near the origin the velocity is nearly zero so this does not pose a problem. The radial dependence of the logarithmic derivatives for $\nu \neq r + \beta c\Delta t$ may be ascertained by making use of the asymptotic expressions for the Hankel functions [15] which are

$$H_\nu^{(j)}(x) \simeq \sqrt{\frac{2}{\pi}}\,(x^2-\nu^2)^{-1/4} \exp -(-1)^j\, i\left(\sqrt{x^2-\nu^2} - \nu \cos^{-1}\frac{\pi}{x}\right) - \frac{\pi}{4} \qquad \nu < x \tag{D-2a}$$

$$H_\beta^{(j)}(x) \simeq (-1)^j \sqrt{\frac{2}{\pi}}\,(\nu^2-x^2)^{-1/4} \exp \nu\, \ell n\, \frac{\nu+\sqrt{\nu^2-x^2}}{x} - \nu^2 - x^2 \qquad \nu > x \tag{D-2b}$$

where ν is a half integer. We find that

$$\frac{d}{dx}\ell n[H_\nu^{(j)}] = -[\frac{1}{2}\frac{x}{(x^2-\nu^2)^{5/4}} + i(-1)^j\sqrt{1-\frac{\nu^2}{x^2}}] \qquad \nu < x \qquad \text{(D-3a)}$$

$$\frac{d}{dx}\ell n[H_\nu^{(j)}] = -[\frac{1}{x} - \frac{1}{2}\frac{x}{(\nu^2-x^2)^{5/4}} + \frac{x}{\nu^2-x^2} + \frac{x}{\sqrt{\nu^2-x^2}+(\nu^2-x^2)}]$$

$$\nu > x \qquad \text{(D-3b)}$$

where $x = \int_0^r k_{eff}^{(j)}\, dr$ and $\nu = \ell - \frac{1}{2}$. For $\nu < x$, (D-3a) varies slowly compared with (D-2a). For $\nu > x$ both the logarithmic derivative and the Hankel function vary slowly (i.e., are not oscillatory) and hence neither will affect the Doppler shift in our analysis.

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